The BCS Critical Temperature at High Density

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Abstract

We investigate the BCS critical temperature T_c in the high-density limit and derive an asymptotic formula, which strongly depends on the behavior of the interaction potential V on the Fermi-surface. Our results include a rigorous confirmation for the behavior of T_c at high densities proposed by Langmann et al. (Phys Rev Lett 122:157001, 2019) and identify precise conditions under which superconducting domes arise in BCS theory.

Keywords BCS theory · Critical temperature · Superconducting domes

Mathematics Subject Classification 81Q10 · 46N50 · 82D55

1 Introduction

The Bardeen–Cooper–Schrieffer (BCS) gap equation [2]

$$\Delta(p) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{V}(p-q) \frac{\Delta(q)}{E_{\Delta,\mu}(q)} \tanh\left(\frac{E_{\Delta,\mu}(q)}{2T}\right) dq, \qquad (1)$$

with dispersion relation $E_{\Delta,\mu}(p) = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}$, has played an important role in physics since its introduction. The function Δ is interpreted as the order parameter describing paired fermions (Cooper pairs) interacting via the local pair potential $2V$, which we assume to be integrable, i.e. $V \in L^1(\mathbb{R}^3)$. In this case, $\hat{V}(p) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} V(x) e^{-ip \cdot x} dx$ denotes its Fourier transform. The positive parameters T and μ are the temperature and the chemical potential, respectively, where the latter controls the density of fermions. Whenever the temperature T is below a certain critical temperature T_c (see Definition 1), the gap equation (1) admits non-

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trivial solutions, above it does not. Physically, this corresponds to the system being in a superconducting state ($T < T_c$) or a normal state ($T \geq T_c$).

BCS theory has previously been studied in the weak-coupling limit [6,9] and low-density limit [10,18]. In the weak-coupling limit one considers a potential λV for a fixed potential V for small coupling constant $\lambda \to 0$. In this limit, it was shown by Hainzl and Seiringer [9] that the critical temperature satisfies $T_c \sim A \exp(-B/\lambda)$ for explicit constants $A, B > 0$. In the low-density limit, $\mu \to 0$, it is shown, again by Hainzl and Seiringer [10], that $T_c \sim \mu A \exp(-B/\sqrt{\mu})$ for some (different) explicit constants $A, B > 0$ (see Eq. (9)).

In this paper we are interested in the critical temperature for the existence of non-trivial solutions of the BCS gap equation (1) in the high-density limit, i.e. $\mu \to \infty$. Studying the high-density limit of the critical temperature is especially relevant for explaining superconducting domes [3,5,14,20,21,24], i.e. a non-monotonic $T_c(\mu)$ exhibiting a maximum value at finite μ and going to zero for large μ. In a recent paper [16], the authors claim the *ubiquity of superconducting domes in BCS theory*, but only for pure s-wave superconductivity (i.e. angular momentum $\ell = 0$, see Remark 3). Their result disproves the conventional wisdom, that the presence of a superconducting dome necessarily indicates some kind of exotic superconductivity, e.g. resulting from competing orders. BCS theory containing a non-monotonic behavior of $T_c(\mu)$ is in particular relevant for understanding superconducting domes in doped band insulators [24] and magic-angle graphene [3], where no competing orders occur, and thus a more conventional explanation is desirable.

There is a simple physical picture arising from an interplay of length scales, that explains the ubiquitous appearance of superconducting domes (see [16]). If the effective range ξ of the interaction is much smaller than the mean interparticle distance $\mu^{-1/2}$, i.e. $\xi \ll \mu^{-1/2}$, the critical temperature T_c increases by increasing μ as predicted by standard BCS theory [2] and rigorously justified in [10]. At high densities, i.e. if $\xi \gg \mu^{-1/2}$, the pairing of electrons near the Fermi surface (with approximately opposite momenta), which is responsible for the superconducting behavior, becomes weaker with increasing μ due to the decay of the interaction in Fourier space, suppressing T_c towards zero. Therefore, at some intermediate density, where $\xi \sim \mu^{-1/2}$, a superconducting dome arises. This simple argument is reflected in our results by the presence of the operator $\mathcal{V}_\mu$, defined in Eq. (2), acting on functions on the (rescaled) Fermi surface.

Our results in Sect. 2 are threefold: first, we confirm a proposed asymptotic formula from [16] for the critical temperature at high densities for s-wave superconductivity (to leading order) by proving a more general result for radially symmetric interaction potentials V (Theorem 2); second, we provide a counterexample, showing that the assumptions on V from [16] are not quite sufficient to conclude a non-monotonic behavior of T_c and need to be slightly strengthened (Proposition 4); third, we use these strengthened assumptions to improve the asymptotics obtained in Theorem 2 to second order with the aid of perturbation theory, and obtain an analogous formula to the ones proven in the weak-coupling and low-density limit (Theorem 7). All proofs are given in Sect. 3.

2 Main Results

2.1 Preliminaries

It was proven in [8] (see also [12] for a more recent review) that the critical temperature for the existence of non-trivial solutions of the BCS gap equation (1) can be characterized as follows.

Definition 1 (*Critical Temperature*) Let $\mu > 0$, $V \in L^{3/2}(\mathbb{R}^3)$ be real-valued and $K_{T,\mu}$ denote the multiplication operator in momentum space

$$K_{T,\mu}(p) = \frac{|p^2 - \mu|}{\tanh\left(\frac{|p^2 - \mu|}{2T}\right)} \, .$$

The critical temperature for the BCS gap equation (1) is given by

$$T_c = \inf \left\{ T > 0 \mid K_{T,\mu}(p) + V(x) \geq 0 \right\} \, .$$

One might think of the operator $K_{T,\mu}(p) + V(x)$ as the Hessian in the BCS functional of superconductivity at a normal state (see [12]), where the positivity corresponds to the "stability" of this normal state, which is directly related to the existence of a non-trivial solution of the BCS gap equation (1). Note that the continuous spectrum of $K_{T,\mu}$ starts at $2T$ and thus T_c is well defined by Sobolev's inequality [19, Thm. 8.3] since $K_{T,\mu} \sim p^2$ for large $|p|$.

Moreover, note that $K_{T,\mu}$ takes its minimum value $2T$ on the codim-1 submanifold $\{p^2 = \mu\}$. Thus, similarly to the weak coupling situation [6] and as pointed out by Laptev, Safronov and Weidl [17] (see also [11]), the spectrum of $K_{T,\mu} + V$ is mainly determined by the behavior of V near $\{p^2 = \mu\}$, i.e. the Fermi sphere. More precisely, as emphasized in the introduction, a crucial role for the investigation of T_c in the high-density limit is played by the (rescaled) operator $\mathcal{V}_\mu : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ where

$$\left(\mathcal{V}_\mu u\right)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{S}^2} \hat{V}(\sqrt{\mu}(p - q)) u(q) \, d\omega(q) \, . \tag{2}$$

Here $d\omega$ denotes the uniform (Lebesgue) measure on the unit sphere $\mathbb{S}^2$. The pointwise evaluation of $\hat{V}$ (and thus also on a codim -1 submanifold) is well defined since $\hat{V}$ is continuous for $V \in L^1(\mathbb{R}^3)$. See Remark 9 for a discussion of the assumption $V \in L^1(\mathbb{R}^3)$ (cf. also [4]). The lowest eigenvalue of $\mathcal{V}_\mu$, which we denote by

$$e_\mu = \inf \text{spec} \, \mathcal{V}_\mu$$

will be of particular importance. Note, that $\mathcal{V}_\mu$ is a compact operator (so $e_\mu \leq 0$), which is in fact trace class (see the argument above Eq. (3.2) in [6]) with

$$\text{tr}(\mathcal{V}_\mu) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} V(x) dx \, .$$

The case $e_\mu < 0$ will be important for our main results as it corresponds to an attractive interaction between (some) electrons on the Fermi sphere. Since $\mathcal{V}_\mu$ is trace class, a sufficient condition for $e_\mu < 0$ is that the trace of $\mathcal{V}_\mu$ is negative, i.e. $\int V < 0$. Moreover, by considering a trial function that is concentrated on two small sets on the rescaled Fermi sphere $\mathbb{S}^2$ separated by a distance $|p| < 2$, one can easily see that $e_\mu < 0$ if $|\hat{V}(p)| > \hat{V}(0)$ for some $|p| < 2\sqrt{\mu}$.

In this work, we restrict ourselves to the special case of radial potentials V depending only on $|x|$, where the spectrum of $\mathcal{V}_\mu$ can be determined more explicitly (see, e.g., Sect. 2.1 in [6]). Indeed, if V is radially symmetric, the eigenfunctions of $\mathcal{V}_\mu$ are spherical harmonics and the corresponding eigenvalues are

$$\frac{1}{2\pi^2} \int_{\mathbb{R}^3} V(x) \left(j_\ell(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x \,, \tag{3}$$

with $\ell \in \mathbb{N}_0$ and where j_ℓ denotes the ℓth-order spherical Bessel function. A few important properties of the spherical Bessel functions used in our proofs are collected in Proposition 16. By Eq. (3), the lowest eigenvalue e_μ is thus given by

$$e_\mu = \frac{1}{2\pi^2} \inf_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} V(x) \left(j_\ell(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x \,.$$

If additionally $\hat{V} \leq 0$, the minimal eigenvalue is attained for the constant eigenfunction (i.e. the spherical harmonic with $\ell = 0$) by the Perron–Frobenius Theorem and we thus have the more concrete expression

$$e_\mu = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} V(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \,. \tag{4}$$

We refer to Remark 8 for a discussion of the radiality assumption on V.

2.2 Results

As desribed in the introduction, our results are threefold: First, we show an asymptotic formula for radial potentials V (Theorem 2), including the rigorous confirmation of the result from [16] to leading order. Afterwards, we provide a counterexample showing that the assumptions made in [16] are not quite sufficient to conclude a non-monotonic behavior of T_c, i.e. a superconducting dome (Proposition 4). Finally, by slightly strengthening the assumptions on V, we provide an asymptotic formula for the critical temperature valid to second order (Theorem 7). All proofs are given in Sect. 3.

Theorem 2 *Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ be real-valued and radially symmetric. Assume that there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ we have $e_\mu < 0$. Then $T_c(\mu) > 0$ for all sufficiently large μ and*

$$\lim_{\mu \to \infty} \sqrt{\mu}\, e_\mu \, \log \frac{\mu}{T_c} = -1 \,. \tag{5}$$

Or in other words, we have the asymptotic behavior

$$T_c = \mu \, e^{(1+o(1))/(\sqrt{\mu}e_\mu)} \tag{6}$$

in the limit of large μ. Note, that the right hand side is the same formula as in the weak-coupling case [6,9] but we have coupling parameter $\lambda = 1$.

Remark 3 (Connection to the result from [16]) Assume that $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ is real-valued, radially symmetric and additionally satisfies $\hat{V} \leq 0$ and $\hat{V}(0) < 0$ (the latter implies that $e_\mu < 0$ for all $\mu > 0$). Note that these conditions, which are identical to the ones required in [16], are included in the more general conditions of Theorem 2. Then we have, using the notation from [16], that

$$\sqrt{\mu}e_\mu = \frac{\sqrt{\mu}}{2\pi^2} \int_{\mathbb{R}^3} V(x) \frac{\sin^2(\sqrt{\mu}|x|)}{\mu|x|^2} dx = \frac{1}{4\pi^2} \frac{f_{-2V}(4\mu)}{4\sqrt{\mu}} =: -\lambda \,,$$

where the first equality follows by Eq. (4) and after inserting the definition of the function f_{-2V} from [16], the second equality is a simple computation using Fubini. By means of Theorem 2, we thus confirmed the validity of Eq. (6) in [16] in the high-density limit to leading order, i.e.

$$T_c = \mu \, e^{-(1+o(1))/\lambda} \,.$$

In full generality, the asymptotic formula proposed in Eq. (6) in [16] reads

$$T_c = \frac{2e^\gamma}{\pi} \mu \exp\left(-\frac{1}{\lambda} + \sum_{n=0}^{\infty} a_n \lambda^n \right) \,,$$

where $\gamma \approx 0.577$ denotes the Euler–Mascheroni constant and $(a_n)_{n \geq 0}$ is a sequence of explicit constants determined by an iterative procedure. The quantity λ is understood as an intrinsic small parameter which encodes either a weak-coupling, low-density, or high-density limit, or an appropriate combination.

In order to obtain a meaningful asymptotic formula of the critical temperature at high densities in a rigorous way, the question to be addressed now is the behavior of $\sqrt{\mu}e_\mu$ in the limit $\mu \to \infty$. In the following Proposition we present a special family of interaction potentials (V_α) showing that the conditions of Theorem 2 (which include the more restricted conditions from [16]) not necessarily lead to a non-monotonic behavior of T_c as claimed in [16], since $|\sqrt{\mu}e_\mu| \gg \log(\mu)^{-1}$ in the limit of large μ for this family of potentials. More precisely, the $L^{3/2}(\mathbb{R}^3)$-condition, which essentially concerns the behavior of the interaction potential near the origin, is not quite sufficient to obtain a dome-shaped behavior of $T_c(\mu)$. Since the potentials (V_α) are perfectly well behaved away from the origin and decay rapidly at infinity, they illustrate the significance of the behavior of interaction potentials near the origin for the asymptotics of the critical temperature. It is natural that the critical temperature is sensitive to the short range behavior of the interaction potential, since the interparticle distance as the

physically relevant length scale that depends on the particle density tends to zero in the high-density limit.

Proposition 4 *Let $\alpha \in (1/3, 1/2)$ and set*

$$V_\alpha(x) = -\frac{\exp(-|x|)}{|x|^2 \left(\log^2(|x|) + 1\right)^\alpha}.$$

Then the critical temperature T_c associated with $K_{T,\mu} + V_\alpha$ approaches infinity as $\mu \to \infty$.

Our observations from Proposition 4 lead to the following definition of "admissible potentials", that are slightly better behaved at the origin and, in particular, allow for an analysis of e_μ (and also all the other eigenvalues of $\mathcal{V}_\mu$) by requiring certain definiteness conditions of V (cf. Lemmas 12 and 13).

Definition 5 (*Admissible Potentials*) Let $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ be a real-valued radial function and define

$$s_\pm^* := \sup \left\{ s \geq 0 : |\cdot|^{-s} V_\pm \in L^1(\mathbb{R}^3) \right\} \qquad s^* := \min(s_+^*, s_-^*), \qquad (7)$$

where $V_\pm = \max\{\pm V, 0\}$ are the positive and negative parts of V. We call V an *admissible potential* if the following is satisfied:

(a) There exists $a > 0$ such that

$$\sup \left\{ r \geq 0 : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm(x) dx = 0 \right\} = \sup \left\{ r \geq 0 : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm|_{B_a}^*(x) \, dx = 0 \right\},$$

where $V_\pm|_{B_a}^*$ denotes the symmetric decreasing rearrangement of $V_\pm|_{B_a}$, the restriction of $V_\pm$ to the ball of radius a around 0,

(b) if $|\cdot|^{-2} V \notin L^1(\mathbb{R}^2)$, we have $s^* = s_-^* < s_+^*$, if $|\cdot|^{-2} V \in L^1(\mathbb{R}^2)$, we have $\int_{\mathbb{R}^3} \frac{V(x)}{|x|^2} dx < 0$,

(c) $s^* > 1$, and

(d) if $s^* \geq 53/27$, we have $V \in L^p(\mathbb{R}^3)$ for some $p > 5/3$.

Condition (d) can be dropped, whenever we have control on the ground state space of $\mathcal{V}_\mu$ in the following sense: There exists $\mu_0 > 0$ and $\mathcal{L} \subset \mathbb{N}_0$ with $|\mathcal{L}| < \infty$, such that for all $\mu \geq \mu_0$, the ground state space of $\mathcal{V}_\mu$ is contained in the subspace of $L^2(\mathbb{S}^2)$ spanned by the spherical harmonics with angular momentum $\ell \in \mathcal{L}$.

In a nutshell, an admissible potential is a radial potential $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, which satisfies the following:

(i) There exists some $a > 0$ such that both, positive and negative part, have their strongest singularity in B_a at the origin.

(ii) It has a dominating attractive part (for short distances), i.e. $s_-^* < s_+^*$ resp. $\int_{\mathbb{R}^3} \frac{V(x)}{|x|^2} dx < 0$.

(iii) It is slightly less divergent at the origin than allowed by the $L^{3/2}(\mathbb{R}^3)$-assumption, i.e. $s^* > 1$.

The most relevant examples for admissible potentials are the attractive Yukawa and Gaussian potential, i.e.

$$V_{\text{Yukawa}}(x) = -\frac{1}{4\pi|x|}e^{-|x|} \quad \text{and} \quad V_{\text{Gauss}}(x) = -(2\pi)^{-3/2}e^{-|x|^2/2}.$$

Remark 6 (On condition (d) for admissible potentials) The additional $L^p(\mathbb{R}^3)$-assumption with $p > 5/3$ for $s^* \geq 53/27$ in condition (d) is due to technical reasons and will we be explained during the proof of Theorem 7, which is formulated below. Note that, since $s^* \geq 53/27$ and $V \in L^1(\mathbb{R}^3)$, this condition is essentially about regularity away from 0 and infinity. However, our proof would work without change if we only had $p > f(s^*)$, where f has some complicated (explicit) expression (see Lemma 12 and Eq. (26)) and is strictly monotonically increasing between 53/27 and 2, and satisfies $f(53/27) = 3/2$ and $f(s^*) = 5/3$ for all $s^* \geq 2$. We do not state Theorem 7 with this slight generalization for simplicity. Whenever we have some control on the ground state space of $\mathcal{V}_\mu$, the $L^p(\mathbb{R}^3)$-assumption is not necessary. For example, in the special case $\hat{V} \leq 0$, one can choose $\mathcal{L} = \{0\}$ by means of Eq. (4) and completely drop condition (d).

We will show in Lemma 13 that for any admissible potential $e_\mu < 0$ for μ large enough. Moreover, for any radial potential $V \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ with $e_\mu < 0$ and $s^* > 1$ (in particular any admissible potential), by application of Theorem 2, the critical temperature decays exponentially fast as $\mu \to \infty$ since

$$|\sqrt{\mu}e_\mu| \leq \frac{1}{2\pi^2}\left\|\frac{V}{|\cdot|^s}\right\|_{L^1}\sup_{\ell\in\mathbb{N}_0}\left\||\cdot|^{s/2}j_\ell\right\|_{L^\infty}^2\mu^{\frac{1-s}{2}} \tag{8}$$

for $s \in (1, s^*)$ and the term involving j_ℓ is finite as long as $s \leq 5/3$ by uniform decay of spherical Bessel functions (see Proposition 16 (iii)). A slightly different bound as given in Lemma 12 allows to improve this threshold. Note that the class of interaction potentials from Proposition 4 is not admissible since $s^* = 1$ for these potentials.

The existence of a maximal critical temperature at some intermediate density (*superconducting dome*), can now be obtained by combining the decay of T_c in the high-density limit from Eqs. (6) and (8) for admissible potentials in the sense of Definition 5 to the decay of T_c in the low-density limit, where

$$T_c = \mu\left(\frac{8}{\pi}e^{\gamma-2} + o(1)\right)e^{\pi/(2\sqrt{\mu}a)} \tag{9}$$

as shown in [10]. This result was obtained for (not necessarily radially symmetric) real valued interaction potentials V, with $V(x)(1 + |x|) \in L^1(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, negative scattering length a, and in the absence of bound states. Thus, we rigorously confirmed

the *ubiquity of superconducting domes in BCS theory* for a general class of interaction potentials, as claimed in [16].

As our next result, we shall derive the second order correction to Eq. (6), i.e. we shall compute the constant in front of the exponential for admissible potentials. For this purpose we define the operator $\mathcal{W}_\mu^{(\kappa)}$ on $u \in L^2(\mathbb{S}^2)$ via its quadratic form

$$\langle u | \mathcal{W}_\mu^{(\kappa)} | u \rangle = \sqrt{\mu} \int_0^\infty \mathrm{d}|p| \left(\frac{|p|^2}{||p|^2 - 1|} \left[\int_{\mathbb{S}^2} \mathrm{d}\omega(p) \left(|\hat{\varphi}(\sqrt{\mu}p)|^2 \right. \right. \right.$$
$$\left. \left. \left. + -|\hat{\varphi}(\sqrt{\mu}p/|p|)|^2 \right) \right] \frac{|p|^2}{|p|^2 + \kappa^2} \int_{\mathbb{S}^2} \mathrm{d}\omega(p) |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2 \right) \quad (10)$$

for fixed $\kappa \geq 0$ (cf. Eq. (13) in [9] for an analogous definition in the weak coupling case with $\kappa = 0$). Here, we denote $\hat{\varphi}(p) = (2\pi)^{-3/2} \int_{\mathbb{S}^2} \hat{V}(p - \sqrt{\mu}q) u(q) \mathrm{d}\omega(q)$, and $(|p|, \omega(p)) \in (0, \infty) \times \mathbb{S}^2$ are spherical coordinates for $p \in \mathbb{R}^3$. Since $V \in L^1(\mathbb{R}^3)$, the map $|p| \mapsto \int_{\mathbb{S}^2} \mathrm{d}\omega(p) |\hat{\varphi}(p)|^2$ is Lipschitz continuous for any $u \in L^2(\mathbb{S}^2)$, such that the radial integral in Eq. (10) is well defined even in the vicinity of $|p| \sim 1$. For large $|p|$ the integral converges since $V \in L^{3/2}(\mathbb{R}^3)$. Although we formulate our result in Theorem 7 only for $\kappa = 0$, the case of a positive parameter $\kappa > 0$ is crucial in the proof of this statement, as it ensures, e.g., that the first term in the decomposition of the Birman–Schwinger operator associated with $K_{T,\mu} + V$ is small (cf. Eq. (15)). Whenever it does not lead to confusion, we refer to some κ-dependent quantity at $\kappa = 0$ by simply dropping the (κ)-superscript.

Now, we define the operator

$$\mathcal{B}_\mu^{(\kappa)} = \frac{\pi}{2} \left(\mathcal{V}_\mu - \mathcal{W}_\mu^{(\kappa)} \right), \quad (11)$$

which measures the strength of the interaction potential near the Fermi surface up to second order and let $b_\mu^{(\kappa)}$ denote its lowest eigenvalue,

$$b_\mu^{(\kappa)} = \inf \operatorname{spec} \mathcal{B}_\mu^{(\kappa)}. \quad (12)$$

We introduced the factor $\pi/2$ in Eq. (11) since $b_\mu = b_\mu^{(0)}$ has the interpretation of an effective scattering length, which is best illustrated in the case of small μ (see Proposition 1 in [9]). Moreover, we will see in the proof of Theorem 7 that if $e_\mu < 0$ then also $b_\mu^{(\kappa)} < 0$ for large enough μ. With the aid of $b_\mu^{(\kappa)}$ we can now state our second main result concerning the asymptotic formula for the critical temperature valid up to second order.

Theorem 7 *Let V be an admissible potential. Then the critical temperature T_c is positive and satisfies*

$$\lim_{\mu \to \infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu} \right) = 2 - \gamma - \log(8/\pi).$$

In other words,

$$T_c = \mu \left(\frac{8}{\pi} e^{\gamma-2} + o(1) \right) e^{\pi/(2\sqrt{\mu}b_\mu)}$$

in the limit $\mu \to \infty$. Similarly to Theorem 2, this formula is in complete analogy to the weak-coupling case [9] (replace $V \to \lambda V$ and take the limit $\lambda \to 0$) but we have coupling parameter $\lambda = 1$ here. As discussed in the introduction, this analogy is not entirely surprising. In physical terms, only those fermions with momenta close to the Fermi surface $\{p^2 = \mu\}$ contribute to the superconductivity. Therefore, due to the decay of the interaction $\hat{V}$ in Fourier space, the high-density limit, $\mu \to \infty$, is effectively a weak-coupling limit.

The constant in front of the exponential is in particular relevant for obtaining the universality of the ratio of the critical temperature and the energy gap, which is achieved in [13], where a similar asymptotic formula for the energy gap is proven.

Remark 8 (Radiality) The assumption of the interaction potential being radially symmetric enters the proofs of our main theorems in a crucial way. On the one hand, the radial symmetry allows an additional averaging over the sphere $\mathbb{S}^2$ in position space in the proof of Theorem 2, which leads to a "decoupling" of the position variables x and y (cf. Eq. (19)) as the arguments of integral kernels of operators that appear after employing the Birman–Schwinger principle [6,8,12]. Without this averaging the supposed error terms in Eq. (15) could not be concluded to be small. On the other hand, the radial symmetry enables us to obtain useful bounds on the quantity e_μ (cf. Lemma 11, Lemma 12, and Lemma 13), which naturally appears in the obtained asymptotics in Theorem 2 and Theorem 7. Although the assumption of a radial potential is a loss of generality compared to the weak coupling [6,9] and low density [10] situation, the case of an isotropic interactions seems physically the most relevant and natural.

Remark 9 (Potentials with slow decay at infinity) The recent work [4] by Cuenin and Merz indicates how to generalize our results to interaction potentials with slow decay at infinity, i.e. which fail to satisfy $V \in L^1(\mathbb{R}^3)$. The main idea is to employ the Tomas–Stein Theorem to define the Fourier transform of the potential on the codim-1 submanifold $\mathbb{S}^2 \subset \mathbb{R}^3$ having non-vanishing curvature. Moreover, by using the methods from [7], where Gontier, Hainzl, and Lewin originally studied a lower bound on the Hartree–Fock energy of the electron gas, one can see that

$$T_c \le C_1 \, \mu \exp \left(-C_2 \, \mu^{1/4} \right) \tag{13}$$

for any real-valued potential V satisfying $|\cdot|V \in L^\infty(\mathbb{R}^3)$. A detailed proof of this estimate is given in Sect. 3. Note that for an admissible potential V that satisfies $|\cdot|V \in L^\infty(\mathbb{R}^3)$, we have $s^* \ge 2$ and infer by Eq. (8) that the bound provided by Eq. (13) is not optimal. Although these results indicate that it is mathematically possible to deal with slow decay at infinity, it seems physically natural to assume fast decay at infinity, at least in the high-density limit for an effective interaction potential, when the phenomenon of screening plays an important role.

3 Proofs

The most important tool for our proofs will be the Birman–Schwinger principle (see [6,8,12]). According to this principle, T_c is determined by the fact that for $T = T_c$ the smallest eigenvalue of

$$B_{T,\mu} = V^{1/2} \frac{1}{K_{T,\mu}} |V|^{1/2}$$

equals -1. Here, we used the notation $V(x)^{1/2} = \mathrm{sgn}(V(x))|V(x)|^{1/2}$. The main simplification is that the study of the spectrum of the unbounded operator $K_{T,\mu} + V$ reduces to identifying the singular part of the compact Birman–Schwinger operator. With this in mind, our proofs will build on a convenient decomposition of $B_{T,\mu}$ in a dominant singular term and other error terms.

Let $\mathfrak{F}_\mu : L^1(\mathbb{R}^3) \to L^2(\mathbb{S}^2)$ denote the rescaled Fourier transform restricted to $\mathbb{S}^2$ with

$$\left(\mathfrak{F}_\mu \psi\right)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\sqrt{\mu}\, p \cdot x} \psi(x) \mathrm{d}x \,,$$

which is well-defined by the Riemann–Lebesgue Lemma. Since $V \in L^1(\mathbb{R}^3)$, the multiplication with $|V|^{1/2}$ is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^1(\mathbb{R}^3)$, and hence $\mathfrak{F}_\mu |V|^{1/2}$ is a bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{S}^2)$. A further important ingredient in our proofs is to study the asymptotic behavior of

$$m_\mu^{(\kappa)}(T) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \kappa^2\mu} \right) \mathrm{d}p$$

for fixed $\kappa > 0$, which was done in a similar way for the low-density and weak-coupling limit of the critical temperature and the energy gap (see [9,10,12,18]). Indeed, using Lemma 1 from [9] one can easily see that

$$m_\mu^{(\kappa)}(T) = \sqrt{\mu} \left(\log \frac{\mu}{T} + \gamma - 2 + \log \frac{8}{\pi} + \kappa \frac{\pi}{2} + o(1) \right) \tag{14}$$

as long as $T/\mu \to 0$. Using the definitions above, we arrive at our convenient decomposition, which we define as

$$B_{T,\mu} = V^{1/2} \frac{1}{p^2 + \kappa^2\mu} |V|^{1/2} + m_\mu^{(\kappa)}(T)\, V^{1/2} \mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2} + A_{T,\mu}^{(\kappa)} \tag{15}$$

for $\kappa > 0$, where $A_{T,\mu}^{(\kappa)}$ is implicitly defined. For the first term to be small, we need that $\kappa > 0$. For the second term, note that

$$V^{1/2} \mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2}$$

is isospectral to $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu{}^\dagger$. In fact, the spectra agree at first except possibly at 0, but 0 is in both spectra as the operators are compact on an infinite dimensional space.

This second term will be the dominant term, which is how the quantity e_μ appears in the asymptotic formulas in our main theorems, whereas the first and third term are negligible error terms in the limit $\mu \to \infty$. Showing this, is the objective of the proofs of Theorems 2 and 7.

A priori, it is not clear, how T_c behaves at high densities. Therefore, before we go to the proofs of Theorems 2 and 7, let us fix the following

Lemma 10 $T_c \leq O(\mu)$ *as* $\mu \to \infty$.

Proof Since $\tanh(t) \leq \min(1, t)$ for $t \geq 0$, we have

$$K_{T,\mu} + V \geq \frac{1}{2}\left(|p^2 - \mu| + 2T\right) + V$$

$$\geq \frac{1}{2}\left(p^2 + \mu + 2V\right) + (T - \mu).$$

The first term is non-negative for sufficiently large μ by application of Sobolev's inequality [19, Thm. 8.3] using $V \in L^{3/2}(\mathbb{R}^3)$. Thus, by Definition 1, we obtain $T_c \leq \mu$.   □

In the proof of Theorem 2 below, we will in fact show that $T_c = o(\mu)$, so Eq. (14) gives the correct asymptotic behavior.

3.1 Proof of Theorem 2

Proof of Theorem 2 Fix $\kappa > 0$. As outlined above, the strategy of the proof is to show that the first and the third term in the decomposition (15) vanish in operator norm in the high-density limit and thus the asymptotic behavior is entirely determined by the spectrum of the operator in the second term. We discuss this in detail now.

For the first term, note that the Fourier transform of $\frac{1}{p^2 + \kappa^2 \mu}$ is given by $\frac{e^{-\kappa\sqrt{\mu}|x|}}{|x|}$, up to a constant. Thus the Hilbert–Schmidt norm $\|\cdot\|_{\mathrm{HS}}$, which is always an upper bound for the operator norm $\|\cdot\|_{\mathrm{op}}$, is given by

$$\left\| V^{1/2} \frac{1}{p^2 + \kappa^2 \mu} |V|^{1/2} \right\|_{\mathrm{HS}}^2 = C \int_{\mathbb{R}^3} \mathrm{d}x \int_{\mathbb{R}^3} \mathrm{d}y \, |V(x)| \frac{e^{-2\kappa\sqrt{\mu}|x-y|}}{|x - y|^2} |V(y)| .$$

This vanishes as $\mu \to \infty$ by an application of the dominated convergence theorem in combination with the Hardy–Littlewood–Sobolev inequality [19, Thm. 4.3]. Here and in the following, we shall use the notation c and C for generic positive small resp. large constants, possibly having a different value in each appearance. If we want to emphasize the dependence on a certain parameter, we add a subscript, e.g. by writing c_δ or $C_{p,a}$.

For the third term, we will heavily use the radiality of V. In fact, since V is radially symmetric, every eigenfunction of $K_{T,\mu}$ and thus B_T will have definite angular

momentum and we can focus on $f, g \in L^2(\mathbb{R}^3)$ of the form $f(x) = f(|x|)Y_\ell^m(\hat{x})$ resp. $g(x) = g(|x|)Y_{\ell'}^{m'}(\hat{x})$, with a slight abuse of notation, where $\hat{x} = x/|x|$ denotes the unit vector in direction x. Now we aim to bound $\langle f | A_{T,\mu}^{(\kappa)} | g \rangle$, uniformly in (ℓ, ℓ') (and (m, m')). As $A_{T,\mu}^{(\kappa)}$ has integral kernel

$$A_{T,\mu}^{(\kappa)}(x, y) = CV^{1/2}(x)|V(y)|^{1/2} \int_{\mathbb{R}^3} \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \kappa^2 \mu} \right) \left(e^{ip \cdot (x-y)} - e^{i\sqrt{\mu}\hat{p} \cdot (x-y)} \right) dp,$$

and using the radial symmetry of V we arrive at

$$\langle f | A_{T,\mu}^{(\kappa)} | g \rangle = C \int_0^\infty d|x| \, |x|^2 \int_0^\infty d|y| \, |y|^2 \bar{f}(|x|) V^{1/2}(|x|) |V(|y|)|^{1/2} g(|y|) \quad (16)$$

$$\times \int_{\mathbb{R}^3} dp \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \kappa^2 \mu} \right) \quad (17)$$

$$\times \int_{\mathbb{S}^2} d\omega(x) \int_{\mathbb{S}^2} d\omega(y) \overline{Y_\ell^m}(\hat{x}) Y_{\ell'}^{m'}(\hat{y}) \left(e^{ip \cdot (x-y)} - e^{i\sqrt{\mu}\hat{p} \cdot (x-y)} \right). \quad (18)$$

Note that the angular integrals of x and y can be performed first only by the radial symmetry of V. If V were not radially symmetric, we would have had to compute the angular integral of p first and would have ended up with completely different integrals to estimate. Now, using the plane wave expansion $e^{ip \cdot x} = 4\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell i^\ell j_\ell(|p||x|) Y_\ell^m(\hat{p}) \overline{Y_\ell^m}(\hat{x})$, the last line (18) evaluates to

$$16\pi^2 \, (-i)^{\ell+\ell'} \left(j_\ell(|p||x|) j_{\ell'}(|p||y|) - j_\ell(\sqrt{\mu}|x|) j_{\ell'}(\sqrt{\mu}|y|) \right) \overline{Y_\ell^m}(\hat{p}) Y_{\ell'}^{m'}(\hat{p}). \quad (19)$$

In order to get a non-zero angular p-integral from the second line (17), we need to have $\ell = \ell'$ and $m = m'$, by orthogonality of spherical harmonics. We can hence focus on that case and write x, y, and p instead of $|x|$, $|y|$, and $|p|$, respectively, such that (17) and (18) combine to (a constant times)

$$\int_0^\infty dp \, p^2 \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \kappa^2 \mu} \right) \left(j_\ell(px) j_\ell(py) - j_\ell(\sqrt{\mu}x) j_\ell(\sqrt{\mu}y) \right). \quad (20)$$

After changing the integration variable $p \to p/\sqrt{\mu}$ and inserting $\pm j_\ell(p\sqrt{\mu}x) j_\ell(\sqrt{\mu}y)$, we use the uniform Lipschitz continuity and the uniform decay of spherical Bessel functions (Proposition 16 (ii) and Proposition 16 (iii)) to obtain

$$|(20)| \leq C\mu^{1/2} \int_0^\infty dp \, p^2 \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2 + \kappa^2} \right| |p - 1|^\varepsilon \left(\frac{1}{p^\varepsilon} + 1 \right)$$

$$\times \left(|j_\ell(p\sqrt{\mu}x)|^{1-11\varepsilon/5} + |j_\ell(\sqrt{\mu}x)|^{1-11\varepsilon/5} \right) \left(|j_\ell(p\sqrt{\mu}y)|^{1-11\varepsilon/5} \right.$$

$$\left. + |j_\ell(\sqrt{\mu}y)|^{1-11\varepsilon/5} \right),$$

for any $\varepsilon \in (0, 2/11)$. Using that and by employing Hölder for the integrals over x and y in Eq. (16), we get

$$\left| \langle f | A_{T,\mu}^{(\kappa)} | g \rangle \right| \leq C \, \|f\|_{L^2} \, \|g\|_{L^2} \int_0^\infty \mathrm{d}p \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2 + \kappa^2} \right| |p - 1|^\varepsilon \left(\frac{1}{p^\varepsilon} + 1 \right) p^2 \tag{21}$$

$$\times \, \mu^{1/2} \int_{\mathbb{R}^3} \mathrm{d}x \, |V(x)| \left(|j_\ell(p\sqrt{\mu}|x|)|^{2-22\varepsilon/5} + |j_\ell(\sqrt{\mu}|x|)|^{2-22\varepsilon/5} \right) . \tag{22}$$

In Lemma 11 below (as $\varepsilon < 2/11$ we have $2 - 22\varepsilon/5 > 6/5$), we show that the last line (22) can be estimated by

$$(22) \leq \left(\frac{1}{p} + 1 \right) |o(1)| \, ,$$

where $o(1)$ is some function of μ that vanishes as $\mu \to \infty$. Thus, we arrive at

$$\left| \langle f | A_{T,\mu}^{(\kappa)} | g \rangle \right| \leq C \, |o(1)| \, \|f\|_{L^2} \, \|g\|_{L^2} \int_0^\infty \mathrm{d}p \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2 + \kappa^2} \right| |p - 1|^\varepsilon p^{1-\varepsilon} (1 + p^{1+\varepsilon}) \, ,$$

where the integral is uniformly bounded (since $\kappa > 0$) as long as $T \leq C\mu$ and we conclude

$$\limsup_{\mu \to \infty} \sup_{0 < T \leq C\mu} \left\| A_{T,\mu}^{(\kappa)} \right\|_{\mathrm{op}} = 0 \, ,$$

since all bounds above are uniform in ℓ. Therefore, as long as $T_c = O(\mu)$, the spectrum of the Birman–Schwinger operator approaches the spectrum of the operator in the second term in Eq. (15) as $\mu \to \infty$.

Summarizing our considerations above, we get that, since by assumption $e_\mu < 0$ for $\mu \geq \mu_0$, $T_c > 0$ for all sufficiently large μ. This is because the second term in the decomposition (15) can be made arbitrarily negative by taking $T \to 0$, whereas the first and the third term are bounded uniformly in $T \leq C\mu$. Thus we get with the aid of Lemma 10 that

$$-1 = \lim_{\mu \to \infty} m_\mu^{(\kappa)}(T_c) \, e_\mu \, .$$

In order to obtain Eq. (5) by means of Eq. (14), it remains to show that $T_c = o(\mu)$. Since it is already shown in Lemma 10 that $T_c = O(\mu)$, we assume that $T_c = \Theta(\mu)$, i.e. there exist $0 < c < C$ such that $c\mu \leq T_c \leq C\mu$. This means that $m_\mu^{(\kappa)}(T_c)$ is of order $\sqrt{\mu}$, which leads to a contradiction since $\sqrt{\mu} e_\mu = o(1)$ by Lemma 11 below. So, Eq. (14) implies Eq. (5) as desired. □

Lemma 11 *Let $V \in L^{3/2}(\mathbb{R}^3)$ and $\alpha > 6/5$. Then*

$$\limsup_{\mu \to \infty} \sqrt{\mu} \sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x \, |V(x)| \left| j_\ell(\sqrt{\mu}|x|) \right|^\alpha = 0 \, .$$

Proof We estimate

$$\sqrt{\mu}\,\sup_{\ell\in\mathbb{N}_0}\int_{\mathbb{R}^3}\mathrm{d}x\,|V(x)|\,\big|j_\ell(\sqrt{\mu}|x|)\big|^\alpha \le C\,\sqrt{\mu}\int_{\mathbb{R}^3}\mathrm{d}x\,|V(x)|\frac{1}{\big(\sqrt{\mu}|x|\big)^{5\alpha/6}+1}\,,$$

(23)

where the inequality follows from the uniform decay of spherical Bessel functions (see Proposition 16 (iii)). By using Hölder, we can further bound

$$(23) \le C\|V-\phi\|_{L^{3/2}}\left\|\frac{1}{|\cdot|^{5\alpha/6}+1}\right\|_{L^3} + C\,\sqrt{\mu}\int_{\mathbb{R}^3}\mathrm{d}x\,|\phi(x)|\frac{1}{\big(\sqrt{\mu}|x|\big)^{5\alpha/6}+1}$$

for any $\phi \in C_0^\infty(\mathbb{R}^3)$. Since $\alpha > 6/5$, the second term vanishes as $\mu \to \infty$ by dominated convergence using $\phi \in C_0^\infty(\mathbb{R}^3)$, and the first term can be made arbitrarily small as $C_0^\infty(\mathbb{R}^3)$ is dense in $L^{3/2}(\mathbb{R}^3)$. Thus, we have proven the claim. $\qquad\square$

3.2 Proof of Proposition 4

Proof of Proposition 4 We check that V_α satisfies the assumptions of Theorem 2. First, V_α is radial and clearly satisfies $V_\alpha \in L^1(\mathbb{R}^3)$. $V_\alpha \in L^{3/2}(\mathbb{R}^3)$ follows since $\alpha > 1/3$. Next, we calculate the derivative of $|x||V_\alpha(x)|$ w.r.t. $|x|$ as

$$\left(\frac{(\log(|x|)+\alpha)^2}{|x|(\log^2(|x|)+1)} + \frac{1-\alpha^2}{|x|(\log^2(|x|)+1)} + 1\right)\frac{\exp(-|x|)}{|x|(\log^2(|x|)+1)}$$

and conclude that $|x|V_\alpha(x)$ is monotonically increasing in $|x|$, since $\alpha < 1/2$. Therefore, by using the radiality of V_α and the argument from Eq. (4) in [23], we find that $\hat{V}_\alpha \le 0$ and infer

$$e_\mu = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}V_\alpha(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2\mathrm{d}x < 0$$

by Eq. (4). Thus, V_α satisfies all conditions of Theorem 2. In order to obtain a lower bound on $T_c(\mu)$ we estimate

$$|\sqrt{\mu}e_\mu| \ge c\int_0^{\sqrt{\mu}/2}\frac{\sin(x)^2}{x^2|\log(x/\sqrt{\mu})|^{2\alpha}}\mathrm{d}x \ge c\,\frac{1}{|\log(\mu)|^{2\alpha}}$$

for some $c > 0$ and μ large enough. Using Theorem 2, this yields

$$T_c \gtrsim \mu\exp(-\log(\mu)^{2\alpha}/c) \to \infty$$

as $\mu \to \infty$ since $\alpha < 1/2$. $\qquad\square$

3.3 Proof of Theorem 7

The proof of Theorem 7 is based on the following two Lemmas providing the necessary estimates for a perturbation theoretic argument yielding the next order correction to the asymptotics obtained in Theorem 2. While Lemma 12 improves the upper bounds on integrals of the interaction potential against spherical Bessel functions obtained in Lemma 11 and Eq. (8), in particular for $s^* > 5/3$, Lemma 13 provides a lower bound on e_μ for admissible potentials. We postpone the proofs of Lemmas 12 and 13 until the end of this Section.

Lemma 12 *Let $V \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p \in [3/2, 9/4]$ with dual $q = \frac{p}{p-1} \in [9/5, 3]$. Assume that $s^* > 1$, with s^* as in Definition 5 and set*

$$\beta_p^* = \begin{cases} \frac{s^*}{2} & \text{for } s^* \in (1, 5/3] \\ \min\left(\frac{(q+1)s^*-4}{3qs^*-7} + \frac{1}{2}, \frac{10q-11}{12q-14}\right) & \text{for } s^* > 5/3. \end{cases} \tag{24}$$

Note that β_p^* depends continuously on s^* and is (strictly) monotonically increasing (between 1 and 2), and $\beta_p^* \le \min(s^*, 2)/2$ for any $s^* > 1$. Then for any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ we have

$$\limsup_{\mu \to \infty} \mu^{\beta_p^*-\delta} \sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x |V(x)| \, |j_\ell(\sqrt{\mu}|x|)|^{2-\varepsilon} = 0.$$

For admissible potentials that satisfy the $L^p(\mathbb{R}^3)$-assumption in condition (d) from Definition 5, we will use this Lemma with

$$\beta^* = \begin{cases} \frac{s^*}{2} & \text{for } s^* \in (1, 5/3] \\ \frac{4s^*-4}{9s^*-7} + \frac{1}{2} & \text{for } s^* \in (5/3, 53/27) \\ \min\left(\frac{7s^*-8}{15s^*-14} + \frac{1}{2}, \frac{7}{8}\right) + \delta_p & \text{for } s^* \ge 53/27, \end{cases}$$

for some $\delta_p > 0$ since $p > 5/3$. For our perturbation theoretic argument to work in the general case, where we have no control on the ground state space of $\mathcal{V}_\mu$, it turns out to be necessary that

$$4\beta^* - \frac{3\min(s^*, 2)}{2} - \frac{1}{2} > 0, \tag{25}$$

which is why we need the $L^p(\mathbb{R}^3)$-assumption in Definition 5 for $s^* \ge 53/27$. The function $f(s^*)$ from Remark 6 can explicitly be determined by requiring that

$$4\beta_{f(s^*)}^* - \frac{3\min(s^*, 2)}{2} - \frac{1}{2} = 0. \tag{26}$$

Lemma 13 *Let V be an admissible potential (cf. Definition 5, condition (d) can be dropped). Then for any $\delta > 0$ there exists $c_\delta > 0$ such that*

$$\liminf_{\mu \to \infty} |\mu^{\min(s^* + \delta, 2)/2} \, e_\mu| \geq c_\delta \, .$$

In particular, for admissible V, we have $e_\mu < 0$ for μ large enough.

The proof of Theorem 7 is divided in two parts. In the first part, Lemma 14, we provide an asymptotic formula for T_c with parameter $\kappa > 0$. In the second part, Lemma 15, we asymptotically compare $1/(\sqrt{\mu} b_\mu^{(\kappa)})$ with $1/(\sqrt{\mu} b_\mu)$. By combining these formulas, we obtain Theorem 7.

Lemma 14 *Let V be an admissible potential and fix $\kappa > 0$. Then the critical temperature T_c is positive and satisfies*

$$\lim_{\mu \to \infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right) = 2 - \gamma - \kappa \frac{\pi}{2} - \log(8/\pi) \, . \tag{27}$$

Lemma 15 *Let V be an admissible potential and $\kappa > 0$. Then*

$$\lim_{\mu \to \infty} \left(\frac{\pi}{2\sqrt{\mu} b_\mu} - \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right) = \kappa \frac{\pi}{2} \, . \tag{28}$$

Proof of Theorem 7 By combining Lemma 14 with Lemma 15 we obtain

$$\lim_{\mu \to \infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu} \right) = \lim_{\mu \to \infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right)$$

$$+ \lim_{\mu \to \infty} \left(\frac{\pi}{2\sqrt{\mu} b_\mu} - \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right)$$

$$= 2 - \gamma - \kappa \frac{\pi}{2} - \log(8/\pi) + \kappa \frac{\pi}{2} = 2 - \gamma - \log(8/\pi) \, .$$

The rest of this Section is devoted to the proofs of the four Lemmas above. We begin with Lemmas 14 and 15.

Proof of Lemma 14 Fix $\kappa > 0$. We first assume condition (d) from Definition 5 and discuss the changes for the special case afterwards. Similarly to the proof of Theorem 2, we show that the first and the third term in the decomposition (15) vanish in operator norm.

For the first term, we need to improve the estimate from Theorem 2, where we employed the easily accessible Hilbert–Schmidt norm as an upper bound to the operator norm. Indeed, using the radial symmetry of V, similarly to the bound of $A_{T,\mu}^{(\kappa)}$ in Eq. (22), the operator norm of the compact operator $L_\mu^{(\kappa)} := V^{1/2}(p^2 + \kappa^2 \mu)^{-1} |V|^{1/2}$

can be estimated as

$$\left\| L_\mu^{(\kappa)} \right\|_{\mathrm{op}} \leq C \mu^{1/2} \int_0^\infty \mathrm{d}p \frac{p^2}{p^2 + \kappa^2} \sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x |V(x)| \, |j_\ell(\sqrt{\mu}p|x|)|^2 \,, \quad (29)$$

which is bounded by $\mu^{-\beta^*+1/2+\delta}$ for any $\delta > 0$ by means of Lemma 12 (note that the p-integral is finite since $s^* > 1$). Recall from the prove of Theorem 2 (in particular Eq. 22) that

$$\left\| A_{T,\mu}^{(\kappa)} \right\|_{\mathrm{op}} \leq C \int_0^\infty \mathrm{d}p \left| \frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2 + \kappa^2} \right| |p-1|^\varepsilon \left(\frac{1}{p^\varepsilon} + 1 \right) p^2$$
$$\times \mu^{1/2} \sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x \, |V(x)| \left(|j_\ell(p\sqrt{\mu}|x|)|^{2-22\varepsilon/5} + |j_\ell(\sqrt{\mu}|x|)|^{2-22\varepsilon/5} \right) .$$
$$(30)$$

for any $\varepsilon \in (0, 5/11)$. Again by Lemma 12 we may bound the x-integral by $\mu^{-\beta^*+\delta}(1+p^{-\beta^*+\delta})$ for any $\delta > 0$ and the p-integral is finite as long as $0 < T \leq C\mu$. We now define, analogously to Eq. (28) in [9],

$$V^{1/2} M_{T,\mu}^{(\kappa)} |V|^{1/2} := V^{1/2} K_{T,\mu}^{-1} |V|^{1/2} - m_\mu^{(\kappa)}(T) V^{1/2} \mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2} = L_\mu^{(\kappa)} + A_{T,\mu}^{(\kappa)}$$

and combine the bounds (29) and (30) from above to obtain

$$\limsup_{\mu \to \infty} \mu^{\beta^*-1/2-\delta} \sup_{0 < T \leq C\mu} \left\| V^{1/2} M_{T,\mu}^{(\kappa)} |V|^{1/2} \right\|_{\mathrm{op}} = 0 \qquad (31)$$

for any $\delta > 0$. Also, since $V^{1/2} \mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2}$ is isospectral to $\mathcal{V}_\mu$, so its eigenvalues are given by Eq. (3), one can easily see, using Lemma 12 again, that

$$\limsup_{\mu \to \infty} \mu^{\beta^*-\delta} \left\| V^{1/2} \mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2} \right\|_{\mathrm{op}} = 0 \qquad (32)$$

for any $\delta > 0$. In particular, since $s^* > 1$, the bound (31) implies that $1 + V^{1/2} M_{T,\mu}^{(\kappa)} |V|^{1/2}$ is invertible for any $0 < T \leq C\mu$ and μ large enough.

We can thus, following the argument around Eq. (30) in [9], conclude that the Birman–Schwinger operator $B_{T,\mu}$ having an eigenvalue -1 is equivalent to the self-adjoint operator

$$\mathfrak{F}_\mu |V|^{1/2} \frac{m_\mu^{(\kappa)}(T)}{1 + V^{1/2} M_{T,\mu}^{(\kappa)} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \,. \qquad (33)$$

acting on $L^2(\mathbb{S}^2)$ having an eigenvalue -1. At $T = T_c$, -1 is the smallest eigenvalue of $B_{T,\mu}$, hence (33) has an eigenvalue -1 for this value of T. By continuity and monotonicity of $m_\mu^{(\kappa)}(T)$ one can in fact show that -1 is the smallest eigenvalue of (33) in this case (cf. the discussion below Eq. (31) in [9]).

Since $\mathfrak{F}_\mu V \mathfrak{F}_\mu^\dagger = \mathcal{V}_\mu$ (see Eq. (2)) and $e_\mu = \inf \operatorname{spec} \mathcal{V}_\mu < 0$ by Lemma 13, it immediately follows that

$$-1 = \lim_{\mu \to \infty} \inf \operatorname{spec} \mathcal{V}_\mu \, m_\mu^{(\kappa)}(T_c) = \lim_{\mu \to \infty} e_\mu m_\mu^{(\kappa)}(T_c) \,,$$

which, in combination with the asymptotics (14) and the argument for $T_c = o(\mu)$ from the proof of Theorem 2, reproves (5) resp. (6), i.e. the asymptotic formula for T_c to leading order. To obtain the next order, we employ first order perturbation theory as in the proof of Theorem 1 in [9] (in particular, see Eq. (32)).

Indeed, using Eqs. (31) and (32), we can expand the geometric series in Eq. (33) to first order and employ first order perturbation theory to arrive at

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) = \frac{-1}{\sqrt{\mu} e_\mu - \sqrt{\mu} \langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle + O(\mu^{-4\beta^* + \min(s^*,2)/2 + 3/2 + \delta})} \,, \tag{34}$$

for any $\delta > 0$. Here, u is the normalized eigenfunction corresponding to the lowest negative eigenvalue e_μ of the compact operator $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu^\dagger$ (see Lemma 13). In case of (finite!) degeneracy, one has to choose the ground state u of $\mathcal{V}_\mu$ that minimizes the second term in the denominator of (34). The error term in Eq. (34) is twofold. The first part comes from expanding the geometric series. The second part comes from first order perturbation theory, where we made use of the fact that

$$|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)/2 + 1/2} \quad \text{and}$$
$$\sup_{0 < T \leq C\mu} \sqrt{\mu} \big\| \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger \big\|_{\mathrm{op}} \leq C_\delta \mu^{-2\beta^* + 1 + \delta} \tag{35}$$

for any $\delta > 0$ by Lemma 13 resp. Eqs. (32) and (31) (recall $T_c = o(\mu)$ from above). The error from the series expansion is of order $O(\mu^{-3\beta^* + 3/2 + \delta})$ and the error from the perturbation argument is of order $O(\mu^{-4\beta^* + \min(s^*,2)/2 + 3/2 + \delta})$ and hence dominates, since $\beta^* \leq \min(s^*, 2)/2$.

Equation (34) is an implicit equation for T_c. By the second estimate in Eq. (35) and since $T_c \to 0$ as $\mu \to \infty$, we need to evaluate the limit of $\langle u | \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle$ as $T \to 0$. By the same arguments as used in Eq. (35) in [9] (dominated convergence and Lipschitz continuity of the angular integrals), this yields

$$\lim_{T \to 0} \langle u | \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle = \langle u | \mathcal{W}_\mu^{(\kappa)} | u \rangle,$$

uniformly in $u \in L^2(\mathbb{S}^2)$ with $\|u\|_{L^2(\mathbb{S}^2)} = 1$, where $\mathcal{W}_\mu^{(\kappa)}$ was defined in (10). Using that T_c is exponentially small (in some positive power of μ) as $\mu \to \infty$ by application of Theorem 2 in combination with Eq. (8), we obtain

$$\big| \langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle - \langle u | \mathcal{W}_\mu^{(\kappa)} | u \rangle \big| \leq C_D \mu^{-D} \tag{36}$$

for any $D > 0$, uniformly in $u \in L^2(\mathbb{S}^2)$ with $\|u\|_{L^2(\mathbb{S}^2)} = 1$. Combining the second estimate in Eq. (35) with Eq. (36) we thus get

$$\left\| \mathcal{W}_\mu^{(\kappa)} \right\|_{\mathrm{op}} \leq C_\delta \mu^{-2\beta^* + 1/2 + \delta} \tag{37}$$

for any $\delta > 0$. Since $|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^* + \delta, 2) + 1/2}$, we have that whenever $e_\mu < 0$ also $b_\mu^{(\kappa)} < 0$ for large enough μ (recall Eqs. (11) and (12)). In particular, combining Eqs. (34), (36) and (37), we have shown that

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} = O(\mu^{-4\beta^* + 3\min(s^*, 2)/2 + 1/2 + \delta}),$$

for any $\delta > 0$. Since $4\beta^* - 3\min(s^*, 2)/2 - 1/2 > 0$ (see Eq. (25)), we conclude Eq. (27) by employing the asymptotics (14).

In case that there exists $\mu_0 > 0$ and $\mathcal{L} \subset \mathbb{N}_0$ with $|\mathcal{L}| < \infty$, such that for all $\mu \geq \mu_0$, the ground state space of $\mathcal{V}_\mu$ is contained in the finite-dimensional subspace

$$\mathcal{I}_\mathcal{L} := \mathrm{span}\left\{ Y_\ell^m : \ell \in \mathcal{L}, \, |m| \leq \ell \right\}$$

of $L^2(\mathbb{S}^2)$, spanned by the spherical harmonics with angular momentum $\ell \in \mathcal{L}$, we can drop condition (d) from Definition 5. In order to see this, take Y_ℓ^m with $\ell \in \mathcal{L}$ and $|m| \leq \ell$ and estimate

$$\begin{aligned}
\left\| |V|^{1/2} \mathfrak{F}_\mu^\dagger Y_\ell^m \right\|_{L^2}^2 &= C \int_{\mathbb{R}^3} |V(x)| \left| \int_{\mathbb{S}^2} e^{i\sqrt{\mu} p \cdot x} Y_\ell^m(p) \mathrm{d}\omega(p) \right|^2 \mathrm{d}x \\
&= C \int_{\mathbb{R}^3} |V(x)| (j_\ell(\sqrt{\mu}|x|))^2 \mathrm{d}x \leq C_{\ell, \delta} \, \mu^{-\min(s^*, 2) + \delta},
\end{aligned}$$

for any $\delta > 0$. The second equality follows by the radiality of V and the final estimate by the decay of spherical Bessel functions (see the first bound in Proposition 16 (iii)). Using Eq. (31) with $\beta_{3/2}^*$ instead of β^* by means of Lemma 12, this implies that

$$\sup_{u \in \mathcal{I}_\mathcal{L}, \, \|u\|_{L^2} = 1} \left| \sqrt{\mu} \langle u | \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle \right| \leq C_{\mathcal{L}, \delta} \, \mu^{-\beta_{3/2}^* - \min(s^*, 2)/2 + 1 + \delta} \tag{38}$$

for any $\delta > 0$ (and μ large enough). Therefore, since $\beta_{3/2}^* \leq \min(s^*, 2)/2$, the error from the geometric expansion dominates the error from the perturbation theory in Eq. (34) and we get

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) = \frac{-1}{\sqrt{\mu} e_\mu - \sqrt{\mu} \langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle + O(\mu^{-3\beta_{3/2}^* + 3/2 + \delta})}, \tag{39}$$

for any $\delta > 0$, instead. Moreover, using Eqs. (38) and (36), we get

$$\left\| \mathcal{W}_\mu^{(\kappa)} \big|_{\mathcal{I}_\mathcal{L}} \right\|_{\mathrm{op}} \le C_{\mathcal{L},\delta} \mu^{-\beta_{3/2}^* - \min(s^*,2)/2 + 1/2 + \delta} \tag{40}$$

for any $\delta > 0$. By combining Eqs. (39), (36) and (40) with $|\sqrt{\mu} e_\mu| \ge c_\delta \mu^{-\min(s^* + \delta, 2) + 1/2}$, we find

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} = O(\mu^{-3\beta_{3/2}^* + \min(s^*,2) + 1/2 + \delta})$$

for any $\delta > 0$, and the proof comes to an end in the same way as above by realizing that $3\beta_{3/2}^* - \min(s^*, 2) - 1/2 > 0$.   □

Proof of Lemma 15 The proof follows a similar perturbation theoretic argument as in the proof of Lemma 14. We first assume condition (d) from Definition 5 and discuss the changes for the special case afterwards. To begin with, we derive a bound on $\|\mathcal{W}_\mu\|_{\mathrm{op}}$. For this purpose, we take a normalized $u \in L^2(\mathbb{S}^2)$ and estimate

$$\begin{aligned}
|\langle u | \mathcal{W}_\mu | u \rangle| &\le |\langle u | \mathcal{W}_\mu | u \rangle - \langle u | \mathcal{W}_\mu^{(\kappa)} | u \rangle| \\
&+ |\langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle - \langle u | \mathcal{W}_\mu^{(\kappa)} | u \rangle| + |\langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \rangle|.
\end{aligned}$$

The second term is smaller than any inverse power of μ by Eq. (36). Using Eqs. (31) and (32), the third term is bounded by $\mu^{-2\beta^* + 1/2 + \delta}$ for any $\delta > 0$, uniformly in $u \in L^2(\mathbb{S}^2)$. Since

$$\langle u | \mathcal{W}_\mu | u \rangle - \langle u | \mathcal{W}_\mu^{(\kappa)} | u \rangle = \sqrt{\mu} \int_0^\infty \mathrm{d}|p| \left(1 - \frac{|p|^2}{|p|^2 + \kappa^2} \right) \|\mathcal{V}_\mu u\|_{L^2}^2 = \kappa \frac{\pi}{2} \sqrt{\mu} \|\mathcal{V}_\mu u\|_{L^2}^2 , \tag{41}$$

we infer by means of Eq. (31) that also the first term is bounded by $\mu^{-2\beta^* + 1/2 + \delta}$, uniformly in $u \in L^2(\mathbb{S}^2)$, and we thus have

$$\|\mathcal{W}_\mu\|_{\mathrm{op}} \le C_\delta \mu^{-2\beta^* + 1/2 + \delta} \tag{42}$$

for any $\delta > 0$. In particular, since $|\sqrt{\mu} e_\mu| \ge c_\delta \mu^{-\min(s^* + \delta, 2) + 1/2}$ for any $\delta > 0$, this shows that, whenever $e_\mu < 0$ also $b_\mu < 0$ for large enough μ. Moreover, using $|\sqrt{\mu} e_\mu| \ge c_\delta \mu^{-\min(s^* + \delta, 2) + 1/2}$ together with Eqs. (37) and (42), first order

perturbation theory implies

$$
\begin{aligned}
\frac{\pi}{2\sqrt{\mu}b_\mu} - \frac{\pi}{2\sqrt{\mu}b_\mu^{(\kappa)}} &= \frac{\pi}{2}\frac{b_\mu^{(\kappa)} - b_\mu}{\sqrt{\mu}\,b_\mu^{(\kappa)}b_\mu} \\
&= \frac{(e_\mu - \langle u|\mathcal{W}_\mu^{(\kappa)}|u\rangle) - (e_\mu - \langle u'|\mathcal{W}_\mu|u'\rangle) + O(\mu^{-4\beta^*+\min(s^*,2)/2+3/2+\delta})}{\sqrt{\mu}e_\mu^2 + O(\mu^{-2\beta^*+1+\delta})} \\
&= \kappa\,\frac{\pi}{2} + O(\mu^{-4\beta^*+3\min(s^*,2)/2+1/2+\delta}).
\end{aligned}
\tag{43}
$$

As in the proof of Lemma 14, u resp. u' is a (the) normalized eigenfunction corresponding to the lowest eigenvalue e_μ of $\mathcal{V}_\mu$. In case of (finite!) degeneracy, one has to choose the ground state u resp. u' of $\mathcal{V}_\mu$ that minimizes the second term in each bracket $(\cdots)$ in Eq. (43). *A priori*, u and u' could be different. But, by application of Eq. (41) we get that $\mathcal{W}_\mu$ and $\mathcal{W}_\mu^{(\kappa)}$ differ only by the constant $(\kappa\pi\sqrt{\mu}e_\mu^2)/2$ on the ground state space of $\mathcal{V}_\mu$. Therefore, $u = u'$ and the last equality in Eq. (43) follows by Eq. (41) in combination with $|\sqrt{\mu}e_\mu| \geq c_\delta\,\mu^{-\min(s^*+\delta,2)+1/2}$. Since $4\beta^* - 3\min(s^*,2)/2 - 1/2 > 0$ (see Eqs. (25)), (43) implies Eq. (28).

In case that there exists $\mu_0 > 0$ and $\mathcal{L} \subset \mathbb{N}_0$ with $|\mathcal{L}| < \infty$, such that for all $\mu \geq \mu_0$, the ground state space of $\mathcal{V}_\mu$ is contained in the finite-dimensional subspace

$$
\mathcal{I}_\mathcal{L} := \operatorname{span}\left\{Y_\ell^m : \ell \in \mathcal{L},\ |m| \leq \ell\right\}
$$

of $L^2(\mathbb{S}^2)$, spanned by the spherical harmonics with angular momentum $\ell \in \mathcal{L}$, we can drop condition (d) from Definition 5. In order to see this, we use Eq. (3) and estimate

$$
\left\|\mathcal{V}_\mu\big|_{\mathcal{I}_\mathcal{L}}\right\|_{\mathrm{op}} = \sup_{\ell \in \mathcal{L}}\left|\frac{1}{2\pi^2}\int_{\mathbb{R}^3} V(x)\left(j_\ell(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x\right| \leq C_{\mathcal{L},\delta}\,\mu^{-\min(s^*,2)+\delta}
$$

for any $\delta > 0$ (and μ large enough) by means of Proposition 16 (iii). Combining this with Eq. (38) and using $\beta_{3/2}^* \leq \min(s^*,2)/2$, we get by the same argument as above that

$$
\left\|\mathcal{W}_\mu\big|_{\mathcal{I}_\mathcal{L}}\right\|_{\mathrm{op}} \leq C_{\mathcal{L},\delta}\mu^{-\beta_{3/2}^*-\min(s^*,2)/2+1/2+\delta}
\tag{44}
$$

for any $\delta > 0$. Using first order perturbation theory, Eqs. (44) and (41) together with $|\sqrt{\mu}e_\mu| \geq c_\delta\,\mu^{-\min(s^*+\delta,2)+1/2}$ imply

$$
\frac{\pi}{2\sqrt{\mu}b_\mu} - \frac{\pi}{2\sqrt{\mu}b_\mu^{(\kappa)}} = \kappa\,\frac{\pi}{2} + O(\mu^{-3\beta_{3/2}^*+\min(s^*,2)+1/2+\delta})
$$

for any $\delta > 0$. Since $3\beta_{3/2}^* - \min(s^*,2) - 1/2 > 0$ we conclude the desired. $\qquad\square$

Finally, we give the proofs of Lemmas 12 and 13.

Proof of Lemma 12 For $s^* \in (1, 5/3]$ the statement easily follows from the uniform decay of spherical Bessel functions (see Proposition 16 (iii)). For $s^* > 5/3$ choose

$$\alpha = \max\left(\frac{5q - 7}{3qs^* - 7}, \frac{5q - 7}{6q - 7}\right) \in (0, 1) \tag{45}$$

and for (small) $\delta > 0$ set $s := \min(s^*, 2) - \delta/\alpha$. Recall that $q = p/(p-1)$ denotes the dual of $p \in [3/2, 9/4]$. We now employ Hölder's inequality to obtain

$$\sup_{\ell \in \mathbb{N}_0} \int_{\mathbb{R}^3} \mathrm{d}x |V(x)| |j_\ell(\sqrt{\mu}|x|)|^{2-\varepsilon}$$

$$\leq C \left\|\frac{V}{|\cdot|^s}\right\|_{L^1}^\alpha \|V\|_{L^p}^{1-\alpha} \sup_{\ell \in \mathbb{N}_0} \left(\int_0^\infty \mathrm{d}x \, x^{\frac{q\alpha s}{1-\alpha}+2} |j_\ell(\sqrt{\mu}x)|^{\frac{q}{1-\alpha}(2-\varepsilon)}\right)^{\frac{1-\alpha}{q}}$$

$$\leq C\mu^{-\frac{\alpha s + 3(1-\alpha)/q}{2}} \left\|\frac{V}{|\cdot|^s}\right\|_{L^1}^\alpha \|V\|_{L^p}^{1-\alpha} \sup_{\ell \in \mathbb{N}_0} \left(\int_0^\infty \mathrm{d}x \, x^{\frac{q\alpha s}{1-\alpha}+2} |j_\ell(x)|^{\frac{q}{1-\alpha}(2-\varepsilon)}\right)^{\frac{1-\alpha}{q}}.$$

For $\varepsilon(\delta) > 0$ small enough, the integral is finite by the uniform L^p-integrability of spherical Bessel functions (see Proposition 16 (iv)) since $\alpha < (5q-7)/(3qs-7)$ and thus the claim follows since $\frac{\alpha s + 3(1-\alpha)/q}{2} = \beta_p^* - \delta/2$ (cf. Eq. (24) for the definition of β_p^*, and Eq. (45)). □

Proof of Lemma 13 To begin with the proof, we have two important observations.

First, recall the definition of $s_\pm^*$ from Eq. (7). We aim to prove that $r_\pm^* = s_\pm^*$, where

$$r_\pm^* := \sup\left\{r \geq 0 : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm(x)\mathrm{d}x = 0\right\}.$$

For this purpose, we define

$$s_\pm^*(a) := \sup\left\{s \geq 0 : |\cdot|^{-s} V_\pm|_{B_a}^* \in L^1(\mathbb{R}^3)\right\}$$

and

$$r_\pm^*(a) := \sup\left\{r \geq 0 : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm|_{B_a}^*(x) \, \mathrm{d}x = 0\right\}$$

for the same $a > 0$, for which we assumed that $r_\pm^* = r_\pm^*(a)$ in Definition 5.

Note that $r_\pm^* \geq s_\pm^*$ by definition. Using that $|\cdot|^{-s}$ is equal to its symmetric decreasing rearrangement, we can employ the basic rearrangement inequality [19, Thm. 3.4] to obtain $s_\pm^* \geq s_\pm^*(a)$. Therefore, since $r_\pm^* = r_\pm^*(a)$ by assumption, we have

$$r_\pm^*(a) = r_\pm^* \geq s_\pm^* \geq s_\pm^*(a).$$

In order to see $r_\pm^* = s_\pm^*$ it is sufficient to prove that $s_\pm^*(a) \geq r_\pm^*(a)$. Assume the contrary, i.e. $s_\pm^*(a) < r_\pm^*(a)$, and let r, $r + \delta \in (s_\pm^*(a), r_\pm^*(a))$ for some $\delta > 0$. We denote $V_{\pm,a}^* \equiv V_\pm|_{B_a}^*$ for short. By definition of $s_\pm^*$ and $r_\pm^*$, we thus have

$$\int_{B_\varepsilon} \frac{V_{\pm,a}^*(x)}{|x|^r} dx \geq c \quad \text{and} \quad \int_{B_\varepsilon} V_{\pm,a}^*(x) dx = o(\varepsilon^{r+\delta}). \tag{46}$$

The first integral actually equals infinity, but we only need that it is uniformly bounded from below by some $c > 0$. Since $V_{\pm,a}^*$ is symmetric-decreasing and thus one-sided limits exist, the auxiliary quantity

$$t_\pm^*(a) := \inf\left\{ t \geq 0 : \lim_{|x|\to 0} |x|^t \, V_{\pm,a}^*(x) = 0 \right\}$$

is well defined. By definition of $t_\pm^*(a)$ we thus get

$$\frac{c_\nu}{|x|^{t_\pm^*(a)-\nu}} \leq V_{\pm,a}^*(x) \leq \frac{C_\nu}{|x|^{t_\pm^*(a)+\nu}}$$

for any $\nu > 0$ and $|x|$ small enough. Inserting this in Eq. (46) we arrive at

$$\varepsilon^{3-t_\pm^*(a)-r-\nu} \geq c_\nu \quad \text{and} \quad \varepsilon^{3-t_\pm^*(a)-r-\delta+\nu} \leq C_\nu$$

which yields a contradiction by choosing $\nu \in (0, \delta/2)$. Therefore, $r_\pm^*(a) = s_\pm^*(a)$, which proves that $r_\pm^* = s_\pm^*$.

Second, note that for any $f \in L^1(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} f(x)(\sin(n|x|))^2 dx = \frac{1}{2}\int_{\mathbb{R}^3} f(x)(1 - \cos(2n|x|)) dx \longrightarrow \frac{1}{2}\int_{\mathbb{R}^3} f(x) dx$$

as $n \to \infty$ by the Riemann–Lebesgue Lemma.

In order to prove Lemma 13, we study the asymptotic behavior of the integral

$$v_\mu := \int_{\mathbb{R}^3} V(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx$$

in three different cases.

Case 1. If $|\cdot|^{-2} V \in L^1(\mathbb{R}^3)$, we get by our second observation that

$$v_\mu = \mu^{-1} \left(\frac{1}{2}\int_{\mathbb{R}^3} \frac{V(x)}{|x|^2} dx + o(1) \right) \leq -c\mu^{-1},$$

which immediately proves the claim.

Case 2. If $|\cdot|^{-2}V \notin L^1(\mathbb{R}^3)$ and $s^* < 2$ we take some $r \in (0, 1/2)$ and estimate

$$\int_{\mathbb{R}^3} V(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx$$

$$= \int_{B_r} V_+(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx - \int_{B_r} V_-(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx$$

$$+ O(\mu^{-1}).$$

The first term can be bounded by $\mu^{-s_+^*/2+\delta}$ for any $\delta > 0$. The second term can be estimated from below as

$$\int_{B_r} V_-(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx \geq \int_{B_{\frac{r}{\sqrt{\mu}}}} V_-(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx$$

$$\geq c_r \int_{B_{\frac{r}{\sqrt{\mu}}}} V_-(x) dx \geq c_{r,\delta} \mu^{-s_-^*/2-\delta}$$

for any $\delta > 0$. Since $s^* = s_-^* < s_+^*$, we get that $v_\mu \leq -c_\delta \mu^{-\min(s^*+\delta,2)/2}$ for any $\delta > 0$.

Case 3. If $|\cdot|^{-2}V \notin L^1(\mathbb{R}^3)$ and $s^* = 2$ we have that $|\cdot|^{-2}V_+ \in L^1(\mathbb{R}^3)$ but $|\cdot|^{-2}V_- \notin L^1(\mathbb{R}^3)$ since $s^* = s_-^* < s_+^*$. On the one hand, this implies that

$$\int_{\mathbb{R}^3} V_+(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx \leq K\mu^{-1}$$

for some $K > 0$ by means of our second observation. On the other hand, let $r > 0$ and estimate

$$\mu \int_{\mathbb{R}^3} V_-(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx \geq \mu \int_{B_r^c} V_-(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 dx \xrightarrow{\mu\to\infty} \frac{1}{2} \int_{B_r^c} \frac{V_-(x)}{|x|^2} dx.$$

By taking $r \to 0$ the right hand side can be made arbitrarily large, in particular greater than K. This implies that $v_\mu \leq -C\mu^{-1}$ for any $C > 0$. $\qquad\square$

3.4 Properties of Spherical Bessel Functions

Proposition 16 (Properties of spherical Bessel functions [1,15,22]) *The spherical Bessel functions $(j_\ell)_{\ell \in \mathbb{N}_0}$ satisfy the following properties:*

(i) uniform boundedness, i.e. $\sup_{\ell \in \mathbb{N}_0} \sup_{x \geq 0} |j_\ell(x)| \leq 1$,
(ii) uniform Lipschitz continuity, i.e. $\sup_{\ell \in \mathbb{N}_0} \sup_{x \geq 0} |j_\ell'(x)| \leq 1$,
(iii) (uniform) decay, i.e. for every $\ell \in \mathbb{N}_0$, we have $\sup_{x \geq 0} |x \, j_\ell(x)| \leq C_\ell$ for some $C_\ell > 0$, and $\sup_{\ell \in \mathbb{N}_0} \sup_{x \geq 0} |x^{5/6} j_\ell(x)| \leq C$ for some universal $C > 0$,

(iv) uniform L^p-integrability, i.e. for $p \in (0, \infty)$ and $a \in (-1, p-1)$ if $p \in (0, 4]$ or $a \in (-1, 5p/6 - 1/3)$ if $p \in (4, \infty)$, we have

$$\sup_{\ell \in \mathbb{N}_0} \int_0^\infty |j_\ell(x)|^p x^a \mathrm{d}x \leq C_{p,a}$$

for some universal constant $C_{p,a} > 0$.

Proof The first statement (i) is an elementary property of the spherical Bessel functions. The second statement (ii) follows from the uniform boundedness in (i) and the recursion relation [1, Eq. 10.1.20]

$$j_\ell' = \frac{1}{2\ell + 1} \left(\ell j_{\ell-1} - (\ell + 1) j_{\ell+1} \right).$$

By noticing that $j_\ell(x) = \sqrt{\pi/(2x)} J_{\ell+1/2}(x)$, the third (iii) and the fourth statement (iv) are easy consequence of [1, Eq. 9.2.1], [15, Eq. 1], and [22, Eq. 3], respectively, where analogous estimates for the standard Bessel functions J_ν with $\nu \geq 0$ are proven. □

3.5 Proof of Equation (13)

Proof of Equation (13) We note that $K_{T,\mu}(p) + V(x) \geq 0$ is equivalent to $K_{T/\mu,1}(p) + \frac{1}{\mu} V(x/\sqrt{\mu}) \geq 0$ and estimate

$$
\begin{aligned}
K_{T/\mu,1}(p) + \frac{1}{\mu} V(x/\sqrt{\mu}) &\geq \frac{1}{2} \left(|p^2 - 1| + \frac{2T}{\mu} \right) - \frac{1}{\mu} V_-(x/\sqrt{\mu}) \\
&\geq \frac{1}{2} \left(|p^2 - 1| + \frac{2T}{\mu} - \frac{2}{\mu} V_-(x/\sqrt{\mu}) \left(e^{-m|x|} + m|x| \right) \right) \\
&\geq \frac{1}{2} \left(|p^2 - 1| + \frac{2T}{\mu} - \frac{2}{\sqrt{\mu}} \| |\cdot| V \|_{L^\infty} \left(\frac{e^{-m|x|}}{|x|} + m \right) \right)
\end{aligned}
$$

for any $m > 0$. By definition of T_c, we have the bound

$$T_c \leq -\frac{\mu}{2} \inf \operatorname{spec} \left(|p^2 - 1| - \frac{2}{\sqrt{\mu}} \| |\cdot| V \|_{L^\infty} \left(\frac{e^{-m|x|}}{|x|} + m \right) \right).$$

After taking $m = (\text{const.}) \mu^{1/4} e^{-\sqrt{\pi/(2\| |\cdot| V \|_{L^\infty})} \mu^{1/4}}$ and using the estimate above Eq. (15) in [7], we get

$$T_c \lesssim \mu \exp \left(-\sqrt{\frac{\pi}{2\| |\cdot| V \|_{L^\infty}}} \mu^{1/4} \right).$$

Data Availability Not applicable.

Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

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