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On the Volume of Sections of the Cube

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Abstract: We study the properties of the maximal volume k-dimensional sections of the n-dimensional cube $[-1, 1]^n$. We obtain a first order necessary condition for a k-dimensional subspace to be a local maximizer of the volume of such sections, which we formulate in a geometric way. We estimate the length of the projection of a vector of the standard basis of $\mathbb{R}^n$ onto a k-dimensional subspace that maximizes the volume of the intersection. We find the optimal upper bound on the volume of a planar section of the cube $[-1, 1]^n$, $n \geq 2$.

Keywords: Tight frame; section of cube; volume; Ball's inequality

MSC: 52A38, 49Q20, 52A40, 15A45

1 Introduction

The problem of volume extrema of the intersection of the standard n-dimensional cube $\square^n = [-1, 1]^n$ with a k-dimensional linear subspace H has been studied intensively. The tight lower bound for all $n \geq k \geq 1$ was obtained by J. Vaaler [17], he showed that

$$\operatorname{vol}_k \square^k \leq \operatorname{vol}_k(\square^n \cap H).$$

A. Akopyan and R. Karasev [1] gave a new proof of Vaaler's inequality in terms of waists. A deep generalization of Vaaler's result for ℓ_p^n balls was made by M. Meyer and A. Pajor [15]. K. Ball in [3], using his celebrated version of the Brascamb-Lieb inequality, found the following upper bounds

$$\operatorname{vol}_k(\square^n \cap H) \leq \left(\frac{n}{k}\right)^{k/2} \operatorname{vol}_k \square^k \qquad \text{and} \qquad \operatorname{vol}_k(\square^n \cap H) \leq \left(\sqrt{2}\right)^{n-k} \operatorname{vol}_k \square^k. \tag{1.1}$$

The leftmost inequality here is tight if and only if $k|n$ (see [11]), and the rightmost one is tight whenever $2k \geq n$. Thus, if k does not divide n and $2k < n$, the maximal volume of a section of $\square^n$ remains unknown.

Before proving inequality (1.1), K. Ball obtained a particular case of it in [2], namely, the hyperplane case $k = n - 1$. Using the Fourier transform, he reduced the hyperplane case to a certain integral inequality. A simpler than original proof of this integral inequality was introduced later by F. Nazarov and A. Podkorytov [16]. Also, for the hyperplane case $k = n - 1$, the rightmost inequality in (1.1) was generalized to certain product measures which include Gaussian type measures by A. Koldobsky and H. König [12]. The Gaussian measure of sections of the cube was studied in [4] and [19]. A. Koldobsky [13] used the Fourier transform of a power of the radial function to express the volume of central hyperplane sections of star bodies in $\mathbb{R}^d$ and confirmed the Meyer and Pajor conjecture from [15] related to the volume of central sections of ℓ_p^n balls. This result was recently generalized by A. Eskenazis [7] (see also [6]). We refer the reader interested in the interaction between Convex Geometry and Fourier Analysis to [14].

In [9], a tight bound on the volume of a section of $\square^n$ by a k-dimensional linear subspace was conjectured for all $n > k \geq 1$.

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Conjecture 1. *If the maximal volume of a section of the cube $\square^n$ by a k-dimensional subspace of $\mathbb{R}^n$ is attained at a subspace H, then $\square^n \cap H$ is an affine cube.*

Let $C_\square(n,k)\operatorname{vol}_k \square^k$ be the maximal volume of a section of $\square^n$ by a k-dimensional subspace L such that $\square^n \cap L$ is an affine cube. Conjecture 1 states that for any k-dimensional subspace H of $\mathbb{R}^n$, one has

$$\operatorname{vol}_k(\square^n \cap H) \leq C_\square(n,k)\operatorname{vol}_k \square^k.$$

A complete description of a k-dimensional subspace L of $\mathbb{R}^n$ such that the section $\square^n \cap L$ is an affine cube of volume $C_\square(n,k)\operatorname{vol}_k \square^k$ is given in the following lemma.

Lemma 1.1. *The constant $C_\square(n,k)$ is given by*

$$C_\square^2(n,k) = \left\lceil \frac{n}{k} \right\rceil^{n-k\lfloor n/k \rfloor} \left\lfloor \frac{n}{k} \right\rfloor^{k-(n-k\lfloor n/k \rfloor)}. \tag{1.2}$$

and is attained at the subspaces given by the following rule.

1. *We partition $\{1,2,\dots,n\}$ into k sets such that the cardinalities of any two sets differ by at most one.*
2. *Let $\{i_1,\dots,i_\ell\}$ be one of the sets of the partition. Then, choosing arbitrary signs, we write the system of linear equations*
$$\pm x[i_1] = \ldots = \pm x[i_\ell],$$
where $x[i]$ denotes the i-th coordinate of x in $\mathbb{R}^n$.
3. *Our subspace is the solution of the system of all equations written for each set of the partition at step* (2).

Since Lemma 1.1 was proven in [9], we provide a sketch of its proof in Appendix A.

In this paper, we continue our study of maximizers of

$$G(H) = \operatorname{vol}_k(\square^n \cap H), \quad H \in \operatorname{Gr}(n,k) \text{ with } n > k > 1. \tag{1.3}$$

Using the approach of [10], which is described in detail below, we get a geometric first order necessary condition for H to be a local maximizer of (1.3).

Theorem 1.1. *Let H be a local maximizer of* (1.3), *v_i be the projection of the i-th vector of the standard basis of $\mathbb{R}^n$ onto H, $i \in \{1,2,\dots,n\}$. Denote $P = \square^n \cap H$; we understand P as a k-dimensional polytope in H. Then*

1. $P = \bigcap\limits_{i=1}^{n} \{x \in H : |\langle x, v_i\rangle| \leq 1\}$.
2. *For every $i \in \{1,2,\dots,n\}$, $v_i \neq 0$ and the intersection of P with the hyperplane $\{\langle x, v_i\rangle = 1\}$ is a facet of P.*
3. *For every $i \in \{1,2,\dots,n\}$, the line* $\operatorname{span}\{v_i\}$ *intersects the boundary of P in the centroid of a facet of P.*
4. *Let F be a facet of P. Denote $P_F = \operatorname{co}\{0, F\}$. Then*
$$\frac{\operatorname{vol}_k P_F}{\operatorname{vol}_k P} = \frac{1}{2}\frac{\sum_\star |v_i|^2}{k},$$
where the summation is over all indices $i \in \{1,2,\dots,n\}$ such that the line $\operatorname{span}\{v_i\}$ *intersects F in its centroid.*

One of the arguments used by K. Ball to prove the rightmost inequality in (1.1) is that the projection of a vector of the standard basis onto a maximizer of (1.3) for $2k \geq n$ has length at least $\sqrt{2}$. We prove the following extension of this result.

Theorem 1.2. *Let $n > k > 1$ and H be a global maximizer of* (1.3), *v be the projection of a vector of the standard basis of $\mathbb{R}^n$ onto H. Then*

$$\frac{k}{n+k} \leq |v|^2 \leq \frac{k}{n-k}.$$

Using these results and some additional geometric observations, we prove the following.

Theorem 1.3. *Conjecture 1 is true for $k = 2$ and $n \geq 3$. That is, for any two-dimensional subspace $H \subset \mathbb{R}^n$ the following inequality holds*

$$\operatorname{Area}(\square^n \cap H) \leq C_\square(n, 2)\operatorname{vol}_2 \square^2 = 4\sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor}.$$

This bound is optimal and is attained if and only if $\square^n \cap H$ is a rectangle with the sides of lengths $2\sqrt{\lceil \frac{n}{2} \rceil}$ and $2\sqrt{\lfloor \frac{n}{2} \rfloor}$.

2 Definitions and Preliminaries

For a positive integer n, we refer to the set $\{1, 2, \dots, n\}$ as $[n]$. The standard n-dimensional cube $[-1, 1]^n$ is denoted by $\square^n$. We use $\langle p, x \rangle$ to denote the *standard inner product* of vectors p and x in $\mathbb{R}^n$. For vectors $u, v \in \mathbb{R}^n$, their *tensor product* (or, diadic product) is the linear operator on $\mathbb{R}^n$ defined as $(u \otimes v)x = \langle u, x \rangle v$ for every $x \in \mathbb{R}^d$. The linear hull of a subset S of $\mathbb{R}^n$ is denoted by span S. For a k-dimensional linear subspace H of $\mathbb{R}^n$ and a body $K \subset H$, we denote by $\operatorname{vol}_k K$ the k-dimensional volume of K. The two-dimensional volume of a body $K \subset \mathbb{R}^2$ is denoted by Area K. We denote the identity operator on a linear subspace $H \subset \mathbb{R}^n$ by I_H. If $H = \mathbb{R}^k$, we use I_k for convenience.

For a non-zero vector $v \in \mathbb{R}^k$, we denote by H_v the affine hyperplane $\{x \in \mathbb{R}^k : \langle x, v \rangle = 1\}$, and by H_v^+ and H_v^- the half-spaces $\{x \in \mathbb{R}^k : \langle x, v \rangle \leq 1\}$ and $\{x \in \mathbb{R}^k : \langle x, v \rangle \geq -1\}$, respectively.

It is convenient to identify a section of the cube with a convex polytope in $\mathbb{R}^k$. Let $\{v_1, \dots, v_n\}$ be the projections of the standard basis of $\mathbb{R}^n$ onto H. Clearly,

$$\square^n \cap H = \bigcap_{i=1}^{n} \{x \in H : |\langle x, v_i \rangle| \leq 1\}.$$

That means that a section of $\square^n$ is determined by the set of vectors $\{v_i\}_1^n \subset H$, which are the projections of the orthogonal basis. Such sets of vectors have several equivalent description and names.

Definition 1. We will say that an ordered n-tuple of vectors $\{v_1, \dots, v_n\} \subset H$ is *a tight frame* (or forms a tight frame) in a vector space H if

$$\left(\sum_1^n v_i \otimes v_i\right)\Bigg|_H = I_H, \tag{2.1}$$

where I_H is the identity operator in H and $A|_H$ is the restriction of an operator A onto H.
We use $\Omega(n, k)$ to denote the set of all tight frames with n vectors in $\mathbb{R}^k$.

Definition 2. An n-tuple of vectors in a linear space H that spans H is called a *frame* in H.

In the following trivial lemma we understand $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of vectors, whose last $n - k$ coordinates are zero. For convenience, we will consider $\{v_i\}_1^n \subset \mathbb{R}^k \subset \mathbb{R}^n$ as k-dimensional vectors.

Lemma 2.1. *The following assertions are equivalent:*

1. *the vectors $\{v_1, \dots, v_n\} \subset \mathbb{R}^k$ form a tight frame in $\mathbb{R}^k$;*
2. *there exists an orthonormal basis $\{f_1, \dots, f_n\}$ of $\mathbb{R}^n$ such that v_i is the orthogonal projection of f_i onto $\mathbb{R}^k$, for any $i \in \{1, 2, \dots, n\}$;*
3. $\operatorname{span}\{v_1, \dots, v_n\} = \mathbb{R}^k$ *and the Gram matrix Γ of vectors $\{v_1, \dots, v_n\} \subset \mathbb{R}^k$ is the matrix of the projection operator from $\mathbb{R}^n$ onto the linear hull of the rows of the matrix $M = (v_1, \dots, v_n)$.*
4. *the $k \times n$ matrix $M = (v_1, \dots, v_n)$ is a sub-matrix of an orthogonal matrix of order n.*

It follows that the tight frames in $\mathbb{R}^k$ are exactly the projections of orthonormal bases onto $\mathbb{R}^k$. This observation allows us to reformulate the problems in terms of tight frames and associated polytopes in $\mathbb{R}^k$. Indeed, identifying H with $\mathbb{R}^k$, we identify the projection of the standard basis onto H with a tight frame $\{v_1, \ldots, v_n\} \in \Omega(n,k)$. Thus, we identify $\square^n \cap H$ with the intersection of slabs of the form $H_{v_i}^+ \cap H_{v_i}^-$, $i \in \{1, 2, \ldots, n\}$. Vice versa, assertion (3) gives a way to reconstruct H from a given tight frame $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^k$.

Definition 3. We will say that an n-tuple $S = \{v_1, \ldots, v_n\} \in \mathbb{R}^k$ generates

1. the polytope
$$\square(S) = \bigcap_{i=1}^{n} \left(H_{v_i}^+ \cap H_{v_i}^-\right), \tag{2.2}$$
which we call the *section of the cube generated by S*;
2. the matrix $\sum\limits_{i=1}^{n} v_i \otimes v_i$. We use A_S to denote this matrix.

To sum up, the global extrema of (1.3) coincide with that of
$$F(S) = \operatorname{vol}_k \square(S), \text{ where } S \in \Omega(n,k) \text{ with } n > k > 1. \tag{2.3}$$

To compare the local extrema of (1.3) and (2.3), we have to define metrics on $\Omega(n,k)$ and on the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$. We endow the set of n-tuples of vectors in $\mathbb{R}^k$ with the metric
$$\operatorname{dist}(\{v_1, \ldots, v_n\}, \{u_1, \ldots, u_n\}) = \left(\sum_{i=1}^{n} |v_i - u_i|^2\right)^{1/2}.$$
This defines the metric on the set of frames in $\mathbb{R}^k$ consisting of n vectors and on $\Omega(n,k)$. The standard metric on the Grassmanian is given by
$$\operatorname{Dist}(H, H') = \left\| P^H - P^{H'} \right\|,$$
where $\|\cdot\|$ denotes the operator norm and P^H and $P^{H'}$ are the orthogonal projections onto the k-dimensional subspaces H and H', respectively. It was shown in [10] that the local extrema of (2.3) coincide with that of (1.3). However, we note that there is an ambiguity when we identify H with $\mathbb{R}^k$. Any choice of orthonormal basis of H gives its own tight frame in $\mathbb{R}^k$, all of them are isometric but different from each other. It is not a problem as there exists a one-to-one correspondence between $\operatorname{Gr}(n,k)$ and $\frac{\Omega(n,k)}{\mathrm{O}(k)}$, where $\mathrm{O}(k)$ is the orthogonal group in dimension k. And, clearly, $F(S_1) = F(S_2)$ whenever $S_2 = U(S_1)$ for some $U \in \mathrm{O}(k)$.

From now on, we will study properties of the maximizers of (2.3) and work with tight frames.

3 Operations on frames

The following approach to problem (2.3) was proposed in [10] and used in [9] to study the properties of projections of the standard cross-polytope.

The main idea is to transform a given tight frame S into a new one S' and compare the volumes of the sections of the cube generated by them. Since it is not very convenient to transform a given tight frame into another one, we add an intermediate step: we transform a tight frame S into a frame $\tilde{S}$, and then we transform $\tilde{S}$ into a new tight frame S' using a linear transformation. The main observation here is that we can always transform any frame $\tilde{S} = \{v_1, \ldots, v_n\}$ into a tight frame S' using a suitable linear transformation L: $S' = L\tilde{S} = \{Lv_1, \ldots, Lv_n\}$. Equivalently, any non-degenerate centrally symmetric polytope in $\mathbb{R}^k$ is an affine image of a section of a high dimension cube.

For a frame S in $\mathbb{R}^k$, by definition put
$$B_S = A_S^{-\frac{1}{2}} = \left(\sum_{v \in S} v \otimes v\right)^{-\frac{1}{2}}.$$

The operator B_S is well-defined as the condition span $S = \mathbb{R}^k$ implies that A_S is a positive definite operator. Clearly, for any frame S, B_S maps S to the tight frame $B_S S$:

$$\sum_{v \in S} B_S v \otimes B_S v = B_S \left(\sum_{v \in S} v \otimes v \right) B_S^T = B_S A_S B_S = I_k.$$

We obtain the following necessary and sufficient condition for a tight frame to be a maximizer of (2.3).

Lemma 3.1. *The maximum of* (2.3) *is attained at a tight frame* $S \in \Omega(n,k)$ *iff for an arbitrary frame* $\tilde{S}$ *in* $\mathbb{R}^k$ *inequality*

$$\frac{\operatorname{vol}_k \square(\tilde{S})}{\operatorname{vol}_k \square(S)} \leq \frac{1}{\sqrt{\det A_{\tilde{S}}}} \tag{3.1}$$

holds.

Proof. For any frame $\tilde{S}$, we have that $B_{\tilde{S}}\tilde{S}$ is a tight frame and $\operatorname{vol}_k \square(B_{\tilde{S}}\tilde{S}) = \operatorname{vol}_k \square(\tilde{S}) / \det B_{\tilde{S}}$. The maximum of (2.3) is attained at a tight frame S iff $\operatorname{vol}_k \square(B_{\tilde{S}}\tilde{S}) \leq \operatorname{vol}_k \square(S)$ for an arbitrary frame $\tilde{S}$. Hence

$$1 \geq \frac{\operatorname{vol}_k \square(B_{\tilde{S}}\tilde{S})}{\operatorname{vol}_k \square(S)} = \frac{\operatorname{vol}_k \square(\tilde{S})}{\det B_{\tilde{S}}} \frac{1}{\operatorname{vol}_k \square(S)} = \frac{\operatorname{vol}_k \square(\tilde{S})}{\operatorname{vol}_k \square(S)} \sqrt{\det A_{\tilde{S}}}$$

Dividing by $\sqrt{\det A_{\tilde{S}}}$, we obtain inequality (3.1). □

Clearly, if $\tilde{S}$ in the assertion of Lemma 3.1 is close to S, then the tight frame $B_{\tilde{S}}\tilde{S}$ is close to S as well. Therefore, inequality (3.1) gives a necessary condition for local maximizers of (2.3). Let us illustrate how we will use it.

Let S be an extremizer of (2.3), and T be a map from a subset of $\Omega(n,k)$ to the set of frames in $\mathbb{R}^k$. In order to obtain properties of extremizers, we consider a composition of two operations:

$$S \xrightarrow{T} \tilde{S} \xrightarrow{B_{\tilde{S}}} S',$$

where $B_{\tilde{S}}$ is as defined above. For example, see Figure 1, where T is the operation of replacing a vector v of S by the origin.

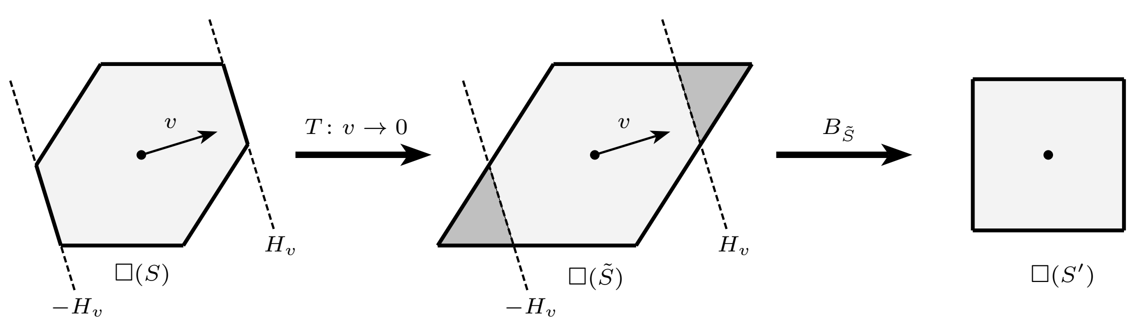

Figure 1: A frame $S \subset \mathbb{R}^2$ consists of three vectors and $\square(S)$ is a hexagon. A vector $v \in S$ is mapped to the origin, yielding the frame $\tilde{S}$. By construction, $\square(\tilde{S})$ is a parallelogram. Since the frame $S' = B_{\tilde{S}}\tilde{S}$ is a tight frame with only two nonzero vectors, these vectors are two orthogonal unit vectors. Hence $\square(S')$ is a square.

Choosing a simple operation T, we may calculate the left-hand side of (3.1) in some geometric terms. We consider several simple operations: Scaling one or several vectors, mapping one vector to the origin, mapping one vector to another. On the other hand, the determinant in the right-hand side of (3.1) can be calculated for the operations listed above.

In particular, the following first-order approximation of the determinant was obtained by the first author in [10, Theorem 1.2]. We provide a sketch of its proof in Appendix A.

Lemma 3.2. *Let $S = \{v_1, \dots, v_n\} \subset \mathbb{R}^k$ be a tight frame and the n-tuple $\tilde{S}$ be obtained from S by substitution $v_i \to v_i + tx_i$, where $t \in \mathbb{R}$, $x_i \in \mathbb{R}^k$, $i \in [n]$. Then*

$$\sqrt{\det A_{\tilde{S}}} = 1 + t\sum_{i=1}^{n}\langle x_i, v_i\rangle + o(t).$$

We state as lemmas several technical facts from linear algebra that will be used later.

Lemma 3.3. *Let A be a positive definite operator on $\mathbb{R}^k$. For any $u \in \mathbb{R}^k$, we have*

$$\det(A \pm u \otimes u) = 1 \pm \left(|A^{-1/2}u|^2\right)\det A.$$

Proof. We have

$$\det(A \pm u \otimes u) = \det A \cdot \det\left(I_k \pm A^{-1/2}u \otimes A^{-1/2}u\right).$$

Diagonalizing the operator $I_k \pm A^{-1/2}u \otimes A^{-1/2}u$, we see that its determinant equals $1 \pm \left|A^{-1/2}u\right|^2$. This completes the proof. □

For any n-tuple S of vectors of $\mathbb{R}^k$ with $v \in S$, we use $S \setminus v$ to denote the $(n-1)$-tuple of vectors obtained from S by removing the first occurrence of v in S.

Lemma 3.4. *Let $S \in \Omega(n,k)$ and $v \in S$ be a vector such that $|v| < 1$. Then $S \setminus v$ is a frame in $\mathbb{R}^k$ and $B_{S\setminus v}$ is the stretch of $\mathbb{R}^k$ by the factor $\left(1-|v|^2\right)^{-1/2}$ along $\operatorname{span}\{v\}$. In particular, for any $u \in \mathbb{R}^k$, we have $\left|B_{S\setminus v}u\right| \geq |u|$.*

Proof. Since $A_{S\setminus v} = I_k - v \otimes v > 0$, we have that $S \setminus v$ is a frame in $\mathbb{R}^k$. Clearly, $A_{S\setminus v}$ stretches the space by the factor $\left(1-|v|^2\right)$ along $\operatorname{span}\{v\}$. Therefore, the operator $B_{S\setminus v} = A_{S\setminus v}^{-1/2}$ stretches the space by the factor $\left(1-|v|^2\right)^{-1/2}$ along the same direction. □

4 Properties of a global maximizer

Theorem 1.2 is formulated in terms of subspaces. For the sake of convenience, we introduce its equivalent reformulation in terms of tight frames.

Theorem 4.1 (Frame version of Theorem 1.2). *Let $S \in \Omega(n,k)$ be a global maximizer of* (2.3) *for $n > k > 1$. Pick an arbitrary $v \in S$. Then*

$$\frac{k}{n+k} \leq |v|^2 \leq \frac{k}{n-k}. \tag{4.1}$$

Proof. The definition of tight frame implies that $\sum_{p\in S}|p|^2 = k$. Hence there is a vector $u \in S$ such that $|u|^2 \leq k/n$ and a vector $w \in S$ such that $|w|^2 \geq k/n$.

We start with the rightmost inequality in (4.1). Let $\tilde{S}$ be the n-tuple obtained from S by substitution $u \to v$. Since $|u|^2 \leq k/n < 1$, we have $A_{\tilde{S}} = I_k - u\otimes u + v \otimes v \geq I_k - u \otimes u > 0$. Hence $\tilde{S}$ is a frame in $\mathbb{R}^k$. By identity (2.2), inclusion $\square(S) \subset \square(\tilde{S})$ holds. Therefore, $\operatorname{vol}_k \square(S) \leq \operatorname{vol}_k \square(\tilde{S})$. Using Lemma 3.1 and Lemma 3.3, we obtain

$$1 \geq \det A_{\tilde{S}} = \left(1 + \left|B_{S\setminus u}v\right|^2\right)\left(1-|u|^2\right).$$

By Lemma 3.4, the right-hand side of this inequality is at least $\left(1+|v|^2\right)\left(1-|u|^2\right)$. Therefore,

$$|v|^2 \leq \frac{|u|^2}{1-|u|^2} \leq \frac{k}{n-k}.$$

Let us prove the leftmost inequality in (4.1). There is nothing to prove if $|v|^2 \geq k/n$. Assume that $|v| < k/n$. Let $\tilde{S}$ be the n-tuple obtained from S by substitution $v \to w$. Since $A_{\tilde{S}} = I - v\otimes v + u \otimes u > 0$, $\tilde{S}$ is a frame in $\mathbb{R}^k$.

By identity (2.2), the inclusion $\square(S) \subset \square(\tilde{S})$ holds. Therefore, $\mathrm{vol}_k \square(S) \le \mathrm{vol}_k \square(\tilde{S})$. Using Lemma 3.1 and Lemma 3.3, we obtain

$$1 \ge \det A_{\tilde{S}} = \left(1 + \left|B_{S\setminus v} w\right|^2\right)\left(1 - |v|^2\right).$$

Again, by Lemma 3.4, the right-hand side of this inequality is at least $\left(1 + |w|^2\right)\left(1 - |v|^2\right)$. It follows that

$$|v|^2 \ge \frac{|w|^2}{1 + |w|^2} \ge \frac{k}{n+k}.$$

This completes the proof. □

Clearly, Theorem 4.1 implies Theorem 1.2.

These theorems can be sharpened in the case of sections by planes, i.e. $k = 2$.

Lemma 4.1. *Let $S \in \Omega(n, 2)$ be a maximizer of (2.3) for $k = 2$ and $n \ge 3$, let $v \in S$. Then*

$$\frac{2}{n+1} \le |v|^2 \le \frac{2}{n-1}. \tag{4.2}$$

Proof. Lemma 1.1 provides an example of a two-dimensional subspace H satisfying

$$\mathrm{Area}\left(\square^n \cap H\right) = 4C_\square(n, 2) = 4\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}.$$

Therefore, the maximal area of a planar section of the cube $\square^n$ is at least $4\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}$. Thus, we have

$$\mathrm{Area}\, \square(S) \ge 4\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}. \tag{4.3}$$

Let us prove the leftmost inequality in (4.2). It is trivial if $|v| \ge 2/n$. Assume that $|v| < 2/n$. Let $S' \in \Omega(n-1, 2)$ be a maximizer of (2.3). By Ball's inequality (1.1), we have

$$\mathrm{Area}\, \square(S') \le 2(n-1). \tag{4.4}$$

Consider $S \setminus v$. It is a frame by Lemma 3.4. Then, by Lemma 3.1 and Lemma 3.3, we get

$$\frac{\mathrm{Area}\, \square(S \setminus v)}{\mathrm{Area}\, \square(S')} \le \frac{1}{\sqrt{\det A_{S\setminus v}}} = \frac{1}{\sqrt{1 - |v|^2}}.$$

By identity (2.2), we have $\square(S) \subset \square(S \setminus v)$. By this and by inequalities (4.4) and (4.3), we get

$$\frac{\mathrm{Area}\, \square(S \setminus v)}{\mathrm{Area}\, \square(S')} \ge \frac{\mathrm{Area}\, \square(S)}{\mathrm{Area}\, \square(S')} \ge \frac{4\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}}{2(n-1)}.$$

Combining the last two inequalities, we obtain

$$|v|^2 \ge 1 - \left(\frac{\mathrm{Area}\, \square(S')}{\mathrm{Area}\, \square(S)}\right)^2 \ge 1 - \left(\frac{n-1}{2\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}}\right)^2 \ge \frac{2}{n+1}.$$

We proceed with the rightmost inequality in (4.2). Let $S' \in \Omega(n+1, 2)$ be a maximizer of (2.3). By Ball's inequality (1.1), we have

$$\mathrm{Area}\, \square(S') \le 2(n+1). \tag{4.5}$$

We use $\tilde{S}$ to denote the $(n+1)$-tuple obtained from S by concatenating S with the vector v. Since S is a frame in $\mathbb{R}^2$, $\tilde{S}$ is a frame in $\mathbb{R}^2$ as well. By Lemma 3.1 and Lemma 3.3, we get

$$\frac{\mathrm{Area}\, \square(\tilde{S})}{\mathrm{Area}\, \square(S')} \le \frac{1}{\sqrt{\det A_{\tilde{S}}}} = \frac{1}{\sqrt{1 + |v|^2}}.$$

By identity (2.2), we have $\square(S) = \square(\tilde{S})$. By this and by inequalities (4.5) and (4.3), we get

$$\frac{\mathrm{Area}\, \square(S)}{\mathrm{Area}\, \square(S')} = \frac{\mathrm{Area}\, \square(\tilde{S})}{\mathrm{Area}\, \square(S')} \ge \frac{4\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}}{2(n+1)}.$$

Combining the last two inequality, we obtain

$$|v|^2 \le \left(\frac{\mathrm{Area}\,\square(S')}{\mathrm{Area}\,\square(S)}\right)^2 - 1 \le \left(\frac{n+1}{2\sqrt{\lceil n/2\rceil\lfloor n/2\rfloor}}\right)^2 - 1 \le \frac{2}{n-1}.$$

□

Remark 1. It is possible to sharpen inequality (4.1) for $k > 2$ and $n > 2k$ using the same approach as in Lemma 4.1. The idea is to remove $n \bmod k$ from or add $n - (n \bmod k)$ vectors to a maximizer and compare the volume of a section of the cube generated by the new frame with the Ball bound (1.1). However, it doesn't give a substantial improvement.

Remark 2. As was pointed out to us by the anonymous reviewer, the results of this and the next section might be proven using the so-called shadow movement technique. Moreover, we expect that combining the result on convexity of the function related to the polar body of a symmetric shadow system [5] and suitable discrete substitutions, one could confirm Conjecture 1.

5 Local properties

In this section, we prove some properties of the local maximizers of (2.3). We will perturb facets of $\square(S)$ of a local maximizer S (that is, we will perturb the vectors of S in a specific way corresponding to a perturbation of some facets of the polytope $\square(S)$). To this end, we need to recall some general properties of polytopes connected to perturbations of a half-space supporting a polytope in its facet.

5.1 Properties of polytopes

Recall that a point c is the *centroid* of a facet F of a polytope $P \subset \mathbb{R}^k$ if

$$c = \frac{1}{\mathrm{vol}_{k-1} F}\int\limits_F x\,d\lambda, \tag{5.1}$$

where $d\lambda$ is the standard Lebesgue measure on the hyperplane containing F.

For a set $W \subset \mathbb{R}^k$, we use $P(W)$ to denote the polytopal set $\cap_{w\in W} H_w^+$. Let W be a set of pairwise distinct vectors such that

- the set $P(W)$ is a polytope;
- for every $w \in W$, the hyperplane H_w supports $P(W)$ in a facet of $P(W)$.

That is, W is the set of scaled outer normals of $P(W)$. Denote $P = P(W)$. We fix $w \in W$ and the facet $F = P \cap H_w$ of P. Let c be the centroid of F.

Transformation 1. We will "shift" a facet of a polytope parallel to itself. Let W' be obtained from W by substitution $w \to w + h\frac{w}{|w|}$, where $h \in \mathbb{R}$. Denote $P' = P(W')$. That is, the polytopal set P' is obtained from P by the shift of the half-space H_w^+ by h in the direction of its outer normal. By the celebrated Minkowski existence and uniqueness theorem for convex polytopes (see, for example, [8, Theorem 18.2]), we have

$$\mathrm{vol}_k P' - \mathrm{vol}_k P = h\,\mathrm{vol}_{k-1} F + o(h). \tag{5.2}$$

Transformation 2. We will rotate a facet around a codimension two subspace. Let u be a unit vector orthogonal to w. Define $c_w = H_w \cap \mathrm{span}\{w\}$ and $L_u = H_w \cap (u^\perp + c_w)$. Note that L_u is a codimension two affine subspace of $\mathbb{R}^k$ and an affine hyperplane in H_w. Clearly, for any non-zero $t \in \mathbb{R}$, $L_u = H_w \cap H_{w+tu} = \{x \in H_w : \langle x - c(w), u\rangle = 0\}$. Thus, L_u divides F into two parts

$$F^+ = F \cap \{x \in H_w : \langle x - c(w), u\rangle \ge 0\} \text{ and } F^- = F \cap \{x \in H_w : \langle x - c(w), u\rangle \le 0\}$$

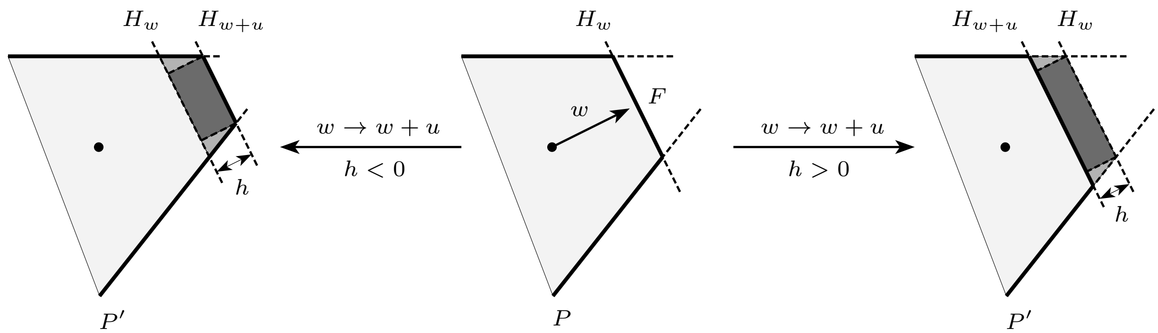

Figure 2: Parallel shift of the facet F by the vector $u = hw/|w|$

(one of the sets F^+ or F^- is empty if $c(w) \notin F$). Let $\alpha \in (-\pi/2, \pi/2)$ be the oriented angle between hyperplanes H_w and H_{w+tu} such that α is positive for positive t.

Let W' be obtained from W by substitution $w \to w + tu$, where $t \in \mathbb{R}$ and u is a unit vector orthogonal to w. Denote $P' = P(W')$. Thus, for a sufficiently small $|t|$, the polytopal set P' is a polytope obtained from P by the rotation of the half-space H_w^+ around the codimension two affine subspace L_u by some angle $\alpha = \alpha(t)$. Clearly, in order to calculate the volume of P', we need to subtract from $\operatorname{vol}_k P$ the volume of the subset of P that is above H_{w+tu} and to add to $\operatorname{vol}_k P$ the volume of the subset of P' that is above H_w. Formally speaking, denote

$$Q^+ = \begin{cases} P' \cap \left(\mathbb{R}^k \setminus H_w^+\right) & \text{for } \alpha \geq 0 \\ P \cap \left(\mathbb{R}^k \setminus H_{w+tu}^+\right) & \text{for } \alpha < 0 \end{cases} \quad \text{and} \quad Q^- = \begin{cases} P \cap \left(\mathbb{R}^k \setminus H_{w+tu}^+\right) & \text{for } \alpha \geq 0 \\ P' \cap \left(\mathbb{R}^k \setminus H_w^+\right) & \text{for } \alpha < 0 \end{cases}$$

Then, we have (see Figure 3)

$$\operatorname{vol}_k P' - \operatorname{vol}_k P = \operatorname{sign} \alpha \left(\operatorname{vol}_k Q^+ - \operatorname{vol}_k Q^-\right). \tag{5.3}$$

There is a nice approximation for $\operatorname{vol}_k Q^+ - \operatorname{vol}_k Q^-$. Let C_α^+ (resp., C_α^-) be the set swept out by F^+ (resp., F^-) while rotating around L_u by the angle α. By routine,

$$\operatorname{vol}_k C_\alpha^+ = |\alpha| \int_0^{+\infty} r \operatorname{vol}_{k-2} \left(F \cap (L_u + ru)\right) dr$$

$$\left(\text{resp.,} \quad \operatorname{vol}_k C_\alpha^- = |\alpha| \int_0^{+\infty} r \operatorname{vol}_{k-2} \left(F \cap (L_u - ru)\right) dr = -|\alpha| \int_{-\infty}^{0} r \operatorname{vol}_{k-2} \left(F \cap (L_u + ru)\right) dr\right).$$

We claim that

$$\operatorname{vol}_k Q^+ = \operatorname{vol}_k C_\alpha^+ + o(\alpha) \quad \text{and} \quad \operatorname{vol}_k Q^- = \operatorname{vol}_k C_\alpha^- + o(\alpha). \tag{5.4}$$

This can be shown in the following way. If F^+ is not a body in H_w, then $\operatorname{vol}_k Q^+ = \operatorname{vol}_k C_\alpha^+ = 0$. Assume F^+ is a body in H_w. Since a polytope is defined by a system of linear inequalities, there is a positive constant b such that for any sufficiently small τ the Hausdorff distance on $H_{w+\tau u}$ between $Q^+ \cap H_{w+\tau u}$ and $C_\alpha^+ \cap H_{w+\tau u}$ is at most $b\alpha$ (recall that the Hausdorff distance between $A, B \subset \mathbb{R}^d$ is $d_H(A,B) = \inf\left\{\varepsilon > 0 : A \subset B + \varepsilon B^d \text{ and } B \subset A + \varepsilon B^d\right\}$, where B^d is the Euclidean unit ball). Note that the section of C_α^+ by $H_{w+\tau u}$ is F^+ rotated around L_u by some small angle $\beta = \beta(\tau)$. Therefore, there exist two homothets $F_1^+ \subset H_w$ and $F_2^+ \subset H_w$ of F^+ satisfying two properties:

- both the Hausdorff distance between F_1^+ and F^+ and the Hausdorff distance between F_2^+ and F^+ are at most $\tilde{b}\alpha$ for a positive constant $\tilde{b}$;
- the body swept out by F_1^+ while rotating around L_u by the angle α is contained in Q^+ and the body swept out by F_2^+ contains Q^+.

By the first property and a simple integration, the volumes of the bodies swept out by F_1^+ and F_2^+ coincide with $\mathrm{vol}_k\, C_\alpha^+$ up to the terms of the first order of α. The second property implies the leftmost identity in (5.4). The rightmost one is obtained similarly.

By identities (5.3) and (5.4), we obtain

$$\mathrm{vol}_k P' - \mathrm{vol}_k P = \alpha \left(\mathrm{vol}_k C_\alpha^+ - \mathrm{vol}_k C_\alpha^-\right) + o(\alpha) = \alpha \int\limits_{\mathbb{R}} r\, \mathrm{vol}_{k-2} \left(F \cap (L_u + ru)\right) dr + o(\alpha).$$

By this and by (5.1), we get

$$\mathrm{vol}_k P' - \mathrm{vol}_k P = \alpha \langle c - c_w, u\rangle \mathrm{vol}_{k-1} F + o(\alpha).$$

Since w and u are orthogonal, we have

$$\alpha = \arctan \frac{|u|}{|w|} t = \frac{|u|}{|w|} t + o(t).$$

Finally, we obtain

$$\mathrm{vol}_k P' - \mathrm{vol}_k P = \frac{\mathrm{vol}_{k-1} F}{|w|} \langle c - c_w, u\rangle\, t + o(t). \tag{5.5}$$

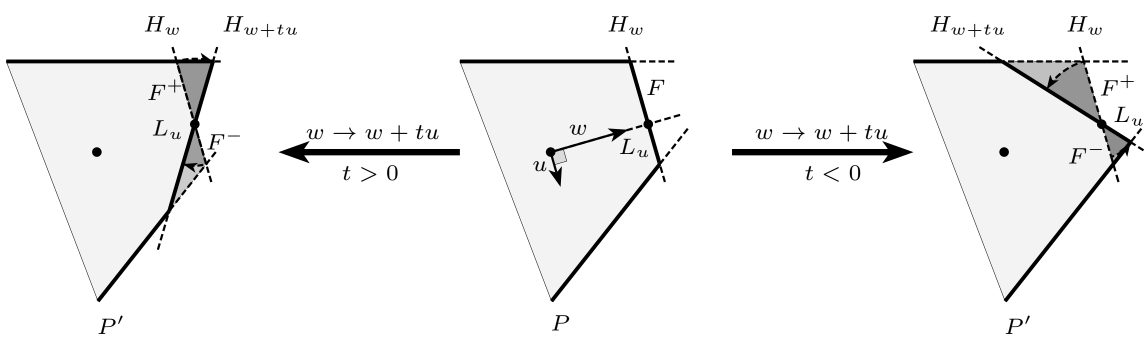

Figure 3: A rotation of the facet F around L_u

5.2 Local properties of sections of the cube

Let S be a frame in $\mathbb{R}^k$. For every $v \in S$, we denote the set $H_v \cap \square(S)$ by F_v. We say that $v \in S$ *corresponds to a facet* F of $\square(S)$ if either $F = F_v$ or $F = -F_v$. Clearly, if some vectors of S correspond to the same facet of $\square(S)$, then they are equal up to a sign. For a given frame S in $\mathbb{R}^k$ and $u \in \mathbb{R}^k$, a facet F of $\square(S)$ and a vector $u \in \mathbb{R}^k$, we define an F-substitution in the direction u as follows:

- each vector v of S such that $F \subset H_v$ is substituted by $v + u$;
- each vector v of S such that $-F \subset H_v$ is substituted by $v - u$;
- all other vectors of S remain the same.

In order to prove Theorem 1.1, we will use F-substitutions.

At first, we simplify the structure of a local maximizer.

Lemma 5.1. *Let S be a local maximizer of* (2.3) *and $v \in S$. Then F_v is a facet of* $\square(S)$.

Proof. Let K be a convex body in $\mathbb{R}^k$, then its polar body is defined by

$$\left\{y \in \mathbb{R}^k : \langle y, x\rangle \le 1 \text{ for all } x \in K\right\}.$$

Since $\square(S)$ is the intersection of half-spaces of the form $\{\langle w, x\rangle \le 1\}$ with $w \in \pm S$, we have that $\mathrm{co}\{\pm S\}$ is polar to $\square(S)$ in $\operatorname{span} S = \mathbb{R}^k$.

By the duality argument, it suffices to prove that $v \in S$ is a vertex of the polytope $\mathrm{co}\{\pm S\}$. Assume that v is not a vertex of $\mathrm{co}\{\pm S\}$.

Clearly, $v \in \mathrm{co}\{\pm(S \setminus v)\}$ and v is not a vertex of the polytope $\mathrm{co}\{\pm(S \setminus v)\}$. Therefore we have that $\operatorname{span}\{S \setminus v\} = \operatorname{span} S = \mathbb{R}^k$. That is, $S \setminus v$ is a frame in $\mathbb{R}^k$. Since $B_{S\setminus v}$ is a nondegenerate linear transformation, $B_{S\setminus v} v$ is not a vertex of the polytope $\mathrm{co}\{\pm B_{S\setminus v}(S \setminus v)\}$. By this and by the triangle inequality, there is a vertex u of $\mathrm{co}\{\pm S\}$ such that $u \in S$ and $|B_{S\setminus v} v| < |B_{S\setminus v} u|$.

Denote by $\tilde{S}$ the n-tuple obtained from S by substitution $v \to v + t(u - v)$, where $t \in (0, 1]$. Since $A_{\tilde{S}} \ge A_{S\setminus v} > 0$, $\tilde{S}$ is a frame in $\mathbb{R}^k$. By the choice of u and identity (2.2), we have $\square(\tilde{S}) = \square(S)$. Hence $\mathrm{vol}_k \square(\tilde{S}) = \mathrm{vol}_k \square(S)$. Lemma 3.1 implies that $\det A_{\tilde{S}} \le 1$.

On the other hand, by Lemma 3.3, we have

$$\det A_{\tilde{S}} = \left(1 + \left|B_{S\setminus v}(v + t(u - v))\right|^2\right)\left(1 - |v|^2\right).$$

Inequality $|B_{S\setminus v} v| < |B_{S\setminus v} u|$ implies that $|B_{S\setminus v} v| < |B_{S\setminus v}(v + t(u - v))|$. By this and by Lemma 3.4, we conclude that $\det A_{\tilde{S}} > 1$. This is a contradiction. Thus, v is a vertex of $\mathrm{co}\{\pm S\}$. The lemma is proven. □

As an immediate corollary of Lemma 5.1 and by the standard properties of polytopes, we have the following statement.

Corollary 5.1. *Let S be a local maximizer of* (2.3) *and $v \in S$. Let $\tilde{S}(t)$ be the n-tuple obtained from S by F_v-substitution in the direction tu with $t \in \mathbb{R}$ and $u \in \mathbb{R}^k$. Then, for a sufficiently small $|t|$, $\tilde{S}(t)$ is a frame, the vector $v + ut$ corresponds to a facet of $\square(\tilde{S}(t))$. Moreover,* $\mathrm{vol}_k \square(\tilde{S}(t))$ *is a smooth function of t at $t = 0$.*

In the following two lemmas, we will perturb a local maximizer by making F-substitutions. Geometrically speaking, making an F-substitution in the direction tu with $u \in \mathbb{R}^k$ and $t \in \mathbb{R}$, we move the opposite facets F and $-F$ of a local maximizer in a symmetric way. Thus, for a sufficiently small t, perturbations of the facets F and $-F$ are independent.

Lemma 5.2. *Let S be a local maximizer of* (2.3). *Let $v \in S$ and d be the number of the vectors of S that correspond to F_v. Then*

$$\frac{2}{|v|} \mathrm{vol}_{k-1} F_v = d|v|^2 \mathrm{vol}_k \square(S). \tag{5.6}$$

Proof. Denote by $\tilde{S}$ the n-tuple obtained from S by F_v-substitution in the direction tv with $t \in \mathbb{R}$. Thus, we apply Transformation 1 to the facets $\pm F_v$ of $\square(S)$. By Lemma 3.1, we have

$$\frac{\mathrm{vol}_k \square(\tilde{S})}{\mathrm{vol}_k \square(S)} \le \frac{1}{\sqrt{\det A_{\tilde{S}}}}. \tag{5.7}$$

By Lemma 3.2 and Corollary 5.1, both sides of this inequality are smooth as functions of t in a sufficiently small neighborhood of $t_0 = 0$. Consider the Taylor expansions of both sides of inequality (5.7) as functions of t about $t_0 = 0$.

By Lemma 3.3, $\det A_{\tilde{S}} = 1 + d(2t + t^2)|v|^2$. Hence

$$\frac{1}{\sqrt{\det A_{\tilde{S}}}} = 1 - td|v|^2 + o(t). \tag{5.8}$$

Geometrically speaking, we shift the half-space H_v^+ (resp., H_v^-) by

$$h = \frac{1}{|(1+t)v|} - \frac{1}{|v|} = -\frac{t}{|v|} + o(t)$$

in the directions of its outer normal. By this and by (5.2), we obtain

$$\mathrm{vol}_k \square(\tilde{S}) - \mathrm{vol}_k \square(S) = -\frac{2t}{|v|} \mathrm{vol}_{k-1} F_v + o(t). \tag{5.9}$$

Using identities (5.9) and (5.8) in (5.7) , we get

$$1 - \frac{2t}{|v|} \frac{\operatorname{vol}_{k-1} F_v}{\operatorname{vol}_k \square(S)} \le 1 - d|v|^2 t + o(t).$$

Since $\tilde{S} = S$ for $t = 0$ and the previous inequality holds for all $t \in (-\varepsilon, \varepsilon)$ for a sufficiently small ε, the coefficients of t in both sides of the previous inequality coincide. That is,

$$\frac{2}{|v|} \frac{\operatorname{vol}_{k-1} F_v}{\operatorname{vol}_k \square(S)} = d|v|^2.$$

This completes the proof. □

Lemma 5.3. *Let S be a local maximizer of* (2.3) *and $v \in S$. Then the line* $\operatorname{span}\{v\}$ *intersects the hyperplane H_v in the centroid of the facet F_v.*

Proof. Denote the centroid of F_v by c and let $c_v = \operatorname{span}\{v\} \cap H_v$. Fix a unit vector u orthogonal to v. Denote by $\tilde{S}$ the n-tuple obtained from S by F_v-substitution in the direction tu with $t \in \mathbb{R}$. Thus, we apply Transformation 2 to the facets $\pm F_v$ of $\square(S)$.

By Lemma 3.1, we have

$$\frac{\operatorname{vol}_k \square(\tilde{S})}{\operatorname{vol}_k \square(S)} \le \frac{1}{\sqrt{\det A_{\tilde{S}}}}. \tag{5.10}$$

By Lemma 3.2 and Corollary 5.1, both sides of this inequality are smooth as functions of t in a sufficiently small neighborhood of $t_0 = 0$. Consider the Taylor expansions of both sides of inequality (5.10) as functions of t about $t_0 = 0$.

By (5.5), we obtain

$$\operatorname{vol}_k \square(\tilde{S}) - \operatorname{vol}_k \square(S) = C \langle c - c_v, u \rangle\, t + o(t),$$

where $C = 2 \operatorname{vol}_{k-1} F_v / |v| > 0$. By Lemma 3.2, $\sqrt{\det A_{\tilde{S}}} = 1 + o(t)$. Therefore, inequality (5.10) takes the following form

$$1 + \frac{C}{\operatorname{vol}_k \square(S)} \langle c - c_v, u \rangle\, t + o(t) \le 1 + o(t).$$

Since $\tilde{S} = S$ for $t = 0$ and the previous inequality holds for all $t \in (-\varepsilon, \varepsilon)$ for a sufficiently small ε, the coefficients of t in both sides of the previous inequality coincide. That is, we conclude

$$\langle c - c_v, u \rangle = 0.$$

Since $c, c_v \in H_v$ and the last identity holds for all unit vectors parallel to H_v, it follows that $c = c_v$. The lemma is proven. □

As a simple consequence of Lemma 5.3, we obtain the following result for the planar case.

Theorem 5.1. *Let $S \in \Omega(n, 2)$ be a local maximizer of* (2.3) *for $k = 2$. Then, the polygon $\square(S)$ is cyclic. That is, there is a circle that passes through all the vertices of $\square(S)$.*

Proof. Denote the origin by o. Let ab be an edge of $\square(S)$ and oh be the altitude of the triangle abo. By Lemma 5.3, h is the midpoint of ab. Hence, the triangle abo is isosceles and $ao = bo$. It follows that $\square(S)$ is cyclic. □

We are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Recall that any identification of H with $\mathbb{R}^k$ identifies the projections of the standard basis $\{v_1, \dots, v_n\}$ with a tight frame, denoted by S, that is a local maximizer of (2.3).

Next, assertion 1 is trivial and holds for any section of the cube. Assertion 2 and assertion 3 are equivalent to Lemma 5.1 and Lemma 5.3, respectively.

By Lemma 5.1, all vectors $v \in S$ such that span v intersects F correspond to F and have the same length that we denote by $|v|$. Then by Lemma 5.3, the span of each of these vectors intersects F in its centroid. Since the length of the altitude of the pyramid P_F is $1/|v|$, we have

$$\mathrm{vol}_k P_F = \frac{1}{k} \frac{\mathrm{vol}_{k-1} F}{|v|}.$$

Hence assertion 4 follows from Lemma 5.2. □

6 Proof of Theorem 1.3

We use the setting of tight frames developed in the previous sections to prove the theorem. More precisely, we use the obtained necessary conditions for a tight frame in $\mathbb{R}^2$ that maximizes (2.3) for $n > k = 2$ to prove that the section of the cube generated by the tight frame is a rectangle of area $4C_{\square}(n, 2) = 4\sqrt{\lceil n/2 \rceil \lfloor n/2 \rfloor}$ with the sides of lengths $2\sqrt{\lfloor n/2 \rfloor}$ and $2\sqrt{\lceil n/2 \rceil}$.

First, let us introduce the notation. Let $S = \{v_1, \dots, v_n\} \in \Omega(n, 2)$ be a global maximizer of (2.3) for $k = 2$ and $n > 2$. Clearly, $\square(S)$ is a centrally symmetric polygon in $\mathbb{R}^2$. The number of edges of $\square(S)$ is denoted by $2f$. Clearly, $f \le n$. By Theorem 5.1, the polygon $\square(S)$ is cyclic; and we denote its circumradius by R. Let $F_1, \dots F_{2f}$ be the edges of $\square(S)$ enumerated in clockwise direction (that is, edges F_i and F_{i+f} are opposite to each other, $i \in [f]$). We reenumerate the vectors of S in such a way that the vector v_i corresponds to the edge F_i for every $i \in [f]$. The central angle subtended by the edge F_i is denoted by $2\varphi_i$, $i \in [f]$.

Clearly, we have the following identities (see Figure 4):

$$\varphi_1 + \dots + \varphi_f = \frac{\pi}{2}, \tag{6.1}$$

$$R \cos \varphi_i = \frac{1}{|v_i|} \quad \text{for all } i \in [f], \tag{6.2}$$

and

$$\mathrm{Area}\,\square(S) = R^2 \sum_{i=1}^{f} \sin 2\varphi_i. \tag{6.3}$$

Also, we note here that

$$\mathrm{Area}\,\square(S) \ge 4C_{\square}(n, 2) = 4\sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor}. \tag{6.4}$$

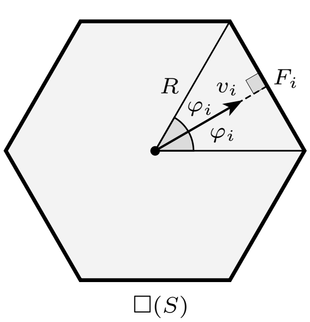

Figure 4: Notation for $\square(S)$

There are several steps in the proof. We explain the main steps briefly. In fact, we want to show that the number of edges of a local maximizer is $2f = 4$. Using the discrete isoperimetric inequality (see below), we obtain an upper bound on Area $\square(S)$ in terms of f. This upper bound yields the desired result for $n \ge 8$ (the bound is less than conjectured volume $4C_{\square}(n, 2)$ for $n \geq 8$). Finally, we deal with the lower-dimension cases using the necessary conditions obtained earlier.

The discrete isoperimetric inequality for cyclic polygons says that among all cyclic f-gons with fixed circummradius there is a unique maximal area polygon – the regular f-gon. We will use a slightly more general form. Namely, fixing one or several central angles of a cyclic polygon, its area is maximized when all other central angles are equal. In our notation fixing the central angle φ_i and its vertically opposite, we have

$$R^2 f \sin\frac{\pi}{f} \geq R^2\left(\sin 2\varphi_i + (f-1)\sin\frac{\pi - 2\varphi_i}{f-1}\right) \geq 4C_{\square}(n, 2). \tag{6.5}$$

This inequality immediately follows from the Jensen inequality and concavity of the sine function on $[0, \pi]$.

6.1 Step 1.

Claim 1. The area of $\square(S)$ such that $f = 2$ is at most $4C_{\square}(n, 2)$. The bound is attained when $\square(S)$ is a rectangle with the sides of lengths $2\sqrt{\lceil n/2 \rceil}$ and $2\sqrt{\lfloor n/2 \rfloor}$.

Proof. Since $f = 2$, the polygon $\square(S)$ is an affine square. Hence the claim is an immediate consequence of Lemma 1.1. □

Thus, it suffices to prove that $f = 2$ for any $n > 2$.

6.2 Step 2.

Claim 2. For any $n > 2$ the following inequality holds

$$R^2 \leq \frac{n+1}{2}\frac{1}{\cos^2\frac{\pi}{2f}}. \tag{6.6}$$

Proof. Let φ_1 be the smallest central angle. By identity (6.1), we have $\cos\varphi_1 \geq \cos\frac{\pi}{2f}$. Combining this with the leftmost inequality in (4.2) and identity (6.2), we obtain

$$R^2 = \frac{1}{|v_1|^2\cos^2\varphi_1} \leq \frac{n+1}{2}\frac{1}{\cos^2\frac{\pi}{2f}}.$$

□

Claim 3. For any $n > 2$ the following inequality holds

$$f\tan\frac{\pi}{2f} \geq \frac{4}{n+1}\sqrt{\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor}. \tag{6.7}$$

Proof. By the discrete isoperimetric inequality (6.5), we have

$$R^2 f\sin\frac{\pi}{f} \geq 4C_{\square}(n, 2).$$

Combining this with inequalities (6.6) and (6.4), we obtain

$$f\frac{\sin\frac{\pi}{f}}{\cos^2\frac{\pi}{2f}} \geq \frac{2}{n+1}C_{\square}(n, 2) = \frac{8}{n+1}\sqrt{\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor}.$$

The claim follows. □

Claim 4. The following bounds on f hold:

1. $f = 2$ if $n \geq 8$;
2. $f \leq 3$ if $n = 7$;

3. $f \le 4$ if $n = 5$.

Proof. We consider the functions in the left- and right-hand sides of (6.7) as functions of f and n respectively. Set $g(f) = f \tan \frac{\pi}{2f}$ and $h(n) = \frac{4}{n+1}\sqrt{\lfloor n/2 \rfloor \lceil n/2 \rceil}$. Thus, inequality (6.7) takes the form $g(f) \ge h(n)$. By routine analysis, we have that g is strictly decreasing and h is increasing on $\{n \in \mathbb{N} : n \ge 2\}$. The first two assertions of the claim follows from this and the identity $g(3) = h(7)$. Inequality $f \le 4$ for $n = 5$ follows from the direct computations of $g(5)$, $g(4)$ and $h(5)$ (see Figure 5). □

f	2	3	4	5
$g(f)$	2	$\sqrt{3}$	$4\left(\sqrt{2}-1\right)$	$\sqrt{5\,(5-2\sqrt{5})}$

n	5	6	7
$h(n)$	$2\sqrt{6}/3$	$12/7$	$\sqrt{3}$

Figure 5: Some values of g and h

Theorem 1.3 is proven for $n \ge 8$. We proceed with the lower-dimensional cases.

6.3 Step 3.

Claim 5. For $n = 7$, we have that $f = 2$.

Proof. We showed that $f \le 3$ for $n = 7$. Assume that $f = 3$. We see that inequality (6.7) with such values turns into an identity. It follows that $\varphi_1 = \varphi_2 = \varphi_3 = \pi/6$ and $\square(S)$ is a regular hexagon. Hence the vectors of S are of the same length. Since $\sum\limits_{v \in S} |v|^2 = \operatorname{tr} I_2 = 2$, we conclude that $|v|^2 = 2/7$ and $R^2 = \frac{1}{|v^2| \cos^2 \pi/6} = 14/3$. However, the volume of such a hexagon is strictly less than $4C_{\square}(7, 2)$. We conclude that $f = 2$ for $n = 7$. □

Claim 6. For $n \in \{3, 4, 6\}$, we have that $f = 2$.

Proof. For $n = 3$, the statement is a simple exercise (see [18]). Conjecture 1 was confirmed in [11] for any $n > k \ge 1$ such that $k|n$, in particular, for $k = 2$ and $n \in \{4, 6\}$. □

Remark 3. The inequality on the area for $n \in \{4, 6\}$ and $k = 2$ is a special case of the leftmost inequality in (1.1) originally proved by K. Ball [3]. In [11], the equality cases in this Ball's inequality are described.

6.4 Step 4.

Claim 7. Let $n = 5$ and either $f = 3$ or $f = 4$. Then $\varphi_i \le \pi/4$ for every $i \in [f]$.

Proof. Assume that there is $i \in [f]$ such that $\varphi_i > \pi/4$. Thus, $\cos \varphi_i < 1/\sqrt{2}$. Using identity (6.2) for i and for any $j \in [f]$, we have that

$$\frac{\cos \varphi_j}{\cos \varphi_i} = \frac{|v_i|}{|v_j|} \le \sqrt{\frac{n+1}{n-1}} = \sqrt{\frac{3}{2}},$$

where the inequality follows from Lemma 4.1. Hence $\cos \varphi_j \le \sqrt{3/2} \cos \varphi_i < \sqrt{3}/2$ and, therefore, $\varphi_j > \pi/6$. This contradicts identity (6.1):

$$\frac{\pi}{2} = \varphi_1 + \cdots + \varphi_f > \frac{\pi}{4} + (f-1)\frac{\pi}{6} > \frac{\pi}{2}.$$

Thus, $\varphi_i \le \pi/4$ for every $i \in [f]$. □

Claim 8. Let $n = 5$ and either $f = 3$ or $f = 4$. Then $\varphi_i \ge \pi/10$ for every $i \in [f]$.

Proof. Fix $i \in [f]$. By identity (6.2) and Lemma 4.1, we have $R^2 \le \frac{3}{2\cos^2\varphi_i}$. By inequality (6.5), we get

$$\frac{3}{\cos^2\varphi_i}\left(\sin 2\varphi_i + (f-1)\sin\frac{\pi - 2\varphi_i}{f-1}\right) \ge 4C_{\square}(5,2) = 4\sqrt{6}.$$

The function of φ_i in the left-hand side of this inequality is increasing on $[0, \pi/2]$. Since the inequality does not hold for $\varphi_i = \pi/10$, we conclude that φ_i is necessarily at least $\pi/10$. □

Claim 9. For $n = 5$, we have that $f = 2$.

Proof. Assume that either $f = 3$ or $f = 4$. Denote by d_i the number of vectors in S that correspond to F_i. We want to rewrite the inequality of Lemma 5.2 using the circumradius and the center angle. Since the length of edge F_i is $2R\sin\varphi_i$ and by identity (6.2), identity (5.6) takes the form

$$2R^2\sin 2\varphi_i = \frac{d_i}{R^2\cos^2\varphi_i}\operatorname{Area}\square(S).$$

Set $q(\varphi) = \cos^2\varphi\sin 2\varphi$. Then for all $i, j \in [f]$, we have

$$\frac{d_i}{d_j} = \frac{q(\varphi_i)}{q(\varphi_i)}.$$

Clearly, there are $i, j \in [f]$ such that $d_i = 2$ and $d_j = 1$. Therefore, $q(\varphi_i)/q(\varphi_j) = 2$. By Claim 7 and Claim 8, we have that $\varphi_i, \varphi_j \in [\pi/10, \pi/4]$. By simple computations, the maximum of q on the segment $[\pi/10, \pi/4]$ is $q(\pi/6) = 3\sqrt{3}/8$ and the minimum is $q(\pi/4) = 1/2$. Hence $\max\limits_{\varphi,\psi\in[\pi/10,\pi/4]} q(\varphi)/q(\psi) = 3\sqrt{3}/4 < 2$ and we come to a contradiction. Thus, $f = 2$. □

Thus, we have proved that for a maximizer of (2.3), then $f = 2$. By Claim 1, the conjectured upper bound for the area of a planar section holds and also is tight. The proof of Theorem 1.3 is complete.

Remark 4. We used the Ball inequality (1.1) to prove Theorem 1.3 for $n \in \{3, 4, 6\}$. However, it can be done by using our approach without the Ball inequality. The proof is technical. Since it is not of great interest, we do not give a proof.

A Sketches of proofs

Sketch of the proof of Lemma 1.1. Let H be a k-dimensional subspace of $\mathbb{R}^n$ such that $C_{\square}(n,k)$ is attained. Denote $P = \square^n \cap H$. Since P is an affine k-dimensional cube, there are vectors $\{a_1, \ldots, a_k\}$ such that $P = \bigcap_{i\in[k]}\left(H_{a_i}^+ \cap H_{a_i}^-\right)$.

Let $\{v_1, \ldots, v_n\}$ be the projection of the vectors of the standard basis onto H. By the same arguments as in Lemma 5.1, the hyperplane H_{v_i} meets the polytope P in a facet of P for every $i \in [n]$. Thus, v_i coincides with $\pm a_j$ for a proper sign and $j \in [k]$. Or, equivalently, we partition $[n]$ into k sets and H is the solution of a proper system of linear equations constructed as in (2) and (3), except we have not proved that (1) holds yet. Let us prove this assertion. Let d_i vectors of the standard basis of $\mathbb{R}^n$ project onto a pair $\pm a_i$. Therefore, a k-tuple of vectors $\{\sqrt{d_i}a_i\}_{i\in[k]}$ is a tight frame. Identifying H with $\mathbb{R}^k$ and by the assertion (4) of Lemma 2.1, we conclude that a_i and a_j are orthogonal whenever $i \ne j$. Therefore, $|a_i|^2 = \frac{1}{d_i}$ and

$$\operatorname{vol}_k \square^n \cap H = 2^k\sqrt{d_1 \cdot \ldots \cdot d_k}. \tag{A.1}$$

Suppose $d_i \ge d_j + 2$ for some $i, j \in [k]$. Then $d_i \cdot d_j \le (d_i - 1)(d_j + 1)$. By this and by (A.1), we showed that (1) holds.

It is easy to see that there are exactly $n - k\lfloor n/k\rfloor$ of d_i's equal $\lceil n/k\rceil$ and all others $k - (n - k\lfloor n/k\rfloor)$ are equal to $\lfloor n/k\rfloor$. That is, $C_{\square}(n,k)$ is given by (1.2). This completes the proof.

□

Sketch of the proof of Lemma 3.2. Recall that the cross product of $k-1$ vectors $\{x_1,\dots,x_{k-1}\}$ of $\mathbb{R}^k$ is the vector x defined by

$$\langle x, y\rangle = \det(x_1,\dots,x_{k-1},y) \quad \text{for all} \quad y\in\mathbb{R}^k.$$

For an ordered $(k-1)$-tuple $L=\{i_1,\dots,i_{k-1}\}\in\binom{[n]}{k-1}$ and a frame $S=\{v_1,\dots,v_n\}$, we use $[v_L]$ to denote the cross product of $v_{i_1},\dots,v_{i_{k-1}}$.

We claim the following property of the tight frames.

Let $S=\{v_1,\dots,v_n\}$ be a tight frame in $\mathbb{R}^k$. Then the set of vectors $\{[v_L]\}_{L\in\binom{[n]}{k-1}}$ is a tight frame in $\mathbb{R}^k$.

We use $\Lambda^k(\mathbb{R}^n)$ to denote the space of exterior k-forms on $\mathbb{R}^n$. By assertion (2) of Lemma 2.1, there exists an orthonormal basis $\{f_i\}_1^n$ of $\mathbb{R}^n$ such that v_i is the orthogonal projection of f_i onto $\mathbb{R}^k$, for any $i\in[n]$. Then the $(k-1)$-form $v_{i_1}\wedge\dots\wedge v_{i_{k-1}}$ is the orthogonal projections of the $(k-1)$-form $f_{i_1}\wedge\dots\wedge f_{i_{k-1}}$ onto $\Lambda^{k-1}(\mathbb{R}^k)\subset\Lambda^{k-1}(\mathbb{R}^n)$, for any ordered $(k-1)$-tuple $L=\{i_1,\dots,i_{k-1}\}\in\binom{[n]}{k-1}$. By Lemma 2.1 and since the $(k-1)$-forms $\{f_{i_1}\wedge\dots\wedge f_{i_{k-1}}\}_{\{i_1,\dots,i_{k-1}\}\in\binom{[n]}{k-1}}$ form an orthonormal basis of $\Lambda^{k-1}(\mathbb{R}^n)$, we have that the set of $(k-1)$-forms $\{v_{i_1}\wedge\dots\wedge v_{i_{k-1}}\}_{\{i_1,\dots,i_{k-1}\}\in\binom{[n]}{k-1}}$ is a tight frame in $\Lambda^{k-1}(\mathbb{R}^k)$. Finally, the Hodge star operator maps $v_{i_1}\wedge\dots\wedge v_{i_{k-1}}$ to the cross product of vectors $v_{i_1},\dots,v_{i_{k-1}}$. Since the Hodge star is an isometry, the set of cross products $\{[v_L]\}_{L\in\in\binom{[n]}{k-1}}$ is a tight frame. The claim is proven.

By linearity of the determinant, it is enough to prove the lemma for $\tilde{S}=\{v_1+tx, v_2,\dots,v_n\}$. Denote $v_1'=v_1+tx$ and $v_i'=v_i$, for $2\le i\le n$.

By the Cauchy–Binet formula, we have

$$\det A_{\tilde{S}} = \det\left(\sum_1^n v_i'\otimes v_i'\right) = \sum_{Q\in\binom{[n]}{k}} \det\left(\sum_{i\in Q} v_i'\otimes v_i'\right). \tag{A.2}$$

By the properties of the Gram matrix, we have

$$\det\left(\sum_1^k v_{i_1}'\otimes v_{i_k}'\right) = \left(\det\left(v_{i_1}',\dots,v_{i_k}'\right)\right)^2.$$

By this, by the definition of cross product and by identity (A.2), we obtain

$$\det A_{\tilde{S}} = 1 + 2t\sum_{Q\in\binom{[n]}{k},1\in Q}\left\langle v_1,[v_{Q\setminus 1}]\right\rangle\left\langle [v_{Q\setminus 1}],x\right\rangle + o(t).$$

Since $\langle v_1,[v_J]\rangle=0$ for any $J\in\binom{[n]}{k-1}$ such that $1\in J$, we have that the linear term of the Taylor expansion of $\det A_{\tilde{S}}$ equals

$$2t\sum_{Q\in\binom{[n]}{k},1\in Q}\left\langle v_1,[v_{Q\setminus 1}]\right\rangle [v_{Q\setminus 1}],x = 2t\sum_{L\in\binom{[n]}{k-1}}\langle v_i,[v_L]\rangle\langle [v_L],x\rangle.$$

Since $\{[v_L]\}_{L\in\binom{[n]}{k-1}}$ is a tight frame in $\mathbb{R}^k$, we have that

$$\sum_{L\in\binom{[n]}{k-1}}\langle v_i,[v_L]\rangle\langle [v_L],x\rangle = \langle v_i,x\rangle.$$

Therefore,

$$\sqrt{\det A_{\tilde{S}}} = \sqrt{\det A_S + 2t\langle v_i,x\rangle + o(t)} = 1 + t\langle v_i,x\rangle + o(t).$$

□

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