Journal of Functional Analysis 282 (2022) 109441

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Full Length Article

Functional John ellipsoids

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A R T I C L E I N F O

Article history:
Received 20 January 2021
Accepted 14 February 2022
Available online 4 March 2022
Communicated by E. Milman

MSC:
primary 52A23
secondary 52A40, 46T12

Keywords:
John ellipsoid
Logarithmically concave function
Helly type theorem

A B S T R A C T

We introduce a new way of representing logarithmically concave functions on $\mathbb{R}^d$. It allows us to extend the notion of the largest volume ellipsoid contained in a convex body to the setting of logarithmically concave functions as follows. For every $s > 0$, we define a class of non-negative functions on $\mathbb{R}^d$ derived from ellipsoids in $\mathbb{R}^{d+1}$. For any log-concave function f on $\mathbb{R}^d$, and any fixed $s > 0$, we consider functions belonging to this class, and find the one with the largest integral under the condition that it is pointwise less than or equal to f, and we call it the *John s-function* of f. After establishing existence and uniqueness, we give a characterization of this function similar to the one given by John in his fundamental theorem. We find that John s-functions converge to characteristic functions of ellipsoids as s tends to zero and to Gaussian densities as s tends to infinity.
As an application, we prove a quantitative Helly type result: the integral of the pointwise minimum of any family of log-concave functions is at least a constant c_d multiple of the integral of the pointwise minimum of a properly chosen subfamily of size $3d + 2$, where c_d depends only on d.

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https://doi.org/10.1016/j.jfa.2022.109441

1. Main results and the structure of the paper

The largest volume ellipsoid contained in a convex body in $\mathbb{R}^d$ and, in particular, John's result [18] characterizing it, plays a fundamental role in convexity. The latter states that the origin-centered Euclidean unit ball is the largest volume ellipsoid contained in the convex body K if and only if it is contained in K and the contact points (that is, the intersection points of the unit sphere and the boundary of K) satisfy a certain algebraic condition.

Alonso-Gutiérrez, Gonzales Merino, Jiménez and Villa [2] extended the notion of the John ellipsoid to the setting of logarithmically concave functions. To any log-concave function f of finite positive integral on $\mathbb{R}^d$, they associate an ellipsoid in $\mathbb{R}^d$, which we call *the AMJV ellipsoid*, in the following manner.

We denote the L_∞ norm of f by $\|f\|$. For every $\|f\| > \beta > 0$, consider the superlevel set $\{x \in \mathbb{R}^d : f(x) \geq \beta\}$ of f. This is a bounded convex set with non-empty interior, we take its largest volume ellipsoid, and multiply the volume of this ellipsoid by β. As shown in [2], there is a unique height $\beta_0 \in [0, \|f\|]$ such that this product is maximal. The AMJV ellipsoid is the ellipsoid E in $\mathbb{R}^d$ obtained for this β_0.

We propose an alternative route to this extension with the introduction of a parameter $s > 0$ that can be chosen arbitrarily. As a limit as s tends to zero, we recover the above described approach of Alonso-Gutiérrez, Gonzales Merino, Jiménez and Villa. The main advantage of our framework is that it implies a John type characterization of the maximal ellipsoid. We present an application of this characterization: a quantitative Helly type result for the integral of the pointwise minimum of a family of logarithmically concave functions.

The paper is organized as follows.

In Section 2, we introduce the notions of s-lifting and s-volume, which will frame our study of logarithmically concave functions, and then, we define our main object of interest, the *John s-ellipsoid* (an ellipsoid in $\mathbb{R}^{d+1}$) and the *John s-function* (a function on $\mathbb{R}^d$) of a log-concave function f on $\mathbb{R}^d$.

The idea is the following. Fix an $s > 0$ and consider the graph of the function $f^{1/s}$, which is a set in $\mathbb{R}^{d+1}$, and turn it into a not necessarily convex body in $\mathbb{R}^{d+1}$, which we call the *s-lifting* of f. We define also a measure-like quantity, the *s-volume* of sets in $\mathbb{R}^{d+1}$. Then we look for the ellipsoid in $\mathbb{R}^{d+1}$ which is contained in the s-lifting of f and is of maximal s-volume. We call this ellipsoid in $\mathbb{R}^{d+1}$ the John s-ellipsoid of f. This ellipsoid defines a function on $\mathbb{R}^d$, which is the John s-function of f. This function is pointwise less than or equal to f.

In Subsection 2.6, we describe our definitions in geometric terms and in Subsection 2.7, in terms of a functional optimization problem, concluding the second introductory section.

In Section 3, we prove some basic inequalities about the quantities introduced before. As an immediate application of these inequalities, we obtain a compactness result that, in the next section, yields that the John s-ellipsoid exists.

Section 4 contains one of our main tools, *interpolation between ellipsoids*. In the classical theory of the John ellipsoid, the uniqueness of the largest volume ellipsoid contained in a convex body K in $\mathbb{R}^d$ may be proved in the following way. Assume that $E_1 = A_1\mathbf{B}^d + a_1$ and $E_2 = A_2\mathbf{B}^d + a_2$ are ellipsoids of the same volume contained in K, where $\mathbf{B}^d$ denotes the Euclidean unit ball, A_1, A_2 are matrices, and $a_1, a_2 \in \mathbb{R}^d$. Then the ellipsoid $\frac{A_1+A_2}{2}\mathbf{B}^d + \frac{a_1+a_2}{2}$ is also contained in K and its volume is larger than that of E_1 and E_2.

One cannot apply this argument in our setting in a straightforward manner, as the set we consider is not convex. However, we show that if two ellipsoids in $\mathbb{R}^{d+1}$ of the same s-volume are contained in the s-lifting of a log-concave function f, then one can define a third ellipsoid "between" the two ellipsoids which is of larger s-volume. This intermediate ellipsoid is obtained as a non-linear combination of the parameters determining the two ellipsoids.

As an immediate application, we obtain that the John s-ellipsoid is unique, see Theorem 4.1.

In Section 5, we state and prove a necessary and sufficient condition for the $(d+1)$-dimensional Euclidean unit ball $\mathbf{B}^{d+1}$ to be the John s-ellipsoid of a log-concave function f on $\mathbb{R}^d$, see Theorem 5.1. Here, we phrase a simplified version of it.

Theorem 1.1. *Let $\overline{K} = \{(x,\xi) \in \mathbb{R}^{d+1} : \ |\xi| \leq f(x)/2\} \subseteq \mathbb{R}^{d+1}$ denote the symmetrized subgraph of an upper semi-continuous log-concave function f on $\mathbb{R}^d$ of positive non-zero integral. Assume that the $(d+1)$-dimensional Euclidean unit ball $\mathbf{B}^{d+1}$ is contained in $\overline{K}$. Then the following are equivalent.*

(1) The ball $\mathbf{B}^{d+1}$ is the unique maximum volume ellipsoid contained in $\overline{K}$.
(2) There are contact points $\overline{u}_1, \ldots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$, and positive weights $c_1, \ldots, c_k$ such that

$$\sum_{i=1}^{k} c_i \overline{u}_i \otimes \overline{u}_i = \overline{I} \quad \text{and} \quad \sum_{i=1}^{k} c_i u_i = 0,$$

where u_i is the orthogonal projection of $\overline{u}_i$ onto $\mathbb{R}^d$ and $\overline{I}$ is the $(d+1)\times(d+1)$ identity matrix.

The implication from (1) to (2) is proved in more or less the same way as John's fundamental theorem about convex bodies, there are hardly any additional difficulties. The converse however, is not straightforward, since $\overline{K}$ is not a convex body in general. That part of the proof relies heavily on the technique of interpolation between ellipsoids described in Section 4.

We note that non-convex sets in place of ellipsoids in a similar context for sets (not functions) were considered in [8]. In our case, however, it is the set which contains the other (the "container set") which is non-convex, and that is the source of difficulties in finding the optimum (maximum volume or integral).

We give also an equivalent, purely functional formulation of Theorem 1.1 without reference to bodies in $(d+1)$-dimensional space, see Theorem 5.2.

In Section 6, we describe the relationship between the approach of Alonso-Gutiérrez, Gonzales Merino, Jiménez and Villa [2] and our approach.

In Theorem 6.1, we show that $\beta_0\chi_E$ is the limit (in a rather strong sense) of our John s-functions as s tends to 0, where β_0 is the height of the AMJV ellipsoid E.

This result is based on the comparison of the s-volumes of John s-ellipsoids for distinct values of s. We compare also these s-volumes and the integral of f obtaining a bound on the integral ratio, the functional analogue of volume ratio.

In Section 7, we study the John s-functions as s tends to infinity. We show that the limit may only be a Gaussian density, see Theorem 7.2. What is perhaps surprising is that the largest integral Gaussian density that is pointwise less than or equal to f is not necessarily unique, see Section 7.2. We show however, that in this case, the two Gaussians are translates of each other, see Theorem 7.1.

Finally, Section 8 contains the proof of our quantitative Helly type result. This is a non-trivial application of the results of the previous sections. We describe it in detail here.

For a positive integer n, we denote by $[n]$ the set $[n] = \{1, 2, \ldots, n\}$. For $m \le n$, the family of subsets of $[n]$ of cardinality at most m is denoted by $\binom{[n]}{\le m}$.

According to Helly's theorem, *if the intersection of a finite family of convex sets in* $\mathbb{R}^d$ *is empty, then it has a subfamily of at most* $d+1$ *members such that the intersection of all members of the subfamily is empty.*

A quantitative variant of Helly's theorem was discovered by Bárány, Katchalski and Pach [7], stating the following. *Let* $K_1, \ldots, K_n$ *be convex sets in* $\mathbb{R}^d$. *Then there is a set* $\sigma \in \binom{[n]}{\le 2d}$ *of at most* $2d$ *indices such that*

$$\operatorname{vol}_d\left(\bigcap_{i\in\sigma} K_i\right) \le c_d \operatorname{vol}_d\left(\bigcap_{i\in[n]} K_i\right),$$

where c_d *depends only on* d.

In [7], it is shown that one can take $c_d = d^{2d^2}$ and it is conjectured that the theorem should hold with $c_d = d^{cd}$ for a proper absolute constant $c > 0$. It was confirmed in [21] with $c_d \approx d^{2d}$, where it is also shown that such result will not hold with $c_d \ll d^{d/2}$. The argument in [21] was refined by Brazitikos [9] who showed that one may take $c_d \approx d^{3d/2}$. For more on quantitative Helly type results, see the surveys [17,12]

Observe that the pointwise minimum of a family of log-concave functions is again log-concave. Our quantitative Helly type result is the following.

Theorem 1.2. *Let* $f_1, \ldots, f_n$ *be upper semi-continuous log-concave functions on* $\mathbb{R}^d$. *For every* $\sigma \subseteq [n]$, *let* f_σ *denote the pointwise minimum:*

$$f_\sigma(x) = \min\{f_i(x) : \ i \in \sigma\}.$$

Then there is a set $\sigma \in \binom{[n]}{\leq 3d+2}$ *of at most* $3d+2$ *indices such that, with the notation* $f = f_{[n]}$*, we have*

$$\int\limits_{\mathbb{R}^d} f_\sigma \leq (100d)^{5d/2} \int\limits_{\mathbb{R}^d} f. \tag{1.1}$$

The characteristic function of a convex set is log-concave, and pointwise minimum of functions corresponds to intersection of sets. Thus, Theorem 1.2 yields a quantitative Helly type result about convex sets as a special case. When comparing quantitative Helly type results, one may consider the Helly number and the bound on the volume (integral). Regarding the Helly number, on the one hand, we show in Subsection 8.6 that in our functional case, it is at least $2d+1$, unlike in the case of convex sets, where it is $2d$. Our bound on the integral is of the right order of magnitude, as it can not be improved beyond $d^{d/2}$ even for convex sets, see [21].

At the expense of obtaining a much worse bound on the integral in place of the multiplicative constant $d^{5d/2}$, we can show a similar result with Helly number $2d+1$ instead of $3d+2$. That result will be part of a sequel to the present paper.

We note also that our proof of this functional result does not make use of the analogous statement for convex sets.

1.1. Notation, basic terminology

We denote the Euclidean unit ball in $\mathbb{R}^n$ by $\mathbf{B}^n$, and we write $|\cdot|$ for the Euclidean norm.

We identify the hyperplane in $\mathbb{R}^{d+1}$ spanned by the first d standard basis vectors with $\mathbb{R}^d$. A set $C \subset \mathbb{R}^{d+1}$ is *d-symmetric* if C is symmetric about $\mathbb{R}^d$, that is, if $(2P-I)C = C$, where $P : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is the orthogonal projection onto $\mathbb{R}^d$.

For a square matrix $A \in \mathbb{R}^{d\times d}$ and a scalar $\alpha \in \mathbb{R}$, we denote by $A \oplus \alpha$ the $(d+1) \times (d+1)$ matrix

$$A \oplus \alpha = \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}.$$

For a function $f : \mathbb{R}^d \to \mathbb{R}$ and a scalar $\alpha \in \mathbb{R}$, we denote the *superlevel set* $\{x \in \mathbb{R}^d : f(x) \geq \alpha\}$ by $[f \geq \alpha]$. The *epigraph* of f is the set $\operatorname{epi}(f) = \{(x,\xi) \in \mathbb{R}^{d+1} : \ \xi \geq f(x)\}$ in $\mathbb{R}^{d+1}$. The L_∞ norm of a function f is denoted by $\|f\|$.

We will say that a function $f_1 : \mathbb{R}^d \to \mathbb{R}$ is *below* a function $f_2 : \mathbb{R}^d \to \mathbb{R}$, and denote it as $f_1 \leq f_2$, if f_1 is pointwise less than or equal to f_2, that is, $f_1(x) \leq f_2(x)$ for all $x \in \mathbb{R}^d$.

A function $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is called *convex* if $\psi((1-\lambda)x+\lambda y) \leq (1-\lambda)\psi(x)+\lambda\psi(y)$ for every $x, y \in \mathbb{R}^d$ and $\lambda \in [0,1]$. A function f on $\mathbb{R}^d$ is *logarithmically concave* (or *log-concave* for short) if $f = e^{-\psi}$ for a convex function ψ on $\mathbb{R}^d$. We say that a log-concave function f on $\mathbb{R}^d$ is a *proper log-concave function* if f is upper semi-continuous and has finite positive integral.

We will use $\prec$ to denote the standard partial order on the cone of positive semi-definite matrices, that is, we will write $A \prec B$ if $B - A$ is positive definite. We recall the additive and the multiplicative form of *Minkowski's determinant inequality*. Let A and B be positive definite matrices of order d. Then, for any $\lambda \in (0,1)$,

$$\left(\det\left(\lambda A + (1-\lambda)B\right)\right)^{1/d} \geq \lambda \left(\det A\right)^{1/d} + (1-\lambda)\left(\det B\right)^{1/d}, \tag{1.2}$$

with equality if and only if $A = cB$ for some $c > 0$; and

$$\det\left(\lambda A + (1-\lambda)B\right) \geq \left(\det A\right)^{\lambda} \cdot \left(\det B\right)^{1-\lambda}, \tag{1.3}$$

with equality if and only if $A = B$.

2. The s-volume, the s-lifting and the s-ellipsoids

2.1. Motivation for the definitions

One way to obtain a log-concave function f on $\mathbb{R}^d$ is to fix a convex body $\overline{K}$ in $\mathbb{R}^{d+s}$ for some positive integer s, take the *uniform measure* on $\overline{K}$ (that is, the absolutely continuous measure whose density is the characteristic function of $\overline{K}$) and take the density of its marginal on $\mathbb{R}^d$. Conversely, it is well known that any log-concave function is a limit of functions obtained this way. This representation of log-concave functions was used by Artstein-Avidan, Klartag and Milman in [1], where a functional form of the Santaló inequality is proved.

If f is obtained this way, then it is natural to consider the largest volume ($(d+s)$-dimensional) ellipsoid contained in $\overline{K}$, and take the uniform measure on this ellipsoid. The marginal on $\mathbb{R}^d$ of this measure could be a candidate for the John ellipsoid function of f. However, for a given f, the convex body $\overline{K}$ in $\mathbb{R}^{d+s}$ described above is not unique, if it exists. One may take the *Schwarz symmetrization* of any such $\overline{K}$ about $\mathbb{R}^d$ (defined in Subsection 2.6) to obtain a new convex body in $\mathbb{R}^{d+s}$ which is now symmetric about $\mathbb{R}^d$ and still has the property that the density of the marginal on $\mathbb{R}^d$ of the uniform measure on it is f. Since the Schwarz symmetrization of an ellipsoid is again an ellipsoid, the John ellipsoid of the Schwarz symmetrization of $\overline{K}$ is at least as large as the John ellipsoid of $\overline{K}$. In summary, the marginal on $\mathbb{R}^d$ of the uniform measure on the John ellipsoid of the Schwartz symmetrization of $\overline{K}$ is a function of special form, and is below f. Moreover, it is of maximal integral among functions of this special form that are below f. This is now a good candidate for the John function of f.

With one more idea, we can reduce the dimension from $d+s$ to $d+1$. In fact, due to the symmetry about $\mathbb{R}^d$, there is no need to consider a body in $\mathbb{R}^{d+s}$. Instead, we may consider the section of this body by the linear subspace spanned by $\mathbb{R}^d$ and any vector which is not in $\mathbb{R}^d$, say e_{d+1}. We just need to remember that the last coordinate in $\mathbb{R}^{d+1}$ represents s coordinates when it comes to computing the marginal of the uniform distribution of a convex body in $\mathbb{R}^{d+1}$.

In what follows, we formalize this reasoning without referring to any $(d+s)$-dimensional convex body. An advantage of the formalism that follows is that it works for non-integer s, as well as for any proper log-concave function f, and not only for functions obtained as the marginals of the uniform measure on some higher dimensional convex set.

We will mostly study objects in $\mathbb{R}^d$ and in $\mathbb{R}^{d+1}$. For an easier reading, we emphasize that a set is in $\mathbb{R}^{d+1}$ by using a bar in its symbol, e.g. $\overline{K}$.

2.2. The s-volume and its s-marginal

Fix a positive real s. For every $x \in \mathbb{R}^d$, we denote the line in $\mathbb{R}^{d+1}$ perpendicular to $\mathbb{R}^d$ at x by ℓ_x.

Let $\overline{C} \subset \mathbb{R}^{d+1}$ be a d-symmetric Borel set. The *s-volume* of $\overline{C}$ is defined by

$$^{(s)}\mu(\overline{C}) = \int\limits_{\mathbb{R}^d} \left[\frac{1}{2}\mathrm{length}\left(\overline{C} \cap \ell_x\right)\right]^s \,\mathrm{d}x.$$

Note that $^{(s)}\mu(\cdot)$ is not a measure on $\mathbb{R}^{d+1}$. However, for any d-symmetric Borel set $\overline{C}$ in $\mathbb{R}^{d+1}$, the *s-marginal* of $\overline{C}$ on $\mathbb{R}^d$ defined for any Borel set B in $\mathbb{R}^d$ by

$$^{(s)}\mathrm{marginal}\left(\overline{C}\right)(B) = \int\limits_{B} \left[\frac{1}{2}\mathrm{length}\left(\overline{C} \cap \ell_x\right)\right]^s \,\mathrm{d}x \tag{2.1}$$

is a measure on $\mathbb{R}^d$.

We note that for any matrix $\overline{A} = A \oplus \alpha$, where $A \in \mathbb{R}^{d\times d}$ and $\alpha \in \mathbb{R}$, any d-symmetric set $\overline{C}$ in $\mathbb{R}^{d+1}$ and any Borel set B in $\mathbb{R}^d$, we have

$$\begin{cases} ^{(s)}\mathrm{marginal}\left(\overline{A}\,\overline{C}\right)(AB) = |\det A| \cdot |\alpha|^s \cdot {}^{(s)}\mathrm{marginal}\left(\overline{C}\right)(B), \\ ^{(s)}\mu\left(\overline{A}\,\overline{C}\right) = |\det A| \cdot |\alpha|^s \cdot {}^{(s)}\mu\left(\overline{C}\right). \end{cases} \tag{2.2}$$

2.3. The s-lifting of a function

Let $f : \mathbb{R}^d \to [0, \infty)$ be a function and $s > 0$. The *s-lifting* of f is a d-symmetric set in $\mathbb{R}^{d+1}$ defined by

$$^{(s)}\overline{f} = \left\{(x, \xi) \in \mathbb{R}^{d+1} :\ |\xi| \leq \left(f(x)\right)^{1/s}\right\}.$$

Note the following *scaling property* of s-lifting: for any $\gamma > 0$,

$$^{(s)}\overline{(\gamma f)} = \left(I \oplus \gamma^{1/s}\right) {}^{(s)}\overline{f}. \tag{2.3}$$

Clearly, for any Borel set B in $\mathbb{R}^d$,

$$\int_B f = {}^{(s)}\mu\left({}^{(s)}\overline{f} \cap (B \times \mathbb{R})\right),$$

that is, $\overset{(s)}{}\mathrm{marginal}\left({}^{(s)}\overline{f}\right)$ is the measure on $\mathbb{R}^d$ with density f.

2.4. Ellipsoids

Let A be a positive definite matrix in $\mathbb{R}^{d\times d}$ and $a \in \mathbb{R}^d$. They determine an *ellipsoid* defined by

$$A\left(\mathbf{B}^d\right) + a. \tag{2.4}$$

Note that $A\left(\mathbf{B}^d\right) + a = \{x \in \mathbb{R}^d : \left\langle A^{-1}x, A^{-1}x\right\rangle \le 1\} + a$.

We will consider d-symmetric ellipsoids in $\mathbb{R}^{d+1}$ (see Section 1.1 for the definition of d-symmetry). To describe them, we introduce the vector space

$$\mathcal{M} = \left\{(\overline{A}, a) : \overline{A} \in \mathbb{R}^{(d+1)\times(d+1)},\ \overline{A}^{\mathsf{T}} = \overline{A},\ a \in \mathbb{R}^d\right\}, \tag{2.5}$$

and the convex cone

$$\mathcal{E} = \left\{(A \oplus \alpha, a) \in \mathcal{M},\ A \in \mathbb{R}^{d\times d} \text{ positive definite}, \alpha > 0\right\}. \tag{2.6}$$

Clearly, any d-symmetric ellipsoid in $\mathbb{R}^{d+1}$ is represented by

$$(A \oplus \alpha)\mathbf{B}^{d+1} + a,$$

in a unique way. Thus, from this point on, we identify $\mathcal{E}$ with the set of all d-symmetric ellipsoids in $\mathbb{R}^{d+1}$, and in particular, we may write ${}^{(s)}\mu((A \oplus \alpha, a))$ to refer to the s-volume of the corresponding ellipsoid. We note that

$$\dim \mathcal{M} = \frac{(d+1)(d+2)}{2} + d. \tag{2.7}$$

2.5. Definition of the John s-ellipsoid of a function

Fix $s > 0$ and let $z(f, s)$ denote the supremum of the s-volumes of all d-symmetric ellipsoids $\overline{E}$ in $\mathbb{R}^{d+1}$ with $\overline{E} \subseteq {}^{(s)}\overline{f}$. Lemma 3.2 and a standard compactness argument yield that this supremum is attained. We will see (Theorem 4.1) that it is attained on a unique ellipsoid. We call this ellipsoid in $\mathbb{R}^{d+1}$ the *John s-ellipsoid* of f and denote it by $\overline{E}(f, s)$. We call the s-marginal of $\overline{E}(f, s)$ the *John s-function* of f, and denote its density by

$$ {}^{(s)}J_f = \text{ the density of } {}^{(s)}\text{marginal}\left(\overline{E}(f, s)\right). $$

As a consequence of (2.3), we note the *scaling property* of s-ellipsoids: for any $s, \gamma > 0$,

$$ \overline{E} \text{ is the John } s\text{-ellipsoid of } f \text{ if and only if } \left(I \oplus (\gamma^{1/s})\right) \overline{E} \text{ is the John } s\text{-ellipsoid of } \gamma f, \tag{2.8} $$

or, equivalently, ${}^{(s)}J_f$ is the John s-function of f if and only if $\gamma \cdot {}^{(s)}J_f$ is the John s-function of γf. Similarly, for any affine map $\mathcal{A} : \mathbb{R}^d \to \mathbb{R}^d$, ${}^{(s)}J_f$ is the John s-function of f if and only if ${}^{(s)}J_f \circ \mathcal{A}$ is the John s-function of $f \circ \mathcal{A}$.

2.6. How the definitions described above implement the idea described in 2.1

We return to the case when s is a positive integer. We first recall a classical definition.

We regard $\mathbb{R}^{d+s}$ as the orthogonal sum $\mathbb{R}^{d+s} = \mathbb{R}^d \oplus \mathbb{R}^s$, and denote by $\mathbf{B}^s$ the unit ball of $\mathbb{R}^s$. Let $\overline{K}$ be a convex body in $\mathbb{R}^{d+s}$. The *Schwarz symmetrization* of $\overline{K}$ about $\mathbb{R}^d$ is the set

$$ \overline{K}' = \bigcup \left\{ r\mathbf{B}^s + x : \; x \in P\left(\overline{K}\right), \mathrm{vol}_s\left(r\mathbf{B}^s\right) = \mathrm{vol}_s\left(\overline{K} \cap (x + \mathbb{R}^s)\right) \right\}, $$

where P denotes the orthogonal projection from $\mathbb{R}^{d+s}$ onto $\mathbb{R}^d$, cf. [10, Section 9.2.1.I]. As a well known consequence of the Brunn–Minkowski inequality, we have that $\overline{K}'$ is a convex body in $\mathbb{R}^{d+s}$. It is immediate from the definition that $\mathrm{vol}_{d+s} \overline{K}' = \mathrm{vol}_{d+s} \overline{K}$, and more generally, the marginal on $\mathbb{R}^d$ of the uniform measure on $\overline{K}$ is identical to the marginal of the uniform measure on $\overline{K}'$.

The following claim follows from our definitions, we leave the proof to the reader.

Proposition 2.1. *Let $d, s > 0$ be positive integers and let the function $f : \mathbb{R}^d \to \mathbb{R}$ be the density of the marginal on $\mathbb{R}^d$ of the uniform measure on a convex body $\overline{K}$ in $\mathbb{R}^{d+s}$. Let $\overline{K}'$ denote the Schwarz symmetrization of $\overline{K}$ about $\mathbb{R}^d$, and let $\overline{E}$ denote the John ellipsoid of $\overline{K}'$. Then the marginal of the uniform measure on $\overline{E}$ is the John s-function of f.*

2.7. The height function of an ellipsoid, and formulation of our problem as functional optimization

For any $(A \oplus \alpha, a) \in \mathcal{E}$, we will say that α is the *height* of the ellipsoid $\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a$. We define the *height function* of $\overline{E}$ as

$$\hbar_{\overline{E}}(x) = \begin{cases} \alpha\sqrt{1 - \langle A^{-1}(x-a), A^{-1}(x-a)\rangle}, & \text{if } x \in A\mathbf{B}^d + a \\ 0, & \text{otherwise.} \end{cases}$$

Note that the height function of an ellipsoid is a proper log-concave function. Clearly, the inclusion $\overline{E} \subset {}^{(s)}\overline{f}$ holds if and only if

$$\hbar_{\overline{E}}(x+a) \le f^{1/s}(x+a) \text{ for all } x \in A\mathbf{B}^d. \tag{2.9}$$

As a closing note of the present introductory section, we rephrase our problem in a less geometric, more analytical language.

The classical John ellipsoid can be introduced as follows. We consider the class of all nonsingular affine images (we may call them *positions*) of the unit ball $\mathbf{B}^d$ contained in a given convex body K. The John ellipsoid is the (unique) largest volume element of this family.

With the notion of height functions, it is easy to extend this approach to the setting of log-concave function. For any $s > 0$, the John s-function of a proper log-concave function f on $\mathbb{R}^d$ is the (unique) solution to the problem

$$\max_h \int_{\mathbb{R}^d} h^s,$$

where the maximum is taken over those positions

$$\left\{h(x) = \alpha \cdot \hbar_{\mathbf{B}^{d+1}}(A^{-1}(x-a)),\ \text{where } a \in \mathbb{R}^d,\ A \in \mathrm{GL}(d),\ \alpha > 0\right\}$$

of the height function

$$\hbar_{\mathbf{B}^{d+1}}(x) = \begin{cases} \sqrt{1-|x|^2}, & \text{if } x \in \mathbf{B}^d \\ 0, & \text{otherwise} \end{cases} \tag{2.10}$$

of the unit ball $\mathbf{B}^{d+1}$ which satisfy $\hbar_{\mathbf{B}^{d+1}}(A^{-1}(x-a)) \le f(x)$ for all $x \in \mathbb{R}^d$.

It follows from the polar decomposition theorem that we may restrict the set of positions to those where A is a positive definite matrix.

3. Some basic inequalities

3.1. The s-volume of ellipsoids

We denote the s-volume of the ball $\mathbf{B}^{d+1}$ of unit radius centered at the origin in $\mathbb{R}^{d+1}$ by ${}^{(s)}\kappa_{d+1}$, and compute it using spherical coordinates.

$${}^{(s)}\kappa_{d+1} = {}^{(s)}\mu\left(\mathbf{B}^{d+1}\right) = \int_{\mathbf{B}^d} \left(\sqrt{1-|x|^2}\right)^s \mathrm{d}x = \mathrm{vol}_{d-1} S \int_0^1 r^{d-1}(\sqrt{1-r^2})^s \,\mathrm{d}r = \tag{3.1}$$

$$\frac{\mathrm{vol}_{d-1} S}{2}\int\limits_0^1 t^{(d-2)/2}(1-t)^{s/2}\,\mathrm{d}t = \frac{d\,\mathrm{vol}_d\,\mathbf{B}^d}{2}\frac{\Gamma(s/2+1)\,\Gamma(d/2)}{\Gamma(s/2+d/2+1)} = \pi^{d/2}\frac{\Gamma(s/2+1)}{\Gamma(s/2+d/2+1)},$$

where $S = \mathrm{bd}\left(\mathbf{B}^d\right)$ denotes the unit sphere in $\mathbb{R}^d$, and $\Gamma(\cdot)$ is Euler's Gamma function.

Note that

$$\lim_{s\to 0^+} {}^{(s)}\kappa_{d+1} = \mathrm{vol}_d\,\mathbf{B}^d. \tag{3.2}$$

Thus, ${}^{(s)}\kappa_{d+1}$, as a function of s on $[0,\infty)$ with ${}^{(0)}\kappa_{d+1} = \mathrm{vol}_d\,\mathbf{B}^d$, is a strictly decreasing continuous function on $[0,\infty)$.

By (2.2), the s-volume of a d-symmetric ellipsoid can be expressed as

$$ {}^{(s)}\mu\big((A\oplus\alpha)\mathbf{B}^{d+1}+a\big) = {}^{(s)}\kappa_{d+1}\alpha^s \det A, \tag{3.3}$$

for any $(A\oplus\alpha, a)\in\mathcal{E}$.

3.2. Bounds on f based on local behavior

Lemma 3.1. *Let ψ_1 and ψ_2 be convex functions on $\mathbb{R}^d$ and $f_1 = e^{-\psi_1}$ and $f_2 = e^{-\psi_2}$. Let $f_2 \le f_1$ and $f_1(x_0) = f_2(x_0) > 0$ at some point x_0 in the interior of the domain of ψ_2. Assume that ψ_2 is differentiable at x_0. Then f_1 and f_2 are differentiable at x_0, $\nabla f_1(x_0) = \nabla f_2(x_0)$ and the following holds*

$$f_1(x) \le f_2(x_0)e^{-\langle\nabla\psi_2(x_0), x-x_0\rangle}$$

for all $x\in\mathbb{R}^d$.

Proof. Since $f_2\le f_1$, the epigraph of ψ_1 contains the epigraph of ψ_2. Next, since $f_1(x_0) = f_2(x_0)$ and $f_2(x_0)$ is differentiable at x_0, we conclude that both f_1 and f_2 have finite values and are continuous in a neighborhood of x_0 (see Proposition 2.2.6 of [11]).

Using the Subdifferential Maximum Rule (see Proposition 2.3.12 of [11]), we see that ψ_1 is differentiable at x_0 and $\nabla\psi_1(x_0) = \nabla\psi_2(x_0)$, since the subdifferential of ψ_2 at x_0 consists of the single vector $\nabla\psi_2(x_0)$.

By the convexity of ψ_1, we have

$$\psi_1(x) \ge \psi_1(x_0) + \langle\nabla\psi_1(x_0), x-x_0\rangle = \psi_2(x_0) + \langle\nabla\psi_2(x_0), x-x_0\rangle$$

for all $x\in\mathbb{R}^d$, and the result follows. $\square$

Corollary 3.1. *Let f be a log-concave function on $\mathbb{R}^d$, and $s>0$. Assume that $\mathbf{B}^{d+1}\subseteq {}^{(s)}\overline{f}$ and $\overline{u}\in\mathbb{R}^{d+1}\setminus\mathbb{R}^d$ is a* contact point *of $\mathbf{B}^{d+1}$ and ${}^{(s)}\overline{f}$, that is, $\overline{u}\in \mathrm{bd}\left(\mathbf{B}^{d+1}\right)\cap\mathrm{bd}\left({}^{(s)}\overline{f}\right)\setminus\mathbb{R}^d$. Then*

$$f(x) \leq w^s e^{-\frac{s}{w^2}\langle u, x-u\rangle} \quad \textit{for all } x \in \mathbb{R}^d, \tag{3.4}$$

where u is the orthogonal projection of $\overline{u}$ onto $\mathbb{R}^d$ and $w = \sqrt{1-|u|^2}$.

Note that since $u \notin \mathbb{R}^d$, we have $w > 0$.

Proof of Corollary 3.1. Applying Lemma 3.1 to the functions $f_1 = f^{1/s}$ and $f_2 = \hbar_{\mathbf{B}^{d+1}}$ at $x_0 = u$, we obtain

$$f^{1/s}(x) \leq w e^{-\langle \nabla[-\log \hbar_{\mathbf{B}^{d+1}}](u), x-u\rangle} \quad \text{for all } x \in \mathbb{R}^d.$$

Since for any $y \in \operatorname{int}\left(\mathbf{B}^d\right)$, we have

$$\nabla\left[-\log \hbar_{\mathbf{B}^{d+1}}\right](y) = -\frac{1}{2}\nabla\left[\log(1-|y|^2)\right] = \frac{y}{1-|y|^2},$$

inequality (3.4) follows. □

3.3. Compactness

We show that ellipsoids of large s-volume contained in ${}^{(s)}\overline{f}$ are contained in a bounded region of $\mathbb{R}^{d+1}$. We phrase the next lemma in a general functional language, but we will apply it mostly for $g = \hbar_{\mathbf{B}^2}$, the height function of the ball $\mathbf{B}^2$ (see (2.10)).

Lemma 3.2 *(Compactness). For any proper log-concave function $f : \mathbb{R}^d \to [0,\infty)$ and any $\delta > 0$, there exist $\vartheta, \rho, \rho_1, \rho_2 > 0$ with the following property. If for a proper even log-concave function $g : \mathbb{R} \to [0,\infty)$ with $g(0) = 1$ and $(A \oplus \alpha, a) \in \mathcal{E}$, the function $\tilde{g} : \mathbb{R}^d \to [0,\infty)$ given by*

$$\tilde{g}(x) = \alpha g\left(\left|A^{-1}(x-a)\right|\right)$$

satisfies $\tilde{g} \leq f$ and $\int_{\mathbb{R}^d} \tilde{g} \geq \delta$, then the following inequalities hold.

$$\vartheta \leq \alpha \leq \|f\| \quad \textit{and} \quad |a| \leq \rho, \tag{3.5}$$

and

$$\rho_1 \frac{\left(\int_{\mathbb{R}} g\right)^{d-1}}{\int_{\mathbb{R}^d} g(|x|)\,\mathrm{d}x} \cdot I \prec A \prec \frac{\rho_2}{\int_{\mathbb{R}} g} \cdot I. \tag{3.6}$$

Proof. Obviously, $\alpha \leq \|f\|$. To bound α from below, we fix ϑ with $\alpha \leq \vartheta$. Then $\tilde{g} \leq \vartheta$, and thus,

$$\int_{\mathbb{R}^d} \tilde{g} \leq \int_{\mathbb{R}^d} \min\{f(x), \vartheta\}\,\mathrm{d}x.$$

Since f is a non-negative function of finite integral, the last expression is less than δ if ϑ is sufficiently small. Thus, the leftmost inequality in (3.5) holds. Since $\tilde{g}(a) = \alpha$, we conclude that $a \in [f \geq \vartheta]$ completing the proof of (3.5).

We proceed with inequality (3.6). Let ℓ be the line passing through a in the direction of an eigenvector of A corresponding to the eigenvalue $\|A\|$. We have

$$\int_{\ell} f \geq \int_{\ell} \tilde{g} = \alpha \|A\| \int_{\mathbb{R}} g.$$

On the other hand, there exists a positive constant C_f such that the integral of the proper log-concave function f over any line is at most C_f. It follows, for example, from the existence of constants $\Theta, \nu > 0$ depending only on f such that

$$f(x) \leq \Theta e^{-\nu |x|}$$

for all $x \in \mathbb{R}^d$, see [6, Lemma 2.2.1]. Thus, the rightmost relation in (3.6) holds with $\rho_2 = 2\frac{C_f}{\vartheta}$.

By the assumption, we have

$$\delta \leq \int_{\mathbb{R}^d} \tilde{g} = \alpha \det A \cdot \int_{\mathbb{R}^d} g(|x|)\, \mathrm{d}x.$$

Let β be the smallest eigenvalue of A. By the previous inequality and since $\alpha \in [\vartheta, \|f\|]$, we have

$$0 < \frac{\delta}{\|f\|} \frac{1}{\int_{\mathbb{R}^d} g(|x|)\, \mathrm{d}x} \leq \det A \leq \beta \, \|A\|^{d-1}.$$

By the rightmost relation in (3.6), the existence of ρ_1 follows. $\square$

4. Interpolation between ellipsoids

In this section, we show that if two ellipsoids are contained in ${}^{(s)}\overline{f}$, then we can define a third ellipsoid that is also contained in ${}^{(s)}\overline{f}$, and we give a lower bound on its s-volume. The latter is a Brunn–Minkowski type inequality for the s-volume of ellipsoids.

After preliminaries, we present the main results of this section in Subsection 4.2, which is followed by immediate applications, one of which is the proof of the existence and uniqueness of the John s-ellipsoid (Theorem 4.1).

4.1. Operations on functions: Asplund sum, epi-product

Following Section 9.5 of [22], we define the *Asplund sum* (or *sup-convolution*) of two log-concave functions f_1 and f_2 on $\mathbb{R}^d$ by

$$(f_1 \star f_2)(x) = \sup_{x_1+x_2=x} f_1(x_1)f_2(x_2),$$

and the *epi-product* of a log-concave function f on $\mathbb{R}^d$ with a scalar $\lambda > 0$ by

$$(\lambda * f)(x) = f\left(\frac{x}{\lambda}\right)^{\lambda}.$$

Clearly,

$$\|f\| = \|f_1\| \, \|f_2\| \,, \text{ where } f = f_1 \star f_2.$$

It is easy to see that for any proper log-concave function f and $\lambda \in [0, 1]$, we have

$$(\lambda * f) \star ((1 - \lambda) * f) = f. \tag{4.1}$$

As motivation for the definitions above, we describe a geometric interpretation of the Asplund sum: analogy with the Minkowski sum of convex bodies in $\mathbb{R}^d$. Let $\psi_1, \psi_2 : \mathbb{R}^d \to \mathbb{R}$ be two convex functions that attain their minimums. Then the Asplund sum of $f_1 = e^{-\psi_1}$ and $f_2 = e^{-\psi_2}$ equals

$$f_1 \star f_2 = e^{-\psi},$$

where ψ is the function defined by taking the Minkowski sum of the epigraphs, that is,

$$\operatorname{epi} \psi = \operatorname{epi} \psi_1 + \operatorname{epi} \psi_2.$$

4.2. A non-linear combination of two ellipsoids

The following two lemmas are our key tools. They allow us to interpolate between two ellipsoids.

Lemma 4.1 *(Containment of the interpolated ellipsoid). Fix $s_1, s_2, \beta_1, \beta_2 > 0$ with $\beta_1 + \beta_2 = 1$. Let f_1 and f_2 be two proper log-concave functions on $\mathbb{R}^d$, and $\overline{E_1}, \overline{E_2}$ be two d-symmetric ellipsoids represented by $(A_1 \oplus \alpha_1, a_1) \in \mathcal{E}$ and $(A_2 \oplus \alpha_2, a_2) \in \mathcal{E}$, respectively, such that*

$$\overline{E_1} \subset {}^{(s_1)}\overline{f_1} \quad \textit{and} \quad \overline{E_2} \subset {}^{(s_2)}\overline{f_2}. \tag{4.2}$$

Define

$$f = (\beta_1 * f_1) \star (\beta_2 * f_2) \quad \textit{and} \quad s = \beta_1 s_1 + \beta_2 s_2.$$

Set

$$(A \oplus \alpha, a) = \left((\beta_1 A_1 + \beta_2 A_2) \oplus (\alpha_1^{\beta_1 s_1} \alpha_2^{\beta_2 s_2})^{1/s}, \beta_1 a_1 + \beta_2 a_2\right) \quad \textit{and}$$
$$\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a.$$

Then,

$$\overline{E} \subset {}^{(s)}\overline{f}. \tag{4.3}$$

Proof. Fix $x \in A\mathbf{B}^d$ and define

$$x_1 = A_1 A^{-1} x, \quad x_2 = A_2 A^{-1} x.$$

Clearly, $x_1 \in A_1\mathbf{B}^d$ and $x_2 \in A_2\mathbf{B}^d$. Thus, by (4.2) and (2.9), we have

$$f_1^{1/s_1}(x_1 + a_1) \geq \hbar_{\overline{E}_1}(x_1 + a_1) \quad \text{and} \quad f_2^{1/s_2}(x_2 + a_2) \geq \hbar_{\overline{E}_2}(x_2 + a_2). \tag{4.4}$$

By our definitions, we have that $\beta_1(x_1 + a_1) + \beta_2(x_2 + a_2) = x + a$. Therefore, by the definition of the Asplund sum, we have that

$$f(x + a) \geq f_1^{\beta_1}(x_1 + a_1) f_2^{\beta_2}(x_2 + a_2),$$

which by (4.4), yields

$$f(x + a) \geq \left(\hbar_{\overline{E}_1}(x_1 + a_1)\right)^{\beta_1 s_1} \left(\hbar_{\overline{E}_2}(x_2 + a_2)\right)^{\beta_2 s_2}.$$

By the definition of the height function, and since $A^{-1}x = A_1^{-1}x_1 = A_2^{-1}x_2$, we have

$$\left(\hbar_{\overline{E}_1}(x_1 + a_1)\right)^{\beta_1 s_1} \left(\hbar_{\overline{E}_2}(x_2 + a_2)\right)^{\beta_2 s_2} =$$
$$\left(\alpha_1 \sqrt{1 - \langle A_1^{-1}x_1, A_1^{-1}x_1 \rangle}\right)^{\beta_1 s_1} \left(\alpha_2 \sqrt{1 - \langle A_2^{-1}x_2, A_2^{-1}x_2 \rangle}\right)^{\beta_2 s_2} =$$
$$\alpha_1^{\beta_1 s_1} \alpha_2^{\beta_2 s_2} \left(\sqrt{1 - \langle A^{-1}x, A^{-1}x \rangle}\right)^{\beta_1 s_1 + \beta_2 s_2} = \left(\alpha \sqrt{1 - \langle A^{-1}x, A^{-1}x \rangle}\right)^s = \left(\hbar_{\overline{E}}(x + a)\right)^s.$$

Combining this with the previous inequality, we obtain inequality (2.9). This completes the proof. □

Lemma 4.2 *(Volume of the interpolated ellipsoid). Under the conditions of Lemma 4.1 with $s = s_1 = s_2$, the following inequality holds.*

$$ {}^{(s)}\mu(\overline{E}) \geq \left({}^{(s)}\mu(\overline{E}_1)\right)^{\beta_1} \left({}^{(s)}\mu(\overline{E}_2)\right)^{\beta_2}, \tag{4.5}$$

with equality if and only if $A_1 = A_2$.

Proof. We set $s = s_1 = s_2$, and observe that by (3.3), inequality (4.5) is equivalent to

$$ {}^{(s)}\kappa_{d+1}(\alpha_1^{\beta_1}\alpha_2^{\beta_2})^s \cdot \det\left(\beta_1 A_1 + \beta_2 A_2\right) \geq {}^{(s)}\kappa_{d+1}(\alpha_1^{\beta_1}\alpha_2^{\beta_2})^s \cdot \left(\det A_1\right)^{\beta_1} \left(\det A_2\right)^{\beta_2}, $$

which holds if and only if

$$ \det\left(\beta_1 A_1 + \beta_2 A_2\right) \geq \left(\det A_1\right)^{\beta_1} \left(\det A_2\right)^{\beta_2}. $$

Finally, (4.5) and its equality condition follow from Minkowski's determinant inequality (1.3) and the equality condition therein. □

4.3. Uniqueness of the John s-ellipsoids

We start with a simple but useful observation.

Lemma 4.3 *(Interpolation between translated ellipsoids). Let f be a proper log-concave function on $\mathbb{R}^d$, and $s > 0$. Assume that the two d-symmetric ellipsoids $\overline{E}_1$ and $\overline{E}_2$ contained in ${}^{(s)}\overline{f}$ are translates of each other by a vector in $\mathbb{R}^d$. More specifically, assume that they are represented by $(A \oplus \alpha, a_1)$ and $(A \oplus \alpha, a_2)$ with $a_1 = -a_2 = \delta A e_1$, where e_1 is the first standard basis vector in $\mathbb{R}^d$, and $\delta > 0$. Then the origin centered ellipsoid*

$$ \overline{E}_0 = (A \oplus \alpha)\overline{M}\mathbf{B}^{d+1}, \quad \textit{where } \overline{M} = \operatorname{diag}(1+\delta, 1, \ldots, 1), $$

is contained in ${}^{(s)}\overline{f}$.

Proof. Since all super-level sets of $f^{1/s}$ are convex sets in $\mathbb{R}^d$, it is easy to see that for any convex set $\overline{H}$ in $\mathbb{R}^{d+1}$ and vector $v \in \mathbb{R}^d$, if $\overline{H} \subseteq {}^{(s)}\overline{f}$ and $\overline{H} + v \subseteq {}^{(s)}\overline{f}$, then $\operatorname{conv}\left(\overline{H} \cup (\overline{H} + v)\right) = \overline{H} + [0, v] \subseteq {}^{(s)}\overline{f}$.

Thus, ${}^{(s)}\overline{f}$ contains the "sausage-like" body

$$ \begin{aligned} \overline{W} &= \operatorname{conv}\left(\overline{E}_1 \cup \overline{E}_2\right) = (A \oplus \alpha)\left(\mathbf{B}^{d+1} + [A^{-1}a_2, A^{-1}a_1]\right) \\ &= (A \oplus \alpha)\left(\mathbf{B}^{d+1} + [-\delta e_1, \delta e_1]\right). \end{aligned} $$

On the other hand, clearly, $\overline{E}_0 \subseteq \overline{W}$ completing the proof of Lemma 4.3. □

As an application of Lemmas 4.1 and 4.2, we show that in the set of d-symmetric ellipsoids in ${}^{(s)}\overline{f}$ with a fixed height, a largest s-volume d-symmetric ellipsoid is unique.

Lemma 4.4 *(Uniqueness for a fixed height). Let f be a proper log-concave function on $\mathbb{R}^d$, and $s > 0$. Then, among all d-symmetric ellipsoids of height $\alpha, 0 < \alpha < \|f\|^{1/s}$, in ${}^{(s)}\overline{f}$, there is a unique one of maximal s-volume. Additionally, if there is a d-symmetric ellipsoid in ${}^{(s)}\overline{f}$ of height $\|f\|^{1/s}$, then among all d-symmetric ellipsoids of height $\alpha = \|f\|^{1/s}$ in ${}^{(s)}\overline{f}$, there is a unique one of maximal s-volume.*

Proof. Clearly, the maximum s-volume among d-symmetric ellipsoids of height α contained in ${}^{(s)}\overline{f}$ is positive. By Lemma 3.2 applied with $g = \hbar_{\mathbf{B}^2}^s$, where $\hbar_{\mathbf{B}^2}$ is the height function of $\mathbf{B}^2$ (see (2.10)), identity (3.3) and a standard compactness argument, this maximum is attained.

We show that such an ellipsoid is unique. Assume that $\overline{E}_1 \subset {}^{(s)}\overline{f}$ and $\overline{E}_2 \subset {}^{(s)}\overline{f}$, represented by $(A_1 \oplus \alpha, a_1) \in \mathcal{E}$ and $(A_2 \oplus \alpha, a_2) \in \mathcal{E}$, are two d-symmetric ellipsoids of the maximal s-volume.

Define a new d-symmetric ellipsoid $\overline{E}$ represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \alpha, \frac{a_1 + a_2}{2}\right) \in \mathcal{E}.$$

Applying (4.1) with $\lambda = 1/2$ and Lemma 4.1, we have $\overline{E} \subset {}^{(s)}\overline{f}$. Next, by the choice of the ellipsoids, we have that

$${}^{(s)}\mu(\overline{E}) \leq {}^{(s)}\mu(\overline{E}_1) = \sqrt{{}^{(s)}\mu(\overline{E}_1)\, {}^{(s)}\mu(\overline{E}_2)} = {}^{(s)}\mu(\overline{E}_2)\,.$$

By Lemma 4.2, we have that ${}^{(s)}\mu(\overline{E}) \geq \sqrt{{}^{(s)}\mu(\overline{E}_1)\, {}^{(s)}\mu(\overline{E}_2)}$, therefore equality holds. Thus, by the equality condition in Lemma 4.2, we conclude that $A_1 = A_2$.

To complete the proof, we need to show that $a_1 = a_2$. Assume the contrary: $a_1 \neq a_2$. By translating the origin and rotating the space $\mathbb{R}^d$, we may assume that $a_1 = -a_2 \neq 0$ and that $A_1^{-1} a_1 = \delta e_1$ for some $\delta > 0$.

By Lemma 4.3, the ellipsoid $\overline{E}_0 = (A_1 \oplus \alpha)\overline{M}\mathbf{B}^{d+1}$ is contained in ${}^{(s)}\overline{f}$, where $\overline{M} = \operatorname{diag}(1 + \delta, 1, \ldots, 1)$. However, ${}^{(s)}\mu(\overline{E}_0) > {}^{(s)}\mu(\overline{E}) = {}^{(s)}\mu(\overline{E}_1)$, which contradicts the choice of $\overline{E}_1$ and $\overline{E}_2$, completing the proof of Lemma 4.4. $\square$

Theorem 4.1 *(Existence and uniqueness of the John s-ellipsoid). Let $s > 0$ and f be a proper log-concave function on $\mathbb{R}^d$. Then, there exists a unique John s-ellipsoid of f.*

Proof of Theorem 4.1. As in the proof of Lemma 4.4, the existence of an s-ellipsoid of maximal s-volume follows from Lemma 3.2 applied with $g = \hbar_{\mathbf{B}^2}^s$, where $\hbar_{\mathbf{B}^2}$ is the height function of $\mathbf{B}^2$ (see (2.10)), identity (3.3) and a standard compactness argument.

Assume that $\overline{E}_1 \subset {}^{(s)}\overline{f}$ and $\overline{E}_2 \subset {}^{(s)}\overline{f}$ are two d-symmetric ellipsoids of maximal s-volume, represented by $(A_1 \oplus \alpha_1, a_1) \in \mathcal{E}$ and $(A_2 \oplus \alpha_2, a_2) \in \mathcal{E}$, respectively. We define a new d-symmetric ellipsoid $\overline{E}$ represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \sqrt{\alpha_1 \alpha_2}, \frac{a_1 + a_2}{2}\right) \in \mathcal{E}.$$

Applying (4.1) with $\lambda = 1/2$ and Lemma 4.1, we have $\overline{E} \subset {}^{(s)}\overline{f}$. Next, by the choice of the ellipsoids, we also have

$$^{(s)}\mu\left(\overline{E}\right) \leq {}^{(s)}\mu\left(\overline{E}_1\right) = \sqrt{{}^{(s)}\mu\left(\overline{E}_1\right)\,{}^{(s)}\mu\left(\overline{E}_2\right)} = {}^{(s)}\mu\left(\overline{E}_2\right),$$

which, combined with Lemma 4.2, yields ${}^{(s)}\mu\left(\overline{E}\right) = {}^{(s)}\mu\left(\overline{E}_1\right) = {}^{(s)}\mu\left(\overline{E}_2\right)$ and $A_1 = A_2$. This implies that $\alpha_1 = \alpha_2$, since the s-volume of $\overline{E}_1$ and $\overline{E}_2$ are equal. Therefore, by Lemma 4.4, the ellipsoids $\overline{E}_1$ and $\overline{E}_2$ coincide, completing the proof of Theorem 4.1. $\square$

4.4. Bound on the height

Recall from Section 2.5 that ${}^{(s)}J_f$ denotes the density of the John s-function of f, that is, the density of the s-marginal of the John s-ellipsoid of f. The following result is an extension of the analogous result on the "height" of the AMJV ellipsoid [2, Theorem 1.1] to the John s-ellipsoid with a similar proof.

Lemma 4.5. *Let f be a proper log-concave function on $\mathbb{R}^d$ and $s > 0$. Then,*

$$\left\|{}^{(s)}J_f\right\| \geq e^{-d}\left\|f\right\|. \tag{4.6}$$

We note that if the John s-ellipsoid of f is represented by $(A_s \oplus \alpha_s, a_s)$ (that is, its height is α_s), then $\left\|{}^{(s)}J_f\right\| = \alpha_s^s$.

Proof of Lemma 4.5. We define a function $\Psi : (0, \|f\|^{1/s}) \to \mathbb{R}^+$ as follows. By Lemma 4.4, for any $\alpha \in (0, \|f\|^{1/s})$, there is a unique d-symmetric ellipsoid of maximal s-volume among d-symmetric ellipsoids of height α in ${}^{(s)}\overline{f}$. Let this ellipsoid be represented by $(A_\alpha \oplus \alpha, a_\alpha) \in \mathcal{E}$. We set $\Psi(\alpha) = \det A_\alpha$.

Claim 4.1. *For any $\alpha_1, \alpha_2 \in (0, \|f\|^{1/s})$ and $\lambda \in [0, 1]$, we have*

$$\Psi\left(\alpha_1^\lambda \alpha_2^{1-\lambda}\right)^{1/d} \geq \lambda \Psi(\alpha_1)^{1/d} + (1-\lambda)\Psi(\alpha_2)^{1/d}. \tag{4.7}$$

Proof of Claim 4.1. Let $(A_1 \oplus \alpha_1, a_1)$ and $(A_2 \oplus \alpha_2, a_2)$ represent the d-symmetric ellipsoids of maximum s-volume contained in ${}^{(s)}\overline{f}$ with the corresponding heights. By Lemma 4.1 and (3.3), we have that

$$\Psi\left(\alpha_1^\lambda \alpha_2^{1-\lambda}\right) \geq \det\left(\lambda A_1 + (1-\lambda)A_2\right).$$

Now, (4.7) follows immediately from Minkowski's determinant inequality (1.2). $\square$

Set $\Phi(t) = \Psi(e^t)^{1/d}$ for all $t \in \left(-\infty, \frac{\log\|f\|}{s}\right)$. Inequality (4.7) implies that Φ is a concave function on its domain.

Let α_0 be the height of the John s-ellipsoid of f. Then, by (3.3), for any α in the domain of Ψ, we have that

$$\Psi(\alpha)\alpha^s \leq \Psi(\alpha_0)\alpha_0^s.$$

Setting $t_0 = \log \alpha_0$ and taking root of order d, we obtain

$$\Phi(t) \leq \Phi(t_0) e^{\frac{s}{d}(t_0 - t)}$$

for any t in the domain of Φ. The expression on the right-hand side is a convex function of t, while Φ is a concave function. Since these functions take the same value at $t = t_0$, we conclude that the graph of Φ lies below the tangent line to graph of $\Phi(t_0)e^{\frac{s}{d}(t_0-t)}$ at point t_0. That is,

$$\Phi(t) \leq \Phi(t_0)\left(1 - \frac{s}{d}(t - t_0)\right).$$

Passing to the limit as $t \to \frac{\log\|f\|}{s}$ and since the values of Φ are positive, we get

$$0 \leq 1 - \frac{\log\|f\|}{d} + \frac{s}{d} t_0.$$

Or, equivalently, $t_0 \geq -\frac{d}{s} + \frac{\log\|f\|}{s}$. Therefore, $\alpha_0 \geq e^{-d/s}\|f\|^{1/s}$ and $\left\|{}^{(s)}J_f\right\| = \alpha_0^s \geq e^{-d}\|f\|$. This completes the proof of Lemma 4.5. $\square$

5. John's condition — proof of Theorem 1.1

Theorem 1.1 is an immediate consequence of the following theorem whose proof is the topic of this section.

Theorem 5.1. *Let $\overline{K}$ be a closed d-symmetric set in $\mathbb{R}^{d+1}$, and let $s > 0$. Assume that $\mathbf{B}^{d+1} \subseteq \overline{K}$. Then the following hold.*

(1) Assume that $\mathbf{B}^{d+1}$ is a locally *maximal s-volume ellipsoid contained in $\overline{K}$, that is, in some neighborhood of $\mathbf{B}^{d+1}$, no ellipsoid contained in $\overline{K}$ is of larger s-volume. Then there are contact points $\overline{u}_1, \dots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$ and positive weights $c_1, \dots, c_k$ such that*

$$\sum_{i=1}^{k} c_i \overline{u}_i \otimes \overline{u}_i = \overline{S} \quad \textit{and} \quad \sum_{i=1}^{k} c_i u_i = 0, \tag{5.1}$$

where u_i is the orthogonal projection of $\overline{u}_i$ onto $\mathbb{R}^d$ and $\overline{S} = \operatorname{diag}(1, \dots, 1, s) = I \oplus s$. Moreover, such contact points and positive weights exist for some k with $d+1 \leq k \leq \frac{(d+1)(d+2)}{2} + d + 1$.

(2) Assume that $\overline{K} = {}^{(s)}\overline{f}$ for a proper log-concave function f, and that there are contact points and positive weights satisfying (5.1).

Then $\mathbf{B}^{d+1}$ *is the unique ellipsoid of (*globally*) maximum* s*-volume among* d*-symmetric ellipsoids contained in* $\overline{K}$*.*

We equip $\mathcal{M}$ (for the definition, see (2.5)) with an inner product (that comes from the Frobenius product on the space of matrices and the standard inner product on $\mathbb{R}^d$) defined by

$$\left\langle (\overline{A}, a), (\overline{B}, b) \right\rangle = \operatorname{trace}\left(\overline{A}\,\overline{B}\right) + \langle a, b\rangle .$$

Thus, we may use the topology of $\mathcal{M}$ on the set $\mathcal{E}$ of ellipsoids in $\mathbb{R}^{d+1}$.

Denote the set of contact points by $C = \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left(\overline{K}\right)$, and consider

$$\widehat{C} = \{(\overline{u} \otimes \overline{u}, u) : \ \overline{u} \in C\} \subset \mathcal{M},$$

where u denotes the orthogonal projection of $\overline{u}$ onto $\mathbb{R}^d$.

The proof of Part (1) of Theorem 5.1 is an adaptation of the argument given in [5] and [16] (see also [14,15,8,19] and [23, Theorem 14.5]) to the s-volume. The idea is that, if there are no contact points and positive weights satisfying (5.1), then there is a path, namely a *line segment* in the space $\mathcal{E}$ of ellipsoids starting from $\mathbf{B}^{d+1}$ such that the s-volume increases along the path and the path stays in the family of ellipsoids contained in $\overline{K}$.

Part (2) on the other hand, needs a finer argument. The idea is that if $\mathbf{B}^{d+1}$ is not the global maximizer of the s-volume, then we will find a path in $\mathcal{E}$ starting from $\mathbf{B}^{d+1}$ such that the s-volume increases along the path, and the path stays in the family of ellipsoids contained in $\overline{K}$. The difficulty is that ${}^{(s)}\overline{f}$ is not necessarily convex. Thus, this path *is not a line segment*. We will, however, be able to differentiate the s-volume along this path, and by doing so, we will show that $(\overline{S}, 0)$ is separated by a hyperplane from the points $\widehat{C}$ in $\mathcal{M}$, which in turn will yield that there are no contact points and positive weights satisfying (5.1).

First, as a standard observation, we state the relationship between (5.1) and separation by a hyperplane of the point $(\overline{S}, 0)$ from the set $\widehat{C}$ in the space $\mathcal{M}$.

Claim 5.1. *The following assertions are equivalent.*

(1) There are contact points and positive weights satisfying (5.1)*.*
(2) There are contact points and positive weights satisfying a modified version of (5.1)*, where in the second equation* u_i *is replaced by* $\overline{u}_i$*.*
(3) $(\overline{S}, 0) \in \operatorname{pos}(\widehat{C})$*.*
(4) $\frac{1}{d+s}(\overline{S}, 0) \in \operatorname{conv}\left(\widehat{C}\right)$*.*
(5) There is no $(\overline{H}, h) \in \mathcal{M}$ *with*

$$\left\langle (\overline{H}, h), (\overline{S}, 0) \right\rangle > 0, \ \ \textit{and} \ \left\langle (\overline{H}, h), (\overline{u} \otimes \overline{u}, u) \right\rangle < 0 \ \textit{for all}\ \overline{u} \in C. \tag{5.2}$$

(6) There is no $(\overline{H}, h) \in \mathcal{M}$ *with*

$$\left\langle (\overline{H}, h), (\overline{S}, 0) \right\rangle > 0, \text{ and } \left\langle (\overline{H}, h), (\overline{u} \otimes \overline{u}, u) \right\rangle \leq 0 \text{ for all } \overline{u} \in C. \tag{5.3}$$

Proof. We leave it to the reader to verify the equivalence of (1) and (2) and (3), as well as that of (5) and (6).

To see that (3) is equivalent to (4), we take trace in (5.1) and notice that $\operatorname{trace}(\overline{u} \otimes \overline{u}) = \operatorname{trace}\left(\frac{1}{d+s}\overline{S}\right) = 1$, which shows that $\sum_{i=1}^{k} c_i = d + s$.

Finally, observe that the convex cone $\operatorname{pos}(\widehat{C})$ in $\mathcal{M}$ does not contain the point $\left(\overline{S}, 0\right) \in \mathcal{M}$ if and only if it is separated from this point by a hyperplane through the origin. This is what (5.2) expresses, showing that (3) is equivalent to (5), and hence, completing the proof of Claim 5.1. □

Claim 5.2. *If contact points and positive weights satisfying* (5.1) *exist for some* k*, then they exist for some* $d + 1 \leq k \leq \frac{(d+1)(d+2)}{2} + d + 1$.

Proof. Since $\overline{u} \otimes \overline{u}$ is of rank 1, the lower bound on k is obvious. The upper bound follows from (4) in Claim 5.1 and Carathéodory's theorem applied in the vector space $\mathcal{M}$. □

Next, we show that if $(\overline{S}, 0)$ and $\widehat{C}$ are separated by a hyperplane in $\mathcal{M}$, then the normal vector of that hyperplane can be chosen to be of a special form.

Claim 5.3. *There is* $(\overline{H}, h) \in \mathcal{M}$ *satisfying* (5.2) *if and only if there is* $(\overline{H}_0, h) \in \mathcal{M}$ *satisfying* (5.2)*, where* $\overline{H}_0 = H_0 \oplus \gamma$ *for some* $H_0 \in \mathbb{R}^{d \times d}$.

Proof. For any $\overline{u} \in \mathbb{R}^{d+1}$, let $\overline{u}'$ denote the reflection of $\overline{u}$ about $\mathbb{R}^d$, that is, $\overline{u}'$ differs from $\overline{u}$ only in the last coordinate, which is the opposite of the last coordinate of $\overline{u}$. Since both $\overline{K}$ and $\mathbf{B}^{d+1}$ are symmetric about $\mathbb{R}^d$, we conclude that, if $\overline{u}$ is in C, then so is $\overline{u}'$.

Let $\overline{H}_0$ denote the matrix obtained from $\overline{H}$ by setting the first d entries of the last row to zero, and the first d entries of the last column to zero. Thus, $\overline{H}_0$ is of the required form. We show that $(\overline{H}_0, h)$ satisfies (5.2). Clearly, $\left\langle \left(\overline{H}, h\right), \left(\overline{S}, 0\right) \right\rangle = \left\langle \left(\overline{H}_0, h\right), \left(\overline{S}, 0\right) \right\rangle$, and thus, the first inequality in (5.2) holds.

For the other inequality in (5.2), consider an arbitrary vector $\overline{u} \in C$. Then the inequalities $0 > \left\langle (\overline{H}, h), (\overline{u} \otimes \overline{u}, u) \right\rangle$ and $0 > \left\langle (\overline{H}, h), (\overline{u}' \otimes \overline{u}', u) \right\rangle$ hold. Note that in the $(d+1) \times (d+1)$ matrix $(\overline{u}' \otimes \overline{u}' + \overline{u} \otimes \overline{u})$, the first d entries of the last row as well as of the last column are 0. Thus,

$$0 > \left\langle (\overline{H}, h), \left((\overline{u}' \otimes \overline{u}' + \overline{u} \otimes \overline{u})/2, u\right) \right\rangle =$$
$$\left\langle (\overline{H}_0, h), \left((\overline{u}' \otimes \overline{u}' + \overline{u} \otimes \overline{u})/2, u\right) \right\rangle = \left\langle (\overline{H}_0, h), (\overline{u} \otimes \overline{u}, u) \right\rangle,$$

completing the proof of Claim 5.3. □

In both parts of the proof of Theorem 5.1, we will consider a path in $\mathcal{E}$ and compute the derivative of the s-volume at the start of this path.

Claim 5.4. *Let $\varepsilon_0 > 0$ and let $\gamma : [0, \varepsilon_0] \to \mathbb{R}$ be a continuous function whose right derivative at 0 exists. Let $H \in \mathbb{R}^{d\times d}$ be an arbitrary symmetric matrix and $h \in \mathbb{R}^d$. Consider the path*

$$\overline{E} : [0, \varepsilon_0] \to \mathcal{M};\ \ t \mapsto \Big(\overline{I} + t\big(H \oplus \gamma(t)\big), th\Big). \tag{5.4}$$

For sufficiently small t, we have that $\overline{E}(t)$ is in $\mathcal{E}$, and the right derivative of the s-volume is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}} = \big\langle (H \oplus \gamma(0), h), (\overline{S}, 0)\big\rangle\,. \tag{5.5}$$

Proof. We apply (3.3),

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \Big[(1+t\gamma(t))^s \det(I+tH)\Big] =$$
$$(1+0\cdot\gamma(0))^s \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \Big[\det(I+tH)\Big] + \det(I+0\cdot H)\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \Big[(1+t\gamma(t))^s\Big] =$$
$$\operatorname{trace}(H) + s\gamma(0),$$

which is equal to the right hand side of (5.5) completing the proof of Claim 5.4. □

Claim 5.5. *If there is $(H\oplus\gamma, h) \in \mathcal{M}$ satisfying (5.2), then $\mathbf{B}^{d+1}$ is not a locally maximal s-volume d-symmetric ellipsoid contained in $\overline{K}$.*

Proof. Let $\gamma(t) = \gamma$ be the constant function for $t \geq 0$, and consider the path (5.4). By Claim 5.4 and (5.2), the s-volume has positive derivative at the start of this path. Clearly, ${}^{(s)}\mu\big(\overline{E}(t)\big)$ is differentiable on some interval $[0, \varepsilon_0]$, and hence, there is an $\varepsilon_1 > 0$ such that for every $0 < t < \varepsilon_1$, we have

$${}^{(s)}\mu\big(\overline{E}(t)\big) > {}^{(s)}\mu\big(\mathbf{B}^{d+1}\big)\,. \tag{5.6}$$

Now, it suffices to establish that there is $\varepsilon_2 > 0$ such that for all $0 < t < \varepsilon_2$, we have

$$\overline{E}(t) \subseteq \overline{K}. \tag{5.7}$$

Set $\overline{H} = H \oplus \gamma$. First, we fix an arbitrary contact point $\overline{u} \in C$. We claim that there is an $\varepsilon(\overline{u}) > 0$ such that for every $0 < t < \varepsilon(\overline{u})$, we have $(\overline{I} + t\overline{H})\overline{u} + th \in \operatorname{int}\big(\mathbf{B}^{d+1}\big)$. Indeed,

$$\left\langle (\overline{I}+t\overline{H})\overline{u}+th, (\overline{I}+t\overline{H})\overline{u}+th \right\rangle = 1+2t\Big(\left\langle \overline{H}\overline{u}, \overline{u}\right\rangle + \langle h, u\rangle \Big) + o(t) =$$
$$1+2t\left\langle (\overline{H}, h), (\overline{u}\otimes\overline{u}, u)\right\rangle + o(t).$$

By (5.2), the latter is less than 1 for a sufficiently small positive t. Next, the compactness of C yields that there is an $\varepsilon_3 > 0$ such that $(\overline{I}+\varepsilon_3\overline{H})C + \varepsilon_3 h \subseteq \operatorname{int}\left(\mathbf{B}^{d+1}\right) \subseteq \overline{K}$.

By the continuity of the map $x \mapsto (\overline{I}+\varepsilon_3\overline{H})x + \varepsilon_3 h$, there is an open neighborhood $\mathcal{W}$ of C in $\mathbf{B}^{d+1}$ such that $(\overline{I}+\varepsilon_3\overline{H})\mathcal{W} + \varepsilon_3 h \subseteq \operatorname{int}\left(\mathbf{B}^{d+1}\right) \subseteq \overline{K}$. The latter combined with $\mathcal{W} \subset \operatorname{int}\left(\mathbf{B}^{d+1}\right)$ and with the convexity of $\mathbf{B}^{d+1}$ yield that for all $0 < t < \varepsilon_3$, we have $(\overline{I}+t\overline{H})\mathcal{W} + th \subseteq \operatorname{int}\left(\mathbf{B}^{d+1}\right) \subseteq \overline{K}$.

On the other hand, the compact set $\mathbf{B}^{d+1}\setminus\mathcal{W}$ is a subset of $\operatorname{int}\left(\overline{K}\right)$, and hence, there is an $\varepsilon_4 > 0$ such that for all $0 < t < \varepsilon_4$, we have $(\overline{I}+t\overline{H})(\mathbf{B}^{d+1}\setminus\mathcal{W}) + th \subseteq \operatorname{int}\left(\overline{K}\right)$. Thus, if $0 < t < \min\{\varepsilon_3, \varepsilon_4\}$, then $(\overline{I}+t\overline{H})(\mathcal{W}) + th \subseteq \operatorname{int}\left(\overline{K}\right)$ and $(\overline{I}+t\overline{H})(\mathbf{B}^{d+1}\setminus\mathcal{W}) + th \subseteq \operatorname{int}\left(\overline{K}\right)$. Thus, (5.7) holds concluding the proof of Claim 5.5. □

5.1. *Proof of part (1) of Theorem 5.1*

Assume that there are no contact points and positive weights satisfying (5.1). By Claims 5.1 and 5.3, there is $(H\oplus\gamma, h) \in \mathcal{M}$ satisfying (5.2). Claim 5.5 yields that $\mathbf{B}^{d+1}$ is not a locally maximal s-volume ellipsoid contained in $\overline{K}$.

The bound on k follows from Claim 5.2, completing the proof of part (1) of Theorem 5.1.

5.2. *Proof of part (2) of Theorem 5.1*

Assume that there is an ellipsoid $\overline{E}$, represented by $(A\oplus\alpha, a)$, contained in $\operatorname{int}\left({}^{(s)}\overline{f}\right)$ with ${}^{(s)}\mu\left(\overline{E}\right) > {}^{(s)}\mu\left(\mathbf{B}^{d+1}\right)$.

Set $G = A - I \in \mathbb{R}^{d\times d}$, and define the function $\gamma(t) = \frac{\alpha^t - 1}{t}$ for $t \in (0,1]$, which, with $\gamma(0) = \ln\alpha$, is a continuous function on $[0,1]$ whose right derivative at 0 exists. Consider the path

$$\overline{E} : [0,1] \to \mathcal{M};\;\; t \mapsto \left(\overline{I} + t\big(G\oplus\gamma(t)\big), ta\right).$$

Clearly, this path is in $\mathcal{E}$, it starts at $\overline{E}(0) = \mathbf{B}^{d+1}$ and ends at $\overline{E}(1) = \overline{E}$.

Claim 5.6.

$$0 \le \left\langle (G\oplus\gamma(0), a), (\overline{S}, 0)\right\rangle. \tag{5.8}$$

Proof. By Lemma 4.2, for every $t \in [0,1]$, we have

$$\frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}} \ge 1,$$

and hence, for the right derivative, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \frac{{}^{(s)}\mu\big(\overline{E}(t)\big)}{{}^{(s)}\kappa_{d+1}} \geq 0.$$

Claim 5.4 now yields the assertion of Claim 5.6. □

We want to have strict inequality in (5.8), thus we modify G a bit. Let

$$H = G + \delta I, \text{ with a small } \delta > 0.$$

By Claim 5.6, we have

$$0 < \big\langle (H \oplus \gamma(0), a), (\overline{S}, 0) \big\rangle . \tag{5.9}$$

Moreover, since $\overline{A}\mathbf{B}^{d+1} + a \subset \operatorname{int}\left({}^{(s)}\overline{f}\right)$, we can fix $\delta > 0$ sufficiently small such that we also have that

$$((I + H) \oplus (1 + \gamma(1)))\, \mathbf{B}^{d+1} + a \subset \operatorname{int}\left({}^{(s)}\overline{f}\right). \tag{5.10}$$

Claim 5.7. *Set $\overline{H}_0 = H \oplus \gamma(0)$. Then*

$$\big\langle (\overline{H}_0, a), (\overline{u} \otimes \overline{u}, u) \big\rangle \leq 0 \tag{5.11}$$

for every contact point $\overline{u} \in C$.

Proof. Fix an $\overline{u} \in C$ and consider the curve $\xi : [0,1] \to \mathbb{R}^{d+1}; t \mapsto \overline{u} + t\big(H \oplus \gamma(t)\big)\overline{u} + ta$ in $\mathbb{R}^{d+1}$. By Lemma 4.1 and (5.10), the ellipsoid represented by $(\overline{I}, 0) + t(H \oplus \gamma(t), a)$ is contained in ${}^{(s)}\overline{f}$ for every $t \in [0,1]$, and in particular, the curve ξ is contained in ${}^{(s)}\overline{f}$. By convexity and (5.10), we have that the projection of ξ onto $\mathbb{R}^d$ is a subset of the closure of the support of f. Further, ξ is a smooth curve and its tangent vector $\xi'(0)$ is given by

$$\begin{aligned}\xi'(0) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \big(\overline{u} + t\big(H \oplus \gamma(t)\big)\overline{u} + ta\big) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0^+} \big((tH \oplus (\alpha^t - 1))\overline{u} + ta\big) = (H \oplus \ln \alpha)\overline{u} + a.\end{aligned}$$

We consider two cases as to whether $\overline{u} \in \mathbb{R}^d$ or not.

First, if $\overline{u} \in \mathbb{R}^d$, then $\overline{u}$ belongs to the boundary of the support of f. Since the support of a log-concave function is a convex set, we conclude that $\overline{u}$ is the outer normal vector to the support of f at $\overline{u}$. Thus, $\langle \xi'(0), \overline{u} \rangle \leq 0$.

Second, if $\overline{u} \notin \mathbb{R}^d$, then Lemma 3.1 implies that bd $\left({}^{(s)}\overline{f}\right)$ is a smooth hypersurface in $\mathbb{R}^{d+1}$ at $\overline{u}$, whose outer unit normal vector at $\overline{u}$ is $\overline{u}$ itself. Thus, the angle between the tangent vector $\xi'(0)$ of the curve ξ and the outer normal vector of the hypersurface bd $\left({}^{(s)}\overline{f}\right)$ at $\overline{u}$ is not acute. That is, $\langle \xi'(0), \overline{u}\rangle \leq 0$.

Hence, in both cases, we have

$$0 \geq \langle \xi'(0), \overline{u}\rangle = \langle (H \oplus \ln \alpha)\overline{u} + a) , \overline{u}\rangle ,$$

which is (5.11) completing the proof of Claim 5.7. $\square$

In summary, (5.9) and Claim 5.7 show that when $\left(\overline{H}_0, a\right)$ is substituted in the place of $(\overline{H}, h)$, then (5.3) holds. Hence, by Claim 5.1, the proof of part (2) of Theorem 5.1 is complete.

We rephrase Theorem 5.1 without any reference to lifting of a function to $\mathbb{R}^{d+1}$ as follows.

Theorem 5.2. *Let f be a proper log-concave function on $\mathbb{R}^d$, $s > 0$. Assume $\hbar^s_{\mathbf{B}^{d+1}} \leq f$. Then the following are equivalent:*

(1) The function $\hbar^s_{\mathbf{B}^{d+1}}$ is the John s-function of f.
(2) There are points $u_1, \dots, u_k \in \mathbf{B}^d \subset \mathbb{R}^d$ and positive weights $c_1, \dots, c_k$ such that
   (a) $f(u_i) = \hbar^s_{\mathbf{B}^{d+1}}(u_i)$ for all $i \in [k]$;
   (b) $\sum_{i=1}^k c_i u_i \otimes u_i = I$;
   (c) $\sum_{i=1}^k c_i f^{1/s}(u_i) \cdot f^{1/s}(u_i) = s$;
   (d) $\sum_{i=1}^k c_i u_i = 0$,
   where I is the $d \times d$ identity matrix.

6. Further inequalities and the limit as s tends to 0

6.1. Comparison of the s-volumes of John s-ellipsoids for distinct values of s

Lemma 6.1. *Let f be a proper log-concave function on $\mathbb{R}^d$, and $0 < s_1 < s_2$. Then,*

$$\sqrt{\left(\frac{s_2}{d+s_2}\right)^{s_2}\left(\frac{d}{d+s_2}\right)^{d}} \cdot \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}} \leq \frac{{}^{(s_1)}\mu\left(\overline{E}(f, s_1)\right)}{{}^{(s_2)}\mu\left(\overline{E}(f, s_2)\right)} \leq \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}}.$$

Proof. We start with the second inequality. We may assume that $\overline{E}(f, s_1) = \mathbf{B}^{d+1}$, and hence, its height function is $\hbar_{\overline{E}(f,s_1)}(x) = \sqrt{1 - |x|^2}$ for $x \in \mathbf{B}^d$. Since $s_1 < s_2$ and $\hbar_{\overline{E}(f,s_1)}(x) \leq 1$, we have

$$\left(\hbar_{\overline{E}(f,s_1)}(x)\right)^{s_2} \leq \left(\hbar_{\overline{E}(f,s_1)}(x)\right)^{s_1} \leq f(x) \quad \text{ for all } x \in \mathbf{B}^d.$$

That is, by (2.9), $\mathbf{B}^{d+1} \subset {}^{(s_2)}\overline{f}$, which yields ${}^{(s_2)}\mu\big(\mathbf{B}^{d+1}\big) \leq {}^{(s_2)}\mu\big(\overline{E}(f, s_2)\big)$. Hence,

$$\frac{{}^{(s_1)}\mu\big(\overline{E}(f, s_1)\big)}{{}^{(s_2)}\mu\big(\overline{E}(f, s_2)\big)} \leq \frac{{}^{(s_1)}\mu\big(\mathbf{B}^{d+1}\big)}{{}^{(s_2)}\mu\big(\mathbf{B}^{d+1}\big)} = \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}}.$$

Next, we prove the first inequality of the assertion of the lemma. Now, we assume that $\overline{E}(f, s_2) = \mathbf{B}^{d+1}$. Therefore, for any $\rho \in (0, 1)$, we have that ${}^{(s_2)}\overline{f}$ contains the cylinder $\rho\mathbf{B}^d \times [-\sqrt{1-\rho^2}, \sqrt{1-\rho^2}]$. Hence, ${}^{(s_1)}\overline{f}$ contains the ellipsoid $\overline{E}$, represented by $\left(\rho I \oplus \left(\sqrt{1-\rho^2}\right)^{s_2/s_1}, 0\right)$, whose s_1-volume by (3.3) is ${}^{(s_1)}\kappa_{d+1} \cdot \rho^d \cdot \big(1-\rho^2\big)^{s_2/2}$. Choosing $\rho = \sqrt{\frac{d}{d+s_2}}$, we obtain

$$\sqrt{\left(\frac{s_2}{d+s_2}\right)^{s_2}\left(\frac{d}{d+s_2}\right)^{d}} \cdot \frac{{}^{(s_1)}\kappa_{d+1}}{{}^{(s_2)}\kappa_{d+1}} = \frac{{}^{(s_1)}\mu\big(\overline{E}\big)}{{}^{(s_2)}\mu\big(\mathbf{B}^{d+1}\big)} \leq \frac{{}^{(s_1)}\mu\big(\overline{E}(f, s_1)\big)}{{}^{(s_2)}\mu\big(\overline{E}(f, s_2)\big)}. \qquad \square$$

6.2. Stability of the John s-ellipsoid

Lemma 6.2. *Fix a dimension d and a positive constant $C > 0$. Then there exist constants $\varepsilon_C > 0$ and $k_C > 0$ with the following property. Let $s \in (0, \infty)$, $\varepsilon \in [0, \varepsilon_C]$ and f be a proper log-concave function on $\mathbb{R}^d$, whose John s-ellipsoid $\overline{E}(f, s)$ is represented by $(A_1 \oplus \alpha_1, a_1)$, and let $\overline{E}_2$ denote another ellipsoid, represented by $(A_2 \oplus \alpha_2, a_2)$, with $\overline{E}_2 \subset {}^{(s)}\overline{f}$. Assume that*

$$ {}^{(s)}\mu\big(\overline{E}(f, s)\big) \geq C - \varepsilon \quad \textit{and} \quad {}^{(s)}\mu\big(\overline{E}(f, s)\big) \geq {}^{(s)}\mu\big(\overline{E}_2\big) \geq {}^{(s)}\mu\big(\overline{E}(f, s)\big) - \varepsilon. \tag{6.1}$$

Then

$$\left\| \frac{A_1}{\|A_1\|} - \frac{A_2}{\|A_2\|} \right\| + \frac{|\alpha_1^s - \alpha_2^s|}{\|f\|} + \frac{|a_1 - a_2|}{\|A_1\| \cdot \|f\|} \leq k_C\sqrt{\varepsilon}. \tag{6.2}$$

In this subsection, we prove Lemma 6.2.

Let $\overline{E}$ denote the ellipsoid represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \sqrt{\alpha_1\alpha_2}, \frac{a_1 + a_2}{2}\right).$$

By Lemma 4.1, $\overline{E} \subset {}^{(s)}\overline{f}$, and, therefore, ${}^{(s)}\mu\big(\overline{E}(f, s)\big) \geq {}^{(s)}\mu\big(\overline{E}\big)$.

Claim 6.1. *There are constants $\varepsilon_0 > 0$ and $k_0 > 0$ such that if the ellipsoids $\overline{E}(f, s)$ and $\overline{E_2}$ satisfy (6.1) for $\varepsilon \in [0, \varepsilon_0]$, then*

$$(1 - k_0\sqrt{\varepsilon})A_1 \prec A_2 \prec (1 + k_0\sqrt{\varepsilon})A_1, \tag{6.3}$$

and

$$\left\| \frac{A_1}{\|A_1\|} - \frac{A_2}{\|A_2\|} \right\| \leq k_0\sqrt{\varepsilon} \quad \textit{and} \quad 1 - k_0\sqrt{\varepsilon} \leq \frac{\det A_1}{\det A_2} \leq 1 + k_0\sqrt{\varepsilon}.$$

Proof. By (3.3), we have

$$\frac{{}^{(s)}\mu\big(\overline{E}\big)}{\sqrt{{}^{(s)}\mu\big(\overline{E}(f,s)\big)\,{}^{(s)}\mu\big(\overline{E}_2\big)}} = \frac{1}{2^d}\frac{\det(A_1 + A_2)}{\sqrt{\det A_1 \det A_2}}.$$

Since ${}^{(s)}\mu\big(\overline{E}(f,s)\big) \geq {}^{(s)}\mu\big(\overline{E}\big)$ and by (6.1), there exist $\varepsilon_1 > 0$ and $k_1 > 0$ such that the left-hand side in the equation above is at most $1 + k_1 \cdot \varepsilon$ for all $\varepsilon \in [0, \varepsilon_1]$. Therefore, we have that

$$1 + k_1 \cdot \varepsilon \geq \frac{1}{2^d}\frac{\det(A_1 + A_2)}{\sqrt{\det A_1 \det A_2}}. \tag{6.4}$$

Let R be the square root of A_1, and U be the orthogonal transformation that diagonalizes $R^{-1}A_2R^{-1}$, that is, the matrix $D = UR^{-1}A_2R^{-1}U^T$ is diagonal. Let $D = \operatorname{diag}(\beta_1, \ldots, \beta_d)$. Then for $S = UR^{-1}$, we have $SA_1S^T = I, SA_2S^T = D$. By the multiplicativity of the determinant, inequality (6.4) is equivalent to

$$1 + k_1 \cdot \varepsilon \geq \prod_1^d \frac{1 + \beta_i}{2\sqrt{\beta_i}}.$$

Since $1 + \beta \geq 2\sqrt{\beta}$ for any $\beta > 0$, this implies that

$$1 + k_1 \cdot \varepsilon \geq \frac{1 + \beta_i}{2\sqrt{\beta_i}}$$

for every $i \in [d]$. If we consider the above formula as a quadratic inequality in the variable $\sqrt{\beta_i}$, then, by the quadratic formula, we obtain that there exist positive constants k_2 and ε_2 such that the inequality

$$1 - k_2\sqrt{\varepsilon} \leq \beta_i \leq 1 + k_2\sqrt{\varepsilon} \tag{6.5}$$

holds for all $\varepsilon \in [0, \varepsilon_2]$.

Clearly, $\frac{\det A_1}{\det A_2} = 1/\prod_{i=1}^d \beta_i$ and hence, the estimate on $\frac{\det A_1}{\det A_2}$ follows from (6.5).

On the other hand, (6.5) yields also that

$$(1 - k_2\sqrt{\varepsilon})I \prec SA_2S^T \prec (1 + k_2\sqrt{\varepsilon})I.$$

Thus, (6.3) follows. Hence, there exist positive constants k_3 and ε_3 such that the inequality

$$\left| \frac{\|A_2\|}{\|A_1\|} - 1 \right| \le k_3 \sqrt{\varepsilon}$$

holds for all $\varepsilon \in [0, \varepsilon_3]$. This and (6.3) yield that

$$\left\| \frac{A_1}{\|A_1\|} - \frac{A_2}{\|A_2\|} \right\| \le \left\| \frac{A_1}{\|A_1\|} - \frac{A_2}{\|A_1\|} \right\| + \left\| \frac{A_2}{\|A_1\|} - \frac{A_2}{\|A_2\|} \right\| \le (k_2 + k_3)\sqrt{\varepsilon}$$

for all $\varepsilon \in [0, \min\{\varepsilon_2, \varepsilon_3\}]$. This completes the proof of Claim 6.1. □

Claim 6.2. *There are constants $\varepsilon_0 > 0$ and $k_0 > 0$ such that if the ellipsoids $\overline{E}(f, s)$ and $\overline{E_2}$ satisfy (6.1) for $\varepsilon \in [0, \varepsilon_0]$, then*

$$|\alpha_1^s - \alpha_2^s| \le \|f\| \, k_0 \sqrt{\varepsilon}.$$

Proof. By identity (3.3) and the inequalities (6.1), we have that

$$1 \ge \frac{{}^{(s)}\mu\big(\overline{E}_2\big)}{{}^{(s)}\mu\big(\overline{E}(f,s)\big)} = \frac{\det A_2 \cdot \alpha_2^s}{\det A_1 \cdot \alpha_1^s} \ge 1 - \frac{\varepsilon}{{}^{(s)}\mu\big(\overline{E}(f,s)\big)}.$$

By this and by Claim 6.1, we get the following inequality

$$\big(1 + k_1\sqrt{\varepsilon}\big)\, \alpha_1^s \ge \alpha_2^s \ge \big(1 - k_1\sqrt{\varepsilon}\big)\, \alpha_1^s$$

for all $\varepsilon \in [0, \varepsilon_1]$, where k_1 and ε_1 are some positive constants. Equivalently, we have that

$$k_1\sqrt{\varepsilon} \cdot \frac{\alpha_1^s}{\|f\|} \ge \frac{\alpha_2^s - \alpha_1^s}{\|f\|} \ge -k_1\sqrt{\varepsilon} \cdot \frac{\alpha_1^s}{\|f\|}.$$

The claim follows since $\frac{\alpha_1^s}{\|f\|} \le 1$. □

To complete the proof of Lemma 6.2, we need to show that a_1 and a_2 are close. By translating the origin and rotating the space $\mathbb{R}^d$, we may assume that $a_1 = -a_2 \ne 0$ and that $A_1^{-1} a_1 = \delta e_1$ for some $\delta > 0$. Consider the origin centered d-symmetric ellipsoid

$$\overline{E}_0 = (A_1 \oplus \alpha_1)\overline{M}\mathbf{B}^{d+1}, \quad \text{where } \overline{M} = \operatorname{diag}(1 + \delta, 1, \ldots, 1).$$

Clearly, ${}^{(s)}\mu\big(\overline{E}_0\big) = {}^{(s)}\mu\big(\overline{E}(f,s)\big)\left(1 + \frac{|A_1^{-1}(a_1 - a_2)|}{2}\right) \ge {}^{(s)}\mu\big(\overline{E}(f,s)\big)\left(1 + \frac{|(a_1 - a_2)|}{2\|A_1\|}\right)$.

By (6.3) and Claim 6.2, we have

$$\left((1 - k_0\sqrt{\varepsilon})A_1 \oplus (1 - k_0 \|f\| \sqrt{\varepsilon})^{1/s} \alpha_1\right) \mathbf{B}^{d+1} + a_2 \subseteq {}^{(s)}\overline{f}.$$

On the other hand, clearly,

$$\left((1-k_0\sqrt{\varepsilon})A_1 \oplus (1-k_0\,\|f\|\,\sqrt{\varepsilon})^{1/s}\alpha_1\right)\mathbf{B}^{d+1}+a_1 \subseteq (A_1\oplus\alpha_1)\,\mathbf{B}^{d+1}+a_1 \subseteq {}^{(s)}\overline{f}.$$

Thus, by Lemma 4.3,

$$\left((1-k_0\sqrt{\varepsilon})A_1 \oplus (1-k_0\,\|f\|\,\sqrt{\varepsilon})^{1/s}\alpha_1\right)\overline{M}\mathbf{B}^{d+1}$$

is contained in ${}^{(s)}\overline{f}$.

By (3.3), the s-volume of this ellipsoid is

$${}^{(s)}\mu\big(\overline{E}(f,s)\big)\left(1-k_0\,\|f\|\,\sqrt{\varepsilon}\right)\left(1-k_0\sqrt{\varepsilon}\right)^d\left(1+\frac{|(a_1-a_2)|}{2\,\|A_1\|}\right)\le {}^{(s)}\mu\big(\overline{E}(f,s)\big)\,.$$

Thus, there exist constants $\varepsilon_1, k_1 > 0$ such that $\frac{|a_1-a_2|}{\|A_1\|} \le k_1\,\|f\|\,\sqrt{\varepsilon}$ for any $\varepsilon\in[0,\varepsilon_1]$. From this and Claims 6.1 and 6.2, Lemma 6.2 follows.

6.3. The limit as $s \to 0$

We recall from Section 1 the approach of Alonso-Gutiérrez, Gonzales Merino, Jiménez and Villa [2].

Let f be a proper log-concave function on $\mathbb{R}^d$. For every $\beta\in(0,\|f\|)$, consider the superlevel set $[f\ge\beta]$ of f. This is a bounded convex set with non-empty interior in $\mathbb{R}^d$, we take its largest volume ellipsoid, and multiply the volume of this ellipsoid by β. As shown in [2], there is a unique $\beta_0\in[0,\|f\|]$ such that this product is maximal. Furthermore, $\beta_0 \ge e^{-d}\,\|f\|$. We call the ellipsoid E in $\mathbb{R}^d$ obtained for this β_0 *the AMJV ellipsoid.*

We refer to a function of the form $\beta\chi_E$, where $E\subset\mathbb{R}^d$ is an ellipsoid in $\mathbb{R}^d$ and $\beta>0$, as a *flat ellipsoid function.* We will say that $(A\oplus\alpha, a)\in\mathcal{E}$ represents the flat ellipsoid function $\alpha\chi_{A\mathbf{B}^d+a}$. Clearly, any flat ellipsoid function is represented by a unique element of $\mathcal{E}$ and the AMJV ellipsoid is the maximal integral flat ellipsoid function among all flat ellipsoid functions that are below f.

Theorem 6.1 *(The AMJV ellipsoid is the John 0-ellipsoid). Let f be a proper log-concave function. Then there exists $(A\oplus\alpha, a)\in\mathcal{E}$ such that*

(1) The function $\alpha\chi_{A\mathbf{B}^d+a}$ is below f.
(2) The functions ${}^{(s)}J_f$ converge uniformly to $\alpha\chi_{A\mathbf{B}^d+a}$ on the complement of any open neighborhood of the boundary in $\mathbb{R}^d$ of $A\mathbf{B}^d+a$ as s tends to 0.
(3) The function $\alpha\chi_{A\mathbf{B}^d+a}$ is the unique flat ellipsoid function of maximal integral among all flat ellipsoid functions that are below f.

In this subsection, we prove Theorem 6.1.

We start with the existence of the limit flat ellipsoid function in (2). Let $\overline{E}(f,s)$ be represented by $(A_s\oplus\alpha_s, a_s)$ for every $s\in(0,1]$.

Claim 6.3. *The following limits exist*

$$\lim_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \mu > 0, \ \lim_{s\to 0^+} A_s = A, \ \lim_{s\to 0^+} \alpha_s^s = \alpha > 0 \quad \textit{and} \quad \lim_{s\to 0^+} a_s = a, \quad (6.6)$$

where A is positive definite.

Proof. Since the John 1-ellipsoid exists, by (3.2) and Lemma 6.1, we have

$$\inf_{s\in(0,1]} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0.$$

Recall from Section 3.1 that ${}^{(s)}\kappa_{d+1}$ as a function of s with ${}^{(0)}\kappa_{d+1} = \mathrm{vol}_d\,\mathbf{B}^d$ is a strictly decreasing continuous function on $[0,\infty)$. Applying Lemma 3.2 with $g = \hbar^s_{\mathbf{B}^2}$ for each $s \in (0,1]$, where $\hbar_{\mathbf{B}^2}$ is the height function of $\mathbf{B}^2$ (see (2.10)), we conclude that for some positive constants $\vartheta, \rho, \rho_1, \rho_2$ and all $s \in (0,1]$, the inequalities $\vartheta \le \alpha_s \le \|f\|$ and $|a_s| \le \rho$, and the relation $\rho_1 I \prec A_s \prec \rho_2 I$ hold. Thus, there exists a sequence of positive reals $\{s_i\}_1^\infty$ with $\lim\limits_{i\to\infty} s_i = 0$ such that

$${}^{(s_i)}\mu\big(\overline{E}(f,s_i)\big) \to \limsup_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big)\,,\ A_{s_i} \to A,\ \alpha_{s_i}^{s_i} \to \alpha \quad \text{and} \quad a_{s_i} \to a$$

for a positive definite matrix $A \in \mathbb{R}^{d\times d}$, $\alpha > 0$ and $a \in \mathbb{R}^d$, as i tends to ∞. definite.

We use J_f to denote the flat ellipsoid function represented by $(A \oplus \alpha, a)$. Clearly, J_f is below f. Consider the ellipsoids $\overline{E}_s$ represented by $(A \oplus \alpha^{1/s}, a)$ for all $s \in (0,1]$. Then, $\overline{E}_s \subset {}^{(s)}\overline{J_f} \subset {}^{(s)}\overline{f}$ for every $s \in (0,1]$. By (3.3) and (3.2), we have

$${}^{(s)}\mu\big(\overline{E}_s\big) = \frac{{}^{(s)}\kappa_{d+1}}{\mathrm{vol}_d\,\mathbf{B}^d} \int_{\mathbb{R}^d} J_f \to \int_{\mathbb{R}^d} J_f \text{ as } s \to 0^+.$$

That is, $\lim\limits_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \int_{\mathbb{R}^d} J_f\,\mathrm{d}x$. As an immediate consequence, Lemma 6.2 implies (6.6). $\square$

Claim 6.4. *J_f, as defined in the proof of Claim 6.3, is the unique flat ellipsoid function that is of maximal integral among those that are below f.*

Proof. Assume that there is a flat ellipsoid function J_E, represented by $(A_0 \oplus \alpha_0, a_0)$, such that $\int_{\mathbb{R}^d} J_E \ge \int_{\mathbb{R}^d} J_f$. Consider the ellipsoids $\overline{E}'_s$ represented by $(A_0 \oplus \alpha_0^{1/s}, a_0)$ for all $s \in (0,1]$. Clearly, $\overline{E}'_s \subset {}^{(s)}\overline{f}$ for every $s \in (0,1]$. By (3.3), ${}^{(s)}\mu\Big(\overline{E}'_s\Big) = \frac{{}^{(s)}\kappa_{d+1}}{\mathrm{vol}_d\,\mathbf{B}^d} \int_{\mathbb{R}^d} J_E$. By (3.2) and by the definition of the John s-ellipsoid, we have that

$$\int_{\mathbb{R}^d} J_E = \lim_{s\to 0^+} {}^{(s)}\mu\Big(\overline{E}'_s\Big) \le \lim_{s\to 0^+} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \int_{\mathbb{R}^d} J_f.$$

Thus, for every positive integer i, there is $s_i > 0$ such that

$$
{}^{(s)}\mu\left(\overline{E}'_s\right) \geq {}^{(s)}\mu\left(\overline{E}(f,s)\right) - \frac{1}{i} \geq \int\limits_{\mathbb{R}^d} J_f - \frac{2}{i}
$$

for all $s \in (0, s_i]$. Finally, by Lemma 6.2, we have that $\lim\limits_{s_i \to 0+} A_0 = A$, $\lim\limits_{s_i \to 0+} \alpha_0 = \alpha$ and $\lim\limits_{s_i \to 0+} a_0 = a$. That is, J_f and J_E coincide. $\square$

Theorem 6.1 is an immediate consequence of Claims 6.3 and 6.4.

6.4. Integral ratio

For any $s \in [0, \infty)$ and positive integer d, it is reasonable to define the *s-integral ratio* of $f : \mathbb{R}^d \to [0, \infty)$ by

$$
{}^{(s)}\mathrm{I.\,rat}(f) = \left(\frac{\int_{\mathbb{R}^d} f}{\int_{\mathbb{R}^d} {}^{(s)}J_f} \right)^{1/d}.
$$

Corollary 1.3 of [2] states that there exists $\Theta > 0$ such that

$$
{}^{(0)}\mathrm{I.\,rat}(f) \leq \Theta\sqrt{d},
$$

for any proper log-concave function $f : \mathbb{R}^d \to [0, \infty)$ and any positive integer d.

Using Lemma 6.1 and Theorem 6.1, we obtain the following.

Corollary 6.1. *Fix $s \in [0, \infty)$. Then there exists Θ_s such that for any positive integer d and any proper log-concave function $f : \mathbb{R}^d \to [0, \infty)$, the following inequality holds.*

$$
{}^{(s)}\mathrm{I.\,rat}(f) \leq \mathrm{B}(s/2 + 1, d/2)^{-\frac{1}{d}} \cdot {}^{(0)}\mathrm{I.\,rat}(f) \leq \Theta_s\sqrt{d},
$$

where $\mathrm{B}(\cdot, \cdot)$ denotes Euler's Beta function.

7. Large s behavior

We will say that a Gaussian density on $\mathbb{R}^d$ defined by $x \mapsto \alpha e^{-\langle A^{-1}(x-a), A^{-1}(x-a)\rangle}$ is *represented* by $(A \oplus \alpha, a) \in \mathcal{E}$. Clearly, any Gaussian density is represented by a unique element of $\mathcal{E}$. We will denote the Gaussian density represented by $(A \oplus \alpha, a)$ as $G[(A \oplus \alpha, a)]$. If a Gaussian density is represented by $(A \oplus \alpha, a) \in \mathcal{E}$, we will call α its *height*. We have that

$$
\int\limits_{\mathbb{R}^d} G[(A \oplus \alpha, a)] = \alpha\pi^{d/2} \det A. \tag{7.1}
$$

We will need the following property of Euler's Gamma function (see [3, 6.1.46])

$$\lim_{s\to\infty}\frac{\Gamma(s+t_1)}{\Gamma(s+t_2)}s^{t_2-t_1}=1.$$

Using this in (3.1), we obtain

$$\lim_{s\to\infty}{}^{(s)}\kappa_{d+1}\cdot\left(\frac{s}{2}\right)^{d/2}=\pi^{d/2}. \tag{7.2}$$

7.1. Existence of a maximal Gaussian

Theorem 7.1. *Let $f:\mathbb{R}^d\to[0,\infty)$ be a proper log-concave function. If there is a Gaussian density below f, then there exists a Gaussian density below f of maximal integral. All Gaussian densities of maximal integral below f are translates of each other.*

Proof of Theorem 7.1. The proof mostly repeats the argument in Section 4.

Lemma 3.2 with $g=e^{-t^2}$, $t\in\mathbb{R}$, implies that if there exists a Gaussian density below f, then there is a Gaussian density of maximal integral among those that are below f. Next, we show that this Gaussian density of maximal integral is unique up to translation.

First, we need the following extension of Lemmas 4.1 and 4.2.

Lemma 7.1 *(Interpolation between Gaussians). Fix $\beta_1,\beta_2>0$ with $\beta_1+\beta_2=1$. Let f_1 and f_2 be two proper log-concave functions on $\mathbb{R}^d$, and G_1,G_2 be two Gaussian densities represented by $(A_1\oplus\alpha_1,a_1)\in\mathcal{E}$ and $(A_2\oplus\alpha_2,a_2)\in\mathcal{E}$, respectively, such that $G_1\le f_1$ and $G_2\le f_2$. With the operation introduced in Section 4.1, define*

$$f=(\beta_1*f_1)\star(\beta_2*f_2),$$

and set

$$(A\oplus\alpha,a)=\big((\beta_1A_1+\beta_2A_2)\oplus\alpha_1^{\beta_1}\alpha_2^{\beta_2},\beta_1a_1+\beta_2a_2\big).$$

Then, $G\left[(A\oplus\alpha,a)\right]\le f$ and the following inequality holds

$$\int_{\mathbb{R}^d}G[(A\oplus\alpha,a)]\ge\left(\int_{\mathbb{R}^d}G_1\right)^{\beta_1}\left(\int_{\mathbb{R}^d}G_2\right)^{\beta_2}, \tag{7.3}$$

with equality if and only if $A_1=A_2$.

Proof. Fix $x\in\mathbb{R}^d$ and define x_1,x_2 by

$$x_1-a_1=A_1A^{-1}(x-a),\quad x_2-a_2=A_2A^{-1}(x-a). \tag{7.4}$$

Since $G_1 \le f_1, G_2 \le f_2$, we have

$$f_1(x_1) \ge \alpha_1 e^{-\left\langle A_1^{-1}(x_1-a_1), A_1^{-1}(x_1-a_1)\right\rangle} \quad \text{and} \quad f_2(x_2) \ge \alpha_2 e^{-\left\langle A_2^{-1}(x_2-a_2), A_2^{-1}(x_2-a_2)\right\rangle}. \tag{7.5}$$

Since $\beta_1 x_1 + \beta_2 x_2 = x$ and by the definition of the Asplund sum, we have that

$$f(x) \ge f_1^{\beta_1}(x_1) f_2^{\beta_2}(x_2).$$

Combining this with inequalities (7.5) and (7.4), we obtain

$$f(x) \ge \alpha_1^{\beta_1}\alpha_2^{\beta_2} e^{-\beta_1\left\langle A_1^{-1}(x_1-a_1), A_1^{-1}(x_1-a_1)\right\rangle} e^{-\beta_2\left\langle A_2^{-1}(x_2-a_2), A_2^{-1}(x_2-a_2)\right\rangle} =$$
$$\alpha_1^{\beta_1}\alpha_2^{\beta_2} e^{-(\beta_1+\beta_2)\langle A^{-1}(x-a), A^{-1}(x-a)\rangle} = \alpha_1^{\beta_1}\alpha_2^{\beta_2} e^{-\langle A^{-1}(x-a), A^{-1}(x-a)\rangle}.$$

Thus, G is below f.

We proceed with showing (7.3). Substituting (7.1), inequality (7.3) takes the form

$$\pi^{d/2}\alpha_1^{\beta_1}\alpha_2^{\beta_2} \cdot \det\left(\beta_1 A_1 + \beta_2 A_2\right) \ge \pi^{d/2}\alpha_1^{\beta_1}\alpha_2^{\beta_2} \cdot \left(\det A_1\right)^{\beta_1}\left(\det A_2\right)^{\beta_2},$$

or, equivalently,

$$\det\left(\beta_1 A_1 + \beta_2 A_2\right) \ge \left(\det A_1\right)^{\beta_1}\left(\det A_2\right)^{\beta_2}.$$

Thus, inequality (7.3) and its equality condition follow from Minkowski's determinant inequality (1.3) and the equality condition therein, completing the proof of Lemma 7.1. □

Let G_1, represented by $(A_1 \oplus \alpha_1, a_1)$, be a maximal integral Gaussian density that is below f. Assume that there is another Gaussian density G_2, represented by $(A_2 \oplus \alpha_2, a_2)$, below f with the same integral as G_1. Consider the Gaussian density G represented by

$$\left(\frac{A_1 + A_2}{2} \oplus \sqrt{\alpha_1\alpha_2}, \frac{a_1 + a_2}{2}\right) \in \mathcal{E}.$$

By (4.1) and Lemma 7.1, we have that G is below f. Next, by the choice of the Gaussian densities, we also have

$$\int_{\mathbb{R}^d} G \le \int_{\mathbb{R}^d} G_1 = \sqrt{\int_{\mathbb{R}^d} G_1 \int_{\mathbb{R}^d} G_2} = \int_{\mathbb{R}^d} G_2,$$

which, combined with Lemma 7.1, yields

$$\int_{\mathbb{R}^d} G = \int_{\mathbb{R}^d} G_1 = \int_{\mathbb{R}^d} G_2, \text{ and } A_1 = A_2.$$

Combined with (7.1), it implies $\alpha_1 = \alpha_2$. This completes the proof of Theorem 7.1. □

7.2. Uniqueness does not hold for $s = \infty$

In this subsection, first, we show that it is possible that two Gaussian densities $G[(A \oplus \alpha, a_1)]$ and $G[(A \oplus \alpha, a_2)]$ with $a_1 \neq a_2$ below a proper log-concave function f are of maximal integral. Next in Proposition 7.1, we show that uniqueness holds for a certain important class of log-concave functions.

Consider the Asplund sum

$$f = G[(A \oplus \alpha, a)] \star \chi_K,$$

where $(A \oplus \alpha, a) \in \mathcal{E}$ and K is a compact convex set in $\mathbb{R}^d$. We claim that the set of the maximal integral Gaussian densities that are below f is

$$\{G[(A \oplus \alpha, a_m)] : \ a_m \in a + K\}.$$

To see this, one observes that if $G[(A' \oplus \alpha', a')] \leq f$, then $A' \preceq A$. The claim now follows from (7.1).

Uniqueness of the maximal Gaussian density below f holds for an important class of log-concave functions.

Proposition 7.1. *Let $K \subset \mathbb{R}^d$ be a compact convex set containing the origin in the interior, and let $\|\cdot\|_K$ denote the gauge function of K, that is, $\|x\|_K = \inf\{\lambda > 0 : \ x \in \lambda K\}$. Let $A\left(\mathbf{B}^d\right)$ be the largest volume origin centered ellipsoid contained in K, where A is a positive definite matrix. Then the Gaussian density represented by $(A \oplus 1, 0)$ is the unique maximal integral Gaussian density below the log-concave function $e^{-\|x\|_K^2}$.*

Proof. Let $(A' \oplus \alpha', a') \in \mathcal{E}$ be such that $G[(A' \oplus \alpha', a')] \leq f$. First, we show that $A'\mathbf{B}^d \subseteq K$. Indeed, we have

$$\left\langle (A')^{-1}(x - a'), (A')^{-1}(x - a') \right\rangle - \ln(\alpha') \geq \|x\|_K^2$$

for every $x \in \mathbb{R}^d$. Suppose for a contradiction that there is a $y \in A'\left(\operatorname{int}\left(\mathbf{B}^d\right)\right) \setminus K$. Consider $x = \vartheta y$, and substitute into the previous inequality. We obtain

$$\vartheta^2 |(A')^{-1} y|^2 - 2\vartheta \left\langle (A')^{-1} a', (A')^{-1} y \right\rangle + 2\left|(A')^{-1} a'\right|^2 - \ln(\alpha') \geq \vartheta^2 \|y\|_K^2 > \vartheta^2.$$

As $\left|(A')^{-1} y\right| < 1$, letting ϑ tend to infinity, we obtain a contradiction. Thus, $A'\mathbf{B}^d \subseteq K$.

Hence, $\det(A') \leq \det(A)$. On the other hand, $\alpha' \leq \|f\| = 1$. The Proposition now easily follows from (7.1). $\square$

7.3. Approximation of a largest Gaussian by John s-ellipsoids

Theorem 7.2. *Let $f : \mathbb{R}^d \to [0, \infty)$ be a proper log-concave function. Then the following hold.*

(1) There is a Gaussian density below f if and only if $\limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0$.

(2) If there is a Gaussian density below f, then $\lim_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)$ and there is a sequence $\{s_i\}_1^\infty$ of positive reals with $\lim_{i\to\infty} s_i = \infty$ such that the John s-functions ${}^{(s_i)}J_f$ converge uniformly to a Gaussian density which is of maximal integral among those Gaussian densities that are below f.

(3) If there is a Gaussian density below f, then any Gaussian density which is of maximal integral among those Gaussian densities that are below f is of height at least $\|f\|\, e^{-d}$.

In the rest of this subsection, we prove Theorem 7.2.

We start by describing the limit of s-marginals of origin centered ellipsoids.

Lemma 7.2. *Let $\{s_i\}_1^\infty$ be a sequence of positive reals with $\lim_{i\to\infty} s_i = \infty$, let $\{A_i\}_1^\infty$ be a sequence of positive definite operators such that $\lim_{i\to\infty} \frac{A_i}{\|A_i\|} = A$, where A is positive definite, and the ellipsoids $\overline{E}_i$, represented by $(A_i \oplus 1, 0)$, satisfy $\lim_{i\to\infty} {}^{(s_i)}\mu\big(\overline{E}_i\big) = \mu > 0$. Then, the s_i-th power of the height functions $\big(\hbar_{\overline{E}_i}\big)^{s_i}$ converge uniformly to the Gaussian density $G[(A_\infty \oplus 1, 0)]$, where*

$$A_\infty = \frac{1}{\sqrt{\pi}} \left(\frac{\mu}{\det A}\right)^{1/d} A.$$

Proof of Lemma 7.2. The limit $\frac{A_i}{\|A_i\|} \to A$ as $i \to \infty$ yields two properties,

$$\lim_{i\to\infty} \frac{\det A_i}{\|A_i\|^d} = \det A, \tag{7.6}$$

and

$$\lim_{i\to\infty} \|A_i\| A_i^{-1} = A^{-1}. \tag{7.7}$$

By (3.3) and (7.2), we obtain

$$\mu = \lim_{i\to\infty} {}^{(s_i)}\mu\big(\overline{E}_i\big) = \lim_{i\to\infty} {}^{(s_i)}\kappa_{d+1} \det A_i = \pi^{d/2} \lim_{i\to\infty} \left(\frac{s_i}{2}\right)^{-d/2} \det A_i.$$

Combining this with (7.6), we get

$$\lim_{i\to\infty} \frac{s_i}{2} \frac{1}{\|A_i\|^2} = \pi \left(\frac{\det A}{\mu}\right)^{2/d}. \tag{7.8}$$

We obtain that $\lim_{i\to\infty} \|A_i\| = \infty$, and hence by (7.7), the smallest eigenvalue of A_i tends to infinity.

It follows that for any fixed $\rho > 0$ and sufficiently large i, we have $\rho\mathbf{B}^d \subset A_i\mathbf{B}^d$, and hence,

$$\left(\hbar_{\overline{E}_i}(x)\right)^{s_i} = \left(1 - \left\langle A_i^{-1}x, A_i^{-1}x\right\rangle\right)^{s_i/2} = \left(1 - \frac{\left\langle \|A_i\|\, A_i^{-1}x, \|A_i\|\, A_i^{-1}x\right\rangle}{\|A_i\|^2}\right)^{\|A_i\|^2 \cdot \frac{s_i}{2} \frac{1}{\|A_i\|^2}}$$

for all $x \in \rho\mathbf{B}^d$.

By (7.8) and (7.7), for any $1 > \varepsilon > 0$, there exists i_ε such that the inequalities

$$\left(1 - \frac{(1+\varepsilon)\left\langle A^{-1}x, A^{-1}x\right\rangle}{\|A_i\|^2}\right)^{\|A_i\|^2\left[\pi\left(\frac{\det A}{\mu}\right)^{2/d}(1+\varepsilon)\right]} \leq \left(\hbar_{\overline{E}_i}(x)\right)^{s_i},$$

$$\left(\hbar_{\overline{E}_i}(x)\right)^{s_i} \leq \left(1 - \frac{(1-\varepsilon)\left\langle A^{-1}x, A^{-1}x\right\rangle}{\|A_i\|^2}\right)^{\|A_i\|^2\left[\pi\left(\frac{\det A}{\mu}\right)^{2/d}(1-\varepsilon)\right]}.$$

hold for all $x \in \rho\mathbf{B}^d$ and for all $i > i_\varepsilon$. Since $\lim\limits_{i\to\infty} \|A_i\| = \infty$, this implies that the sequence of functions $\left\{\left(\hbar_{\overline{E}_i}(x)\right)^{s_i}\right\}_{i=1}^{\infty}$ converges uniformly to $g(x) = e^{-\pi\left(\frac{\det A}{\mu}\right)^{2/d}\left\langle A^{-1}x, A^{-1}x\right\rangle}$ on $\rho\mathbf{B}^d$. Since $\sup_{x\in\mathbb{R}^d\setminus\rho\mathbf{B}^d} g(x)$ tends to zero as $\rho \to \infty$ and $\lim\limits_{i\to\infty} {}^{(s_i)}\mu\left(\overline{E}_i\right) = \int_{\mathbb{R}^d} g$, we conclude that $\left\{\left(\hbar_{\overline{E}_i}(x)\right)^{s_i}\right\}$ is uniformly convergent on $\mathbb{R}^d$ This completes the proof of Lemma 7.2. □

Lemma 7.3. *Let G be a Gaussian density. Then, the John s-functions ${}^{(s)}J_G$ converge uniformly to G on $\mathbb{R}^d$ as $s \to \infty$.*

In order to prove Lemma 7.3, we assume that $G(x) = e^{-|x|^2/2}$. First, we relax the condition and prove that it suffices to approximate G by any sequence of suitable height functions of d-ellipsoids.

Claim 7.1. *If there is a function $c : [1, \infty) \to [1, \infty)$ such that*

$$\lim_{s\to\infty} \int_{\mathbb{R}^d} \hbar^s_{(c(s)I\oplus 1,0)} = \int_{\mathbb{R}^d} G = (2\pi)^{d/2}, \tag{7.9}$$

and $\hbar^s_{(c(s)I\oplus 1,0)} \leq G$ for all $s \geq 1$, then the functions ${}^{(s)}J_G$ converge uniformly to G on $\mathbb{R}^d$ as $s \to \infty$.

Proof of Claim 7.1. By Theorem 4.1, the John s-function ${}^{(s)}J_G$ of G exists and is unique for any positive s. By symmetry, we see that ${}^{(s)}J_G$ is of the form $\hbar^s_{(\beta(s)I\oplus\alpha(s),0)}$, where $\beta : [1, \infty) \to (0, \infty)$ and $\alpha : [1, \infty) \to (0, 1)$. By (7.9), we obtain that

$$\lim_{s\to\infty} \int\limits_{\mathbb{R}^d} {}^{(s)}J_G = \int\limits_{\mathbb{R}^d} G.$$

This implies that $\alpha(s) \to 1$ as $s \to \infty$. Hence, the functions ${}^{(s)}J_G = \hbar^s_{(\beta(s)I\oplus\alpha(s),0)}$ converge uniformly to the same function as the functions $\hbar^s_{(\beta(s)I\oplus 1,0)}$ as $s \to \infty$ (if the latter sequence converges). However, by Lemma 7.2, the functions $\hbar^s_{(\beta(s)I\oplus 1,0)}$ converge uniformly to G as $s \to \infty$. $\square$

It is not hard to find a suitable function $c(s)$.

Claim 7.2. *Let $c(s) = \sqrt{s}$. Then, $\hbar^s_{(c(s)I\oplus 1,0)} \le G$ for all $s \ge 1$, and identity (7.9) holds.*

Proof of Claim 7.2. Identity (7.9) is an immediate consequence of (3.3) and (7.2).

Inequality $\hbar^s_{(c(s)I\oplus 1,0)} \le G$ is purely technical. By a routine calculation, for any $x \in \mathbb{R}^d$, we have

$$\lim_{s\to\infty} \hbar^s_{(c(s)I\oplus 1,0)}(x) = \lim_{s\to\infty} \left(1 - \frac{|x|^2}{s}\right)^{s/2} = G(x).$$

As is easily seen, $\hbar^s_{(c(s)I\oplus 1,0)}(x)$ is an increasing function of $s \in [1,\infty)$ for any fixed $x \in \mathbb{R}^d$. $\square$

Lemma 7.3 follows from Claims 7.1 and 7.2.

Lemma 7.4. *If $\limsup\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) > 0$, then there exists a sequence $\{s_i\}_1^\infty$ of positive reals with $\lim\limits_{i\to\infty} s_i = \infty$ such that the John s-functions ${}^{(s_i)}J_f$ converge uniformly on $\mathbb{R}^d$ to a Gaussian density, which we denote by $G[(A_\infty \oplus \alpha_\infty, a_\infty)]$, which is below f and is of maximal integral among Gaussian densities below f. Moreover, we have*

$$\int\limits_{\mathbb{R}^d} G[(A_\infty \oplus \alpha_\infty, a_\infty)] = \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big), \text{ and } \alpha_\infty \in [e^{-d}\|f\|, \|f\|].$$

In the proof of Lemma 7.4, we will need the following immediate consequence of Lemma 3.2 and (7.2).

Claim 7.3. *Let $f : \mathbb{R}^d \to [0,\infty)$ be a proper log-concave function, and $\delta, s_0 > 0$. Then there exist $\rho_1, \rho_2 > 0$ such that for any $s \ge s_0$, if $\overline{E} = (A \oplus \alpha)\mathbf{B}^{d+1} + a$, where $(A \oplus \alpha, a) \in \mathcal{E}$, is a d-symmetric ellipsoid in $\mathbb{R}^{d+1}$ with $\overline{E} \subseteq {}^{(s)}\overline{f}$ and ${}^{(s)}\mu(\overline{E}) \ge \delta$, then we have*

$$\rho_1 I \prec \frac{A}{\sqrt{s}} \prec \rho_2 I. \tag{7.10}$$

Proof of Lemma 7.4. Let $(A_s \oplus \alpha_s, a_s)$ represent $\overline{E}(f,s)$. By Lemma 4.5, we have that $\left\| {}^{(s)}J_f \right\|$ belongs to the interval $[e^{-d}\|f\|, \|f\|]$. Thus, $[f \geq e^{-d}\|f\|] \supseteq \{a_s\}_{s>0}$. Since f is a proper log-concave function, the set $[f \geq e^{-d}\|f\|]$ is a bounded subset of $\mathbb{R}^d$, and thus, so is $\{a_s\}_{s>0}$. Thus, there exists a sequence $\{s_i\}_1^\infty$ with $\lim\limits_{i\to\infty} s_i = \infty$ such that

$$\begin{aligned} &{}^{(s_i)}\mu\big(\overline{E}(f,s_i)\big) \to \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big), \quad \frac{A_{s_i}}{\|A_{s_i}\|} \to A, \\ &\left\| {}^{(s_i)}J_f \right\| \to \alpha_\infty > 0 \quad \text{and} \quad a_{s_i} \to a_\infty \end{aligned} \tag{7.11}$$

for some positive semidefinite matrix $A \in \mathbb{R}^{d\times d}$, an $\alpha_\infty > 0$ and $a_\infty \in \mathbb{R}^d$, as i tends to ∞.

Claim 7.3 implies that A is positive definite. Hence by (7.11), we may apply Lemma 7.2 to obtain that the sequence $\left\{ {}^{(s_i)}J_f \right\}_{i=1}^\infty$ converges uniformly to the Gaussian density $G[(A_\infty \oplus \alpha_\infty, a_\infty)]$, where

$$A_\infty = \frac{1}{\sqrt{\pi}} \left(\frac{\limsup\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)}{\det A} \right)^{1/d} A.$$

Clearly, $G[(A_\infty \oplus \alpha_\infty, a_\infty)] \leq f$ and, by the uniform convergence,

$$\lim_{i\to\infty} {}^{(s_i)}\mu\big(\overline{E}(f,s_i)\big) = \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \int\limits_{\mathbb{R}^d} G[(A_\infty \oplus \alpha_\infty, a_\infty)].$$

The latter implies that there is no Gaussian density below f with the integral strictly greater than $\int_{\mathbb{R}^d} G[(A_\infty \oplus \alpha_\infty, a_\infty)]$, since, by Lemma 7.3, any Gaussian density G' is the limit of ${}^{(s)}J_{G'}$ as $s \to \infty$. $\square$

Proof of Theorem 7.2. First, assume that there is a Gaussian density G below f. Then, by Lemma 7.3,

$$\limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big) = \limsup_{s\to\infty} \int\limits_{\mathbb{R}^d} {}^{(s)}J_f \geq \limsup_{s\to\infty} \int\limits_{\mathbb{R}^d} {}^{(s)}J_G = \int\limits_{\mathbb{R}^d} G > 0.$$

The converse in part (1) follows from Lemma 7.4.

To prove part (2), assume again that there is a Gaussian density below f. By part (1), we may apply Lemma 7.4 and obtain a Gaussian density $G[(A_\infty \oplus \alpha_\infty, a_\infty)]$ with all the desired properties. We need to verify only that the limit in part (2) exists and is equal to the lim sup. We have by Lemma 7.3 that

$${}^{(s)}\mu\big(\overline{E}(f,s)\big) \geq {}^{(s)}\mu\big(\overline{E}(G[(A_\infty \oplus \alpha_\infty, a_\infty)], s)\big)$$

$$\xrightarrow{s\to\infty} \int_{\mathbb{R}^d} G[(A_\infty \oplus \alpha_\infty, a_\infty)] = \limsup_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)\,,$$

and hence, the limit $\lim\limits_{s\to\infty} {}^{(s)}\mu\big(\overline{E}(f,s)\big)$ exists completing the proof of part (2).

Part (3) follows immediately from Lemma 7.4 and Theorem 7.1. □

8. The Helly type result — proof of Theorem 1.2

In this section, we prove Theorem 1.2.

8.1. Assumption: the functions are supported on $\mathbb{R}^d$

We claim that we may assume that the support of each f_i is $\mathbb{R}^d$. Indeed, any log-concave function can be approximated in the L_1-norm by log-concave functions whose support is $\mathbb{R}^d$. Recall that f_σ denotes the pointwise minimum of functions $\{f_i\}_{i\in\sigma}$ for $\sigma \subseteq [n]$. We may approximate each function so that the f_σ are also all well approximated. One way to achieve this is to take the Asplund sum $f_i \star (e^{-\delta|x|^2})$ for a sufficiently large $\delta > 0$ (see Section 4.1).

8.2. Assumption: John position

Consider the s-lifting of our functions with $s = 1$. Clearly, the s-lifting of a pointwise minimum of a family of functions is the intersection of the s-liftings of the functions.

From our assumption in Subsection 8.1, it follows that $\int_{\mathbb{R}^d} f > 0$. By applying a linear transformation on $\mathbb{R}^d$, we may assume that, with $s = 1$, the largest s-volume ellipsoid in the s-lifting ${}^{(1)}\overline{f}$ of f is $\mathbf{B}^{d+1} \subset {}^{(1)}\overline{f}$.

By Theorem 5.1, there are contact points $\overline{u}_1, \ldots, \overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right) \cap \operatorname{bd}\left({}^{(1)}\overline{f}\right)$, and positive weights $c_1, \ldots, c_k$ satisfying (5.1) with $s = 1$. For each $j \in [k]$, we denote by u_j the orthogonal projection of the contact point $\overline{u}_j$ onto $\mathbb{R}^d$ and by $w_j = \sqrt{1 - |u_j|^2}$.

8.3. Reduction of the problem to finding P and η

Claim 8.1. *With the assumptions in Subsections 8.1 and 8.2, we can find a set of indices $\eta \in \binom{[k]}{\le 2d+1}$ and an origin-symmetric convex body P in $\mathbb{R}^d$ with the following two properties.*

$$\operatorname{vol}_d P \le 8 \cdot 4^d \cdot d^d (d+2)^d \left(\operatorname{vol}_d \mathbf{B}^d\right)^2 \tag{8.1}$$

and

$$\|x\|_P \le \max\left\{ \langle x, u_j\rangle : \ j \in \eta\right\} \quad \textit{for every } x \in \mathbb{R}^d, \tag{8.2}$$

where $\|\cdot\|_P$ is the gauge function of P, that is, $\|x\|_P = \inf\{\lambda > 0 : \ x \in \lambda P\}$.

We will prove Claim 8.1 in Subsection 8.5.

In the present subsection, we show that Claim 8.1 yields the existence of the desired index set $\sigma \in \binom{[n]}{\leq 3d+2}$ *that satisfies* (1.1).

The *polar* of a set K in $\mathbb{R}^n$ is defined by $K^\circ = \{p \in \mathbb{R}^n : \langle y, p\rangle \leq 1 \text{ for all } y \in K\}$. Set $T = \{u_j : j \in \eta\}$. It is easy to see that (8.2) is equivalent to

$$T^\circ \subseteq P. \tag{8.3}$$

Notice that

$$\text{for any } x \in \mathbb{R}^d \setminus T^\circ, \text{ there is } j \in \eta \text{ such that } \langle u_j, x - u_j\rangle \geq 0. \tag{8.4}$$

We will split the integral in (1.1) into two parts: the integral on $\mathbb{R}^d \setminus T^\circ$ and the integral on T°.

First, we find a set σ_1 of indices in $[n]$ that will help us bound the integral in (1.1) on $\mathbb{R}^d \setminus T^\circ$.

Fix a $j \in \eta$. Since $\overline{u}_j \in \operatorname{bd}\left({}^{(1)}\overline{f}\right)$, there is an index $i(j) \in [n]$ such that $\overline{u}_j \in \operatorname{bd}\left({}^{(1)}\overline{f}_{i(j)}\right)$. Let σ_1 be the set of these indices, that is, $\sigma_1 = \{i(j) : j \in \eta\}$.

By (3.4), for each $j \in \eta$, we have

$$f_{i(j)}(x) \leq w_j e^{-\frac{1}{w_j^2}\langle u_j, x-u_j\rangle} \leq e^{-\frac{1}{w_j^2}\langle u_j, x-u_j\rangle} \tag{8.5}$$

for all $x \in \mathbb{R}^d$.

Next, we find a set σ_2 of indices in $[n]$ that will help us bound the integral in (1.1) on T°.

It is easy to see that there is a $\sigma_2 \in \binom{[n]}{\leq d+1}$ such that $\|f\| = \|f_{\sigma_2}\|$. Indeed, for any $i \in [n]$, consider the following convex set in $\mathbb{R}^d$: $[f_i > \|f\|]$. By the definition of f, the intersection of these n convex sets in $\mathbb{R}^d$ is empty. Helly's theorem yields the existence of σ_2.

We combine the two index sets: let $\sigma = \sigma_1 \cup \sigma_2$. Clearly, σ is of size at most $3d + 2$. We need to show that σ satisfies (1.1).

Note that by (4.6) and Assumption 8.2, we have $\|f_{\sigma_2}\| = \|f\| \leq e^d$. Hence,

$$\int_{\mathbb{R}^d} f_\sigma \leq \int_{T^\circ} \|f_\sigma\| + \int_{\mathbb{R}^d\setminus T^\circ} f_\sigma \leq \int_{T^\circ} e^d + \int_{\mathbb{R}^d\setminus T^\circ} f_\sigma \overset{(8.3)}{\leq} e^d \operatorname{vol}_d P + \int_{\mathbb{R}^d\setminus T^\circ} f_{\sigma_1}$$

Next, we bound the second summand using the tail bound (8.5).

$$\int_{\mathbb{R}^d\setminus T^\circ} f_{\sigma_1} \overset{(8.5)}{\leq} \int_{\mathbb{R}^d\setminus T^\circ} \exp\left(-\max\left\{\frac{1}{w_j^2}\langle u_j, x-u_j\rangle : j \in \eta\right\}\right) \mathrm{d}x \overset{(8.4)}{\leq}$$

$$\int\limits_{\mathbb{R}^d\setminus T^\circ} \exp\left(-\max\left\{\langle u_j, x-u_j\rangle : j\in\eta\right\}\right) \mathrm{d}x \le e \int\limits_{\mathbb{R}^d\setminus T^\circ} \exp\left(-\max\left\{\langle u_j, x\rangle : j\in\eta\right\}\right) \mathrm{d}x.$$

By property (8.2), the latter is at most

$$e \int\limits_{\mathbb{R}^d\setminus T^\circ} \exp\left(-\|x\|_P\right) \mathrm{d}x \le e \int\limits_{\mathbb{R}^d} \exp\left(-\|x\|_P\right) \mathrm{d}x = e\cdot d!\operatorname{vol}_d P.$$

Hence,

$$\int\limits_{\mathbb{R}^d} f_\sigma \le \left(e^d + e\cdot d!\right)\operatorname{vol}_d P \le 10\cdot d^{d-1}\operatorname{vol}_d P.$$

Using, the fact that

$$\operatorname{vol}_d \mathbf{B}^d \le d \operatorname{vol}_{d+1}\mathbf{B}^{d+1} \le d\int\limits_{\mathbb{R}^d} f,$$

and inequality (8.1) here, we obtain

$$\int\limits_{\mathbb{R}^d} f_\sigma \le 80\cdot 4^d\cdot d^{2d}(d+2)^d \operatorname{vol}_d \mathbf{B}^d \cdot \int\limits_{\mathbb{R}^d} f.$$

This inequality directly implies inequality (1.1) in the case $d=1$. Consider $d\ge 2$. Then, since $d+2\le 2d$ and $\operatorname{vol}_d \mathbf{B}^d \le 10^d d^{-d/2}$, we conclude that

$$\int\limits_{\mathbb{R}^d} f_\sigma \le 80\cdot 80^d\cdot d^{5d/2}\int\limits_{\mathbb{R}^d} f \le (100d)^{5d/2}\int\limits_{\mathbb{R}^d} f,$$

completing the proof of inequality (1.1).

8.4. The Dvoretzky–Rogers lemma

One key tool in proving Claim 8.1 is the Dvoretzky–Rogers lemma [13].

Lemma 8.1 *(Dvoretzky–Rogers lemma). Assume that the points $\overline{u}_1,\dots,\overline{u}_k \in \operatorname{bd}\left(\mathbf{B}^{d+1}\right)$, satisfy (5.1) for $s=1$ with some positive weights $c_1,\dots,c_k$. Then there is a sequence $j_1,\dots,j_{d+1}$ of $d+1$ distinct indices in $[k]$ such that*

$$\operatorname{dist}\left(\overline{u}_{j_t}, \operatorname{span}\{\overline{u}_{j_1},\dots,\overline{u}_{j_{t-1}}\}\right) \ge \sqrt{\frac{d-t+2}{d+1}} \quad \text{for all } t=2,\dots,d+1,$$

where dist *denotes the shortest Euclidean distance between a vector and a subspace.*

It follows immediately that the determinant of the $(d+1)\times(d+1)$ matrix with columns $\overline{u}_{j_1},\dots\overline{u}_{j_{d+1}}$ is at least

$$\left|\det\left[\overline{u}_{j_1},\dots\overline{u}_{j_{d+1}}\right]\right| \geq \frac{\sqrt{(d+1)!}}{(d+1)^{(d+1)/2}}. \tag{8.6}$$

8.5. Finding P and η

In this subsection, we prove Claim 8.1, that is, we show that with the assumptions in Subsections 8.1 and 8.2, there is an origin symmetric convex body P and a set of indices $\eta \in \binom{[k]}{\leq 2d+1}$ satisfying (8.1) and (8.2). Once it is shown, by Subsection 8.3, the proof of Theorem 1.2 is complete.

The proof in this section follows very closely the proof of the main result in [21] as refined by Brazitikos in [9].

Let $\eta_1 \in \binom{[k]}{d+1}$ be the set of $d+1$ indices in $[k]$ given by Lemma 8.1, and let $\overline{\Delta}$ be the simplex $\overline{\Delta} = \operatorname{conv}\left(\{\overline{u}_j : j \in \eta_1\} \cup \{0\}\right)$ in $\mathbb{R}^{d+1}$. Let $\overline{z} = \frac{\sum_{j\in\eta_1}\overline{u}_j}{d+1}$ denote the centroid of $\overline{\Delta}$, and $\overline{P}_1$ denote the intersection of $\overline{\Delta}$ and its reflection about $\overline{z}$, that is, $\overline{P}_1 = \overline{\Delta}\cap(2\overline{z}-\overline{\Delta})$, a polytope which is centrally symmetric about $\overline{z}$. It is well known [20, Corollary 3] (see also [4, Section 4.3.5]), that $\operatorname{vol}_{d+1}\overline{P}_1 \geq 2^{-(d+1)}\operatorname{vol}_{d+1}\overline{\Delta}$, and hence, by (8.6), we have

$$\operatorname{vol}_{d+1}\overline{P}_1 \geq 2^{-(d+1)}\frac{|\det[\overline{u}_j : j \in \eta_1]|}{(d+1)!} \geq \frac{1}{2^{d+1}\sqrt{(d+1)!}(d+1)^{(d+1)/2}}$$

Let P_1 denote the orthogonal projection of $\overline{P}_1$ onto $\mathbb{R}^d$. Since $\overline{P}_1 \subset P_1\times[-1,1]$, we have that

$$\operatorname{vol}_d P_1 \geq \frac{1}{2^{d+2}\sqrt{(d+1)!}(d+1)^{(d+1)/2}}. \tag{8.7}$$

Moreover, P_1 is symmetric about the orthogonal projection z of $\overline{z}$ onto $\mathbb{R}^d$.

Let $\overline{Q}$ denote the convex hull of the contact points, $\overline{Q} = \operatorname{conv}\left(\operatorname{bd}\left({}^{(s)}\overline{f}\right)\cap\operatorname{bd}\left(\mathbf{B}^{d+1}\right)\right)$, and Q denote the orthogonal projection of $\overline{Q}$ onto $\mathbb{R}^d$. As a well known consequence of (5.1) for $s=1$ [5], we have $\frac{1}{d+1}\mathbf{B}^{d+1} \subset \overline{Q}$, and hence, $\frac{1}{d+1}\mathbf{B}^{d} \subset Q$.

Let ℓ be the ray in $\mathbb{R}^d$ emanating from the origin in the direction of the vector $-z$, and let y be the point of intersection of ℓ with the boundary (in $\mathbb{R}^d$) of Q, that is, $\{y\} = \ell\cap\operatorname{bd}(Q)$. Now, $\frac{1}{d+1}\mathbf{B}^d \subset Q$ yields that $|y| \geq 1/(d+1)$.

We apply a contraction with center y and ratio $\lambda = \frac{|y|}{|y-z|}$ on P_1 to obtain the polytope P_2. Clearly, P_2 is a convex polytope in $\mathbb{R}^d$ which is symmetric about the origin. Furthermore,

$$\lambda = \frac{|y|}{|y-z|} \geq \frac{|y|}{1+|y|} \geq \frac{1}{d+2}. \tag{8.8}$$

Let P be the polar $P = P_2^\circ$ of P_2 taken in $\mathbb{R}^d$. By the Santaló inequality [16, Theorem 9.5], we obtain

$$\operatorname{vol}_d P \leq \frac{\left(\operatorname{vol}_d \mathbf{B}^d\right)^2}{\operatorname{vol}_d P_2} = \frac{\left(\operatorname{vol}_d \mathbf{B}^d\right)^2}{\lambda^d \operatorname{vol}_d P_1},$$

which, by (8.7), the inequality $d+1 \leq 2d$ and (8.8), yields that P satisfies (8.1).

To complete the proof, we need to find $\eta \in \binom{[k]}{\leq 2d+1}$ such that P and η satisfy (8.2).

Since y is on $\operatorname{bd}(Q)$, by Carathéodory's theorem, y is in the convex hull of some subset of at most d vertices of Q. Let this subset be $\{u_j : j \in \eta_2\}$, where $\eta_2 \in \binom{[k]}{\leq d}$.

We set $\eta = \eta_1 \cup \eta_2$, and claim that P and η satisfy (8.2).

Indeed, since $P_2 \subseteq \operatorname{conv}(\{u_j : j \in \eta_1\} \cup \{y\})$ and $y \in \operatorname{conv}(\{u_j : j \in \eta_2\})$, we have

$$P_2 \subseteq \operatorname{conv}(\{u_j : j \in \eta\}).$$

Taking the polar of both sides in $\mathbb{R}^d$, we obtain $P \supseteq \{u_j : j \in \eta\}^\circ$, which is equivalent to (8.2).

Thus, P and η satisfy (8.1) and (8.2), and hence, the proof of Theorem 1.2 is complete.

8.6. Lower bound on the Helly number

The number of functions selected in Theorem 1.2 is $3d+2$. In this subsection, we show that it cannot be decreased to $2d$. In fact, for any dimension d and any $\Delta > 0$, we give an example of $2d+1$ log-concave functions $f_1, \dots, f_{2d+1}$ such that $\int f_{[n]} = 2^d$, but for any $I \in \binom{[2d+1]}{\leq 2d}$, the integral is $\int f_I > \Delta$. Our example is a simple extension of the standard one (the $2d$ supporting half-spaces of a cube) for convex sets.

Set

$$\varphi(t) = \begin{cases} 0, & \text{if } t < 0 \\ e^{\Delta}, & \text{otherwise.} \end{cases}$$

Clearly, φ is upper semi-continuous. Let $e_1, \dots, e_d$ denote the standard basis in $\mathbb{R}^d$, and for each $i \in [d]$, define the functions $f_i(x) = \varphi(\langle e_i, x + e_i \rangle)$ and $f_{d+i} = \varphi(-\langle e_i, x - e_i \rangle)$, and let $f_{2d+1} = 1$. These functions are proper log-concave functions. The bounds on the integrals are easy.

Acknowledgment

G.I. was supported by the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926. M.N. was supported by the National Research, Development and Innovation Fund (NRDI) grants K119670 and KKP-133864 as well as the Bolyai Scholarship of the Hungarian Academy of Sciences and the New National Excellence Programme and the TKP2020-NKA-06 program provided by the NRDI.

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