Finite Fields and Their Applications 83 (2022) 102085

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

The Bertini irreducibility theorem for higher codimensional slices

Philip Kmentt, Alec Shute *

Institute of Science and Technology, Am Campus 1, 3400 Klosterneuburg, Austria

ARTICLE INFO

Article history:
Received 23 November 2021
Received in revised form 18 June 2022
Accepted 22 June 2022
Available online 15 July 2022
Communicated by Michel Lavrauw

MSC:
14A10
14D05
14G15
11G25

Keywords:
Bertini theorems
Irreducible varieties
Finite fields
Lang–Weil bound
Random sampling

ABSTRACT

In [3], Poonen and Slavov recently developed a novel approach to Bertini irreducibility theorems over an arbitrary field, based on random hyperplane slicing. In this paper, we extend their work by proving an analogous bound for the dimension of the exceptional locus in the setting of linear subspaces of higher codimensions.

© 2022 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

Bertini theorems are a family of results which typically state that if a projective variety $X \subseteq \mathbb{P}^n_F$ over a field F has a certain nice property (such as smoothness, or

* Corresponding author.
E-mail addresses: philipkmentt@hotmail.com (P. Kmentt), alec.shute@ist.ac.at (A. Shute).

https://doi.org/10.1016/j.ffa.2022.102085
1071-5797/© 2022 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

geometric irreducibility), then a generic hyperplane section of X has this property too. We define

$$\widehat{\mathbb{P}}_F^n = \{\text{hyperplanes } H \subseteq \mathbb{P}_F^n\}$$
$$\mathfrak{M}_{\text{bad}}^{(1)} = \{H \in \widehat{\mathbb{P}}_F^n : X \cap H \text{ is not geometrically irreducible}\}.$$

The set of bad hyperplanes $\mathfrak{M}_{\text{bad}}^{(1)}$ is a constructible locus in $\mathbb{P}_F^n$, and so it makes sense to ask about its dimension. A classical Bertini irreducibility theorem would state that $\dim \mathfrak{M}_{\text{bad}}^{(1)} \leqslant n-1$. However, much more is true, as demonstrated by the following theorem of Benoist.

Theorem 1.1. *[1, Théorème 1.4] We have* $\dim \mathfrak{M}_{bad}^{(1)} \leqslant n - \dim X + 1$.

In 2020, Poonen and Slavov reproved Theorem 1.1 using a novel approach based on estimates for the mean and variance of random hyperplane slices of X over a finite field. In this paper, we demonstrate that Poonen and Slavov's methods can be generalised to deal with linear subspaces H of a general codimension k. For a fixed $1 \leqslant k \leqslant n-1$, we define

$$V = \mathbb{G}(n-k, n) = \{\text{linear subspaces } H \subseteq \mathbb{P}_F^n \text{ of codimension } k\} \tag{1.1}$$
$$\mathfrak{M}_{\text{bad}}^{(k)} = \{H \in V : X \cap H \text{ is not geometrically irreducible}\}. \tag{1.2}$$

The main result of this paper is the following theorem.

Theorem 1.2. *Let $X \subseteq \mathbb{P}_F^n$ be a geometrically irreducible variety over an arbitrary field F. Let V and $\mathfrak{M}_{bad}^{(k)}$ be as in (1.1) and (1.2). Then*

$$\dim \mathfrak{M}_{bad}^{(k)} \leqslant \dim V - \dim X + k.$$

We recall that for any $k \leqslant n$, $\dim(\mathbb{G}(n-k,n)) = (n-k+1)k$. Therefore, on taking $k=1$ in Theorem 1.2, we recover the bound $\dim \mathfrak{M}_{\text{bad}}^{(1)} \leqslant n - \dim X + 1$ from Theorem 1.1.

Remark 1.3. Suppose that $\dim X \leqslant k$. If X is not linear, then a generic intersection $X \cap H$ for $H \in V$ will either be empty or a union of at least two points, and so not irreducible. Hence we cannot expect a Bertini irreducibility theorem to hold in this setting. This is consistent with the fact that the bound in Theorem 1.2 becomes trivial when $\dim X \leqslant k$.

In the case $k=1$, Benoist proves in [1, Proposition 3.1] that Theorem 1.1 is best possible, by exhibiting an irreducible variety X with the property $\dim \mathfrak{M}_{\text{bad}}^{(1)} = n - \dim X + 1$. The following theorem, which we prove in Section 4, is a generalisation of this argument, and shows that Theorem 1.2 is sharp whenever $\dim X \geqslant k$.

Theorem 1.4. *Let $1 \leqslant k \leqslant r \leqslant n-1$. Then there exists an irreducible variety $X \subseteq \mathbb{P}_F^n$ of dimension r such that*

$$\dim \mathfrak{M}_{bad}^{(k)} = \dim V - r + k.$$

Remark 1.5. Rather than considering intersections $X \cap H$, Poonen and Slavov work in [3] with a morphism $\varphi : X \to \mathbb{P}_F^n$ whose nonempty fibres all have the same dimension. They then study the exceptional locus of the hyperplanes $H \subseteq \mathbb{P}_F^n$ such that $\varphi^{-1}(H)$ is not geometrically irreducible. The reader comparing our arguments to [3] should consider φ to be the embedding $X \subseteq \mathbb{P}_F^n$ from the statement of Theorem 1.2, so that $\varphi^{-1}(H) = X \cap H$. However, we expect that Theorem 1.2 could easily be extended to the more general choice of φ used in [3].

Acknowledgments. The authors would like to thank Tim Browning for suggesting this project and for many helpful discussions during the development of this paper. We are also grateful to the anonymous referees for providing helpful feedback on an earlier version of this work.

2. Statistics of random linear slices

In this section, we work over a fixed finite field $\mathbb{F}_q$, and estimate the mean and variance of the number of $\mathbb{F}_q$-points on random linear slices of X.

Lemma 2.1. *Fix a variety $X \subseteq \mathbb{P}_{\mathbb{F}_q}^n$. Let $V = \mathbb{G}(n-k, n)$. For $H \in V(\mathbb{F}_q)$ chosen uniformly at random, define the random variable $Z := \#(X \cap H)(\mathbb{F}_q)$. Let μ and σ^2 denote the mean and variance of Z respectively. Then*

$$\begin{aligned} \mu &= \#X(\mathbb{F}_q)(q^{-k} + O(q^{-k-1})), \\ \sigma^2 &= O(q^{-k}\#X(\mathbb{F}_q)). \end{aligned}$$

Proof. We begin by considering the mean. We have

$$\begin{aligned} \mu &= \frac{1}{\#V(\mathbb{F}_q)} \sum_{H \in V(\mathbb{F}_q)} \#(X \cap H)(\mathbb{F}_q) \\ &= \frac{1}{\#V(\mathbb{F}_q)} \sum_{H \in V(\mathbb{F}_q)} \sum_{x \in (X \cap H)(\mathbb{F}_q)} 1 \\ &= \frac{1}{\#V(\mathbb{F}_q)} \sum_{x \in X(\mathbb{F}_q)} \sum_{\substack{H \in V(\mathbb{F}_q) \\ x \in H(\mathbb{F}_q)}} 1. \end{aligned} \tag{2.1}$$

We observe that the inner sum in (2.1) is independent of x. Consequently, we can average the value of this sum over a dummy variable $w \in \mathbb{P}^n(\mathbb{F}_q)$ to obtain

$$\sum_{\substack{H\in V(\mathbb{F}_q)\\ x\in H(\mathbb{F}_q)}} 1 = \frac{1}{\#\mathbb{P}^n(\mathbb{F}_q)} \sum_{w\in\mathbb{P}^n(\mathbb{F}_q)} \sum_{\substack{H\in V(\mathbb{F}_q)\\ w\in H(\mathbb{F}_q)}} 1.$$

Returning to (2.1), we conclude that

$$\begin{aligned}\mu &= \frac{1}{\#\mathbb{P}^n(\mathbb{F}_q)\#V(\mathbb{F}_q)} \sum_{x\in X(\mathbb{F}_q)} \sum_{H\in V(\mathbb{F}_q)} \sum_{w\in H(\mathbb{F}_q)} 1\\ &= \frac{\#X(\mathbb{F}_q)\#\mathbb{P}^{n-k}(\mathbb{F}_q)}{\#\mathbb{P}^n(\mathbb{F}_q)} &(2.2)\\ &= \#X(\mathbb{F}_q)(q^{-k} + O(q^{-k-1})). &(2.3)\end{aligned}$$

A similar argument can be applied for the variance. We have

$$\begin{aligned}\sigma^2 &= \left(\frac{1}{\#V(\mathbb{F}_q)} \sum_{H\in V(\mathbb{F}_q)} (\#(X\cap H)(\mathbb{F}_q))^2\right) - \mu^2\\ &= \left(\frac{1}{\#V(\mathbb{F}_q)} \sum_{H\in V(\mathbb{F}_q)} \sum_{x,y\in (X\cap H)(\mathbb{F}_q)} 1\right) - \mu^2\\ &= \left(\frac{1}{\#V(\mathbb{F}_q)} \sum_{x,y\in X(\mathbb{F}_q)} \sum_{\substack{H\in V(\mathbb{F}_q)\\ x,y\in H(\mathbb{F}_q)}} 1\right) - \mu^2. &(2.4)\end{aligned}$$

The contribution to (2.4) from the case $x = y$ is simply μ. Therefore, $\sigma^2 = B - \mu^2 + \mu$, where

$$B = \frac{1}{\#V(\mathbb{F}_q)} \sum_{\substack{x,y\in X(\mathbb{F}_q)\\ x\neq y}} \sum_{\substack{H\in V(\mathbb{F}_q)\\ x,y\in H(\mathbb{F}_q)}} 1. \tag{2.5}$$

A similar trick to above can be applied to the inner sum of (2.5), by averaging over two dummy variables u and v, ranging over the entirety of $\mathbb{P}^n(\mathbb{F}_q)$, but this time, with the added condition $u \neq v$. We obtain

$$\begin{aligned}B &= \frac{1}{\#V(\mathbb{F}_q)\#\mathbb{P}^n(\mathbb{F}_q)(\#\mathbb{P}^n(\mathbb{F}_q)-1)} \sum_{\substack{x,y\in X(\mathbb{F}_q)\\ x\neq y}} \sum_{\substack{u,v\in\mathbb{P}^n(\mathbb{F}_q)\\ u\neq v}} \sum_{\substack{H\in V(\mathbb{F}_q)\\ u,v\in H(\mathbb{F}_q)}} 1\\ &= \frac{1}{\#V(\mathbb{F}_q)\#\mathbb{P}^n(\mathbb{F}_q)(\#\mathbb{P}^n(\mathbb{F}_q)-1)} \sum_{\substack{x,y\in X(\mathbb{F}_q)\\ x\neq y}} \sum_{H\in V(\mathbb{F}_q)} \sum_{\substack{u,v\in H(\mathbb{F}_q)\\ u\neq v}} 1\\ &= \frac{\#X(\mathbb{F}_q)(\#X(\mathbb{F}_q)-1)\#\mathbb{P}^{n-k}(\mathbb{F}_q)(\#\mathbb{P}^{n-k}(\mathbb{F}_q)-1)}{\#\mathbb{P}^n(\mathbb{F}_q)(\#\mathbb{P}^n(\mathbb{F}_q)-1)}. &(2.6)\end{aligned}$$

Now, inspecting every term in (2.6) and comparing it to the corresponding term in the expression for μ^2 obtained from squaring (2.2), we see that $B \leqslant \mu^2$, implying that $\sigma^2 \leqslant \mu$. Combining with (2.3), we deduce that in particular, $\sigma^2 = O(q^{-k}\#X(\mathbb{F}_q))$. $\square$

3. Proof of Theorem 1.2

For the remainder of this paper, we allow all implied constants to depend on X but not on q. (In fact, the implied constants need only depend on the *geometric complexity* of X, as defined in [3, Section 5].)

The following classical theorem of Lang and Weil provides an estimate for the number of $\mathbb{F}_q$-points on a projective variety X.

Lemma 3.1 *([2]). Let X be a projective variety over $\mathbb{F}_q$ of dimension r. Let a be the number of irreducible components of X that are geometrically irreducible and have dimension r. Then $\#X(\mathbb{F}_q) = aq^r + O(q^{r-1/2})$.*

We can now deduce Theorem 1.2 from Lemma 2.1 and Lemma 3.1 by following a similar argument to [3, Sections 5, 6]. By applying Poonen and Slavov's reduction from [3, Section 3], we may assume that the ground field F is finite. Moreover, we are free to pass to a sufficiently large finite extension $\mathbb{F}_q \supseteq F$.

We call $H \in V(\mathbb{F}_q)$ *very bad* if the number of $\mathbb{F}_q$-irreducible components of $X \cap H$ which are geometrically irreducible is not 1. Let $\mathscr{A} \subseteq V(\mathbb{F}_q)$ denote the set of very bad linear spaces H. The strategy will be to deduce Theorem 1.2 from appropriate upper and lower bounds for $\#\mathscr{A}$.

We obtain a lower bound for $\#\mathscr{A}$ by applying [3, Lemma 6.2]. We choose an irreducible variety $B \subseteq \mathfrak{M}^{(k)}_{\text{bad}}$ with the same dimension as $\mathfrak{M}^{(k)}_{\text{bad}}$. We define a variety $Y \subseteq X \times B$ via the incidence relation $Y = \{(x, H) \in X \times B : x \in H\}$. Let $\psi : Y \to B$ be the morphism sending (x, H) to H. Then the fibres $\psi^{-1}(H)$ are isomorphic to $X \cap H$, and so by definition of B the generic fibre of ψ is not geometrically irreducible. Applying [3, Lemma 6.2] with this choice of B and ψ, we deduce that

$$\#\mathscr{A} \gg q^{\dim B} \gg \#\mathfrak{M}^{(k)}_{\text{bad}}(\mathbb{F}_q). \tag{3.1}$$

In order to obtain an upper bound for $\#\mathscr{A}$, we follow a similar approach to [3, Lemma 6.1]. The idea is to show that if $H \in \mathscr{A}$, then the random variable $Z = \#(X \cap H)(\mathbb{F}_q)$ introduced in Section 2 differs considerably from the mean μ. Hence there cannot be many such H by the upper bound for the variance obtained in Lemma 2.1.

Let $r = \dim X$. Combining Lemma 2.1 and Lemma 3.1, we have

$$\begin{aligned}\mu &= q^{r-k} + O(q^{r-k-1/2})\\ \sigma^2 &= O(q^{r-k}).\end{aligned}$$

If $H \in \mathscr{A}$, then by Lemma 3.1, $\#(X \cap H)(\mathbb{F}_q)$ is either $O(q^{r-k-1/2})$ or at least $2q^{r-k} - O(q^{r-k-1/2})$. Consequently,

$$| \#(X \cap H)(\mathbb{F}_q) - \mu | \geqslant q^{r-k} - O(q^{r-k-\frac{1}{2}}) \geqslant \frac{1}{2}q^{r-k},$$

for sufficiently large q. Define t such that $\frac{1}{2}q^{r-k} = t\sigma$. Then,

$$\begin{aligned}\mathrm{Prob}(H \in \mathscr{A}) &\leqslant \mathrm{Prob}(| \#(X \cap H)(\mathbb{F}_q) - \mu | \geqslant t\sigma) \\ &\leqslant \frac{1}{t^2} \qquad \text{(by Chebyshev's inequality)} \\ &= \frac{4\sigma^2}{q^{2(r-k)}} \\ &= O(q^{-r+k}).\end{aligned}$$

Multiplying by $\#V(\mathbb{F}_q)$, we obtain

$$\#\mathscr{A} = O(q^{\dim V - r + k}). \tag{3.2}$$

Therefore, combining (3.1) and (3.2), we conclude that

$$\dim \mathfrak{M}_{\mathrm{bad}}^{(k)} \leqslant \dim V - r + k.$$

4. Proof of Theorem 1.4

If $r = 1$, then $k = 1$, and so by Remark 1.3 it suffices to take X to be any curve of degree at least 2. From now on, we assume that $r \geqslant 2$. We use the same construction of X as in [1, Proposition 3.1], which we now recall for convenience. Fix an integral curve C of degree at least 2, and a linear space L of dimension $r - 2$ not containing C. Let $X \subseteq \mathbb{P}^n_F$ be a cone with base C and vertex L. Then X is an irreducible variety of dimension r.

Let $\mathscr{H}$ denote the locus of hyperplanes H which contain L. We denote by $\mathscr{U}$ the locus of hyperplanes $H \in \mathscr{H}$ satisfying the following additional properties.

(1) $(H \cap C) \nsubseteq L$.
(2) $\dim(X \cap H) = r - 1$.
(3) $\deg(X \cap H) \geqslant 2$.

These properties hold for a generic $H \in \mathscr{H}$, so $\dim \mathscr{U} = \dim \mathscr{H}$. Suppose that $H \in \mathscr{U}$. We choose a point P in $H \cap C$ not contained in L. Let N denote the linear span of L and P. Then $N \subseteq H$. Furthermore, X contains all lines between P and L, and hence also contains N. Therefore $N \subseteq X \cap H$. From the above properties, we deduce that N

is a proper closed subset of $X \cap H$ with $\dim N = r-1 = \dim(X \cap H)$, and hence $X \cap H$ is not irreducible. We conclude that

$$\dim \mathfrak{M}_{\text{bad}}^{(1)} \geqslant \dim \mathscr{U} = \dim \mathscr{H} = n - r + 1,$$

and in fact, equality holds by Theorem 1.1.

We now generalise to an arbitrary $k \leqslant r$. Fix $H \in \mathscr{U}$, and let N be chosen as above. Let $\mathscr{M}_H$ denote the locus of linear spaces $M \in \mathbb{G}(n-k, n)$ satisfying $M \subseteq H$. Then

$$\dim \mathscr{M}_H = \dim(\mathbb{G}(n-k, n-1)) = \dim(\mathbb{G}(n-k, n)) - (n-k+1). \tag{4.1}$$

For a generic $M \in \mathscr{M}_H$, we have

(1) $\dim(X \cap M) = r - k = \dim(N \cap M)$.
(2) $\deg(X \cap M) \geqslant 2$.
(3) The linear span of M and L is H.

To see that property (3) holds generically, we note that the linear span of M and L has dimension $\dim M + \dim L - \dim(M \cap L)$. (This is true even when $M \cap L = \emptyset$ if we use the convention $\dim \emptyset = -1$.) Every $M \in \mathscr{M}_H$ has codimension $k-1$ in H. Therefore, for generic $M \in \mathscr{M}_H$, we have that $\dim(M \cap L) = \dim L - (k-1)$, and hence the linear span of M and L has dimension $\dim M + (k-1) = n - 1 = \dim H$.

For M satisfying the above three properties, we have that $N \cap M$ is a proper closed subset of $X \cap M$ with the same dimension as $X \cap M$, and hence $M \in \mathfrak{M}_{\text{bad}}^{(k)}$. Property (3) ensures that H is the unique element of $\mathscr{U}$ containing M. Combining with (4.1), we deduce that

$$\begin{aligned} \dim \mathfrak{M}_{\text{bad}}^{(k)} &\geqslant \dim(\mathbb{G}(n-k, n)) - (n-k+1) + \dim \mathscr{U} \\ &= \dim(\mathbb{G}(n-k, n)) + k - r, \end{aligned} \tag{4.2}$$

and from Theorem 1.2, (4.2) is in fact an equality.

References

[1] O. Benoist, Le théorème de Bertini en famille, Bull. Soc. Math. Fr. 139 (2011) 555–569.
[2] S. Lang, A. Weil, Number of points of varieties in finite fields, Am. J. Math. 76 (1954) 819–827.
[3] B. Poonen, K. Slavov, The exceptional locus in the Bertini irreducibility theorem for a morphism, Int. Math. Res. Not. 2022 (2022) 4503–4513.