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Weak-strong Uniqueness for the Navier–Stokes Equation for Two Fluids with Ninety Degree Contact Angle and Same Viscosities

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Abstract. We consider the flow of two viscous and incompressible fluids within a bounded domain modeled by means of a two-phase Navier–Stokes system. The two fluids are assumed to be immiscible, meaning that they are separated by an interface. With respect to the motion of the interface, we consider pure transport by the fluid flow. Along the boundary of the domain, a complete slip boundary condition for the fluid velocities and a constant ninety degree contact angle condition for the interface are assumed. In the present work, we devise for the resulting evolution problem a suitable weak solution concept based on the framework of varifolds and establish as the main result a weak-strong uniqueness principle in 2D. The proof is based on a relative entropy argument and requires a non-trivial further development of ideas from the recent work of Fischer and the first author (Arch. Ration. Mech. Anal. 236, 2020) to incorporate the contact angle condition. To focus on the effects of the necessarily singular geometry of the evolving fluid domains, we work for simplicity in the regime of same viscosities for the two fluids.

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1. Introduction

1.1. Context

The question of uniqueness or non-uniqueness of weak solution concepts in the context of classical fluid mechanics models has seen a series of intriguing breakthroughs throughout the last three decades. In case of the Euler equations, the journey started with the seminal works of Scheffer [23] and Shnirelman [28] providing the construction of compactly supported nonzero weak solutions. The first example of an energy dissipating weak solution to the Euler equations is again due to Shnirelman [29]. Later, De Lellis and Székelyhidi Jr. not only strengthened these results in their groundbreaking works (see, e.g., [8] and [9]), but in retrospect even more importantly introduced a novel perspective on the problem: their proofs are based on a nontrivial transfer of convex integration techniques from typically geometric PDEs to the framework of the Euler equations. Indeed, their ideas eventually culminated in the resolution of Onsager's conjecture by Isett [17]; see also the work of Buckmaster, De Lellis, Székelyhidi Jr. and Vicol [7].

By now, these developments also generated spectacular results for the Navier–Stokes equations. For instance, Buckmaster and Vicol [5] as well as Buckmaster, Colombo and Vicol [6] establish that mild solutions in the energy class are non-unique. The constructed solutions are not Leray–Hopf solutions, i.e., it is not proven that they are subject to the energy dissipation inequality. However, Albritton, Brué and Colombo [2] even show in a very recent preprint that one can construct an external force such that there exists a finite time horizon so that one may construct at least two distinct Leray–Hopf solutions for the associated forced full-space Navier–Stokes equations in 3D (both starting from zero initial data).

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Hence, in terms of uniqueness of weak solutions the best one can expect in general is essentially a weak-strong uniqueness principle. Roughly speaking, this refers to uniqueness of weak solutions within a class of sufficiently regular solutions. In the context of the incompressible Navier–Stokes equations, such results are classical and can be traced back to the works of Leray [19], Prodi [20] and Serrin [25]. In the case of the compressible Navier–Stokes equations, we mention the works of Germain [15], Feireisl, Jin and Novotný [10], as well as Feireisl and Novotný [11]. The usual strategy to establish these results is based on a by now widely used method which infers weak-strong uniqueness from a quantitative stability estimate for a suitable distance measure between two solutions, the so-called relative entropy (or relative energy). We refer to the survey article by Wiedemann [33] for an overview on the relative entropy method in the context of mathematical fluid mechanics.

In the present work, we are concerned with the question of weak-strong uniqueness with respect to a two-phase free boundary fluid problem within a physical domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$. More precisely, we study this question in terms of a suitably devised concept of varifold solutions for the evolution problem of the flow of two incompressible Navier–Stokes fluids separated by a sharp interface. Along the boundary of the domain, a complete slip boundary condition for the fluid velocities as well as a constant ninety degree contact angle condition for the interface are assumed. For the precise PDE formulation of the model, we refer to Sect. 1.2. For a discussion of the weak solution concept and its precise definition, we instead refer to Sect. 1.3 and Definition 11, respectively. The main result of the present work establishes in 2D a weak-strong uniqueness principle for the above introduced two-phase free boundary fluid problem. We refer to Theorem 1 for the precise mathematical formulation of our result. In the spirit of [13], we also derive a conditional weak-strong uniqueness result in the three-dimensional setting; cf. Proposition 4 for the precise statement. To the best of the authors' knowledge, the present work is the first to establish weak-strong uniqueness in the context of an interface evolution problem incorporating contact point dynamics in combination with a fluid mechanical coupling.

Even when neglecting the fluid mechanics, uniqueness of weak solutions in form of a weak-strong uniqueness principle is in general the best one can expect also for interface evolution problems. In this context, this is due to the formation of singularities and topology changes; see already, for instance, the work of Brakke [4] for mean curvature flow of networks of interfaces in $\mathbb{R}^2$ or the work of Angenent, Ilmanen and Chopp [3] for mean curvature flow of surfaces in $\mathbb{R}^3$. When restricting to the full-space setting $\Omega = \mathbb{R}^d$ and thus neglecting non-trivial boundary effects, Fischer and the first author [12] recently established a weak-strong uniqueness principle up to the first topology change for the corresponding two-phase free boundary fluid problem considered in this work. Their approach relies on a suitable extension of the relative entropy method to get control on the difference in the underlying geometries of two solutions; cf. Sect. 1.4 for a discussion in this direction. Their ideas were later generalized by Fischer, Laux, Simon and the first author [13] to derive a weak-strong uniqueness principle for BV solutions of Laux and Otto [18] to mean curvature flow of networks of interfaces in $\mathbb{R}^2$, or even for canonical multiphase Brakke flows of Stuvard and Tonegawa [31] (cf. also [16]).

The main challenges of the present work are twofold. First, we need to devise a weak solution concept for the above introduced two-phase free boundary fluid problem. We emphasize that this is not already contained in the work of Abels [1] which in the presence of surface tension only deals with the full-space setting. Even though our notion of varifold solutions is clearly directly inspired by Abels' formulation, some additional thoughts are necessary in the present setting of a bounded domain with contact point dynamics (cf. again Sect. 1.3 for a discussion). Indeed, the point is to formulate a solution concept which on one side is weak enough to allow for a satisfactory global-in-time existence theory (cf. Appendix A for a sketch of an existence proof along the lines of the argument of Abels [1]), but on the other side is still strong enough to support a weak-strong uniqueness principle. To obtain the latter, the second challenge of the present work is to further develop parts of the analysis of Fischer and the first author [12] to deal with the non-trivial boundary effects and the necessarily singular geometry of the evolving fluid domains. Due to the latter two, it turns out to be beneficial to implement the relative entropy argument based on a two-step procedure rather in the spirit of [13] than the more direct approach from [12] (cf. Sect. 2.2 for further discussion).

1.2. Strong PDE Formulation of the Two-phase Fluid Model

We start with a description of the underlying evolving geometry. Denoting by Ω a bounded domain in $\mathbb{R}^d$ with smooth and orientable boundary $\partial\Omega$, $d \in \{2,3\}$, each of the two fluids is contained within a time-evolving domain $\Omega^+(t) \subset \Omega$ resp. $\Omega^-(t) \subset \Omega$, $t \in [0,T)$. The interface separating both fluids is given as the common boundary between the two fluid domains. Denoting it at time $t \in [0,T)$ by $I(t) \subset \overline{\Omega}$, we then have a disjoint decomposition of $\overline{\Omega}$ in form of $\overline{\Omega} = \Omega^+(t) \cup \Omega^-(t) \cup (I(t) \cap \Omega) \cup \partial\Omega$ for every $t \in [0,T)$. We write $n_{\partial\Omega}$ to refer to the inner pointing unit normal vector field of $\partial\Omega$, as well as $n_I(\cdot,t)$ to denote the unit normal vector field along $I(t)$ pointing towards $\Omega^+(t)$, $t \in [0,T)$.

With respect to internal boundary conditions along the separating interface, first, a no-slip boundary condition is assumed. This in fact allows to represent the two fluid velocity fields by a single continuous vector field v. We also consider a single scalar field p as the pressure, which in contrast may jump across the interface. Second, along the interface the internal forces of the fluids have to match a surface tension force. Denoting by $\chi(\cdot,t)$ the characteristic function associated with the domain $\Omega^+(t)$, $t \in [0,T)$, and defining $\mu(\chi) := \mu^+\chi + \mu^-(1-\chi)$ with μ^+ and μ^- being the viscosities of the two fluids, the stress tensor $\mathbb{T} := \mu(\chi)(\nabla v + \nabla v^\mathsf{T}) - p\,\mathrm{Id}$ is required to satisfy

$$[[\mathbb{T} n_I]](\cdot,t) = \sigma\, \mathrm{H}_I(\cdot,t) \quad \text{along } I(t) \tag{1}$$

for all $t \in [0,T)$, where moreover $[[\cdot]]$ denotes the jump in normal direction, $\sigma > 0$ is the fixed surface tension coefficient of the interface, and $\mathrm{H}_I(\cdot,t)$ represents the mean curvature vector field along the interface $I(t)$, $t \in [0,T)$.

With respect to boundary conditions along $\partial\Omega$, we assume in terms of the two fluids a complete slip boundary conditions. In terms of the evolving geometry, a ninety degree contact angle condition at the contact set of the fluid-fluid interface with the boundary of the domain is imposed. Mathematically, this amounts to

$$v(\cdot,t) \cdot n_{\partial\Omega} = 0 \qquad \text{along } \partial\Omega, \tag{2}$$

$$\big(n_{\partial\Omega} \cdot \mu(\chi)(\nabla v + \nabla v^\mathsf{T})(\cdot,t)B\big) = 0 \qquad \text{along } \partial\Omega \tag{3}$$

for all $t \in [0,T)$ and all tangential vector fields B along $\partial\Omega$, as well as

$$n_I(\cdot,t) \cdot n_{\partial\Omega} = 0 \qquad \text{along } I(t) \cap \partial\Omega \tag{4}$$

for all $t \in [0,T)$. These boundary conditions not only prescribe that the fluid cannot exit from the domain and that it can move only tangentially to its boundary, but they also exclude any external contribution to the viscous stress and any friction effect with the boundary. Observe also that the ninety degree contact angle condition is consistent with the complete slip boundary conditions (2) and (3), in the sense that (4) together with (2) implies (3). Furthermore, the ninety degree contact angle may be imposed only as an initial condition: for later times it can be deduced using (2) and (3) and a Gronwall-type argument. For details, see the remark after Definition 10.

Now, defining $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$ with ρ^+ and ρ^- representing the densities of the two fluids, the fluid motion is given by the incompressible Navier–Stokes equation, which by (1) and (3) can be formulated as

$$\partial_t\big(\rho(\chi)v\big) + \nabla \cdot \big(\rho(\chi)v \otimes v\big) = -\nabla p + \nabla \cdot \big(\mu(\chi)(\nabla v + \nabla v^\mathsf{T})\big) + \sigma\, \mathrm{H}_I\, |\nabla\chi| \llcorner \Omega, \tag{5}$$

$$\nabla \cdot v = 0, \tag{6}$$

where $|\nabla\chi|(\cdot,t)\llcorner\Omega$ represents the surface measure $\mathcal{H}^{d-1}\llcorner(I(t) \cap \Omega)$, $t \in [0,T)$. Second, the interface is assumed to be transported along the fluid flow. In other words, the associated normal velocity of the interface is given by the normal component of the fluid velocity v. Thanks to (2), (4) and (6), this is formally equivalent to

$$\partial_t\chi + (v \cdot \nabla)\chi = 0. \tag{7}$$

Finally, from a modeling perspective, the total energy of the PDE system (5)–(7) is given by the sum of kinetic and surface tension energies

$$E[\chi, v] := \int_\Omega \frac{1}{2}\rho(\chi)|v|^2 \,\mathrm{d}x + \sigma \int_\Omega 1 \,\mathrm{d}|\nabla\chi| + \sigma^+ \int_{\partial\Omega} \chi \,\mathrm{d}S + \sigma^- \int_{\partial\Omega} (1-\chi) \,\mathrm{d}S, \tag{8}$$

where σ^+ and σ^- are the surface tension coefficients of $\partial\Omega \cap \overline{\Omega_t^+}$ and $\partial\Omega \cap \overline{\Omega_t^-}$, respectively. Note that the ninety degree contact angle condition (4) corresponds to $\sigma^- = \sigma^+$. Indeed, a general constant contact angle $\alpha \in (0, \pi)$ is prescribed by Young's equation which in our notation reads as follows

$$\sigma \cos\alpha = \sigma^+ - \sigma^-.$$

In particular, by subtracting the constant $\int_{\partial\Omega} 1 \,\mathrm{d}S$ from (8) we see that the relevant part of the total energy does not contain a surface energy contribution along $\partial\Omega$ in our special case of a constant ninety degree contact angle. By formal computations, one finally observes that this energy satisfies an energy dissipation inequality

$$E[\chi, v](T') + \int_0^{T'} \int_\Omega \frac{\mu(\chi)}{2} |\nabla v + \nabla v^T|^2 \,\mathrm{d}x \,\mathrm{d}t \leq E[\chi, v](0), \quad T' \in [0, T). \tag{9}$$

1.3. Varifold Solutions for Two-phase Fluid Flow with 90° Contact Angle

In terms of weak solution theories for the evolution problem (5)–(7), the energy dissipation inequality suggests to consider velocity fields in the space $L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)) \cap L^2(0, T; H^1(\Omega; \mathbb{R}^d))$, and the evolving geometry may be modeled based on a time-evolving set of finite perimeter so that the associated characteristic function χ is an element of $L^\infty(0, T; BV(\Omega; \{0, 1\}))$.

However, a well-known problem arises when considering limit points of a sequence of pairs $(\chi_k, v_k)_{k\in\mathbb{N}}$ representing solutions originating from an approximation scheme for (5)–(7). Ignoring the time variable for the sake of the discussion, the main point is that a uniform bound of the form $\sup_{k\in\mathbb{N}} \|\chi_k\|_{BV(\Omega)} < \infty$ in general does not suffice to pass to the limit (not even subsequentially) in the surface tension force $\sigma\, \mathrm{H}_{I_k} |\nabla\chi_k| \llcorner \Omega$. Recalling that we work in a setting with a ninety degree angle condition, this term is represented in distributional form by

$$\int_\Omega \mathrm{H}_{I_k} \cdot B \,\mathrm{d}|\nabla\chi_k| = -\int_\Omega (\mathrm{Id} - n_k \otimes n_k) : \nabla B \,\mathrm{d}|\nabla\chi_k| \tag{10}$$

for all smooth vector fields B which are tangential along $\partial\Omega$, where $n_k = \frac{\nabla\chi_k}{|\nabla\chi_k|}$ denotes the measure-theoretic interface unit normal. One may pass to the limit on the right hand side of the previous display provided $|\nabla\chi_k|(\Omega) \to |\nabla\chi|(\Omega)$. However, for standard approximation schemes there is in general no reason why this should be true. For instance, hidden boundaries may be generated within Ω in the limit. Furthermore, but now specific to the setting of a bounded domain, nontrivial parts of the approximating interfaces may converge towards the boundary $\partial\Omega$.

The upshot is that one has to pass to an even weaker representation of the surface tension force than (10). A popular workaround is based on the concept of (oriented) varifolds. In the setting of the present work and in view of the preceding discussion, this in fact amounts to consider the space of finite Radon measures on the product space $\overline{\Omega} \times \mathbb{S}^{d-1}$. Indeed, introducing the varifold lift $V_k := |\nabla\chi_k| \llcorner \Omega \otimes (\delta_{n_k(x)})_{x\in\Omega}$ one may equivalently express the right hand side of (10) in terms of the functional $B \mapsto -\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s \otimes s) : \nabla B \,\mathrm{d}V_k(x, s)$ which is now stable with respect to weak* convergence in the space of finite Radon measures on $\overline{\Omega} \times \mathbb{S}^{d-1}$. Note also that by the choice of working in a varifold setting, one expects $\sigma \int_{\overline{\Omega}} 1 \,\mathrm{d}|V|_{\mathbb{S}^{d-1}}$ instead of $\sigma \int_\Omega 1 \,\mathrm{d}|\nabla\chi|$ as the interfacial energy contribution in (8), where the finite Radon measure $|V|_{\mathbb{S}^{d-1}}$ denotes the mass of the varifold V.

Motivated by the previous discussion, we give a full formulation of a varifold solution concept to two-phase fluid flow with surface tension and constant ninety degree contact angle in Definition 11 below. This definition is nothing else but the suitable analogue of the definition by Abels [1], who provides for the full-space setting a global-in-time existence theory for such varifold solutions with respect to rather general initial data. Unfortunately, in the bounded domain case with non-zero interfacial surface tension, to the best of our knowledge a global-in-time existence result for varifold solutions is missing. In particular, such a result is not contained in the work of Abels [1]. For this reason, we include in this work at least a sketch of an existence proof. To this end, one may follow on one side the higher-level structure of the argument given by Abels [1] for the full-space setting. On the other side, additional arguments are of course necessary due to the specified boundary conditions for the geometry and the fluids, respectively. These additional arguments are outlined in Appendix A.

1.4. Weak-strong Uniqueness for Varifold Solutions of Two-phase Fluid Flow

In case the two fluids occupy the full space $\mathbb{R}^d$, $d \in \{2,3\}$, a weak-strong uniqueness result for Abels' [1] varifold solutions of the system (5)–(7) was recently established by Fischer and the first author [12]. Given sufficiently regular initial data, it is shown that on the time horizon of existence of the associated unique strong solution, any varifold solution in the sense of Abels [1] starting from the same initial data has to coincide with this strong solution.

This result is achieved by extending a by now several decades old idea in the analysis of classical PDE models from continuum mechanics to a previously not covered class of problems: a relative entropy method for surface tension driven interface evolution. The gist of this method can be described as follows. Based on a dissipated energy functional, one first tries to build an error functional — the relative entropy — which penalizes the difference between two solutions in a sufficiently strong sense. A minimum requirement is to ensure that the error functional vanishes if and only if the two solutions coincide. In a second step, one proceeds by computing the time evolution of this error functional. In a third step, one tries to identify all the terms appearing in this computation as contributions which either are controlled by the error functional itself or otherwise may be absorbed into a residual quadratic term represented essentially by the difference of the dissipation energies. One finally concludes by an application of Gronwall's lemma.

The novelty of the work [12] consists of an implementation of this strategy for the full-space version of the energy functional (8). More precisely, the relative entropy as it was originally constructed in the full-space setting in [12] essentially consists of two contributions. The first aims for a penalization of the difference of the underlying geometries of the two solutions. This in fact is performed at the level of the interfaces by introducing a tilt-excess type error functional with respect to the two associated unit normal vector fields. To this end, the construction of a suitable extension of the unit normal vector field of the interface of the strong solution in the vicinity of its space-time trajectory is required. Furthermore, the length of this vector field is required to decrease quadratically fast as one moves away from the interface of the strong solution. The merit of this is that one also obtains a measure of the interface error in terms of the distance between them.

Due to the inclusion of contact point dynamics in form of a constant ninety degree contact angle, some additional ingredients are needed for the present work. We refer to Sect. 2.2 below for a detailed and mathematical account on the geometric part of the relative entropy functional. There are however two notable additional difficulties in comparison to [12] which are worth emphasizing already at this point. Both are related to the required extension ξ of the unit normal vector field associated with the evolving interface of the strong solution. The first is concerned with the correct boundary condition for the extension ξ along $\partial\Omega$. Since along the contact set the interface intersects the boundary of the domain orthogonally, it is natural to enforce ξ to be tangential along $\partial\Omega$. This indeed turns out to be the right condition as it allows by an integration by parts to rewrite the interfacial part of the relative entropy as the sum of interfacial energy of the weak solution and a linear functional with respect to the characteristic function χ of the weak solution. This is crucial to even attempt computing the time evolution.

The second difference concerns the actual construction of the extension ξ. In contrast to [12], where only a finite number of sufficiently regular closed curves ($d = 2$) or closed surfaces ($d = 3$) are allowed at the level of the strong solution, this results in a nontrivial and subtle task in the context of the present work due to the necessarily singular geometry in contact angle problems. The main difficulty roughly speaking is to provide a construction which on one side respects the required boundary condition and on the other side is regular enough to support the computations and estimates in the Gronwall-type argument. For a complete list of the required conditions for the extension ξ, we refer to Definition 2 below.

We finally turn to a brief discussion of the second contribution in the total relative entropy functional from [12]. In principle, this term on first sight should be nothing else than the relative entropy analogue to the kinetic part of the energy of the system, thus controlling the squared L^2-distance between the fluid velocities of the two solutions. However, as recognized in [12] a major problem arises for the two-phase fluid problem in the regime of different viscosities $\mu^+ \neq \mu^-$: without performing a very careful (and in its implementation highly technical) perturbation of this naive ansatz for the fluid velocity error, a Gronwall-type argument will not be realizable; cf. for more details the discussion in [12, Sect. 3.4]. Since the main focus of the present work lies on the inclusion of the ninety degree contact angle condition, we do not delve into these issues and simply assume for the rest of this work that the viscosities of the two fluids coincide: $\mu := \mu^+ = \mu^-$. We emphasize, however, that at least for the construction of the extension ξ and the verification of its properties we in fact do not rely on this assumption.

2. Main Results

2.1. Weak-strong Uniqueness and Stability of Evolutions

The main result of this work reads as follows.

Theorem 1. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on a time interval $[0, T_w)$. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0, T_s)$ where $T_s \leq T_w$.*

Then, for every $T \in (0, T_s)$ there exists a constant $C = C(\chi_v, v, T) > 0$ such that the relative entropy functional (29) and the bulk error functional (31) satisfy stability estimates of the form

$$E[\chi_u, u, V|\chi_v, v](t) \leq Ce^{Ct}\big(E[\chi_u, u, V|\chi_v, v](0) + E_{\mathrm{vol}}[\chi_u|\chi_v](0)\big), \tag{11}$$

$$E_{\mathrm{vol}}[\chi_u|\chi_v](t) \leq Ce^{Ct}\big(E[\chi_u, u, V|\chi_v, v](0) + E_{\mathrm{vol}}[\chi_u|\chi_v](0)\big) \tag{12}$$

for almost every $t \in [0, T]$.

In particular, in case the initial data for the varifold solution and strong solution coincide, it follows that

$$\chi_u(\cdot, t) = \chi_v(\cdot, t), \quad u(\cdot, t) = v(\cdot, t) \qquad \textit{a.e. in } \Omega \textit{ for a.e. } t \in [0, T_s), \tag{13}$$

$$V_t = (|\nabla\chi_u(\cdot, t)|\llcorner\Omega) \otimes \big(\delta_{\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)}\big)_{x\in\Omega} \qquad \textit{for a.e. } t \in [0, T_s). \tag{14}$$

Before proceeding with a discussion on the proof of Theorem 1, we comment on its validity in the regime of different shear viscosities $\mu^+ \neq \mu^-$ of the two fluids (cf. also the detailed discussion in [12, Sect. 3.4]). In this case, one would have to deal with an additional term in the derivation of the Gronwall inequality (11) of the form

$$-\int_0^{T'} \int_\Omega (\mu^+ - \mu^-)(\chi_u - \chi_v)2(\nabla^{\mathrm{sym}} u - \nabla^{\mathrm{sym}} v) : \nabla v \, dx. \tag{15}$$

A major problem then results from the observation that, even for strong solutions, the normal derivative of the tangential velocity is discontinuous across the associated interface in case $\mu^+ \neq \mu^-$. As a consequence, the term (15) is in fact only of linear order in our error functionals which makes the derivation of a stability estimate as in (11) infeasible (cf. the example given in [12, Sect. 3.4]).

The key idea for the weak-strong uniqueness result in the different viscosities regime in the full space setting [12] was to adapt the kinetic energy contribution of the relative entropy: instead of directly comparing u with v, one carefully constructs an auxiliary divergence free vector field w and compares u with $v+w$. The two desired main properties of w are as follows. First, the L^2 norm of w shall be controlled by the interfacial error contribution of the relative entropy, so that the adapted relative entropy does not lose coercivity with respect to the error in the velocity fields. Second, ∇w should be designed such that it essentially compensates the linear order error term (15). The main idea for the latter is to adapt ∇v through ∇w to the different location of the interface of the varifold solution.

Of course, also in our bounded domain setting with constant 90° degree contact angle and pure slip condition, this additional adaptation of the relative entropy is needed to conclude about the validity of Theorem 1 in case of different viscosities for the two fluids. In principle, we expect this to be possible in the setting of the present work. However, adapting the construction of the compensating vector field w from [12] in the vicinity of the domain boundary (in order to satisfy required boundary conditions) together with then verifying all of its desired properties may certainly require a substantial amount of technical work (e.g., due to the singular nature of the geometry at the contact set). For this reason, we omit the rigorous study of the different viscosities regime in this work and we leave it as a possible further development of our result. Finally, as for the validity of Theorem 1 for non-Newtonian fluids, we mention that this is an open problem in both the full space setting and the setting of the present work.

Returning to the regime of same viscosities $\mu^+ = \mu^-$, we explain throughout the next two subsections the key ideas underlying the proof of Theorem 1.

2.2. Quantitative Stability by a Relative Entropy Approach

Following the general strategy of [12], our weak-strong uniqueness result essentially relies on two ingredients: *i)* the construction of a suitable extension ξ of the unit normal vector field of the interface of a strong solution, and *ii)* based on this extension, the introduction of a suitably defined error functional penalizing the interface error between a varifold and a strong solution in a sufficiently strong sense. In comparison to [12], the extension of the unit normal has to be carefully constructed in the sense that the vector field ξ is required to be tangent to the domain boundary $\partial\Omega$ (which is the natural boundary condition in case of a 90° contact angle). Due to the singular nature of the geometry at the contact set, this is a nontrivial task. The precise conditions on the extension ξ are summarized as follows.

Definition 2 *(Boundary adapted extension of the interface unit normal).* Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let $T \in (0,\infty)$ be a finite time horizon. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on the time interval $[0,T]$.

In this setting, we call a vector field $\xi \colon \overline{\Omega} \times [0,T] \to \mathbb{R}^d$ a *boundary adapted extension of n_{I_v} for two-phase fluid flow (χ_v, v) with 90° contact angle* if the following conditions are satisfied:

- In terms of regularity, it holds $\xi \in \big(C_t^0 C_x^2 \cap C_t^1 C_x^0\big)\big(\overline{\Omega \times [0,T]} \setminus (I_v \cap (\partial\Omega \times [0,T]))\big)$.
- The vector field ξ extends the unit normal vector field n_{I_v} (pointing inside Ω_v^+) of the interface I_v subject to the conditions

$$|\xi| \leq \max\big\{0, 1 - C \operatorname{dist}^2(\cdot, I_v)\big\} \qquad \text{in } \Omega \times [0,T], \tag{16a}$$

$$\xi \cdot n_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times [0,T], \tag{16b}$$

$$\nabla \cdot \xi = -H_{I_v} \qquad \text{on } I_v, \tag{16c}$$

  for some $C > 0$. Here, H_{I_v} denotes the scalar mean curvature of the interface I_v (oriented with respect to the normal n_{I_v}).
- The fluid velocity approximately transports the vector field ξ in form of

$$\partial_t \xi + (v \cdot \nabla)\xi + (\mathrm{Id} - \xi \otimes \xi)(\nabla v)^{\mathsf{T}} \xi = O(\operatorname{dist}(\cdot, I_v) \wedge 1) \qquad \text{in } \Omega \times [0,T], \tag{16d}$$

$$\partial_t|\xi|^2 + (v\cdot\nabla)|\xi|^2 = O(\mathrm{dist}^2(\cdot, I_v)\wedge 1) \qquad \text{in } \Omega\times[0,T]. \tag{16e}$$

Let us comment on the motivation behind this definition. Given a vector field ξ with respect to a fixed strong solution (χ_v, v) as in the previous definition, we may introduce for any varifold solution (χ_u, u, V) and for all $t\in[0,T]$ a functional

$$E[\chi_u, V|\chi_v](t) := \sigma\int_{\overline{\Omega}} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} - \sigma\int_{I_u(t)} \frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}\cdot\xi(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{17}$$

where $I_u(t) := \mathrm{supp}|\nabla\chi_u(\cdot,t)|\cap\Omega$ denotes the interface associated to the varifold solution. The functional $E[\chi_u, V|\chi_v]$ is a measure for the interfacial error between the two solutions for the following reasons. First of all, it is a consequence of the definition of a varifold solution, cf. the compatibility condition (42), that for almost every $t\in[0,T]$ it holds $|\nabla\chi_u(\cdot,t)|\llcorner\Omega \le |V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega$ in the sense of measures on Ω. In particular, it follows that the functional $E[\chi_u, V|\chi_v]$ controls its "BV-analogue"

$$0 \le E[\chi_u|\chi_v](t) := \sigma\int_{I_u(t)} 1 - \frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}\cdot\xi(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1} \le E[\chi_u, V|\chi_v](t). \tag{18}$$

Introducing the Radon–Nikodým derivative $\theta_t := \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|\llcorner\Omega}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega}$, one can be even more precise in the sense that

$$E[\chi_u, V|\chi_v](t) = \sigma\int_{\partial\Omega} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} + \sigma\int_{\Omega} 1-\theta_t\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} + E[\chi_u|\chi_v](t). \tag{19}$$

This representation of the functional $E[\chi_u, V|\chi_v]$ as well as the length constraint (16a) for the vector field ξ lead to the following two observations. First, the functional $E[\chi_u, V|\chi_v]$ controls the mass of hidden boundaries and higher multiplicity interfaces (i.e., where $\theta_t\in[0,1)$) in the sense of

$$\sigma\int_{\partial\Omega} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} + \sigma\int_{\Omega} 1-\theta_t\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \le E[\chi_u, V|\chi_v](t). \tag{20}$$

Second, because of (16a) it measures the interface error in the sense that

$$\sigma\int_{I_u(t)} \frac{1}{2}\left|\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|} - \xi\right|^2 \mathrm{d}\mathcal{H}^{d-1} \le E[\chi_u|\chi_v](t), \tag{21}$$

$$\sigma\int_{I_u(t)} \min\left\{1, C\,\mathrm{dist}^2(\cdot, I_v(t))\right\}\mathrm{d}\mathcal{H}^{d-1} \le E[\chi_u|\chi_v](t). \tag{22}$$

On a different note, the compatibility condition (42) satisfied by a varifold solution together with the boundary condition (16b) also allows to represent the error functional $E[\chi_u, V|\chi_v]$ in the alternative form

$$E[\chi_u, V|\chi_v](t) = \sigma\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} 1 - s\cdot\xi\,\mathrm{d}V_t, \tag{23}$$

which then entails as a consequence of (16a)

$$\sigma\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \frac{1}{2}|s-\xi|^2\,\mathrm{d}V_t \le E[\chi_u, V|\chi_v](t), \tag{24}$$

$$\sigma\int_{\overline{\Omega}} \min\left\{1, C\,\mathrm{dist}^2(\cdot, I_v(t))\right\}\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \le E[\chi_u, V|\chi_v](t). \tag{25}$$

Finally, let us quickly discuss what is implied by $E[\chi_u, V|\chi_v](t) = 0$. We claim that (14) and $I_u(t)\subset I_v(t)$ up to $\mathcal{H}^{d-1}$-negligible sets have to be satisfied. Indeed, the latter follows directly from (18) and (22). The former is best seen when representing the varifold $V_t\llcorner(\Omega\times\mathbb{S}^{d-1})$ by its disintegration $(|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega)\otimes(\nu_{x,t})_{x\in\Omega}$. Then, it follows on one side from (20) that $|V_t|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega = 0$ and $|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega = |\nabla\chi_u(\cdot,t)|\llcorner\Omega$ as measures on $\partial\Omega$ and Ω, respectively, and then on the other side that $\nu_{x,t} = \delta_{\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)}$ for $|\nabla\chi_u(\cdot,t)|$-a.e. $x\in\Omega$ due to

$$\int_\Omega\int_{\mathbb{S}^{d-1}} \frac{1}{2}\left|s - \frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)\right|^2 \mathrm{d}\nu_{x,t}(s)\,\mathrm{d}(|\nabla\chi_u(\cdot,t)|\llcorner\Omega)(x)$$

$$= \int_\Omega \int_{\mathbb{S}^{d-1}} 1 - s \cdot \frac{\nabla \chi_u(\cdot,t)}{|\nabla \chi_u(\cdot,t)|}(x) \,\mathrm{d}\nu_{x,t}(s) \,\mathrm{d}(|\nabla \chi_u(\cdot,t)|\llcorner\Omega)(x) = 0,$$

where for the last equality we simply plugged in the compatibility condition (42) and again $|V_t|_{\mathbb{S}^{d-1}}\llcorner\partial\Omega = 0$ as well as $|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega = |\nabla\chi_u(\cdot,t)|\llcorner\Omega$.

Apart from these coercivity conditions, it is equally important to be able to estimate the time evolution of the error functional $E[\chi_u, V|\chi_v]$. The main observation in this regard is that the functional can be rewritten as a perturbation of the interface energy $E[\chi_u, V](t) := \sigma \int_{\overline{\Omega}} 1 \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}$ which is linear in the dependence on the indicator function χ_u. Indeed, thanks to the boundary condition (16b) for the extension ξ, a simple integration by parts readily reveals

$$E[\chi_u, V|\chi_v](t) = E[\chi_u, V](t) + \sigma \int_\Omega \chi_u(\cdot,t)(\nabla\cdot\xi)(\cdot,t)\,\mathrm{d}x. \tag{26}$$

This structure is in fact the very reason why we call $E[\chi_u, V|\chi_v]$ a relative entropy. Computing the time evolution of $E[\chi_u, V|\chi_v]$ then only requires to exploit the dissipation of energy and using $\nabla\cdot\xi$ as a test function in the evolution equation of the phase indicator χ_u of the varifold solution. The latter in turn requires knowledge on the time evolution of ξ itself, which is encoded in terms of the fluid velocity v through the Eqs. (16d) and (16e). The condition (16c) is natural in view of the interpretation of ξ as an extension of the unit normal n_{I_v} away from the interface I_v.

Even though all of this may already be quite promising, there is one small caveat: obviously, one can not deduce from $E[\chi_u, V|\chi_v] = 0$ that $\chi_u = \chi_v$ (e.g., χ_u representing an empty phase is consistent with having vanishing relative entropy). This lack of coercivity in the regime of vanishing interface measure motivates to introduce a second error functional which directly controls the deviation of χ_u from χ_v. The main input to such a functional is captured in the following definition.

Definition 3 *(Transported weight).* Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let $T \in (0,\infty)$ be a finite time horizon, consider a solenoidal vector field $v \in L^2([0,T]; H^1(\Omega;\mathbb{R}^d))$ with $(v\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, and let $(\Omega_v^+(t))_{t\in[0,T]}$ be a family of sets of finite perimeter in Ω. Denote by $I_v(t)$, $t\in[0,T]$, the reduced boundary of $\Omega_v^+(t)$ in Ω. Writing $\chi_v(\cdot,t)$ for the indicator function associated to $\Omega_v^+(t)$, assume that $\partial_t\chi_v = -\nabla\cdot(\chi_v v)$ in a weak sense.

In this setting, we call a map $\vartheta\colon \overline{\Omega}\times[0,T]\to[-1,1]$ a *transported weight with respect to* (χ_v, v) if the following conditions are satisfied:

- (Regularity) It holds $\vartheta \in W^{1,\infty}_{x,t}(\Omega\times[0,T])$.
- (Coercivity) Throughout the essential interior of Ω_v^+ (relative to Ω) it holds $\vartheta < 0$, throughout the essential exterior of Ω_v^+ (relative to Ω) it holds $\vartheta > 0$, and along $I_v \cup \partial\Omega$ we have $\vartheta = 0$. There also exists $C > 0$ such that

$$\mathrm{dist}(\cdot,\partial\Omega)\wedge\mathrm{dist}(\cdot,I_v)\wedge 1 \le C|\vartheta| \quad \text{in } \Omega\times[0,T]. \tag{27}$$

- (Transport equation) There exists $C > 0$ such that

$$|\partial_t\vartheta + (v\cdot\nabla)\vartheta| \le C|\vartheta| \quad \text{in } \Omega\times[0,T]. \tag{28}$$

The merit of the previous two definitions is now the following result. It reduces the proof of Theorem 1 to the existence of a boundary adapted extension ξ of the interface unit normal and a transported weight ϑ with respect to a strong solution (χ_v, v), respectively.

Proposition 4 (Conditional weak-strong uniqueness principle). *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on a time interval $[0,T]$. Consider in addition a strong solution (χ_v, v) to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$.*

Assume there exists a boundary adapted extension ξ of the unit normal n_{I_v} as well as a transported weight ϑ with respect to (χ_v, v) in the sense of Definitions 2 and 3, respectively. Then the stability estimates (11) and (12) for the relative entropy functional (29) and the bulk error functional (31) are satisfied, *respectively. Moreover, if the initial data of the varifold solution and the strong solution coincide, we may conclude that*

$$\chi_u(\cdot,t) = \chi_v(\cdot,t), \quad u(\cdot,t) = v(\cdot,t) \qquad \text{a.e. in } \Omega \text{ for a.e. } t \in [0,T],$$
$$V_t = (|\nabla\chi_u(\cdot,t)|\llcorner\Omega) \otimes \big(\delta_{\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}(x)}\big)_{x\in\Omega} \qquad \text{for a.e. } t \in [0,T].$$

A proof of this conditional weak-strong uniqueness principle is presented in Sect. 3.3 below. We emphasize again that it is valid for $d \in \{2,3\}$. The key ingredient to the stability estimate (11) is the following relative entropy inequality. We refer to Sect. 3.1 for a proof.

Proposition 5 (Relative entropy inequality in case of a 90° contact angle). *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a smooth and bounded domain. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on a time interval $[0,T]$. In particular, let θ be the density $\theta_t := \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|\llcorner\Omega}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega}$ as defined in (43). Furthermore, let (χ_v, v) be a strong solution in the sense of Definition 10 on the same time interval $[0,T]$, and assume there exists a boundary adapted extension ξ of the interface unit normal n_{I_v} with respect to (χ_v, v) as in Definition 2.*

Then, the total relative entropy defined by (recall the definition (17) of the interface contribution $E[\chi_u, V|\chi_v]$)

$$E[\chi_u,u,V|\chi_v,v](t) := \int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,t))|u(\cdot,t)-v(\cdot,t)|^2\,\mathrm{d}x + E[\chi_u,V|\chi_v](t) \tag{29}$$

satisfies the relative entropy inequality

$$\begin{aligned} &E[\chi_u,u,V|\chi_v,v](T') + \int_0^{T'}\int_\Omega \frac{\mu}{2}|\nabla(u-v)+\nabla(u-v)^{\mathsf{T}}|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\leq E[\chi_u,u,V|\chi_v,v](0) + R_{dt} + R_{adv} + R_{surTen}, \end{aligned} \tag{30}$$

for almost every $T' \in [0,T]$, where we made use of the abbreviations (denote by $n_u := \frac{\nabla\chi_u}{|\nabla\chi_u|}$ the measure-theoretic unit normal)

$$\begin{aligned} R_{dt} &= -\int_0^{T'}\int_\Omega (\rho(\chi_v)-\rho(\chi_u))(u-v)\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t, \\ R_{adv} &= -\int_0^{T'}\int_\Omega (\rho(\chi_u)-\rho(\chi_v))(u-v)\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad -\int_0^{T'}\int_\Omega \rho(\chi_u)(u-v)\cdot((u-v)\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

as well as

$$\begin{aligned} R_{surTen} = &-\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_\Omega (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_\Omega (\chi_u-\chi_v)((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &-\sigma\int_0^{T'}\int_\Omega (n_u-\xi)\cdot(\partial_t\xi+(v\cdot\nabla)\xi+(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &-\sigma\int_0^{T'}\int_\Omega ((n_u-\xi)\cdot\xi)(\xi\otimes\xi:\nabla v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \end{aligned}$$

$$- \sigma \int_0^{T'} \int_\Omega \Big(\partial_t \frac{1}{2}|\xi|^2 + (v\cdot\nabla)\frac{1}{2}|\xi|^2 \Big) \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t$$
$$+ \sigma \int_0^{T'} \int_\Omega (1 - n_u\cdot\xi)(\nabla\cdot v) \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t.$$

The stability estimate (12) for the bulk error functional is in turn based on the following auxiliary result; see Sect. 3.2 for a proof.

Lemma 6 (Time evolution of the bulk error). *Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a smooth and bounded domain. Let $T \in (0,\infty)$ be a finite time horizon, and let (χ_v, v) be as in Definition 3 of a transported weight. Let (χ_u, u, V) be a varifold solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 11 on $[0,T]$. Assume there exists a transported weight ϑ with respect to (χ_v, v) in the sense of Definition 3, and define the bulk error functional*

$$E_{\mathrm{vol}}[\chi_u|\chi_v](t) := \int_\Omega |\chi_u(\cdot,t) - \chi_v(\cdot,t)||\vartheta(\cdot,t)| \,\mathrm{d}x. \tag{31}$$

Then the following identity holds true for almost every $T' \in [0,T]$

$$\begin{aligned} E_{\mathrm{vol}}[\chi_u|\chi_v](T') = E_{\mathrm{vol}}[\chi_u|\chi_v](0) + \int_0^{T'} \int_\Omega (\chi_u - \chi_v)(\partial_t\vartheta + (v\cdot\nabla)\vartheta) \,\mathrm{d}x \,\mathrm{d}t \\ + \int_0^{T'} \int_\Omega (\chi_u - \chi_v)\big((u - v)\cdot\nabla\big)\vartheta \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{32}$$

2.3. Existence of Boundary Adapted Extensions of the Interface Unit Normal and Transported Weights in Planar Case

To upgrade the conditional weak-strong uniqueness principle of Proposition 4 to the statement of Theorem 1, it remains to construct a boundary adapted extension ξ of n_{I_v} and a transported weight ϑ associated to a given strong solution (χ_v, v). In the context of the present work, we perform this task for simplicity in the planar regime $d = 2$. However, it is expected that the principles of the construction carry over to the case $d = 3$ involving contact lines.

Proposition 7. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Then there exists a boundary adapted extension ξ of n_{I_v} w.r.t. (χ_v, v) in the sense of Definition 2.*

A proof of this result is presented in Sect. 6.2 below. One major step in the proof consists of reducing the global construction to certain local constructions being supported in the bulk Ω or in the vicinity of contact points along $\partial\Omega$, respectively. The main ingredients for this reduction argument are provided in Sect. 6.1. The construction of suitable local vector fields subject to conditions as in Definition 2 is in turn relegated to Sect. 4 (bulk construction) and Sect. 5 (construction near contact points). We finally provide the construction of a transported weight in Sect. 7.

Lemma 8. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Then there exists a transported weight ϑ w.r.t. (χ_v, v) in the sense of Definition 3.*

2.4. Definition of Varifold and Strong Solutions

In this subsection, we present definitions of strong and varifold solutions for the free-boundary problem of the evolution of two immiscible, incompressible, viscous fluids separated by a sharp interface with surface tension inside a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, with smooth and orientable boundary. Recall in this context that we restrict ourselves to the case of a $90°$ contact angle between the interface and the boundary of the domain Ω. In order to define a notion of strong solutions, we first introduce the notion of a smoothly evolving domain within Ω.

Definition 9 *(Smoothly evolving domains and smoothly evolving interfaces with* $90°$ *contact angle).* Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let $T \in (0,\infty)$ be a finite time horizon. Consider an open subset $\Omega_0^+ \subset \Omega$ subject to the following regularity conditions:

- Denoting by I_0 the closure of $\partial\Omega_0^+ \cap \Omega$ in $\overline{\Omega}$, we require I_0 to be a $(d{-}1)$-dimensional uniform C^3_x submanifold of $\overline{\Omega}$ with or without boundary. Moreover, I_0 is compact and consists of finitely many connected components.
- Interior points of I_0 are contained in Ω, whereas boundary points of I_0 are contained in $\partial\Omega$. In particular, $I_0 \cap \partial\Omega$ is a $(d{-}2)$-dimensional uniform C^3_x submanifold of $\partial\Omega$.
- Whenever I_0 intersects with $\partial\Omega$, it does so by forming an angle of $90°$.

Now, consider a set $\Omega^+ = \bigcup_{t\in[0,T]} \Omega^+(t)\times\{t\}$ represented in terms of open subsets $\Omega^+(t) \subset \Omega$ for all $t \in [0,T]$. Denote by $I(t)$ the closure of $\partial\Omega^+(t) \cap \Omega$ in $\overline{\Omega}$, $t \in [0,T]$. We call Ω^+ a *smoothly evolving domain in* Ω, and $I = \bigcup_{t\in[0,T]} I(t)\times\{t\}$ a *smoothly evolving interface with* $90°$ *contact angle*, if there exists a flow map $\psi\colon \overline{\Omega} \times [0,T] \to \overline{\Omega}$ such that the following requirements are satisfied:

- $\psi(\cdot,0) = \mathrm{Id}$. For any $t \in [0,T]$, the map $\psi_t := \psi(\cdot,t)\colon \overline{\Omega} \to \overline{\Omega}$ is a C^3_x diffeomorphism such that $\psi_t(\Omega) = \Omega$, $\psi_t(\partial\Omega) = \partial\Omega$ and $\sup_{t\in[0,T]} \|\psi_t\|_{W^{3,\infty}_x(\overline{\Omega})} < \infty$.
- For all $t \in [0,T]$, it holds $\Omega^+(t) = \psi_t(\Omega_0^+)$ and $I(t) = \psi_t(I_0)$.
- $\partial_t\psi \in C([0,T];C^1(\overline{\Omega}))$ such that $\sup_{t\in[0,T]} \|\partial_t\psi(\cdot,t)\|_{W^{1,\infty}_x(\overline{\Omega})} < \infty$.
- Whenever $I(t)$, $t \in [0,T]$, intersects $\partial\Omega$ it does so by forming an angle of $90°$.

With the geometric setup in place, we can proceed with our notion of strong solutions to two-phase Navier–Stokes flow with $90°$ contact angle.

Definition 10 *(Strong solution).* Let $d \in \{2,3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with orientable and smooth boundary. Let a surface tension constant $\sigma > 0$, the densities and shear viscosity of the two fluids $\rho^\pm, \mu > 0$, and a finite time $T_s > 0$ be given. Let χ_0 denote the indicator function of an open subset $\Omega_0^+ \subset \Omega$ subject to the conditions of Definition 9. Denoting the associated initial interface by $I_v(0)$, let a solenoidal initial velocity profile $v_0 \in L^2(\Omega;\mathbb{R}^d)$ be given such that it holds $v_0 \in C^2(\overline{\Omega} \setminus I_v(0))$. (Of course, additional compatibility conditions in terms of an initial pressure p_0 have to be satisfied by v_0 to allow for the below required regularity of the solution.)

A pair (χ_v, v) consisting of a velocity field $v\colon \overline{\Omega} \times [0,T_s) \to \mathbb{R}^d$ and an indicator function $\chi_v\colon \overline{\Omega} \times [0,T_s) \to \{0,1\}$ is called a *strong solution to the free boundary problem for the Navier–Stokes equation for two fluids with* $90°$ *contact angle and initial data* (χ_0, v_0) if for all $T \in (0,T_s)$ it is a *strong solution on* $[0,T]$ in the following sense:

- It holds
$$
\begin{aligned}
&v \in W^{1,\infty}([0,T];W^{1,\infty}(\Omega;\mathbb{R}^d)),\\
&\nabla v \in L^1([0,T];\mathrm{BV}(\Omega;\mathbb{R}^{d\times d})),\\
&\chi_v \in L^\infty([0,T];\mathrm{BV}(\Omega;\{0,1\})).
\end{aligned}
$$
- Define $\Omega_v^+(t) := \{x \in \Omega : \chi_v(x,t) = 1\}$. Then, $\Omega_v^+ = \bigcup_{t\in[0,T]} \Omega_v^+(t)\times\{t\}$ is a smoothly evolving domain in Ω in the sense of Definition 9 with $\Omega_v^+(0) = \Omega_0^+$. Denoting by $I_v(t)$ the closure of $\partial\Omega_v^+(t) \cap \Omega$ in $\overline{\Omega}$ for all $t \in [0,T]$, the set $I_v = \bigcup_{t\in[0,T]} I_v(t)\times\{t\}$ is a smoothly evolving interface

with 90° contact angle in the sense of Definition 9. In particular, for every $t \in [0,T]$ and every contact point $c(t) \in I_v(t) \cap \partial\Omega$

$$n_{\partial\Omega}(c(t)) \cdot n_{I_v}(c(t),t) = 0. \tag{33}$$

Moreover, for every $t \in [0,T]$ and every $c(t) \in I_v(t) \cap \partial\Omega$ the following higher-order compatibility condition is required to hold:

$$-\big((n_{\partial\Omega} \cdot \nabla)(n_{I_v} \cdot v)\big)(c(t),t) = H_{\partial\Omega}(c(t))(n_{I_v} \cdot v)(c(t),t), \tag{34}$$

where $H_{\partial\Omega}$ denotes the scalar mean curvature of $\partial\Omega$ (with respect to the inward pointing unit normal $n_{\partial\Omega}$).

- The velocity field v has vanishing divergence $\nabla \cdot v = 0$, and it satisfies the boundary conditions

$$v(\cdot,t) \cdot n_{\partial\Omega} = 0 \qquad \text{along } \partial\Omega, \tag{35}$$

$$\big(n_{\partial\Omega} \cdot \mu(\nabla v + \nabla v^{\mathsf{T}})(\cdot,t)B\big) = 0 \qquad \text{along } \partial\Omega \tag{36}$$

for all $t \in [0,T]$ and all tangential vector fields B along $\partial\Omega$.

Moreover, the equation for the momentum balance

$$\begin{aligned} &\int_\Omega \rho(\chi_v(\cdot,T'))v(\cdot,T') \cdot \eta(\cdot,T')\,\mathrm{d}x - \int_\Omega \rho(\chi_0))v_0 \cdot \eta(\cdot,0)\,\mathrm{d}x \\ &= \int_0^{T'}\int_\Omega \rho(\chi_v)v \cdot \partial_t\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)v \otimes v : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^{T'}\int_\Omega \mu(\nabla v + \nabla v^{\mathsf{T}}) : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \sigma\int_0^{T'}\int_{I_v(t)} \mathrm{H}_{I_v} \cdot \eta\,\mathrm{d}S\,\mathrm{d}t \end{aligned} \tag{37}$$

holds true for almost every $T' \in [0,T]$ and every $\eta \in C^\infty(\overline{\Omega} \times [0,T];\mathbb{R}^d)$ such that $\nabla \cdot \eta = 0$ as well as $(\eta \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. Here, $\mathrm{H}_{I_v}(\cdot,t)$ denotes the mean curvature vector of the interface $I_v(t)$. For the sake of brevity, we have used the abbreviation $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$.

- The indicator function χ_v is transported by the fluid velocity v in form of

$$\int_\Omega \chi_v(\cdot,T')\varphi(\cdot,T')\,\mathrm{d}x - \int_\Omega \chi_0\varphi(\cdot,0)\,\mathrm{d}x = \int_0^{T'}\int_\Omega \chi_v(\partial_t\varphi + (v \cdot \nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{38}$$

for almost every $T' \in [0,T]$ and all $\varphi \in C^\infty(\overline{\Omega} \times [0,T])$.

- It holds $v \in C_t^1C_x^0(\overline{\Omega\times[0,T]} \setminus I_v) \cap C_t^0C_x^2(\overline{\Omega\times[0,T]} \setminus I_v)$.

Short-time existence of strong solutions in the precise sense of the previous definition may in principle be established based on the results of Wilke [34] resp. Watanabe [32], which in turn are based on a maximal L_x^p-L_t^p resp. L_x^q-L_t^p regularity approach (cf. [27], [22] and [26] for further maximal L_x^q-L_t^p regularity results in the context of two-phase Navier–Stokes flow with surface tension). In these works, the evolving interface is represented in terms of the graph of a time-dependent height function over the initial interface, whereas the evolving phase of one of the fluids is represented in terms of the associated Hanzawa transform.

However, it has to be said that the results of [34] and [32] are not immediately sufficient to guarantee the required higher regularity of the interface and the fluid velocity from Definition 10 (in particular, the regularity up to time $t = 0$). One may expect that this higher regularity can be derived along the lines of [12, Remark 7, Remark 36, and Remark 37], where for our purposes next to the higher regularity of the fluid velocity from each side of the evolving interface one also has to provide similar arguments near the domain boundary. Needless to say, one has to be particularly careful in the vicinity of contact points or contact lines, for which our mathematically idealized setting of pure slip and constant ninety degree contact angle may prove beneficial (cf. the discussion in [24] or [14]). In summary, a detailed proof of the required higher regularity is certainly worth a paper on its own and thus out of the scope of this article.

We conclude the discussion on strong solutions with a series of remarks. First, by standard arguments one may deduce from (38), the solenoidality of v, and the boundary condition $(v \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$ that

$V_{I_v} = v \cdot n_{I_v}$ holds true along the interface I_v for the normal speed V_{I_v} of I_v (oriented with respect to n_{I_v}). Second, as a consequence of the contact point condition (33) it holds for all $t \in [0, T_s)$

$$\int_{I_v(t)} \mathrm{H}_{I_v} \cdot \eta \, \mathrm{d}S = -\int_{I_v(t)} \big(\mathrm{Id} - n_{I_v}(\cdot,t) \otimes n_{I_v}(\cdot,t)\big) : \nabla\eta \, \mathrm{d}S$$

for all test fields $\eta \in C^\infty(\overline{\Omega}; \mathbb{R}^d)$ subject to $\nabla \cdot \eta = 0$ and $(\eta \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. Third, note that Definition 10 implies that all pairs of two distinct contact points at the initial time remain distinct at all later times within a finite time horizon. This in fact is a consequence of the regularity of the velocity field and the evolving interface. Indeed, denoting by $t \mapsto c(t) \in I_v(t) \cap \partial\Omega$ resp. $t \mapsto \hat{c}(t) \in I_v(t) \cap \partial\Omega$ the trajectories of two distinct contact points, we may estimate the time evolution of their squared distance $\alpha(t) := \frac{1}{2}|c(t) - \hat{c}(t)|^2$ by means of

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \big(c(t) - \hat{c}(t)\big) \cdot \big(v(c(t),t) - v(\hat{c}(t),t)\big) \geq -2\|\nabla v\|_{L^\infty_{x,t}} \alpha(t).$$

Using Gronwall's Lemma, we can conclude that $\alpha(t) \geq \alpha(0)\exp(-2\|\nabla v\|_{L^\infty_{x,t}} t)$.

Fourth, we remark that it actually suffices to require the compatibility conditions (33) and (34) at the initial time $t = 0$ only. For later times $t \in (0, T]$, they are in fact consequences of the regularity of a strong solution, which can be seen as follows. For the sake fo simplicity, consider the case $d = 2$. By means of the chain rule, the fact that $v \cdot n_{\partial\Omega} = 0$ along $\partial\Omega$, and the formulas for $\nabla n_{\partial\Omega}$ and $\nabla \tau_{\partial\Omega}$ from Lemma 19, we may rewrite the boundary condition $(\mu(\nabla v + \nabla v^{\mathsf{T}}) : n_{\partial\Omega} \otimes \tau_{\partial\Omega}) = 0$ along $\partial\Omega$ as

$$H_{\partial\Omega}(v \cdot \tau_{\partial\Omega}) + (n_{\partial\Omega} \cdot \nabla)(v \cdot \tau_{\partial\Omega}) = 0 \quad \text{along } \partial\Omega,$$

which holds in particular at a contact point $c(t)$ for any $t \in [0, T]$. Then, since the quantities $|\tau_{\partial\Omega} \cdot \tau_{I_v}| = |n_{I_v} \cdot n_{\partial\Omega}|$, $|\tau_{\partial\Omega} - n_{I_v}|$, $|n_{\partial\Omega} + \tau_{I_v}|$ evaluated at a contact point can all be bounded from above by $\sqrt{1 - n_{I_v} \cdot \tau_{\partial\Omega}}$, we may compute by adding zeros (see also the formulas for $\nabla n_{\partial\Omega}$ and $\nabla \tau_{\partial\Omega}$ as well as the expressions for $\frac{\mathrm{d}}{\mathrm{d}t}\tau_{\partial\Omega}(c(t))$ and $\frac{\mathrm{d}}{\mathrm{d}t}n_{I_v}(c(t),t)$ from Lemmas 19 and 20, respectively)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[1 - n_{I_v}(c(t),t) \cdot \tau_{\partial\Omega}(c(t))] \\
&\quad = -\big((n_{I_v} \cdot n_{\partial\Omega})((n_{\partial\Omega} \cdot \nabla)(v \cdot \tau_{\partial\Omega}) + (\tau_{I_v} \cdot \nabla)(v \cdot n_{I_v}))\big)\big|_{(c(t),t)} \\
&\quad = -\big((n_{I_v} \cdot n_{\partial\Omega})(\nabla v : (\tau_{\partial\Omega} - n_{I_v}) \otimes n_{\partial\Omega} + \nabla v : n_{I_v} \otimes (n_{\partial\Omega} + \tau_{I_v}) \\
&\qquad\quad - H_{I_v}(v \cdot \tau_{I_v})(\tau_{\partial\Omega} \cdot \tau_{I_v}))\big)\big|_{(c(t),t)} \\
&\quad \leq C\|\nabla v\|_{L^\infty_{x,t}}[1 - n_{I_v}(c(t),t) \cdot \tau_{\partial\Omega}(c(t))]
\end{aligned}$$

for some $C > 0$ and any $t \in [0, T]$. From an application of a Gronwall-type argument and the validity of the contact angle condition (33) at the initial time $t = 0$, we may conclude that (33) is indeed satisfied for any $t \in [0, T]$. The compatibility condition (34) in turn follows from differentiating in time the angle condition (33) along a smooth trajectory $t \mapsto c(t) \in I_v(t) \cap \partial\Omega$ of a contact point, see for details the proof of Lemma 20.

We proceed with the notion of a varifold solution.

Definition 11 *(Varifold solution in case of* 90° *contact angle condition).* Let a surface tension constant $\sigma > 0$, the densities and shear viscosity of the two fluids $\rho^\pm, \mu > 0$, a finite time $T_w > 0$, a solenoidal initial velocity profile $u_0 \in L^2(\Omega; \mathbb{R}^d)$, and an indicator function $\chi_0 \in \mathrm{BV}(\Omega)$ be given.

A triple (χ_u, u, V) consisting of a velocity field u, an indicator function χ_u, and an oriented varifold V with

$$\begin{aligned}
&u \in L^2([0,T_w]; H^1(\Omega;\mathbb{R}^d)) \cap L^\infty([0,T_w]; L^2(\Omega;\mathbb{R}^d)), \\
&\chi_u \in L^\infty([0,T_w]; \mathrm{BV}(\Omega;\{0,1\})), \\
&V \in L^\infty_w([0,T_w]; \mathcal{M}(\overline{\Omega} \times \mathbb{S}^{d-1})),
\end{aligned}$$

is called a *varifold solution to the free boundary problem for the Navier-Stokes equation for two fluids with* 90° *contact angle and initial data* (χ_0, u_0) if the following conditions are satisfied:

- The velocity field u has vanishing divergence $\nabla\cdot u = 0$, its trace a vanishing normal component on the boundary of the domain $(u\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, and the equation for the momentum balance

$$\begin{aligned}&\int_\Omega \rho(\chi_u(\cdot,T))u(\cdot,T)\cdot\eta(\cdot,T)\,\mathrm{d}x - \int_\Omega \rho(\chi_0))u_0\cdot\eta(\cdot,0)\,\mathrm{d}x\\ &= \int_0^T\int_\Omega \rho(\chi_u)u\cdot\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \rho(\chi_u)u\otimes u : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_0^T\int_\Omega \mu(\nabla u + \nabla u^{\mathsf{T}}) : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \sigma\int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla\eta\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \end{aligned}\tag{39}$$

  is satisfied for almost every $T\in[0,T_w)$ and for every test vector field η subject to $\eta\in C^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$, $\nabla\cdot\eta = 0$ as well as $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$. We again made use of the abbreviation $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$.
- The indicator χ_u satisfies the weak formulation of the transport equation

$$\int_\Omega \chi_u(\cdot,T)\varphi(\cdot,T)\,\mathrm{d}x - \int_\Omega \chi_0\varphi(\cdot,0)\,\mathrm{d}x = \int_0^T\int_\Omega \chi_u(\partial_t\varphi + (u\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{40}$$

  for almost every $T\in[0,T_w)$ and all $\varphi\in C^\infty(\overline{\Omega}\times[0,T_w))$.
- The energy dissipation inequality

$$\begin{aligned}&\int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,T))|u(\cdot,T)|^2\,\mathrm{d}x + \sigma|V_T|_{\mathbb{S}^{d-1}}(\overline{\Omega}) + \int_0^T\int_\Omega \frac{\mu}{2}|\nabla u + \nabla u^{\mathsf{T}}|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \int_\Omega \frac{1}{2}\rho(\chi_0(\cdot))|u_0(\cdot)|^2\,\mathrm{d}x + \sigma|\nabla\chi_0|(\Omega)\end{aligned}\tag{41}$$

  is satisfied for almost every $T\in[0,T_w)$.
- The phase boundary $\partial^*\{\chi_u(\cdot,t)=0\}\cap\Omega$ and the varifold V_t satisfy the compatibility condition

$$\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \psi(x)\cdot s\,\mathrm{d}V_t(x,s) = \int_\Omega \psi(x)\cdot\,\mathrm{d}\nabla\chi_u(x,t) \tag{42}$$

  for almost every $t\in[0,T_w)$ and every smooth function $\psi\in C^\infty(\overline{\Omega};\mathbb{R}^d)$ such that $(\psi\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$.

Finally, if (χ_u, V) satisfy (14) we call the pair (χ_u, u) a BV *solution to the free boundary problem for the Navier-Stokes equation for two fluids with* 90° *contact angle and initial data* (χ_0, u_0).

We conclude with a remark concerning the notion of varifold solutions. Denote by $V_t\in\mathcal{M}(\overline{\Omega}\times\mathbb{S}^{d-1})$ the non-negative measure representing at time $t\in[0,T_w)$ the varifold associated to a varifold solution (χ_u,u,V). The compatibility condition (42) entails that $|\nabla\chi_u(\cdot,t)|\llcorner\Omega$ is absolutely continuous with respect to $|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega$; in fact, $|\nabla\chi_u(\cdot,t)|\llcorner\Omega \leq |V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega$ in the sense of measures on Ω. Hence, we may define the Radon–Nikodym derivative

$$\theta_t := \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|\llcorner\Omega}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega}, \tag{43}$$

which is a $(|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega)$-measurable function with $|\theta_t|\leq 1$ valid $(|V_t|_{\mathbb{S}^{d-1}}\llcorner\Omega)$-almost everywhere in Ω. In other words, the quantity $\frac{1}{\theta_t}$ represents the multiplicity of the varifold (in the interior). With this notation in place, it then holds

$$\int_\Omega f(x)\,\mathrm{d}|\nabla\chi_u(\cdot,t)|(x) = \int_\Omega \theta_t(x)f(x)\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x) \tag{44}$$

for every $f\in L^1(\Omega,|\nabla\chi_u(\cdot,t)|)$ and almost every $t\in[0,T_w)$.

2.5. Summary of Strategy

To summarize, the proof of our weak-strong uniqueness result (Theorem 1) is divided into two parts. The first part is concerned with the derivation of Gronwall stability estimates (cf. Proposition 4, Proposition 5 and Lemma 6) of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\chi_u,u,V|\chi_v,v] \leq C(E[\chi_u,u,V|\chi_v,v]+E_{\mathrm{vol}}[\chi_u|\chi_v]),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{vol}}[\chi_u|\chi_v] \leq C(E[\chi_u,u,V|\chi_v,v]+E_{\mathrm{vol}}[\chi_u|\chi_v]),$$

where $E[\chi_u,u,V|\chi_v,v]$ and $E_{\mathrm{vol}}[\chi_u|\chi_v]$ are two suitably constructed error functionals between a varifold (cf. Definition 11) and a strong solution (cf. Definition 10). The functional $E[\chi_u,u,V|\chi_v,v]$ has the form of a relative entropy and penalizes, amongst other things, the error in the two velocity fields (cf. (29)) and the error in the locations of the two interfaces (cf. (18)–(25)). The other functional $E_{\mathrm{vol}}[\chi_u|\chi_v]$ in turn directly controls the difference between the phase indicator functions of the respective first fluids of the two solutions (cf. (31)). These coercivity properties are not only sufficient to establish the above Gronwall estimates, but also that the two solutions have to coincide if both error functionals are zero.

The argument in the first part of the proof is conditional in the sense that both error functionals rely on suitable inputs which have to be constructed from the strong solution. More precisely, the interfacial part of the relative entropy $E[\chi_u,u,V|\chi_v,v]$ is defined in terms of a suitable extension ξ of the interface unit normal n_{I_v} of the strong solution (cf. Definition 2), whereas $E_{\mathrm{vol}}[\chi_u|\chi_v]$ is defined based on a suitable weight ϑ essentially representing a truncated signed distance function with respect to the phase of the first fluid of the strong solution (cf. Definition 3). Once these two inputs are rigorously constructed, one may infer our main result Theorem 1 from the corresponding conditional one of Proposition 4.

The second part of the proof therefore takes care of establishing the existence of such ξ and ϑ for strong solutions (cf. Proposition 7 and Lemma 8). In the following, we provide some comments on the construction of the former (which is the more challenging task). Away from the domain boundary, and therefore in particular away from contact points, one may simply follow the ansatz from [12] which is

$$\xi(x,t) := \eta_{I_v}(\mathrm{sdist}(x,I_v(t)))n_{I_v}(P_{I_v}(x,t),t), \tag{45}$$

where η_{I_v} is a quadratic cutoff localizing to the width of a regular tubular neighborhood of the interface $I_v(t)$, $\mathrm{sdist}(\cdot,I_v(t))$ denotes the signed distance to $I_v(t)$, and $P_{I_v}(\cdot,t)$ represents the nearest point projection onto $I_v(t)$.

Near contact points $\partial I_v(t)$, the above ansatz (45) requires a careful adaptation because one of the main requirements for ξ is to be tangential along the domain boundary: $(\xi\cdot n_{\partial\Omega})|_{\partial\Omega}\equiv 0$. To achieve this, it is first easiest to think about fixing the values of ξ along either the interface I_v or the domain boundary $\partial\Omega$:

$$\xi(x,t) := \eta_{\partial I_v(t)}(x,t)\widetilde{\xi}^{I_v}(x,t), \quad \widetilde{\xi}^{I_v}(x,t) = n_{I_v}(x,t) \qquad \text{along } I_v(t),$$
$$\xi(x,t) := \eta_{\partial I_v(t)}(x,t)\widetilde{\xi}^{\partial\Omega}(x,t), \quad \widetilde{\xi}^{\partial\Omega}(x,t) = \tau_{\partial\Omega}(x,t) \qquad \text{along } \partial\Omega,$$

where $\eta_{\partial I_v(t)}$ is a quadratic cutoff localizing to a neighborhood of the contact points $\partial I_v(t)$, and where $\tau_{\partial\Omega}(\cdot,t)$ is a tangent vector field along $\partial\Omega$ extending locally for each contact point $c\in\partial I_v(t)$ the normal $n_{I_v}(c,t)$. Due to the 90° degree contact angle condition, this is indeed meaningful and guarantees continuity of ξ along $I_v(t)\cup\partial\Omega$.

Now, in order to define ξ in a full neighborhood of the contact points $\partial I_v(t)$, the basic idea is to interpolate between the two auxiliary fields $\widetilde{\xi}^{I_v}$ and $\widetilde{\xi}^{\partial\Omega}$. However, some care has to be taken here due to the required regularity of ξ. This is the reason why we employ an expansion ansatz for both $\widetilde{\xi}^{I_v}$ and $\widetilde{\xi}^{\partial\Omega}$ of the structure

$$\widetilde{\xi}^{I_v} := n_{I_v} + \alpha_{I_v}\,\mathrm{sdist}(\cdot,I_v)\tau_{I_v} - \frac{1}{2}\alpha_{I_v}^2\,\mathrm{sdist}^2(\cdot,I_v)n_{I_v},$$

$$\widetilde{\xi}^{\partial\Omega} := \tau_{\partial\Omega} + \alpha_{\partial\Omega}\,\mathrm{sdist}(\cdot,\partial\Omega)n_{\partial\Omega} - \frac{1}{2}\alpha_{\partial\Omega}^2\,\mathrm{sdist}^2(\cdot,\partial\Omega)\tau_{\partial\Omega},$$

where the normal-tangent frames (n_{I_v},τ_{I_v}) and $(n_{\partial\Omega},\tau_{\partial\Omega})$ as well as the coefficients α_{I_v} and $\alpha_{\partial\Omega}$ are extended constantly in the respective normal directions. The point then is to choose the coefficients in a suitable way such that $\nabla\widetilde{\xi}^{I_v}$ and $\nabla\widetilde{\xi}^{\partial\Omega}$ agree at contact points. With this in place, one may then interpolate between the two constructions so that the resulting vector field ξ satisfies the required regularity. The second-order terms in the above expansions are only needed for a length correction of the first-order perturbations. We finally remark that controlling the time evolution of the interpolation construction requires the higher-order compatibility condition at contact points following from differentiating in time the 90° contact angle condition.

With the constructions of suitable candidates for ξ in place, one technical problem remains. Namely, the domains of definition for the above two outlined constructions away and near contact points overlap. The solution for this technicality consists of carefully designing the quadratic cutoff functions η_{I_v} and $\eta_{\partial I_v}$ so that they form on one side a partition of unity along the interface of the strong solution, and that they on the other side get transported along the fluid flow. Once this is established, the construction of ξ is finished.

In terms of organization, the remaining parts of the paper are structured as follows. The first part of the proof as outlined above is carried out in Sect. 3. The construction of the vector field ξ, which is the main step of the second part of the proof, is distributed across Sect. 4 (construction away from contact points), Sect. 5 (construction near contact points) and Sect. 6 (global construction by partition of unity). We conclude the paper with the construction of the weight ϑ in Sect. 7.

2.6. Notation

Throughout the present work, we employ the notational conventions of [12]. A notable addition is the following convention. If $D \subset \mathbb{R}^d$ is an open subset and $\Gamma \subset D$ a closed subset of Hausdorff-dimension $k \in \{0,\ldots,d{-}1\}$, we write $C^k(\overline{D}\setminus\Gamma)$ for all maps $f\colon D \to \mathbb{R}$ which are k-times continuously differentiable throughout $D\setminus\Gamma$ such that the function together with all its derivatives stays bounded throughout $D\setminus\Gamma$. Analogously, one defines the space $C^k_tC^m_x(\overline{D}\setminus\Gamma)$ for $D = \bigcup_{t\in[0,T]} D(t)\times\{t\}$ and $\Gamma = \bigcup_{t\in[0,T]}\Gamma(t)\times\{t\}$, where $(D(t))_{t\in[0,T]}$ is a family of open subsets of $\mathbb{R}^d$ and $(\Gamma(t))_{t\in[0,T]}$ is a family of closed subsets $\Gamma(t) \subset D(t)$ of constant Hausdorff-dimension $k \in \{0,\ldots,d{-}1\}$.

3. Proof of Main Results

3.1. Relative Entropy Inequality: Proof of Proposition 5

The general structure of the proof is in parts similar to the proof of [12, Proposition 10]. In what follows, we thus mainly focus on how to exploit the boundary conditions for the velocity fields (u,v) and a boundary adapted extension ξ of the strong interface unit normal in these computations.

Step 1: Since $\rho(\chi_v)$ is an affine function of χ_v, it consequently satisfies

$$\int_\Omega \rho(\chi_v(\cdot,T'))\varphi(\cdot,T')\,\mathrm{d}x - \int_\Omega \rho(\chi_v^0)\varphi(\cdot,0)\,\mathrm{d}x = \int_0^T\int_\Omega \rho(\chi_v)(\partial_t\varphi + (v\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{46}$$

for almost every $T' \in [0,T]$ and all $\varphi \in C^\infty(\overline{\Omega}\times[0,T])$. By the regularity of v and an approximation argument, we may test this equation with $v\cdot\eta$ for any $\eta \in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$, yielding

$$\begin{aligned}&\int_\Omega \rho(\chi_v(\cdot,T'))v(\cdot,T')\cdot\eta(\cdot,T')\,\mathrm{d}x - \int_\Omega \rho(\chi_v^0)v(\cdot,0)\cdot\eta(\cdot,0)\,\mathrm{d}x\\ &= \int_0^{T'}\int_\Omega \rho(\chi_v)(v\cdot\partial_t\eta + \eta\cdot\partial_t v)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$+\int_0^{T'}\int_\Omega \rho(\chi_v)(\eta\cdot(v\cdot\nabla)v + v\cdot(v\cdot\nabla)\eta)\,\mathrm{d}x\,\mathrm{d}t \tag{47}$$

for almost every $T'\in[0,T]$. Next, we subtract from (47) the equation for the momentum balance (37) of the strong solution. It follows that the velocity field v of the strong solution satisfies

$$\begin{aligned}0 &= \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^{T'}\int_\Omega \mu(\nabla v+\nabla v^T):\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_{I_v(t)} \mathrm{H}_{I_v}\cdot\eta\,\mathrm{d}S\,\mathrm{d}t\end{aligned} \tag{48}$$

for almost every $T'\in[0,T]$ and every test vector field $\eta\in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$ such that $\nabla\cdot\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$. For any such test vector field η, note that by means of (16c), the incompressibility of η as well as $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$, we may rewrite

$$\begin{aligned}-\sigma\int_0^{T'}\int_{I_v(t)} \mathrm{H}_{I_v}\cdot\eta\,\mathrm{d}S\,\mathrm{d}t &= \sigma\int_0^{T'}\int_{I_v(t)} (\nabla\cdot\xi)\eta\cdot n_{I_v}\,\mathrm{d}S\,\mathrm{d}t\\ &= -\sigma\int_0^{T'}\int_\Omega \chi_v(\eta\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{49}$$

Hence, we deduce from inserting (49) back into (48) that

$$\begin{aligned}0 &= \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^{T'}\int_\Omega \mu(\nabla v+\nabla v^T):\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_\Omega \chi_v(\eta\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{50}$$

for almost every $T'\in[0,T]$ and every test vector field $\eta\in C^\infty(\overline{\Omega}\times[0,T];\mathbb{R}^d)$ such that $\nabla\cdot\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$. The merit of rewriting (48) into the form (50) consists of the following observation. Consider a test vector field $\eta\in C^\infty([0,T];H^1(\Omega;\mathbb{R}^d))$ such that $\nabla\cdot\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$. Denoting by ψ a standard mollifier, for every $k\in\mathbb{N}$ by $\psi_k := k^d\psi(k\cdot)$ its usual rescaling, and by P_Ω the Helmholtz projection associated with the smooth domain Ω, it follows from standard theory (e.g., by a combination of [30] and standard $W^{m,2}(\Omega)$-elliptic regularity theory – see also Appendix A) that $\eta_k := P_\Omega(\psi_k * \eta)$ is an admissible test vector field for (50). Moreover, taking the limit $k\to\infty$ in (50) with η_k as test vector fields is admissible and results in

$$\begin{aligned}0 &= \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)\eta\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^{T'}\int_\Omega \mu(\nabla v+\nabla v^T):\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_\Omega \chi_v(\eta\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{51}$$

for almost every $T'\in[0,T]$ and every test vector field $\eta\in C^\infty([0,T];H^1(\Omega;\mathbb{R}^d))$ such that $\nabla\cdot\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$. As an important consequence, because of the boundary condition for the velocity fields (u,v) and their solenoidality, we may choose (after performing a mollification argument in the time variable) $\eta=u-v$ as a test function in (51) which entails for almost every $T'\in[0,T]$

$$\begin{aligned}0 &= \int_0^{T'}\int_\Omega \rho(\chi_v)(u-v)\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_v)(u-v)\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^{T'}\int_\Omega \mu(\nabla v+\nabla v^\mathsf{T}):\nabla(u-v)\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_\Omega \chi_v((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{52}$$

We proceed by testing the analogue of (46) for the phase-dependent density $\rho(\chi_u)$ with the test function $\frac{1}{2}|v|^2$, obtaining for almost every $T' \in [0,T]$

$$\int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,T'))|v(\cdot,T')|^2 \,\mathrm{d}x - \int_\Omega \frac{1}{2}\rho(\chi_u^0)|v_0(\cdot)|^2 \,\mathrm{d}x$$
$$= \int_0^{T'}\int_\Omega \rho(\chi_u)v\cdot\partial_t v \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int_\Omega \rho(\chi_u)v\cdot(u\cdot\nabla)v \,\mathrm{d}x\,\mathrm{d}t. \tag{53}$$

We next want to test (39) with the fluid velocity v. Modulo a mollification argument in the time variable, we have to argue that ∇v does not jump across the interface so that v is an admissible test function. Indeed, since the tangential derivative $(\tau_{I_v}\cdot\nabla)v$ is continuous across the interface it follows from $\nabla\cdot v = 0$ that also $n_{I_v}\cdot(n_{I_v}\cdot\nabla)v$ does not jump across I_v. The only component which may jump is thus $\tau_{I_v}\cdot(n_{I_v}\cdot\nabla)v$. However, this is ruled out by the equilibrium condition for the stresses along I_v together with having $\mu_+ = \mu_-$. In summary, using v in (39) implies

$$-\int_\Omega \rho(\chi_u(\cdot,T'))u(\cdot,T')\cdot v(\cdot,T')\,\mathrm{d}x + \int_\Omega \rho(\chi_u^0))u_0\cdot v_0(\cdot)\,\mathrm{d}x$$
$$-\int_0^{T'}\int_\Omega \mu(\nabla u + \nabla u^\mathsf{T}) : \nabla v \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^{T'}\int_\Omega \rho(\chi_u)u\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t - \int_0^{T'}\int_\Omega \rho(\chi_u)u\cdot(u\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id} - s\otimes s) : \nabla v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \tag{54}$$

for almost every $T' \in [0,T]$. We finally use $\sigma(\nabla\cdot\xi)$ as a test function in the transport equation (40) for the indicator function χ_u of the varifold solution. Hence, we obtain

$$\sigma\int_\Omega \chi_u(\cdot,T')(\nabla\cdot\xi)(\cdot,T')\,\mathrm{d}x - \int_\Omega \chi_u^0(\nabla\cdot\xi)(\cdot,0)\,\mathrm{d}x$$
$$= \sigma\int_0^{T'}\int_\Omega \chi_u(\nabla\cdot\partial_t\xi + (u\cdot\nabla)(\nabla\cdot\xi))\,\mathrm{d}x\,\mathrm{d}t.$$

for almost every $T' \in [0,T]$. Based on the boundary condition (16b), which in turn in particular implies $(\partial_t\xi\cdot n_{\partial\Omega})|_{\partial\Omega} = \partial_t(\xi\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, we may integrate by parts to upgrade the previous display to

$$-\sigma\int_\Omega n_u(\cdot,T')\cdot\xi(\cdot,T')\,\mathrm{d}|\nabla\chi_u(\cdot,T)| + \int_\Omega n_u^0\cdot\xi(\cdot,0)\,\mathrm{d}|\nabla\chi_u(\cdot,0)|$$
$$= -\sigma\int_0^{T'}\int_\Omega n_u\cdot\partial_t\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t + \sigma\int_0^{T'}\int_\Omega \chi_u(u\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \tag{55}$$

for almost every $T' \in [0,T]$.

Step 2: Summing (52), (53), (41) as well as (54), we obtain

$$LHS_{kin}(T') + LHS_{visc} + LHS_{surEn}(T')$$
$$\leq RHS_{kin}(0) + RHS_{surEn}(0) + RHS_{dt} + RHS_{adv} + RHS_{surTen}, \tag{56}$$

where the individual terms are given by (cf. the proof of [12, Proposition 10])

$$LHS_{kin}(T') := \int_\Omega \frac{1}{2}\rho(\chi_u(\cdot,T'))|u-v|^2(\cdot,T')\,\mathrm{d}x, \tag{57}$$

$$RHS_{kin}(0) := \int_\Omega \frac{1}{2}\rho(\chi_u^0)|u_0 - v_0|^2\,\mathrm{d}x, \tag{58}$$

$$LHS_{surEn}(T') := \sigma|\nabla\chi_u(\cdot,T')|(\Omega) + \sigma\int_\Omega (1-\theta_{T'})\,\mathrm{d}|V_{T'}|_{\mathbb{S}^{d-1}}(x), \tag{59}$$

$$RHS_{surEn}(0) := \sigma|\nabla\chi_u^0(\cdot)|(\Omega), \tag{60}$$

$$LHS_{visc} := \int_0^{T'}\int_\Omega \frac{\mu}{2}|\nabla(u-v)+\nabla(u-v)^{\mathsf{T}}|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{61}$$

$$RHS_{dt} := -\int_0^{T'}\int_\Omega (\rho(\chi_v)-\rho(\chi_u))(u-v)\cdot\partial_t v \,\mathrm{d}x\,\mathrm{d}t, \tag{62}$$

$$\begin{aligned} RHS_{adv} := &-\int_0^{T'}\int_\Omega (\rho(\chi_u)-\rho(\chi_v))(u-v)\cdot(v\cdot\nabla)v \,\mathrm{d}x\,\mathrm{d}t \\ &-\int_0^{T'}\int_\Omega \rho(\chi_u)(u-v)\cdot((u-v)\cdot\nabla)v \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{63}$$

$$\begin{aligned} RHS_{surTen} := &-\sigma\int_0^{T'}\int_\Omega \chi_v((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &+\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s):\nabla v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t. \end{aligned} \tag{64}$$

Adding zeros, $\nabla\cdot v = 0$, the boundary condition $n_{\partial\Omega}\cdot(\nabla v+(\nabla v)^{\mathsf{T}})\xi = n_{\partial\Omega}\cdot(\nabla v+(\nabla v)^{\mathsf{T}})(\mathrm{Id}-n_{\partial\Omega}\otimes n_{\partial\Omega})\xi = 0$ due to (36) and (16b), and the compatibility condition (42) allow to rewrite the second term of (64) as follows

$$\begin{aligned} &\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s):\nabla v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &= -\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &\quad -\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} s\cdot(\nabla v+(\nabla v)^{\mathsf{T}})\xi \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &\quad +\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &= -\sigma\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &\quad -\sigma\int_0^{T'}\int_\Omega \xi\cdot(n_u\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t - \sigma\int_0^{T'}\int_\Omega n_u\cdot(\xi\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &\quad +\sigma\int_0^{T'}\int_{\overline{\Omega}} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t. \end{aligned} \tag{65}$$

Furthermore, because of (44) we obtain

$$\begin{aligned} &\sigma\int_0^{T'}\int_{\overline{\Omega}} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &= \sigma\int_0^{T'}\int_\Omega (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t + \sigma\int_0^{T'}\int_\Omega \theta_t\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &\quad +\sigma\int_0^{T'}\int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &= \sigma\int_0^{T'}\int_\Omega (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t + \sigma\int_0^{T'}\int_\Omega \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &\quad +\sigma\int_0^{T'}\int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t. \end{aligned} \tag{66}$$

The combination of (64), (65) and (66) together with $\nabla \cdot v = 0$ then implies

$$\begin{aligned} RHS_{surTen} = &- \sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &+ \sigma \int_0^{T'} \int_{\Omega} (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+ \sigma \int_0^{T'} \int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} \chi_v((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} \xi\cdot((n_u-\xi)\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} (n_u-\xi)\cdot(\xi\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &+ \sigma \int_0^{T'} \int_{\Omega} (\mathrm{Id}-\xi\otimes\xi):\nabla v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t. \end{aligned} \tag{67}$$

In summary, plugging back (57)–(63) and (67) into (56), and then summing (55) to the resulting inequality yields in view of the definition (29) of the relative entropy

$$\begin{aligned} &E[\chi_u, u, V|\chi_v, v](T') + \int_0^{T'} \int_{\Omega} \frac{\mu}{2}|\nabla(u-v)+\nabla(u-v)^{\mathsf{T}}|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\le E[\chi_u, u, V|\chi_v, v](0) + R_{dt} + R_{adv} + R^{(1)}_{surTen} + R^{(2)}_{surTen} \end{aligned} \tag{68}$$

for almost every $T' \in [0,T]$, where in addition to the notation of Proposition 5 we also defined the two auxiliary quantities

$$\begin{aligned} R^{(1)}_{surTen} := &-\sigma \int_0^{T'} \int_{\overline{\Omega}\times\mathbb{S}^{d-1}} (s-\xi)\cdot((s-\xi)\cdot\nabla)v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &+ \sigma \int_0^{T'} \int_{\Omega} (1-\theta_t)\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+ \sigma \int_0^{T'} \int_{\partial\Omega} \xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t, \end{aligned} \tag{69}$$

$$\begin{aligned} R^{(2)}_{surTen} := &\,\sigma \int_0^{T} \int_{\Omega} \chi_u(u\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} \chi_v((u-v)\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} \xi\cdot((n_u-\xi)\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} (n_u-\xi)\cdot(\xi\cdot\nabla)v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &+ \sigma \int_0^{T'} \int_{\Omega} (\mathrm{Id}-\xi\otimes\xi):\nabla v \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &- \sigma \int_0^{T'} \int_{\Omega} n_u\cdot\partial_t\xi \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t. \end{aligned} \tag{70}$$

The remainder of the proof is concerned with the post-processing of the term $R^{(2)}_{surTen}$.

Step 3: By adding zeros, we can rewrite the last right hand side term of (70) as

$$\begin{aligned}
&-\sigma\int_0^{T'}\int_\Omega n_u\cdot\partial_t\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&=-\sigma\int_0^{T'}\int_\Omega (n_u-\xi)\cdot(\partial_t\xi+(v\cdot\nabla)\xi+(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad-\sigma\int_0^{T'}\int_\Omega ((n_u-\xi)\cdot\xi)(\xi\otimes\xi:\nabla v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad-\sigma\int_0^{T'}\int_\Omega \Big(\partial_t\frac{1}{2}|\xi|^2+(v\cdot\nabla)\frac{1}{2}|\xi|^2\Big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad+\sigma\int_0^{T'}\int_\Omega \xi\otimes(n_u-\xi):\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad+\sigma\int_0^{T'}\int_\Omega n_u\cdot((v\cdot\nabla)\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}\tag{71}$$

We proceed by manipulating the last term in the latter identity. To this end, we compute applying the product rule in the first step and then adding zero

$$\begin{aligned}
&\sigma\int_0^{T'}\int_\Omega n_u\cdot((v\cdot\nabla)\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&=\sigma\int_0^{T'}\int_\Omega n_u\cdot(\nabla\cdot(\xi\otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad+\sigma\int_0^{T'}\int_\Omega (1-n_u\cdot\xi)(\nabla\cdot v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t-\sigma\int_0^{T'}\int_\Omega \mathrm{Id}:\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}\tag{72}$$

Noting that for symmetry reasons $\nabla\cdot(\nabla\cdot(\xi\otimes v))=\nabla\cdot(\nabla\cdot(v\otimes\xi))$, an integration by parts based on the boundary conditions (16b) and $(v\cdot n_{\partial\Omega})|_{\partial\Omega}=0$ entails

$$\begin{aligned}
&\sigma\int_0^{T'}\int_\Omega n_u\cdot(\nabla\cdot(\xi\otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&=-\sigma\int_0^{T'}\int_\Omega \chi_u\nabla\cdot(\nabla\cdot(v\otimes\xi))\,\mathrm{d}x\,\mathrm{d}t-\sigma\int_0^{T'}\int_{\partial\Omega}\chi_u(n_{\partial\Omega}\otimes v:\nabla\xi)\,\mathrm{d}S\,\mathrm{d}t\\
&=\sigma\int_0^{T'}\int_\Omega n_u\cdot(\nabla\cdot(v\otimes\xi))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad+\sigma\int_0^{T'}\int_{\partial\Omega}\chi_u(n_{\partial\Omega}\cdot((\xi\cdot\nabla)v-(v\cdot\nabla)\xi))\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned}$$

We next observe that the last right hand side term of the previous display is zero. Indeed, note first that thanks to the boundary conditions (16b) and $(v\cdot n_{\partial\Omega})|_{\partial\Omega}=0$ the involved gradients are in fact tangential gradients along $\partial\Omega$. Since the tangential gradient of a function only depends on its definition along the manifold, we are free to substitute $(\xi\cdot\tau_{\partial\Omega})\tau_{\partial\Omega}$ for ξ resp. $(v\cdot\tau_{\partial\Omega})\tau_{\partial\Omega}$ for v, obtaining in the process

$$\begin{aligned}
&\int_0^{T'}\int_{\partial\Omega}\chi_u(n_{\partial\Omega}\cdot((\xi\cdot\nabla)v-(v\cdot\nabla)\xi))\,\mathrm{d}S\,\mathrm{d}t\\
&=\int_0^{T'}\int_{\partial\Omega}\chi_u[(\xi\cdot\nabla)(v\cdot\tau_{\partial\Omega})-(v\cdot\nabla)(\xi\cdot\tau_{\partial\Omega})](\tau_{\partial\Omega}\cdot n_{\partial\Omega})\,\mathrm{d}S\,\mathrm{d}t
\end{aligned}$$

$$+\int_0^{T'}\int_{\partial\Omega}\chi_u[((v\cdot\tau_{\partial\Omega})\xi-(\xi\cdot\tau_{\partial\Omega})v)\cdot\nabla)\tau_{\partial\Omega}]\cdot n_{\partial\Omega}\,\mathrm{d}S\,\mathrm{d}t=0.$$

The combination of the previous two displays together with an integration by parts and an application of the product rule thus yields

$$\begin{aligned}&\sigma\int_0^{T'}\int_\Omega n_u\cdot(\nabla\cdot(\xi\otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&=\sigma\int_0^{T'}\int_\Omega(n_u\cdot v)(\nabla\cdot\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t+\sigma\int_0^{T'}\int_\Omega n_u\otimes\xi:\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.\end{aligned}$$

By another integration by parts, relying in the process also on $\nabla\cdot v=0$ and $(v\cdot n_{\partial\Omega})|_{\partial\Omega}=0$, we may proceed computing

$$\begin{aligned}&\sigma\int_0^{T'}\int_\Omega n_u\cdot(\nabla\cdot(\xi\otimes v))\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&=-\sigma\int_0^{T'}\int_\Omega\chi_u\nabla\cdot(v(\nabla\cdot\xi))\,\mathrm{d}x\,\mathrm{d}t+\sigma\int_0^{T'}\int_\Omega n_u\otimes\xi:\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&=-\sigma\int_0^{T'}\int_\Omega\chi_u(v\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t+\sigma\int_0^{T'}\int_\Omega n_u\otimes\xi:\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.\end{aligned}\tag{73}$$

In summary, taking together (71)–(73) and adding for a last time zero yields

$$\begin{aligned}&-\sigma\int_0^{T'}\int_\Omega n_u\cdot\partial_t\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&=-\sigma\int_0^{T'}\int_\Omega\chi_u(v\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\sigma\int_0^{T'}\int_\Omega(n_u-\xi)\cdot(\partial_t\xi+(v\cdot\nabla)\xi+(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}\xi)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&\quad-\sigma\int_0^{T'}\int_\Omega((n_u-\xi)\cdot\xi)(\xi\otimes\xi:\nabla v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&\quad-\sigma\int_0^{T'}\int_\Omega\Big(\partial_t\frac{1}{2}|\xi|^2+(v\cdot\nabla)\frac{1}{2}|\xi|^2\Big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&\quad+\sigma\int_0^{T'}\int_\Omega(1-n_u\cdot\xi)(\nabla\cdot v)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&\quad+\sigma\int_0^{T'}\int_\Omega(n_u-\xi)\otimes\xi:\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t+\sigma\int_0^{T'}\int_\Omega\xi\otimes(n_u-\xi):\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\&\quad-\sigma\int_0^{T'}\int_\Omega(\mathrm{Id}-\xi\otimes\xi):\nabla v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.\end{aligned}\tag{74}$$

Inserting (74) into (70) then implies that $R^{(1)}_{surTen}+R^{(2)}_{surTen}$ combines to the desired term R_{surTen}. In particular, the estimate (68) upgrades to (30) as asserted. □

3.2. Time Evolution of the Bulk Error: Proof of Lemma 6

Note that the sign conditions for the transported weight ϑ, see Definition 3, ensure that

$$E_{\mathrm{vol}}[\chi_u|\chi_v](t)=\int_\Omega\big(\chi_u(\cdot,t)-\chi_v(\cdot,t)\big)\vartheta(\cdot,t)\,\mathrm{d}x$$

for all $t \in [0,T]$. Hence, as a consequence of the transport equations for χ_v and χ_u (see Definitions 10 and 11, respectively) one obtains

$$\begin{aligned} E_{\mathrm{vol}}[\chi_u|\chi_v](T') &= E_{\mathrm{vol}}[\chi_u|\chi_v](0) \\ &\quad + \int_0^{T'} \int_\Omega (\chi_u - \chi_v)\partial_t \vartheta \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'} \int_\Omega (\chi_u u - \chi_v v)\cdot\nabla\vartheta \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{75}$$

for almost every $T' \in [0,T]$. Note that for any sufficiently regular solenoidal vector field F with $(F \cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, since $\vartheta = 0$ along I_v (see Definition 3), an integration by parts yields

$$\int_\Omega \chi_v (F\cdot\nabla)\vartheta \,\mathrm{d}x = 0. \tag{76}$$

Adding zero in (75) and making use of (76) with respect to the choices $F = u$ and $F = v$ in form of $\int_\Omega \chi_v \big((u-v)\cdot\nabla\big)\vartheta \,\mathrm{d}x = 0$ then updates (75) to (32). This concludes the proof of Lemma 6. □

3.3. Conditional Weak-strong Uniqueness: Proof of Proposition 4

Starting point for a proof of the conditional weak-strong uniqueness principle is the following important coercivity estimate (cf. [12, Lemma 20]).

Lemma 12. *Let the assumptions and notation of Proposition 4 be in place. Then there exists a constant $C = C(\chi_v, v, T) > 0$ such that for all $\delta \in (0,1]$ it holds*

$$\begin{aligned} \int_0^{T'} \int_\Omega |\chi_v - \chi_u||u - v| \,\mathrm{d}x\,\mathrm{d}t &\le \frac{C}{\delta} \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t \\ &\quad + \delta \int_0^{T'} \int_\Omega |\nabla u - \nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{77}$$

for all $T' \in [0,T]$.

Proof. It turns out to be convenient to introduce a decomposition of the interface I_v into its topological features: the connected components of $I_v \cap \Omega$ and the connected components of $I_v \cap \partial\Omega$. Let $N \in \mathbb{N}$ denote the total number of such topological features of I_v, and split $\{1,\ldots,N\} =: \mathcal{I} \cup \mathcal{C}$ as follows. The subset $\mathcal{I}$ enumerates the space-time connected components of $I_v \cap \Omega$ (being time-evolving connected *interfaces*), whereas the subset $\mathcal{C}$ enumerates the space-time connected components of $I_v \cap \partial\Omega$ (being time-evolving *contact points* if $d = 2$, or time-evolving connected *contact lines* if $d = 3$). If $i \in \mathcal{I}$, we let $\mathcal{T}_i$ denote the space-time trajectory in Ω of the corresponding connected interface. Furthermore, for every $c \in \mathcal{C}$ we write $\mathcal{T}_c$ representing the space-time trajectory in $\partial\Omega$ of the corresponding contact point (if $d = 2$) or line (if $d = 3$). Finally, let us write $i \sim c$ for $i \in \mathcal{I}$ and $c \in \mathcal{C}$ if and only if $\mathcal{T}_i$ ends at $\mathcal{T}_c$. With this language and notation in place, the proof is now split into five steps.

Step 1: (Choice of a suitable localization scale) Denote by $n_{\partial\Omega}$ the unit normal vector field of $\partial\Omega$ pointing into Ω, and by $n_{I_v}(\cdot,t)$ the unit normal vector field of $I_v(t)$ pointing into $\Omega_v(t)$. Because of the uniform C^2_x regularity of the boundary $\partial\Omega$ and the uniform $C_tC^2_x$ regularity of the interface $I_v(t)$, $t \in [0,T]$, we may choose a scale $r \in (0, \frac{1}{2}]$ such that for all $t \in [0,T]$ and all $i \in \mathcal{I}$ the maps

$$\Psi_{\partial\Omega} \colon \partial\Omega \times (-3r, 3r) \to \mathbb{R}^d, \quad (x,y) \mapsto x + y n_{\partial\Omega}(x), \tag{78}$$

$$\Psi_{\mathcal{T}_i(t)} \colon \mathcal{T}_i(t) \times (-3r, 3r) \to \mathbb{R}^d, \quad (x,y) \mapsto x + y n_{I_v}(x,t) \tag{79}$$

are C^1 diffeomorphisms onto their image. By uniform regularity of $\partial\Omega$ and I_v (the latter in space-time), we have bounds

$$\sup_{\partial\Omega\times[-r,r]} |\nabla\Psi_{\partial\Omega}| \le C, \qquad \sup_{\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])} |\nabla\Psi^{-1}_{\partial\Omega}| \le C, \tag{80}$$

$$\sup_{t\in[0,T]}\sup_{\mathcal{T}_i(t)\times[-r,r]}|\nabla\Psi_{\mathcal{T}_i(t)}|\leq C,\quad \sup_{t\in[0,T]}\sup_{\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])}|\nabla\Psi^{-1}_{\mathcal{T}_i(t)}|\leq C \tag{81}$$

for all $i\in\mathcal{I}$. By possibly choosing $r\in(0,\frac{1}{2}]$ even smaller, we may also guarantee that for all $t\in[0,T]$ and all $i\in\mathcal{I}$ it holds

$$\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\cap\Psi_{\mathcal{T}_{i'}(t)}(\mathcal{T}_{i'}(t)\times[-r,r])=\emptyset \text{ for all } i'\in\mathcal{I},\ i'\neq i, \tag{82}$$

$$\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\cap\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\neq\emptyset\ \Leftrightarrow\ \exists c\in\mathcal{C}\colon i\sim c, \tag{83}$$

$$\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\cap\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\subset B_{2r}(\mathcal{T}_c(t)) \text{ if } \exists c\in\mathcal{C}\colon i\sim c \tag{84}$$

$$B_{2r}(\mathcal{T}_c(t))\cap B_{2r}(\mathcal{T}_{c'}(t))=\emptyset \text{ for all } c,c'\in\mathcal{C},\ c'\neq c. \tag{85}$$

Note finally that because of the 90° contact angle condition and by possibly choosing $r\in(0,\frac{1}{2}]$ even smaller, we can furthermore ensure that

$$\begin{aligned}&\Omega\setminus\Big(\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\cup\bigcup_{i\in\mathcal{I}}\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\Big)\\&\subset\Omega\cap\{x\in\mathbb{R}^d\colon\ \mathrm{dist}(x,\partial\Omega)\wedge\mathrm{dist}(x,I_v(t))>r\}\end{aligned} \tag{86}$$

for all $t\in[0,T]$. Indeed, for $x\in\Omega\setminus\big(\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\cup\bigcup_{i\in\mathcal{I}}\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\big)$ it follows that $\mathrm{dist}(x,\partial\Omega)>r$. In case the interface $I_v(t)$ intersects $\partial\Omega$ it may not be immediately clear that also $\mathrm{dist}(x,I_v(t))>r$ holds true. Assume there exists a point $x\in\Omega\setminus\big(\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\cup\bigcup_{i\in\mathcal{I}}\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\big)$ such that $\mathrm{dist}(x,I_v(t))\leq r$. Then necessarily $x\in(\Omega\cap B_r(c(t)))\setminus\bigcup_{i\in\mathcal{I}}\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])$ for some boundary point $c(t)\in\partial\Omega\cap I_v(t)$. Hence, because of the uniform C^2_x regularity of $\partial\Omega$ and $I_v(t)$ intersecting $\partial\Omega$ at an angle of 90°, one may choose $r\in(0,\frac{1}{2}]$ small enough such that $x\in(\Omega\cap B_r(c(t)))$ implies $\mathrm{dist}(x,\partial\Omega)\leq r$. As we have already seen, this contradicts $x\in\Omega\setminus\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])$.

Step 2: (A reduction argument) We may estimate by a union bound and (86)

$$\begin{aligned}&\int_0^{T'}\int_\Omega|\chi_v-\chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t\\&\leq\int_0^{T'}\int_{\Omega\cap\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\setminus\bigcup_{c\in\mathcal{C}}B_{2r}(\mathcal{T}_c(t))}|\chi_v-\chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\sum_{i\in\mathcal{I}}\int_0^{T'}\int_{\Omega\cap\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\setminus\bigcup_{c\in\mathcal{C}}B_{2r}(\mathcal{T}_c(t))}|\chi_v-\chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t\\&\quad+C\sum_{c\in\mathcal{C}}\int_0^{T'}\int_{\Omega\cap B_{2r}(\mathcal{T}_c(t))}|\chi_v-\chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^{T'}\int_{\Omega\cap\{\mathrm{dist}(\cdot,\partial\Omega)\wedge\mathrm{dist}(\cdot,I_v(t))>r\}}|\chi_v-\chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{87}$$

An application of Hölder's inequality and Young's inequality, the definition (29) of the relative entropy functional, the coercivity estimate (27) for the transported weight, and the definition (31) of the bulk error functional further imply

$$\begin{aligned}&\int_0^{T'}\int_{\Omega\cap\{\mathrm{dist}(\cdot,\partial\Omega)\wedge\mathrm{dist}(\cdot,I_v(t))>r\}}|\chi_v-\chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t\\&\leq C\int_0^{T'}\int_{\Omega\cap\{\mathrm{dist}(\cdot,\partial\Omega)\wedge\mathrm{dist}(\cdot,I_v(t))>r\}}|\chi_v-\chi_u|\,\mathrm{d}x\,\mathrm{d}t+C\int_0^{T'}E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t\\&\leq C\int_0^{T'}E[\chi_u,u,V|\chi_v,v](t)+E_{\mathrm{vol}}[\chi_u,\chi_v](t)\,\mathrm{d}t.\end{aligned}$$

Hence, it remains to estimate the first three terms on the right hand side of (87).

Step 3: (Estimate near the interface but away from contact points) First of all, because of the localization properties (82)–(84) it holds for all $i \in \mathcal{I}$

$$\mathrm{dist}(\cdot, \mathcal{T}_i) = \mathrm{dist}(\cdot, \partial\Omega) \wedge \mathrm{dist}(\cdot, I_v(t)) \tag{88}$$

in $\Omega \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r, r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$. Hence, the local interface error height as measured in the direction of n_{I_v} on $\mathcal{T}_i$

$$h_{\mathcal{T}_i}(x,t) := \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\mathcal{T}_i(t)}(x,y), t)\,\mathrm{d}y, \quad x \in \mathcal{T}_i(t),\ t \in [0,T],$$

is, because of (88) and the coercivity estimate (27) of the transported weight ϑ, subject to the estimate

$$\begin{aligned} h^2_{\mathcal{T}_i}(x,t) &\leq C \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\mathcal{T}_i(t)}(x,y), t) y\,\mathrm{d}y \\ &\leq C \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\mathcal{T}_i(t)}(x,y), t) |\vartheta|(\Psi_{\mathcal{T}_i(t)}(x,y), t)\,\mathrm{d}y \end{aligned} \tag{89}$$

for all $x \in \mathcal{T}_i(t) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$, all $t \in [0,T]$ and all $i \in \mathcal{I}$. Carrying out the slicing argument of the proof of [12, Lemma 20] in $\Omega \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t) \times [-r, r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$ by means of $\Psi_{\mathcal{T}_i(t)}$, which is indeed admissible thanks to (79), (81) and (89), shows that one obtains an estimate of required form

$$\begin{aligned} &\sum_{i\in\mathcal{I}} \int_0^{T'} \int_{\Omega\cap\Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times[-r,r])\setminus\bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\delta} \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t + \delta \int_0^{T'} \int_\Omega |\nabla u - \nabla v|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Step 4: (Estimate near the boundary of the domain but away from contact points) The argument is similar to the one of the previous step, with the only major difference being that the slicing argument of the proof of [12, Lemma 20] is now carried out in $\Omega \cap \Psi_{\partial\Omega}(\partial\Omega \times [-r, r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$ by means of $\Psi_{\partial\Omega}$. This in turn is facilitated by the following facts. First, the localization properties (82)–(84) ensure

$$\mathrm{dist}(\cdot, \partial\Omega) = \mathrm{dist}(\cdot, \partial\Omega) \wedge \mathrm{dist}(\cdot, I_v(t)) \tag{90}$$

in $\Omega \cap \Psi_{\partial\Omega}(\partial\Omega \times [-r, r]) \setminus \bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))$. Second, as a consequence of (90) and the coercivity estimate (27) of the transported weight ϑ, the local interface error height as measured in the direction of $n_{\partial\Omega}$

$$h_{\partial\Omega}(x,t) := \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\partial\Omega}(x,y), t)\,\mathrm{d}y, \quad x \in \partial\Omega,\ t \in [0,T],$$

satisfies the estimate

$$\begin{aligned} h^2_{\partial\Omega}(x,t) &\leq C \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\partial\Omega}(x,y), t) y\,\mathrm{d}y \\ &\leq C \int_{-r}^{r} |\chi_u - \chi_v|(\Psi_{\partial\Omega}(x,y), t) |\vartheta|(\Psi_{\partial\Omega}(x,y), t)\,\mathrm{d}y. \end{aligned} \tag{91}$$

Hence, we obtain

$$\begin{aligned} &\int_0^{T'} \int_{\Omega\cap\Psi_{\partial\Omega}(\partial\Omega\times[-r,r])\setminus\bigcup_{c\in\mathcal{C}} B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\delta} \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t + \delta \int_0^{T'} \int_\Omega |\nabla u - \nabla v|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Step 5: (Estimate near contact points) Fix $c \in \mathcal{C}$, and let $i \in \mathcal{I}$ denote the unique connected interface $\mathcal{T}_i$ such that $i \sim c$. Because of the regularity of $\partial\Omega$, the regularity of $\mathcal{T}_i$, and the 90° contact angle condition we may decompose the neighborhood $\Omega \cap B_{2r}(\mathcal{T}_c(t))$—by possibly reducing the localization scale $r \in (0, \frac{1}{2}]$ even further—into three pairwise disjoint open sets $W_{\partial\Omega}(t)$, $W_{\mathcal{T}_i}(t)$ and $W_{\partial\Omega\sim\mathcal{T}_i}(t)$ such that $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \setminus \big(W_{\partial\Omega}(t) \cup W_{\mathcal{T}_i}(t) \cup W_{\partial\Omega\sim\mathcal{T}_i}(t)\big)$ is an $\mathcal{H}^d$ null set and

$$\operatorname{dist}(\cdot,\partial\Omega) = \operatorname{dist}(\cdot,\partial\Omega) \wedge \operatorname{dist}(\cdot, I_v(t)) \qquad \text{in } W_{\partial\Omega}(t), \tag{92}$$

$$\operatorname{dist}(\cdot,\mathcal{T}_i(t)) = \operatorname{dist}(\cdot,\partial\Omega) \wedge \operatorname{dist}(\cdot, I_v(t)) \qquad \text{in } W_{\mathcal{T}_i}(t), \tag{93}$$

$$\operatorname{dist}(\cdot,\partial\Omega) \sim \operatorname{dist}(\cdot,\mathcal{T}_i(t)) \sim \operatorname{dist}(\cdot, I_v(t)) \qquad \text{in } W_{\partial\Omega\sim\mathcal{T}_i}(t), \tag{94}$$

as well as

$$W_{\partial\Omega}(t) \subset \Psi_{\partial\Omega}(\partial\Omega\times(-3r,3r)), \tag{95}$$

$$W_{\mathcal{T}_i}(t) \subset \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times(-3r,3r)), \tag{96}$$

$$W_{\partial\Omega\sim\mathcal{T}_i}(t) \subset \Psi_{\partial\Omega}(\partial\Omega\times(-3r,3r)) \cap \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t)\times(-3r,3r)). \tag{97}$$

(Up to a rigid motion, these sets can in fact be defined independent of $t \in [0,T]$.) Hence, applying the argument of *Step 3* based on (93) and (96) with respect to $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \cap W_{\mathcal{T}_i}(t)$, the argument of *Step 4* based on (92) and (95) with respect to $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \cap W_{\partial\Omega}(t)$, and either the argument of *Step 3* or *Step 4* based on (94) and (97) with respect to $\Omega \cap B_{2r}(\mathcal{T}_c(t)) \cap W_{\partial\Omega\sim\mathcal{T}_i}(t)$ entails

$$\begin{aligned}&\sum_{c\in\mathcal{C}} \int_0^{T'} \int_{\Omega\cap B_{2r}(\mathcal{T}_c(t))} |\chi_v - \chi_u||u-v| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \frac{C}{\delta}\int_0^{T'} E[\chi_u,u,V|\chi_v,v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t + \delta\int_0^{T'}\int_\Omega |\nabla u - \nabla v|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This in turn concludes the proof of Lemma 12. □

Proof of Proposition 4. The proof proceeds in three steps.

Step 1: (Post-processing the relative entropy inequality (30)*)* It follows immediately from the $L^\infty_{x,t}$-bound for $\partial_t v$ and $\rho(\chi_v) - \rho(\chi_u) = (\rho^+ - \rho^-)(\chi_v - \chi_u)$ that

$$|R_{dt}| \leq C\int_0^{T'}\int_\Omega |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t \tag{98}$$

for almost every $T' \in [0,T]$. Furthermore, the $L^\infty_t W^{1,\infty}_x$-bound for v, the definition (29) of the relative entropy functional, and again the identity $\rho(\chi_v) - \rho(\chi_u) = (\rho^+ - \rho^-)(\chi_v - \chi_u)$ imply that

$$|R_{adv}| \leq C\int_0^{T'}\int_\Omega |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t + C\int_0^{T'} E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t \tag{99}$$

for almost every $T' \in [0,T]$. For a bound on the interface contribution R_{surTen}, we rely on the $L^\infty_t W^{1,\infty}_x$-bound for v, the $L^\infty_t W^{2,\infty}_x$-bound for ξ, the $L^\infty_t W^{1,\infty}_x$-bound for B, the definition (29) of the relative entropy functional, as well as the estimates (16d) and (16e) of a boundary adapted extension ξ of n_{I_v} to the effect that

$$\begin{aligned}|R_{surTen}| \leq{}& C\int_0^{T'}\int_\Omega |\chi_v - \chi_u||u-v|\,\mathrm{d}x\,\mathrm{d}t \\ &+ C\int_0^{T'}\int_{\overline{\Omega}\times\mathbb{S}^{d-1}} |s-\xi|^2\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &+ C\int_0^{T'}\int_\Omega 1-\theta_t\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+ C\int_0^{T'}\int_{\partial\Omega} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t \\ &+ C\int_0^{T'}\int_\Omega |n_u - \xi|^2\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+ C \int_0^{T'} \int_\Omega \operatorname{dist}^2(\cdot, I_v) \wedge 1 \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t \\
&+ C \int_0^{T'} \int_\Omega |\xi \cdot (\xi - n_u)| \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t \\
&+ C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t
\end{aligned} \tag{100}
$$

for almost every $T' \in [0, T]$. It follows from property (16a) of a boundary adapted extension ξ and the trivial estimates $|\xi \cdot (\xi - n_u)| \leq (1-|\xi|^2) + (1-n_u \cdot \xi) \leq 2(1-|\xi|) + (1-n_u \cdot \xi)$ and $1 - |\xi| \leq 1 - n_u \cdot \xi$ that

$$
\begin{aligned}
&\int_0^{T'} \int_\Omega \operatorname{dist}^2(\cdot, I_v) \wedge 1 \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t + \int_0^{T'} \int_\Omega |\xi \cdot (\xi - n_u)| \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t \\
&\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t.
\end{aligned} \tag{101}
$$

Moreover, the trivial estimate $|n_u - \xi|^2 \leq 2(1 - n_u \cdot \xi)$ implies

$$
\int_0^{T'} \int_\Omega |n_u - \xi|^2 \,\mathrm{d}|\nabla\chi_u| \,\mathrm{d}t \leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t. \tag{102}
$$

Recall finally from (24) and (20) that

$$
\begin{aligned}
\int_0^{T'} \int_{\overline{\Omega} \times \mathbb{S}^{d-1}} |s - \xi|^2 \,\mathrm{d}V_t(x, s) \,\mathrm{d}t &\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t, \\
\int_0^{T'} \int_\Omega 1 - \theta_t \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \,\mathrm{d}t + \int_0^{T'} \int_{\partial\Omega} 1 \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \,\mathrm{d}t &\leq C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t.
\end{aligned} \tag{103}
$$

By inserting back the estimates (98)–(103) into the relative entropy inequality (30), then making use of the coercivity estimate (77) and Korn's inequality, and finally carrying out an absorption argument, it follows that there exist two constants $c = c(\chi_v, v, T) > 0$ and $C = C(\chi_v, v, T) > 0$ such that for almost every $T' \in [0, T]$

$$
\begin{aligned}
&E[\chi_u, u, V|\chi_v, v](T') + c \int_0^{T'} \int_\Omega |\nabla(u-v) + \nabla(u-v)^\mathsf{T}|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\leq E[\chi_u, u, V|\chi_v, v](0) + C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t) \,\mathrm{d}t.
\end{aligned} \tag{104}
$$

Step 2: (Post-processing the identity (32)*)* By the $L_t^\infty W_x^{1,\infty}$-bound for the transported weight ϑ, the estimate (28) on the advective derivative of the transported weight ϑ, and the definition (31) of the bulk error functional we infer that

$$
\begin{aligned}
E_{\mathrm{vol}}[\chi_u|\chi_v](T') &\leq E_{\mathrm{vol}}[\chi_u|\chi_v](0) + C \int_0^{T'} E_{\mathrm{vol}}[\chi_u|\chi_v](t) \,\mathrm{d}t \\
&\quad + C \int_0^{T'} \int_\Omega |\chi_v - \chi_u||u-v| \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

for almost every $T' \in [0, T]$. Adding (104) to the previous display, and making use of the coercivity estimate (77) in combination with Korn's inequality and an absorption argument thus implies that for almost every $T' \in [0, T]$

$$
\begin{aligned}
&E[\chi_u, u, V|\chi_v, v](T') + E_{\mathrm{vol}}[\chi_u|\chi_v](T') + c \int_0^{T'} \int_\Omega |\nabla(u-v) + \nabla(u-v)^\mathsf{T}|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\leq E[\chi_u, u, V|\chi_v, v](0) + E_{\mathrm{vol}}[\chi_u|\chi_v](0)
\end{aligned}
$$

$$+ C \int_0^{T'} E[\chi_u, u, V|\chi_v, v](t) + E_{\mathrm{vol}}[\chi_u|\chi_v](t)\,\mathrm{d}t. \tag{105}$$

Step 3: (Conclusion) The stability estimates (11) and (12) are an immediate consequence of the estimate (105) by an application of Gronwall's lemma. In case of coinciding initial conditions, it follows that $E_{\mathrm{vol}}[\chi_u|\chi_v](t) = 0$ for almost every $t \in [0,T]$. This in turn implies that $\chi_u(\cdot,t) = \chi_v(\cdot,t)$ almost everywhere in Ω for almost every $t \in [0,T]$. The asserted representation of the varifold follows from the fact that $E[\chi_u, u, V|\chi_v, v](t) = 0$ for almost every $t \in [0,T]$. This concludes the proof of the conditional weak-strong uniqueness principle. □

3.4. Proof of Theorem 1

This is now an immediate consequence of Proposition 4 and the existence results of Proposition 7 and Lemma 8, respectively. □

4. Bulk Extension of the Interface Unit Normal

The aim of this short section is the construction of an extension of the interface unit normal in the vicinity of a space-time trajectory in Ω of a connected component of the interface I_v corresponding to a strong solution in the sense of Definition 10 on a time interval $[0,T]$.

Mainly for reference purposes in later sections, it turns out to be beneficial to introduce already at this stage some notation in relation to a decomposition of the interface I_v into its topological features: the connected components of $I_v \cap \Omega$ and the connected components of $I_v \cap \partial\Omega$. Denoting by $N \in \mathbb{N}$ the total number of such topological features present in the interface I_v we split $\{1,\ldots,N\} =: \mathcal{I} \mathbin{\dot\cup} \mathcal{C}$ by means of two disjoint subsets. In particular, the subset $\mathcal{I}$ enumerates the space-time connected components of $I_v \cap \Omega$, i.e., time-evolving connected *interfaces*, whereas the subset $\mathcal{C}$ enumerates the space-time connected components of $I_v \cap \partial\Omega$, i.e., time-evolving *contact points*. If $i \in \mathcal{I}$, we denote by $\mathcal{T}_i := \bigcup_{t\in[0,T]} \mathcal{T}_i(t) \times \{t\} \subset I_v \cap (\Omega \times [0,T])$ the space-time trajectory of the corresponding connected interfaces $\mathcal{T}_i(t) \subset I_v(t) \cap \Omega$, $t \in [0,T]$.

For each $i \in \mathcal{I}$, we want to define a vector field ξ^i subject to conditions as in Definition 2; at least in a suitable neighborhood of $\mathcal{T}_i$. We first formalize what we mean by the latter in form of the following definition.

Definition 13. Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a two-phase interface $i \in \mathcal{I}$. We call $r_i \in (0,1]$ an *admissible localization radius for the interface* $\mathcal{T}_i \subset I_v \cap (\Omega \times [0,T])$ if the map

$$\Psi_{\mathcal{T}_i} \colon \mathcal{T}_i \times (-2r_i, 2r_i) \to \mathbb{R}^2 \times [0,T], \quad (x,t,s) \mapsto \big(x + s n_{I_v}(x,t), t\big) \tag{106}$$

is bijective onto its image $\mathrm{im}(\Psi_{\mathcal{T}_i}) := \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i \times (-2r_i, 2r_i)\big)$, and its inverse is a diffeomorphism of class $C^0_t C^2_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_t C^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$.

In case such a scale $r_i \in (0,1]$ exists, we may express the inverse by means of $\Psi^{-1}_{\mathcal{T}_i} =: (P_{\mathcal{T}_i}, \mathrm{Id}, s_{\mathcal{T}_i})\colon \mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathcal{T}_i \times (-2r_i, 2r_i)$. Hence, the map $P_{\mathcal{T}_i}$ represents in each time slice the *nearest-point projection* onto the interface $\mathcal{T}_i(t) \subset I_v(t) \cap \Omega$, $t \in [0,T]$, whereas $s_{\mathcal{T}_i}$ bears the interpretation of a *signed distance function* with orientation fixed by $\nabla s_{\mathcal{T}_i} = n_{I_v}$. In particular, $s_{\mathcal{T}_i} \in C^0_t C^3_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_t C^1_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$ as well as $P_{\mathcal{T}_i} \in C^0_t C^2_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_t C^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$.

By a slight abuse of notation, we extend to $\mathrm{im}(\Psi_{\mathcal{T}_i})$ the definition of the normal vector field resp. the scalar mean curvature of $\mathcal{T}_i$ by means of

$$n_{I_v} \colon \mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{S}^1, \quad (x,t) \mapsto n_{I_v}(P_{\mathcal{T}_i}(x,t), t) = \nabla s_{\mathcal{T}_i}(x,t), \tag{107}$$

$$H_{I_v}\colon \mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{R}, \quad (x,t) \mapsto -(\Delta s_{\mathcal{T}_i})(P_{\mathcal{T}_i}(x,t),t). \tag{108}$$

Hence, we may register that $n_{I_v} \in C^0_t C^2_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})}) \cap C^1_t C^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$ as well as $H_{I_v} \in C^0_t C^1_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i})})$.

It is clear from Definition 10 of a strong solution to the incompressible Navier–Stokes equation for two fluids, in particular Definition 9 of smoothly evolving domains and interfaces, that all interfaces admit an admissible localization radius in the sense of Definition 13 as a consequence of the tubular neighborhood theorem.

Construction 14. Let $d=2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a two-phase interface $i \in \mathcal{I}$ and let $r_i \in (0,1]$ be an admissible localization radius for the interface $\mathcal{T}_i \subset I_v$ in the sense of Definition 13. Then *a bulk extension of the unit normal* n_{I_v} *along a smooth interface* $\mathcal{T}_i$ is the vector field ξ^i defined by

$$\xi^i(x,t) := n_{I_v}(x,t), \qquad (x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega \times [0,T]). \tag{109}$$

We record the required properties of the vector field ξ^i.

Proposition 15. *Let the assumptions and notation of Construction 14 be in place. Then, in terms of regularity it holds that* $\xi^i \in C^0_t C^2_x \cap C^1_t C^0_x(\overline{\mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega \times [0,T])})$. *Moreover, we have*

$$\nabla \cdot \xi^i + H_{I_v} = O(\mathrm{dist}(\cdot, \mathcal{T}_i)), \tag{110}$$

$$\partial_t \xi^i + (v \cdot \nabla)\xi^i + (\mathrm{Id} - \xi^i \otimes \xi^i)(\nabla v)^{\mathsf{T}} \xi^i = O(\mathrm{dist}(\cdot, \mathcal{T}_i)), \tag{111}$$

$$\partial_t |\xi^i|^2 + (v \cdot \nabla)|\xi^i|^2 = 0 \tag{112}$$

throughout the space-time domain $\mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega \times [0,T])$.

Proof. The asserted regularity of ξ^i is a direct consequence of its definition (109) and the regularity of n_{I_v} from Definition 13. In view of the definitions (109), (107) and (108), the estimate (110) is directly implied by a Lipschitz estimate based on the regularity of H_{I_v} from Definition 13. The Eq. (112) is trivially fulfilled because ξ^i is a unit vector, cf. the definition (109).

For a proof of (111), we first note that $\partial_t s_{\mathcal{T}_i}(x,t) = -\big(v(P_{\mathcal{T}_i}(x,t),t) \cdot \nabla\big) s_{\mathcal{T}_i}(x,t)$ for all $(x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}) \cap (\Omega \times [0,T])$. Indeed, $\partial_t s_{\mathcal{T}_i}$ equals the normal speed (oriented with respect to $-n_{I_v}$) of the nearest point on the connected interface $\mathcal{T}_i$, which in turn by $n_{I_v} = \nabla s_{\mathcal{T}_i}$ is precisely given by the asserted right hand side term. Differentiating the equation for the time evolution of $s_{\mathcal{T}_i}$ then yields (111) by means of $\nabla P_{\mathcal{T}_i} = \mathrm{Id} - n_{I_v} \otimes n_{I_v} - s_{\mathcal{T}_i} \nabla n_{I_v}$, the chain rule, and the regularity of v. Note carefully that this argument is actually valid regardless of the assumption $\mu_- = \mu_+$ since $(\tau_{I_v} \cdot \nabla)v$ does not jump across the interface $\mathcal{T}_i$. □

5. Extension of the Interface Unit Normal at a 90° Contact Point

This section constitutes the core of the present work. We establish the existence of a boundary adapted extension of the interface unit normal in the vicinity of a space-time trajectory of a 90° contact point on the boundary $\partial\Omega$.

The vector field from the previous section serves as the main building block for an extension of n_{I_v} away from the domain boundary $\partial\Omega$. However, it is immediately clear that the bulk construction in general does not respect the necessary boundary condition $n_{\partial\Omega} \cdot \xi = 0$ along $\partial\Omega$. (Even more drastically, on non-convex parts of $\partial\Omega$ the domain of definition for the bulk construction from the previous section may not even include $\partial\Omega$!) Hence, in the vicinity of contact points a careful perturbation of the rather trivial construction from the previous section is required to enforce the boundary condition. That this can indeed be achieved is summarized in the following Proposition 16, representing the main result of this section.

For its formulation, it is convenient for the purposes of Sect. 6 to recall the notation in relation to the decomposition of the interface I_v in terms of its topological features. More precisely, denoting by $N \in \mathbb{N}$ the total number of such topological features present in the interface I_v, we split $\{1, \ldots, N\} =: \mathcal{I} \cup \mathcal{C}$, where $\mathcal{I}$ enumerates the time-evolving connected *interfaces* of $I_v \cap \Omega$, whereas $\mathcal{C}$ enumerates the time-evolving *contact points* of $I_v \cap \partial\Omega$. If $i \in \mathcal{I}$, $\mathcal{T}_i := \bigcup_{t\in[0,T]} \mathcal{T}_i(t)\times\{t\} \subset I_v \cap (\Omega\times[0,T])$ denotes the space-time trajectory of the corresponding connected interface, whereas if $c \in \mathcal{C}$, we denote by $\mathcal{T}_c := \bigcup_{t\in[0,T]} \mathcal{T}_c(t)\times\{t\} \subset I_v \cap (\partial\Omega\times[0,T])$ the space-time trajectory of the corresponding contact point. Finally, we write $i \sim c$ for $i \in \mathcal{I}$ and $c \in \mathcal{C}$ if and only if $\mathcal{T}_i$ ends at $\mathcal{T}_c$; otherwise $i \not\sim c$.

Proposition 16. *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary $\partial\Omega$. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a contact point $c \in \mathcal{C}$ and let $i \in \mathcal{I}$ be such that $i \sim c$. Let $r_c \in (0,1]$ be an associated admissible localization radius in the sense of Definition 17 below.*

There exists a potentially smaller radius $\widehat{r}_c \in (0, r_c]$, and a vector field

$$\xi^c \colon \mathcal{N}_{\widehat{r}_c,c}(\Omega) \to \mathbb{S}^1$$

defined on the space-time domain $\mathcal{N}_{\widehat{r}_c,c}(\Omega) := \bigcup_{t\in[0,T]} \big(B_{\widehat{r}_c}(\mathcal{T}_c(t)) \cap \Omega\big)\times\{t\}$, such that the following conditions are satisfied:

(i) *It holds $\xi^c \in \big(C^0_t C^2_x \cap C^1_t C^0_x\big)\big(\overline{\mathcal{N}_{\widehat{r}_c,c}(\Omega)} \setminus \mathcal{T}_c\big)$.*
(ii) *We have $\xi^c(\cdot,t) = n_{I_v}(\cdot,t)$ and $\nabla\cdot\xi^c(\cdot,t) = -H_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t) \cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$.*
(iii) *The required boundary condition is satisfied even away from the contact point, namely $\xi^c \cdot n_{\partial\Omega} = 0$ along $\mathcal{N}_{\widehat{r}_c,c}(\Omega) \cap (\partial\Omega\times[0,T])$.*
(iv) *The following estimates on the time evolution of ξ^c hold true in $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$*

$$\partial_t \xi^c + (v\cdot\nabla)\xi^c + (\mathrm{Id} - \xi^c \otimes \xi^c)(\nabla v)^\mathsf{T}\xi^c = O\big(\operatorname{dist}(\cdot,\mathcal{T}_i)\big), \tag{113}$$

$$\partial_t |\xi^c|^2 + (v\cdot\nabla)|\xi^c|^2 = 0. \tag{114}$$

(v) *Let $r_i \in (0,1]$ be an admissible localization radius for the interface $\mathcal{T}_i$, and let ξ^i be the bulk extension of the interface unit normal on scale r_i as provided by Proposition 15. The vector field ξ^c is a perturbation of the bulk extension ξ^i in the sense that the following compatibility bounds hold true*

$$|\xi^i(\cdot,t) - \xi^c(\cdot,t)| + |\nabla\cdot\xi^i(\cdot,t) - \nabla\cdot\xi^c(\cdot,t)| \le C \operatorname{dist}(\cdot,\mathcal{T}_i(t)), \tag{115}$$

$$|\xi^i(\cdot,t)\cdot(\xi^i - \xi^c)(\cdot,t)| \le C \operatorname{dist}^2(\cdot,\mathcal{T}_i(t)) \tag{116}$$

within $B_{\widehat{r}_c \wedge r_i}(\mathcal{T}_c(t)) \cap \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big)$ for all $t \in [0,T]$, cf. Definition 17.

A vector field ξ^c subject to these requirements will be referred to as a contact point extension of the interface unit normal on scale $\widehat{r}_c$.

A proof of Proposition 16 is provided in Sect. 5.4. The preceding three subsections collect all the ingredients required for the construction.

5.1. Description of the Geometry Close to a Moving Contact Point, Choice of Orthonormal Frames, and a Higher-order Compatibility Condition

We provide a suitable decomposition for a space-time neighborhood of a moving contact point $\mathcal{T}_c$, $c \in \mathcal{C}$. The main ingredient is given by the following notion of an admissible localization radius. Though rather technical and lengthy in appearance, all requirements in the definition are essentially a direct consequence of the regularity of a strong solution. The main purpose of the notion of an admissible localization radius is to collect in a unified way notation and properties which will be referred to numerous times in the sequel.

Definition 17. Let $d=2$, and let $\Omega\subset\mathbb{R}^2$ be a bounded domain with orientable and smooth boundary $\partial\Omega$. Let (χ_v,v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Fix a contact point $c\in\mathcal{C}$ and let $i\in\mathcal{I}$ be such that $i\sim c$. Let $r_i\in(0,1]$ be an admissible localization radius for the connected interface $\mathcal{T}_i$ in the sense of Definition 13. We call $r_c\in(0,r_i]$ an *admissible localization radius for the moving* 90° *contact point* $\mathcal{T}_c$ if the following list of properties is satisfied:

(i) Let the map $\Psi_{\partial\Omega}\colon\partial\Omega\times(-2r_c,2r_c)\to\mathbb{R}^2$ be given by $(x,s)\mapsto x+sn_{\partial\Omega}(x)$. We require $\Psi_{\partial\Omega}$ to be bijective onto its image $\mathrm{im}(\Psi_{\partial\Omega}):=\Psi_{\partial\Omega}\big(\partial\Omega\times(-2r_c,2r_c)\big)$, and its inverse $\Psi^{-1}_{\partial\Omega}$ is a diffeomorphism of class $C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$. We may express the inverse by means of $\Psi^{-1}_{\partial\Omega}=:(P_{\partial\Omega},s_{\partial\Omega})\colon\mathrm{im}(\Psi_{\partial\Omega})\to\partial\Omega\times(-2r_c,2r_c)$. Hence, $P_{\partial\Omega}$ represents the *nearest-point projection* onto $\partial\Omega$, whereas $s_{\partial\Omega}$ is the *signed distance function* with orientation fixed by $\nabla s_{\partial\Omega}=n_{\partial\Omega}$. In particular, $s_{\partial\Omega}\in C^3_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$ and $P_{\partial\Omega}\in C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$.

By a slight abuse of notation, we extend to $\mathrm{im}(\Psi_{\partial\Omega})$ the definition of the normal vector field resp. the scalar mean curvature of $\partial\Omega$ by means of

$$n_{\partial\Omega}\colon\mathrm{im}(\Psi_{\partial\Omega})\to\mathbb{S}^1,\quad(x,t)\mapsto n_{\partial\Omega}(P_{\partial\Omega}(x))=\nabla s_{\partial\Omega}(x),\tag{117}$$
$$H_{\partial\Omega}\colon\mathrm{im}(\Psi_{\partial\Omega})\to\mathbb{R},\quad(x,t)\mapsto-(\Delta s_{\partial\Omega})(P_{\partial\Omega}(x)).\tag{118}$$

Hence, we note that $n_{\partial\Omega}\in C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$ and $H_{\partial\Omega}\in C^1_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$.

(ii) There exist sets $W^c_{\mathcal{T}_i}=\bigcup_{t\in[0,T]}W^c_{\mathcal{T}_i}(t)\times\{t\}$, $W^c_{\Omega^\pm_v}=\bigcup_{t\in[0,T]}W^c_{\Omega^\pm_v}(t)\times\{t\}$ and $W^{\pm,c}_{\partial\Omega}=\bigcup_{t\in[0,T]}W^{\pm,c}_{\partial\Omega}(t)\times\{t\}$ with the following properties:

First, for every $t\in[0,T]$, the sets $W^c_{\mathcal{T}_i}(t)$, $W^c_{\Omega^\pm_v}(t)$ and $W^{\pm,c}_{\partial\Omega}(t)$ are non-empty subsets of $\overline{B_{r_c}(\mathcal{T}_c(t))}$ with pairwise disjoint interior. For all $t\in[0,T]$, each of these sets is represented by a cone with apex at the contact point $\mathcal{T}_c(t)$ intersected with $\overline{B_{r_c}(\mathcal{T}_c(t))}$. More precisely, there exist six time-dependent pairwise distinct unit-length vectors $X^\pm_{\mathcal{T}_i}$, $X_{\Omega^\pm_v}$ and $X^\pm_{\partial\Omega}$ of class $C^1_t([0,T])$ such that for all $t\in[0,T]$ it holds

$$W^c_{\mathcal{T}_i}(t)=\big(\mathcal{T}_c(t)+\{\alpha X^+_{\mathcal{T}_i}(t)+\beta X^-_{\mathcal{T}_i}(t)\colon\alpha,\beta\in[0,\infty)\}\big)\cap\overline{B_{r_c}(\mathcal{T}_c(t))},\tag{119}$$
$$W^c_{\Omega^\pm_v}(t)=\big(\mathcal{T}_c(t)+\{\alpha X_{\Omega^\pm_v}(t)+\beta X^\pm_{\mathcal{T}_i}(t)\colon\alpha,\beta\in[0,\infty)\}\big)\cap\overline{B_{r_c}(\mathcal{T}_c(t))},\tag{120}$$
$$W^{\pm,c}_{\partial\Omega}(t)=\big(\mathcal{T}_c(t)+\{\alpha X^\pm_{\partial\Omega}(t)+\beta X_{\Omega^\pm_v}(t)\colon\alpha,\beta\in[0,\infty)\}\big)\cap\overline{B_{r_c}(\mathcal{T}_c(t))}.\tag{121}$$

The opening angles of these cones are constant, and numerically fixed by

$$X^\pm_{\partial\Omega}\cdot X_{\Omega^\pm_v}=X^+_{\mathcal{T}_i}\cdot X^-_{\mathcal{T}_i}=\cos(\pi/3),\quad X_{\Omega^\pm_v}\cdot X^\pm_{\mathcal{T}_i}=\cos(\pi/6).\tag{122}$$

Second, for every $t\in[0,T]$, the sets $W^c_{\mathcal{T}_i}(t)$, $W^c_{\Omega^\pm_v}(t)$ and $W^{\pm,c}_{\partial\Omega}(t)$ provide a decomposition of $B_{r_c}(\mathcal{T}_c(t))$ in form of

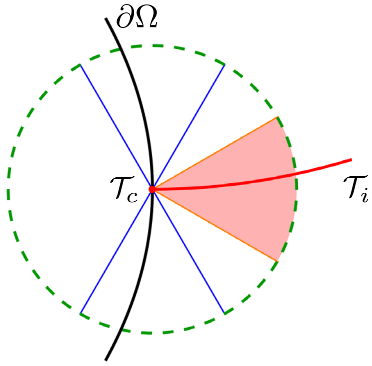
$$\begin{aligned}&\overline{B_{r_c}(\mathcal{T}_c(t))}\cap\overline{\Omega}\\&=\big(W^c_{\mathcal{T}_i}(t)\cup W^c_{\Omega^+_v}(t)\cup W^c_{\Omega^-_v}(t)\cup W^{+,c}_{\partial\Omega}(t)\cup W^{-,c}_{\partial\Omega}(t)\big)\cap\overline{\Omega}.\end{aligned}\tag{123}$$

Third, for each $t\in[0,T]$, the following inclusions hold true (recall from Definition 13 the notation for the diffeomorphism $\Psi_{\mathcal{T}_i}$):

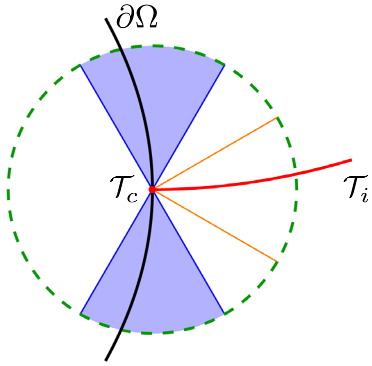
$$\overline{B_{r_c}(\mathcal{T}_c(t))}\cap\mathcal{T}_i(t)\subset\big(W^c_{\mathcal{T}_i}(t)\setminus\mathcal{T}_c(t)\big)\subset\{x\in\Omega\colon(x,t)\in\mathrm{im}(\Psi_{\mathcal{T}_i})\},\tag{124}$$
$$\overline{B_{r_c}(\mathcal{T}_c(t))}\cap\partial\Omega\subset W^{+,c}_{\partial\Omega}(t)\cup W^{-,c}_{\partial\Omega}(t),\tag{125}$$
$$W^{\pm,c}_{\partial\Omega}(t)\subset\{x\in\mathbb{R}^2\colon x\in\mathrm{im}(\Psi_{\partial\Omega})\},\tag{126}$$
$$W^c_{\Omega^\pm_v}(t)\setminus\mathcal{T}_c(t)\subset\Omega^\pm_v(t)\cap\{x\in\Omega\colon(x,t)\in\mathrm{im}(\Psi_{\mathcal{T}_i}),\,x\in\mathrm{im}(\Psi_{\partial\Omega})\}.\tag{127}$$

(iii) Finally, there exists a constant $C>0$ such that

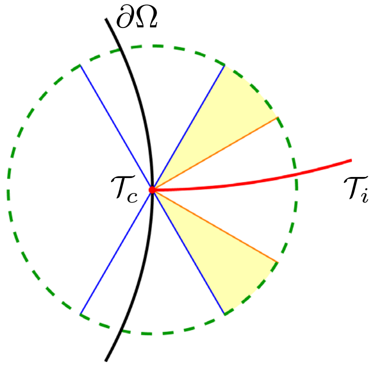
$$\mathrm{dist}(\cdot,\mathcal{T}_c)\vee\mathrm{dist}(\cdot,\partial\Omega)\leq C\,\mathrm{dist}(\cdot,\mathcal{T}_i)\qquad\text{on }W^c_{\Omega^\pm_v}\cup W^{\pm,c}_{\partial\Omega},\tag{128}$$



(A) Interface wedge $W^c_{\mathcal{T}_i}$.



(B) Boundary wedges $W^{\pm}_{\partial\Omega}$.

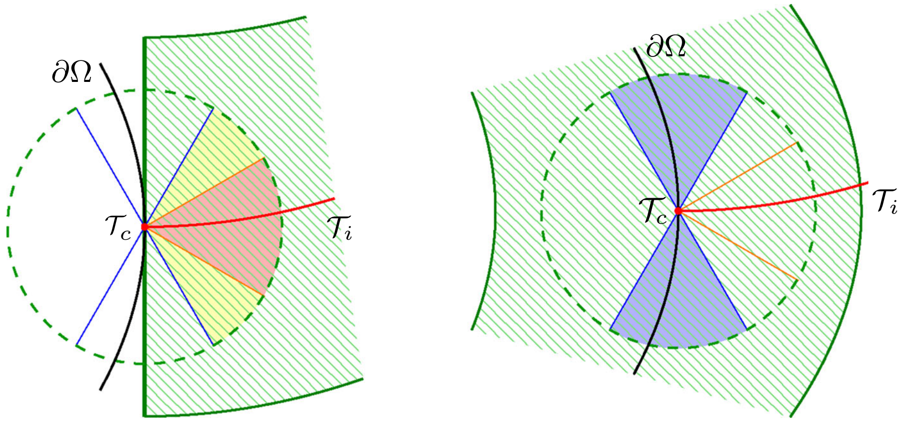


(C) Interpolation wedges $W^c_{\Omega^{\pm}_v}$.

Fig. 1. Decomposition for a space-time neighborhood of $\mathcal{T}_c$

We refer from here onwards to $W^c_{\mathcal{T}_i}$ as the *interface wedge*, $W^{\pm,c}_{\partial\Omega}$ as *boundary wedges*, and $W^c_{\Omega^{\pm}_v}$ as *interpolation wedges*.

Figures 1, 2 and 3 contain several illustrations of the previous definition. Before moving on, we briefly discuss the existence of an admissible localization radius.



(A) Inclusion in the image of $\Psi_{\mathcal{T}_i}$. (B) Inclusion in the image of $\Psi_{\partial\Omega}$.

FIG. 2. Inclusion properties of diffeomorphisms

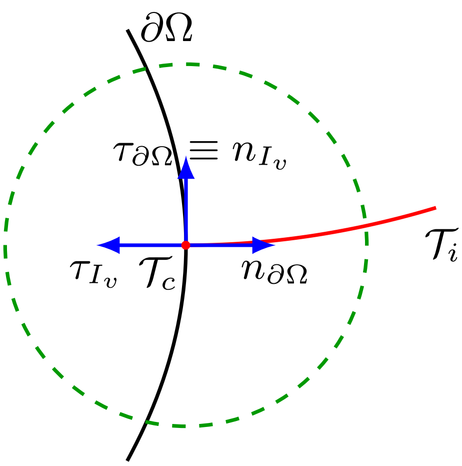


FIG. 3. Orientation of normal and tangential vectors at $\mathcal{T}_c$

Lemma 18. *Let the assumptions and notation of Definition 17 be in place. There exists a constant $C = C(\partial\Omega, \chi_v, v, T) \geq 1$ such that each $r_c \in (0, \frac{1}{C}]$ is an admissible localization radius for the contact point $\mathcal{T}_c$ in the sense of Definition 17.*

Proof. The first item in the definition of an admissible localization radius is an immediate consequence of the tubular neighborhood theorem, which in turn is facilitated by the regularity of the domain boundary $\partial\Omega$.

For a construction of the wedges, we only have to provide a definition for the vectors $X^{\pm}_{\mathcal{T}_i}$, $X_{\Omega^{\pm}_v}$ and $X^{\pm}_{\partial\Omega}$ A possible choice is the following. Fix $t \in [0, T]$ and let $\{c(t)\} = \mathcal{T}_c(t)$. The desired unit vectors are obtained through rotation of the inward-pointing unit normal $n_{\partial\Omega}(c(t))$. Note that $\big(n_{\partial\Omega}(c(t)),\, n_{I_v}(c(t),t)\big)$ form an orthonormal basis of $\mathbb{R}^2$ thanks to the contact angle condition (33). We then let $X^{\pm}_{\mathcal{T}_i}(t)$ be the unique unit vector with $X^{\pm}_{\mathcal{T}_i}(t) \cdot n_{\partial\Omega}(c(t)) = \frac{\sqrt{3}}{2}$ as well as $\mathrm{sign}\big(X^{\pm}_{\mathcal{T}_i}(t) \cdot n_{I_v}(c(t),t)\big) = \pm 1$. Similarly, $X_{\Omega^{\pm}_v}(t)$ represents the unique unit vector with $X_{\Omega^{\pm}_v}(t) \cdot n_{\partial\Omega}(c(t)) = \frac{1}{2}$ and $\mathrm{sign}\big(X_{\Omega^{\pm}_v}(t) \cdot n_{I_v}(c(t),t)\big) = \pm 1$. Finally,

$X^{\pm}_{\partial\Omega}(t)$ denotes the unique unit vector with $X^{\pm}_{\Omega}(t)\cdot n_{\partial\Omega}(c(t)) = -\frac{1}{2}$ and $\mathrm{sign}\big(X^{\pm}_{\Omega}(t)\cdot n_{I_v}(c(t),t)\big) = \pm 1$. For an illustration, we refer again to Fig. 1.

The wedges $W^c_{\mathcal{T}_i}(t)$, $W^c_{\Omega^{\pm}_v}(t)$ and $W^{\pm,c}_{\partial\Omega}(t)$ may now be defined through the right hand sides of (119), (120) and (121), respectively. The properties (123)–(128) are then obviously valid for sufficiently small radii as a consequence of the regularity of the domain boundary $\partial\Omega$, the regularity of the interface I_v due to Definition 10 of a strong solution, as well as the 90° contact angle condition (33). $\square$

A main step in the construction of a contact point extension of the interface unit normal consists of perturbing the bulk construction of Sect. 4 by introducing suitable tangential terms, cf. Sect. 5.2 below. (This in turn becomes necessary due to the boundary constraint $n_{\partial\Omega}\cdot\xi^c = 0$ along $\partial\Omega$.) To this end, the following constructions and formulas will be of frequent use.

Lemma 19. *Let the assumptions and notation of Definitions 13 and 17 be in place. Let r_c be an admissible localization radius of a contact point $\mathcal{T}_c$ and let $i\in\mathcal{I}$ such that $i\sim c$. Define $\mathcal{N}_{r_c,c}(\Omega) := \bigcup_{t\in[0,T]}\big(B_{r_c}(\mathcal{T}_c(t))\cap\overline{\Omega}\big)\times\{t\}$. We fix unit-length tangential vector fields $\widetilde{\tau}_{I_v}$ resp. $\widetilde{\tau}_{\partial\Omega}$ along $\mathcal{N}_{r_c,c}(\Omega)\cap\mathcal{T}_i$ resp. $\partial\Omega$ with orientation chosen such that $\widetilde{\tau}_{I_v} = -n_{\partial\Omega}$ resp. $\widetilde{\tau}_{\partial\Omega} = n_{I_v}$ hold true at the contact point $\mathcal{T}_c$. We then define extensions*

$$\tau_{I_v}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{S}^1, \quad (x,t)\mapsto \widetilde{\tau}_{I_v}(P_{\mathcal{T}_i}(x,t),t),$$
$$\tau_{\partial\Omega}\colon \mathrm{im}(\Psi_{\partial\Omega}) \to \mathbb{S}^1, \quad x\mapsto \widetilde{\tau}_{\partial\Omega}(P_{\partial\Omega}(x)),$$

Then, it holds $\tau_{I_v}\in C^0_tC^2_x(\overline{\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})})\cap C^1_tC^0_x(\overline{\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})})$ as well as $\tau_{\partial\Omega}\in C^2_x(\overline{\mathrm{im}(\Psi_{\partial\Omega})})$. Moreover,

$$\nabla n_{I_v} = -H_{I_v}\tau_{I_v}\otimes\tau_{I_v} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \qquad \text{in } \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i}), \tag{129}$$
$$\nabla \tau_{I_v} = H_{I_v}n_{I_v}\otimes\tau_{I_v} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \qquad \text{in } \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i}). \tag{130}$$

Analogous formulas hold on $\mathrm{im}(\Psi_{\partial\Omega})$ for the orthonormal frame $(n_{\partial\Omega},\tau_{\partial\Omega})$.

Proof. By the choice of the orientations, there exists a constant matrix R representing rotation by 90° so that $n_{I_v} = R\tau_{I_v}$ and $n_{\partial\Omega} = R\tau_{\partial\Omega}$. The regularity of the tangential fields τ_{I_v} and $\tau_{\partial\Omega}$ thus follows from Definition 13 and Definition 17, respectively. Moreover, the formula (130) simply follows from (129) and the product rule. For a proof of (129), note first that $(n_{I_v}\cdot\nabla)n_{I_v} = \nabla\frac{1}{2}|n_{I_v}|^2 = 0$ and, as a consequence of $\nabla n_{I_v} = \nabla^2 s_{\mathcal{T}_i}$ being symmetric, that $(\nabla n_{I_v})^{\mathsf{T}}n_{I_v} = (n_{I_v}\cdot\nabla)n_{I_v} = 0$. The only surviving component of ∇n_{I_v} is thus the one in the direction of $\tau_{I_v}\otimes\tau_{I_v}$, which on the interface in turn evaluates to $-H_{I_v}$, see (108). The regularity of the map H_{I_v} from Definition 13 then entails (129). Of course, the exact same argument works in terms of the orthonormal frame $(n_{\partial\Omega},\tau_{\partial\Omega})$. $\square$

The values of a contact point extension in the sense of Proposition 16 are highly constrained along the domain boundary $\partial\Omega$ (i.e., $n_{\partial\Omega}\cdot\xi^c = 0$) or along the interface $\mathcal{T}_i$ (i.e., $\xi^c = n_{I_v}$), respectively. This will be reflected in the construction by stitching together certain local building blocks (i.e., $\xi^c_{\partial\Omega}$ and $\xi^c_{\mathcal{T}_i}$, see Sect. 5.2 below) which in turn take care of these restrictions on an individual basis (i.e., $n_{\partial\Omega}\cdot\xi^c_{\partial\Omega} = 0$ along $\partial\Omega$, or $\xi^c_{\mathcal{T}_i} = n_{I_v}$ along $\mathcal{T}_i$, in the vicinity of the contact point). These local building blocks will be unified into a single vector field by interpolation (see Sect. 5.3 below). With this in mind, it is of no surprise that compatibility conditions (including a higher-order one) at the contact point are needed to implement this procedure. Indeed, recall from Proposition 16 that a contact point extension requires a certain amount of regularity in combination with a control on its time evolution. We therefore collect for reference purposes the necessary compatibility conditions in the following result.

Lemma 20. *Let the assumptions and notation of Definitions 13 and 17 and Lemma 19 be in place. Then it holds*

$$n_{I_v}(\cdot,t) = \tau_{\partial\Omega}(\cdot), \quad \tau_{I_v}(\cdot,t) = -n_{\partial\Omega}(\cdot) \qquad \text{at } \mathcal{T}_c(t),\ t\in[0,T], \tag{131}$$
$$\big(\tau_{I_v}(\cdot,t)\cdot\nabla\big)(n_{I_v}\cdot v)(\cdot,t) = H_{\partial\Omega}(\cdot)(n_{I_v}\cdot v)(\cdot,t) \qquad \text{at } \mathcal{T}_c(t),\ t\in[0,T]. \tag{132}$$

Proof. The relations (131) are immediate from the choices made in the statement of Lemma 19. Let $\{c(t)\} = \mathcal{T}_c(t)$ for all $t \in [0,T]$. The compatibility condition (132) follows from differentiating in time the condition $n_{I_v}(c(t),t) = \tau_{\partial\Omega}(c(t))$. Indeed, one one side we may compute by means of the chain rule, the analogue of (130) for $\tau_{\partial\Omega}$, (131), and $\frac{\mathrm{d}}{\mathrm{d}t}c(t) = \big(n_{I_v}(c(t),t)\cdot v(c(t),t)\big)n_{I_v}(c(t),t)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_{\partial\Omega}(c(t)) = H_{\partial\Omega}(c(t))\big(n_{I_v}(c(t),t)\cdot v(c(t),t)\big)n_{\partial\Omega}(c(t)).$$

On the other side, it follows from an application of the chain rule, the formula (129), the previous expression of $\frac{\mathrm{d}}{\mathrm{d}t}c(t)$, $\partial_t s_{\mathcal{T}_i}(\cdot,t) = -n_{I_v}(\cdot,t)\cdot v(P_{\mathcal{T}_i}(\cdot,t),t)$, as well as $n_{I_v} = \nabla s_{\mathcal{T}_i}$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}n_{I_v}(c(t),t) = -\big(\tau_{I_v}(c(t),t)\cdot\nabla\big)\big(n_{I_v}\cdot v\big)(c(t),t)\tau_{I_v}(c(t),t).$$

The second condition of (131) together with the previous two displays thus imply the compatibility condition (132) as asserted. □

5.2. Construction and Properties of Local Building Blocks

We have everything in place to proceed on with the first major step in the construction of a contact point extension in the sense of Proposition 16. We define auxiliary extensions $\xi^c_{\mathcal{T}_i}$ resp. $\xi^c_{\partial\Omega}$ of the unit normal vector field in the space-time domains $\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})$ resp. $\mathcal{N}_{r_c,c}(\Omega)\cap(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T])$. In other words, we construct the extensions separately in the regions close to the interface or close to the boundary (but always near to the contact point).

5.2.1. Definition and Regularity Properties of Local Building Blocks for the Extension of the Unit Normal.

A suitable ansatz for the two vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ may be provided as follows.

Construction 21. Let the assumptions and notation of Definition 13, Definition 17 and Lemma 19 be in place. Expressing $\{c(t)\} = \mathcal{T}_c(t)$ for all $t\in[0,T]$, we define coefficients

$$\alpha_{\mathcal{T}_i}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{R}, \quad (x,t)\mapsto -H_{\partial\Omega}(c(t),t), \tag{133}$$

$$\alpha_{\partial\Omega}\colon \mathcal{N}_{r_c,c}(\Omega)\cap(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T]) \to \mathbb{R}, \quad (x,t)\mapsto -H_{I_v}(c(t),t). \tag{134}$$

Based on these coefficient functions, we then define extensions

$$\xi^c_{\mathcal{T}_i}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i}) \to \mathbb{R}^2, \quad \xi^c_{\partial\Omega}\colon \mathcal{N}_{r_c,c}(\Omega)\cap\big(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T]\big) \to \mathbb{R}^2$$

of the normal vector field n_{I_v} by means of an expansion ansatz

$$\xi^c_{\mathcal{T}_i} := n_{I_v} + \alpha_{\mathcal{T}_i}s_{\mathcal{T}_i}\tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}n_{I_v}, \tag{135}$$

$$\xi^c_{\partial\Omega} := \tau_{\partial\Omega} + \alpha_{\partial\Omega}s_{\partial\Omega}n_{\partial\Omega} - \frac{1}{2}\alpha^2_{\partial\Omega}s^2_{\partial\Omega}\tau_{\partial\Omega}. \tag{136}$$

Regularity properties of $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$, in particular compatibility up to first order at the contact point, are the content of the following result.

Lemma 22. *Let the assumptions and notation of Construction 21 be in place. Then the auxiliary vector fields satisfy* $\xi^c_{\mathcal{T}_i} \in (C^0_tC^2_x\cap C^1_tC^0_x)(\overline{\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})})$ *and* $\xi^c_{\partial\Omega} \in (C^0_tC^2_x\cap C^1_tC^0_x)(\overline{\mathcal{N}_{r_c,c}(\Omega)\cap(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T])})$*, with corresponding estimates for* $k\in\{0,1,2\}$

$$|\nabla^k\xi^c_{\mathcal{T}_i}| + |\partial_t\xi^c_{\mathcal{T}_i}| \leq C, \quad \text{on } \overline{\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})}, \tag{137}$$

$$|\nabla^k\xi^c_{\partial\Omega}| + |\partial_t\xi^c_{\partial\Omega}| \leq C, \quad \text{on } \overline{\mathcal{N}_{r_c,c}(\Omega)\cap(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T])}. \tag{138}$$

Moreover, the constructions are compatible to first order at the contact point in the sense that

$$\xi^c_{\mathcal{T}_i}(\cdot,t) = \xi^c_{\partial\Omega}(\cdot,t), \quad \nabla\xi^c_{\mathcal{T}_i}(\cdot,t) = \nabla\xi^c_{\partial\Omega}(\cdot,t) \quad \text{at } \mathcal{T}_c(t),\ t\in[0,T]. \tag{139}$$

Proof. Step 1 (Regularity estimates): Note first that $\alpha_{\mathcal{T}_i}, \alpha_{\partial\Omega} \in C^1_t([0,T])$ due to the regularity of the maps H_{I_v} resp. $H_{\partial\Omega}$ from (108) resp. (118). The asserted bounds (137) and (138) for the derivatives of the vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ can thus be inferred from the definitions (135) and (136) in combination with the regularity of $s_{\mathcal{T}_i}, n_{I_v}$ from Definition 13, the regularity of $s_{\partial\Omega}, n_{\partial\Omega}$ from Definition 17, as well as the regularity of $\tau_{I_v}, \tau_{\partial\Omega}$ from Lemma 19.

Step 2 (First order compatibility at the contact point): The zeroth order condition of (139) is a direct consequence of the definitions (135) and (136) in combination with the compatibility condition (131). In order to prove the first order condition, it directly follows from (129)–(130) and their analogues for the frame $(n_{\partial\Omega}, \tau_{\partial\Omega})$, as well as the definitions (135) and (136) that

$$\nabla\xi^c_{\mathcal{T}_i} = -H_{I_v}\tau_{I_v}\otimes\tau_{I_v} + \alpha_{\mathcal{T}_i}\tau_{I_v}\otimes n_{I_v} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)), \tag{140}$$

$$\nabla\xi^c_{\partial\Omega} = H_{\partial\Omega}n_{\partial\Omega}\otimes\tau_{\partial\Omega} + \alpha_{\partial\Omega}n_{\partial\Omega}\otimes n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)). \tag{141}$$

Finally, since we have (131) due to the conventions adopted, using (133) and (134) we can deduce the first order compatibility condition of (139). □

5.2.2. Evolution Equations for Local Building Blocks. The following lemma provides the approximate evolution equations for our local constructions $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$, which will eventually lead us to (113)–(114).

Lemma 23. *Let the assumptions and notation of Construction 21 be in place. Then it holds*

$$\partial_t\xi^c_{\mathcal{T}_i} + (v\cdot\nabla)\xi^c_{\mathcal{T}_i} + (\mathrm{Id} - \xi^c_{\mathcal{T}_i}\otimes\xi^c_{\mathcal{T}_i})(\nabla v)^{\mathsf{T}}\xi^c_{\mathcal{T}_i} = O(\mathrm{dist}(\cdot,\mathcal{T}_i)), \tag{142}$$

$$\partial_t|\xi^c_{\mathcal{T}_i}|^2 + (v\cdot\nabla)|\xi^c_{\mathcal{T}_i}|^2 = O(\mathrm{dist}^3(\cdot,\mathcal{T}_i)), \tag{143}$$

$$|1 - |\xi^c_{\mathcal{T}_i}|^2| = O(\mathrm{dist}^4(\cdot,\mathcal{T}_i)) \tag{144}$$

throughout the space-time domain $\mathcal{N}_{r_c,c}(\Omega)\cap\mathrm{im}(\Psi_{\mathcal{T}_i})$. Moreover, we have

$$\partial_t\xi^c_{\partial\Omega} + (v\cdot\nabla)\xi^c_{\partial\Omega} + (\mathrm{Id} - \xi^c_{\partial\Omega}\otimes\xi^c_{\partial\Omega})(\nabla v)^{\mathsf{T}}\xi^c_{\partial\Omega} = O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c)), \tag{145}$$

$$\partial_t|\xi^c_{\partial\Omega}|^2 + (v\cdot\nabla)|\xi^c_{\partial\Omega}|^2 = O(\mathrm{dist}^3(\cdot,\partial\Omega)), \tag{146}$$

$$|1 - |\xi^c_{\partial\Omega}|^2| = O(\mathrm{dist}^4(\cdot,\partial\Omega)) \tag{147}$$

throughout the space-time domain $\mathcal{N}_{r_c,c}(\Omega)\cap\big(\mathrm{im}(\Psi_{\partial\Omega})\times[0,T]\big)$.

Proof. Step 1 (Proof of (142)): Note that because of the definitions (109) and (135), it holds $\xi^c_{\mathcal{T}_i} = \xi^i + \alpha_{\mathcal{T}_i}s_{\mathcal{T}_i}\tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}n_{I_v}$. Since we already proved (111), we only need to show that

$$\alpha_{I_v}(\partial_t s_{\mathcal{T}_i})\tau_{I_v} + \alpha_{I_v}(v\cdot\nabla s_{\mathcal{T}_i})\tau_{I_v} = O(\mathrm{dist}(\cdot,\mathcal{T}_i)).$$

However, the above relation is an immediate consequence of the identity $\partial_t s_{\mathcal{T}_i}(x,t) = -\big(v(P_{\mathcal{T}_i}(x,t),t)\cdot\nabla\big)s_{\mathcal{T}_i}(x,t)$ and the regularity of v, see Definition 10 of a strong solution, through a Lipschitz estimate. This proves (142).

Step 2 (Proof of (145)): From the definition (136) and $\alpha_{\partial\Omega}\in C^1_t([0,T])$ it directly follows

$$\partial_t\xi^c_{\partial\Omega} = (\partial_t\alpha_{\partial\Omega})s_{\partial\Omega}n_{\partial\Omega} = O(\mathrm{dist}(\cdot,\partial\Omega)).$$

Having $\xi^c_{\partial\Omega} = \tau_{\partial\Omega} + \alpha_{\partial\Omega}s_{\partial\Omega}n_{\partial\Omega} - \frac{1}{2}\alpha^2_{\partial\Omega}s^2_{\partial\Omega}\tau_{\partial\Omega}$, cf. the definition (136), it follows from $\nabla s_{\partial\Omega} = n_{\partial\Omega}$, the analogues of (129)–(130) for the frame $(n_{\partial\Omega},\tau_{\partial\Omega})$, as well as the boundary condition $v\cdot n_{\partial\Omega} = 0$ along $\partial\Omega$ that

$$\begin{aligned}(v\cdot\nabla)\xi^c_{\partial\Omega} &= (v\cdot\nabla)(\tau_{\partial\Omega} + \alpha_{\partial\Omega}s_{\partial\Omega}n_{\partial\Omega}) + O(\mathrm{dist}(\cdot,\partial\Omega))\\ &= (v\cdot\tau_{\partial\Omega})\tau_{\partial\Omega}\cdot(H_{\partial\Omega}\tau_{\partial\Omega}\otimes n_{\partial\Omega} + \alpha_{\partial\Omega}n_{\partial\Omega}\otimes n_{\partial\Omega}) + O(\mathrm{dist}(\cdot,\partial\Omega))\\ &= (v\cdot\tau_{\partial\Omega})H_{\partial\Omega}n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)).\end{aligned}$$

Moreover, based on $\xi^c_{\partial\Omega} = \tau_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega))$ due to (136), $v(c(t),t) = \big(v(c(t),t)\cdot n_{I_v}(c(t),t)\big)n_{I_v}(c(t),t)$ along the moving contact point $\{c(t)\} = \mathcal{T}_c(t)$, the formula (129), and the compatibility conditions (131)–(132) we infer that

$$
\begin{aligned}
&(\mathrm{Id} - \xi^c_{\partial\Omega}\otimes\xi^c_{\partial\Omega})(\nabla v)^{\mathsf{T}}\xi^c_{\partial\Omega}\\
&= (\mathrm{Id} - \tau_{\partial\Omega}\otimes\tau_{\partial\Omega})(\nabla v)^{\mathsf{T}}\tau_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega))\\
&= (\tau_{\partial\Omega}\cdot(n_{\partial\Omega}\cdot\nabla)v)n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega))\\
&= -\big(n_{I_v}(c(t),t)\cdot\big(\tau_{I_v}(c(t),t)\cdot\nabla\big)v(c(t),t)\big)n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c))\\
&= -\big(\big(\tau_{I_v}(c(t),t)\cdot\nabla\big)(v\cdot n_{I_v})(c(t),t)\big)n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c))\\
&= -(v\cdot\tau_{\partial\Omega})H_{\partial\Omega}n_{\partial\Omega} + O(\mathrm{dist}(\cdot,\partial\Omega)\vee\mathrm{dist}(\cdot,\mathcal{T}_c)).
\end{aligned}
$$

Hence, the estimate (145) follows as a consequence of the previous three displays.

Step 3 (Proof of (143)–(144) *and* (146)–(147)*):* Simply note that (143)–(144) as well as (146)–(147) directly follow from the definitions (135) resp. (136) of the vector field $\xi^c_{\mathcal{T}_i}$ resp. the vector field $\xi^c_{\partial\Omega}$ in form of

$$
|\xi^c_{\mathcal{T}_i}|^2 = \Big(1 - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}\Big)^2 + \alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i} = 1 + \frac{1}{4}\alpha^4_{\mathcal{T}_i}s^4_{\mathcal{T}_i}, \tag{148}
$$

$$
|\xi^c_{\partial\Omega}|^2 = \Big(1 - \frac{1}{2}\alpha^2_{\partial\Omega}s^2_{\partial\Omega}\Big)^2 + \alpha^2_{\partial\Omega}s^2_{\partial\Omega} = 1 + \frac{1}{4}\alpha^4_{\partial\Omega}s^4_{\partial\Omega}. \tag{149}
$$

This concludes the proof of Lemma 23. □

5.3. From Building Blocks to Contact Point Extensions by Interpolation

As we discussed in the previous subsections, the auxiliary vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ provide main building block for a contact point extension of the interface unit normal near the connected interface $\mathcal{T}_i$ or near the domain boundary $\partial\Omega$, respectively. More precisely, we will make use of the auxiliary vector field $\xi^c_{\mathcal{T}_i}$ on the wedges $W^c_{\mathcal{T}_i}\cup W^c_{\Omega^+_v}\cup W^c_{\Omega^-_v}$, and of the auxiliary vector field $\xi^c_{\partial\Omega}$ on the wedges $W^{+,c}_{\partial\Omega}\cup W^{-,c}_{\partial\Omega}\cup W^c_{\Omega^+_v}\cup W^c_{\Omega^-_v}$. Note that this is indeed admissible thanks to the inclusions (124), (126) and (127). As the domains of definition for the auxiliary vector fields overlap, we adopt an interpolation procedure on the interpolation wedges $W^c_{\Omega^\pm_v}$. To this end, we first define suitable interpolation functions.

Lemma 24. *Let the assumptions and notation of Definition 17 be in place. Then there exists a pair of interpolation functions*

$$
\lambda^\pm_c\colon \bigcup_{t\in[0,T]} \big(W^c_{\Omega^\pm_v}(t)\setminus\mathcal{T}_c(t)\big)\times\{t\} \to [0,1]
$$

which satisfies the following list of properties:

(i) *On the boundary of the interpolation wedges $W^c_{\Omega^\pm_v}$ intersected with $B_{r_c}(\mathcal{T}_c)$, the values of $\lambda^\pm_c$ and its derivatives up to second order are given by*

$$
\lambda^\pm_c(\cdot,t) = 0 \qquad \text{on } \big(\partial W^c_{\Omega^\pm_v}(t)\cap\partial W^{\pm,c}_{\partial\Omega}(t)\big)\setminus\mathcal{T}_c(t), \tag{150}
$$

$$
\lambda^\pm_c(\cdot,t) = 1 \qquad \text{on } \big(\partial W^c_{\Omega^\pm_v}(t)\cap\partial W^c_{\mathcal{T}_i}(t)\big)\setminus\mathcal{T}_c(t), \tag{151}
$$

$$
\nabla\lambda^\pm_c(\cdot,t) = 0, \qquad \text{on } \big(\partial W^c_{\Omega^\pm_v}(t)\cap B_{r_c}(\mathcal{T}_c(t))\big)\setminus\mathcal{T}_c(t), \tag{152}
$$

$$
\nabla^2\lambda^\pm_c(\cdot,t) = 0,\ \partial_t\lambda^\pm_c(\cdot,t) = 0 \qquad \text{on } \big(\partial W^c_{\Omega^\pm_v}(t)\cap B_{r_c}(\mathcal{T}_c(t))\big)\setminus\mathcal{T}_c(t) \tag{153}
$$

for all $t\in[0,T]$.

(ii) *There exists a constant C such that the estimates*

$$
|\partial_t\lambda^\pm_c(\cdot,t)| + |\nabla\lambda^\pm_c(\cdot,t)| \le C|\,\mathrm{dist}(\cdot,\mathcal{T}_c(t))|^{-1}, \tag{154}
$$

$$|\nabla\partial_t\lambda_c^\pm(\cdot,t)| + |\nabla^2\lambda_c^\pm(\cdot,t)| \le C|\operatorname{dist}(\cdot,\mathcal{T}_c(t))|^{-2} \tag{155}$$

hold true on $W^c_{\Omega_v^\pm}(t)\setminus\mathcal{T}_c(t)$ *for all* $t\in[0,T]$.

(iii) *We have an improved estimate on the advective derivative in form of*

$$\big|\partial_t\lambda_c^\pm(\cdot,t) + \big(v\cdot\nabla\big)\lambda_c^\pm(\cdot,t)\big| \le C \tag{156}$$

on $W^c_{\Omega_v^\pm}(t)\setminus\mathcal{T}_c(t)$ *for all* $t\in[0,T]$.

Proof. We fix a smooth function $\widetilde{\lambda}\colon\mathbb{R}\to[0,1]$ such that $\widetilde{\lambda}\equiv 0$ on $[\frac{2}{3},\infty)$ and $\widetilde{\lambda}\equiv 1$ on $(-\infty,\frac{1}{3}]$. Recall the representation (120) of the interpolation wedges $W_{\Omega_v^\pm}$, and that their opening angle is determined via $X^\pm_{\mathcal{T}_i}\cdot X_{\Omega_v^\pm} = \cos(\pi/6)$ along $\mathcal{T}_c$, see (122). We then define a function $\lambda\colon[-1,1]\to[0,1]$ by $\lambda(u) := \widetilde{\lambda}(\frac{1-u}{1-\cos(\pi/6)})$, and set

$$\lambda_c^\pm(x,t) := \lambda\bigg(X^\pm_{\mathcal{T}_i}(t)\cdot\frac{x-c(t)}{|x-c(t)|}\bigg),\quad t\in[0,T],\, x\in W_{\Omega_v^\pm}(t)\setminus\mathcal{T}_c(t).$$

The assertions of the first two items of Lemma 24 are now immediate consequences of the definitions due to $\frac{\mathrm{d}}{\mathrm{d}t}X^\pm_{\mathcal{T}_i}\in C^0([0,T])$, cf. Definition 17.

It remains to prove the estimate (156) on the advective derivative. To this end, abbreviating $u^\pm := X^\pm_{\mathcal{T}_i}(t)\cdot\frac{x-c(t)}{|x-c(t)|}$ we compute

$$\begin{aligned}
\partial_t\lambda_c^\pm(x,t) &= \lambda'(u^\pm)X^\pm_{\mathcal{T}_i}(t)\cdot\partial_t\frac{x-c(t)}{|x-c(t)|} + \lambda'(u^\pm)\frac{x-c(t)}{|x-c(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}X^\pm_{\mathcal{T}_i}(t)\\
&= \lambda'(u^\pm)X^\pm_{\mathcal{T}_i}(t)\cdot\frac{1}{|x-c(t)|}\Big(\mathrm{Id}-\frac{x-c(t)}{|x-c(t)|}\otimes\frac{x-c(t)}{|x-c(t)|}\Big)\frac{\mathrm{d}}{\mathrm{d}t}c(t)\\
&\quad + \lambda'(u^\pm)\frac{x-c(t)}{|x-c(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}X^\pm_{\mathcal{T}_i}(t)\\
&= -\Big(\frac{\mathrm{d}}{\mathrm{d}t}c(t)\cdot\nabla\Big)\lambda_c^\pm(x,t) + \lambda'(u^\pm)\frac{x-c(t)}{|x-c(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}X^\pm_{\mathcal{T}_i}(t).
\end{aligned}$$

This in turn yields the asserted estimate (156) due to $\frac{\mathrm{d}}{\mathrm{d}t}X^\pm_{\mathcal{T}_i}\in C^0([0,T])$, cf. Definition 17, $\frac{\mathrm{d}}{\mathrm{d}t}c(t) = v(c(t),t)$, and a Lipschitz estimate based on the regularity of the fluid velocity v from Definition 10 (which counteracts the blow-up (154) of $\nabla\lambda_c^\pm$). This concludes the proof. □

We have by now everything in place to state the definition of a vector field which in the end will give rise to a contact point extension of the interface unit normal in the precise sense of Proposition 16.

Construction 25. Let the assumptions and notation of Definition 17, Construction 21 and Lemma 24 be in place. In particular, let $r_c\in(0,1]$ be an admissible localization radius for the contact point $\mathcal{T}_c$. We define a vector field

$$\widehat{\xi}^c\colon\mathcal{N}_{r_c,c}(\Omega)\to\mathbb{R}^2$$

on the space-time domain $\mathcal{N}_{r_c,c}(\Omega) := \bigcup_{t\in[0,T]}\big(B_{r_c}(\mathcal{T}_c(t))\cap\Omega\big)\times\{t\}$ as follows (recall the decomposition (123) of the neighborhood $B_r(\mathcal{T}_c(t))\cap\overline{\Omega}$):

$$\widehat{\xi}^c(\cdot,t) := \begin{cases}\xi^c_{\mathcal{T}_i}(\cdot,t) & \text{on } W^c_{\mathcal{T}_i}(t)\cap\overline{\Omega},\\ \xi^c_{\partial\Omega}(\cdot,t) & \text{on } W^{\pm,c}_{\partial\Omega}(t)\cap\overline{\Omega},\\ \lambda_c^\pm(\cdot,t)\xi^c_{\mathcal{T}_i}(\cdot,t) + \big(1-\lambda_c^\pm(\cdot,t)\big)\xi^c_{\partial\Omega}(\cdot,t) & \text{on } W_{\Omega_v^\pm}(t)\setminus\mathcal{T}_c(t)\cap\overline{\Omega},\end{cases} \tag{157}$$

for all $t\in[0,T]$. Note that the vector field $\widehat{\xi}^c$ is not yet normalized to unit length, which is the reason for denoting it by $\widehat{\xi}^c$ instead of ξ^c. Observe also that (157) is well-defined in view of the inclusions (124), (126) and (127).

5.4. Proof of Proposition 16

The proof proceeds in several steps. We first establish the required properties in terms of the vector field $\widehat{\xi}^c$. The penultimate step is devoted to fixing $\widehat{r}_c \in (0, r_c]$ such that $|\widehat{\xi}^c| \geq \frac{1}{2}$ on $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, so that one may define $\xi := |\widehat{\xi}^c|^{-1}\widehat{\xi}^c \in \mathbb{S}^1$ throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$ and transfer the properties of $\widehat{\xi}^c$ to ξ^c. Finally, in the last step we verify the asserted compatibility conditions between a contact point extension and a bulk extension of the interface unit normal.

Step 1: Regularity of $\widehat{\xi}^c$ and properties i)–iii). Because of the inclusion (124) as well as the definitions (135) and (157), it follows that $\widehat{\xi}^c(\cdot,t) = n_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t) \cap B_{r_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. By the same reasons, relying also on $\xi^c_{\mathcal{T}_i} = \xi^i + \alpha_{\mathcal{T}_i} s_{\mathcal{T}_i} \tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i} s^2_{\mathcal{T}_i} n_{I_v}$, cf. the definitions (109) and (135), $\nabla s_{\mathcal{T}_i} = n_{I_v}$ and (110), we deduce that $\nabla \cdot \widehat{\xi}^c(\cdot,t) = -H_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t) \cap B_{r_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. Moreover, in view of the inclusion (125) as well as the definitions (136) and (157), we obtain $\widetilde{\xi}^c(\cdot,t) \cdot n_{\partial\Omega} = \tau_{\partial\Omega} \cdot n_{\partial\Omega} = 0$ along $B_{r_c}(\mathcal{T}_c(t)) \cap \partial\Omega$. This yields the asserted properties i)–iii) of a contact point extension in terms of $\widehat{\xi}^c$ on scale r_c.

The vector fields $\widehat{\xi}^c$, $\partial_t\widehat{\xi}^c$, $\nabla\widehat{\xi}^c$ and $\nabla^2\widehat{\xi}^c$ exist in a pointwise sense and are continuous throughout $\mathcal{N}_{r_c,c}(\Omega) \setminus \mathcal{T}_c$ due to the definition (157) of $\widehat{\xi}^c$, the regularity of the local building blocks $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$ as provided by Lemma 22, as well as the regularity of the interpolation parameter $\lambda^\pm_c$ from Lemma 24. Note in this context that no jumps occur across the boundaries of the interpolation wedges as a consequence of the conditions (150)–(153). It remains to prove the bounds

$$
|\partial_t\widehat{\xi}^c(\cdot,t)| + |\nabla^k\widehat{\xi}^c(\cdot,t)| \leq C \quad \text{on } \big(\overline{B_{r_c}(\mathcal{T}_c(t))} \setminus \mathcal{T}_c\big) \cap \overline{\Omega} \tag{158}
$$

for $k \in \{0,1,2\}$, for all $t \in [0,T]$ and some constant $C > 0$.

In the wedges $W^c_{\mathcal{T}_i}$ and $W^{\pm,c}_{\partial\Omega}$ containing the interface or the boundary of the domain, respectively, the estimate follows directly from the estimates (137)–(138) and the definition (157). On interpolation wedges $W^c_{\Omega^\pm_v}$, we compute recalling (157)

$$
\begin{aligned}
\partial_t\widehat{\xi}^c &= \lambda^\pm_c \partial_t\xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)\partial_t\xi^c_{\partial\Omega} + (\xi^c_{\mathcal{T}_i} - \xi^c_{\partial\Omega})\partial_t\lambda^\pm_c \\
\nabla\widehat{\xi}^c &= \lambda^\pm_c \nabla\xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)\nabla\xi^c_{\partial\Omega} + (\xi^c_{\mathcal{T}_i} - \xi^c_{\partial\Omega}) \otimes \nabla\lambda^\pm_c, \\
\nabla^2\widehat{\xi}^c &= \lambda^\pm_c \nabla^2\xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)\nabla^2\xi^c_{\partial\Omega} + (\nabla\lambda^\pm_c \otimes \nabla^{\mathrm{sym}})(\xi^c_{\mathcal{T}_i} - \xi^c_{\partial\Omega}) + (\xi^c_{\mathcal{T}_i} - \xi^c_{\partial\Omega}) \otimes \nabla^2\lambda^\pm_c.
\end{aligned}
$$

Then we recall the bounds (154) and (155) for the derivatives of the interpolation functions, the estimates (137) and (138) as well as the compatibility conditions (139) for the auxiliary vector fields $\xi^c_{\mathcal{T}_i}$ and $\xi^c_{\partial\Omega}$. Feeding these into the previous display establishes (158) on the interpolation wedges.

Step 2: Evolution equation in terms of $\widehat{\xi}^c$. We claim that

$$
\partial_t\widehat{\xi}^c + (v \cdot \nabla)\widehat{\xi}^c + (\nabla v)^\mathsf{T}\widehat{\xi}^c = O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \quad \text{in } \mathcal{N}_{r_c,c}(\Omega). \tag{159}
$$

The validity of (159) on the wedges $W^c_{\mathcal{T}_i}$ and $W^{\pm,c}_{\partial\Omega}$ follows directly from the estimates (142) resp. (145), the definition (157) and the bound (128). Hence, we only need to prove the bound (159) on the interpolation wedges $W^c_{\Omega^\pm_v}$.

To this end, recall first that on the interpolation wedges $W^c_{\Omega^\pm_v}$ the distance with respect to the contact point $\mathcal{T}_c$ or the distance with respect to the domain boundary $\partial\Omega$ is dominated by the distance to the connected interface $\mathcal{T}_i$, see (128). Writing $\widehat{\xi}^c = \xi^c_{\mathcal{T}_i} + (1-\lambda^\pm_c)(\xi^c_{\partial\Omega} - \xi^c_{\mathcal{T}_i})$, and resp. $\widehat{\xi}^c = \xi^c_{\partial\Omega} + \lambda^\pm_c(\xi^c_{I_v} - \xi^c_{\partial\Omega})$, we then immediately see that

$$
\widehat{\xi}^c \otimes \widehat{\xi}^c = \xi^c_{\mathcal{T}_i} \otimes \xi^c_{\mathcal{T}_i} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)), \tag{160}
$$

$$
\widehat{\xi}^c \otimes \widehat{\xi}^c = \xi^c_{\partial\Omega} \otimes \xi^c_{\partial\Omega} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)), \tag{161}
$$

due to compatibility (139) up to first order at the contact point $\mathcal{T}_c$, and the regularity estimates (137)–(138). Using the product rule and the definition (157) of $\widehat{\xi}^c$ on $W^c_{\Omega^\pm_v}$, we thus obtain

$$\begin{aligned}
&\partial_t\widehat{\xi}^c + (v\cdot\nabla)\widehat{\xi}^c + (\mathrm{Id} - \widehat{\xi}^c\otimes\widehat{\xi}^c)(\nabla v)^\mathsf{T}\widehat{\xi}^c \\
&= \lambda_c^\pm\big(\partial_t + (v\cdot\nabla) + (\mathrm{Id} - \xi^c_{\mathcal{T}_i}\otimes\xi^c_{\mathcal{T}_i})(\nabla v)^\mathsf{T}\big)\xi^c_{\mathcal{T}_i} \\
&\quad + (1-\lambda_c^\pm)\big(\partial_t + (v\cdot\nabla) + (\mathrm{Id} - \xi^c_{\partial\Omega}\otimes\xi^c_{\partial\Omega})(\nabla v)^\mathsf{T}\big)\xi^c_{\partial\Omega} \\
&\quad + (\partial_t\lambda_c^\pm + (v\cdot\nabla)\lambda_c^\pm)(\xi^c_{\mathcal{T}_i} - \xi^c_{\partial\Omega}) + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)).
\end{aligned} \tag{162}$$

Hence, we obtain (159) on interpolation wedges as a consequence of the estimates (142) resp. (145), the bound (156) on the advective derivative of the interpolation parameter, as well as the compatibility condition (139).

Step 3: We next claim that

$$\partial_t\big|\widehat{\xi}^c\big|^2 + (v\cdot\nabla)\big|\widehat{\xi}^c\big|^2 = O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \qquad \text{in } \mathcal{N}_{r_c,c}(\Omega), \tag{163}$$

$$\Big|\nabla\big|\widehat{\xi}^c\big|^2\Big| = O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \qquad \text{in } \mathcal{N}_{r_c,c}(\Omega). \tag{164}$$

Outside of interpolation wedges, both claims are already established in view of the estimates (143)–(144) resp. (146)–(147), the estimate (128) as well as the definition (157). Using the latter, we may compute on interpolation wedges $W^c_{\Omega^\pm_v}$

$$\begin{aligned}
|\widehat{\xi}^c|^2 - 1 &= \lambda_c^{\pm\,2}(|\xi^c_{\mathcal{T}_i}|^2 - 1) + (1-\lambda_c^\pm)^2(|\xi^c_{\partial\Omega}|^2 - 1) \\
&\quad + 2\lambda_c^\pm(1-\lambda_c^\pm)(\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} - 1),
\end{aligned} \tag{165}$$

and thus

$$\begin{aligned}
\big(\partial_t + (v\cdot\nabla)\big)\big|\widehat{\xi}^c\big|^2 &= \big(\partial_t + (v\cdot\nabla)\big)\big((\lambda_c^\pm)^2|\xi^c_{\mathcal{T}_i}|^2 + (1-\lambda_c^\pm)^2|\xi^c_{\partial\Omega}|^2 + 2\lambda_c^\pm(1-\lambda_c^\pm)\big) \\
&\quad + (\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} - 1)\big(\partial_t + (v\cdot\nabla)\big)\big(2\lambda_c^\pm(1-\lambda_c^\pm)\big) \\
&\quad + 2\lambda_c^\pm(1-\lambda_c^\pm)\big(\partial_t + (v\cdot\nabla)\big)(\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} - 1).
\end{aligned} \tag{166}$$

Because of (143)–(144) and (146)–(147), the first right hand side term of (166) is of required order. For an estimate of the second and third right hand side term of (166), observe that it suffices to prove $\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} - 1 = O(\mathrm{dist}^2(\cdot,\mathcal{T}_i))$ on interpolation wedges as the advective derivative of the interpolation parameter is bounded, see (156). However, it follows immediately from the definitions (135) and (136), the formulas (140) and (141), as well as the compatibility condition (139), that at the contact point $\mathcal{T}_c$ it holds $\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} = 1$, $(\nabla\xi^c_{\mathcal{T}_i})^\mathsf{T}\xi^c_{\partial\Omega} = 0$ and $(\nabla\xi^c_{\partial\Omega})^\mathsf{T}\xi^c_{\mathcal{T}_i} = 0$. Hence, $\xi^c_{\mathcal{T}_i}\cdot\xi^c_{\partial\Omega} - 1 = O(\mathrm{dist}^2(\cdot,\mathcal{T}_i))$ is a consequence of a Lipschitz estimate making use of the estimates (137)–(138) and the bound (128).

In summary, the above arguments upgrade (166) to (163), and analogous considerations based on (165) also entail (164) on interpolation wedges.

Step 4: Choice of $\widehat{r}_c$ and definition of the normalized vector field ξ^c. By the definition (157) of the vector field $\widehat{\xi}^c$ we have $|\widehat{\xi}^c(\cdot,t)| = 1$ on $B_{r_c}(\mathcal{T}_c(t))\cap(\partial\Omega\cup\mathcal{T}_i(t))$ for all $t\in[0,T]$. Due to its Lipschitz continuity, see *Step 1* of the proof, we may choose a radius $\widehat{r}_c\le r_c$ such that $|\widehat{\xi}^c|\ge\frac{1}{2}$ holds true in the space-time domain $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$. We then define $\xi^c := \big|\widehat{\xi}^c\big|^{-1}\widehat{\xi}^c\in\mathbb{S}^1$ throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, so that it remains to argue that the properties of $\widehat{\xi}^c$ are inherited by ξ^c.

Since $\xi^c(\cdot,t) = \widehat{\xi}^c(\cdot,t)$ on $B_{r_c}(\mathcal{T}_c(t))\cap(\partial\Omega\cup\mathcal{T}_i(t))$ for all $t\in[0,T]$, it immediately follows that $\xi^c(\cdot,t) = n_{I_v}(\cdot,t)$ along $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ as well as $\xi^c(\cdot,t)\cdot n_{\partial\Omega}(\cdot) = 0$ along $\partial\Omega\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. Moreover, $\nabla\cdot\xi^c = |\widehat{\xi}^c|^{-1}\nabla\cdot\widehat{\xi}^c - \frac{(\widehat{\xi}^c\cdot\nabla)|\widehat{\xi}^c|^2}{2|\widehat{\xi}^c|^3}$ so that $\nabla\cdot\xi^c = -H_{I_v}(\cdot,t)$ holds true on $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$ because of (164), the validity of this equation in terms of $\widehat{\xi}^c$, and the fact that $|\widehat{\xi}^c(\cdot,t)| = 1$ on $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. In summary, properties *ii)–iii)* are satisfied.

The required regularity is obtained by the choice of the radius $\widehat{r}_c$, the definition $\xi^c := \big|\widehat{\xi}^c\big|^{-1}\widehat{\xi}^c$, and the fact that the vector field $\widehat{\xi}^c$ already satisfies it as argued in *Step 1* of this proof. Since $\xi^c \in \mathbb{S}^1$ throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, (114) holds true for trivial reasons. For a proof of (113), one may argue as follows. Recalling that $|\widehat{\xi}^c| \geq \frac{1}{2}$ holds true in $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$, adding zero and using the product rule yields

$$
\begin{aligned}
&\partial_t\xi^c + (v\cdot\nabla)\xi^c + (\mathrm{Id} - \xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c \\
&= \partial_t\xi^c + (v\cdot\nabla)\xi^c + (\mathrm{Id} - \widehat{\xi}^c\otimes\widehat{\xi}^c)(\nabla v)^{\mathsf{T}}\xi^c - (1-|\widehat{\xi}^c|^2)(\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c \\
&= \frac{1}{|\widehat{\xi}^c|}\big(\partial_t\widehat{\xi}^c + (v\cdot\nabla)\widehat{\xi}^c + (\mathrm{Id} - \widehat{\xi}^c\otimes\widehat{\xi}^c)(\nabla v)^{\mathsf{T}}\widehat{\xi}^c\big) - \frac{\widehat{\xi}^c}{2|\widehat{\xi}^c|^3}\big(\partial_t|\widehat{\xi}^c|^2 + (v\cdot\nabla)|\widehat{\xi}^c|^2\big) \\
&\quad - (1-|\widehat{\xi}^c|^2)(\xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c
\end{aligned}
$$

throughout $\mathcal{N}_{\widehat{r}_c,c}(\Omega)$. Observe that the first right hand side term is estimated by (159), the second by (163), and the third by a Lipschitz estimate based on the fact $|\widehat{\xi}^c(\cdot,t)| = 1$ along $\mathcal{T}_i(t)\cap B_{\widehat{r}_c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. Hence, (113) holds true.

Step 5: Contact point extensions as perturbations of bulk extensions. As a preparation for the proof of the compatability estimates, we claim that

$$
|\xi^c - \widehat{\xi}^c| \leq C\,\mathrm{dist}^2(\cdot,\mathcal{T}_i). \tag{167}
$$

Note that because of the definition (157), the compatibility conditions (139) at the contact point, the regularity estimates (137)–(138) for the local building blocks, the controlled blow-up (154) , the coercivity estimate (144) , and the estimate (128), it holds

$$
\begin{aligned}
\nabla\frac{1}{|\widehat{\xi}^c|} = -\frac{(\widehat{\xi}^c\cdot\nabla)\widehat{\xi}^c}{|\widehat{\xi}^c|^3} &= -\frac{(\xi^c_{\mathcal{T}_i}\cdot\nabla)\widehat{\xi}^c}{|\widehat{\xi}^c|^3} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)) \\
&= -\frac{(\xi^c_{\mathcal{T}_i}\cdot\nabla)\xi^c_{\mathcal{T}_i}}{|\widehat{\xi}^c|^3} + O(\mathrm{dist}(\cdot,\mathcal{T}_i)) = O(\mathrm{dist}(\cdot,\mathcal{T}_i)).
\end{aligned}
$$

Hence, the asserted estimate (167) follows from $\xi^c - \widehat{\xi}^c = (|\widehat{\xi}^c|^{-1}-1)\widehat{\xi}^c$, the fact that $\xi^c(\cdot,t) = \widehat{\xi}^c(\cdot,t) \equiv n_{I_v}(\cdot,t)$ along the local interface patch $\mathcal{T}_i(t)\cap B_{\widehat{r}^c}(\mathcal{T}_c(t))$ for all $t\in[0,T]$, and the previous display.

We exploit (167) as follows. Within the interface wedge $W^c_{\mathcal{T}_i}$, it now follows from the definitions (109), (135) and (157) that

$$
\xi^c - \xi^i = \xi^c_{\mathcal{T}_i} - \xi^i + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)) = \alpha_{\mathcal{T}_i}s_{\mathcal{T}_i}\tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}n_{I_v} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)).
$$

Within interpolation wedges, we have the same representation thanks to the first-order compatibility (139) in form of

$$
\begin{aligned}
\xi^c - \xi^i &= \widehat{\xi}^c - \xi^i + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)) \\
&= (\xi^c_{\mathcal{T}_i} - \xi^i) + (1-\lambda^{\pm}_c)(\xi^c_{\partial\Omega} - \xi^c_{\mathcal{T}_i}) + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)) \\
&= \alpha_{\mathcal{T}_i}s_{\mathcal{T}_i}\tau_{I_v} - \frac{1}{2}\alpha^2_{\mathcal{T}_i}s^2_{\mathcal{T}_i}n_{I_v} + O(\mathrm{dist}^2(\cdot,\mathcal{T}_i)).
\end{aligned}
$$

In particular, the compatibility bounds (115) and (116) are satisfied within interface and interpolation wedges, respectively. □

6. Existence of Boundary Adapted Extensions of the Unit Normal

6.1. From Local to Global Extensions

The idea for proving Proposition 7 consists of stitching together the local extensions from the previous two sections by means of a suitable partition of unity on the interface I_v. For a construction of the latter, recall first the decomposition of the interface I_v into its topological features, namely, the connected components of $I_v \cap \Omega$ and the connected components of $I_v \cap \partial\Omega$. Denoting by $N \in \mathbb{N}$ the total number of such topological features present in the interface I_v we split $\{1, \ldots, N\} =: \mathcal{I} \cup \mathcal{C}$ by means of two disjoint subsets. Here, the subset $\mathcal{I}$ enumerates the space-time connected components of $I_v \cap \Omega$ (being time-evolving connected *interfaces*), whereas the subset $\mathcal{C}$ enumerates the space-time connected components of $I_v \cap \partial\Omega$ (being time-evolving *contact points*). If $i \in \mathcal{I}$, we let $\mathcal{T}_i \subset I_v$ denote the space-time trajectory in Ω of the corresponding connected interface. Furthermore, for every $c \in \mathcal{C}$ we write $\mathcal{T}_c$ representing the space-time trajectory in $\partial\Omega$ of the corresponding contact point. Finally, let us write $i \sim c$ for $i \in \mathcal{I}$ and $c \in \mathcal{C}$ if and only if $\mathcal{T}_i$ ends at $\mathcal{T}_c$; otherwise $i \not\sim c$.

Lemma 26 (Construction of a partition of unity). *Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0, T]$. For each $i \in \mathcal{I}$ let r_i be the localization radius of Definition 13, and for each $c \in \mathcal{C}$ denote by $\widehat{r}_c$ the localization radius of Proposition 16. There then exists a family $(\eta_1, \ldots, \eta_N)$ of cutoff functions*

$$\begin{aligned} &\eta_n : \mathbb{R}^2 \times [0,T] \to [0,1], \quad n \in \{1, \ldots, N\}, \\ &\textit{with the regularity } \eta_n \in (C_t^0 C_x^2 \cap C_t^1 C_x^0)\Big(\mathbb{R}^2 \times [0,T] \setminus \bigcup_{c \in \mathcal{C}} \mathcal{T}_c\Big), \end{aligned} \tag{168}$$

and a localization radius $\widehat{r} \in (0, \min_{i \in \mathcal{I}} r_i \wedge \min_{c \in \mathcal{C}} \widehat{r}_c)$, which together are subject to the following list of conditions:

- *The family $(\eta_1, \ldots, \eta_N)$ is a partition of unity along the interface I_v. Defining a bulk cutoff by means of $\eta_{\mathrm{bulk}} := 1 - \sum_{n=1}^N \eta_n$, it holds $\eta_{\mathrm{bulk}} \in [0,1]$. On top we have coercivity estimates in form of*

$$\frac{1}{C}(\mathrm{dist}^2(\cdot, I_v) \wedge 1) \leq \eta_{\mathrm{bulk}} \leq C(\mathrm{dist}^2(\cdot, I_v) \wedge 1) \qquad \textit{in } \mathbb{R}^2 \times [0,T], \tag{169}$$

$$|\nabla \eta_{\mathrm{bulk}}| \leq C(\mathrm{dist}(\cdot, I_v) \wedge 1) \qquad \textit{in } \mathbb{R}^2 \times [0,T], \tag{170}$$

- *For all two-phase interfaces $i \in \mathcal{I}$ it holds*

$$\mathrm{supp}\, \eta_i(\cdot, t) \subset \Psi_{\mathcal{T}_i}(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]) \quad \textit{for all } t \in [0,T], \tag{171}$$

with $\Psi_{\mathcal{T}_i}$ denoting the change of variables from Definition 13. For contact points $c \in \mathcal{C}$, it is required that

$$\mathrm{supp}\, \eta_c(\cdot, t) \subset B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \quad \textit{for all } t \in [0,T]. \tag{172}$$

- *For all distinct two-phase interfaces $i, i' \in \mathcal{I}$ it holds*

$$\mathrm{supp}\, \eta_i(\cdot, t) \cap \mathrm{supp}\, \eta_{i'}(\cdot, t) = \emptyset \quad \textit{for all } t \in [0,T]. \tag{173}$$

The same is required for all distinct contact points $c, c' \in \mathcal{I}$

$$\mathrm{supp}\, \eta_c(\cdot, t) \cap \mathrm{supp}\, \eta_{c'}(\cdot, t) = \emptyset \quad \textit{for all } t \in [0,T]. \tag{174}$$

- *Let a two-phase interface $i \in \mathcal{I}$ and a contact point $c \in \mathcal{C}$ be fixed. Then $\mathrm{supp}\, \eta_i \cap \mathrm{supp}\, \eta_c \neq \emptyset$ if and only if $i \sim c$, and in that case it holds*

$$\mathrm{supp}\, \eta_i(\cdot, t) \cap \mathrm{supp}\, \eta_c(\cdot, t) \subset B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega_v^\pm}(t)\big) \tag{175}$$

for all $t \in [0,T]$, with the wedges $W^c_{\mathcal{T}_i}$ and $W^c_{\Omega_v^\pm}$ introduced in Definition 17.

Proof. The proof proceeds in several steps.

Step 1: (Definition of auxiliary cutoff functions) Fix a smooth cutoff function $\theta : \mathbb{R} \to [0,1]$ with the properties that $\theta(r) = 1$ for $|r| \leq \frac{1}{2}$ and $\theta(r) = 0$ for $|r| \geq 1$. Define

$$\zeta(r) := (1 - r^2)\theta(r^2), \quad r \in \mathbb{R}. \tag{176}$$

Based on this quadratic profile, we may introduce two classes of cutoff functions associated to the two different natures of topological features present in the interface I_v. To this end, let $\widehat{r} \in (0, \min_{i \in \mathcal{I}} r_i \wedge$

$\min_{c\in\mathcal{C}} \widehat{r}_c)$. Moreover, let $\delta \in (0,1]$ be a constant. Both constants $\widehat{r}$ and δ will be determined in the course of the proof.

For two-phase interfaces $\mathcal{T}_i \subset I_v$, $i \in \mathcal{I}$, we may then define

$$\zeta_i(x,t) := \zeta\Big(\frac{\mathrm{sdist}(x,\mathcal{T}_i(t))}{\delta\widehat{r}}\Big),\ (x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}) := \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i{\times}(-2r_i,2r_i)\big) \tag{177}$$

where the change of variables $\Psi_{\mathcal{T}_i}$ and the associated signed distance $\mathrm{sdist}(\cdot,\mathcal{T}_i)$ are from Definition 13 of the admissible localization radius r_i. Furthermore, for contact points $\mathcal{T}_c$, $c \in \mathcal{C}$, we define

$$\zeta_c(x,t) := \zeta\Big(\frac{\mathrm{dist}(x,\mathcal{T}_c(t))}{\delta\widehat{r}}\Big), \quad (x,t) \in \mathbb{R}^2 \times [0,T]. \tag{178}$$

Step 2: (Choice of the constant $\widehat{r} \in (0, \min_{i\in\mathcal{I}} r_i \wedge \min_{c\in\mathcal{C}} \widehat{r}_c)$) It is a consequence of the uniform regularity of the interface I_v in space-time that one may choose $\widehat{r} \in (0, \min_{i\in\mathcal{I}} r_i \wedge \min_{c\in\mathcal{C}} \widehat{r}_c)$ small enough such that the following localization properties hold true

$$\Psi_{\mathcal{T}_i}(\mathcal{T}_i(t){\times}\{t\}{\times}[-\widehat{r},\widehat{r}]) \cap \Psi_{\mathcal{T}_{i'}}(\mathcal{T}_{i'}(t){\times}\{t\}{\times}[-\widehat{r},\widehat{r}]) = \emptyset \quad \forall i' \in \mathcal{I},\ i' \neq i, \tag{179}$$

$$\Psi_{\mathcal{T}_i}(\mathcal{T}_i(t){\times}\{t\}{\times}[-\widehat{r},\widehat{r}]) \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \neq \emptyset \ \Leftrightarrow\ \exists c \in \mathcal{C}\colon i \sim c, \tag{180}$$

$$B_{\widehat{r}}(\mathcal{T}_c(t)) \cap B_{\widehat{r}}(\mathcal{T}_{c'}(t)) = \emptyset \quad \forall c,c' \in \mathcal{C},\ c' \neq c. \tag{181}$$

for all $t \in [0,T]$ and all $i \in \mathcal{I}$.

Step 3: (Construction of the partition of unity, part I) We start with the construction of the cutoffs η_i for two-phase interfaces $i \in \mathcal{I}$. Away from contact points, we set

$$\eta_i(x,t) := \zeta_i(x,t), \quad (x,t) \in \mathrm{im}(\Psi_{\mathcal{T}_i}) \setminus \bigcup_{c\in\mathcal{C}} \bigcup_{t'\in[0,T]} B_{\widehat{r}}\big(\mathcal{T}_c(t')\big){\times}\{t'\}, \tag{182}$$

which is well-defined due to the choice of $\widehat{r}$.

Assume now there exists $c \in \mathcal{C}$ such that $i \sim c$. Recall from Definition 17 of the admissible localization radius r_c that for all $t \in [0,T]$ we decomposed $\Omega \cap B_{r_c}(\mathcal{T}_c(t))$ by means of five pairwise disjoint open wedges $W^{\pm,c}_{\partial\Omega}(t), W^c_{\mathcal{T}_i}(t), W^c_{\Omega^\pm_v}(t) \subset \mathbb{R}^2$. In the wedge $W^c_{\mathcal{T}_i}$ containing the two-phase interface $\mathcal{T}_i \subset I_v$, we define

$$\eta_i(x,t) := (1-\zeta_c(x,t))\zeta_i(x,t),\ (x,t) \in \bigcup_{t'\in[0,T]} \big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \cap W^c_{\mathcal{T}_i}(t')\big){\times}\{t'\}. \tag{183}$$

This is indeed well-defined by the choice of $\widehat{r}$ and having

$$B_{r_c}(\mathcal{T}_c(t)) \cap W^c_{\mathcal{T}_i}(t) \subset \Psi_{\mathcal{T}_i}(\mathcal{T}_i(t){\times}\{t\}{\times}(-2r_c,2r_c))$$

for all $t \in [0,T]$; the latter in turn being a consequence of Definition 17 of the admissible localization radius r_c.

Within the ball $B_{\widehat{r}}(\mathcal{T}_c(t))$, we aim to restrict the support of $\eta_i(\cdot,t)$ to the region $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big)$ for all $t \in [0,T]$. This will be done by means of the interpolation functions $\lambda^\pm_c$ of Lemma 24. Recall in this context the convention that $\lambda^\pm_c(\cdot,t)$ was set equal to one on $\big(\partial W^c_{\Omega^\pm_v}(t) \cap \partial W^c_{\mathcal{T}_i}(t)\big) \setminus \mathcal{T}_c(t)$ and set equal to zero on $\big(\partial W^c_{\Omega^\pm_v}(t) \cap \partial W^{\pm,c}_{\partial\Omega}(t)\big) \setminus \mathcal{T}_c(t)$ for all $t \in [0,T]$. In particular, we may define in the interpolation wedges $W^c_{\Omega^\pm_v}$

$$\begin{aligned} &\eta_i(x,t) := \lambda^\pm_c(x,t)(1-\zeta_c(x,t))\zeta_i(x,t), \\ &(x,t) \in \bigcup_{t'\in[0,T]} \big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \cap W^c_{\Omega^\pm_v}(t)\big){\times}\{t'\}. \end{aligned} \tag{184}$$

Again, this is well-defined because of the choice of $\widehat{r}$ and the fact that

$$B_{r_c}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t) \subset \Psi_{\mathcal{T}_i(t)}(\mathcal{T}_i(t){\times}\{t\}{\times}(-2r_c,2r_c))$$

for all $t \in [0,T]$ due to Definition 17 of the admissible localization radius r_c.

Outside of the space-time domains appearing in the definitions (182)–(184), we simply set η_i equal to zero.

In view of the definitions (176)–(178) and the definitions (182)–(184), it now suffices to choose $\delta \in (0,1]$ sufficiently small such that (171) holds true, and in case there exists $c \in \mathcal{C}$ such that $i \sim c$ one may on top achieve

$$\operatorname{supp}\eta_i(\cdot,t) \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \subset B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big) \tag{185}$$

for all $t \in [0,T]$. Moreover, in light of (171) and (179) we also obtain (173).

Step 4: (Construction of the partition of unity, part II) We proceed with the construction of the cutoffs η_c for contact points $c \in \mathcal{C}$. To this end, let $i \in \mathcal{I}$ be the unique two-phase interface such that $i \sim c$. In the wedge $W^c_{\mathcal{T}_i}$ containing the two-phase interface $\mathcal{T}_i \subset I_v$ we set

$$\eta_c(x,t) := \zeta_c(x,t)\zeta_i(x,t),\ (x,t) \in \bigcup_{t' \in [0,T]} \big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \cap W^c_{\mathcal{T}_i}(t')\big) \times \{t'\}, \tag{186}$$

which is well-defined based on the same reason as for (183).

Moreover, in the interpolation wedges $W^c_{\Omega^\pm_v}$ we define

$$\begin{aligned}&\eta_c(x,t) := \lambda^\pm_c(x,t)\zeta_c(x,t)\zeta_i(x,t) + (1-\lambda^\pm_c(x,t))\zeta_c(x,t),\\ &(x,t) \in \bigcup_{t' \in [0,T]} \big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \cap W^c_{\Omega^\pm_v}(t)\big) \times \{t'\}.\end{aligned} \tag{187}$$

By the same argument as for (184), this is again well-defined.

Outside of the space-time domains appearing in the previous two definitions we simply set $\eta_c := \zeta_c$. In particular, we register for reference purposes that

$$\eta_c(x,t) := \zeta_c(x,t),\ (x,t) \in \bigcup_{t' \in [0,T]} \big(B_{\widehat{r}}\big(\mathcal{T}_c(t')\big) \setminus \big(W^c_{\mathcal{T}_i}(t') \cup W^c_{\Omega^\pm_v}(t)\big)\big) \times \{t'\}. \tag{188}$$

It now immediately follows from the definition (178) that (172) is satisfied. In particular, for pairs $i \in \mathcal{I}$ and $c \in \mathcal{C}$ such that $i \sim c$, $\operatorname{supp}\eta_i \cap \operatorname{supp}\eta_c \neq \emptyset$ and we obtain (175) as an update of (185). Moreover, by (172) and (181) we deduce the validity of (174). In the case of pairs $i \in \mathcal{I}$ and $c \in \mathcal{C}$ with $i \not\sim c$, due to (180), (171) and (172), we can conclude that $\operatorname{supp}\eta_i \cap \operatorname{supp}\eta_c = \emptyset$.

Step 5: (Partition of unity property along the interface) Fix $t \in [0,T]$, and consider first the case of $x \in I_v(t) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}(\mathcal{T}_c(t))$. The combination of the support properties (171) and (172) with the localization property (179) implies there exists a unique two-phase interface $i_* = i_*(x) \in \mathcal{I}$ such that $\sum_{n=1}^N \eta_n(x,t) = \eta_{i_*}(x,t)$. Hence, we may deduce from (182) that $\sum_{n=1}^N \eta_n(x,t) = 1$ for all $t \in [0,T]$ and all $x \in I_v(t) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}(\mathcal{T}_c(t))$.

Fix a contact point $c \in \mathcal{C}$ and a point $x \in I_v(t) \cap B_{\widehat{r}}(\mathcal{T}_c(t))$. Let $i \in \mathcal{I}$ be the unique two-phase interface such that $i \sim c$. By the support properties (171) and (172) in combination with the localization properties (179)–(181) it follows that $\sum_{n=1}^N \eta_n(x,t) = \eta_c(x,t) + \eta_i(x,t)$. In particular $\sum_{n=1}^N \eta_n(x,t) = 1$ due to the definitions (183) and (186). The two discussed cases thus imply that

$$\sum_{n=1}^N \eta_n(x,t) = 1, \quad (x,t) \in \bigcup_{t' \in [0,T]} I_v(t') \times \{t'\}. \tag{189}$$

Step 6: (Regularity) Outside of interpolation wedges, the required regularity is an immediate consequence of the uniform regularity of the interface I_v and the definitions (182), (183), (186) and (187).

In interpolation wedges, one has to argue based on the definitions (184) and (187). In terms of regularity, the critical cases originating from an application of the product rule consist of those when derivatives hit the interpolation parameter. However, the by (154)–(155) controlled blow-up of the derivatives of the interpolation parameter is always counteracted by the presence of the term $1-\zeta_c$ (cf. (184) and (187)) which is of second order in the distance to the contact point due to (176) and (178). In other words, the required regularity also holds true within interpolation wedges.

The two considered cases taken together entail the asserted regularity.

Step 7: (Estimate for the bulk cutoff) In the course of establishing the desired coercivity estimates (169) and (170), we also convince ourselves of the fact that

$$\eta_{\mathrm{bulk}} = 1 - \sum_{n=1}^{N} \eta_n \in [0,1] \tag{190}$$

throughout $\mathbb{R}^2 \times [0,T]$. By the support properties (171) and (172), in both cases it suffices to argue for points contained in $\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ or $B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i \in \mathcal{I}$, all $c \in \mathcal{C}$ and all $t \in [0,T]$.

We start with the latter and fix $i \in \mathcal{I}$ as well as $t \in [0,T]$. Due to the localization property (179) and subsequently plugging in (182), we get

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1-\eta_i(\cdot,t) = 1-\zeta_i(\cdot,t) \text{ in } \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \setminus \bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big). \tag{191}$$

The validity of (169), (170) and (190) in $\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ thus follows immediately from definition (177).

Fix $c \in \mathcal{C}$, and let $i \in \mathcal{I}$ be the unique two-phase interface with $i \sim c$. Due to (171), (172) as well as (179)–(181) we have

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1 - \eta_c(\cdot,t) - \eta_i(\cdot,t) \quad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \cap \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big). \tag{192}$$

Plugging in (183) and (186) or (184) and (187), respectively, yields

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1 - \zeta_i(\cdot,t) \quad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \cap W^c_{\mathcal{T}_i}(t), \tag{193}$$

as well as

$$\eta_{\mathrm{bulk}}(\cdot,t) = \lambda^\pm_c(\cdot,t)(1-\zeta_i(\cdot,t)) + (1-\lambda^\pm_c(\cdot,t))(1-\zeta_c(\cdot,t)) \qquad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \cap W^c_{\Omega^\pm_v}(t). \tag{194}$$

Hence, we can infer by means of (177) and (178) that (169), (170) and (190) hold true in the domain $B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \cap \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big)$. Finally, we have

$$\eta_{\mathrm{bulk}}(\cdot,t) = 1 - \eta_c(\cdot,t) = 1 - \zeta_c(\cdot,t) \quad \text{in } B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \setminus \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big) \tag{195}$$

as a consequence of (171), (172), (179)–(181) and (188). The previous display in turn implies (169), (170) and (190) in $B_{\widehat{r}}\big(\mathcal{T}_c(t)\big) \setminus \big(W^c_{\mathcal{T}_i}(t) \cup W^c_{\Omega^\pm_v}(t)\big)$ because of (178). This eventually concludes the proof of Lemma 26. □

Construction 27 (From local to global extensions). Let $d = 2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Let $(\eta_1,\ldots,\eta_N)$ be a partition of unity along the interface I_v as given by the proof of Lemma 26. For each two-phase interface $i \in \mathcal{I}$ denote by ξ^i the bulk extension of Proposition 15, and for each contact point $c \in \mathcal{C}$ let ξ^c be the contact point extension of Proposition 16.

We then define a vector field $\xi\colon \overline{\Omega} \times [0,T] \to \mathbb{R}^2$ with regularity

$$\xi \in \big(C^0_t C^2_x \cap C^1_t C^0_x\big)\big(\overline{\Omega\times[0,T]} \setminus (I_v \cap (\partial\Omega\times[0,T]))\big) \tag{196}$$

by means of the formula

$$\xi := \sum_{n=1}^{N} \eta_n \xi^n. \tag{197}$$

Before we proceed on with a proof of Proposition 7, we first deduce that the bulk cutoff η_{bulk} of Lemma 26 is transported by the fluid velocity v up to an admissible error in the distance to the interface of the strong solution.

Lemma 28 (Transport equation for bulk cutoff). *Let $d=2$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with orientable and smooth boundary. Let (χ_v, v) be a strong solution to the incompressible Navier–Stokes equation for two fluids in the sense of Definition 10 on a time interval $[0,T]$. Let $(\eta_1,\ldots,\eta_N)$ be a partition of unity along the interface I_v as given by the proof of Lemma 26.*

The bulk cutoff $\eta_{\mathrm{bulk}} = 1-\sum_{n=1}^N \eta_n$ is then transported by the fluid velocity v to second order in form of

$$|\partial_t \eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}}| \leq C(1\wedge \operatorname{dist}^2(\cdot, I_v)) \quad \text{in } \Omega\times[0,T]. \tag{198}$$

Proof. Let $\widehat{r}\in(0,\frac12]$ be the localization radius of Lemma 26. In view of the regularity estimate (168) and the fact that

$$\Omega\setminus\bigg(\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}(\mathcal{T}_c(t))\cup\bigcup_{i\in\mathcal{I}}\operatorname{im}(\Psi_{\mathcal{T}_i})\bigg)\subset\Omega\cap\big\{x\in\mathbb{R}^2\colon\ \operatorname{dist}(x,I_v(t))>\widehat{r}\big\}$$

for all $t\in[0,T]$, it suffices to establish (198) within $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ or $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i\in\mathcal{I}$, all $c\in\mathcal{C}$ and all $t\in[0,T]$.

Step 1: (Estimate near the interface but away from contact points) Fix a two-phase interface $i\in\mathcal{I}$. As a consequence of the two identities in (191), we may compute

$$\partial_t\eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} = -\big(\partial_t\zeta_i + (v\cdot\nabla)\zeta_i\big) + \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_i \tag{199}$$

in $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ for all $t\in[0,T]$. Recall that the signed distance to the two-phase interface $\mathcal{T}_i\subset I_v$ is transported to first order by the fluid velocity v, and that the profile ζ from (176) is quadratic around the origin. Hence, by the chain rule and the definition (177) we obtain

$$\big|\partial_t\zeta_i + (v\cdot\nabla)\zeta_i\big| \leq C\operatorname{dist}^2(\cdot, I_v) \quad \text{in } \Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big) \tag{200}$$

for all $t\in[0,T]$. Since we also have the coercivity estimate (169) for the bulk cutoff at our disposal, we may thus upgrade (199) to (198) in $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ for all $t\in[0,T]$.

Step 2: (Estimate near contact points, part I) Fix $c\in\mathcal{C}$, and denote by $i\in\mathcal{I}$ the unique two-phase interface such that $i\sim c$. This step is devoted to the proof of (198) in the wedge $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^c_{\mathcal{T}_i}(t)$ containing the interface $\mathcal{T}_i(t)\subset I_v(t)$, $t\in[0,T]$. Because of (192), (193) and (197) we have

$$\partial_t\eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} = -\big(\partial_t\zeta_i + (v\cdot\nabla)\zeta_i\big) + \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_i \tag{201}$$

in $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^c_{\mathcal{T}_i}(t)$ for all $t\in[0,T]$. Due to Definition 17 of the admissible localization radius r_c and $\widehat{r}\leq r_c$ by Lemma 26, it holds $B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^c_{\mathcal{T}_i}(t)\subset\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)$ for all $t\in[0,T]$. In particular, the estimate (200) is applicable in $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^c_{\mathcal{T}_i}(t)$ for all $t\in[0,T]$. Hence, the estimate (200) in combination with the coercivity estimate (169) for the bulk cutoff allow to deduce (198) from (201) in $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^c_{\mathcal{T}_i}(t)$ for all $t\in[0,T]$.

Step 3: (Estimate near contact points, part II) Fix a contact point $c\in\mathcal{C}$. The goal of this step is to prove (198) in the wedges $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^{\pm,c}_{\partial\Omega}(t)$ containing the boundary $\partial\Omega$ for all $t\in[0,T]$. To this end, it follows from (195) and (197) that

$$\partial_t\eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} = -\big(\partial_t\zeta_c + (v\cdot\nabla)\zeta_c\big) + \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_c \tag{202}$$

in $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))\cap W^{\pm,c}_{\partial\Omega}(t)$ for all $t\in[0,T]$. Note that because of (176) one can view the profile ζ_c from (178) as a smooth function of the contact point $\mathcal{T}_c$. Performing a slight yet convenient abuse of notation $\mathcal{T}_c(t)=\{c(t)\}$, we obtain as a consequence of $\frac{\mathrm{d}}{\mathrm{d}t}c(t)=v(c(t),t)$ and an application of the chain rule that $\partial_t\zeta_c(\cdot,t)+\big(v(c(t),t)\cdot\nabla\big)\zeta_c(\cdot,t)=0$ at $c(t)$ for all $t\in[0,T]$. Furthermore, proceeding similarly as done in the proof of [12, Lemma 11], we can also deduce that $\partial_t\zeta_c(\cdot,t)+\big(v(c(t),t)\cdot\nabla\big)\zeta_c(\cdot,t)=0$ in $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $t\in[0,T]$. By the regularity of the fluid velocity v, this in turn implies by adding zero (and exploiting the quadratic behaviour of the profile ζ from (176) around the origin) that

$$\big|\partial_t\zeta_c + (v\cdot\nabla)\zeta_c\big| \leq C\operatorname{dist}^2(\cdot,\mathcal{T}_c) \quad \text{in } \Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t)) \tag{203}$$

for all $t \in [0,T]$. Since $\widehat{r} \leq r_c$ by Lemma 26, we can infer from Definition 17 of the admissible localization radius r_c that $\operatorname{dist}(\cdot, \mathcal{T}_c)$ is dominated by $\operatorname{dist}(\cdot, I_v)$ in $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^{\pm,c}_{\partial\Omega}(t) \cup W^c_{\Omega^\pm_v}(t)\big)$ for all $t \in [0,T]$. Hence, we deduce from (203) that

$$\big|\partial_t \zeta_c + (v\cdot\nabla)\zeta_c\big| \leq C \operatorname{dist}^2(\cdot, I_v) \text{ in } \Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^{\pm,c}_{\partial\Omega}(t) \cup W^c_{\Omega^\pm_v}(t)\big) \tag{204}$$

for all $t \in [0,T]$. Inserting the estimate (204) and the coercivity estimate (169) for the bulk cutoff into (202) thus yields (198) in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^{\pm,c}_{\partial\Omega}(t)$ for all $t \in [0,T]$.

Step 4: (Estimate near contact points, part III) Fix $c \in \mathcal{C}$, and denote by $i \in \mathcal{I}$ the unique two-phase interface such that $i \sim c$. We aim to verify (198) in the interpolation wedges $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t)$ for all $t \in [0,T]$. To this end, we may employ (192), (194) and (197) to argue that

$$\begin{aligned}
&\partial_t \eta_{\mathrm{bulk}} + (v\cdot\nabla)\eta_{\mathrm{bulk}} \\
&= -\lambda^\pm_c \Big\{ \big(\partial_t \zeta_i + (v\cdot\nabla)\zeta_i\big) - \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_i \Big\} \\
&\quad - (1-\lambda^\pm_c)\Big\{ \big(\partial_t \zeta_c + (v\cdot\nabla)\zeta_c\big) - \eta_{\mathrm{bulk}}(v\cdot\nabla)\zeta_c \Big\} \\
&\quad + \big(\partial_t \lambda^\pm_c + (v\cdot\nabla)\lambda^\pm_c\big)(\zeta_c - \zeta_i)
\end{aligned} \tag{205}$$

in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t)$ for all $t \in [0,T]$. Due to Definition 17 of the admissible localization radius r_c and $\widehat{r} \leq r_c$ by Lemma 26, it holds $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t) \subset \Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t) \times \{t\} \times [-\widehat{r}, \widehat{r}]\big)$ for all $t \in [0,T]$. The estimates (200) and (169) therefore imply that the first term on the right hand side of (205) is of required order. For the second term on the right hand side of (205), we may instead rely on the estimates (204) and (169).

Note that in view of the definitions (176)–(178), the auxiliary cutoffs ζ_i and ζ_c are compatible to second order in the sense that $|\zeta_i - \zeta_c| \leq C \operatorname{dist}^2(\cdot, \mathcal{T}_c)$ in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t)$ for all $t \in [0,T]$. Recall from the previous step that $\operatorname{dist}(\cdot, \mathcal{T}_c)$ is dominated by $\operatorname{dist}(\cdot, I_v)$ in $B_{\widehat{r}}(\mathcal{T}_c(t)) \cap \big(W^{\pm,c}_{\partial\Omega}(t) \cup W^c_{\Omega^\pm_v}(t)\big)$ for all $t \in [0,T]$. Hence,

$$|\zeta_i - \zeta_c| \leq C \operatorname{dist}^2(\cdot, I_v) \tag{206}$$

in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t)$ for all $t \in [0,T]$. In particular, together with (156) the bound (206) allows to upgrade (205) to the desired estimate (198) in $\Omega \cap B_{\widehat{r}}(\mathcal{T}_c(t)) \cap W^c_{\Omega^\pm_v}(t)$ for all $t \in [0,T]$.

Step 5: (Conclusion) Recall from Definition 17 of the admissible localization radius r_c that for all $t \in [0,T]$ the set $\Omega \cap B_{r_c}(\mathcal{T}_c(t))$ is decomposed by means of the five pairwise disjoint open wedges $W^{\pm,c}_{\partial\Omega}(t), W^c_{\mathcal{T}_i}(t), W^c_{\Omega^\pm_v}(t) \subset \mathbb{R}^2$. Hence, the previous three steps entail the validity of (198) in $\Omega \cap B_{r_c}(\mathcal{T}_c(t))$ for all $t \in [0,T]$. In particular, based on the discussion at the beginning of this proof and the argument in the vicinity of the interface but away from contact points (see *Step 1*), we may conclude the proof of Lemma 26. □

6.2. Proof of Proposition 7

All ingredients are in place to proceed with the proof of the main result of this section, i.e., that the vector field ξ of Construction 27 gives rise to a boundary adapted extension of the interface unit normal for two-phase fluid flow in the sense of Definition 2 with respect to (χ_v, v).

Proof of (16a). This is an easy consequence of the lower bound in the coercivity estimate (169) for the bulk cutoff, the definition (197) of the global vector field ξ, the fact that the local vector fields $(\xi^n)_{n\in\{1,\ldots,N\}}$ as provided by Proposition 15 and Proposition 16 are of unit length, and the triangle inequality in form of $|\xi| = |\sum_{n=1}^N \eta_n \xi_n| \leq \sum_{n=1}^N \eta_n |\xi^n| = \sum_{n=1}^N \eta_n = 1 - \eta_{\mathrm{bulk}}$ in $\Omega \times [0,T]$. □

Proof of (16b). By definition (197) of the candidate extension ξ and the localization properties (171)–(175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26, it suffices to verify (16b) in terms of $\xi=\eta_c\xi^c$ in the associated region $B_{\widehat{r}}(\mathcal{T}_c(t))\cap\partial\Omega$ for all contact points $c\in\mathcal{C}$ and all $t\in[0,T]$. However, this in turn is an immediate consequence of Proposition 16. □

Proof of (16c). For a proof of (16c), we start computing based on the definition (197) of the global vector field ξ that $\nabla\cdot\xi=\sum_{n=1}^N\eta_n\nabla\cdot\xi^n+\sum_{n=1}^N(\xi^n\cdot\nabla)\eta_n$. As a consequence of the corresponding local versions of (16c) from Proposition 15 and Proposition 16, and the fact that $(\eta_1,\ldots,\eta_n)$ is a partition of unity along the interface I_v by Lemma 26 we obtain $\sum_{n=1}^N\eta_n\nabla\cdot\xi^n=-H_{I_v}$ along $I_v\cap\Omega$. Moreover, by adding zero and subsequently relying on the definition (197) of the global vector field ξ, the localization properties (171)–(175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26, the compatibility estimate (115) and the estimates (169) and (170) for the bulk cutoff we may infer that

$$
\begin{aligned}
\sum_{n=1}^N(\xi^n\cdot\nabla)\eta_n &= -(\xi\cdot\nabla)\eta_{\mathrm{bulk}}-\sum_{n=1}^N((\xi-\xi^n)\cdot\nabla)\eta_n\\
&= -(\xi\cdot\nabla)\eta_{\mathrm{bulk}}+\eta_{\mathrm{bulk}}\sum_{n=1}^N(\xi^n\cdot\nabla)\eta_n\\
&\quad+\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c\big((\xi^i-\xi^c)\cdot\nabla\eta_i\big)+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i\big((\xi^c-\xi^i)\cdot\nabla\eta_c\big)\\
&= O(1\wedge\mathrm{dist}(\cdot,I_v))\quad\text{in }\Omega\times[0,T].
\end{aligned}
$$

In summary, we thus obtain (16c). □

Proof of (16d). For a proof of (16d), we start estimating based on the definition (197) of the global vector field ξ as well as the corresponding local versions of (16d) from Proposition 15 and Proposition 16

$$
\begin{aligned}
\partial_t\xi &= \sum_{n=1}^N\eta_n\partial_t\xi^n+\sum_{n=1}^N\xi^n\partial_t\eta_n\\
&= -\sum_{n=1}^N\eta_n(v\cdot\nabla)\xi^n+\sum_{n=1}^N\xi^n\partial_t\eta_n\\
&\quad-\sum_{n=1}^N\eta_n(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n+O(1\wedge\mathrm{dist}(\cdot,I_v))\quad\text{in }\Omega\times[0,T].
\end{aligned}\tag{207}
$$

Adding zero twice and applying the product rule, we may further rewrite based on the definition (197) of the candidate extension ξ and the localization properties (171)–(175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26

$$
\begin{aligned}
&-\sum_{n=1}^N\eta_n(v\cdot\nabla)\xi^n+\sum_{n=1}^N\xi^n\partial_t\eta_n\\
&= -(v\cdot\nabla)\xi+\sum_{n=1}^N\xi^n\big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\big)\\
&= -(v\cdot\nabla)\xi-\xi\big(\partial_t\eta_{\mathrm{bulk}}+(v\cdot\nabla)\eta_{\mathrm{bulk}}\big)+\sum_{n=1}^N(\xi^n-\xi)\big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\big)\\
&= -(v\cdot\nabla)\xi-\xi\big(\partial_t\eta_{\mathrm{bulk}}+(v\cdot\nabla)\eta_{\mathrm{bulk}}\big)+\eta_{\mathrm{bulk}}\sum_{n=1}^N\xi^n\big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\big)\\
&\quad+\sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c(\xi^i-\xi^c)\big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\big)+\sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i(\xi^c-\xi^i)\big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\big)
\end{aligned}
$$

in $\Omega \times [0,T]$. Hence, estimating based on the compatibility estimate (115) as well as the estimates (169) and (198) for the bulk cutoff yields the bound

$$-\sum_{n=1}^{N} \eta_n (v\cdot\nabla)\xi^n + \sum_{n=1}^{N} \xi^n \partial_t \eta_n = -(v\cdot\nabla)\xi + O(1\wedge \mathrm{dist}(\cdot, I_v)) \text{ in } \Omega\times[0,T]. \tag{208}$$

Adding zero twice and making use of the definition (197) of the candidate extension ξ together with the localization properties (171)–(175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26, we next compute

$$\begin{aligned}
&\mathbb{1}_{\mathrm{supp}\eta_n}\xi^n\otimes\xi^n \\
&= \mathbb{1}_{\mathrm{supp}\eta_n}\xi\otimes\xi + \mathbb{1}_{\mathrm{supp}\eta_n}(\xi^n-\xi)\otimes\xi^n + \mathbb{1}_{\mathrm{supp}\eta_n}\xi\otimes(\xi^n-\xi) \\
&= \mathbb{1}_{\mathrm{supp}\eta_n}\xi\otimes\xi \\
&\quad + \mathbb{1}_{\mathrm{supp}\eta_n}\eta_{\mathrm{bulk}}\xi^n\otimes\xi^n + \mathbb{1}_{\mathrm{supp}\eta_n}\eta_{\mathrm{bulk}}\xi\otimes\xi^n \\
&\quad + \mathbb{1}_{n=i\in\mathcal{I}}\mathbb{1}_{\mathrm{supp}\eta_i}\sum_{c\in\mathcal{C},i\sim c}\eta_c(\xi^i-\xi^c)\otimes\xi^i + \mathbb{1}_{n=c\in\mathcal{C}}\mathbb{1}_{\mathrm{supp}\eta_c}\sum_{i\in\mathcal{I},i\sim c}\eta_i(\xi^c-\xi^i)\otimes\xi^c \\
&\quad + \mathbb{1}_{n=i\in\mathcal{I}}\mathbb{1}_{\mathrm{supp}\eta_i}\sum_{c\in\mathcal{C},i\sim c}\eta_c\xi\otimes(\xi^i-\xi^c) + \mathbb{1}_{n=c\in\mathcal{C}}\mathbb{1}_{\mathrm{supp}\eta_c}\sum_{i\in\mathcal{I},i\sim c}\eta_i\xi\otimes(\xi^c-\xi^i)
\end{aligned} \tag{209}$$

in $\Omega \times [0,T]$. Relying on the same ingredients as for the previous computation we also have

$$\begin{aligned}
-\sum_{n=1}^{N}\eta_n(\nabla v)^{\mathsf{T}}\xi^n &= -(\nabla v)^{\mathsf{T}}\xi - \sum_{n=1}^{N}\eta_n(\nabla v)^{\mathsf{T}}(\xi^n-\xi) + \eta_{\mathrm{bulk}}(\nabla v)^{\mathsf{T}}\xi \\
&= -(\nabla v)^{\mathsf{T}}\xi + \eta_{\mathrm{bulk}}(\nabla v)^{\mathsf{T}}\xi - \eta_{\mathrm{bulk}}\sum_{n=1}^{N}\eta_n(\nabla v)^{\mathsf{T}}\xi^n \\
&\quad - \sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_i\eta_c(\nabla v)^{\mathsf{T}}(\xi^i-\xi^c) - \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i(\nabla v)^{\mathsf{T}}(\xi^c-\xi^i)
\end{aligned}$$

in $\Omega \times [0,T]$. The compatibility estimate (115) as well as the estimates (169) and (198) therefore imply in view of the previous two displays that

$$\begin{aligned}
&-\sum_{n=1}^{N}\eta_n(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n \\
&= -(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}\xi + O(1\wedge\mathrm{dist}(\cdot,I_v)) \quad \text{in } \Omega\times[0,T].
\end{aligned} \tag{210}$$

The combination of the bounds (207)–(210) now immediately entails the desired estimate (16d) on the time evolution of the global vector field ξ. $\square$

Proof of (16e). We get as a consequence of the product rule and inserting the local versions of (16e) from Proposition 15 and Proposition 16

$$\begin{aligned}
\xi\cdot\partial_t\xi &= \sum_{n=1}^{N}\eta_n\xi\cdot\partial_t\xi^n + \sum_{n=1}^{N}(\xi\cdot\xi^n)\partial_t\eta_n \\
&= -\sum_{n=1}^{N}\eta_n\xi^n\cdot(v\cdot\nabla)\xi^n + \sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot\partial_t\xi^n \\
&\quad + \sum_{n=1}^{N}(\xi\cdot\xi^n)\partial_t\eta_n + O(\mathrm{dist}(\cdot,I_v)^2\wedge 1) \quad \text{in } \Omega\times[0,T].
\end{aligned}$$

Adding zero to produce the left hand sides of the local versions of (16d) from Proposition 15 and Proposition 16 further updates the previous display to

$$\begin{aligned}\xi\cdot\partial_t\xi &= -\sum_{n=1}^{N}\eta_n\xi\cdot(v\cdot\nabla)\xi^n + \sum_{n=1}^{N}(\xi\cdot\xi^n)\partial_t\eta_n \\ &\quad - \sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n \\ &\quad + \sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot\big(\partial_t\xi^n+(v\cdot\nabla)\xi^n+(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\big) \\ &\quad + O(\mathrm{dist}(\cdot,I_v)^2\wedge 1) \quad \text{in } \Omega\times[0,T].\end{aligned}$$

We then continue with adding zeros to obtain

$$\begin{aligned}\xi\cdot\partial_t\xi &= -\xi\cdot(v\cdot\nabla)\xi \\ &\quad + \sum_{n=1}^{N}\big(\xi\cdot(\xi^n-\xi)\big)\big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\big) - |\xi|^2\big(\partial_t\eta_{\mathrm{bulk}}+(v\cdot\nabla)\eta_{\mathrm{bulk}}\big) \\ &\quad - \sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\xi\otimes\xi-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n \\ &\quad - \sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot(\mathrm{Id}-\xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi^n-\xi) \\ &\quad + \sum_{n=1}^{N}\eta_n(\xi-\xi^n)\cdot\big(\partial_t\xi^n+(v\cdot\nabla)\xi^n+(\mathrm{Id}-\xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\big) \\ &\quad + O(\mathrm{dist}(\cdot,I_v)^2\wedge 1) \quad \text{in } \Omega\times[0,T].\end{aligned} \tag{211}$$

As it is by now routine, we may employ the localization properties (171)–(175) of the partition of unity $(\eta_1,\ldots,\eta_N)$ from Lemma 26 and the estimates (169) and (198) for the bulk cutoff to reduce the task of estimating the right hand side terms of (211) to an application of the compatibility estimates (115)–(116). More precisely, we obtain by straightforward applications of these two ingredients that

$$\begin{aligned}&\sum_{n=1}^{N}\big(\xi\cdot(\xi-\xi^n)\big)\big(\partial_t\eta_n+(v\cdot\nabla)\eta_n\big) \\ &= \sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c^2\big((\xi^c-\xi^i)\cdot(\xi^c-\xi^i)\big)\big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\big) \\ &\quad + \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i\big((\xi^c-\xi^i)\cdot(\xi^i-\xi^c)\big)\big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\big) \\ &\quad + \sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_c^2\big(\xi^i\cdot(\xi^c-\xi^i)\big)\big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\big) \\ &\quad + \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_c\eta_i\big(\xi^i\cdot(\xi^i-\xi^c)\big)\big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\big) \\ &\quad + \sum_{i\in\mathcal{I}}\sum_{c\in\mathcal{C},i\sim c}\eta_i\eta_c\big(\xi^i\cdot(\xi^c-\xi^i)\big)\big(\partial_t\eta_i+(v\cdot\nabla)\eta_i\big) \\ &\quad + \sum_{c\in\mathcal{C}}\sum_{i\in\mathcal{I},i\sim c}\eta_i^2\big(\xi^i\cdot(\xi^i-\xi^c)\big)\big(\partial_t\eta_c+(v\cdot\nabla)\eta_c\big) \\ &\quad + O(\mathrm{dist}(\cdot,I_v)^2\wedge 1) \quad \text{in } \Omega\times[0,T],\end{aligned} \tag{212}$$

$$
\begin{aligned}
&\sum_{n=1}^{N} \eta_n(\xi - \xi^n) \cdot (\mathrm{Id} - \xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi - \xi^n) \\
&= \sum_{i\in\mathcal{I}} \sum_{c\in\mathcal{C}, i\sim c} \eta_i\eta_c^2(\xi^c - \xi^i) \cdot (\mathrm{Id} - \xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi^c - \xi^i) \\
&\quad + \sum_{c\in\mathcal{C}} \sum_{i\in\mathcal{I}, i\sim c} \eta_c\eta_i^2(\xi^i - \xi^c) \cdot (\mathrm{Id} - \xi\otimes\xi)(\nabla v)^{\mathsf{T}}(\xi^i - \xi^c) \\
&\quad + O(\mathrm{dist}(\cdot, I_v)^2 \wedge 1) \quad \text{in } \Omega\times[0,T],
\end{aligned} \tag{213}
$$

$$
\begin{aligned}
&\sum_{n=1}^{N} \eta_n(\xi - \xi^n) \cdot \big(\partial_t\xi^n + (v\cdot\nabla)\xi^n + (\mathrm{Id} - \xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n\big) \\
&= \sum_{i\in\mathcal{I}} \sum_{c\in\mathcal{C}} \eta_i\eta_c(\xi^c - \xi^i) \cdot \big(\partial_t\xi^i + (v\cdot\nabla)\xi^i + (\mathrm{Id} - \xi^i\otimes\xi^i)(\nabla v)^{\mathsf{T}}\xi^i\big) \\
&\quad + \sum_{c\in\mathcal{C}} \sum_{i\in\mathcal{I}, i\sim c} \eta_c\eta_i(\xi^i - \xi^c) \cdot \big(\partial_t\xi^c + (v\cdot\nabla)\xi^c + (\mathrm{Id} - \xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c\big) \\
&\quad + O(\mathrm{dist}(\cdot, I_v)^2 \wedge 1) \quad \text{in } \Omega\times[0,T],
\end{aligned} \tag{214}
$$

and finally

$$
\begin{aligned}
&\sum_{n=1}^{N} \eta_n(\xi - \xi^n) \cdot (\xi\otimes\xi - \xi^n\otimes\xi^n)(\nabla v)^{\mathsf{T}}\xi^n \\
&= \sum_{i\in\mathcal{I}} \sum_{c\in\mathcal{C}, i\sim c} \eta_c(\xi^c - \xi^i) \cdot (\xi\otimes\xi - \xi^i\otimes\xi^i)(\nabla v)^{\mathsf{T}}\xi^i \\
&\quad + \sum_{c\in\mathcal{C}} \sum_{i\in\mathcal{I}, i\sim c} \eta_i(\xi^i - \xi^c) \cdot (\xi\otimes\xi - \xi^c\otimes\xi^c)(\nabla v)^{\mathsf{T}}\xi^c \\
&\quad + O(\mathrm{dist}(\cdot, I_v)^2 \wedge 1) \quad \text{in } \Omega\times[0,T].
\end{aligned} \tag{215}
$$

We then exploit the compatibility estimates (115) and (116) for an estimate of (212), the compatibility estimate (115) for an estimate of (213), the local versions of (16d) from Proposition 15 and Proposition 16 in combination with the compatibility estimate (115) for an estimate of (214), and finally (209) together with the estimate for the bulk cutoff (169) and the compatibility estimate (115) to estimate (215). In summary, using also the bound on the advection derivative (198) as well as the coercivity estimate (169), we may upgrade (211) to the desired estimate (16e). □

7. Existence of Transported Weights: Proof of Lemma 8

We decompose the argument for the construction of a transported weight ϑ in the sense of Definition 3 in several steps.

Step 1: (Choice of suitable profiles) Let $\bar\vartheta\colon \mathbb{R}\to\mathbb{R}$ be chosen such that it represents a smooth truncation of the identity in the sense that $\bar\vartheta(r) = r$ for $|r| \leq \frac{1}{2}$, $\bar\vartheta(r) = -1$ for $r \leq -1$, $\bar\vartheta(r) = 1$ for $r \geq 1$, $0 \leq \bar\vartheta' \leq 2$ as well as $|\bar\vartheta''| \leq C$.

For each two-phase interface $i\in\mathcal{I}$ present in the interface I_v of the strong solution, we then define an auxiliary weight

$$
\bar\vartheta_i(x,t) := -\bar\vartheta\Big(\frac{\mathrm{sdist}(x, \mathcal{T}_i(t))}{\delta\widehat{r}}\Big), \quad (x,t)\in \mathrm{im}(\Psi_{\mathcal{T}_i}) \tag{216}
$$

where the change of variables $\Psi_{\mathcal{T}_i}$ and the associated signed distance $\mathrm{sdist}(\cdot, \mathcal{T}_i)$ are the ones from Definition 13 of the admissible localization radius r_i. Moreover, $\widehat{r}$ represents the localization scale of Lemma 26 and $\delta\in(0,1]$ denotes a constant to be chosen in the course of the proof.

Recalling also from Definition 17 of the admissible localization radii $(r_c)_{c\in\mathcal{C}}$ the definition of the change of variables $\Psi_{\partial\Omega}$ with associated signed distance $\mathrm{sdist}(\cdot,\partial\Omega)$ we define another two auxiliary weights by means of

$$\begin{aligned}&\bar{\vartheta}^{\pm}_{\partial\Omega}(x,t) := \mp\bar{\vartheta}\Big(\frac{\mathrm{sdist}(x,\partial\Omega)}{\delta\widehat{r}}\Big),\\ &(x,t)\in\bigcup_{t'\in[0,T]}\big(\Omega^{\pm}_v(t')\cap\Psi_{\partial\Omega}\big(\partial\Omega\times(-2\widehat{r},2\widehat{r})\big)\big)\times\{t'\}.\end{aligned}\tag{217}$$

Step 2: (Construction of the transported weight) Away from contact points and the interface but in the vicinity of the domain boundary, we introduce the following notational shorthand

$$\mathcal{U}_{\widehat{r}}(t) := \bigcup_{i\in\mathcal{I}}\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\cup\bigcup_{c\in\mathcal{C}}B_{\widehat{r}}\big(\mathcal{T}_c(t)\big),\quad t\in[0,T],\tag{218}$$

and then define

$$\begin{aligned}&\vartheta(x,t) := \bar{\vartheta}^{\pm}_{\partial\Omega}(x,t),\\ &(x,t)\in\bigcup_{t'\in[0,T]}\big(\Omega^{\pm}_v(t')\cap\Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big)\setminus\mathcal{U}_{\widehat{r}}(t')\big)\times\{t'\}.\end{aligned}\tag{219}$$

Fix next a two-phase interface $i\in\mathcal{I}$. Away from contact points but in the vicinity of the interface, we then define

$$\begin{aligned}&\vartheta(x,t) := \bar{\vartheta}_i(x,t),\\ &(x,t)\in\bigcup_{t'\in[0,T]}\Big(\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t')\times\{t'\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}}B_{\widehat{r}}\big(\mathcal{T}_c(t')\big)\Big)\times\{t'\}.\end{aligned}\tag{220}$$

Let now a contact point $c\in\mathcal{C}$ be fixed, and denote by $i\in\mathcal{I}$ the unique two-phase interface with $i\sim c$. Recall from Definition 17 of the admissible localization radius r_c that for all $t\in[0,T]$ we decomposed $\Omega\cap B_{r_c}(\mathcal{T}_c(t))$ by means of five pairwise disjoint open wedges $W^{\pm,c}_{\partial\Omega}(t), W^c_{\mathcal{T}_i}(t), W^c_{\Omega^{\pm}_v}(t)\subset\mathbb{R}^2$. In the wedge $W^c_{\mathcal{T}_i}$ containing the two-phase interface $\mathcal{T}_i\subset I_v$, we still define

$$\vartheta(x,t) := \bar{\vartheta}_i(x,t),\quad (x,t)\in\bigcup_{t'\in[0,T]}\big(\Omega\cap B_{\widehat{r}}\big(\mathcal{T}_c(t')\big)\cap W^c_{\mathcal{T}_i}(t')\big)\times\{t'\}.\tag{221}$$

In the wedges $W^{\pm,c}_{\partial\Omega}$ containing the domain boundary $\partial\Omega$, we instead set

$$\vartheta(x,t) := \bar{\vartheta}^{\pm}_{\partial\Omega}(x,t),\quad (x,t)\in\bigcup_{t'\in[0,T]}\big(\Omega\cap B_{\widehat{r}}\big(\mathcal{T}_c(t')\big)\cap W^{\pm,c}_{\partial\Omega}(t')\big)\times\{t'\}.\tag{222}$$

In the interpolation wedges $W^c_{\Omega^{\pm}_v}$, we make use of the interpolation parameter $\lambda^{\pm}_c$ of Lemma 24 to interpolate between the two constructions near the interface (221) and near the domain boundary (222). Recall in this context the convention that $\lambda^{\pm}_c(\cdot,t)$ was set equal to one on $\big(\partial W^c_{\Omega^{\pm}_v}(t)\cap\partial W^c_{\mathcal{T}_i}(t)\big)\setminus\mathcal{T}_c(t)$ and set equal to zero on $\big(\partial W^c_{\Omega^{\pm}_v}(t)\cap\partial W^{\pm,c}_{\partial\Omega}(t)\big)\setminus\mathcal{T}_c(t)$ for all $t\in[0,T]$. With this notation in place, we define on the interpolation wedges

$$\begin{aligned}&\vartheta(x,t) := \lambda^{\pm}_c(x,t)\bar{\vartheta}_i(x,t) + (1-\lambda^{\pm}_c(x,t))\bar{\vartheta}^{\pm}_{\partial\Omega}(x,t),\\ &(x,t)\in\bigcup_{t'\in[0,T]}\big(\Omega\cap B_{\widehat{r}}\big(\mathcal{T}_c(t')\big)\cap W^c_{\Omega^{\pm}_v}(t')\big)\times\{t'\}.\end{aligned}\tag{223}$$

Finally, choosing δ small enough in the definition (216) of the auxiliary weights $(\vartheta_i)_{i\in\mathcal{I}}$ and recalling the localization properties (179)–(181) of the scale $\widehat{r}$, it is safe to define in the space-time domain not captured by the definitions (219)–(223)

$$\vartheta(x,t) := \mp 1,$$
$$(x,t) \in \bigcup_{t'\in[0,T]} \big(\Omega_v^{\pm}(t') \setminus \big(\mathcal{U}_{\widehat{r}}(t') \cup \Psi_{\partial\Omega}(\partial\Omega\times[-\widehat{r},\widehat{r}])\big)\big)\times\{t'\}. \tag{224}$$

Recall for this definition also the notation (218).

Step 3: (Regularity and coercivity) The validity of the asserted sign conditions in Definition 3 are immediate from (219)–(224). Since the first-order derivatives of the interpolation parameter $\lambda_c^{\pm}$ feature controlled blow-up (154), it is also a direct consequence of the definitions (219)–(224) that $\vartheta \in W^{1,\infty}_{x,t}(\Omega\times[0,T])$ as asserted.

In view of the definition (224) of the weight in the bulk it suffices to establish (27) in the regions $\Omega\cap\Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big)\setminus\mathcal{U}_{\widehat{r}}(t)$, $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ and $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i\in\mathcal{I}$, all $c\in\mathcal{C}$ and all $t\in[0,T]$. However, in these regions the asserted estimate (27) is immediately implied by the properties of the truncation of unity $\bar{\vartheta}$ from *Step 1* of this proof and the definitions (219)–(223).

Step 4: (Advection equation) Because of the definition (224) of the weight ϑ in the bulk, it suffices to establish (28) in the regions $\Omega\cap\Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big)\setminus\mathcal{U}_{\widehat{r}}(t)$, $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ and $\Omega\cap B_{\widehat{r}}(\mathcal{T}_c(t))$ for all $i\in\mathcal{I}$, all $c\in\mathcal{C}$ and all $t\in[0,T]$.

Observe first that it follows from the definitions (217), (219) and (222) as well as the boundary condition for the fluid velocity $(v\cdot n_{\partial\Omega})|_{\partial\Omega}=0$ that

$$\partial_t\vartheta + (v\cdot\nabla)\vartheta = 0 \quad \text{along } \partial\Omega\setminus\bigcup_{c\in\mathcal{C}}\mathcal{T}_c(t) \tag{225}$$

for all $t\in[0,T]$. By a Lipschitz estimate together with the coercivity estimate (27), the desired estimate (28) follows in $\Omega\cap\Psi_{\partial\Omega}\big(\partial\Omega\times[-\widehat{r},\widehat{r}]\big)\setminus\mathcal{U}_{\widehat{r}}(t)$ for all $t\in[0,T]$.

Fix next a two-phase interface $i\in\mathcal{I}$. We then claim that

$$\big|\partial_t\bar{\vartheta}_i + (v\cdot\nabla)\bar{\vartheta}_i\big| \le C\,\mathrm{dist}(\cdot,I_v) \quad \text{in } \Omega\cap\Psi_{\mathcal{T}_i(t)}\big(\mathcal{T}_i(t)\times[-\widehat{r},\widehat{r}]\big) \tag{226}$$

for all $t\in[0,T]$. Indeed, one only needs to recall that the signed distance to the two-phase interface $\mathcal{T}_i\subset I_v$ is transported by the fluid velocity v to first order in the distance to the interface. In particular, combining (226) with the definition (220) and the coercivity estimate (27) entails (28) in $\Omega\cap\Psi_{\mathcal{T}_i}\big(\mathcal{T}_i(t)\times\{t\}\times[-\widehat{r},\widehat{r}]\big)\setminus\bigcup_{c\in\mathcal{C}} B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)$ for all $t\in[0,T]$.

Let now a contact point $c\in\mathcal{C}$ be given, and let $i\in\mathcal{I}$ be the unique two-phase interface such that $i\sim c$. The desired estimate (28) follows immediately from (226) and (221) in the wedge $\Omega\cap B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap W^c_{\mathcal{T}_i}(t)$ for all $t\in[0,T]$. For the wedges containing the domain boundary $\partial\Omega$, the estimate (28) in form of

$$\big|\partial_t\bar{\vartheta}^{\pm}_{\partial\Omega} + (v\cdot\nabla)\bar{\vartheta}^{\pm}_{\partial\Omega}\big| \le C\,\mathrm{dist}(\cdot,\partial\Omega) \text{ in } \Omega\cap B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap\big(W^c_{\Omega_v^{\pm}}(t)\cup W^{\pm,c}_{\partial\Omega}(t)\big) \tag{227}$$

for all $t\in[0,T]$, is satisfied because of the analogue of (225) and a Lipschitz estimate. Finally, in the interpolation wedges one may estimate

$$\begin{aligned}|\partial_t\vartheta+(v\cdot\nabla)\vartheta| &\le |\bar{\vartheta}_i-\bar{\vartheta}^{\pm}_{\partial\Omega}||\partial_t\lambda_c^{\pm}+(v\cdot\nabla)\lambda_c^{\pm}| \\ &\quad + \lambda_c^{\pm}|\partial_t\bar{\vartheta}_i+(v\cdot\nabla)\bar{\vartheta}_i| + (1-\lambda_c^{\pm})|\partial_t\bar{\vartheta}^{\pm}_{\partial\Omega}+(v\cdot\nabla)\bar{\vartheta}^{\pm}_{\partial\Omega}|.\end{aligned}$$

The desired bound thus follows from the estimate (156) for the advective derivative of the interpolation parameter $\lambda_c^{\pm}$, the estimates (226) and (227), and the fact that the auxiliary weights from (216) and (217) are compatible in the sense

$$|\bar{\vartheta}_i-\bar{\vartheta}^{\pm}_{\partial\Omega}| \le C(\mathrm{dist}(\cdot,\partial\Omega)\wedge\mathrm{dist}(\cdot,I_v))$$

in $\Omega\cap B_{\widehat{r}}\big(\mathcal{T}_c(t)\big)\cap W^c_{\Omega_v^{\pm}}(t)$ for all $t\in[0,T]$. This concludes the proof of Lemma 8. □

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Declarations

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Appendix A. Existence of Varifold Solutions to two-phase Fluid Flow With Surface Tension

The aim of this Appendix is to give a sketch of a proof regarding existence of varifold solutions to two-phase fluid flow with surface tension and with ninety degree contact angle (see Definition 11). Note that this is not treated by the work of Abels [1] in which the existence of a varifold solution in the presence of surface tension is only established in a full space setting. However, in principle it still suggests itself to follow, where possible, the structure of the proof for the case of an unbounded domain by Abels [1]. In this regard, we first discuss two tools which are needed due to the different setting of the present work, i.e., geometric evolution with a ninety degree contact angle condition and the associated boundary conditions for the solenoidal fluid velocity. These tools concern an existence result for weak solutions to the required transport equation (for sufficiently regular transport velocities) and elliptic regularity estimates for the Helmholtz decomposition associated with the bounded and smooth domain Ω. In a second step, we present the corresponding approximate problem, focusing again on the key steps of the proof which differ with respect to the case of an unbounded domain studied by Abels [1]. Note that analogous to the existence theory of [1], we will assume some regularity for the geometry of the initial data and, for simplicity, that the densities of the two fluids coincide and are normalized to 1.

Transport equation. In order to construct approximate solutions of the two-phase flow with surface tension and with ninety degree contact angle, one first needs an existence result for weak solutions to the transport equation in a bounded domain. In particular, it suffices to motivate the validity of [1, Lemma 2.3, $\Omega \equiv \mathbb{R}^d$] in case of a smooth and bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$.

To this aim, let the open subset $\Omega_0^+ \subset \Omega$ be subject to the regularity conditions in Definition 9, let $\chi_0 := \chi_{\Omega_0^+} \in \mathrm{BV}(\Omega;\{0,1\})$, let $T \in (0,\infty)$, and consider a sufficiently regular fluid velocity $v \in C([0,T];C^2_b(\Omega)) \cap C(\overline{\Omega}\times[0,T])$ such that $\operatorname{div} v = 0$ in Ω and $(n_{\partial\Omega}\cdot v)|_{\partial\Omega} = 0$. Consider any $C([0,T];C^2_b(\mathbb{R}^d))$

extension of v which we denote by $\widetilde{v}$. Then, a solution $\widetilde{\chi}$ to the transport equation associated with $\widetilde{v}$ can be constructed on $\mathbb{R}^d$ by the usual method of characteristics (see, e.g., [1, Proof of Lemma 2.3]). The associated flow map is a C^1-diffeomorphism at any time $t \in [0,T]$. However, note that it maps $\partial\Omega$ onto itself, due to $v|_{\partial\Omega} = \widetilde{v}|_{\partial\Omega}$ being tangential along $\partial\Omega$. Moreover, since the flow map is a global diffeomorphism (and since continuous images of connected sets are connected), it also maps Ω onto itself. Then, one can conclude by means of the same computations as in the proof of [1, Lemma 2.3] — using in the process the fact that $\operatorname{div} v = 0$ in Ω — that the restriction $\chi := \widetilde{\chi}|_{\Omega\times[0,T]} \in L^\infty(0,T;\mathrm{BV}(\Omega;\{0,1\}))$ is a weak solution of the transport equation associated with v in the sense of

$$\int_0^T \int_\Omega \chi\,(\partial_t\varphi + v\cdot\nabla\varphi)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \chi_0\varphi(x,0)\mathrm{d}x = 0 \tag{228}$$

for any $\varphi \in C^1_c([0,T);C(\overline{\Omega})) \cap C_c([0,T);C^1(\overline{\Omega}))$. Moreover, we have

$$\|\chi\|_{L^\infty(0,T;BV(\Omega))} \leqslant M\left(\|v\|_{C([0,T];C^2_b(\Omega))}\right)\|\chi_0\|_{BV(\Omega)}\,, \tag{229}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\chi(\cdot,t)|\,(\Omega) = -\left\langle H_{\chi(\cdot,t)}, v(\cdot,t)\right\rangle \qquad \text{for all } t\in(0,T) \tag{230}$$

for some continuous function M. Note that the latter holds because the 90 degree contact angle condition is preserved by sufficiently regular transport velocities (see, e.g., the remark after Definition 10).

Helmholtz decomposition associated with bounded domains. We recall properties of the Helmholtz projection P_Ω associated with the smooth bounded domain Ω, referring the reader to [21, Corollaries 7.4.4-5] (see also [30]).

Define $W_p(\Omega) := \{g \in W^{1,p}(\Omega;\mathbb{R}^d) : \operatorname{div} g = 0, (g\cdot n_{\partial\Omega})|_{\partial\Omega} = 0\}$. Given $f \in W^{1,p}(\Omega;\mathbb{R}^d)$, $2 \le p < \infty$, there are unique functions $\phi \in W^{2,p}(\Omega)$ and $w \in W_p(\Omega)$ such that $f = \nabla\phi + w$. The bounded linear operator $P_\Omega \in \mathcal{B}(W^{1,p}(\Omega;\mathbb{R}^d), W_p(\Omega))$ defined by $P_\Omega f := w$ is a projection, which is the Helmholtz projection associated with the smooth bounded domain Ω. Moreover, if $f \in W^{2,p}(\Omega;\mathbb{R}^d)$ it holds $\phi \in W^{3,p}(\Omega)$ and

$$\|P_\Omega f\|_{W^{2,p}(\Omega;\mathbb{R}^d)} \le C\|f\|_{W^{2,p}(\Omega;\mathbb{R}^d)}, \tag{231}$$

and if $f \in W^{k,2}(\Omega;\mathbb{R}^d)$, $k \ge 2$, then $\phi \in W^{k,2}(\Omega)$ and

$$\|P_\Omega f\|_{W^{k,2}(\Omega;\mathbb{R}^d)} \le C\|f\|_{W^{k,2}(\Omega;\mathbb{R}^d)}. \tag{232}$$

This follows from existence and regularity theory of the associated Neumann problem (see for the case $p > 2$ the result of [21, Corollary 7.4.5])

$$\begin{aligned} \Delta\phi &= \operatorname{div} f && \text{in } \Omega,\\ (n_{\partial\Omega}\cdot\nabla)\phi &= f\cdot n_{\partial\Omega} && \text{on } \partial\Omega. \end{aligned}$$

Solutions to approximate two-phase fluid flow. In order to formulate the approximate equations, let ψ be a standard mollifier, for every $k \in \mathbb{N}$ we denote by $\psi_k := k^d\psi(k\cdot)$ its usual rescaling, and by P_Ω the Helmholtz projection associated with the smooth domain Ω. Moreover, let $\Psi_k\cdot = P_\Omega(\Psi_k * \cdot)$. Consider the initial data $v_0 \in L^2(\Omega)$ with $\operatorname{div} v_0 = 0$ and $(n_{\partial\Omega}\cdot v_0)|_{\partial\Omega} = 0$, and let $\chi_0 := \chi_{\Omega_0^+} \in \mathrm{BV}(\Omega;\{0,1\})$, where $\Omega_0^+ \subset \Omega$ is subject to the regularity conditions in Definition 9. Let $\mu,\sigma > 0$. Then, we consider an approximate two-phase flow on $(0,T_w)$, $T_w \in (0,\infty)$. This is a pair (v_k,χ_k) consisting on one side of a fluid velocity field $v_k \in L^\infty([0,T_w];L^2(\Omega)) \cap L^2([0,T_w];W_2(\Omega))$ solving

$$\begin{aligned} &\int_\Omega v_k(\cdot,T)\cdot\eta(\cdot,T)\,\mathrm{d}x - \int_\Omega v_0\cdot\eta(\cdot,0)\,\mathrm{d}x - \int_0^T\int_\Omega v_k\cdot\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_0^T\int_\Omega \Psi_k v_k\otimes\psi_k * v_k : \nabla(\psi_k*\eta)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \mu(\nabla v_k + \nabla v_k^\mathsf{T}) : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &= \sigma\int_0^T\int_{\partial^*\{\chi_k=1\}\cap\Omega} \mathrm{H}_{\chi_k}\cdot\Psi_k\eta\,\mathrm{d}S\,\mathrm{d}t \end{aligned} \tag{233}$$

for a.e. $T \in [0, T_w)$ and every $\eta \in C^\infty([0, T_w); C^1(\overline{\Omega}; \mathbb{R}^d) \cap \bigcap_{p \geq 2} W^{2,p}(\Omega; \mathbb{R}^d))$ with $\operatorname{div} \eta = 0$ and $(n_{\partial\Omega} \cdot \eta)_{\partial\Omega} = 0$, and on the other side an evolving phase indicator $\chi_k \in L^\infty([0, T_w]; \mathrm{BV}(\Omega; \{0, 1\}))$ which is the unique weak solution — in the sense of (228) — to the transport equation

$$\begin{aligned} \partial_t \chi_k + (\Psi_k v_k) \cdot \nabla \chi_k &= 0 && \text{in } (0, T_w) \times \Omega, \\ \chi_k|_{t=0} &= \chi_0 && \text{in } \Omega. \end{aligned}$$

The existence of approximate solutions (v_k, χ_k) satisfying the energy equality

$$\begin{aligned} &\frac{1}{2}\|v_k(\cdot, T)\|^2_{L^2(\Omega)} + \sigma|\nabla \chi_k(\cdot, T)|(\Omega) + \frac{\mu}{2}\|\nabla v_k\|^2_{L^2(\Omega \times (0,T))} \\ &= \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \sigma|\nabla \chi_0|(\Omega), \quad T \in (0, T_w), \end{aligned} \tag{234}$$

and satisfying

$$\text{the map} \quad (0, T_w) \ni t \mapsto |\nabla \chi_k(\cdot, t)|(\Omega) \quad \text{is absolutely continuous,} \tag{235}$$

can then be proved by means of a fixed-point argument as done in [1, Proof of Theorem 4.2], relying in the process on the above two ingredients corresponding to the different setting of the present work: the existence result for weak solutions to the transport Eq. (228) with sufficiently regular transport velocity, and the elliptic regularity estimates (232) for the Helmholtz projection associated with Ω. In particular, one obtains uniform bounds

$$\sup_{k \in \mathbb{N}} \sup_{t \in (0, T_w)} \|v_k(\cdot, t)\|^2_{L^2(\Omega)} + \sup_{k \in \mathbb{N}} \|\nabla v_k\|^2_{L^2(\Omega \times (0, T_w))} < \infty, \tag{236}$$

$$\sup_{k \in \mathbb{N}} \sup_{t \in (0, T_w)} |\nabla \chi_k(\cdot, t)|(\Omega) < \infty. \tag{237}$$

Limit passage in the approximation scheme to a varifold solution. As for the passage to the limit, we only discuss the surface tension term on the right hand side of the approximate problem (233) as well as the validity of the energy inequality (41). The other terms as well as the passage to the limit in the transport equation can be treated as in [1]. First, we define a varifold $V_k \in \mathcal{M}((0, T_w) \times \overline{\Omega} \times \mathbb{S}^{d-1})$ by

$$V_k := \mathcal{L}^1 \llcorner (0, T_w) \otimes (V_k(t))_{t \in (0, T_w)}, \tag{238}$$

where

$$V_k(t) := |\nabla \chi_k(\cdot, t)| \llcorner \Omega \otimes \big(\delta_{\frac{\nabla \chi_k(\cdot, t)}{|\nabla \chi_k(\cdot, t)|}}\big)_{x \in \Omega} \in \mathcal{M}(\overline{\Omega} \times \mathbb{S}^{d-1}) \quad \text{for any } t \in (0, T_w).$$

Since $\chi_k \in L^\infty([0, T_w]; \mathrm{BV}(\Omega; \{0, 1\}))$ is uniformly bounded in the sense of (237), there then exists $\chi \in L^\infty([0, T_w]; \mathrm{BV}(\Omega; \{0, 1\}))$ such that, up to taking a subsequence,

$$\chi_k \rightharpoonup^* \chi \qquad \text{in } L^\infty(\Omega \times (0, T_w)), \tag{239}$$

$$\nabla \chi_k \rightharpoonup^* \nabla \chi \qquad \text{in } L^\infty([0, T_w]; \mathcal{M}(\Omega)). \tag{240}$$

Moreover, we have $\sup_k \|V_k\|_{\mathcal{M}} < \infty$ due to (237) and the definition of V_k. In particular, there exists $V \in \mathcal{M}((0, T_w) \times \overline{\Omega} \times \mathbb{S}^{d-1})$ such that, up to taking a subsequence,

$$V_k \rightharpoonup^* V \qquad \text{in } \mathcal{M}((0, T_w) \times \overline{\Omega} \times \mathbb{S}^{d-1}). \tag{241}$$

Note that the compatibility condition (42) then simply follows from exploiting (240) and (241). As a preparation for the remaining arguments, note also that thanks to the condition (235) a careful inspection of the argument of [16, Lemma 2] reveals that one may disintegrate the limit varifold V in form of

$$V = \mathcal{L}^1 \llcorner (0, T_w) \otimes (V_t)_{t \in (0, T_w)}, \quad V_t \in \mathcal{M}(\overline{\Omega} \times \mathbb{S}^{d-1}), \, t \in (0, T_w), \tag{242}$$

and that the limit interface energy satisfies

$$|V_t|_{\mathbb{S}^{d-1}}(\overline{\Omega}) \leq \liminf_k |\nabla \chi_k(\cdot, t)|(\Omega) \quad \text{for a.e. } t \in [0, T_w). \tag{243}$$

For any $\eta \in C^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$ such that $\operatorname{div}\eta = 0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega} = 0$, we discuss the limit of

$$\int_0^T\int_\Omega\Big(\mathrm{Id}-\frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big):\nabla(\Psi_k\eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t \quad \text{for } k\to\infty,$$

for almost every $T\in[0,T_w)$. By adding a zero, we obtain

$$\begin{aligned}&\int_0^T\int_\Omega\Big(\mathrm{Id}-\frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big):\nabla(\Psi_k\eta-\eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t\\ &+\int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla\eta\,\mathrm{d}V_k(t,x,s),\end{aligned}$$

where the second term converges to $\int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla\eta\,\mathrm{d}V_t(x,s)$ for $k\to\infty$ for any $\eta\in C_0^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$. Indeed, the latter guarantees $(\mathrm{Id}-s\otimes s):\nabla\eta\in C_0((0,T_w)\times\overline{\Omega}\times\mathbb{S}^{d-1})$ so that one may use (241) for such η. However, the additional support assumption on the time variable can be removed by means of a standard truncation argument relying on the disintegration formulas (238) and (242), respectively, and the uniform bound $\sup_k\|V_k\|_{\mathcal{M}}<\infty$. As for the first term, we exploit the regularity properties of the Helmholtz projection. More precisely, we may estimate for any $p>3$ based on (231) and the Sobolev embedding $W^{1,p}(\Omega)\hookrightarrow C(\overline{\Omega})$, $d\in\{2,3\}$,

$$\begin{aligned}&\left|\int_0^T\int_\Omega\Big(\mathrm{Id}-\frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big):\nabla(\Psi_k\eta-\eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t\right|\\ &\leq C\int_0^T\|\nabla(\Psi_k\eta-\eta)\|_{C(\overline{\Omega};\mathbb{R}^{d\times d})}\,\mathrm{d}t\\ &\leq C\int_0^T\|\nabla P_\Omega(\psi_k*\eta-\eta)\|_{C(\overline{\Omega};\mathbb{R}^{d\times d})}\,\mathrm{d}t\\ &\leq C\int_0^T\|\psi_k*\eta-\eta\|_{W^{2,p}(\Omega;\mathbb{R}^d)}\,\mathrm{d}t.\end{aligned}$$

The right hand side obviously goes to zero by letting $k\to\infty$. In summary, we obtain as desired

$$\begin{aligned}&\int_0^T\int_\Omega\Big(\mathrm{Id}-\frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big):\nabla(\Psi_k\eta)\,\mathrm{d}|\nabla\chi_k|\,\mathrm{d}t\\ &\to\int_0^T\int_{\overline{\Omega}\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla\eta\,\mathrm{d}V_t(x,s)\quad\text{for } k\to\infty,\end{aligned}$$

for almost every $T\in[0,T_w)$ and all $\eta\in C^\infty([0,T_w);C^1(\overline{\Omega};\mathbb{R}^d)\cap\bigcap_{p\geq 2}W^{2,p}(\Omega;\mathbb{R}^d))$ such that $\operatorname{div}\eta=0$ and $(\eta\cdot n_{\partial\Omega})|_{\partial\Omega}=0$.

At last, we comment how to recover the energy inequality (41).

This can be obtained from combining the energy equality (234) with the lower-semicontinuity property (243) and the convergence properties of v_k to its limit v (i.e., up to a subsequence, $v_k\rightharpoonup v$ in $L^2(0,T_w;H^1(\Omega))$ and $v_k\rightharpoonup^* v$ in $L^\infty(0,T_w;L^2(\Omega))$ due to the uniform bound (236)).

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