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ABSTRACT

We consider the many-body time evolution of weakly interacting bosons in the mean field regime for initial coherent states. We show that bounded k-particle operators, corresponding to dependent random variables, satisfy both a law of large numbers and a central limit theorem.

I. INTRODUCTION AND MAIN RESULTS

We consider N weakly interacting bosons in the mean-field regime described on $L^2_s(\mathbb{R}^{3N})$, the symmetric subspace of $L^2(\mathbb{R}^{3N})$, by the Hamilton operator

$$H_N = \sum_{i=1}^N (-\Delta_i) + \frac{1}{N}\sum_{i<j=1}^N v(x_i - x_j), \tag{1.1}$$

with the two-body interaction potential v satisfying

$$v^2 \leq C(1-\Delta) \tag{1.2}$$

for a positive constant $C > 0$. The mean-field regime is characterized through weak and long-range interactions of particles. Trapped Bose gases at extremely low temperatures, as prepared in the experiments, are known to relax to the ground state. The ground state ψ_N^{gs} of (1.1), if it exists, exhibits Bose–Einstein condensation,[17] i.e., the associated ℓ-particle reduced density

$$\gamma^{(\ell)}_{\psi_N^{\mathrm{gs}}} := \mathrm{tr}\,_{\ell+1,\ldots,N}|\psi_N^{\mathrm{gs}}\rangle\langle\psi_N^{\mathrm{gs}}| \tag{1.3}$$

converges in the trace norm to

$$\gamma^{(\ell)}_{\psi_N^{\mathrm{gs}}} \to |\varphi\rangle\langle\varphi|^{\otimes \ell} \qquad \text{as} \quad N\to\infty \tag{1.4}$$

for all $\ell \in \mathbb{N}$, where $\varphi \in L^2(\mathbb{R}^3)$ denotes the condensate wave function, known to be the Hartree minimizer. However, we remark that the factorized state $\varphi^{\otimes N}$ does not approximate the ground state due to correlations of particles.[11]

A. Law of large numbers

Turning to the probabilistic picture, the property of Bose–Einstein condensation (1.4) implies a law of large numbers for bounded one-particle operators.[3] To be more precise, for $k \in \mathbb{N}$, we denote with $O^{(k)}$ a bounded, self-adjoint k-particle operator on $L^2(\mathbb{R}^{3k})$ and with $\underline{i}_k$ the multi-index

$$\underline{i}_k = (i_1, \ldots, i_k) \in \mathcal{I}_N^{(k)}, \quad \mathcal{I}_N^{(k)} := \left\{(i_1, \ldots, i_k) \in \{1, \ldots, N\}^k \middle| \ i_j \neq i_m \qquad \text{for} j \neq m\right\}. \tag{1.5}$$

Then, we define for fixed $k \leq N$ the N-particle operator

$$O_{\underline{i}_k}^{(k)} \quad \text{with} \quad \underline{i}_k \in \mathcal{I}_N^{(k)}, \tag{1.6}$$

acting as $O^{(k)}$ on particles $i_1, \ldots, i_k$ and as identity elsewhere. We consider the operator $O_{\underline{i}_k}^{(k)}$ as a random variable with probability distribution determined through ψ_N by

$$\mathbb{P}_{\psi_N}\left[O_{\underline{i}_k}^{(k)} \in A\right] = \mathbb{E}_{\psi_N}\left[\chi_A\left(O_{\underline{i}_k}^{(k)}\right)\right] = \langle \psi_N | \chi_A\left(O_{\underline{i}_k}^{(k)}\right) | \psi_N \rangle, \tag{1.7}$$

where χ_A denotes the characteristic function of the set $A \subset \mathbb{R}$.

For one-particle operators, factorized states correspond to i.i.d. random variables as for any subsets $A_1, A_2 \subset \mathbb{R}$ and $i, j \in \mathcal{I}_N^{(1)}$ with $i \neq j$,

$$\begin{aligned}
\mathbb{P}_{\varphi^{\otimes N}}\left[O_i^{(1)} \in A_1,\ O_j^{(1)} \in A_2\right] &= \langle \varphi^{\otimes N} | \chi_{A_1}\left(O_i^{(1)}\right) \chi_{A_2}\left(O_j^{(1)}\right) | \varphi^{\otimes N} \rangle \\
&= \langle \varphi | \chi_{A_1}\left(O^{(1)}\right) | \varphi \rangle \langle \varphi | \chi_{A_2}\left(O^{(1)}\right) | \varphi \rangle \\
&= \langle \varphi^{\otimes N} | \chi_{A_1}\left(O_i^{(1)}\right) | \varphi^{\otimes N} \rangle \langle \varphi^{\otimes N} | \chi_{A_2}\left(O_j^{(1)}\right) | \varphi^{\otimes N} \rangle \\
&= \mathbb{P}_{\varphi^{\otimes N}}\left[O_i^{(1)} \in A_1\right] \mathbb{P}_{\varphi^{\otimes N}}\left[O_j^{(1)} \in A_2\right].
\end{aligned} \tag{1.8}$$

In particular, for factorized states, Chebychef's inequality implies a law of large numbers for the centered averaged sum,

$$\frac{1}{N} O_N^{(1)} := \frac{1}{N} \sum_{i=1}^{N} \left(O_i^{(1)} - \langle \varphi | O^{(1)} | \varphi \rangle\right). \tag{1.9}$$

In contrast to one-particle operators for k-particle operators with $k \geq 2$, factorized states do not correspond to i.i.d. random variables. In fact, for $k \geq 2$, we have

$$\mathbb{E}_{\varphi^{\otimes N}}\left[\left(O_{\underline{i}_k}^{(k)} - \langle \varphi^{\otimes k} | O^{(k)} | \varphi^{\otimes k} \rangle\right)\left(O_{\underline{j}_k}^{(k)} - \langle \varphi^{\otimes k} | O^{(k)} | \varphi^{\otimes k} \rangle\right)\right] \neq 0 \tag{1.10}$$

for all $\underline{i}_k \neq \underline{j}_k$ for which $\underline{i}_k$ contains at least one element of $\underline{j}_k$. We conclude that in this case, the random variables are correlated and, thus, dependent. In contrast, whenever $\underline{i}_k$ does not intersect with $\underline{j}_k$, the random variables $O_{\underline{i}_k}^{(k)}, O_{\underline{j}_k}^{(k)}$ are independent [following from arguments similarly to (1.8)]. Consequently, for factorized states, the random variables $\{O_{\underline{i}_k}^{(k)}\}_{\underline{i}_k \in \mathcal{I}_N^{(k)}}$ denote a sequence of m-dependent random variables with $m \in \mathbb{R}$. Still, as in Theorem 1.1, the centered averaged sum

$$\frac{1}{\binom{N}{k}} O_N^{(k)} := \frac{1}{\binom{N}{k}} \sum_{\underline{i}_k \in \mathcal{I}_N^{(k)}} \left(O_{\underline{i}_k}^{(k)} - \langle \varphi^{\otimes k} | O^{(k)} | \varphi^{\otimes k} \rangle\right) \tag{1.11}$$

satisfies a law of large numbers.

Theorem 1.1 *(law of large numbers). For $k \in \mathbb{N}$, let $O^{(k)}$ denote a self-adjoint bounded k-particle operator, $\varphi \in L^2(\mathbb{R}^3)$, and $\psi_N \in L_s^2(\mathbb{R}^{3N})$ be a bosonic wave function satisfying*

$$\gamma_{\psi_N}^{(\ell)} \to |\varphi\rangle\langle\varphi|^{\otimes \ell} \qquad \text{as} \quad N \to \infty \tag{1.12}$$

for all $\ell \in \mathbb{N}$. Then, for any fixed $k \in \mathbb{N}$ and $\delta > 0$, the averaged sum $O_N^{(k)}$ defined in (1.11) satisfies

$$\mathbb{P}_{\psi_N}\left[\left|\frac{1}{\binom{N}{k}} O_N^{(k)}\right| > \delta\right] \to 0 \qquad \text{as} \quad N \to \infty. \tag{1.13}$$

For factorized states, we have $\gamma_{\varphi^{\otimes N}}^{(\ell)} = |\varphi\rangle\langle\varphi|^{\otimes \ell}$, and a law of large numbers follows from Theorem 1.1.

In particular, Theorem 1.1 shows that the property of condensation (1.12) implies a law of large numbers for bounded k-particle operators for fixed $k \in \mathbb{N}$. Thus, Theorem 1.1 generalizes known results from Ref. 3 for bounded one-particle operators to k-particle operators with fixed $k \in \mathbb{N}$. We recall that the ground state ψ_N^{gs} of (1.1) cannot be approximated by a factorized state; nonetheless, the condensation property (1.4) ensures that bounded k-particle operators satisfy a law of large numbers for ψ_N^{gs}, too.

1. Generalization to the Fock space

In order to generalize Theorem 1.1 to any Fock space vector $\psi \in \mathcal{F}$ of the bosonic Fock space $\mathcal{F} = \bigoplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}$, we introduce some more notation.

For any vector $\psi \in \mathcal{F}$, we have the following identity for the operator $\widetilde{O}^{(k)} = O^{(k)} - \langle \varphi^{\otimes k}|O^{(k)}|\varphi^{\otimes k}\rangle$ on the N-particle sector:

$$\left(d\Gamma^{(k)}(\widetilde{O}^{(k)})\psi\right)^{(N)} = O_N \psi^{(N)}, \tag{1.14}$$

where we introduced the second quantization for any integral operator $O^{(k)}$ on $L^2(\mathbb{R}^{3k})$,

$$d\Gamma^{(k)}(O^{(k)}) = \int dx_1 \ldots dx_k dy_1 \ldots dy_k \; O^{(k)}(x_1,\ldots,x_k; y_1,\ldots y_k) a^*_{x_1} \ldots a^*_{x_k} a_{y_1} \ldots a_{y_k}. \tag{1.15}$$

Note that we can generalize the definition of the probability distribution (1.7) to the Fock space: For any $\psi \in \mathcal{F}$, integral operator $O^{(k)}$ on $L^2(\mathbb{R}^{3k})$, and $A \subset \mathbb{R}$, we have

$$\mathbb{P}_\psi\left[d\Gamma^{(k)}(O^{(k)}) \in A\right] = \mathbb{E}_\psi\left[\chi_A\left(d\Gamma^{(k)}(O^{(k)})\right)\right] = \langle \psi|\chi_A\left(d\Gamma^{(k)}(O^{(k)})\right)|\psi\rangle. \tag{1.16}$$

On the Fock space, the k-particle reduced density $\gamma_\psi^{(k)}$ associated with $\psi \in \mathcal{F}$ is given by the integral operator with kernel

$$\gamma_\psi^{(k)}(x_1,\ldots,x_k; y_1,\ldots y_k) := \langle \psi|a^*_{y_1} \ldots a^*_{y_k} a_{x_1} \ldots a_{x_k}|\psi\rangle. \tag{1.17}$$

It follows from a generalization of Theorem 1.1's proof in Sec. II that for $\psi \in \mathcal{F}$ satisfying (1.12), we have for any $\delta > 0$,

$$\mathbb{P}_{\psi_N}\left[\left|\frac{1}{\binom{N}{k}} d\Gamma^{(k)}(\widetilde{O}^{(k)})\right| > \delta\right] \to 0 \qquad \text{as} \quad N \to \infty. \tag{1.18}$$

2. Dynamics

We are interested in the dynamics of initially trapped Bose gases. Removing the trap, the bosons evolve with respect to the Schrödinger equation,

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}, \tag{1.19}$$

with H_N being the mean-field Hamiltonian given in (1.1). In the following, we consider coherent initial data, i.e., initial data of the form

$$\psi_{N,0} = W(\sqrt{N}\varphi)\Omega, \tag{1.20}$$

where Ω denotes the vacuum of the bosonic Fock space $\mathcal{F} = \bigoplus_{n\geq 0} L^2(\mathbb{R}^3)^{\otimes_s^n}$ equipped with creation and annihilation operators $a^*(f), a(f)$ for $f \in L^2(\mathbb{R}^3)$, $W(f) = e^{a^*(f) - a(f)}$ denotes the Weyl operator, and $f \in H^1(\mathbb{R}^3)$ denotes the condensate wave function. Coherent states of the form (1.20) exhibit Bose–Einstein condensation in the quantum state φ, i.e., they satisfy (1.4).

Thus, it follows from Theorem 1.1 that initially a law of large numbers holds true. The property of condensation is preserved along the many-body time evolution (Ref. 4, Theorem 3.1), i.e., the ℓ-particle reduced density $\gamma_{N,t}^{(\ell)}$ associated with $\psi_{N,t}$ satisfies

$$\gamma_{N,t}^{(\ell)} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes \ell} \qquad \text{as} \quad N \to \infty \qquad \text{for all} \quad \ell \in \mathbb{N}, \tag{1.21}$$

where $\varphi_t \in H^1(\mathbb{R}^3)$ denotes the solution to the Hartree equation,

$$i\partial_t \varphi_t = h_{\mathrm{H}}(t)\varphi_t \quad \text{with} \quad h_{\mathrm{H}}(t) = -\Delta + (v * |\varphi_t|^2), \tag{1.22}$$

with initial data $\varphi_0 = \varphi \in H^1(\mathbb{R}^3)$ (for further references, see, e.g., Refs. 1, 2, 7, 9, 10, 15, 22, and 23). Theorem 1.1 and (1.21) show that

$$\frac{1}{\binom{N}{k}} d\Gamma(O_t^{(k)}) := \frac{1}{\binom{N}{k}} d\Gamma\left(O^{(k)} - \langle \varphi_t^{\otimes k}|O^{(k)}|\varphi_t^{\otimes k}\rangle\right) \tag{1.23}$$

satisfies a law of large numbers for positive times $t > 0$ too, i.e., for any $\delta > 0$,

$$\mathbb{P}_{\psi_{N,t}}\left[\left|\frac{1}{\binom{N}{k}} d\Gamma(O_t^{(k)})\right| > \delta\right] \to 0 \qquad \text{as} \quad N \to \infty \qquad \text{for all} \quad t \in \mathbb{R}. \tag{1.24}$$

B. Central limit theorem

While the law of large numbers characterizes the mean of the probability distribution, fluctuations around the mean are governed through the central limit theorem. Before stating our result on a central limit theorem for fluctuations of order $O(N^{k-1/2})$, we introduce some notations. For a bounded k-particle integral operator $O^{(k)}$ and $\varphi \in L^2(\mathbb{R}^3)$, we define

$$\begin{aligned}&\left(\overline{\varphi}_t^{\otimes(k-1)} O^{(k)} \varphi_t^{\otimes k}\right)_j(x)\\ &\quad := \int dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_k dy_1 \dots dy_k \, O^{(k)}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k; y_1, \dots y_k) \prod_{\substack{i=1\\ i\neq j}}^k \overline{\varphi}_t(x_i) \prod_{m=1}^k \varphi_t(y_m),\end{aligned} \tag{1.25}$$

and furthermore, for $t \in \mathbb{R}$, $0 \leq s \leq t$, and $j \in \{1, \dots, k\}$, the function $f_{s;t}^{(j)}$ is given by

$$i\partial_s f_{s;t}^{(j)} = (h_{\mathrm{H}}(s) + K_{1,s} - K_{2,s} J) f_{s;t}^{(j)} \quad \text{with} \quad f_{t;t}^{(j)} = q_t \left(\overline{\varphi}_t^{\otimes(k-1)} O^{(k)} \varphi_t^{\otimes k}\right)_j, \tag{1.26}$$

with the anti-linear operator $Jf = \overline{f}$ for any $f \in L^2(\mathbb{R}^3)$, $q_t = 1 - |\varphi_t\rangle\langle\varphi_t|$, the Hartree Hamiltonian h_{H} defined in (1.22), and the operators

$$K_{1,t}(x;y) = \varphi_t(x) v(x-y) \overline{\varphi}_t(y), \quad K_{2,t}(x;y) = \varphi_t(x) v(x-y) \varphi_t(y). \tag{1.27}$$

Theorem 1.2 *(central limit theorem). For $k, N \in \mathbb{N}$ with $k \leq N$, let $O^{(k)}$ be a self-adjoint, bounded k-particle integral operator and φ_t be the solution to the Hartree equation (1.22) with initial datum $\varphi_0 = \varphi \in H^1(\mathbb{R}^3)$. Let $\psi_{N,t} \in L_s^2(\mathbb{R}^{3N})$ denote the solution to the Schrödinger equation (1.19) with the initial datum of the form $\psi_{N,0} = W(\sqrt{N}\varphi)\Omega$.*

Let $a, b \in \mathbb{R}$ with $a < b$; then, there exists a constant $C_{a,b,k} > 0$ such that the centered averaged sum $d\Gamma(O_t^{(k)})$ defined in (1.23) satisfies

$$\left| \mathbb{P}_{\psi_{N,t}}\left[N^{-k+1/2} d\Gamma(O_t^{(k)}) \in [a,b] \right] - \mathbb{P}[\mathcal{G}_t \in [a,b]] \right| \leq C_{a,b,k} \, e^{C|t|} N^{-1/12}, \tag{1.28}$$

where $\mathcal{G}_t$ denotes the centered Gaussian random variable with variance given by

$$\sigma_t^2 = \sum_{i,j=1}^k \langle f_{0;t}^{(i)} || f_{0;t}^{(j)} \rangle. \tag{1.29}$$

We remark that for a factorized state, we can explicitly compute the variance

$$\begin{aligned}\sigma_N^2 &= \mathbb{E}_{\varphi^{\otimes N}}\left[\left(O_N^{(k)} \right)^2 \right] - \mathbb{E}_{\varphi^{\otimes N}}\left[O_N^{(k)} \right]^2 \\ &= \sum_{\underline{i}_k, \underline{j}_k \in I_N^{(k)}} \mathbb{E}_{\varphi^{\otimes N}}\left[\widetilde{O}_{\underline{i}_k}^{(k)} \widetilde{O}_{\underline{j}_k}^{(k)} \right] - \left(\sum_{\underline{i}_k \in I_N^{(k)}} \mathbb{E}_{\varphi^{\otimes N}}\left[\widetilde{O}_{\underline{i}_k}^{(k)} \right] \right)^2,\end{aligned} \tag{1.30}$$

where we introduced the centered k-particle operator

$$\widetilde{O}^{(k)} = O^{(k)} - \langle \varphi^{\otimes k} | O^{(k)} | \varphi^{\otimes k} \rangle. \tag{1.31}$$

The last sum of the rhs of (1.30) vanishes. Furthermore, the first sum vanishes whenever $\underline{j}_k$ does not intersect with $\underline{i}_k$, and for the remaining terms, we find

$$\sigma_N^2 = \sum_{i,j=1}^k \frac{N \cdots (N-2k+1)}{k!(k-1)!} \mathcal{M}_{\varphi^{\otimes N}}(i,j) + O\left(N^{2k-2}\right) \tag{1.32}$$

using the definition

$$\mathcal{M}_{\varphi^{\otimes N}}(i,j) = \left\langle \left(\overline{\varphi}^{\otimes(k-1)}\widetilde{O}^{(k)}\varphi^{\otimes k}\right)_i \middle| \left(\overline{\varphi}^{\otimes(k-1)}\widetilde{O}^{(k)}\varphi^{\otimes k}\right)_j \right\rangle_{L^2(\mathbb{R}^3)}$$
$$= \left\langle q\left(\overline{\varphi}^{\otimes(k-1)}O^{(k)}\varphi^{\otimes k}\right)_i \middle| q\left(\overline{\varphi}^{\otimes(k-1)}O^{(k)}\varphi^{\otimes k}\right)_j \right\rangle_{L^2(\mathbb{R}^3)} \tag{1.33}$$

with $q = 1 - |\varphi\rangle\langle\varphi|$ and (1.25). In particular, we observe that the variance scales as $\sigma_N^2 = O(N^{2k-1})$, and thus, we expect fluctuations to be $O(N^{k-1/2})$.

We observe that Theorem 1.2 shows that the fluctuations of the many-body dynamics scale similarly to the fluctuations of a factorized state. Moreover, for $t = 0$, the variance σ_0^2 of the many-body dynamics defined in (1.29) agrees with the covariance matrix $\mathcal{M}_{\varphi^{\otimes N}}(i,j)$ in (1.33) of a factorized state.

We remark that for $k = 1$, i.e., considering bounded one-particle observables, Theorem 1.2 generalizes known results[3,6] to more general one-particle observables. This generalization is due to a different strategy of the Proof of Theorem 1.3 than in Refs. 3 and 6. We follow the ideas of Ref. 6; however, we directly use as a first step in Lemma 4.1 the norm approximation (4.1) of the many-body time evolution (for more details, see Sec. IV B). Furthermore, the authors of Ref. 6 proved a multivariate central limit theorem: it is shown that the expectation value of products of functions $f_1, \ldots, f_k$ of bounded, self-adjoint, and centered one-particle operators $O_1, \ldots, O_k$ [i.e., operators of the special form (1.9)] can be approximated with the integral of $f_1, \ldots, f_k$ against a complex-valued Gaussian density.

Recently, for one-particle operators, the probability distribution's tails were characterized through large deviation estimates,[14,21] showing that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\psi_{N,t}}\left[\frac{1}{N} d\Gamma(O_t^{(1)}) > x\right] = -\frac{x^2}{\|\widetilde{f}^{(1)}_{0;t}\|_2^2} + O(x^{5/2}) \tag{1.34}$$

for sufficiently small $x \le Ce^{-e^{C|t|}}$, where $\widetilde{f}^{(1)}_{t,0}$ is defined similarly to (1.26), but using the projected kernels $\widetilde{K}_{j,s}(x,y) = q_s K_{j,s}(x,y) q_s$.

Furthermore, for one-particle operators, a central limit theorem is proven for stronger particles' interactions in the intermediate regime,[19] interpolating between the mean-field and the Gross–Pitaevski regime. In the Gross–Pitaevski regime of singular particles' interaction, a central limit theorem is proven for quantum fluctuations in the ground state,[20] too.

Theorem 1.2 follows from an approximation of the random variable's characteristic function given in the following.

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, we have*

$$\left| \mathbb{E}_{\psi_{N,t}}\left[e^{iN^{-k+1/2} d\Gamma(O_t^{(k)})}\right] - e^{-\sigma_t^2/2} \right| \le C_k e^{C|t|} \|O^{(k)}\|_{\mathrm{op}} \sum_{\ell=1}^{2k-1} N^{-\ell/2} \left(1 + \sum_{i,j} \|O^{(k)}\|_{\mathrm{op}}^2\right)^{(\ell+1)/2}. \tag{1.35}$$

In the following, we will now first turn to the Proof of Theorem 1.1 in Sec. II, then prove Theorem 1.2 from Theorem 1.3 in Sec. III, and finally prove Theorem 1.3 in Sec. IV.

II. PROOF OF THEOREM 1.1

We generalize ideas from Ref. 3 on a law of large numbers for bounded one-particle observables to the case of k-particle operators.

Proof. By Chebycheff's inequality, we have

$$\mathbb{P}_{\psi_N}\left[\left|\frac{1}{\binom{N}{k}} O_N^{(k)}\right| > \delta\right] \le \frac{1}{\binom{N}{k}^2 \delta^2} \mathbb{E}_{\psi_N}\left[\left|\sum_{\underline{i}_k \in \mathcal{I}_N^{(k)}} \widetilde{O}^{(k)}_{\underline{i}_k}\right|^2\right], \tag{2.1}$$

where we used the notation $\widetilde{O}^{(k)}$ defined in (1.31). Furthermore, we denote with $\sharp\{\underline{i}_k, \underline{j}_k\}$ the number of elements of $\underline{i}_k$ agreeing with $\underline{j}_k$. Then, we can write

$$\left|\sum_{\underline{i}_k \in \mathcal{I}_N^{(k)}} \widetilde{O}^{(k)}_{\underline{i}_k}\right|^2 = \sum_{\ell=0}^{k} \sum_{\substack{\underline{i}_k, \underline{j}_k \in \mathcal{I}_N^{(k)} \\ \sharp\{\underline{i}_k, \underline{j}_k\} = \ell}} \widetilde{O}^{(k)}_{\underline{i}_k} \widetilde{O}^{(k)}_{\underline{j}_k}. \tag{2.2}$$

We can express the rhs of (2.1) in terms of j-particle reduced density matrices defined in (1.3) and find

$$\mathbb{E}_{\psi_N}\left[\left|\sum_{i_1,\ldots,i_k \in I_N} \widetilde{O}^{(k)}_{\{i_1,\ldots,i_k\}}\right|^2\right] = \sum_{\ell=0}^{k} \sum_{\substack{\underline{i}_k, \underline{j}_k \in \mathcal{I}_N^{(k)} \\ \sharp\{\underline{i}_k, \underline{j}_k\} = \ell}} \mathrm{tr}\, \gamma_{\psi_N}^{(2k-\ell)} \widetilde{O}^{(k)}_{\underline{i}_k} \widetilde{O}^{(k)}_{\underline{j}_k}. \tag{2.3}$$

Plugging (2.3) into the rhs of (2.1), we find

$$\mathbb{P}_{\psi_N}\Big[\big|O_N^{(k)}\big|>\delta\Big]\le\frac{1}{\binom{N}{k}^2\delta^2}\sum_{\ell=0}^{k}\sum_{\substack{\underline{i}_k,\underline{j}_k\in\mathcal{I}_N^{(k)}\\ \sharp\{\underline{i}_k,\underline{j}_k\}=\ell}}\operatorname{tr}\,\gamma_{\psi_N}^{(2k-\ell)}\Big(\widetilde{O}_{\underline{i}_k}^{(k)}\widetilde{O}_{\underline{j}_k}^{(k)}\Big). \tag{2.4}$$

For $\ell=0$, the term of the sum of the rhs of (2.4) is given by

$$\frac{1}{\binom{N}{k}^2\delta^2}\sum_{\substack{\underline{i}_k,\underline{j}_k\in\mathcal{I}_N^{(k)}\\ \sharp\{\underline{i}_k,\underline{j}_k\}=0}}\operatorname{tr}\,\gamma_{\psi_N}^{(2k}\widetilde{O}_{\underline{i}_k}^{(k)}\widetilde{O}_{\underline{j}_k}^{(k)}=\frac{\binom{N}{2k}}{\binom{N}{k}^2\delta^2}\operatorname{tr}\,\gamma_{\psi_N}^{(2k)}\Big(\widetilde{O}^{(k)}\otimes\widetilde{O}^{(k)}\Big)\le C_k\operatorname{tr}\,\gamma_{\psi_N}^{(2k)}\Big(\widetilde{O}^{(k)}\otimes\widetilde{O}^{(k)}\Big). \tag{2.5}$$

Since ψ_N exhibits Bose–Einstein condensation, it follows by assumption (1.12),

$$\operatorname{tr}\ \gamma_{\psi_N}^{(2k)}\Big(\widetilde{O}^{(k)}\otimes\widetilde{O}^{(k)}\Big)^{(2k)}\to\operatorname{tr}\,|\varphi\rangle\langle\varphi|^{\otimes(2k)}\Big(\widetilde{O}^{(k)}\otimes\widetilde{O}^{(k)}\Big)^{(2k)}\qquad\text{as}\quad N\to\infty, \tag{2.6}$$

and by definition (1.31) of $\widetilde{O}^{(k)}$, we arrive at

$$\operatorname{tr}\,|\varphi\rangle\,\langle\varphi|^{\otimes(2k)}\Big(\widetilde{O}^{(k)}\otimes\widetilde{O}^{(k)}\Big)^{(2k)}=0. \tag{2.7}$$

For $\ell\ge1$, the terms of the sum of the rhs of (2.4) consist of $(2k-\ell)$-particle operators whose expectation values are computed with $(2k-\ell)$-particle operators. In particular, we find

$$\frac{1}{\binom{N}{k}^2\delta^2}\sum_{\ell=1}^{k}\sum_{\substack{\underline{i}_k,\underline{j}_k\in\mathcal{I}_N^{(k)}\\ \sharp\{\underline{i}_k,\underline{j}_k\}=\ell}}\operatorname{tr}\ \gamma_{\psi_N}^{(2k-\ell)}\Big(\widetilde{O}_{\underline{i}_k}^{(k)}\widetilde{O}_{\underline{j}_k}^{(k)}\Big)\le C_k\|O^{(k)}\|_{\mathrm{op}}^2\sum_{\ell=1}^{k}\frac{\binom{N}{2k-\ell}}{\binom{N}{k}^2}\le C_k\|O^{(k)}\|_{\mathrm{op}}^2\sum_{\ell=1}^{k}\frac{1}{N^\ell}\to0 \tag{2.8}$$

as $N\to\infty$.

We conclude with (2.4), (2.7), and (2.8) by

$$\mathbb{P}_{\psi_N}\Bigg[\bigg|\frac{1}{\binom{N}{k}}O_N^{(k)}\bigg|>\delta\Bigg]\to0\qquad\text{as}\quad N\to\infty. \tag{2.9}$$

□

III. PROOF OF THEOREM 1.2

We use standard arguments from probability theory to prove Theorem 1.2 from Theorem 1.3. We follow the arguments from Ref. 6, Corollary 1.2.

Proof. We consider the difference

$$\begin{aligned}&\mathbb{P}_{\psi_{N,t}}\Big[N^{-k+1/2}d\Gamma(O_t^{(k)})\in[a,b]\Big]-\mathbb{P}[\mathcal{G}_t\in[a,b]]\\&\quad=\langle\psi_{N,t}|\chi_{[a,b]}\Big(N^{-k+1/2}d\Gamma(O_t^{(k)})\Big)|\psi_{N,t}\rangle-\frac{1}{\sqrt{2\pi}\sigma_t}\int dx\,e^{-\frac{x^2}{2\sigma_t^2}}\chi_{[a,b]}(x)\\&\quad=\mathbb{E}_{\psi_{N,t}}\Big[\chi_{[a,b]}\Big(N^{-k+1/2}d\Gamma(O_t^{(k)})\Big)\Big]-\mathbb{E}\big[\chi_{[a,b]}(\mathcal{G}_t)\big],\end{aligned} \tag{3.1}$$

where $\chi_{[a,b]}$ denotes the characteristic function of the set $[a,b]$. We observe that for $g\in L^1(\mathbb{R})$ with Fourier transform $\widehat{g}\in L^1\big(\mathbb{R},(1+s^{2k})\,ds\big)$, we have, on the one hand,

$$\begin{aligned}\mathbb{E}_{\psi_{N,t}}\Big[g\Big(N^{-k+1/2}d\Gamma(O_t^{(k)})\Big)\Big]&=\langle\psi_{N,t}|g\Big(N^{-k+1/2}d\Gamma(O_t^{(k)})\Big)|\psi_{N,t}\rangle\\&=\int d\tau\,\widehat{g}(\tau)\langle\psi_{N,t}|e^{i\tau N^{-k+1/2}d\Gamma(O_t^{(k)})}|\psi_{N,t}\rangle\end{aligned} \tag{3.2}$$

and, on the other hand,

$$\mathbb{E}[g(\mathcal{G}_t)]=\frac{1}{\sqrt{2\pi\sigma^2}}\int dx\,g(x)\,e^{-\frac{x^2}{2\sigma_t^2}}=\int d\tau\,\widehat{g}(\tau)\,e^{-\frac{\tau^2\sigma_t^2}{2}}, \tag{3.3}$$

and, in particular, by Theorem 1.3,

$$\begin{aligned}&\Big|\mathbb{E}_{\psi_{N,t}}\Big[g\Big(N^{-k+1/2}d\Gamma(O_t^{(k)})\Big)\Big]-\mathbb{E}[g(\mathcal{G}_t)]\Big|\\&\quad\le C_k e^{C|t|}\|O^{(k)}\|_{\mathrm{op}}\int d\tau\,|\widehat{g}(\tau|\sum_{\ell=1}^{2k-1}N^{-\ell/2}\Big(1+\tau^2\|O^{(k)}\|_{\mathrm{op}}^2\Big)^{(\ell+1)/2}.\end{aligned}\tag{3.4}$$

Thus, in order to find an estimate for (3.1), we shall find an approximation from above $f_{+,\varepsilon}$ and from below $f_{-,\varepsilon}$ of the characteristic function $\chi_{[a,b]}$, which satisfy $f_{-,\varepsilon}, f_{+,\varepsilon}\in L^1(\mathbb{R}^3)$ and $\widehat{f}_{-,\varepsilon},\widehat{f}_{+,\varepsilon}\in L^1(\mathbb{R},(1+s^{2k})ds)$. For this, let $\eta\in C_0^\infty(\mathbb{R})$ with $\eta\ge 0$, $\eta(s)=0$ for all $|s|\ge 1$ and $\int ds\,\eta(s)=1$. Furthermore, for $\varepsilon>0$, let $\eta_\varepsilon(s)=\varepsilon^{-1}\eta(s/\varepsilon)$. Then, for any $\varepsilon>0$, we define

$$f_{-,\varepsilon}:=\chi_{[a+\varepsilon,b-\varepsilon]}*\eta_\varepsilon\quad\text{and}\quad f_{+,\varepsilon}:=\chi_{[a-\varepsilon,b+\varepsilon]}*\eta_\varepsilon,\tag{3.5}$$

which satisfy

$$f_{-,\varepsilon}\le\chi_{[a,b]}\le f_{+,\varepsilon}.\tag{3.6}$$

Moreover, the Fourier transform is given by

$$\widehat{f}_{-,\varepsilon}(\tau)=-i\tau^{-1}\Big(e^{i\tau(b-\varepsilon)}-e^{i\tau(a+\varepsilon)}\Big)\widehat{\eta}(\varepsilon\tau).\tag{3.7}$$

Thus, it follows from (3.1) and (3.6) that

$$\begin{aligned}&\mathbb{P}_{\psi_{N,t}}\Big[N^{-k+1/2}d\Gamma(O_t^{(k)})\in[a,b]\Big]-\mathbb{P}[\mathcal{G}_t\in[a,b]]\\&\quad\ge-\Big|\mathbb{E}[f_{-,\varepsilon}(\mathcal{G}_t)]-\mathbb{E}_{\psi_{N,t}}\Big[f_{-,\varepsilon}\Big(N^{-k+1/2}d\Gamma(O_t^{(k)})\Big)\Big]\Big|-\Big|\mathbb{E}[f_{-,\varepsilon}(\mathcal{G}_t)]-\mathbb{E}\big[\chi_{[a,b]}(\mathcal{G}_t)\big]\Big|,\end{aligned}\tag{3.8}$$

and with (3.4) and (3.7), we arrive at

$$\begin{aligned}&\mathbb{P}_{\psi_{N,t}}\Big[N^{-k+1/2}d\Gamma(O_t^{(k)})\in[a,b]\Big]-\mathbb{P}[\mathcal{G}_t\in[a,b]]\\&\quad\ge-C_k e^{C|t|}\sum_{\ell=1}^{2k-1}N^{-\ell}\Big(|a-b|\varepsilon^{-1}+\varepsilon^{-2}\Big)^{(\ell+1)/2)}-C\varepsilon.\end{aligned}\tag{3.9}$$

Similarly, using $f_{+,\varepsilon}$, we have

$$\begin{aligned}&\mathbb{P}_{\psi_{N,t}}\Big[N^{-k+1/2}d\Gamma(O_t^{(k)})\in[a,b]\Big]-\mathbb{P}[\mathcal{G}_t\in[a,b]]\\&\quad\le C_k e^{C|t|}\sum_{\ell=1}^{2k-1}N^{-\ell}\Big(|a-b|\varepsilon^{-1}+\varepsilon^{-2}\Big)^{(\ell+1)/2)}+C\varepsilon.\end{aligned}\tag{3.10}$$

Now, we optimize with respect to $\varepsilon>0$ and arrive at (1.28). □

IV. PROOF OF THEOREM 1.3

A. Fluctuations around the Hartree dynamics

In the following, we consider the bosonic N-particle wave function $\psi_{N,t}$ as an element of the bosonic Fock space $\mathcal{F}=\bigoplus_{n\ge0}L^2(\mathbb{R}^3)^{\otimes_s^n}$ with creation and annihilation operators $a^*(f), a(f)$ for $f\in L^2(\mathbb{R}^3)$. Theorem 1.3 characterizes the fluctuations around the Hartree dynamics, which are well described by the approximation of the many-body time evolution (Ref. 4, Theorem 4.1, and Ref. 6, Proposition 3.3) in the $L^2(\mathbb{R}^{3N})$-norm,

$$\|\psi_{N,t}-W(\sqrt{N}\varphi_t)\mathcal{U}_\infty(t;0)\Omega\|\le C|t|\,N^{-1/2},\tag{4.1}$$

where the limiting dynamics $\mathcal{U}_\infty(t;0)$ is given by

$$i\partial_t\mathcal{U}_\infty(t;0)=\mathcal{L}(t)\mathcal{U}_\infty(t;0),\quad \mathcal{U}_\infty(0;0)=1,\tag{4.2}$$

with the generator

$$\mathcal{L}(t)=d\Gamma(h_{\mathrm{H}}(t)+K_{1,t})+\int dxdy\Big(K_{2,t}(x;y)a_x^*a_y^*+\overline{K}_{2,t}(x;y)a_xa_y\Big).\tag{4.3}$$

Here, $d\Gamma(A) = \int dxdy\, A(x;y)a_x^* a_y$ denotes the second quantization of an operator A on $L^2(\mathbb{R}^3)$, $h_H(t)$ denotes the Hartree Hamiltonian defined in (1.22), and $K_{j,t}$ denote the operators defined in (1.27). For further references, see also Refs. 8, 12, 13, 16, and 18. The generator $\mathcal{L}_\infty(t)$ is quadratic in creation and annihilation operators and thus (Ref. 3, Theorem 2.2) (see also Refs. 5 and 19) gives rise to a Bogoliubov dynamics, i.e., there exist bounded operators $U(t;\,0)$, $V(t;\,0)$ on $L^2(\mathbb{R}^3)$ such that for $f, g \in L^2(\mathbb{R}^3)$ and the operator $A(f,g) = a(f) + a^*(\overline{g})$, we have

$$\mathcal{U}_\infty^*(t;0)A(f,g)\mathcal{U}_\infty(t;0) = A(\Theta(t;0)(f,g)) \quad \text{with} \quad \Theta(t;0) = \begin{pmatrix} U(t;0) & JV(t;0)J \\ V(t;0) & JU(t;0)J \end{pmatrix}, \tag{4.4}$$

where $Jf = \overline{f}$ for any $f \in L^2(\mathbb{R}^3)$. In particular, for the operator

$$\phi(f) = a(f) + a^*(f) \qquad \text{for} \quad f \in L^2(\mathbb{R}^3), \tag{4.5}$$

from (4.4), we have

$$\mathcal{U}_\infty^*(t;0)\phi(f)\mathcal{U}_\infty(t;0) = \phi((U(t;0) + JV(t;0))f), \tag{4.6}$$

and it follows from Ref. 3, Theorem 2.2, and the subsequent remark that

$$i\partial_s(U(t;s) + JV(t;s))f = (h_H(s) + K_{1,s} - K_{2,s}J)(U(t;s) + JV(t;s))f. \tag{4.7}$$

Compared with (1.26), we note that the variance σ_t defined in (1.29) is determined by the limiting Bogoliubov dynamics (4.2), i.e., the fluctuations' quasi-free approximation.

B. Proof of Theorem 1.3

Proof 1.3 is split into three steps covered by Lemmas 4.1–4.3.

For the first step, Lemma 4.1, we use a strategy similar to the strategy in Refs. 19 and 20, directly the norm approximation (4.1). This allows us to consider more general k (respectively, one) particle operators than in Refs. 3 and 6 where the difference of the limiting fluctuation dynamics $\mathcal{U}_\infty(t;0)$ defined in (4.2) to the full many-body dynamics was estimated in (4.10) by Duhamel's formula and a Gronwall estimate. The remaining steps use the same ideas as in Refs. 3 and 6.

Lemma 4.1. Under the same assumptions as in Theorem 1.2, let

$$\xi_{N,t} = W(\sqrt{N}\varphi_t)\mathcal{U}_\infty(t;0)\Omega. \tag{4.8}$$

Then, there exists $C > 0$ such that

$$\left|\mathbb{E}_{\psi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right] - \mathbb{E}_{\xi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right]\right| \le C|t|N^{-1/2}. \tag{4.9}$$

Proof. We have

$$\begin{aligned}\mathbb{E}_{\psi_{N,t}}&\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right] - \mathbb{E}_{\xi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right] \\ &= |\langle\psi_{N,t}|e^{i\tau N^{-k+1/2}d\Gamma(O_t^{(k)})}|\psi_{N,t} - \xi_{N,t}\rangle| + |\langle\psi_{N,t} - \xi_{N,t}|e^{i\tau N^{-k+1/2}d\Gamma\,(O_t^{(k)}))}|\xi_{N,t}\rangle|.\end{aligned} \tag{4.10}$$

The operator $O^{(k)}$ is a self-adjoint operator; thus, $\|e^{iN^{-k+1/2}O_N^k}\|_{\text{op}} \le 1$, and we find with (4.1) and (4.8),

$$\left|\mathbb{E}_{\psi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right] - \mathbb{E}_{\xi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right]\right| \le C|t|N^{-1/2}. \tag{4.11}$$

□

Lemma 4.2. Under the same assumptions as in Theorem 1.2, let $\phi(f)$ be defined as in (4.5) and $h_t = \sum_{j=1}^k h_{j,t} \in L^2(\mathbb{R}^3)$ be defined with (1.25) by

$$h_{j,t} = \left(\overline{\varphi}_t^{\otimes(k-1)}O^{(k)}\varphi_t^{\otimes k}\right)_j. \tag{4.12}$$

Then, there exists $C > 0$ such that

$$\left|\mathbb{E}_{\xi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right] - \mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]\right| \le C_k\, e^{C|t|}\|O^{(k)}\|_{\text{op}}\sum_{j=1}^{2k-1}N^{-j/2}\left(1 + \|O^{(k)}\|_{\text{op}}^2\right)^{(j+1)/2}. \tag{4.13}$$

Proof. Recalling (4.8), we shall estimate the expectation value

$$\mathbb{E}_{\xi_{N,t}}\Big[e^{N^{-k+1/2}d\Gamma(O_t^{(k)})}\Big] = \Big\langle W^*(\sqrt{N}\varphi_t)e^{i\tau N^{-k+1/2}d\Gamma^{(k)}(\widetilde{O}^{(k)})}W(\sqrt{N}\varphi_t)\Big\rangle\mathcal{U}_\infty(t;0)\Omega$$
$$= \mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\Big[W^*(\sqrt{N}\varphi_t)e^{i\tau N^{-k+1/2}d\Gamma^{(k)}(\widetilde{O}^{(k)})}W(\sqrt{N}\varphi_t)\Big]. \tag{4.14}$$

In order to compute the operator

$$\mathcal{O}_{N,t} = N^{-k+1/2}W^*(\sqrt{N}\varphi_t)\, d\Gamma^{(k)}(\widetilde{O}_t^{(k)})W(\sqrt{N}\varphi_t), \tag{4.15}$$

we use the Weyl operators' shifting properties on creation and annihilation operators, i.e.,

$$W^*(\sqrt{N}\varphi_t)a_x W(\sqrt{N}\varphi_t) = a_x + \sqrt{N}\varphi_t(x), \quad W^*(\sqrt{N}\varphi_t)a_x^* W(\sqrt{N}\varphi_t) = a_x^* + \sqrt{N}\,\overline{\varphi}_t(x). \tag{4.16}$$

With $O_t^{(k)} := O^{(k)} - \langle\varphi_t^{\otimes k}|O^{(k)}|\varphi_t^{\otimes k}\rangle$, we find

$$\mathcal{O}_{N,t} = N^{-k+1/2}\int dx_1\ldots dx_k dy_1\ldots dy_k\, O_t^{(k)}(x_1,\ldots,x_k;y_1,\ldots y_k)$$
$$\times\Big(a_{x_1}^* + \sqrt{N}\,\overline{\varphi}_t(x_1)\Big)\cdots\Big(a_{x_k}^* + \sqrt{N}\,\overline{\varphi}_t(x_k)\Big)\Big(a_{y_1} + \sqrt{N}\varphi_t(y_1)\Big)\cdots\Big(a_{y_k} + \sqrt{N}\varphi_t(y_k)\Big). \tag{4.17}$$

We observe that the leading order term $O(N^k)$ vanishes by the definition of $\widetilde{O}^{(k)}$ in (1.31). Thus, the first non-vanishing leading order term is $O(N^{k-1/2})$, and in particular, we have with (4.5) and (4.12),

$$\mathcal{O}_{N,t} = \int dx_1\ldots dx_k dy_1\ldots dy_k\,\widetilde{O}^{(k)}(x_1,\ldots,x_k;y_1,\ldots y_k)$$
$$\times\left(\prod_{m=1}^k\varphi_t(y_m)\sum_{j=1}^k a_{x_j}^*\prod_{\substack{i=1\\i\neq j}}^k\overline{\varphi}_t(x_i) + \prod_{m=1}^k\overline{\varphi}_t(x_m)\sum_{j=1}^k a_{y_j}\prod_{\substack{i=1\\i\neq j}}^k\varphi_t(y_i)\right) + \mathcal{R}_N$$
$$= \phi(h_t) + \mathcal{R}_N. \tag{4.18}$$

The remainder

$$\mathcal{R}_N = \mathcal{O}_{N,t} - \phi(h_t) \tag{4.19}$$

is the sum of $(2^k - 2k)$ terms. The estimates

$$\|a(f)\xi\| \le \|f\|_2\|\mathcal{N}^{1/2}\xi\|, \quad \|a^*(f)\xi\| \le \|f\|_2\|(\mathcal{N}+1)^{1/2}\xi\| \tag{4.20}$$

for any $f\in L^2(\mathbb{R}^3)$ and any Fock space vector $\xi\in\mathcal{F}$ together with (4.17) yield the upper bound

$$\|\mathcal{R}_N\xi\| \le C_k\|O^{(k)}\|_{\mathrm{op}}\sum_{j=1}^{2k}N^{-j/2}\|(\mathcal{N}+1)^{(j+1)/2}\xi\| \tag{4.21}$$

for any $\xi\in\mathcal{F}$. We use the fundamental theorem of calculus to write

$$\mathbb{E}_{\xi_{N,t}}\Big[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\Big] - \mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\Big[e^{i\phi(h_t)}\Big]$$
$$= \langle e^{i\tau N^{-k+1/2}\mathcal{O}_{N,t}}\rangle\mathcal{U}_\infty(t;0)\Omega - \langle\mathcal{U}_\infty(t;0)\Omega|e^{i\tau\phi(h_t)}|\mathcal{U}_\infty(t;0)\Omega\rangle$$
$$= -\int_0^1 ds\,\frac{d}{ds}\langle\mathcal{U}_\infty(t;0)\Omega|e^{i(1-s)\mathcal{O}_{N,t}}e^{is\phi(h_t)}|\mathcal{U}_\infty(t;0)\Omega\rangle$$
$$= \int_0^1 ds\langle\mathcal{U}_\infty(t;0)\Omega|e^{i(1-s)\mathcal{O}_{N,t}}\mathcal{R}_N e^{is\phi(h_t)}|\mathcal{U}_\infty(t;0)\Omega\rangle. \tag{4.22}$$

Then, it follows from (4.21) that

$$\begin{aligned}
&\left|\mathbb{E}_{\xi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right]-\mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]\right| \\
&\quad\le \int_0^1 ds\, \|\mathcal{R}_N e^{is\phi(h_t)}\mathcal{U}_\infty(t;0)\Omega\| \\
&\quad\le C_k\|O^{(k)}\|_{\text{op}}\int_0^1 ds\sum_{j=1}^{2k-1}N^{-j/2}\|(\mathcal{N}+1)^{(j+1)/2}e^{is\phi(h_t)}\mathcal{U}_\infty(t;0)\Omega\|. \qquad (4.23)
\end{aligned}$$

With Ref. 6, Proposition 3.4 and $\|h_t\|_2^2\le\|O^{(k)}\|_{\text{op}}^2$ by definition (1.25), we have

$$\|(\mathcal{N}+1)^{(j+1)/2}e^{is\phi(h_t)}\mathcal{U}_\infty(t;0)\Omega\|\le C\left\|\left(\mathcal{N}+1+s^2\|O^{(k)}\|_{\text{op}}^2\right)^{(j+1)/2}\mathcal{U}_\infty(t;0)\Omega\right\|, \qquad (4.24)$$

and furthermore, with Ref. 6, Lemma 3.2,

$$\begin{aligned}
&\|(\mathcal{N}+1)^{(j+1)/2}e^{is\phi(h_t)}\mathcal{U}_\infty(t;0)\Omega\| \\
&\quad\le Ce^{C|t|}\left\|\left(\mathcal{N}+1+s^2\|O^{(k)}\|_{\text{op}}^2\right)^{(j+1)/2}\Omega\right\|\le Ce^{C|t|}\left(1+s^2\|O^{(k)}\|_{\text{op}}^2\right)^{(j+1)/2}. \qquad (4.25)
\end{aligned}$$

We use now estimate (4.25) for (4.23) and arrive at

$$\begin{aligned}
&\left|\mathbb{E}_{\xi_{N,t}}\left[e^{iN^{-k+1/2}d\Gamma(O_t^{(k)})}\right]-\mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]\right| \\
&\quad\le C_k\, e^{C|t|}\,\|O^{(k)}\|_{\text{op}}\sum_{j=1}^{2k-1}N^{-j/2}\int_0^1 ds\left(1+s^2\|O^{(k)}\|_{\text{op}}^2\right)^{(j+1)/2} \\
&\quad\le C_k\, e^{C|t|}\|O^{(k)}\|_{\text{op}}\sum_{j=1}^{2k-1}N^{-j/2}\left(1+\|O^{(k)}\|_{\text{op}}^2\right)^{(j+1)/2}, \qquad (4.26)
\end{aligned}$$

which proves the lemma. □

Lemma 4.3. Under the same assumptions as in Theorem 1.2, let $f_{t;s}=\sum_{i=1}^k f_{t;s}^{(i)}\in L^2(\mathbb{R}^3)$ *be given by (1.26). Then, we have*

$$\mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]=e^{-\|f_{t;0}\|_2^2/2}. \qquad (4.27)$$

Proof. We need to compute the expectation value

$$\mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]=\langle\mathcal{U}_\infty(t;0)\Omega|e^{i\phi(h_t)}|\mathcal{U}_\infty(t;0)\Omega\rangle. \qquad (4.28)$$

We recall that the limiting dynamics $\mathcal{U}_\infty(t;0)$ defined in (4.2) acts as a Bogoliubov transform. In particular, in follows from (4.4) and (4.7) and the notations introduced therein that

$$\mathcal{U}_\infty^*(t;0)\phi(h_t)\mathcal{U}_\infty(t;0)=\phi([U(t;0)+JV(t;0)]h_t)=\phi(f_{0;t}), \qquad (4.29)$$

with $f_{0;t}$ defined in (1.26). Hence, we have

$$\mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]=\langle\Omega|e^{i\mathcal{U}_\infty(t;0)^*\phi(h_t)\mathcal{U}_\infty(t;0)}|\Omega\rangle=\langle\Omega|e^{i\phi(f_{0;t})}|\Omega\rangle. \qquad (4.30)$$

With the Baker–Campbell–Hausdorff formulas, we split the sum in the exponential and arrive at

$$\mathbb{E}_{\mathcal{U}_\infty(t;0)\Omega}\left[e^{i\phi(h_t)}\right]=e^{-\|f_{0;t}\|_2^2/2}\langle\Omega|e^{ia^*(f_{0;t})}e^{ia(f_{0;t})}|\Omega\rangle=e^{-\|f_{0;t}\|_2^2/2}. \qquad (4.31)$$

□

Proof of Theorem 1.3. Combining now Lemmas 4.1–4.3, we arrive at Theorem 1.3. □

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

[1] Adami, R., Golse, F., and Teta, A., "Rigorous derivation of the cubic NLS in dimension one," J. Stat. Phys. **127**(6), 1193–1220 (2007).
[2] Ammari, Z., Falconi, M., and Pawilowski, B., "On the rate of convergence for the mean field approximation of bosonic many-body quantum dynamics," Commun. Math. Sci. **14**(5), 1417–1442 (2016).
[3] Arous, G. B., Kirkpatrick, K., and Schlein, B., "A central limit theorem in many-body quantum dynamics," Commun. Math. Phys. **321**(2), 371–417 (2013).
[4] Benedikter, N., Porta, M., and Schlein, B., *Effective Evolution Equations from Quantum Dynamics*, Springer Briefs in Mathematical Physics Vol. 7 (Springer, Cham, 2016).
[5] Bossmann, L., Petrat, S., Pickl, P., and Soffer, A., "Beyong Bogoliubov dynamics," arXiv:1912.11004 (2019).
[6] Buchholz, S., Saffirio, C., and Schlein, B., "Multivariate central limit theorem in quantum dynamics," J. Stat. Phys. **154**(1–2), 113–152 (2014).
[7] Chen, L., Lee, J. O., and Schlein, B., "Rate of convergence towards Hartree dynamics," J. Stat. Phys. **144**(4), 872–903 (2011).
[8] Chen, X., "Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions," Arch. Ration. Mech. Anal. **203**, 455–497 (2012).
[9] Fröhlich, J., Knowles, A., and Schwarz, S., "On the mean-field limit of bosons with Coulomb two-body interaction," Commun. Math. Phys. **288**(3), 1023–1059 (2009).
[10] Ginibre, J. and Velo, G., "The classical field limit of scattering theory for non-relativistic many-boson systems. I," Commun. Math. Phys. **66**, 37–76 (1979).
[11] Grech, P. D. and Seiringer, R., "The excitation spectrum for weakly interacting bosons in a trap," Commun. Math. Phys. **322**, 559–591 (2012).
[12] Grillakis, M. and Machedon, M., "Pair excitations and the mean field approximation of interacting Bosons, II," Commun. Partial Differ. Equations **42**(1), 24–67 (2017).
[13] Hepp, K., "The classical limit for quantum mechanics correlation functions," Commun. Math. Phys. **35**, 265–277 (1974).
[14] Kirkpatrick, K., Rademacher, S., and Schlein, B., "A large deviation principle for many–body quantum dynamics," Ann. Henri Poincaré **22**, 2595–2618 (2021).
[15] Knowles, A. and Pickl, P., "Mean-field dynamics: Singular potentials and rate of convergence," Commun. Math. Phys. **298**(1), 101–138 (2010).
[16] Lewin, M., Nam, P. T., and Schlein, B., "Fluctuations around Hartree states in the mean-field regime," Am. J. Math. **137**, 1613–1650 (2013).
[17] Lieb, E. H. and Seiringer, R., "Proof of Bose-Einstein condensation for dilute trapped gases," Phys. Rev. Lett. **88**(17), 170409 (2002).
[18] Mitrouskas, D., Petrat, S., and Pickl, P., "Bogoliubov corrections and trace norm convergence for the Hartree dynamics," Rev. Math. Phys. **31**(08), 1950024 (2019).
[19] Rademacher, S., "Central limit theorem for Bose gases interacting through singular potentials," Lett. Math. Phys. **110**, 2143 (2019).
[20] Rademacher, S. and Schlein, B., "Central limit theorem for Bose-Einstein condensates," J. Math. Phys. **60**, 071902 (2019).
[21] Rademacher, S. and Seiringer, R., "Large deviation estimates for weakly interacting bosons," arXiv:2112.01999.
[22] Rodnianski, I. and Schlein, B., "Quantum fluctuations and rate of convergence towards mean filed dynamics," Commun. Math. Phys. **291**(1), 31–61 (2009).
[23] Spohn, H., "Kinetic equations from Hamiltonian dynamics," Rev. Mod. Phys. **52**(3), 569–615 (1980).