An Updated Survey of Bidding Games on Graphs

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Abstract

A graph game is a two-player zero-sum game in which the players move a token throughout a graph to produce an infinite path, which determines the winner or payoff of the game. In *bidding games*, both players have budgets, and in each turn, we hold an "auction" (bidding) to determine which player moves the token. In this survey, we consider several bidding mechanisms and their effect on the properties of the game. Specifically, bidding games, and in particular bidding games of infinite duration, have an intriguing equivalence with *random-turn* games in which in each turn, the player who moves is chosen randomly. We summarize how minor changes in the bidding mechanism lead to unexpected differences in the equivalence with random-turn games.

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Related Version This paper is a short, updated version of [3]. In particular, results from [7] have been added.
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1 Introduction

Games on graphs are a central class of games in formal verification [2] and have deep connections to foundations of logic [17]. They have numerous applications including reactive synthesis [16], verification [10], and reasoning about multi-agent systems [1]. Theoretically, graph games give rise to interesting and challenging problems. For example, solving parity games is a rare problem that is in NP and coNP [11], not known to be in P, and for which a quasi-polynomial algorithm was only recently discovered [8].

A graph game proceeds as follows. We place a token on one of the vertices of a graph and allow the players to move it to produce an infinite path that determines the winner or payoff of the game. Several *modes of moving* the token have been studied [2], and the most popular is *turn-based* graph games in which the players alternate turns in moving the token. We study the *bidding* mode of moving [13, 12]: players have budgets and in each turn, an "auction" (bidding) determines which player moves the token. Bidding games are a class of concurrent graph games [1]. They combine graph games with auctions, a central topic of research in algorithmic game theory (e.g., [14]).

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Objectives. We stress that bidding is the mode of moving the token and it is orthogonal to the players' objectives. We consider the qualitative objectives reachability and parity, and the quantitative objective mean-payoff.

2 Bidding Mechanisms

In all the bidding mechanisms that we consider, in each turn, both players simultaneously submit bids that do not exceed their available budget, and the higher bidder "wins" the bidding and moves the token. The bidding mechanisms differ in two orthogonal properties:

1. Who pays: in *first-price* bidding only the higher bidder pays; in *all-pay* bidding both players pay their bids.
2. Where are the bids paid: in *Richman* bidding (named after David Richman), bids are paid to the opponent; in *poorman* bidding bids are paid to the "bank", thus the budget is lost.

▶ Remark 1. A well-known auction mechanism is *second-price bidding* in which the highest bidder pays the second highest bid. We point out that bidding games under first- and second-price bidding coincide, since the players in second-price bidding can follow the same optimal strategies they use in first-price bidding to guarantee the same values.

The central quantity in bidding games regards the ratio between the two players' budget, formally defined as follows.

▶ **Definition 2** (Budget ratio). *Assuming Player i's budget is B_i, for $i \in \{0, 1\}$, then Player 1's ratio is $B_1/(B_1 + B_2)$.*

Random-turn games. A *random-turn game* [15] is similar to a bidding game only that instead of bidding for moving, in each turn, we decide which player moves according to a (possibly biased) coin toss. For a bidding game $\mathcal{G}$ and $p \in [0, 1]$, we denote by $\mathsf{RT}(\mathcal{G}, p)$ the random-turn game that is obtained from $\mathcal{G}$ using a coin with bias p. Formally, random-turn games are a subclass of stochastic games [9]. To obtain a random-turn game from $\mathcal{G}$, we proceed as follows. For every vertex v in $\mathcal{G}$, we add two vertices v_{Max} and v_{Min} owned by the respective players. To simulate the coin toss at v, we add probabilistic edges from v to v_{Max} with probability p and to v_{Min} with probability $1 - p$. To simulate the choice of the player who wins the bidding, we add edges from both v_{Max} and v_{Min} to every u that is a neighbor of v in $\mathcal{G}$. The objective of $\mathsf{RT}(\mathcal{G}, p)$ coincides with the objective of $\mathcal{G}$.

3 Qualitative bidding games

The main question considered in qualitative bidding games regards the threshold budgets, which intuitively represent a necessary and sufficient initial budget ratio that suffices for winning the game. Formally,

▶ **Definition 3** (Threshold ratio). *Consider a qualitative bidding game $\mathcal{G}$ and a vertex v in $\mathcal{G}$. The* threshold ratio *in v, denoted $\mathit{Th}(v)$, is such that:*

- *If Player 1's initial ratio is strictly greater than $\mathit{Th}(v)$, he has a winning strategy from v.*
- *If Player 1's initial ratio is strictly less than $\mathit{Th}(v)$, Player 2 has a winning strategy from v.*

3.1 Reachability first-price bidding games

The focus in [13, 12] was on first-price reachability bidding games. An intriguing equivalence between reachability games with first-price Richman bidding with random-turn games, and, interestingly, only for this bidding mechanism. We formally state the result below and illustrate it in Fig. 1.

▶ **Theorem 4** ([13, 12]). *Consider a reachability bidding game with target states t_1 and t_2. Threshold ratios exist. Moreover, $\mathit{Th}(t_1) = 0$ and $\mathit{Th}(t_2) = 1$, and for any other vertex v, let v^- and v^+ be the neighbors of v such that $\mathit{Th}(v^-) \leq \mathit{Th}(v') \leq \mathit{Th}(v^+)$, for every neighbor v' of v. Then:*

- *Richman bidding: $\mathit{Th}(v) = \frac{1}{2}(\mathit{Th}(v^+) + \mathit{Th}(v^-))$ and $\mathit{Th}(v)$ coincides with the value of the vertex v in the random-turn game $\mathsf{RT}(\mathcal{G}, 0.5)$.*
- *Poorman bidding: $\mathit{Th}(v) = \frac{\mathit{Th}(v^+)}{1-\mathit{Th}(v^-)+\mathit{Th}(v^+)}$.*

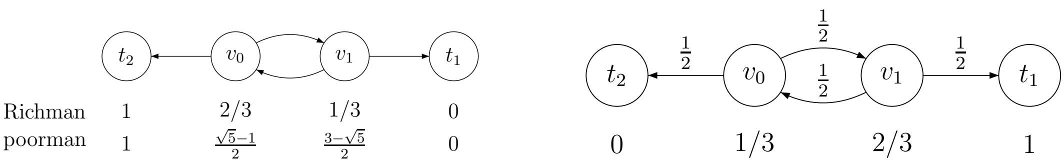


Figure 1 Left: A reachability bidding game $\mathcal{G}$ with threshold ratios, under first-price Richman and poorman bidding.
Right: The (simplified) unbiased random-turn game $\mathsf{RT}(\mathcal{G}, 0.5)$ is a Markov chain. The value of a vertex is the probability of reaching t_1. Note that under Richman bidding, for every vertex v, we have $\mathtt{Th}(\mathcal{G}, v) = 1 - \mathit{val}\big(\mathsf{RT}(\mathcal{G}, 0.5), v\big)$. Moreover, under poorman bidding, ratios are irrational thus such an equivalence is unlikely to exist.

3.2 Reachability all-pay bidding games

In [6], reachability games under all-pay bidding are shown to be technically much more challenging than under first-price bidding. Some positive results are shown; namely, an approximation algorithm based on discretization in games played on DAGs and results on the threshold for surely winning. Most results, however, are negative and fundamental problems, including proving that the value of the game always exists, remain open.

▶ **Theorem 5** ([6]). *Optimal strategies in reachability all-pay poorman bidding are sometimes mixed and draw bids from infinite-support distributions.*

3.3 Parity bidding games

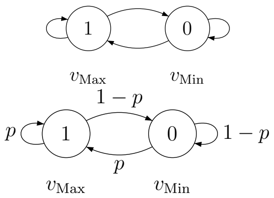
We state a key property of parity bidding games played on strongly-connected graphs.

▶ **Theorem 6** ([4, 5, 7]). *Consider a parity game $\mathcal{G}$ played on a strongly-connected graph in which the highest parity index is odd. Under first-price Richman and poorman bidding, the threshold ratios are 0 in all the vertices; namely, Player 1 can win with any positive initial budget. Under all-pay Richman and poorman bidding, with any positive initial ratio, Player 1 has a mixed strategy that guarantees satisfying the parity objective with probability 1.*

For first-price bidding games, Theorem 6 gives rise to the following simple reduction from parity to reachability bidding games. Given a parity game $\mathcal{G}$, first reason about the bottom strongly-connected components and classify them into those that are "winning" for Player 1 and those that are "winning" for Player 2. Then, construct a reachability bidding game in which each player's goal is to force the game to a winning bottom strongly-connected component. A similar reduction applies to all-pay bidding, however reachability games under those bidding mechanisms are not yet understood.

4 Mean-Payoff Bidding Games

In this section we consider mean-payoff games played on strongly-connected graphs. We show intricate equivalences between mean-payoff bidding games and random-turn games.



	Richman		poorman	
First-price	$\mathsf{RT}(\mathcal{G}, \frac{1}{2})$ [4]		$\mathsf{RT}(\mathcal{G}, r)$ [5]	
All-pay [7]	Pure $\mathsf{RT}(\mathcal{G}, 0)$	Mixed $\mathsf{RT}(\mathcal{G}, \frac{1}{2})$	Pure $\mathsf{RT}(\mathcal{G}, \frac{2r-1}{r})$	Mixed $\mathsf{RT}(\mathcal{G}, \frac{3r-1}{r})$

Figure 2 Left: On top, the mean-payoff bidding game $\mathcal{G}_{\bowtie}$. The payoff of a player in $\mathcal{G}_{\bowtie}$ is the long-run ratio of the biddings won. On bottom, for $p \in [0, 1]$, the (simplified) random-turn game $\mathsf{RT}(\mathcal{G}_{\bowtie}, p)$ is a weighted Markov chain. The expected payoff in $\mathsf{RT}(\mathcal{G}_{\bowtie}, p)$ is p; we expect that a random walk stays ratio p of the time in v_{Max}.
Right: The equivalence relates the optimal payoff in a strongly-connected mean-payoff game with the expected payoff in a random-turn game, where for all-pay poorman bidding we omit the cases of $r \leq 0.5$.

Mean-payoff value. Each play of a mean-payoff game has a payoff, which is Player 1's (Max) reward and Player 2's (Min) cost. We illustrate the definition of the mean-payoff objective. Consider the game $\mathcal{G}_{\bowtie}$ that is depicted in the top left of Fig. 2. It models the following setting. Max and Min represent two advertisers. In each day, an auctioneer holds an auction to determine which ad shows that day. Max's goal is to maximize the payoff, which coincides with the number of days that his ad appears in a very long time (say, a year). Alternatively, the payoff in $\mathcal{G}_{\bowtie}$ can be seen as the ratio of the biddings that Max wins in the long run. Formally, the payoff of an infinite sequence of weights $w_1, w_2, \ldots$ is $\liminf_{n\to\infty} \frac{1}{n} \sum_{1\leq i\leq n} w_i$.

▶ **Definition 7** (Mean-payoff value in bidding games). *Consider a strongly-connected mean-payoff bidding game $\mathcal{G}$ and a budget ratio $r \in (0, 1)$. The* mean-payoff value *of $\mathcal{G}$ w.r.t. r, denoted $\mathsf{MP}(\mathcal{G}, r)$, is $c \in \mathbb{R}$ if independent of the initial vertex,*

- *when Max's initial ratio exceeds r, he has a strategy that guarantees a payoff of $c - \varepsilon$, for every $\varepsilon > 0$, and*
- *Max cannot do better: with a ratio that exceeds $1 - r$, Min can guarantee a payoff of at most $c + \varepsilon$, for every $\varepsilon > 0$.*

Similarly, we use asMP to denote the almost-sure value, which is defined as the payoff that Max can guarantee with a mixed strategy with probability 1.

Consider a strongly-connected mean-payoff game $\mathcal{G}$ and $p \in [0, 1]$. Recall that $\mathsf{RT}(\mathcal{G}, p)$ denotes the random-turn game that is constructed from $\mathcal{G}$ in which in each turn, Max moves the token with probability p. We denote by $\mathsf{MP}(\mathsf{RT}(\mathcal{G}, p))$ the *mean-payoff value* of $\mathsf{RT}(\mathcal{G}, p)$, which is defined as the expected payoff when both players play optimally, and it is well-known to exist in stochastic games.

▶ **Theorem 8.** *Consider a strongly-connected mean-payoff game $\mathcal{G}$ and a ratio $r \in (0,1)$.*

- *First-price Richman bidding [4]: For all $r \in (0,1)$, we have $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, 0.5)\big)$.*
- *First-price poorman bidding [5]: $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, r)\big)$.*
- *All-pay Richman bidding [7]:*
  - *Under pure strategies, $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, 0)\big)$.*
  - *Under mixed strategies, $\mathit{asMP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, 0.5)\big)$.*
- *All-pay poorman bidding [7]:*
  - *Under pure strategies, if $r > 0.5$, then $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, \frac{2r-1}{r})\big)$, and if $r \leq 0.5$, then $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, 0)\big)$.*
  - *Under mixed strategies, if $r > 0.5$, then $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, \frac{3r-1}{r})\big)$, and if $r \leq 0.5$, then $\mathit{MP}(\mathcal{G}, r) = \mathit{MP}\big(\mathit{RT}(\mathcal{G}, \frac{1-r}{r})\big)$.*

In the following example, we illustrate the results stated in Theorem 8 on $\mathcal{G}_{\bowtie}$ that is depicted in Fig. 2.

▶ **Example 9.** First, let $p \in (0,1)$. The mean-payoff value of $\mathsf{RT}(\mathcal{G}_{\bowtie}, p)$ is p since intuitively, a random walk is expected to stay portion p in vertex v_{Max}.

We start with first-price bidding. Here, optimal strategies are pure (deterministic). Under Richman bidding, the initial ratio does not matter and the optimal payoff is 0.5, matching the mean-payoff value of $\mathsf{RT}(\mathcal{G}_{\bowtie}, 0.5)$. That is, for every $\varepsilon > 0$, Max can guarantee a payoff of at least 0.5 and he cannot do better even when his ratio is $1 - \varepsilon$.

The equivalence for mean-payoff Richman-bidding games can be seen as an extension of the equivalence of reachability Richman-bidding games. Since no equivalence is known for reachability poorman-bidding games, we find the equivalence for mean-payoff poorman-bidding particularly surprising. The optimal payoff Max can guarantee with a ratio of $r \in (0,1)$ in $\mathcal{G}_{\bowtie}$ is r. For example, when the initial budgets are $\langle 3, 1\rangle$, Max's ratio is $\frac{3}{4}$, and he can guarantee a payoff arbitrarily close to $\frac{3}{4}$ (in a similar manner to Richman bidding above). This means that in the long-run, Max can win 3 times more biddings than Min. Thus, given the option to choose between first-price Richman and poorman bidding, Max prefers using first-price poorman bidding when his budget exceeds Min's budget.

We turn to illustrate the results for all-pay bidding. Again, since reachability all-pay bidding games are technically involved, we find the equivalences in mean-payoff games to be particularly good news. First, under all-pay Richman, pure (deterministic) strategies are "useless". Using mixed strategies, first-price and all-pay coincide. Specifically, using a pure strategy, Max cannot guarantee a payoff greater than 0 in $\mathcal{G}_{\bowtie}$, and he has a mixed strategy that guarantees an almost-sure payoff of 0.5. Thus, given a choice between all-pay and first-price Richman, Max would not have a preference between the two bidding mechanisms.

Surprisingly, the properties of all-pay poorman bidding are quite different. First, in contrast to all-pay Richman bidding, pure strategies are "useful" under all-pay poorman when $r > \frac{1}{2}$. For example, when $r = \frac{3}{4}$, Max can guarantee a payoff of $\frac{2}{3}$ with a pure strategy. Not too far from $\frac{3}{4}$, the optimal payoff under first-price poorman bidding. Second, when allowing mixed strategies, given the choice between all-pay and first-price poorman bidding, when $r > \frac{1}{2}$, Max prefers all-pay poorman! With a ratio of $r = \frac{3}{4}$, Max has a mixed strategy that can guarantee an almost-sure payoff of $\frac{5}{6}$; higher than the optimal payoff under first-price poorman.

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