The Design Space of Kirchhoff Rods Supplemental Material

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1 EQUILIBRIUM EQUATIONS OF A KIRCHHOFF ROD

We assume *kinematic*, or double-clamped, boundary conditions: both γ and F are fixed at $s = 0$ and $s = \ell$. These boundary conditions are often encountered in architectural and interior design applications, and they make for the richest design space of equilibrium states.

To set up the variational problem, we choose F as the primary variable, so fixing F at both ends imposes Dirichlet boundary conditions. Assuming that $\gamma(0)$ coincides with the origin, we can express γ as a function of F via $\gamma(s) = \int_0^s \gamma' = \int_0^s Fe_3$. Thus, the endpoint constraint $\gamma(\ell) = \gamma_\ell$ takes the form of an integral constraint $\int_0^\ell Fe_3 = \gamma_\ell$. Constrained extremals of the Kirchhoff energy are characterized by extremals of the Lagrangian

$$\mathcal{L} = \int_0^\ell \left(\frac{1}{2} \langle k, Kk \rangle - \langle c, Fe_3 \rangle \right),$$

with Lagrange multiplier $c \in \mathbb{R}^3$.

To derive the Euler–Lagrange equations, we first discuss admissible variations of F. Any one-parametric family of variations takes the form $\tilde{F}(s, \varepsilon)$, such that $\tilde{F}(s, 0) = F(s)$, and $\tilde{F}(s, \varepsilon) \in \mathrm{SO}(3)$ for all $s \in (0, \ell)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. To characterize the variation $\delta F(s) := (\partial/\partial\varepsilon)\tilde{F}(s, \varepsilon)|_{\varepsilon=0}$, we differentiate the equation $\tilde{F}\tilde{F}^T = \mathrm{id}$ with respect to ε. This shows that $\delta F\, F^T$ is skew-symmetric, so there exists some $\eta : (0, \ell) \to \mathbb{R}^3$ such that $\delta F = [\eta]_\times F$.

Next, we discuss variations δk induced by η. From differentiating δF with respect to s, we get $\delta F' = [\eta']_\times F + [\eta]_\times F'$, and, from skew-symmetry of $[\eta]_\times$, we arrive at $\delta F^T = -F^T [\eta]_\times$. Then, we take the variation of $[k]_\times = F^T F'$:

$$\begin{aligned} [\delta k]_\times &= -F^T [\eta]_\times F' + F^T \left([\eta']_\times F + [\eta]_\times F' \right) \\ &= F^T [\eta']_\times F = [F^T \eta']_\times. \end{aligned}$$

This implies $\delta k = F^T \eta'$.

Now, we can compute the variation of the Lagrangian:

$$\delta\mathcal{L} = \int_0^\ell (\langle F^T \eta', Kk \rangle - \langle c, [\eta]_\times Fe_3 \rangle) = \int_0^\ell (\langle FKk, \eta' \rangle + \langle c \times \gamma', \eta \rangle).$$

According to the fundamental lemma of variational calculus, we have $\delta\mathcal{L} = 0$ for all test functions η if and and only if $(FKk)' = c \times \gamma'$. We can integrate this equation to arrive at the equilibrium equation

$$FKk = c \times \gamma + \bar{c}, \tag{1}$$

with integration constant $\bar{c} \in \mathbb{R}^3$. This shows that (γ, F) represents a static equilibrium of a Kirchhoff rod with stiffness K and kinematic boundary conditions if and only if Eq. 1 holds for some $c, \bar{c} \in \mathbb{R}^3$.

2 PROOF OF PROPOSITION 1

We show that zero is the tight lower bound of the torsional rigidity for any given bending rigidity. In other words, given $I \in S^2_{++}$ and $\varepsilon > 0$, we can find a bounded (and simply-connected) domain $\mathcal{D} \subset \mathbb{R}^2$ with bending rigidity equal to I and torsional rigidity $J \leq \varepsilon$. For convenience, we repeat the definitions of I and J:

$$\begin{aligned} I &= \int_{\mathcal{D}} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} dA(x, y), \\ J &= 4 \int_{\mathcal{D}} \|\nabla\chi\|^2, \quad \text{with} \quad \begin{aligned} \Delta\chi &= -1 \quad \text{in } \mathcal{D}, \\ \chi &= 0 \quad \text{on } \partial\mathcal{D}. \end{aligned} \end{aligned}$$

Proof. Choose $I \in S^2_{++}$ and $r > 0$, and construct a domain $\Omega \subset \mathbb{R}^2$ as follows: Starting from an elliptical disk with bending rigidity I, add linear cuts from the boundary to the interior in such a way that the domain remains simply connected and that the incircle radius (the supremum of the radii of all circles contained in Ω) falls below r, as illustrated in Fig. 6 (right) of the main text. Let $\chi \in H^1_0(\Omega)$ be the solution to $\Delta\chi = -1$ in Ω and $\chi = 0$ on $\partial\Omega$, where $H^1_0(\Omega)$ denotes the Sobolev space of weakly differentiable functions in $L^2(\Omega)$ supported in Ω.

Partition the axis-aligned bounding rectangle of Ω into a rectilinear grid such that every cell has side lengths greater than $2r$ and at most $3r$ (which is always possible for small enough r). This guarantees that every cell intersects Ω^c in a set containing a line segment of positive length, on which $\chi = 0$. By the Poincaré–Friedrichs inequality [Braess 2007, II.1.5-6], we have

$$\|\chi\|_{L^2(C)} \leq 3r\, \|\nabla\chi\|_{L^2(C)}$$

for every cell C of the partition, and by summing over all cells,

$$\|\chi\|_{L^2(\Omega)} \leq 3r\, \|\nabla\chi\|_{L^2(\Omega)}.$$

By Green's first identity, we have

$$\|\nabla\chi\|^2_{L^2(\Omega)} = \int_{\partial\Omega} \chi\, \partial_n \chi - \int_\Omega \chi\, \Delta\chi = \int_\Omega \chi \leq \|\chi\|_{L^1(\Omega)},$$

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and by the Cauchy–Schwarz inequality,

$$\|1 \cdot \chi\|_{L^1(\Omega)} \leq \|1\|_{L^2(\Omega)} \|\chi\|_{L^2(\Omega)} = \sqrt{\mu(\Omega)} \|\chi\|_{L^2(\Omega)},$$

where μ denotes the Lebesgue measure. Altogether, this gives

$$\|\nabla \chi\|^2_{L^2(\Omega)} \leq \|\chi\|_{L^1(\Omega)} \leq \sqrt{\mu(\Omega)} \|\chi\|_{L^2(\Omega)} \leq 3r\sqrt{\mu(\Omega)} \|\nabla \chi\|_{L^2(\Omega)}.$$

Cancelling $\|\nabla \chi\|_{L^2(\Omega)}$ and squaring gives

$$J = 4\|\nabla \chi\|^2_{L^2(\Omega)} \leq 36r^2 \mu(\Omega).$$

Choosing r such that $36r^2\mu(\Omega) \leq \varepsilon$ gives the statement. □

3 PROOF OF LEMMA 2

We show that the map $\psi : S^2_{++} \to \mathbb{R}_{>0} : X \mapsto \frac{\det X}{\operatorname{tr} X}$ is concave.

Proof. Let $X, Y \in S^2_{++}$ and $t \in (0, 1)$. We need to show that $(1 - t)\psi(X) + t\psi(Y) \leq \psi((1 - t)X + tY)$, which expands to

$$\frac{(1-t) \operatorname{tr} Y \det X + t \operatorname{tr} X \det Y}{\operatorname{tr} X \operatorname{tr} Y} \leq \frac{\det((1-t)X + tY)}{(1-t)\operatorname{tr} X + t \operatorname{tr} Y}.$$

We multiply through by the product of the denominators, which is strictly positive, expand $\det((1 - t)X + tY)$ in terms of the components of X and Y, and divide by $t(1 - t)$, which is also positive. Most terms cancel, and we arrive at the equivalent statement

$$(\operatorname{tr} X)^2 \det Y + (\operatorname{tr} Y)^2 \det X \leq \operatorname{tr} X \operatorname{tr} Y (X_{11}Y_{22} - 2X_{12}Y_{12} + X_{22}Y_{11}),$$

which can be factorized to give

$$0 \leq (X_{11}Y_{22} - X_{22}Y_{11})^2 + (Y_{12} \operatorname{tr} X - X_{12} \operatorname{tr} Y)^2.$$

This shows that ψ is concave. □

4 PROOF OF PROPOSITION 7

For convenience, we restate the proposition:

Proposition 7. *Let γ be an arc-length parametrized curve in $\mathbb{R}^3$. Then the following are equivalent:*

(1) *There exist $c, \bar{c} \in \mathbb{R}^3$ such that γ satisfies*

$$\langle \gamma', c \times \gamma + \bar{c} \rangle = 0, \tag{2a}$$

$$\langle \gamma' \times \gamma'', c \times \gamma + \bar{c} \rangle > 0, \tag{2b}$$

(2) *There exist $c, \bar{c} \in \mathbb{R}^3$ such that the ordered set $\{\gamma', \gamma'', c \times \gamma + \bar{c}\}$ is a right-handed orthogonal basis at every point.*

(3) *There exist $c, \bar{c} \in \mathbb{R}^3$ and $m : (0, \ell) \to \mathbb{R}_{>0}$ such that*

$$\gamma''(s) = m(s) \cdot (c \times \gamma(s) + \bar{c}) \times \gamma'(s) \tag{3}$$

and $\langle \gamma'(0), c \times \gamma(0) + \bar{c} \rangle = 0$. It holds that $\kappa = m \cdot \|c \times \gamma + \bar{c}\|$.

Proof. (1) $\Rightarrow$ (2): By differentiating $\langle \gamma', \gamma' \rangle = 1$, we get $\langle \gamma', \gamma'' \rangle = 0$, and by differentiating Eq. 2a,

$$0 = \langle \gamma'', c \times \gamma + \bar{c} \rangle + \langle \gamma', c \times \gamma' \rangle = \langle \gamma'', c \times \gamma + \bar{c} \rangle.$$

Eq. 2a and these two new equations give the three orthogonality conditions. Eq. 2b shows right-handedness.

(2) $\Rightarrow$ (3): Eq. 3 with $m > 0$ and the initial condition are immediately implied by the right-handed orthogonal basis assumption. Using $\langle \gamma', c \times \gamma + \bar{c} \rangle = 0$, we compute

$$\kappa = \|\gamma' \times \gamma''\| = m \cdot \|\gamma' \times ((c \times \gamma + \bar{c}) \times \gamma')\| = m \cdot \|c \times \gamma + \bar{c}\|.$$

(3) $\Rightarrow$ (1): To show that $\langle \gamma', c \times \gamma + \bar{c} \rangle$ remains constant, compute

$$\langle \gamma', c \times \gamma + \bar{c} \rangle' = \langle \gamma'', c \times \gamma + \bar{c} \rangle = m \cdot \langle (c \times \gamma + \bar{c}) \times \gamma', c \times \gamma + \bar{c} \rangle = 0,$$

so Eq. 2a follows from the initial condition. Eqs. 2a and 3 imply Eq. 2b because γ' and $c \times \gamma + \bar{c}$ are both non-zero. □

5 HELICAL SYMMETRY OF CONSTANT-CURVATURE PARALLEL EQUILIBRIUM CURVES

We give a sketch of the proof that solutions to

$$\gamma''(s) = B(\gamma(s)) \times \gamma'(s), \quad \text{with} \quad B(x) = \kappa \frac{c \times x + \bar{c}}{\|c \times x + \bar{c}\|},$$

with $c = e_3$, $\bar{c} = pe_3$, and constant $\kappa > 0$ have a discrete helical symmetry with axis e_3, i.e., there exist $h, \zeta \in \mathbb{R}$ and $\sigma > 0$, such that, for all $s \in \mathbb{R}$,

$$\gamma(s + \sigma) = \begin{pmatrix} \cos\zeta & -\sin\zeta & 0 \\ \sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \gamma(s) + \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

To show this, we note that B is divergence-free, so the equation $\gamma'' = B \times \gamma'$ describes the trajectory of a charged particle in a magnetic field. In cylindrical coordinates with radius ϱ, azimuth θ, and height z, we define the vector potential

$$A(\varrho, \theta) = \frac{\kappa p}{\varrho} \left(\sqrt{\varrho^2 + p^2} - 1 \right) e_\theta(\theta) + \kappa \left(\sqrt{\varrho^2 + p^2} - 1 \right) e_3,$$

such that $\operatorname{div} A = 0$ and $\operatorname{curl} A = B$, where B is given by

$$B(\varrho, \theta) = \frac{\kappa}{\sqrt{\varrho^2 + p^2}} \left(\varrho e_\theta(\theta) + p e_3 \right)$$

in cylindrical coordinates. The Lagrangian for a charged particle in a magnetic field is given by $\frac{1}{2}\langle \gamma', \gamma' \rangle + \langle \gamma', A \circ \gamma \rangle$, which we can use to extract invariants of our differential equation by using Noether's theorem. Symmetry under time translation gives the arc-length condition $\varrho'^2 + (\varrho\theta')^2 + z'^2 = 1$, and symmetry under translation along e_3 and rotation around e_3 gives two new invariants

$$I_z = z' + \kappa\sqrt{\varrho^2 + p^2}, \quad I_\theta = \varrho^2\theta' - \kappa p \sqrt{\varrho^2 + p^2},$$

which show that z' and θ' only depend on ϱ (but not on θ and z). We compute $0 = \langle \gamma', c \times \gamma + \bar{c} \rangle = \varrho^2\theta' + pz' = I_\theta + pI_z$, showing that $I_\theta = -pI_z$. Next, we substitute θ' and z' in the arc-length condition for the invariants, which gives (for $\varrho \neq 0$),

$$\varrho'^2 = 1 - \frac{\varrho^2 + p^2}{\varrho^2} \left(I_z - \kappa\sqrt{\varrho^2 + p^2} \right)^2, \tag{4}$$

showing that ϱ' only depends on ϱ, up to sign. To determine zeros of ϱ', substitute $\bar{\varrho} = \sqrt{\varrho^2 + p^2} \geq |p|$, which gives

$$\bar{\varrho}^2(1 - (I_z - \kappa\bar{\varrho})^2) = p^2.$$

On $\bar{\varrho} > |p|$, this equation has either two distinct real solutions, or one real double solution (which corresponds to the special case of a helical solution). In the former case, we have $\varrho'(\varrho) = 0$ exactly for some $\varrho = R_1$ and $\varrho = R_2$, with $0 < R_1 < R_2$. Then, ϱ consists of alternating, mirror-symmetric segments, on which ϱ monotonically increases from R_1 to R_2, and then monotonically decreases from R_2 to R_1. To show that the sign of ϱ' actually changes at R_1 and R_2, one verifies $\varrho'' \neq 0$ at these points. Furthermore, θ' and z' only depend on ϱ, so every pair of alternating segments will give a copy of the same curve segment, which is translated along and rotated around e_3 with respect to the previous one. This shows the discrete helical symmetry of γ.

REFERENCES

Dietrich Braess. 2007. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics* (3 ed.). Cambridge University Press, Cambridge, England. https://doi.org/10.1017/CBO9780511618635