The Design Space of Kirchhoff Rods
Cheat Sheet

CHRISTIAN HAFNER and BERND BICKEL, Institute of Science and Technology Austria (ISTA), Austria

ACM Reference Format:

Some of the symbols refer to quantities that vary across the length of a beam with arc-length parameter \( s \in (0, \ell) \). In the paper, we will often omit the parameter \( s \) for brevity, whenever we make an argument that is true at every parameter location. Sometimes, we will also write, e.g., \( I \in S^2_{+} \) instead of \( I : (0, \ell) \to S^2_{+} \), when it is clear from context that a choice \( I(s) \in S^2_{+} \) is made for every \( s \in (0, \ell) \).

<table>
<thead>
<tr>
<th>Sym.</th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\cdot)' )</td>
<td>((\cdot)' : \mathbb{C}^{d} \to \mathbb{C}^{d-1} )</td>
<td>First derivative with respect to arc-length parameter ( s )</td>
</tr>
<tr>
<td>( [\cdot]_{\times} )</td>
<td>([\cdot]_{\times} : \mathbb{R}^{3} \to \mathbb{R}^{3 \times 3} )</td>
<td>Transforms a vector ( v \in \mathbb{R}^{3} ) into its &quot;cross product matrix&quot;, the skew-symmetric matrix ([v]<em>{\times} ) such that ([v]</em>{\times} x = v \times x ) for all ( x \in \mathbb{R}^{3} )</td>
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<tr>
<td>( a )</td>
<td>( a \in \mathbb{R}_{&gt;0} )</td>
<td>Radius of an ellipse, associated with the first semi-axis ((\cos \varphi, \sin \varphi)^{t} )</td>
</tr>
<tr>
<td>( b )</td>
<td>( b \in \mathbb{R}_{&gt;0} )</td>
<td>Radius of an ellipse, associated with the second semi-axis ((- \sin \varphi, \cos \varphi)^{t} )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( \beta : (0, \ell) \to \mathbb{R} )</td>
<td>Rotation of the normal plane, relating two frames ( F ) and ( F_{\beta} ) adapted to the same curve ( y ) via ( F_{\beta, n} = F_{n} Q_{\beta} ), with ( Q_{\beta} = \begin{pmatrix} \cos \beta &amp; - \sin \beta \ \sin \beta &amp; \cos \beta \end{pmatrix} )</td>
</tr>
<tr>
<td>( (c, \bar{c}) )</td>
<td>( c, \bar{c} \in \mathbb{R}^{3} )</td>
<td>Homogeneous coordinates of the linear complex ( C )</td>
</tr>
<tr>
<td>( C )</td>
<td>( C \subset \Lambda_{3d} )</td>
<td>Set of all lines in ( \mathbb{R}^{3} ) whose Plücker coordinates ((l, \bar{l}) ) satisfy ((l, \bar{c}) + (\bar{l}, c) = 0 )</td>
</tr>
<tr>
<td>( \mathcal{D} )</td>
<td>( \mathcal{D}(s) \subset \mathbb{R}^{2} )</td>
<td>Cross section of the Kirchhoff rod at a particular ( s \in (0, \ell) ); often assumed to be elliptical</td>
</tr>
<tr>
<td>( E )</td>
<td>( E \in \mathbb{R}_{&gt;0} )</td>
<td>Young’s modulus of the base material</td>
</tr>
<tr>
<td>( e_{i} )</td>
<td>( e_{i} \in \mathbb{R}^{3} )</td>
<td>Standard basis vectors ( e_{1} = (1, 0, 0)^{t}, e_{2} = (0, 1, 0)^{t}, ) and ( e_{3} = (0, 0, 1)^{t} )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F : (0, \ell) \to SO(3) )</td>
<td>Moving frame adapted to ( y ); encodes the twist of the Kirchhoff rod deformation; the columns of ( F ) are given by ( F(s) = (n_{1}(s), n_{2}(s), y'(s)) )</td>
</tr>
<tr>
<td>( F_{n} )</td>
<td>( F_{n} : (0, \ell) \to \mathbb{R}^{3 \times 2} )</td>
<td>The matrix of material normals of ( F ), so ( F_{n} = FS = (n_{1}, n_{2}) )</td>
</tr>
<tr>
<td>( f_{i} )</td>
<td>( f_{i} \in \mathbb{R}^{3}, i = 1, \ldots, n )</td>
<td>Concentrated point load ( f_{i} ) is applied to the centerline of a rod at ( y(s_{i}) )</td>
</tr>
<tr>
<td>( y )</td>
<td>( y : (0, \ell) \to \mathbb{R}^{3} )</td>
<td>Arc-length parametrized curve that gives the centerline of a deformed Kirchhoff rod; assumed at least twice continuously differentiable</td>
</tr>
</tbody>
</table>
\[ I \quad I : (0, \ell) \to \mathbb{S}^2_{++} \]
Area moment of inertia tensor of the cross section of the Kirchhoff rod, at a particular \( s \in (0, \ell) \), given by \( I(s) = \int_{\mathcal{D}(s)} \left( y^2 - x y \right) \, d(x, y) \)

\[ J \quad J : (0, \ell) \to \mathbb{R}_{>0} \]
Torsional rigidity of the cross section of the Kirchhoff rod, at a particular \( s \in (0, \ell) \); computed as \( J(s) = 4 \int_{\mathcal{D}(s)} \|\nabla \chi\|^2 \), where \( \chi \) is the solution to \( \Delta \chi = -1 \) in \( \mathcal{D}(s) \), and \( \chi = 0 \) on \( \partial \mathcal{D}(s) \)

\[ K \quad K : (0, \ell) \to \mathbb{R}^{3 \times 3} \]
Stiffness matrix of the Kirchhoff rod, at a particular \( s \in (0, \ell) \); the upper-left two-by-two block is given by \( E I \), and the lower-right entry by \( \mu J \); we often use \( K \) and the pair \((I, J)\) interchangeably, because \( E \) and \( \mu \) are assumed fixed

\[ k \quad k : (0, \ell) \to \mathbb{R}^3 \]
Curvature vector of the framed curve \((\gamma, F)\), with components \( k = (k_1, k_2, \tau) \); related to \( F \) and \( \omega \) via \( \omega = F k \) and \( [k]_{\chi} = F' \)

\[ k_n \quad k_n : (0, \ell) \to \mathbb{R}^2 \]
Vector of material curvatures of \( F \), so \( k_n = S' k = (k_1, k_2)' \)

\[ \mathcal{K} \quad \mathcal{K} \subset \mathbb{S}^2_{++} \times \mathbb{R} \]
Set of admissible stiffnesses \((I, J)\) that satisfy \( 0 < J \leq 4 \psi'(I) \); by abuse of notation, we write \( K \in \mathcal{K} \) and \((I, J) \in \mathcal{K} \) interchangeably

\[ \mathcal{K}^* \quad \mathcal{K}^* \subset \mathcal{K} \]
Set of stiffnesses induced by elliptical cross sections, i.e., \( J = 4 \psi(I) \)

\[ \kappa_i \quad \kappa_i : (0, \ell) \to \mathbb{R} \]
Material curvatures \( k_1 \) and \( k_2 \) of \( F \); measure bending of the Kirchhoff rod around the material normals \( n_1 \) and \( n_2 \), respectively

\[ \kappa \quad \kappa : (0, \ell) \to \mathbb{R}_{\geq 0} \]
Total (Frenet) curvature of \( \gamma \), given by \( \kappa = \|y''\| \); for any frame \( F \) adapted to \( \gamma \), it holds that \( \kappa = \sqrt{k_1^2 + k_2^2} \)

\[ t \quad t \in \mathbb{R}_{>0} \]
Length of the Kirchhoff rod

\[ \lambda \quad \lambda(l, \bar{l}) \subset \mathbb{R}^3 \]
Map from the Plücker coordinates \( l, \bar{l} \in \mathbb{R}^3 \) with \( \langle l, \bar{l} \rangle = 0 \) to the line in \( \mathbb{R}^3 \) incident to the point \( \frac{lx}{\|l\|^2} \) and with direction \( l \)

\[ \Lambda_{kl} \quad \Lambda_{kl} \subset \mathbb{P}^5 \]
Klein quadric, the set of all points with homogeneous coordinates \((l, \bar{l}) \in \mathbb{R}^6 \) satisfying \( \langle l, \bar{l} \rangle = 0 \); we interpret these points as Plücker coordinates of a line in \( \mathbb{R}^3 \) with direction \( l \) and incident to a point \( x \), such that \((l, \bar{l}) = (a, x \times v)\)

\[ M \quad M : (0, \ell) \to \mathbb{R}^3 \]
Accumulated moment on a deformed rod, given by \( M = \int_0^\ell r \times q \)

\[ \mu \quad \mu \in \mathbb{R}_{>0} \]
Shear modulus of the base material

\[ n_i \quad n_i : (0, \ell) \to \mathbb{R}^3 \]
Material normals of the moving frame \( F \), so \( n_i = Fe_i \) for \( i = 1, 2 \)

\[ \nu \quad \nu \in (-1, 1/2) \]
Poisson’s ratio of the base material

\[ \omega \quad \omega : (0, \ell) \to \mathbb{R}^3 \]
Darboux vector of the moving frame \( F \); related to \( F \) and \( k \) via \( \omega = F k \) and \( F' = [\omega]_{\chi} F \)

\[ p \quad p : (0, \ell) \to \mathbb{R}^3 \]
Line load applied to the centerline of a rod, where \( p(s) \) gives the load density at \( y(s) \)

\[ \varphi \quad \varphi \in \mathbb{R} \]
Orientation of ellipse with respect to reference frame; first and second semi-axes are given by \((\cos \varphi, \sin \varphi)^t\) and \((-\sin \varphi, \cos \varphi)^t\) respectively

\[ \psi \quad \psi : \mathbb{S}^2_{++} \to \mathbb{R}_{>0} \]
The determinant-over-trace function \( \psi(X) := \frac{\det X}{\tr X} \)

\[ Q \quad Q : (0, \ell) \to \mathbb{R}^3 \]
Accumulated load on a rod, given by \( Q(s) = \int_0^s q \)

\[ q \quad q \in \mathcal{D}'((0, \ell); \mathbb{R}^3) \]
Load distribution applied to the centerline of a rod, consisting of a line load \( p \) and point loads \( f_i \)

\[ S \quad S \in \mathbb{R}^{3 \times 2} \]
Selection matrix \( S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) that extracts the first two columns of a three-column matrix by multiplication from the right, i.e., \((x_1, x_2, x_3)S = (x_1, x_2)\)
$S_2^+ \subset \mathbb{R}^{2 \times 2}$ 
Set of all symmetric positive-definite 2-by-2 matrices

$SO(3) \subset \mathbb{R}^{3 \times 3}$ 
Set of all rotations of $\mathbb{R}^3$ about the origin

$s \in (0, \ell)$ 
Arc-length parameter of $\gamma$

$s_i \in (0, \ell), i = 1, \ldots, n$ 
Concentrated point load $f_i$ is applied to the centerline of a rod at $\gamma(s_i)$

$\tau : (0, \ell) \rightarrow \mathbb{R}$ 
Twist of the moving frame $F$; measures rotation per arc-length unit around $\gamma'$