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Eigenstate Thermalisation Hypothesis for Translation Invariant Spin Systems

Shoki Sugimoto[1] · **Joscha Henheik**[2] · **Volodymyr Riabov**[2] · **László Erdős**[2]

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Abstract
We prove the Eigenstate Thermalisation Hypothesis (ETH) for local observables in a typical translation invariant system of quantum spins with L-body interactions, where L is the number of spins. This mathematically verifies the observation first made by Santos and Rigol (Phys Rev E 82(3):031130, 2010, https://doi.org/10.1103/PhysRevE.82.031130) that the ETH may hold for systems with additional translational symmetries for a naturally restricted class of observables. We also present numerical support for the same phenomenon for Hamiltonians with local interaction.

Keywords Eigenstate thermalisation · Microcanonical ensemble · Translational invariance · Quantum spin systems

Mathematics Subject Classification 60B20, 82B44, 82D30

1 Introduction

Recent experiments have demonstrated thermalisation of isolated quantum systems under unitary time evolution [1–6]. In this context, thermalisation means that, after a long-time evolution, observables attain their equilibrium (thermal) values determined by statistical mechanics. The primary mechanism behind this thermalisation of isolated quantum systems is

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✉ Shoki Sugimoto
sugimoto@cat.phys.s.u-tokyo.ac.jp

Joscha Henheik
joscha.henheik@ist.ac.at

Volodymyr Riabov
volodymyr.riabov@ist.ac.at

László Erdős
lerdos@ist.ac.at

[1] Department of Physics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

[2] IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria

an even stronger concept, the *Eigenstate Thermalisation Hypothesis (ETH)* [7–9]. Informally, the ETH asserts that (i) *physical* observables A take their thermal value on every eigenstate of a many-body quantum system and (ii) off-diagonal elements of A in the energy eigenbasis are vanishingly small. In particular, the ETH ensures the thermalisation of A for any initial state with a macroscopically definite energy, given no massive degeneracy in the energy spectrum [10–12].

The ETH has numerically been verified for individual models with several local or few-body observables [13–23]. On the other hand, recent studies have revealed several classes of systems for which the ETH breaks down: examples include systems with an extensive number of local conserved quantities [24–29], many-body localisation [30–33], and quantum many-body scars [34–36].

As another approach to this question, it has been proven that the ETH holds true for *any* deterministic observable for *almost all* Hamiltonians H [7, 37–39] sampled from a Wigner matrix ensemble which has no further unitary symmetry (see also [40, 41] for ETH for more general mean-field ensembles). If the Hamiltonian has some unitary symmetry, the ETH clearly breaks down for conserved quantities related to those symmetries because we can find simultaneous eigenstates of the Hamiltonian and conserved quantities. However, Ref. [17] observed an interesting phenomenon, namely that local quantities still satisfy the ETH even in a system with translational symmetry. Therefore, the question of how *generically* and for what class of observables the ETH holds true in realistic situations has yet to be fully resolved.

In this paper, we mathematically rigorously prove an instance of the observation first made in Ref. [17] and confirmed in other numerical studies [18, 20, 21]. More precisely, we show that, for the L-body case of an ensemble with translational symmetry, where L is the number of spins, the ETH typically holds for quantities whose support does not exceed half of the system size with the optimal speed of convergence. The ETH also typically holds for quantities whose support exceeds half the system size but with a slower convergence speed, while it typically breaks down for some observables whose support extends to the entire system. We complement our analytical results for the L-body case with a numerical simulation for an ensemble of more realistic Hamiltonians with local interactions.

2 Setup

We consider a one-dimensional periodic quantum spin system on the $L \in \mathbb{N}$ sites of the standard discrete torus

$$\mathbb{T}_L := {}^{\mathbb{Z}}\!/_{L\mathbb{Z}}.$$

On each vertex $j \in \mathbb{T}_L$, the one-particle Hilbert space $\mathcal{H}_j$ is given by $\mathbb{C}^2$ and we denote its canonical basis by $\{|\uparrow\rangle, |\downarrow\rangle\}$. The corresponding L-particle Hilbert space

$$\mathcal{H} := \bigotimes_{j=1}^{L} \mathbb{C}^2$$

is simply given by the tensor product with dimension 2^L. For simplicity, we restrict ourselves to the spin-$1/2$ case, but our results can straightforwardly be extended to general spin s with one-particle Hilbert space being $\mathbb{C}^{2s+1}$.

Next, we introduce the ensemble of Hamiltonians, which is first introduced in Ref. [42] and shall be studied in this article. The main parameter in the definition is a tunable range

$\ell \leq L$ of interactions, which allows us to consider how generically the ETH holds in realistic situations.

Definition 2.1 (*Hamiltonian*) Let $T = T_L$ be the (left) translation operator acting on L spins at the vertices of $\mathbb{T}_L$. We define the ensemble of Hamiltonians with local interactions as

$$H_L^{(\ell)} := \sum_{j=0}^{L-1} T_L^{-j} \left(h_\ell \otimes I_{L-\ell}\right) T_L^j \,, \quad h_\ell := \sum_{p_1,\ldots,p_\ell=0}^{3} J_{p_1,\ldots,p_\ell} \sigma_1^{(p_1)} \ldots \sigma_\ell^{(p_\ell)} \tag{2.1}$$

where $\ell \leq L$ is the interaction range, $I_{L-\ell}$ is the identity on the sites $\ell+1, \ldots, L$. Here $\sigma_j^{(p)}$ is the p^{th} Pauli matrix $\sigma^{(p)}$ acting on the site $j \in \mathbb{T}_L$, where we recall the we recall the standard Pauli matrices,

$$\sigma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^{(2)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.2}$$

The 4^ℓ coefficients $J_{p_1,\ldots,p_\ell}$ are independent, identically distributed real Gaussian random variables with zero mean, $\mathbb{E} J_{p_1,\ldots,p_\ell} = 0$, and variance

$$v_\ell^2 := \mathbb{E} \left| J_{p_1,\ldots,p_\ell} \right|^2 .$$

The ensemble of Hamiltonians h_ℓ (2.1) contains prototypical spin models such as the XYZ model, $h_\ell = \sum_{p=1}^3 J_p \sigma_1^{(p)} \sigma_2^{(p)}$.

Observe that the Hamiltonian $H_L^{(\ell)}$ is a shifted version of the same local Hamiltonian h_ℓ. In particular, $H_L^{(\ell)}$ is translation invariant by construction, i.e., $T_L H_L^{(\ell)} T_L^{-1} = H_L^{(\ell)}$. We impose this structure to study a Hamiltonian with a symmetry. In the sequel we shall exploit this feature of $H_L^{(\ell)}$ by switching from position space to momentum space.

Lemma 2.2 *Let*

$$\Pi_k := \frac{1}{L} \sum_{j=1}^{L} e^{2\pi i \frac{kj}{L}} T_L^{-j} \quad \textit{for} \quad k = 0, \ldots, L-1 \tag{2.3}$$

be the projection operator onto the k-momentum space, i.e., $T_L \Pi_k = e^{2\pi i \frac{k}{L}} \Pi_k$. *Then* $H_L^{(\ell)}$ *is block-diagonal in the momentum space representation, i.e., in the eigenbasis of* T_L, *since we have*

$$H_L^{(\ell)} = L \sum_{k=0}^{L-1} \Pi_k \left(h_\ell \otimes I_{L-\ell}\right) \Pi_k \,. \tag{2.4}$$

Proof This follows immediately by substituting the spectral decomposition of T given by $T = \sum_{k=0}^{L-1} e^{2\pi i k/L} \Pi_k$ into (2.1). □

As we will show in Lemma 3.4, the dimensions of each of the L momentum sectors are almost equal to each other, $\mathrm{tr}_L \Pi_k \approx 2^L/L$.

In order to present our main result, the ETH in translation-invariant systems (Theorem 3.1), in a concise form, we need to introduce the *microcanonical average*. Below, we denote by $|E_\alpha^{(k)}\rangle$ the normalised eigenvector of $H_L^{(\ell)}$ in the k-momentum sector with eigenvalue E_α, i.e., $H_L^{(\ell)} |E_\alpha^{(k)}\rangle = E_\alpha |E_\alpha^{(k)}\rangle$ and $\Pi_k |E_\alpha^{(k)}\rangle = |E_\alpha^{(k)}\rangle$. It is easy to see that the spectrum of H in each momentum sector is simple almost surely.

Definition 2.3 (*Microcanonical ensemble*) For every energy $E \in \mathbb{R}$ and energy window $\Delta > 0$, we define the *microcanonical energy shell* $\mathcal{H}_{E,\Delta}$ centred at energy E with width 2Δ by

$$\mathcal{H}_{E,\Delta} := \bigoplus_{k=0}^{L-1} \mathcal{H}_{E,\Delta}^{(k)}, \quad \text{where} \quad \mathcal{H}_{E,\Delta}^{(k)} := \text{span}\left\{ |E_\alpha^{(k)}\rangle : \left| E_\alpha^{(k)} - E \right| \leq \Delta \right\}.$$

We denote the dimension of $\mathcal{H}_{E,\Delta}^{(k)}$ by $d_{E,\Delta}^{(k)}$ and that of $\mathcal{H}_{E,\Delta}$ by $d_{E,\Delta} = \sum_{k=0}^{L-1} d_{E,\Delta}^{(k)}$.

Whenever $d_{E,\Delta} > 0$, we define the *microcanonical average* of any self-adjoint observable $A \in \mathbb{C}^{N\times N}$ within $\mathcal{H}_{E,\Delta}$ by

$$\langle A \rangle_\Delta^{(\text{mc})}(E) := \frac{1}{d_{E,\Delta}} \sum_{k=0}^{L-1} \sum_{|E_\alpha^{(k)}\rangle \in \mathcal{H}_{E,\Delta}^{(k)}} \langle E_\alpha^{(k)} | A | E_\alpha^{(k)} \rangle . \tag{2.5}$$

Remark 2.4 In order to be thermodynamically meaningful, there are two natural requirements on the energy shell $\mathcal{H}_{E,\Delta}$:

(i) The density of states is approximately constant in the interval $[E - \Delta, E + \Delta]$.
(ii) The microcanonical energy shell contains $\gg 1$ states, i.e., $d_{E,\Delta} \to \infty$ as $L \to \infty$.

Note that for any fixed energy E, (i) corresponds to an upper bound, and (ii) corresponds to a lower bound on Δ, both being dependent on E. We point out that very close to the spectral edges with only a few states, it is not guaranteed that both requirements can be satisfied simultaneously. However, our main result, Theorem 3.1, will even hold for an arbitrary Δ.

We set

$$N := 2^L = \dim \mathcal{H}$$

for the total Hilbert space dimension. Our analytic results below will always be understood in the limit of large system size, i.e., $L \to \infty$, or, equivalently $N \to \infty$. We shall also use the following common notion (see, e.g., [43]) of stochastic domination.

Definition 2.5 Given two families of non-negative random variables

$$X := \left(X^{(N)}(u) : N \in \mathbb{N},\ u \in U^{(N)} \right) \quad \textit{and} \quad Y := \left(Y^{(N)}(u) : N \in \mathbb{N},\ u \in U^{(N)} \right)$$

indexed by N, we say that X is stochastically dominated by Y, if for all $\xi, D > 0$, we have

$$\sup_{u \in U^{(N)}} \mathbb{P}\left[X^{(N)}(u) > N^\xi Y^{(N)}(u) \right] \leq N^{-D}$$

for any sufficiently large $N \geq N_0(\xi, D)$ and use the notation $X \prec Y$ or $X = \mathcal{O}_\prec(Y)$ in that case.

3 Main Result in the *L*-Body Case

Throughout the entire section, we are in the L-body case, i.e., $\ell = L$. For any $q \leq L$ we also introduce the concept of *q-local observables* for self-adjoint operators of the form $A = A_q \otimes I_{L-q}$, i.e., A_q is self-adjoint and only acts on the first q sites.

Our main result in this setting is the following theorem.

Theorem 3.1 (ETH in translation-invariant systems) *Let $\ell = L$ and consider the Hamiltonian $H_L^{(L)}$ from (2.1) with eigenvalues $E_\alpha^{(k)}$ and associated normalised eigenvectors $|E_\alpha^{(k)}\rangle$. Then, for every $\Delta > 0$ and bounded q-local observable $A = A_q \otimes I_{L-q}$, $\|A\| \lesssim 1$, it holds that*

$$\max_{\alpha,\beta} \max_{k,k'} \left| \langle E_\alpha^{(k)} |A| E_\beta^{(k')} \rangle - \delta_{\alpha\beta}\delta_{k,k'} \langle A \rangle_\Delta^{(mc)} (E_\alpha^{(k)}) \right| \prec \frac{1}{2^{\min\{L/2, L-q\}}}, \tag{3.1}$$

where the maxima are taken over all indices labelling the eigenvectors of $H_L^{(L)}$. In particular, for $q \leq L/2$ the ETH holds with optimal speed of convergence of order $1/\sqrt{N}$.

An extension of Theorem 3.1 to arbitrary dimension $d \geq 2$ is provided in Theorem A.3 in the Appendix.

Remark 3.2 (Typicality of ETH) Theorem 3.1 asserts that for any fixed local observable A the ETH in the form (3.1) holds with a very high probability, i.e., apart from an event of probability $N^{-D} = 2^{-LD}$, for any fixed D, see the precise Definition 2.5. This exceptional event may depend on the observable A. However, as long as q is L-independent (in fact some mild logarithmic increase is allowed), it also holds that

$$\max_{\alpha,\beta} \max_{k,k'} \max_{A} \left| \langle E_\alpha^{(k)} |A| E_\beta^{(k')} \rangle - \delta_{\alpha\beta}\delta_{k,k'} \langle A \rangle_\Delta^{(\mathrm{mc})} (E_\alpha^{(k)}) \right| \prec \frac{1}{2^{L/2}}, \tag{3.2}$$

i.e., we may take the supremum over all bounded q-local observables A in (3.1). This extension is a simple consequence of choosing a sufficiently fine grid in the unit ball of the $4^q \times 4^q$ dimensional space of q-local observables and taking the union bound. The estimate (3.2) can be viewed as a very strong form of the typicality of the ETH within our class of translation-invariant L-body operators $H_L^{(L)}$. It asserts that apart from an exceptional set of the coupling constants $J_{p_1,\ldots,p_L}$ the Hamiltonian $H_L^{(L)}$ satisfies the ETH with the optimal speed of convergence, uniformly in the entire spectrum and tested against all finite range (q-local) observables. The exceptional set has exponentially small measure of order 2^{-LD} for any D if L is sufficiently large.

In Lemma 3.5 we will see that in the L-body case the Hamiltonian in each momentum sector is a GUE matrix, in particular the density of states of H follows Wigner's semicircle law. An elementary calculation shows that the radius of this semicircle is given by

$$R := 2 \cdot 2^L \sqrt{L} v_L (1 + O(2^{-L})).$$

In light of Remark 2.4 we also mention that $\langle A \rangle_\Delta^{(\mathrm{mc})} (E)$ in (3.1) is thermodynamically meaningful at energy $|E| \leq R$ if

$$\frac{R}{N^{2/3}} \ll \Delta \ll R - |E|. \tag{3.3}$$

The upper bound in (3.3) comes from requirement (i) in Remark 2.4, while the lower bound in (3.3) stems from (ii) using that the eigenvalue spacing near the spectral edge for Wigner matrices is of order $R/N^{2/3}$.

For the sequel we introduce the notation

$$\langle A \rangle := \frac{\mathrm{tr} A}{\mathrm{tr} I}$$

for the normalised trace of an operator A on any finite-dimensional Hilbert space, where I is the identity on that space. In particular, if $A = A_q \otimes I_{L-q}$ is a q-local observable, then $\langle A \rangle = \langle A_q \rangle$.

The proof of Theorem 3.1 crucially relies on the fact that in our L-body case $\langle E_\alpha^{(k)} \,|A|\, E_\beta^{(k')}\rangle$ converges to $\delta_{\alpha\beta}\delta_{k,k'}\,\langle A\rangle$. In other words, the thermodynamics of the system is trivial; the thermal value of A is always given by its normalised trace. This is formalised in the following main proposition:

Proposition 3.3 *Under the assumptions of Theorem 3.1 it holds that*

$$\max_{\alpha,\beta}\max_{k,k'}\left|\langle E_\alpha^{(k)} \,|A|\, E_\beta^{(k')}\rangle - \delta_{\alpha\beta}\delta_{k,k'}\,\langle A\rangle\right| \prec \frac{1}{2^{\min\{L/2,L-q\}}}\,. \tag{3.4}$$

Having Proposition 3.3 at hand, we can readily prove Theorem 3.1.

Proof of Theorem 3.1 Averaging (3.4) for $\alpha=\beta$ and $k=k'$ according to the microcanonical average (2.5), we find that

$$\max_{\alpha}\max_{k}\left|\langle A\rangle_\Delta^{(\mathrm{mc})}\,(E_\alpha^{(k)}) - \langle A\rangle\right| \prec \frac{1}{2^{\min\{L/2,L-q\}}}\,.$$

Combining this with (3.4), the claim immediately follows. □

The rest of this section is devoted to the proof of Proposition 3.3, which is conducted in four steps.

1. The momentum sectors are all of the same size with very high precision (Lemma 3.4).
2. In each momentum sector the L-body Hamiltonian $H_L^{(L)}$, represented in the eigenbasis of the translation operator T, is a GUE matrix (Lemma 3.5).
3. The ETH holds within each momentum sector separately (Lemma 3.6).
4. The normalised trace on each momentum sector and the total normalised trace are close to each other – at least for local observables (Lemma 3.7).

We shall first formulate all the four lemmas precisely and afterwards conclude the proof of Proposition 3.3.

Lemma 3.4 (Step 1: Dimensions of momentum sectors) *The dimension* $\mathrm{tr}_L\,\Pi_k$ *of the* k*-momentum sectors* ($k=0,\dots,L-1$) *is almost equal to each other in the sense that we have*

$$\mathrm{tr}_L\,\Pi_k = \frac{2^L}{L} + \mathcal{O}\left(L^{1/2}2^{L/2}\right)\,.$$

The proof is given in Sect. 3.1

Lemma 3.5 (Step 2: GUE in momentum blocks) *Each momentum block of the* L*-body Hamiltonian* $H_L^{(L)}$*, represented in an eigenbasis of* T*, is an i.i.d. complex Gaussian Wigner matrix (GUE), whose entries have mean zero and variance* $2^L\,L^2\,v_L^2$*. Recall that* $v_L^2 = \mathbb{E}\left|J_{p_1,\dots,p_L}\right|^2$ *from Definition 2.1.*

Proof In the L-body case $\ell=L$, a simple direct calculation of all first and second moments of the matrix elements shows that the interaction matrix h_ℓ is a complex Gaussian Wigner matrix whose entries have variance $2^L v_L^2$. Since the transformation from the standard basis to an eigenbasis of T is unitary, and the Gaussian distribution is invariant under unitary transformation, h_ℓ represented in an eigenbasis of T is again a Gaussian Wigner matrix. Finally, the projection operators Π_k in (2.4) set the off-diagonal blocks to zero. Incorporating the additional factor L in (2.4) into the variance proves Lemma 3.5. □

As the next step, we show that the ETH holds within each momentum sector.

Lemma 3.6 (Step 3: ETH within each momentum sector) *For an arbitrary deterministic observable A with $\|A\| \lesssim 1$ it holds that*

$$\max_{\alpha,\beta} \max_{k} \left| \langle E_\alpha^{(k)} \,|A|\, E_\beta^{(k)} \rangle - \delta_{\alpha\beta} \frac{tr_L\,(\Pi_k A \Pi_k)}{tr_L\,\Pi_k} \right| \prec \frac{1}{2^{L/2}}\,. \tag{3.5}$$

Here, the size of the right-hand side is optimal.

Proof For any fixed k, Lemma 3.5 asserts that $\Pi_k H_L^{(L)} \Pi_k$ is a standard GUE matrix (up to normalisation by v_L). Using [39, Theorem 1], therefore its eigenvectors $|E_\alpha\rangle = |E_\alpha^{(k)}\rangle$ satisfy the ETH in the form that $\langle E_\alpha \,|A|\, E_\beta \rangle$ is approximately given by the normalised trace of A in the k-momentum sector

$$\langle A \rangle_k := \frac{\mathrm{tr}_L\,(\Pi_k A \Pi_k)}{\mathrm{tr}_L\,\Pi_k}$$

with very high probability and with the optimal size of the error given by the square root of the inverse of the dimension of the k-momentum sector, $1/\sqrt{\mathrm{tr}_L\,\Pi_k}$. This holds in the sense of stochastic domination given in Definition 2.5. Using that $\mathrm{tr}_L\,\Pi_k \approx 2^L/L$ from Lemma 3.4, we obtain that (3.5) holds for each fixed k, uniformly in all eigenvectors. Finally, the very high probability control in the stochastic domination allows us to take the maximum over $k = 1, 2, \ldots, L$ by a simple union bound. This completes the proof of (3.5). □

We remark that the essential ingredient of this proof, the Theorem 1 from [39], applies not only for the Gaussian ensemble but for arbitrary Wigner matrices with i.i.d. entries (with some moment condition on their entry distribution) and its proof is quite involved. However, the ETH for GUE, as needed in Lemma 3.6, can also be proven with much more elementary methods using that the eigenvectors are columns of a Haar unitary matrix. Namely, moments of $\langle E_\alpha \,|A|\, E_\beta \rangle$ can be directly computed using Weingarten calculus [44]. Since in (3.5) we aim at a control with very high probability, this would require to compute arbitrary high moments of $\langle E_\alpha \,|A|\, E_\beta \rangle - \delta_{\alpha,\beta}\,\langle A \rangle_k$. The Weingarten formalism gives the exact answer but it is somewhat complicated for high moments, so identifying their leading order (given by the "ladder" diagrams) requires some elementary efforts. For brevity, we therefore relied on [39, Theorem 1] in the proof of Lemma 3.6 above.

Finally, we formulate the fourth and last step of the proof of Proposition 3.3 in the following lemma, the proof of which is given in Sect. 3.2.

Lemma 3.7 (Step 4: Traces within momentum sectors) *Let $A = A_q \otimes I_{L-q}$ be an arbitrary q-local observable with $\|A\| \lesssim 1$. Then it holds that*

$$\max_{k} \left| \frac{\mathrm{tr}_L\,(\Pi_k A \Pi_k)}{\mathrm{tr}_L\,\Pi_k} - \langle A \rangle \right| \leq \mathcal{O}\left(\frac{L}{2^{\min\{L-q,\,L/2\}}} \right)\,. \tag{3.6}$$

Moreover, for $q > L/2 + 1$ this bound is optimal (up to the factor L).

Armed with these four lemmas, we can now turn to the proof of Proposition 3.3.

Proof of Proposition 3.3 First, for any q-local observable $A = A_q \otimes I_{L-q}$, we conclude from Lemma 3.6 and Lemma 3.7 that

$$\max_{\alpha,\beta} \max_{k} \left| \langle E_\alpha^{(k)} \,|A|\, E_\beta^{(k)} \rangle - \delta_{\alpha\beta} \langle A \rangle \right| \prec \frac{1}{2^{\min\{L-q,\,L/2\}}}\,. \tag{3.7}$$

For the element $\langle E_\alpha^{(k)}|A|E_\beta^{(k')}\rangle$ with $k \neq k'$, i.e., in off-diagonal blocks, $|E_\alpha^{(k)}\rangle$ and $|E_\beta^{(k')}\rangle$ are normalised Gaussian vectors independent of each other. This follows from that each momentum block of $\hat{H}_L^{(L)}$ is an *i.i.d.* complex Gaussian Wigner matrix (GUE) (Lemma 3.5) and that each eigenvector of a GUE matrix is a normalised Gaussian vector. Therefore standard concentration estimate shows that

$$\max_{k\neq k'} \left|\langle E_\alpha^{(k)}\,|A|\,E_\beta^{(k')}\rangle\right| \prec \frac{1}{2^{L/2}}\,. \tag{3.8}$$

Combining (3.7) with (3.8), we have proven Proposition 3.3. □

3.1 Dimensions of Momentum Sectors: Proof of Lemma 3.4

In this section we prove Lemma 3.4, and establish that the sizes of the momentum sectors are almost equal. To this end, we show that the leading term in the size of each of the momentum blocks is given by the number of aperiodic elements in the product basis of $\mathcal{H}$.

We present the proof using group theory notation, which is not strictly necessary for the one-dimensional case under consideration since the translation group of the torus $\mathbb{T}_L$ is cyclic. Nevertheless, we do it to allow for a more straightforward generalisation to the d-dimensional case (cf. Lemma A.4).

Proof of Lemma 3.4 We introduce the following objects. Let $\mathfrak{S}$ denote the canonical product basis of $\mathcal{H}$,

$$\mathfrak{S}(L) := \{\sigma : \mathbb{T}_L \to \{|\uparrow\rangle, |\downarrow\rangle\}\}\,, \tag{3.9}$$

and let $\mathcal{G}$ be the group of translations of $\mathbb{T}_L$ generated by $T = T_L$. Note that $\mathcal{G}$ is a finite cyclic group of size $|\mathcal{G}| = L$. The action of $\mathcal{G}$ on $\mathfrak{S}(L)$ is defined by

$$(g\sigma)(x) := \sigma(g^{-1}(x))\,, \quad x \in \mathbb{T}_L\,, \quad \sigma \in \mathfrak{S}(L)\,, \quad g \in \mathcal{G}\,. \tag{3.10}$$

In particular, the set $\mathfrak{S}(L)$ is a disjoint union of sets $\mathfrak{S}_b(L)$ defined by

$$\mathfrak{S}_b(L) := \{\sigma \in \mathfrak{S}(L) \,:\, |\mathcal{G}_\sigma| = b\}\,, \quad b = 1, 2, \ldots, L\,,$$

where $\mathcal{G}_\sigma \subset \mathfrak{S}(L)$ is the stabiliser of σ under the action (3.10). By the orbit-stabiliser theorem, $\mathfrak{S}_b(L) = \emptyset$ for all b that do not divide L. Since the group $\mathcal{G}$ is cyclic, it has a unique subgroup of size b for all $b|L$, given explicitly by

$$\mathcal{G}^{(b)} := \{T^{L/b}, T^{2L/b}, \ldots, T^L\}\,.$$

Observe that each $\sigma \in \mathfrak{S}_b(L)$ corresponds to a unique map $\widetilde{\sigma}$ on a reduced torus $\mathfrak{S}(L/b) := {}^{\mathbb{T}_L}\!/_{\mathcal{G}^{(b)}}$, which is defined by

$$\widetilde{\sigma}([x]) := \sigma(x)\,, \quad [x] \in \mathfrak{S}(L/b)\,. \tag{3.11}$$

Since σ is stabilised by $\mathcal{G}^{(b)}$, the map $\sigma \mapsto \widetilde{\sigma}$ in (3.11) is well-defined and injective. In particular, $|\mathfrak{S}_b(L)| \leq 2^{L/b}$, and hence

$$2^L = \sum_{b|L} |\mathfrak{S}_b(L)| = M(L) + \sum_{b|L, b\geq 2} |\mathfrak{S}_b(L)| \leq M(L) + \mathcal{O}\left(L^{1/2} 2^{L/2}\right)\,, \tag{3.12}$$

where $M(L) := |\mathfrak{S}_1(L)|$ denotes the number of elements in $\mathfrak{S}(L)$ with a trivial stabiliser. The last inequality follows from the fact that L has at most $\mathcal{O}(L^{1/2})$ divisors.

Since $M(L) \leq 2^L$, we conclude from (3.12) that

$$M(L) = 2^L + \mathcal{O}\left(L^{1/2} 2^{L/2}\right) . \tag{3.13}$$

For any $k \in \{0, \ldots, L-1\}$, we can construct an eigenvector of T corresponding to the eigenvalue $e^{2\pi i k/L}$ by defining

$$\mathbf{v}(\sigma, k) := \Pi_k \sigma = \frac{1}{L} \sum_{j=0}^{L-1} e^{2\pi i \frac{kj}{L}} T^{-j} \sigma , \quad \sigma \in \mathfrak{S}_1(L) . \tag{3.14}$$

Since the orbit of σ under T consists of L distinct basis elements, the vector $\mathbf{v}(\sigma, k)$ is non-zero. Furthermore, the vectors $\mathbf{v}(\sigma, k)$ and $\mathbf{v}(\sigma', k)$ corresponding to σ and σ' in disjoint orbits are linearly independent because they share no basis element. Therefore, the dimension of the k-th momentum space is bounded from below by the number of disjoint orbits in $\mathfrak{S}_1(L)$, that is

$$\mathrm{tr}_L \Pi_k \geq \frac{2^L}{L} + \mathcal{O}\left(L^{-1/2} 2^{L/2}\right) , \tag{3.15}$$

where we used inequality (3.13) and the fact that all orbits in $\mathfrak{S}_1(L)$ have size L. By means of (3.15), we obtain the following chain of inequalities

$$\mathrm{tr}_L \Pi_k = 2^L - \sum_{j \neq k} \mathrm{tr}_L \Pi_j \leq \frac{2^L}{L} + \mathcal{O}\left(L^{1/2} 2^{L/2}\right) , \tag{3.16}$$

which, together with (3.15) concludes the proof of Lemma 3.4. □

3.2 Traces Within Momentum Sectors: Proof of Lemma 3.7

In this section, we give a proof of Lemma 3.7, which evaluates the difference of the normalised trace $\mathrm{tr}_L(\Pi_k A \Pi_k)/\mathrm{tr}_L \Pi_k$ on a momentum sector and the full normalised trace $\langle A \rangle$ for a q-local observable $A = A_q \otimes I_{L-q}$. We separate A into the tracial part $\langle A \rangle I$ and the traceless part $\mathring{A} := A - \langle A \rangle I$.

Proof of Lemma 3.7 Substituting $\Pi_k := \frac{1}{L} \sum_{j=1}^{L} e^{2\pi i \frac{kj}{L}} T_L^{-j}$, we obtain

$$\begin{aligned} \mathrm{tr}_L(\Pi_k A \Pi_k) &= \langle A \rangle \, \mathrm{tr}_L \Pi_k + \mathrm{tr}_L(\Pi_k \mathring{A}) \\ &= \langle A \rangle \, \mathrm{tr}_L \Pi_k + \frac{1}{L} \sum_{j=1}^{L-1} e^{2\pi i \frac{kj}{L}} \mathrm{tr}_L(T_L^{-j} \mathring{A}) . \end{aligned} \tag{3.17}$$

Then, the task is to evaluate the size of the quantity $\mathrm{tr}_L(T_L^{-j} \mathring{A})$.

Lemma 3.8 *Let $A := A_q \otimes I_{L-q}$ be a q-local observable with $\|A\| \lesssim 1$. Then, for any $j = 1, \ldots, L-1$, we have*

$$\left| \mathrm{tr}_L(T_L^{-j} A) \right| \lesssim 2^{\max\{q, \gcd(j, L)\}} , \tag{3.18}$$

where gcd *stands for the greatest common divisor.*

Combining (3.18) with $\gcd(j, L) \leq L/2$ for $j = 1, \ldots, L-1$ and Lemma 3.4 gives the bound (3.6). The optimality of (3.6) for $q > L/2 + 1$ is proven in Lemma 3.9 below. □

It remains to give the proof of Lemma 3.8.

Proof of Lemma 3.8 We choose a product basis $\left\{|s_1 \dots s_L\rangle \mid s_j \in \{\uparrow, \downarrow\}\right\}$ to calculate the trace. Then, we obtain

$$\left|\mathrm{tr}_L(T_L^{-j}A)\right| = \left|\sum_{s_1\dots s_L} \langle s_{1+j}\dots s_{q+j}\,|A_q|\,s_1\dots s_q\rangle \prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}\right|$$
$$\lesssim \sum_{s_1\dots s_L} \prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}\,. \tag{3.19}$$

Because of the product $\prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$ of Kronecker deltas, not all of the summation variables $s_1, \dots, s_L$ are independent.

To count the number of independent summations in the right-hand side of (3.19) and obtain an upper bound for $\left|\mathrm{tr}_L(T_L^{-j}A)\right|$ with $j = 1, \dots, L-1$, we count the number of independent deltas in the product

$$\mathcal{G}_{q,j}^{(L)} := \prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}\,. \tag{3.20}$$

Here, not all of the delta functions in $\mathcal{G}_{q,j}^{(L)}$ are independent in the sense that we may express $\mathcal{G}_{q,j}^{(L)}$ with a fewer number of deltas. For example, we have $\mathcal{G}_{1,2}^{(4)} = \delta_{s_2 s_4}\delta_{s_3 s_1}\delta_{s_4 s_2} = \delta_{s_3 s_1}\delta_{s_4 s_2}$.

To obtain an expression of $\mathcal{G}_{q,j}^{(L)}$ with the minimal number of deltas, we graphically represent the product $\prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$ by arranging the sites on a circle and representing the $\delta_{s_m s_{m+j}}$'s with a line connecting the site m and $m+j$ (Fig. 1). A minimal representation of $\mathcal{G}_{q,j}^{(L)}$ is obtained by removing exactly one delta for every occurrence of a loop in the graph of $\prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$.

The graph of $\prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$ can be obtained in two steps: First, in step (i), drawing the graph of $\prod_{m=1}^{L} \delta_{s_m s_{m+j}}$ and second, in step (ii), removing the lines corresponding to the delta functions $\delta_{s_m s_{m+j}}$ $(m = 1, \dots, q)$, which are depicted with red dashed lines in Fig. 1.

In the first step (i), there are exactly $\gcd(j, L)$ loops each starting from the sites $1, \dots, \gcd(j, L)$. If $q > \gcd(j, L)$, there is no loop remaining after the second step (ii). Thus, we obtain a minimal representation of $\mathcal{G}_{q,j}^{(L)}$ as $\mathcal{G}_{q,j}^{(L)} = \prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$. If $q \le \gcd(j, L)$, the loops starting from the sites $q+1, \dots, \gcd(j, L)$ remain after the second step (ii), for each of which we remove one delta to obtain a minimal representation of $\mathcal{G}_{q,j}^{(L)}$ as $\mathcal{G}_{q,j}^{(L)} = \prod_{m=\gcd(j,L)+1}^{L} \delta_{s_m s_{m+j}}$.

In summary, we obtain a minimal representation of $\mathcal{G}_{q,j}^{(L)}$ as

$$\mathcal{G}_{q,j}^{(L)} = \prod_{m=\max\{q,\gcd(j,L)\}+1}^{L} \delta_{s_m s_{m+j}}\,. \tag{3.21}$$

By substituting (3.21) into (3.19) we obtain

$$\left|\mathrm{tr}_L(T_L^{-j}A)\right| \lesssim \sum_{s_1\dots s_L} \prod_{m=\max\{q,\gcd(j,L)\}+1}^{L} \delta_{s_m s_{m+j}} = 2^{\max\{q,\gcd(j,L)\}}\,.$$

□

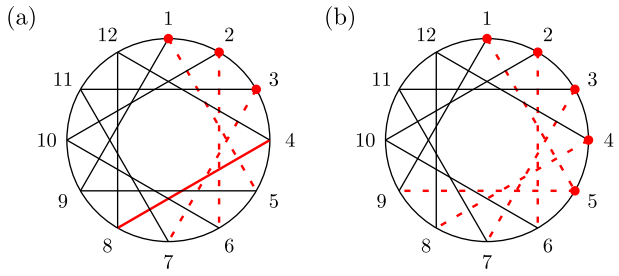

Fig. 1 Graphical representation of the product $\prod_{m=q+1}^{L} \delta_{s_{m+j} s_m}$ for **a** $L = 12$, $q = 3$, $j = 4$ and **b** $L = 12$, $q = 5$, $j = 4$. For the first case **(a)** where $q < \gcd(j, L)$, there is a loop 4-8-12-4 remaining after the step (ii), which contains exactly one redundant delta function $\delta_{s_4 s_8}$ depicted with a solid red line. In general, exactly one redundant delta function appears for every occurrence of a loop in the graph of $\prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$ (Color figure online)

Finally, we prove the optimality of (3.6) in the regime $q > L/2 + 1$.

Lemma 3.9 *Let $B_q := T_q + T_q^{-1} - 2^{2-q} I_q$, where T_q is the (left) translation operator acting only on the first q spins arranged on the torus $\mathbb{T}_q$. Observe that B_q is Hermitian and traceless. Then, for $q > L/2 + 1$, the normalised trace of $B := B_q \otimes I_{L-q}$ within the k-momentum sector is given by*

$$\frac{\mathrm{tr}_L(\Pi_k B \Pi_k)}{\mathrm{tr}_L \Pi_k} = \frac{2}{2^{L-q}} \cos\left(\frac{2\pi k}{L}\right) + \mathcal{O}\left(\frac{L}{2^{L/2}}\right). \tag{3.22}$$

This shows that the q-local observable $B_q := T_q + T_q^{-1} - 2^{2-q} I_q$ saturates the bound (3.6) when $q > L/2+1$. It also shows that the deviation of the normalised trace within a momentum sector, $\mathrm{tr}_L(\Pi_k B \Pi_k)/\mathrm{tr}_L \Pi_k$, from $\langle B \rangle = 0$, which is of order $2^{-(L-q)}$, becomes the dominant source of error in the ETH whenever $q > L/2 + 1$.

Proof of Lemma 3.9 We first reduce the range of the summation over j in the generally valid expression (3.17) applied to B. To do so, we introduce the parity operator P_L defined by $P_L |s_1 s_2 \ldots s_L\rangle := |s_L \ldots s_2 s_1\rangle$. It satisfies $P_L T_L P_L = T_L^{-1}$ and $P_L A P_L = I_{L-q} \otimes (P_q A_q P_q)$ for any $A = A_q \otimes I_{L-q}$. Since B_q is invariant under the parity transformation, we have

$$\mathrm{tr}_L(T_L^{-j} B) = \mathrm{tr}_L[T_L^{+j}(I_{L-q} \otimes B_q)] = \mathrm{tr}_L(T_L^{-(L-j)} B),$$

Therefore, we can rewrite (3.17) with the aid of (2.3) as

$$\mathrm{tr}_L(\Pi_k B \Pi_k) = \frac{2}{L} \sum_{j=1}^{\lfloor \frac{L}{2} \rfloor} \mathrm{tr}_L(T_L^{-j} \mathring{B}) \cos\left(\frac{2\pi k j}{L}\right) + \begin{cases} \frac{(-1)^k}{L} \mathrm{tr}_L(T_L^{-\frac{L}{2}} \mathring{B}) & L \text{ even} \\ 0 & L \text{ odd}. \end{cases} \tag{3.23}$$

When $q > L/2 + 1$, we have $j < q$ and cannot skip over the region $1, \ldots, j$ when going along the lines in the graph of $\mathcal{G}_{q,j}^{(L)}$ (recall (3.20)). Therefore, each line starting at one of the sites $p \in \{q + 1, \ldots, q + j\}$ passes through a point in $\{1, \ldots, j\}$. Moreover, the correspondence between p and the first intersection of the line starting at p with $\{1, \ldots, j\}$ is one-to-one. Therefore, there exists a permutation τ_j on $1, \ldots, j$ such that $s_{q+i} = s_{\tau(i)}$ for

$i = 1, \ldots, j$ due to $\mathcal{G}^{(L)}_{q,j}$. With this permutation τ, we obtain

$$\begin{aligned}
\mathrm{tr}_L(T_L^{-j}\mathring{B}) &= \sum_{s_1\ldots s_L} \left\langle s_{q+1}\ldots s_q s_{q+1} s_{q+j} \left| \mathring{B}_q \right| s_1 \ldots s_q \right\rangle \mathcal{G}^{(L)}_{q,j} \\
&= \sum_{s_1\ldots s_q} \left\langle s_{q+1}\ldots s_q s_{\tau(1)} s_{\tau(j)} \left| \mathring{B}_q \right| s_1 \ldots s_q \right\rangle \\
&= \mathrm{tr}_q(\tau_j^\dagger T_q^{-j} \mathring{B}_q) \\
&= \mathrm{tr}_q(\tau_j^\dagger T_q^{-(j-1)}) + \mathrm{tr}_q(\tau_j^\dagger T_q^{-(j+1)}) - 2^{2-q}\mathrm{tr}_q(\tau_j^\dagger T_q^{-j}) \,. \qquad (3.24)
\end{aligned}$$

Because τ_j is a j-local operator (not necessarily self-adjoint) on the q-site chain, we can apply Lemma 3.8 to each term in (3.24). Combined with $j < q-1$ and $\gcd(j,q) \le j$, we obtain

$$\mathrm{tr}_L(T_L^{-j}\mathring{B}) = \delta_{j1} 2^q + \mathcal{O}\left(2^j\right) = \delta_{j1} 2^q + \mathcal{O}\left(2^{L/2}\right) .$$

Substituting this result into (3.23) and employing $\mathrm{tr}_L \Pi_k = \frac{2^L}{L} + \mathcal{O}\left(L^{1/2} 2^{L/2}\right)$ from Lemma 3.4, we obtain the result (3.22). □

4 Numerical Verification of Theorem 3.1 for $\ell = \mathcal{O}(1)$

In this section, we numerically demonstrate that Theorem 3.1 also holds for the case of $\ell = 2$. For that purpose, we adopt the following measure of the ETH used in Refs [42, 45]. For any self-adjoint operator A we define

$$\Lambda = \Lambda(A) := \mathbf{E} \max_k {\max_\alpha}' \frac{\left| \langle E^{(k)}_\alpha | A | E^{(k)}_\alpha \rangle - \langle A \rangle^{(\mathrm{mc})}_\Delta (E^{(k)}_\alpha) \right|}{a_{\max} - a_{\min}} , \qquad (4.1)$$

where $a_{\max(\min)}$ is the maximum (minimum) eigenvalue of A. Here, $\mathbf{E}$ denotes the average over the realisations of the Hamiltonian (2.1), and $\max_\alpha{}'$ denotes the maximum over the eigenstates $|E^{(k)}_\alpha\rangle$ in the energy shell at the centre of the spectrum, i.e., those α for which

$$\left| E^{(k)}_\alpha - \langle H \rangle \right| \le \Delta \,.$$

The width Δ of the energy interval is set to be $\Delta = 0.4 v_\ell / L$ such that it satisfies the two physical requirements mentioned in Remark 2.4 for $L \ge 6$. With this choice of Δ, the microcanonical energy shell $\mathcal{H}_{\langle H\rangle,\Delta}$ defined by (4.1) typically contains more than 10 states, while the density of states does not change too much within $\mathcal{H}_{\langle H\rangle,\Delta}$.

As the observable, we choose $A = B_q \otimes I_{L-q}$ with $B_q := T_q + T_q^{-1} - 2^{2-q} I_q$ for $q = 2, \ldots, L$, which saturates the upper bound in (3.6) and thus also saturates that of (3.2). With this choice we have $a_{\max} - a_{\min} \simeq 4$ for any L and q. Therefore, the ETH measure Λ is essentially the same as the diagonal part of the left-hand side of (3.2) in Theorem 3.1 – except that the maximum over α is now taken only at the centre of the spectrum (and we do not take maximum over all A). This is because the eigenstate expectation value $\langle E^{(k)}_\alpha | A | E^{(k)}_\alpha \rangle$ of a local observable $A = A_q \otimes I_{L-q}$ with $q \ll L$ typically acquires an energy dependence when $\ell \ll L$ [46], and the number of states becomes not enough to calculate the microcanonical average near the edges for the computationally accessible system size.

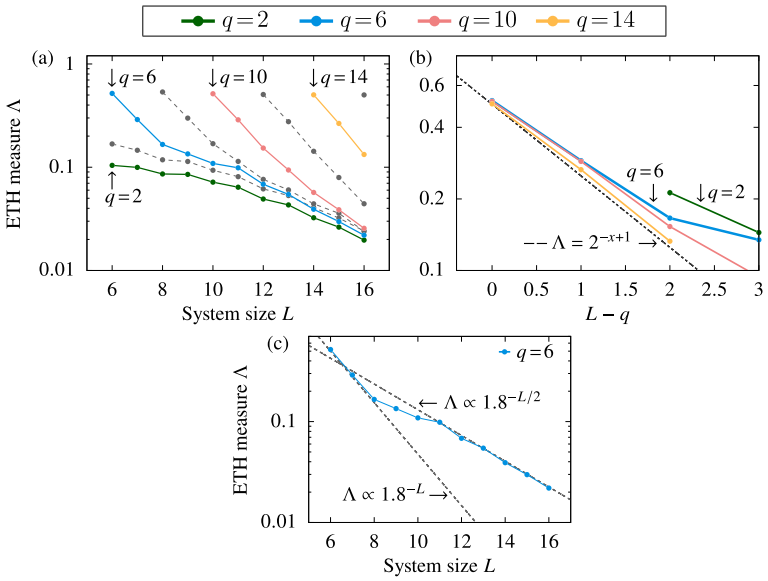

Fig. 2 **a** System-size dependence of the ETH measure Λ for the observable $A = B_q \otimes I_{L-q}$ with $B_q := T_q + T_q^{-1} - 2^{2-q} I_q$. Grey curves between coloured curves show intermediate values of q, i.e., $q = 4, 8, 12$. **b** The same data as the panel (a) for $q = 2, 6, 10$ and 14 plotted against $L-q$. When $L \simeq q$ so that $L-q < L/2$, Λ decreases as $\propto 2^{-L}$. **c** The same data as the panel **(a)** for $q = 6$. It decreases as $\propto 1.8^{-L}$ when $L-q \ll L/2$. When $L/2 \gtrsim L-q$, the decrease in Λ becomes slower and follows a different exponential decay with a base of $1.8^{-1/2}$, instead of 1.8^{-1}. Aside from the value of the base, this behavior is consistent with (3.2), which predicts that the exponent of the exponential decrease of Λ in L in the region $L-q \ll L/2$ should be twice as large as that in the region $L-q \gtrsim L/2$. The standard errors are smaller than the size of the data points. The number of samples lies between 1000 and 10000 for each datum (Color figure online)

The ETH measure Λ satisfies reasonable thermodynamical properties. It is (i) invariant under the linear transformation $A \mapsto aA+b$, (ii) dimensionless, and (iii) thermodynamically intensive for additive observables A [42].

Figure 2a–c depict the L-dependence of the ETH measure Λ for different values of the parameter q. In particular, Fig. 2b illustrates that, whenever L is approximately equal to q so that $L-q < L/2$, the ETH measure Λ decays as $\propto 2^{-L}$. The rate of this decay is slower for smaller values of q, but approaches 2^{-L} as q becomes larger. In Fig. 2c, we take a closer look at the L-dependence of Λ for $q = 6$. The data indicates that for $L-q \ll L/2$, Λ decays as $\propto 1.8^{-L}$, whereas for $L \gtrsim 2q$, Λ decays as $\propto 1.8^{-L/2}$. These numerical observations are in agreement with our analytical results for the L-body case in Theorem 3.1, which predicts that the exponent of the exponential decrease of Λ in L in the region $L-q \ll L/2$ should be twice as large as that in the region $L-q \gtrsim L/2$. This fact suggests that the theorem remains qualitatively valid for $\ell = \mathcal{O}(1)$ in the bulk of the spectrum as long as the energy shell width is appropriately chosen.

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Data Availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Competing Interests The authors declare that there is no conflict of interest

Appendix A: Extension to Higher Dimensions

In this appendix, we extend our main result, Theorem 3.1, to the d-dimensional case.

A.1 Multidimensional Setup

Let $\mathbf{L} := (L_1, \dots, L_d)$ be a vector of positive integers and set $V := \prod_{s=1}^{d} L_s$. We consider a d-dimensional system with V quantum spins at the vertices of the classical discrete torus

$$\mathbb{T}_{\mathbf{L}} := \bigtimes_{s=1}^{d} \mathbb{Z}/L_s\mathbb{Z}.$$

As before, on each vertex, the one-particle Hilbert space is given by $\mathbb{C}^2$ with canonical basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. The corresponding V-particle Hilbert space is given by

$$\mathcal{H} := \bigotimes_{s=1}^{V} \mathbb{C}^2 \quad \text{with dimension} \quad \dim\mathcal{H} = 2^V .$$

For a vector $\mathbf{q} = (q_1, \dots, q_d) \in \mathbb{T}_{\mathbf{L}}$, we introduce a rectangular subregion $\mathcal{R}_{\mathbf{q}} \subset \mathbb{T}_{\mathbf{L}}$ by

$$\mathcal{R}_{\mathbf{q}} := \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{T}_{\mathbf{L}} \colon 1 \le x_s \le q_s \,,\ s = 1, \dots, d\} \,.$$

A self-adjoint operator of the form $A = A_{\mathbf{q}} \otimes I_{\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}}$ is called a $\mathbf{q}$-*local observable*, where $A_{\mathbf{q}}$ is self-adjoint and acts on the Hilbert space of the spins in $\mathcal{R}_{\mathbf{q}}$, and $I_{\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}}$ is the identity on $\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}$.

Finally, let T_s be the (left) translation operator along the s-th coordinate acting on $\mathbb{T}_{\mathbf{L}}$. For a vector $\mathbf{j} := (j_1, \dots, j_d) \in \mathbb{T}_{\mathbf{L}}$, we introduce $T^{\mathbf{j}} := \prod_{s=1}^{d} T_s^{j_s}$.

The d-dimensional version of our model in Definition 2.1 is given as follows.

Definition A.1 Set the vector $\boldsymbol{\ell} := (\ell_1, \dots, \ell_d) \in \mathbb{T}_{\mathbf{L}}$ that determines the interaction range in each coordinate direction. We define the ensemble of Hamiltonians with local interactions as

$$H_{\mathbf{L}}^{(\boldsymbol{\ell})} := \sum_{\mathbf{j} \in \mathbb{T}_{\mathbf{L}}} T^{-\mathbf{j}} \left(h_{\boldsymbol{\ell}} \otimes I_{\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\boldsymbol{\ell}}} \right) T^{\mathbf{j}} \quad \text{with} \quad h_{\boldsymbol{\ell}} := \sum_{p_{\mathbf{1}}, \dots, p_{\boldsymbol{\ell}} = 0}^{3} J_{p_{\mathbf{1}}, \dots, p_{\boldsymbol{\ell}}} \sigma_{\mathbf{1}}^{(p_{\mathbf{1}})} \dots \sigma_{\boldsymbol{\ell}}^{(p_{\boldsymbol{\ell}})} \quad \text{(A1)}$$

where the symbols $\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{\ell}$ label the elements of $\mathcal{R}_{\boldsymbol{\ell}}$ in an arbitrary order. As in (2.1), $\sigma^{(p)}$ for $p \in \{0, 1, 2, 3\}$ are the Pauli matrices (2.2).

The $4^{|\mathcal{R}_{\boldsymbol{\ell}}|}$ coefficients $J_{p_{\mathbf{1}},\ldots,p_{\boldsymbol{\ell}}}$ are i.i.d. real Gaussian random variables with zero mean, $\mathbb{E}J_{p_{\mathbf{1}},\ldots,p_{\boldsymbol{\ell}}} = 0$, and variance $v_{\boldsymbol{\ell}}^2 := \mathbb{E}\left|J_{p_{\mathbf{1}},\ldots,p_{\boldsymbol{\ell}}}\right|^2$.

We have the following multidimensional analogue of Lemma 2.2.

Lemma A.2 *Let*

$$\Pi_{\mathbf{k}} := \frac{1}{V} \sum_{\mathbf{j} \in \mathbb{T}_{\mathbf{L}}} e^{2\pi i \sum_{s=1}^{d} \frac{k_s j_s}{L_s}} T^{-\mathbf{j}} \quad \textit{for} \quad \mathbf{k} \in \mathbb{T}_{\mathbf{L}}$$

be the projection operator onto the $\mathbf{k}$*-momentum space, i.e.,* $T_s \Pi_{\mathbf{k}} = e^{2\pi i \frac{k_s}{L_s}} \Pi_{\mathbf{k}}$ *for all* $s = 1, \ldots, d$*. Then we have*

$$H_{\mathbf{L}}^{(\boldsymbol{\ell})} = V \sum_{\mathbf{k} \in \mathbb{T}_{\mathbf{L}}} \Pi_{\mathbf{k}} \left(h_{\boldsymbol{\ell}} \otimes I_{\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\boldsymbol{\ell}}} \right) \Pi_{\mathbf{k}} \,.$$

Proof of Lemma 3.4 This follows by Lemma 2.2 coordinatewise. □

Denoting by $|E_\alpha^{(\mathbf{k})}\rangle$ the normalised eigenvector of $H_{\mathbf{L}}^{(\boldsymbol{\ell})}$ belonging to an eigenvalue E_α and the $\mathbf{k}$-momentum sector, i.e., $H_{\mathbf{L}}^{(\ell)}|E_\alpha^{(\mathbf{k})}\rangle = E_\alpha |E_\alpha^{(\mathbf{k})}\rangle$ and $\Pi_{\mathbf{k}}|E_\alpha^{(\mathbf{k})}\rangle = |E_\alpha^{(\mathbf{k})}\rangle$, the definition of the *microcanocical average* is completely analogous to Definition 2.3.

Moreover, whenever we use the notation $\prec$ for stochastic domination (Definition 2.5), it is always understood with $N := 2^V$.

A.2 Multidimensional Version of the Main Result

The d-dimensional version of Theorem 3.1 is then given as follows.

Theorem A.3 (ETH in d-dimensional translation-invariant systems) *Let* $\boldsymbol{\ell} = \mathbf{L}$ *and consider the Hamiltonian* $H_{\mathbf{L}}^{(\mathbf{L})}$ *from* (A1) *with eigenvalues* $E_\alpha^{(\mathbf{k})}$ *and normalised eigenvectors* $|E_\alpha^{(\mathbf{k})}\rangle$*. Then, for every* $\Delta > 0$ *and bounded* $\mathbf{q}$*-local observable* $A = A_{\mathbf{q}} \otimes I_{\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}}$ *with* $q_s \le L_s/2$ *for all* $s = 1, \ldots, d$*, it holds that*

$$\max_{\alpha,\beta} \max_{\boldsymbol{k},\boldsymbol{k'}} \left| \langle E_\alpha^{(\boldsymbol{k})} | A | E_\beta^{(\boldsymbol{k'})} \rangle - \delta_{\alpha\beta} \delta_{\boldsymbol{k},\boldsymbol{k'}} \langle A \rangle_\Delta^{(mc)} (E_\alpha^{(\boldsymbol{k})}) \right| \prec \frac{1}{2^{V/2}} \,. \tag{A2}$$

That is, the ETH holds with optimal speed of convergence.

The principal strategy for proving Theorem A.3 is exactly the same as for Theorem 3.1, which has been outlined right below Proposition 3.3. We shall hence only discuss the differences compared to the proof in Section 3, which consist solely of Step 1 (generalising Lemma 3.4, cf. Lemma A.4) and Step 4 (generalising Lemma 3.7, cf. Lemma A.5).

Lemma A.4 (Step 1: Dimensions of momentum sectors) *The dimension* $\mathrm{tr}_{\mathbf{L}} \Pi_{\mathbf{k}}$ *of the* $\mathbf{k}$*-momentum sectors for* $\mathbf{k} \in \mathbb{T}_{\mathbf{L}}$ *is almost equal to each other in the sense that we have*

$$\mathrm{tr}_{\mathbf{L}} \Pi_{\mathbf{k}} = \frac{2^V}{V} + \mathcal{O}\left(2^{V/2 + (\log_2 V)^2}\right) .$$

Proof of Lemma 3.4 Let $\mathfrak{S} = \mathfrak{S}(\mathbf{L})$ denote the canonical product basis of $\mathcal{H}$, as in (3.9), and let $\mathcal{G}$ be the commutative group generated by the translation operators $\{T_s\}_{s=1}^d$. The action of $\mathcal{G}$ on $\mathfrak{S}$ is defined by (3.10).

In general, the group $\mathcal{G}$ is not cyclic, hence the subgroups of $\mathcal{G}$ are not uniquely determined by their size. However, $\mathfrak{S}$ can be decomposed into a disjoint union of sets $\mathfrak{S}_\mathcal{K} = \mathfrak{S}_\mathcal{K}(\mathbf{L})$ defined by

$$\mathfrak{S}_\mathcal{K} := \{\sigma \in \mathfrak{S} \,:\, \mathcal{G}_\sigma = \mathcal{K}\}\,,$$

where $\mathcal{G}_\sigma \subset \mathcal{G}$ is the stabiliser of σ under the action (3.10), and $\mathcal{K} \leq \mathcal{G}$ is a subgroup of $\mathcal{G}$. Similarly to (3.11), for any subgroup $\mathcal{K}$ of $\mathcal{G}$, we define the map

$$\varphi_\mathcal{K} : \mathfrak{S}_\mathcal{K} \to \left({}^{\mathbb{T}_\mathbf{L}}\!/_\mathcal{K} \to \{\uparrow, \downarrow\}\right)\,, \quad \big(\varphi_\mathcal{K}(\sigma)\big)([x]) := \sigma(x)\,, \quad [x] \in {}^{\mathbb{T}_\mathbf{L}}\!/_\mathcal{K}\,,$$

which is easily seen to be an injection and hence, $|\mathfrak{S}_\mathcal{K}| \leq 2^{V/|\mathcal{K}|}$. Therefore, denoting the number of elements in $\mathfrak{S}$ with a trivial stabiliser by $M(\mathbf{L})$, we obtain

$$2^V = \sum_{\mathcal{K}\leq G} |\mathfrak{S}_\mathcal{K}| = M(\mathbf{L}) + \sum_{K\leq G, |\mathcal{K}|\geq 2} |\mathfrak{S}_\mathcal{K}| \leq M(\mathbf{L}) + s(\mathcal{G})2^{V/2}\,,$$

where $s(\mathcal{G})$ denotes the number of subgroups of $\mathcal{G}$. Combining this with the following well-known bound[1]

$$s(\mathcal{G}) \leq |\mathcal{G}|^{\log_2|\mathcal{G}|} \tag{A3}$$

and the trivial estimate $M(\mathbf{L}) \leq 2^V$, we conclude that

$$M(\mathbf{L}) = 2^V + \mathcal{O}\left(2^{V/2+(\log_2 V)^2}\right). \tag{A4}$$

The construction of linearly independent vectors with a fixed momentum $\mathbf{k} \in \mathbb{T}_\mathbf{L}$ for each disjoint orbit with a trivial stabiliser is analogous to (3.14). Using estimates analogous to (3.15) and (3.16) together with (A4) concludes the proof of Lemma A.4. □

Finally, we discuss the generalisation of Step 4, i.e., Lemma 3.7.

Lemma A.5 (Step 4: Traces within momentum sectors) *Let $A = A_\mathbf{q} \otimes I_{\mathbb{T}_\mathbf{L}\setminus\mathcal{R}_\mathbf{q}}$ be a bounded $\mathbf{q}$-local observable with $q_s \leq L_s/2$ for all $s = 1, \ldots, d$. Then it holds that*

$$\max_\mathbf{k} \left| \frac{\mathrm{tr}_\mathbf{L}\,(\Pi_\mathbf{k} A \Pi_\mathbf{k})}{\mathrm{tr}_\mathbf{L}\,\Pi_\mathbf{k}} - \langle A\rangle \right| \leq \mathcal{O}\left(\frac{V}{2^{V/2}}\right). \tag{A5}$$

Proof of Lemma 3.4 Substituting $\Pi_\mathbf{k} := \frac{1}{V}\sum_{\mathbf{j}\in\mathbb{T}_\mathbf{L}} e^{2\pi i \sum_{s=1}^d \frac{k_s j_s}{L_s}} T^{-\mathbf{j}}$ for $\mathbf{k} \in \mathbb{T}_\mathbf{L}$, we obtain

$$\mathrm{tr}_\mathbf{L}\,(\Pi_\mathbf{k} A \Pi_\mathbf{k}) = \langle A\rangle\, \mathrm{tr}_\mathbf{L}\,\Pi_\mathbf{k} + \frac{1}{V}\sum_{\mathbf{j}\in\mathbb{T}_\mathbf{L}\setminus\{\mathbf{0}\}} e^{2\pi i \sum_{s=1}^d \frac{k_s j_s}{L_s}}\, \mathrm{tr}_\mathbf{L}(T^{-\mathbf{j}}\mathring{A})\,.$$

Then, the task is to evaluate the size of the quantity $\mathrm{tr}_\mathbf{L}(T^{-\mathbf{j}}\mathring{A})$.

[1] More precisely, in order to see that (A3) holds, observe that for any subgroup $\mathcal{K}$ of $\mathcal{G}$ and any $g \in \mathcal{G}\setminus\mathcal{K}$, the size of the subgroup generated by $\mathcal{K}$ and g is at least $2\,|\mathcal{K}|$. Therefore, any subgroup $\mathcal{K}$ is generated by at most $\log_2|\mathcal{G}|$ elements, hence the set of all subgroups of $\mathcal{G}$ can be injectively mapped to $\mathcal{G}^{\log_2|\mathcal{G}|}$.

Lemma A.6 *Let* $A := A_{\mathbf{q}} \otimes I_{\mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}}$ *be a* $\mathbf{q}$*-local observable with* $q_s \leq L_s/2$ *for all* $s = 1, \ldots, d$ *and* $\|A\| \lesssim 1$. *Then, for any* $\mathbf{j} \in \mathbb{T}_{\mathbf{L}} \setminus \{\mathbf{0}\}$, *we have that*

$$\left|\mathrm{tr}_{\mathbf{L}}(T^{-\mathbf{j}}A)\right| \lesssim 2^{V/2}. \tag{A6}$$

Combining (A6) with Lemma A.4 gives the bound (A5). □

It remains to prove Lemma A.6.

Proof of Lemma A.6 The $d = 1$ case is proven in Lemma 3.8. Thus, we assume $d \geq 2$ in the following. We choose an orthonormal basis of the Hilbert space on $\mathbb{T}_{\mathbf{L}}$ as $\{|s\rangle \mid s : \mathbb{T}_{\mathbf{L}} \to \{\uparrow, \downarrow\}\}$ to calculate the trace. Then, similarly to (3.19), we obtain

$$\left|\mathrm{tr}_{\mathbf{L}}(T^{-\mathbf{j}}A)\right| \lesssim \sum_{s\,:\,\mathbb{T}_{\mathbf{L}} \to \{\uparrow,\downarrow\}} \prod_{\mathbf{x} \in \mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}} \delta_{s(\mathbf{x}),s(\mathbf{x}+\mathbf{j})}. \tag{A7}$$

Next, analogously to (3.20), we count the number of independent summations on the right-hand side of (A7). To do so, we consider a graph $\mathcal{G}_{\mathbf{L},\mathbf{q},\mathbf{j}} = (V, E)$, whose vertices and edges are given by $V := \mathbb{T}_{\mathbf{L}}$ and $E := \left\{(\mathbf{x}, \mathbf{x}+\mathbf{j}) : \mathbf{x} \in \mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}\right\}$, respectively. Exactly one redundant delta function appears in the product $\prod_{\mathbf{x} \in \mathbb{T}_{\mathbf{L}} \setminus \mathcal{R}_{\mathbf{q}}} \delta_{s(\mathbf{x}),s(\mathbf{x}+\mathbf{j})}$ for every occurrence of a loop in $\mathcal{G}_{\mathbf{L},\mathbf{q},\mathbf{j}}$. Thus, by denoting the number of loops in $\mathcal{G}_{\mathbf{L},\mathbf{q},\mathbf{j}}$ by $N(\mathbf{L}, \mathbf{q}, \mathbf{j})$, we obtain

$$\left|\mathrm{tr}_{\mathbf{L}}(T^{-\mathbf{j}}\mathring{A})\right| \lesssim 2^{|\mathcal{R}_{\mathbf{q}}|+N(\mathbf{L},\mathbf{q},\mathbf{j})}. \tag{A8}$$

As in the one-dimensional case, Lemma 3.8, the graph $\mathcal{G}_{\mathbf{L},\mathbf{q},\mathbf{j}}$ is obtained from $\mathcal{G}_{\mathbf{L},\mathbf{0},\mathbf{j}}$ by removing the edge $(\mathbf{x}, \mathbf{x}+\mathbf{j})$ for all $\mathbf{x} \in \mathcal{R}_{\mathbf{q}}$. Therefore, we have $N(\mathbf{L}, \mathbf{q}, \mathbf{j}) \leq N(\mathbf{L}, \mathbf{0}, \mathbf{j})$ for all $\mathbf{q}$. Now, the number of loops in $\mathcal{G}_{\mathbf{L},\mathbf{0},\mathbf{j}}$ can be counted by considering the orbits of the cyclic group $\langle T^{\mathbf{j}} \rangle$ on $\mathbb{T}_{\mathbf{L}}$. It is clear that the size of each orbit is equal to one another. Denoting it by $g(\mathbf{j})$, the number of loops in $\mathcal{G}_{\mathbf{L},\mathbf{0},\mathbf{j}}$ is given by

$$N(\mathbf{L}, \mathbf{0}, \mathbf{j}) = \frac{V}{g(\mathbf{j})}.$$

Note that, since $\mathbf{j} \neq \mathbf{0}$ by assumption, we have $g(\mathbf{j}) \geq 2$.

If $g(\mathbf{j}) \geq 4$, the bound (A6) is already proven because

$$\left|\mathrm{tr}_{\mathbf{L}}(T^{-\mathbf{j}}\mathring{A})\right| \lesssim 2^{|\mathcal{R}_{\mathbf{q}}|+N(\mathbf{L},\mathbf{q},\mathbf{j})} \leq 2^{\frac{V}{4}+\frac{V}{4}} = 2^{\frac{V}{2}},$$

where we used $\left|\mathcal{R}_{\mathbf{q}}\right| \leq V/2^d$ and $d \geq 2$ by assumption.

If $g(\mathbf{j}) = 2$ or $g(\mathbf{j}) = 3$, we must have $g(\mathbf{j})\, j_s \equiv 0 \pmod{L_s}$ for all s. Using again that $\mathbf{j} \neq \mathbf{0}$, there exists a non-zero component j_t. For such a coordinate direction $t \in \{1, ..., d\}$, we must have $g(\mathbf{j}) \mid L_t$ because $g(\mathbf{j}) \in \{2, 3\}$ is prime. We hence have a decomposition

$$\mathbb{T}_{\mathbf{L}} = \mathcal{A}_t \sqcup T^{\mathbf{j}}\mathcal{A}_t \sqcup T^{2\mathbf{j}}\mathcal{A}_t, \quad \mathcal{A}_t := \left\{\mathbf{x} \in \mathbb{T}_{\mathbf{L}} : 1 \leq x_t \leq \frac{L_t}{g(\mathbf{j})}\right\}.$$

Every loop in $\mathcal{G}_{\mathbf{L},\mathbf{0},\mathbf{j}}$ can be considered to start from a site in $\mathcal{A}_t$. Therefore, removing the edge $(\mathbf{x}, \mathbf{x}+\mathbf{j})$ for all $\mathbf{x} \in \mathcal{R}_{\mathbf{q}}$ from $\mathcal{G}_{\mathbf{L},\mathbf{0},\mathbf{j}}$ decreases the number of loops at least by $\left|\mathcal{R}_{\mathbf{q}} \cap \mathcal{A}_t\right|$, which implies

$$N(\mathbf{L}, \mathbf{q}, \mathbf{j}) \leq N(\mathbf{L}, \mathbf{0}, \mathbf{j}) - \left|\mathcal{R}_{\mathbf{q}} \cap \mathcal{A}_t\right|.$$

Thus, from (A8), we obtain

$$\left|\mathrm{tr}_{\mathbf{L}}(T^{-\mathbf{j}}\mathring{A})\right| \lesssim 2^{|\mathcal{R}_{\mathbf{q}}|-|\mathcal{R}_{\mathbf{q}}\cap\mathcal{A}_t|+N(\mathbf{L},\mathbf{0},\mathbf{j})} = 2^{|\mathcal{R}_{\mathbf{q}}\setminus\mathcal{A}_t|+N(\mathbf{L},\mathbf{0},\mathbf{j})}.$$

Finally, we have

$$\left|\mathcal{R}_{\mathbf{q}} \setminus \mathcal{A}_t\right| + N(\mathbf{L}, \mathbf{0}, \mathbf{j}) \leq \left(\frac{L_t}{2} - \frac{L_t}{g(\mathbf{j})}\right) \prod_{s(\neq t)} \frac{L_s}{2} + \frac{V}{g(\mathbf{j})} = \frac{V}{2^d}\left(1 - \frac{2}{g(\mathbf{j})}\right) + \frac{V}{g(\mathbf{j})} \leq \frac{V}{2},$$

which completes the proof of Lemma A.6. □

References

1. Trotzky, S., Chen, A. Yu.-Ao., Flesch, A., McCulloch, I.P., Schollwöck, U., Eisert, J., Bloch, I.: Probing the relaxation towards equilibrium in an isolated strongly correlated one-dimensional Bose gas. Nat. Phys. **4**(8), 325–330 (2012). https://doi.org/10.1038/nphys2232
2. Langen, T., Geiger, R., Kuhnert, M., Rauer, B., Schmiedmayer, J.: Local emergence of thermal correlations in an isolated quantum many-body system. Nat. Phys. **9**(10), 640–643 (2013). https://doi.org/10.1038/nphys2739
3. Clos, G., Porras, D., Warring, U., Schaetz, T.: Time-resolved observation of thermalization in an isolated quantum system. Phys. Rev. Lett. **117**(17), 170401 (2016). https://doi.org/10.1103/PhysRevLett.117.170401
4. Kaufman, A.M., Tai, M.E., Lukin, A., Rispoli, M., Schittko, R., Preiss, P.M., Greiner, M.: Quantum thermalization through entanglement in an isolated many-body system. Science **353**(6301), 794–800 (2016). https://doi.org/10.1126/science.aaf6725
5. Neill, C., Roushan, P., Fang, M., Chen, Y., Kolodrubetz, M., Chen, Z., Megrant, A., Barends, R., Campbell, B., Chiaro, B., Dunsworth, A., Jeffrey, E., Kelly, J., Mutus, J., O'Malley, P.J.J., Quintana, C., Sank, D., Vainsencher, A., Wenner, J., White, T.C., Polkovnikov, A., Martinis, J.M.: Ergodic dynamics and thermalization in an isolated quantum system. Nat. Phys. **12**(11), 1037–1041 (2016). https://doi.org/10.1038/nphys3830
6. Tang, Y., Kao, W., Li, K.-Y., Seo, S., Mallayya, K., Rigol, M., Gopalakrishnan, S., Lev, B.L.: Thermalization near integrability in a dipolar quantum Newton's cradle. Phys. Rev. X **8**(2), 021030 (2018). https://doi.org/10.1103/PhysRevX.8.021030
7. Neumann, J.: Proof of the ergodic theorem and the H-theorem in quantum mechanics. Eur. Phys. J. H **35**(2), 201–237 (2010). https://doi.org/10.1140/epjh/e2010-00008-5
8. Deutsch, J.M.: Quantum statistical mechanics in a closed system. Phys. Rev. A **43**(4), 2046 (1991). https://doi.org/10.1103/PhysRevA.43.2046
9. Srednicki, M.: Chaos and quantum thermalization. Phys. Rev. E **50**(2), 888 (1994). https://doi.org/10.1103/PhysRevE.50.888
10. D'Alessio, L., Kafri, Y., Polkovnikov, A., Rigol, M.: From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. Adv. Phys. **65**(3), 239–362 (2016). https://doi.org/10.1080/00018732.2016.1198134
11. Mori, T., Ikeda, T.N., Kaminishi, E., Ueda, M.: Thermalization and prethermalization in isolated quantum systems: a theoretical overview. J. Phys. B **51**(11), 112001 (2018). https://doi.org/10.1088/1361-6455/aabcdf
12. Deutsch, J.M.: Eigenstate thermalization hypothesis. Rep. Prog. Phys. **81**(8), 082001 (2018). https://doi.org/10.1088/1361-6633/aac9f1
13. Rigol, M., Dunjko, V., Olshanii, M.: Thermalization and its mechanism for generic isolated quantum systems. Nature **452**(7189), 854–858 (2008). https://doi.org/10.1038/nature06838
14. Rigol, M.: Breakdown of thermalization in finite one-dimensional systems. Phys. Rev. Lett. **103**(10), 100403 (2009). https://doi.org/10.1103/PhysRevLett.103.100403
15. Rigol, M.: Quantum quenches and thermalization in one-dimensional fermionic systems. Phys. Rev. A **80**(5), 053607 (2009). https://doi.org/10.1103/PhysRevA.80.053607
16. Biroli, G., Kollath, C., Läuchli, A.M.: Effect of rare fluctuations on the thermalization of isolated quantum systems. Phys. Rev. Lett. **105**(25), 250401 (2010). https://doi.org/10.1103/PhysRevLett.105.250401
17. Santos, L.F., Rigol, M.: Localization and the effects of symmetries in the thermalization properties of one-dimensional quantum systems. Phys. Rev. E **82**(3), 031130 (2010). https://doi.org/10.1103/PhysRevE.82.031130
18. Steinigeweg, R., Herbrych, J., Prelovšek, P.: Eigenstate thermalization within isolated spin-chain systems. Phys. Rev. E **87**(1), 012118 (2013). https://doi.org/10.1103/PhysRevE.87.012118
19. Beugeling, W., Moessner, R., Haque, M.: Finite-size scaling of eigenstate thermalization. Phys. Rev. E **89**(4), 042112 (2014). https://doi.org/10.1103/PhysRevE.89.042112

20. Kim, H., Ikeda, T.N., Huse, D.A.: Testing whether all eigenstates obey the eigenstate thermalization hypothesis. Phys. Rev. E **90**(5), 052105 (2014). https://doi.org/10.1103/PhysRevE.90.052105
21. Mondaini, R., Fratus, K.R., Srednicki, M., Rigol, M.: Eigenstate thermalization in the two-dimensional transverse field Ising model. Phys. Rev. E **93**(3), 032104 (2016). https://doi.org/10.1103/PhysRevE.93.032104
22. Garrison, J.R., Grover, T.: Does a single eigenstate encode the full Hamiltonian? Phys. Rev. X **8**(2), 021026 (2018). https://doi.org/10.1103/PhysRevX.8.021026
23. Dymarsky, A., Lashkari, N., Liu, H.: Subsystem eigenstate thermalization hypothesis. Phys. Rev. E **97**(1), 12140 (2018). https://doi.org/10.1103/PhysRevE.97.012140
24. Rigol, M., Dunjko, V., Yurovsky, V., Olshanii, M.: Relaxation in a completely integrable many-body quantum system: an ab initio study of the dynamics of the highly excited states of 1D lattice hard-core bosons. Phys. Rev. Lett. **98**(5), 050405 (2007). https://doi.org/10.1103/PhysRevLett.98.050405
25. Cassidy, A.C., Clark, C.W., Rigol, M.: Generalized thermalization in an integrable lattice system. Phys. Rev. Lett. **106**(14), 140405 (2011). https://doi.org/10.1103/PhysRevLett.106.140405
26. Ikeda, T.N., Watanabe, Y., Ueda, M.: Finite-size scaling analysis of the eigenstate thermalization hypothesis in a one-dimensional interacting Bose gas. Phys. Rev. E **87**(1), 012125 (2013). https://doi.org/10.1103/PhysRevE.87.012125
27. Alba, V.: Eigenstate thermalization hypothesis and integrability in quantum spin chains. Phys. Rev. B **91**(15), 155123 (2015). https://doi.org/10.1103/PhysRevB.91.155123
28. Hamazaki, R., Ikeda, T.N., Ueda, M.: Generalized Gibbs ensemble in a nonintegrable system with an extensive number of local symmetries. Phys. Rev. E **93**(3), 032116 (2016). https://doi.org/10.1103/PhysRevE.93.032116
29. Dymarsky, A., Pavlenko, K.: Generalized Eigenstate thermalization hypothesis in 2D conformal field theories. Phys. Rev. Lett. **123**(11), 111602 (2019). https://doi.org/10.1103/PhysRevLett.123.111602
30. Basko, D.M., Aleiner, I.L., Altshuler, B.L.: Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states. Ann. Phys. **321**(5), 1126–1205 (2006). https://doi.org/10.1016/j.aop.2005.11.014
31. Pal, A., Huse, D.A.: Many-body localization phase transition. Phys. Rev. B **82**(17), 174411 (2010). https://doi.org/10.1103/PhysRevB.82.174411
32. Imbrie, J.Z.: On Many-body localization for quantum spin chains. J. Stat. Phys. **163**(5), 998–1048 (2016). https://doi.org/10.1007/s10955-016-1508-x
33. Nandkishore, R., Huse, D.A.: Many-body localization and thermalization in quantum statistical mechanics. Annu. Rev. Condens. Matter Phys. **6**(1), 15–38 (2015). https://doi.org/10.1146/annurev-conmatphys-031214-014726
34. Shiraishi, N., Mori, T.: Systematic construction of counterexamples to the Eigenstate thermalization hypothesis. Phys. Rev. Lett. **119**(3), 030601 (2017). https://doi.org/10.1103/PhysRevLett.119.030601
35. Turner, C.J., Michailidis, A.A., Abanin, D.A., Serbyn, M., Papić, Z.: Weak ergodicity breaking from quantum many-body scars. Nat. Phys. **14**(7), 745–749 (2018). https://doi.org/10.1038/s41567-018-0137-5
36. Bull, K., Martin, I., Papić, Z.: Systematic construction of scarred many-body dynamics in 1D lattice models. Phys. Rev. Lett. **123**(3), 030601 (2019). https://doi.org/10.1103/PhysRevLett.123.030601
37. Goldstein, S., Lebowitz, J.L., Mastrodonato, C., Tumulka, R., Zanghi, N.: Approach to thermal equilibrium of macroscopic quantum systems. Phys. Rev. E **81**(1), 011109 (2010). https://doi.org/10.1103/PhysRevE.81.011109
38. Reimann, P.: Generalization of von Neumann's approach to thermalization. Phys. Rev. Lett. **115**(1), 010403 (2015). https://doi.org/10.1103/PhysRevLett.115.010403
39. Cipolloni, G., Erdős, L., Schröder, D.: Eigenstate thermalization hypothesis for Wigner matrices. Commun. Math. Phys. **2**, 1005–1048 (2021). https://doi.org/10.1007/s00220-021-04239-z
40. Cipolloni, G., Erdős, L., Henheik, J., Kolupaiev, O.: Gaussian fluctuations in the equipartition principle for Wigner matrices (2023) arXiv:2301.05181
41. Adhikari, A., Dubova, S., Xu, C., Yin, J.: Eigenstate thermalization hypothesis for generalized Wigner matrices (2023) arXiv:2302.00157
42. Sugimoto, S., Hamazaki, R., Ueda, M.: Test of the eigenstate thermalization hypothesis based on local random matrix theory. Phys. Rev. Lett. **126**(12), 120602 (2021). https://doi.org/10.1103/PhysRevLett.126.120602
43. Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: The local semicircle law for a general class of random matrices. Electron. J. Probab. **18**(59), 1 (2013). https://doi.org/10.1214/EJP.v18-2473
44. Collins, B., Matsumoto, S., Novak, J.: The Weingarten calculus. Not. Am. Math. Soc. **69**(05), 1 (2022). https://doi.org/10.1090/noti2474

45. Sugimoto, S., Hamazaki, R., Ueda, M.: Eigenstate thermalization in long-range interacting systems. Phys. Rev. Lett. **129**(3), 030602 (2022). https://doi.org/10.1103/PhysRevLett.129.030602
46. Hamazaki, R., Ueda, M.: Atypicality of most few-body observables. Phys. Rev. Lett. **120**(8), 080603 (2018). https://doi.org/10.1103/PhysRevLett.120.080603