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COBOUNDARY EXPANSION, EQUIVARIANT OVERLAP, AND CROSSING NUMBERS OF SIMPLICIAL COMPLEXES

BY

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To Nati Linial on his seventieth birthday

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ABSTRACT

We prove the following quantitative Borsuk–Ulam-type result (an equivariant analogue of Gromov's Topological Overlap Theorem): Let X be a free $\mathbb{Z}/2$-complex of dimension d with coboundary expansion at least η_k in dimension $0 \leq k < d$. Then for every equivariant map $F\colon X \to_{\mathbb{Z}/2} \mathbb{R}^d$, the fraction of d-simplices σ of X with $0 \in F(\sigma)$ is at least $2^{-d}\prod_{k=0}^{d-1}\eta_k$.

As an application, we show that for every sufficiently thick d-dimensional spherical building Y and every map $f\colon Y \to \mathbb{R}^{2d}$, we have $f(\sigma) \cap f(\tau) \neq \emptyset$ for a constant fraction $\mu_d > 0$ of pairs $\{\sigma, \tau\}$ of d-simplices of Y. In particular, such complexes are non-embeddable into $\mathbb{R}^{2d}$, which proves a conjecture of Tancer and Vorwerk for sufficiently thick spherical buildings.

We complement these results by upper bounds on the coboundary expansion of two families of simplicial complexes; this indicates some limitations to the bounds one can obtain by straighforward applications of the quantitative Borsuk–Ulam theorem. Specifically, we prove

- an upper bound of $(d+1)/2^d$ on the normalized $(d-1)$-th coboundary expansion constant of complete $(d+1)$-partite d-dimensional complexes (under a mild divisibility assumption on the sizes of the parts); and
- an upper bound of $(d+1)/2^d + \varepsilon$ on the normalized $(d-1)$-th coboundary expansion of the d-dimensional spherical building associated with $\mathrm{GL}_{d+2}(\mathbb{F}_q)$ for any $\varepsilon > 0$ and sufficiently large q. This disproves, in a rather strong sense, a conjecture of Lubotzky, Meshulam and Mozes.

1. Introduction

Crossing numbers of graphs (which provide a quantitative measure of non-planarity) are a fundamental and extensively studied notion in graph theory and discrete and computational geometry (see, e.g., [38, Chs. 4–5] and [41]). The goal of this paper is to present a general approach to study crossing numbers—of graphs as well as of higher-dimensional simplicial complexes—through the lens of high-dimensional expansion properties of natural configuration spaces associated with the crossing number problem, deleted joins.

Let X be a finite d-dimensional simplicial complex with underlying polyhedron (also called **geometric realization**) $|X|$, and let $X(k)$ denote the set of k-dimensional simplices of X, $-1 \leq k \leq d$.[1] Given a continuous

[1] All simplicial complexes in this paper are assumed to be finite.

map $f\colon |X| \to \mathbb{R}^{2d}$ we define its **independent pair crossing number** as

$$\mathrm{ipcr}(f) := \frac{1}{2}|\{(\sigma,\tau) \in X(d) \times X(d)\colon \sigma \cap \tau = \emptyset, f(\sigma) \cap f(\tau) \neq \emptyset\}|,$$

and the **independent pair crossing number of X** as

$$\mathrm{ipcr}(X) := \min\{\mathrm{ipcr}(f) : f\colon |X| \to \mathbb{R}^{2d} \text{ continuous}\}.$$

Every d-dimensional complex X can be embedded into $\mathbb{R}^{2d+1}$ (by mapping the vertices to points in general position and extending linearly on each simplex, see, e.g., [33, Theorem 1.6.1]), but there are d-dimensional complexes that do not embed into $\mathbb{R}^{2d}$ (e.g., the complete d-dimensional complex on $2d+3$ vertices, by the van Kampen–Flores theorem [33, Theorem 5.1.1]); thus, embeddability of d-complexes into $\mathbb{R}^{2d}$ is the first non-trivial instance of the embeddability problem.

Clearly, $\mathrm{ipcr}(X) > 0$ implies that X is not embeddable into $\mathbb{R}^{2d}$. (Moreover, the converse holds for the case $d = 1$ of graphs, and for $d \geq 3$, by completeness of the classical van Kampen obstruction, see, e.g., [13].) We are interested in proving lower bounds for $\mathrm{ipcr}(X)$, as a quantitative measure of non-embeddability.

A classical approach to study embeddability problems is via deleted joins and induced equivariant maps (this is a special instance of the more general configuration space/test map method, see [33, 46] for a detailed introduction and further background). For finite simplicial complexes X and Y, let $X * Y$ denote their **join** (the simplicial complex whose simplices are joins $\sigma * \tau$ of pairs of simplices σ of X and τ of Y, where $\dim(\sigma * \tau) = \dim(\sigma) + \dim(\tau) + 1$).[2]

The **deleted join** X^{*2}_Δ of a simplicial complex X is the subcomplex of $X^{*2} := X * X$ given by

$$X^{*2}_\Delta := \{\sigma * \tau : \sigma, \tau \in X, \sigma \cap \tau = \emptyset\}.$$

Points $x \in |X^{*2}_\Delta|$ can be written as formal convex combinations

$$x = tx_1 \oplus (1-t)x_2,$$

where $t \in [0,1]$ and $x_1, x_2 \in |X|$ lie in disjoint simplices of X.[3]

[2] If we view simplicial complexes abstractly, as finite set systems closed under inclusion, the simplices of the join are disjoint unions $\sigma \uplus \tau$ of pairs of simplices of X and Y, respectively (this is the notation used in [33]).

[3] Here, by convention, $0x_1 \oplus 1x_2 = 0x_1' \oplus 1x_2$ and $1x_1 \oplus 0x_2 = 1x_1 \oplus 0x_2'$ for all $x_1, x_1', x_2, x_2' \in |X|$.

The group $\mathbb{Z}/2$ acts on the deleted join X^{*2}_Δ by interchanging components,

$$\begin{aligned}\nu\colon |X^{*2}_\Delta| &\to |X^{*2}_\Delta|\\ x = tx_1 \oplus (1-t)x_2 &\mapsto \nu(x) = (1-t)x_2 \oplus tx_1.\end{aligned}$$

This action is **free** (has no fixed points) and **simplicial** (i.e., the involution ν is a simplicial map), turning the deleted join into a **free $\mathbb{Z}/2$-complex**.

Given a continuous map $f\colon |X| \to \mathbb{R}^{2d}$ we obtain a map $F\colon |X^{*2}_\Delta| \to \mathbb{R}^{2d+1}$ given by

$$tx_1 \oplus (1-t)x_2 \mapsto \begin{pmatrix} 1-2t \\ tf(x_1) - (1-t)f(x_2)\end{pmatrix}.$$

This map is **equivariant**, i.e., $F(\nu(x)) = -F(x)$ for all $x \in |X^{*2}_\Delta|$. Moreover, $F(tx_1 \oplus (1-t)x_2) = 0$ if and only if $t = 1/2$ and $f(x_1) = f(x_2)$. Thus,

$$\mathrm{ipcr}(f) = \frac{1}{2}|\{\sigma * \tau \in X^{*2}_\Delta\colon\ \dim(\sigma*\tau) = 2d+1,\ 0 \in F(\sigma*\tau)\}|.$$

A well-known generalization of the classical Borsuk–Ulam Theorem (see, e.g., [45]) asserts that if X is a nonempty free $\mathbb{Z}/2$-complex with vanishing reduced cohomology groups $\tilde{H}^k(X;\mathbb{F}_2) = 0$ for $0 \leq k < d$, then every equivariant map $F\colon |X| \to \mathbb{R}^d$ has a zero.[4] Our first result is a quantitative version of this theorem, formulated in terms of the coboundary expansion properties of X. The notion of coboundary expansion, which arose independently in the work of Linial, Meshulam and Wallach [27, 34] and of Gromov [16], is a generalization of edge expansion of graphs and provides a quantitative measure of vanishing cohomology.

To define this notion, let X be a **pure** d-dimensional simplicial complex (i.e., all inclusion-maximal simplices of X have dimension d). Endow X with the weight function $w\colon X \to \mathbb{R}_{\geq 0}$ on its simplices given by

$$\sigma \mapsto w(\sigma) = \frac{|\{\tau \in X(d) : \sigma \subseteq \tau\}|}{\binom{d+1}{|\sigma|}|X(d)|}.$$

These weights, often called **Garland weights**, induce a norm $\|\cdot\|$ on the simplicial cochain groups $C^k(X;\mathbb{F}_2)$ (see Definition 7 below) by

$$\|c\| = \sum_{\sigma\in X(k), c(\sigma)\neq 0} w(\sigma).$$

[4] The Borsuk–Ulam Theorem is the case where X is (an antipodally symmetric triangulation of) the sphere $\mathbb{S}^d$, with the antipodal action.

We get an induced quotient norm $\|[\cdot]\|$ on $C^k(X;\mathbb{F}_2)/B^k(X;\mathbb{F}_2)$ given by

$$\|[c]\| = \min\{\|c+b\| : b \in B^k(X;\mathbb{F}_2)\}.$$

Definition 1 (Coboundary expansion): The **k-th coboundary expansion constant** $\eta_k(X)$ of X (with respect to $\|\cdot\|$-norm and $\mathbb{F}_2$-coefficients) is defined as

$$\eta_k(X) := \min_{c \in C^k(X;\mathbb{F}_2)\setminus B^k(X;\mathbb{F}_2)} \frac{\|\delta c\|}{\|[c]\|}.$$

Note that $\eta_k(X) > 0$ if and only if $\tilde{H}^k(X;\mathbb{F}_2) = 0$. We are now ready to state our first result.

Theorem 2 (Quantitative Borsuk–Ulam theorem): *Let $d \in \mathbb{N}$. Let X be a d-dimensional free $\mathbb{Z}/2$-complex. Then for any equivariant map $F\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ we have*

$$|\{\sigma \in X(d) : 0 \in F(\sigma)\}| \geq \frac{\prod_{i=0}^{d-1} \eta_i(X)}{2^d} |X(d)|.$$

In particular, whenever we can prove good lower bounds for the expansion constants of the deleted join X^{*2}_Δ of a d-dimensional complex X then we get a lower bound on the independent pair crossing number $\operatorname{ipcr}(X)$ of the complex, and hence in particular a proof that X is not embeddable into $\mathbb{R}^{2d}$. Therefore, it is natural to ask for conditions on X that guarantee lower bounds on the expansion constants of X^{*2}_Δ. In particular, can we bound the coboundary expansion constants of the deleted join X^{*2}_Δ in terms of the coboundary expansion constants of X? We are still quite far from a fully satisfying answer to this question in general. One particular family of simplicial complexes for which we know how to show expansion for the deleted join are spherical buildings (whose definition we recall in Section 5 below), provided they are sufficiently thick; here, by definition, a d-dimensional simplicial complex X is **δ-thick** if every $(d-1)$-simplex of X is contained in at least δ d-simplices of X.

Theorem 3 (Quantitative non-embeddability for sufficiently thick spherical buildings): *For every d there exist $\delta_d > 0$ and $\mu_d > 0$ such that for every d-dimensional δ_d-thick spherical building X*

$$\operatorname{ipcr}(X) \geq \mu_d \cdot \binom{|X(d)|}{2}.$$

Vorwerk and Tancer [43, Conjecture 8.1] conjectured that no d-dimensional 3-thick spherical building embeds into $\mathbb{R}^{2d}$. Theorem 3 shows that this is true, in a strong quantitative sense, under the stronger assumption of sufficiently large thickness. It is worth mentioning here that spherical buildings of arbitrary large thickness do indeed exist (e.g., the spherical buildings $A_d(\mathbb{F}_q)$ described below are $(1+q)$-thick, where q can be any prime power).

One initial motivation for the present work was to use Theorem 2 to attack various old problems regarding crossing numbers of graphs. Arguably the oldest and most prominent of these is Turán's Brick Factory Problem (see, e.g., [38, Ch. 5] or [41, Ch. 1]), which asks for the crossing number $\operatorname{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$. A classical construction due to Zarankiewicz shows that

$$\operatorname{cr}(K_{m,n}) \leq \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \sim \frac{m^2 n^2}{16} \tag{1}$$

and it is a long-standing conjecture (known as Zarankiewicz's conjecture) that equality holds in (1), but even the asymptotics of $\operatorname{cr}(K_{m,n})$ as $m, n \to \infty$ remains elusive.

The deleted join of the complete bipartite graph is

$$(K_{m,n})^{*2}_{\Delta} = ([m]^{*2}_{\Delta}) * ([n]^{*2}_{\Delta}).$$

For large m and n, this complex is roughly the complete 4-partite complex $\Lambda^3_{m,m,n,n} = [m]^{*2} * [n]^{*2}$, in the sense that the two complexes differ only in an asymptotically negligible number of faces, and it is easy to show that their expansion constants are asymptotically the same (see Proposition 21 below for a general result of this flavor). Thus, Theorem 2 would imply the asymptotic version of Zarankiewicz's conjecture

$$\operatorname{cr}(K_{m,n}) \geq \operatorname{ipcr}(K_{m,n}) \geq \Big(\frac{1}{16} + o(1)\Big) m^2 n^2,$$

if we could show that $\eta_k(\Lambda^3_{m,m,n,n}) \geq 1$ for $0 \leq k \leq 2$.

Unfortunately, this turns out to be false (at least for $k = 2$). To state this result, let us fix some notation. For integers $d \geq 1$ and $n_0, n_1, \ldots, n_d \geq 2$, let $\Lambda^d_{n_0,n_1,\ldots,n_d} = [n_0] * \cdots * [n_d]$ denote the **complete $(d+1)$-partite complex** with parts of size $n_0, n_1, \ldots, n_{d-1}, n_d$. More concretely, $\Lambda^d_{n_0,\ldots,n_d}$ has vertex set $V = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_d$, where the V_i are pairwise disjoint sets of cardinality $|V_i| = n_i$, and $\sigma \subseteq V$ forms a simplex in $\Lambda^d_{n_0,\ldots,n_d}$ iff $|\sigma \cap V_i| \leq 1$ for $0 \leq i \leq d$. (We remark that $\Lambda^d_{n_0,\ldots,n_d}$ is the simplicial complex of independent sets of the so-called **partition matroid** associated with $[n_0], \ldots, [n_d]$.)

If $n_0 = n_1 = \cdots = n_d = n$, we will write Λ_n^d instead of $\Lambda^d_{n_0,\ldots,n_d}$. With this notation, we have

Theorem 4 (Upper bound on $\eta_{d-1}(\Lambda_n^d)$): *If 2^d divides n_i for all $0 \le i \le d$, then*

$$\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d}) \le \frac{d+1}{2^d}.$$

The proof of Theorem 4 is by an explicit construction of a family of d-coboundaries with some extra algebraic structure (closely related to the notion of sum complexes [28]).

By a probabilistic argument, this construction also yields an upper bound on the $(d-1)$-th expansion constant of the spherical building $A_d(\mathbb{F}_q)$ associated with $\mathrm{GL}_{d+2}(\mathbb{F}_q)$ for sufficiently large q. We recall that, given a prime power q, $A_d(\mathbb{F}_q)$ is the simplicial complex whose vertices are the non-trivial, proper subspaces of $\mathbb{F}_q^{d+2}$ and whose k-simplices correspond to chains

$$\{0\} \neq U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_k \subsetneq \mathbb{F}_q^{d+2}$$

of subspaces. In particular, $A_1(\mathbb{F}_q)$ is the points vs. lines graph of the Desarguesian projective plane of order q. It is known that the edge expansion of $A_1(\mathbb{F}_q)$ satisfies

$$\eta_0(A_1(\mathbb{F}_q)) \ge 1 - \frac{2\sqrt{q}}{q+1}$$

(see [29, Section 8.3]). Lubotzky, Meshulam and Mozes conjectured in [31, Conjecture 5.1] that, more generally, $\eta_{d-1}(A_d(\mathbb{F}_q)) = 1 + o(1)$ as $q \to +\infty$ for any $d \ge 2$. Our final result disproves this conjecture for any $d \ge 2$.

Theorem 5 (Upper bound $\eta_{d-1}(A_d(\mathbb{F}_q))$): *For any dimension d and $\varepsilon > 0$ there is a positive integer $Q = Q(d, \varepsilon)$ such that for all prime powers $q \ge Q$ we have*

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \le \frac{d+1}{2^d} + \varepsilon.$$

Previously, it was known that (see [31, Claim 3.4]) $\eta_{d-1}(\Lambda_n^d) \le 1$ whenever n is divisible by $d+1$. (For $\eta_{d-1}(A_d(\mathbb{F}_q))$ we are not aware of any upper bound explicitly stated in the literature, but Theorem 5.3 in [26] is applicable to $A_d(\mathbb{F}_q)$ and yields that $\eta_{d-1}(A_d(\mathbb{F}_q)) \le 1 + o(1)$ as $q \to +\infty$.)

As for lower bounds, it is known (see [10, Proposition 5.7] and [31, Theorem 3.3]) that for any dimension $d \geq 1$ and $n \geq 2$ we have[5]

$$\eta_{d-1}(\Lambda_n^d) \geq \frac{d+1}{3 \cdot 2^{d-1} - 1}.$$

For $A_d(\mathbb{F}_q)$ it is known ([26, Corollary 3.9], building up on the work in [16] and in [31]) that

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \geq \frac{1}{\sum_{j=1}^{d+1} j!}.$$

Thus, our new upper bounds in Theorem 4 and Theorem 5 make significant progress in closing the gap between the known upper and lower bounds on $\eta_{d-1}(A_d(\mathbb{F}_q))$ and $\eta_{d-1}(\Lambda_n^d)$.

Remarks 6: (i) A remarkable result due to Gromov [16], known as Gromov's Topological Overlap Theorem, asserts, informally speaking, that for every $d \in \mathbb{N}$ and every vector $\eta = (\eta_0, \ldots, \eta_{d-1})$ of positive real numbers, there exists a constant $c_d = c_d(\eta)$ depending only on d and η such that every sufficiently large d-dimensional simplicial complex X with coboundary expansion $\eta_k(X) \geq \eta_k$, $0 \leq k < d$, satisfies the following overlap property: For every continuous map $F\colon |X| \to \mathbb{R}^d$, there exists a point $p \in \mathbb{R}^d$ such that

$$|\{\sigma \in X(d)\colon p \in F(\sigma)\}| \geq c_d \cdot |X(d)|$$

(see, e.g., [11] for the precise statement of the result and a streamlined proof). Theorem 2 and its proof are inspired by this result and can be seen as an analogue of Gromov's result in the setting of free $\mathbb{Z}/2$-spaces and equivariant maps.

(ii) Our proof of Theorem 2 is quite robust (and arguably simpler than the proof of the non-equivariant overlap theorem, partly because the origin 0 is a canonical candidate for the heavily covered point in the equivariant setting). In particular, one can use other norms (Definition 7 below) than the Garland norm to measure the size of cochains (see Theorem 14 below for a more general statement).

[5] Strictly speaking the bound stated here is slightly stronger than the bound given in [10] and [31]. But for instance in [10] a recursion (page 511 under item (5)) is not solved exactly. Taking a bit more care, one can easily obtain the lower bound claimed here.

(iii) Moreover, Theorem 2 can be generalized to the setting where a more general group G acts freely and simplicially on X and by linear transformations on $\mathbb{R}^d$, e.g., to the case $G=\mathbb{Z}/p$, p a prime, which plays an important role in the study of Tverberg-type problems (see [33, 46] for more background); we plan to present these results in a companion paper.

Related work on high-dimensional expansion. The present paper fits into the broader context of high-dimensional expanders (HDXs), an emerging research area that aims to generalize the well-developed theory of expander graphs to higher dimensions.[6] One interesting aspect is that even the definition of higher-dimensional expansion is not at all obvious and, unlike in the case of graphs, there is a rich array of mutually non-equivalent notions of high-dimensional expansion, each of interest in its own right and with its own applications. While still in a formative stage, the theory of high-dimensional expanders has already led to a number of striking results over the past decade, of which we will just highlight a few here (referring to [30] for a more thorough survey, including many aspects that we will neglect).

One striking application of the study of HDX is a fully polynomial-time randomized approximation scheme for sampling and counting bases of matroids due to Anari, Liu, Gharan and Vinzant in [3], building upon earlier work on random walks on simplicial complexes by Dinur, Kaufman, Mass and Oppenheim among others (see, e.g., [9, 23, 36] and [24]). A key role is played by local-to-global arguments that allow one to deduce global expansion properties from local ones, such as expansion of links. This method can be traced back to the work of Garland [14].

Another, very recent, breakthrough coming out of the study of HDXs is the construction of locally testable codes with constant rate, constant distance and constant number of queries due to Dinur, Evra, Livne, Lubotzky and Mozes in [7] and independently by Panteleev and Kalachev in [39]. These works can be seen as part of a whole line of research which links HDXs to computer science, in particular to property testing, error correcting codes and probabilistically checkable proofs (see [22, 9, 6, 2, 8], to name a few results in this direction).

[6] This, in turn, can be seen as part of an even more general program of developing high-dimensional combinatorics (a term coined by Nati Linial), i.e., of generalizing the combinatorial theory of graphs and other 1-dimensional objects, such as permutations, to higher-dimensional objects such as simplicial complexes/hypergraphs, Latin squares and designs, etc.

The aforementioned results mainly focus on notions of high-dimensional expansion formulated in terms of spectral gaps for combinatorial Laplacians and related operators. By contrast, the notion of coboundary expansion has a more combinatorial flavor and is, at the same time, more closely attuned to certain topological applications, including overlap properties of continuous maps as in Gromov's work [16] and as in Theorem 2 or for measuring the geometric complexity of embeddings as in [17], and to the thresholds for the vanishing of the (co)homology of random complexes in the work of Linial, Meshulam, and Wallach [27, 34]. We hope that the present paper might stimulate more research into topological aspects and applications of high-dimensional expanders.

We remark that the connection between coboundary expansion and spectral expansion is quite subtle; it is known that neither one implies the other, [18, 42], but nonetheless, ideas originating in the theory of spectral expansion, in particular partial analogues of local-to-global arguments, play a central role in the proof of existence of simplicial complexes (d-skeleta of $(d+1)$-dimensional Ramanujan complexes) with uniformly bounded degree that have the topological overlap property [12, 21].

Outline of paper. The structure of the remaining parts is as follows: After reviewing some preliminaries and fixing some notation in Section 2, we present the proof of Theorem 2 in Section 3. The lower bound for the pair-crossing number for spherical buildings is proved in Section 5; as a technical tool, we relate expansion properties of the join X^{*2} and the deleted join X^{*2}_{Δ} of a simplicial complex X in Section 4. The upper bounds on $\eta_{d-1}(\Lambda_n^d)$ and $\eta_{d-1}(A_d(\mathbb{F}_q))$ are proven in Section 6 before we close with some final remarks in Section 7.

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2. Preliminaries

We refer the reader to the textbook [33] for a detailed introduction to simplicial complexes and many of the notions used in the present paper. Here, we review some definitions and facts not covered in [33], and fix some notation and conventions that we will use throughout (in particular, some that deviate from [33]).

Throughout this paper, all simplicial complexes will be considered finite and to include the empty face $\emptyset$ as the unique (-1)-dimensional simplex. Given an (abstract) simplicial complex X and a simplex $\sigma \in X$, the **link** X_σ of X at σ is the simplicial complex

$$X_\sigma := \{\tau \setminus \sigma : \tau \in X, \sigma \subseteq \tau\}.$$

Note that $X_\emptyset = X$. A simplicial complex X is s**-partite**, for a positive integer s, if there is a labeling

$$\lambda\colon X(0) \to \{1, \ldots, s\}$$

of the vertices such that for every $\sigma \in X$ and $1 \leq i \leq s$ we have $|\sigma \cap \lambda^{-1}(\{i\})| \leq 1$.

2.1. (Co)cochains, (co)homology, norms, and coboundary expansion. Let X be a d-dimensional finite simplicial complex, and let $-1 \leq k \leq d$. We denote by $C^k(X) = \mathbb{F}_2^{X(k)}$ the vector space of k-dimensional simplicial cochains of X with coefficients in the field $\mathbb{F}_2$ with two elements. Thus, a k-dimensional **simplicial cochain** is a function $c\colon X(k) \to \mathbb{F}_2$. There is a one-to-one correspondence between k-cochains and subsets $S \subseteq X(k)$: we identify each k-cochain c with its support $\mathrm{supp}(c) = \{\sigma \in X(k)\colon c(\sigma) \neq 0\}$; conversely, we associate to every $S \subseteq X(k)$ the k-cochain $\mathbb{1}_S \in C^k(X)$ given by

$$\mathbb{1}_S(\sigma) = \begin{cases} 1 & \text{if } \sigma \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Occasionally, we will abuse notation and simply write S instead of $\mathbb{1}_S$.

For a pair $(\sigma, \tau) \in X(k+1) \times X(k)$ we set $[\sigma : \tau] = 1$ if $\tau \subseteq \sigma$ and 0 otherwise. The **coboundary operator** $\delta_k\colon C^k(X) \to C^{k+1}(X)$ is defined by

$$\delta_k c(\sigma) := \sum_{\tau \in X(k)} [\sigma : \tau] c(\tau)$$

for $c \in C^k(X)$ and $\sigma \in X(k+1)$. Thus, given $S \subseteq X(k)$, the support of $\delta_k \mathbb{1}_S$ are precisely those $(k+1)$-simplices that have an odd number of boundary k-simplices in S. In what follows, we will often omit the subscript k in δ_k and write δ instead of δ_k.

The image $B^k(X) = \mathrm{Im}\delta_{k-1}$ is the **space of k-coboundaries** of X, and the kernel $Z^k(X) = \ker \delta_k$ is called the **space of k-cocycles** of X. Given $\beta \in B^k(X)$ we call $\alpha \in C^{k-1}(X)$ a **cofilling** of β if $\delta\alpha = \beta$. It is easy to check that $\delta_k \circ \delta_{k-1} = 0$, i.e., $B^k(X) \subseteq Z^k(X)$. Thus, we can define the **k-th (reduced) cohomology group** $\tilde{H}^k(X)$ of X (with $\mathbb{F}_2$-coefficients) as

$$\tilde{H}^k(X) := Z^k(X)/B^k(X).$$

Note that given a $\mathbb{Z}/2$-complex X with $\mathbb{Z}/2$-action ν there is an induced $\mathbb{Z}/2$-action on the cochain group $C^k(X)$ given by $(\nu c)(\sigma) = c(\nu\sigma)$ for all $\sigma \in X(k)$, $c \in C^k(X)$. This action commutes with the coboundary operators, i.e., $\nu\delta c = \delta(\nu c)$ for all $c \in C^k(X)$.

Dually to the spaces of cochains, we have the spaces of k-chains $C_k(X) = \mathbb{F}_2^{X(k)}$,[7] which we identify with subsets (or, equivalently, as formal $\mathbb{F}_2$-linear combinations) of k-simplices, and **boundary maps**

$$\partial_k \colon C_k(X) \to C_{k-1}(X)$$

given on basis vectors $\sigma \in X(k)$ by

$$\partial_k(\sigma) = \sum_{\tau \in X(k-1), \tau \subseteq \sigma} \tau.$$

As for the coboundary maps, we will usually drop the subscript and write ∂ instead of ∂_k. The spaces $B_k(X) = \mathrm{Im}\partial_{k+1}$ and $Z_k(X) = \ker \partial_k$ are called the spaces of **k-boundaries** and **k-cycles** of X, respectively. They satisfy $B_k(X) \subseteq Z_k(X)$, and the quotient $\tilde{H}_k(X)$ is the k-th reduced homology of X. Since we work with coefficients in a field (and all complexes considered in this paper are assumed to be finite), we have $\tilde{H}_k(X) \cong \tilde{H}^k(X)$ (see [40, Theorem 1.15] for an elementary proof of this fact).

As mentioned in the introduction, the notion of coboundary expansion, which quantifies the vanishing of the cohomology groups $\tilde{H}^k(X)$, can be defined whenever we have a suitable way of measuring the "size" of cochains. The formal definition is as follows:

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[7] Formally, chains and cochains are dual vector spaces, which we identify here due to our choice of a fixed basis corresponding to the k-dimensional simplices.

Definition 7 (Norm on cochains): A **norm** on $C^k(X)$ is a function $\|\cdot\|\colon C^k(X) \to \mathbb{R}_{\geq 0}$ such that

(i) $\|0\| = 0$,
(ii) $\|c + c'\| \leq \|c\| + \|c'\|$ for all $c, c' \in C^k(X)$,
(iii) $\|\cdot\|$ is monotone in the sense that $\|c\| \leq \|c'\|$ whenever $c, c' \in C^k(X)$ with $\mathrm{supp}(c) \subseteq \mathrm{supp}(c')$,
(iv) $\|c\| > 0$ for all $c \in C^k(X) \setminus B^k(X)$.

If X is a free $\mathbb{Z}/2$-complex with $\mathbb{Z}/2$-action ν, we will additionally require that the norm is invariant, i.e., $\|c\| = \|\nu c\|$ for all $c \in C^k(X)$.

Note that any norm $\|\cdot\|$ on $C^k(X)$ induces a function $\|[\cdot]\|$ on the quotient $C^k(X)/B^k(X)$ by

$$\|[c]\| := \min_{b \in B^k(X)} \|c + b\|,$$

which we will sometimes refer to as the **quotient norm**.

We are ready to define coboundary expansion constants.

Definition 8 (Coboundary expansion constant): Let X be a d-dimensional simplicial complex and $0 \leq k \leq d-1$. Let $\|\cdot\|$ be a norm on $C^k(X)$ and $C^{k+1}(X)$. The **k-th coboundary expansion constant** $\eta_k^{\|\cdot\|}(X)$ of X (with respect to $\mathbb{F}_2$-coefficients and the norm $\|\cdot\|$) is defined as

$$\eta_k^{\|\cdot\|}(X) := \min_{c \in C^k(X) \setminus B^k(X)} \frac{\|\delta c\|}{\|[c]\|}.$$

Let us mention various common choices for a norm $\|\cdot\|$. A standard way to get a norm on $C^k(X)$ is by fixing a weight function $w\colon X(k) \to \mathbb{R}_{>0}$ and defining

$$\|c\|_w = \sum_{\sigma \in X(k), c(\sigma) \neq 0} w(\sigma).$$

If X is a $\mathbb{Z}/2$-complex with $\mathbb{Z}/2$-action ν, we additionally require that $w(\sigma) = w(\nu\sigma)$ for all $\sigma \in X(k)$ to ensure that the resulting norm is invariant.

If $w(\sigma) = 1$ for all $\sigma \in X(k)$, then $\|c\|_w$ is the **Hamming norm** of c, which we will simply denote by $|c|$.

For most of the paper, however, it will be convenient to work with the Garland weights and the resulting Garland norm, as defined in the introduction,[8] and unless explicitly stated otherwise, the notation w and $\|\cdot\|$ will refer to the Garland weights and the Garland norm throughout this paper. Moreover, we will write w_σ for the Garland weights on the link X_σ of X at a simplex σ and $\eta_k(X)$ instead of $\eta_k^{\|\cdot\|}(X)$ for the coboundary expansion constant with respect to the Garland norm.

For some of our computations, it will be helpful to work with the norm $\|\cdot\|_w$ on $C^k(X_\Delta^{*2})$ induced by the weights on X_Δ^{*2} obtained by restricting the Garland weights $w\colon X^{*2} \to \mathbb{R}_{\geq 0}$ to X_Δ^{*2}. In order to distinguish these weights and this norm from the norm induced by Garland weights on X_Δ^{*2}, we will denote these weights by w_*, the induced norm by $\|\cdot\|_*$ and the corresponding coboundary expansion constants by $\eta_k^*(X)$ instead of $\eta_k^{\|\cdot\|_*}(X)$.

Remarks 9: (i) For the Garland norm, we have $\eta_k(X) > 0$ if and only if $\tilde{H}^k(X) = 0$. Thus, $\eta_k(X)$ quantifies the vanishing of $\tilde{H}^k(X)$. (Intuitively, the larger $\eta_k(X)$ is, the more $(k+1)$-simplices we have to remove in order to create a non-trivial k-cocycle.) More generally, Conditions (iv) and (i) in Definition 7 ensure that for an arbitrary norm, $\eta_k^{\|\cdot\|}(X) > 0$ implies $\tilde{H}^k(X) = 0$.

(ii) Note that $\eta_k(X) \geq \eta$ for some $\eta > 0$ is equivalent to the statement that $\tilde{H}^k(X) = 0$ and that for every coboundary $\beta \in B^{k+1}(X)$ there is $\alpha \in C^k(X)$ with $\delta\alpha = \beta$ and $\|\alpha\| \leq \frac{1}{\eta}\|\beta\|$.

(iii) Given $\beta \in B^k(X)$ we call $\alpha \in C^{k-1}(X)$ a **minimal cofilling** of β if α has the smallest norm among all cofillings of β (for some previously fixed norm). Note that for a minimal cofilling α we have $\|[\alpha]\| = \|\alpha\|$.

(iv) Let $G = (V, E)$ be a graph. Recall that the **edge expansion** or **Cheeger constant** of G is

$$h(G) := \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}.$$

Here $E(S, V \setminus S)$ denotes the set of edges of G with one vertex in S and one vertex in $V \setminus S$. If we view G as a 1-dimensional simplicial complex, then $h(G) = \eta_0^{|\cdot|}(G)$. Moreover, if G is d-regular, then $\eta_0(G) = \frac{2}{d}h(G)$. Thus, coboundary expansion generalizes edge expansion.

[8] We note that we restrict ourselves to pure complexes when working with Garland weights, to avoid technical difficulties.

3. Proof of the quantitative Borsuk–Ulam theorem

For our proof of Theorem 2, we approximate an arbitrary continuous map by a piecewise-linear map in general position, which allows us to define and work with algebraic intersection numbers (as in the streamlined proof of Gromov's topological overlap theorem in [11]), and we combine this with the idea of using the standard $\mathbb{Z}/2$-invariant cell structure on spheres (similarly to Walker's proof of the Borsuk–Ulam theorem for $\mathbb{Z}/2$-spaces in [45]).

3.1. Approximation by a PL map. Given a finite simplicial complex X and a continuous map $F\colon |X| \to \mathbb{R}^d$, by compactness, the image $F(|X|)$ is contained in the interior of some closed ball $\mathbb{B}^d = B(0,R)$ of finite radius R.

We will use special triangulations of $\mathbb{B}^d$ which refine a particular CW-complex structure on the boundary sphere $\mathbb{S}^{d-1} = \partial\mathbb{B}^d$. This structure has two cells in each dimension and can be inductively obtained by decomposing a d-dimensional sphere into a $(d-1)$-dimensional equitorial sphere with two d-dimensional cells (upper and lower hemisphere) attached. We illustrate this cell structure for $\mathbb{S}^d$, $d \in \{0,1,2\}$, in Figure 1.

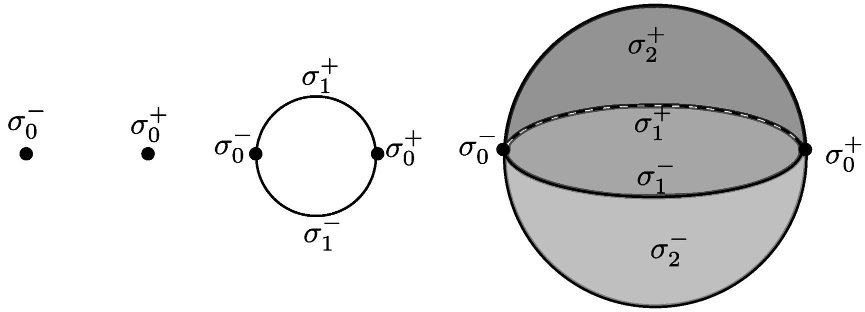

Figure 1. The cell structure for $\mathbb{S}^d$, $d \in \{0,1,2\}$ with 2 cells in each dimension. For $d=1$ we attach two semicircles σ_1^- and σ_1^+ to the two points σ_0^- and σ_0^+. For $d=2$ we start with the cell structure for $\mathbb{S}^1$ and attach two hemispheres σ_2^- and σ_2^+ along this $\mathbb{S}^1$.

Given $\mathbb{S}^{d-1}$ and $0 \leq i \leq d-1$ we write σ_i^+ and σ_i^- for the two cells in dimension i in this decomposition. This cell structure gives rise to a cell structure on $\mathbb{B}^d$ by adding the origin 0 as an additional 0-cell and then coning each σ_i^+ and σ_i^- with 0. In other words, we add all cells of the form $\tau_{i+1}^\varepsilon = 0 * \sigma_i^\varepsilon$ for all $0 \leq i \leq d-1, \varepsilon \in \{-,+\}$. Below we will only consider triangulations T of $\mathbb{B}^d$ which subdivide the cell structure on $\mathbb{B}^d$ given by the cells

$$\{0\} \cup \{\sigma_i^\varepsilon : 0 \leq i \leq d-1, \varepsilon \in \{-,+\}\} \cup \{\tau_i^\varepsilon : 1 \leq i \leq d, \varepsilon \in \{-,+\}\}$$

and such that T is a $\mathbb{Z}/2$-complex with respect to the antipodal map on $\mathbb{B}^d$. We call such a triangulation T of $\mathbb{B}^d$ **good**.

The first step in the proof of Theorem 2 is a (standard) limiting argument that allows us to replace arbitrary continuous maps by piecewise-linear maps in general position; for completeness, we include the details of this argument. Let X be a finite simplicial complex. We recall that a map $f\colon |X| \to \mathbb{R}^d$ is **piecewise-linear** (**PL**) if there is a subdivision X' of X on which f is simplexwise affine (i.e., the restriction of f to every simplex of X' is an affine map).

We need the following notion of general position of affine spaces, points in Euclidean maps and PL maps:

Definition 10 (General position): Two affine spaces $A_1, A_2 \subseteq \mathbb{R}^d$ are in **general position** if $\dim(A_1 \cap A_2) = \max\{-1, \dim(A_1) + \dim(A_2) - d\}$. A subset of points $S \subseteq \mathbb{R}^d$ is in **general position** if for any two disjoint subsets $S_1, S_2 \subseteq S$ the affine hulls $\mathrm{aff}(S_1)$ and $\mathrm{aff}(S_2)$ are in general position. A simplexwise affine map $f\colon |X| \to \mathbb{R}^d$ is in **general position** if it is injective on the vertices of X and $\{f(v) : v \in Y(0)\} \subseteq \mathbb{R}^d$ is in general position.

Let X be a d-dimensional simplicial complex and T a triangulation of $\mathbb{B}^d = B(0,R)$. Let $f\colon |X| \to \mathbb{R}^d$ be a piecewise-linear map such that $f(X)$ is contained in the interior of $\mathbb{B}^d$. We say that f is **in general position with respect to** T if f as a simplexwise affine map $f\colon |X'| \to \mathbb{R}^d$ is in general position and for all $\sigma \in X'$ and $\tau \in T$ we have $\dim(f(\sigma) \cap \tau) \leq \max\{-1, \dim \sigma + \dim \tau - d\}$.[9]

Lemma 11: *Let X be a d-dimensional free $\mathbb{Z}/2$-complex. Let $f\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ be an equivariant map such that $f(|X|)$ is contained in the interior of the closed ball $\mathbb{B}^d = B(0,R)$. Let T be a good triangulation of $\mathbb{B}^d$. Then for any $\varepsilon > 0$*

[9] Note that both $f(\sigma)$ and τ are geometric simplices in $\mathbb{R}^d$, so their intersection is a convex polytope (possibly empty), whose dimension is that of its affine hull.

there is an equivariant piecewise-linear map $g\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ *such that the image of* g *is contained in the interior of* $\mathbb{B}^d$, g *is in general position with respect to* T, *and* $\operatorname{dist}(f(x), g(x)) \leq \varepsilon$ *for all* $x \in X$.[10]

Proof. It is not difficult to see that the classical simplicial approximation theorem extends to the equivariant setting considered here (see [40, Theorem 3.49] or [5, I, Exercise 6]). Thus, there is an equivariant piecewise-linear map $\tilde{g}\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ such that $\operatorname{dist}(\tilde{g}(x), f(x)) \leq \varepsilon/2$ for all $x \in X$ and the image of $\tilde{g}$ is contained in the interior of $\mathbb{B}^d$. Let X' be a subdivision of X on which $\tilde{g}$ is simplexwise affine. The map $\tilde{g}$ might not yet be in general position with respect to T. In order to fix this, we partition $X'(0) = V_+(X') \sqcup V_-(X')$ such that $V_+(X')$ contains precisely one vertex from each ν-orbit (here we use that X is a free $\mathbb{Z}/2$-complex). For each $v \in V_+(X')$ we pick a vector ε_v in $\mathbb{B}^d_{\varepsilon/2}(0) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, 0) \leq \varepsilon/2\}$ uniformly at random. Let $g\colon |X'| \to_{\mathbb{Z}/2} \mathbb{R}^d$ be the simplexwise affine map given by

$$g(v) = \begin{cases} \tilde{g}(v) + \varepsilon_v & \text{if } v \in V_+(X'), \\ \tilde{g}(v) - \varepsilon_{\nu v} & \text{if } v \in V_-(X'). \end{cases}$$

Since ν acts freely, we have $\sigma \cap \nu\sigma = \emptyset$ for all $\sigma \in X'$. In particular, every simplex σ contains at most one vertex from each ν-orbit and the added noise ε_v (or $-\varepsilon_{\nu v}$) is independent for the vertices of a given simplex. This implies that with probability 1, g is in general position with respect to T. Since g and $\tilde{g}$ are both simplexwise affine and $\operatorname{dist}(g(v), \tilde{g}(v)) \leq \varepsilon/2$ for all $v \in X'(0)$ we get by triangle inequality that for $x \in X$

$$\operatorname{dist}(f(x), g(x)) \leq \operatorname{dist}(f(x), \tilde{g}(x)) + \operatorname{dist}(\tilde{g}(x), g(x)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as desired. ∎

Combining this with the following lemma, we see that it suffices to prove Theorem 2 for PL maps $F\colon X \to_{\mathbb{Z}/2} \mathbb{R}^d$ in general position with respect to a good triangulation T of $\mathbb{B}^d$.

Lemma 12: *Let* X *be a* $\mathbb{Z}/2$*-complex and let* $F\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ *be an equivariant map. Let* $F_n\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ *be a sequence of equivariant maps with* $\lim_{n\to+\infty} \sup_{x\in X} \operatorname{dist}(F(x), F_n(x)) = 0$, *i.e.,* F_n *converges uniformly to* F.

[10] Here, for $a, b \in \mathbb{R}^d$ we write $\operatorname{dist}(a, b)$ for the Euclidean distance between a and b.

Let $\|\cdot\|$ be a norm on $C^d(X)$. If there is $\mu \geq 0$ such that

$$\|\{\sigma \in X(d) : 0 \in F_n(\sigma)\}\| \geq \mu$$

for all $n \in \mathbb{N}$ then

$$\|\{\sigma \in X(d) : 0 \in F(\sigma)\}\| \geq \mu.$$

(Here, as before, we identify $S \subseteq X(k)$ with its characteristic function $\mathbb{1}_S \in C^k(X)$.)

Proof. By compactness the infimum in

$$\rho := \inf\{\operatorname{dist}(F(x), 0) : \sigma \in X(d), x \in \sigma, 0 \notin F(\sigma)\}$$

is attained and $\rho > 0$. Let n be large enough such that $\operatorname{dist}(F_n(x), F(x)) < \rho$ for all $x \in X$. Then let $\sigma \in X(d)$ with $x \in \sigma$ such that $F_n(x) = 0$. We get $\operatorname{dist}(F(x), 0) = \operatorname{dist}(F(x), F_n(x)) < \rho$. By choice of ρ this implies $0 \in F(\sigma)$. We have shown that

$$\{\sigma \in X(d) : 0 \in F_n(\sigma)\} \subseteq \{\sigma \in X(d) : 0 \in F(\sigma)\}$$

from which the conclusion of the lemma follows by monotonicity of the norm $\|\cdot\|$ (Property (iii) in Definition 7). ∎

3.2. Intersection numbers. The advantage of working with PL maps in general position with respect to a good triangulation of $\mathbb{B}^d$ is that it allows to define (algebraic) intersection numbers. Indeed, let X be a d-dimensional simplicial complex, T a good triangulation of $\mathbb{B}^d$. Given a PL map $F\colon |X| \to \mathbb{B}^d$ in general position with respect to T, we have that $\sigma \cap F^{-1}(\tau)$ is a set of finitely many points for any k-simplex $\sigma \in X$ and $(d-k)$-simplex $\tau \in T$.[11] We define the **intersection number**

$$F(\sigma) \cdot \tau := |\sigma \cap F^{-1}(\tau)| \mod 2 \in \mathbb{F}_2$$

and extend this (bi)linearly to define $F(a) \cdot b \in \mathbb{F}_2$ for arbitrary chains $a \in C_k(X)$ and $b \in C_{d-k}(T)$. For any $0 \leq k \leq d$ the intersection number induces an **intersection homomorphism** $F^{\pitchfork}\colon C_k(T) \to C^{d-k}(X)$ by

$$c \mapsto F^{\pitchfork}(c)(\sigma) := F(\sigma) \cdot c.$$

[11] To see this, consider a finite subdivison X' of X such that F is simplexwise affine on X'. If σ is subdivided into $\sigma = \sigma_0 \cup \cdots \cup \sigma_l$, then by general position with respect to T the image of every σ_i intersects τ in at most one point. Hence, $\sigma \cap F^{-1}(\tau)$ contains at most $l+1$ points.

The following lemma asserts that $F^{\pitchfork}$ is a so-called chain-cochain map and is a well-known fact (see [32, Section 2.2] for a detailed review on intersection numbers).

Lemma 13: *For any $1 \leq k \leq d$ and $c \in C_k(T)$ it holds that*

$$F^{\pitchfork}(\partial c) = \delta F^{\pitchfork}(c).$$

3.3. Pagodas and the proof of Theorem 2. In this section we prove the following generalization of Theorem 2.

Theorem 14 (Quantitative Borsuk–Ulam): *Let (X, ν) be a d-dimensional free $\mathbb{Z}/2$-complex. Let $\|\cdot\|$ be an invariant norm on cochains of X (i.e., we assume $\|c\| = \|\nu c\|$ for all cochains c). Then for any equivariant map $F\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ we have*

$$\|\{\sigma \in X(d) : 0 \in F(\sigma)\}\| \geq \frac{\|\mathbb{1}_{X(0)}\|}{2^d} \prod_{i=0}^{d-1} \eta_i^{\|\cdot\|}(X).$$

The statement of the theorem is trivial if $\eta_k^{\|\cdot\|}(X)=0$ for some $k\in\{0,\dots,d-1\}$. Thus, we may assume that $\eta_k^{\|\cdot\|}(X) > 0$ and hence

$$\tilde{H}^k(X) = 0$$

for all $0 \leq k \leq d-1$.

By the discussion above, it suffices to prove Theorem 14 for PL maps $F\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}^d$ such that $F(|X|)$ is contained in the interior of $\mathbb{B}^d$ and such that F is in general position with respect to a good triangulation T of $\mathbb{B}^d$. Let us fix such a map $F\colon |X| \to_{\mathbb{Z}/2} T$ and denote by $\nu\colon |X| \to |X|$ the $\mathbb{Z}/2$-action on X. Recall that we assume that the origin 0 is a vertex of the good triangulation T. Clearly,

$$\{\sigma \in X(d) : F^{\pitchfork}(0)(\sigma) = 1\} \subseteq \{\sigma \in X(d) : 0 \in F(\sigma)\}.$$

Thus, it suffices to give a lower bound on $\|F^{\pitchfork}(0)\|$. (Here, we again use monotonicity of the norm $\|\cdot\|$, i.e., Property (iii) in Definition 7.) To this end, the notion of a pagoda will be useful. We call a sequence of cochains $(b^{(d)}, a^{(d-1)}, b^{(d-1)}, \dots, a^{(0)}, b^{(0)})$ a **pagoda** for F if the following conditions hold:

(i) $b^{(d)} = F^{\pitchfork}(0)$,
(ii) $b^{(i)}, a^{(i)} \in C^i(X)$ for all $0 \leq i \leq d-1$,
(iii) $b^{(i)} = a^{(i)} + \nu a^{(i)}$ for $0 \leq i \leq d-1$,
(iv) $b^{(i)} = \delta a^{(i-1)}$ for $1 \leq i \leq d$.

There is always a pagoda coming from pulling-back the special cell decomposition of the good triangulation of $\mathbb{B}^d$. We illustrate this pagoda for an equivariant PL map from the octahedron Λ_2^2 to $\mathbb{B}^2$ in Figure 2.

Lemma 15: *There exists a pagoda for F with $b^{(0)} = \mathbb{1}_{X(0)} \in B^0(X)$.*

Proof. To keep notation simple, let us also write σ_i^ε or τ_i^ε, $\varepsilon \in \{-,+\}$, for the i-chain corresponding to the subsets of i-simplices in T refining the i-cell σ_i^ε or τ_i^ε, respectively.

Now, define $b^{(d)} := F^{\pitchfork}(0)$ and for $0 \le i \le d-1$ let

$$a^{(i)} := F^{\pitchfork}(\tau_{d-i}^+) \quad \text{and} \quad b^{(i)} := a^{(i)} + \nu a^{(i)}.$$

Since the image of F is contained in the interior of T, we have $F^{\pitchfork}(\sigma_k^+) = 0$ for all $0 \le k \le d-1$. Using this and Lemma 13, we compute

$$\delta a^{(d-1)} = \delta F^{\pitchfork}(\tau_1^+) = F^{\pitchfork}(\partial \tau_1^+) = F^{\pitchfork}(0) + F^{\pitchfork}(\sigma_0^+) = F^{\pitchfork}(0) = b^{(d)}$$

and for $1 \le i \le d-1$ we get

$$\begin{aligned}\delta a^{(i-1)} &= \delta F^{\pitchfork}(\tau_{d-i+1}^+) = F^{\pitchfork}(\partial \tau_{d-i+1}^+)\\ &= F^{\pitchfork}(\sigma_{d-i}^+ + \tau_{d-i}^+ + \tau_{d-i}^-) = F^{\pitchfork}(\tau_{d-i}^+) + \nu F^{\pitchfork}(\tau_{d-i}^+) = b^{(i)}.\end{aligned}$$

Finally

$$b^{(0)} = F^{\pitchfork}(\tau_d^+) + F^{\pitchfork}(\tau_d^-) = \mathbb{1}_{X(0)},$$

since every vertex of X is mapped to the interior of a unique d-simplex of T. ∎

Lemma 16: *Every pagoda $(b^{(d)}, a^{(d-1)}, b^{(d-1)}, \ldots, a^{(0)}, b^{(0)})$ for F satisfies*

$$b^{(0)} = \mathbb{1}_{X(0)}.$$

Proof. Let $(b_*^{(d)}, a_*^{(d-1)}, b_*^{(d-1)}, \ldots, a_*^{(0)}, b_*^{(0)})$ be a pagoda for F with $b_*^{(0)} = \mathbb{1}_{X(0)}$ whose existence is guaranteed by the previous lemma. Recall that we assume, without loss of generality, that $\tilde{H}^k(X) = 0$ for $0 \le k \le d-1$. Given any other pagoda $(b^{(d)}, a^{(d-1)}, b^{(d-1)}, \ldots, a^{(0)}, b^{(0)})$, we use this to argue by downward induction on i that for all $0 \le i \le d$, $b^{(i)} + b_*^{(i)}$ is the coboundary of an $(i-1)$-cochain of the form $c^{(i-1)} + \nu c^{(i-1)}$ for some $c^{(i-1)} \in C^{i-1}(X)$. For $i = 0$, the only (-1)-dimensional cochain of the form $c^{(-1)} + \nu c^{(-1)}$ is the cochain $0 \in C^{-1}(X)$; thus we must have $b^{(0)} + b_*^{(0)} = 0$, hence $b^{(0)} = b_*^{(0)} = \mathbb{1}_{X(0)}$, as desired.

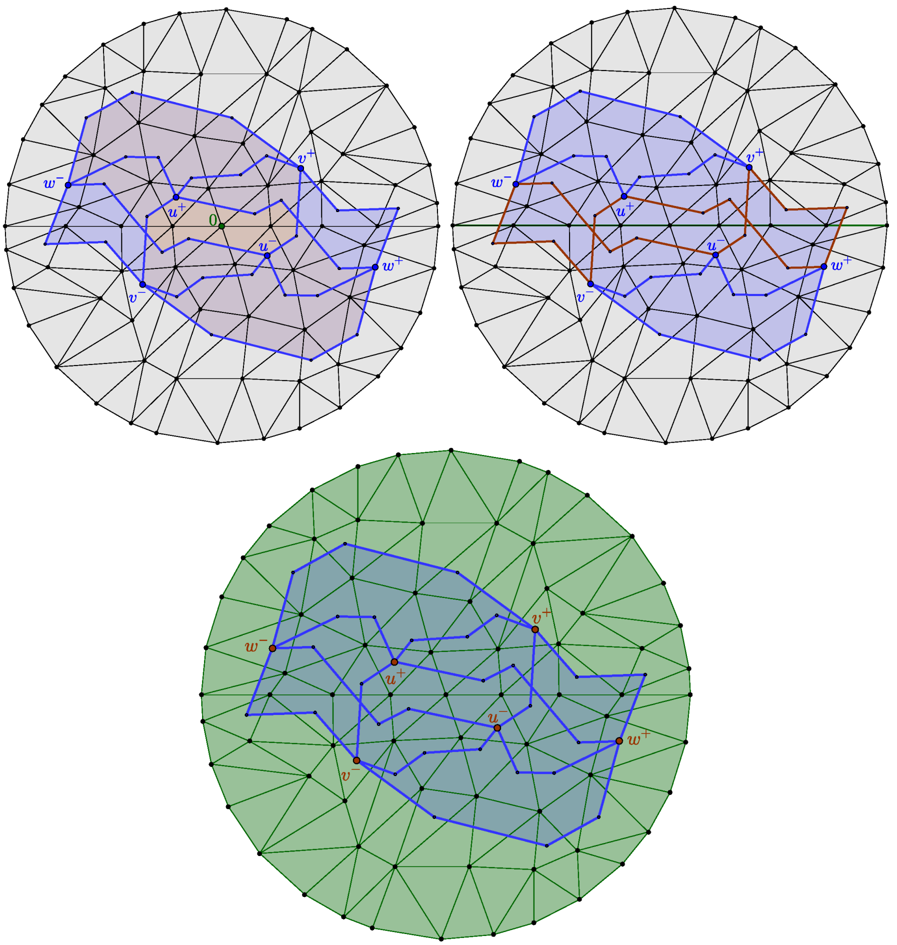

Figure 2. We illustrate the pagoda as constructed in the proof of Lemma 15 for an equivariant PL map $F\colon X \to_{\mathbb{Z}/2} \mathbb{B}^2$ in general position with respect to a good triangulation T where $X = \Lambda_2^2$ is an octahedron. In blue we show the image of X under F. At the top left, we have that $F^{\pitchfork}(0)$ are the two triangles $u^+v^-w^+$ and $u^-v^+w^-$ marked in red. At the top right, we depict $b^{(1)} = F^{\pitchfork}(\tau_1^-)+F^{\pitchfork}(\tau_1^+)$ in red and the chain $\tau_1^- + \tau_1^+$ in green. We see that the support of $b^{(1)}$ consists of all edges in $F(X)$ that the green line intersects an odd number of times. Finally, at the bottom we have $b^{(0)} = \mathbb{1}_{X(0)} = F^{\pitchfork}(\mathbb{1}_{T(2)})$.

The base case ($i = d$) of the induction is clear since $b^{(d)} = b_*^{(d)} = F^{\pitchfork}(0)$ by the definition of a pagoda, hence $b^{(d)} + b_*^{(d)} = 0 = \delta(0 + \nu 0)$.

Assume now that $0 \le k \le d-1$ and, inductively, that there exists a cochain $c^{(k)} \in C^k(X)$ with $b^{(k+1)} + b_*^{(k+1)} = \delta(c^{(k)} + \nu c^{(k)})$. Thus,

$$\delta(a^{(k)} + a_*^{(k)} + c^{(k)} + \nu c^{(k)}) = 0.$$

Using $\tilde{H}^k(X) = 0$, we find $c^{(k-1)} \in C^{k-1}(X)$ with $\delta c^{(k-1)} = a^{(k)} + a_*^{(k)} + c^{(k)} + \nu c^{(k)}$, hence

$$a^{(k)} + a_*^{(k)} = c^{(k)} + \nu c^{(k)} + \delta c^{(k-1)}.$$

Thus,

$$\begin{aligned} b^{(k)} + b_*^{(k)} &= a^{(k)} + \nu a^{(k)} + a_*^{(k)} + \nu a_*^{(k)} \\ &= (c^{(k)} + \nu c^{(k)} + \delta c^{(k-1)}) + \nu(c^{(k)} + \nu c^{(k)} + \delta c^{(k-1)}) \\ &= \delta(c^{(k-1)} + \nu c^{(k-1)}), \end{aligned}$$

as desired. ∎

With all these preparations we can wrap-up the proof of Theorem 14 by inductively constructing a pagoda using minimal cofillings along the way. Coboundary expansion then guarantees that $F^{\pitchfork}(0)$ is large.

Proof of Theorem 14. We construct a pagoda for F by inductively defining $a^{(i)} \in C^i(X)$. Then $b^{(i)}$ is determined through the condition $b^{(i)} = a^{(i)} + \nu a^{(i)}$.

We know that $b^{(d)} := F^{\pitchfork}(0)$ is a coboundary. We choose $a^{(d-1)} \in C^{d-1}(X)$ to be a minimal cofilling of $b^{(d)}$. Assume that $a^{(i)}$ has already been constructed. Then $b^{(i)} = a^{(i)} + \nu a^{(i)}$ satisfies

$$\delta b^{(i)} = \delta a^{(i)} + \nu \delta a^{(i)} = b^{(i+1)} + \nu b^{(i+1)} = 0.$$

Since $\tilde{H}^i(X) = 0$, $b^{(i)}$ is a coboundary and we choose $a^{(i-1)} \in C^{d-1}(X)$ to be a minimal cofilling of $b^{(i)}$. By construction $\|b^{(i)}\| \ge \eta_{i-1}^{\|\cdot\|}(X)\|a^{(i-1)}\|$ for all $1 \le i \le d$. Also,

$$\|b^{(i)}\| = \|a^{(i)} + \nu a^{(i)}\| \le 2\|a^{(i)}\|$$

by the triangle inequality and the invariance of $\|\cdot\|$ under ν. By Lemma 16 we have $b^{(0)} = \mathbb{1}_{X(0)}$. Combining all these we conclude that

$$\|\mathbb{1}_{X(0)}\| = \|b^{(0)}\| \le 2\|a^{(0)}\| \le \frac{2}{\eta_0^{\|\cdot\|}(X)}\|b^{(1)}\| \le \cdots \le \frac{2^d}{\prod_{i=0}^{d-1} \eta_i^{\|\cdot\|}(X)}\|b^{(d)}\|,$$

as desired. ∎

4. Expansion of join versus expansion of deleted join

It seems that the expansion properties of the join X^{*2} of a simplicial complex X with itself are a bit easier to analyze than the expansion properties of the deleted join X^{*2}_{Δ}. In fact, X^{*2} has vanishing cohomology provided that the cohomology groups of X vanish. More precisely, we have the following (simple) special case of the Künneth theorem (see for instance [19, Chapter V] for the general Künneth theorem for products and [35] for the relation between products and joins).

Theorem (Künneth theorem): *Let X be a d-dimensional complex. Then for all $0 \leq k \leq 2d+1$*

$$\tilde{H}^k(X^{*2}) = \bigoplus_{i+j=k-1} \tilde{H}^i(X) \otimes \tilde{H}^j(X).$$

In particular, if $\tilde{H}^i(X) = 0$ for all $0 \leq i \leq d-1$ then $\tilde{H}^k(X^{*2}) = 0$ for all $0 \leq k \leq 2d$.

We do not know how to prove a quantitative version of this result in general. But assuming we could prove strong lower bounds on $\eta_k(X^{*2})$, we would expect to be able to deduce lower bounds for $\eta_k(X^{*2}_{\Delta})$. Intuitively speaking, (at least for large complexes) X^{*2} and X^{*2}_{Δ} look alike and it is reasonable to think that $\eta_k(X^{*2})$ and $\eta_k(X^{*2}_{\Delta})$ do not differ by too much.

The goal of this section is to make this intuition precise and give a quantitative relation between $\eta_k(X^{*2})$ and $\eta^*_k(X^{*2}_{\Delta})$, using the notion of thickness that was mentioned in the introduction. Recall that a d-dimensional simplicial complex X is called δ-thick if every $(d-1)$-simplex of X is contained in at least δ d-simplices. We define the **thickness** $\delta(X)$ to be the maximal integer δ such that X is δ-thick.

By definition, if X is δ-thick, then for all $\sigma \in X(d-1)$ and $v \in X_\sigma(0)$, we have

$$w_\sigma(v) = \frac{1}{|X_\sigma(0)|} \leq \frac{1}{\delta}.$$

For our estimates below, we will need such a bound for all $-1 \leq k \leq d-1$, $\sigma \in X(k)$ and $v \in X_\sigma(0)$. Fortunately, δ-thickness implies such bounds, as the following lemma shows.

LEMMA 17: *Let X be a pure d-dimensional simplicial complex. Assume that X is δ-thick for some $\delta > 0$. Then*

(i) *for any $\sigma \in X(k), -1 \leq k \leq d-1$ the link X_σ is a $(d-|\sigma|)$-dimensional, δ-thick simplicial complex,*
(ii) *for every $v \in X(0)$ we have $w(v) \leq \frac{1}{\delta}$,*
(iii) *for every $\sigma \in X(k)$, $-1 \leq k \leq d-1$ and $v \in X_\sigma(0)$ we have $w_\sigma(v) \leq \frac{1}{\delta}$.*

Proof. For (i) we simply observe that for $\tau \in X_\sigma(d-|\sigma|-1)$ we have

$$|(X_\sigma)_\tau(0)| = |X_{\sigma\cup\tau}(0)| \geq \delta,$$

since X is δ-thick.

For (ii) we first note that since X is δ-thick

$$\delta|X(d-1)| \leq \sum_{\sigma\in X(d-1)} |X_\sigma(0)| = (d+1)|X(d)|.$$

Then for $v \in X(0)$ we compute

$$w(v) = \frac{|X_v(d-1)|}{(d+1)|X(d)|} \leq \frac{|X_v(d-1)|}{\delta|X(d-1)|} \leq \frac{1}{\delta},$$

where we used $X_v(d-1) \subseteq X(d-1)$ for the last inequality.

Finally, (iii) follows by combining (i) and (ii). ∎

Let us provide some intuition for the condition that $w_\sigma(u) \leq \varepsilon$ for $u \in X_\sigma(0)$ and some $\varepsilon > 0$. To this end, we first note that for $\sigma \in X(k)$, $u \in X_\sigma(0)$ we have

$$w(\sigma) = \frac{1}{k+2} \sum_{v\in X_\sigma(0)} w(\sigma \sqcup v) \quad \text{and} \quad w_\sigma(u) = \frac{w(\sigma\sqcup u)}{(k+2)w(\sigma)}.$$

Thus, the condition $w_\sigma(u) \leq \varepsilon$ for some $\varepsilon > 0$ is equivalent to

$$\frac{1}{k+2} w(\sigma \sqcup u) \leq \varepsilon w(\sigma).$$

That is to say, that every $(k+1)$-simplex containing σ contributes only a small fraction to the weight of σ.

The following consequence of δ-thickness will help us to relate the expansion properties of X^{*2} to the expansion properties of X^{*2}_Δ.

Lemma 18: *Let X be a pure d-dimensional simplicial complex. Assume that X is δ-thick for some $\delta > 0$. Then for all $0 \le k \le 2d$ and $\tau \in X^{*2}_{\Delta}(k)$ we have*

$$\sum_{\sigma \in X^{*2}(k+1)\setminus X^{*2}_{\Delta}(k+1), \tau \subseteq \sigma} w(\sigma) \le \frac{1}{\delta}(k+2)(k+1)w_*(\tau).$$

For the proof of Lemma 18 we need the following identities.

Claim 19: *Let X be a d-dimensional simplicial complex. For $-1 \le i, j \le d$ let*

$$c_{i,j} = \frac{\binom{d+1}{i+1}\binom{d+1}{j+1}}{\binom{2d+2}{i+j+2}}.$$

Then:

(i) *For all $\sigma, \tau \in X$ we have for $\sigma \uplus \tau \in X^{*2}$ that*

$$w(\sigma \uplus \tau) = c_{|\sigma|-1,|\tau|-1} w(\sigma) w(\tau).$$

(ii) *For all $-1 \le i, j \le d$ we have*

$$c_{i,j} = c_{j,i}.$$

(iii) *For all $-1 \le i \le d$, $0 \le j \le d$ we have*

$$\frac{c_{i,j}}{c_{i,j-1}} = \frac{d+1-j}{j+1}\frac{i+j+2}{2d-i-j+1}.$$

Also, if $\sigma \in X$ and $v \in X_\sigma(0)$ we have

$$w(\sigma \sqcup v) = w_\sigma(v) w(\sigma)(|\sigma| + 1).$$

The proof of this claim is a straightforward computation which we omit. We turn to the proof of Lemma 18.

Proof of Lemma 18. Let $\tau = \tau' \uplus \tau'' \in X^{*2}_{\Delta}(k)$ with $\tau', \tau'' \in X, \tau' \cap \tau'' = \emptyset$. It will be convenient to extend the weight function w to arbitrary subsets of $X^{*2}(0)$ and set $w(s) = 0$ for $s \subseteq X^{*2}(0)$ if $s \notin X^{*2}$. Similarly, for $u \in X(0)$ we interpret $w_\sigma(u)$ as 0 if u is not a vertex of X_σ. Write $\tau' = \{v_0, \dots v_l\}$ (we allow $l = -1$ if $\tau' = \emptyset$) and $\tau'' = \{v_{l+1}, \dots, v_k\}$. Using the identities in Claim 19 and

Lemma 17 (iii), we compute

$$\begin{aligned}
&\sum_{\sigma\in X^{*2}(k+1)\setminus X^{*2}_{\Delta}(k+1),\tau\subseteq\sigma} w(\sigma)\\
&\quad= \sum_{i=0}^{l} w(\tau'\uplus(\tau''\cup v_i)) + \sum_{i=l+1}^{k} w((\tau'\cup v_i)\uplus\tau'')\\
&\quad= \sum_{i=0}^{l} c_{l,k-l}w(\tau')w_{\tau''}(v_i)w(\tau'')(|\tau''|+1)\\
&\qquad+ \sum_{i=l+1}^{k} c_{l+1,k-l-1}w_{\tau'}(v_i)w(\tau')(|\tau'|+1)w(\tau'')\\
&\quad\le \frac{1}{\delta}((|\tau''|+1)c_{l,k-l}(l+1)+(|\tau'|+1)c_{l+1,k-l-1}(k-l))w(\tau')w(\tau'')\\
&\quad= \frac{1}{\delta}\frac{(l+1)(k-l+1)c_{l,k-l}+(l+2)(k-l)c_{l+1,k-l-1}}{c_{l,k-l-1}}w_*(\tau)\\
&\quad= \frac{1}{\delta}\Big(\frac{(l+1)(k+2)(d-k+l+1)+(k+2)(d-l)(k-l)}{2d-k+1}\Big)w_*(\tau)\\
&\quad\le \frac{1}{\delta}\frac{k+2}{2d-k+1}((k+1)(d-k+l+1+(d-l))w_*(\tau)\\
&\quad= \frac{1}{\delta}(k+2)(k+1)w_*(\tau).
\end{aligned}$$

This finishes the proof. ∎

We need one last lemma before we are ready to relate $\eta^*_k(X^{*2}_\Delta)$ to $\eta_k(X^{*2})$.

Lemma 20: *Let X be a d-dimensional simplicial complex. Let $c\in C^k(X^{*2}_\Delta)$ be such that $\|c\|_*\le\|c+\delta a\|_*$ for all $a\in C^{k-1}(X^{*2}_\Delta)$. Let $\bar c\in C^k(X^{*2})$ be the extension by 0 of c, i.e., $\bar c(\sigma)=c(\sigma)$ for $\sigma\in X^{*2}_\Delta(k)\subseteq X^{*2}(k)$ and 0 otherwise. Then $\bar c$ satisfies $\|\bar c\|\le\|\bar c+\delta a\|$ for all $a\in C^{k-1}(X^{*2})$.*[12]

Proof. Write $i\colon X^{*2}_\Delta\to X^{*2}$ for the inclusion map and $i^\sharp\colon C^\bullet(X^{*2})\to C^\bullet(X^{*2}_\Delta)$ for the induced restriction map on cochains. Let $a\in C^{k-1}(X^{*2})$. Since $\|\cdot\|_*$ is obtained by restricting the Garland weights on X^{*2} to X^{*2}_Δ, we have

$$\|\bar c+\delta a\|\ge\|i^\sharp(\bar c+\delta a)\|_*.$$

[12] Here it is crucial that we do not use the norm on X^{*2}_Δ induced by Garland weights on X^{*2}_Δ but by the weights obtained by restricting the Garland weights on X^{*2}.

Note that $i^\sharp$ is linear, $i^\sharp(\bar{c}) = c$ and that $i^\sharp \delta a = \delta i^\sharp a$. Hence,

$$\|i^\sharp(\bar{c} + \delta a)\|_* = \|c + \delta i^\sharp a\|_*.$$

By assumption $\|c + \delta i^\sharp a\|_* \geq \|c\|_*$. Finally, $\|c\|_* = \|\bar{c}\|$, by definition. We conclude that $\|\bar{c} + \delta a\| \geq \|\bar{c}\|$, as desired. ∎

With all these preparations we can finally show:

Proposition 21 ($\eta_k^*(X_\Delta^{*2})$ vs. $\eta_k(X^{*2})$): *Let X be a pure d-dimensional simplicial complex. Assume that X is δ-thick for some $\delta > 0$. Then for $0 \leq k \leq 2d$ we have*

$$\eta_k^*(X_\Delta^{*2}) \geq \eta_k(X^{*2}) - (k+2)(k+1)\frac{1}{\delta}.$$

Proof. Let $c \in C^k(X_\Delta^{*2})$ with $\|c + \delta a\|_* \geq \|c\|_*$ for all $a \in C^{k-1}(X_\Delta^{*2})$. As in the previous lemma, let $\bar{c} \in C^k(X^{*2})$ be the extension by 0 of c. By Lemma 20, $\bar{c}$ satisfies $\|\bar{c}\| \leq \|\bar{c} + \delta a\|$ for all $a \in C^{k-1}(X^{*2})$. Thus

$$\|\delta \bar{c}\| \geq \eta_k(X^{*2})\|c\|_*.$$

Let $\Delta(k+1) = X^{*2}(k+1) \setminus X_\Delta^{*2}(k+1)$. We have

$$\|\delta \bar{c}\| = \|\delta c\|_* + \|(\delta \bar{c})|_{\Delta(k+1)}\|.$$

We estimate

$$\begin{aligned} \|(\delta \bar{c})|_{\Delta(k+1)}\| &\leq \sum_{\tau \in c} \sum_{\sigma \in \Delta(k+1), \tau \subseteq \sigma} w(\sigma) \\ &\leq \sum_{\tau \in \mathrm{supp}(c)} \frac{1}{\delta}(k+2)(k+1) w_*(\tau) = \frac{1}{\delta}(k+2)(k+1)\|c\|_*, \end{aligned}$$

where we used Lemma 18 and that X is δ-thick. Combining these we get

$$\eta_k(X^{*2})\|c\|_* \leq \|\delta c\|_* + \frac{1}{\delta}(k+2)(k+1)\|c\|_*.$$

The proposition follows after rearranging. ∎

We close this section with the remark that it is likely that in some situations or with a more careful analysis one can tighten the relationship between $\eta_k^*(X_\Delta^{*2})$ and $\eta_k(X^{*2})$. But this would most likely introduce quite an amount of technicalities and additional parameters. We preferred to stick with a potentially looser but simple to state relationship between $\eta_k^*(X_\Delta^{*2})$ and $\eta_k(X^{*2})$ involving a single, fairly natural parameter—the thickness $\delta(X)$ of X.

5. Quantitative non-embeddability of spherical buildings

We give a very brief introduction to spherical buildings. Buildings are highly symmetric (combinatorial) structures that have been extensively studied since their introduction by Jacques Tits in the 1960s. We will only need very few basic facts and refer the interested reader to the books [1], [15] or [44]. We start with the definition of a (spherical) building.

Definition 22 (Building): A d-dimensional (**thick**) **building** X is a d-dimensional simplicial complex X for which there is a family $\mathcal{A}$ of subcomplexes, called **apartments**, such that

(i) X is pure and every $\sigma \in X(d-1)$ is contained in at least three d-simplices.
(ii) Any two simplices of X are contained in a common apartment $A \in \mathcal{A}$.
(iii) Any $(d-1)$-simplex in an apartment A is contained in precisely two d-simplices of A.
(iv) For any two d-simplices σ, σ' in an apartment A there is a sequence of d-simplices $\sigma_0, \dots, \sigma_n \in A$ such that $\sigma = \sigma_0, \sigma' = \sigma_n$ and $|\sigma_i \cap \sigma_{i+1}| = d$ for all $0 \le i \le n-1$.
(v) If $\sigma, \tau \in X$ are contained in apartments $A, A' \in \mathcal{A}$ then there is a simplicial isomorphism $\phi\colon A \to A'$ which fixes σ and τ pointwise.

A building is **spherical** if every apartment is finite.

It turns out that for a given building X (see [1, Theorem 4.131]) there is a Coxeter system (W, S) such that every apartment A is isomorphic to the Coxeter complex associated with (W, S).[13] In particular, every apartment of X has the same number of d-simplices, namely $|A(d)| = |W|$. We will denote this number by $w_d(X)$ and call it the **width** of X. Elaborating on the work of Gromov in [16] the following lower bound on the coboundary expansion constants of spherical buildings was shown in [31].

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[13] It is not important to us what these exactly are. Let us just mention that a Coxeter system (W, S) is a group W with a generating set S satisfying special types of relations. The associated Coxeter complex (W, S) is a triangulation of a $(|S|-1)$-dimensional sphere if W is finite and reflects the group structure of W geometrically.

Theorem 23 (Expansion of spherical buildings (Corollary 3.6 in [31])): *Let X be a d-dimensional spherical building. Then for any $0 \le k \le d-1$ we have*

$$\eta_k(X) \ge \frac{1}{\binom{d+1}{k+2}^2 w_d(X)}.$$

It is not hard to see that the join X^{*2} of a d-dimensional spherical building X with itself is a $(2d+1)$-dimensional spherical building with width

$$w_{2d+1}(X^{*2}) = w_d(X)^2.$$

Indeed, given apartments $\mathcal{A}$ for X, let $\mathcal{A} * \mathcal{A}$ be the set of subcomplexes of X^{*2} of the form $A * A'$ with $A, A' \in \mathcal{A}$. Now, it is a straightforward (but somewhat tedious) exercise to check that $\mathcal{A} * \mathcal{A}$ satisfies properties (i)–(v) in Definition 22. We immediately deduce

Corollary 24: *Let X be a d-dimensional spherical building. Then for all $0 \le k \le 2d$ we have*

$$\eta_k(X^{*2}) \ge \frac{1}{\binom{2d+2}{k+2}^2 w_d(X)^2}.$$

We are ready to prove the following slightly refined version of Theorem 3 from the introduction.

Theorem 25 (Quantitative non-embeddability spherical buildings): *Let X be a d-dimensional building such that $\delta(X) > (k+2)(k+1)\binom{2d+2}{k+2}^2 w_d(X)^2$ for all $0 \le k \le 2d$. Then*

$$\operatorname{ipcr}(X) \ge \left(\frac{1}{2^{2d+1}} \prod_{k=0}^{2d} \left(\frac{1}{\binom{2d+2}{k+2}^2 w_d(X)^2} - (k+2)(k+1)\frac{1}{\delta(X)} \right) \right) \binom{|X(d)|}{2}.$$

Proof. We apply the quantitative Borsuk–Ulam theorem (Theorem 14) to X^{*2}_{Δ} where we use the norm on cochains obtained by restricting the Garland weights on X^{*2} to X^{*2}_{Δ}. Then the result follows by plugging-in the bounds from Corollary 24 and Proposition 21. ∎

We remark that there is some constant w_d such that $w_d(X) \le w_d$ for all d-dimensional spherical buildings. Thus, if one wished, one could make the assumption on the thickness of X in the previous theorem not to depend on $w_d(X)$.

6. Upper bounds on expansion constants for Λ_n^d and $A_d(\mathbb{F}_q)$

In this section we prove the upper bounds on $\eta_{d-1}(A_d(\mathbb{F}_q))$ and $\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$. We start with some preparations including a characterization of coboundaries in terms of cycles and a basis for the d-dimensional cycles of the complete $(d+1)$-partite d-dimensional complex. Then we give the construction of many coboundaries in Λ_n^d with some extra algebraic structure. We use these coboundaries to deduce our upper bounds on $\eta_{d-1}(A_d(\mathbb{F}_q))$ and $\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$.

6.1. Cycles vs. coboundaries. There is a nice characterization of coboundaries using a pairing between chain and cochains. This bilinear pairing $\langle\cdot,\cdot\rangle\colon C^k(X)\times C_k(X)\to\mathbb{F}_2$ is given by

$$(c,\sigma)\mapsto(\langle c,\sigma\rangle := c(\sigma))$$

extended linearly to all of $C^k(X)\times C_k(X)$. Note that $\langle\delta c,a\rangle=\langle c,\partial a\rangle$ for all $c\in C^{k-1}(X)$ and $a\in C_k(X)$. The following lemma is an easy consequence of standard facts from linear algebra:

Lemma 26: *Let $c\in C^k(X)$. Then the following are equivalent:*

(i) $c\in B^k(X)$.
(ii) $\langle c,z\rangle=0$ *for all cycle* $z\in Z_k(X)$.
(iii) $\langle c,z\rangle=0$ *for* $z\in\mathcal{Z}$ *where* $\mathcal{Z}\subseteq Z_k(X)$ *is a generating set.*

A basis for $Z_d(\Lambda^d_{n_0,\ldots,n_d})$ is easy to describe.

Lemma 27: *Let $\Lambda^d_{n_0,\ldots,n_d}=U_0*U_1*\cdots*U_d$ with $|U_i|=n_i$. Given pairwise distinct vertices $u_i^+,u_i^-\in U_i$, $0\le i\le d$, let*

$$\diamond^d((u_i^+,u_i^-)_{0\le i\le d}) := \{u_0^+,u_0^-\}*\cdots*\{u_d^+,u_d^-\}$$

be the octahedral sphere spanned by these vertices. We think of

$$\diamond^d((u_i^+,u_i^-)_{0\le i\le d})(d)$$

as a chain in $C_d(\Lambda^d_{n_0,\ldots,n_d})$. Then for any fixed $u_i^+\in U_i, 0\le i\le d$ the set

$$\{\diamond^d((u_i^+,u_i^-)_{0\le i\le d})(d)\in C_d(\Lambda^d_{n_0,\ldots,n_d}) : u_i^-\in U_i\setminus\{u_i^+\}, 0\le i\le d\}$$

is a basis for $Z_d(\Lambda^d_{n_0,\ldots,n_d})$.

Proof. Fix $u_i^+ \in U_i, 0 \leq i \leq d$ and let

$$\mathcal{Z} = \{\Diamond^d((u_i^+, u_i^-)_{0\leq i\leq d})(d) \in C_d(\Lambda^d_{n_0,\ldots,n_d}) : u_i^- \in U_i \setminus \{u_i^+\}, 0 \leq i \leq d\}.$$

Clearly, every $z \in \mathcal{Z}$ is a cycle.

Note that for any choice of $u_i^- \in U_i \setminus \{u_i^+\}, 0 \leq i \leq d$, there is precisely one $z \in \mathcal{Z}$ which contains $\{u_0^-, \ldots, u_d^-\}$ in its support. This implies that the cycles in $\mathcal{Z}$ are linearly independent.

Note that $|\mathcal{Z}| = \prod_{i=0}^d (n_i - 1)$.

On the other hand, it is well-known that $\tilde{H}_k(\Lambda^d_{n_0,\ldots,n_d}) = 0$ for all $-1 \leq k \leq d-1$. This can be deduced from [33, Proposition 4.4.3] concerning the connectivity of joins; alternatively, it also follows from the fact that $\Lambda^d_{n_0,\ldots,n_d}$ is the matroid complex of a partition matroid and hence shellable (see, e.g., [4, Theorem 7.3.3]). We get by the rank–nullity theorem that

$$\begin{aligned}\dim Z_d(\Lambda^d_{n_0,\ldots,n_d}) &= \sum_{i=0}^{d+1} (-1)^i \dim C_{d-i}(\Lambda^d_{n_0,\ldots,n_d}) \\ &= \sum_{i=0}^{d+1} (-1)^i \sum_{0\leq i_0<\cdots<i_{d-i}\leq d} \prod_{l=0}^{d-i} n_{i_l} = \prod_{i=0}^{d} (n_i - 1).\end{aligned}$$

Thus, $\mathcal{Z}$ generates all of $Z_d(\Lambda^d_{n_0,\ldots,n_d})$. ∎

6.2. A wealth of coboundaries. The following proposition provides us with a wealth of coboundaries.

Proposition 28: *Let $d \in \mathbb{Z}_{>0}$ be a dimension. Let $n_0, n_1, \ldots, n_d \in \mathbb{Z}$ with $n_i \geq 2$ for all $0 \leq i \leq d$. Let $X = \Lambda^d_{n_0,\ldots,n_d}$. Given $\varphi\colon X(0) \to \mathbb{F}_2^d$ define $c^\varphi \in C^d(X)$ by*

$$c^\varphi(\{v_0, \ldots, v_d\}) := \begin{cases} 1 & \text{if } \sum_{i=0}^d \varphi(v_i) = 0 \in \mathbb{F}_2^d, \\ 0 & \text{otherwise.} \end{cases}$$

Then c^φ is a coboundary, i.e., $c^\varphi \in B^d(X)$.

Proof. Write $X = V_0 * \cdots * V_d$ with $V_i = [n_i]$. By Lemma 26 and Lemma 27 it suffices to check that for every collection of pairs $\{u_i^+, u_i^-\} \in \binom{V_i}{2}, 0 \leq i \leq d$, the crosspolytope $\Diamond^d = \{u_0^+, u_0^-\} * \cdots * \{u_d^+, u_d^-\}$ contains an even number of d-simplices from c^φ.

So let us fix a choice of pairs $\{u_i^+, u_i^-\} \in \binom{V_i}{2}$, $0 \le i \le d$, and consider the corresponding crosspolytope $\diamond^d = \{u_0^+, u_0^-\} * \cdots * \{u_d^+, u_d^-\}$. First we reduce to the case when

$$\varphi(u_0^+) = \varphi(u_1^+) = \cdots = \varphi(u_d^+) = 0.$$

If $\diamond^d$ does not contain a d-simplex from c^φ, we are done. Otherwise we can assume (after relabeling the vertices in $\diamond^d$) that

$$\sum_{i=0}^{d} \varphi(u_i^+) = 0.$$

Now consider $\tilde{\varphi}\colon X(0) \to \mathbb{F}_2^d$ given by

$$\tilde{\varphi}(v_i) = \varphi(v_i) + \varphi(u_i^+)$$

for any $v_i \in V_i$, $0 \le i \le d$. Note that $\sum_{i=0}^{d} \varphi(u_i^+) = 0$ implies that $c^{\tilde{\varphi}} = c^\varphi$. Moreover $\tilde{\varphi}(u_i^+) = 0$ for all $0 \le i \le d$ by construction. Hence, we are left with the case when $\varphi(u_i^+) = 0$ for all $0 \le i \le d$. In this case there is a one-to-one correspondence between d-simplices in $\diamond^d$ from c^φ and vectors $(\alpha_0, \ldots, \alpha_d) \in \mathbb{F}_2^{d+1}$ for which

$$\sum_{i=0}^{d} \alpha_i \varphi(u_i^-) = 0.$$

The number of such vectors equals $2^{\dim\ker A}$ where $A \in \mathbb{F}_2^{d\times(d+1)}$ is the matrix with columns $\varphi(u_0^-), \ldots, \varphi(u_d^-)$. Note that $\dim\ker A \ge 1$ (we consider linear dependencies of $d+1$ vectors in the d-dimensional vector space $\mathbb{F}_2^d$), hence $2^{\dim\ker A}$ is even. This finishes the proof. ∎

6.3. Upper bound for the spherical building $A_d(\mathbb{F}_q)$. In this subsection we prove Theorem 5. Recall that for a given prime power q, $A_d(\mathbb{F}_q)$ is the d-dimensional simplicial complex whose vertices correspond to non-trivial, proper subspaces of a $(d+2)$-dimensional vector space $\mathbb{F}_q^{d+2}$ over the finite field $\mathbb{F}_q$ with q elements and whose k-simplices correspond to flags $0 \ne U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_k \subsetneq \mathbb{F}_q^{d+2}$ of subspaces.

We will need a few very basic combinatorial properties of $A_d(\mathbb{F}_q)$. To state them, we need some notation. For $k \ge 1$ let

$$[k]_q = \sum_{i=0}^{k-1} q^i \quad \text{and} \quad [k]_q! = [k]_q \cdot [k-1]_q \cdot \cdots \cdot [1]_q.$$

We have

LEMMA 29: *Let $d \in \mathbb{N}$, q a prime power.*

(i) $|A_d(\mathbb{F}_q)(d)| = [d+2]_q!$.
(ii) *The number $N_{l,d,q}$ of l-dimensional subspaces of $\mathbb{F}_q^d$ is $\frac{[d]_q!}{[l]_q![d-l]_q!}$.*

Proof. For (i) we note that picking a d-simplex in $A_d(\mathbb{F}_q)$ corresponds to picking a chain of non-trivial subspaces $0 \neq U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_d \subsetneq \mathbb{F}_q^{d+2}$. The 1-dimensional space U_0 is spanned by a non-zero vector of $\mathbb{F}_q^{d+2}$. There are $q^{d+2}-1$ non-zero vectors in $\mathbb{F}_q^{d+2}$ but each 1-dimensional space U_0 contains $q-1$ of them. Hence, there are $\frac{q^{d+2}-1}{q-1} = [d+2]_q$ choices for U_0. Once we have chosen U_0, we can choose U_1 by choosing any vector in $\mathbb{F}_q^{d+2} \setminus U_0$. There are $q^{d+2}-q$ of such vectors and any 2-dimensional subspace of $\mathbb{F}_q^{d+2}$ containing U_0 has q^2-q vectors not in U_0. Hence, there are $\frac{q^{d+2}-q}{q^2-q} = \frac{q^{d+1}-1}{q-1} = [d+1]_q$ choices for U_1 containing a fixed U_0. Continuing this argument, we see that given $U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_k$ there are $[d+1-k]_q$ choices for U_{k+1}. The expression for $|A_d(\mathbb{F}_q)(d)|$ follows.

For (ii) we note that $N_{l,d,q}$ is equal to the number of ordered l-tuples of linear independent vectors in $\mathbb{F}_q^d$ divided by the number of ordered basis in $\mathbb{F}_q^l$. Note that the number of ordered k-tuples of linear independent vectors in $\mathbb{F}_q^n$ is given by $(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})$. Indeed, for the first vector we have to pick a non-zero vector. There are q^n-1 choices. For the second vector we can choose any of the q^n-q in the complement of the span of the first vector and so on. It follows that

$$N_{l,d,q} = \frac{(q^d-1)(q^d-q)\cdots(q^d-q^l)}{(q^l-1)(q^l-q)\cdots(q^l-q^{l-1})} = \frac{(q^d-1)(q^{d-1}-1)\cdots(q^{d-l}-1)}{(q^l-1)(q^{l-1}-1)\cdots(q-1)}.$$

Now, every factor in the numerator and denominator is divisible by $q-1$. Dividing out these factors, we get

$$N_{l,d,q} = \frac{[d]_q[d-1]_q\cdots[d-l]_q}{[l]_q[l-1]_q\cdots[1]_q} = \frac{[d]_q!}{[d-l]_q![l]_q!},$$

as desired. ∎

Note that every $(d-1)$-simplex of $A_d(\mathbb{F}_q)$ is contained in precisely $q+1$ (the number of 1-dimensional subspaces in a 2-dimensional vector space over $\mathbb{F}_q$) d-simplices. Thus, using Lemma 29(i),

$$|A_d(\mathbb{F}_q)(d-1)| = \frac{d+1}{q+1}[d+2]_q!$$

which is a polynomial in q with leading term $(d+1)q^{\frac{d(d+3)}{2}}$. In particular, for sufficiently large q ($q \geq (d+2)!$ suffices) we have

$$|A_d(\mathbb{F}_q)(d-1)| \leq 2(d+1)q^{\frac{d(d+3)}{2}}.$$

Note that the map $\lambda\colon A_d(\mathbb{F}_q)(0) \to \{1, 2, \ldots, d+1\}$ given by

$$U \mapsto \lambda(U) := \dim(U)$$

is a labeling of the vertices of $A_d(\mathbb{F}_q)$ showing that $A_d(\mathbb{F}_q)$ is $(d+1)$-partite. In particular, there is an embedding $\iota\colon A_d(\mathbb{F}_q) \to \Lambda^d_{n_0,\ldots,n_d}$ where, according to Lemma 29(ii), $n_k = \frac{[d+2]_q!}{[k+1]_q![d+1-k]_q!}$ is the number of $(k+1)$-dimensional subspaces of $\mathbb{F}_q^{d+2}$.

Outline of proof of Theorem 5. We first observe that since the restriction of a coboundary to a subcomplex is a coboundary, Proposition 28 also provides a wealth of coboundaries in $A_d(\mathbb{F}_q)$.

Corollary 30: *Let $\varphi\colon A_d(\mathbb{F}_q)(0) \to \mathbb{F}_2^d$. Let $c^\varphi \in C^d(A_d(\mathbb{F}_q))$ be given by*

$$c^\varphi(\{u_0, \ldots, u_d\}) = \begin{cases} 1 & \text{if } \sum_{i=0}^d \varphi(u_i) = 0 \in \mathbb{F}_2^d, \\ 0 & \text{otherwise.} \end{cases}$$

Then c^φ is a coboundary, i.e., $c^\varphi \in B^d(A_d(\mathbb{F}_q))$.

Now the idea is to pick φ uniformly at random and consider c^φ. That is for every vertex $v \in A_d(\mathbb{F}_q)(0)$ we choose $\varphi(v) \in \mathbb{F}_2^d$ independently and uniformly at random. It will turn out that as $q \to +\infty$, with positive probability, there is a coboundary $b = c^\varphi$ for which every $(d-1)$-simplex in $A_d(\mathbb{F}_q)$ is contained in at most $\frac{q+1}{2^d} + o(q)$ d-simplices of b. Thus, every cofilling c of b must satisfy

$$\Big(\frac{q+1}{2^d} + o(q)\Big)|c| \geq |b|$$

giving us a cochain $c \in C^{d-1}(A_d(\mathbb{F}_q))$ for which

$$\frac{|\delta c|}{|[c]|} \leq \frac{q+1}{2^d} + o(q).$$

Normalizing, we get

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \leq \frac{d+1}{q+1}\Big(\frac{q+1}{2^d} + o(q)\Big) = \frac{d+1}{2^d} + o(1)$$

as $q \to +\infty$.

Proof of Theorem 5. We add some more details to the proof outline above. To this end, let $(\Omega, \mathcal{B}, \mathbb{P})$ be the probability space with $\Omega = (\mathbb{F}_2^d)^{A_d(\mathbb{F}_q)(0)}$, i.e., the set of maps $\varphi\colon A_d(\mathbb{F}_q)(0) \to \mathbb{F}_2^d$, $\mathcal{B} = 2^\Omega$ and $\mathbb{P}$ the uniform distribution. For $\omega \in \Omega$ we let

$$b(\omega) := c^\omega \in B^d(A_d(\mathbb{F}_q))$$

as defined in Corollary 30. For $\tau \in A_d(\mathbb{F}_q)(d)$ let $b^{(\tau)}\colon \Omega \to \mathbb{R}$ be given by

$$b^{(\tau)}(\omega) := \begin{cases} 1 & \text{if } b(\omega)(\tau) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $\sigma \in A_d(\mathbb{F}_q))(d-1)$ let $d^{(\sigma)}\colon \Omega \to \mathbb{R}$ be given by

$$d^{(\sigma)}(\omega) := \sum_{\tau \in A_d(\mathbb{F}_q)(d), \sigma \subseteq \tau} b^{(\tau)}(\omega),$$

i.e., $d^{(\sigma)}(\omega)$ is the number of d-simplices incident to σ which are contained in $b(\omega)$. We have

Lemma 31: (i) $\mathbb{P}(b^{(\tau)} = 1) = \mathbb{E}[b^{(\tau)}] = \frac{1}{2^d}$ *for all* $\tau \in A_d(\mathbb{F}_q)(d)$.
(ii) $\mathbb{E}[d^{(\sigma)}] = \frac{q+1}{2^d}$ *for all* $\sigma \in A_d(\mathbb{F}_q)(d-1)$.

Proof. (i) follows from the fact that for any fixed $a_0, a_1, \ldots, a_{d-1} \in \mathbb{F}_2^d$ the equation

$$a_0 + a_1 + \cdots + a_{d-1} + x = 0$$

has precisely one solution for x. (ii) then follows from (i) by linearity of expectation using that every $(d-1)$-simplex of $A_d(\mathbb{F}_q)$ is contained in exactly $q+1$ d-simplices. ∎

The following observation is crucial, since it will allow us to use Hoeffding's inequality for $d^{(\sigma)}$.

Lemma 32: *Fix* $\sigma \in A_d(\mathbb{F}_q)(d-1)$. *Let* $\tau_1, \ldots, \tau_{q+1}$ *be the* $q+1$ *d-simplices incident to* σ. *Then the random variables* $b^{(\tau_1)}, \ldots, b^{(\tau_{q+1})}$ *are independent.*

Proof. Let $\sigma = \{v_0, \ldots, v_{d-1}\}$. When randomly picking $\varphi\colon A_d(\mathbb{F}_q)(0) \to \mathbb{F}_2^d$ we can think that the values of φ on the vertices of σ have already been picked. Then the value of $b^{(\tau_i)}$ solely depends on the choice of φ on the remaining vertex $v \in \tau_i \setminus \sigma$. These choices are independent. ∎

Recall Hoeffding's inequality

Theorem 33 (Hoeffding's inequality, [20, Theorem 1]): *Let $X_1, \ldots, X_n$ be $\{0,1\}$-valued independent identically distributed (i.i.d.) random variables with $p = \mathbb{E}X_i$. Then for any $t \geq 0$ we have*

$$\mathbb{P}\bigg(\sum_{i=1}^{n} X_i \geq (p+t)n\bigg) \leq e^{-2t^2 n}.$$

By Lemma 31(i) and Lemma 32, $d^{(\sigma)}$ is a sum of $\{0,1\}$-valued i.i.d. random variables with success probability $p = \frac{1}{2^d}$. Thus, we can apply Hoeffding's inequality to $d^{(\sigma)}$ with

$$n = q+1, \quad p = \frac{1}{2^d}, \quad t = \sqrt{\frac{(d(d+3)+2)\log q}{4(q+1)}}$$

and combine it with a union bound over all $(d-1)$-simplices $\sigma \in A_d(\mathbb{F}_q)(d-1)$ to get (for $q \geq (d+2)!$) that

$$\begin{aligned}\mathbb{P}\Big(\exists \sigma \in A_d(\mathbb{F}_q)(d-1) \text{ with } d^{(\sigma)} \geq \Big(\frac{1}{2^d}+t\Big)(q+1)\Big)&\\ \leq |A_d(\mathbb{F}_q)(d-1)|e^{-2t^2(q+1)}&\\ \leq 2(d+1)q^{\frac{d(d+3)}{2}} e^{-(\frac{d(d+3)}{2}+1)\log q}&\\ = \frac{2(d+1)}{q}.&\end{aligned}$$

For the last inequality we used that $|A_d(\mathbb{F}_q)(d-1)| \leq 2(d+1)q^{\frac{d(d+3)}{2}}$ whenever $q \geq (d+2)!$. In particular, for $q \geq (d+2)!$ there is some $\omega \in \Omega$ such that for all $\sigma \in A_d(\mathbb{F}_q)(d-1)$ it holds that

$$\begin{aligned} d^{(\sigma)}(\omega) &\leq \frac{q+1}{2^d} + (q+1)\sqrt{\frac{(d(d+3)+2)\log q}{4(q+1)}}\\ &= \frac{q+1}{2^d} + \frac{1}{2}\sqrt{(d(d+3)+2)(q+1)\log q}.\end{aligned}$$

As we noticed earlier, this implies that every $c \in C^{d-1}(A_d(\mathbb{F}_q))$ with $\delta c = b(\omega)$ must satisfy

$$\Big(\frac{q+1}{2^d} + \frac{1}{2}\sqrt{(d(d+3)+2)(q+1)\log q}\Big)|c| \geq |b(\omega)|.$$

It follows that

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \leq \frac{d+1}{2^d} + \frac{(d+1)\sqrt{(d(d+3)+2)(q+1)\log q}}{2(q+1)}.$$

Since

$$\lim_{q\to+\infty} \frac{(d+1)\sqrt{(d(d+3)+2)(q+1)\log q}}{2(q+1)} = 0$$

this finishes the proof of Theorem 5. ∎

6.4. Upper bound for complete multipartite complexes. Let $d \in \mathbb{N}$ be a dimension, $n_0, n_1, \ldots, n_d \geq 2$ integers. Recall that we write $\Lambda^d_{n_0,n_1,\ldots,n_d}$ for the complete $(d+1)$-partite d-dimensional complex with parts of sizes $n_0, n_1, \ldots, n_d$. In this subsection we prove the following slightly refined version of Theorem 4.

Theorem 34 (Upper bound on $\eta_{d-1}(\Lambda^d_n)$): *If 2^d divides n_i for all $0 \leq i \leq d$, then*

$$\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d}) \leq \frac{d+1}{2^d}.$$

Moreover, let $\varepsilon > 0$. If $\min\{n_0, \ldots, n_d\} \geq 2^d + \frac{d+1}{\varepsilon}$, then

$$\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d}) \leq \frac{d+1}{2^d} + \varepsilon.$$

Proof. Let $X = \Lambda^d_{n_0,n_1,\ldots,n_d}$. We have

$$X = V_0 * V_1 * \cdots * V_d$$

with $V_i = [n_i]$. Write $n_i = l_i 2^d + r_i$ with $0 \leq r_i < 2^d$, $l_i \in \mathbb{Z}_{\geq 0}$, $0 \leq i \leq d$. Partition $V_i = \bigsqcup_{j=1}^{2^d} V_{ij}$ as equally as possible, i.e., such that $||V_{ij}| - |V_{ij'}|| \leq 1$ for all $j, j' \in \{1, \ldots, 2^d\}$. Let $\psi\colon [2^d] \to \mathbb{F}_2^d$ be a bijection. Define $\varphi\colon X(0) \to \mathbb{F}_2^d$ by $\varphi(v) := \psi(j)$ for $v \in V_{ij}$, $1 \leq j \leq 2^d, 0 \leq i \leq d$. Let $b = c^\varphi \in B^d(X)$ as defined in Proposition 28.

Given $\sigma \in X(d-1)$ there is a unique $i \in \{0, 1, \ldots, d\}$ for which $\sigma \cap V_i = \emptyset$. We call this i the **type** of σ.

First assume that $r_i = 0$ for all $0 \leq i \leq d$, i.e., that 2^d divides n_i for all $0 \leq i \leq d$. Then every $(d-1)$-simplex of type i is contained in exactly l_i d-simplices in the support of b. So, if $c \in C^{d-1}(X)$ with $\delta c = b$ and we decompose $c = \sum_{i=0}^d c^{(i)}$ where $\operatorname{supp}(c^{(i)}) = \{\sigma \in c : \sigma \text{ has type } i\}$, then

$$\sum_{i=0}^{d} l_i |c^{(i)}| \geq |b|.$$

Note that a $(d-1)$-simplex σ of type i has Garland weight $w(\sigma) = \frac{n_i}{(d+1)|X(d)|}$. It follows that

$$\begin{aligned}\|c\| &= \sum_{i=0}^{d} \frac{n_i}{(d+1)|X(d)|}|c^{(i)}| \\ &= \frac{2^d}{(d+1)|X(d)|}\sum_{i=0}^{d} l_i|c^{(i)}| \\ &\geq \frac{2^d}{(d+1)|X(d)|}|b| \\ &= \frac{2^d}{d+1}\|b\|.\end{aligned}$$

This shows that

$$\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d}) \leq \frac{d+1}{2^d},$$

whenever 2^d divides all n_i, $0 \leq i \leq d$.

The second part follows by a similar analysis using that even if not all of the n_i's are divisible by 2^d, we still have that every $(d-1)$-simplex σ of type i is contained in at most $l_i + 1$ d-simplices from b. ∎

7. Concluding remarks

We presented a quantitative Borsuk–Ulam theorem, Theorem 2, and illustrated its potential for applications to quantitative non-embeddability questions by giving a lower bound on the pair-crossing number of sufficiently thick spherical buildings.

At the same time, we noted that for some applications to classical crossing number problems, such as Turán's Brick Factory problem, where we care about *exact* bounds (at least up to lower-order terms), there are limits to the lower bounds we can obtain by applying Theorem 2 in a straightforward way, due to the fact that the expansion constants of the relevant deleted join are strictly less than 1 in dimension 2.

In general, it would be interesting to achieve a better understanding of the expansion constants of the join $X^{*2} = X * X$ of a d-dimensional complex with itself. The Künneth formula guarantees that if $\tilde{H}^k(X) = 0$ for $0 \leq k < d$ then $\tilde{H}^k(X^{*2}) = 0$ for $0 \leq k \leq 2d$. Is there a quantitative version of this?

Question 35: Is it possible to prove lower bounds for the expansion constants $\eta_k(X^{*2})$, $0 \le k \le 2d$, in terms of the expansion constants $\eta_k(X)$, $0 \le k < d$?

At the moment, we do not know the answer even for the case $d = 1$ of graphs. Even this special case would be of interest: A classical lower bound for the crossing number in terms of bisection width [37] implies that for a bounded-degree expander graph X on n vertices, the crossing number of X satisfies $\mathrm{cr}(X) = \Omega(n^2)$. A positive answer to Question 35 would imply that $\mathrm{ipcr}(X) = \Omega(n^2)$ as well, which is an open problem—the best lower bound in the literature seems to be $\mathrm{ipcr}(X) = \Omega(n^2/\log(n)^2)$, due to Kolman and Matoušek [25].[14] Building on the ideas presented in this paper, we can improve this lower bound to $\mathrm{ipcr}(X) = \Omega(n^2/\log n)$, but currently we do not know whether the remaining factor of $1/\log n$ can be removed as well.

We remark that if the answer to Question 35 is positive, the proof might be quite subtle and will not generalize further to joins $X * Y$ of two different complexes: We can construct infinite families of graphs $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ for which

$$\lim_{n\to+\infty} \frac{\eta_2(X_n * Y_n)}{\eta_0(X_n)\eta_0(Y_n)} = 0.$$

These examples are very unbalanced though, in the sense that Y_n has exponentially many more vertices than X_n, so they do not provide a negative answer to Question 35.

Another natural question is to what extent the assumptions in Theorem 2 can be weakened. For instance, for Gromov's Topological Overlap Theorem, the assumption of coboundary expansion can be relaxed to the weaker property of X being a **cosystolic expander**, which means that there are constants $\eta, \theta > 0$ such that the following conditions are satisfied for $0 \le k \le d-1$: For every $\beta \in B^{k+1}(X)$ there exists $\alpha \in C^k(X)$ with $\delta\alpha$ and $\|\alpha\| \le \frac{1}{\eta}\|\beta\|$, and for every $z \in Z^k(X) \setminus B^k(X)$ we have $\|z\| \ge \theta$. It is easy to see that cosystolic expansion is not a suitable replacement for coboundary expansion as an assumption in Theorem 2: For instance, if the coboundary expansion constants of X are bounded away from zero, then the disjoint union $X \sqcup X$ of two copies

[14] We remark that the results in [25] are stated for the *pair crossing number* $\mathrm{pcr}(X)$, which also counts crossings between pairs of edges that share vertices. However, for bounded degree graphs, $\mathrm{pcr}(X) \le \mathrm{ipcr}(X) + O(n)$ (and $\mathrm{pcr}(X) \ge \mathrm{ipcr}(X)$ for all graphs), so the difference does not matter for the present discussion.

of X is a cosystolic expander and a free $\mathbb{Z}/2$-complex (with respect to the action that interchanges the two copies of X), yet at the same time X admits an equivariant map to $\mathbb{R}^1$ without zeros by mapping one copy of X to $+1$ and the other to -1. On the other hand, if $G = (V, E)$ is a graph with a free $\mathbb{Z}/2$-action ν such that every subset $S \subset V$ of vertices with precisely one vertex of each ν-orbit is expanding then $|E \cap f^{-1}(0)|$ has to be large for every equivariant map $f\colon G \to_{\mathbb{Z}/2} \mathbb{R}$. This is a much weaker condition than G to be an expander graph. It would be interesting to find such weaker conditions in higher dimensions, too.

In a different direction, while our upper bounds on $\eta_{d-1}(A_d(\mathbb{F}_q))$ and $\eta_{d-1}(\Lambda_n^d)$ make significant progress in closing the gap between the best known upper and lower bounds of these coboundary expansion constant, finding their exact value remains challenging. We are happy to conjecture that

$$\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d}) \geq \frac{d+1}{2^d}$$

for all $n_0, \ldots, n_d \in \mathbb{Z}_{>0}, d \geq 2$.

Our construction yielding the upper bound on $\eta_{d-1}(\Lambda_n^d)$ can be extended to show that (for n divisible by 2^d) $\eta_k(\Lambda_n^d) \leq \frac{k+2}{2^{\lfloor (k+1)/(d-k) \rfloor}}$. In particular, for constant codimension $d-k$, $\eta_k(\Lambda_n^d)$ is exponentially small in d. The exact value of $\eta_k(\Lambda_n^d)$ for $0 < k < d-1$ seems even more elusive and we do not even dare to make a precise conjecture.

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