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RESEARCH ARTICLE

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Quantitative Steinitz theorem: A polynomial bound

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Abstract

The classical Steinitz theorem states that if the origin belongs to the interior of the convex hull of a set $S \subset \mathbb{R}^d$, then there are at most $2d$ points of S whose convex hull contains the origin in the interior. Bárány, Katchalski, and Pach proved the following quantitative version of Steinitz's theorem. Let Q be a convex polytope in $\mathbb{R}^d$ containing the standard Euclidean unit ball $\mathbf{B}^d$. Then there exist at most $2d$ vertices of Q whose convex hull Q' satisfies

$$r\mathbf{B}^d \subset Q'$$

with $r \geqslant d^{-2d}$. They conjectured that $r \geqslant cd^{-1/2}$ holds with a universal constant $c > 0$. We prove $r \geqslant \frac{1}{5d^2}$, the first polynomial lower bound on r. Furthermore, we show that r is not greater than $\frac{2}{\sqrt{d}}$.

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1 | INTRODUCTION

The goal of this paper is to establish a quantitative version of the following classical result of E. Steinitz [11].

Proposition 1.1 (Steinitz theorem). *Let the origin belong to the interior of the convex hull of a set $S \subset \mathbb{R}^d$. Then there are at most $2d$ points of S whose convex hull contains the origin in the interior.*

The first quantitative version of this result was obtained in [3], where the following statement was proven.

Proposition 1.2 (Quantitative Steinitz theorem). *There exists a constant $r = r(d) > 0$ such that for any subset Q of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball $\mathbf{B}^d$, there exists a subset F of Q of size at most $2d$ whose convex hull contains the ball $r\mathbf{B}^d$.*

It was also shown that $r(d) > d^{-2d}$.

With the exception of the planar case $d = 2$ [2, 6, 10], no significant improvement on $r(d)$ has been obtained (see also [8]).

Now we state the main result of this paper in which we obtain a polynomial bound on $r(d)$.

Theorem 1 (Q.S.T. with polynomial bound). *Let Q be a subset of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball $\mathbf{B}^d$. Then, there exist at most $2d$ points of Q whose convex hull Q' satisfies*

$$\frac{1}{6d^2}\mathbf{B}^d \subset Q'.$$

We conjecture the following.

Conjecture 1.1. *There is a constant $c > 0$ such that in any subset Q of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball $\mathbf{B}^d$, there are at most $2d$ points whose convex hull Q' satisfies*

$$\frac{c}{\sqrt{d}}\mathbf{B}^d \subset Q'.$$

We provide an upper bound on $r(d)$.

Theorem 2. *Let $u_1, \dots, u_n$ be unit vectors in $\mathbb{R}^d$. Then, their absolute convex hull, that is, the convex hull of $\pm u_1, \dots, \pm u_n$, does not contain the ball $(\frac{\sqrt{n}}{d} + \varepsilon)\mathbf{B}^d$ for any positive ε.*

It follows that if $u_1, \dots, u_m$ form a sufficiently dense subset of the unit sphere (with a large m), then their convex hull is almost the unit ball, whereas for any n of them with $n \leqslant 2d$, we have that their convex hull does not contain the ball $\frac{2}{\sqrt{d}}\mathbf{B}^d$, which shows that the order of magnitude of $r(d)$ in Conjecture 1.1 is sharp if the conjecture holds.

We mention the following conjecture that is closely related to Theorem 2. It can be found in a different formulation in [4, p. 194].

Conjecture 1.2. *Let $\{u_1, \dots, u_{2d}\}$ be unit vectors in $\mathbb{R}^d$. Then there is a point in the set*

$$\bigcap_{i=1}^{2d}\{x \in \mathbb{R}^d \,:\, \langle u_i, x\rangle \leqslant 1\}$$

with norm $\sqrt{d}$.

2 | THE MAIN STEPS IN THE PROOF OF THEOREM 1

Since $r(1) = 1$, we will assume that $d \geqslant 2$ throughout the paper.

First, we reduce the problem to the polytopal case. By the classical Carathéodory theorem [7, p. 200], any point of a convex hull of a subset Q of $\mathbb{R}^d$ can be represented as a convex combination of at most $d+1$ points of Q. Thus, taking a sufficiently dense subset of the unit sphere, we observe that for any $\epsilon \in (0,1)$ and any set $Q \subset \mathbb{R}^d$ whose convex hull contains $\mathbf{B}^d$, there is a finite subset Q_f of Q whose convex hull contains the ball $(1-\epsilon)\mathbf{B}^d$. Hence, Theorem 1 follows from the following polytopal version.

Theorem 3. *Let Q be a convex polytope in $\mathbb{R}^d$ containing the Euclidean unit ball $\mathbf{B}^d$. Then there are at most $2d$ vertices of Q whose convex hull Q' satisfies*

$$\frac{1}{5d^2}\mathbf{B}^d \subset Q'.$$

Proposition 1.2 was used in [3] to prove certain quantitative versions of the Helly theorem. The connection between the quantitative Steinitz result and the quantitative Helly-type result is via polar duality. Recently, the authors of this paper [9] have proposed a new approach to quantitative Helly-type results via sparse approximation of polytopes. The connection between the sparse approximation of polytopes and the quantitative Helly-type result is via polar duality again. We state a refined version of the result on the sparse approximation of polytopes obtained by Almendra–Hernández, Ambrus, and Kendall in [1, Theorem 1].

Proposition 2.1 (Almendra–Hernández et. al.). *Let $\lambda > 0$, and $L \subset \mathbb{R}^d$ be a convex polytope such that $L \subset -\lambda L$. Then, there exist at most $2d$ vertices of L whose convex hull L' satisfies*

$$L \subset -(\lambda+2)d \cdot L'.$$

Choosing the origin smartly, one can achieve $\lambda = d$. For instance, the following statement holds.

Proposition 2.2. *Let K be a convex body (i.e., a compact convex set with nonempty interior) in $\mathbb{R}^d$. Then, the inclusion $(K-c) \subset -d(K-c)$ holds for some point c in the interior of K, for example, if c is the centroid of K or of a maximal volume simplex within K.*

We recall that the *polar* of a set $S \subset \mathbb{R}^d$ is defined by

$$S^\circ = \left\{x \in \mathbb{R}^d \,:\, \langle x, s\rangle \leqslant 1 \quad \text{for all} \quad s \in S\right\}.$$

Our idea of the proof of Theorem 3 is to use duality twice: We will start with translating the assertion of the theorem in terms of the polar polytope Q° of Q. Then, we will choose a point c "deep" in Q° and consider $(Q^\circ - c)^\circ$. Roughly speaking, by changing the center of polarity, we obtain a more well-structured convex polytope. Next, we use Proposition 2.1 to obtain a sufficiently reasonable bound on $r(d)$, which is not destroyed on the way back to Q° and then to Q.

We use $[n]$ to denote the sets $\{1, \ldots, n\}$. The convex hull of a set S is denoted by $\operatorname{conv} S$. For a nonzero vector $v \in \mathbb{R}^d$, H_v denotes the half-space

$$H_v = \left\{x \in \mathbb{R}^d \;:\; \langle x, v\rangle \leqslant 1\right\}.$$

We use $\operatorname{vert} P$ to denote the vertex set of a polytope P.

For the sake of completeness, we provide a shortened original proof of Proposition 2.1.

Proof of Proposition 2.1. Without loss of generality, we may assume that the interior of L is nonempty. The condition $L \subseteq -\lambda L$ ensures that the origin belongs to the interior of L. Among all simplices with d vertices from the set of vertices of L and one vertex at the origin, consider a simplex $S = \operatorname{conv}\{0, v_1, \ldots, v_d\}$ with maximal volume. The simplex S can be represented as

$$S = \left\{x \in \mathbb{R}^d \;:\; x = \alpha_1 v_1 + \cdots + \alpha_d v_d \quad \text{for } \alpha_i \geqslant 0 \text{ and } \sum_{i=1}^{d} \alpha_i \leqslant 1\right\}. \tag{1}$$

Define $P = \sum_{i\in[d]} [-v_i, v_i]$. It is easy to see that P is a paralletope that can be represented as

$$P = \{x \in \mathbb{R}^d \;:\; x = \beta_1 v_1 + \cdots + \beta_d v_d \quad \text{for} \beta_i \in [-1, 1]\}. \tag{2}$$

Since S is chosen maximally, Equation (2) shows that for any vertex v of L, $v \in P$. By convexity,

$$L \subset P. \tag{3}$$

Let $S' = -2dS + (v_1 + \cdots + v_d)$. By (1),

$$S' = \left\{x \in \mathbb{R}^d \;:\; x = \gamma_1 v_1 + \cdots + \gamma_d v_d \quad \text{for} \gamma_i \leqslant 1 \text{ and } \sum_{i=1}^{d} \gamma_i \geqslant -d\right\},$$

which, together with (2), yields

$$P \subseteq S'. \tag{4}$$

Let y be the intersection of the ray emanating from 0 in the direction $-(v_1 + \cdots + v_d)$ and the boundary of L. By Carathéodory's theorem, we can choose $k \leqslant d$ vertices $\{v_1', \ldots, v_k'\}$ of L such that $y \in \operatorname{conv}\{v_1', \ldots, v_k'\}$. Set $L' = \operatorname{conv}\{v_1, \ldots, v_d, v_1', \ldots, v_k'\}$. Clearly, $\frac{v_1+\cdots+v_d}{d} \in S \subset L$. Thus, $0 \in L'$, and consequently,

$$S \subseteq L'. \tag{5}$$

Since $L \subset -\lambda L$, we also have that

$$\frac{v_1 + \cdots + v_d}{d} \in -\lambda[y, 0] \subset -\lambda L'.$$

Combining it with (3)–(5), we obtain

$$L \subset P \subset S' = -2dS + (v_1 + \cdots + v_d) \subset -2d\,L' - \lambda d\,L' = -(\lambda + 2)d\,L', \tag{6}$$

completing the proof of Proposition 2.1. □

3 | PROOF OF THEOREM 1

As was explained in the previous section, it suffices to prove Theorem 3, which we proceed to work with.

Set $K = Q^\circ$. Since $Q \supset \mathbf{B}^d$, $K \subset \mathbf{B}^d$. Also, it is easy to see that K is a convex polytope of the form

$$K = \bigcap_{v \in \operatorname{vert} Q} H_v, \tag{7}$$

containing the origin in its interior. By duality, it suffices to show that there are at most $2d$ half-spaces H_v with $v \in \operatorname{vert} Q$, whose intersection is contained in the ball $5d^2\mathbf{B}^d$.

Let c be a point in the interior of K such that the inclusion

$$K - c \subset -d(K - c)$$

holds. The existence of c follows from Proposition 2.2. Set $L = (K - c)^\circ$. Clearly,

$$L \subset -dL.$$

Now, we use Proposition 2.1 with $\lambda = d$. We obtain that there are $w_1, \ldots, w_m \in \operatorname{vert} L$ for some integer m satisfying $m \leqslant 2d$ such that

$$L \subset -(d+2)d \cdot \operatorname{conv}\{w_i \;:\; i \in [m]\}.$$

Since $c \in K \subset \mathbf{B}^d$, one has that $K - c \subset 2\mathbf{B}^d$. Consequently, $L \supset \frac{1}{2}\mathbf{B}^d$. So,

$$\frac{1}{2}\mathbf{B}^d \subset L \subset -(d+2)d \cdot \operatorname{conv}\{w_i \;:\; i \in [m]\}.$$

Considering the polar sets, we get

$$(\operatorname{conv}\{w_i \;:\; i \in [m]\})^\circ \subset 2(d+2)d\mathbf{B}^d.$$

Recall that c is an interior point of the polytope K. By (7), one has that for any $w \in \operatorname{vert} L$, $H_w = H_v - c$ for some $v \in \operatorname{vert} Q$. It means that

$$(\operatorname{conv}\{w_i \;:\; i \in [m]\})^\circ = \bigcap_{i \in [m]} \left(H_{v_i} - c\right)$$

for corresponding $v_i \in \operatorname{vert} Q$. Thus,

$$\bigcap_{i\in[m]} H_{v_i} = \bigcap_{i\in[m]} \left(H_{v_i} - c\right) + c \subset 2(d+2)d\mathbf{B}^d + c \subset (2(d+2)d+1)\mathbf{B}^d.$$

Since $d \geqslant 2$, the desired bound for $Q' = \operatorname{conv}\{v_i \,:\, i \in [m]\}$ follows. The proof of Theorem 3 is complete, which implies Theorem 1 as was discussed earlier.

4 | PROOF OF THEOREM 2

In this section, we prove Theorem 2, which is a dual version of [9, Theorem 1.4] and immediately follows from it. For the sake of completeness, we prove Theorem 2 here. We first state the main ingredient of the proof obtained by K. Ball and M. Prodromou.

Proposition 4.1 [5, Theorem 1.4]. *Let vectors* $\{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ *satisfy* $\sum_1^n v_i \otimes v_i = \mathrm{Id}$. *Then for any positive semidefinite operator* $T : \mathbb{R}^d \to \mathbb{R}^d$, *there is a point* p *in the intersection of the strips* $\{x \in \mathbb{R}^d \,:\, |\langle x, v_i\rangle| \leqslant 1\}$ *satisfying* $\langle p, Tp\rangle \geqslant \operatorname{trace} T$.

Proof of Theorem 2. There is nothing to prove if the absolute convex hull $\operatorname{conv}\{\pm u_i \,:\, i \in [n]\}$ does not contain the origin in its interior. So, assume that $\operatorname{conv}\{\pm u_i \,:\, i \in [n]\}$ contains the origin in its interior. Set $K = (\operatorname{conv}\{\pm u_i \,:\, i \in [n]\})^\circ$. By duality, it suffices to show that K contains a point of Euclidean norm $\frac{d}{\sqrt{n}}$.

Clearly, $\{u_i \,:\, i \in [n]\}$ spans $\mathbb{R}^d$. Consider $A = \sum_{i\in[n]} u_i \otimes u_i$. Since the vectors span the space, A is positive definite. Using Proposition 4.1 with $v_i = A^{-1/2}u_i, i \in [n]$, and $T = A^{-1}$, we find a point p in

$$\bigcap_{i\in[n]} \{x \,:\, |\langle v_i, x\rangle| \leqslant 1\}.$$

such that

$$\left\langle p, A^{-1}p\right\rangle \geqslant \operatorname{trace} A^{-1}.$$

Denote $q = A^{-1/2}p$. Then, by the choice of p,

$$1 \geqslant \left|\left\langle p, A^{-1/2}u_i\right\rangle\right| = \left|\left\langle A^{-1/2}p, u_i\right\rangle\right| = |\langle q, u_i\rangle|.$$

That is, $q \in K$. On the other hand,

$$|q|^2 = \left\langle A^{-1/2}p, A^{-1/2}p\right\rangle = \left\langle p, A^{-1}p\right\rangle \geqslant \operatorname{trace} A^{-1}.$$

Finally, since $\operatorname{trace} A = n$ and by the Cauchy–Schwarz inequality, one sees that $\operatorname{trace} A^{-1}$ is at least $\frac{d^2}{n}$. Thus, $|q| \geqslant \frac{d}{\sqrt{n}}$. This completes the proof of Theorem 2. □

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