

# Absence of excited eigenvalues for Fröhlich type polaron models at weak coupling

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**Abstract.** We consider a class of polaron models, including the Fröhlich model, at zero total momentum, and show that at sufficiently weak coupling there are no excited eigenvalues below the essential spectrum.

## 1. Main result

We consider general polaron models of the form

$$H = P^2 + \Phi(v) + \mathbb{N} \tag{1.1}$$

acting on the (bosonic) Fock space  $\mathcal{F}(L^2(\mathbb{R}^d))$  for  $d \geq 1$ , with  $P = \int_{\mathbb{R}^d} k a_k^\dagger a_k dk$  the field momentum,  $\mathbb{N}$  the number operator and  $\Phi(v) = a(v) + a^\dagger(v)$ . The Hamiltonian  $H$  in (1.1) arises as the restriction of the usual polaron models (describing an electron coupled to a phonon quantum field) to total momentum zero [10, 13]. The most studied model of this kind is presumably the Fröhlich model [2], corresponding to  $d = 3$  and  $v_k = g|k|^{-1}$  for some  $g \in \mathbb{R}$ . We adopt the standard notation that  $a^\dagger(v) = \int_{\mathbb{R}^d} a_k^\dagger v_k dk$ , with the canonical commutation relations  $[a_k, a_l^\dagger] = \delta(k - l)$ .

We shall assume that  $v$  is even, i.e.,  $v_k = v_{-k}$ , and that  $(1 + |\cdot|)^{-1}v \in L^2(\mathbb{R}^d)$ , which in particular ensures that  $\Phi(v)$  is infinitesimally form-bounded with respect to  $P^2 + \mathbb{N}$ , hence  $H$  is well defined through its quadratic form and is bounded from below [3, 6, 7, 12]. We shall actually make the slightly stronger assumption that

$$\|v\| := \sup_{k \in \mathbb{R}^d} \|(1 + |\cdot - k|)^{-1}v\| < \infty$$

(with  $\|\cdot\|$  denoting the  $L^2(\mathbb{R}^d)$  norm).

In the following, we shall write  $v = gw$  with  $g \geq 0$  and  $w$  fixed, and study the spectrum of  $H$  for small  $g$ . Let  $E_0$  denote the ground state energy of  $H$ . It is well

known that the essential spectrum of  $H$  equals  $[E_0 + 1, \infty)$  [5, 10]. Our main result is as follows.

**Theorem 1.1.** *There exists a  $g_0 > 0$  such that for  $0 \leq g < g_0$ ,  $H$  has only one eigenvalue below its essential spectrum. In particular, the spectrum of  $H$  equals  $\sigma(H) = \{E_0\} \cup [E_0 + 1, \infty)$ .*

The proof given below shows that the smallness condition can be quantified in terms of  $\|v\|$ . In other words,  $g_0 \geq C \|v\|^{-1}$  for some universal constant  $C > 0$  (see Remark 2.4 at the end of the next Section).

In the infrared regular case when  $|\cdot|^{-1}v \in L^2(\mathbb{R}^d)$ , the corresponding result in Theorem 1.1 is much easier to obtain via perturbation theory and is actually known (see [1, 8]). The smallness condition depends on  $\| |\cdot|^{-1}v \|$ , however, and hence the general result cannot be obtained via a limiting argument. Our main contribution thus concerns the infrared singular case when  $|\cdot|^{-1}v \notin L^2(\mathbb{R}^d)$ , which is in particular the case for the usual Fröhlich model. The proof shows that in this case the result is non-perturbative in a certain sense, to be made precise in Remark 2.2 below. In fact, the relevant Birman–Schwinger eigenvalue (whose negativity would imply the existence of an excited eigenvalue) turns out to be identically zero, hence obtaining it only to a finite order in  $g$  would not allow to draw a conclusion.

In the case of the Fröhlich model, it was recently shown in [9] that excited eigenvalues do appear below the essential spectrum for larger values of  $g$ . In fact, their number goes to infinity as  $g \rightarrow \infty$ . Our result thus complements that work by proving that a minimal threshold on the coupling constant is needed for the existence of excited eigenstates.

We expect that a result as in Theorem 1.1 holds also for non-zero total momentum, where  $P$  in (1.1) has to be replaced by  $P - \xi$  for  $\xi \in \mathbb{R}^d$ , but our proof does not extend to the case  $\xi \neq 0$  in an obvious way. As shown in [1], it is possible to prove the absence of excited eigenvalues outside a small window  $|\xi| \leq O(g)$ .

## 2. Proof of Theorem 1.1

In this section we shall prove absence of excited eigenvalues of  $H$  for  $g$  small enough. Before starting the proof, let us introduce some notation. We shall denote by  $\Pi^{\geq n}$  the projection onto the subspace of  $\mathcal{F}$  where  $\mathbb{N} \geq n$ , and also by  $\Pi^n$  the projection onto the  $n$ -particle space where  $\mathbb{N} = n$ .

For convenience, we shall assume that  $v$  is real-valued, which is not a restriction and can always be achieved by a unitary transformation, replacing  $a_k$  by  $e^{i\theta_k} a_k$  for suitable  $\theta_k \in [0, 2\pi)$ .

Throughout the proof we shall assume that  $g$  is suitably small. In particular, we shall assume that  $E_0 > -1$ , and also that  $\nu_2 > 0$ , where we denote

$$\nu_n := \inf \text{spec } \Pi^{\geq n}(H - 1 - E_0)\Pi^{\geq n}$$

(viewed as an operator on  $\Pi^{\geq n}\mathcal{F}$ ). Since  $\nu_2$  is equal to 1 at  $g = 0$ , this is the case for small  $g$  by continuity.

A key tool in our analysis is the Schur complement formula, which we recall here for the convenience of the reader. If  $A$  is a self-adjoint operator and  $P$  is an orthogonal projection such that  $Q A Q$  is invertible on the range of  $Q = 1 - P$ , then the existence of an eigenvalue 0 of  $A$  is equivalent to

$$\hat{A} := P A P - P A Q [Q A Q]^{-1} Q A P$$

having an eigenvalue 0 on the range of  $P$ . In fact, if  $A\psi = 0$ , then  $\hat{A}P\psi = 0$ . On the other hand, if  $\hat{A}\varphi = 0$  with  $P\varphi = \varphi$ , then  $A\psi = 0$  with  $\psi = \varphi - [Q A Q]^{-1} Q A P\varphi$ .

The main strategy of the proof is strongly inspired by the work in [1]. Assume that  $H$  has an eigenvalue  $E_0 + 1 - \varepsilon$  for  $0 < \varepsilon < 1$ . By continuity,  $\varepsilon$  is small if  $g$  is small. In fact, we claim that  $\varepsilon \leq -\nu_1$ , which is a quantity that vanishes as  $g \rightarrow 0$ . Otherwise, if  $\nu_1 + \varepsilon > 0$  we can apply the Schur complement formula to the vacuum sector  $\Pi^0\mathcal{F}$  to obtain the identity

$$\varepsilon - E_0 - 1 = \langle v | [\Pi^{\geq 1}(H - 1 - E_0 + \varepsilon)\Pi^{\geq 1}]^{-1} | v \rangle. \tag{2.1}$$

Since the right side is decreasing in  $\varepsilon$  for  $\varepsilon > -\nu_1$ , there can be only one solution to this equation, given by  $\varepsilon = 1$  and corresponding to the ground state.

We can thus assume that  $\varepsilon$  is small. By the Schur complement formula, applied to the one-particle sector  $\Pi^1\mathcal{F}$ , the existence of an eigenvalue  $E_0 + 1 - \varepsilon$  of  $H$  is equivalent to the operator  $\mathcal{O}^{(\varepsilon)}$  having an eigenvalue 0, where

$$\mathcal{O}^{(\varepsilon)} = \varepsilon + k^2 - E_0 - D^{(\varepsilon)} + \frac{1}{1 + E_0 - \varepsilon} |v\rangle\langle v|$$

acting on the one-particle space  $L^2(\mathbb{R}^d) = \Pi^1\mathcal{F}$ . Here we denote

$$D^{(\varepsilon)} = \Pi^1 a(v) X^{(\varepsilon)} a^\dagger(v) \Pi^1 \quad \text{with } X^{(\varepsilon)} = [\Pi^{\geq 2}(H - E_0 - 1 + \varepsilon)\Pi^{\geq 2}]^{-1}.$$

Note that  $X^{(\varepsilon)}$  is well defined, positive and bounded for  $\varepsilon \geq 0$ , by our assumption that  $\nu_2 > 0$ . Since  $\mathcal{O}^{(\varepsilon)} \geq \varepsilon + \mathcal{O}^{(0)}$ , Theorem 1.1 follows if  $\mathcal{O}^{(0)} \geq 0$ , which we shall show in the following. For simplicity of notation, we shall drop superscripts (0) from now on.

We start by taking a closer look at the structure of  $D = D^{(0)}$ . The canonical commutation relations imply that

$$\begin{aligned} X^{-1}a_k^\dagger &= \Pi^{\geq 2}(H - E_0 - 1)a_k^\dagger \\ &= a_k^\dagger \Pi^{\geq 1}((P + k)^2 + \Phi(v) + \mathbb{N} - E_0) + \Pi^{\geq 2}v_k \\ &= a_k^\dagger Y_k^{-1} + \Pi^{\geq 2}v_k \end{aligned}$$

where, for general  $k \in \mathbb{R}^d$ , we denote

$$Y_k = [\Pi^{\geq 1}((P + k)^2 + \Phi(v) + \mathbb{N} - E_0)\Pi^{\geq 1}]^{-1}.$$

Since  $\inf \text{spec}((P + k)^2 + \Phi(v) + \mathbb{N}) \geq E_0$  for all  $k \in \mathbb{R}^d$  [4,5,11], and the ground state of  $H$  is not orthogonal to the Fock space vacuum,  $Y_k$  is well defined. This leads to the pull-through formula

$$Xa_k^\dagger = a_k^\dagger Y_k - v_k X Y_k. \tag{2.2}$$

Similarly,

$$\begin{aligned} a_l Y_k^{-1} &= a_l((P + k)^2 + \Phi(v) + \mathbb{N} - E_0)\Pi^{\geq 1} \\ &= ((P + k + l)^2 + \Phi(v) + \mathbb{N} + 1 - E_0)a_l + v_l \Pi^{\geq 1} \end{aligned}$$

and hence

$$a_l Y_k = Z_{k+l} a_l - v_l Z_{k+l} Y_k \tag{2.3}$$

where we denote

$$Z_k = [(P + k)^2 + \Phi(v) + \mathbb{N} + 1 - E_0]^{-1}.$$

With  $\Omega \in \mathcal{F}$  denoting the vacuum vector, the kernel of  $D$  can be expressed as

$$D(k, l) = \langle \Omega | a_k a(v) X a^\dagger(v) a_l^\dagger | \Omega \rangle = \langle v | a_k X a_l^\dagger | v \rangle.$$

With the identities (2.2) and (2.3) above, we have

$$\begin{aligned} a_k X a_l^\dagger &= a_k (a_l^\dagger Y_l - v_l X Y_l) \\ &= \delta(k - l) Y_l + a_l^\dagger a_k Y_l - v_l (Y_k a_k - v_k Y_k X) Y_l \\ &= \delta(k - l) Y_l + a_l^\dagger (Z_{k+l} a_k - v_k Z_{k+l} Y_l) - v_l (Y_k a_k - v_k Y_k X) Y_l \\ &= \delta(k - l) Y_l + a_l^\dagger Z_{k+l} a_k - v_k a_l^\dagger Z_{k+l} Y_l - v_l Y_k Z_{k+l} a_k \\ &\quad + v_l v_k Y_k Z_{k+l} Y_l + v_l v_k Y_k X Y_l \end{aligned}$$

and hence

$$D(k, l) = \delta(k - l)\langle v|Y_l|v\rangle + v_k v_l \langle \Omega|(1 - a(v)Y_k)Z_{k+l}(1 - Y_l a^\dagger(v))|\Omega\rangle + v_k v_l \langle v|Y_k X Y_l|v\rangle.$$

In the following we shall denote

$$E_k = -\langle v|Y_k|v\rangle$$

for general  $k \in \mathbb{R}^d$ . Note that indeed  $E_0 = -\langle v|Y_0|v\rangle$  equals the ground state energy of  $H$ , again by the Schur complement formula ((2.1) for  $\varepsilon = 1$ ). We thus have

$$\mathcal{O} = k^2 + E_k - E_0 + R$$

where  $k^2$  and  $E_k$  are understood as multiplication operators, and  $R$  is the operator with integral kernel  $R(k, l) = v_k v_l C(k, l)$  with

$$C(k, l) = \frac{1}{1 + E_0} - \langle \Omega|(1 - a(v)Y_k)Z_{k+l}(1 - Y_l a^\dagger(v))|\Omega\rangle - \langle v|Y_k X Y_l|v\rangle. \tag{2.4}$$

We shall further need that  $|E_k - E_0| \leq C g^2 |k|^2$  for small  $g$  and a suitable constant  $C > 0$ . This can easily be proved using the resolvent identity in the form  $Y_0 - Y_k = Y_0(k^2 + 2k \cdot P)Y_k$ ; the details are carried out in the appendix.

Of particular relevance will be the constant  $c_0 = C(0, 0)$ , which turns out to be positive, at least for small  $g$ . In fact,

$$c_0 = \langle v|G^2 P^2|v\rangle + O(g^4)$$

where we introduced the notation  $G = (P^2 + 1 - E_0)^{-1}$ . Let us write

$$C(k, l) = c_0 + \psi_k + \psi_l + F(k, l)$$

with  $\psi_k = C(k, 0) - C(0, 0)$ . This leads to the decomposition

$$\begin{aligned} \mathcal{O} &= k^2 + E_k - E_0 + c_0|v\rangle\langle v| + |v\rangle\langle v\psi| + |v\psi\rangle\langle v| + vFv \\ &= k^2 + E_k - E_0 + c_0|v\rangle\langle v| + c_0^{-1}v\psi\rangle\langle v + c_0^{-1}v\psi| - c_0^{-1}|v\psi\rangle\langle v\psi| + vFv \end{aligned} \tag{2.5}$$

where  $vFv$  is short for the operator with integral kernel  $v_k v_l F(k, l)$ .

We shall now distinguish two cases. If  $k \mapsto |k|^{-1}v_k$  is in  $L^2(\mathbb{R}^d)$ , we can argue in a perturbative way and simply take the identity in the first line in (2.5) and write it as

$$\begin{aligned} \mathcal{O} &= |k| \left( 1 + \frac{E_k - E_0}{|k|^2} + c_0|k|^{-1}|v\rangle\langle v||k|^{-1} + |k|^{-1}|v\rangle\langle v\psi||k|^{-1} \right. \\ &\quad \left. + |k|^{-1}|v\psi\rangle\langle v||k|^{-1} + |k|^{-1}vFv|k|^{-1} \right) |k|. \end{aligned}$$

Since all the terms besides 1 in the parentheses are bounded and  $O(g^2)$ , one readily deduces that  $\mathcal{O} \geq 0$  for  $g$  small enough.

We can thus assume from now on that  $k \mapsto |k|^{-1}v_k$  is not in  $L^2(\mathbb{R}^d)$ . As long as  $c_0 > 0$  we can drop the first rank-one projection in the second line of (2.5) for a lower bound, and obtain  $\mathcal{O} \geq |k|\mathcal{S}|k|$  with

$$\mathcal{S} = 1 + \frac{E_k - E_0}{|k|^2} + |k|^{-1}vFv|k|^{-1} - |\varphi\rangle\langle\varphi| =: 1 + A - |\varphi\rangle\langle\varphi|$$

where  $\varphi_k = c_0^{-1/2}|k|^{-1}v_k\psi_k$ . Note that  $\varphi \in L^2(\mathbb{R}^d)$  even if  $k \mapsto |k|^{-1}v_k$  is not, since  $\psi_k$  vanishes at  $k = 0$  at least linearly. For the same reason,  $A$  is bounded. In fact,  $\|\varphi\| = O(1)$  while  $\|A\| = O(g^2)$ . This can easily be shown by controlling the derivative of  $k \mapsto C(k, l)$ ; the details are carried out in the appendix.

For  $g$  small enough we can thus further write

$$\mathcal{S} = \sqrt{1 + A}(1 - (1 + A)^{-1/2}|\varphi\rangle\langle\varphi|(1 + A)^{-1/2})\sqrt{1 + A}$$

and positivity of  $\mathcal{S}$  is equivalent to the bound  $\|(1 + A)^{-1/2}\varphi\| \leq 1$ . Remarkably, this latter norm is identically equal to 1, as shown in the following Lemma.

**Lemma 2.1.** *For small  $g$  we have*

$$\langle\varphi|(1 + A)^{-1}|\varphi\rangle = 1.$$

This readily implies that  $\inf \text{spec } \mathcal{S} = 0$ , and thus with the above proves Theorem 1.1.

**Remark 2.2.** It was already observed in [1] that

$$\lim_{g \rightarrow 0} \|(1 + A)^{-1/2}\varphi\| = \lim_{g \rightarrow 0} \|\varphi\| = 1$$

and thus that a perturbative investigation would require going to higher order in  $g$ . Lemma 2.1 shows that such a perturbative strategy is bound to fail, however, as all higher order terms in  $g$  vanish. In this sense, the result is non-perturbative.

We also note that the positive first rank-one projection in the second line of (2.5), which was dropped to obtain the lower bound  $\mathcal{O} \geq |k|\mathcal{S}|k|$ , cannot be used to obtain a stronger lower bound in the infrared singular case  $|\cdot|^{-1}v \notin L^2(\mathbb{R}^d)$ . In fact, since the vector in question, when divided by  $|k|$ , is not in  $L^2(\mathbb{R}^d)$ , one can easily check that

$$\liminf_{\varepsilon \rightarrow 0} \text{spec}(k^2 + \varepsilon)^{-1/2}\mathcal{O}(k^2 + \varepsilon)^{-1/2} = \inf \text{spec } \mathcal{S}$$

so  $\mathcal{S}$  is indeed the relevant Birman–Schwinger operator.

*Proof of Lemma 2.1.* We shall show that

$$\sqrt{c_0}\varphi = \frac{1 + A}{1 + E_0}\Pi^1|P|Y_0|v\rangle \tag{2.6}$$

as well as

$$\sqrt{c_0}\langle\varphi||P|Y_0|v\rangle = (1 + E_0)c_0 \tag{2.7}$$

which together obviously imply the statement.

By definition we have

$$\sqrt{c_0}\varphi_k = \frac{v_k}{|k|}(\lambda_0 - \lambda_k) \tag{2.8}$$

where

$$\lambda_k = \langle\Omega|(1 - a(v)Y_k)Z_k(1 - Y_0a^\dagger(v))|\Omega\rangle + \langle v|Y_kXY_0|v\rangle.$$

The key observation is contained in the following lemma.

**Lemma 2.3.**  $v_k\lambda_k = \langle\Omega|a_k(1 + a(v) + D)Y_0|v\rangle.$

As a consequence of the resolvent identity

$$\Pi^1Y_0\Pi^1 = \Pi^1G - \Pi^1Ga(v)Y_0\Pi^1$$

we have

$$\begin{aligned} \langle\Omega|a_k a(v)Y_0|v\rangle &= (k^2 + 1 - E_0)\langle\Omega|a_kGa(v)Y_0|v\rangle \\ &= v_k - (k^2 + 1 - E_0)\langle\Omega|a_kY_0|v\rangle \end{aligned}$$

and hence we obtain from Lemma 2.3 that

$$v_k\lambda_k = v_k - (k^2 - E_0)\langle\Omega|a_kY_0|v\rangle + \langle\Omega|a_kDY_0|v\rangle.$$

The identity

$$\begin{aligned} &\frac{1}{|k|}(k^2 - E_0 - D) \\ &= (1 + A)|k| - \left(\frac{1}{1 + E_0} - c_0\right)\frac{1}{|k|}|v\rangle\langle v| + \frac{\sqrt{c_0}}{|k|}|v\rangle\langle\varphi||k| + \sqrt{c_0}|\varphi\rangle\langle v| \end{aligned}$$

thus implies that

$$\begin{aligned} \sqrt{c_0}\varphi_k &= \frac{v_k}{|k|}\left(\lambda_0 - 1 + \frac{E_0}{1 + E_0} - E_0c_0 + \sqrt{c_0}\langle\varphi||P|Y_0|v\rangle\right) \\ &\quad + \langle\Omega|a_k(1 + A)|P|Y_0|v\rangle - E_0\sqrt{c_0}\varphi_k. \end{aligned} \tag{2.9}$$

Now,  $\varphi \in L^2(\mathbb{R}^d)$  and so is  $\Pi^1|P|Y_0|v\rangle$ , since  $(1 + P^2 + \mathbb{N})^{1/2}Y_0(1 + P^2 + \mathbb{N})^{1/2}$  is a bounded operator. But  $k \mapsto v_k|k|^{-1}$  is not in  $L^2(\mathbb{R}^d)$ , hence the term in parentheses in the first line of (2.9) has to vanish. This in particular implies the first identity (2.6), and also the second in (2.7) since

$$\lambda_0 - 1 + \frac{E_0}{1 + E_0} - E_0c_0 = -c_0(1 + E_0)$$

using that  $c_0 = 1/(1 + E_0) - \lambda_0$ . ■

It remains to give the Proof of Lemma 2.3.

*Proof of Lemma 2.3.* Besides the identities (2.2) and (2.3) we are going to use that

$$Y_k Z_k = Y_k - Z_k - Y_k|v\rangle\langle\Omega|Z_k + |\Omega\rangle\langle\Omega|Z_k \tag{2.10}$$

as well as

$$XY_0 = X - Y_0 + \Pi^1Y_0 - Xa^\dagger(v)\Pi^1Y_0 \tag{2.11}$$

which can easily be obtained by evaluating the differences  $X^{-1} - Y_0^{-1}$  and  $Y_k^{-1} - Z_k^{-1}$ , respectively. With the aid of (2.3) and (2.10), we have (using that  $a(v)|\Omega\rangle = 0$ )

$$\begin{aligned} &(1 - a(v)Y_k)Z_k(1 - Y_0a^\dagger(v)) \\ &= Z_k - Z_kY_0a^\dagger(v) - a(v)Y_kZ_k(1 - Y_0a^\dagger(v)) \\ &= Z_k + \frac{1}{v_k}(a_kY_0 - Z_k a_k)a^\dagger(v) - a(v)(Y_k - Z_k - Y_k|v\rangle\langle\Omega|Z_k)(1 - Y_0a^\dagger(v)) \\ &= Z_k + \frac{1}{v_k}(a_kY_0 - Z_k a_k)a^\dagger(v) - a(v)(Y_k - Z_k - Y_k|v\rangle\langle\Omega|Z_k) \\ &\quad + a(v)Y_kY_0a^\dagger(v) + \frac{1}{v_k}a(v)(1 + Y_k|v\rangle\langle\Omega|)(a_kY_0 - Z_k a_k)a^\dagger(v). \end{aligned}$$

Similarly, with (2.11) and (2.2) we have

$$\begin{aligned} &a(v)Y_kXY_0a^\dagger(v) \\ &= a(v)Y_k(X - Y_0 + \Pi^1Y_0 - Xa^\dagger(v)\Pi^1Y_0)a^\dagger(v) \\ &= -\frac{1}{v_k}a(v)(a_kX - Y_k a_k)(1 - a^\dagger(v)\Pi^1Y_0)a^\dagger(v) - a(v)Y_k(Y_0 - \Pi^1Y_0)a^\dagger(v). \end{aligned}$$

Adding up these two identities yields

$$\begin{aligned} &(1 - a(v)Y_k)Z_k(1 - Y_0a^\dagger(v)) + a(v)Y_kXY_0a^\dagger(v) \\ &= Z_k - a(v)(Y_k - Z_k - Y_k|v\rangle\langle\Omega|Z_k) \\ &\quad + \frac{1}{v_k}(1 + a(v)(1 + Y_k|v\rangle\langle\Omega|))(a_kY_0 - Z_k a_k)a^\dagger(v) \\ &\quad - \frac{1}{v_k}a(v)(a_kX - Y_k a_k)(1 - a^\dagger(v)\Pi^1Y_0)a^\dagger(v) + a(v)Y_k\Pi^1Y_0a^\dagger(v). \end{aligned}$$



Taking the vacuum expectation value and using that  $a_k a^\dagger(v) = v_k + a^\dagger(v) a_k$  and  $X \Pi^1 = 0$  as well as  $Y_k \Pi^0 = 0$  one readily checks that

$$\begin{aligned} \lambda_k &= \langle \Omega | (1 - a(v) Y_k) Z_k (1 - Y_0 a^\dagger(v)) + a(v) Y_k X Y_0 a^\dagger(v) | \Omega \rangle \\ &= \frac{1}{v_k} \langle \Omega | a_k (1 + a(v)) Y_0 | v \rangle + \frac{1}{v_k} \langle v | a_k X a^\dagger(v) \Pi^1 Y_0 | v \rangle \end{aligned}$$

as claimed. ■

**Remark 2.4.** One can check that all the smallness conditions assumed, namely  $E_0 > -1$ ,  $v_2 > 0$ ,  $c_0 > 0$  and  $\|A\| < 1$ , can be expressed as a bound on  $\|v\|$ , which quantifies the relative form bound of  $\Phi(v)$  with respect to  $(P + k)^2 + \mathbb{N}$ , uniformly in  $k \in \mathbb{R}^d$  (see the Appendix). This leads to the claimed lower bound on  $g_0$  stated after Theorem 1.1, at least in the infrared singular case when  $|\cdot|^{-1} v \notin L^2(\mathbb{R}^d)$ . To extend this statement to all  $v$ , we shall now give an alternative proof of Lemma 2.1 that equally holds in the infrared regular case.

We start from (2.9) and shall show that the parenthesis in the first line vanishes, even if  $|\cdot|^{-1} v \in L^2(\mathbb{R}^d)$ . By (2.8) and Lemma 2.3, we have

$$\sqrt{c_0} \langle \varphi_0 | P | Y_0 | v \rangle = -\lambda_0 E_0 - \langle v | Y_0 (1 + a^\dagger(v) + D) \Pi^1 Y_0 | v \rangle.$$

Thus, the desired identity (2.7) follows if

$$c_0 = -\frac{E_0}{1 + E_0} - \langle v | Y_0 (1 + a^\dagger(v) + D) \Pi^1 Y_0 | v \rangle. \tag{2.12}$$

In order to show (2.12), we start from (2.4) and observe that

$$Z_0 (1 - Y_0 a^\dagger(v)) | \Omega \rangle = (1 - Y_0 a^\dagger(v)) | \Omega \rangle$$

since this vector is actually equal to the ground state of  $H$ . Hence

$$c_0 = \frac{1}{1 + E_0} - 1 - \langle v | Y_0 (1 + X) Y_0 | v \rangle.$$

That this indeed equals (2.12) is then an easy consequence of (2.11).

### A. Technical bounds

In this appendix we shall show the bound  $\|A\| \leq O(g^2)$  claimed in the text. We start by showing that

$$\sup_{k \in \mathbb{R}^d} \|(1 + |P + k|) Y_k (1 + |P + k|)\| < \infty. \tag{A.1}$$

By explicitly designating the dependence on  $v$  and writing  $E_0(v)$  for the ground state energy of  $H$  with interaction  $\Phi(v)$ , we can bound

$$\begin{aligned} Y_k^{-1} &= \Pi^{\geq 1}((P+k)^2 + \Phi(v) + \mathbb{N} - E_0(v))\Pi^{\geq 1} \\ &\geq \Pi^{\geq 1}(\delta + \delta(P+k)^2 - E_0(v) + (1-\delta)E_0(v(1-\delta)^{-1})) \end{aligned}$$

for any  $0 \leq \delta \leq 1$ , which readily implies the desired bound, at least for small  $g$ .

From (A.1) one immediately deduces that  $\sup_{k \in \mathbb{R}^d} |E_k| \leq C \|v\|$  for some constant  $C > 0$ . Similarly, one can show that  $|E_k - E_0| \leq C \|v\| |k|^2$ . In fact, the resolvent identity implies that  $E_k$  is twice differentiable, and

$$\partial_{k_i} \partial_{k_j} E_k = 2\delta_{ij} \langle v | Y_k^2 | v \rangle - 4 \langle v | Y_k (P_i + k_i) Y_k (P_j + k_j) Y_k | v \rangle$$

for  $1 \leq i, j \leq d$ . Since

$$\partial_{k_i} E_k = 2 \langle v | Y_k (P_i + k_i) Y_k | v \rangle$$

vanishes at  $k = 0$  (since  $v$  is even), the desired bound follows.

In a similar way, one shows that  $F$  is bounded and has bounded derivatives. Since  $F(k, l)$  vanishes by construction if either  $k = 0$  or  $l = 0$ , this implies the desired bound on the norm of  $|k|^{-1} v F v |k|^{-1}$  (in fact, one obtains a bound on its Hilbert–Schmidt norm this way).

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