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EQUIDISTRIBUTION OF PRIMITIVE LATTICES IN $\mathbb{R}^n$

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Abstract

We count primitive lattices of rank d inside $\mathbb{Z}^n$ as their covolume tends to infinity, with respect to certain parameters of such lattices. These parameters include, for example, the subspace that a lattice spans, namely its projection to the Grassmannian; its homothety class and its equivalence class modulo rescaling and rotation, often referred to as a *shape*. We add to a prior work of Schmidt by allowing sets in the spaces of parameters that are general enough to conclude the joint equidistribution of these parameters. In addition to the primitive d-lattices Λ themselves, we also consider their orthogonal complements in $\mathbb{Z}^n$, $\Lambda^\perp$, and show that the equidistribution occurs jointly for Λ and $\Lambda^\perp$. Finally, our asymptotic formulas for the number of primitive lattices include an explicit bound on the error term.

1. Introduction

The aim of this paper is to extend classical counting and equidistribution results for *primitive vectors* to their higher-rank counterparts: *primitive lattices*. A primitive vector is an n-tuple of integers $(a_1, \dots, a_n)$ with $\gcd(a_1, \dots, a_n) = 1$, and the set of primitive vectors in $\mathbb{R}^n$ is denoted by $\mathbb{Z}^n_{\text{prim}}$. We can associate with each vector $0 \neq v \in \mathbb{R}^n$ the discrete subgroup that it spans, $\mathbb{Z}v$; following this logic, a rank d ($1 \leq d \leq n$) analogue for a vector is a *lattice of rank d* in $\mathbb{R}^n$, namely

$$\Lambda = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_d,$$

where $v_1, \dots, v_d \in \mathbb{R}^n$ are linearly independent. We will refer to it briefly as a *d-lattice*. We say that a d-lattice Λ is *integral* if $\Lambda \subset \mathbb{Z}^n$ and *primitive* if $\Lambda = V \cap \mathbb{Z}^n$, where V is a d-dimensional rational subspace of $\mathbb{R}^n$. For example, a primitive 1-lattice is simply all the integral points on a rational line, or, equivalently, $\mathbb{Z}v$ where v is a primitive vector.

Questions about counting primitive vectors date back to the days of Gauss and Dirichlet, for example with the *Gauss Class Number problem*. In the 20th century, questions about equidistribution of integral vectors began to arise, with the principle example being *Linnik-type problems* [**10**, **13**–**16**, **19**, **34**]. These questions and others generalize naturally to primitive lattices, as we now describe.

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The Primitive Circle Problem.

The well-known Gauss circle problem concerns the asymptotic number of integral vectors with (Euclidean) norm at most $X > 0$. The analogous question for primitive vectors, namely the asymptotic amount of primitive vectors up to norm X, is often referred to as the *primitive circle problem* [**38**, **47**, **48**]. In lattices, the role of a norm is played by the *covolume*: the covolume of Λ, denoted $\mathrm{covol}(\Lambda)$, is the volume of a fundamental parallelopiped for Λ in the linear space

$$V_\Lambda := \Lambda \otimes \mathbb{R} = \mathrm{span}_{\mathbb{R}}(\Lambda).$$

Thus, the *primitive circle problem for lattices* is to estimate the asymptotics of

$$\#\{\text{primitive } d-\text{lattices in } \mathbb{R}^n \text{ of covolume up to } X\} \tag{1.1}$$

as $X \to \infty$. Note that for 1-lattices, the notions of norm and covolume coincide: $\mathrm{covol}(\mathbb{Z}v) = \|v\|$, hence when $d = 1$ the above recovers the 'original' primitive circle problem. Schmidt [**44**] showed that the amount in (1.1) equals

$$c_{d,n} X^n + O(X^{n-\max\{\frac{1}{d},\frac{1}{n-d}\}}), \tag{1.2}$$

where

$$c_{d,n} = \frac{1}{n}\binom{n}{d} \cdot \frac{\prod_{i=n-d-1}^{n} \mathfrak{B}(i)}{\prod_{j=1}^{d} \mathfrak{B}(j)} \cdot \frac{\prod_{i=2}^{d} \zeta(i)}{\prod_{j=n-d+1}^{n} \zeta(j)},$$

and $\mathfrak{B}(i)$ the Lebesgue volume of the unit ball in $\mathbb{R}^i$. Thunder [**46**, Thm. 5] proved a variation on this result for lattices over a general number field that trivially intersect a certain subspace, and Kim [**31**, Thm. 1.3] has found a more concrete presentation for the error term in Schmidt's result. We remark that the optimal exponent in the error term of the circle problem (primitive or not) is established only in dimensions $n \geq 4$ and that this case ($d = 1$, $n \geq 4$) is the only case where an optimal error exponent is known for the lattice circle problem (primitive or not).

Linnik-type problems.

This is a unifying name for questions on the distribution of the projections of integral vectors to the unit sphere, that is of $v/\|v\|$ when $v \in \mathbb{Z}^n$or $\mathbb{Z}^n_{\mathrm{prim}}$. Viewing the unit sphere as the space of oriented lines in $\mathbb{R}^n$, the analogous object when considering d-lattices would be the *Grassmannian* of oriented d-dimensional subspaces in $\mathbb{R}^n$, denoted $\mathrm{Gr}(d,n)$ (see Section 2). Accordingly, we will view our lattices as carrying an orientation, which simplifies our discussion on the technical level but has no effect on the results. In particular, the two-to-one correspondence between primitive vectors and primitive 1-lattices (arising from the fact that v and $-v$ span the same lattice) becomes a one-to-one correspondence between primitive vectors and oriented primitive 1-lattices. The *average Linnik problem for primitive lattices* is to study the distribution of the (oriented) spaces V_Λ in $\mathrm{Gr}(d,n)$ as $\mathrm{covol}(\Lambda) \leq X \to \infty$. Note that V_Λ are exactly the rational subspaces in $\mathrm{Gr}(d,n)$.

Shapes of orthogonal lattices.

More recently, with the rise of dynamical approaches in number theory, another type of equidistribution questions for primitive vectors arose. To a primitive vector v we associate the $(n-1)$-lattice $v^\perp \cap \mathbb{Z}^n$, referred to as the *orthogonal lattice* of v, where $v^\perp$ is the orthogonal hyperplane to v. Several recent papers (by Marklof [37], Aka Einsiedler and Shapira [2, 1], Einsiedler, Mozes, Shah and Shapira [18] and Einsiedler Rühr and Wirth [20]) studied the equidistribution of *shapes* of the orthogonal lattices to primitive vectors as their norm tends to infinity, where the shape of a lattice is its similarity class modulo rotation and homothety. The space of shapes of d-lattices is a double coset space of $\mathrm{SL}_d(\mathbb{R})$, denoted $\mathcal{X}_d$ and defined explicitly in Section 2, and the aforementioned papers show that the shapes of the orthogonal lattices to $v \in \mathbb{Z}^n_{\mathrm{prim}}$ equidistribute in $\mathcal{X}_{n-1}$ as $\|v\| \to \infty$ with respect to the uniform measure arriving from the Haar measure on $\mathrm{SL}_{n-1}(\mathbb{R})$. In the works of Einsiedler et al., they in fact show that the shapes of $v^\perp \cap \mathbb{Z}^n$ equidistribute in $\mathcal{X}_{n-1}$ *jointly* with the directions $v/\|v\|$ in $\mathbb{S}^{n-1}$.

Just like for (primitive) vectors, orthogonal lattices can be defined for (primitive) lattices as well: for a primitive d-lattice Λ, we let:

$$\Lambda^\perp := V_\Lambda^\perp \cap \mathbb{Z}^n,$$

where $V_\Lambda^\perp$ is the orthogonal complement of V_Λ in $\mathbb{R}^n$. Note that $\Lambda^\perp$ is primitive by definition and has rank $n-d$. Also note that $\Lambda \mapsto \Lambda^\perp$ defines a bijection between primitive lattices of ranks d and $n-d$. This bijection extends to a bijection between oriented lattices, with a natural choice of orientation on the orthogonal lattice (Def@. 2.2).

One could then ask about the equidistribution of shapes of the orthogonal lattices $\Lambda^\perp$ to primitive lattices Λ, where the one-dimensional case $\Lambda = \mathbb{Z}v$ recovers the question studied in the aforementioned papers about the equidistribution of shapes of $v^\perp \cap \mathbb{Z}^n$.

Equidistribution of a sequence in a finite-volume space can be deduced from counting in 'sufficiently general' subsets of this space. Indeed, we will count in subsets that have *controlled boundary* (Def. 3.1), which is a notion that generalizes the property of having a C^1 boundary. We denote by $\|\mu\|$ the total mass of a finite measure ν.

Theorem 1.1 *Let $n \geq 3$ and $1 \leq d < n$ and assume that $\Phi \subseteq \mathrm{Gr}(d,n)$ and $\mathcal{E} \times \mathcal{F} \subseteq \mathcal{X}_d \times \mathcal{X}_{n-d}$ have controlled boundary. Then, the number of primitive d-lattices Λ with $\mathrm{covol}(\Lambda) \leq X$, $V_\Lambda \in \Phi$ and $(\mathrm{shape}(\Lambda), \mathrm{shape}(\Lambda^\perp)) \in \mathcal{E} \times \mathcal{F}$ is*

$$c_{d,n} \cdot \frac{\mathrm{vol}_{\mathcal{X}_d}(\mathcal{E})}{\|\mathrm{vol}_{\mathcal{X}_d}\|} \frac{\mathrm{vol}_{\mathcal{X}_{n-d}}(\mathcal{F})}{\|\mathrm{vol}_{\mathcal{X}_{n-d}}\|} \frac{\mathrm{vol}_{\mathrm{Gr}(d,n)}(\Phi)}{\|\mathrm{vol}_{\mathrm{Gr}(d,n)}\|} \cdot X^n + O_\epsilon(X^{n-\kappa+\epsilon})$$

for every $\epsilon > 0$, where

$$\kappa = \begin{cases} X^{n-\frac{1}{2n(n+1)}+\epsilon} & \text{when both } \mathcal{E}, \mathcal{F} \text{ are bounded}, \\ X^{n-\frac{1}{4n(n^2-1)}+\epsilon} & \text{otherwise}. \end{cases}$$

In this theorem, $\mathrm{vol}_{\mathrm{space}}$ stands for the standard uniform measure on the relevant space (independent of the normalization), so it implies the joint uniform distribution of the directions and shapes of the orthogonal lattices of primitive lattices as their covolume tends to infinity. In particular, it is interesting to observe that the shapes of Λ and of $\Lambda^{\perp}$ are independent parameters, meaning that there is no way to know the shape of $\Lambda^{\perp}$ given the shape of Λ, even though the latter lattice determines the first.

The strength of Theorem 1.1 compared to previous work about counting and equidistribution of d-lattices lies both in being the first to consider the shapes of lattices in parallel to the shapes of their orthogonal lattices and in the quality of the error term in the bounded case. A non-quantitative version of Theorem 1.1 (for Λ only) was obtained by Schmidt in [**45**], and prior to that, for $d=2$, by Maass [**35**, **36**] and Roelcke [**42**] (who considered integral lattices without restricting to primitive ones). The case of $d=n-1$ was handled using a dynamical approach by Marklof in [**37**], as well as by the authors in [**27**]. Almost 50 years after the result in (1.2), Schmidt proved yet another effective result for primitive d-lattices that project to certain $\Phi \subseteq \mathrm{Gr}(d,n)$ and $\mathcal{E} \subset \mathcal{X}_d$, but the admissible $\Phi, \mathcal{E}$ were not general enough to achieve equidistribution (more on that later). The error term there is $\ll X^{n-\frac{1}{d}}$, upon which the error term in Theorem 1.1 improves where $\mathcal{E}$ is bounded (and otherwise it is of similar quality). In addition, the counting in Theorem 1.1 allows sets $\Phi, \mathcal{E}$ that are general enough to deduce equidistribution, as we now turn to describe.

Equidistribution.

Theorem 1.1 can also be formulated in terms of convergence of measures, namely that for every compactly supported Lipschitz functions $f_1 \in C_C(\mathcal{X}_d)$, $f_2 \in C_C(\mathcal{X}_{n-d})$, $f_3 \in C_C(\mathrm{Gr}(d,n))$ one has that

$$\frac{1}{\#|\Lambda \text{primitive} : \mathrm{covol}(\Lambda) \leq \mathrm{X}|} \sum_{\Lambda} f_1(\mathrm{shape}(\Lambda)) f_2(\mathrm{shape}(\Lambda^{\perp})) f_3(V_\Lambda)$$

converges as $X \to \infty$ to

$$\frac{1}{\|\mathrm{vol}_{\mathcal{X}_d}\|}\left(\int f_1 d\,\mathrm{vol}_{\mathcal{X}_d}\right) \cdot \frac{1}{\|\mathrm{vol}_{\mathcal{X}_{n-d}}\|}\left(\int f_2 d\,\mathrm{vol}_{\mathcal{X}_{n-d}}\right) \cdot \frac{1}{\|\mathrm{vol}_{\mathrm{Gr}(d,n)}\|}\left(\int f_3 d\,\mathrm{vol}_{\mathrm{Gr}(d,n)}\right).$$

But in fact, a stronger statement holds.

THEOREM 1.2 *For every compactly supported function $f \in C_C(\mathcal{X}_d \times \mathcal{X}_{n-d} \times \mathrm{Gr}(d,n))$ one has that*

$$\frac{1}{\#|\Lambda \text{primitive} : \mathrm{covol}(\Lambda) \leq \mathrm{X}|} \sum_{\Lambda} f(\mathrm{shape}(\Lambda), \mathrm{shape}(\Lambda^{\perp}), V_\Lambda)$$

converges as $X \to \infty$ to

$$\frac{1}{\|\mathrm{vol}_{\mathcal{X}_d}\| \|\mathrm{vol}_{\mathcal{X}_{n-d}}\| \|\mathrm{vol}_{\mathrm{Gr}(d,n)}\|} \cdot \int f d\,\mathrm{vol}_{\mathcal{X}_d} d\,\mathrm{vol}_{\mathcal{X}_{n-d}} d\,\mathrm{vol}_{\mathrm{Gr}(d,n)}.$$

In principle, the above equidistribution can be made effective when the function f is Lipschitz, where in accordance with Theorem 1.1, the rate of convergence would be $\ll_\epsilon X^{-\frac{1}{2n(n+1)}+\epsilon}$. Indeed, the proofs for the counting results in this paper rely on the work [**25**] of Gorodnik and Nevo, which produces effective counting results 'for sets rather than for functions' (namely, for counting the number of lattice points in a given set rather than summing the values of a given function on these points); while it is possible to extend their work for the functions setting (see [29, proof of Cor. 1.2]), it has not been done anywhere.

There are two types of equidistribution statements: one is 'on level sets', for example where $\mathrm{covol}(\Lambda) = X$ and $X \to \infty$, and the other is 'on average', for example where $\mathrm{covol}(\Lambda) \leq X$ and $X \to \infty$. Theorem 1.2 clearly belongs to the second type, where indeed results often appear in the formulation of counting. Results of the first type are in a sense more delicate, but in many cases they do not imply, nor follow from, counting (however, such an implication is proved in the appendix of [3]). Equidistribution of shapes for lattices of covolume X (namely equidistribution of the first type) was obtained for the case $d = n-1$ in [1, 2, 7, 18, 20], and recently (while a previous version of the present work was already available) for a general d by Aka, Musso and Wieser in [4]. The latter includes the conjecture that the equidistribution in Theorem 1.1 occurs also when $\mathrm{covol}(\Lambda) = X$ and not just on average, namely that a Linnik-type phenomenon occurs for lattices.

Counting d-lattices: Beyond shapes.

There exists a wide body of work on equidistribution problems in the space

$$\mathcal{L}_n := \mathrm{SL}_n(\mathbb{R}) / \mathrm{SL}_n(\mathbb{Z}),$$

which is the space of unimodular (that is with covolume one and positive orientation) full lattices in $\mathbb{R}^n$, as well as in the space $\mathcal{X}_n$ of their shapes. The restriction to covolume one is necessary because the space $\mathcal{L}_n$ (and therefore $\mathcal{X}_n$) has finite volume, while the space of all lattices, $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$, does not. Comparing the spaces $\mathcal{L}_n$ and $\mathcal{X}_n$, the space $\mathcal{L}_n$ naturally contains 'more information' than $\mathcal{X}_n$, which is obtained by modding $\mathcal{L}_n$ by rotations. This brings up the question of whether one can define a space of 'unimodular d-lattices in $\mathbb{R}^n$' so as to consider the unimodular d-lattices in $\mathbb{R}^n$ without modding out by rotations. In Section 2 we introduce two such spaces, which are homogeneous spaces of $\mathrm{SL}_n(\mathbb{R})$. We will prove a stronger statement than Theorem 1.1, namely Theorem 3.2, where we count primitive d-lattices according to their projections to these more refined spaces.

Organization of the paper and strategy of proof.

The paper is organized as follows: In Section 2 we define the two aforementioned spaces that project to the product space $\mathcal{X}_d \times \mathrm{Gr}(d,n)$. In Section 3 we state our main result, Theorem 3.2, which concerns counting primitive d-lattices w.r.t. their projections to these two spaces. We explain how this theorem implies Theorems 1.1 and 1.2, and then the rest of the paper is devoted to proving Theorem 3.2, where the strategy is to translate counting primitive d-lattices to counting points of the lattice $\mathrm{SL}_n(\mathbb{Z})$ inside an increasing family of subsets in $\mathrm{SL}_n(\mathbb{R})$. In order to define these subsets—namely, sets in $\mathrm{SL}_n(\mathbb{R})$ that capture the integral matrices corresponding to d-lattices with a certain shape, covolume, etc.—we define in Section 4 a refinement of the Iwasawa coordinates on $\mathrm{SL}_n(\mathbb{R})$. In Section 5, we reduce Theorem 3.2 (and therefore Theorems 1.1 and 1.2) to one of the four statements in Theorem 3.2—the one which concerns counting d-lattices in the most refined space (the one that projects to all the others). In Section 6, we make explicit the translation of our results to a problem of counting $\mathrm{SL}_n(\mathbb{Z})$ elements in $\mathrm{SL}_n(\mathbb{R})$, by associating with each primitive d-lattice a unique element in $\mathrm{SL}_n(\mathbb{Z})$. In Section 7, we describe a method developed by Gorodnik and Nevo in

[25] for counting lattice points in semi-simple Lie groups, which will be our main tool in approaching the counting problem at hand. However, this method cannot be applied directly to our counting problem, since the sets in $\mathrm{SL}_n(\mathbb{R})$ that we are concerned with are not well rounded. Our solution is to split each set into a well-rounded part and a ‘tail’: in Section 8 we show that the ‘tails’ contribute a negligible amount of points, and in Section 9 we apply the method from [25] for the counting in the well-rounded subsets, hence completing the proof of Theorem 3.2. This article also includes an appendix, in which we expand upon the spaces that are introduced in Section 2, prove some auxiliary claims that are needed throughout the paper and put together some useful facts about lattices in $\mathbb{R}^n$ that are known but do not appear in the literature.

2. Spaces of lattices

We begin by explicitly defining the spaces $\mathcal{X}_d$ and $\mathrm{Gr}(d,n)$ appearing in Theorem 1.1. An oriented subspace on $\mathbb{R}^n$ is a subspace with a sign attached, and the Grassmannian $\mathrm{Gr}(d,n)$ is the set of all d-dimensional oriented subspaces of $\mathbb{R}^n$. It can also be defined as the following quotients:

$$\mathrm{Gr}(d,n) = \mathrm{SO}_n(\mathbb{R}) / \left\{ \begin{bmatrix} \mathrm{SO}_d(\mathbb{R}) & 0_{d,n-d} \\ 0_{n-d,d} & \mathrm{SO}_{n-d}(\mathbb{R}) \end{bmatrix} \right\} = \mathrm{SL}_n(\mathbb{R}) / \left\{ \begin{bmatrix} g_1 & \mathbb{R}^{d,n-d} \\ 0_{n-d,d} & g_2 \end{bmatrix} : \det(g_1 g_2) = 1 \right\}.$$

A coset in SO_n (or SL_n) represents an oriented d-dimensional subspace V if the first d columns of the matrices in this coset span V with the right orientation.

Recall that the shape of (an oriented) lattice is its equivalent class modulo homothety and rotation. The space of shapes of (oriented) d-lattices is

$$\mathcal{X}_d = \mathrm{SO}_d(\mathbb{R}) \backslash \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z}).$$

Each of the above spaces is equipped with a natural measure that is unique up to rescaling, and although these measures are rather standard, we recall their definition. In general, there is a natural way to define a measure on the space of orbits of a unimodular group:

THEOREM ([30, Thm. 2.2]). *Let $\mathbf{G}$ be a unimodular Radon lcsc group, which acts on an lcsc space Y strongly properly. Assume that $\mu_{\mathbf{G}}$ is a Haar measure on $\mathbf{G}$ and that μ_Y is a $\mathbf{G}$-invariant Radon measure on Y. Then there exists a unique Radon measure $\mu_{\mathbf{G}\backslash Y}$ on $\mathbf{G}\backslash Y$ such that for all $f \in L^1(Y,\nu)$,*

$$\int_Y f(y)\, d\mu_Y(y) = \int_{\mathbf{G}\backslash Y} \left(\int_{\mathbf{G}} f(gy)\, d\mu_{\mathbf{G}}(g) \right) d\mu_{\mathbf{G}\backslash Y}(\mathbf{G}y). \tag{2.1}$$

For a measure on the Grassmannian, take (in the notations of (2.1)) $Y = \mathrm{SO}_n(\mathbb{R})$ and $\mathbf{G} = \mathrm{SO}_d(\mathbb{R}) \times \mathrm{SO}_{n-d}(\mathbb{R})$ to obtain a unique SO_n– invariant measure $\mathrm{vol}_{\mathrm{Gr}(d,n)}$ on the quotient which satisfies

$$\|\mathrm{vol}_{\mathrm{Gr}(d,n)}\| = \frac{\mathrm{Haar}(\mathrm{SO}_n(\mathbb{R}))}{\mathrm{Haar}(\mathrm{SO}_d(\mathbb{R})) \cdot \mathrm{Haar}(\mathrm{SO}_{n-d}(\mathbb{R}))} = \frac{2n!}{d!(n-d)!} \cdot \frac{\prod_{i=1}^{n} i\mathfrak{B}(i)}{\prod_{i=1}^{d} i\mathfrak{B}(i) \cdot \prod_{i=1}^{n-d} i\mathfrak{B}(i)}. \tag{2.2}$$

For a measure on the space of shapes, use (2.1) to obtain an $\mathrm{SL}_d(\mathbb{R})$-invariant measure on $\mathrm{SO}_d(\mathbb{R})\backslash\mathrm{SL}_d(\mathbb{R})$ and then let $\mathrm{vol}_{\mathcal{X}_d}$ be its restriction to a fundamental domain of $\mathrm{SL}_d(\mathbb{Z})$, normalized such that

$$\|\mathrm{vol}_{\mathcal{X}_d}\| = \prod_{i=2}^{d} \zeta(i)/\Upsilon(d), \tag{2.3}$$

where

$$\Upsilon(d) = \frac{\frac{d!}{2}\prod_{i=1}^{d}\mathfrak{V}(i)}{\#\,|Z(\mathrm{SO}_d(\mathbb{R}))|} \tag{2.4}$$

(here the denominator is the cardinality of the center of $\mathrm{SO}_d(\mathbb{R})$, which is 2 if d is even and 1 if d is odd).

We now proceed to define a space of d-lattices in $\mathbb{R}^n$ that encodes both their shapes and their directions (that is their projections to the Grassmannian).

2.1. *Space of homothety classes of d-lattices*

A *unimodular* lattice is an oriented lattice with positive orientation and covolume one. Recall our notation for the space of rank n unimodular lattices

$$\mathcal{L}_n = \mathrm{SL}_n(\mathbb{R})/\,\mathrm{SL}_n(\mathbb{Z}),$$

where a matrix in $\mathrm{SL}_n(\mathbb{R})$ lies in the coset that represents a full unimodular lattice in $\mathbb{R}^n$ if its columns span this lattice. This space is equipped with a natural left $\mathrm{SL}_n(\mathbb{R})$ invariant measure, which is the left Haar measure on $\mathrm{SL}_n(\mathbb{R})$ restricted to a fundamental domain of $\mathrm{SL}_n(\mathbb{Z})$. The typical normalization of this Haar measure is

$$\|\mathrm{vol}_{\mathcal{L}_n}\| = \prod_{i=2}^{n} \zeta(i). \tag{2.5}$$

As the space of shapes $\mathcal{X}_n$ is obtained from $\mathcal{L}_n$ via modding by SO_n, the space $\mathcal{L}_n$ is more refined, containing not only the information about the shape of a lattice but also about its position in $\mathbb{R}^n$. To define the analogous space for d-lattices in $\mathbb{R}^n$, notice that since $\mathcal{L}_n$ consists of a unique representative from any equivalence class of n-lattices in $\mathbb{R}^n$ modulo homothety, one can identify $\mathcal{L}_n$ with the space of such equivalence classes. The space of homothety classes of oriented d-lattices inside $\mathbb{R}^n$ is

$$\mathcal{L}_{d,n} := \mathrm{SL}_n(\mathbb{R})/\left(\begin{bmatrix}\mathrm{SL}_d(\mathbb{Z}) & \mathbb{R}^{d,n-d}\\ 0_{n-d\times d} & \mathrm{SL}_{n-d}(\mathbb{R})\end{bmatrix}\times\left\{\begin{bmatrix}\alpha^{\frac{1}{d}}\,\mathrm{I}_d & 0_{d\times n-d}\\ 0_{n-d\times d} & \alpha^{-\frac{1}{n-d}}\,\mathrm{I}_{n-d}\end{bmatrix} : \alpha > 0\right\}\right),$$

where a matrix in $\mathrm{SL}_n(\mathbb{R})$ lies in the coset that represents an equivalence class of a d-lattice in $\mathbb{R}^n$ if and only if its first d columns span a positive scalar multiplication of this lattice, with the corresponding orientation. The need to mod out by the block-scalar group follows from the fact that the first d columns of a matrix in SL_n span a lattice that is hardly ever of covolume one, so one needs to divide by the covolume (hence an element in this quotient space is an equivalence class of d-lattices up to homothety). However, modding out the block-scalar group comes with a price, which is that the space $\mathcal{L}_{d,n}$ does not carry an $\mathrm{SL}_n(\mathbb{R})$-invariant measure since the acting group is not unimodular. To fix this flaw, let us consider a different presentation of $\mathcal{L}_{d,n}$. Notice that for every d-lattice Λ there exist (non-unique) $g_d \in \mathrm{SL}_d(\mathbb{R})$, $k \in \mathrm{SO}_n(\mathbb{R})$ and (a unique) $\alpha > 0$ such that

$$\Lambda = \alpha^{\frac{1}{d}} k^{-1}(g_d \mathbb{Z}^d \times \{0_{n-d}\}).$$

The element g_d can be replaced by any other element in $g_d\,\mathrm{SL}_d(\mathbb{Z})$, and the element k can be replaced by any other element in $(\mathrm{SO}_d(\mathbb{R}) \times \mathrm{I}_{n-d})k$, if the element g_d is adjusted accordingly. As a result, where by $\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}}$ we mean the diagonal embedding of $\mathrm{SO}_d(\mathbb{R})$ in $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$, in which

$$\mathcal{L}_{d,n} \simeq \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z}) \tag{2.6}$$

the embedding in $\mathrm{SL}_d(\mathbb{R})$ is the identity map, and the embedding in $\mathrm{SO}_n(\mathbb{R})$ is $\mathrm{SO}_d(\mathbb{R}) \times \mathrm{I}_{n-d}$. Then (2.1) allows us to define a measure on the manifold $\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$, and $\mathrm{vol}_{\mathcal{L}_{d,n}}$ is the restriction of this measure to a fundamental domain of $\mathrm{SL}_d(\mathbb{Z})$, normalized such that

$$\|\mathrm{vol}_{\mathcal{L}_{d,n}}\| = \|\mathrm{vol}_{\mathrm{Gr}(d,n)}\| \|\mathrm{vol}_{\mathcal{L}_d}\| \tag{2.7}$$

(this normalization is natural in view of Proposition 2.5). Note that we would not have gotten a finite volume space had we considered the space of d-lattices on $\mathbb{R}^n$ without 'modding out the covolume', just like in the case of $\mathcal{L}_n$.

2.2. *Factor lattices and the space of pairs*

Recall that an integral lattice Λ is primitive if it is of the form $\Lambda = \mathbb{Z}^n \cap V_\Lambda$; this is equivalent to the fact that any basis of Λ can be completed to a basis of $\mathbb{Z}^n$. Theorem 1.1 consists of a joint equidistribution result for primitive lattices Λ and their orthogonal complements $\Lambda^\perp = \mathbb{Z}^n \cap V_\Lambda^\perp$. It is a consequence of the stronger Theorem 3.2, in which $\Lambda^\perp$ is replaced by another $(n-d)$-lattice in the space $V_\Lambda^\perp$:

Definition 2.1 The *factor lattice* Λ^π of a primitive d-lattice Λ is the orthogonal projection of $\mathbb{Z}^n$ to $V_\Lambda^\perp$.

The factor lattice Λ^π is isometric to the quotient $\mathbb{Z}^n/\Lambda$ (Prop. B.3), so one should think of Λ^π as a realization of $\mathbb{Z}^n/\Lambda$ inside $\mathbb{R}^n$. Notice that, like $\Lambda^\perp$, Λ^π is a full lattice inside $V_\Lambda^\perp$, and in particular is of rank $n-d$. The relation between $\Lambda^\perp$ and Λ^π is that they are *dual* to one another (for the definition of dual lattices, see [**11**, I.5], or Appendix A in the present paper; for the duality of $\Lambda^\perp$ and Λ^π, see Claim B.5). It holds that

$$\mathrm{covol}(\Lambda^\pi) = \mathrm{covol}(\Lambda)^{-1} = \mathrm{covol}(\Lambda^\perp)^{-1}$$

([**43**, **44**]; see also Prop. B.4 and Cor. B.6 in the Appendix). Since we view d-lattices as carrying an orientation, we need to define an orientation on Λ^π and $\Lambda^\perp$, which is done as follows:

DEFINITION 2.2 Let L be a full lattice in $V_\Lambda^\perp$. A basis $\mathbf{C}$ for L is positively oriented if $\det(\mathbf{B}|\mathbf{C}) = 1$ for a positively oriented basis $\mathbf{B}$ of Λ.

The last space we introduce is the space of pairs of oriented lattices (Λ, L) such that (i) Λ is a d-lattice, (ii) L is a full lattice in $V_\Lambda^\perp$ (hence is of rank $n-d$ and its orientation is given in Definition 2.2) and (iii) $\operatorname{covol}(\Lambda)\operatorname{covol}(L) = 1$. In fact, it is the space of homothety classes of such pairs, where (Λ, L) is equivalent to $(\alpha^{\frac{1}{d}}\Lambda, \alpha^{-\frac{1}{n-d}}L)$ for every $\alpha > 0$. This space is given as the quotient

$$\mathcal{P}_{d,n} := \mathrm{SL}_n(\mathbb{R}) / \left(\begin{bmatrix} \mathrm{SL}_d(\mathbb{Z}) & \mathbb{R}^{d,n-d} \\ 0_{n-d\times d} & \mathrm{SL}_{n-d}(\mathbb{Z}) \end{bmatrix} \times \left\{ \begin{bmatrix} \alpha^{\frac{1}{d}} \mathrm{I}_d & 0_{d\times n-d} \\ 0_{n-d\times d} & \alpha^{-\frac{1}{n-d}} \mathrm{I}_{n-d} \end{bmatrix} : \alpha > 0 \right\} \right),$$

where a matrix $g \in \mathrm{SL}_n(\mathbb{R})$ lies in the coset that represents the homothety class of (Λ, L) if and only if the lattice Λ' spanned by the first d columns of g and the lattice L' which is the projection of $g\mathbb{Z}^n$ onto $V_\Lambda^\perp$ satisfy that (Λ', L') and (Λ, L) are in the same homothety class. Note the similarity to the definition of $\mathcal{L}_{d,n}$ above; here also, modding out the block-scalar group results in having no $\mathrm{SL}_n(\mathbb{R})$-invariant measure, and it is preferable to present this space as a quotient of a manifold by a discrete group. For every pair (Λ, L) as above there exist (non-unique) $g_d \in \mathrm{SL}_d(\mathbb{R})$, $g_{n-d} \in \mathrm{SL}_{n-d}(\mathbb{R})$, $k \in \mathrm{SO}_n(\mathbb{R})$ and (a unique) $\alpha > 0$ such that

$$\Lambda = \alpha^{\frac{1}{d}} k^{-1}(g_d\mathbb{Z}^d \times \{0_{n-d}\}), \quad L = \alpha^{-\frac{1}{n-d}} k^{-1}(\{0_d\} \times g_{n-d}\mathbb{Z}^{n-d}).$$

For similar considerations as in the case of $\mathcal{L}_{d,n}$,

$$\mathcal{P}_{d,n} = \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \times \mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}} \backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_{n-d}(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z}) \times \mathrm{SL}_{n-d}(\mathbb{Z}), \tag{2.8}$$

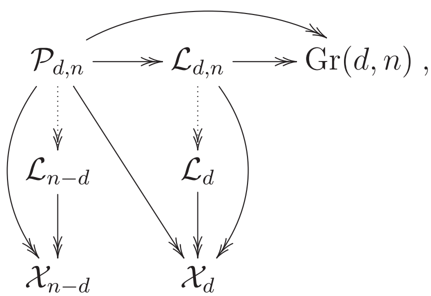
where $\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}}$ is the diagonal embedding of $\mathrm{SO}_d(\mathbb{R})$ in $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$ (as in (2.6)) and $\mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}}$ is the analogous diagonal embedding of $\mathrm{SO}_{n-d}(\mathbb{R})$ in $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_{n-d}(\mathbb{R})$. Then we may apply (2.1) to define a measure on the manifold that is the left quotient in (2.8) and set $\mathrm{vol}_{\mathcal{P}_{\mathrm{d,n}}}$ as the restriction of this measure to a fundamental domain of $\mathrm{SL}_d(\mathbb{Z}) \times \mathrm{SL}_{n-d}(\mathbb{Z})$, normalized such that

$$\|\mathrm{vol}_{\mathcal{P}_{d,n}}\| = \|\mathrm{vol}_{\mathcal{L}_{d,n}}\| \|\mathrm{vol}_{\mathcal{L}_{n-d}}\|$$

(this normalization is natural in view of Proposition 2.5). More details on dual lattices and factor lattices can be found in the Appendix.

2.3. *Relation between the spaces of lattices and their measures*

While the spaces $\mathcal{L}_d$, $\mathcal{X}_d$ and $\mathrm{Gr}(d,n)$ are well known, the spaces $\mathcal{L}_{d,n}$ and $\mathcal{P}_{\mathrm{d,n}}$, as far as we are aware, make their first appearance here (at least for the case $d > 1$; the case $d = 1$ was introduced in [**27**] and later appeared in [**7**]). It therefore seems appropriate to explain how $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$ add to the more 'familiar' spaces $\mathcal{L}_d$, $\mathcal{X}_d$ and $\mathrm{Gr}(d,n)$. The relation between all the different spaces is that they naturally project to one another as depicted in the following commutative diagram: Let $\pi_{\mathcal{Y}\to\mathcal{Z}}$ denote the projection from a space $\mathcal{Y}$ to a space $\mathcal{Z}$. All the projections except for $\pi_{\mathcal{L}_{d,n}\to\mathcal{L}_d}$ and $\pi_{\mathcal{P}_{d,n}\to\mathcal{L}_{n-d}}$ are

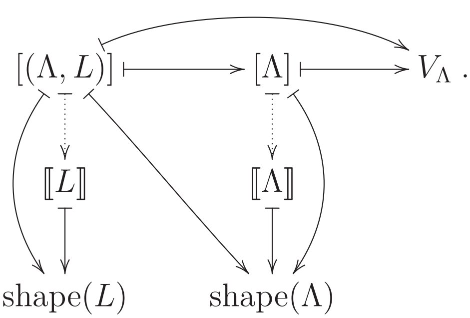


(2.9)

of the form $G/H_1 \to G/H_2$ with $H_1 < H_2$ and are therefore well defined and continuous. However, the projections into $\mathcal{L}_d$ and $\mathcal{L}_{n-d}$ are not canonical, as they depend on a choice of coordinates on the d-dimensional (resp. $n-d$ dimensional) vector spaces in $\mathbb{R}^n$. We will fix a choice of coordinates in Section 4.1 and define these maps explicitly in Section 5.1; meanwhile, we state the following for future reference:

Notation 2.3 *Given an oriented d-lattice* $\Lambda < \mathbb{R}^n$ *we denote its homothety class by* $[\Lambda] \in \mathcal{L}_{d,n}$*, and its shape by* $\mathrm{shape}(\Lambda) \in \mathcal{X}_d$*. Given a pair* (Λ, L) *of lattices in orthogonal subspaces where* Λ *is primitive of rank d, we denote its homothety class by* $[(\Lambda, L)] \in \mathcal{P}_{d,n}$*. The image of* Λ *in* $\mathcal{L}_d$ *(resp. of* L *in* $\mathcal{L}_{n-d}$*) is denoted* $[\![\Lambda]\!]$ *(resp.* $[\![L]\!]$*).*

Proposition 2.4 *The projections in Diagram (2.9) are well defined and given by*



All the maps, except for the ones with the dotted arrows, are continuous. The maps from $\mathcal{L}_d$ *and* $\mathcal{L}_{d,n}$ *to* $\mathcal{X}_d$*, the map from* $\mathcal{L}_{n-d}$ *to* $\mathcal{X}_{n-d}$ *and the map from* $\mathcal{P}_{d,n}$ *to* $\mathcal{X}_d \times \mathcal{X}_{n-d}$ *are proper (namely, they have the property that the preimage of a bounded set is bounded).*

Proof. The projections are the obvious ones; for example, $\pi_{\mathcal{P}_{d,n}\to\mathcal{L}_{d,n}}$ is the projection from (2.8) to the two left components, (2.6), $\pi_{\mathcal{P}_{d,n}\to\mathrm{Gr}(d,n)}$ is the projection to the most left component, $(\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \times \mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}})\backslash \mathrm{SO}_n(\mathbb{R})$, composed with the inverse map $k \mapsto k^{-1}$, etc. Notice that $\pi_{\mathcal{P}_{d,n}\to\mathcal{L}_d\times\mathcal{L}_{n-d}}$, namely the projection from (2.8) to the two right components $\mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_{n-d}(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z}) \times \mathrm{SL}_{n-d}(\mathbb{Z})$ is not well defined unless a choice of a section of $(\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \times \mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}})\backslash \mathrm{SO}_n(\mathbb{R})$ is fixed. This projection is therefore not canonical and *a priori* not continuous. Same holds for $\pi_{\mathcal{L}_{d,n}\to\mathcal{L}_d}$. The fact that the maps into the spaces of shapes are proper is clear, since they are obtained by projecting modulo a compact component. □

PROPOSITION 2.5 *Considering the projections from Proposition 2.4, the following maps are one to one*

$$\begin{aligned}(\pi_{\mathcal{P}_{d,n}\to\mathcal{L}_{d,n}},\pi_{\mathcal{P}_{d,n}\to\mathcal{L}_{n-d}}):\quad & \mathcal{P}_{d,n}\to\quad \mathcal{L}_{d,n}\times\mathcal{L}_{n-d},\\ (\pi_{\mathcal{L}_{d,n}\to\mathcal{L}_d},\pi_{\mathcal{L}_{d,n}\to\mathrm{Gr}(d,n)}):\quad & \mathcal{L}_{d,n}\to\quad \mathcal{L}_d\times\mathrm{Gr}(d,n),\end{aligned}$$

and each satisfies that the measure on the range is the pullback of the measure on the domain. Moreover, the map $\pi_{\mathcal{L}_n\to\mathcal{X}_n}$ satisfies that the volume of the preimage of a subset $\mathcal{E}\subseteq\mathcal{X}_n$ is $\mathrm{vol}_{\mathcal{X}_n}(\mathcal{E})\Upsilon(n)$.

The proof of Proposition 2.5 is in Section 5.1.

3. Counting lattices: Main theorem

The goal of this section is to introduce our main result, Theorem 3.2, which implies Theorems 1.1 and 1.2. Counting results always assume a certain regularity condition on the sets in question, and indeed we require that $[(\Lambda,\Lambda^\pi)]$ and all the other parameters of Λ from Notation 2.3 fall in sets that satisfy the following property:

DEFINITION 3.1 A subset B of an orbifold $\mathcal{M}$ will be called a *boundary controllable set*, or BCS, if for every $x\in\mathcal{M}$ there is an open neighborhood U_x of x such that $U_x\cap\partial B$ is contained in a finite union of embedded C^1 submanifolds of $\mathcal{M}$, whose dimension is strictly smaller than $\dim\mathcal{M}$. In particular, B is a BCS if its (topological) boundary consists of finitely many subsets of embedded C^1 submanifolds.

THEOREM 3.2 *Let $n\geq 3$ and assume that $\Phi\subseteq\mathrm{Gr}(d,n)$, $\mathcal{E}\times\mathcal{F}\subseteq\mathcal{X}_d\times\mathcal{X}_{n-d}$, $\tilde{\mathcal{E}}\times\widetilde{\mathcal{F}}\subseteq\mathcal{L}_d\times\mathcal{L}_{n-d}$, $\Psi\subseteq\mathcal{L}_{d,n}$ and $\Xi\subseteq\mathcal{P}_{d,n}$ are boundary controllable. Then:*

1. The number of primitive d-lattices Λ of covolume at most X with $V_\Lambda\in\Phi$ and $(\mathrm{shape}(\Lambda),\mathrm{shape}(\Lambda^\pi))\in\mathcal{E}\times\mathcal{F}$ is

$$\frac{\mathrm{vol}_{\mathcal{X}_d}(\mathcal{E})\,\mathrm{vol}_{\mathcal{X}_{n-d}}(\mathcal{F})\,\mathrm{vol}_{\mathrm{Gr}(d,n)}(\Phi)}{n\prod_{i=2}^{n}\zeta(i)}\Upsilon(d)\Upsilon(n-d)\cdot X^n+\text{error term}.$$

2. The number of primitive d-lattices Λ of covolume at most X with $V_\Lambda\in\Phi$ and $([\![\Lambda]\!],[\![\Lambda^\pi]\!])\in\tilde{\mathcal{E}}\times\widetilde{\mathcal{F}}$ is

$$\frac{\mathrm{vol}_{\mathcal{L}_d}(\tilde{\mathcal{E}})\,\mathrm{vol}_{\mathcal{L}_{n-d}}(\widetilde{\mathcal{F}})\,\mathrm{vol}_{\mathrm{Gr}(d,n)}(\Phi)}{n\prod_{i=2}^{n}\zeta(i)}\cdot X^n+\text{error term}.$$

3. The number of primitive d-lattices Λ of covolume at most X with $[\Lambda]\in\Psi$ and $[\![\Lambda^\pi]\!]\in\widetilde{\mathcal{F}}$ is

$$\frac{\mathrm{vol}_{\mathcal{L}_{d,n}}(\Psi)\,\mathrm{vol}_{\mathcal{L}_{n-d}}(\widetilde{\mathcal{F}})}{n\prod_{i=2}^{n}\zeta(i)}\cdot X^n+\text{error term}.$$

4. The number of primitive d-lattices Λ of covolume at most X with $[(\Lambda,\Lambda^{\pi})]\in\Xi$ is

$$\frac{\mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)}{n\prod_{i=2}^{n}\zeta(i)}\cdot X^{n}+\text{error term.}$$

For $\tau_n=\left(4n^2\lceil(n-1)/2\rceil\right)^{-1}$ and every $\epsilon>0$, the error term is $\ll_{\epsilon} X^{n(1-\tau_n+\epsilon)}<X^{n-\frac{1}{2n(n+1)}+\epsilon}$ when the sets in question $(\mathcal{E},\mathcal{F},\tilde{\mathcal{E}},\tilde{\mathcal{F}},\Psi,\Xi)$ are bounded, and $\ll_{\epsilon} X^{n(1-\frac{\tau_n}{2n-2})+\epsilon}\leq X^{n-\frac{1}{4n(n^2-1)}+\epsilon}$ when they are not.

Remark 3.3

1. It can be easily shown that the leading constants in any of the part of Theorem 3.2 can also be written as

$$2c_{d,n}\cdot\ \text{product of probability measures of the sets involved,}$$

where $c_{d,n}$ is Schmidt's constant; for example, the leading constant in part (4) is

$$2c_{d,n}\cdot\frac{\mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)}{\mathrm{vol}_{\mathcal{P}_{d,n}}(\mathcal{P}_{d,n})}.$$

Below we demonstrate this computation for part (1) of the theorem, where we prove Theorem 1.1 based on Theorem 3.2. The 2 factor is due to the fact that we count lattices with orientation, so every non-oriented lattice is counted twice.
2. Again comparing to the work of Schmidt, we note that a boundary controllable set is Jordan measurable, and indeed Schmidt (in [**45**]) provides an example for how the asymptotic formula for number of d-lattices with shapes in $\mathcal{E}$ fails when $\mathcal{E}$ is not Jordan measurable.
3. Primitive d-lattices are in one-to-one correspondence with rational subspaces in $\mathbb{R}^n$. These spaces are the rational points on the Grassmanian variety: the projective variety consisting of all the d– dimensional spaces in $\mathbb{R}^n$. Therefore, the aforementioned result of Schmidt can be read as the counting of rational points up to a bounded height in the Grassmannian variety (the height being the covolume of the unique primitive lattice in the space). As such, it provides yet another example where the Manin conjecture [**22**, **39**] on counting rational points in varieties holds. The more refined counting we suggest in Theorem 3.2 (Part (4)) plays a key role in the intensive study of rational points on the Grassmannian conducted in [**8**]. In that paper, the authors confirm a modification to the Manin conjecture suggested by Peyre [**40**] and obtain an equidistribution result for the integral lattices in the rational tangent bundle of the Grassmannian.

In order to deduce Theorems 1.1 and 1.2 from Theorem 3.2, we need to reformulate the latter with $\Lambda^{\perp}$ instead of Λ^{π}. As we have already mentioned, these lattices are dual to one another; denote the dual of a lattice Λ by Λ^{*}.

Theorem 3.4 *A version of Theorem 3.2 holds when replacing (Λ,Λ^{π}) by each of $(\Lambda,\Lambda^{\perp})$, $(\Lambda^{*},\Lambda^{\perp})$ and $(\Lambda^{*},\Lambda^{\pi})$.*

Indeed, note that in all of the pairs above, the right-hand lattice spans the orthogonal subspace to the one spanned by the left-hand lattice. In other words, their homothety classes are elements in $\mathcal{P}_{d,n}$.

Proof of Theorem 3.4. Observe that the three pairs $(\Lambda, \Lambda^{\perp})$, $(\Lambda^{*}, \Lambda^{\perp})$ and $(\Lambda^{*}, \Lambda^{\pi})$ are obtained from the pair (Λ, Λ^{π}) in Theorem 3.2 by taking the dual of Λ, of Λ^{π}, or of both. According to Propositions A.7 and A.8, passing to the dual in either of the lattices is a measure preserving auto-diffeomorphism of $\mathcal{P}_{d,n}$. □

We can now prove the theorems from the introduction.

Proof of Theorem 1.1 *based on Theorem* 3.2. By Theorem 3.4 and part (1) of Theorem 3.2, the number of oriented primitive d-lattices Λ of covolume at most X with $V_\Lambda \in \Phi$ and $(\text{shape}(\Lambda), \text{shape}(\Lambda^{\perp})) \in \mathcal{E} \times \mathcal{F}$ is asymptotic to

$$\text{vol}_{\mathcal{X}_d}(\mathcal{E})\, \text{vol}_{\mathcal{X}_{n-d}}(\mathcal{F})\, \text{vol}_{\text{Gr}(d,n)}(\Phi) \frac{\Upsilon(d)\Upsilon(n-d)}{n \cdot \text{vol}(\mathcal{L}_n)} \cdot X^n,$$

where a bound on the error term is provided in Theorem 3.2. It is only left to show that the leading constant above coincides twice with the one in Theorem 1.1, namely

$$2c_{d,n} \cdot \frac{\text{vol}_{\mathcal{X}_d}(\mathcal{E})}{\|\text{vol}_{\mathcal{X}_d}\|} \cdot \frac{\text{vol}_{\mathcal{X}_{n-d}}(\mathcal{F})}{\|\text{vol}_{\mathcal{X}_{n-d}}\|} \cdot \frac{\text{vol}_{\text{Gr}(d,n)}(\Phi)}{\|\text{vol}_{\text{Gr}(d,n)}\|}.$$

Notice first that, by (2.2) and (2.3),

$$c_{d,n} = \frac{\|\text{vol}_{\text{Gr}(d,n)}\|}{2n} \cdot \frac{\|\text{vol}_{\mathcal{L}_d}\|\|\text{vol}_{\mathcal{L}_{n-d}}\|}{\|\text{vol}_{\mathcal{L}_n}\|}.$$

Now,

$$\frac{\Upsilon(d)\Upsilon(n-d)}{n \cdot \|\text{vol}_{\mathcal{L}_n}\|} =$$

$$= \frac{1}{n} \frac{\|\text{vol}_{\mathcal{L}_d}\|\|\text{vol}_{\mathcal{L}_{n-d}}\|}{\|\text{vol}_{\mathcal{L}_n}\|} \cdot \frac{\Upsilon(d)}{\|\text{vol}_{\mathcal{L}_d}\|} \cdot \frac{\Upsilon(n-d)}{\|\text{vol}_{\mathcal{L}_{n-d}}\|} \cdot \frac{\|\text{vol}_{\text{Gr}(d,n)}\|}{2} \cdot \frac{2}{\|\text{vol}_{\text{Gr}(d,n)}\|}$$

$$= 2c_{d,n} \cdot \frac{1}{\|\text{vol}_{\mathcal{X}_d}\|} \cdot \frac{1}{\|\text{vol}_{\mathcal{X}_{n-d}}\|} \cdot \frac{1}{\|\text{vol}_{\text{Gr}(d,n)}\|}.$$

□

To prove Theorem 1.2 from Theorem 3.2, we need the following claim, which is a follow-up to Proposition 2.5.

PROPOSITION 3.5 *The three maps from Proposition 2.5 have the property that the preimage of a boundary controllable set is also boundary controllable.*

The proof is in Section 5.2.

Proof of Theorem 1.2 based on Theorem 3.2 and Proposition 3.5. In light of Propositions 2.5 and 3.5, it is sufficient to prove equidistribution in the space $\mathcal{P}_{d,n}$, namely that

$$\frac{1}{\#|\Lambda\text{primitive} : \text{covol}(\Lambda)\leq \text{X}|} \cdot \sum_{\{\Lambda:\text{covol}(\Lambda)\leq X\}} f([(\Lambda,\Lambda^{\pi})]) \underset{\longrightarrow}{X\to\infty} \frac{1}{\|\text{vol}_{\mathcal{P}_{d,n}}\|} \cdot \int f d\,\text{vol}_{\mathcal{P}_{d,n}},$$

for every $f \in C_C(\mathcal{P}_{d,n})$. Given such f, it is uniformly continuous, and hence for every $\epsilon > 0$ there exists $\delta > 0$ such that if two points $x, y \in \mathcal{P}_{d,n}$ are of Riemannian distance at most δ, then $|f(x) - f(y)| < \epsilon$. Fix ϵ and the associated δ and consider a finite cover $\{B_i\}$ of supp(f) by balls of radius $\delta/2$. By letting $U_i = B_i - \cup_{j>i} B_j$ for every i, we obtain that $\{U_i\}$ is a finite disjoint cover of supp(f). For a choice of $x_i \in U_i$, notice that for every $x \in \text{supp}(f)$ there exists a unique i such that $x \in U_i$ and in particular x is δ-close to x_i. Thus, $|f(x) - f(x_i)| < \epsilon$. It follows that

$$f = \sum_i f(x_i)\mathbf{1}_{U_i} + O_f(\epsilon),$$

and it is therefore sufficient to prove the claim for $f = \mathbf{1}_U$ with $U \in \{U_i\}$. Clearly

$$\frac{1}{\#|\Lambda\text{primitive} : \text{covol}(\Lambda)\leq \text{X}|} \sum_{\{\Lambda:\text{covol}(\Lambda)\leq X\}} \mathbf{1}_U([(\Lambda,\Lambda^{\pi})]) = \frac{\#|\Lambda\text{primitive} : \text{covol}(\Lambda)\leq \text{X},\ [(\Lambda,\Lambda^{\pi})]\in \text{U}|}{\#|\Lambda\text{primitive} : \text{covol}(\Lambda)\leq \text{X}|}.$$

Notice that the boundary of $U \subset \mathcal{P}_{d,n}$ is a finite union of boundary pieces of balls, so U is boundary controllable. Thus, by part (4) of Theorem 3.2, the above converges as $X \to \infty$ to

$$\frac{\text{vol}_{\mathcal{P}_{d,n}}(U)}{\|\text{vol}_{\mathcal{P}_{d,n}}\|} = \frac{\int \mathbf{1}_U d\,\text{vol}_{\mathcal{P}_{d,n}}}{\|\text{vol}_{\mathcal{P}_{d,n}}\|}.$$

□

4. Refined Iwasawa components of $\mathbf{SL}_n(\mathbb{R})$

4.1. *Refining the Iwasawa decomposition of* $\text{SL}_n(\mathbb{R})$

Set $G = G_n := \text{SL}_n(\mathbb{R})$ and let $G = KAN$ be the Iwasawa decomposition of G, meaning that $K = K_n$ is $\text{SO}_n(\mathbb{R})$, $A = A_n$ is the diagonal subgroup in G with positive entries and $N = N_n$ is the subgroup of upper unipotent matrices. We also let $P_n = A_n N_n$, the group of upper triangular matrices of determinant one with positive diagonal entries. Consider the following isomorphic copy of

$\mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_{n-d}(\mathbb{R})$ inside G,

$$G'' := \left[\begin{array}{c|c} G_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & G_{n-d} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{SL}_d(\mathbb{R}) & 0_{d,n-d} \\ \hline 0_{n-d,d} & \mathrm{SL}_{n-d}(\mathbb{R}) \end{array}\right].$$

Write $G'' = K''A''N''$ for the Iwasawa decomposition of G'', namely

$$\begin{aligned} K'' &:= K \cap G'' = \left[\begin{array}{c|c} K_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & K_{n-d} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{SO}_d(\mathbb{R}) & 0_{d,n-d} \\ \hline 0_{n-d,d} & \mathrm{SO}_{n-d}(\mathbb{R}) \end{array}\right], \\ A'' &:= A \cap G'' = \left[\begin{array}{c|c} A_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & A_{n-d} \end{array}\right], \\ N'' &:= N \cap G'' = \left[\begin{array}{c|c} N_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & N_{n-d} \end{array}\right]. \end{aligned}$$

Let

$$P'' := A''N'' \text{ and } Q := KP''$$

(note that Q is not a group, but it is a smooth manifold). To conclude the definition of the refined Iwasawa decomposition, we define K', A', N' that complete K'', A'', N'' to K, A and N, respectively. Let

$$N' := \left[\begin{array}{c|c} \mathrm{I}_d & \mathbb{R}^{d,n-d} \\ \hline 0_{n-d,d} & \mathrm{I}_{n-d} \end{array}\right], \quad A' := \left[\begin{array}{c|c} a^{\frac{1}{d}}\,\mathrm{I}_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & a^{-\frac{1}{n-d}}\,\mathrm{I}_{n-d} \end{array}\right],$$

and observe that $N = N''N'$, $A = A''A'$, and that A' is a one-parameter subgroup of A which commutes with G''. Often we would like to restrict to the upper $d \times d$ (resp. lower $(n-d) \times (n-d)$) block of G'', hence we denote

$$G''_d := \left[\begin{array}{c|c} G_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & \mathrm{I}_{n-d} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{SL}_d(\mathbb{R}) & 0_{d,n-d} \\ \hline 0_{n-d,d} & \mathrm{I}_{n-d} \end{array}\right]$$

(resp. $G''_{n-d} := \left[\begin{array}{c|c} \mathrm{I}_d & 0_{d,n-d} \\ \hline 0_{n-d,d} & G_{n-d} \end{array}\right]$). Similarly for K'', P'' and A''.

Fix a transversal (Borel measurable set of representatives) K' of the diffeomorphism $K/K'' \to \mathrm{Gr}(d,n)$, meaning that $K = K'K''$; by [27, Lem. 3.4 (ii)], it is possible to choose K' such that if $\Phi \subseteq \mathrm{Gr}(d,n)$ and $\mathcal{B} \subseteq K''$ are boundary controllable, and K'_Φ is the image of Φ in K', then $K'_\Phi \mathcal{B} \subseteq K$ is boundary controllable in K. Note that Q is also $K'G''$, and we let

$$Q_d := K'G''_d.$$

Note that $Q = Q_d G''_{n-d}$ and that Q_d is not a manifold. Then the refined Iwasawa (or RI, for short) decomposition is given by

$$G = K'K''A''A'N''N' = K'G''A'N' = KP''A'N' = QA'N' = Q_d G''_{n-d} A'N'. \tag{4.1}$$

4.2. *Refined Iwasawa decomposition of the Haar measure*

It is well known (for example [32, Prop. 8.43]) that a Haar measure μ on $\mathrm{SL}_n(\mathbb{R})$ can be decomposed w.r.t. the Iwasawa components of $\mathrm{SL}_n(\mathbb{R})$:

$$\mu = \mu_K \times \mu_A \times \mu_N$$

($K = K_n$, $A = A_n$, $N = N_n$), where μ_K and μ_N are Haar measures on K and N, and μ_A is absolutely continuous w.r.t. the Haar measure on A. Specifically, it is given by

$$\mu_A = \mu_{A_n} = \prod_{j=1}^{n} e^{-h_j} dh_j,$$

where for $\underline{h} = (h_1, \dots, h_{n-1}) \in \mathbb{R}^{n-1}$ we let

$$a_{\underline{h}} = (e^{-h_1/2}, e^{(h_1-h_2)/2}, \dots, e^{(h_{n-2}-h_{n-1})/2}, e^{h_{n-1}/2}) \in A.$$

In this subsection, we extend the decomposition of μ to the RI coordinates. Set

$$\mu_{A'} = e^{nt} dt,$$

where for $t \in \mathbb{R}$ we let

$$a'_t = \begin{bmatrix} e^{\frac{t}{d}} \mathrm{I}_d & 0 \\ 0 & e^{-\frac{t}{n-d}} \mathrm{I}_{n-d} \end{bmatrix},$$

and let $\mu_{A''} = \mu_{A_d} \times \mu_{A_{n-d}} = \mu_{A''_d} \times \mu_{A''_{n-d}}$. Then μ_A satisfies

$$\mu_A = \mu_{A'} \times \mu_{A''},$$

where neither of these measures is Haar. Let μ_N, $\mu_{N'}$, μ_{N_i} and $\mu_{N''}$ be the Lebesgue measures on the associated spaces $\mathbb{R}^{\dim N}$, $\mathbb{R}^{\dim N'}$, etc. Then,

$$\mu_N = \mu_{N''} \times \mu_{N'}.$$

Setting $K_i = \mathrm{SO}_i(\mathbb{R})$, we let μ_{K_i} for $i = n, d, n-d$ be the Haar measure on K_i satisfying

$$\|\mu_{K_i}\| = \prod_{j=1}^{i-1} \mathrm{Leb}(\mathbb{S}^j) = \frac{i!}{2} \prod_{i=1}^{i} \mathfrak{B}(i), \tag{4.2}$$

where Leb is the Lebesgue measure and $\mathbb{S}^j$ is the unit sphere in $\mathbb{R}^{j+1}$. In particular, $\|\mu_{K''}\| = \|\mu_{K_d}\| \|\mu_{K_{n-d}}\|$. Since K' parameterizes $\mathrm{Gr}(d,n) = K/K''$, we can endow it with a measure $\mu_{K'}$ that is the pullback of a K-invariant Radon measure on $\mathrm{Gr}(d,n)$ so that $\mu_K = \mu_{K'} \times \mu_{K''}$. All in all, our choices of μ_K, μ_N determine μ, and similarly the choices of $\mu_{K''}, \mu_{A''}$ determine the Haar measure $\mu_{G''}$:

$$\mu_{G''} = \mu_{K''} \times \mu_{A''} \times \mu_{N''} = \mu_{G''_d} \times \mu_{G''_{n-d}}, \text{ where } \mu_{G''_i} = \mu_{G_i} = \mu_{K_i} \times \mu_{A_i} \times \mu_{N_i},$$

and therefore also

$$\mu = \mu_{K'} \times \mu_{G''} \times \mu_{A'} \times \mu_{N'}.$$

Naturally, $\mu_P = \mu_A \times \mu_N$ and $\mu_{P''} = \mu_{A''} \times \mu_{N''} = \mu_{P''_d} \times \mu_{P''_{n-d}}$, where the measures on the Borel subgroups are right Haar measures.

It is left to determine the measures μ_Q and μ_{Q_d}. Since $Q = KP''$, we endow it with the measure $\mu_Q = \mu_K \times \mu_{P''}$. As $\mu_K = \mu_{K'} \times \mu_{K''}$, $\mu_{P''} = \mu_{A''} \times \mu_{N''}$ and $\mu_{G''} = \mu_{K''} \times \mu_{A''} \times \mu_{N''}$, we also have that $\mu_Q = \mu_{K'} \times \mu_{G''}$. Since $Q_d = K'G''_d$, we endow it with the measure $\mu_{Q_d} = \mu_{K'} \times \mu_{G''_d}$. All in all, we have that

$$\mu = \mu_Q \times \mu_{A'} \times \mu_{N'} = \mu_{Q_d} \times \mu_{G''_{n-d}} \times \mu_{A''} \times \mu_{N''},$$

meaning that the measure μ decomposes in a manner that it compatible with (4.1).

4.3. *RI components and parameters of lattices*

The RI components of an element $g \in \mathrm{SL}_n(\mathbb{R})$ encode certain information regarding the lattices spanned by its columns, as explained in the proposition below. To state it, we extend the definition of primitiveness and of factor lattices from d-lattices inside $\mathbb{Z}^n$, to d-lattices inside any full lattice of $\mathbb{R}^n$.

Definition 4.1 Assume that a d-lattice Λ is contained inside a full lattice $\Delta < \mathbb{R}^n$. We say that Λ is *primitive* inside Δ if $\Lambda = \Delta \cap V_\Lambda$—in other words, if there is no subgroup of Δ that lies inside V_Λ and properly contains Λ. Given a d-lattice Λ that is primitive inside Δ, we define the *factor lattice of Λ (w.r.t. Δ)*, denoted $\Lambda^{\pi,\Delta}$, as the orthogonal projection of Δ into the space $(V_\Lambda)^\perp$. When Λ is primitive inside $\mathbb{Z}^n$, we omit the explicit mentioning of $\mathbb{Z}^n$ and say just that Λ is primitive. Accordingly, we denote its factor lattice in $\mathbb{Z}^n$ by just Λ^π and say that it is the factor lattice of Λ.

Let us also introduce the following notation:

Notation 4.2 For $g \in \mathrm{SL}_n(\mathbb{R})$ and $d \in \{1, \dots, n-1\}$, let Λ_g denote the lattice spanned by the columns of g and let Λ_g^d denote the lattice spanned by the first d columns of g. Let $\Lambda_g{}^{\underline{d}}$ denote the lattice spanned by the last d columns of Λ_g. Finally, given a lattice Λ in $\mathbb{R}^n$, recall that V_Λ denotes the linear space spanned by Λ.

Proposition 4.3 *Let $g \in \mathrm{SL}_n(\mathbb{R})$ and $1 \le d \le n-1$. Denote $\Lambda = \Lambda_g^d$, the lattice spanned by the first d columns of g, and $\Lambda^\sharp = \Lambda^{\pi,\Lambda_g}$, the factor lattice of Λ w.r.t. Λ_g. Write $g = kan = qa'_tn'$ with $q = k'_U g'' = k'_U k'' a''_{\underline{s},\underline{w}} n''$, where $a''_{\underline{s},\underline{w}} = \begin{bmatrix} a_d \in A_d & 0 \\ 0 & a_{n-d} \in A_{n-d} \end{bmatrix}$ and k'_U is the image of $U \in \mathrm{Gr}(d,n)$. Let $g'' = \begin{bmatrix} g_d & 0 \\ 0 & g_{n-d} \end{bmatrix}$,*

$g''_d = \left[\begin{smallmatrix} g_d & 0 \\ 0 & \mathrm{I}_{n-d}\end{smallmatrix}\right]$ and $g''_{n-d} = \left[\begin{smallmatrix} \mathrm{I}_d & 0 \\ 0 & g_{n-d}\end{smallmatrix}\right]$, and similarly for a'' and p''. The RI components of g represent parameters related to Λ in the following way:

$$\begin{array}{llclllcl}
(i) & U & = & V_\Lambda & (i)^\sharp & U^\perp & = & V_{\Lambda^\sharp} \\
(ii) & e^t & = & \mathrm{covol}(\Lambda) & (ii)^\sharp & e^{-t} & = & \mathrm{covol}(\Lambda^\sharp) \\
(iii) & e^{\frac{it}{d}-\frac{s_i}{2}} & = & \mathrm{covol}(\Lambda^i) & (iii)^\sharp & e^{-\frac{jt}{n-d}-\frac{w_j}{2}} & = & \mathrm{covol}(\left(\Lambda^\sharp\right)^j) \\
(iv) & [\Lambda_q^d] = [\Lambda_{k' g''_d}] & = & [\Lambda] & (iv)^\sharp & [\Lambda_q^{n-d}] & = & [\Lambda^\sharp] \\
(v) & \left[\!\left[\Lambda_{g_d}\right]\!\right] & = & [\![\Lambda]\!] & (v)^\sharp & \left[\!\left[\Lambda_{g_{n-d}}\right]\!\right] & = & \left[\!\left[\Lambda^\sharp\right]\!\right] \\
(vi) & \mathrm{shape}(\Lambda_{p_d}) & = & \mathrm{shape}(\Lambda) & (vi)^\sharp & \mathrm{shape}(\Lambda_{p_{n-d}}) & = & \mathrm{shape}(\Lambda^\sharp)
\end{array}$$

for every $1 \le i \le d$ and $1 \le j \le n-d$, and (vii) $[(\Lambda_q^d, \Lambda_q^{n-d})] = [(\Lambda_g^d, (\Lambda_g^d)^{\pi,\Lambda_g})]$.

Proof. Since the columns of k are obtained by performing the Gram–Schmidt orthogonalization procedure on the columns of g, we have that the first d columns of k span V_Λ. Since $k' = k\,(k'')^{-1}$ where $k'' \in \left[\begin{smallmatrix} \mathrm{SO}_d(\mathbb{R}) & 0 \\ 0 & \mathrm{SO}_{n-d}(\mathbb{R})\end{smallmatrix}\right]$, the first d columns of k' span (and are in fact an orthonormal basis to) the same space as the first d columns of k, which is V_Λ. This proves (i) and $(i)^\sharp$, by definition of orientation on $V_\Lambda^\perp$.

Write $g(n')^{-1}(a')^{-1} = k'g'' = q$; right multiplication by an element of N' does not change the first d columns of g, and right multiplication by $(a')^{-1}$ multiplies each of these columns by $e^{-\frac{t}{d}}$. This means that $e^{-\frac{t}{d}}\Lambda = \Lambda_q^d$, proving (iv), and (v), (vi) directly follow. Also, notice that Λ_q^d has covolume one, since it is a rotation of the unimodular $\Lambda^d_{g''_d}$; then, by considering the covolumes of the lattices on both sides, we obtain $e^{-t}\,\mathrm{covol}(\Lambda) = 1$, proving (ii).

Considering g'', it is clear that the lattice $\Lambda_{g_{n-d}}$ is the factor lattice of Λ_{g_d} w.r.t. $\Lambda_{g''}$. Rotating it by left multiplication by k', we have that the lattice Λ_q^{n-d} is the factor lattice of Λ_q^d w.r.t. Λ_q. But since $q = g(n')^{-1}(a')^{-1}$, we may also say that Λ_q^{n-d} is the factor lattice of $\Lambda^d_{g(n')^{-1}(a')^{-1}} < V_\Lambda$ w.r.t. $\Lambda_{g(n')^{-1}(a')^{-1}}$, namely it is the projection of $\Lambda_{g(n')^{-1}(a')^{-1}}$ to $V_\Lambda^\perp$. Noticing that this projection kills the contribution of $(n')^{-1}$, as well as the first d columns of $g(n')^{-1}(a')^{-1}$, we remain only with the projection of the lattice spanned by the last $n-d$ columns of g, on which $(a')^{-1}$ acts as multiplication by $e^{-\frac{t}{n-d}}$. In other words, Λ_q^{n-d} is in fact $e^{-\frac{t}{n-d}}\Lambda^\sharp$. This proves $(iv)^\sharp$, from which $(v)^\sharp$ and $(vi)^\sharp$ follow, and then similarly to how we proved (ii) we also obtain $(ii)^\sharp$ (The fact that $\mathrm{covol}(\Lambda^\sharp) = \mathrm{covol}(\Lambda_g)/\mathrm{covol}(\Lambda)$ is also proved in the Appendix, Proposition B.4). Since Λ_q^d and Λ_q^{n-d} span orthogonal subspaces and are both unimodular, then (iv) and $(iv)^\sharp$ imply (vii). It is well known that if $g = kan$ and $a = \mathrm{diag}(\alpha_1, \dots, \alpha_r)$, then $\prod_1^i \alpha_j = \mathrm{covol}(\Lambda_g^i)$. Since the lattice $\Lambda^d_{g''_d}$ is a rotation of $e^{-t/d}\Lambda$, it has the same covolume and partial covolumes; writing $g''_d = k''_d a''_d n''_d$, we have that the product of the first $1 \le i \le d$ entries of a''_d is $e^{-\frac{it}{d}}\,\mathrm{covol}(\Lambda^i_{g''_d}) = e^{-\frac{it}{d}}\,\mathrm{covol}(\Lambda_g^i)$. On the other hand, since we have $a''_d = \mathrm{diag}(e^{-\frac{s_1}{2}}, e^{\frac{s_1-s_2}{2}}, \dots, e^{\frac{s_{d-2}-s_{d-1}}{2}}, e^{\frac{s_{d-1}}{2}}, 1, \dots, 1)$, comparing the products of the first i elements implies $\mathrm{covol}(\Lambda^i)\cdot e^{-\frac{it}{d}} = e^{-\frac{s_i}{2}}$ and proves (iii). Doing the same for $a''_{n-d} = \mathrm{diag}(1, \dots, 1, e^{-\frac{w_1}{2}}, e^{\frac{w_1-w_2}{2}}, \dots, e^{\frac{w_{n-d-2}-w_{n-d-1}}{2}}, e^{\frac{w_{n-d-1}}{2}})$, while recalling that $\Lambda^{n-d}_{g''_{n-d}}$ is a rotation of $e^{\frac{t}{n-d}}\Lambda^\sharp$, implies $e^{\frac{jt}{n-d}}\,\mathrm{covol}((\Lambda^\sharp)^j) = e^{-\frac{w_j}{2}}$ for every $1 \le j \le n-d$ and proves $(iii)^\sharp$. □

5. Spaces of lattices and their sets of representatives

The spaces $\mathcal{X}_d$ and $\mathcal{L}_d$, as well as $\mathcal{L}_{d,n}$ (by (2.6)) and $\mathcal{P}_{d,n}$ (by (2.8)), are quotients of smooth manifolds by discrete subgroups and can therefore be represented by fundamental domains for the action of these discrete groups on the said manifolds. In this section we will establish that these manifolds correspond (through a diffeomorphism, except for the case of $\mathcal{L}_{d,n}$) to certain RI components in $\mathrm{SL}_n(\mathbb{R})$, implying that the spaces in question can be identified with 'fundamental domains' inside the associated RI components. In other words, we shall identify the spaces $\mathcal{X}_d$, $\mathcal{L}_d$, etc. with nice measurable subsets lying inside the RI components of $\mathrm{SL}_n(\mathbb{R})$, and this will allow us to translate the proof of Theorem 3.2 into a problem of counting integral matrices in $\mathrm{SL}_n(\mathbb{R})$. Moreover, this identification will allow us to interpret the projections from Prop. 2.4 very concretely, thus completing the analysis of the interactions between the spaces $\mathcal{P}_{d,n}$, $\mathcal{L}_{d,n}$, $\mathcal{L}_d$, $\mathcal{X}_d$ and $\mathrm{Gr}(d,n)$: we will finally be able to prove Propositions 2.5 and 3.5 and then reduce the proof of Theorem 3.2 to proving only its fourth part.

5.1. *Constructing fundamental domains*

Consider the smooth manifolds

$$\mathcal{M}_{\mathcal{X}_i} := \mathrm{SO}_i(\mathbb{R})\backslash \mathrm{SL}_i(\mathbb{R}),\ \mathcal{M}_{\mathcal{L}_{d,n}} := \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}}\backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R})$$

and

$$\mathcal{M}_{\mathcal{P}_{\mathrm{d,n}}} := \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \times \mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}}\backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_{n-d}(\mathbb{R}).$$

As we have seen in Section 2, each of the spaces $\mathcal{X}_i$, $\mathcal{L}_{d,n}$ and $\mathcal{P}_{\mathrm{d,n}}$ is the orbit space of a discrete group in the associated manifold. It is not hard to see (but yet we prove in Prop. 5.1) that these manifolds are identified with certain RI components of $\mathrm{SL}_n(\mathbb{R})$—$\mathcal{M}_{\mathcal{X}_i}$ with P_i (group of unimodular upper triangular matrices), $\mathcal{M}_{\mathcal{L}_{d,n}}$ with Q_d and $\mathcal{M}_{\mathcal{P}_{\mathrm{d,n}}}$ with Q. Moreover, the measures on these manifolds that were defined in 2 using (2.1) correspond to the measures μ_{P_i}, μ_{Q_d} and μ_Q, respectively. As a result, the action of the discrete groups on the manifolds $\mathcal{M}$ can be pushed to the corresponding RI components, thus representing the spaces $\mathcal{X}_i$, $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$ with fundamental domains inside P_i, Q_d and Q, respectively.

As a first step, let us recall the standard construction for fundamental domains of $\mathrm{SL}_i(\mathbb{Z})$ inside $\mathrm{SL}_i(\mathbb{R})$ and in $\mathrm{SO}_i(\mathbb{R})\backslash\mathrm{SL}_i(\mathbb{R}) \simeq P_i$. The one in P_i is essentially due to Siegel and is made explicit in [**26**, **28**, **45**Sec. 7]. We will not require the exact definition but just mention the fact that boundary of F_i is a finite union of (strict) submanifolds of P_i ([**45**, pp. 48–49]), making it boundary controllable, and that in the case of $i=2$, where P_2 is diffeomorphic with the hyperbolic upper half plane and F_2 is the well-known subset depicted in Fig. 1.

From F_i, construct a fundamental domain for $\mathrm{SL}_i(\mathbb{Z})$ in $\mathrm{SL}_i(\mathbb{R})$ as follows: Let

$$\widetilde{F}_i = \bigcup_{z\in F_i} K_z \cdot z \tag{5.1}$$

([**28**, Thm. 7.10 and Prop. 7.13]), where for every $z \in F_i$, $K_z \subset \mathrm{SO}_i(\mathbb{R})$ is a boundary controllable fundamental domain for $\mathrm{Sym}^+(\Lambda_z)$, the finite group of rotations that preserve Λ_z (the lattice spanned by the columns of z). There is a finite number of possible symmetry groups up to conjugation, and for

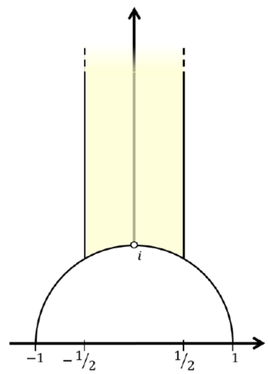


Figure 1. F_2, a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ in P_2 (the hyperbolic upper half plane).

every $z \in \mathrm{int}(F_i)$ this group is the center, $Z(\mathrm{SO}_i(\mathbb{R}))$ [**45**]. Denote the generic fiber (above $\mathrm{int}(F_i)$) by K_{gen}. Then

$$\widetilde{F}_i = K_{\mathrm{gen}} \cdot \mathrm{int}(F_i) \cup \bigcup_{j=1}^{\kappa(n)} K_{z_j} \cdot \{z \in \partial F_n : \mathrm{Sym}^+(\Lambda_z) = \mathrm{Sym}^+(\Lambda_{z_j})\}. \tag{5.2}$$

Now we are ready to prove that the spaces appearing in Theorem 3.2 are represented by fundamental domains in the corresponding RI components.

Proposition 5.1 *The following are measurable sets of representatives for the corresponding spaces:*

- $F_i \subset P_i$ *for the spaces* $\mathcal{X}_i$.
- $\widetilde{F}_i \subset G_i$ *for the spaces* $\mathcal{L}_i$.
- $K'\widetilde{F_d} \subset Q_d$ *for the space* $\mathcal{L}_{d,n}$.
- $K'(\widetilde{F_d} \times \widetilde{F_{n-d}}) \subset Q$ *for the space* $\mathcal{P}_{d,n}$.

The measures vol_{space} *defined on these spaces in Section 2 are pushforwards of the measures* $\mu_{component}$ *defined in Section 4.2 on the ambient RI component, restricted to the corresponding set of representatives.* (*For example,* $\mathrm{vol}_{\mathcal{P}_{d,n}}$ *is the pullback of* μ_Q *restricted to* $K'(\widetilde{F_d} \times \widetilde{F_{n-d}})$). *Moreover, the projections from Diagram* (2.9) *correspond to the projections between these sets of representatives* (*For example,* $\pi_{\mathcal{P}_{d,n} \to \mathcal{L}_{d,n}}$ *corresponds to* $k' g''_d g''_{n-d} \mapsto k' g''_d$ *from* $K'(\widetilde{F_d} \times \widetilde{F_{n-d}}) \to K'\widetilde{F_d}$ *or* $\pi_{\mathcal{L}_{d,n} \to \mathcal{X}_d}$ *corresponds to* $k' g''_d = k' k''_d p''_d \mapsto p''_d$ *from* $K'\widetilde{F_d} \to F_d$*, etc.*).

Recall that $\mu_{K'}$ is by definition the pullback of $\mathrm{vol}_{\mathrm{Gr}(d,n)}$, so the statement about the measures from Proposition 5.1 is valid also for K' and $\mathrm{Gr}(d,n)$.

Proof. The manifolds $\mathcal{M}_{\mathcal{X}_i}$, $\mathcal{M}_{\mathcal{L}_{d,n}}$ and $\mathcal{M}_{\mathcal{P}_{d,n}}$ are easily identified with the RI components P_i, Q_d and Q, respectively. The identifications $\mathcal{M}_{\mathcal{X}_i} \leftrightarrow P_i$ and $\mathcal{M}_{\mathcal{P}_{d,n}} \leftrightarrow Q$ are diffeomorphisms, and the identification $\mathcal{M}_{\mathcal{L}_{d,n}} \leftrightarrow Q_d$ is a Borel bijection (it cannot be a homeomorphism since Q_d is not a manifold). We claim that under the said identifications, the measures on the manifolds $\mathcal{M}$ (as defined in Section 2) correspond to the measures on the associated RI components. For example, the measure on $\mathcal{M}_{\mathcal{X}_i} = \mathrm{SO}_i(\mathbb{R})\backslash\mathrm{SL}_i(\mathbb{R})$ was defined as the unique (up to rescaling) measure on $\mathcal{M}_{\mathcal{X}_i}$ that satisfies (2.1) with $Y = \mathrm{SL}_i(\mathbb{R})$ and $\mathbf{G} = \mathrm{SO}_i(\mathbb{R})$ and the associated Haar measures; but, since μ_{P_i} satisfies $\mu_{\mathrm{SL}_i(\mathbb{R})} = \mu_{\mathrm{SO}_i(\mathbb{R})} \times \mu_{P_i}$, μ_{P_i} (or rather, its pushforward to $\mathcal{M}_{\mathcal{X}_i}$) must be a positive scalar multiple of the invariant measure on $\mathcal{M}_{\mathcal{X}_i}$. Similar considerations apply for $\mathcal{M}_{\mathcal{L}_{d,n}}$ with μ_{Q_d} and $\mathcal{M}_{\mathcal{P}_{d,n}}$ with μ_Q. To see that the scalars equal one, we must show that the normalizations correspond, namely that $\mu_{P_i}(F_i) = \|\mathrm{vol}_{\mathcal{X}_i}\|$, $\mu_{G_i}(\widetilde{F}_i) = \|\mathrm{vol}_{\mathcal{L}_i}\|$, $\mu_{Q_d}(K'\widetilde{F_d}) = \|\mathrm{vol}_{\mathcal{L}_{d,n}}\|$ and $\mu_Q(K'(\widetilde{F_d} \times \widetilde{F_{n-d}})) = \|\mathrm{vol}_{\mathcal{P}_{d,n}}\|$. For $\widetilde{F}_i$, we refer to [**23**] for a computation showing that

$$\mu_{G_i}(\widetilde{F}_i) = \prod_{i=1}^{d-1} \mathrm{Leb}(\mathbb{S}^i) \cdot \prod_{i=2}^{d} \zeta(i) / \mu_{\mathrm{SO}_i(\mathbb{R})}(\mathrm{SO}_i(\mathbb{R})).$$

Then by our choice (4.2) of $\mu_{K_i} = \mu_{\mathrm{SO}_i(\mathbb{R})}$ we have that

$$\mu_{G_i}(\widetilde{F}_i) = \prod_{i=2}^{d} \zeta(i),$$

corresponding to $\|\mathrm{vol}_{\mathcal{L}_i}\|$ by (2.5). For F_i, we have from (5.2) that

$$\mu_{\mathrm{SL}_n(\mathbb{R})}(\widetilde{F}_i) = \mu_{P_i}(F_n) \cdot \mu_{\mathrm{SO}_i(\mathbb{R})}(\mathrm{SO}_i(\mathbb{R}))/[Z(\mathrm{SO}_i(\mathbb{R})) : \mathrm{SO}_i(\mathbb{R})],$$

coinciding with $\|\mathrm{vol}_{\mathcal{X}_i}\|$ by (2.3). For $K'\widetilde{F_d}$, we have that $\mu_{Q_d} = \mu_{K'} \times \mu_{G_d}$ and therefore

$$\mu_{Q_d}(K'\widetilde{F_d}) = \mu_{K'}(K')\mu_{G_d}(\widetilde{F_d}).$$

Since $\|\mu_{K'}\|$ is the volume of $\mathrm{Gr}(d,n)$, the above agrees with $\|\mathrm{vol}_{\mathcal{L}_{d,n}}\|$ by (2.7). The proof for $K'(\widetilde{F_d} \times \widetilde{F_{n-d}})$ is similar.

The fact that the projections between the spaces (Diagram (2.9)) correspond to the projections between the fundamental domains is a consequence of Prop. 4.3. □

The fundamental domains corresponding to the spaces $\mathrm{Gr}(d,n)$, $\mathcal{X}_d$, $\mathcal{L}_d$, $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$ allow us to complete the analysis of the interactions between these spaces, by proving Propositions 2.5 and 3.5. Let us introduce a notation for the image of a subset of any of these spaces in the associated set of representatives:

$Q_\Xi \subseteq K'(\widetilde{F_d} \times \widetilde{F_{n-d}}) \subset Q$	is the image of	$\Xi \subseteq \mathcal{P}_{d,n}$,
$(Q_d)_\Psi \subseteq K'(\widetilde{F_d} \times \mathrm{I}_{n-d}) \subset Q_d$	is the image of	$\Psi \subseteq \mathcal{L}_{d,n}$,
$G''_{\tilde{\mathcal{E}}\times\tilde{\mathcal{F}}} \subseteq \widetilde{F_d} \times \widetilde{F_{n-d}} \subset G''$	is the image of	$\tilde{\mathcal{E}} \times \tilde{\mathcal{F}} \subseteq \mathcal{L}_d \times \mathcal{L}_{n-d}$,
$P''_{\mathcal{E}\times\mathcal{F}} \subseteq F_d \times F_{n-d}$	is the image of	$\mathcal{E} \times \mathcal{F} \subseteq \mathcal{X}_d \times \mathcal{X}_{n-d}$,
$K'_\Phi \subseteq K'$	is the image of	$\Phi \subseteq \mathrm{Gr}(d,n)$.

Proof of Proposition 2.5. The fact that the maps from $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$ are one to one and pushforward, the measures is a consequence of Prop. 5.1. As for the map $\pi_{\mathcal{L}_n\to\mathcal{X}_n}$, let $\tilde{\mathcal{E}} \subseteq \mathcal{L}_n$ be the inverse image of $\mathcal{E} \subseteq \mathcal{X}_n$, and let $G = PK$ where $G = \mathrm{SL}_n(\mathbb{R})$, $P = P_n$ the subgroup of upper triangular matrices, and $K = \mathrm{SO}_n(\mathbb{R})$. By the construction (5.1) of $\widetilde{F_n}$ from F_n, we have that

$$G_{\widetilde{\mathcal{E}}} = \bigcup_{z \in P_{\mathcal{E}} \subseteq F_n} K_z \cdot z,$$

and as a result (similarly to (5.2)),

$$G_{\widetilde{\mathcal{E}}} = K_{\mathrm{gen}} \cdot (P_{\mathcal{E}} \cap \mathrm{int}(F_n)) \cup \bigcup_{i=1}^{\varkappa(n)} K_{z_i} \cdot \{z \in P_{\mathcal{E}} \cap \partial F_n : \mathrm{Sym}^+(\Lambda_z) = \mathrm{Sym}^+(\Lambda_{z_i})\}.$$

Apart from the first set in the union they all have measure zero, and therefore, using $\mu_G = \mu_K \times \mu_P$,

$$\mu_G(G_{\widetilde{\mathcal{E}}}) = \mu_K(K_{\mathrm{gen}}) \cdot \mu_P(P_{\mathcal{E}} \cap \mathrm{int}(F_n)) = \mu_K(K)\mu_P(P_{\mathcal{E}})/\#|Z(K)|.$$

This concludes the proof, since from Prop. 5.1 we have that $\mathrm{vol}_{\mathcal{L}_n}(\tilde{\mathcal{E}}) = \mu_G(G_{\widetilde{\mathcal{E}}})$ and $\mathrm{vol}_{\mathcal{X}_n}(\mathcal{E}) = \mu_P(P_{\mathcal{E}})$. □

5.2. *Reduction to the space of pairs*

The goal of this section is to reduce the proof of Theorem 3.2 to its fourth part, the one about equidistribution in $\mathcal{P}_{d,n}$. The main ingredient is Proposition 3.5, which we now prove. The proof will make use of the following:

LEMMA 5.2 *The image of a BCS in any of the spaces appearing in Proposition 5.1 inside the associated set of representatives is also a BCS. The opposite holds as well (For example, $\Xi \subseteq \mathcal{P}_{d,n}$ is boundary controllable if and only if Q_Ξ is).*

The proof is not hard, and we omit it (for a proof in much greater generality, see [**28**, Prop. 6.5 and 6.7]).

Proof of Proposition 3.5. We begin with the map $\pi_{\mathcal{L}_n \to \mathcal{X}_n}$. Let $\tilde{\mathcal{E}} \subseteq \mathcal{L}_n$ be the preimage of $\mathcal{E} \subseteq \mathcal{X}_n$ and then, as in the proof of Proposition 2.5,

$$G_{\widetilde{\mathcal{E}}} = K_{\mathrm{gen}} \cdot (P_{\mathcal{E}} \cap \mathrm{int}(F_n)) \cup \bigcup_{i=1}^{\varkappa(n)} K_{z_i} \cdot \overbrace{\{z \in P_{\mathcal{E}} \cap \partial F_n : \mathrm{Sym}^+(\Lambda_z) = \mathrm{Sym}^+(\Lambda_{z_i})\}}^{P_{\mathcal{E}}(i)}.$$

We claim that $G_{\widetilde{\mathcal{E}}}$ is a BCS. First of all, every $P_{\mathcal{E}}(i)$ is contained in ∂F_n and is therefore a BCS of measure zero in P. Moreover, $\mathrm{int}(F_n)$ is a BCS, because F_n is, and $P_{\mathcal{E}}$ is a BCS by Lemma 5.2, hence $P_{\mathcal{E}} \cap \mathrm{int}(F_n)$ is a BCS. As every K_z (including K_{gen}) is a BCS, we have that the image of $G_{\widetilde{\mathcal{E}}}$ in $K \times P$ under the diffeomorphism

$$g = kp \mapsto (k, p)$$

is boundary controllable, as a finite union of direct product of such. Hence $G_{\widetilde{\mathcal{E}}}$ is boundary controllable, which by Lemma 5.2 means that $\tilde{\mathcal{E}}$ is also.

We proceed to consider the map $(\pi_{\mathcal{L}_{d,n}\to\mathcal{L}_d}, \pi_{\mathcal{L}_{d,n}\to\mathrm{Gr}(d,n)}) : \mathcal{L}_{d,n} \to \mathcal{L}_d \times \mathrm{Gr}(d,n)$. Assume that $\Psi \subseteq \mathcal{L}_{d,n}$ is the inverse image of $\tilde{\mathcal{E}} \times \Phi$, which (by Prop. 5.1) means that

$$(Q_d)_\Psi = K'_\Phi (G''_d)_{\tilde{\mathcal{E}}}.$$

Let $\mathcal{E} = \pi^{-1}_{\mathcal{L}_n\to\mathcal{X}_n}(\tilde{\mathcal{E}}) \subseteq \mathcal{X}_d$. Then

$$K'_\Phi (G''_d)_{\tilde{\mathcal{E}}} = K'_\Phi \cdot \bigcup_{z\in P_\mathcal{E}} K_z \cdot z = K'_\Phi K_{\mathrm{gen}} \cdot (P_\mathcal{E} \cap \mathrm{int}(F_n)) \cup \bigcup_{i=1}^{\kappa(n)} K'_\Phi K_{z_i} \cdot P_\mathcal{E}(i).$$

The sets $K'_\Phi K_z$ are boundary controllable in K, by [**27**, Lem. 3.4 (ii)] (See also [**28**, Prop. 6.15 (ii)], which roughly says that the 'product' of a BCS in a subgroup $L < H$ with a BCS in the space H/L is a BCS in H); from here we proceed as in the case of the first map. The map $(\pi_{\mathcal{P}_{d,n}\to\mathcal{L}_{d,n}}, \pi_{\mathcal{P}_{d,n}\to\mathcal{L}_{n-d}}) : \mathcal{P}_{d,n} \to \mathcal{L}_{d,n} \times \mathcal{L}_{n-d}$ is handled similarly. □

We are now ready for the first step in the proof of Theorem 3.2, which is reducing to proving only the last part of this theorem.

Proof of parts (1), (2), (3) *of Theorem* 3.2 *from part* (4). Let us first see how part (1) of the theorem follows from part (2). Assume $\Phi \subseteq \mathrm{Gr}(d,n)$ and $\mathcal{E} \times \mathcal{F} \subseteq \mathcal{X}_d \times \mathcal{X}_{n-d}$ are boundary controllable, and let $\tilde{\mathcal{E}} \times \widetilde{\mathcal{F}} \subseteq \mathcal{L}_d \times \mathcal{L}_{n-d}$ be the inverse image of $\mathcal{E} \times \mathcal{F}$ under the projection $\mathcal{L}_d \times \mathcal{L}_{n-d} \to \mathcal{X}_d \times \mathcal{X}_{n-d}$. By Proposition 3.5, $\tilde{\mathcal{E}} \times \widetilde{\mathcal{F}}$ is also boundary controllable, and by Proposition 2.5

$$\mathrm{vol}_{\mathcal{L}_d}(\tilde{\mathcal{E}})\, \mathrm{vol}_{\widetilde{\mathcal{F}}}(\widetilde{\mathcal{F}}) = \mathrm{vol}_{\mathcal{X}_d}(\mathcal{E})\, \mathrm{vol}_{\mathcal{X}_{n-d}}(\mathcal{F}) \Upsilon(d) \Upsilon(n-d).$$

Part (2) of the theorem has that the number of Λ with $V_\Lambda \in \Phi$ and $([\![\Lambda]\!], [\![\Lambda^\pi]\!]) \in \tilde{\mathcal{E}} \times \widetilde{\mathcal{F}}$ is asymptotic to

$$\frac{\mathrm{vol}(\tilde{\mathcal{E}})\, \mathrm{vol}(\widetilde{\mathcal{F}})\, \mathrm{vol}(\Phi)}{n \cdot \prod_{i=2}^n \zeta(i)} \cdot X^n = \frac{\mathrm{vol}_{\mathcal{X}_d}(\mathcal{E})\, \mathrm{vol}_{\mathcal{X}_{n-d}}(\mathcal{F})\, \mathrm{vol}(\Phi)}{n \cdot \prod_{i=2}^n \zeta(i)} \cdot \Upsilon(d)\Upsilon(n-d) \cdot X^n,$$

hence we are done. Similarly, part (2) of Theorem 3.2 follows from part (3) thereof, and part (3) follows from part (4). □

6. Correspondence between integral matrices and primitive lattices

The goal of this section is to translate Theorem 3.2 (or rather, its part (4), since we have reduced to proving only this statement) into a counting problem of integral matrices. The first step is to establish a correspondence between primitive d-lattices and integral matrices in a fundamental domain of the following discrete group of $\mathrm{SL}_n(\mathbb{R})$:

$$\Gamma' := (N' \rtimes G'')(\mathbb{Z}) = \begin{bmatrix} \mathrm{SL}_d(\mathbb{Z}) & \mathbb{Z}^{d,n-d} \\ 0 & \mathrm{SL}_{n-d}(\mathbb{Z}) \end{bmatrix}.$$

PROPOSITION 6.1 *There exists a bijection $\Lambda \leftrightarrow \gamma_\Lambda$ between oriented primitive d-lattices and integral matrices in a fundamental domain of $\mathrm{SL}_n(\mathbb{R}) \curvearrowleft \Gamma'$, where γ_Λ is the unique integral matrix in the fundamental domain whose first d columns span Λ.*

Proof. Let $\Omega \subset \mathrm{SL}_n(\mathbb{R})$ be any fundamental domain for Γ'. The direction $\Leftarrow$ is simple: given $\gamma \in \Omega \cap \mathrm{SL}_n(\mathbb{Z})$, its columns span $\mathbb{Z}^n$, hence by definition its first d columns span a primitive d– lattice. In the opposite direction, let $\mathbf{B}$ be the basis for Λ. Since Λ is primitive, $\mathbf{B}$ can be completed to a basis of $\mathbb{Z}^n$; let $\gamma \in \mathrm{SL}_n(\mathbb{Z})$ be a matrix having this basis in its columns, with $\mathbf{B}$ in the first d columns. The orbit $\gamma \cdot \Gamma$, whose elements consist of integral matrices having a basis for Λ in their first d columns, meets Ω in a single point, γ_Λ. □

Let us construct an explicit fundamental domain for Γ'. Denote

$$\square := \text{the unit cube}(-1/2, 1/2]^{d(n-d)},$$

and then a fundamental domain for the right action of Γ' on $\mathrm{SL}_n(\mathbb{R})$ is

$$\Omega := K' G''_{\widetilde{F_d} \times \widetilde{F_{n-d}}} A' N'_{\square}, \tag{6.1}$$

where $N'_{\square}$ is the image of $\square$ under a natural diffeomorphism $\mathbb{R}^{d(n-d)} \to N'$.

By Proposition 6.1, the integral matrices γ_Λ in Ω correspond to the primitive d-lattices Λ spanned by the first d columns of the matrix. But Proposition 4.3 teaches us that the properties (shape, covolume, etc.) of Λ are encoded in the RI components of γ_Λ. We combine the two to obtain:

Corollary 6.2 *Consider the correspondence $\Lambda \leftrightarrow \gamma_\Lambda$ between primitive d– lattices and matrices in $\Omega \cap \mathrm{SL}_n(\mathbb{Z})$. For $\Xi \subseteq \mathcal{P}_{d,n}$ and $T > 0$, let*

$$\Omega_T(\Xi) = \Omega \cap \left\{ g = qa'n' : q \in Q_\Xi, a' \in A'_{[0,T]}, n' \in N'_{\square} \right\} = Q_\Xi A'_{[0,T]} N'_{\square}.$$

Then

$$\Omega_T(\Xi) \cap \mathrm{SL}_n(\mathbb{Z}) = \{\gamma_\Lambda : \mathrm{covol}(\Lambda) \le e^T, [(\Lambda, \Lambda^\pi)] \in \Xi\}.$$

We conclude with the following computation.

Lemma 6.3 *For a measurable $\Xi \subseteq \mathcal{P}_{d,n}$ and $T > 0$,*

$$\mu(\Omega_T(\Xi)) = n^{-1} \mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)(e^{nT} - 1).$$

Proof. Since $\Omega_T(\Xi) = Q_\Xi A'_{[0,T]} N'_{\square}$ and $\mu = \mu_Q \times \mu_{A'} \times \mu_{N'}$,

$$\mu(\Omega_T(\Xi)) = \mu_Q(Q_\Xi)\mu_{A'}((A')_T \mu_{N'}(N'_{\square}).$$

By definition of $\mu_{N'}$, $\mu_{N'}(N'_{\square}) = \mathrm{Leb}(\square) = 1$, and by Proposition 5.1, $\mu_Q(Q_\Xi) = \mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)$. By definition of $\mu_{A'}$,

$$\mu_{A'}((A')_T = \int_0^T e^{nt} dt = n^{-1}(e^{nT} - 1).$$

□

7. Ergodic method to count lattice points

We are now at the point where we have reduced the proof of Theorem 3.2 to a problem of counting integral matrices inside the subsets of $\mathrm{SL}_n(\mathbb{R})$ that are defined in Corollary 6.2. This counting problem will be approached using an ergodic method that was developed by Gorodnik and Nevo in [**25**] and whose roots lie in the celebrated work of Eskin and McMullen [**17**]. This section is devoted to describing it.

Every counting result assumes a certain regularity condition on the sets in which the counting takes place. In the classical setting of counting integral points in $\mathbb{R}^n$, one always assumes that the boundaries of the sets in question satisfy some level of smoothness; however, in the semisimple setting (semisimple Lie groups and their homogeneous spaces), the boundary presents a fundamentally different behavior, and the notion of regularity of the sets should be defined accordingly. Eskin and McMullen have coined the term *well roundedness* for this type of regularity; here is how it was defined in [**25**]:

Definition 7.1 Let $\mathbf{G}$ be a Lie group with a Borel measure μ, and let $\{\mathcal{O}_\epsilon\}_{\epsilon>0}$ be a family of identity neighborhoods in $\mathbf{G}$. Assume $\{\mathcal{B}_T\}_{T>0} \subset \mathbf{G}$ is a family of measurable domains and denote

$$\mathcal{B}_T^+(\epsilon) := \mathcal{O}_\epsilon \mathcal{B}_T \mathcal{O}_\epsilon = \bigcup_{u,v\in\mathcal{O}_\epsilon} u\,\mathcal{B}_T\,v,$$

$$\mathcal{B}_T^-(\epsilon) := \bigcap_{u,v\in\mathcal{O}_\epsilon} u\,\mathcal{B}_T\,v.$$

The family $\{\mathcal{B}_T\}$ is *Lipschitz well-rounded (or LWR)* with (positive) parameters $(\mathcal{C}, T_0)$ if for every $0 < \epsilon < 1/\mathcal{C}$ and $T > T_0$:

$$\mu\left(\mathcal{B}_T^+(\epsilon)\right) \le (1 + \mathcal{C}\epsilon)\,\mu\left(\mathcal{B}_T^-(\epsilon)\right). \tag{7.1}$$

The parameter $\mathcal{C}$ is called the *Lipschitz constant* of the family $\{\mathcal{B}_T\}$.

A *lattice subgroup* Γ of a locally compact second countable group is a discrete subgroup whose space of cosets has finite volume w.r.t. to the Haar measure on the ambient group. In certain Lie groups, among which are algebraic simple non-compact Lie groups $\mathbf{G}$, there exists $p \in \mathbb{N}$ for which the matrix coefficients $\langle \pi^0_{\mathbf{G}/\Gamma}(g)u, v\rangle$ are in $L^{p+\epsilon}(\mathbf{G})$ for every $\epsilon > 0$, with u, v lying in a dense subspace of $L^2_0(\mathbf{G}/\Gamma)$ (see [**24**, Thm. 5.6]). (In other words, the unitary representation $\pi^0_{\mathbf{G}/\Gamma}$ is $\frac{2}{p}$-tempered). Let $p(\Gamma)$ be the smallest among these p's, and denote

$$m(\Gamma) = \begin{cases} 1 & \text{if } p = 2, \\ 2\lceil p(\Gamma)/4 \rceil & \text{otherwise.} \end{cases}$$

Define:

$$\tau(\Gamma) = \frac{1}{2m(\Gamma)(1 + \dim \mathbf{G})} \in (0,1).$$

Then $\tau(\Gamma)$ is a parameter that depends on the rate of decay of the matrix coefficients of the **G**-representation on $L_0^2(\mathbf{G}/\Gamma)$, and specifically it is larger when the decay is faster. We remark that for $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $n \geq 3$ it was estimated in [33, 12] that $p(\Gamma) = 2n-2$, and therefore $\tau(\mathrm{SL}_n(\mathbb{Z})) = (4n^2\lceil\frac{n-1}{2}\rceil)^{-1}$. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, it is known that $p(\Gamma) = 2$ and therefore $\tau(\mathrm{SL}_2(\mathbb{Z})) = 1/8$.

THEOREM 7.2 ([25, Thm. 1.9, Thm. 4.5 and Rem. 1.10]). *Let* **G** *be an algebraic simple Lie group with Haar measure* μ*, and let* $\Gamma < \mathbf{G}$ *be a lattice subgroup. Assume that* $\{\mathcal{B}_T\} \subset \mathbf{G}$ *is a family of finite-measure domains which satisfy* $\mu(\mathcal{B}_T) \to \infty$ *as* $T \to \infty$*. If the family* $\{\mathcal{B}_T\}$ *is LWR with parameters* $(C_{\mathcal{B}}, T_0)$*, then* $\exists T_1 > 0$ *such that for every* $\delta > 0$ *and* $T > T_1$*:*

$$\#(\mathcal{B}_T \cap \Gamma) - \mu(\mathcal{B}_T)/\mu(\mathbf{G}/\Gamma) \ll_{\mathbf{G},\Gamma,\delta} (C_{\mathcal{B}}^{\frac{\dim \mathbf{G}}{1+\dim \mathbf{G}}} \cdot \mu(\mathcal{B}_T)^{1-\tau(\Gamma)+\delta}),$$

where $\mu(\mathbf{G}/\Gamma)$ *is the measure of a fundamental domain of* Γ *in* **G** *and the parameter* T_1 *is such that* $T_1 \geq T_0$ *and such that for every* $T \geq T_1$

$$C_{\mathcal{B}}^{\frac{\dim \mathbf{G}}{1+\dim \mathbf{G}}} \ll_{\mathbf{G},\Gamma} (\mu(\mathcal{B}_T)^{\tau(\Gamma)}). \tag{7.2}$$

However, our goal of counting in $\{\Omega_T(\Xi)\}_{T>0}$ cannot be approached in its current form by Theorem 7.2, since the sets $\Omega_T(\Xi)$ are not (Lipschitz) well rounded and in fact may not even be compact. This issue is handled in the next section.

8. Neglecting lattices up the cusp: reducing to well-rounded sets

This section is devoted to handling the obstacle arising from the fact that the sets $\Omega_T(\Xi)$ are (mostly) not LWR and in fact not necessarily compact. Their compactness is equivalent to the compactness of Ξ, which is equivalent to the compactness of $A'' \cap Q_\Xi$. To overcome this, we will reduce to counting in compact subsets of $\Omega_T(\Xi)$, namely subsets that are obtained by truncating the A'' coordinate of Q_Ξ, and hence of $\Omega_T(\Xi)$.

REMARK 8.1 Geometrically, matrices in $\mathrm{SL}_n(\mathbb{R})$ with a very large A'' coordinate (where 'very large' means comparable with the covolume of the d-lattice that they span) correspond to d-lattices which have a very small successive minimum (if the large A'' coordinate is among the first $d-1$ ones), or their factor lattice has a very large successive minimum (if the large coordinate is among the last $n-d-1$ ones). Namely, d-lattices whose shape is 'up the cusp' of the space $\mathcal{X}_d$, or that the shape of their factor lattice lies 'up the cusp' of $\mathcal{X}_{n-d}$.

For every

$$\underline{S} = (S_1, \ldots, S_d) > \underline{0}, \ \underline{W} = (W_1, \ldots, W_{n-d}) > \underline{0} \tag{8.1}$$

and a subset $\mathcal{B} \subset G$*, let* $\mathcal{B}^{\underline{S},\underline{W}}$ *denote the set*

$$\mathcal{B} \cap \{g : a'' = a''_{\underline{s},\underline{w}} \text{ with } s_i \leq S_i, w_j \leq W_j \ ; \forall i,j\},$$

where a'' *is the* A'' *component of* g*.*

Recall the fundamental domain $\Omega \subset G$ for Γ' appearing in (6.1). Accordingly with Notation 8.2, we let

$$\Omega_T^{\underline{S},\underline{W}} := K' \begin{bmatrix} \widetilde{F_d}^{\underline{S}} & \\ & \widetilde{F_{n-d}}^{\underline{W}} \end{bmatrix} A'_T N'_{\square}.$$

We claim that

$$\mu(\Omega_T - \Omega_T^{\underline{S},\underline{W}}) \ll (e^{nT - \min\{S_{\min}, W_{\min}\}}) \tag{8.2}$$

with $S_{\min} = \min_j S_j$ and $W_{\min} = \min_j W_j$; indeed, let $\underline{Z} = (\underline{S}, \underline{W})$ namely $Z_j = S_j$ for $1 \le j \le d$ and $Z_j = W_{j-d}$ for $d+1 \le j \le n$. Then

$$\begin{aligned}
\mu\left(\Omega_T - \Omega_T^{\underline{Z}}\right) &\sim \left(\int_{[0,T]} d\mu_{A'}\right)\left(\int_{[0,\infty)^{n-2} - \prod_{j=1}^{d}[0,Z_j]} d\mu_{A''}\right) \\
&\le e^{nT} \cdot \sum_{j=1}^{n-2} \int_{(Z_j \le z_j < \infty) \times \prod_{i \ne j}(0 \le z_i < \infty)} \frac{dz_1 \cdots dz_{n-2}}{e^{z_1 + \cdots z_{n-2}}} \\
&\sim e^{nT}\left(e^{-Z_1} + \cdots + e^{-Z_{n-2}}\right) \\
&\le e^{nT} \cdot e^{-\min_j\{Z_{i,}\}}.
\end{aligned}$$

The goal of this section is to prove the following (a similar claim appears in [**27**, Prop. 6.2]):

Proposition 8.3 *Let Ω be as in (6.1), and let $\underline{\sigma} = (\sigma_1, \ldots, \sigma_{d-1})$, $\underline{\omega} = (\omega_1, \ldots, \omega_{n-d-1})$ where $0 < \sigma_i, \omega_i < 1$ $\forall i$. Then for every $\epsilon > 0$*

$$\#\left|\left(\Omega_T - \Omega_T^{\underline{\sigma}T, \underline{\omega}T}\right) \cap \mathrm{SL}_n(\mathbb{Z})\right| \ll_\epsilon \left(e^{T(n - \min\{\sigma_{\min}, \omega_{\min}\} + \epsilon)}\right),$$

where $\sigma_{\min} = \min \sigma_i$, $\omega_{\min} = \min \omega_i$.

Proof. Let $\gamma_\Lambda = k a''_{\underline{s},\underline{w}} a'_t n \in \Omega_T \cap \mathrm{SL}_n(\mathbb{Z})$. In what follows, Λ^i is the lattice spanned by the first i columns of γ_Λ, $(\Lambda^\pi)^j < \Lambda^\pi$ is the lattice spanned by the first j columns of the $n \times (n-d)$ matrix obtained by projecting the columns of γ_Λ to $V_\Lambda^\perp$, and $\left(\Lambda^\perp\right)^k < \Lambda^\perp$ is the lattice spanned by the first k columns of the matrix that represents $\Lambda^\perp$ inside $\widetilde{F_{n-d}}$.

Recall that $\gamma_\Lambda \in \Omega_T - \Omega_T^{\underline{\sigma}T,\underline{\infty}}$ if and only if $\exists i \in \{1, \ldots, d-1\}$ for which $s_i \ge \sigma_i T$. This implies (using Proposition 4.3 and the fact that Λ^i is integral) that

$$1 \le \mathrm{covol}(\Lambda^i) = e^{\frac{it}{d} - \frac{s_i}{2}} \le e^{\frac{it}{d} - \frac{\sigma_i t}{2}} \le e^{(\frac{i}{d} - \frac{\sigma_i}{2})T}.$$

For the remaining $i \ne i' \in \{1, \ldots, d-1\}$, $1 \le \mathrm{covol}(\Lambda^{i'}) = e^{\frac{it}{d}} \le e^{\frac{i}{d}T}$.

By [27, Prop. 6.4], the number of such possible lattices Λ is $e^{T(n-\sigma+\epsilon)}$ where $\sigma = \sum \sigma_i$. Thus, the number of $\mathrm{SL}_n(\mathbb{Z})$ elements γ in $\Omega_T - \Omega_T^{\underline{\sigma}T,\underline{\infty}}$ is bounded by

$$\#\Big(\bigcup_{\substack{\underline{u}=(u_1,\dots,u_{d-1})\\ \in\{0,1\}^{d-1}-\{0\}}} \{\Lambda : \forall i, \mathrm{covol}(\Lambda_v^i) \in [1, e^{T(\frac{i}{d}-\frac{\sigma_i u_i}{2})}]\}\Big)$$
$$= \sum_{\underline{u}\in\{0,1\}^{d-1}-\{0\}} O_\epsilon(e^{T(n-\sigma+\epsilon)}) = O_\epsilon(e^{T(n-\sigma_{\min}+\epsilon)}),$$

where $\sigma_{\min} = \min\{\sigma_i\}$, as ϵ is arbitrary.

Now recall that $\gamma_\Lambda \in \Omega_T - \Omega_T^{\underline{\infty},\underline{\omega}T}$ if and only if $\exists j \in \{1,\dots,n-d-1\}$ for which $w_j \geq \omega_j T$. By Lemma A.13 and Proposition B.5, $\mathrm{covol}((\Lambda^\pi)^j)$ is proportional to the quotient $\mathrm{covol}((\Lambda^\perp)^{n-d-j})/\mathrm{covol}(\Lambda)$, with a positive constant that depends only on d and n. Then, by Proposition 4.3 and Lemma A.13,

$$\widetilde{C}e^{-t}\cdot \mathrm{covol}((\Lambda^\perp)^{n-d-j}) = \widetilde{C}\cdot \frac{\mathrm{covol}((\Lambda^\perp)^{n-d-j})}{\mathrm{covol}(\Lambda)} \leq \mathrm{covol}((\Lambda^\pi)^j) = e^{-\frac{jt}{n-d}-\frac{w_j}{2}},$$

where $\widetilde{C} > 0$. Hence, up to an additive constant that becomes negligible when t is large, $w_j \geq \omega_j T$ implies

$$1 \leq \mathrm{covol}((\Lambda^\perp)^{n-d-j}) \leq e^{t-\frac{jt}{n-d}-\frac{w_j}{2}} \leq e^{T(1-\frac{j}{n-d}-\frac{\omega_j}{2})}.$$

For the remaining $j \neq j' \in \{1,\dots,n-d-1\}$,

$$1 \leq \mathrm{covol}(\Lambda^{n-d-j'}) \leq e^{t(1-\frac{j'}{n-d})} \leq e^{T(1-\frac{j'}{n-d})}.$$

By [27, Cor. 6.4], the number of such possible lattices Λ is $e^{T(n-\omega+\epsilon)}$ where $\omega = \sum \omega_i$. Thus, by similar considerations, the number $\#|(\Omega_T - \Omega_T^{\underline{\infty},\underline{\omega}T}) \cap \mathrm{SL}_n(\mathbb{Z})|$ is in $O(e^{T(n-\omega_{\min}+\epsilon)})$, where $\omega_{\min} = \min \omega_i$. □

9. Proof of the Theorem 3.2

For the sets $\Omega_T^{\underline{S},\underline{W}}(\Xi)$, we can use the method described in Section 7 to produce the following counting statement:

Proposition 9.1 *Let $n \geq 2$ and $\Gamma < \mathrm{SL}_n(\mathbb{R})$ a lattice subgroup with $\tau = \tau(\Gamma)$. Set $\lambda_n = \frac{n^2}{2(n^2-1)}$. Let $\underline{S}, \underline{W}$ as in (8.1), and*

$$\mathbf{S} = \text{sum of components of } \underline{S},\ \mathbf{W} = \text{sum of components of } \underline{W}.$$

Let $\underline{S}(T), \underline{W}(T)$ denote vectors like $\underline{S}, \underline{W}$, but with components that depend on T, and let $\mathbf{S}(T), \mathbf{W}(T)$ be their sums of components. If $\Xi \subseteq \mathcal{P}_{d,n}$ has controlled boundary, then the following holds for every $0 < \epsilon < \tau$:

1. *For $T \geq \frac{\mathbf{S}+\mathbf{W}}{n\lambda_n\tau} + O_{\Xi}(1)$:*

$$\#(\Omega_T^{\underline{S},\underline{W}}(\Xi) \cap \Gamma) = \frac{\mu(\Omega_T^{\underline{S},\underline{W}}(\Xi))}{\mu(G/\Gamma)} + O_{\Xi,\epsilon}(e^{(\mathbf{S}+\mathbf{W})/\lambda_n} e^{nT(1-\tau+\epsilon)}).$$

2. *Let $n \geq 3$. For every $\delta \in (0, \tau - \epsilon)$, $T \geq O_{\Xi}(1)$, $\underline{S}(T)$ and $\underline{W}(T)$ such that $\mathbf{S}(T) + \mathbf{W}(T) \leq n\delta\lambda_n T + O_{\Xi}(1)$:*

$$\#(\Omega_T^{\underline{S}(T),\underline{W}(T)}(\Xi) \cap \Gamma) = \frac{\mu(\Omega_T^{\underline{S}(T),\underline{W}(T)}(\Xi))}{\mu(G/\Gamma)} + O_{\Xi,\epsilon}(e^{nT(1-\tau+\delta+\epsilon)}).$$

When only $\underline{S}$ is bounded the condition on $\mathbf{S}(T) + \mathbf{W}(T)$ becomes $\mathbf{W}(T) \leq n\delta\lambda_n T + O_{\Xi}(1)$, and when only $\underline{W}$ is bounded the condition becomes $\mathbf{S}(T) \leq n\delta\lambda_n T + O_{\Xi}(1)$.

We will prove this proposition at the end of the section, by showing the sets $\Omega_T^{\underline{S},\underline{W}}(\Xi)$ are Lipschitz well rounded.

Proposition 9.1 counts $\mathrm{SL}_n(\mathbb{Z})$ elements in $\Omega_T^{\underline{S}(T),\underline{W}(T)}$, while Proposition 8.3 counts $\mathrm{SL}_n(\mathbb{Z})$ elements in the complement of $\Omega_T^{\underline{S}(T),\underline{W}(T)}$ inside $\Omega_T(\Xi)$. combining the two, we obtain counting in $\Omega_T(\Xi)$, thus proving Theorem 3.2. Here are the details:

Proof of Theorem 3.2 assuming Proposition 9.1. Recall that it is sufficient to prove part (4) of the theorem. By Corollary 6.2, when setting $T = \log X$, the quantity we seek to estimate in this part is the amount of $\mathrm{SL}_n(\mathbb{Z})$ elements inside $\Omega_T(\Xi)$. Suppose first that Ξ is bounded. Then, by the properness of $\pi_{\mathcal{P}_{d,n} \to \mathcal{X}_d \times \mathcal{X}_{n-d}}$ from Proposition 2.4, there exist $\underline{S} = \underline{S}_{\Xi}$ and $\underline{W} = \underline{W}_{\Xi}$ such that $\Xi = \Xi^{\underline{S},\underline{W}}$, namely $\Omega_T(\Xi) = \Omega_T^{\underline{S},\underline{W}}(\Xi)$. Then, by part (i) of Proposition 9.1,

$$\#(\Omega_T^{\underline{S},\underline{W}}(\Xi) \cap \mathrm{SL}_n(\mathbb{Z})) = \frac{\mu(\Omega_T^{\underline{S},\underline{W}}(\Xi))}{\mu(\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{SL}_n(\mathbb{R}))} + O_{\Xi,\epsilon}(e^{nT(1-\tau_n+\epsilon)})$$

for T large enough, where

$$\tau_n = \tau(\mathrm{SL}_n(\mathbb{Z})).$$

Taking into account Lemma 6.3 and the volume of $\widetilde{F_n}$, this equals

$$\frac{\mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)}{n \cdot \prod_{i=2}^{n} \zeta(i)} e^{nT} + O_{\Xi,\epsilon}(e^{nT(1-\tau_n+\epsilon)})$$

and the proof of this case is completed.

Assume now that Ξ is not bounded. Let $0 < \epsilon < \tau_n$ and $0 < \delta < \tau_n - \epsilon$, and recall $\lambda_n = \frac{n^2}{2(n^2-1)}$ and $\tau_n = (4n^2\lceil (n-1)/2\rceil)^{-1}$. Set

$$\begin{aligned} \underline{\sigma}_d &= \frac{\delta n\lambda_n - \epsilon}{n-2} \cdot \underline{1}_{d-1} \cdot \in \mathbb{R}^{d-1}, \\ \underline{\omega}_{n-d} &= \frac{\delta n\lambda_n - \epsilon}{n-2} \cdot \underline{1}_{n-d-1}. \end{aligned}$$

If $\underline{S}(T) = \underline{\sigma}_d T$ and $\underline{W}(T) = \underline{\omega}_{n-d} T$, then (in the notation of Prop. 9.1, where $\mathbf{S}(T)$ is the sum of coordinates of $\underline{S}(T)$ and $\mathbf{W}(T)$ is the sum of coordinates of $\underline{S}(T)$), for sufficiently large T

$$\mathbf{S}(T) + \mathbf{W}(T) = (\delta n\lambda_n - \epsilon)T \le \delta n\lambda_n T + O(1). \tag{9.1}$$

Since $\Omega_T(\Xi)$ equals

$$\Omega_T(\Xi) = \Omega_T^{\underline{\sigma}_d T, \underline{\omega}_{n-d} T}(\Xi) \sqcup \left(\Omega_T(\Xi) - \Omega_T^{\underline{\sigma}_d T, \underline{\omega}_{n-d} T}(\Xi)\right),$$

then the amount of lattice points in the left-hand side (LHS) is the sum of lattice points in the sets on the right-hand side (RHS). Considering the first set on the RHS: by (the second part of) Proposition 9.1, which we may use thanks to (9.1),

$$\#\left(\mathrm{SL}_n(\mathbb{Z}) \cap \Omega_T^{\underline{\sigma}_d T, \underline{\omega}_{n-d} T}(\Xi)\right) = \frac{\mu(\Omega_T^{\underline{\sigma}_d T, \underline{\omega}_{n-d} T}(\Xi))}{\mu(\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{SL}_n(\mathbb{R}))} + O_{\Xi,\epsilon}(e^{nT(1-\tau_n+\delta+\epsilon)}).$$

By (8.2) and Lemma 6.3, this equals

$$= \frac{\mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)}{n \cdot \prod_{i=2}^n \zeta(i)} e^{nT} + O(e^{nT\left(1-\frac{1}{n}\min\{\sigma_{\min},\omega_{\min}\}+\epsilon\right)}) + O_{\Xi,\epsilon}(e^{nT(1-\tau_n+\delta+\epsilon)}).$$

As for the second set on the RHS, by Proposition 8.3,

$$\begin{aligned} \#\left(\mathrm{SL}_n(\mathbb{Z}) \cap (\Omega_T(\Xi) - \Omega_T^{\underline{\sigma}_d T, \underline{\omega}_{n-d} T}(\Xi))\right) &\ll_\epsilon e^{nT\left(1-\frac{1}{n}\min\{\sigma_{\min},\omega_{\min}\}+\epsilon\right)} \\ &\le e^{nT\left(1-\frac{\delta\lambda_n}{n-2}+\epsilon\right)}, \end{aligned}$$

and therefore

$$\#(\mathrm{SL}_n(\mathbb{Z}) \cap \Omega_T(\Xi)) = \frac{\mathrm{vol}_{\mathcal{P}_{d,n}}(\Xi)}{n \cdot \prod_{i=2}^n \zeta(i)} e^{nT} + O_{\Xi,\epsilon}(e^{nT(1-\tau_n+\delta+\epsilon)}) + O_\epsilon\left(e^{nT\left(1-\frac{\delta\lambda_n}{n-2}+\epsilon\right)}\right).$$

Finally, we choose δ that will balance the two error terms above, that is δ that satisfies: $1 - \tau_n + \delta = 1 - \frac{\delta\lambda_n}{n-2}$. This δ is

$$\delta = \tau_n \cdot (1 - \frac{\lambda_n}{n-2+\lambda_n}) = \tau_n \cdot (1 - \frac{n^2}{2(n-2)(n^2-1)+n^2}) \le \tau_n \cdot (1 - \frac{1}{2n-2}),$$

hence we get an error term of

$$\ll_{\Xi,\epsilon} e^{nT(1-(\tau_n-\delta)+\epsilon)} \ll_{\Xi,\epsilon} (e^{nT(1-\frac{\tau_n}{2n-2}+\epsilon)}).$$

This concludes the proof of the case where Ξ is not bounded. □

We conclude by proving Proposition 9.1.

Proof of Proposition 9.1. To prove part 1 of the proposition using Theorem 7.2, we must show that the family $\{\Omega_T^{(\underline{S},\underline{W})}(\Xi)\}_{T>0}$ is LWR. Consider the map which projects to the RI coordinates,

$$r: \begin{array}{ccc} \mathrm{SL}_n(\mathbb{R}) & \to & K\times A'\times A''\times N''\times N' \\ g=ka'a''n''n' & \mapsto & (k,a',a'',n'',n') \end{array}.$$

In [27, Cor. 10.5] we have shown that the well roundedness of $\Omega_T^{(\underline{S},\underline{W})}(\Xi)$ relates to the well roundedness of its image under r in the following manner. If each of the components of the image is LWR with T_0 and a Lipschitz constant that do not depend on $\underline{s},\underline{w}$, then $\Omega_T^{(\underline{S},\underline{W})}(\Xi)$ is also LWR, with T_0 that does not depend on $\underline{s},\underline{w}$, and a Lipschitz constant that is of order $O_\Xi(e^{2(\mathbf{S}+\mathbf{W})})$. Clearly,

$$r(\Omega_T^{\underline{S},\underline{W}}(\Xi)) = r(Q_{\Xi\underline{S},\underline{W}})\times A'_{[0,T]}\times N'_{\square}$$

with $r(Q_{\Xi\underline{S},\underline{W}})\subset K\times A''\times N''$. Each of the three factors in the above direct product is LWR with a Lipschitz constant that is $O(1)$: $N'_{\square}$ is LWR since it is diffeomorphic to a cube in $\mathbb{R}^{d(n-d)}$, and $A'_{[0,T]}$ is LWR by [27, Prop. 9.6]. As for $r(Q_{\Xi\underline{S},\underline{W}})$, notice that Q_Ξ is boundary controllable, by Lemma 5.2. Then, by [27, Lem. 11.1] (This lemma is actually for $d=n-1$, but the proof is valid for any $1<d<n$), $r(Q_{\Xi\underline{S},\underline{W}})$ is LWR in $K\times A''\times N''$ with parameters that do not depend on $\underline{S},\underline{W}$. Therefore, $\Omega_T^{(\underline{S},\underline{W})}(\Xi)$ is LWR with Lipschitz constant that is of order $O_\Xi(e^{2(\mathbf{S}+\mathbf{W})})$.

The claimed error term is obtained from Theorem 7.2, while recalling that the volume of $\Omega_T^{(\underline{S},\underline{W})}(\Xi)$ in $\mathrm{SL}_n(\mathbb{R})$ is of order e^{nT}. The lower bound on T is obtained from substituting the LWR parameters of $\Omega_T^{(\underline{S},\underline{W})}(\Xi)$ into (7.2).

For the proof of the second part, let us compute a bound on $\mathbf{S},\mathbf{W}$ for which the error term established in part 1 of the proposition remains smaller than the main term. Namely, that there exists $\gamma\in(0,1)$ for which

$$(\mathbf{S}+\mathbf{W})/\lambda_n + (1-\tau+\epsilon)\cdot nT = \gamma\cdot nT.$$

If δ denotes the number $\gamma+\tau-\epsilon-1$, we have that $\gamma=\delta+1+\epsilon-\tau$. Then $\gamma<1$ if and only if $\delta<\tau-\epsilon$, where $\tau-\epsilon$ is positive since $\tau>\epsilon$. We conclude that for $0<\delta<\tau-\epsilon$ and $\mathbf{S}+\mathbf{W}=\mathbf{S}(T)+\mathbf{W}(T)<\delta\lambda_n nT$, the counting applies with an error term of order $e^{\gamma nT}=e^{nT(1-\tau+\delta+\epsilon)}$. As for the lower bound T_1 on T, in part 1 we got $\mathbf{S}+\mathbf{W}\le n\lambda_n\tau T+O_\Xi(1)$; so, combining both bounds on $\mathbf{S}+\mathbf{W}$ we get

$$\mathbf{S}+\mathbf{W}\le \min\{n\lambda_n\delta T,\, n\lambda_n\tau T\}+O_\Xi(1)=n\lambda_n\delta T+O_\Xi(1)$$

for T large enough and $\delta\in(0,\tau-\epsilon)$. This completes the proof. □

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Appendix: Dual lattices, factor lattices

This appendix, which is independent from the paper, has two goals. The first is to further investigate the spaces $\mathcal{P}_{d,n}$ and $\mathcal{L}_{d,n}$, especially from the differential perspective of smooth maps between spaces, hence proving some claims that have been used throughout the paper; the second is to be a source of facts about dual lattices and factor lattices that are not necessarily hard, but cannot be found in the literature (at least not easily). By that, we hope to assist future authors.

In what follows, if the columns of a matrix $\mathbf{B} = \mathbf{B}_{n\times d}$ span a d-lattice $\Lambda < \mathbb{R}^n$ (resp. a subspace $V < \mathbb{R}^n$), we say that $\mathbf{B}$ is a basis for the lattice Λ (resp. the subspace V). Recall that the covolume of a d-lattice Λ, which is the volume of the fundamental parallelopiped for Λ in the linear subspace that it spans, equals $(|\det(\mathbf{B}^{\mathrm{t}}\mathbf{B})|)^{1/2}$ where $\mathbf{B}$ is (any) a basis for Λ. We will use an underscore to denote that an object is being spanned by a set, so that $\Lambda_{\mathbf{B}}$ is the lattice spanned (over $\mathbb{Z}$) by $\mathbf{B}$, V_Λ is the linear space spanned (over $\mathbb{R}$) by Λ, etc.

Appendix A: Dual lattices

The claims A.1 until A.3 can be found in the book by Cassels [**11**] (where dual lattices are referred to as *polar* lattices), or in the helpful notes [**41**].

Given a d-lattice $\Lambda < \mathbb{R}^n$, we define the *dual lattice* of Λ to be

$$\Lambda^* = \left\{ y \in \operatorname{span}_{\mathbb{R}}(\Lambda) : \forall x \in \Lambda, \langle x, y\rangle \in \mathbb{Z} \right\}.$$

Note that it is contained in V_Λ and that *a priori*, it is unclear that Λ^* is indeed a lattice.

Proposition A.1 *The matrix* $\mathbf{D}_{n\times d} := \mathbf{B}(\mathbf{B}^t\mathbf{B})^{-1}$ *is basis for* Λ^*.

This proposition motivates the notation $\mathbf{D} = \mathbf{B}^*$ for $\mathbf{D}$ as above, as well as the name *the dual basis* of $\mathbf{B}$.

Proof. Since $(\mathbf{B}^{\mathrm{t}}\mathbf{B})^{-1} \in \mathrm{GL}_d(\mathbb{R})$, we have that $\operatorname{span}_{\mathbb{R}}(\mathbf{D}) = \operatorname{span}_{\mathbb{R}}(\mathbf{B}) = V_\Lambda$. For the inclusion $\operatorname{span}_{\mathbb{Z}}(\mathbf{D}) \subseteq \Lambda^*$, take $v \in \mathbb{Z}^d$ and we want to show that $\mathbf{D}v \in \Lambda^*$, namely that $\langle \mathbf{B}w, \mathbf{D}v\rangle_{\mathbb{R}^n} \in \mathbb{Z}$ for every $w \in \mathbb{Z}^d$. Indeed,

$$\langle \mathbf{B}w, \mathbf{D}v\rangle_{\mathbb{R}^n} = \left\langle \mathbf{B}w, \mathbf{B}(\mathbf{B}^{\mathrm{t}}\mathbf{B})^{-1}v \right\rangle_{\mathbb{R}^n} = \left\langle w, \mathbf{B}^{\mathrm{t}}\mathbf{B}(\mathbf{B}^{\mathrm{t}}\mathbf{B})^{-1}v \right\rangle_{\mathbb{R}^d} = \langle w, v\rangle_{\mathbb{R}^d} \in \mathbb{Z}.$$

But this inclusion is in fact an equality, since according to the above $\langle \mathbf{B}w, \mathbf{D}v\rangle_{\mathbb{R}^n} = \langle w, v\rangle_{\mathbb{R}^d}$, and this value is integral for every $w \in \mathbb{Z}^d$ if and only if $v \in \mathbb{Z}^d$, namely if and only if the element $\mathbf{D}v$ of V_Λ is in fact in Λ^*. □

COROLLARY A.2 *The dual lattice Λ^* is a d-lattice, and $V_{\Lambda^*} = V_\Lambda$.*

COROLLARY A.3 *One has that $(\Lambda^*)^* = \Lambda$, and $\operatorname{covol}(\Lambda^*) = \operatorname{covol}(\Lambda)^{-1}$.*

Proof. Check that $\mathbf{D}(\mathbf{D}^t\mathbf{D})^{-1} = \mathbf{B}$ and that $(|\det(\mathbf{D}^t\mathbf{D})|)^{1/2} = 1/(|\det(\mathbf{B}^t\mathbf{B})|)^{1/2}$. □

EXAMPLE A.4 The dual of $\mathbb{Z}^n$ is $\mathbb{Z}^n$, but in general the dual of an integral (and even primitive) lattice must not be integral. For example, the dual of $\mathbb{Z}(1,1)$ is $\mathbb{Z}(\frac{1}{2},\frac{1}{2})$. More generally, when $\Lambda < \mathbb{Z}^n$, the entries of Λ^* are in the ring $\mathbb{Z}[\frac{1}{\operatorname{covol}(\Lambda)^2}]$. To see this, recall that (Proposition A.1) if $\mathbf{B}$ is a basis for Λ, then $\mathbf{B}(\mathbf{B}^t\mathbf{B})^{-1}$ is a basis for Λ^*. Since $(\mathbf{B}^t\mathbf{B})^{-1} = \operatorname{adj}\left(\mathbf{B}^t\mathbf{B}\right)/\det(\mathbf{B}^t\mathbf{B}) = \operatorname{adj}\left(\mathbf{B}^t\mathbf{B}\right)/\operatorname{covol}(\Lambda)^2$ where $\operatorname{adj}\left(\mathbf{B}^t\mathbf{B}\right)$ is the adjucate matrix of $\mathbf{B}$, and since $\operatorname{adj}\left(\mathbf{B}^t\mathbf{B}\right)$ is integral when $\mathbf{B}$ is, we get that the entries of $(\mathbf{B}^t\mathbf{B})^{-1}$ (and therefore also the entries of $\mathbf{B}(\mathbf{B}^t\mathbf{B})^{-1}$) are in $\mathbb{Z}[\frac{1}{\operatorname{covol}(\Lambda)^2}]$.

Consider the following two spaces:

$$\widetilde{\mathcal{L}_{d,n}} := \mathrm{GL}_n(\mathbb{R}) / \left(\begin{bmatrix} \mathrm{GL}_d(\mathbb{Z}) & \mathbb{R}^{d,n-d} \\ 0_{n-d\times d} & \mathrm{GL}_{n-d}(\mathbb{R}) \end{bmatrix} \right) = \text{space of } d\text{−lattices inside } \mathbb{R}^n,$$

$$\widetilde{\mathcal{P}_{d,n}} := \mathrm{GL}_n(\mathbb{R}) / \left(\begin{bmatrix} \mathrm{GL}_d(\mathbb{Z}) & \mathbb{R}^{d,n-d} \\ 0_{n-d\times d} & \mathrm{GL}_{n-d}(\mathbb{Z}) \end{bmatrix} \right) = \begin{array}{c} \text{space of pairs } (\Lambda, L)\, where \Lambda a d - lattice in \mathbb{R}^n \\ \text{and } L \text{ is an } n-d \text{lattice in } V_\Lambda^\perp \end{array}.$$

Let us justify why these quotients are indeed the spaces of lattices we claim they are. A coset of a group element g inside $\widetilde{\mathcal{L}_{d,n}}$ corresponds to the d-lattice spanned by the first d columns of g. A coset of a group element g inside $\widetilde{\mathcal{P}_{d,n}}$ corresponds to the pair (Λ, L), where Λ is the d-lattice spanned by the first d columns of g and L is the lattice spanned by the orthogonal projections of the last $n-d$ columns of g to $V_\Lambda^\perp$. Note that since the columns of g are independent and the first d columns span V_Λ, the projections of the last $n-d$ columns to $V_\Lambda^\perp$ must be independent; in particular, L is an $(n-d)$-lattice. Proving that these two identifications are well defined and bijective is an easy exercise.

The spaces $\widetilde{\mathcal{L}_{d,n}}$ and $\widetilde{\mathcal{P}_{d,n}}$ can be described more geometrically as follows: Since for every d-lattice $\Lambda < \mathbb{R}^n$ there exists a (non-unique) $k \in \mathrm{SO}_n(\mathbb{R})$ which rotates V_Λ to the space $\operatorname{span}_\mathbb{R}(e_1,\dots,e_d)$, then Λ is of the form

$$\Lambda = k^{-1} \begin{pmatrix} g_d \\ 0_{n-d\times d} \end{pmatrix} \begin{pmatrix} \mathbb{Z}^d \\ 0_{n-d\times 1} \end{pmatrix}$$

for some $g_d \in \mathrm{GL}_d(\mathbb{R})$. The element g_d can be replaced by any other element in the coset $g_d\,\mathrm{GL}_d(\mathbb{Z})$, and the element k can be replaced by any other element in $(\mathrm{SO}_d(\mathbb{R}) \times \mathrm{I}_{n-d})k$, if the element g_d is adjusted accordingly. As a result, where $\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}}$ is the diagonal embedding of $\mathrm{SO}_d(\mathbb{R})$ in

$$\widetilde{\mathcal{L}_{d,n}} \simeq \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \diagdown \mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R}) \diagup \mathrm{SL}_d(\mathbb{Z}) \tag{A1}$$

$\mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R})$. Similarly, every full lattice L in $V_\Lambda^\perp$ can be written as

$$L = k^{-1} \begin{pmatrix} 0_{d\times n-d} \\ g_{n-d} \end{pmatrix} \begin{pmatrix} 0_{d\times 1} \\ \mathbb{Z}^{n-d} \end{pmatrix},$$

where g_{n-d} can be replaced by any other element in the coset $g_{n-d}\,\mathrm{GL}_{n-d}(\mathbb{Z})$, and the element k can be replaced by any other element in $(\mathrm{I}_d \times \mathrm{SO}_{n-d}(\mathbb{R}))k$, if the element g_{n-d} is adjusted accordingly. As a result, where $\mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}}$ is as before, and $\mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}}$ is the analogous diagonal embedding

$$\widetilde{\mathcal{P}_{d,n}} \simeq \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \times \mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}} \backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_{n-d}(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z}) \times \mathrm{SL}_{n-d}(\mathbb{Z}), \tag{A2}$$

of $\mathrm{SO}_{n-d}(\mathbb{R})$ in $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_{n-d}(\mathbb{R})$. The spaces $\widetilde{\mathcal{L}_{d,n}}$ and $\widetilde{\mathcal{P}_{d,n}}$ are equipped with natural measures that are induced by the chosen Haar measures on the ambient groups ($\mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R})$ for $\widetilde{\mathcal{L}_{d,n}}$ and $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_{n-d}(\mathbb{R})$ for $\widetilde{\mathcal{P}_{d,n}}$)). With the presentations in (A1) and (A2), the map that sends a lattice to its dual has the geometric interpretation of being induced by the Cartan involution on $\mathrm{GL}(\mathbb{R})$:

Lemma A.5 *The map $\Lambda \mapsto \Lambda^*$ from $\widetilde{\mathcal{L}_{d,n}}$ to itself descends from the involution $(k, g_d) \mapsto (k, g_d^{-t})$ of $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R})$, where $\widetilde{\mathcal{L}_{d,n}}$ is viewed as the quotient* (A1).

Proof. This is a consequence of (A1) along with Proposition A.1, which says that a basis to the dual basis to the lattice spanned by the columns of g_d is $g_d^{-\mathrm{t}}$. □

The following two propositions are a consequence of this lemma:

Proposition A.6 *The map $\Lambda \mapsto \Lambda^*$ is a measure preserving diffeomorphism from $\widetilde{\mathcal{L}_{d,n}}$ to itself.*

Proof. By Lemma A.5, the map $\Lambda \mapsto \Lambda^*$ descends from a map that is the identity on $\mathrm{SO}_n(\mathbb{R})$ and the Cartan involution on $\mathrm{GL}_d(\mathbb{R})$—both are (Haar) measures preserving diffeomorphisms. □

Proposition A.7 *The three maps*

$$(\Lambda, L) \mapsto (\Lambda^*, L), \quad (\Lambda, L) \mapsto (\Lambda, L^*), \quad (\Lambda, L) \mapsto (\Lambda^*, L^*)$$

from $\widetilde{\mathcal{P}_{d,n}}$ to itself are measures preserving diffeomorphisms.

Proof. Similarly to Lemma A.5, when considering (A2), the three maps in the statement descend from the following involutions of $\mathrm{SO}_n(\mathbb{R}) \times \mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_{n-d}(\mathbb{R})$:

$$(k, g_d, g_{n-d}) \mapsto (k, g_d^{-\mathrm{t}}, g_{n-d}),$$
$$(k, g_d, g_{n-d}) \mapsto (k, g_d, g_{n-d}^{-\mathrm{t}}),$$
$$(k, g_d, g_{n-d}) \mapsto (k, g_d^{-\mathrm{t}}, g_{n-d}^{-\mathrm{t}}),$$

respectively. These involutions all consist of combinations of the identity map and the Cartan involution on the groups involved and are therefore diffeomorphisms that preserve the Haar measure. □

Spaces of unimodular lattices

The space $\mathcal{L}_{d,n} = \mathrm{SL}_n(\mathbb{R}) / \left(\begin{bmatrix} \mathrm{SL}_d(\mathbb{Z}) & \mathbb{R}^{d,n-d} \\ 0_{n-d\times d} & \mathrm{SL}_{n-d}(\mathbb{R}) \end{bmatrix} \times A' \right)$, where

$$A' = \left\{ \begin{bmatrix} \alpha^{\frac{1}{d}} \mathrm{I}_d & 0_{d\times n-d} \\ 0_{n-d\times d} & \alpha^{-\frac{1}{n-d}} \mathrm{I}_{n-d} \end{bmatrix} : \alpha > 0 \right\},$$

is the space of oriented d-lattices in $\mathbb{R}^n$, up to homothety. Similarly, the space $\mathcal{P}_{d,n} = \mathrm{SL}_n(\mathbb{R}) / \left(\begin{bmatrix} \mathrm{SL}_d(\mathbb{Z}) & \mathbb{R}^{d,n-d} \\ 0_{n-d\times d} & \mathrm{SL}_{n-d}(\mathbb{Z}) \end{bmatrix} \times A' \right)$ is the space of pairs (Λ, L) of oriented lattices satisfying $\mathrm{covol}(\Lambda)\,\mathrm{covol}(L) = 1$, where Λ is a d-lattice and L is an $(n-d)$-lattice in $V_\Lambda^\perp$. More accurately, it is the space of equivalence classes of such pairs, modulo the equivalence relation $(\Lambda', L') \sim (\Lambda, L)$ if and only if there exists $\alpha > 0$ such that $(\Lambda', L') = \left(\alpha^{\frac{1}{d}}\Lambda, \alpha^{-\frac{1}{n-d}} L\right)$. In analogy with (A1) and (A2), the spaces $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$ can be presented as and The spaces $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$, unlike $\widetilde{\mathcal{L}_{d,n}}$ and $\widetilde{\mathcal{P}_{d,n}}$,

$$\mathcal{L}_{d,n} \simeq \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z})$$

$$\mathcal{P}_{d,n} = \mathrm{SO}_d(\mathbb{R})^{\mathrm{diag}} \times \mathrm{SO}_{n-d}(\mathbb{R})^{\mathrm{diag}} \backslash \mathrm{SO}_n(\mathbb{R}) \times \mathrm{SL}_d(\mathbb{R}) \times \mathrm{SL}_{n-d}(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z}) \times \mathrm{SL}_{n-d}(\mathbb{Z}).$$

have finite volume.

PROPOSITION A.8 *Propositions A.6 and A.7 hold also when replacing $\widetilde{\mathcal{L}_{d,n}}$ and $\widetilde{\mathcal{P}_{d,n}}$ by $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$.*

Proof. The dual of a unimodular lattice is also unimodular, by Corollary A.3; so the map $\Lambda \mapsto \Lambda^*$ is an involution of $\mathcal{L}_{d,n}$, and similarly the maps in Proposition A.7 are involutions of $\mathcal{P}_{d,n}$. The proofs for the adaptations of Propositions A.6 and A.7 to unimodular lattices are obtained by adjusting the proofs of these propositions: the appearances of GL are replaced by SL. □

Since the spaces $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$ are unbounded but of finite volume, it is desirable to have a criterion for determining when a set is compact, or alternatively, when does a sequence of elements in the space diverge to infinity. For the more familiar space of full unimodular lattices in $\mathbb{R}^d$, $\mathrm{SL}_d(\mathbb{R}) / \mathrm{SL}_d(\mathbb{Z})$, which is also non-compact but of finite measure, the answer is provided by Mahler's compactness criterion [9, V.3]. The latter states that a set is compact if and only if there exists $\delta > 0$ such that all the lattices in this set have shortest non-zero vector of length $> \delta$. Equivalently, a sequence of lattices $\{\Lambda_m\}$ diverges if and only if the lengths of the shortest vectors in $\{\Lambda_m\}$ is a sequence of positive real numbers that converges to zero. The purpose of the following is to state an analogous criterion for compactness in the spaces $\mathcal{L}_{d,n}$ and $\mathcal{P}_{d,n}$. For every equivalence class $[\Lambda_0] \in \mathcal{L}_{d,n}$ (resp.

$[(\Lambda_0, L_0)] \in \mathcal{P}_{d,n}$), a *unimodular representative* is the unique representative $\Lambda \in [\Lambda_0]$ of covolume one (resp. $(\Lambda, L) \in [(\Lambda_0, L_0)]$ such that Λ and L are of covolume one).

PROPOSITION A.9 *A subset Ψ of $\mathcal{L}_{d,n}$ is bounded iff there is some $\delta > 0$ such that for every $[\Lambda_0] \in \Psi$, its unimodular representative $\Lambda \in [\Lambda_0]$ has the property that the shortest vector of Λ is of length $\geq \delta$.*

Similarly, subset Ξ of $\mathcal{P}_{d,n}$ is bounded iff there is some $\delta > 0$ such that for every $[(\Lambda_0, L_0)] \in \Xi$, its unimodular representative $(\Lambda, L) \in [(\Lambda_0, L_0)]$ has the property that both shortest vectors of Λ and L are of length $\geq \delta$.

Proof. We will prove the claim for $\mathcal{P}_{d,n}$, since the case of $\mathcal{L}_{d,n}$ is similar. The projection $(\pi_{\mathcal{P}_{d,n}\to\mathcal{X}_d}, \pi_{\mathcal{P}_{d,n}\to\mathcal{X}_{n-d}}) : \mathcal{P}_{d,n} \to \mathcal{X}_d \times \mathcal{X}_{n-d}$ is continuous and proper, by Proposition 2.4. Hence, a set Ξ in $\mathcal{P}_{d,n}$ is bounded if and only if its image under this projection is bounded. Also by Proposition 2.4, the image of an element $[(\Lambda, L)] \in \mathcal{P}_{d,n}$ under $\pi_{\mathcal{P}_{d,n}\to\mathcal{X}_d}$ is $\mathrm{shape}(\Lambda) \in \mathcal{X}_d$, and the image of this element under $\pi_{\mathcal{P}_{d,n}\to\mathcal{X}_{n-d}}$ is $\mathrm{shape}(L) \in \mathcal{X}_{n-d}$. Hence, Ξ is bounded iff for all $[(\Lambda, L)] \in \Xi$ we have that $\mathrm{shape}(\Lambda)$ is restricted to a bounded set in $\mathcal{X}_d$, and $\mathrm{shape}(L)$ is restricted to a bounded set in $\mathcal{X}_{n-d}$. Now Mahler's compactness criterion completes the proof. □

Connection between the shape of a lattice and of its dual

A *shape* of a lattice (sometimes referred to as a *type* of a lattice) is its similarity class modulo rotation and rescaling. It is a very common parameter to study in the context of lattices and appears for example in crystallography and the theory of periodic tilings. Naturally, the shape of a lattice is connected to its symmetries, namely the set of orthogonal transformations of $\mathbb{R}^n$ under which the lattice is preserved.

DEFINITION A.10 Let $1 \leq d \leq n$, and a d-lattice Λ. The finite group $\mathrm{Sym}(\Lambda) := \{g \in \mathrm{O}(V_\Lambda) : g\Lambda = \Lambda\}$ is called the *symmetry group* of Λ.

If two lattices in $\mathbb{R}^n$ have the same shape, then in particular their group symmetries are conjugated. The opposite does not hold, for example the lattices $\mathbb{Z}e_1 \oplus \mathbb{Z}\alpha e_2$ and $\mathbb{Z}e_1 \oplus \mathbb{Z}\beta e_2$ for $1 < \alpha < \beta$ have different shapes, but their symmetries are identical: $\pm\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \pm\left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right] \in \mathrm{O}_2(\mathbb{R})$. Below we observe the surprising fact that even though a lattice and its dual in general do not have the same shape (for example, $\mathbb{Z}e_1 \oplus \mathbb{Z}\alpha e_2$ and $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2/\alpha$ are dual), they do share the same group of symmetries.

CLAIM A.11. *The symmetry groups of Λ and Λ^* are identical.*

Proof. To show $\mathrm{Sym}(\Lambda) \subseteq \mathrm{Sym}(\Lambda^*)$, take $\gamma \in \mathrm{Sym}(\Lambda)$ and we need to show that $\gamma y \in \Lambda^*$ for every $y \in \Lambda^*$. Since $\mathrm{Sym}(\Lambda) \subset \mathrm{O}_n(\mathbb{R})$, we have $\gamma^{\mathrm{t}} = \gamma^{-1} \in \mathrm{Sym}(\Lambda)$, and therefore $\gamma^t x \in \Lambda$ for every $x \in \Lambda$. Thus,

$$\langle x, \gamma y\rangle = \langle \gamma^t x, y\rangle \in \mathbb{Z},$$

which proves $\gamma y \in \Lambda^*$. Equality follows from the fact that $(\Lambda^*)^* = \Lambda$ (Cor. A.3). □

We conclude our introduction to dual lattices with a short discussion on the successive minima of the dual lattice.

THEOREM A.12 ([**5**, Thm. 2.1], see also [**11**, VIII.5, Thm. VI]). *If $\lambda_1,\dots,\lambda_k$ are the successive minima of a k-lattice and $\lambda_1^*,\dots,\lambda_k^*$ of its dual, then $1\le\lambda_j\lambda_{k-j+1}^*\le k$ for every $1\le j\le k$.*

From this theorem, we conclude the following lemma. By Minkowski's second theorem [**11**, VIII.2, Thm. 1], if L is a k-dimensional lattice with successive minima $\lambda_1,\dots,\lambda_k$ then

$$\frac{1}{k!}\lambda_1\cdots\lambda_k\le\operatorname{covol}(L)\le\lambda_1\cdots\lambda_k. \tag{A3}$$

Keeping the notation from the proof of Prop. 8.3, we let L^j for $1\le j\le k$ denote the lattice spanned by the first j columns of the matrix in $\widetilde{F_k}$ that represents L and let $(L^*)^{k-j}$ denote the lattice spanned by the first $k-j$ elements in the basis that is dual to the columns of this matrix.

LEMMA A.13 *Let L be a k-lattice in $\mathbb{R}^n$, with $1\le k\le n-1$. Then for every $1\le j\le k$,*

$$\frac{\operatorname{covol}(L^j)}{(k-j)!k^j}\le\frac{\operatorname{covol}\left((L^*)^{k-j}\right)}{\operatorname{covol}(L^*)}\le j!k!\operatorname{covol}(L^j).$$

Proof. If L is spanned by $\mathbf{B}=[v_1\dots v_k]\in\widetilde{F_k}$, and for every $1\le j\le k$ the projection of v_j to the orthogonal complement of $\operatorname{span}_{\mathbb{R}}(v_1,\dots,v_{j-1})$ has length $a_j\ne0$, then it is well known that $a_j\asymp\lambda_j(L)$ (see for example [**21**]), and more specifically $a_j\le\lambda_j(L)\le ja_j$ (This is essentially because $\mathbf{B}$ is reduced in the sense of Korkine–Zolotarev). Since L^j is spanned by $v_1,\dots,v_j$ then $\operatorname{covol}(L^j)=\prod_{i=1}^j a_i$ and therefore we obtain that, similarly to A3,

$$\frac{\lambda_1(L)\cdots\lambda_j(L)}{j!}\le\operatorname{covol}(L^j)\le\lambda_1(L)\cdots\lambda_j(L).$$

Using this, Theorem A.12 and A3, we have that for every $1\le j\le k$,

$$\operatorname{covol}(L^j)\le\lambda_1\cdots\lambda_j\le\frac{k^j}{\lambda_k^*\cdots\lambda_{k-j+1}^*}$$

$$\le\frac{k^j}{\lambda_k^*\cdots\lambda_{k-j+1}^*}\cdot\frac{\lambda_1^*\cdots\lambda_k^*}{\operatorname{covol}(L^*)}=k^j\cdot\frac{\lambda_1^*\cdots\lambda_{k-j}^*}{\operatorname{covol}(L^*)}\le\frac{k^j\cdot(k-j)!\operatorname{covol}((L^*)^{k-j})}{\operatorname{covol}(L^*)},$$

which proves the LHS inequality in the statement of the lemma. The RHS inequality is proved similarly. □

Appendix B: Factor lattices

The term *factor lattice* was coined by Schmidt in [**44**], and we extend it from integral lattices to general lattices. For this, we begin by extending the definition of primitiveness to lattices that are not necessarily integral.

Definition B.1 Assume Δ is a full lattice in $\mathbb{R}^n$, and Λ is a d-lattice that is contained in Δ. We say that Λ is *primitive* inside Δ if $\Lambda = \Delta \cap V_\Lambda$.

Note that when $\Delta = \mathbb{Z}^n$, this definition agrees with the standard definition of a primitive lattice. Indeed, in this case, we will call Λ primitive (and omit the 'w.r.t. $\mathbb{Z}^n$').

Definition B.2 Given a d-lattice Λ that is primitive inside Δ, define the factor lattice of Λ (w.r.t. Δ), denoted $\Lambda^{\pi,\Delta}$, as the orthogonal projection of Δ into the space $(\mathrm{span}_{\mathbb{R}}(\Lambda))^\perp$. When Λ is primitive inside $\mathbb{Z}^n$, we omit the 'w.r.t. $\mathbb{Z}^n$' from the name and the notation: we denote Λ^π and refer to it as the factor lattice of Λ.

For example, the factor lattice of $\mathbb{Z}(1,-1)$ is $\mathbb{Z}(\frac{1}{2},\frac{1}{2})$. The following proposition reveals the motivation behind defining the factor lattice, which is that it represents the quotient lattice Δ/Λ. It also ensures that $\Lambda^{\pi,\Delta}$ is indeed a lattice, also this is not hard to show.

Proposition B.3 *For a d-lattice Λ that is primitive inside a full lattice Δ, consider the inner product on the quotient $\mathbb{R}^n/V_\Lambda$:*

$$\langle x + V_\Lambda, y + V_\Lambda \rangle_{\mathbb{R}^n/V_\Lambda} := \langle \pi(x), \pi(y) \rangle ,$$

where $\pi : \mathbb{R}^n \to V_\Lambda^\perp$ is the orthogonal projection and $\langle \cdot,\cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. Then the quotient lattice Δ/Λ with $\langle \cdot,\cdot \rangle_{\mathbb{R}^n/V_\Lambda}$ is isometric to the factor lattice $\Lambda^{\pi,\Delta}$ with $\langle \cdot,\cdot \rangle$.

Proof. The group isomorphism $x + \Lambda \mapsto \pi(x)$ from Δ/Λ to $\Lambda^{\pi,\Delta}$ is an isometry by definition. □

Then $\Lambda^{\pi,\Delta}$ is a lattice, because it is isometric to one. The above also demonstrates that it is necessary to require that Λ is primitive inside Δ; otherwise, Δ/Λ (and therefore $\Lambda^{\pi,\Delta}$) has torsion.

Proposition B.4 $\mathrm{covol}(\Lambda^{\pi,\Delta}) = \mathrm{covol}(\Delta)/\mathrm{covol}(\Lambda)$.

Proof. Since covolumes do not change under rotations, we may assume that $V_\Lambda = \mathbb{E}_d := \mathrm{span}_{\mathbb{R}}(e_1,\dots,e_d)$, where $d = \mathrm{rank}(\Lambda)$. Let $\mathbf{B}_{d\times n}$ be a basis for Λ; by primitiveness, it can be completed to a basis $g := [\mathbf{B}|\mathbf{C}]$ of Δ, where $g \in \mathrm{GL}_n(\mathbb{R})$. Since $\mathbf{B} \subset \mathbb{E}_d$, the matrix g is of the form $g = \left[\begin{smallmatrix} g_1 & * \\ 0 & g_2 \end{smallmatrix}\right]$ with $g_1 \in \mathrm{GL}_d(\mathbb{R})$ and $g_2 \in \mathrm{GL}_{n-d}(\mathbb{R})$. Clearly $\Lambda = [g_1|0_{d\times n}]\mathbb{Z}^n$ and therefore $\mathrm{covol}(\Lambda)^2 = \det(\left[\begin{smallmatrix} g_1 & 0_{d\times n-d} \end{smallmatrix}\right]\left[\begin{smallmatrix} g_1 \\ 0_{n-d\times d} \end{smallmatrix}\right]) = \det(g_1)^2$, so $\mathrm{covol}(\Lambda) = \det(g_1)$. Note that the projection of g to $\mathbb{E}_d^\perp$ is $\left[\begin{smallmatrix} 0_{d\times n-d} \\ g_2 \end{smallmatrix}\right]$, so the projection of $\Lambda_g = \Delta$ to $\mathbb{E}_d^\perp = V_\Lambda^\perp$ is the lattice spanned by $\left[\begin{smallmatrix} 0_{d\times n-d} \\ g_2 \end{smallmatrix}\right]$. By definition, this lattice is $\Lambda^{\pi,\Delta}$. We conclude that $\mathrm{covol}(\Lambda^{\pi,\Delta})^2 = \det(\left[\begin{smallmatrix} 0_{d\times n-d} \\ g_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0_{n-d\times d} & g_2 \end{smallmatrix}\right]) = \det(g_2)^2$, namely $\mathrm{covol}(\Lambda^{\pi,\Delta}) = \det(g_2)$. Finally, $\mathrm{covol}(\Delta) = \det(g) = \det(g_1)\cdot\det(g_2)$. □

In the case where $\Delta = \mathbb{Z}^n$, namely of primitive integral lattices, the concepts of a dual and a factor lattice are related in the following way: Given a primitive d-lattice $\Lambda < \mathbb{Z}^n$, we define the *orthogonal* *lattice* of Λ to be

$$\Lambda^{\perp} = \mathbb{Z}^n \cap V_{\Lambda}^{\perp}$$

(see for example [11, 6]). Note that, by definition, $\Lambda^{\perp} < \mathbb{Z}^n$ is a primitive $n-d$-lattice.

Proposition B.5 ([45]). *For Λ primitive,* $(\Lambda^{\pi})^{*} = \Lambda^{\perp}$.

Proof. Let $\pi : \mathbb{R}^n \to (V_{\Lambda})^{\perp}$ denote the orthogonal projection; then $\Lambda^{\pi} = \pi(\mathbb{Z}^n)$. Notice that for every $x \in (V_{\Lambda})^{\perp}$ and $z \in \mathbb{Z}^n$,

$$\langle x, \pi(z)\rangle_{\mathbb{R}^n} = \left\langle \pi^{t}(x), z\right\rangle_{\mathbb{R}^n} = \langle \pi(x), z\rangle_{\mathbb{R}^n} = \langle x, z\rangle_{\mathbb{R}^n}.$$

Then $x \in (\Lambda^{\pi})^{*}$ if and only if $\langle x, z\rangle_{\mathbb{R}^n} \in \mathbb{Z}$ for every $z \in \mathbb{Z}^n$, which holds if and only if $x \in \mathbb{Z}^n$—namely, $x \in \mathbb{Z}^n \cap (V_{\Lambda})^{\perp} = \Lambda^{\perp}$. □

Corollary B.6 *For Λ primitive,* $\operatorname{covol}(\Lambda^{\perp}) = \operatorname{covol}(\Lambda)$.

Proof. This is a direct consequence of Corollary A.3, Proposition B.4 and Proposition B.5, while noticing that $\operatorname{covol}(\mathbb{Z}^n) = 1$. □

Remark B.7 One could wonder if the result of Proposition B.5 can be extended to d-lattices $\Lambda < \mathbb{Z}^n$ that are not necessarily primitive, or even to d-lattices $\Lambda < \Delta$ where Δ is not $\mathbb{Z}^n$. Indeed, it is quite natural to extend the definition of $\Lambda^{\perp}$ to general lattices Λ such that if Λ is primitive inside any full lattice Δ, then the orthogonal lattice to Λ with respect to Δ is $\Lambda^{\perp,\Delta} = \Delta \cap V_{\Lambda}^{\perp}$. Then the question becomes, is it true that the dual of $\Lambda^{\pi,\Delta}$ is $\Lambda^{\perp,\Delta}$. The answer to this is no! Let us consider two counter examples.

1. Consider the lattice Δ spanned by $\left[\begin{smallmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{smallmatrix}\right]$, and the lattice $\Lambda = \operatorname{span}_{\mathbb{Z}}(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right])$ which is primitive inside it. Then by definition $\Lambda^{\perp,\Delta} = \Delta \cap V_{\Lambda}^{\perp} = \Delta \cap \operatorname{span}_{\mathbb{R}}(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]) = \{0\}$.
2. Take $\Delta := \mathbb{Z}e_1 \oplus \mathbb{Z}\alpha e_2$ for some $\alpha > 0$. Clearly Δ has covolume α. The lattice $\Lambda = \mathbb{Z}e_1$ is primitive inside Δ and has covolume 1. Then $\Lambda^{\perp,\Delta} = \mathbb{Z}\alpha e_2$ has covolume α, and in particular it cannot be that $\Lambda^{\perp,\Delta}$ is the dual of $\Lambda^{\pi,\Delta}$, because otherwise it would have covolume $(\operatorname{covol}(\Delta)/\operatorname{covol}(\Lambda))^{-1} = \alpha^{-1}$.

The first example shows that the definition of $\Lambda^{\perp,\Delta}$ is in a sense meaningless; the second example shows us that $\Lambda^{\perp,\Delta}$ being the dual of $\Lambda^{\pi,\Delta}$ fails even when Δ is integral (which happens when we take $\alpha \in \mathbb{Z}$ in the second example), unless it is the whole of $\mathbb{Z}^n$.

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