

Tamely Ramified Geometric Langlands Correspondence in Positive Characteristic

Shiyu Shen*

The Institute of Science and Technology Austria, Klosterneuburg 3400, Austria

*Correspondence to be sent to: e-mail: shiyu.shen@ist.ac.at

Communicated by Prof. Xinwen Zhu

We prove a version of the tamely ramified geometric Langlands correspondence in positive characteristic for $GL_n(k)$, where k is an algebraically closed field of characteristic $p > n$. Let X be a smooth projective curve over k with marked points, and fix a parabolic subgroup of $GL_n(k)$ at each marked point. We denote by $\text{Bun}_{n,p}$ the moduli stack of (quasi-)parabolic vector bundles on X , and by $\mathcal{L}oc_{n,p}$ the moduli stack of parabolic flat connections such that the residue is nilpotent with respect to the parabolic reduction at each marked point. We construct an equivalence between the bounded derived category $D^b(\text{QCoh}(\mathcal{L}oc_{n,p}^0))$ of quasi-coherent sheaves on an open substack $\mathcal{L}oc_{n,p}^0 \subset \mathcal{L}oc_{n,p}$, and the bounded derived category $D^b(\mathcal{D}_{\text{Bun}_{n,p}}^0\text{-mod})$ of $\mathcal{D}_{\text{Bun}_{n,p}}^0$ -modules, where $\mathcal{D}_{\text{Bun}_{n,p}}^0$ is a localization of $\mathcal{D}_{\text{Bun}_{n,p}}$ the sheaf of crystalline differential operators on $\text{Bun}_{n,p}$. Thus, we extend the work of Bezrukavnikov–Braverman [8] to the tamely ramified case. We also prove a correspondence between flat connections on X with regular singularities and meromorphic Higgs bundles on the Frobenius twist $X^{(1)}$ of X with first-order poles.

1 Introduction

1.1 Geometric Langlands in positive characteristic

Let X be a smooth projective curve over \mathbb{C} . Let G be a reductive group over \mathbb{C} , and let \check{G} be its Langlands dual group. The geometric Langlands correspondence (GLC), as proposed by Beilinson and Drinfeld in [7], is a conjectural equivalence between the (appropriately defined) category of \mathcal{D} -modules on the moduli stack Bun_G of G -bundles on X , and the (appropriately defined) category of quasi-coherent sheaves on the moduli stack $\mathcal{L}oc_{\check{G}}$ of \check{G} -local systems on X . A precise statement of this conjecture can be found in [2].

In [8], a generic version of the GLC in positive characteristic is established for $G = GL_n(k)$. The \mathcal{D} -modules are interpreted in terms of crystalline differential operators. Using the Azumaya property of crystalline differential operators and a twisted version of the Fourier–Mukai transform, the authors prove a generic version of the GLC over the open subset of the Hitchin base where the spectral curves are smooth. In the case of $G = GL_n(k)$, the results of [8] are generalized in various directions. In [23], the mirabolic version of this correspondence is established. In [28], the author proved the quantum version of this correspondence. In [17], the equivalence in [8] is extended to the Hitchin base of reduced and irreducible spectral curves. The results of [8] were extended to arbitrary reductive groups in [10] and [11].

Received: February 9, 2023. Revised: September 16, 2023. Accepted: January 5, 2024

© The Author(s) 2024. Published by Oxford University Press.

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted reuse, distribution, and reproduction in any medium, provided the original work is properly cited.

1.2 Tamely ramified geometric Langlands correspondence

The main purpose of this paper is to establish the tamely ramified version of the GLC proved in [8], that is, we allow the flat connections to have regular singularities. The term “tamely ramified” comes from analogy with the local Langlands program. See [15] Section 8 for a discussion of the tamely ramified GLC over \mathbb{C} . Let k be an algebraically closed field of characteristic p , and let X be a smooth projective curve over k . We will work on the case of $G = GL_n(k)$ and assume $p > n$. Let $D = q_1 + q_2 + \dots + q_m$ be an effective reduced divisor on X , and let $P_D = (P_1, P_2, \dots, P_m)$ be an ordered m -tuple of parabolic subgroups of $GL_n(k)$. We assume that we are in one of the following three cases:

- (1) $g_X \geq 2$,
- (2) $g_X = 1, m \geq 2$, and at least two P_i are proper parabolic subgroups; or $m = 1, P$ is a Borel subgroup and $n \geq 3$,
- (3) $g_X = 0, m \geq 4$, and all P_i are Borel subgroups.

Compared to the unramified version of the GLC, instead of considering \mathbf{Bun}_n and $\mathcal{L}oc_n$, we consider the moduli stack \mathbf{Bun}_{n,P_D} of (quasi-)parabolic vector bundles (vector bundles of rank n with a P_i -reduction at each q_i), and the moduli stack $\mathcal{L}oc_{n,P_D}$ of flat connections on parabolic vector bundles with regular singularities at q_1, q_2, \dots, q_m such that the residue at each q_i is nilpotent with respect to the P_i -reduction. Note that the cotangent bundle $T^* \mathbf{Bun}_{n,P_D}$ is isomorphic to the moduli stack $\mathcal{H}iggs_{n,P_D}$ of parabolic Higgs bundles such that the residue of the Higgs field is nilpotent with respect to the parabolic reduction at each q_i . We denote by $\mathcal{D}_{\mathbf{Bun}_{n,P_D}}$ the sheaf of crystalline differential operators on \mathbf{Bun}_{n,P_D} (Defined in the sense of [8], Section 3.13. See Section 3.3 and Section 5.1.). We will define a localization $\mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0$ of $\mathcal{D}_{\mathbf{Bun}_{n,P_D}}$ and an open substack $\mathcal{L}oc_{n,P_D}^0$ of $\mathcal{L}oc_{n,P_D}$ (see Section 5.1 for precise definitions). We will construct an $\mathcal{O}_{\mathcal{L}oc_{n,P_D}^0} \boxtimes \mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0$ -module \mathcal{P} (see Section 5.5) and consider the Fourier–Mukai functor with kernel \mathcal{P}

$$\Phi_{\mathcal{P}} : D^b(\mathbf{QCoh}(\mathcal{L}oc_{n,P_D}^0)) \longrightarrow D^b(\mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0\text{-mod})$$

from the bounded derived category of quasi-coherent sheaves on $\mathcal{L}oc_{n,P_D}^0$ to the bounded derived category of $\mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0$ -modules. The main theorem of the paper is the following:

Theorem 1.1. $\Phi_{\mathcal{P}}$ is an equivalence of derived categories.

There are natural functors from both sides of the equivalence: the Hecke functor $H_{P_D}^0$ (see Section 5.6)

$$H_{P_D}^0 : D^b(\mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0\text{-mod}) \longrightarrow D^b(\mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0 \boxtimes \mathcal{D}_{X \setminus D}\text{-mod})$$

and the functor $W_{P_D}^0$

$$W_{P_D}^0 : D^b(\mathcal{O}_{\mathcal{L}oc_{n,P_D}^0}\text{-mod}) \longrightarrow D^b(\mathcal{O}_{\mathcal{L}oc_{n,P_D}^0} \boxtimes \mathcal{D}_{X \setminus D}\text{-mod})$$

defined by tensoring with the universal flat connection. Let $\Phi_{\mathcal{P},X \setminus D}$ be the Fourier–Mukai equivalence induced by the pull-back of \mathcal{P} :

$$\Phi_{\mathcal{P},X \setminus D} : D^b(\mathcal{O}_{\mathcal{L}oc_{n,P_D}^0} \boxtimes \mathcal{D}_{X \setminus D}\text{-mod}) \xrightarrow{\cong} D^b(\mathcal{D}_{\mathbf{Bun}_{n,P_D}}^0 \boxtimes \mathcal{D}_{X \setminus D}\text{-mod}).$$

The equivalence in Theorem 1.1 satisfies the following Hecke eigenvalue property:

Theorem 1.2. There is an isomorphism of functors:

$$H_{P_D}^0 \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{P},X \setminus D} \circ W_{P_D}^0.$$

Now let (E, ∇) be a k -point of $\mathcal{L}oc_{n,P_D}^0$. We denote by $\mathcal{M}_{E,\nabla}$ the image of (E, ∇) under $\Phi_{\mathcal{P}}$. By Theorem 1.2, $\mathcal{M}_{E,\nabla}$ satisfies

$$H_{P_D}^0(\mathcal{M}_{E,\nabla}) \cong \mathcal{M}_{E,\nabla} \boxtimes E.$$

1.3 Summary of the proof

We fix a k -point $q \in X$ and a parabolic subgroup P of $GL_n(k)$. For the purpose of simplifying notations, our exposition will be restricted to the case of $D = q$ and $P_D = P$ from now on. The only proof that will be different in the more general setting is the proof of Proposition 2.7 in the case of $X = \mathbf{P}^1$, $m \geq 4$, and all P_i are Borel subgroups. We discuss this case in Remark 2.11.

Our proof of Theorem 1.1 is based on the same strategy as used in [8], but some new ingredients come into play. Note that in [8], the GLC is established over the open subset of the Hitchin base where the spectral curves are smooth. Compared to the unramified case in [8], one of the main difficulties in the tamely ramified case is that unless P is a Borel subgroup of $GL_n(k)$, there are no smooth spectral curves. We resolve this situation by considering the normalization of the spectral curves. It is observed in [25] that under generic restrictions on the spectral curves, a fiber of the Hitchin map

$$h_P : \mathcal{Higgs}_{n,P} \longrightarrow B_P$$

is isomorphic to the Picard stack of the normalization of the corresponding spectral curve. In Section 2.3, we extend this observation to a family version. More precisely, we prove:

Theorem 1.3. There exists a Zariski open dense subset $B_P^0 \subset B_P$ and a flat family of smooth projective curves $\tilde{\Sigma} \rightarrow B_P^0$ such that

$$\mathcal{Higgs}_{n,P} \times_{B_P} B_P^0 \cong \text{Pic}(\tilde{\Sigma}/B_P^0).$$

For each $b \in B_P^0(k)$, $\tilde{\Sigma}_b$ is the normalization of the spectral curve Σ_b .

In Section 4, we establish a correspondence between flat connections on X with regular singularity at q and $\Omega_{X^{(1)}}(q)$ -twisted Higgs bundles on the Frobenius twist $X^{(1)}$ of X , which can be thought of as a characteristic p version of the non-abelian Hodge correspondence in [26]. Let \underline{a} be an unordered n -tuple of elements in k . We denote by $\mathcal{Higgs}_{n,\underline{a}}(X^{(1)})$ the moduli stack of $\Omega_{X^{(1)}}(q)$ -twisted Higgs bundles (E, ϕ) on $X^{(1)}$ such that the tuple of eigenvalues of the residue $\text{res}_q(\phi)$ of the Higgs field at q is \underline{a} . Let $B_{\underline{a}}^{(1)}$ be the image of $\mathcal{Higgs}_{n,\underline{a}}(X^{(1)})$ under the Hitchin map $h^{(1)}$. We fix a set-theoretic section σ of the Artin–Schreier map $k \rightarrow k$ that maps t to $t^p - t$. We denote by $\mathcal{Loc}_{n,\sigma(\underline{a})}$ the moduli stack of flat connections (E, ∇) with regular singularity at q such that the tuple of eigenvalues of $\text{res}_q(\nabla)$ is $\sigma(\underline{a})$. The p -curvature of (E, ∇) (see Section 4.1) defines the Hitchin map h' for flat connections with regular singularity at q :

$$h' : \mathcal{Loc}_{n,\sigma(\underline{a})} \longrightarrow B_{\underline{a}}^{(1)}.$$

We will define an open substack (see Section 4.2)

$$\mathcal{Loc}_{n,\sigma(\underline{a})}^r \subset \mathcal{Loc}_{n,\sigma(\underline{a})}$$

and prove the following theorem:

Theorem 1.4.

- (1) $\mathcal{Loc}_{n,\sigma(\underline{a})}^r$ is a $\text{Pic}(\Sigma^{(1)}/B_{\underline{a}}^{(1)})$ -torsor,
- (2) $\mathcal{Loc}_{n,\sigma(\underline{a})} \cong \mathcal{Loc}_{n,\sigma(\underline{a})}^r \times^{\text{Pic}(\Sigma^{(1)}/B_{\underline{a}}^{(1)})} \mathcal{Higgs}_{n,\underline{a}}(X^{(1)})$.

Note that for an arbitrary reductive group G , a similar construction is used in [10] to establish the characteristic p version of the non-abelian Hodge correspondence for flat connections without singularities.

One of the key steps in our proof of Theorem 1.4 is to show that the map

$$h' : \mathcal{Loc}_{n,\sigma(\underline{a})}^r \longrightarrow B_{\underline{a}}^{(1)}$$

is surjective. Since we consider flat connections with singularity at q , we cannot apply the Azumaya property of differential operators on X directly. Instead, we construct a flat connection on $X \setminus q$ using the Azumaya property, construct a flat connection on the formal disk around q by explicitly solving a differential equation for the connection form, and glue them together using the Beauville–Laszlo theorem [5].

Note that $\mathcal{L}oc_{n,(0)}$ is the moduli stack of flat connections with regular singularity and nilpotent residue at q . Restricting the isomorphism in Theorem 1.4(2) to $(B_p^0)^{(1)}$ and combining Theorem 1.3, we deduce that $\mathcal{L}oc_{n,p}^0 := \mathcal{L}oc_{n,p} \times_{B_p^0} (B_p^0)^{(1)}$ is a $\text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$ -torsor.

It is proved in [8] that for a smooth algebraic stack Z that is good in the sense of [7] (i.e., Z satisfies $\dim T^*Z = 2 \dim Z$), there is a natural sheaf of algebras \mathcal{D}_Z on $T^*Z^{(1)}$ that satisfies $\pi_*^{(1)} \mathcal{D}_Z \cong \text{Fr}_* D_Z$, and the restriction of \mathcal{D}_Z to the maximal smooth open substack $(T^*Z^0)^{(1)} \subseteq (T^*Z)^{(1)}$ is an Azumaya algebra of rank $p^{2 \dim Z}$ (see Section 3.3 for a review of this construction). Here $\pi^{(1)} : T^*Z^{(1)} \rightarrow Z^{(1)}$ is the projection and $\text{Fr} : Z \rightarrow Z^{(1)}$ is the relative Frobenius. The stack $\text{Bun}_{n,p}$ “almost” satisfies those two properties, and we can still construct a sheaf of algebras $\mathcal{D}_{\text{Bun}_{n,p}}$ that satisfies $\pi_*^{(1)} \mathcal{D}_{\text{Bun}_{n,p}} \cong \text{Fr}_* D_{\text{Bun}_{n,p}}$. See Section 5.1 for details.

The restriction $\mathcal{D}_{\text{Bun}_{n,p}}^0$ of $\mathcal{D}_{\text{Bun}_{n,p}}$ to

$$\text{Higgs}_{n,p}^{(1)} \times_{B_p^0} (B_p^0)^{(1)} \cong \text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$$

is an Azumaya algebra. We associate with $\mathcal{D}_{\text{Bun}_{n,p}}^0$ its stack of splittings $\mathcal{Y}_{\mathcal{D}_{\text{Bun}_{n,p}}^0}$, which is a \mathbb{G}_m -gerbe over the Picard stack $\text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$. In Section 5.2, we show that $\mathcal{D}_{\text{Bun}_{n,p}}^0$ has a tensor structure, therefore $\mathcal{Y}_{\mathcal{D}_{\text{Bun}_{n,p}}^0}$ has the structure of a commutative group stack, and there is a short exact sequence

$$0 \rightarrow B\mathbb{G}_m \rightarrow \mathcal{Y}_{\mathcal{D}_{\text{Bun}_{n,p}}^0} \rightarrow \text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)}) \rightarrow 0.$$

By taking dual, we get another short exact sequence:

$$0 \rightarrow \text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)}) \rightarrow \mathcal{Y}_{\mathcal{D}_{\text{Bun}_{n,p}}^0}^\vee \xrightarrow{\pi} \mathbb{Z} \rightarrow 0.$$

In Section 5.5, we prove that $(\mathcal{Y}_{\mathcal{D}_{\text{Bun}_{n,p}}^0}^\vee)_1 := \pi^{-1}(1)$ is isomorphic to $\mathcal{L}oc_{n,p}^0$ as $\text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$ -torsors, therefore we can apply a twisted version of the Fourier–Mukai transform (reviewed in Section 5.4) to prove the equivalence in Theorem 1.1. For the proof of this isomorphism, we show that the tautological 1-form $\theta^{(1)}$ on $T^*(X \setminus q)^{(1)}$ extends to a 1-form $\tilde{\theta}^{(1)}$ on $\tilde{\Sigma}^{(1)}$, and both $\text{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$ -torsors are isomorphic to the moduli stack of rank one flat connections on $\tilde{\Sigma}$ with p -curvature $\tilde{\theta}^{(1)}$.

1.4 Structure of the article

In Section 2, we first review some basic constructions related to the Hitchin fibration. Then we define the Zariski open dense subset $B_p^0 \subset B_p$ and establish the correspondence between parabolic Higgs bundles and the Picard stack of the normalization of spectral curves over B_p^0 . In Section 3, we first review some properties of crystalline differential operators in positive characteristic, including the Azumaya property and the Cartier descent. Then we describe the correspondence between modules over an Azumaya algebra and twisted sheaves associated to its \mathbb{G}_m -gerbe of splittings. Finally, we review the definition of tensor structures on Azumaya algebras over group stacks. In Section 4, we first construct the Hitchin map for flat connections with regular singularities. Then we prove the non-abelian Hodge correspondence between $\mathcal{L}oc_{n,\sigma(\underline{a})}$ and $\text{Higgs}_{n,\underline{a}}(X^{(1)})$. In Section 5, we first define the sheaf of algebras $\mathcal{D}_{\text{Bun}_{n,p}}$ and construct a tensor structure on $\mathcal{D}_{\text{Bun}_{n,p}}^0$. Then we review the Fourier–Mukai transforms on commutative group stacks and use this framework to prove the main theorem. Finally, we discuss the Hecke eigenvalue property of this equivalence.

1.5 Notations and definitions

Unless otherwise mentioned, k is an algebraically closed field of characteristic $p > 0$. We consider the general linear group $GL_n(k)$ and assume $p > n$. Let $\mathfrak{gl}_n(k)$ be the Lie algebra of $GL_n(k)$. We denote by \mathcal{N} the nilpotent cone in $\mathfrak{gl}_n(k)$. Let P be a parabolic subgroup of $GL_n(k)$. The Lie algebra of P decomposes as $\text{Lie}(P) \cong \mathfrak{l} \oplus \mathfrak{n}_P^+$. We denote by \mathcal{O}_P the Richardson orbit corresponding to P , which is the unique nilpotent orbit in $\mathfrak{gl}_n(k)$ such that the intersection with \mathfrak{n}_P^+ is open dense in \mathfrak{n}_P^+ . Let X be a smooth projective algebraic curve over k . Let g_X be the genus of X . We fix a k -point $q \in X$. For any k -scheme S , we denote by $\iota_q : S \rightarrow S \times X$ the base change of $q : \text{Spec}(k) \rightarrow X$. We denote by p_X the projection from $S \times X$ to X , and by p_S the projection to S .

Definition 1.5. An S -family of (quasi-)parabolic vector bundles on X is a vector bundle E of rank n on $S \times X$ with a P -reduction along $S \times q$. We denote the moduli stack of such objects by $\mathbf{Bun}_{n,P}$. To be more precise, $\mathbf{Bun}_{n,P}$ classifies triples (E, E_P, τ) , where E_P is a P -bundle on $S \times q$ and τ is an isomorphism

$$\tau : E_P \times_P k^n \xrightarrow{\cong} t_q^* E.$$

Let \mathbf{Bun}_n be the moduli stack of rank n vector bundles on X . There is a canonical map from $\mathbf{Bun}_{n,P}$ to \mathbf{Bun}_n , which is defined by forgetting the P -reduction.

Remark 1.6. Let B be the Borel subgroup of $GL_n(k)$ that consists of upper triangular matrices. There is a one-to-one correspondence between the set of parabolic subgroups of $GL_n(k)$ containing B and the set of ordered n -tuples of positive integers $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ such that $\sum_{i=1}^s \mu_i = n$. This correspondence can be described as follows. We consider the standard representation of $GL_n(k)$ acting on k^n . Let e_1, e_2, \dots, e_n be the standard basis of k^n . For $i = 1, 2, \dots, s$, let $V_i = \bigoplus_{j=1}^{m_i} ke_j$ where $m_i = \sum_{k=1}^i \mu_k$. Then the parabolic subgroup P_μ corresponding to μ is identified with

$$\{g \in GL_n(k) | g(V_i) \subseteq V_i, 1 \leq i \leq s\}.$$

Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ be the conjugate partition to μ . The Richardson orbit corresponding to P_μ consists of M_λ the nilpotent matrix with Jordan blocks of sizes $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. Let E be a rank n vector bundle on $S \times X$. A P_μ -reduction of the structure group along $S \times q$ corresponds to a partial flag structure:

$$0 = E_q^0 \subset E_q^1 \subset E_q^2 \subset \dots \subset E_q^s = t_q^* E,$$

where E_i is a vector bundle of rank m_i on S .

Remark 1.7. In the work of Mehta–Seshadri [22], a parabolic vector bundle is defined as a quasi-parabolic vector bundle together with a set of real numbers $(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_r)$ satisfying

$$1 = \alpha_0 > \alpha_1 > \dots > \alpha_r \geq 0$$

called parabolic weights. The parabolic weights can be used to define a stability condition on such objects, which is necessary for the construction of a moduli space. Since we focus on studying the moduli stack of such objects, we do not introduce the parabolic weights in this paper.

Definition 1.8. An S -family of (quasi-)parabolic Higgs bundles on X is a parabolic vector bundle (E, E_P, τ) together with a Higgs field

$$\phi \in \Gamma(\mathcal{E}nd(E) \otimes p_X^*(\Omega_X(q))),$$

such that the residue of ϕ at q , which we denote by $\mathbf{res}_q(\phi) \in \mathbf{End}(t_q^* E)$, lies in $\Gamma(S, E_P \times_P n_P^+)$. In other words, if the parabolic reduction gives the following partial flag structure:

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_s = t_q^* E,$$

we require $\mathbf{res}_q(\phi)(E_i) \subseteq E_{i-1}$. We denote the moduli stack of such objects by $\mathcal{Higgs}_{n,P}$. We denote by $\mathcal{Higgs}_{n,q}$ the moduli stack of $\Omega_X(q)$ -twisted Higgs bundles (E, ϕ) ,

$$\phi \in \Gamma(\mathcal{E}nd(E) \otimes p_X^*(\Omega_X(q))).$$

There is a canonical map from $\mathcal{Higgs}_{n,P}$ to $\mathcal{Higgs}_{n,q}$, which is defined by forgetting the P -reduction.

Remark 1.9. $\mathcal{Higgs}_{n,P} \cong T^*\text{Bun}_{n,P}$.

Remark 1.10. A parabolic version of the Hitchin moduli stack is previously considered in the work of Yun [29]. Definition 2.1.1 in [29] is different from our Definition 1.8 in two aspects: the marked point q on X is allowed to move in [29], and the Higgs field ϕ is only required to preserve the flag structure instead of being nilpotent with respect to the flag structure.

Definition 1.11. An S -family of parabolic flat connections on X is a parabolic vector bundle (E, E_P, τ) together with a flat connection with regular singularity at q

$$\nabla : E \longrightarrow E \otimes p_X^*(\Omega_X(q)),$$

(i.e., ∇ is a \mathcal{O}_S -linear map of sheaves that satisfies the Leibniz rule), such that the residue $\text{res}_q \nabla$ of ∇ at q lies in $\Gamma(S, E_P \times_P n_P^+)$. We denote the moduli stack of such objects by $\mathcal{L}oc_{n,P}$.

We denote by $\mathcal{L}oc_{n,q}$ the moduli stack of flat connections of rank n on X with regular singularity at q . There is a canonical map from $\mathcal{L}oc_{n,P}$ to $\mathcal{L}oc_{n,q}$, which is defined by forgetting the P -reduction.

2 Spectral Data of Parabolic Higgs Bundles

2.1 Basic constructions

In this Subsection, we discuss the construction of the Hitchin map, spectral curves, and spectral sheaves in [19] and [6] in the parabolic setting. By taking the coefficients of the characteristic polynomial of the Higgs field, we get the Hitchin map:

$$h : \mathcal{Higgs}_{n,q} \longrightarrow B,$$

where $B = \bigoplus_{i=1}^n \Gamma(X, \Omega_X(q)^i)$ (More precisely, B is the affine space associated to the k -vector space $\bigoplus_{i=1}^n \Gamma(X, \Omega_X(q)^i)$, i.e., $B = \text{Spec}(\text{Sym}(\bigoplus_{i=1}^n \Gamma(X, \Omega_X(q)^i))^\vee)$). If we require the residue of the Higgs field to be nilpotent, the image of this map lies in $B_{\mathcal{N}} := \bigoplus_{i=1}^n \Gamma(X, \Omega_X^{\otimes i}((i-1)q))$.

Let $T^*X(q) = \text{Spec}_X(\text{Sym}_{\mathcal{O}_X} \mathcal{T}_X(-q))$, where $\mathcal{T}_X(-q)$ is the sheaf of vector fields on X that vanish at q . Let π be the projection $\pi : T^*X(q) \longrightarrow X$. We denote by y the tautological section of $\pi^*(\Omega_X(q))$. For $b = (b_1, b_2, \dots, b_n)$, $b_i \in \Gamma(X, \Omega_X(q)^i)$, we define the spectral curve Σ_b to be the zero-subscheme of the section

$$y^n + b_1 y^{n-1} + \dots + b_{n-1} y + b_n$$

of $\pi^*(\Omega_X(q)^n)$. By abuse of notation, we also denote by π the projection from Σ_b to X . Since $\pi_* \mathcal{O}_{\Sigma_b} = \bigoplus_{i=0}^{n-1} \mathcal{T}_X(-q)^{\otimes i}$, we can compute the genus of Σ_b

$$g_{\Sigma_b} = \frac{n(n-1)}{2}(2g-1) + n(g-1) + 1.$$

Let (E, ϕ) be a k -point of $\mathcal{Higgs}_{n,q}$ such that $h(E, \phi) = b$. We can think of ϕ as a morphism

$$\phi : \mathcal{T}_X(-q) \longrightarrow \mathcal{E}nd(E).$$

By Cayley–Hamilton, there is a coherent sheaf \mathcal{F} on Σ_b such that $\pi_*(\mathcal{F}) = E$. We call \mathcal{F} the spectral sheaf corresponding to (E, ϕ) . Conversely, let \mathcal{G} be a coherent sheaf on Σ_b , there is a canonical section $\phi_{\text{can}} \in \Gamma(X, \mathcal{E}nd(\pi_*(\mathcal{G})) \otimes \Omega_X(q))$ obtained by adjunction. It is proved in [6] that if Σ_b is reduced, the Hitchin fiber $h^{-1}(b)$ is isomorphic to the stack of torsion free sheaves on Σ_b , and if Σ_b is smooth, $h^{-1}(b)$ is isomorphic to the Picard stack $\text{Pic}(\Sigma_b)$ of Σ_b .

If we require the residue of ϕ to be nilpotent, then $\pi^{-1}(q)$ is a single point q' that lies in the zero-section of $T^*X(q)$. Let $V = \text{spec}(A)$ be an affine open neighborhood of q in X . Let x be an element of A that is mapped to a local parameter of X at q . Shrinking V if necessary, we assume $\frac{dx}{x}$ is a nowhere vanishing section of $\Omega_V(q)$. Let $U = \pi^{-1}(V)$. The section $\frac{dx}{x}$ gives a trivialization of $T^*X(q)|_V$ and $\pi^*(T^*X(q))|_U$. Under this trivialization, the tautological section y is equal to $x\partial_x$ considered as an element in \mathcal{O}_U . Let $\Sigma_b(V) := V \times_X \Sigma_b$, then $\mathcal{O}_{\Sigma_b(V)}$ is isomorphic to $\mathcal{O}_U/(f_b)$, where $f_b = y^n + b_1y^{n-1} + \dots + b_{n-1}y + b_n$, $b_i \in \mathcal{O}_V$. We denote by \hat{f}_b the image of f_b in $\hat{\mathcal{O}}_{U,q'} \cong k[[x, y]]$, then $\hat{\mathcal{O}}_{\Sigma_b,q'} \cong k[[x, y]]/(\hat{f}_b)$.

2.2 The parabolic Hitchin base B_p

Now let P be a parabolic subgroup of $GL_n(k)$, and we assume the Richardson orbit \mathcal{O}_P of P contains the nilpotent matrix with Jordan blocks of sizes $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$, $\sum_{i=1}^r \lambda_i = n$. Composing h with the forgetful map from $\text{Higgs}_{n,P}$ to $\text{Higgs}_{n,q}$, we get

$$h_p : \text{Higgs}_{n,P} \longrightarrow B.$$

In order to describe the image of h_p , we define the following sets of formal power series.

Definition 2.1. Let $\eta = (\eta_1, \eta_2, \dots, \eta_s)$, $\eta_1 \geq \eta_2 \geq \dots \geq \eta_s$ be a decreasing sequence of positive integers. Let $\gamma_i = \sum_{j=i+1}^s \eta_j$ for $i = 0, 1, 2, \dots, s-1$ and $\gamma_s = 0$. We denote by P_η the set of formal power series of the form

$$y^{\gamma_0} + \sum_{i=1}^s a_i(x, y)x^i y^{\eta_i}, \text{ where } a_i(x, y) \in k[[x, y]],$$

and by P_η^0 the subset of elements in P_η that satisfy $a_i(x, y) \in k[[x, y]]^\times$.

In particular, if $\eta = (m)$, P_m^0 is the set of formal power series of the form

$$y^m + a(x, y)x, \text{ where } a(x, y) \in k[[x, y]]^\times.$$

Lemma 2.2. Let (E, E_P, τ, ϕ) be a k -point of $\text{Higgs}_{n,P}$ such that $h_p(E, \phi) = b$. Let \hat{f}_b be the element in $k[[x, y]]$ such that $\hat{\mathcal{O}}_{\Sigma_b,q'} \cong k[[x, y]]/(\hat{f}_b)$ as above, then $\hat{f}_b \in P_\lambda$.

This lemma follows from a direct computation, see Proposition 22 in [3]. It follows from Lemma 2.2 that h_p factors through the affine space

$$B_p := \bigoplus_{i=1}^n \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q)),$$

here $m_i = j$ if $\gamma_j \leq n - i < \gamma_{j-1}$. To be more explicit, we have $\mathbf{m} = 1^{\lambda_1} 2^{\lambda_2} \dots r^{\lambda_r}$, meaning that the first λ_1 terms are 1, the next λ_2 terms are 2, ..., and the last λ_r terms are r .

Lemma 2.3.

- (1) Let \hat{f} be a formal power series in P_η^0 , $\eta = \eta_1^{\lambda_1} \eta_2^{\lambda_2} \dots \eta_t^{\lambda_t}$, $\eta_1 \geq \eta_2 \geq \dots \geq \eta_t$. Then \hat{f} factorizes uniquely as $\hat{f} = f_1 f_2 \dots f_t$, where each f_i is a formal power series in $P_{\eta_i}^0$,
- (2) Let $\hat{g} = y^{\eta_1} + a_1(x, y)xy^{\eta_1-1} + a_2(x, y)x^2y^{\eta_1-2} + \dots + a_i(x, y)x^i$ be a power series in $P_{\eta_1}^0$. We write $a_i(0, 0)$ for the constant term of $a_i \in k[[x, y]]$. Assume the polynomial $y^{\eta_1} + a_1(0, 0)y^{\eta_1-1} + a_2(0, 0)y^{\eta_1-2} + \dots + a_i(0, 0)$ has distinct roots. Then \hat{g} factorizes uniquely as $\hat{g} = g_1 g_2 \dots g_i$, where each $g_i \in P_{\eta_i}^0$.

Proof. We start with Part (1). The uniqueness part follows from the fact that $k[[x, y]]$ is a UFD. We prove the existence part by induction on t . Let $s = \sum_{i=1}^t \lambda_i$ be the length of η . Let

$$\hat{f} = y^{\gamma_0} + \sum_{i=1}^s a_i(x, y)x^i y^{\eta_i}, \text{ where } a_i(x, y) \in k[[x, y]]^\times.$$

In order to show that \hat{f} factorizes as required, it is enough to show that f factorizes as $\hat{f} = gh$, where

$$g = y^{\eta_0 - \eta_l} + \sum_{i=1}^{s-l} b_i(x, y)x^i y^{\eta_i - \eta_l} \in P_{\eta_1^0, \eta_2^0, \dots, \eta_{l-1}^0}^0$$

and

$$h = y^{\eta_l} + c_1(x, y)xy^{\eta_l(l-1)} + c_2(x, y)x^2y^{\eta_l(l-2)} + \dots + c_l(x, y)x^l \in P_{\eta_l^0}^0.$$

Comparing coefficients, we have

$$\begin{cases} b_1 + c_1 y^\square = a_1 \\ b_2 + b_1 c_1 y^\square + c_2 y^\square = a_2 \\ \dots \\ b_{s-l} + b_{s-l-1} c_1 y^\square + b_{s-l-2} c_2 y^\square + \dots + b_{s-2l} c_l y^\square = a_{s-l} \\ b_{s-l} c_1 + b_{s-l-1} c_2 y^\square + b_{s-l-2} c_3 y^\square + \dots + b_{s-2l+1} c_l y^\square = a_{s-l+1} \\ \dots \\ b_{s-l} c_{l-1} + b_{s-l-1} c_l y^\square = a_{s-1} \\ b_{s-l} c_l = a_s, \end{cases}$$

where y^\square stands for raising y to some positive integer power. Since a_s is invertible, b_{s-l} is also invertible by the last equation. Solving this system of equations is equivalent to solving a single equation with variable b_{s-l} . Indeed, we can solve c_l, c_{l-1}, \dots, c_1 in turn as functions of b_{s-l} from the last l equations; then we can solve $b_1, b_2, \dots, b_{s-l-1}$ in turn as functions of b_{s-l} from the first $s - l - 1$ equations; then we get the desired equation with variable b_{s-l} by substituting the other variables as functions of b_{s-l} in the $(s - l)$ -th equation. This equation has a solution by Hensel's lemma. Indeed, after reduction to k , this equation has a unique solution $b_{s-l}(0, 0) = a_{s-l}(0, 0)$.

For Part (2), since we assume $y^l + a_1(0, 0)y^{l-1} + a_2(0, 0)y^{l-2} + \dots + a_l(0, 0)$ has distinct roots, by Hensel's lemma, they lift to distinct roots of the polynomial

$$y^l + a_1(x, y)y^{l-1} + a_2(x, y)y^{l-2} + \dots + a_l(x, y),$$

which we denote by $b_1(x, y), b_2(x, y), \dots, b_l(x, y)$. Then

$$\hat{g} = \prod_{i=1}^l (y^\eta - b_i(x, y)x)$$

gives the desired factorization. Since $a_i(x, y) \in k[[x, y]]^\times$, we have $b_i(x, y) \in k[[x, y]]^\times$ for each i . ■

In order to obtain a spectral description of parabolic Hitchin fibers, we define the following open subset of the Hitchin base B_P .

Definition 2.4. We define B_P^0 to be the subset of B_P such that $b \in B_P^0$ is characterized by the following properties:

- (2.1) $\Sigma_b \setminus q'$ is smooth,
- (2.2) \hat{f} lies in P_λ^0 , and all components in the factorization of \hat{f} in Lemma 2.3 Part (1) satisfy the assumption in Lemma 2.3 Part (2). It follows that \hat{f} factorizes as $\hat{f} = f_1 f_2 \dots f_r$, where $f_i = y^{\lambda_i} + a_i(x, y)x$, $a_i(x, y) \in k[[x, y]]^\times$. If $\lambda_s = \lambda_t$ for some $s \neq t$, then the constant terms of a_s and a_t are not equal to each other.

In particular, if P is a Borel subgroup of $GL_n(k)$, B_P^0 is characterized by the spectral curve being smooth.

Lemma 2.5. For every $b \in B_p^0$, there exists a k -point of $\mathcal{Higgs}_{n,p}$ that is mapped to b under the Hitchin map h_p .

Proof. Let $\tilde{\Sigma}_b \rightarrow \Sigma_b$ be the normalization of the spectral curve Σ_b and let $\tilde{\pi} : \tilde{\Sigma}_b \rightarrow X$ be the projection to X . Let $D = \text{Spec } \hat{\mathcal{O}}_{X,q}$ be the formal disk around q . By (2.2), $\tilde{\Sigma}_b \times_X D$ is the disjoint union of Σ_i , where

$$\Sigma_i \cong \text{Spec } k[[x, y]] / (y^{k_i} + a_i(x, y)x), a_i(x, y) \in k[[x, y]]^*.$$

Let \mathcal{L} be an invertible sheaf on $\tilde{\Sigma}_b$, then $\tilde{\pi}_*(\mathcal{L})$ defines a k -point $(\tilde{\pi}_*(\mathcal{L}), \phi)$ of $\mathcal{Higgs}_{n,q}$ such that $h(\tilde{\pi}_*(\mathcal{L}), \phi) = b$ and $\text{res}_q \phi \in \mathcal{O}_p$ the Richardson orbit of P . Therefore, we can find a partial flag structure on $\tilde{\pi}_*(\mathcal{L})_q$ such that $\text{res}_q \phi$ is nilpotent with respect to this partial flag structure. ■

Remark 2.6. Let (E, E_p, τ, ϕ) be a k -point of $\mathcal{Higgs}_{n,p}$ that is mapped to $b \in B_p^0$. Condition (2.2) on Σ_b enforces that $\text{res}_q(\phi)$ lies in the Richardson orbit \mathcal{O}_p . Note that $\mathcal{O}_p \cap n_p^+$ consists of a single P -orbit. Since we are in type A, for any $x \in \mathcal{O}_p \cap n_p^+$, the centralizer of x in $GL_n(k)$ lies in P . Therefore, there is a unique partial flag structure on E_q that is compatible with $\text{res}_q(\phi)$.

Proposition 2.7. In the following two cases:

- (1) $g_X \geq 2$,
- (2) $g_X = 1$, $n \geq 3$ and $P \subseteq G$ is a Borel subgroup,

B_p^0 is Zariski open dense in B_p . Moreover, B_p is the scheme-theoretic image of the Hitchin map h_p , that is, the smallest closed subscheme of B through which h_p factors.

Proof. The first statement together with Lemma 2.5 implies the second statement. For the first statement, we only need to show that both (2.1) and (2.2) define a non-empty open subset in B_p .

We start by showing that (2.1) defines a non-empty open subset in B_p . We denote by B_p^{sm} the locus in B_p where the spectral curves are smooth away from q' . Since $B_p^{sm} \subset B_p$ is open, it is enough to show that it is non-empty.

Case 1. $g_X \geq 2$, except for the case when $g_X = 2$, $n = 2$, $P = GL_2(k)$. We use the following version of Bertini's theorem in [12]: ■

Theorem 2.8 (cf. [12], Corollary 1). Let V be a smooth algebraic variety over an algebraically closed field k . Let S be a finite-dimensional linear system on V . Assume that the rational map $V \dashrightarrow P^N$ corresponding to S induces (whenever defined) separably generated residue field extensions. Then a generic element of S defines a subscheme of V that is smooth away from the base locus of S .

Let π be the projection $\pi : T^*X(q) \rightarrow X$. We denote by y the tautological section of $\pi^*(\Omega_X(q))$. Let S be the linear system of sections in $\pi^*(\Omega_X(q)^n)$ spanned by y^n and $\pi^*(b_i)y^{n-i}$ for all $b_i \in \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q))$, $i = 1, 2, \dots, n$. The section y^n is not contained in the span of $\pi^*(b_i)y^{n-i}$. The set of spectral curves Σ_b with $b \in B_p$ corresponds to the open subset of S defined by the coefficient of y^n being non-zero. Let $N = \dim(S) - 1$. We denote by $f_S : T^*X(q) \dashrightarrow P^N$ the map induced by S . In order to apply Theorem 2.8, we show that f_S is unramified away from $\pi^{-1}(q)$, which will imply that f_S induces finite separable extensions on the residue fields when restricted to $T^*(X \setminus q)$. By the exact sequence

$$f_S^* \Omega_{P^N} |_{T^*(X \setminus q)} \xrightarrow{v} \Omega_{T^*(X \setminus q)} \rightarrow \Omega_{T^*(X \setminus q)/P^N} \rightarrow 0,$$

it is enough to show that for any k -point p' on $T^*X(q)$ such that $\pi(p') = p \neq q$, the map v induces a surjection onto the fiber of $\Omega_{T^*(X \setminus q)}$ at p' .

Let $V = \text{Spec}(A)$ be an affine open neighborhood of p in X . Let x be an element of A that is mapped to a local parameter of X at p . Shrinking V if necessary, we assume $q \notin V$ and dx is a nowhere vanishing section of Ω_V^1 . Let $U = \pi^{-1}(V)$. The section dx gives a trivialization of $T^*X(q)|_U$ and $\pi^*(T^*X(q))|_U$. Under this trivialization, the tautological section y is equal to ∂_x considered as an element in \mathcal{O}_U , and Ω_U is

a free \mathcal{O}_U -module generated by dx and dy . The fiber of Ω_U at p' is a k -vector space of dimension two spanned by dx and dy .

Under our assumptions on g_X , n and P , we have

$$\begin{aligned} \dim_k \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q)) - \dim_k \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q - p)) &= 1, \text{ for } i \geq 1, \\ \dim_k \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q - p)) - \dim_k \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q - 2p)) &= 1, \text{ for } i \geq 2. \end{aligned}$$

Take

$$\begin{aligned} s_1 &\in \Gamma(X, \Omega_X^{\otimes n}((n - m_n)q)) \setminus \Gamma(X, \Omega_X^{\otimes n}((n - m_n)q - p)), \\ s_2 &\in \Gamma(X, \Omega_X^{\otimes n}((n - m_n)q - p)) \setminus \Gamma(X, \Omega_X^{\otimes n}((n - m_n)q - 2p)), \\ s_3 &\in \Gamma(X, \Omega_X^{\otimes(n-1)}((n - 1 - m_{n-1})q)) \setminus \Gamma(X, \Omega_X^{\otimes(n-1)}((n - 1 - m_{n-1})q - p)), \end{aligned}$$

then $d(s_2/s_1)$ and $d(s_3y/s_1)$ span the fiber of Ω_U at p' .

Now we apply Theorem 2.8 to the restriction of the linear system S to $T^*(X \setminus q)$. Since $q' \in \pi^{-1}(q)$ is the only base point of S , a spectral curve Σ_b is smooth away from q' for a generic $b \in B_P$.

Case 2. $g_X = 2, n = 2, P = GL_2(k)$. By the same arguments as in Case 1, the map $f_S : T^*(X \setminus q) \rightarrow P^N$ is unramified away from the union of $\pi^{-1}(p)$ for all $p \in X \setminus q$ that satisfies $\mathcal{O}(2p) \cong \Omega_X$. There are finite many points of X with this property, therefore the fact that a generic spectral curve is smooth away from q' follows from the following lemma:

Lemma 2.9. Let $p \in X \setminus q$. For a generic $b \in B_P$, the spectral cover $\Sigma_b \rightarrow X$ is étale around p .

Proof. This follows easily from the calculation

$$\dim_k \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q)) - \dim_k \Gamma(X, \Omega_X^{\otimes i}((i - m_i)q - p)) = 1, \text{ for } i \geq 1. \quad \blacksquare$$

Case 3. $g_X = 1$. We consider the subspace $\bigoplus_{i=1}^n \Gamma(X, \Omega_X^{\otimes i}) \subseteq B_P$. Since Ω_X is isomorphic to \mathcal{O}_X , it is easy to find $b \in \bigoplus_{i=1}^n \Gamma(X, \Omega_X^{\otimes i})$ such that the spectral cover $\Sigma_b \rightarrow X$ is étale away from $\pi^{-1}(q)$.

Now we turn to (2.2). Let $b \in B_P$. The condition $\hat{f}_b \in P_\lambda^0$ is equivalent to the condition that for $i = 1, 2, \dots, r$, the $(n - \gamma_i)$ -th component of b lies in

$$\Gamma(X, \Omega_X^{\otimes(n-\gamma_i)}((n - \gamma_i - i)q)) \setminus \Gamma(X, \Omega_X^{\otimes(n-\gamma_i)}((n - \gamma_i - i - 1)q)).$$

This condition defines a non-empty open subset of B_P since

$$\dim_k \Gamma(X, \Omega_X^{\otimes(n-\gamma_i)}((n - \gamma_i - i)q)) - \dim_k \Gamma(X, \Omega_X^{\otimes(n-\gamma_i)}((n - \gamma_i - i - 1)q)) = 1$$

under our assumptions on g_X, n and P . The fact that the second condition in (2.2) defines a non-empty open subset follows easily from the uniqueness part of Lemma 2.3.

Remark 2.10. The second statement in Proposition 2.7 was previously obtained in [3] using different methods.

Remark 2.11. Proposition 2.7 also holds for the case of $X = P^1$ with ramification at $D = q_1 + q_2 + \dots + q_m, m \geq 4$ and each parabolic subgroup P_i is a Borel subgroup. We need to show that for a generic $b \in B_{P_D} = \bigoplus_{i=1}^n \Gamma(P^1, \Omega_{P^1}^{\otimes i}((i - 1)D))$, the spectral curve Σ_b is smooth. For each $i = 1, 2, \dots, m$, a generic spectral curve is smooth above q_i since

$$\dim_k \Gamma(P^1, \Omega_{P^1}^{\otimes n}((n - 1)D)) - \dim_k \Gamma(P^1, \Omega_{P^1}^{\otimes n}((n - 1)D - q_i)) = 1.$$

Therefore, it is enough to show that there exists $b \in B_{P_D}(k)$ such that Σ_b is smooth away from $\pi^{-1}(q_i)$. If $n \geq 3$, the same arguments as in Case 1 of the proof of Proposition 2.7 would work.

If $n = 2$, we consider the subspace

$$\Gamma(\mathbf{P}^1, \Omega_{\mathbf{P}^1}) \oplus \Gamma(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^{\otimes 2}(q_1 + q_2 + q_3 + q_4)) \subseteq B_{\mathbb{P}^D}.$$

Since $\Gamma(\mathbf{P}^1, \Omega_{\mathbf{P}^1}) = 0$ and $\Gamma(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^{\otimes 2}(q_1 + q_2 + q_3 + q_4)) \cong \Gamma(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = k$, the spectral curve Σ_b is étale away from $\pi^{-1}(q_i)$ for any $b \in k^\times$.

2.3 Spectral data of parabolic Higgs bundles

The next theorem describes the spectral data of parabolic Higgs bundles.

Theorem 2.12 (cf. [25], Theorem 5.16). For $b \in B_{\mathbb{P}}^0(k)$, the fiber of the Hitchin map $h_{\mathbb{P}}^{-1}(b)$ is isomorphic to the Picard stack $\text{Pic}(\tilde{\Sigma}_b)$. Here $\sigma : \tilde{\Sigma}_b \rightarrow \Sigma_b$ is the normalization of the spectral curve Σ_b .

Proof. We've already constructed a map $\text{Pic}(\tilde{\Sigma}_b) \rightarrow h_{\mathbb{P}}^{-1}(b)$ in the proof of Lemma 2.5, therefore it is enough to construct the inverse map. Let $(E, \phi) \in h_{\mathbb{P}}^{-1}(b)$, and we denote by $\mathcal{F} \in \text{Coh}(\Sigma_b)$ the corresponding spectral sheaf. Our goal is to show that there is a natural sheaf $\mathcal{L} \in \text{Coh}(\tilde{\Sigma}_b)$ such that $\sigma_*(\mathcal{L}) = \mathcal{F}$.

Let $\hat{\Sigma}_b = \text{Spec } \hat{\mathcal{O}}_{\Sigma_b, q'}$. We write $\lambda = \lambda_1^{l_1} \lambda_2^{l_2} \cdots \lambda_t^{l_t}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t$. Note that by condition (2.2) in the definition of $B_{\mathbb{P}}^0$,

$$\mathcal{O}_{\tilde{\Sigma}_b} \cong k[[x, y]]/(\hat{f}), \text{ and } \hat{f} = \prod_{i=1}^t \prod_{j=1}^{l_i} (y^{\lambda_i} - a_{ij}x), a_{ij} \in k[[x, y]]^\times.$$

Therefore,

$$\tilde{\Sigma}_b \times_{\Sigma_b} \hat{\Sigma} \cong \prod_{i=1}^t \prod_{j=1}^{l_i} \Sigma_{ij}, \text{ where } \mathcal{O}_{\Sigma_{ij}} \cong k[[x, y]]/(y^{\lambda_i} - a_{ij}x).$$

Each Σ_{ij} is a formal disk such that the closed point is mapped to q' under $\sigma : \tilde{\Sigma}_b \rightarrow \Sigma_b$.

Note that since the action of y on $\mathcal{F}/x\mathcal{F}$ as a matrix with Jordan blocks of type λ , the element $v_1 := y^{\lambda_1}/x$ acts on the spectral sheaf \mathcal{F} sheafifies over $\hat{\Sigma}_b^1$ defined by

$$\mathcal{O}_{\hat{\Sigma}_b^1} = k[[x, y]][v_1]/(\hat{f}, y^{\lambda_1} - xv_1).$$

This new curve $\hat{\Sigma}_b^1$ is a disjoint union of $l_1 + 1$ components

$$\hat{\Sigma}_b^1 = \prod_{j=1}^{l_1} \text{Spec } k[[v_1, y]]/(v_1 - a_{1j}) \amalg \prod_{i=2}^t \prod_{j=1}^{l_i} \text{Spec } k[[x, y]][v_1]/(\prod_{i=2}^t \prod_{j=1}^{l_i} (y^{\lambda_i} - a_{ij}x), y^{\lambda_1} - xv_1). \tag{2.1}$$

The first l_1 components are formal disks that correspond to $\Sigma_{1j}, j = 1, 2, \dots, l_1$ in the normalization curve. The spectral sheaf over those components must be line bundles, and each contributes a Jordan block of size λ_1 to the residue of the Higgs field at the marked point q . Let \mathcal{F}_1 be the spectral sheaf over the last component of $\hat{\Sigma}_b^1$ in (2.1). Since y acts on $\mathcal{F}_1/x\mathcal{F}_1$ as a matrix with Jordan blocks of type $\lambda_2^{l_2} \lambda_3^{l_3} \cdots \lambda_t^{l_t}$, the element $v_2 := y^{\lambda_2}/x$ acts on \mathcal{F}_1 , therefore \mathcal{F}_1 sheafifies over $\hat{\Sigma}_b^2$ defined by

$$\mathcal{O}_{\hat{\Sigma}_b^2} = k[[x, y]][v_2]/(\prod_{i=2}^t \prod_{j=1}^{l_i} (y^{\lambda_i} - a_{ij}x), y^{\lambda_2} - xv_2).$$

Repeating the same procedure for t times, the spectral sheaf \mathcal{F} over $\hat{\Sigma}_b$ that we start with decomposes as

$$\mathcal{F} = \bigoplus_{i=1}^t \bigoplus_{j=1}^{l_i} \mathcal{L}_{ij},$$

where each \mathcal{L}_{ij} is a line bundle over Σ_{ij} . Since the normalization curve $\tilde{\Sigma}_b$ locally is the disjoint union of those Σ_{ij} , we get the desired statement that the spectral sheaf \mathcal{F} sheafifies over $\tilde{\Sigma}_b$. ■

For the purpose of this paper, we need to develop a family version of Theorem 2.12. The first step is to construct a simultaneous normalization of the family of spectral curves above B_p^0 . This can be done since the spectral curves above B_p^0 are equisingular. To be more precise, let $\Sigma \subseteq B_p^0 \times T^*X(q)$ be the global spectral curve above B_p^0 ; we will construct a new family of curves $\tilde{\Sigma} \rightarrow B_p^0$ with a proper birational morphism $\sigma : \tilde{\Sigma} \rightarrow \Sigma$ such that for each $b \in B_p^0(k)$, the morphism $\sigma_b : \tilde{\Sigma}_b \rightarrow \Sigma_b$ is the normalization of Σ_b .

The construction is as follows. Recall that q' is the closed point of $T^*X(q)$ above $q \in X$ that lies in the zero section of $T^*X(q)$. We blow up $B_p^0 \times T^*X(q)$ along $B_p^0 \times q'$, and denote the strict transform of Σ by Σ^1 . Let V be an open neighborhood of q and $U = \pi^{-1}(V)$. For $b \in B_p^0(k)$, let $\Sigma_b(V) := V \times_X \Sigma_b$, then $\mathcal{O}_{\Sigma_b(V)}$ is isomorphic to $\mathcal{O}_U/(f)$ for some $f = y^n + b_1y^{n-1} + \dots + b_{n-1}y + b_n$, $b_i \in \mathcal{O}_V$. Since $b \in B_p^0(k)$, \hat{f} factorizes as $\hat{f} = f_1 f_2 \dots f_r$, $f_i \in P_{\lambda_i}^0$. We write

$$f_i = y^{\lambda_i} + a_i(x, y)x, \text{ where } a_i(x, y) \in k[[x, y]]^\times.$$

We denote $V \times_X \Sigma_b^1$ by $\Sigma_b^1(V)$, then $\Sigma_b^1(V)$ is a closed subvariety of

$$\text{Spec}(\mathcal{O}_U[u]/(x - yu)).$$

We denote by q_1^1 the point defined by $y = u = 0$. By assumption (2.1) and the second part of assumption (2.2) in the definition of B_p^0 , Σ_b^1 is smooth away from q_1^1 . Let $\hat{\Sigma}_b^1 \cong \text{Spec}(\hat{\mathcal{O}}_{\Sigma_b^1, q_1^1})$, then

$$\mathcal{O}_{\hat{\Sigma}_b^1} \cong k[[u, y]] / \left(\prod_{i=1}^t (y^{\lambda_i-1} + a_i(yu, y)u) \right),$$

where t is the largest integer so that $\lambda_t - 1 > 0$. Let

$$g = \prod_{i=1}^t (y^{\lambda_i-1} + a_i(yu, y)u) \text{ and } g_i = y^{\lambda_i-1} + a_i(yu, y)u,$$

so g factorizes as $g = g_1 g_2 \dots g_t$. In each g_i , there is a unique monomial of the form y^m , and the degree of such monomial is in decreasing order. Compared to f_1 , the degree of such monomial in g_1 is lower by 1. This observation guarantees that the family of curves Σ can be resolved simultaneously by λ_1 steps of blow-ups. Now we blow up $\text{Spec}(\mathcal{O}_U[u]/(x - yu))$ along $B_p^0 \times q_1^1$, and denote the strict transform of Σ^1 by Σ^2 . Repeating this procedure, we get a series of families of curves above B_p^0 :

$$\Sigma^{\lambda_1} \rightarrow \Sigma^{\lambda_1-1} \rightarrow \dots \Sigma^1 \rightarrow \Sigma.$$

It follows from the observation above that $\Sigma_b^{\lambda_1}$ is smooth for each $b \in B_p^0(k)$. The morphism $\Sigma^{\lambda_1} \rightarrow B_p^0$ is flat since each $\Sigma_b^{\lambda_1}$ is a projective curve of the same genus. The morphism $\Sigma^{\lambda_1} \rightarrow \Sigma$ is proper and birational by properties of strict transforms. We set $\tilde{\Sigma} \cong \Sigma^{\lambda_1}$.

Remark 2.13. After our paper appeared on the arXiv, similar results as in Theorem 2.12 were also obtained in [27], see Theorem 1.1. In [27], the authors also considered the generic fiber of so-called weak parabolic fibrations, in which the residue of the Higgs field is not required to

be nilpotent. We will prove Theorem 2.14 the family version of Theorem 2.12, which did not appear in [27].

Now we are ready to state the following theorem, which is a family version of Theorem 2.12. We denote $\mathcal{Higgs}_{n,p} \times_{B_p} B_p^0$ by $\mathcal{Higgs}_{n,p}^0$.

Theorem 2.14. The correspondence between Higgs bundles and spectral sheaves induces an isomorphism of stacks over B_p^0 :

$$\mathcal{Higgs}_{n,p}^0 \cong \text{Pic}(\tilde{\Sigma}/B_p^0).$$

Proof. Let S be a k -scheme. Since both $\mathcal{Higgs}_{n,p}^0$ and B_p^0 are locally of finite type over k , we can assume S is locally of finite type over k . Let (E, ϕ) be an S -point of $\mathcal{Higgs}_{n,p}^0$ such that $h(E, \phi) = b \in B_p^0(S)$. We denote by \mathcal{F} the corresponding spectral sheaf on Σ_b . The goal is to construct a sheaf $\tilde{\mathcal{F}}$ on $\tilde{\Sigma}_b$ such that $(\sigma_b)_* \tilde{\mathcal{F}} = \mathcal{F}$. We set $\Sigma_b^0 = \Sigma_b$, $\mathcal{F}_0 = \mathcal{F}$. The strategy is to construct by induction a series of sheaves \mathcal{F}_k on Σ_b^k , $k = 1, 2, \dots, \lambda_1$, such that $(p_k)_* \mathcal{F}_k = \mathcal{F}_{k-1}$, where p_k is the map $p_k : \Sigma_b^k \rightarrow \Sigma_b^{k-1}$. We assume that we already have $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{t-1}$ with the required property and aim to obtain \mathcal{F}_t . Note that above V an open neighborhood of q , while obtaining Σ_b^k , we add a new variable u_k to $\mathcal{O}_{\Sigma_b^{k-1}}$ and impose $u_{k-1} = u_k y$, starting from $u_0 = x$. Therefore, in order to construct \mathcal{F}_t so that $(p_t)_* \mathcal{F}_t = \mathcal{F}_{t-1}$, all we need to do is to define an action of u_{t-1}/y on \mathcal{F}_{t-1} . Note that for any $s : \text{Spec}(k) \rightarrow S$ a closed point of S , $s^* \mathcal{F}_{t-1}$ is a torsion-free sheaf on $(\Sigma_b^{t-1})_s := \Sigma_b^{t-1} \times_{S,s} \text{Spec}(k)$, therefore if such an action exists, it is unique. For the existence of such an action, we consider the coherent sheaf $\mathcal{G} = u_{t-1} \mathcal{F}_{t-1} / u_{t-1} \mathcal{F}_{t-1} \cap y \mathcal{F}_{t-1}$ on Σ_b^{t-1} . There exists an action of u_{t-1}/y on \mathcal{F}_{t-1} if and only if $\mathcal{G} = 0$. By Theorem 2.12, such an action exists when restricted to s , so $s^* \mathcal{G} = 0$ for all closed points s of S . Therefore $\mathcal{G} = 0$.

We set $\tilde{\mathcal{F}} = \mathcal{F}_{\lambda_1}$. Since $\tilde{\Sigma}_b$ is smooth, $\tilde{\mathcal{F}}$ is an invertible sheaf. Now let (E_1, ϕ_1) and (E_2, ϕ_2) be two S -points of $\mathcal{Higgs}_{n,p}^0$, both mapped to b under the Hitchin map, and we denote the corresponding spectral sheaves by \mathcal{F}_1 and \mathcal{F}_2 . The construction of $\tilde{\mathcal{F}}$ implies that there is an isomorphism $\text{Hom}_{\mathcal{O}_{\Sigma_b}}(\mathcal{F}_1, \mathcal{F}_2) \cong \text{Hom}_{\mathcal{O}_{\tilde{\Sigma}_b}}(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$. Therefore we have a morphism of stacks $\mathcal{Higgs}_{n,p}^0 \rightarrow \text{Pic}(\tilde{\Sigma}^0/B_p^0)$.

The inverse of this morphism is constructed as follows. Let \mathcal{L} be an invertible sheaf on $\tilde{\Sigma}_b$. Since $\Sigma_b \subseteq S \times T^*X(q)$, there is a morphism

$$\mathcal{O}_S \boxtimes \pi^* \mathcal{T}_X(-q) \rightarrow \mathcal{E}nd_{\mathcal{O}_{\Sigma_b}}(\sigma_* \mathcal{L}).$$

By adjunction, we get a morphism

$$\mathcal{O}_S \boxtimes \mathcal{T}_X(-q) \rightarrow \pi_* \mathcal{E}nd_{\mathcal{O}_{\Sigma_b}}(\sigma_* \mathcal{L}) \rightarrow \mathcal{E}nd_{\mathcal{O}_X}(\tilde{\pi}_* \mathcal{L}).$$

By Remark 2.6, there is a unique parabolic reduction of $\tilde{\pi}_* \mathcal{L}$ at q that is compatible with this Higgs field. ■

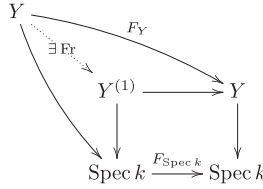
3 Azumaya Property of Differential Operators in Positive Characteristic

3.1 Frobenius twist of a k -scheme

Let Y be a scheme over an algebraically closed field k of characteristic p . Recall that the absolute Frobenius $F_Y : Y \rightarrow Y$ is the map that fixes the underlying topological space and takes f to f^p on regular functions. The Frobenius twist $Y^{(1)}$ of Y is the k -scheme that fits into the following pull-back diagram:

$$\begin{array}{ccc} Y^{(1)} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \text{Spec } k & \xrightarrow{F_{\text{Spec } k}} & \text{Spec } k \end{array}$$

The relative Frobenius $\text{Fr} : Y \rightarrow Y^{(1)}$ is the unique map that makes the following diagram commute.



Since Fr induces a bijection on k -points, we will not distinguish between k -points on Y and k -points on $Y^{(1)}$.

Let \mathcal{F} and \mathcal{G} be two \mathcal{O}_Y -modules. A map $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is called p -linear if it is additive and satisfies $\varphi(fs) = f^p\varphi(s)$ for any $f \in \mathcal{O}_U, s \in \mathcal{F}(U)$ and open $U \subseteq Y$. For any \mathcal{O}_Y -module \mathcal{F} , there is a natural p -linear map $\mathcal{F} \rightarrow (F_Y)^*\mathcal{F}$. This map is “universally p -linear” in the sense that any p -linear map $\mathcal{F} \rightarrow \mathcal{G}$ factors through $\mathcal{F} \rightarrow (F_Y)^*\mathcal{F}$ and gives a unique \mathcal{O}_Y -linear map $(F_Y)^*\mathcal{F} \rightarrow \mathcal{G}$.

3.2 Azumaya property of differential operators

In this section we review the Azumaya property of crystalline differential operators in characteristic p , following [8]. Let Y be a smooth variety over k . We denote by D_Y the sheaf of crystalline differential operators on Y , that is, the sheaf of algebras generated by \mathcal{O}_Y and \mathcal{T}_Y subject to the relations: $\partial f - f\partial = \partial(f), \partial_1\partial_2 - \partial_2\partial_1 = [\partial_1, \partial_2]$, for any $f \in \mathcal{O}_U, \partial, \partial_1, \partial_2 \in \mathcal{T}_Y(U)$ and $U \subseteq Y$ open. Since we are in characteristic p , for any $\partial \in \mathcal{T}_Y(U), \partial^p \in D_Y$ acts as a derivation on \mathcal{O}_U , and we denote this derivation by $\partial^{[p]} \in \mathcal{T}_Y(U)$. There is a p -linear map $\mathcal{T}_Y \rightarrow D_Y$ defined by $\iota(\partial) = \partial^p - \partial^{[p]}$. By the discussion above, ι induces an \mathcal{O}_Y -linear map $\text{Fr}^* \mathcal{T}_{Y^{(1)}} \cong \text{Fr}^* \mathcal{T}_Y \rightarrow D_Y$. By adjunction, we have an $\mathcal{O}_{Y^{(1)}}$ -linear map

$$\iota : \mathcal{T}_{Y^{(1)}} \rightarrow \text{Fr}_* D_Y.$$

Therefore, $\text{Fr}_* D_Y$ sheafifies on $T^*Y^{(1)}$, that is, there exists a sheaf of algebras \mathcal{D}_Y on $T^*Y^{(1)}$ that satisfies $\pi_*^{(1)} \mathcal{D}_Y \cong \text{Fr}_* D_Y$.

The following theorem is proved in [9].

Theorem 3.1 (cf. [8] Theorem 3.3 and [9] Theorem 2.2.3).

- (1) The map ι induces an isomorphism of sheaves from $\mathcal{O}_{T^*Y^{(1)}}$ to the center of \mathcal{D}_Y .
- (2) The sheaf of algebras \mathcal{D}_Y is an Azumaya algebra over $T^*Y^{(1)}$ of rank p^{2d} , where d is the dimension of Y .

Let \mathcal{A} be an Azumaya algebra on Y . A splitting of \mathcal{A} is defined to be a pair (E, ρ) , where E is a locally free sheaf on Y and $\rho : \mathcal{A} \xrightarrow{\sim} \mathcal{E}nd(E)$ is an isomorphism of \mathcal{O}_Y -algebras. Such a (E, ρ) induces an equivalence between the category $\text{QCoh}(Y)$ of quasi-coherent sheaves on Y and the category $\mathcal{A}\text{-mod}$ of \mathcal{A} -modules, which maps $\mathcal{F} \in \text{QCoh}(Y)$ to $E \otimes \mathcal{F}$. We define an equivalence from an Azumaya algebra \mathcal{A} to another Azumaya algebra \mathcal{B} to be a splitting of $\mathcal{A}^{op} \otimes \mathcal{B}$. Such a splitting induces an equivalence from the category of \mathcal{A} -modules to the category of \mathcal{B} -modules. Note that if there is a locally free sheaf E that gives a splitting of $\mathcal{A}^{op} \otimes \mathcal{B}$, then $\text{Hom}_{\mathcal{O}_Y}(E, \mathcal{O}_Y)$ gives a splitting of $\mathcal{A} \otimes \mathcal{B}^{op}$.

Let $f : Z \rightarrow Y$ be a morphism between smooth k -varieties. We denote by $df^{(1)}$ the Frobenius twist of the map induced by the differential of f :

$$df^{(1)} : Z^{(1)} \times_{Y^{(1)}} T^*Y^{(1)} \rightarrow T^*Z^{(1)}.$$

Let p_2 be the projection to $T^*Y^{(1)}$. Then we have:

Proposition 3.2 (cf. [8] Proposition 3.7). The Azumaya algebras $(df^{(1)})^* \mathcal{D}_Z$ and $p_2^* \mathcal{D}_Y$ are canonically equivalent.

Following [8], we define $f^! : \mathcal{D}_Y\text{-mod} \rightarrow \mathcal{D}_Z\text{-mod}$ to be the composition of the pull-back functor $\mathcal{D}_Y\text{-mod} \rightarrow p_2^* \mathcal{D}_Y\text{-mod}$, the equivalence in Proposition 3.2, and the push-forward functor

$df^*\mathcal{D}_Z\text{-mod} \rightarrow \mathcal{D}_Z\text{-mod}$. Similarly, we define $f_* : \mathcal{D}_Z\text{-mod} \rightarrow \mathcal{D}_Y\text{-mod}$ to be the composition of the pull-back functor $\mathcal{D}_Z\text{-mod} \rightarrow df^*\mathcal{D}_Z\text{-mod}$, the equivalence in Proposition 3.2, and the push-forward functor $p_2^*\mathcal{D}_Y\text{-mod} \rightarrow \mathcal{D}_Y\text{-mod}$.

Let $\theta_Y^{(1)}$ be the tautological 1-form on $T^*Y^{(1)}$. We think of $\theta_Y^{(1)}$ as a map:

$$\theta_Y^{(1)} : T^*Y^{(1)} \rightarrow T^*(T^*Y)^{(1)}.$$

Corollary 3.3 (cf. [8] Proposition 3.11 and Corollary 3.12).

- (1) The Azumaya algebra $(\theta_Y^{(1)})^*\mathcal{D}_{T^*Y}$ is canonically equivalent to \mathcal{D}_Y .
- (2) Let $\theta_1^{(1)} \in \Gamma(Z^{(1)}, \Omega_{Z^{(1)}})$ and $\theta_2^{(1)} \in \Gamma(Y^{(1)}, \Omega_{Y^{(1)}})$. If $(f^{(1)})^*(\theta_2^{(1)}) = \theta_1^{(1)}$, then the two Azumaya algebras $(\theta_1^{(1)})^*\mathcal{D}_Z$ and $(\theta_2^{(1)} \circ f^{(1)})^*\mathcal{D}_Y$ are canonically equivalent.

Let \mathcal{M} be a D_Y -module. We denote by $\nabla_{\mathcal{M}}$ the corresponding flat connection $\mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_Y$. There is a p -linear map $\mathcal{T}_Y \rightarrow \mathcal{E}nd(\mathcal{M})$ defined by

$$\psi_{\nabla_{\mathcal{M}}}(\partial) = \nabla_{\mathcal{M}}(\partial)^p - \nabla_{\mathcal{M}}(\partial^{[p]}).$$

By the discussion in Section 3.1, we can associate with it a \mathcal{O}_Y -linear map

$$\psi_{\nabla_{\mathcal{M}}} : \mathbf{Fr}^* \mathcal{T}_{Y^{(1)}} \rightarrow \mathcal{E}nd(\mathcal{M}),$$

which we call the p -curvature of \mathcal{M} .

We review the Cartier descent for flat connections with zero p -curvature. Let \mathcal{F} be a quasi-coherent sheaf on $Y^{(1)}$. There is a canonical D_Y -action on $\mathbf{Fr}^*(\mathcal{F}) \cong \mathcal{O}_Y \otimes_{\mathcal{O}_{Y^{(1)}}} \mathcal{F}$, which comes from the canonical action of D_Y on \mathcal{O}_Y . Therefore we have a flat connection $(\mathbf{Fr}^* \mathcal{F}, \nabla_{\text{can}})$. This construction induces a functor from the category of quasi-coherent sheaves on $Y^{(1)}$ to the category of D_Y -modules on Y with zero p -curvature.

Theorem 3.4 (Cartier descent, cf. [20] Theorem 5.1). Let Y be a smooth variety over k . Then the construction of $(\mathbf{Fr}^* \mathcal{F}, \nabla_{\text{can}})$ induces an equivalence between the category of quasi-coherent sheaves on $Y^{(1)}$ and the category of D_Y -modules on Y with zero p -curvature.

3.3 Differential operators on smooth stacks

Let Y be a smooth irreducible algebraic stack over an algebraically closed field k . When k is the field of complex numbers \mathbb{C} , for Y that is good in the sense that it satisfies $\dim T^*Y = 2 \dim Y$, the sheaf of differential operators on Y is defined in [7] as a sheaf of algebras D_Y on the smooth topology Y_{sm} . We review this definition as follows. The objects of Y_{sm} are k -schemes S together with a smooth morphism $f_S : S \rightarrow Y$, and the morphisms between (S, f_S) and $(S', f_{S'})$ are pairs (ϕ, α) containing a smooth morphism $\phi : S \rightarrow S'$ and $\alpha : f_S \xrightarrow{\sim} f_{S'} \circ \phi$. Let (S, f_S) be an object of Y_{sm} . We denote by \mathcal{I} the left ideal $D_S \mathcal{T}_{S/Y} \subset D_S$ generated by the relative tangent sheaf $\mathcal{T}_{S/Y}$. We define $(D_Y)_S^\sharp := D_S/\mathcal{I}$. It has a D_S -action by left multiplication. Let $\mathcal{N}_{D_S}(\mathcal{I})$ be the normalizer of \mathcal{I} in D_S . We define $(D_Y)_S := \mathcal{N}_{D_S}(\mathcal{I})/\mathcal{I}$. In other words, we set $(D_Y)_S = \mathcal{E}nd_{D_S}((D_Y)_S^\sharp)^{op}$. For any smooth morphism $\phi : S \rightarrow S'$ over Y , we have a canonical isomorphism

$$\phi^*((D_Y)_{S'}^\sharp) \xrightarrow{\sim} (D_Y)_S^\sharp, \tag{3.1}$$

which restricts to an isomorphism

$$\phi^{-1}((D_Y)_{S'}) \xrightarrow{\sim} (D_Y)_S, \tag{3.2}$$

where ϕ^{-1} is the sheaf-theoretic inverse image. We call D_Y the sheaf of differential operators on Y .

It is observed in [8] that the isomorphism (3.2) no longer holds when k is of characteristic $p > 0$. But meanwhile, $\mathbf{Fr}_* D_Y$ is a quasi-coherent sheaf on $Y^{(1)}$, and the authors constructed a coherent sheaf of algebras \mathcal{D}_Y on $T^*Y^{(1)}$ that satisfies $\pi_*^{(1)} \mathcal{D}_Y \cong \mathbf{Fr}_* D_Y$. The construction of \mathcal{D}_Y is as follows. For any k -scheme S with a smooth morphism $f_S : S \rightarrow Y$, we need to define a coherent sheaf of algebras $(\mathcal{D}_Y)_S$ on $(T^*Y)_S^{(1)} := S^{(1)} \times_{Y^{(1)}} T^*Y^{(1)}$. We consider the D_S -module $(D_Y)_S^\sharp$, and denote by $(\mathcal{D}_Y)_S^\sharp$ the corresponding

coherent sheaf on $T^*S^{(1)}$. Since we mod out the left ideal generated by $\mathcal{T}_{S/Y}$ when defining $(D_Y^\sharp)_S$, the support of $(D_Y^\sharp)_S$ lies in the closed substack $(T^*Y)_S^{(1)} \xrightarrow{df_S^{(1)}} T^*S^{(1)}$. We set $(\mathcal{D}_Y)_S := \mathcal{E}nd_{\mathcal{D}_S}((D_Y^\sharp)_S)^{op}$. For any smooth morphism $\phi : S \rightarrow S'$ over Y , isomorphism (3.1) induces an isomorphism $(\tilde{\phi}^{(1)})^*(\mathcal{D}_Y)_S \cong (\mathcal{D}_Y)_{S'}$, where $\tilde{\phi}$ is the map $(T^*Y)_S \rightarrow (T^*Y)_{S'}$. Therefore, $(\mathcal{D}_Y)_S$ sheafifies to be a coherent sheaf of algebras \mathcal{D}_Y on $T^*Y^{(1)}$. We have the following proposition:

Proposition 3.5 (cf. [8] Lemma 3.14 and [28] Proposition 2.7). The coherent sheaf of algebras \mathcal{D}_Y satisfies $\pi_*^{(1)}\mathcal{D}_Y \cong \mathbf{Fr}_* \mathcal{D}_Y$. If the stack Y is good in the sense that $\dim T^*Y = 2 \dim Y$, and we denote by T^*Y^0 the maximal smooth open substack of T^*Y , then the restriction of \mathcal{D}_Y to $(T^*Y^0)^{(1)}$ is an Azumaya algebra of rank $p^{2 \dim Y}$.

3.4 \mathcal{D} -modules, Azumaya algebras, and \mathbb{G}_m -gerbes

Let k be an algebraically closed field. Let \mathcal{B} be a k -scheme locally of finite type. Let Y be a stack locally of finite type over \mathcal{B} . Let $\tilde{Y} \rightarrow Y$ be a \mathbb{G}_m -gerbe over Y . We denote by $\mathbf{QCoh}(\tilde{Y})$ the category of quasi-coherent sheaves on \tilde{Y} . We say \tilde{Y} splits if there is an isomorphism $\tilde{Y} \cong Y \times B\mathbb{G}_m$ of \mathbb{G}_m -gerbes. In this case, there is a decomposition

$$\mathbf{QCoh}(\tilde{Y}) \cong \prod_{n \in \mathbb{Z}} \mathbf{QCoh}(\tilde{Y})_n$$

given by the weight of the \mathbb{G}_m -action. If \tilde{Y} does not split, we still have such a decomposition by pulling back along the action map $a : B\mathbb{G}_m \times \tilde{Y} \rightarrow \tilde{Y}$. We call $\mathbf{QCoh}(\tilde{Y})_1$ the category of twisted quasi-coherent sheaves associated to \tilde{Y} .

Let \mathcal{A} be an Azumaya algebra on Y . We associate with it a \mathbb{G}_m -gerbe $\tilde{Y}_{\mathcal{A}}$ over Y , which is defined as follows. For $f : S \rightarrow \mathcal{B}$ a map of schemes, $\tilde{Y}_{\mathcal{A}}(S)$ classifies triples (y, E, σ) where $y \in Y(S)$, E is a vector bundle on S , and $\sigma : y^*\mathcal{A} \xrightarrow{\sim} \mathcal{E}nd(E)$ is an isomorphism of algebras over S . We call $\tilde{Y}_{\mathcal{A}}$ the stack of splittings of \mathcal{A} . We have the following lemma:

Lemma 3.6 (cf. [8] Lemma 2.3 and [14] Example 2.6). There is a canonical equivalence between the category $\mathcal{A}\text{-mod}$ of \mathcal{A} -modules on Y and $\mathbf{QCoh}(\tilde{Y}_{\mathcal{A}})_1$.

Now let Y be a smooth irreducible algebraic stack over k . A (crystalline) \mathcal{D} -module \mathcal{M} on Y is the datum of a \mathcal{D} -module \mathcal{M}_S on S for each object (S, f_S) in Y_{sm} , and an isomorphism $\phi^! \mathcal{M}_{S'} \xrightarrow{\sim} \mathcal{M}_S$ of \mathcal{D} -modules for each morphism $(\phi, \alpha), \phi : S \rightarrow S'$ in Y_{sm} . Here $\phi^!$ denotes the \mathcal{O} -module pull-back with the natural \mathcal{D} -module structure. Those isomorphisms need to satisfy the cocycle condition for compositions. When k is of characteristic $p > 0$, \mathcal{D} -modules on Y correspond to twisted quasi-coherent sheaves associated to a certain \mathbb{G}_m -gerbe \mathcal{G}_Y on $T^*Y^{(1)}$, which is defined as follows. For any smooth morphism $f_S : S \rightarrow Y$, we associate with it a \mathbb{G}_m -gerbe $(\mathcal{G}_Y)_S$ on $(T^*Y)_S^{(1)} := S^{(1)} \times_{Y^{(1)}} T^*Y^{(1)}$, which is defined to be the pull-back of the \mathbb{G}_m -gerbe of splittings of the Azumaya algebra \mathcal{D}_S along $df_S^{(1)} : (T^*Y)_S^{(1)} \rightarrow T^*S^{(1)}$. For any smooth morphism $\phi : S \rightarrow S'$ over Y , we have an isomorphism $(\tilde{\phi}^{(1)})^*(\mathcal{G}_Y)_{S'} \rightarrow (\mathcal{G}_Y)_S$ since df_S factorizes as

$$S \times_Y T^*Y = S \times_{S'} S' \times_Y T^*Y \xrightarrow{\text{Id} \times df_{S'}} S \times_{S'} T^*S' \xrightarrow{d\phi} T^*S,$$

and the two Azumaya algebras $(d\phi^{(1)})^*\mathcal{D}_S$ and $p_2^*\mathcal{D}_{S'}$ are equivalent by Proposition 3.2. It is shown in [28] that the category of \mathcal{D} -modules on Y is equivalent to the category of twisted quasi-coherent sheaves associated to \mathcal{G}_Y , see Theorem 2.3 in [28].

Now we assume Y satisfies $\dim T^*Y = 2 \dim Y$, and denote by T^*Y^0 the maximal smooth open substack of T^*Y . Recall that in Section 3.3, we defined a coherent sheaf of algebras \mathcal{D}_Y on $T^*Y^{(1)}$, such that its restriction to $(T^*Y^0)^{(1)}$ is an Azumaya algebra of rank $p^{2 \dim Y}$. The \mathbb{G}_m -gerbe of splittings of this Azumaya algebra is isomorphic to the restriction of \mathcal{G}_Y to $(T^*Y^0)^{(1)}$, see Proposition 2.7 in [28]. Therefore, the category of $\mathcal{D}_Y|_{(T^*Y^0)^{(1)}}$ -modules is a localization of the category of (crystalline) \mathcal{D} -modules on Y .

3.5 Tensor structures on Azumaya algebras

Let \mathcal{G} be a commutative group stack over \mathcal{B} , and \mathcal{A} an Azumaya algebra over \mathcal{G} . We denote the multiplication on \mathcal{G} by $\mu : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$. Following [24], we define a tensor structure on \mathcal{A} to be an

equivalence of Azumaya algebras from $\mu^*\mathcal{A}$ to $\mathcal{A} \boxtimes \mathcal{A}$, which is a bimodule \mathcal{M} that induces a Morita equivalence, together with an isomorphism

$$\mathcal{M} \boxtimes \mathcal{A} \otimes_{\mu^*\mathcal{A} \boxtimes \mathcal{A}} (\mu, p_3)^*\mathcal{M} \cong \mathcal{A} \boxtimes \mathcal{M} \otimes_{\mathcal{A} \boxtimes \mu^*\mathcal{A}} (p_1, \mu)^*\mathcal{M}$$

of bimodules that satisfies the pentagon condition [13](1.0.1).

A tensor structure on the Azumaya algebra \mathcal{A} induces a group structure on the stack $\mathcal{Y}_{\mathcal{A}}$ of splittings of \mathcal{A} as follows. Let S be a k -scheme. An S -point of $\mathcal{Y}_{\mathcal{A}}$ is a pair (a, E) , where $a \in \mathcal{G}(S)$ and E is a splitting module for $a^*\mathcal{A}$. Let (a, E) and (b, F) be two such pairs. The locally free sheaf $E \boxtimes F$ is a splitting module for $a^*\mathcal{A} \boxtimes b^*\mathcal{A}$. Applying the equivalence between $\mu^*\mathcal{A}$ and $\mathcal{A} \boxtimes \mathcal{A}$ and then pulling-back along the diagonal map $\Delta_S : S \rightarrow S \times S$, we get a splitting module for $\mu(a, b)^*\mathcal{A}$. The construction of this group structure implies that the projection map $\mathcal{Y}_{\mathcal{A}} \rightarrow \mathcal{G}$ is a group homomorphism, therefore we have a short exact sequence:

$$0 \rightarrow \mathrm{BG}_m \rightarrow \mathcal{Y}_{\mathcal{A}} \rightarrow \mathcal{G} \rightarrow 0.$$

4 A Non-Abelian Hodge Correspondence Between $\mathrm{Loc}_{n,q}$ and $\mathrm{Higgs}_{n,q}$

4.1 Spectral data for flat connections with regular singularities

Let (E, ∇) be a flat connection of rank n on X with regular singularity at q . We associate with it the p -curvature ψ_{∇} , which is a \mathcal{O}_X -linear map

$$\psi_{\nabla} : \mathrm{Fr}^* \mathcal{T}_{X^{(1)}}(-q) \rightarrow \mathcal{E}nd(E).$$

It is associated with the p -linear map

$$\psi_{\nabla} : \mathcal{T}_X(-q) \rightarrow \mathcal{E}nd(E)$$

defined by $\psi_{\nabla}(\partial) = \nabla(\partial)^p - \nabla(\partial^{[p]})$ for any $\partial \in \mathcal{T}_X(-q)(U)$ and $U \subseteq X$ open. We can think of ψ_{∇} as a twisted Higgs field

$$\psi_{\nabla} : E \rightarrow E \otimes \mathrm{Fr}^* \Omega_{X^{(1)}}(q).$$

The coefficients of its characteristic polynomial define a point b of

$$\bigoplus_{i=1}^n \Gamma(X, (\mathrm{Fr}^* \Omega_{X^{(1)}}(q))^i).$$

Let $\mathrm{Fr}^* : \bigoplus_{i=1}^n \Gamma(X^{(1)}, \Omega_{X^{(1)}}(q)^i) \hookrightarrow \bigoplus_{i=1}^n \Gamma(X, (\mathrm{Fr}^* \Omega_{X^{(1)}}(q))^i)$ be the pull-back map. It follows from a similar argument as in [21] Proposition 3.2 that b actually lies in the image of Fr^* , and we also denote by b the corresponding point in $B^{(1)} \cong \bigoplus_{i=1}^n \Gamma(X^{(1)}, \Omega_{X^{(1)}}(q)^i)$. We call this map $h' : \mathrm{Loc}_{n,q} \rightarrow B^{(1)}$ the Hitchin map for flat connections with regular singularity at q . The corresponding spectral curve Σ'_b lies in the total space of $\mathrm{Fr}^* \Omega_{X^{(1)}}(q)$, which is isomorphic to $X \times_{X^{(1)}} T^*X(q)^{(1)}$. Since $b \in B^{(1)}$, Σ'_b fits into the following pull-back square:

$$\begin{array}{ccc} \Sigma'_b & \xrightarrow{\rho} & \Sigma_b^{(1)} \\ \downarrow \pi' & & \downarrow \pi^{(1)} \\ X & \xrightarrow{\mathrm{Fr}} & X^{(1)} \end{array}$$

where $\Sigma_b^{(1)} \subset T^*X(q)^{(1)}$ is the spectral curve above $b \in B^{(1)}$ as defined in Section 2.1. We denote by $E' \in \mathrm{Coh}(\Sigma'_b)$ the spectral sheaf corresponding to ψ_{∇} , so E' satisfies $\pi'_*(E') \cong E$.

Let x be a local parameter of $\mathcal{O}_{X,q}$. Let (E, ∇) be a flat connection with regular singularity at q . Restricting $\psi_{\nabla}(x\partial_x)$ to q , we get $\mathrm{res}_q(\psi_{\nabla}) \in \mathrm{End}(E_q)$, which we call the residue of ψ_{∇} at q .

Lemma 4.1. $\text{res}_q(\psi_\nabla) = (\text{res}_q \nabla)^p - \text{res}_q \nabla$.

Proof. This equation follows from the computation $(x\partial_x)^{[p]} = x\partial_x$. ■

Remark 4.2. If we assume $\text{res}_q \nabla$ is nilpotent, since $p > n$, $(\text{res}_q \nabla)^p = 0$. So $\text{res}_q(\psi_\nabla) = -\text{res}_q \nabla$. In particular, they lie in the same nilpotent orbit.

4.2 Statement of the theorem

Let \underline{a} be an unordered n -tuple of elements in k . We denote by $\mathcal{Higgs}_{n,\underline{a}}(X^{(1)})$ the moduli stack of $\Omega_{X^{(1)}}(q)$ -twisted Higgs bundles (E, ϕ) on $X^{(1)}$ such that the unordered n -tuple of eigenvalues of $\text{res}_q(\phi)$ is \underline{a} . Let $B_{\underline{a}}^{(1)}$ be the image of $\mathcal{Higgs}_{n,\underline{a}}(X^{(1)})$ under the Hitchin map $h^{(1)}$. Note that when $\underline{a} = (0, 0, \dots, 0)$, $B_{\underline{a}}^{(1)} = B_{\mathcal{N}}^{(1)}$. We fix a set-theoretic section σ of the Artin–Schreier map $k \rightarrow k$ that maps t to $t^p - t$. Let $\mathcal{L}oc_{n,\sigma(\underline{a})}$ be the substack of $\mathcal{L}oc_{n,q}$ that classifies flat connections (E, ∇) such that the unordered n -tuple of eigenvalues of $\text{res}_q(\nabla)$ is $\sigma(\underline{a})$. Note that by Lemma 4.1, $h'(\mathcal{L}oc_{n,\sigma(\underline{a})}) \subseteq B_{\underline{a}}^{(1)}$.

We denote by $\mathcal{L}oc'_{n,\sigma(\underline{a})}$ the substack of $\mathcal{L}oc_{n,\sigma(\underline{a})}$ that classifies flat connections (E, ∇) such that the corresponding spectral sheaf $E' \in \text{Coh}(\Sigma'_b)$ is invertible. We have the following theorem:

Theorem 4.3.

- (1) $\mathcal{L}oc'_{n,\sigma(\underline{a})}$ has a natural structure of a $\text{Pic}(\Sigma^{(1)}/B_{\underline{a}}^{(1)})$ -torsor.
- (2) $\mathcal{L}oc_{n,\sigma(\underline{a})} \cong \mathcal{L}oc'_{n,\sigma(\underline{a})} \times^{\text{Pic}(\Sigma^{(1)}/B_{\underline{a}}^{(1)})} \mathcal{Higgs}_{n,\underline{a}}(X^{(1)})$.

Before getting into the proof of Theorem 4.3, we state two corollaries.

Corollary 4.4. There exists an étale cover $U \rightarrow B_{\underline{a}}^{(1)}$, such that

$$\mathcal{L}oc_{n,\sigma(\underline{a})} \times_{B_{\underline{a}}^{(1)}} U \cong \mathcal{Higgs}_{n,\underline{a}}(X^{(1)}) \times_{B_{\underline{a}}^{(1)}} U.$$

We denote by $\mathcal{Higgs}_{\mathcal{N}}(X^{(1)})$ the moduli stack of $\Omega_{X^{(1)}}(q)$ -twisted Higgs bundles (E, ϕ) on $X^{(1)}$ such that $\text{res}_q(\phi)$ is nilpotent, and by $\mathcal{L}oc_{\mathcal{N}}$ the substack of $\mathcal{L}oc_{n,q}$ that classifies (E, ∇) with nilpotent $\text{res}_q(\nabla)$. Then we have:

Corollary 4.5.

- (1) $\mathcal{L}oc'_{\mathcal{N}}$ has a natural structure of a $\text{Pic}(\Sigma^{(1)}/B_{\mathcal{N}}^{(1)})$ -torsor,
- (2) $\mathcal{L}oc_{\mathcal{N}} \cong \mathcal{L}oc'_{\mathcal{N}} \times^{\text{Pic}(\Sigma^{(1)}/B_{\mathcal{N}}^{(1)})} \mathcal{Higgs}_{\mathcal{N}}(X^{(1)})$.

Our definition of $\mathcal{L}oc'_{n,\sigma(\underline{a})}$ and formulation of Theorem 4.3 is motivated by the work of Chen–Zhu [10] on the characteristic p version of the non-abelian Hodge correspondence for flat connections without singularities. The strategy of proof is similar to [10] besides the proof of the surjectivity result Proposition 4.6. The rest of this section is devoted to the proof of Theorem 4.3. We start by showing:

Proposition 4.6. The map $h' : \mathcal{L}oc'_{n,\sigma(\underline{a})} \rightarrow B_{\underline{a}}^{(1)}$ is surjective.

We need to show that for any $b \in B_{\underline{a}}^{(1)}(k)$, there exists $(E, \nabla) \in \mathcal{L}oc'_{n,\sigma(\underline{a})}(k)$ that is mapped to b under the Hitchin map. The idea of constructing (E, ∇) is as follows: we construct a flat connection (E_0, ∇_0) on $X \setminus q$ and a flat connection $(\hat{E}, \hat{\nabla})$ on the formal disk around q , such that both flat connections have the correct p -curvature. Then we glue (E_0, ∇_0) and $(\hat{E}, \hat{\nabla})$ together using the Beauville–Laszlo theorem [5].

4.3 Proof of Proposition 4.6

Let $b \in B_{\underline{a}}^{(1)}(k)$. Let $\pi' : \Sigma' \rightarrow X$ and $\pi^{(1)} : \Sigma^{(1)} \rightarrow X^{(1)}$ be the corresponding spectral covers as described in Section 4.1. We will construct (E, ∇) such that $h'(E, \nabla) = b$ and the spectral sheaf E' is invertible.

Step 1. In this step, we show that there exists a flat connection (E_0, ∇_0) on $X \setminus q$ such that the spectral curve of ψ_{∇_0} is $\Sigma' \setminus (\pi')^{-1}(q)$ and the spectral sheaf E'_0 is invertible. Such a (E_0, ∇_0) is equivalent to a splitting of the Azumaya algebra $i_0^* \mathcal{D}_{X \setminus q}$, where i_0 is the embedding $i_0 : \Sigma^{(1)} \setminus (\pi^{(1)})^{-1}(q) \rightarrow T^*(X \setminus q)^{(1)}$. Note that for any rank p vector bundle F on $\Sigma^{(1)} \setminus (\pi^{(1)})^{-1}(q)$ such that $\mathcal{E}nd(F) \cong i_0^* \mathcal{D}_{X \setminus q}$, the corresponding spectral sheaf on $\Sigma' \setminus (\pi')^{-1}(q)$ is invertible. This is because for any $p \in X \setminus q$ and x a local parameter at p , x acts as a regular nilpotent matrix on the fiber $F_{p'}$ for any $p' \in \Sigma^{(1)} \setminus (\pi^{(1)})^{-1}(q)$ such that $\pi^{(1)}(p') = p$, see the proof of Lemma 2.2.1 in [9]. The existence of such a splitting is guaranteed by the following theorem.

Theorem 4.7 (cf. [17] Theorem 3.21). Let Y be a scheme of finite type over an algebraically closed field. Assume $\dim(Y) \leq 1$. Then $H_{\text{ét}}^2(Y, \mathbb{G}_m) = 0$. In particular, every Azumaya algebra on Y splits.

Step 2. In this step, we construct a flat connection $(\hat{E}, \hat{\nabla})$ on the formal disk $D = \text{Spec}(\hat{\mathcal{O}}_{X,q})$ around q that satisfies the following three properties:

- (5.1) $(\hat{E}, \hat{\nabla})$ has regular singularity and the unordered n -tuple of eigenvalues of $\text{res}(\hat{\nabla})$ is $\sigma(\underline{a})$,
- (5.2) the spectral curve of $\psi_{\hat{\nabla}}$ is $\hat{\Sigma}' := D \times_X \Sigma'$,
- (5.3) the spectral sheaf \hat{E}' is invertible.

Now let x be a local parameter of X at q , then $D \cong \text{Spec}(k[[x]])$. We denote $\iota(x\partial_x) = x^p \partial_x^p$ by y , so

$$\mathcal{O}_{\hat{\Sigma}'} \cong k[[x]][y]/(f), \text{ where } f = y^n + b_1(x)y^{n-1} + \dots + b_{n-1}(x)y + b_n(x), b_i(x) \in k[[x]].$$

Since $b \in B_{\underline{a}}^{(1)}$, $b_i(x)$ actually lies in $k[[x^p]]$. We assume that \underline{a} consists of t distinct elements a_1, a_2, \dots, a_t , each appearing m_i times, then $\tilde{f} \in k[[x^p]][y]/x^p k[[x^p]][y]$ factorizes as

$$\tilde{f} = \prod_{i=1}^t (y - a_i)^{m_i},$$

therefore f factorizes as $f = f_1 f_2 \dots f_t$, where $f_i \in k[[x^p]][y]$ is monic and $\tilde{f}_i = (y - a_i)^{m_i}$. Therefore $\hat{\Sigma}'$ is the disjoint union of $\hat{\Sigma}'_i := \text{Spec} k[[x]][y]/(f_i)$. It is enough to construct flat connections $(\hat{E}_i, \hat{\nabla}_i)$ with the following properties:

- (5.1') $(\hat{E}_i, \hat{\nabla}_i)$ has regular singularity and the eigenvalues of $\text{res}(\psi_{\hat{\nabla}_i})$ are all $\sigma(a_i)$.
- (5.2') the spectral curve of $\psi_{\hat{\nabla}_i}$ is $\hat{\Sigma}'_i$,
- (5.3') the spectral sheaf \hat{E}'_i is invertible.

Since $\mathcal{O}_{\hat{\Sigma}'_i}$ is a local ring, (5.3') implies that \hat{E}'_i is isomorphic to $\mathcal{O}_{\hat{\Sigma}'_i}$. Let e be its generator. A meromorphic flat connection with spectral curve $\hat{\Sigma}'_i$ is determined by the connection acting on e , which can be written as $\nabla(e) = g e dx$, $g \in \mathcal{O}_{\hat{\Sigma}'_i}[[x^{-1}]]$. By (5.1') and (5.2'), ∇ need to satisfy the following:

- (5.1'') $(\nabla(x\partial_x) - \sigma(a_i))(e) \subseteq (x, y - a_i)e$,
- (5.2'') $(\nabla(x\partial_x)^p - \nabla((x\partial_x)^{[p]}))(e) = ye$.

Since $(\nabla(x\partial_x)^p - \nabla((x\partial_x)^{[p]}))(e) = x^p(\partial_x^{p-1}(g) + g^p)e$, (5.2'') is equivalent to the following equation in $\mathcal{O}_{\hat{\Sigma}'_i}$:

$$x^p(\partial_x^{p-1}(g) + g^p) = y.$$

We look for solutions of the form

$$g = -(y - a_i - \sigma(a_i))/x + g_1, g_1 \in k[[x]][y],$$

so (5.1'') is automatically satisfied, and (5.2'') is equivalent to

$$\partial_x^{p-1}(g_1) + g_1^p = (y - a_i)^p/x^p. \tag{4.1}$$

Note that since $f_i \equiv (y - a_i)^{m_i} \pmod{x^p k[[x^p]][y]}$, there exists a polynomial $h \in k[[x^p]][y]$ such that $(y - a_i)^p/x^p = \tilde{h}$ in $\mathcal{O}_{\hat{\Sigma}'_i}$. By a substitution $y' = y - a_i$, we can assume that $f_i \equiv y^{m_i} \pmod{x^p k[[x^p]][y]}$. In this case, $\mathcal{O}_{\hat{\Sigma}'_i} \cong k[[x, y]]/(f_i)$, therefore it is enough to find a solution $g_1 \in k[[x, y]]$. It is easy to see that

for any $h \in k[[x^p]][y]$, the equation $\partial_x^{p-1}(g_1) + g_1^p = h$ has solutions in $k[[x, y]]$. We look for solutions of the form $g_1 = x^{p-1} \cdot g_2$, where $g_2 \in k[[x^p, y]]$. Equation 4.1 becomes

$$g_2^p \cdot x^{p^2-p} - g_2 = h,$$

for which the existence of solutions follows from Hensel’s lemma.

Step 3. Let $D^\times = \text{Spec}(k((x)))$ be the punctured disk around q , and let $\hat{\Sigma}^\times = D^\times \times_D \hat{\Sigma}$ be the spectral curve above D^\times . Both $(E_0|_{D^\times}, \nabla_0|_{D^\times})$ and $(\hat{E}|_{D^\times}, \hat{\nabla}|_{D^\times})$ give splittings of the Azumaya algebra $\mathcal{D}_X|_{\hat{\Sigma}^\times}$. Since all invertible sheaves on $\hat{\Sigma}^\times$ are trivial, we have an isomorphism of connections

$$(E_0|_{D^\times}, \nabla_0|_{D^\times}) \cong (\hat{E}|_{D^\times}, \hat{\nabla}|_{D^\times}).$$

We fix such an isomorphism. By the theorem of Beauville–Laszlo [5], E_0 and \hat{E} can be glued together to get a rank n vector bundle E on X . Since the gluing data is compatible with the connections, ∇_0 and $\hat{\nabla}$ are glued together to get a flat connection ∇ on E with regular singularity at q . This connection (E, ∇) satisfies all the properties we need.

4.4 D_X -modules on spectral covers of X

Let $b \in B^{(1)}$, and let $\pi' : \Sigma' \rightarrow X$ and $\pi^{(1)} : \Sigma^{(1)} \rightarrow X^{(1)}$ be the corresponding spectral covers as described in Section 4.1. We have the following pull-back square:

$$\begin{CD} \Sigma' @>\rho>> \Sigma^{(1)} \\ @V\pi'VV @VV\pi^{(1)}V \\ X @>\text{Fr}>> X^{(1)} \end{CD}$$

There is a canonical D_X -action on $\mathcal{O}_{\Sigma'} = \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{O}_{\Sigma^{(1)}}$, which comes from the canonical action of D_X on \mathcal{O}_X . Similarly, for any quasi-coherent sheaf \mathcal{M} on $\Sigma^{(1)}$, the pull-back sheaf $\rho^*\mathcal{M} \cong \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \mathcal{M}$ has a canonical D_X -action. We denote by ∇_{can} the corresponding map

$$\nabla_{\text{can}} : \rho^*\mathcal{M} \rightarrow \rho^*\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X.$$

Definition 4.8. We define a D_X -module on Σ' to be a quasi-coherent sheaf \mathcal{F} on Σ' together with a k -linear map

$$\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X$$

that satisfies $\nabla(fs) = \nabla_{\text{can}}(f)(s) + f\nabla(s)$ for $f \in \mathcal{O}_U$, $s \in \mathcal{F}(U)$ and $U \subseteq \Sigma'$ open. Let $D_X(-q)$ be the subsheaf of algebras of D_X generated by \mathcal{O}_X and $\mathcal{T}_X(-q)$. Similarly, we define $D_X(-q)$ -modules on Σ' . The only difference is that now ∇ is a map

$$\nabla : \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_X(q).$$

We have the following lemma concerning this definition.

Lemma 4.9.

- (1) The structure sheaf $\mathcal{O}_{\Sigma'}$ is a D_X -module on Σ' . For any quasi-coherent sheaf \mathcal{M} on $\Sigma^{(1)}$, the pull-back $\rho^*\mathcal{M}$ is a D_X -module on Σ' .
- (2) Let (E, ∇) be a flat connection with regular singularity at q such that $h'(E, \nabla) = b$. Let $E' \in \text{Coh}(\Sigma')$ be the corresponding spectral sheaf. Then (E', ∇) is a $D_X(-q)$ -module on Σ' .
- (3) Let $(\mathcal{F}_1, \nabla_1)$ and $(\mathcal{F}_2, \nabla_2)$ be two $D_X(-q)$ -modules on Σ' , then $\mathcal{F}_1 \otimes_{\mathcal{O}_{\Sigma'}} \mathcal{F}_2$ and $\text{Hom}_{\mathcal{O}_{\Sigma'}}(\mathcal{F}_1, \mathcal{F}_2)$ have canonical structures of $D_X(-q)$ -modules on Σ' .

In all of the cases above, we denote by ∇_{can} the corresponding map induced by the action of $\mathcal{T}_X(-q)$.

Let (\mathcal{F}, ∇) be a $D_X(-q)$ -module on Σ' such that $\pi'_*(\mathcal{F})$ is locally free, then $\pi'_*(\mathcal{F})$ has the structure of a flat connection with regular singularity at q .

4.5 Proof of Theorem 4.3

Now we construct the map Φ that induces the isomorphism in Theorem 4.3. Let $b \in B_{\underline{a}}^{(1)}$. Let $(E, \nabla_E) \in \mathcal{L}oc_{n,\sigma(\underline{a})}^r$ and $(M, \phi) \in \mathcal{H}iggs_{n,\underline{a}}(X^{(1)})$, both mapped to b under the Hitchin map. We denote the spectral sheaf of (E, ∇_E) by $E' \in \mathbf{Coh}(\Sigma')$ and the spectral sheaf of (M, ϕ) by $\mathcal{M} \in \mathbf{Coh}(\Sigma^{(1)})$.

Lemma 4.10. Let $\mathcal{G} \in \mathbf{Coh}(\Sigma^{(1)})$ and let \mathcal{L} be an invertible sheaf on Σ' . The push-forward $\pi'_*(\mathcal{L} \otimes \rho^*(\mathcal{G}))$ is a locally free sheaf of rank n on X if and only if $\pi_*^{(1)}(\mathcal{G})$ is a locally free sheaf of rank n on $X^{(1)}$.

By Lemma 4.9 and Lemma 4.10, we get a flat connection $(\pi'_*(E' \otimes \rho^*(\mathcal{M})), \nabla_{can})$ on X with regular singularity at q .

Lemma 4.11.

- (1) The flat connection $(\pi'_*(E' \otimes \rho^*(\mathcal{M})), \nabla_{can})$ is mapped to b under the Hitchin map h' ,
- (2) The residue $\mathbf{res}_q(\nabla_{can})$ has eigenvalues $\sigma(\underline{a})$.

The construction of $(\pi'_*(E' \otimes \rho^*(\mathcal{M})), \nabla_{can})$ is functorial. Therefore, we have a morphism of stacks over $B_{\underline{a}}^{(1)}$:

$$\Phi : \mathcal{L}oc_{n,\sigma(\underline{a})}^r \times_{B_{\underline{a}}^{(1)}} \mathcal{H}iggs_{n,\underline{a}}(X^{(1)}) \longrightarrow \mathcal{L}oc_{n,\sigma(\underline{a})}.$$

Now we construct a map Ψ in the inverse direction. Let (F, ∇_F) be a point of $\mathcal{L}oc_{n,\sigma(\underline{a})}$ such that $h'(F, \nabla_F) = b$. Let $F' \in \mathbf{Coh}(\Sigma')$ be the spectral sheaf. Then by Lemma 4.9, there is a canonical $D_X(-q)$ -action on $\pi'_*(\mathcal{H}om_{\mathcal{O}_{\Sigma'}}(E', \mathcal{O}_{\Sigma'}) \otimes F')$. We denote $\mathcal{H}om_{\mathcal{O}_{\Sigma'}}(E', \mathcal{O}_{\Sigma'}) \otimes F'$ by \mathcal{F} .

Lemma 4.12.

- (1) The flat connection $(\pi'_*(\mathcal{F}), \nabla_{can})$ has zero p -curvature.
- (2) The residue $\mathbf{res}_q(\nabla_{can})$ is nilpotent.

By Lemma 4.1, the residue $\mathbf{res}_q(\nabla_{can})$ of $(\pi'_*(\mathcal{F}), \nabla_{can})$ at q satisfies

$$\mathbf{res}_q(\nabla_{can})^p - \mathbf{res}_q(\nabla_{can}) = 0.$$

This implies $\mathbf{res}_q(\nabla_{can})$ is a semisimple matrix with integer eigenvalues. But meanwhile, $\mathbf{res}_q(\nabla_{can})$ needs to be nilpotent, so $\mathbf{res}_q(\nabla_{can})$ must be the zero, therefore $(\pi'_*(\mathcal{F}), \nabla_{can})$ is a flat connection without singularities. By the Cartier descent (Theorem 3.4), there is a canonical quasi-coherent sheaf \mathcal{N} on $X^{(1)}$ such that $(\pi'_*(\mathcal{F}), \nabla_{can})$ is isomorphic to $(\mathbf{Fr}^*(\mathcal{N}), \nabla_{can})$. Note that \mathcal{N} can be identified with elements in $\pi'_*(\mathcal{F})$ that vanish under ∇_{can} . The action of $\mathcal{O}_{\Sigma^{(1)}}$ preserve those elements, therefore there is a canonical quasi-coherent sheaf \mathcal{M} on $\Sigma^{(1)}$ such that $(\mathcal{F}, \nabla_{can})$ is isomorphic to $(\rho^*(\mathcal{M}), \nabla_{can})$ as $D_X(-q)$ -modules on Σ' . Since E' is an invertible sheaf, $(F', \nabla_F) \cong (E' \otimes \rho^*(\mathcal{M}), \nabla_{can})$. The construction of \mathcal{M} is functorial. Therefore, we have a morphism Ψ of stacks over $B_{\underline{a}}^{(1)}$:

$$\Psi : \mathcal{L}oc_{n,\sigma(\underline{a})}^r \times_{B_{\underline{a}}^{(1)}} \mathcal{L}oc_{n,\sigma(\underline{a})} \longrightarrow \mathcal{H}iggs_{n,\underline{a}}(X^{(1)}).$$

Let $(E, \nabla_E) \in \mathcal{L}oc_{n,\sigma(\underline{a})}^r$ such that $h'(E, \nabla_E) = b$, and denote the corresponding spectral sheaf by $E' \in \mathbf{Coh}(\Sigma')$. Let \mathcal{L} be an invertible sheaf on $\Sigma^{(1)}$. Then by Lemma 4.11,

$$(\pi'_*(E' \otimes \rho^*(\mathcal{L})), \nabla_{can}) \in \mathcal{L}oc_{n,\sigma(\underline{a})}^r.$$

This construction defines an action of $\mathbf{Pic}(\Sigma^{(1)}/B_{\underline{a}}^{(1)})$ on $\mathcal{L}oc_{n,\sigma(\underline{a})}^r$.

Proposition 4.13. This action induces the structure of a pseudo $\mathbf{Pic}(\Sigma^{(1)}/B_{\underline{a}}^{(1)})$ -torsor on $\mathcal{L}oc_{n,\sigma(\underline{a})}^r$.

Proof. Let S be a k -scheme. Let b be an S -point of $B_{\underline{a}}^{(1)}$. We need to show that the action of $\text{Pic}(\Sigma_b^{(1)})$ on the fiber $(\mathcal{L}oc_{n,\sigma(\underline{a})}^r)_b := \mathcal{L}oc_{n,\sigma(\underline{a})}^r \times_{B_{\underline{a}}^{(1)},b} S$ is simply transitive when $(\mathcal{L}oc_{n,\sigma(\underline{a})}^r)_b$ is non-empty. Let (E, ∇_E) and (F, ∇_F) be two points of $(\mathcal{L}oc_{n,\sigma(\underline{a})}^r)_b$ that is mapped to b under the Hitchin map. We denote the corresponding spectral sheaves by E' and F' . By the discussion after Lemma 4.12, there exists a quasi-coherent sheaf \mathcal{M} on $\Sigma_b^{(1)}$ such that $(F', \nabla_F) \cong (E' \otimes \rho^*(\mathcal{M}), \nabla_{can})$. Since ρ is faithfully flat, E' and F' being invertible sheaves on $\Sigma_b^{(1)}$ implies that \mathcal{M} is an invertible sheaf on $\Sigma_b^{(1)}$. The map Φ induces a map

$$\text{Hom}_{\mathcal{O}_{\Sigma_b^{(1)}}}(\mathcal{O}_{\Sigma_b^{(1)}}, \mathcal{M}) \xrightarrow{\cong} \text{Hom}_{D_X(-q)}((E, \nabla_E), (F, \nabla_F)),$$

which is an isomorphism since Ψ produces its inverse. ■

We denote by $\mathcal{L}oc_{n,q}$ is the moduli stack of flat connections on X with regular singularity at q , without constraints on the eigenvalues of the residue. Let $\mathcal{L}oc_{n,q}^r \subset \mathcal{L}oc_{n,q}$ be the substack characterized by the spectral sheaf being invertible. We have the following proposition:

Proposition 4.14. The map $h' : \mathcal{L}oc_{n,q}^r \rightarrow B_{\underline{a}}^{(1)}$ is smooth.

Before getting into the proof of Proposition 4.14, we state a corollary that is going to be used in the proof of Theorem 4.3.

Corollary 4.15. The map $h' : \mathcal{L}oc_{n,\sigma(\underline{a})}^r \rightarrow B_{\underline{a}}^{(1)}$ is smooth.

Proof. The map $\mathcal{L}oc_{n,q}^r \times_{B_{\underline{a}}^{(1)}} B_{\underline{a}}^{(1)} \rightarrow B_{\underline{a}}^{(1)}$ is smooth by base change, and the fiber product $\mathcal{L}oc_{n,q}^r \times_{B_{\underline{a}}^{(1)}} B_{\underline{a}}^{(1)}$ is the disjoint union of $\mathcal{L}oc_{n,\underline{c}}^r$, where \underline{c} ranges from all unordered n -tuples of elements in k that maps to \underline{a} under the Artin–Schreier map. ■

We denote by $\widetilde{\mathcal{L}oc}_{n,q}$ the stack that classifies triples (E, ∇, θ) , where E is a vector bundle of rank n on X , $\nabla : E \rightarrow E \otimes \Omega_X(q)$ is a flat connection with regular singularity at q and $\theta : E_q \xrightarrow{\cong} k^n$ is a frame of E at q . The natural action of GL_n on the frame θ gives $\widetilde{\mathcal{L}oc}_{n,q}$ the structure of a GL_n -torsor over $\mathcal{L}oc_{n,q}$.

Lemma 4.16. $\mathcal{L}oc_{n,q}$ and $\widetilde{\mathcal{L}oc}_{n,q}$ are algebraic stacks locally of finite type over k .

Proof. The 1-morphism $\mathcal{L}oc_{n,q} \rightarrow \text{Bun}_n$ is representable and locally of finite presentation. Since Bun_n is an algebraic stack locally of finite type over k and $\widetilde{\mathcal{L}oc}_{n,q}$ is a GL_n -torsor over $\mathcal{L}oc_{n,q}$, both $\mathcal{L}oc_{n,q}$ and $\widetilde{\mathcal{L}oc}_{n,q}$ are algebraic stacks locally of finite type over k . ■

Lemma 4.17. $\mathcal{L}oc_{n,q}^r$ and $\widetilde{\mathcal{L}oc}_{n,q}^r$ are smooth.

Proof. In order to show that $\widetilde{\mathcal{L}oc}_{n,q}^r$ is smooth, all we need to show is that for any small extension of finite-generated Artinian local k -algebras $A' \rightarrow A$, an A -point of $\widetilde{\mathcal{L}oc}_{n,q}^r$ can be lifted to an A' -point of $\widetilde{\mathcal{L}oc}_{n,q}^r$, that is, we want to produce the dashed arrow for the following commutative diagram:

$$\begin{array}{ccc} \text{Spec}(A) & \longrightarrow & \widetilde{\mathcal{L}oc}_{n,q}^r \\ \downarrow & \nearrow \text{dotted} & \\ \text{Spec}(A') & & \end{array}$$

We denote by (E, ∇, θ) the k -point

$$\text{Spec}(A/m_A A) \rightarrow \text{Spec}(A) \rightarrow \widetilde{\mathcal{L}oc}_{n,q}^r.$$

The obstruction to the existence of such liftings lies in the second hypercohomology $\mathbb{H}^2(\mathcal{F}_{E,\nabla}^\bullet)$ of the complex

$$\mathcal{F}_{E,\nabla}^\bullet : \mathcal{E}nd(E)(-q) \xrightarrow{\nabla_{\mathcal{E}nd(E)}} \mathcal{E}nd(E) \otimes \Omega_X(q),$$

where $\nabla_{\mathcal{E}nd(E)}$ is the canonical connection on $\mathcal{E}nd(E)$ induced by ∇ . By Serre duality, $\mathbb{H}^2(\mathcal{F}_{E,\nabla}^\bullet) \cong \mathbb{H}^0(\mathcal{F}_{E,\nabla}^\bullet)$. Note that $\mathbb{H}^0(\mathcal{F}_{E,\nabla}^\bullet)$ is isomorphic to the Lie algebra of $\mathbf{Aut}(E, \nabla, \theta)$. Since $(E, \nabla) \in \mathcal{L}oc_{n,q}^r$, we have $\mathbf{Aut}(E, \nabla) = k^\times$ by Proposition 4.13. But multiplication by scalars does not preserve the framing θ , therefore $\mathbf{Aut}(E, \nabla, \theta)$ is the trivial group. This implies $\mathbb{H}^2(\mathcal{F}_{E,\nabla}^\bullet) \cong \mathbb{H}^0(\mathcal{F}_{E,\nabla}^\bullet) = 0$. ■

A by-product of the proof of Lemma 4.17 is the following computation of the dimension of $\mathcal{L}oc_{n,q}^r$. Since $\mathbb{H}^0(\mathcal{F}_{E,\nabla}^\bullet) = \mathbb{H}^2(\mathcal{F}_{E,\nabla}^\bullet) = 0$,

$$\begin{aligned} \dim \widetilde{\mathcal{L}oc}_{n,q}^r &= \dim \mathbb{H}^1(\mathcal{F}_{E,\nabla}^\bullet) \\ &= 2(\dim H^0(X, \mathcal{E}nd(E) \otimes \Omega_X(q)) - \dim H^0(X, \mathcal{E}nd(E)(-q))). \end{aligned}$$

By Riemann–Roch,

$$\dim H^0(X, \mathcal{E}nd(E) \otimes \Omega_X(q)) - \dim H^0(X, \mathcal{E}nd(E)(-q)) = n^2g.$$

Therefore,

$$\begin{aligned} \dim \mathcal{L}oc_{n,q}^r &= \dim \widetilde{\mathcal{L}oc}_{n,q}^r - \dim GL_n(k) \\ &= n^2(2g - 1). \end{aligned}$$

Proof Proposition 4.14. Let $b \in B^{(1)}(k)$. By Proposition 4.6 and 4.13, the fiber $(\mathcal{L}oc_{n,q}^r)_b := \mathcal{L}oc_{n,q}^r \times_{B^{(1)},b} \text{Spec}(k)$ is a $\text{Pic}(\Sigma_b^{(1)})$ -torsor. We compute that

$$\dim B^{(1)} = n(n+1)(2g-1)/2 + n(1-g)$$

and

$$\dim \text{Pic}(\Sigma_b^{(1)}) = g_{\Sigma_b^{(1)}} - 1 = n(n-1)(2g-1)/2 + n(g-1),$$

therefore

$$\dim \text{Pic}(\Sigma_b^{(1)}) = \dim \mathcal{L}oc_{n,q}^r - \dim B^{(1)}.$$

Since both $\mathcal{L}oc_{n,q}^r$ and $B^{(1)}$ are smooth, the map h' is flat by miracle flatness. Furthermore, since $\text{Pic}(\Sigma_b^{(1)})$ is smooth, $(\mathcal{L}oc_{n,q}^r)_b$ is smooth, therefore h' is smooth. ■

Proof of Theorem 4.3. The first part follows from Proposition 4.13 and Corollary 4.15. For the second part, it is easy to see that the morphism Φ defined above induces a morphism

$$\Phi : \mathcal{L}oc_{n,\sigma(\underline{q})}^r \times^{\text{Pic}(\Sigma^{(1)}/B_{\underline{q}}^{(1)})} \mathcal{H}iggs_{n,\underline{q}}(X^{(1)}) \longrightarrow \mathcal{L}oc_{n,\sigma(\underline{q})},$$

and Ψ induces the inverse. ■

Now we discuss how the residues of Higgs bundles and flat connections match under Φ .

Proposition 4.18. Let $(E, \nabla_E) \in \mathcal{L}oc_{n,\sigma(\underline{q})}^r$ and $(M, \phi_M) \in \mathcal{H}iggs_{n,\underline{q}}(X^{(1)})$ such that

$$h'(E, \nabla_E) = h^{(1)}(M, \phi_M) = b \in B_{\underline{q}}^{(1)}(k).$$

Denote the image of (E, ∇_E) and (M, ϕ_M) under Φ by (F, ∇_F) . Then $\text{res}_q(\psi_{\nabla_F})$ and $\text{res}_q(\phi_M)$ lie in the same adjoint orbit.

Proof. Let $E' \in \text{Coh}(\Sigma')$ be the spectral sheaf of (E, ∇_E) , and let $\mathcal{M} \in \text{Coh}(\Sigma^{(1)})$ be the spectral sheaf of (M, ϕ_M) . Let x be a local parameter of X at q . Note that $\text{res}_q(\phi_M)$ is the action of $x^p \partial_x^p$ on the fiber $\pi_*^{(1)}(\mathcal{M})|_q$, and $\text{res}_q(\psi_{\nabla_E})$ is the action of $x^p \partial_x^p$ on the fiber $\pi'_*(E' \otimes \rho^*(\mathcal{M}))|_q$. Since $\pi_*^{(1)}(\mathcal{M})|_q \cong \pi'_*(\rho^*(\mathcal{M}))|_q$ with the same $x^p \partial_x^p$ action, it suffices to show that the action of $x^p \partial_x^p$ on $\pi'_*(\rho^*(\mathcal{M}))|_q$ and $\pi'_*(E' \otimes \rho^*(\mathcal{M}))|_q$ lie in the same adjoint orbit. This follows from the assumption that E' is an invertible sheaf. ■

In particular, if $\sigma(\mathfrak{a}) = (\mathfrak{O})$, Proposition 4.18 together with Remark 4.2 implies that $\text{res}_q(\nabla_E)$ and $\text{res}_q(\phi_M)$ lie in the same nilpotent orbit. Therefore, we have the following:

Corollary 4.19. The scheme-theoretic image of $\text{Loc}_{n,p}$ under the Hitchin map h' is $B_p^{(1)}$.

5 Tamely Ramified Geometric Langlands Correspondence in Positive Characteristic

5.1 The algebra $\mathcal{D}_{\text{Bun}_{n,p}}$

In this subsection, we clarify what we mean by $\mathcal{D}_{\text{Bun}_{n,p}}$. Since the stack $\text{Bun}_{n,p}$ does not satisfy the property required in Proposition 3.5, we cannot apply this proposition directly. In order to solve this problem, we introduce a new stack $\underline{\text{Bun}}_{n,p}$ similar to the stack $\underline{\text{Bun}}_n$ introduced in [8]. The stack $\underline{\text{Bun}}_{n,p}$ classifies the same objects as $\text{Bun}_{n,p}$, but the morphisms are different. Let S be a k -scheme, and let (E, E_q^\bullet) and (F, F_q^\bullet) be two rank n vector bundles on $S \times X$ with partial flag structures of type P (see Remark 1.6) along $S \times q$, then the set of morphisms between (E, E_q^\bullet) and (F, F_q^\bullet) are defined to be the set of isomorphic classes of pairs (ι, \mathcal{L}) , where \mathcal{L} is a line bundle on S and ι is an isomorphism $\iota : (E, E_q^\bullet) \xrightarrow{\sim} (F \otimes p_S^*(\mathcal{L}), F_q^\bullet \otimes \mathcal{L})$. By taking $\mathcal{L} = \mathcal{O}_S$, we get a natural map $\text{Bun}_{n,p} \rightarrow \underline{\text{Bun}}_{n,p}$, and $\text{Bun}_{n,p}$ is a \mathbb{G}_m -gerbe over $\underline{\text{Bun}}_{n,p}$.

Proposition 5.1. The stack $\underline{\text{Bun}}_{n,p}$ satisfies $\dim T^*\underline{\text{Bun}}_{n,p} = 2 \dim \underline{\text{Bun}}_{n,p}$.

Proof. We apply the same strategy as in [16]. The main goal is to show that the nilpotent cone $\mathcal{N}ilp := h_p^{-1}(0) \subset T^*\underline{\text{Bun}}_{n,p}$ is isotropic. Then the argument used in the proof of Propositions 7 and 8 in [16] applies here to deduce the desired equality. Let B be a Borel subgroup of $GL_n(k)$ that is contained in P . We denote by Bun_B the moduli stack of B -bundles on X . By Lemma 23 in [18], the natural map $f : \text{Bun}_B \rightarrow \underline{\text{Bun}}_{n,p}$ is surjective. In order to apply Lemma 5 in [16] to show that $\mathcal{N}ilp$ is isotropic, all we need to show is that for any $(E, E_q^\bullet, \phi) \in \mathcal{N}ilp(k)$, there exists $E_B \in \text{Bun}_B(k)$ such that $f(E_B) = (E, E_q^\bullet)$ and $f^*(\phi) = 0 \in T_{E_B}^* \text{Bun}_B$, that is, there exists a complete flag structure of E over X such that its restriction to q is compatible with the partial flag structure E_q^\bullet , and the Higgs field ϕ is nilpotent with respect to this complete flag structure. We choose a basis (e_1, e_2, \dots, e_n) of E_q such that the complete flag structure

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, e_2, \dots, e_n \rangle = E_q$$

is compatible with E_q^\bullet . Let $U = \text{Spec } A$ be an open neighborhood of q over which E and $\Omega_X^1(q)$ trivializes. Fixing such trivializations, the Higgs field ϕ corresponds to an A -linear map $A^n \rightarrow A^n$. Since $\text{res}_q(\phi)$ is nilpotent with respect to E_q^\bullet , $\phi_q(e_i)$ lies in the k -vector space spanned by e_1, e_2, \dots, e_{i-1} . Shrinking U if necessary, the basis (e_1, e_2, \dots, e_n) of E_q can be lifted to a basis $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n)$ of E over U that still satisfies $\phi(\tilde{e}_i) \in \langle \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{i-1} \rangle$. The B -reduction of E over U given by

$$0 \subset \langle \tilde{e}_1 \rangle \subset \langle \tilde{e}_1, \tilde{e}_2 \rangle \subset \dots \subset \langle \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n \rangle = E|_U$$

can be extended to a B -reduction over X since $GL_n(k)/B$ is projective. Such a B -reduction satisfies all the properties we need. ■

Remark 5.2. Over \mathbb{C} the field of complex numbers, the analogue of Proposition 5.1 was proved in [4] (see Theorem 6, 7) for a general reductive group G and parahoric P . It is not clear to the author if their arguments can be adapted to the characteristic p setting.

Now we apply Proposition 3.5 to $\underline{\text{Bun}}_{n,p}$ and get $\mathcal{D}_{\underline{\text{Bun}}_{n,p}}$. The sheaf of algebras $\mathcal{D}_{\text{Bun}_{n,p}}$ is defined to be the pull-back of $\mathcal{D}_{\underline{\text{Bun}}_{n,p}}$ to $\text{Bun}_{n,p}$.

We denote $\mathcal{Higgs}_{n,p} \times_{B_p} B_p^0$ by $\mathcal{Higgs}_{n,p}^0$ and $\mathcal{Loc}_{n,p} \times_{B_p^{(1)}} (B_p^0)^{(1)}$ by $\mathcal{Loc}_{n,p}^0$. Since $\mathcal{Higgs}_{n,p}^0$ is smooth, $\mathcal{D}_{\mathbf{Bun}_{n,p}}$ restricts to an Azumaya algebra $\mathcal{D}_{\mathbf{Bun}_{n,p}}^0$ on

$$(\mathcal{Higgs}_{n,p}^0)^{(1)} \subseteq \mathcal{Higgs}_{n,p}^{(1)} \cong T^*\mathbf{Bun}_{n,p}^{(1)}.$$

Now we are in the position to state the main theorem of the paper:

Theorem 5.3. There exists an $\mathcal{O}_{\mathcal{Loc}_{n,p}^0} \boxtimes \mathcal{D}_{\mathbf{Bun}_{n,p}}^0$ -module \mathcal{P} , such that the Fourier–Mukai transform $\Phi_{\mathcal{P}}$ with kernel \mathcal{P} induces an equivalence

$$D^b(\mathrm{QCoh}(\mathcal{Loc}_{n,p}^0)) \xrightarrow{\cong} D^b(\mathcal{D}_{\mathbf{Bun}_{n,p}}^0\text{-mod})$$

between the bounded derived category of quasi-coherent sheaves on $\mathcal{Loc}_{n,p}^0$ and the bounded derived category of $\mathcal{D}_{\mathbf{Bun}_{n,p}}^0$ -modules.

5.2 The tensor structure on $\mathcal{D}_{\mathbf{Bun}_{n,p}}^0$

Recall that in Section 2, we constructed a family of curves $\tilde{\Sigma} \rightarrow B_p^0$ such that

$$\mathcal{Higgs}_{n,p}^0 \cong \mathrm{Pic}(\tilde{\Sigma}/B_p^0).$$

In this subsection we show that there is a natural tensor structure on the Azumaya algebra $\mathcal{D}_{\mathbf{Bun}_{n,p}}^0$, in the sense of [24] (see Section 3.5). We denote $\tilde{\Sigma} \setminus \tilde{\pi}^{-1}(B_p^0 \times q)$ by $\tilde{\Sigma}^0$, where $\tilde{\pi} : \tilde{\Sigma} \rightarrow B_p^0 \times X$ is the universal spectral cover. Let i be the natural inclusion

$$i : \tilde{\Sigma}^0 \rightarrow B_p^0 \times T^*X.$$

We denote by a the morphism

$$a : \tilde{\Sigma} \times_{B_p^0} \mathrm{Pic}(\tilde{\Sigma}/B_p^0) \rightarrow \mathrm{Pic}(\tilde{\Sigma}/B_p^0)$$

that maps (\tilde{x}, L) to $L(\tilde{x})$. We denote by κ the Abel–Jacobi map

$$\kappa : \tilde{\Sigma}/B_p^0 \rightarrow \mathrm{Pic}(\tilde{\Sigma}/B_p^0)$$

that maps $\tilde{x} \in \tilde{\Sigma}/B_p^0$ to $\mathcal{O}_{\tilde{\Sigma}}(\tilde{x})$.

Let θ_X be the tautological 1-form on T^*X and $\theta_{\mathbf{Bun}_{n,p}}$ the tautological 1-form on $T^*\mathbf{Bun}_{n,p}$. By similar arguments as in Theorem 4.12 in [8], we have

Proposition 5.4. When restricted to $\tilde{\Sigma}^0 \times_{B_p^0} \mathrm{Pic}(\tilde{\Sigma}/B_p^0)$,

$$i^*\theta_X \boxtimes \theta_{\mathbf{Bun}_{n,p}} = a^*\theta_{\mathbf{Bun}_{n,p}}|_{\tilde{\Sigma}^0 \times_{B_p^0} \mathrm{Pic}(\tilde{\Sigma}/B_p^0)}.$$

In particular,

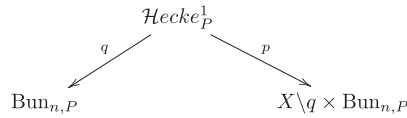
$$i^*\theta_X = \kappa^*\theta_{\mathbf{Bun}_{n,p}}|_{\tilde{\Sigma}^0}.$$

For the proof of Proposition 5.4, we consider the moduli stack \mathcal{Hecke}_1^p of quadruples

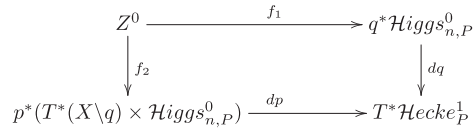
$$((E, E_q^*), (F, F_q^*), x, i : E \hookrightarrow F),$$

where $x \in X \setminus q$, $(E, E_q^*), (F, F_q^*) \in \mathbf{Bun}_{n,p}$ such that F/E is the simple skyscraper sheaf at x , and the partial flag structures E_q^* and F_q^* coincide under i . By considering $\mathbf{Im}(i_x) \subset F_x$, this data is equivalent to a triple

$((F, F_q^\bullet), x, V \subset F_x)$, where V is a dimension $n - 1$ subspace of F_x . We consider the following projections:



where q maps the quadruple to (F, F_q^\bullet) and p maps the quadruple to $((E, E_q^\bullet), x)$. Both p and q are smooth. Consider the following pull-back diagram:



We define α_1 to be the map:

$$\alpha_1 = \text{pr}_2 \circ f_1 : Z^0 \longrightarrow \mathcal{H}iggs_{n,P}^0,$$

where pr_2 is the projection $q^* \mathcal{H}iggs_{n,P}^0 = \mathcal{H}ecke_p^1 \times_{q, \text{Bun}_{n,P}} \mathcal{H}iggs_{n,P}^0 \longrightarrow \mathcal{H}iggs_{n,P}^0$. Similarly, we define

$$\alpha_2 = \text{pr}_2 \circ f_2 : Z^0 \longrightarrow T^*(X \setminus q) \times \mathcal{H}iggs_{n,P}^0.$$

The stack Z^0 and the maps α_1, α_2 can be described as follows:

Lemma 5.5. The stack Z^0 is isomorphic to $\tilde{\Sigma}^0 \times_{B_p^0} \mathcal{H}iggs_{n,P}^0$. Under this isomorphism, α_1 corresponds to the addition map a , and α_2 corresponds to the product of the projection map $\tilde{\Sigma}^0 \subset T^*(X \setminus q) \times B_p^0 \xrightarrow{\text{pr}_1} T^*(X \setminus q)$ with the identity map of $\mathcal{H}iggs_{n,P}^0$.

Proof. Let $((E, E_q^\bullet), (F, F_q^\bullet), x, i : E \hookrightarrow F)$ be a k -point of $\mathcal{H}ecke_p^1$, which we denote by τ . There is a short exact sequence of cotangent spaces

$$0 \longrightarrow T_x^* X \xrightarrow{p_X} T_\tau^* \mathcal{H}ecke_p^1 \xrightarrow{\pi} T_\tau^* p_X^{-1}(x) \longrightarrow 0,$$

where p_X is the projection $\mathcal{H}ecke_p^1 \longrightarrow X$. The fiber $p_X^{-1}(x)$ classifies $(F, F_q^\bullet) \in \text{Bun}_{n,P}$ together with a subspace $V \subset F_x$ of dimension $n - 1$. Therefore, $T_\tau^* p_X^{-1}(x)$ is the subspace of twisted Higgs fields $\phi \in \Gamma(X, \mathcal{E}nd(F) \otimes \Omega_X(q + x))$ such that $\text{res}_q(\phi)$ is nilpotent with respect to the partial flag structure F_q^\bullet and $\text{res}_x(\phi)$ is nilpotent with respect to $V \subset F_x$. The composite $\pi \circ dq$ maps $(\tau, (F, F_q^\bullet, \phi_F))$ to ϕ_F , and $\pi \circ dp$ maps $(\tau, (E, E_q^\bullet, \phi_E), (x, \xi))$ to the unique extension of ϕ_E to F . Therefore, Z^0 classifies triples

$$((F, F_q^\bullet, \phi_F), x, E \subset F),$$

where $(F, F_q^\bullet, \phi_F) \in \mathcal{H}iggs_{n,P}$, $x \in X \setminus q$ such that $F/E = k_x$ and ϕ_F restricts to a twisted Higgs field on E with no pole at x . Since (F, F_q^\bullet, ϕ_F) is isomorphic to (E, E_q^\bullet, ϕ_E) away from x , they are mapped to the same point $b \in B_p^0$ under the Hitchin map. Let \mathcal{L} resp. $\mathcal{L}' \in \text{Pic}(\tilde{\Sigma}_b)$ be the invertible sheaf corresponding to (E, E_q^\bullet, ϕ_E) resp. (F, F_q^\bullet, ϕ_F) under the isomorphism in Theorem 2.14. Since $F/E = k_x$, $\mathcal{L}'/\mathcal{L} = k_{x'}$ for some $x' \in \tilde{\Sigma}_b^0$ that maps to x under the spectral cover map. Therefore having a triple $((F, F_q^\bullet, \phi_F), x, E \subset F)$ as above is equivalent to having (b, \mathcal{L}, x') , where $b \in B_p^0$, $\mathcal{L} \in \text{Pic}(\tilde{\Sigma}_b)$ and $x' \in \tilde{\Sigma}_b^0$. ■

Proof of Proposition 5.4. The goal is to show $\alpha_1^* \theta_{\text{Bun}_{n,P}} = \alpha_2^*(\theta_{\text{Bun}_{n,P}} \boxtimes \theta_X)$. Both 1-forms are equal to the pull-back of the tautological 1-form on $T^* \mathcal{H}ecke_p^1$ to Z^0 . ■

Let $\theta_{\text{Bun}_{n,P}}^0$ be the restriction of $\theta_{\text{Bun}_{n,P}}$ to $\text{Higgs}_{n,P}^0$. By Lemma 3.14 in [8], in order to construct a tensor structure on $\mathcal{D}_{\text{Bun}_{n,P}}^0$, it is enough to show that for the addition map

$$m : \text{Pic}(\tilde{\Sigma}/B_P^0) \times \text{Pic}(\tilde{\Sigma}/B_P^0) \longrightarrow \text{Pic}(\tilde{\Sigma}/B_P^0),$$

the 1-form $\theta_{\text{Bun}_{n,P}}^0$ satisfies the following equality:

$$m^* \theta_{\text{Bun}_{n,P}}^0 = \theta_{\text{Bun}_{n,P}}^0 \boxtimes \theta_{\text{Bun}_{n,P}}^0. \tag{5.1}$$

We denote by $\text{Pic}^d(\tilde{\Sigma}/B_P^0)$ the degree d component of $\text{Pic}(\tilde{\Sigma}/B_P^0)$. Since there are isomorphisms between components of $\text{Pic}(\tilde{\Sigma}/B_P^0)$ that preserve $\theta_{\text{Bun}_{n,P}}^0$, it is enough to prove the equality (5.1) for large enough d, d' and

$$m_{d,d'} : \text{Pic}^d(\tilde{\Sigma}/B_P^0) \times \text{Pic}^{d'}(\tilde{\Sigma}/B_P^0) \longrightarrow \text{Pic}^{d+d'}(\tilde{\Sigma}/B_P^0).$$

We denote by κ_d the map

$$\kappa_d : (\tilde{\Sigma}/B_P^0)^d \longrightarrow \text{Pic}^d(\tilde{\Sigma}/B_P^0)$$

that maps $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_d) \in (\tilde{\Sigma}/B_P^0)^d$ to $\mathcal{O}_{\tilde{\Sigma}}(\tilde{x}_1 + \tilde{x}_2 + \dots + \tilde{x}_d)$. For $d > 2g_{\Sigma_b} - 2$, there is an open subset of $\tilde{\Sigma}^d$ such that κ_d is smooth and dominant. Therefore, it is enough to show that

$$\kappa_{d+d'}^* \theta_{\text{Bun}_{n,P}}^0 = \kappa_d^* \theta_{\text{Bun}_{n,P}}^0 \boxtimes \kappa_{d'}^* \theta_{\text{Bun}_{n,P}}^0.$$

By Proposition 5.4, this equality holds on $(\tilde{\Sigma}^0)^{d+d'}$, therefore it holds on $\tilde{\Sigma}^{d+d'}$.

5.3 Torsor structure on $\mathcal{L}oc_{n,P}^0$

By Corollary 4.5, Proposition 4.18, Remark 4.2, Remark 2.6, and Theorem 2.14, we have the following:

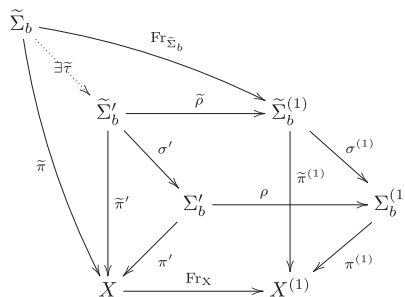
Proposition 5.6.

(1) The isomorphism in Corollary 4.5 induces

$$\mathcal{L}oc_{n,P}^0 \cong \mathcal{L}oc_{\mathcal{N}}^r \times_{B^{(1)}} (B_P^0)^{(1)} \times^{\text{Pic}(\Sigma^{(1)}/(B_P^0)^{(1)})} (\text{Higgs}_{n,P}^0)^{(1)},$$

(2) the action of $\text{Pic}(\tilde{\Sigma}^{(1)}/(B_P^0)^{(1)})$ on $(\text{Higgs}_{n,P}^0)^{(1)}$ gives $\mathcal{L}oc_{n,P}^0$ the structure of a $\text{Pic}(\tilde{\Sigma}^{(1)}/(B_P^0)^{(1)})$ -torsor.

Let S be a k -scheme. Let b be an S -point of $(B_P^0)^{(1)}$. Consider the following commutative diagram:



Here $\tilde{\Sigma}'_b := X \times_{X^{(1)}} \tilde{\Sigma}_b^{(1)}$. There exists a unique map from $\tilde{\Sigma}_b$ to $\tilde{\Sigma}'_b$ that makes the diagram commute. We call this map $\tilde{\tau}$. Note that $\tilde{\tau}$ is finite since $\text{Fr}_{\tilde{\Sigma}_b}$ is finite and $\bar{\rho}$ is separated.

Let $(E, E_q^\bullet, \nabla_E)$ be an S -point of $\mathcal{L}oc_{n,p}$ such that $h'(E, \nabla_E) = b$. Let $E' \in \mathbf{Coh}(\Sigma'_b)$ be the spectral sheaf. We associate with it an invertible sheaf $\tilde{E}' \in \mathbf{Coh}(\tilde{\Sigma}'_b)$ that satisfies $\sigma'_*(\tilde{E}') = E'$ as follows. Let $(E_1, \nabla_1) \in \mathcal{L}oc'_{n,r}$ and $(E_2, \phi_2) \in (\mathit{Higgs}^0_{n,p})^{(1)}$ such that $h'(E_1, \nabla_1) = h^{(1)}(E_2, \phi_2) = b$ and they are mapped to (E, ∇_E) under the isomorphism in Proposition 5.6(1). Let $E'_1 \in \mathbf{Coh}(\Sigma'_b)$ be the spectral sheaf of (E_1, ∇_1) and $\mathcal{L} \in \mathbf{Coh}(\Sigma_b^{(1)})$ the spectral sheaf of (E_2, ϕ_2) , then we have $E' \cong E'_1 \otimes \rho^* \mathcal{L}$. By Theorem 2.14, there exists a unique invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{\Sigma}_b^{(1)}$ such that $\sigma_*^{(1)} \tilde{\mathcal{L}} = \mathcal{L}$. Now we define

$$\tilde{E}' = \sigma'^*(E'_1) \otimes \tilde{\rho}^* \tilde{\mathcal{L}}.$$

This construction does not depend on the choice of (E_1, ∇_1) and (E_2, ϕ_2) .

Lemma 5.7. The flat connection ∇_{can} on $\tilde{\tau}^* \tilde{E}' = \mathcal{O}_{\tilde{\Sigma}_b} \otimes_{\mathcal{O}_{\tilde{\Sigma}'_b}} \tilde{E}'$ defined by

$$\nabla_{can}(\partial)(f \otimes s) = \partial(f) \otimes s + f \nabla_E(d\tilde{\pi}^*(\partial))(s)$$

for any $\partial \in \mathcal{T}_U, f \in \mathcal{O}_U, s \in \tilde{E}'(\tilde{\tau}(U))$ and open $U \subseteq \tilde{\Sigma}_b$, has no singularities. Here $d\tilde{\pi}^*$ is the tangent map $\mathcal{T}_{\tilde{\Sigma}_b} \rightarrow \tilde{\pi}^* \mathcal{T}_X$.

Proof. Since $b \in B_p^0, \tilde{\pi}^{-1}(q)$ consists of r points q_1, q_2, \dots, q_r . The only places that ∇_{can} might have singularities are q_1, q_2, \dots, q_r . Note that $\tilde{\pi} : \tilde{\Sigma}_b \rightarrow X$ has ramification index λ_i at q_i . Let t be a local parameter at $q_i \in \tilde{\Sigma}_b$ and x a local parameter at $q \in X$ such that $\tilde{\pi}^*(x) = t^{\lambda_i}$. Let U be an open neighborhood of q_i . Since (E, ∇_E) has nilpotent residue at q , there exists a positive integer N such that for any $m \geq N$ and $s \in \tilde{E}'(\tilde{\tau}(U)), (\nabla_E(x\partial_x))^m(s) \in x\tilde{E}'$. We compute that

$$(\nabla_{can}(t\partial_t))^{pN}(f \otimes s) = (t\partial_t)^{pN}(f) \otimes s + f(\nabla_E(d\tilde{\pi}^*(t\partial_t)))^{pN}(s).$$

Since $(t\partial_t)^{pN}(f) \in t\mathcal{O}_{\tilde{\Sigma}_b}$ and $(\nabla_E(d\tilde{\pi}^*(t\partial_t)))^{pN}(s) = (\nabla_E(\lambda_i x \partial_x))^{pN}(s) \in x\tilde{E}'$, the sum lies in $t\mathcal{O}_{\tilde{\Sigma}_b} \otimes \tilde{E}'$. Therefore, the residue of $(\tilde{\tau}^* \tilde{E}', \nabla_{can})$ at q' is nilpotent. But since $\tilde{\tau}^* \tilde{E}'$ is an invertible sheaf on $\tilde{\Sigma}_b$, the residue must be zero, so the flat connection ∇_{can} has no singularity at q_i . ■

We denote by $\tilde{\theta}_b$ the restriction of $\kappa^* \theta_{\mathbf{Bun}_{n,p}}$ to $\tilde{\Sigma}_b$. Then we have the following:

Lemma 5.8. The connection $(\tilde{\tau}^* \tilde{E}', \nabla_{can})$ is mapped to $\tilde{\theta}_b^{(1)}$ under the Hitchin map h' .

Proof. Let $p' \in \tilde{\Sigma}_b$ such that $\tilde{\pi}(p') = p \neq q$. Let x be a local parameter at p . We denote ∂_x by y , so near p the spectral curve $\tilde{\Sigma}_b$ is the vanishing scheme of a polynomial of the form

$$y^n + b_1(x)y^{n-1} + \dots + b_{n-1}(x)y + b_n(x).$$

Since $\tilde{\Sigma}_b$ is smooth, $y - y(p')$ is a local parameter of $\tilde{\Sigma}_b$ at p' . Since $\theta_x = ydx$, we have

$$i^* \theta_x = \partial_y(x)ydy \text{ and } (i^* \theta_x)^{(1)} = (\partial_y(x))^p y^p dy^p.$$

Let U be an open neighborhood of p' . For $f \in \mathcal{O}_U$ and $s \in \tilde{E}'(\tilde{\tau}(U))$, we compute

$$\begin{aligned} \partial_y^p(f \otimes s) &= f \otimes (d\pi(\partial_y))^p(s) \\ &= f \otimes (\partial_y(x))^p \partial_x^p(s) \\ &= f \otimes (\partial_y(x))^p y^p(s) \\ &= \langle \partial_y^p, (i^* \theta_x)^{(1)} \rangle (f \otimes s). \end{aligned}$$

Therefore, $h'(\tilde{\tau}^* \tilde{E}', \nabla_{can})$ is equal to $(i^* \theta_x)^{(1)}$ when restricted to $(\tilde{\Sigma}_b^0)^{(1)}$. By Proposition 5.4, $h'(\tilde{\tau}^* \tilde{E}', \nabla_{can}) = \tilde{\theta}_b^{(1)}$. ■

Now recall that for a smooth variety Y over k , \mathcal{D}_Y is the Azumaya algebra on $T^*Y^{(1)}$ that satisfies $\mathrm{Fr}_*(\mathcal{D}_Y) = \pi_*^{(1)}(\mathcal{D}_Y)$.

Proposition 5.9. Let b be an S -point of $(\mathbb{B}_p^0)^{(1)}$. The construction of $(\tilde{\tau}^*\tilde{E}', \nabla_{can})$ induces an isomorphism of stacks between $(\mathcal{L}oc_{n,p}^0)_b := \mathcal{L}oc_{n,p}^0 \times_{(\mathbb{B}_p^0)^{(1)}, b} S$ and the stack of splittings of the Azumaya algebra $(\tilde{\theta}_b^{(1)})^*\mathcal{D}_{\tilde{\Sigma}_b}$. Here we think of $\tilde{\theta}_b^{(1)}$ as a map

$$\tilde{\theta}_b^{(1)} : \tilde{\Sigma}_b^{(1)} \longrightarrow T^*\tilde{\Sigma}_b^{(1)}.$$

Proof. Both stacks are $\mathrm{Pic}(\tilde{\Sigma}_b^{(1)})$ -torsors. Since $\tilde{\tau}^*$ is compatible with the $\mathrm{Pic}(\tilde{\Sigma}_b^{(1)})$ -actions, it induces an isomorphism between those two stacks. ■

5.4 Fourier–Mukai transforms on commutative group stacks

In this subsection, we review the Fourier–Mukai transforms on commutative group stacks, following [8]. Let k be an algebraically closed field. Let \mathcal{B} be an irreducible k -scheme that is locally of finite type. Let \mathcal{G} be a commutative group stack locally of finite type over \mathcal{B} . The dual commutative group stack \mathcal{G}^\vee classifies 1-morphisms of group stacks from \mathcal{G} to $B\mathbb{G}_m$. The main examples we are going to consider are:

Examples 5.10.

- (1) $\mathcal{G} = \mathbb{Z}$, $\mathcal{G}^\vee = B\mathbb{G}_m$,
- (2) $\mathcal{G} = B\mathbb{G}_m$, $\mathcal{G}^\vee = \mathbb{Z}$,
- (3) $\mathcal{G} = \mathbb{Z}_n$, $\mathcal{G}^\vee = B\mu_n$. Here $\mu_n = \mathrm{Spec}(\mathbb{Z}[x]/(x^n - 1))$,
- (4) $\mathcal{G} = \mu_n$, $\mathcal{G}^\vee = B\mathbb{Z}_n$,
- (5) $\mathcal{G} = A$ is an abelian scheme, then $\mathcal{G}^\vee = A^\vee$ is the dual abelian scheme.

By the definition of \mathcal{G}^\vee , there is a universal \mathbb{G}_m -torsor on $\mathcal{G} \times \mathcal{G}^\vee$, which gives rise to the Poincaré line bundle $\mathcal{P}_{\mathcal{G}}$.

In [8], a commutative group stack \mathcal{G} is called very nice, if locally in smooth topology, \mathcal{G} is a finite product of stacks in the examples above. Under this assumption, the natural map $\mathcal{G} \rightarrow \mathcal{G}^{\vee\vee}$ is an isomorphism. Therefore, there is another Poincaré line bundle $\mathcal{P}_{\mathcal{G}^\vee}$ on $\mathcal{G}^\vee \times \mathcal{G}$.

Theorem 5.11 (cf. [8] Theorem 2.7). Let \mathcal{G} be a very nice commutative group stack and let \mathcal{G}^\vee be its dual. Then the Fourier–Mukai functor $\Phi_{\mathcal{P}_{\mathcal{G}}}$ with kernel $\mathcal{P}_{\mathcal{G}}$ induces an equivalence of derived categories

$$D^b(\mathrm{QCoh}(\mathcal{G})) \xrightarrow{\sim} D^b(\mathrm{QCoh}(\mathcal{G}^\vee)).$$

Now Let $\tilde{\mathcal{G}}$ and \mathcal{G} be very nice commutative group stacks that fit into a short exact sequence of group stacks:

$$0 \longrightarrow B\mathbb{G}_m \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{G} \longrightarrow 0.$$

By taking dual, we get another short exact sequence:

$$0 \longrightarrow \mathcal{G}^\vee \longrightarrow \tilde{\mathcal{G}}^\vee \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0.$$

Let $\tilde{\mathcal{G}}_1^\vee = \pi^{-1}(1)$.

Remark 5.12. Note that $\tilde{\mathcal{G}}_1^\vee$ classifies maps of group stacks $\tilde{\mathcal{G}} \rightarrow B\mathbb{G}_m$ such that the composition

$$B\mathbb{G}_m \longrightarrow \tilde{\mathcal{G}} \longrightarrow B\mathbb{G}_m$$

is the identity. Such a map gives a splitting of $\tilde{\mathcal{G}}$ considered as a \mathbb{G}_m -gerbe over \mathcal{G} .

Recall that the $B\mathbb{G}_m$ -action on $\tilde{\mathcal{G}}$ gives a decomposition

$$D^b(\mathrm{QCoh}(\tilde{\mathcal{G}})) \cong \prod_{n \in \mathbb{Z}} D^b(\mathrm{QCoh}(\tilde{\mathcal{G}}))_n.$$

Proposition 5.13 (cf. [1] Proposition A.7 and [8] Proposition 2.9). The Fourier–Mukai functor $\Phi_{\mathcal{P}_{\tilde{\mathcal{G}}}}$ restricts to an equivalence of derived categories

$$D^b(\mathrm{QCoh}(\tilde{\mathcal{G}}_1^\vee)) \xrightarrow{\sim} D^b(\mathrm{QCoh}(\tilde{\mathcal{G}}))_1.$$

5.5 Proof of Theorem 5.3

Let $\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}$ be the \mathbb{G}_m -gerbe (defined in Subsection 3.4) over $\mathcal{Higgs}_{n,p}^0 \cong \mathrm{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$ that classifies splittings of the Azumaya algebra $\mathcal{D}_{\mathrm{Bun}_{n,p}}^0$. As discussed in Section 3.5, the tensor structure on $\mathcal{D}_{\mathrm{Bun}_{n,p}}^0$ gives $\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}$ the structure of a commutative group stack, and it fits into a short exact sequence

$$0 \longrightarrow B\mathbb{G}_m \longrightarrow \mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0} \longrightarrow \mathrm{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)}) \longrightarrow 0.$$

By taking dual, we get another short exact sequence:

$$0 \longrightarrow \mathrm{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)}) \longrightarrow \mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}^\vee \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0.$$

Proposition 5.14. $(\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}^\vee)_1 := \pi^{-1}(1)$ is isomorphic to $\mathcal{L}oc_{n,p}^0$ as $\mathrm{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$ -torsors.

Proof. It is enough to construct a morphism from $(\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}^\vee)_1$ to $\mathcal{L}oc_{n,p}^0$ that is compatible with the $\mathrm{Pic}(\tilde{\Sigma}^{(1)}/(B_p^0)^{(1)})$ -actions. Let S be a k -scheme. Let b be an S -point of $(B_p^0)^{(1)}$. By Remark 5.12, an S -point of $(\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}^\vee)_1$ lying above b gives a splitting of the Azumaya algebra $\mathcal{D}_{\mathrm{Bun}_{n,p}}^0|_{\mathrm{Pic}(\tilde{\Sigma}_b^{(1)})}$. Pulling back along the Abel–Jacobi map

$$\kappa^{(1)} : \tilde{\Sigma}_b^{(1)} \longrightarrow \mathrm{Pic}(\tilde{\Sigma}_b^{(1)}),$$

by Corollary 3.3(2) and Proposition 5.4, such a splitting gives a splitting of the Azumaya algebra $(\tilde{\theta}_b^{(1)})^* \mathcal{D}_{\tilde{\Sigma}_b}$, which in turn gives a point of $(\mathcal{L}oc_{n,p})_b$ by Proposition 5.9. This map is clearly compatible with the $\mathrm{Pic}(\tilde{\Sigma}_b^{(1)})$ -actions. ■

Now let $\mathcal{P}_{\mathcal{Y}^\vee}$ be the Poincaré line bundle on $\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}^\vee \times \mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}_{n,p}}^0}$. By Lemma 3.6 and Proposition 5.14, $\mathcal{P}_{\mathcal{Y}^\vee}$ restricts to an $\mathcal{O}_{\mathcal{L}oc_{n,p}^0} \boxtimes \mathcal{D}_{\mathrm{Bun}_{n,p}}^0$ -module \mathcal{P} . By Proposition 5.13, the Fourier–Mukai transform with kernel \mathcal{P}

$$\Phi_{\mathcal{P}} : D^b(\mathrm{QCoh}(\mathcal{L}oc_{n,p}^0)) \longrightarrow D^b(\mathcal{D}_{\mathrm{Bun}_{n,p}}^0\text{-mod})$$

induces an equivalence of derived categories. This completes the proof of Theorem 5.3.

5.6 The Hecke functor

Recall that we define $\mathcal{H}ecke_1^1$ to be the moduli stack of quadruples

$$((E, E_q^*), (F, F_q^*), \chi, i : E \hookrightarrow F),$$

where $x \in X \setminus q$, $(E, E_q^\bullet), (F, F_q^\bullet) \in \mathbf{Bun}_{n,P}$ such that F/E is the simple skyscraper sheaf at x , and the partial flag structures E_q^\bullet and F_q^\bullet coincide under i . We consider the following projections:

$$\begin{array}{ccc} & \text{Hecke}_P^1 & \\ q \swarrow & & \searrow p \\ \text{Bun}_{n,P} & & \text{Bun}_{n,P} \times X \setminus q \end{array}$$

where q maps the quadruple to (F, F_q^\bullet) and p maps the quadruple to $((E, E_q^\bullet), x)$. The Hecke functor H_P^0 is defined by

$$H_P^0 : D^b(\mathcal{D}_{\text{Bun}_{n,P}}^0\text{-mod}) \longrightarrow D^b(\mathcal{D}_{\text{Bun}_{n,P}}^0 \boxtimes \mathcal{D}_{X \setminus q}\text{-mod})$$

$$\mathcal{M} \mapsto p_* q^! \mathcal{M}.$$

Let \mathcal{E} be the universal $\mathcal{O}_{\mathcal{L}oc_{n,P}^0} \boxtimes \mathcal{D}_{X \setminus q}$ -module. We define another functor W_P^0 :

$$W_P^0 : D^b(\mathcal{O}_{\mathcal{L}oc_{n,P}^0}\text{-mod}) \longrightarrow D^b(\mathcal{O}_{\mathcal{L}oc_{n,P}^0} \boxtimes \mathcal{D}_{X \setminus q}\text{-mod})$$

$$\mathcal{F} \mapsto p_1^* \mathcal{F} \otimes \mathcal{E},$$

where p_1 is the projection $\mathcal{L}oc_n \times X \rightarrow \mathcal{L}oc_n$. Let $\Phi_{\mathcal{P}, X \setminus q}$ be the Fourier–Mukai equivalence induced by the pull-back of \mathcal{P} :

$$\Phi_{\mathcal{P}, X \setminus q} : D^b(\mathcal{O}_{\mathcal{L}oc_{n,P}^0} \boxtimes \mathcal{D}_{X \setminus q}\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{D}_{\text{Bun}_{n,P}}^0 \boxtimes \mathcal{D}_{X \setminus q}\text{-mod}).$$

then we have:

Theorem 5.15. There is an isomorphism of functors:

$$H_P^0 \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{P}, X \setminus q} \circ W_P^0.$$

Proof. The proof is similar to the proof of Theorem 5.4 in [8]. Since the equivalence $\Phi_{\mathcal{P}} : D^b(\mathcal{O}_{\mathcal{L}oc_{n,P}^0}\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{D}_{\text{Bun}_{n,P}}^0\text{-mod})$ is the Fourier–Mukai functor with kernel the $\mathcal{D}_{\text{Bun}_{n,P}}^0 \boxtimes \mathcal{O}_{\mathcal{L}oc_{n,P}^0}$ -module \mathcal{P} , it is enough to show that $H_P^0(\mathcal{P})$ and $W_P^0(\mathcal{P})$ are isomorphic as $\mathcal{D}_{\text{Bun}_{n,P}}^0 \boxtimes \mathcal{D}_{X \setminus q} \boxtimes \mathcal{O}_{\mathcal{L}oc_{n,P}^0}$ -modules. Recall that in the proof of Proposition 5.4, we considered the pull-back diagram

$$\begin{array}{ccc} Z^0 & \xrightarrow{f_1} & q^* \text{Higgs}_{n,P}^0 \\ \downarrow f_2 & & \downarrow dq \\ p^*(\text{Higgs}_{n,P}^0 \times T^*(X \setminus q)) & \xrightarrow{dp} & T^* \text{Hecke}_P^1 \end{array}$$

and two maps $\alpha_1 = \text{pr}_2 \circ f_1 : Z^0 \rightarrow \text{Higgs}_{n,P}^0$ and $\alpha_2 = \text{pr}_2 \circ f_2 : Z^0 \rightarrow \text{Higgs}_{n,P}^0 \times T^*(X \setminus q)$. Since $\alpha_1^* \theta_{\text{Bun}_{n,P}} = \alpha_2^*(\theta_{\text{Bun}_{n,P}} \boxtimes \theta_X)$, we have a canonical equivalence of Azumaya algebras by Corollary 3.3 (2):

$$(\alpha_1^{(1)})^* \mathcal{D}_{\text{Bun}_{n,P}}^0 \sim (\alpha_2^{(1)})^* (\mathcal{D}_{\text{Bun}_{n,P}}^0 \boxtimes \mathcal{D}_{X \setminus q}). \tag{5.2}$$

For any $\mathcal{M} \in \mathcal{D}_{\text{Bun}_{n,P}}^0\text{-mod}$, $H_P^0(\mathcal{M})$ can be obtained by pulling-back along $\alpha_1^{(1)}$, applying equivalence (5.2), then pushing-forward along $\alpha_2^{(1)}$. For any $\sigma \in \mathcal{L}oc_{n,P}^0$, the $\mathcal{D}_{\text{Bun}_{n,P}}^0$ -module \mathcal{P}_σ is a splitting of the Azumaya

algebra $\mathcal{D}_{\text{Bun}_{n,P}^0|_{\text{Pic}(\tilde{\Sigma}_b^{(1)})}}$ that is compatible with the tensor structure defined in Section 5.2. There is a canonical equivalence of Azumaya algebras

$$(\kappa_b^{(1)})^* \mathcal{D}_{\text{Bun}_{n,P}^0|_{\text{Pic}(\tilde{\Sigma}_b^{(1)})}} \sim (i_b^{(1)})^* \mathcal{D}_{X \setminus q} \quad (5.3)$$

induced by the equality in Proposition 5.4, where $b = h'(\sigma)$, κ_b is the Abel–Jacobi map $\tilde{\Sigma}_b \rightarrow \text{Pic}(\tilde{\Sigma}_b)$ and i_b is the inclusion $\tilde{\Sigma}_b^0 \subset T^*(X \setminus q)$. The $\mathcal{D}_{X \setminus q}$ -module \mathcal{E}_σ can be obtained from \mathcal{P}_σ by pulling-back along κ_b and applying equivalence (5.3). Since the stack Z^0 is isomorphic to $\tilde{\Sigma}^0 \times_{\text{B}\mathbb{G}} \mathcal{Higgs}_{n,P}^0$ and α_1 corresponds to the addition map a , we have $H_P^0(\mathcal{P}_\sigma) \cong \mathcal{P}_\sigma \boxtimes \mathcal{E}_\sigma$, which is what we wish to show. ■

Now let (E, ∇) be a k -point of $\mathcal{L}oc_{n,P}^0$. We denote by $\mathcal{M}_{E,\nabla}$ the image of (E, ∇) under $\Phi_{\mathcal{P}}$. By Theorem 5.15, $\mathcal{M}_{E,\nabla}$ satisfies

$$H_P^0(\mathcal{M}_{E,\nabla}) \cong \mathcal{M}_{E,\nabla} \boxtimes E.$$

Funding

This work was supported by the NSF [DMS-1502125 to S.S.]; and the European Union’s Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement [101034413 to S.S.].

Acknowledgments

I would like to thank my advisor Tom Nevins for many helpful discussions on this subject and for his comments on this paper. I would like to thank Christopher Dodd, Michael Groechenig, and Tamas Hausel for helpful conversations. I would like to thank Tsao-Hsien Chen for useful comments on an earlier version of this paper.

References

1. Arinkin, D. “Appendix to [14].”
2. Arinkin, D. and D. Gaiitsgory. “Singular support of coherent sheaves, and the geometric Langlands conjecture.” *Selecta Math.* **21**, no. 1 (2015): 1–199. <https://doi.org/10.1007/s00029-014-0167-5>.
3. Baraglia, D. and M. Kamgarpour. “On the image of the parabolic Hitchin map.” *Q. J. Math.* **69**, no. 2 (2018): 681–708. <https://doi.org/10.1093/qmath/hax055>.
4. Baraglia, D., M. Kamgarpour, and R. Varma. “Complete integrability of the parahoric Hitchin system.” (2017): Preprint, arXiv:1608.05454.
5. Beauville, A. and Y. Laszlo. “Un lemme de descente.” *C. R. Acad. Sci. Paris Sér. I Math.* **320**, no. 3 (1995): 335–40.
6. Beauville, A., M. S. Narasimhan, and S. Ramanan. “Spectral curves and the generalised theta divisor.” *J. Reine Angew. Math.* **398** (1989): 169–79.
7. Beilinson, A. and V. Drinfeld. “Quantization of Hitchin’s integrable system and Hecke eigensheaves.” (1991): Preprint.
8. Bezrukavnikov, R. and A. Braverman. “Geometric Langlands correspondence for $\{D\}$ -modules in prime characteristic: the $GL(n)$ case.” *Pure Appl. Math. Q.* **3** (2007): 153–79. <https://doi.org/10.4310/PAMQ.2007.v3.n1.a5>.
9. Bezrukavnikov, R., I. Mirković, D. Rumynin, and S. Riche. “Localization of modules for a semisimple Lie algebra in prime characteristic.” *Ann. Math.* **167**, no. 3 (2008): 945–91. <https://doi.org/10.4007/annals.2008.167.945>.
10. Chen, T.-H. and X. Zhu. “Non-abelian Hodge theory for curves in characteristic p .” *Geom. Funct. Anal.* **25** (2015): 1706–33. <https://doi.org/10.1007/s00039-015-0343-6>.
11. Chen, T.-H. and X. Zhu. “Geometric Langlands in prime characteristic.” *Compositio Math.* **153** (2017): 395–452. <https://doi.org/10.1112/S0010437X16008113>.
12. Cumino, C., S. Greco, and M. Manaresi. “An axiomatic approach to the second theorem of Bertini.” *J. Algebra* **98**, no. 1 (1986): 153–79.

13. Deligne, P., J. Milne, A. Ogus, K. Y. Shih, P. Deligne, and J. S. Milne. "Tannakian categories." In *Hodge Cycles, Motives, and Shimura Varieties*. Springer Lecture Notes in Mathematics 900. 101–228. Heidelberg: Springer, 1982.
14. Donagi, R. and T. Pantev. "Torus fibrations, gerbes, and duality (with an appendix by Dmitry Arinkin)." *Mem. Amer. Math. Soc.* **193**, no. 901 (2008).
15. Frenkel, E. "Ramifications of the geometric Langlands Program." In *Proceedings of the CIME Summer School "Representation Theory and Complex Analysis"*. Venice (2004).
16. Ginzburg, V. "The global nilpotent variety is Lagrangian." *Duke Math. J.* **109**, no. 3 (2001): 511–9. <https://doi.org/10.1215/S0012-7094-01-10933-2>.
17. Groechenig, M. "Moduli of flat connections in positive characteristic." *Math. Res. Lett.* **23** (2016): 989–1047. <https://doi.org/10.4310/MRL.2016.v23.n4.a3>.
18. Heinloth, J. "Uniformization of $\{G\}$ -bundles." *Math. Ann.* **347** (2010): 499–528. <https://doi.org/10.1007/s00208-009-0443-4>.
19. Hitchin, N. "Stable bundles and integrable systems." *Duke Math. J.* **54**, no. 1 (1987): 91–114.
20. Katz, N. "Nilpotent connections and the monodromy theorem: applications of a result of Turrittin." *Publ. Math. Inst. Hautes Études Sci.* **39** (1970): 175–232. <https://doi.org/10.1007/BF02684688>.
21. Laszlo, Y. and C. Pauly. "On the Hitchin morphism in positive characteristic." *Internat. Math. Res. Notices* **3** (2001): 129–43.
22. Mehta, V. and C. Seshadri. "Moduli of vector bundles on curves with parabolic structures." *Math. Ann.* **248** (1980): 205–39. <https://doi.org/10.1007/BF01420526>.
23. Nevins, T. "Mirabolic Langlands duality and the quantum Calogero–Moser systems." *Transform. Groups* **14** (2009): 931–83. <https://doi.org/10.1007/s00031-009-9068-7>.
24. Ogus, A. and V. Vologodsky. "Nonabelian Hodge theory in characteristic p ." *Publ. Math. Inst. Hautes Études Sci.* **106** (2007): 1–138. <https://doi.org/10.1007/s10240-007-0010-z>.
25. Scheinost, P. and M. Schottenloher. "Metaplectic quantization of the moduli spaces of flat and parabolic bundles." *J. Reine Angew. Math.* **466** (1995): 145–219.
26. Simpson, C. T. "Harmonic bundles on noncompact curves." *J. Amer. Math. Soc.* **3**, no. 3 (1990): 713–70. <https://doi.org/10.1090/S0894-0347-1990-1040197-8>.
27. Su, X., X. Wen and B. Wang. "Parabolic Hitchin maps and their generic fibers." *Math. Z.* **301** (2022): 343–72. <https://doi.org/10.1007/s00209-021-02896-3>.
28. Travkin, R. "Quantum geometric Langlands in positive characteristic: the GL_N case." *Duke Math. J.* **165**, no. 7 (2016): 1283–361.
29. Yun, Z. "Global Springer theory." *Adv. Math.* **228** (2011): 266–328. <https://doi.org/10.1016/j.aim.2011.05.012>.