



Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Optimal lower bound on eigenvector overlaps for non-Hermitian random matrices



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ARTICLE INFO

ABSTRACT

Article history:

Received 26 January 2023

Accepted 26 April 2024

Available online 7 May 2024

Communicated by Benjamin Schlein

MSC:

60B20

15B52

65F22

Keywords:

Eigenvalue condition number

Non-Hermitian perturbation theory

Quantum unique ergodicity

We consider large non-Hermitian $N \times N$ matrices with an additive independent, identically distributed (i.i.d.) noise for each matrix elements. We show that already a small noise of variance $1/N$ completely thermalises the bulk singular vectors, in particular they satisfy the strong form of Quantum Unique Ergodicity (QUE) with an optimal speed of convergence. In physics terms, we thus extend the Eigenstate Thermalisation Hypothesis, formulated originally by Deutsch [34] and proven for Wigner matrices in [23], to arbitrary non-Hermitian matrices with an i.i.d. noise. As a consequence we obtain an optimal *lower* bound on the diagonal overlaps of the corresponding non-Hermitian eigenvectors. This quantity, also known as the (square of the) eigenvalue condition number measuring the sensitivity of the eigenvalue to small perturbations, has notoriously escaped rigorous treatment beyond the explicitly computable Ginibre ensemble apart from the very recent *upper* bounds given in [7] and [45]. As a key tool, we develop a new systematic decomposition of general observables in random matrix theory that governs the

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¹ Supported by ERC Advanced Grant “RMTBeyond” No. 101020331.

² Supported by the SNSF Ambizione Grant PZ00P2_209089.

size of products of resolvents with deterministic matrices in between.

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1. Introduction

Traditional random matrix theory focuses on statistics of eigenvalues, where spectacular universality phenomena arise: the local spectral statistics tend to become universal as the dimension goes to infinity with new distributions arising; most importantly the celebrated *Wigner-Dyson-Mehta bulk statistics* and the *Tracy-Widom edge statistics* in the *Hermitian* spectrum and the *Ginibre statistics* in the *non-Hermitian* spectrum. More recently eigenvectors of *Hermitian* ensembles received considerable attention. They also become universal, albeit in a more conventional way: they tend to be entirely randomised, i.e. Haar distributed [14,17,49,11,25,27,10]. In this paper we study two related questions: how do eigenvectors and singular vectors of a typical *non-Hermitian* random matrix in high dimension look like? To answer them, we introduce a new decomposition of general observables that identifies correlations of the Hermitised resolvents as *entire matrices* at different spectral parameters. This captures correlations of the singular vectors well beyond correlations of traces of resolvents that govern only the singular *values*. Somewhat surprisingly, we are then able to transfer information on singular vectors to the non-Hermitian eigenvectors.

1.1. Non-Hermitian eigenvector overlaps

To be specific, we consider non-Hermitian $N \times N$ matrices of the form $\Lambda + X$, where Λ is an arbitrary deterministic matrix and X is random. We assume that the norm of Λ is bounded independently of N and X has independent, identically distributed (i.i.d.) centred matrix elements with variance $\mathbf{E} |x_{ij}|^2 = \frac{1}{N}$ with some further moment conditions. This normalisation guarantees that $\|X\| \leq 2 + o(1)$ and the spectrum of X lies essentially in the unit disk (*circular law*) with very high probability, hence Λ and X remain of comparable size as N increases. Note that X perturbs each matrix elements of Λ by a small random amount of order $1/\sqrt{N}$, however the spectra of Λ and $\Lambda + X$ substantially differ.

The analysis of non-Hermitian random matrices is typically much harder than that of the Hermitian ones. Non-Hermitian matrices have two different sets of spectral data: eigenvalues/vectors and singular values/vectors which cannot be directly related. In particular, the study of singular vectors and eigenvectors substantially differ: while singular vectors can still be understood from a Hermitian theory, there is no such route for eigenvectors. Unlike for non-Hermitian *eigenvalues*, where Girko's formula translates their linear statistics into a Hermitian problem, no similar "Hermitisation" relation is known

for non-Hermitian *eigenvectors*. Furthermore, left and right eigenvectors differ and their relation is very delicate. Assuming that each eigenvalue μ_i of $\Lambda + X$ is simple, we denote the corresponding left and right eigenvectors by $\mathbf{l}_i, \mathbf{r}_i$, i.e.

$$(\Lambda + X)\mathbf{r}_i = \mu_i\mathbf{r}_i, \quad \mathbf{l}_i^t(\Lambda + X) = \mu_i\mathbf{l}_i^t,$$

under the standard bi-orthogonality relation $\langle \bar{\mathbf{l}}_j, \mathbf{r}_i \rangle = \mathbf{l}_j^t \mathbf{r}_i = \delta_{i,j}$. Note that this relation leaves a large freedom in choosing the normalisation of each eigenvector. The key invariant quantity is the *eigenvector overlap*

$$\mathcal{O}_{ij} := \langle \mathbf{r}_j, \mathbf{r}_i \rangle \langle \mathbf{l}_j, \mathbf{l}_i \rangle,$$

which emerges in many problems where non-Hermitian eigenvectors are concerned, see e.g. [3,19,20,8,13,40]. Two prominent examples are

- (i) in numerical linear algebra; where $\sqrt{\mathcal{O}_{ii}}$ is the *eigenvalue condition number* determining how fast μ_i moves under small perturbation in the worst case using the formula

$$\sqrt{\mathcal{O}_{ii}} = \lim_{t \rightarrow 0} \sup \left\{ \left| \frac{\mu_i(\Lambda + X + tE) - \mu_i(\Lambda + X)}{t} \right| : E \in \mathbf{C}^{N \times N}, \|E\| = 1 \right\} \quad (1.1)$$

(see, e.g. [7]);

- (ii) in the theory of the *Dyson Brownian motion* for non-Hermitian matrices; where \mathcal{O}_{ij} gives the correlation of the martingale increments for the stochastic evolution of the eigenvalues μ_i and μ_j as the matrix evolves by the natural Ornstein-Uhlenbeck flow (see [42], [13, Appendix A]).

The main result of this paper is an almost optimal *lower* bound of order N on the diagonal overlap \mathcal{O}_{ii} , with very high probability. In the context of numerical linear algebra this means that non-Hermitian eigenvalues of $\Lambda + X$ still move at a speed of order \sqrt{N} under the “worst” perturbation E in (1.1), despite having added a random smoothing component X to Λ . Note that in numerics one typically views the random smoothing as a tool to reduce the overlap of Λ in order to enhance the stability of its eigenvalues; our result shows a natural limitation for such reduction. It still does not exclude the possibility that a very specially chosen X reduces the eigenvalue condition numbers much more than a typical random one does, in particular it does not disprove the Davidson-Herrero-Salinas conjecture (see [33, Problem 2.11]). However, our N -dependent lower bound on \mathcal{O}_{ii} shows that a naive randomisation argument is not sufficient for resolving this conjecture. Complementary *upper* bounds on \mathcal{O}_{ii} have recently been proven in [7] and [45]. These hold only in expectation sense, as \mathcal{O}_{ii} has a fat-tail, and they are off by a factor N . Very recently this factor was removed in [37]. We remark, however, that N is the most relevant parameter of the problem only from our random matrix theory point

of view. Works motivated by numerical analysis, such as [7,45] and references therein, often focus on tracking the γ -dependence for the problem $\Lambda + \gamma X$ in the small noise regime $\gamma \ll 1$ in order to reduce the effect of the random perturbation. In this setup the non-optimality of the N -power may be considered less relevant.³

In the context of the Dyson Brownian motion, our lower bound on \mathcal{O}_{ii} implies a diffusive lower bound on the eigenvalues of the Ornstein-Uhlenbeck (OU) matrix flow, generalizing the analogous result of Bourgade and Dubach [13, Corollary 1.6] from Ginibre ensemble to arbitrary i.i.d. ensemble (see (2.14) later).

1.2. Thermalisation of singular vectors

The key step to our lower bound on \mathcal{O}_{ii} is a *thermalisation* result on the singular vectors that is of independent interest. Namely, we show that singular vectors of $\Lambda + X$ are fully randomised in the large N limit in the sense that their quadratic forms with arbitrary test matrices have a deterministic limit with an optimal $N^{-1/2}$ speed of convergence. This holds with very high probability which enables us to make such statement for matrices of the form $(\Lambda - z) + X$ *simultaneously* for any shift parameter z , even for random ones. We will use this for $z = \mu$, an eigenvalue of $\Lambda + X$. This allows us to gain access to eigenvectors of $\Lambda + X$, by noticing that singular vectors and eigenvectors are unrelated in general with an obvious exception: if μ is an eigenvalue of $\Lambda + X$, then any vector in the kernel of $\Lambda + X - \mu$ is an eigenvector of $\Lambda + X$ with eigenvalue μ , and a singular vector of $\Lambda + X - \mu$ with singular value 0. Hence high probability statements for singular vectors can be converted into similar statements for eigenvectors – this key idea may be viewed as the eigenvector version of the transfer principle between eigenvalues and singular values encoded in Girko’s formula.

Our thermalisation result for singular vectors may be viewed as the non-Hermitian analogue of the *Quantum Unique Ergodicity* (QUE) for Hermitian Wigner matrices proven in [23]. We now briefly explain the QUE phenomenon and its physics background in the simplest Hermitian context before we consider the singular vectors of $\Lambda + X$. In fact, via a standard Hermitisation procedure we will turn the singular vector problem to a Hermitian eigenvector problem.

For Hermitian random matrices H , that can be considered as the Hamilton operator of a disordered quantum system, a major motivation comes from physics, where the randomisation of the eigenvectors is interpreted as a *thermalisation* effect. The *Eigenstate Thermalisation Hypothesis* (ETH) by Deutsch [34] and Srednicki [53] (see also [32,35]) asserts that any deterministic Hermitian matrix A (observable), becomes essentially diagonal in the eigenbasis of a “sufficiently chaotic” Hamiltonian, where chaos may come

³ As long as γ is N independent, one may set $\gamma = 1$ by a simple rescaling so we refrain from carrying this extra factor in the current paper. We remark that our methods would allow to trace the polynomial γ -dependence in all our main estimates as well, albeit not with an optimal power.

from an additional randomness or from the ergodicity of the underlying classical dynamics. In other words,

$$\langle \mathbf{u}_i, A\mathbf{u}_j \rangle - \delta_{ij} \langle A \rangle_i \rightarrow 0, \quad \text{as } N \rightarrow \infty, \tag{1.2}$$

where $\{\mathbf{u}_i\}$ is an orthonormal eigenbasis of H and the deterministic “averaged” coefficient $\langle A \rangle_i$ is to be computed from the statistics of H .

In the mathematics literature the same problem is known as the Quantum (Unique) Ergodicity, originally formulated for the Laplace-Beltrami operator on surfaces with ergodic geodesic flow, see [51,30,58], on regular graphs [5] and on special arithmetic surfaces [50,18,48,52]. In [23] we proved QUE in the strongest form with an optimal speed of convergence for the eigenvectors of Wigner matrices that, by E. Wigner’s vision, can be viewed as the “most random” Hamiltonian. In this case, the diagonal limit $\langle A \rangle_i$ in (1.2) is independent of i and given by the normalised trace $\langle A \rangle := \frac{1}{N} \text{Tr } A$. In fact, in subsequent papers [25,27] (see also [11]) even the normal fluctuation of $\sqrt{N}[\langle \mathbf{u}_i, A\mathbf{u}_i \rangle - \langle A \rangle]$ was proven, followed by the proof of joint Gaussianity of finite many overlaps in [10]. Previously QUE results were proven for rank one observables (see [46,54] under four moment matching and [14] in general) and finite rank observables [49], see also [9] for deformed Wigner matrices and [17] for band matrices. The proofs crucially used that H is Hermitian, heavily relying on sophisticated Hermitian techniques (such as *local laws* and *Dyson Brownian Motion*) developed in the last decade for eigenvalue universality questions.

Back to our non-Hermitian context, we consider the singular vectors $\{\mathbf{u}_i, \mathbf{v}_i\}_{i=1}^N$ of $\Lambda + X$,

$$(X + \Lambda)(X + \Lambda)^* \mathbf{u}_i = \sigma_i^2 \mathbf{u}_i, \quad (X + \Lambda)^*(X + \Lambda) \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i,$$

belonging to the singular value σ_i . We view them as the two N -dimensional components of the eigenvectors $\mathbf{w}_i = (\mathbf{u}_i, \mathbf{v}_i)$ of the $2N$ -dimensional *Hermitisation* of $\Lambda + X$, defined as

$$H = H^\Lambda := W + \hat{\Lambda}, \quad W := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \quad \hat{\Lambda} := \begin{pmatrix} 0 & \Lambda \\ \Lambda^* & 0 \end{pmatrix}. \tag{1.3}$$

In particular, from the overlaps $\langle \mathbf{w}_i, A\mathbf{w}_j \rangle$ of eigenvectors for the Hermitised problem with a general $(2N) \times (2N)$ matrix A one may read off all the singular vector overlaps of the form $\langle \mathbf{u}_i, B\mathbf{u}_j \rangle$, $\langle \mathbf{v}_i, B\mathbf{v}_j \rangle$ and $\langle \mathbf{u}_i, B\mathbf{v}_j \rangle$ with any $N \times N$ matrix B . Therefore our goal is to show the general thermalisation phenomenon, the convergence of $\langle \mathbf{w}_i, A\mathbf{w}_j \rangle$ (cf. (1.2)), for the Hermitised matrix H^Λ thus generalizing the ETH proven in [23] beyond Wigner matrices and with an additional arbitrary matrix Λ . Unlike in the Wigner case, the limit $\langle A \rangle_i$ genuinely depends on the index i and part of the task is to determine its precise form. Note that due to the large zero blocks, W has about half as many random degrees of freedom as a Wigner matrix of the same dimension has, moreover the block

structure gives rise to potential instabilities, thus the ETH for H^Λ is considerably more involved than for Wigner matrices. In the next section we explain the main new method of this paper that systematically handles all these instabilities.

1.3. Structural decomposition of observables

We introduce a new concept for splitting general observables into “regular” and “singular” components; where the singular component gives the leading contribution and the regular component is estimated. In the case of Wigner matrices H in [23,29] we used the decomposition $A = \langle A \rangle + \mathring{A}$, where the traceless part of A , $\mathring{A} := A - \langle A \rangle$, is the regular component and the projection⁴ of A onto the one dimensional space spanned by the identity matrix is the singular component. This gave rise to the following decomposition of resolvent $G = G(w) = (H - w)^{-1}$ for any $w \in \mathbf{C} \setminus \mathbf{R}$:

$$\langle GA \rangle = m \langle A \rangle + \langle A \rangle \langle G - m \rangle + \langle G \mathring{A} \rangle, \quad (1.4)$$

where $m = m(w)$ is the Stieltjes transform of the semicircle law. The second term in (1.4) is asymptotically Gaussian of size $\langle G - m \rangle \sim (N\eta)^{-1}$ [43] and the last term is also Gaussian, but of much smaller size $\langle G \mathring{A} \rangle \sim \langle \mathring{A} \mathring{A}^* \rangle^{1/2} / (N\eta^{1/2})$ in the interesting regime of small $\eta := |\operatorname{Im} w| \ll 1$ [29].

Similar decomposition governs the traces of longer resolvent chains of Wigner matrices, for example

$$\langle GAG^*B \rangle = \langle GG^* \rangle = \frac{1}{\eta} \langle \operatorname{Im} G \rangle \sim \frac{1}{\eta} \gg 1$$

if $A = B = I$, i.e. both observable matrices are purely singular, while for regular (and bounded) observables $A = \mathring{A}$, $B = \mathring{B}$ we have

$$\langle GAG^*B \rangle \sim 1. \quad (1.5)$$

Both examples indicate the $\sqrt{\eta}$ -rule (see (3.16) and Remark 4.6 later), informally asserting that each regular observable renders the size of a resolvent chain smaller by a factor $\sqrt{\eta}$ than its singular counterpart. In [26,27] we obtained the deterministic leading terms and optimal error estimates on the fluctuation for resolvent chains of arbitrary length

$$\langle G(w_1)A_1G(w_2)A_2 \dots \rangle \quad (1.6)$$

with arbitrary observables in between. The answer followed the $\sqrt{\eta}$ -rule hence it heavily depended on the $A_i = \langle A_i \rangle + \mathring{A}_i$ decomposition for each observable.

⁴ We equip the space of matrices with the usual normalised Hilbert-Schmidt scalar product, $\langle A, B \rangle := \frac{1}{N} \operatorname{Tr} A^*B = \langle A^*B \rangle$.

In particular, in order to estimate $\langle \mathbf{u}_i, A\mathbf{u}_j \rangle - \delta_{ij}\langle A \rangle = \langle \mathbf{u}_i, \mathring{A}\mathbf{u}_j \rangle$ for ETH in (1.2), we had

$$N|\langle \mathbf{u}_i, \mathring{A}\mathbf{u}_j \rangle|^2 \lesssim \langle \text{Im } G(w_1) \mathring{A} \text{Im } G(w_2) \mathring{A} \rangle \lesssim 1,$$

where we first used spectral decomposition of both G 's and then used a version of (1.5). Here the spectral parameters $w_k = e_k + i\eta$ are chosen such that e_1 and e_2 be close to the eigenvalues corresponding to \mathbf{u}_i and \mathbf{u}_j , respectively, and $\eta \sim N^{-1}$ in order to resolve the spectrum on the fine scale of the individual eigenvalues.⁵

The key point in all these analyses for Wigner matrices was that the regular/singular concept was *independent* of the spectral parameter: the same universal decomposition into tracial and traceless parts worked in every instance along the proofs. One consequence is the i -independence of the limiting overlap $\langle\langle A \rangle\rangle_i := \langle A \rangle$ in (1.2).⁶

For more complicated ensembles, like H^Λ in (1.3), especially if an arbitrary matrix Λ is involved, the correct decomposition depends on the location in the spectrum of H where we work. To guess it, first we recall the *single resolvent local law* (Theorem 2.6) for the resolvent $G = G^\Lambda(w) = (H^\Lambda - w)^{-1}$, asserting that $\langle GA \rangle \approx \langle MA \rangle$, where $M = M^\Lambda(w)$ solves a nonlinear deterministic equation, the *Matrix Dyson Equation (MDE)*, see (2.20) later. Then a heuristic calculation (see Appendix A.1) shows that for $w = e + i\eta \in \mathbf{C}_+$ we have

$$\mathbf{E} |\langle (G - M)A \rangle|^2 \approx \frac{|\langle \text{Im } MA \rangle|^2}{(N\eta)^2} + \frac{|\langle \text{Im } MAE_- \rangle|^2}{N^2\eta(|e| + \eta)} + \mathcal{O}\left(\frac{1}{N^2\eta}\right), \quad E_- := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.7}$$

indicating that the singular component of A is *two dimensional*, depends on w , and for any A orthogonal to the two singular directions $\text{Im } M$ and $E_- \text{Im } M$ the size of $\langle (G - M)A \rangle$ is smaller by a factor $\sqrt{\eta}$. The first singular direction is always present. The second singular direction is a consequence of the block structure of H and it is manifested only for w near the imaginary axis. For energies $|e| \sim 1$, only the first singular direction, namely the one involving $\text{Im } M$ plays a role.

What about longer chains (1.6)? Each matrix A_i is sandwiched between two resolvents with different spectral parameters w_i, w_{i+1} . We find that the correct decomposition of any A between two resolvents in a chain $\dots G(w)AG(w') \dots$ depends only on w, w' and it has the form

$$A = \langle V_+, A \rangle U_+ + \langle V_-, A \rangle U_- + \mathring{A}, \quad V_\pm = V_\pm^{w, w'}, \quad \mathring{A} = \mathring{A}^{w, w'}, \tag{1.8}$$

where the first two terms form the singular component of A , and \mathring{A} , defined by this equation, is the regular component. We will establish that both V_+ and V_- are the right

⁵ Strictly speaking we used $\eta = N^{-1+\xi}$ with any small $\xi > 0$, and all estimates held up to an N^ξ factor but we ignore these technicalities in the introduction.

⁶ A quick direct way to see this independence is the special case of Gaussian Wigner matrices (GUE or GOE), where the eigenvectors are Haar distributed, independently of their eigenvalue.

eigenvectors of a certain *stability operator* \mathcal{B} acting on $\mathbf{C}^{2N \times 2N}$ that corresponds to the Dyson equation, and U_{\pm} will be explained later. For example, if $\text{Im } w$ and $\text{Im } w'$ have opposite signs then V_+ is the right eigenvector of

$$\mathcal{B}[\cdot] = 1 - M(\bar{w})\mathcal{S}[\cdot]M(\bar{w}'),$$

where \mathcal{S} is covariance operator for the matrix W in (1.3) (see (2.21)). V_{\pm} with other sign combinations are defined very similarly (in Appendix A.3 we present all cases). Actually, the special directions $\text{Im } M$ and $E_- \text{Im } M$ that we found by direct variance calculation in (1.7) also emerge *canonically* as eigenvectors of a certain stability operator! Similar variance calculation for longer chains would reveal the same consistency: the variance of the chain (1.6) is the smallest if each A_i is regular with respect to the two neighbouring spectral parameters w_i, w_{i+1} .

Note that the choice of V_{\pm} is basically dictated by variance calculations like (1.7). However, the matrices U_{\pm} in (1.8) can still be chosen freely up to their linear independence and the normalisation requirement $\langle V_{\sigma}, U_{\tau} \rangle = \delta_{\sigma, \tau}$. The latter guarantees that the sum of the singular terms in (1.8) is actually a (non-orthogonal) projection $|U_+\rangle\langle V_+| + |U_-\rangle\langle V_-|$ acting on A . Since V_{\pm} are the right eigenvectors of a stability operator, one may be tempted to choose U_{\pm} as certain left eigenvectors but we did not find this guiding principle helpful. Instead, we use this freedom to simplify the calculation of the singular terms. Substituting the singular part of A into $\dots G(w)AG(w') \dots$, we need to compute $G(w)U_{\pm}G(w')$ and quite pragmatically we choose U_{\pm} such that the entity could be applied and thus reduce the length of the chain. Thanks to the spectral symmetry of $H = H^{\Lambda}$, for its resolvent we have $E_-G(-w)E_- = -G(w)$, and we find that $U_+ = I, U_- = E_-$ do the job, which accidentally coincide with the left eigenvectors of the stability operator for the special case of i.i.d. matrices.

In Appendix A.3 we present the canonical choices of V_{\pm} and U_{\pm} in a more general situation and explain at which stage of the proof their correct choice emerges. In our current application only V_{\pm} are nontrivial (in particular energy dependent), while U_{\pm} are very simple. This is due to the fact that the chain (1.6) consists of resolvents of the *same* operator. In more general problems one may take resolvents with two different Λ 's in the chain, in which case U_{\pm} are also nontrivial.

This decomposition scheme is the really novel ingredient of our proofs. Several other tools we use, such as *recursive Dyson equations*, hierarchy of *master inequalities* and *reduction inequalities* have been introduced before (especially in our related works on Wigner matrices [23,29]), but the dependence of the decomposition on the spectral parameters in the current setup requires quite different new estimates along the arguments. We informally explain the prototype of such an estimate at the beginning of Section 4.1.

1.4. Notations

We define the $2N \times 2N$ matrices $E_{\pm} := E_1 \pm E_2$, where

$$E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.9}$$

Each entry of the matrix is understood as a multiple of the $N \times N$ -identity. By $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ we denote the upper and lower integer part, respectively, i.e. for $x \in \mathbf{R}$ we define $\lceil x \rceil := \min\{m \in \mathbf{Z}: m \geq x\}$ and $\lfloor x \rfloor := \max\{m \in \mathbf{Z}: m \leq x\}$. We denote $[k] := \{1, \dots, k\}$ for $k \in \mathbf{N}$ and $\langle A \rangle := d^{-1}\text{Tr}(A)$, $d \in \mathbf{N}$, is the normalised trace of a $d \times d$ -matrix. For positive quantities A, B we write $A \lesssim B$ resp. $A \gtrsim B$ and mean that $A \leq CB$ resp. $A \geq cB$ for some N -independent constants $c, C > 0$. We denote vectors by bold-faced lower case Roman letters $\mathbf{x}, \mathbf{y} \in \mathbf{C}^{2N}$, for some $N \in \mathbf{N}$, and define

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_i \bar{x}_i y_i, \quad A_{\mathbf{x}\mathbf{y}} := \langle \mathbf{x}, A\mathbf{y} \rangle.$$

Matrix entries are indexed by lower case Roman letters a, b, c, \dots from the beginning of the alphabet and unrestricted sums over a, b, c, \dots are always understood to be over $\{1, \dots, N, N + 1, \dots, 2N\}$. Analogously, unrestricted sums over lower case Roman letters i, j, k, \dots from the middle of the alphabet are always understood to be over $\{-N, \dots, -1, 1, \dots, N\}$. Finally, the lower case Greek letters σ and τ as indices indicate a sign, i.e. $\sigma, \tau \in \{+, -\}$, and unrestricted sums over σ, τ are understood to be over $\{+, -\}$.

We will use the concept of ‘with very high probability’, meaning that any fixed $D > 0$, the probability of an N -dependent event is bigger than $1 - N^{-D}$ for all $N \geq N_0(D)$. Also, we will use the convention that $\xi > 0$ denotes an arbitrarily small constant, independent of N . Moreover, we introduce the common notion of *stochastic domination* (see, e.g., [38]): For two families

$$X = \left(X^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)} \right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)} \right)$$

of non-negative random variables indexed by N , and possibly a parameter u , then we say that X is stochastically dominated by Y , if for all $\varepsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P} \left[X^{(N)}(u) > N^\varepsilon Y^{(N)}(u) \right] \leq N^{-D}$$

for large enough $N \geq N_0(\varepsilon, D)$. In this case we write $X \prec Y$. If for some complex family of random variables we have $|X| \prec Y$, we also write $X = O_{\prec}(Y)$.

Acknowledgement: The authors are grateful to Oleksii Kolupaiev for valuable discussions, especially about the choice of contours in Lemma 5.1. We thank Nikhil Srivastava for bringing [33] to our attention.

2. Main results

We consider *real or complex i.i.d. matrices* X , i.e. $N \times N$ matrices whose entries are independent and identically distributed as $x_{ab} \stackrel{d}{=} N^{-1/2}\chi$ for some real or complex random variable χ satisfying the following assumptions:

Assumption 2.1. *We assume that $\mathbf{E}\chi = 0$ and $\mathbf{E}|\chi|^2 = 1$. Furthermore, we assume the existence of high moments, i.e., that there exist constants $C_p > 0$, for any $p \in \mathbf{N}$, such that*

$$\mathbf{E}|\chi|^p \leq C_p.$$

Additionally, in the complex case, we assume that $\mathbf{E}\chi^2 = 0$.

For definiteness, in the sequel we perform our entire analysis for the complex case; the real case being completely analogous and hence omitted.

2.1. Non-Hermitian singular vectors and eigenvectors

Fix a deterministic matrix $\Lambda \in \mathbf{C}^{N \times N}$, with N -independent norm bound, $\|\Lambda\| \lesssim 1$. Let $\{\sigma_i\}_{i \in [N]}$ be the singular values of $X + \Lambda$ with corresponding (normalised) left- and right-singular vectors $\{\mathbf{u}_i\}_{i \in [N]}$ and $\{\mathbf{v}_i\}_{i \in [N]}$, respectively, i.e.

$$(X + \Lambda)\mathbf{v}_i = \sigma_i\mathbf{u}_i \quad \text{and} \quad (X + \Lambda)^*\mathbf{u}_i = \sigma_i\mathbf{v}_i. \tag{2.1}$$

All these objects naturally depend on Λ , but we will omit this fact from the notation.

Let $\nu_i, i \in [N]$, be the increasingly ordered singular values of Λ . Define the *Hermitisation* of Λ as

$$\hat{\Lambda} := \begin{pmatrix} 0 & \Lambda \\ \Lambda^* & 0 \end{pmatrix}. \tag{2.2}$$

Due to its block structure, the spectrum of $\hat{\Lambda}$ is symmetric with respect to zero, with eigenvalues $\{\nu_i\}_{0 \neq |i| \leq N}$ such that $\nu_{-i} = -\nu_i$ for all $i \in [N]$. The empirical density of states of $\hat{\Lambda}$ is denoted by

$$\mu_{\hat{\Lambda}} := \frac{1}{2N} \sum_{0 \neq |i| \leq N} \delta_{\nu_i}.$$

Let μ_{sc} be the Wigner semicircle distribution with density $\rho_{\text{sc}}(x) := (2\pi)^{-1}\sqrt{[4 - x^2]_+}$, where $[\cdot]_+$ is the positive part of a real number. Define the free additive convolution

$$\mu = \mu^\Lambda := \mu_{\text{sc}} \boxplus \mu_{\hat{\Lambda}}, \tag{2.3}$$

which is a probability distribution on \mathbf{R} . We now briefly recall basic facts about the free convolution with the semicircle density (see, e.g. the classical paper by P. Biane [12]). Most conveniently μ may be defined by inverting its Stieltjes transform

$$m(w) = m^\Lambda(w) := \int_{\mathbf{R}} \frac{\mu(\mathrm{d}e)}{e - w}, \quad w \in \mathbf{C} \setminus \mathbf{R},$$

where m satisfies the implicit equation

$$m(w) = \int_{\mathbf{R}} \frac{\mu_\Lambda(\mathrm{d}e)}{e - (w + m(w))}. \tag{2.4}$$

With the additional constraint $\mathrm{Im} m(w) \cdot \mathrm{Im} w > 0$ this equation has a unique solution that is analytic away from the real axis with $m(\bar{w}) = \overline{m(w)}$. Since μ_Λ is symmetric to zero with bounded support, μ is also symmetric with support bounded independently of N . Moreover μ is absolutely continuous with respect to Lebesgue measure with density denoted by $\rho = \rho^\Lambda$. The density ρ may be obtained⁷ as the boundary value of $\mathrm{Im} m$ at any e on the real line, i.e.

$$\rho(e) := \lim_{\eta \downarrow 0} \rho(e + i\eta), \quad \rho(w) := \frac{1}{\pi} |\mathrm{Im} m(w)|. \tag{2.5}$$

In fact m itself has a continuous extension to the real axis from the upper half plane $m(e) := \lim_{\eta \downarrow 0} m(e + i\eta)$. Proving the existence of these limits is standard from (2.4).

Next, for any (small) $\kappa > 0$, we define the κ -bulk of the density ρ as

$$\mathbf{B}_\kappa = \mathbf{B}_\kappa^\Lambda := \{x \in \mathbf{R} : \rho(x) \geq \kappa^{1/3}\} \tag{2.6}$$

which is symmetric to the origin. Finally, we denote a (modified) i^{th} quantile of the density ρ by γ_i , i.e.

$$\frac{i + N}{2N} = \int_{-\infty}^{\gamma_i} \rho(e) \mathrm{d}e, \quad |i| \leq N, \tag{2.7}$$

and we immediately conclude by symmetry of ρ that $\gamma_i = -\gamma_{-i}$ for every $|i| \leq N$.

Our first main result establishes the *thermalisation of singular vectors* of $X + \Lambda$ in the bulk, i.e. for indices i, j with quantiles γ_i, γ_j uniformly in the *bulk* of the density ρ .

Theorem 2.2. (Thermalisation of Singular Vectors)

⁷ For orientation of the reader we mention that ρ is the deterministic approximation, the so-called *self-consistent density of states (scDos)*, for the empirical eigenvalue density of the Hermitisation of $X + \Lambda$. This connection will be explained in the next Section 2.2.

Fix a bounded $\Lambda \in \mathbf{C}^{N \times N}$ and $\kappa > 0$ independent of N . Let $\{\mathbf{u}_i\}_{i \in [N]}$ and $\{\mathbf{v}_i\}_{i \in [N]}$ be the (normalised) left- and right-singular vectors of $X + \Lambda$, respectively, where X is an i.i.d. matrix satisfying Assumption 2.1. Then, for any deterministic matrix $B \in \mathbf{C}^{N \times N}$ with $\|B\| \lesssim 1$ it holds that⁸

$$\max_{i,j} \left| \langle \mathbf{u}_i, B \mathbf{u}_j \rangle - \delta_{j,i} \frac{\left\langle \operatorname{Im} \left[\frac{\gamma_j + m(\gamma_j)}{\Lambda \Lambda^* - (\gamma_j + m(\gamma_j))^2} \right] B \right\rangle}{\pi \rho(\gamma_j)} \right| \prec \frac{1}{\sqrt{N}}, \tag{2.8a}$$

$$\max_{i,j} \left| \langle \mathbf{v}_i, B \mathbf{v}_j \rangle - \delta_{j,i} \frac{\left\langle \operatorname{Im} \left[\frac{\gamma_j + m(\gamma_j)}{\Lambda^* \Lambda - (\gamma_j + m(\gamma_j))^2} \right] B \right\rangle}{\pi \rho(\gamma_j)} \right| \prec \frac{1}{\sqrt{N}}, \tag{2.8b}$$

$$\max_{i,j} \left| \langle \mathbf{u}_i, B \mathbf{v}_j \rangle - \delta_{j,i} \frac{\left\langle \Lambda \operatorname{Im} \left[(\Lambda^* \Lambda - (\gamma_j + m(\gamma_j))^2)^{-1} \right] B \right\rangle}{\pi \rho(\gamma_j)} \right| \prec \frac{1}{\sqrt{N}}, \tag{2.8c}$$

where the maximum is taken over all $i, j \in [N]$ such that the quantiles $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$ are in the κ -bulk of the density ρ .

The thermalisation of singular vectors will be a simple corollary to the *Eigenstate Thermalisation Hypothesis (ETH)* for the Hermitisation H^Λ of $X + \Lambda$, which is formulated in Theorem 2.7 below. The proof of Theorem 2.2 will be given in Section 3.

Our second main result concerns the bi-orthonormal left and right eigenvectors $\{\mathbf{l}_i\}_{i \in [N]}$ and $\{\mathbf{r}_i\}_{i \in [N]}$, respectively, of $X + \Lambda$, with corresponding eigenvalues $\{\mu_i\}_{i \in [N]}$, i.e.

$$(X + \Lambda) \mathbf{r}_i = \mu_i \mathbf{r}_i, \quad \mathbf{l}_i^t (X + \Lambda) = \mu_i \mathbf{l}_i^t, \tag{2.9}$$

where t denotes the transpose of a vector. More precisely, the following theorem provides a lower bound on the diagonal part of the *overlaps matrix*

$$\mathcal{O}_{ij} := \langle \mathbf{r}_j, \mathbf{r}_i \rangle \langle \mathbf{l}_j, \mathbf{l}_i \rangle, \tag{2.10}$$

defined subject to the standard normalisation

$$\langle \bar{\mathbf{l}}_j, \mathbf{r}_i \rangle = \mathbf{l}_j^t \mathbf{r}_i = \delta_{i,j}. \tag{2.11}$$

We restrict our results to eigenvalues μ_i in the *bulk* of $X + \Lambda$ in the following sense.

Definition 2.3. We say that $z \in \mathbf{C}$ is in the *bulk* of $X + \Lambda$ if and only if there exists an N -independent $\kappa > 0$ for which

$$0 \in \mathbf{B}_\kappa^{\Lambda - z} = \{x \in \mathbf{R} : \rho^{\Lambda - z}(x) \geq \kappa^{1/3}\}.$$

⁸ The deterministic terms following the Kronecker symbol $\delta_{j,i}$ in (2.8) will be shown to be bounded.

There is no simple characterisation of the bulk of $X + \Lambda$ in terms of the spectrum of Λ . However, taking the imaginary part of (2.4) at $w = 0 + i0$ shows that $0 \in \mathbf{B}_\kappa^{\Lambda-z}$ is equivalent to

$$\frac{1}{N} \sum_{i=1}^N \frac{1}{\nu_i(\Lambda - z)^2 + \kappa^{2/3}} \geq 1,$$

where $\nu_i(\Lambda - z)$ are the singular values of $\Lambda - z$.

Theorem 2.4. *Consider $X + \Lambda$, with Λ being a deterministic matrix as in (2.2) and with X being an i.i.d. matrix satisfying Assumption 2.1. Then*

$$\mathcal{O}_{ii} \succ N, \tag{2.12}$$

where the index $i \in [N]$ is such that μ_i is in the bulk of $X + \Lambda$.

In the introduction we already mentioned the consequence of this result on the sensitivity of an eigenvalue of $X + \Lambda$ under small perturbations. Now we explain its other consequence on the diffusivity of the Dyson-type eigenvalue dynamics. Let each entry of $X = X(t)$ evolve as an independent complex OU process,

$$dX_{ij} = \frac{dB_{ij}}{\sqrt{N}} - \frac{1}{2}X_{ij}dt,$$

where B_{ij} are independent standard complex Brownian motions and the initial condition $X(0)$ satisfies Assumption 2.1. A direct calculation [13, Proposition A.1] shows that the eigenvalues $\mu_i = \mu_i(t)$ follow the Dyson-type stochastic dynamics

$$d\mu_i = dM_i - \frac{1}{2}\mu_i dt, \quad \{\mu_i(0)\} = \text{Spec}X(0), \quad 1 \leq i \leq N, \tag{2.13}$$

where the martingales M_i have brackets $\langle M_i, M_j \rangle = 0$ and $d\langle M_i, M_j \rangle_t = \frac{1}{N}\mathcal{O}_{ij}(t)dt$. In particular, we immediately obtain, for any $\epsilon > 0$ that

$$\mathbf{E} \left[|\mu_i(t) - \mu_i(0)|^2 \mathbf{1}(\mu_i(0) \in \mathbf{B}_\kappa) \right] \geq tN^{-\epsilon} \tag{2.14}$$

up to some time $t \leq T(\kappa)$, where \mathbf{B}_κ denotes the κ -bulk of $X(0)$. For Ginibre initial condition $X(0)$ (2.14) was established in [13, Corollary 1.6], we now generalise it to i.i.d. initial conditions. We remark that (2.13) is similar to its Hermitian counterpart, the standard Dyson Brownian motion (DBM) on the real line, with some notable differences. In particular, in the latter process the eigenvalues cannot cross each other, hence they are quite rigid and confined to an interval of size essential $1/N$, so they are not diffusive beyond a time-scale $1/N$. Along the evolution (2.13) the non-Hermitian eigenvalues still repel each other (encoded in the typically negative off-diagonal overlaps, see [13, Theorem

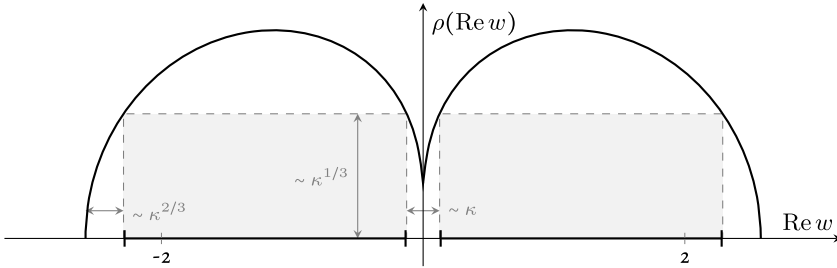


Fig. 1. Depicted is the density ρ for the deformation $\Lambda = -z$ with $|z|$ slightly less than one. On the horizontal axis, we indicated the two components of the bulk \mathbf{B}_κ . The distance between \mathbf{B}_κ and a regular edge scales like $\kappa^{2/3}$, while near an (approximate) cusp the distance between the two components scales linearly (see also (2.6) and (2.23)).

1.3] in the Gaussian case), but they still can pass by each other and not hindering the diffusive behaviour (2.14).

Example 2.5. The most prominently and extensively studied [41,6,56,15,16,57,55,21,28,22] deformation is $\Lambda = -z$ with $z \in \mathbf{C}$, since it plays a key role in Girko’s formula [41] expressing linear statistics of non-Hermitian eigenvalues of X in terms of the Hermitisation of $X - z$. In this case, the self-consistent equation (2.4) reduces to the well-known cubic relation

$$-\frac{1}{m} = w + m - \frac{|z|^2}{w + m}.$$

As a consequence, the deterministic terms in (2.8) drastically simplify (e.g., the fractions in (2.8a) and (2.8b) are simply replaced by $\langle B \rangle$) and one also has explicit formulas for the bulk (2.6) in terms of solution of a cubic equation. In particular, for $|z| < 1 - \epsilon_\kappa$, the κ -bulk \mathbf{B}_κ consists of a single interval, while for $|z| \geq 1 - \epsilon_\kappa$ it consists of two intervals, where $\epsilon_\kappa \sim \kappa^{2/3}$ (Fig. 1). In the former case $0 \in \mathbf{B}_\kappa$. Consequently, Theorem 2.4 gives the lower bound (2.12) for all the diagonal overlaps \mathcal{O}_{ii} of eigenvectors of X whose eigenvalue μ_i lies in a disk of radius $1 - \epsilon$ with some $\epsilon > 0$ independent of N .

In the next section we explain the key technical result behind our main theorems, the eigenstate thermalisation for the Hermitisation of $X + \Lambda$.

2.2. Eigenstate thermalisation hypothesis for the hermitisation of $X + \Lambda$

The key to access the non-Hermitian singular vectors of $X + \Lambda$ is to study its *Hermitisation* [41], which is defined as

$$H = H^\Lambda := \begin{pmatrix} 0 & X + \Lambda \\ (X + \Lambda)^* & 0 \end{pmatrix} =: W + \hat{\Lambda}, \tag{2.15}$$

where $\hat{\Lambda}^* = \hat{\Lambda}$ was defined in (2.2) and can also be viewed as the matrix of expectation $\hat{\Lambda} = \mathbf{E}H^\Lambda$.

We denote by $\{\mathbf{w}_i\}_{|i| \leq N}$ the (normalised) eigenvectors of H and by $\{\lambda_i\}_{|i| \leq N}$ the corresponding eigenvalues.⁹ By means of the singular value decomposition in (2.1), the eigenvalues and eigenvectors of H are related to the singular values and singular vectors of $X + \Lambda$ as follows:

$$\mathbf{w}_i = (\mathbf{u}_i, \mathbf{v}_i)^t \quad \text{and} \quad \lambda_i = \sigma_i \quad \text{for} \quad i \in [N],$$

up to a normalisation, since now $\|\mathbf{u}_i\|^2 = \|\mathbf{v}_i\|^2 = \frac{1}{2}$. Moreover, the block structure of H induces a symmetric spectrum around zero, i.e. $\lambda_{-i} = -\lambda_i$ for any $i \in [N]$. This symmetry for the eigenvalues is also reflected in the eigenvectors, which satisfy $\mathbf{w}_{-i} = E_- \mathbf{w}_i$ for any $i \in [N]$. By spectral decomposition, this immediately shows the *chiral symmetry*

$$E_- G(w) = -G(-w)E_-, \quad \text{with} \quad E_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.16)$$

for the resolvent $G(w) = G^\Lambda(w) := (H^\Lambda - w)^{-1}$, with spectral parameter $w \in \mathbf{C} \setminus \mathbf{R}$. We also have $\langle G(w)E_- \rangle = 0$ for any w since $\langle \mathbf{w}_i, E_- \mathbf{w}_i \rangle = \|\mathbf{u}_i\|^2 - \|\mathbf{v}_i\|^2 = 0$.

A basic feature of a very large class of random matrices is that their resolvent becomes approximately deterministic in the large N limit, often even for any spectral parameter with $|\text{Im } w| \geq N^{-1+\epsilon}$; these statements are called *local laws*. In our case the deterministic approximation of the resolvent $G(w)$ is given by

$$M(w) = M^\Lambda(w) := \begin{pmatrix} M_{11}(w) & \frac{\Lambda M_{22}(w)}{w+m(w)} \\ \frac{\Lambda^* M_{11}(w)}{w+m(w)} & M_{22}(w) \end{pmatrix} \in \mathbf{C}^{2N \times 2N}, \quad w \in \mathbf{C} \setminus \mathbf{R}, \quad (2.17)$$

with each block being understood as a matrix in $\mathbf{C}^{N \times N}$, where the diagonal entries are defined via

$$M_{11}(w) := \frac{w + m(w)}{\Lambda \Lambda^* - (w + m(w))^2}, \quad M_{22}(w) := \frac{w + m(w)}{\Lambda^* \Lambda - (w + m(w))^2}. \quad (2.18)$$

Here we require $m(w) = \langle M(w) \rangle$, which is an implicit equation for the function $m(w)$. Simple calculation shows that this implicit equation is exactly (2.4). Moreover, one can easily check that $M(w)$ also satisfies the chiral symmetry (2.16), i.e.

$$E_- M(w) = -M(-w)E_-. \quad (2.19)$$

⁹ In the definition of the eigenvectors and eigenvalues, we omitted 0 in the set of indices, i.e. $|i| \leq N$ really means $i \in \{-N, \dots, -1, 1, \dots, N\}$.

To derive these formulas systematically, we recall that the deterministic approximation to $G(w)$ is obtained as the unique solution to the *matrix Dyson equation (MDE)* (introduced first in [44] and extensively studied in [1,2,4]). The MDE corresponding to the random matrix H is given by

$$-\frac{1}{M(w)} = w - \hat{\Lambda} + \mathcal{S}[M(w)] \tag{2.20}$$

under the constraint $\text{Im } M(w) \cdot \text{Im } w > 0$, where $\text{Im } M(w) := \frac{1}{2i} [M(w) - (M(w))^*]$. Here $\mathcal{S}[\cdot]$, the *self-energy operator*, is defined via

$$\mathcal{S}[T] := \tilde{\mathbf{E}}(\tilde{H} - \mathbf{E}H)T(\tilde{H} - \mathbf{E}H)$$

for any $T \in \mathbf{C}^{2N \times 2N}$, where \tilde{H} denotes an independent copy of H . In our case we can write \mathcal{S} in the following two ways

$$\mathcal{S}[T] = 2E_1 \langle TE_2 \rangle + 2 \langle E_1 T \rangle E_2 = \sum_{\sigma=\pm} \sigma \langle TE_\sigma \rangle E_\sigma, \tag{2.21}$$

with E_1, E_2 defined as in (1.9), and $E_\pm = E_1 \pm E_2$. Using $\langle M_{11}(w) \rangle = \langle M_{22}(w) \rangle$ that directly follows from (2.18), it is straightforward to check that $M(w)$ as defined in (2.17) satisfies the MDE (2.20). Since the MDE has a unique solution, we see that the density ρ defined via free convolution in Section 2.1 coincides with the *self-consistent density of states (scDos)* corresponding to the MDE, defined as the boundary value of $\frac{1}{\pi} \langle \text{Im } M \rangle$ on the real axis in the theory of MDE [2,4].

For the reader’s convenience in Appendix B.1 of the arXiv:2301.03549 version of this article, we will collect a few facts about M , in particular we will show that it has a continuous extension as a matrix valued function to the real axis, i.e. the limit $M(e) := \lim_{\eta \downarrow 0} M(e + i\eta)$ exists for any $e \in \mathbf{R}$. This extends the similar result on its trace mentioned in (2.5). Moreover, we will also show that for spectral parameters $w \in \mathbf{C} \setminus \mathbf{R}$ with $\text{Re } w \in \mathbf{B}_\kappa$, we have

$$\|M(w)\| \lesssim 1. \tag{2.22}$$

In particular, together with (2.17)–(2.18) this implies that the deterministic terms in (2.8) are bounded. Finally, we will also prove an important regularity property of the κ -bulk, namely that

$$\text{dist}(\partial \mathbf{B}_{\kappa'}, \mathbf{B}_\kappa) \geq \mathfrak{c}(\kappa - \kappa') \tag{2.23}$$

for any small $0 < \kappa' < \kappa$ and some N -independent constant $\mathfrak{c} = \mathfrak{c}(\|\Lambda\|) > 0$. In fact, for our proof it is sufficient if $\mathfrak{c} = \mathfrak{c}(\kappa, \|\Lambda\|)$, i.e. an additional κ dependence is allowed – in Appendix B.1 of the arXiv:2301.03549 version of this article we will explain that this weaker result is considerably easier to obtain (see Remark B.3 there). We will also show

that \mathbf{B}_κ is a finite disjoint union of compact intervals; the number of these *components* depends only on κ and $\|\Lambda\|$.

The above mentioned concentration of G around M is the content of the following single resolvent *local law*, both in *averaged* and *isotropic form*, which we prove in Appendix C of the arXiv:2301.03549 version of this article.

Theorem 2.6. (Single resolvent local law for the Hermitisation H)

Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N \times N}$ and $\kappa > 0$ independent of N . Then, for any $w \in \mathbf{C} \setminus \mathbf{R}$ with $|w| \leq N^{100}$ and $\operatorname{Re} w \in \mathbf{B}_\kappa$, we have

$$|\langle (G(w) - M(w))B \rangle| \prec \frac{1}{N\eta}, \quad |\langle \mathbf{x}, (G(w) - M(w))\mathbf{y} \rangle| \prec \frac{1}{\sqrt{N\eta}},$$

where $\eta := |\operatorname{Im} w|$, for any bounded deterministic matrix $\|B\| \lesssim 1$ and vectors $\|\mathbf{x}\|, \|\mathbf{y}\| \lesssim 1$.

Our main result for the Hermitised random matrix H is the *Eigenstate Thermalisation Hypothesis (ETH)*, that in mathematical terms is the proof of an optimal convergence rate of the strong *Quantum Unique Ergodicity (QUE)* for general observables A , uniformly in the *bulk* (2.6) of the spectrum of H , i.e. in the bulk of the *scDos* ρ .

Theorem 2.7. (Eigenstate Thermalisation Hypothesis for the Hermitisation H)

Fix some bounded $\Lambda \in \mathbf{C}^{N \times N}$ and $\kappa > 0$ independent of N . Let $\{\mathbf{w}_i\}_{|i| \leq N}$ be the orthogonal eigenvectors of the Hermitisation H of $X + \Lambda$, where X is an *i.i.d.* matrix satisfying Assumption 2.1. Then, for any deterministic matrix $A \in \mathbf{C}^{2N \times 2N}$ with $\|A\| \lesssim 1$ it holds that

$$\max_{i,j} \left| \langle \mathbf{w}_i, A\mathbf{w}_j \rangle - \delta_{j,i} \frac{\langle \operatorname{Im} M(\gamma_j)A \rangle}{\langle \operatorname{Im} M(\gamma_j) \rangle} - \delta_{j,-i} \frac{\langle \operatorname{Im} M(\gamma_j)E - A \rangle}{\langle \operatorname{Im} M(\gamma_j) \rangle} \right| \prec \frac{1}{\sqrt{N}}, \quad (2.24)$$

where the maximum is taken over all $|i|, |j| \leq N$, such that the quantiles $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$ defined in (2.7) are in the bulk of the *scDos* ρ .

The main technical result underlying Theorem 2.7 is an averaged local law for *two* resolvents with different spectral parameters, which we will formulate in Theorem 4.4 later.

Remark 2.8. Given the optimal bound (2.24), following a Dyson Brownian Motion (DBM) analysis similar to [25,27], it is possible to prove a CLT for single diagonal overlaps $\langle \mathbf{w}_i, A\mathbf{w}_i \rangle$. However, for the sake of brevity, we do not present this argument here and defer the CLT analysis to future work.

In the following Section 3 we precisely define the regularisation and we will prove our main results formulated above assuming the key technical Proposition 3.4. This

proposition is obtained from a *local law*, which we prove in Section 4. Local laws are proved by a hierarchy of *master and reduction inequalities*, that are derived in Sections 5 and 6, respectively. Appendix A contains two motivating calculations for the correct regularisation. Several technical and auxiliary results are deferred to the appendices of the arXiv:2301.03549 version of this article.

3. Proof of the main results

The key to understanding the eigenvector overlaps and showing our main results is an improved bound on the averaged trace of *two* resolvents with *regular* (see Section 3.1 below for the precise definition) deterministic matrices A_1, A_2 in between, i.e. for

$$\langle G(w_1)A_1G(w_2)A_2 \rangle. \tag{3.1}$$

Naively, for arbitrary A_1, A_2 , estimating (3.1) via a trivial Schwarz inequality and Ward identity yields the upper bound $|\langle G(w_1)A_1G(w_2)A_2 \rangle| \prec 1/\eta$, where $\eta := \min_j |\text{Im } w_j|$. However, this bound drastically improves, whenever the matrices A_1, A_2 are *regular*, i.e. orthogonal to certain *critical* eigenvectors V_\pm of the associated two-body stability operators (B.2), which is denoted as $A_j = \mathring{A}_j$; see (3.2) and Definitions 3.1 and 4.2. In this case, in our key Proposition 3.4 we will show that

$$|\langle G(w_1)\mathring{A}_1G(w_2)\mathring{A}_2 \rangle| \prec 1$$

even for very small $\eta \sim N^{-1+\epsilon}$ as a consequence of a more precise *local law* for (3.1), which we present in Section 4. We find that (see Theorem 4.4 and Remark 4.6) both the size of its deterministic approximation and the fluctuation around this mean heavily depend on whether (one or both of) the matrices A_1, A_2 are regular, i.e. satisfy $\langle V_\pm, A_j \rangle = 0$, or not.

Therefore, the general rather *structural* regularizing decomposition (or *regularisation*) of a matrix A shall be conducted as

$$\boxed{A^\circ \equiv \mathring{A} := A - \langle V_+, A \rangle U_+ - \langle V_-, A \rangle U_-} \tag{3.2}$$

for $U_\sigma, V_\sigma \in \mathbf{C}^{2N \times 2N}$ satisfying $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma, \tau}$ and the normalisation $\langle U_\sigma, U_\sigma \rangle = 1$, where recall that $\langle R, T \rangle := \langle R^* T \rangle$ denotes the (normalised) Hilbert-Schmidt scalar product. The *regularisation map*

$$(1 - \Pi) : \mathbf{C}^{2N \times 2N} \rightarrow \mathbf{C}^{2N \times 2N}, \quad A \mapsto \mathring{A}, \tag{3.3}$$

where Π is a *two-dimensional* (non-orthogonal) projection,¹⁰ is closely related to the built-in chiral symmetry (2.16) of our model. Indeed, for other ensembles without this special structure only *one* of the terms $\langle V_\sigma, A \rangle U_\sigma$ in (3.2) would be present.

As mentioned above, the matrices V_\pm are determined as *critical* eigenvectors (with corresponding small eigenvalue) of naturally associated two-body stability operators with their precise form worked out in Appendix A and given in (A.19). In Appendix A we also give two different calculations that helped us guess these formulas. However, for the directions U_\pm there are *a priori* no further constraints (apart from orthogonality and normalisation). Hence, as it turns out to be convenient for our proofs, we will *choose* the matrices U_σ (in principle, even allowing for two different deformations $\Lambda_1 \neq \Lambda_2$) in such a way, that a resolvent identity

$$G^{\Lambda_1}(w_1)U_\sigma G^{\Lambda_2}(w_2) \approx (G^{\Lambda_1}(w_1) - G^{\Lambda_2}(\sigma w_2))U_\sigma, \tag{3.4}$$

can be applied (here, the symbol ‘ \approx ’ neglects lower order terms). This is used to reduce the number of resolvents in a chain. Note that, again due to the eminent chiral symmetry (2.16) for the resolvents, there are in fact *two* matrices U_σ for which a resolvent identity (3.4) can be applied.

Although the regularisation (3.2) shall be motivated for arbitrary deformations Λ_1, Λ_2 in Appendix A, we will henceforth choose a single bounded deformation $\Lambda \in \mathbf{C}^{N \times N}$, which remains fixed with the just mentioned exception in Appendix A. For a single deformation Λ , this restricts the matrices U_\pm satisfying (3.4) to be given by E_\pm .

In case that the spectral parameters (w_1, w_2) appearing in (3.1) (with a single fixed deformation Λ) are such that *none* of the eigenvectors of the stability operator is *critical* (cf. Appendix B), we consider *every* matrix A as regular. The distinction between these two scenarios is regulated by cutoff functions $\mathbf{1}_\delta^\pm$ introduced in (3.6) below.

3.1. Regular observables: a bound on (3.1)

As already mentioned above, our main result for the Hermitised random matrix, Theorem 2.7, shall be derived from a bound on (3.1), where we assume the (real parts of the) spectral parameters w_1, w_2 to be in the *bulk* of the scDos ρ (recall (2.6)).

We now specify the concept of regularisation (3.2) to our setting. The eigenvectors V_\pm will be computed in Appendix A, the matrices U_\pm are simply chosen as E_\pm .

Definition 3.1. (Regular observables) Given $\kappa > 0$, let¹¹

$$\delta = \delta(\kappa, \|\Lambda\|) > 0 \tag{3.5}$$

¹⁰ The condition $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma, \tau}$ guarantees that the regularisation is idempotent, i.e. $(\mathring{A})^\circ = \mathring{A}$ and $\Pi^2 = \Pi$.

¹¹ Note that the parameter $\delta > 0$ is independent of the matrix size N .

be sufficiently small (to be chosen below, see (4.22)) and let $w, w' \in \mathbf{C} \setminus \mathbf{R}$ with $\operatorname{Re} w, \operatorname{Re} w' \in \mathbf{B}_\kappa$ be spectral parameters. Furthermore, we introduce the (symmetric) cutoff functions

$$\mathbf{1}_\delta^\pm(w, w') := \phi_\delta(\operatorname{Re} w \mp \operatorname{Re} w') \phi_\delta(\operatorname{Im} w) \phi_\delta(\operatorname{Im} w'), \tag{3.6}$$

where $0 \leq \phi_\delta \leq 1$ is a smooth symmetric bump function on \mathbf{R} satisfying $\phi_\delta(x) = 1$ for $|x| \leq \delta/2$ and $\phi_\delta(x) = 0$ for $|x| \geq \delta$.

(a) We define the (w, w') -regular component or (w, w') -regularisation $\mathring{A}^{w, w'}$ of a matrix A as¹²

$$\mathring{A}^{w, w'} := A - \sum_{\tau=\pm} \mathbf{1}_\delta^{\tau s}(w, w') \frac{\langle M(\operatorname{Re} w + i\operatorname{Im} w)AM(\operatorname{Re} w' + \tau i\operatorname{Im} w')E_{\tau s} \rangle}{\langle M(\operatorname{Re} w + i\operatorname{Im} w)E_{\tau s}M(\operatorname{Re} w' + \tau i\operatorname{Im} w')E_{\tau s} \rangle} E_{\tau s},$$

(3.7)

where the relative sign of the imaginary parts is defined as

$$s \equiv s_{w, w'} := -\operatorname{sgn}(\operatorname{Im} w \operatorname{Im} w'). \tag{3.8}$$

(b) We say that A is (w, w') -regular if and only if $A = \mathring{A}^{w, w'}$.

The regularisation shall be revisited in Definition 4.2, where we tailor it to certain averaged (4.5) or isotropic (4.6) resolvent chains.

Remark 3.2. We have several comments concerning the above definition.

(i) In case that at least one of the spectral parameters is away from the imaginary axis, say $|\operatorname{Re} w| > \delta$ w.l.o.g., then the regularisation in (3.7) contains at most one summand: If $\mathbf{1}_\delta^+(w, w') = 1$, i.e. $\operatorname{Re} w$ is close to $\operatorname{Re} w'$, then we have that

$$\mathring{A}^{w, w'} := A - \frac{\langle M(w)AM(\operatorname{Re} w' + s i\operatorname{Im} w') \rangle}{\langle M(w)M(\operatorname{Re} w' + s i\operatorname{Im} w') \rangle} E_+,$$

whereas if $\mathbf{1}_\delta^-(w, w') = 1$, i.e. if $\operatorname{Re} w$ is close to $-\operatorname{Re} w'$, then we have that

¹² Putting the summation parameter τ at the second spectral parameter w' (and not at w) is simply a free choice, which we made here. More precisely, defining the regularisation as

$$\tilde{\mathring{A}}^{w, w'} := A - \sum_{\tau=\pm} \mathbf{1}_\delta^{\tau s}(w, w') \frac{\langle M(\operatorname{Re} w + \tau i\operatorname{Im} w)AM(\operatorname{Re} w' + i\operatorname{Im} w')E_{\tau s} \rangle}{\langle M(\operatorname{Re} w + \tau i\operatorname{Im} w)E_{\tau s}M(\operatorname{Re} w' + i\operatorname{Im} w')E_{\tau s} \rangle} E_{\tau s}$$

would equally work in our proofs (see Appendices A and B for details).

$$\mathring{A}^{w,w'} := A - \frac{\langle M(w)AE_-M(-\operatorname{Re} w' + \mathfrak{s}i\operatorname{Im} w') \rangle}{\langle M(w)M(-\operatorname{Re} w' + \mathfrak{s}i\operatorname{Im} w') \rangle} E_- ,$$

where we used that $M(w)E_- = -E_-M(-w)$ (see (2.19))

(ii) The cutoff functions in (3.6) satisfy the basic symmetry properties

$$\mathbf{1}_\delta^\pm(w, w') = \mathbf{1}_\delta^\pm(\bar{w}, \bar{w}') = \mathbf{1}_\delta^\pm(w, \bar{w}') = \mathbf{1}_\delta^\pm(\bar{w}, w').$$

However, \mathring{A} is *not* symmetric in its two spectral parameters, i.e. $\mathring{A}^{w,w'} \neq \mathring{A}^{w',w}$ in general

(iii) For spectral parameters satisfying $\mathbf{1}_\delta^\pm(w, w') > 0$, it will be shown in Appendix B that the respective denominators in (3.7) are bounded away from zero. In particular, the linear map $A \mapsto \mathring{A}$ is bounded with a bound depending only on δ and $\|\Lambda\|$.

(iv) Whenever it holds that $\mathbf{1}_\delta^\pm(w, w') = 0$ then also $\mathbf{1}_{\delta'}^\pm(w, w') = 0$ for every $0 < \delta' < \delta$. Conversely, whenever it holds that $\mathbf{1}_\delta^\pm(w, w') = 1$ then also $\mathbf{1}_{\delta'}^\pm(w, w') = 1$ for every $0 < \delta < \delta'$.

(v) We point out that the notion of regularity implicitly depends on κ and δ and hence also on the (norm of the) deformation Λ .

Moreover, the regularisation defined above satisfies the following elementary properties. The identities in (3.10) and (3.9) are immediate from the definition, the perturbative statements are proven in Appendix B.

Lemma 3.3. *Fix a bounded deterministic deformation $\Lambda \in \mathbf{C}^{N \times N}$ and let $A \in \mathbf{C}^{2N \times 2N}$ be an arbitrary bounded deterministic matrix.*

(i) *Let $w, w' \in \mathbf{C} \setminus \mathbf{R}$ with $\operatorname{Re} w, \operatorname{Re} w' \in \mathbf{B}_\kappa$. Then, we have the identities*

$$(\mathring{A}^{w,w'})^* = (\mathring{A}^*)^{\bar{w}, \bar{w}}, \quad \mathring{A}^{w,w'} E_- = (A \mathring{E}_-)^{w, -w'}, \quad E_- \mathring{A}^{w,w'} = (E_- A)^{-w, w'}. \tag{3.9}$$

(ii) *Moreover, by definition it holds that*

$$\mathring{A}^{w, \bar{w}'} = \mathring{A}^{w, w'}, \tag{3.10}$$

and setting $\mathfrak{s} := -\operatorname{sgn}(\operatorname{Im} w \operatorname{Im} w')$, we have the perturbative estimate¹³

$$\mathring{A}^{\bar{w}, w'} = \mathring{A}^{w, w'} + \mathcal{O}(|w - \mathfrak{s}\bar{w}'| \wedge 1) E_\mathfrak{s} + \mathcal{O}(|w + \mathfrak{s}w'| \wedge 1) E_{-\mathfrak{s}}. \tag{3.11}$$

(iii) *Let $w_1, w'_1, w_2, w'_2 \in \mathbf{C} \setminus \mathbf{R}$ with $\operatorname{Re} w_1, \operatorname{Re} w'_1, \operatorname{Re} w_2, \operatorname{Re} w'_2 \in \mathbf{B}_\kappa$ as well as $\operatorname{Im} w_1 \cdot \operatorname{Im} w_2 > 0$ and $\operatorname{Im} w'_1 \cdot \operatorname{Im} w'_2 > 0$ be spectral parameters. Then we have that*

¹³ Note that the asymmetry between (3.11) and (3.10) stems from the asymmetry imposed in the definition of the regularisation, namely by placing the summation index τ in (3.7) at the second spectral parameter.

$$\mathring{A}^{w_2, w'_1} = \mathring{A}^{w_1, w'_1} + \mathcal{O}(|w_1 - w_2| \wedge 1)E_+ + \mathcal{O}(|w_1 - w_2| \wedge 1)E_-, \quad (3.12)$$

$$\mathring{A}^{w_1, w'_2} = \mathring{A}^{w_1, w'_1} + \mathcal{O}(|w'_1 - w'_2| \wedge 1)E_+ + \mathcal{O}(|w'_1 - w'_2| \wedge 1)E_-. \quad (3.13)$$

We can now state the bound on (3.1) for regular observables, which shall be proven in Section 4 as an immediate corollary to a local law for (3.1) given in Theorem 4.4 and the bound from Lemma 4.3.

Proposition 3.4. Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N \times N}$, $\epsilon > 0$, $\kappa > 0$, and let $w_1, w_2 \in \mathbf{C}$ with $|w_1|, |w_2| \leq N^{100}$, $\text{Re } w_1, \text{Re } w_2 \in \mathbf{B}_\kappa$, and $|\text{Im } w_1|, |\text{Im } w_2| \geq N^{-1+\epsilon}$. Moreover, let $A_1 \in \mathbf{C}^{2N \times 2N}$ be a (w_1, w_2) -regular and $A_2 \in \mathbf{C}^{2N \times 2N}$ a (w_2, w_1) -regular deterministic matrix, both satisfying $\|A_1\|, \|A_2\| \lesssim 1$. Then we have

$$|\langle G(w_1)\mathring{A}_1^{w_1, w_2}G(w_2)\mathring{A}_2^{w_2, w_1} \rangle| \prec 1. \quad (3.14)$$

3.2. Estimating (3.1) for general observables

Armed with the correct regularisation, we can now present a systematic analysis of $\langle G(w_1)A_1G(w_2)A_2 \rangle$ from (3.1) for arbitrary bounded deterministic matrices A_1, A_2 . Decomposing A_1, A_2 according to Definition 3.1 as

$$\begin{aligned} A_1 &= \mathring{A}_1^{w_1, w_2} + \langle\langle A_1 \rangle\rangle_{w_1, w_2}^+ E_+ + \langle\langle A_1 \rangle\rangle_{w_1, w_2}^- E_-, \\ A_2 &= \mathring{A}_2^{w_2, w_1} + \langle\langle A_2 \rangle\rangle_{w_2, w_1}^+ E_+ + \langle\langle A_2 \rangle\rangle_{w_2, w_1}^- E_-, \end{aligned} \quad (3.15)$$

(where $\langle\langle \cdot \rangle\rangle_{w, w'}$ can be read off as the coefficients in (3.7)) and plugging (3.15) into (3.1), we find that

$$\begin{aligned} \langle G(w_1)A_1G(w_2)A_2 \rangle &= \sum_{\sigma, \tau} \langle\langle A_1 \rangle\rangle_{w_1, w_2}^\sigma \langle\langle A_2 \rangle\rangle_{w_2, w_1}^\tau \langle G(w_1)E_\sigma G(w_2)E_\tau \rangle \\ &\quad + \sum_\sigma \langle\langle A_1 \rangle\rangle_{w_1, w_2}^\sigma \langle G(w_1)E_\sigma G(w_2)\mathring{A}_2^{w_2, w_1} \rangle \\ &\quad + \sum_\tau \langle\langle A_2 \rangle\rangle_{w_2, w_1}^\tau \langle G(w_1)\mathring{A}_1^{w_1, w_2} G(w_2)E_\tau \rangle \\ &\quad + \langle G(w_1)\mathring{A}_1^{w_1, w_2} G(w_2)\mathring{A}_2^{w_2, w_1} \rangle. \end{aligned} \quad (3.16)$$

Which terms in (3.16) are effectively present depends on the coefficients $\langle\langle \cdot \rangle\rangle_{w, w'}$, i.e. on the singular components of A_1, A_2 . For terms with nonzero coefficients the following rule of thumb applies. Denoting $\eta := \min(|\text{Im } w_1|, |\text{Im } w_2|) \geq N^{-1+\epsilon}$, the terms $\langle GEGE \rangle$ in the first line of (3.16) are bounded by $1/\eta$, the terms $\langle GEG\mathring{A} \rangle$ in the two middle lines of (3.16) are bounded by $1/\sqrt{\eta}$, and $\langle G\mathring{A}G\mathring{A} \rangle$ in the last line is of order one (Proposition 3.4). This is in perfect agreement with the $\sqrt{\eta}$ -rule mentioned in the Introduction (see also Remark 4.6 below). Some of these bounds are actually sharp for special values of w_1, w_2 , for example

$$\langle G(w)E_+G(\bar{w})E_+ \rangle = \frac{\langle \text{Im } G(w) \rangle}{\eta} \sim \frac{1}{\eta}, \quad \text{or} \quad \langle G(w)E_-G(-\bar{w})E_- \rangle = -\frac{\langle \text{Im } G(w) \rangle}{\eta},$$

where we used the chiral symmetry (2.16). In fact, two terms with $\sigma\tau = -1$ in the first line of (3.16) are identically zero by applying the chiral symmetry, followed by the resolvent identity and $\langle GE_- \rangle = 0$. For a middle term in (3.16) we have

$$\langle G(w)E_+G(\bar{w})\mathring{A}^{\bar{w},w} \rangle = \frac{1}{\eta} \langle \text{Im } G(w)\mathring{A}^{\bar{w},w} \rangle \lesssim 1 + \frac{1}{N\eta} \frac{1}{\sqrt{\eta}}.$$

In the very last relation we treated $\langle G(w)\mathring{A}^{\bar{w},w} \rangle$ and $\langle G(\bar{w})\mathring{A}^{\bar{w},w} \rangle$ separately. In both cases we first used Lemma 3.3 to adjust the regularisation to $\mathring{A}^{w,w}$ and $\mathring{A}^{\bar{w},\bar{w}}$, respectively, to match the new single-resolvent setup and then we applied the corresponding single-resolvent local law with regular observable (see Theorem 4.5 below).

Note that the most critical estimate concerns the last line of (3.16), i.e. the regular part for both observable matrices. The bound (3.14) is obtained from a local law with *two* resolvents and *two* regular matrices, while the first and the middle terms in (3.16) can be understood already from an improved local law for *one* resolvent and *one* regular matrix (see Theorem 4.5 below) after applying resolvent identities and adjusting the regularisation by Lemma 3.3. Furthermore, observe that the sizes of the first three lines in (3.16) are sensitive to w_1, w_2 via the usual resolvent identity (see (3.21) below) and the chiral symmetry (2.16), for example

$$\begin{aligned} |\langle G(w_1)E_+G(w_2)E_+ \rangle| &= \left| \frac{\langle G(w_1) - G(w_2) \rangle}{w_1 - w_2} \right| \lesssim \frac{1}{|w_1 - w_2|}, \\ \text{or } |\langle G(w_1)E_-G(w_2)E_- \rangle| &= |\langle G(w_1)G(-w_2) \rangle| \lesssim \frac{1}{|w_1 + w_2|}, \end{aligned}$$

while the last line in (3.16) is typically order one.

Summarizing, the singular parts of $\langle G(w_1)A_1G(w_2)A_2 \rangle$ can be explicitly computed (using single-resolvent local laws) as explicit functions of w_1, w_2 , while the regular part remains of order one. A combination of our decomposition (3.7), the perturbation formulas from Lemma 3.3, and our single- and two-resolvent local laws together with their explicit deterministic terms from the subsequent Section 4 provide an effective recipe to compute $\langle G(w_1)A_1G(w_2)A_2 \rangle$ with high precision in all cases. We refrain from formulating it as a comprehensive theorem due to the large number of cases.

3.3. Proof of the main results

We will first focus on the proof of Theorem 2.7 and turn to the proofs of Theorem 2.2 and Theorem 2.4 afterwards.

3.3.1. *Proof of Theorem 2.7*

As a first step towards the proof of Theorem 2.7, we show that (2.24) indeed follows from a bound similar to (3.14), where G is replaced by $\text{Im } G$. The proof of the following simple lemma is given after completion of the proof of Theorem 2.7.

Lemma 3.5. *Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N \times N}$, $\epsilon > 0$, $\kappa > 0$, and let $B \in \mathbf{C}^{2N \times 2N}$. Then, for any bulk indices $|i|, |j| \leq N$, i.e. with $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$, and $\eta \geq N^{-1+\epsilon}$, we have*

$$N |\langle \mathbf{w}_i, B \mathbf{w}_j \rangle|^2 \prec (N\eta)^2 \langle \text{Im } G(\gamma_i + i\eta) B \text{Im } G(\gamma_j + 2i\eta) B^* \rangle. \tag{3.17}$$

The same bound still holds without the factor of two in the argument of the second resolvent in (3.17). However, we chose to have it, in order to ensure that the spectral parameters of the two resolvents are always forced to be different.

Proof of Theorem 2.7. Having Lemma 3.5 at hand, we are left with estimating the rhs. of (3.17) for

$$B = A - \frac{\langle \text{Im } M(\gamma_j) A \rangle}{\langle \text{Im } M(\gamma_j) \rangle} E_+ - \frac{\langle \text{Im } M(\gamma_j) E_- A \rangle}{\langle \text{Im } M(\gamma_j) \rangle} E_- \tag{3.18}$$

using Proposition 3.4. Note that the two terms in (2.24) carrying a δ -symbol arise from the orthogonality relations $\langle \mathbf{w}_i, E_\pm \mathbf{w}_j \rangle = \delta_{j, \pm i}$, following from the spectral symmetry described around (2.16).

We now write out $\text{Im } G(w) = (G(w) - G(\bar{w}))/2i$, such that (3.17) leaves us with four different terms, each of which can be bounded individually. Since their treatment is completely analogous, we focus on the exemplary term

$$\langle G(\gamma_i + i\eta) B G(\gamma_j - 2i\eta) B^* \rangle \tag{3.19}$$

with the deterministic matrix B being defined in (3.18). We rely on the following simple perturbative lemma, which follows from Lemma 3.3 by invoking Lemma B.1.

Lemma 3.6. *Using the notation introduced in (3.7), the matrix $B \in \mathbf{C}^{2N \times 2N}$ from (3.18) satisfies*

$$\begin{aligned} B &= \mathring{A}^{\gamma_i + i\eta, \gamma_j - 2i\eta} + \mathcal{O}(|\gamma_i - \gamma_j| + \eta) E_+ + \mathcal{O}(|\gamma_i + \gamma_j| + \eta) E_- , \\ B^* &= (\mathring{A}^*)^{\gamma_j - 2i\eta, \gamma_i + i\eta} + \mathcal{O}(|\gamma_i - \gamma_j| + \eta) E_+ + \mathcal{O}(|\gamma_i + \gamma_j| + \eta) E_- . \end{aligned} \tag{3.20}$$

Hence, plugging (3.20) into (3.19), we get a sum of several terms, which can all be estimated separately. For the ‘leading term’, we use Proposition 3.4 to get that

$$|\langle G(\gamma_i + i\eta) \mathring{A}^{\gamma_i + i\eta, \gamma_j - 2i\eta} G(\gamma_j - 2i\eta) (\mathring{A}^*)^{\gamma_j - 2i\eta, \gamma_i + i\eta} \rangle| \prec 1 .$$

Two further representative terms are given by

$$\mathcal{O}(|\gamma_i \mp \gamma_j| + \eta) \langle G(\gamma_i + i\eta)E_{\pm}G(\gamma_j - 2i\eta)C \rangle,$$

where $C \in \mathbf{C}^{2N \times 2N}$ is some generic bounded matrix. Now, by using (2.16), these terms can be rewritten as

$$\mathcal{O}(|\gamma_i \mp \gamma_j| + \eta) \langle G(\gamma_i + i\eta)G(\pm(\gamma_j - 2i\eta))E_{\pm}C \rangle.$$

For either sign choice (due to the factor two), we can now employ a simple resolvent identity

$$G(w_1)G(w_2) = \frac{G(w_1) - G(w_2)}{w_1 - w_2}, \tag{3.21}$$

leaving us with

$$\frac{\mathcal{O}(|\gamma_i - \gamma_j| + \eta)}{(\gamma_i \mp \gamma_j) + (1 \pm 2)i\eta} \langle [G(\gamma_i + i\eta) - G(\pm(\gamma_j - 2i\eta))]C \rangle,$$

which is surely stochastically dominated by one by means of Theorem 2.6. Thus, collecting all the terms, we find that |(3.19)| < 1.

Finally, we choose $\eta = N^{-1+\xi}$ for an arbitrarily small $\xi > 0$, such that Lemma 3.5 with B as in (3.18) yields Theorem 2.7. \square

We conclude with giving a proof of Lemma 3.5.

Proof of Lemma 3.5. By spectral decomposition we write

$$\begin{aligned} \langle \text{Im } G(\gamma_i + i\eta)B \text{Im } G(\gamma_j + 2i\eta)B^* \rangle &= \frac{1}{2N} \sum_{k,l} \frac{2\eta^2 |\langle \mathbf{w}_k, B\mathbf{w}_l \rangle|^2}{[(\lambda_k - \gamma_i)^2 + \eta^2][(\lambda_l - \gamma_j)^2 + 4\eta^2]} \\ &\asymp \frac{|\langle \mathbf{w}_i, B\mathbf{w}_j \rangle|^2}{N\eta^2}, \end{aligned}$$

which proves (3.17). We point out that in the last inequality we used rigidity of the eigenvalues [2,39]:

$$|\lambda_i - \gamma_i| \prec \frac{1}{N}, \tag{3.22}$$

which holds for bulk indices as a standard consequence of the single-resolvent local law, Theorem 2.6. \square

3.3.2. *Proof of Theorem 2.2*

The bounds in (2.8a), (2.8b), and (2.8c) follow from Theorem 2.7 by choosing

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix},$$

respectively, and invoking (2.17)–(2.18). \square

3.3.3. *Proof of Theorem 2.4*

By the definition

$$H^z := \begin{pmatrix} 0 & X + \Lambda - z \\ (X + \Lambda - z)^* & 0 \end{pmatrix}$$

it follows that $\mu \in \text{Spec}(X + \Lambda)$ if and only if $\lambda_1^\mu = 0$. Here by $\{\lambda_i^z\}_{i \in [N]}$ we denoted the increasingly ordered non-negative eigenvalues of H^z . We remark that Λ is omitted by the notation since it is fixed throughout the proof. In particular, using the bound for products of two resolvents and two regular matrices in (3.14), we will now prove the lower bound in (2.12) for the overlap of left and right eigenvectors corresponding to eigenvalues μ which lies in the bulk of the spectrum of $X + \Lambda$.

Proof of Theorem 2.4. Define

$$F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathbf{C}^{2N \times 2N},$$

then by (3.14), for $\eta \geq N^{-1}$, we conclude

$$\sup_{z \in \text{bulk}} \langle \text{Im } G^z(i\eta) F \text{Im } G^z(i\eta) F^* \rangle \prec 1, \tag{3.23}$$

where the supremum is taken over the bulk as given in Definition 2.3. The fact that (3.23) holds for the supremum over the z 's with very high probability follows by a standard grid argument together with the Lipschitz continuity of $z \mapsto \text{Im } G^z$. Here we used that F is regular in the sense of (3.7); this immediately follows from the fact that F is (block) off-diagonal and $\text{Im } M(i\eta)$ is (block) diagonal as follows from (2.17)–(2.18) and the symmetry of ρ . We now want to show that if we choose $z = \mu_i$ to be a bulk eigenvalue of $X + \Lambda$ the upper bound (3.23) implies a lower bound on \mathcal{O}_{ii} . To make the notation simpler, from now on we denote $\mu = \mu_i$.

Consider the non-Hermitian left/right-eigenvectors l, r , with corresponding eigenvalue μ , defined as in (2.9) and set

$$\mathcal{P} := \begin{pmatrix} \frac{\bar{l}l^*}{\|l\|^2} & 0 \\ 0 & \frac{rr^*}{\|r\|^2} \end{pmatrix}.$$

Clearly \mathcal{P} is a rank two orthogonal projection whose range lies in the kernel of H^μ , recalling that up to scalar multiples the non-Hermitian eigenvectors $\bar{\mathbf{l}}, \mathbf{r}$ coincide with some singular vectors \mathbf{u}, \mathbf{v} of $X + \Lambda - \mu$, respectively, forming an eigenvector $\mathbf{w} = (\mathbf{u}, \mathbf{v})$ in the kernel of H^μ . Note that $\text{Ker}(H^\mu)$ has dimension two if μ is a simple eigenvalue, but in general the multiplicity of μ and the multiplicity of $\lambda_1^\mu = 0$ may differ. Let \mathcal{Q} be the orthogonal projection onto the kernel of H^μ , then $\mathcal{P} \leq \mathcal{Q}$. Then, almost surely, by spectral decomposition (and by the spectral symmetry of H^μ)

$$\text{Im } G^\mu(i\eta) = \frac{\mathcal{Q}}{\eta} + \sum_{i:\lambda_i^\mu \neq 0} \frac{\eta}{(\lambda_i^\mu)^2 + \eta^2} \begin{pmatrix} \mathbf{u}_i^\mu \\ \mathbf{v}_i^\mu \end{pmatrix} \begin{pmatrix} \mathbf{u}_i^\mu \\ \mathbf{v}_i^\mu \end{pmatrix}^* \geq \frac{\mathcal{P}}{\eta}.$$

By (3.23) we thus obtain

$$1 \succ \sup_{z \in \text{bulk}} \langle \text{Im } G^z(i\eta) F \text{Im } G^z(i\eta) F^* \rangle \succ \frac{1}{\eta^2} \langle \mathcal{P} F \mathcal{P} F^* \rangle = \frac{|\langle \bar{\mathbf{l}}, \mathbf{r} \rangle|^2}{N \eta^2 \|\mathbf{r}\|^2 \|\mathbf{l}\|^2},$$

which, by (2.11), implies

$$\mathcal{O}_{ii} = \|\mathbf{r}\|^2 \|\mathbf{l}\|^2 \succ \frac{1}{N \eta^2}.$$

Choosing $\eta = N^{-1+\epsilon/2}$, this concludes the proof. \square

4. Local laws with regular observables

The goal of the present section is to establish the key Proposition 3.4 by proving an averaged local law for a product of two resolvents (of the Hermitisation (2.15)) in the bulk of the scDos ρ with *regular* (recall Definition 3.1 and see Definition 4.2 below) deterministic matrices A_1, A_2 in between. Throughout the rest of this paper, we consider the case of several spectral parameters w_1, w_2, \dots and fixed bounded deformations $\Lambda_1 = \Lambda_2 = \dots \equiv \Lambda \in \mathbf{C}^{N \times N}$, which we continue to omit from the notation.

Using the abbreviations $G_i := G(w_i) := G^\Lambda(w_i)$ (and analogously for M_i), the deterministic approximation to the resolvent chain

$$G_1 B_1 \cdots B_{k-1} G_k$$

for arbitrary deterministic B_1, \dots, B_k ¹⁴ is denoted by

$$M(w_1, B_1, \dots, B_{k-1}, w_k) \tag{4.1}$$

and defined recursively in the length k of the chain.

¹⁴ We will use the notational convention, that the letter B denotes arbitrary (generic) matrices, while A is reserved for *regular* matrices, in the sense of Definition 4.2.

Definition 4.1. Fix $k \in \mathbf{N}$ and let $w_1, \dots, w_k \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters. As usual, the corresponding solutions to the MDE (2.20) are denoted by $M(w_j)$, $j \in [k]$. Then, for deterministic matrices B_1, \dots, B_{k-1} we recursively define

$$M(w_1, B_1, \dots, B_{k-1}, w_k) = (\mathcal{B}_{1k})^{-1} \left[M(w_1) B_1 M(w_2, \dots, w_k) + \sum_{\sigma=\pm} \sum_{l=2}^{k-1} \sigma M(w_1) \langle M(w_1, \dots, w_l) E_\sigma \rangle E_\sigma M(w_l, \dots, w_k) \right], \tag{4.2}$$

where we introduced the shorthand notation

$$\mathcal{B}_{mn} \equiv \mathcal{B}(w_m, w_n) = 1 - M(w_m) \mathcal{S}[\cdot] M(w_n) \tag{4.3}$$

for the so-called stability operator, discussed later in Appendix B.

Note that the recursion (4.2) is well defined, since on the rhs. of (4.2), there are only $M(w_m, \dots, w_n)$ appearing for which the number of spectral parameters is strictly smaller than on the lhs. of (4.2), i.e. $n - m + 1 < k$. We may call these formulas (4.2) *recursive Dyson equations* as they provide us with the correct deterministic quantity for longer resolvent chains. As an example, we have that

$$M(w_1, B_1, w_2) = \mathcal{B}_{12}^{-1} [M_1 B_1 M_2] = M_1 \mathcal{X}_{12} [B_1] M_2, \tag{4.4}$$

where \mathcal{B}_{12}^{-1} is the inverse stability operator (4.3) and $\mathcal{X}_{12} = (1 - \mathcal{S}[M_1 \cdot M_2])^{-1}$. We remark that M satisfies several different recursions besides (4.2); they are presented in Lemma D.1 of the arXiv:2301.03549 version of this article (see also [24, Lemma 5.4] for a simpler setup of Wigner matrices). The equivalence of these recursions will be proved via the so-called *meta-argument*, see e.g. [31].

As already mentioned above, we are aiming at local laws for expressions of the form

$$\langle G_1 A_1 \cdots G_k A_k \rangle \tag{4.5}$$

in the averaged case, or

$$(G_1 A_1 \cdots A_k G_{k+1})_{\mathbf{x}\mathbf{y}} \tag{4.6}$$

in the isotropic case, where the deterministic matrices A_1, \dots, A_k are assumed to be *regular*.

The general concept of *regularity* depending on two spectral parameters w and w' has already been introduced in Definition 3.1. In the following definition we tailor this concept to observables in chains (4.5) and (4.6). It basically says that observable A_j , located between $G_j = G(w_j)$ and $G_{j+1} = G(w_{j+1})$ in these chains will naturally be regularised using the spectral parameters w_j and w_{j+1} .

Definition 4.2. (Regular observables in chains)

Fix $\kappa > 0$ and let $\delta = \delta(\kappa, \|\Lambda\|) > 0$ be small enough (see (3.5) and (4.22)). Consider one of the two expressions (4.5) or (4.6) for some fixed length $k \in \mathbf{N}$ and bounded matrices $\|A_i\| \lesssim 1$ and let $w_1, \dots, w_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters with $\text{Re } w_i \in \mathbf{B}_\kappa$. For any $j \in [k]$, analogously to (3.6), we denote

$$\mathbf{1}_\delta^\pm(w_j, w_{j+1}) := \phi_\delta(\text{Re } w_j \mp \text{Re } w_{j+1}) \phi_\delta(\text{Im } w_j) \phi_\delta(\text{Im } w_{j+1}) \tag{4.7}$$

and $\mathfrak{s}_j := -\text{sgn}(\text{Im } w_j \text{Im } w_{j+1})$, where, here and in the following, in case of (4.5), the indices are understood cyclically modulo k .

- (a) For $i \in [k]$ we define the *regular component* or *regularisation* of A_i from (4.5) or (4.6) (w.r.t. the pair of spectral parameters (w_i, w_{i+1})) as

$$\mathring{A}_i := \mathring{A}_i^{w_i, w_{i+1}}. \tag{4.8}$$

- (b) Moreover, we call A_i *regular* (w.r.t. (w_i, w_{i+1})) if and only if $\mathring{A}_i = A_i$.

For example, in case of (4.5) for $k = 1$ with spectral parameter $w_1 \in \mathbf{C} \setminus \mathbf{R}$ satisfying $\text{Re } w_1 \in \mathbf{B}_\kappa$, $|\text{Re } w_1| \leq \delta/4$ and $|\text{Im } w_1| \leq \delta/2$ (recall (3.5) and (4.7)), the regular component of A_1 is given by

$$\mathring{A}_1 := A_1 - \frac{\langle \text{Im } M_1 A_1 \rangle}{\langle \text{Im } M_1 \rangle} E_+ - \frac{\langle M_1 A_1 M_1 E_- \rangle}{\langle M_1 E_- M_1 E_- \rangle} E_-. \tag{4.9}$$

Here we used the short-hand notation $M_1 := M(w_1)$.

We emphasise, that our notation $\mathring{\cdot}$ for the regular component of A_i does *not* have an overall fixed meaning but depends on the spectral parameters of the resolvents ‘surrounding’ the deterministic matrix A_i under consideration, i.e.

$$\langle \dots G_i A_i G_{i+1} \dots \rangle \quad \text{or} \quad \left(\dots G_i A_i G_{i+1} \dots \right)_{\mathbf{xy}},$$

or in case of (4.5) for $k = 1$ the single spectral parameter involved. However, if we aim at specifying the spectral parameters defining the operation $\mathring{\cdot}$, we add them (or their indices) as a subscript, i.e. write

$$\mathring{A}_i^{w_i, w_{i+1}} \equiv \mathring{A}_i^{i, i+1} \equiv \mathring{A}_i \equiv A_i^\circ \equiv A_i^{\circ_{i, i+1}} \equiv A_i^{\circ_{w_i, w_{i+1}}},$$

as done in Definition 3.1, and do not use imprecise notation \mathring{A}_i .

The just explained caveats are in stark contrast to the case of Wigner matrices [23, 26, 27], where the regular component of a matrix A is simply its traceless part, i.e. $\mathring{A} = A - \langle A \rangle$, irrespective of the spectral parameters involved. Apart from this independence of the location in the spectrum, there is a one further important difference to our case, which

we already mentioned in Section 3: For Wigner matrices, the condition for A being regular is one-dimensional and hence restricts A to a $(N^2 - 1)$ -dimensional subspace of $\mathbf{C}^{N \times N}$ (the traceless matrices), whereas in our case, the regularity condition is two-dimensional (if $\mathbf{1}_\delta^g(\cdot, \cdot) = 1$) and hence restricts a regular matrix A to a $((2N)^2 - 2)$ -dimensional subspace of $\mathbf{C}^{2N \times 2N}$, which depends on the ‘surrounding’ spectral parameters.

We now give bounds on the size of the deterministic term $M(w_1, B_1, \dots, B_{k-1}, w_k)$ from (4.1), where all B_i are regular in the sense of Definition 4.2. The proof of this lemma is presented in Appendix D of the arXiv:2301.03549 version of this article.

Lemma 4.3. (Bounds on M , see [26, Lemma 2.4])

Fix $\kappa > 0$. Let $k \in [4]$ and $w_1, \dots, w_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters with $\text{Re } w_j \in \mathbf{B}_\kappa$. Then, for bounded regular deterministic matrices A_1, \dots, A_k (in the sense of Definition 4.2), we have the bounds

$$\|M(w_1, A_1, \dots, A_k, w_{k+1})\| \lesssim \begin{cases} \frac{1}{\eta^{\lfloor k/2 \rfloor}} & \text{if } \eta \leq 1 \\ \frac{1}{\eta^{k+1}} & \text{if } \eta > 1 \end{cases}, \tag{4.10}$$

$$|\langle M(w_1, A_1, \dots, A_{k-1}, w_k) A_k \rangle| \lesssim \begin{cases} \frac{1}{\eta^{\lfloor k/2 \rfloor - 1}} \vee 1 & \text{if } \eta \leq 1 \\ \frac{1}{\eta^k} & \text{if } \eta > 1 \end{cases}, \tag{4.11}$$

for the deterministic approximation (4.1) of a resolvent chain, where $\eta := \min_j |\text{Im } w_j|$.

For the presentation of our main results, we would only need (4.10) and (4.11) for $k \in [2]$ from the previous lemma. However, the remaining bounds covered by Lemma 4.3 will be instrumental in our proofs of Theorems 4.5 and 4.4 below (see Sections 5 and 6).

The main result of the present section and most important input for our proofs in Section 3 is the following averaged local law in the bulk of the spectrum for two resolvents and regular matrices.

Theorem 4.4. (Local laws with two regular matrices)

Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N \times N}$, $\epsilon > 0$ and $\kappa > 0$. Then, for spectral parameters $w_1, w_2, w_3 \in \mathbf{C}$ satisfying $\max_j |w_j| \leq N^{100}$, $\text{Re } w_j \in \mathbf{B}_\kappa$ and $\eta := \min_j |\text{Im } w_j| \geq N^{-1+\epsilon}$, deterministic vectors \mathbf{x}, \mathbf{y} with $\|\mathbf{x}\|, \|\mathbf{y}\| \lesssim 1$, and any regular deterministic matrices $A_1, A_2 \in \mathbf{C}^{2N \times 2N}$ (cf. Definition 4.2), we have the averaged local law

$$|\langle G_1 A_1 G_2 A_2 - M(w_1, A_1, w_2) A_2 \rangle| \prec \begin{cases} \frac{1}{\sqrt{N}\eta} & \text{if } \eta \leq 1 \\ \frac{1}{N\eta^3} & \text{if } \eta > 1 \end{cases} \tag{4.12a}$$

and the isotropic law

$$|\langle \mathbf{x}, (G_1 A_1 G_2 A_2 G_3 - M(w_1, A_1, w_2, A_2, w_3)) \mathbf{y} \rangle| \prec \begin{cases} \frac{1}{\eta} & \text{if } \eta \leq 1 \\ \frac{1}{\sqrt{N}\eta^4} & \text{if } \eta > 1 \end{cases}. \tag{4.12b}$$

Together with (4.11) for $k = 2$, this proves Proposition 3.4. Moreover, as a byproduct of our proof of Theorem 4.4, we obtain the following optimal local laws with a single regular matrix.

Theorem 4.5. (Optimal local laws with one regular matrix)

Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N \times N}$, $\epsilon > 0$ and $\kappa > 0$. Then, for spectral parameters $w_1, w_2 \in \mathbf{C}$ satisfying $\max_j |w_j| \leq N^{100}$, $\operatorname{Re} w_j \in \mathbf{B}_\kappa$ and $\eta := \min_j |\operatorname{Im} w_j| \geq N^{-1+\epsilon}$, deterministic vectors \mathbf{x}, \mathbf{y} with $\|\mathbf{x}\|, \|\mathbf{y}\| \lesssim 1$, and any regular deterministic matrix A_1 (cf. Definition 4.2), we have the optimal averaged local law

$$|\langle (G_1 - M_1)A_1 \rangle| \prec \begin{cases} \frac{1}{N\eta^{1/2}} & \text{if } \eta \leq 1 \\ \frac{1}{N\eta^2} & \text{if } \eta > 1 \end{cases} \quad (4.13a)$$

and the optimal isotropic local law

$$|\langle \mathbf{x}, (G_1 A_1 G_2 - M(w_1, A_1, w_2)) \mathbf{y} \rangle| \prec \begin{cases} \frac{1}{\sqrt{N\eta^2}} & \text{if } \eta \leq 1 \\ \frac{1}{\sqrt{N\eta^3}} & \text{if } \eta > 1 \end{cases}. \quad (4.13b)$$

Remark 4.6. We have several comments.

- (i) The above local laws are in agreement with the $\sqrt{\eta}$ -rule first established for Wigner matrices for traceless matrices in [26, Theorem 2.5]: Every regular deterministic matrix A_i reduces both the size of the deterministic approximation and the error term by a factor $\sqrt{\eta}$.
- (ii) The error terms in Theorem 4.4 dealing with two regular matrices can still be improved by a factor $1/\sqrt{N\eta}$, as shown in [26]. A similar analysis could have been conducted here, but we refrain from doing so, as it is not needed for our main results from Section 2. However, the error bounds in (4.13) with one regular matrix are in fact optimal.
- (iii) Given Theorem 2.6, and Theorems 4.4–4.5, it is possible to deduce similar bounds for averaged and isotropic chains as in (4.12), where not both matrices A_1, A_2 are regular (see (3.16)).

In the rest of this paper, we give a detailed proof of Theorem 4.4 in the much more involved $\eta \leq 1$ regime. For $\eta > 1$, the bound simply follows by induction on the number of resolvents in chain by invoking the trivial $\|M(w)\| \lesssim 1/|\operatorname{Im} w|$. The detailed argument has been carried out in [26, Appendix B] for the case of Wigner matrices. However, at a certain technical point (within the proof of the master inequalities in Proposition 4.9 and the reduction inequalities in Lemma 4.10), the proof uses Theorems 4.4 and 4.5 (and even its analogues for longer chains) for the $\eta > 1$ regime. But the master and reduction inequalities are not needed for proving the above estimates in the $\eta > 1$

regime, hence the argument is not circular. Throughout the rest of this paper we assume that $\min_j |\operatorname{Im} w_j| =: \eta \leq 1$.

4.1. Basic control quantities and proof of Theorems 4.4 and 4.5

Our strategy for proving Theorem 4.4 (and thereby Theorem 4.5 as a byproduct) is to derive a system of *master inequalities* (Proposition 4.9) for the errors in the local laws by cumulant expansion, then use an iterative scheme to gradually improve their estimates. The cumulant expansion introduces longer resolvent chains, potentially leading to an uncontrollable hierarchy, so our master inequalities are complemented by a set of *reduction inequalities* (Lemma 4.10) to estimate longer chain in terms of shorter ones. We have used a similar strategy in [26,27] for Wigner matrices, but now many new error terms due to regularisations need to be handled.

Before entering the detailed proof, we explain the main mechanism of the new type of error terms. Cumulant expansions applied to chains $\dots G_i A_i G_{i+1} \dots$ with regular A_i 's introduce more resolvent factors, for example $\dots G_i G_i A_i G_{i+1} \dots$ or $\dots G_i E_- G_i A_i G_{i+1} \dots$ without introducing more A 's. Multiple G factors without intermediate A 's appear which we wish to reduce to fewer G factors using resolvent identities (3.21) or contour integral representations; in the example above we will use

$$G_i G_i = G(w_i)^2 = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(z)}{(z - w_i)^2} dz, \tag{4.14}$$

where Γ is an appropriate contour (see Lemma 5.1). When this formula is inserted into the chain, we have $\dots G(z) A_i G_{i+1} \dots$, i.e. A_i is not regular any more with respect to the neighbouring spectral parameters (z, w_{i+1}) since w_i has been changed to z . We need to regularise A_i to the new situation. Fortunately, the regularisation is Lipschitz continuous by Lemma 3.3, so roughly speaking we make an error of order $|z - w_i|$ when we regularise A_i from (w_i, w_{i+1}) to (z, w_{i+1}) . This error exactly compensates the higher power of $z - w_i$ in the denominator in (4.14), making eventually the adjustment of regularisations harmless in the estimates. We need to meticulously implement this strategy for longer chains and also taking into account the chiral symmetry to reduce $G_i E_- G_i$ in chains like $\dots G_i E_- G_i A_i G_{i+1} \dots$. The precise form of the error terms in Lemma 3.3 is essential. It is remarkable that the signs appearing in (3.11), (3.12), and (3.13) exactly match those that arise in the denominators of the contour integral formulas like (4.14). We now start the actual proof.

As the basic control quantities in the sequel of the proof, we introduce the normalised differences

$$\Psi_k^{\text{av}}(\mathbf{w}_k, \mathbf{A}_k) := N\eta^{k/2} |\langle G_1 A_1 \cdots G_k A_k - M(w_1, A_1, \dots, w_k) A_k \rangle|, \tag{4.15}$$

$$\Psi_k^{\text{iso}}(\mathbf{w}_{k+1}, \mathbf{A}_k, \mathbf{x}, \mathbf{y}) := \sqrt{N\eta^{k+1}} \left| \langle (G_1 A_1 \cdots A_k G_{k+1} - M(w_1, A_1, \dots, A_k, w_{k+1}))_{\mathbf{x}\mathbf{y}} \rangle \right| \tag{4.16}$$

for $k \in \mathbf{N}$, where we used the short hand notations

$$G_i := G(w_i), \quad \eta := \min_i |\operatorname{Im} w_i|, \quad \mathbf{w}_k := (w_1, \dots, w_k), \quad \mathbf{A}_k := (A_1, \dots, A_k).$$

The deterministic matrices $\|A_i\| \leq 1, i \in [k]$, are assumed to be *regular* (i.e., $A_i = \mathring{A}_i$, see Definition 4.2) and the deterministic counterparts

$$M(w_1, A_1, \dots, A_{k-1}, w_k)$$

used in (4.15) and (4.16) (see also (4.1)) are given recursively in Definition 4.1.

For convenience, we extend the above definitions to $k = 0$ by

$$\Psi_0^{\text{av}}(w) := N\eta |\langle G(w) - M(w) \rangle|, \quad \Psi_0^{\text{iso}}(w, \mathbf{x}, \mathbf{y}) := \sqrt{N\eta} |(G(w) - M(w))_{\mathbf{x}\mathbf{y}}|$$

and observe that

$$\Psi_0^{\text{av}} + \Psi_0^{\text{iso}} \prec 1 \tag{4.17}$$

is the usual single-resolvent local law (in the bulk) from Theorem 2.6, where here and in the following the arguments of $\Psi_k^{\text{av}/\text{iso}}$ shall occasionally be omitted. We remark that the index k counts the number of regular matrices in the sense of Definition 4.2.

Throughout the entire argument, let $\epsilon > 0$ and $\kappa > 0$ be *arbitrary* but fixed, and let

$$\mathbf{D}^{(\epsilon, \kappa)} := \{w \in \mathbf{C} : \operatorname{Re} w \in \mathbf{B}_\kappa, N^{100} \geq |\operatorname{Im} w| \geq N^{-1+\epsilon}\} \tag{4.18}$$

be the *target spectral domain*, where the κ -bulk \mathbf{B}_κ has been introduced in (2.6). This target spectral domain $\mathbf{D}^{(\epsilon, \kappa)}$ will be reached by shrinking a larger *initial spectral domain*

$$\mathbf{D}^{(\epsilon_0, \kappa_0)} := \{w \in \mathbf{C} : \operatorname{Re} w \in \mathbf{B}_{\kappa_0}, N^{100} \geq |\operatorname{Im} w| \geq N^{-1+\epsilon_0}\} \tag{4.19}$$

many times, say $(L - 1)$ times, during our whole argument, where $L = L(\epsilon)$ is an N -independent positive integer to be determined below (see Remark 4.12). In (4.19), we set $\epsilon_0 := \epsilon/2$ and chose the initial bulk parameter

$$\kappa_0 = \kappa_0(\epsilon, \kappa) = \frac{\kappa}{L(\epsilon)} > 0 \tag{4.20}$$

The just mentioned shrinking of domains will be conducted alongside the decreasing family $(\mathbf{D}_\ell^{(\epsilon_0, \kappa_0)})_{\ell \in [L]}$ of spectral domains, defined via (Fig. 2)

$$\mathbf{D}_\ell^{(\epsilon_0, \kappa_0)} := \{w \in \mathbf{C} : \operatorname{Re} w \in \mathbf{B}_{\ell\kappa_0}, N^{100} \geq |\operatorname{Im} w| \geq \ell N^{-1+\epsilon_0}\} \subset \mathbf{D}^{(\epsilon_0, \kappa_0)}. \tag{4.21}$$

Finally, the cut-off parameter $\delta > 0$ used in the definition of the regular component of an observable (see (3.5) and (4.8) in Definition 4.2) shall be chosen by the following

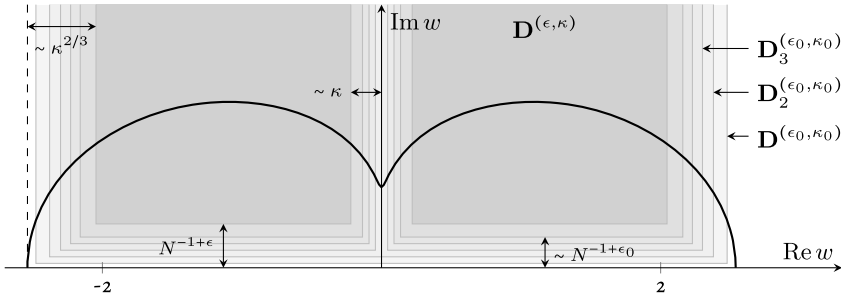


Fig. 2. Depicted are the target spectral domain (4.18), the initial spectral domain (4.19) and four intermediate domains from the family (4.21). The solid black curve represents the symmetric scDos ρ for the perturbation $\Lambda = -z$ with $|z|$ slightly less than one (see Example 2.5). Close to a regular edge of the scDos, the horizontal distance between two domains scales like $\kappa^{2/3}$. Near an (approximate) cusp, the scaling agrees with the linear lower bound given in (2.23).

two requirements: First, it has to be much smaller than the initial bulk-parameter κ_0 from (4.20), i.e.

$$0 < \delta \ll \mathfrak{c}\kappa_0, \tag{4.22}$$

where $\mathfrak{c} > 0$ is the same constant as introduced in (2.23). Second, it has to be small enough such that the denominators in (4.8) (see also Appendix B) as well as in Lemmas 5.5, 5.7 are uniformly bounded away from zero – in case that $\mathbf{1}_\delta^\sigma(w_i, w_{i+1}) = 1$. Note that these requirements also depend on the deformation $\Lambda \in \mathbf{C}^{N \times N}$ (but only via the norm $\|\Lambda\| \lesssim 1$) as it determines the scDos ρ .

Definition 4.7. (Uniform bounds in domains)

Let $\epsilon > 0$ and $\kappa > 0$ as above and let $k \in \mathbf{N}$. We say that the bounds

$$\begin{aligned} & \left| \langle G(w_1)B_1 \cdots G(w_k)B_k - M(w_1, B_1, \dots, w_k)B_k \rangle \right| \prec \mathcal{E}^{\text{av}}, \\ & \left| (G(w_1)B_1 \cdots B_k G(w_{k+1}) - M(w_1, B_1, \dots, B_k, w_{k+1}))_{\mathbf{xy}} \right| \prec \mathcal{E}^{\text{iso}} \end{aligned} \tag{4.23}$$

hold (ϵ, κ) -uniformly for some deterministic control parameters $\mathcal{E}^{\text{av/iso}} = \mathcal{E}^{\text{av/iso}}(N, \eta)$, depending only on N and $\eta := \min_i |\text{Im } w_i|$, if the implicit constant in (4.23) is uniform in bounded deterministic matrices $\|B_j\| \leq 1$, deterministic vectors $\|\mathbf{x}\|, \|\mathbf{y}\| \leq 1$, and admissible spectral parameters $w_j \in \mathbf{D}^{(\epsilon, \kappa)}$ satisfying $1 \geq \eta := \min_j |\text{Im } w_j|$.

Similarly, we use the phrase that a bound holds $(\epsilon_0, \kappa_0, \ell)$ -uniformly (or simply ℓ -uniformly), if the above statement is true with $\mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$ in place of $\mathbf{D}^{(\epsilon, \kappa)}$.

Moreover, we may allow for additional restrictions on the deterministic matrices. For example, we may talk about uniformity under the additional assumption that some (or all) of the matrices are *regular* (in the sense of Definition 4.2).

Note that (4.23) is stated for a fixed choice of spectral parameters w_j in the left hand side, but it is in fact equivalent to an apparently stronger statement, when the same

bound holds with a supremum over the spectral parameters (with the same constraints). More precisely, if $\mathcal{E}^{\text{iso}} \geq N^{-C}$ for some constant $C > 0$, then (4.23) implies

$$\sup_{w_1, \dots, w_{k+1}} \left| \left(G(w_1)B_1 \cdots B_k G(w_{k+1}) - M(w_1, B_1, \dots, B_k, w_{k+1}) \right)_{\mathbf{xy}} \right| \prec \mathcal{E}^{\text{iso}} \quad (4.24)$$

(and analogously for the averaged bound), where the supremum is taken over all choices of w_j 's in the admissible spectral domain $\mathbf{D}^{(\epsilon, \kappa)}$ or $\mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$. This bound follows from (4.23) by a standard *grid argument* (see, e.g., the discussion after [26, Def. 3.1]). Throughout the entire paper, we will frequently use the equivalence between (4.23) and (4.24), e.g. when integrating such bounds over some spectral parameters as done in Sections 5 and 6.

We can now formulate our main results of the present section, Theorem 4.4 and Theorem 4.5, in the language of our basic control quantities $\Psi_k^{\text{av/iso}}$.

Lemma 4.8. (Estimates on $\Psi_1^{\text{av/iso}}$ and $\Psi_2^{\text{av/iso}}$) *For any $\epsilon > 0$ and $\kappa > 0$ we have*

$$\Psi_1^{\text{av}} + \Psi_1^{\text{iso}} \prec 1 \quad \text{and} \quad \Psi_2^{\text{av}} + \Psi_2^{\text{iso}} \prec \sqrt{N\eta}$$

(ϵ, κ) -uniformly in regular matrices (i.e. for spectral parameters $w_j \in \mathbf{D}^{(\epsilon, \kappa)}$ with $1 \geq \eta := \min_j |\text{Im } w_j|$).

Proof of Theorems 4.4 and 4.5. These immediately follow from Lemma 4.8. \square

The rest of the proof is structured as follows: First, in Section 4.2, we state the *master inequalities* and corresponding *reduction inequalities* on the $\Psi_k^{\text{av/iso}}$ parameters, which we then use in Section 4.3 to prove Lemma 4.8. Afterwards, in Section 5, we prove the master inequalities and, finally, the proof of the reduction inequalities is presented in Section 6.

4.2. Master inequalities and reduction lemma

We now state the relevant part of a non-linear infinite hierarchy of coupled master inequalities for Ψ_k^{av} and Ψ_k^{iso} . In fact, for our purposes, it is sufficient to have only the inequalities for $k \in [2]$, but with fairly more effort (despite closely following the arguments in Section 5) it is possible to obtain analogous estimates for general $k \in \mathbf{N}$.

Proposition 4.9. (Master inequalities) *Assume that*

$$\Psi_i^{\text{av/iso}} \prec \psi_i^{\text{av/iso}}, \quad i \in [4], \quad (4.25)$$

ℓ -uniformly (i.e. for spectral parameters $w_j \in \mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$ and $1 \geq \min_j |\text{Im } w_j|$) in regular matrices, for some deterministic control parameters $\psi_i^{\text{av/iso}}$, which are independent of the spectral parameters w_j . Then it holds that

$$\Psi_1^{\text{av}} \prec 1 + \frac{\psi_1^{\text{av}}}{N\eta} + \frac{\psi_1^{\text{iso}} + (\psi_2^{\text{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{4.26a}$$

$$\Psi_1^{\text{iso}} \prec 1 + \frac{\psi_1^{\text{iso}} + \psi_1^{\text{av}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{4.26b}$$

$$\Psi_2^{\text{av}} \prec 1 + \frac{(\psi_1^{\text{av}})^2 + (\psi_1^{\text{iso}})^2 + \psi_2^{\text{av}}}{N\eta} + \frac{\psi_2^{\text{iso}} + (\psi_4^{\text{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{1/2} + (\psi_4^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{4.26c}$$

$$\Psi_2^{\text{iso}} \prec 1 + \psi_1^{\text{iso}} + \frac{\psi_1^{\text{av}}\psi_1^{\text{iso}} + (\psi_1^{\text{iso}})^2}{N\eta} + \frac{\psi_2^{\text{iso}} + (\psi_1^{\text{iso}}\psi_3^{\text{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{1/2} + (\psi_4^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{4.26d}$$

now $(\ell + 1)$ -uniformly (i.e. for spectral parameters $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ with $1 \geq \eta := \min_j |\text{Im } w_j|$) in regular matrices.

As shown in the above proposition, resolvent chains of length k are estimated by resolvent chains up to length $2k$. In order to avoid the indicated infinite hierarchy of master inequalities with higher and higher k indices, we will need the following *reduction lemma*.

Lemma 4.10. (Reduction inequalities) *Assume that $\Psi_n^{\text{av}/\text{iso}} \prec \psi_n^{\text{av}/\text{iso}}$ holds for $1 \leq n \leq 4$, ℓ -uniformly (i.e. for spectral parameters $w_j \in \mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$ with $1 \geq \eta := \min_j |\text{Im } w_j|$) in regular matrices (cf. Definition 4.7). Then we have*

$$\Psi_4^{\text{av}} \prec (N\eta)^2 + (\psi_2^{\text{av}})^2, \tag{4.27}$$

on the same domain. Furthermore, we have

$$\begin{aligned} \Psi_3^{\text{iso}} &\prec N\eta \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right) \left(1 + \frac{\psi_2^{\text{av}}}{N\eta} \right)^{1/2}, \\ \Psi_4^{\text{iso}} &\prec (N\eta)^{3/2} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right) \left(1 + \frac{\psi_2^{\text{av}}}{N\eta} \right) \end{aligned} \tag{4.28}$$

again uniformly in $w_j \in \mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$ and in regular matrices.

4.3. Proof of Lemma 4.8

Within the proof, we repeatedly use a simple argument, which we call *iteration*.

Lemma 4.11. (Iteration) *For every $D > 0$, $\nu > 0$, and $\alpha \in (0, 1)$, there exists some $K = K(D, \nu, \alpha)$, such that whenever (i) $X \prec N^D$ on $\mathbf{D}_1^{(\epsilon_0, \kappa_0)}$ and (ii) $X \prec x$ on $\mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$ for some $\ell \in \mathbf{N}$, implies that*

$$X \prec A + \frac{x}{B} + x^{1-\alpha}C^\alpha \quad \text{on } \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$$

for some constants $B \geq N^\nu$ and $A, C > 0$, it also holds that

$$X \prec A + C \quad \text{on } \mathbf{D}_{\ell+K}^{(\epsilon_0, \kappa_0)}.$$

We can now turn to the proof of Lemma 4.8.

Proof of Lemma 4.8. Assume that

$$\Psi_j^{\text{av/iso}} \prec \psi_j^{\text{av/iso}}, \quad j \in [4],$$

ℓ -uniformly, for some fixed $\ell > 0$, i.e. it holds on the domain $\mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$. Then, by (4.26a)–(4.26d), using that $N\eta \geq 1$ to remove some lower order terms, we immediately obtain

$$\begin{aligned} \Psi_1^{\text{av}} + \Psi_1^{\text{iso}} &\prec 1 + \frac{\psi_1^{\text{av}} + \psi_1^{\text{iso}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{av}})^{1/2} + (\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}} \\ \Psi_2^{\text{av}} + \Psi_2^{\text{iso}} &\prec 1 + \psi_1^{\text{iso}} + \frac{(\psi_1^{\text{av}})^2 + (\psi_1^{\text{iso}})^2}{N\eta} + \frac{\psi_2^{\text{av}} + \psi_2^{\text{iso}}}{(N\eta)^{1/2}} \\ &\quad + \frac{(\psi_4^{\text{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_1^{\text{iso}}\psi_3^{\text{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{1/2} + (\psi_4^{\text{iso}})^{1/2}}{(N\eta)^{1/4}} \end{aligned} \tag{4.29}$$

on the domain $\mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$. We point out that to get the second bound in (4.29) we also used $\psi_1^{\text{iso}}\psi_1^{\text{av}} \leq (\psi_1^{\text{iso}})^2 + (\psi_1^{\text{av}})^2$. Now, given the first estimate in (4.29) we obtain a bound for $\Psi_1^{\text{av}} + \Psi_1^{\text{iso}}$ which is better than the original a priori bound (4.25). We can thus replace the $\psi_1^{\text{av/iso}}$ from (4.25) with the rhs. of the first line of (4.29). Using iteration in both lines, we thus get

$$\begin{aligned} \Psi_1^{\text{av}} + \Psi_1^{\text{iso}} &\prec 1 + \frac{(\psi_2^{\text{av}})^{1/2} + (\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \\ \Psi_2^{\text{av}} + \Psi_2^{\text{iso}} &\prec 1 + \frac{(\psi_4^{\text{av}})^{1/2}}{\sqrt{N\eta}} + \frac{(\psi_2^{\text{av}})^{1/4} + (\psi_2^{\text{iso}})^{1/4}}{(N\eta)^{1/8}} \cdot \frac{(\psi_3^{\text{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{1/2} + (\psi_4^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \end{aligned} \tag{4.30}$$

on the domain $\mathbf{D}_{\ell+K}^{(\epsilon_0, \kappa_0)}$, for some K as in Lemma 4.11. We now use the reduction inequalities from Lemma 4.10 in the second line of (4.30):

$$\begin{aligned} \Psi_1^{\text{av}} + \Psi_1^{\text{iso}} &\prec 1 + \frac{(\psi_2^{\text{av}})^{1/2} + (\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}} \\ \Psi_2^{\text{av}} + \Psi_2^{\text{iso}} &\prec (N\eta)^{1/2} + \frac{\psi_2^{\text{av}}}{\sqrt{N\eta}} + (N\eta)^{1/4}(\psi_2^{\text{iso}})^{1/2} + (\psi_2^{\text{av}})^{1/2} + \frac{(\psi_2^{\text{av}}\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \\ &+ \left((N\eta)^{1/4} + \frac{(\psi_2^{\text{av}})^{1/4} + (\psi_2^{\text{iso}})^{1/4}}{(N\eta)^{1/8}} \right) \left(1 + \frac{(\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}} + \frac{(\psi_2^{\text{av}})^{1/4}}{(N\eta)^{1/8}} + \frac{(\psi_2^{\text{iso}})^{1/2}(\psi_2^{\text{av}})^{1/4}}{(N\eta)^{3/8}} \right), \end{aligned} \tag{4.31}$$

on the domain $\mathbf{D}_{\ell+K}^{(\epsilon_0, \kappa_0)}$. Next, using iteration once again in the second line of (4.31), we obtain

$$\Psi_1^{\text{av}} + \Psi_1^{\text{iso}} \prec 1 + \frac{(\psi_2^{\text{av}})^{1/2} + (\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}}, \quad \Psi_2^{\text{av}} + \Psi_2^{\text{iso}} \prec (N\eta)^{1/2}$$

on the domain $\mathbf{D}_{\ell+K+K'}^{(\epsilon_0, \kappa_0)}$, for some K' as in Lemma 4.11. We point out that here we used Schwarz and Young inequalities for several terms. Finally, using iteration one last time we conclude

$$\Psi_1^{\text{av}} + \Psi_1^{\text{iso}} \prec 1, \quad \Psi_2^{\text{av}} + \Psi_2^{\text{iso}} \prec (N\eta)^{1/2}$$

on the domain $\mathbf{D}_{\ell+K+K'+K''}^{(\epsilon_0, \kappa_0)}$, for some K'' as in Lemma 4.11. This concludes the proof. \square

Remark 4.12. We observe that in every application of Lemma 4.11 during the proof of Lemma 4.8, the parameter D is uniformly bounded by, say, $D \leq 10$, as follows by estimating every resolvent in $\Psi_k^{\text{av}/\text{iso}}$ by norm and using the trivial $1/\eta$ -bound on inverse of the stability operator in the iterative definition of $M(w_1, \dots, w_k)$ given in Definition 4.1. A further quick inspection of the above proof shows, that α can be chosen as fixed $\alpha = 1/2$. Finally, the parameter ν is lower bounded by (some universal positive constant times) ϵ , since $N\eta \geq N\epsilon^{1/2}$ by construction of the initial domain (4.19). Hence, the constants $K, K',$ and K'' only depend on ϵ and therefore also the maximal number $L = L(\epsilon)$ of domain shrinkings.

5. Proof of the master inequalities, Proposition 4.9

Before going into the proofs of the master inequalities, we state a simple lemma, which will frequently be used in the following. Recall that the deformation $\Lambda \in \mathbf{C}^{N \times N}$ is fixed and hence omitted from the notation.

Lemma 5.1. (Integral representations for products of resolvents)

Let $k \in \mathbf{N}$ and $w_1, \dots, w_k \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters, whose imaginary parts have equal sign, i.e. $\text{sgn}(\text{Im } w_1) = \dots = \text{sgn}(\text{Im } w_k) =: \tau$. Then, for any $J \subset \mathbf{R}$ being a union

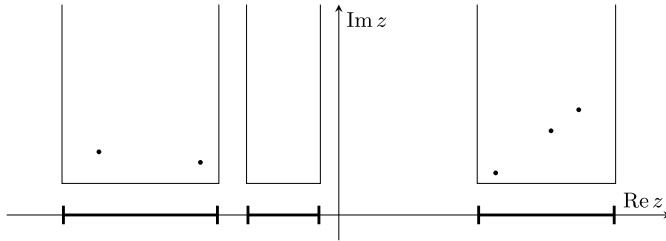


Fig. 3. Depicted is the scenario from Lemma 5.1 with five spectral parameters represented as dots in the upper half plane. Moreover, we indicated the union of compact intervals J on the real axis and the contour Γ as described in (5.2). Note that one of the three intervals constituting J does not contain any $\operatorname{Re} w_j$.

of compact intervals such that $\operatorname{Re} w_i \in \overset{\circ}{J}$ (the interior) for all $i \in [k]$ and $0 < \tilde{\eta} < \eta := \min_j |\operatorname{Im} w_j|$, we have the integral representation

$$\prod_{j=1}^k G(w_j) = \frac{1}{2\pi i} \int_{\Gamma} G(z) \prod_{j=1}^k \frac{1}{z - w_j} dz, \tag{5.1}$$

where the contour Γ from (5.1) is defined as (see Fig. 3)

$$\Gamma \equiv \Gamma_{\tilde{\eta}}^{\tau}(J) := \begin{cases} \partial(J \times [i\tilde{\eta}, i\infty)) & \text{if } \tau = + \\ \partial(J \times (-i\infty, -i\tilde{\eta}]) & \text{if } \tau = - \end{cases} \tag{5.2}$$

and the boundary is parameterised in counter-clockwise orientation.

Proof. This easily follows from residue calculus. For example, for $k = 2$, we have that

$$\frac{1}{\lambda_i - w_1} \frac{1}{\lambda_i - w_2} = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda_i - z} \frac{1}{z - w_1} \frac{1}{z - w_2} dz$$

for every eigenvalue λ_i of H by definition of the contour. This implies (5.1) for $k = 2$ by spectral decomposition for H ; the argument for general k is analogous. \square

We recall the definition of the *second order renormalisation*, denoted by underline, from [23]. For functions $f(W), g(W)$ of the random matrix W (see (2.15)), we define

$$\underline{f(W)Wg(W)} := f(W)Wg(W) - \tilde{\mathbb{E}}[(\partial_{\tilde{W}} f)(W)\tilde{W}g(W) + f(W)\tilde{W}(\partial_{\tilde{W}} g)(W)], \tag{5.3}$$

where $\partial_{\tilde{W}}$ denotes the directional derivative in the direction of

$$\tilde{W} := \begin{pmatrix} 0 & \tilde{X} \\ \tilde{X}^* & 0 \end{pmatrix},$$

where \tilde{X} is a complex Ginibre matrix that is independent of W . The expectation is taken w.r.t. the matrix \tilde{X} . Note that, if W itself consists of a complex Ginibre matrix

X , then $\mathbf{E} \underline{f(W)Wg(W)} = 0$, while for X with a general distribution this expectation is independent of the first two moments of X . In other words, the underline renormalises the product $f(W)Wg(W)$ to second order. We remark that underline (5.3) is a well-defined notation, if the ‘middle’ W to which the renormalisation refers is unambiguous. This is always be the case in all our proofs, since the functions f, g will be products of resolvents, never involving explicitly monomials in W .

We note that for any deterministic matrix $R \in \mathbf{C}^{2N \times 2N}$ we have

$$\widetilde{\mathbf{E}}\widetilde{WR}\widetilde{W} = 2\langle RE_2 \rangle E_1 + 2\langle RE_1 \rangle E_2 = \sum_{\sigma} \sigma \langle RE_{\sigma} \rangle E_{\sigma} = \mathcal{S}[R]$$

and furthermore, that the directional derivative of the resolvent is given by

$$\partial_{\widetilde{W}} G = -G\widetilde{W}G.$$

For example, in the special case $f(W) = 1$ and $g(W) = (W + \hat{\Lambda} - w)^{-1} = G$, we thus have

$$\underline{WG} = WG + \mathcal{S}[G]G$$

by definition of the underline in (5.3).

Using this underline notation in combination with the identity $G(W + \hat{\Lambda} - w) = E_+$ and the defining equation (2.20) for M , we have

$$G = M - \underline{MWG} + M\mathcal{S}[G - M]G = M - \underline{GWM} + G\mathcal{S}[G - M]M. \tag{5.4}$$

Recall that $\langle GE_- \rangle = 0$ (see below (2.16)) which immediately yields that $\mathcal{S}[G] = \sum_{\sigma} \sigma \langle GE_{\sigma} \rangle E_{\sigma} = \langle G \rangle$. Moreover, we have that $\mathcal{S}[M] = \langle M \rangle$, as follows from (2.17)–(2.18), and hence $\mathcal{S}[\cdot]$ effectively acts like a trace on G and M , i.e.

$$\mathcal{S}[G - M] = \langle G - M \rangle. \tag{5.5}$$

Now, similarly to [26], the key idea of the proof of Proposition 4.9 is using (5.4) for some G_j in a chain $G_1 A_1 \cdots A_k G_{k+1}$ and extending the renormalisation (5.3) to the whole product at the expense of adding resolvent products of shorter length. This will be done for each of the four estimates from Proposition 4.9 separately and presented in an *underlined lemma* in the beginning of each of the following subsections. Afterwards, the renormalisation of the whole product will be handled by cumulant expansion, exploiting that its expectation vanishes up to second order. While the proofs of the underlined lemmas for $\Psi_1^{\text{av/iso}}$ are presented in detail, we defer the analogous arguments for $\Psi_2^{\text{av/iso}}$ to Appendix E of the arXiv:2301.03549 version of this article.

5.1. Proof of the first master inequality (4.26a)

Let $w \equiv w_1$ be a spectral parameter in $\mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ (in particular in the bulk of the scDos, recall (4.21)) and $A \equiv A_1$ a (w, w) -regular matrix (cf. Definition 4.2). We use the notation $w = e + i\eta$ and we assume without loss of generality (by conjugation with E_- , see (2.16)) that $1 \geq \eta > 0$. We also assume that (4.25) holds (in this subsection we will need it only for Ψ_1^{av} and Ψ_2^{av}).

Lemma 5.2. (Representation as full underlined)

For any regular matrix $A = \mathring{A}$ we have that

$$\langle\langle (G - M)\mathring{A} \rangle\rangle = -\langle\langle \underline{WGA\mathring{A}'} \rangle\rangle + \mathcal{O}_{\prec}(\mathcal{E}_1^{\text{av}}) \tag{5.6}$$

for some other regular matrix $A' = \mathring{A}'$, which linearly depends on A (see (5.24) for the precise formula for A'). For the error term in (5.6), we used the shorthand notation

$$\mathcal{E}_1^{\text{av}} := \frac{1}{N\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{av}}}{N\eta} \right). \tag{5.7}$$

Having this approximate representation of $\langle\langle (G - M)\mathring{A} \rangle\rangle$ as a full underlined term at hand, we turn to the proof of (4.26a) via a (minimalistic) cumulant expansion:

$$\mathbf{E} w_{ab} f(W) = \sum_{l_1+l_2 \geq 1} \frac{1}{l_1!l_2!} \kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2}) \mathbf{E} \partial_{ab}^{l_1} \partial_{ba}^{l_2} f(W), \tag{5.8}$$

where $\kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2})$ is the cumulant $l_1 + 1$ copies of the random variable w_{ab} and l_2 copies of the random variable w_{ba} , and $\partial_{ab}^{l_1} \partial_{ba}^{l_2}$ denotes the l_1 -th derivative in the ab -entry and the l_2 -th derivative in the ba -entry.

Proof of (4.26a). Let $p \in \mathbf{N}$. Then, starting from (5.6), and recalling the second order renormalisation (5.3), we have

$$\begin{aligned} \mathbf{E} \langle\langle \underline{WGA} \rangle\rangle \langle\langle (G - M)A \rangle\rangle^{2p-1} &= \frac{1}{N} \sum_{ab} \mathbf{E} \underline{w_{ab}(GA)_{ba}} \langle\langle (G - M)A \rangle\rangle^{2p-1} \\ &= \frac{1}{N} \sum_{ab} \mathbf{E} (GA)_{ba} \left(\kappa(w_{ab}, w_{ba}) \partial_{ba} \langle\langle (G - M)A \rangle\rangle^{2p-1} \right. \\ &\quad \left. + \kappa(w_{ab}, w_{ab}) \partial_{ab} \langle\langle (G - M)A \rangle\rangle^{2p-1} \right) \\ &\quad + \frac{1}{N} \sum_{ab} \sum_{l_1+l_2 \geq 2} \frac{1}{l_1!l_2!} \kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2}) \mathbf{E} \partial_{ab}^{l_1} \partial_{ba}^{l_2} [(GA)_{ba} \langle\langle (G - M)A \rangle\rangle^{2p-1}]. \end{aligned} \tag{5.9}$$

By computing the resolvent derivatives explicitly as

$$\partial_{ab} \langle (G - M)A \rangle = -\frac{1}{N} (GAG)_{ba} \tag{5.10}$$

and using that $\kappa(w_{ab}, w_{ba}) = R_{ab}/N$, the first term in the middle line of (5.9) simplifies to

$$\frac{1}{N^3} \sum_{ab} (GA)_{ba} (GAG)_{ab} = \frac{\langle E_1 G A E_2 G A G \rangle + \langle E_2 G A E_1 G A G \rangle}{N^2}$$

(up to the factor of $\langle (G - M)A \rangle^{2p-2}$), the second term being similar up to an additional transposition. Here

$$R_{ab} := \mathbf{1}(a \leq N, b \geq N + 1 \text{ or } b \leq N, a \geq N + 1)$$

is the indicator function for the off-diagonal blocks of W . For the remaining term in (5.9) we simply estimate the cumulants by their size $|\kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2})| \leq N^{-(l_1+l_2+1)/2} R_{ab}$ to obtain

$$\begin{aligned} & \mathbf{E} |\langle (G - M)A \rangle|^{2p} \\ &= |-\mathbf{E} \langle \underline{WGA}' \rangle \langle (G - M)A \rangle^{p-1} \langle (G - M)^* A^* \rangle^p| + \mathcal{O}_{\prec}((\mathcal{E}_1^{\text{av}})^{2p}) \tag{5.11} \\ &\lesssim \mathbf{E} \frac{|\sum_{\sigma} \sigma \langle G E_{\sigma} G A' E_{\sigma} G A \rangle| + |\sum_{\sigma} \sigma \langle G^* E_{\sigma} G A' E_{\sigma} G^* A^* \rangle|}{N^2} |\langle (G - M)A \rangle|^{2p-2} \\ &\quad + \sum_{|\mathbf{l}| + \sum(J \cup J_*) \geq 2} \mathbf{E} \Xi_1^{\text{av}}(\mathbf{l}, J, J_*) |\langle (G - M)A \rangle|^{2p-1-|J \cup J_*|} + \mathcal{O}_{\prec}((\mathcal{E}_1^{\text{av}})^{2p}), \end{aligned}$$

where $\Xi_1^{\text{av}}(\mathbf{l}, J, J_*)$ is defined as

$$\Xi_1^{\text{av}} := N^{-(|\mathbf{l}| + \sum(J \cup J_*) + 3)/2} \sum_{ab} R_{ab} |\partial^{\mathbf{l}} (GA')_{ba}| \prod_{j \in J} |\partial^j \langle GA \rangle| \prod_{j \in J_*} |\partial^j \langle G^* A^* \rangle| \tag{5.12}$$

and the summation in the last line of (5.11) is taken over tuples¹⁵ $\mathbf{l} \in \mathbf{Z}_{\geq 0}^2$ and multisets of tuples $J, J_* \subset \mathbf{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$. Moreover, we set $\partial^{\mathbf{l}} = \partial^{(l_1, l_2)} := \partial_{ab}^{l_1} \partial_{ba}^{l_2}$, similarly $\partial^{\mathbf{j}} = \partial^{(j_1, j_2)} := \partial_{ab}^{j_1} \partial_{ba}^{j_2}$ and we define $|\mathbf{l}| = |(l_1, l_2)| = l_1 + l_2$, $\sum J = \sum_{j \in J} |\mathbf{j}|$. In the remainder of the proof, we need to analyze the rhs. of the inequality derived in (5.11). We begin with the third line and study the terms involving Ξ_1^{av} from (5.12) afterwards.

Before going into the proof, we note that, due to the cumulant expansion in (5.11), there are chains of resolvents G and deterministic matrices A appearing, where some of the A 's are *not* necessarily regular w.r.t. the spectral parameters of the surrounding G 's. The principal idea is to decompose such A with the aid of Lemma 3.3 and carefully track the resulting errors. As a rule of thumb, potentially small denominators resulting from

¹⁵ Note that the role played by (l_1, l_2) here is slightly different than in (5.9) above. Here the derivatives are applied to the individual factors according to Leibniz' rule, resulting in \mathbf{l}, J, J_* , and \mathbf{l} encodes only the derivatives hitting the $(GA')_{ab}$ factor.

resolvent identities (3.21) or the integral representation in Lemma 5.1 are balanced with the linear perturbative estimates from Lemma 3.3. See also Remark 5.3 below.

Gaussian contribution: third line of (5.11). In order to do so, we need to analyze in total four terms, each of which carries a factor of

$$\langle GE_\sigma GA'E_\sigma GA \rangle \quad \text{or} \quad \langle G^* E_\sigma GA'E_\sigma G^* A^* \rangle, \quad \text{for } \sigma = \pm.$$

Since their treatment is very similar, we focus on the two exemplary terms

$$(i) \quad \langle GGA'GA \rangle \quad \text{and} \quad (ii) \quad \langle G^* GA'G^* A^* \rangle. \tag{5.13}$$

In the analysis of the Gaussian contribution in Section 5.2, we will discuss the analogs of the other two terms in more detail.

First term. For the first term in (5.13), we apply the integral representation from Lemma 5.1 to GG with

$$\tau = +, \quad J = \mathbf{B}_{\ell\kappa_0}, \quad \text{and} \quad \tilde{\eta} = \frac{\ell}{\ell + 1}\eta,$$

for which we recall that $w \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$, i.e. in particular $\eta \geq (\ell + 1)N^{-1+\epsilon_0}$ and hence $\tilde{\eta} \geq \ell N^{-1+\epsilon_0}$. The fact that J is a union of compact intervals follows from the fact the support of the density of H^Λ has finitely many components. In particular, $\Gamma \equiv \Gamma_{\tilde{\eta}}^\tau(J) \subset \mathbf{D}_\ell^{(\epsilon_0, \kappa_0)}$. Now, we split the contour Γ in three parts,¹⁶ i.e.

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3. \tag{5.14}$$

As depicted in Fig. 4, the first part of the contour consists of the entire horizontal part of Γ . The second part, Γ_2 , covers the vertical components up to $|\text{Im } z| \leq N^{100}$. Finally, Γ_3 consists of the remaining part with $|\text{Im } z| > N^{100}$.

Now, the contribution coming from Γ_3 can easily be estimated by one via a trivial norm bound on G . For $z \in \Gamma_2$, we use that $\mathbf{1}_\delta^\pm(z, w) = 0$ for every $w \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ (recall (2.23) and (4.22)) and hence every matrix is (z, w) -regular. Therefore, after adding and subtracting the corresponding deterministic approximation, we can bound this part by $(1 + \psi_2^{\text{av}}/(N\eta))$ with the aid of Lemma 4.3. Hence, after splitting the contour integral and bounding each contribution as just described, we find

$$|\langle GGA'GA \rangle| \prec \left(1 + \frac{\psi_2^{\text{av}}}{N\eta}\right) + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{|\langle G(x + i\tilde{\eta})A'G(e + i\eta)A \rangle|}{(x - e)^2 + \eta^2} dx. \tag{5.15}$$

¹⁶ In the case of several w_1, \dots, w_k , the second part might require a further decomposition: If the spectral parameters of the resolvents which are *not* involved in such an integral representation have spectral parameters with imaginary parts of absolute value greater than one, we need to split Γ_2 according to $|\text{Im } z| \leq 1$ and $|\text{Im } z| > 1$. While the former will be treated exactly as Γ_2 here, the latter shall be estimated by means of the $\eta > 1$ -laws, which we discussed after Remark 4.6.

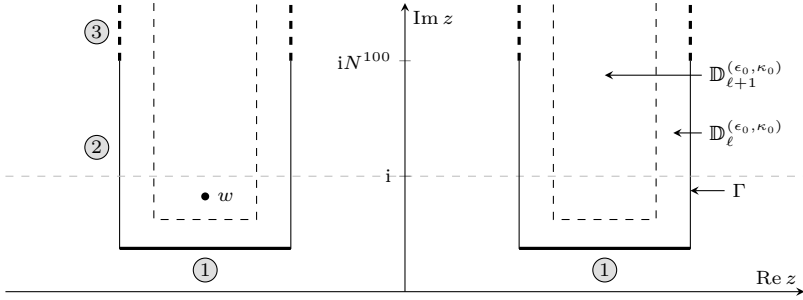


Fig. 4. The contour Γ is split into three parts (see (5.14)). In case of multiple spectral parameters, the second part might require a further decomposition at the level indicated by the dashed horizontal line (see Footnote 16). Depicted is the situation, where the bulk $\mathbf{B}_{\ell\kappa_0}$ consists of two components.

Next, we decompose $A = \mathring{A} = \mathring{A}^{e+i\eta, e+i\eta}$ and $A' = \mathring{A}' = (\mathring{A}')^{e+i\eta, e+i\eta}$ according to Lemma 3.3 as

$$\begin{aligned} \mathring{A}^{e+i\eta, e+i\eta} &= \mathring{A}^{e+i\eta, x+i\tilde{\eta}} + \mathcal{O}(|x - e| + \eta)E_+ + \mathcal{O}(|x - e| + \eta)E_- , \\ (\mathring{A}')^{e+i\eta, e+i\eta} &= (\mathring{A}')^{x+i\tilde{\eta}, e+i\eta} + \mathcal{O}(|x - e| + \eta)E_+ + \mathcal{O}(|x - e| + \eta)E_- . \end{aligned}$$

Plugging this into (5.15), we obtain several terms contributing to the integral. Adding and subtracting the deterministic approximation, the leading term accounts for

$$\int_{\mathbf{B}_{\ell\kappa_0}} \frac{|\langle G(x+i\tilde{\eta})(\mathring{A}')^{x+i\tilde{\eta}, e+i\eta} G(e+i\eta)\mathring{A}^{e+i\eta, x+i\tilde{\eta}} \rangle|}{(x - e)^2 + \eta^2} dx \prec \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{av}}}{N\eta} \right)$$

by means of Lemma 4.3. Here the “1” on the right-hand side is due to the contribution of the deterministic approximation $\langle M(x+i\tilde{\eta}, (A')^\circ, e+i\eta)\mathring{A}^{e+i\eta, x+i\tilde{\eta}} \rangle$, while the “ $\psi_2^{\text{av}}/(N\eta)$ ” is due to the definition of Ψ_2^{av} and the bound $\Psi_2^{\text{av}} \prec \psi_2^{\text{av}}$. The error terms can be dealt with by simple resolvent identities (3.21) in combination with the usual single-resolvent local law, Theorem 2.6, proving them to be bounded by η^{-1} . Indeed, for a generic $B \in \mathbf{C}^{2N \times 2N}$, we consider the exemplary term

$$\begin{aligned} \int_{\mathbf{B}_{\ell\kappa_0}} |\langle G(x+i\tilde{\eta})E_+ G(e+i\eta)B \rangle| \frac{|x - e| + \eta}{(x - e)^2 + \eta^2} dx \\ \lesssim \int_{\mathbf{B}_{\ell\kappa_0}} \frac{|\langle (G(x+i\tilde{\eta}) - G(e+i\eta))B \rangle|}{(x - e)^2 + \eta^2} dx \prec \frac{1}{\eta} . \end{aligned}$$

Second term. The second term in (5.13) is much simpler than the first. After writing $GG^* = \text{Im } G/\eta$, it suffices to realise that, by means of Lemma 3.3,

$$A' = (\mathring{A}')^{e+i\eta, e-i\eta} , \quad (\mathring{A}')^{e-i\eta, e-i\eta} = A' + \mathcal{O}(|e|)E_- , \quad \text{and} \quad A^* = (\mathring{A}^*)^{e-i\eta, e\pm i\eta}$$

in order to bound

$$\begin{aligned} |\langle G^*GA'G^*A^* \rangle| &\prec \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{av}}}{N\eta} \right) + \frac{|e|}{\eta} \frac{|\langle [G(-e + i\eta) - G(e - i\eta)]A^*E_- \rangle|}{|e| + \eta} \\ &\prec \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{av}}}{N\eta} \right) \end{aligned}$$

with the aid of Lemma 4.3, the chiral symmetry (2.16), a resolvent identity (3.21) and Theorem 2.6.

This finishes the estimate for the Gaussian contribution from the third line of (5.11), for which we have shown that

$$\frac{1}{N^2} \sum_{\sigma} (|\langle GE_{\sigma}GA'E_{\sigma}GA \rangle| + |\langle G^*E_{\sigma}GA'E_{\sigma}G^*A^* \rangle|) \prec \frac{1}{N^2\eta} \left(1 + \frac{\psi_2^{\text{av}}}{N\eta} \right). \tag{5.16}$$

We are now left with the terms from the last line (5.11) resulting from higher order cumulants.

Higher order cumulants and conclusion. The terms stemming from higher order cumulants are estimated in Section 5.5, the precise bound being given in (5.74a). Indeed, plugging (5.16) and (5.74a) into (5.11) we obtain

$$\begin{aligned} \mathbf{E}|\langle (G - M)A \rangle|^{2p} &\prec (\mathcal{E}_1^{\text{av}})^{2p} \\ &+ \sum_{m=1}^p \left[\frac{1}{N\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}} + (\psi_2^{\text{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/8}}{(N\eta)^{1/8}} \right) \right]^m (\mathbf{E}|\langle (G - M)A \rangle|^{2p})^{1-m/2p} \end{aligned}$$

and get the appropriate estimate $\mathbf{E}|\dots|^{2p}$ using Young inequalities. Since p was arbitrary, it follows that

$$\Psi_1^{\text{av}} \prec 1 + \frac{\psi_1^{\text{av}}}{N\eta} + \frac{\psi_1^{\text{iso}} + (\psi_2^{\text{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/4}}{(N\eta)^{1/8}}.$$

The bound given in Proposition 4.9 is an immediate consequence after a further trivial Young inequality. \square

Remark 5.3. Although the proof of the first master inequality (4.26a) is rather short, it already reveals a general strategy for dealing with a generic (not strictly) alternating chain

$$\dots GGAGAGE_AGE_GA \dots \tag{5.17}$$

of resolvents G and deterministic matrices A .

- (i) Apply resolvent identities (3.21) and the integral representation from Lemma 5.1 in order to reduce a product of resolvents to a linear combination (discrete or continuous, respectively). For terms of the form GE_G instead of GG this additionally requires an application of the chiral symmetry (2.16).
- (ii) In the resulting strictly alternating chain, decompose every deterministic A according to the regularisation from Definition 4.2 w.r.t. the spectral parameters of its surrounding resolvents by using Lemma 3.3.
- (iii) Estimate the regular parts coming from this decomposition in terms of $\Psi_k^{av/iso} \prec \psi_k^{av/iso}$. Carefully track the resulting errors stemming from the other parts.

These steps shall be applied repeatedly until the entire chain (5.17) has been examined. The first two steps outlined above will be performed mechanically without any complication. However, the third step is non-trivial and requires careful analysis on a case-by-case basis.

We have already mentioned that, as a rule of thumb, potentially small denominators resulting from Step (i) are balanced with the linear perturbative numerators from Step (ii).

It remains to give a proof of Lemma 5.2.

Proof of Lemma 5.2. Similarly as in (5.6), we suppress the indices of $G \equiv G_1$, $M \equiv M_1$ etc.

We start with the first identity in (5.4), such that, after defining the one-body stability operator

$$\mathcal{B} := 1 - MS[\cdot]M$$

we find

$$\mathcal{B}[G - M] = -M\underline{W}G + MSG - M$$

and consequently, by inversion, multiplication by $A = \mathring{A}$ (in the sense of (4.8), see also (4.9)) and taking a trace

$$\langle (G - M)A \rangle = -\langle \underline{W}G\mathcal{X}[A]M \rangle + \langle \mathcal{S}G - M\mathcal{X}[A]M \rangle, \tag{5.18}$$

where we introduced the linear operator

$$\mathcal{X}[B] := ((\mathcal{B}^*)^{-1}[B^*])^* = (1 - \mathcal{S}[M \cdot M])^{-1}[B] \quad \text{for } B \in \mathbf{C}^{2N \times 2N}.$$

Then, it is important to note that the condition $\mathbf{1}_\delta^+ \langle \text{Im } MA \rangle = 0$ (the first of the two imposed via (4.9); recall the definition of the cutoff function $\mathbf{1}_\delta^+$ from (3.6) and (4.7)), is stable under the linear operation $A \mapsto \mathcal{X}[A]M$.

Lemma 5.4. *For a generic $B \in \mathbf{C}^{2N \times 2N}$, we find*

$$\langle \mathcal{X}[B]M \text{Im } M \rangle = \langle B B^{-1} [M \text{Im } M] \rangle = \frac{i}{2} \frac{\langle B \text{Im } M \rangle}{\langle \text{Im } M \rangle} + \mathcal{O}(\eta). \tag{5.19}$$

Proof. Using (B.3), we compute

$$B^{-1} [M \text{Im } M] = \frac{B^{-1} [M^2 - M M^*]}{2i} = \frac{i}{2} \frac{\text{Im } M}{\eta + \langle \text{Im } M \rangle} + \frac{1}{2i} \frac{1 - \langle M M^* \rangle}{1 - \langle M^2 \rangle} M^2.$$

Now, by means of Lemma B.1 and Lemma B.2, we find that

$$|1 - \langle M M^* \rangle| = \mathcal{O}(\eta) \quad \text{and} \quad |1 - \langle M^2 \rangle| \gtrsim 1, \quad \text{respectively.} \quad \square$$

Recall from (5.5) that $\mathcal{S}[G - M] = \langle G - M \rangle$. Therefore, by means of the usual averaged local law, Theorem 2.6, which in particular shows that $|\langle \underline{WGB} \rangle| \prec \frac{1}{N\eta}$ for arbitrary $\|B\| \lesssim 1$ (see also Appendix C of the arXiv:2301.03549 version of this article and [39]), we can write (5.18) as

$$\begin{aligned} \langle (G - M)A \rangle &= - \langle \underline{WG}(\mathcal{X}[A]M)^\circ \rangle + \langle G - M \rangle \langle (G - M)(\mathcal{X}[A]M)^\circ \rangle \\ &\quad - \mathbf{1}_\delta^- c_-(\mathcal{X}[A]M) \langle \underline{WGE}_- \rangle + \mathcal{O}_\prec(N^{-1}), \end{aligned} \tag{5.20}$$

where in the underlined term, we used that the E_+ component of the regularisation of $\mathcal{X}[A]M$ is negligible thanks to Lemma 5.4 and the regularity of A , and we introduced the short hand notation

$$c_-(\mathcal{X}[A]M) := \frac{\langle M \mathcal{X}[A] M M E_- \rangle}{\langle M E_- M E_- \rangle}.$$

Next, with the aid of $WG = I - \hat{\Lambda}G + wG$ and using $\langle GE_- \rangle = 0$ from (5.5), we undo the underline in the second to last term, such that we infer

$$\langle \underline{WGE}_- \rangle = - \langle GE_- \hat{\Lambda} \rangle = - \langle (G - M)E_- \hat{\Lambda} \rangle = - \langle (G - M)(E_- \hat{\Lambda})^\circ \rangle.$$

In the second equality, we used that $\langle M E_- \hat{\Lambda} \rangle = 0$, which follows by a simple computation using the explicit form of M given in (2.17)–(2.18). For the last equality, we note that

$$(E_- \hat{\Lambda})^\circ = E_- \hat{\Lambda} - \mathbf{1}_\delta^+ \frac{\langle \text{Im } M E_- \hat{\Lambda} \rangle}{\langle \text{Im } M \rangle} E_+ - \mathbf{1}_\delta^- \frac{\langle M E_- \hat{\Lambda} M E_- \rangle}{\langle M E_- M E_- \rangle} E_- = E_- \hat{\Lambda},$$

which again follows after a simple computation using the fact that $\hat{\Lambda}$ is off-diagonal together with (2.17)–(2.18).

We can now write (5.20) for $A = \hat{A} = (E_- \hat{\Lambda})^\circ = E_- \hat{\Lambda}$ and solve the resulting equation for $\langle (G - M)E_- \hat{\Lambda} \rangle$. Plugging this back into (5.20) yields

$$\begin{aligned} \langle (G - M)A \rangle &= - \langle \underline{WG}(\mathcal{X}[A]M)^\circ \rangle + \langle G - M \rangle \langle (G - M)(\mathcal{X}[A]M)^\circ \rangle + \mathcal{O}_\prec(N^{-1}) \\ &+ \frac{\mathbf{1}_\delta^- c_-(\mathcal{X}[A]M)}{1 - \mathbf{1}_\delta^- c_-(\mathcal{X}[E_- \hat{\Lambda}]M)} \left[- \langle \underline{WG}(\mathcal{X}[E_- Z]M)^\circ \rangle \right. \\ &\left. + \langle G - M \rangle \langle (G - M)(\mathcal{X}[E_- Z]M)^\circ \rangle + \mathcal{O}_\prec(N^{-1}) \right]. \end{aligned} \tag{5.21}$$

Since $\|\mathcal{X}[\hat{A}]\| \lesssim 1$ (see Lemma B.3), the only thing left to check is, that the denominator in (5.21) is bounded away from zero.

Lemma 5.5. *For small enough $\delta > 0$, we have that*

$$|1 - \mathbf{1}_\delta^-(w, w) c_-(\mathcal{X}[E_- \hat{\Lambda}]M)| \gtrsim 1.$$

Proof. The statement is trivial for $\mathbf{1}_\delta^-(w, w) = 0$ and we hence focus on the case where $\lambda := \mathbf{1}_\delta^-(w, w) \in (0, 1]$. First, we note that $\mathcal{X}[E_- \hat{\Lambda}] = E_- \hat{\Lambda}$, which follows from the explicit form of M given in (2.17)–(2.18) using the fact that $\hat{\Lambda}$ is purely off-diagonal. Next, we use the MDE (2.20), the chiral symmetry (2.19), and Lemma B.1 (a) to infer

$$1 - c_-(\mathcal{X}[E_- \hat{\Lambda}]M) = 1 - \frac{\langle ME_- \hat{\Lambda} M M E_- \rangle}{\langle M E_- M E_- \rangle} = \frac{1}{2} \left[1 - \frac{w + m}{m} \langle M^2 \rangle \right].$$

Now, specialising to $w = i\eta$ with sufficiently small η , we find that, to leading order,

$$\text{Re} \left[1 - \frac{\eta + \text{Im } m}{\text{Im } m} \langle M^2 \rangle \right] \sim \text{Re} [1 - \langle M^2 \rangle] = 1 - \langle M M^* \rangle + 2\langle (\text{Im } M)^2 \rangle \geq 2\langle \text{Im } M \rangle^2 \gtrsim 1 \tag{5.22}$$

by direct computation. Using Lipschitz continuity of this expression in w , this principal lower bound on $\text{Re} [1 - c_-(\mathcal{X}[E_- \hat{\Lambda}]M)]$ of order one persists after a small perturbation of w allowing for a non-zero real part, but as long as $\lambda = \mathbf{1}_\delta^-(w, w) > 0$ for some $\delta > 0$ small enough. Hence, we conclude the lower bound

$$|1 - \lambda c_-(\mathcal{X}[E_- \hat{\Lambda}]M)| \geq (1 - \lambda)1 + \lambda \text{Re} [1 - c_-(\mathcal{X}[E_- \hat{\Lambda}]M)] \gtrsim 1 \tag{5.23}$$

for the convex combination, by separately considering smaller and larger values of $\lambda \in (0, 1]$. \square

From the expansion (5.21) it is apparent, that the main terms for understanding the size of $\langle (G - M)A \rangle$ are the underlined ones, the rest carrying additional $\langle G - M \rangle$ -factors, hence they will become negligible errors. In fact, summarizing our investigations, we have shown that

$$\langle (G - M)\hat{A} \rangle = - \langle \underline{WG\hat{A}'} \rangle + \mathcal{O}_\prec(\mathcal{E}_1^{\text{av}}),$$

where we used the shorthand notation

$$\mathring{A}' := (\mathcal{X}[\mathring{A}]M)^\circ + \frac{\mathbf{1}_\delta^- c_-(\mathcal{X}[\mathring{A}]M)}{1 - \mathbf{1}_\delta^- c_-(\mathcal{X}[E_- \mathring{\Lambda}]M)} (\mathcal{X}[E_- \mathring{\Lambda}]M)^\circ \tag{5.24}$$

in the underlined term. Using the usual averaged local law (4.17) and (4.25), we collected all the error terms from (5.21) in $\mathcal{E}_1^{\text{av}}$, defined in (5.7). \square

5.2. Proof of the second master inequality (4.26b)

Let $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ for $j \in [2]$ be spectral parameters and A_1 a regular matrix w.r.t. the pair of spectral parameters (w_1, w_2) (see Definition 4.2). By conjugation with E_- , we will assume w.l.o.g. that $\text{Im } w_1 > 0$ and $\text{Im } w_2 < 0$. Moreover, we use the notations $e_j \equiv \text{Re } w_j$, $\eta_j := |\text{Im } w_j|$ for $j \in [2]$ and define $1 \geq \eta := \min_j |\text{Im } w_j|$. We also assume that (4.25) holds.

Lemma 5.6. (Representation as full underlined)

For $\|\mathbf{x}\|, \|\mathbf{y}\| \leq 1$ and any (w_1, w_2) -regular matrix $A_1 = \mathring{A}_1$, we have that

$$(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2))_{\mathbf{xy}} = -\underline{(G_1 \mathring{A}'_1 W G_2)}_{\mathbf{xy}} + \mathcal{O}_\prec(\mathcal{E}_1^{\text{iso}}) \tag{5.25}$$

for some (w_1, w_2) -regular matrix $A'_1 = \mathring{A}'_1$, which linearly depends on $A_1 = \mathring{A}_1$ (see (5.57)). For the error term in (5.25), we used the shorthand notation

$$\mathcal{E}_1^{\text{iso}} := \frac{1}{\sqrt{N\eta^2}} \left(1 + \frac{\psi_1^{\text{av}}}{(N\eta)^{1/2}} + \frac{\psi_1^{\text{iso}}}{N\eta} \right). \tag{5.26}$$

Note that unlike in Section 5.1, now in (5.25) the second resolvent G_2 was expanded instead of G_1 rendering the W factor in the middle of the underlined term. This prevents the emergence of resolvent chains in the proof of (4.26b), which are ‘too long’ to be handled within our hierarchical framework of master inequalities (e.g., a chain involving four resolvents would appear in $\tilde{\Xi}_1^{\text{iso}}$ defined below).

Having this approximate representation of $(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2))_{\mathbf{xy}}$ as a full underlined term at hand, we turn to the proof of (4.26b) via a (minimalistic) cumulant expansion.

Proof of (4.26b). Let $p \in \mathbf{N}$. Then, starting from (5.25) and using the same notations as in the proof of (4.26a), we obtain

$$\begin{aligned} & \mathbf{E} \left| (G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2))_{\mathbf{xy}} \right|^{2p} \tag{5.27} \\ & \lesssim \mathbf{E} \tilde{\Xi}_1^{\text{iso}} \left| (G_1 \mathring{A}_1 G_2 - M(\dots))_{\mathbf{xy}} \right|^{2p-2} \\ & \quad + \sum_{|l|+\sum(J \cup J_*) \geq 2} \mathbf{E} \Xi_1^{\text{iso}}(l, J, J_*) \left| (G_1 \mathring{A}_1 G_2 - M(\dots))_{\mathbf{xy}} \right|^{2p-1-|J \cup J_*|} + \mathcal{O}_\prec((\mathcal{E}_1^{\text{iso}})^{2p}), \end{aligned}$$

where

$$\begin{aligned} \tilde{\Xi}_1^{\text{iso}} &:= \frac{\sum_{\sigma} \left[\left| (G_1 \mathring{A}'_1 E_{\sigma} G_1 \mathring{A}_1 G_2)_{\mathbf{x}\mathbf{y}} (G_1 E_{\sigma} G_2)_{\mathbf{x}\mathbf{y}} \right| + \left| (G_1 \mathring{A}'_1 E_{\sigma} G_2)_{\mathbf{x}\mathbf{y}} (G_1 \mathring{A}_1 G_2 E_{\sigma} G_2)_{\mathbf{x}\mathbf{y}} \right| \right]}{N} \\ &+ \frac{\sum_{\sigma} \left[\left| (G_1 \mathring{A}'_1 E_{\sigma} G_2^* (\mathring{A}_1)^* G_1^*)_{\mathbf{x}\mathbf{x}} (G_2^* E_{\sigma} G_2)_{\mathbf{y}\mathbf{y}} \right| + \left| (G_1 \mathring{A}'_1 E_{\sigma} G_1^*)_{\mathbf{x}\mathbf{x}} (G_2^* (\mathring{A}_1)^* G_1^* E_{\sigma} G_2)_{\mathbf{y}\mathbf{y}} \right| \right]}{N} \end{aligned}$$

and $\Xi_1^{\text{iso}}(\mathbf{l}, J, J_*)$ is defined via

$$\begin{aligned} \Xi_1^{\text{iso}} &:= N^{-(|\mathbf{l}| + \sum(J \cup J_*) + 1)/2} \sum_{ab} R_{ab} \left| \partial^{\mathbf{l}} \left[(G_1 \mathring{A}'_1)_{\mathbf{x}\mathbf{a}} (G_2)_{\mathbf{b}\mathbf{y}} \right] \right| \\ &\quad \times \prod_{j \in J} \left| \partial^j (G_1 \mathring{A}_1 G_2)_{\mathbf{x}\mathbf{y}} \right| \prod_{j \in J_*} \left| \partial^j (G_2^* (\mathring{A}_1)^* G_1^*)_{\mathbf{y}\mathbf{x}} \right|. \end{aligned} \tag{5.28}$$

In the remainder of the proof, we need to analyze the rhs. of the inequality derived in (5.27). Following the general strategy outlined in Remark 5.3, we begin with the second line and study the terms involving Ξ_1^{iso} from (5.28) afterwards.

Gaussian contribution: third line of (5.27). In order to do so, following Remark 5.3, we need to analyze in total eight terms, each of which carries one of the summands in the definition of $\tilde{\Xi}_1^{\text{iso}}$ as a factor. Since their treatment is very similar, we focus on the two exemplary terms

$$(i) (G_1 \mathring{A}'_1 E_- G_1 \mathring{A}_1 G_2)_{\mathbf{x}\mathbf{y}} (G_1 E_- G_2)_{\mathbf{x}\mathbf{y}}, \quad (ii) (G_1 \mathring{A}'_1 E_- G_1^*)_{\mathbf{x}\mathbf{x}} (G_2^* (\mathring{A}_1)^* G_1 E_- G_2)_{\mathbf{y}\mathbf{y}}, \tag{5.29}$$

showcasing the key difficulties. Recall that, in the analysis of the Gaussian term in Section 5.1 we discussed analogs of the above terms with the choice $\sigma = +$.

Term (i) in (5.29). For the first term, we decompose, similarly to Lemma 3.3,

$$(\mathring{A}'_1)^{1,2} E_- = ((\mathring{A}'_1)^{1,2} E_-)^{\circ_{1,1}} + \mathcal{O}(|e_1 + e_2| + |\eta_1 - \eta_2|) E_+ + \mathcal{O}(|e_1 + e_2| + |\eta_1 - \eta_2|) E_- . \tag{5.30}$$

Inserting this into the first term in (5.29) and using Lemma 4.3, we find

$$\begin{aligned} \left| (G_1 \mathring{A}'_1 E_- G_1 \mathring{A}_1 G_2)_{\mathbf{x}\mathbf{y}} \right| &< \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right) \\ &+ (|e_1 + e_2| + |\eta_1 - \eta_2|) \sum_{\sigma} \left| (G_1 E_{\sigma} G_1 \mathring{A}_1 G_2)_{\mathbf{x}\mathbf{y}} \right|. \end{aligned} \tag{5.31}$$

In the last term, we focus on $\sigma = -$, while $\sigma = +$ can be dealt with by Lemma 5.1. In fact, using (2.16) and a resolvent identity (3.21), we obtain

$$\begin{aligned} |(G_1 E_- G_1 \mathring{A}_1 G_2)_{\mathbf{x}\mathbf{y}}| &= \left| \frac{1}{w_1} ([G(-w_1) - G(w_1)] \mathring{A}_1^{w_1, w_2} G(w_2))_{(E_- \mathbf{x})\mathbf{y}} \right| \\ &\prec \frac{1}{\eta^2} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right), \end{aligned}$$

where in the last step we used Lemma 4.3 and the trivial approximation

$$\mathring{A}_1^{-w_1, w_2} = \mathring{A}_1^{w_1, w_2} + \mathcal{O}(1)E_+ + \mathcal{O}(1)E_-.$$

For the second factor in the first term in (5.29), we use (2.16) and employ the integral representation from Lemma 5.1 with

$$\tau = +, \quad J = \mathbf{B}_{\ell\kappa_0}, \quad \text{and} \quad \tilde{\eta} = \frac{\ell}{\ell + 1}\eta,$$

for which we recall that $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$, i.e. in particular $\eta \geq (\ell + 1)N^{-1+\epsilon_0}$ and hence $\tilde{\eta} \geq \ell N^{-1+\epsilon_0}$. After splitting the contour integral and estimating the contribution as described around (5.14), we find, with the aid of Lemma 4.3 and absorbing logarithmic corrections into ‘ \prec ’, that

$$\begin{aligned} |(G_1 E_- G_2)_{\mathbf{x}\mathbf{y}}| &\prec 1 + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{|(G(x + i\tilde{\eta}))_{\mathbf{x}(E_- \mathbf{y})}|}{|(x - e_1 - i(\eta_1 - \tilde{\eta}))(x + e_2 - i(\eta_2 - \tilde{\eta}))|} dx \\ &\prec 1 + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2} \end{aligned} \tag{5.32}$$

where in the last step we used the usual single resolvent local law from Theorem 2.6. Notice the key cancellation of the $|e_1 + e_2|$ factor in (5.31) and (5.32). Collecting all the estimates, we have shown that

$$|(5.29) \text{ (i)}| \prec \frac{1}{\eta^2} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right). \tag{5.33}$$

Term (ii) in (5.29). In the first factor in the second term in (5.29), we again employ the decomposition (5.30) to find

$$|(G_1 \mathring{A}'_1 E_- G_1^*)_{\mathbf{x}\mathbf{x}}| \prec \frac{1}{\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) + \frac{|e_1 + e_2| + |\eta_1 - \eta_2|}{\eta} \tag{5.34}$$

with the aid of Theorem 2.6 and Lemma 4.3 as well as a resolvent identity (3.21) and Lemma 5.1 for the E_+ and E_- in (5.30), respectively.

In the second factor, similarly to (5.32) above, we use Lemma 5.1 together with the decomposition¹⁷

¹⁷ The notation $\sum_{\sigma} \mathcal{O}_{\sigma}(\alpha)E_{\sigma}$ means that, for each σ , the term $\mathcal{O}_{\sigma}(\alpha)$ is a scalar function $g_{\sigma}(\alpha)$ of order $\mathcal{O}(\alpha)$.

$$(\mathring{A}_1^{w_1, w_2})^* = (\mathring{A}_1^*)^{\bar{w}_2, \bar{w}_1} = (\mathring{A}_1^*)^{\bar{w}_2, w_1} = (\mathring{A}_1^*)^{\bar{w}_2, x+i\tilde{\eta}} + \sum_{\sigma} \mathcal{O}_{\sigma}(|x - e_1| + |\eta_1 - \tilde{\eta}|)E_{\sigma}$$

from Lemma 3.3 for arbitrary x to find

$$\begin{aligned} \left| (G_2^*(\mathring{A}_1)^*G_1E_-G_2)_{\mathbf{y}\mathbf{y}} \right| &\prec \frac{1}{\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) \\ &+ \int_{\mathbf{B}_{\ell\kappa_0}} \frac{|(G(\bar{w}_2)(\mathring{A}_1^*)^{\bar{w}_2, x+i\tilde{\eta}}G(x+i\tilde{\eta}))_{\mathbf{y}(E-\mathbf{y})}|}{|(x - e_1 - i(\eta_1 - \tilde{\eta})) (x + e_2 - i(\eta_2 - \tilde{\eta}))|} dx \\ &+ \int_{\mathbf{B}_{\ell\kappa_0}} \frac{\sum_{\sigma} |(G(\bar{w}_2)E_{\sigma}G(x+i\tilde{\eta}))_{\mathbf{y}(E-\mathbf{y})}|}{|x + e_2 - i(\eta_2 - \tilde{\eta})|} dx \tag{5.35} \\ &\prec \frac{1}{\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) \left(1 + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2} \right) + \frac{1}{\eta}. \end{aligned}$$

Now, combining (5.34) and (5.35), we obtain

$$|(5.29) \text{ (ii)}| \prec \frac{1}{\eta^2} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right)^2. \tag{5.36}$$

This finishes the estimate for the Gaussian contribution from the third line of (5.27), for which we have shown that

$$\tilde{\Xi}_1^{\text{iso}} \prec \frac{1}{N\eta^2} \left(1 + \frac{(\psi_1^{\text{iso}})^2}{N\eta} + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right) \tag{5.37}$$

as easily follows by combining (5.33) with (5.36) and using a Schwarz inequality.

We are now left with the terms from the last line (5.27) resulting from higher order cumulants.

Higher order cumulants and conclusion. The estimate stemming from higher order cumulants is given in (5.74b). Then, plugging (5.37) and (5.74b) into (5.27), we find, similarly to Section 5.1, that

$$\Psi_1^{\text{iso}} \prec 1 + \frac{\psi_1^{\text{iso}}}{N\eta} + \frac{\psi_1^{\text{iso}} + \psi_1^{\text{av}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}} + \frac{(\psi_2^{\text{iso}})^{1/4}}{(N\eta)^{1/8}}.$$

The bound given in Proposition 4.9 is an immediate consequence after a trivial Young inequality. \square

It remains to give a proof of Lemma 5.6. This is much more involved than for the previous underlined Lemma 5.2. The proof of Lemma 5.2 crucially used that the orthogonality $\langle \text{Im } MA \rangle = 0$ is (almost) preserved under the operation $A \mapsto \mathcal{X}[A]M$ (see

Lemma 5.4). This is simply not available here, since we deal with two spectral parameters w_1, w_2 .

Proof of Lemma 5.6. We denote $A_1 \equiv \mathring{A}_1$, except we wish to emphasise A_1 being regular. Just as in Section 5.1, we start with

$$G_2 = M_2 - M_2 \underline{W} G_2 + M_2 \mathcal{S}[G_2 - M_2] G_2,$$

such that we get

$$G_1 \tilde{A}_1 G_2 = G_1 \tilde{A}_1 M_2 - G_1 \tilde{A}_1 M_2 \underline{W} G_2 + G_1 \tilde{A}_1 M_2 \mathcal{S}[G_2 - M_2] G_2$$

for $\tilde{A}_1 = \mathcal{X}_{12}[A_1]$ and $A_1 = \mathring{A}_1$ (note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ by Lemma B.3), where we introduced the linear operator

$$\mathcal{X}_{12}[B] := (1 - \mathcal{S}[M_1 \cdot M_2])^{-1}[B] \quad \text{for } B \in \mathbf{C}^{2N \times 2N}. \tag{5.38}$$

Extending the underline to the whole product, we obtain

$$\begin{aligned} G_1 \tilde{A}_1 G_2 &= M_1 \tilde{A}_1 M_2 + (G_1 - M_1) \tilde{A}_1 M_2 - \underline{G_1 \tilde{A}_1 M_2 W G_2} \\ &\quad + G_1 \tilde{A}_1 M_2 \mathcal{S}[G_2 - M_2] G_2 + G_1 \mathcal{S}[G_1 \tilde{A}_1 M_2] G_2, \end{aligned}$$

from which we conclude that

$$\begin{aligned} G_1 (\tilde{A}_1 - \mathcal{S}[M_1 \tilde{A}_1 M_2]) G_2 &= M_1 \tilde{A}_1 M_2 + (G_1 - M_1) \tilde{A}_1 M_2 - \underline{G_1 \tilde{A}_1 M_2 W G_2} \\ &\quad + G_1 \tilde{A}_1 M_2 \mathcal{S}[G_2 - M_2] G_2 + G_1 \mathcal{S}[(G_1 - M_1) \tilde{A}_1 M_2] G_2 \end{aligned}$$

and thus

$$\begin{aligned} G_1 A_1 G_2 &= M_1 \mathcal{X}_{12}[A_1] M_2 + (G_1 - M_1) \mathcal{X}_{12}[A_1] M_2 - \underline{G_1 \mathcal{X}_{12}[A_1] M_2 W G_2} \tag{5.39} \\ &\quad + G_1 \mathcal{X}_{12}[A_1] M_2 \mathcal{S}[G_2 - M_2] G_2 + G_1 \mathcal{S}[(G_1 - M_1) \mathcal{X}_{12}[A_1] M_2] G_2. \end{aligned}$$

We note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ by means of Lemma B.3.

Then, we need to further decompose $\mathcal{X}_{12}[A_1] M_2$ in the last three terms in (5.39) as

$$\mathcal{X}_{12}[A_1] M_2 = (\mathcal{X}_{12}[A_1] M_2)^\circ + \sum_{\sigma} \mathbf{1}_{\delta}^{\sigma} c_{\sigma}(\mathcal{X}_{12}[A_1] M_2) E_{\sigma}, \tag{5.40}$$

where we suppressed the spectral parameters (and the relative sign of their imaginary parts, which has been fixed by $\text{Im } w_1 > 0$ and $\text{Im } w_2 < 0$) in the notation for the linear functionals $c_{\sigma}(\cdot)$ on $\mathbf{C}^{2N \times 2N}$ defined as

$$c_+(B) := \frac{\langle M_1 B M_2 \rangle}{\langle M_1 M_2 \rangle} \quad \text{and} \quad c_-(B) := \frac{\langle M_1 B M_2^* E_- \rangle}{\langle M_1 E_- M_2^* E_- \rangle}. \tag{5.41}$$

Plugging (5.40) into (5.39) we find $G_1A_1G_2$ to equal

$$\begin{aligned}
 & M_1\mathcal{X}_{12}[A_1]M_2 + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1(\mathcal{X}_{12}[A_1]M_2)^\circ WG_2} \tag{5.42} \\
 & + G_1(\mathcal{X}_{12}[A_1]M_2)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)(\mathcal{X}_{12}[A_1]M_2)^\circ]G_2 \\
 & + \sum_{\sigma} \mathbf{1}_{\delta}^{\sigma} c_{\sigma}(\mathcal{X}_{12}[A_1]M_2) [-\underline{G_1E_{\sigma}WG_2} + G_1E_{\sigma}\mathcal{S}[G_2 - M_2]G_2 \\
 & + G_1\mathcal{S}[(G_1 - M_1)E_{\sigma}]G_2].
 \end{aligned}$$

Recall that the regular component is defined w.r.t. the pair of spectral parameters (w_1, w_2) . In particular, $(\mathcal{X}_{12}[A_1]M_2)^\circ = (\mathcal{X}_{12}[A_1]M_2)^{\circ_{1,2}}$ in the last term in the second line of (5.42) is *not* regular as defined via the conditions with one resolvent (4.9).

In the last line of (5.42) we now undo the underline and find the bracket $[\dots]$ to equal (the negative of)

$$\begin{aligned}
 & G_1E_{\sigma}WG_2 + G_1E_{\sigma}\mathcal{S}[M_2]G_2 + G_1\mathcal{S}[M_1E_{\sigma}]G_2 \\
 & = G_1E_{\sigma} + G_1(E_{\sigma}(w_2 - \hat{\Lambda} + \mathcal{S}[M_2]) + \mathcal{S}[M_1E_{\sigma}])G_2 \\
 & = G_1E_{\sigma} - G_1(E_{\sigma}M_2^{-1} - \mathcal{S}[M_1E_{\sigma}])G_2 =: G_1E_{\sigma} - G_1\Phi_{\sigma}G_2,
 \end{aligned}$$

where we used $WG_2 = E_+ + w_2G_2 - \hat{\Lambda}G_2$ in the first step and the MDE (2.20) in the second step. Moreover, we introduced the shorthand notation

$$\Phi_{\sigma} := E_{\sigma} \frac{1}{M_2} - \mathcal{S}[M_1E_{\sigma}]. \tag{5.43}$$

From the expansion (5.42) it is apparent (and it can also be checked by hand using the explicit form of (5.43)) that

$$M_1E_{\sigma} = M_1(E_{\sigma}M_2^{-1})M_2 = M_1\mathcal{X}_{12}[\Phi_{\sigma}]M_2 = M(w_1, \Phi_{\sigma}, w_2),$$

where in the last step we used (4.4). This finally yields that $G_1A_1G_2$ equals

$$\begin{aligned}
 & M(w_1, A_1, w_2) + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1(\mathcal{X}_{12}[A_1]M_2)^\circ WG_2} \tag{5.44} \\
 & + G_1(\mathcal{X}_{12}[A_1]M_2)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)(\mathcal{X}_{12}[A_1]M_2)^\circ]G_2 \\
 & + \sum_{\sigma} \mathbf{1}_{\delta}^{\sigma} c_{\sigma}(\mathcal{X}_{12}[A_1]M_2) [-(G_1 - M_1)E_{\sigma} + (G_1\Phi_{\sigma}G_2 - M(w_1, \Phi_{\sigma}, w_2))].
 \end{aligned}$$

The last term in the last line of (5.44) requires further decomposition of Φ_{σ} from (5.43) (completely analogous to (5.40) and (5.41)) as

$$\Phi_{\sigma} = \mathring{\Phi}_{\sigma} + \sum_{\tau} \mathbf{1}_{\delta}^{\tau} c_{\tau}(\Phi_{\sigma})E_{\tau}.$$

Using the explicit form of Φ_σ , we further observe that

$$c_\tau(\Phi_\sigma) \sim \delta_{\sigma,\tau} \quad \text{and} \quad c_\tau(\mathcal{X}_{12}[\Phi_\sigma]M_2) \sim \delta_{\sigma,\tau}. \tag{5.45}$$

Therefore, by means of the first relation in (5.45), the expansion (5.44) can be carried out further as

$$\begin{aligned} M(w_1, A_1, w_2) &+ (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1(\mathcal{X}_{12}[A_1]M_2)^\circ WG_2} \\ &+ G_1(\mathcal{X}_{12}[A_1]M_2)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)(\mathcal{X}_{12}[A_1]M_2)^\circ]G_2 \\ &+ \sum_\sigma \mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[A_1]M_2) \left[- (G_1 - M_1)E_\sigma + (G_1\mathring{\Phi}_\sigma G_2 - M(w_1, \mathring{\Phi}_\sigma, w_2)) \right. \\ &\quad \left. + c_\sigma(\Phi_\sigma)(G_1E_\sigma G_2 - M(w_1, E_\sigma, w_2)) \right]. \end{aligned} \tag{5.46}$$

Next, we write (5.46) for both, $A_1 = \mathring{A}_1 = \mathring{\Phi}_+$ and $A_1 = \mathring{A}_1 = \mathring{\Phi}_-$, and solve the two resulting linear equations for $G_1\mathring{\Phi}_\pm G_2 - M(w_1, \mathring{\Phi}_\pm, w_2)$. Observe that by means of the second relation in (5.45) the original system of linear equations boils down to two separate ones. Thus, plugging the solutions for $G_1\mathring{\Phi}_\pm G_2 - M(w_1, \mathring{\Phi}_\pm, w_2)$ back into (5.46) we arrive at

$$\begin{aligned} G_1A_1G_2 &= M(w_1, A_1, w_2) + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1(\mathcal{X}_{12}[A_1]M_2)^\circ WG_2} \\ &+ G_1(\mathcal{X}_{12}[A_1]M_2)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)(\mathcal{X}_{12}[A_1]M_2)^\circ]G_2 \\ &+ \sum_\sigma \frac{\mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[A_1]M_2)}{1 - \mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)} \left[(G_1 - M_1)\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2 \right. \\ &- \underline{G_1(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)^\circ WG_2} \\ &+ G_1(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)^\circ]G_2 \\ &\left. - (G_1 - M_1)E_\sigma + c_\sigma(\Phi_\sigma)(G_1E_\sigma G_2 - M(w_1, E_\sigma, w_2)) \right]. \end{aligned} \tag{5.47}$$

We now need to check that the denominators in (5.47) are bounded away from zero.

Lemma 5.7. *For small enough $\delta > 0$, we have that*

$$\left| 1 - \mathbf{1}_\delta^\sigma(w_1, w_2) c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2) \right| \gtrsim 1 \quad \text{for } \sigma = \pm.$$

Proof. The statements are trivial for $\mathbf{1}_\delta^\sigma(w_1, w_2) = 0$ and we hence focus on cases where $\lambda^\sigma := \mathbf{1}_\delta^\sigma(w_1, w_2) \in (0, 1]$. First, we compute

$$1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2) = \langle M_1 \rangle \frac{\langle M_1 M_2 M_2 \rangle}{\langle M_1 M_2 \rangle^2} \quad \text{and} \tag{5.48}$$

$$1 - c_-(\mathcal{X}_{12}[\mathring{\Phi}_-]M_2) = \frac{\langle M_1 E_- M_2^* M_2^{-1} E_- \rangle + \langle M_1 \rangle \langle M_1 E_- M_2^* E_- \rangle}{1 + \langle M_1 E_- M_2 E_- \rangle} \frac{\langle M_1 E_- M_2 M_2^* E_- \rangle}{\langle M_1 E_- M_2^* E_- \rangle^2}$$

for arbitrary spectral parameters w_1, w_2 . Recall that we assumed the two spectral parameters to be on different halfplanes, i.e. $\mathfrak{s}_1 = -\text{sgn}(\text{Im } w_1 \text{Im } w_2) = +$, hence we shall specialise (i) the first expression in (5.48) to $w_2 = \bar{w}_1$ and (ii) the second expression in (5.48) to $w_2 = -w_1$.

In this case, for the first expression in (5.48), using Lemma B.1 and $\text{Im } M_1 \text{Im } w_1 > 0$, we obtain

$$|1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2)| = |\langle M_1 \rangle \frac{\langle \text{Im } M_1 M_1^* \rangle}{\langle \text{Im } M_1 \rangle^2} (\langle \text{Im } M_1 \rangle + \text{Im } w_1)| \geq \langle \text{Im } M_1 \rangle^2 \gtrsim 1 \tag{5.49}$$

in the bulk of the spectrum. This principal lower bound of order one persists after a small perturbation of w_2 around the special case $w_2 = \bar{w}_1$, but as long as $\lambda^+ = 1$ (for some $\delta > 0$ small enough), which proves the claim for $\sigma = +$ and $\lambda^+ = 1$. A further direct computation by estimating real and imaginary part of $1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2)$ instead of its absolute value in (5.49), similarly to (5.22) shows that also the convex combination

$$(1 - \lambda^+)1 + \lambda^+[1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2)]$$

is bounded away from zero (in absolute value), by separately considering small and large values of $\lambda^+ \in (0, 1)$. For the second expression in (5.48), the argument is similar and hence omitted. \square

Next, we take the scalar product of (5.47) with two deterministic vectors \mathbf{x}, \mathbf{y} satisfying $\|\mathbf{x}\|, \|\mathbf{y}\| \leq 1$. In the resulting expression, there are two particular terms, namely the ones of the form

$$(G_1 \mathcal{S}[(G_1 - M_1) \mathring{A}_1^{1,2}] G_2)_{\mathbf{x}\mathbf{y}} \quad \text{and} \tag{5.50}$$

$$c_\sigma(\mathcal{X}_{12}[\mathring{A}_1^{1,2}] M_2) c_\sigma(\Phi_\sigma) (G_1 E_\sigma G_2 - M(w_1, E_\sigma, w_2))_{\mathbf{x}\mathbf{y}}, \tag{5.51}$$

whose direct (naive) estimates are $1/(N\eta^2)$ and $1/\eta$, respectively, and thus do not match the target size. Hence, they have to be discussed in more detail. In our notation, we emphasised that the regularisation is defined w.r.t. the spectral parameters (w_1, w_2) , i.e., in particular, $A_1^\circ = A_1^{\circ 1,2}$.

Estimating (5.50). For the term (5.50), we expand

$$(G_1 \mathcal{S}[(G_1 - M_1) \mathring{A}_1^{1,2}] G_2)_{\mathbf{x}\mathbf{y}} = \sum_{\sigma} \sigma \langle (G_1 - M_1) \mathring{A}_1^{1,2} E_\sigma \rangle (G_1 E_\sigma G_2)_{\mathbf{x}\mathbf{y}} \tag{5.52}$$

and observe that, by definition of \cdot° in (4.8), we have, similarly to Lemma 3.3 (see also (5.30)),

$$\mathring{A}_1^{1,2} E_\sigma = (\mathring{A}_1^{1,2} E_\sigma)^{\circ 1,1} + \mathcal{O}(|e_1 - \sigma e_2| + |\eta_1 - \eta_2|) E_+ + \mathcal{O}(|e_1 - \sigma e_2| + |\eta_1 - \eta_2|) E_- . \tag{5.53}$$

Now, in the second term in (5.52) for $\sigma = +$ and $E_\sigma = E_+$, we use a resolvent identity (3.21) and the usual isotropic local law (4.17) to estimate it as

$$|(G_1 G_2)_{\mathbf{xy}}| \prec 1 + \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2} . \tag{5.54}$$

Furthermore, in the second term in (5.1) for $\sigma = -$ and $E_\sigma = E_-$, we employ the integral representation from Lemma 5.1 in combination with the usual isotropic local law (4.17) (see also (5.32)) to infer

$$|(G_1 E_- G_2)_{\mathbf{xy}}| \prec 1 + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2} . \tag{5.55}$$

Combining (5.54) and (5.55) with the decomposition (5.53) and the usual averaged local law (4.17), we find that (5.52) can be bounded by

$$\sum_\sigma \left(\left| \langle (G_1 - M_1)(\mathring{A}_1^{1,2} E_\sigma)^{\circ 1,1} \rangle \right| + \frac{|e_1 - \sigma e_2| + |\eta_1 - \eta_2|}{N\eta_1} \right) \left(1 + \frac{1}{|e_1 - \sigma e_2| + \eta_1 + \eta_2} \right) .$$

Using the definition of Ψ_1^{av} in (4.15) and the apriori bound $\Psi_1^{\text{av}} \prec \psi_1^{\text{av}}$, this immediately implies the estimate

$$|(5.50)| \prec \frac{1}{N\eta} + \frac{1}{\sqrt{N}\eta} \frac{\psi_1^{\text{av}}}{(N\eta)^{1/2}} . \tag{5.56}$$

Estimating (5.51). For the term (5.51), we first note that the two prefactors $c_\sigma(\mathcal{X}_{12}[A_1^{\circ 1,2}]M_2)$ and $c_\sigma(\Phi_\sigma)$ are bounded. However, in each of the two cases $\sigma = \pm$, the bound on *one* of the prefactors needs to be improved: In the first case, $\sigma = +$, we use (B.4) and compute

$$c_+(\Phi_+) = \frac{\langle M_1 \rangle (1 - \langle M_1 M_2 \rangle)}{\langle M_1 M_2 \rangle} = \mathcal{O}(|e_1 - e_2| + \eta_1 + \eta_2)$$

from (5.41) and (5.43). Combining this with the bound

$$\left| (G_1 G_2 - M(w_1, E_+, w_2))_{\mathbf{xy}} \right| \prec \left(\frac{1}{\sqrt{N}\eta_1} + \frac{1}{\sqrt{N}\eta_2} \right) \cdot \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2}$$

which is obtained completely analogous to (5.54), we conclude that (5.51) for $\sigma = +$ can be estimated by $1/\sqrt{N\eta}$ (recall $\eta := \min\{\eta_1, \eta_2\}$). Similarly, in the second case, $\sigma = -$, we perform a computation similar to the one leading to (5.19) and use (B.4) in order to obtain that $c_-(\mathcal{X}_{12}[\mathring{A}_1^{1,2}]M_2)$ equals

$$\frac{i \langle M_1 \mathring{A}_1^{1,2} M_2^* E_- \rangle}{2 \langle M_1 E_- M_2^* E_- \rangle} + \frac{1 \langle M_1 \mathring{A}_1^{1,2} M_2 E_- \rangle}{2i \langle M_1 E_- M_2^* E_- \rangle} \frac{1 + \langle M_1 E_- M_2^* E_- \rangle}{1 + \langle M_1 E_- M_2 E_- \rangle} = \mathcal{O}(|e_1 + e_2| + \eta_1 + \eta_2)$$

Combining this with the bound

$$\left| \langle (G_1 E_- G_2 - M(w_1, E_-, w_2))_{\mathbf{xy}} \rangle \right| \prec \frac{1}{\sqrt{N\eta}} \cdot \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2}$$

which is obtained completely analogous to (5.55), we conclude that (5.51) can be estimated by $1/\sqrt{N\eta}$ – now in both cases $\sigma = \pm$.

Conclusion. Summarizing our investigations, we have shown that

$$(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2))_{\mathbf{xy}} = -\langle \underline{G_1 \mathring{A}'_1 W G_2} \rangle_{\mathbf{xy}} + \mathcal{O}_\prec(\mathcal{E}_1^{\text{iso}}),$$

where we used the shorthand notation

$$\mathring{A}'_1 := (\mathcal{X}_{12}[\mathring{A}_1]M_2)^\circ + \sum_\sigma \frac{\mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[\mathring{A}_1]M_2)}{1 - \mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)} (\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)^\circ \tag{5.57}$$

in the underlined term. Combining (5.56) and the bound on (5.51) established above with the usual single resolvent local laws (4.17) and the bounds on deterministic approximations in Lemma 4.3, we collected all the error terms from (5.47) in (5.26). \square

5.3. Proof of the third master inequality (4.26c)

Let $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ for $j \in [2]$ be spectral parameters and A_1 a regular matrix w.r.t. (w_1, w_2) and A_2 a regular matrix w.r.t. (w_2, w_1) (see Definition 4.2). By conjugation with E_- , we again assume w.l.o.g. that $\text{Im } w_1 > 0$ and $\text{Im } w_2 < 0$. Just as in Section 5.2, we use the notations $e_j \equiv \text{Re } w_j$, $\eta_j := |\text{Im } w_j|$ for $j \in [2]$ and define $1 \geq \eta := \min_j |\text{Im } w_j|$. We also assume that (4.25) holds.

Lemma 5.8. (Representation as full underlined)

For any (w_1, w_2) -regular matrix $A_1 = \mathring{A}_1$ and (w_2, w_1) -regular matrix $A_2 = \mathring{A}_2$, we have that

$$\langle \langle (G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)) \mathring{A}_2 \rangle \rangle = -\langle \underline{W G_1 \mathring{A}'_1 G_2 \mathring{A}'_2} \rangle + \mathcal{O}_\prec(\mathcal{E}_2^{\text{av}}) \tag{5.58}$$

for some (w_2, w_1) -regular matrix $A'_2 = \mathring{A}'_2$, which linearly depends on $A_2 = \mathring{A}_2$ (analogously to (5.57), see Eq. (E.18) in the arXiv:2301.03549 version of this article for an explicit formula). For the error term in (5.58), we used the shorthand notation

$$\mathcal{E}_2^{\text{av}} := \frac{1}{N\eta} \left(1 + \frac{(\psi_1^{\text{av}})^2}{N\eta} + \frac{\psi_2^{\text{av}}}{N\eta} \right). \tag{5.59}$$

Note that similarly to Lemma 5.2 but contrary to Lemma 5.6, we again expanded the first resolvent G_1 . Otherwise, the proof of Lemma 5.8, given in Appendix E of the arXiv:2301.03549 version of this article, is very similar to the one of Lemma 5.6. We only mention that the quadratic error $(\psi_1^{\text{av}})^2$ stems from terms of the form

$$\langle \mathcal{S}[G_1 \mathring{A}_1^{1,2} G_2](G_2 - M_2) \mathring{A}_2^{2,1} \rangle,$$

appearing in the analogue of (5.47) (see Eq. (E.9) in the arXiv:2301.03549 version of this article). Having the approximate representation (5.58), we turn to the proof of (4.26c) via cumulant expansion of the full underlined term.

Proof of (4.26c). Let $p \in \mathbb{N}$. Starting from (5.6), we obtain, as in the proofs of (4.26a) and (4.26b),

$$\begin{aligned} & \mathbf{E} \left| \langle (G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)) \mathring{A}_2 \rangle \right|^{2p} \tag{5.60} \\ & \lesssim \mathbf{E} \tilde{\Xi}_2^{\text{av}} \left| \langle (G_1 \mathring{A}_1 G_2 - M(\dots)) \mathring{A}_2 \rangle \right|^{2p-2} \\ & + \sum_{|\mathbf{l}| + \sum(J \cup J_*) \geq 2} \mathbf{E} \Xi_2^{\text{av}}(\mathbf{l}, J, J_*) \left| \langle (G_1 \mathring{A}_1 G_2 - M(\dots)) \mathring{A}_2 \rangle \right|^{2p-1-|J \cup J_*|} + \mathcal{O}_{\prec}((\mathcal{E}_2^{\text{av}})^{2p}), \end{aligned}$$

where

$$\tilde{\Xi}_2^{\text{av}} := \frac{1}{N^2} \sum_{\sigma} \left| \langle G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_1 E_{\sigma} G_1 \mathring{A}_1 G_2 \mathring{A}'_2 E_{\sigma} \rangle \right| + \dots$$

with the other terms being analogous, just 1 and 2 in the first half $G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_1$ of the chain interchanged or the entire half taken as adjoint, and $\Xi_2^{\text{av}}(\mathbf{l}, J, J_*)$ is defined as

$$\begin{aligned} \Xi_2^{\text{av}} := & N^{-(|\mathbf{l}| + \sum(J \cup J_*) + 3)/2} \sum_{ab} R_{ab} |\partial^{\mathbf{l}} \langle G_1 \mathring{A}_1 G_2 \mathring{A}'_2 \rangle_{ba}| \tag{5.61} \\ & \times \prod_{j \in J} |\partial^j \langle G_1 \mathring{A}_1 G_2 \mathring{A}_2 \rangle| \prod_{j \in J_*} |\partial^j \langle G_2^* \mathring{A}'_2 G_1^* \mathring{A}_1^* \rangle|. \end{aligned}$$

As in Sections 5.1 and 5.2, in the remainder of the proof, we need to analyze the rhs. of (5.60). We begin with the second line and study the terms involving Ξ_2^{av} from (5.61) afterwards.

Gaussian contribution: second line of (5.60). Along the principal strategy outlined in Remark 5.3, we need to analyze in total eight terms, each of which carries one of the summands in the definition of $\tilde{\Xi}_2^{\text{av}}$ as a factor. Since their treatment is very similar, we focus on the exemplary term

$$\langle G_1 \mathring{A}_1^{w_1, w_2} G_2 \mathring{A}_2^{w_2, w_1} G_1 G_1 \mathring{A}_1^{w_1, w_2} G_2 (\mathring{A}'_2)^{w_2, w_1} \rangle. \tag{5.62}$$

Now, we represent $G_1 G_1$ via the integral representation from Lemma 5.1 with

$$\tau = +, \quad J = \mathbf{B}_{\ell\kappa_0}, \quad \text{and} \quad \tilde{\eta} = \frac{\ell}{\ell + 1} \eta,$$

for which we recall that $w \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$, i.e. in particular $\eta \geq (\ell + 1)N^{-1+\epsilon_0}$ and hence $\tilde{\eta} \geq \ell N^{-1+\epsilon_0}$. After splitting the contour integral and bounding the individual contributions as described in (5.14), we obtain, with the aid of Lemma 4.3,

$$|(5.62)| \prec \frac{1}{\eta^2} \left(1 + \frac{\psi_4^{\text{av}}}{N\eta} \right) + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{|\langle G_1 \mathring{A}_1^{w_1, w_2} G_2 \mathring{A}_2^{w_2, w_1} G(x + i\tilde{\eta}) \mathring{A}_1^{w_1, w_2} G_2 (\mathring{A}_2')^{w_2, w_1} \rangle|}{(x - e_1)^2 + \eta_1^2} dx.$$

Next, we decompose $\mathring{A}_2^{w_2, w_1}$ and $\mathring{A}_1^{w_1, w_2}$ in the integrand as (recall the notation in Footnote 17)

$$\begin{aligned} \mathring{A}_2^{w_2, x+i\tilde{\eta}} &= \mathring{A}_2^{w_2, w_1} + \sum_{\sigma} \mathcal{O}_{\sigma}(|x - e_1| + |\eta_1 - \tilde{\eta}|) E_{\sigma} \\ \mathring{A}_1^{x+i\tilde{\eta}, w_2} &= \mathring{A}_1^{w_1, w_2} + \sum_{\sigma} \mathcal{O}_{\sigma}(|x - e_1| + |\eta_1 - \tilde{\eta}|) E_{\sigma}. \end{aligned} \tag{5.63}$$

While the properly regularised term contributes an $\eta^{-2}(1 + \psi_4^{\text{av}}/(N\eta))$ -error, a typical cross term shall be estimated as

$$\int_{\mathbf{B}_{\ell\kappa_0}} \frac{|\langle G_1 \mathring{A}_1^{w_1, w_2} G_2 \mathring{A}_2^{w_2, x+i\tilde{\eta}} [G(x + i\tilde{\eta}) - G_2] (\mathring{A}_2')^{w_2, w_1} \rangle|}{(|x - e_1| + \eta_1) (|x - e_2| + \eta_2)} \prec \frac{1}{\eta^2} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right) \tag{5.64}$$

where in the second step we wrote out the averaged trace and estimated each summand in isotropic form with the aid of Lemma 4.3, using ψ_2^{iso} instead of ψ_3^{av} .

Finally, for ‘error × error’-type terms are bounded by η^{-2} , simply by using a trivial Schwarz inequality in combination with a Ward identity and the usual local law from Theorem 2.6 to infer

$$|\langle G_1 B_1 G_2 B_2 \rangle| \leq \sqrt{\langle G_1 B_1 B_1^* G_1^* \rangle \langle G_2 B_2 B_2^* G_2^* \rangle} \leq \frac{1}{\eta} \sqrt{\langle \text{Im } G_1 B_1 B_1^* \rangle \langle \text{Im } G_2 B_2 B_2^* \rangle} \prec \frac{1}{\eta},$$

which is valid for arbitrary bounded matrices $\|B_1\|, \|B_2\| \lesssim 1$.

This finishes the estimate for the Gaussian contribution from the second line of (5.60), for which, collecting the above estimates, we have shown that

$$\tilde{\Xi}_2^{\text{av}} \prec \frac{1}{N^2 \eta^2} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} + \frac{\psi_4^{\text{av}}}{N\eta} \right). \tag{5.65}$$

We are now left with the terms from the last line of (5.60) resulting from higher order cumulants.

Higher order cumulants and conclusion. The estimate stemming from higher order cumulants is given in (5.74c) in Section 5.5. Then, plugging (5.65) and (5.74c) into (5.60), we find, similarly to Section 5.1, that

$$\Psi_2^{\text{av}} \prec 1 + \frac{(\psi_1^{\text{av}})^2 + (\psi_1^{\text{iso}})^2 + \psi_2^{\text{av}}}{N\eta} + \frac{\psi_2^{\text{iso}} + (\psi_4^{\text{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/2}}{(N\eta)^{1/4}} + \frac{(\psi_3^{\text{iso}})^{3/8} + (\psi_4^{\text{iso}})^{3/8}}{(N\eta)^{3/16}}.$$

The bound given in Proposition 4.9 is an immediate consequence after a trivial Young inequality. \square

5.4. Proof of the fourth master inequality (4.26d)

Let $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ for $j \in [3]$ be spectral parameters and A_1 a regular matrix w.r.t. (w_1, w_2) and A_2 a regular matrix w.r.t. (w_2, w_3) (see Definition 4.2). By conjugation with E_- , we will assume w.l.o.g. that $\text{Im } w_1 > 0$, $\text{Im } w_2 < 0$, and $\text{Im } w_3 > 0$. As before, we use the notations $e_j \equiv \text{Re } w_j$, $\eta_j := |\text{Im } w_j|$ for $j \in [3]$ and define $1 \geq \eta := \min_j |\text{Im } w_j|$. We also assume that (4.25) holds.

Lemma 5.9. (Representation as full underlined)

For $\|\mathbf{x}\|, \|\mathbf{y}\| \leq 1$ and any (w_1, w_2) -regular matrix $A_1 = \mathring{A}_1$ and (w_2, w_3) -regular matrix $A_2 = \mathring{A}_2$, we have that

$$(G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_3 - M(w_1, \mathring{A}_1, w_2, \mathring{A}_2, w_3))_{\mathbf{xy}} = -(\underline{G_1 \mathring{A}'_1 W G_2 \mathring{A}_2 G_3})_{\mathbf{xy}} + \mathcal{O}_{\prec}(\mathcal{E}_2^{\text{iso}}) \tag{5.66}$$

for some other (w_1, w_2) -regular matrix $A'_1 = \mathring{A}'_1$, which linearly depends on $A_1 = \mathring{A}_1$ (analogously to (5.57), see Eq. (E.33) in the arXiv:2301.03549 version of this article for an explicit formula). For the error term in (5.66), we used the shorthand notation

$$\mathcal{E}_2^{\text{iso}} := \frac{1}{\sqrt{N\eta^3}} \left(1 + \psi_1^{\text{iso}} + \frac{\psi_1^{\text{av}} \psi_1^{\text{iso}}}{N\eta} + \frac{\psi_2^{\text{iso}}}{N\eta} \right). \tag{5.67}$$

Note that similarly to (5.25), we again expanded the second resolvent. The proof of Lemma 5.9, given in Appendix E of the arXiv:2301.03549 version of this article, is very similar to the one of Lemma 5.6. We only mention that the errors carrying $\psi_1^{\text{iso}} \psi_1^{\text{av}}$ and ψ_1^{iso} stem from terms of the form

$$(G_1 \mathcal{S}[(G_1 - M_1) A_1^{\sigma_{1,2}}] G_2 \mathring{A}_2 G_3)_{\mathbf{xy}} \quad \text{and} \\ c_{\sigma}(\mathcal{X}_{12}[\mathring{A}_1] M_2) c_{\sigma}(\Phi_{\sigma})(G_1 E_{\sigma} G_2 \mathring{A}_2 G_3 - M(w_1, E_{\sigma}, w_2, \mathring{A}_2, w_3))_{\mathbf{xy}},$$

respectively, appearing in the analogue of (5.47) (see Eqs. (E.24)–(E.26) in Appendix E of the arXiv:2301.03549 version of this article). Having the representation (5.66) we turn to the proof of (4.26d) via cumulant expansion of the underlined term.

Proof of (4.26d). Let $p \in \mathbf{N}$. Then, starting from (5.66), we obtain

$$\begin{aligned} & \mathbf{E} \left| (G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_3 - M(w_1, \mathring{A}_1, w_2, \mathring{A}_2, w_3))_{\mathbf{xy}} \right|^{2p} \tag{5.68} \\ & \lesssim \mathbf{E} \tilde{\Xi}_2^{\text{iso}} \left| (G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_3 - M(\dots))_{\mathbf{xy}} \right|^{2p-2} + \mathcal{O}_{\prec}((\mathcal{E}_1^{\text{iso}})^{2p}) \\ & \quad + \sum_{|\mathbf{l}| + \sum(J \cup J_*) \geq 2} \mathbf{E} \Xi_2^{\text{iso}}(\mathbf{l}, J, J_*) \left| (G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_3 - M(\dots))_{\mathbf{xy}} \right|^{2p-1-|J \cup J_*|}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\Xi}_2^{\text{iso}} & := \frac{\sum_{\sigma} \sum_{j=1}^3 \left| (G_1 \mathring{A}'_1 E_{\sigma} G_j \mathring{A}_j \dots G_3)_{\mathbf{xy}} (G_1 \mathring{A}_1 \dots \mathring{A}_{j-1} G_j E_{\sigma} G_2 \mathring{A}_2 G_3)_{\mathbf{xy}} \right|}{N} \\ & \quad + \frac{\sum_{\sigma} \sum_{j=1}^3 \left| (G_1 \mathring{A}'_1 E_{\sigma} G_j^* \mathring{A}_{j-1}^* \dots \mathring{A}_1^* G_1^*)_{\mathbf{xx}} (G_3^* \dots \mathring{A}_j^* G_j^* E_{\sigma} G_2 \mathring{A}_2 G_3)_{\mathbf{yy}} \right|}{N} \end{aligned}$$

and $\Xi_2^{\text{iso}}(\mathbf{l}, J, J_*)$ is defined as

$$\begin{aligned} \Xi_2^{\text{iso}} & := N^{-(|\mathbf{l}| + \sum(J \cup J_*) + 1)/2} \sum_{ab} R_{ab} \left| \partial^{\mathbf{l}} \left[(G_1 \mathring{A}'_1)_{\mathbf{xa}} (G_2 \mathring{A}_2 G_3)_{\mathbf{by}} \right] \right| \tag{5.69} \\ & \quad \times \prod_{j \in J} \left| \partial^j (G_1 \mathring{A}_1 G_2 \mathring{A}_2 G_3)_{\mathbf{xy}} \right| \prod_{j \in J_*} \left| \partial^j (G_3^* \mathring{A}_2^* G_2^* \mathring{A}_1^* G_1^*)_{\mathbf{yx}} \right|. \end{aligned}$$

We need to analyze the rhs. of the inequality derived in (5.68). We begin with the second line.

Gaussian contribution: second line of (5.68). Following Remark 5.3, we need to analyze in total twelve terms, each of which carries one of the summands in the definition of $\tilde{\Xi}_2^{\text{iso}}$ as a factor. Again, using Lemma 3.3 for the A 's, we pick two exemplary terms

$$(G_1 \mathring{A}_1^{w_1, w_2} G_2 \mathring{A}_2^{w_2, w_3} G_3 E_{-} G_2 \mathring{A}_2^{w_2, w_3} G_3)_{\mathbf{xy}} (G_1 (\mathring{A}'_1)^{w_1, w_2} E_{-} G_3)_{\mathbf{xy}} \tag{5.70}$$

$$(G_1 (\mathring{A}'_1)^{w_1, w_2} G_2^* (\mathring{A}_1^*)^{\bar{w}_2, \bar{w}_1} G_1^*)_{\mathbf{xx}} (G_3^* (\mathring{A}_2^*)^{\bar{w}_3, \bar{w}_2} G_2^* \mathring{A}_2^{w_2, w_3} G_3)_{\mathbf{yy}} \tag{5.71}$$

which shall be treated in more detail. The other terms are analogous and hence omitted. The term (5.70). In the first factor, we use (2.16), Lemma 3.3, Lemma 4.3 and Lemma 5.1 with parameters

$$\tau = +, \quad J = \mathbf{B}_{(\ell + \frac{1}{2})\kappa_0}, \quad \text{and} \quad \tilde{\eta} = \frac{2\ell}{2\ell + 1}\eta,$$

(in order to have some flexibility before approaching the boundary of the domain $\mathbf{D}_{\ell}^{(\epsilon_0, \kappa_0)}$) to bound it as

$$\begin{aligned} & |(G_1 \mathring{A}_1^{w_1, w_2} G_2 \mathring{A}_2^{w_2, w_3} G_3 E_- G_2 \mathring{A}_2^{w_2, w_3} G_3)_{\mathbf{xy}}| \\ & \prec \frac{1}{\eta^{3/2}} \left(1 + \frac{\psi_3^{\text{iso}}}{\sqrt{N\eta}} \right) + \int_{\mathbf{B}_{(\ell + \frac{1}{2})\kappa_0}} \frac{|(G_1 \mathring{A}_1^{w_1, w_2} G_2 \mathring{A}_2^{w_2, w_3} G(x + i\tilde{\eta})(E_- \mathring{A}_2)^{-w_2, w_3} G_3)_{\mathbf{xy}}|}{(|x - e_3| + \eta_3)(|x + e_2| + \eta_2)} dx. \end{aligned}$$

Next, we decompose $\mathring{A}_2^{w_2, w_3}$ and $(E_- \mathring{A}_2)^{-w_2, w_3}$ according to the integration variable with the aid of Lemma 3.3 (iii), analogously to (5.63). This leaves us with four terms, which shall be estimated separately. While the fully regularised term gives

$$\frac{1}{\eta^{3/2}} \left(1 + \frac{\psi_3^{\text{iso}}}{\sqrt{N\eta}} \right) \left(1 + \frac{1}{|e_2 + e_3| + \eta_2 + \eta_3} \right),$$

the cross terms can be estimated as

$$\frac{1}{\eta^2} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right),$$

analogously to (5.64). As an exemplary error term, we consider

$$\int_{\mathbf{B}_{(\ell + \frac{1}{2})\kappa_0}} |(G_1 \mathring{A}_1^{w_1, w_2} G_2 E_+ G(x + i\tilde{\eta}) E_- G_3)_{\mathbf{xy}}| dx \tag{5.72}$$

and use Lemma 5.1 with new parameters

$$\tau = -, \quad J = \mathbf{B}_{\ell\kappa_0}, \quad \tilde{\eta} = \frac{\ell}{\ell + 1} \eta,$$

to find, dropping the integration domains for ease of notation,

$$\begin{aligned} |(5.72)| & \prec \frac{1}{\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) + \int dx \int dy \frac{|(G_1 \mathring{A}_1^{w_1, w_2} G(y - i\tilde{\eta}))_{\mathbf{x}(E_- \mathbf{y})}|}{(|y - e_2| + \eta_2)(|y + x| + \eta)(|y + e_3| + \eta_3)} \\ & \prec \frac{1}{\eta^{3/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) \left(1 + \frac{1}{|e_2 + e_3| + \eta_2 + \eta_3} \right), \end{aligned}$$

where in the last step we used Lemma 3.3 for decomposing $\mathring{A}_1^{w_1, w_2}$ accordingly, and Lemma 4.3.

This finishes the bound on the first factor in (5.70). The second factor can easily be estimated as

$$|(G_1 (\mathring{A}_1')^{w_1, w_2} E_- G_3)_{\mathbf{xy}}| \prec \frac{1}{\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) + \frac{|e_2 + e_3| + \eta_2 + \eta_3}{\eta}$$

using (2.16), Lemma 3.3, and Lemma 4.3. Notice the cancellation of $|e_2 + e_3|$ between the two factors.

The term (5.71). For the first factor in (5.71), we realise that $(\mathring{A}'_1)^{w_1, w_2} = (\mathring{A}'_1)^{w_1, \bar{w}_2}$, which without approximation immediately yields that

$$|(G_1(\mathring{A}'_1)^{w_1, w_2} G_2^*(\mathring{A}'_1)^{\bar{w}_2, \bar{w}_1} G_1^*)_{xx}| \prec \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right)$$

with the aid of Lemma 4.3.

In the second factor, we apply a Ward identity to $G_2^* G_2$ and again use that the regularisation is insensitive to complex conjugation in the second spectral parameter. In this way, and decomposing

$$\mathring{A}_2^{w_2, w_3} = \mathring{A}_2^{\bar{w}_2, w_3} + \mathcal{O}(|e_2 - e_3| + |\eta_2 - \eta_3|)E_+ + \mathcal{O}(|e_2 + e_3| + |\eta_2 - \eta_3|)E_-$$

by means of Lemma 3.3 (ii), we find that the second factor is stochastically dominated by

$$\frac{1}{\eta^2} \left(1 + \frac{\psi_1^{\text{iso}} + \psi_2^{\text{iso}}}{\sqrt{N\eta}} \right).$$

This finishes the estimate for the Gaussian contribution from the second line of (5.68), for which, collecting the above estimates, we have shown that

$$\tilde{\Xi}_2^{\text{iso}} \prec \frac{1}{N\eta^3} \left[\left(1 + \frac{\psi_3^{\text{iso}}}{\sqrt{N\eta}} \right) \left(1 + \frac{\psi_1^{\text{iso}}}{\sqrt{N\eta}} \right) + \left(1 + \frac{\psi_1^{\text{iso}} + \psi_2^{\text{iso}}}{\sqrt{N\eta}} \right)^2 \right]. \tag{5.73}$$

We are now left with the terms from the last line of (5.68) resulting from higher order cumulants.

Higher order cumulants and conclusion. The estimate stemming from higher order cumulants is given in (5.74d) in Section 5.5. Then, plugging (5.73) and (5.74d) into (5.68), we find, similarly to Section 5.1, that

$$\Psi_2^{\text{iso}} \prec 1 + \psi_1^{\text{iso}} + \frac{\psi_1^{\text{av}} \psi_1^{\text{iso}} + (\psi_1^{\text{iso}})^2 + \psi_2^{\text{iso}}}{N\eta} + \frac{\psi_2^{\text{iso}} + (\psi_1^{\text{iso}} \psi_3^{\text{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{3/8} + (\psi_4^{\text{iso}})^{3/8}}{(N\eta)^{3/16}}$$

The bound given in Proposition 4.9 is an immediate consequence after a trivial Young inequality. \square

5.5. Contributions from higher order cumulants

The goal of the present section is to estimate the terms originating from higher order cumulants in (5.11), (5.27), (5.60), and (5.68). In order to do so, we assume that (4.25) holds.

Lemma 5.10. For any $J, J_* \subset \mathbf{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$, $\mathbf{l} \in \mathbf{Z}_{\geq 0}^2$ with $|\mathbf{l}| + \sum(J \cup J_*) \geq 2$ it holds that

$$(\Xi_1^{\text{av}})^{1/(1+\sum(J \cup J_*))} \prec \frac{1}{N\eta^{1/2}} \left(1 + \frac{\psi_1^{\text{iso}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/4}}{(N\eta)^{1/8}} \right), \tag{5.74a}$$

$$(\Xi_1^{\text{iso}})^{1/(1+\sum(J \cup J_*))} \prec \frac{1}{\sqrt{N\eta^2}} \left(1 + \frac{\psi_1^{\text{iso}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\text{iso}})^{1/4}}{(N\eta)^{1/8}} \right), \tag{5.74b}$$

$$(\Xi_2^{\text{av}})^{1/(1+\sum(J \cup J_*))} \prec \frac{1}{N\eta} \left(1 + \frac{(\psi_1^{\text{iso}})^2}{N\eta} + \frac{\psi_2^{\text{iso}}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{3/8} + (\psi_4^{\text{iso}})^{3/8}}{(N\eta)^{3/16}} \right), \tag{5.74c}$$

$$(\Xi_2^{\text{iso}})^{1/(1+\sum(J \cup J_*))} \prec \frac{1}{\sqrt{N\eta^3}} \left(1 + \frac{(\psi_1^{\text{iso}})^2}{N\eta} + \frac{\psi_2^{\text{iso}}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\text{iso}})^{3/8} + (\psi_4^{\text{iso}})^{3/8}}{(N\eta)^{3/16}} \right). \tag{5.74d}$$

For $k = 1, 2$, $\mathbf{l} \in \mathbf{Z}_{\geq 0}^2$ and a multiset $J \subset \mathbf{Z}_{\geq 0}^2 \setminus \{(0, 0)\}$ we now define slightly (notationally) simplified versions of $\Xi_k^{\text{av/iso}}$, namely

$$\Xi_k^{\text{av}}(\mathbf{l}, J) := N^{-(|\mathbf{l}|+\sum J+3)/2} \sum_{ab} |\partial^{\mathbf{l}}((GA)^{k-1}GA')_{ba}| \prod_{j \in J} |\partial^{\mathbf{j}} \langle (GA)^k \rangle|, \tag{5.75}$$

$$\Xi_k^{\text{iso}}(\mathbf{l}, J) := N^{-(|\mathbf{l}|+\sum J+1)/2} \sum_{ab} |\partial^{\mathbf{l}}[(GA)_{xa}(G(AG)^{k-1})_{by}]| \prod_{j \in J} |\partial^{\mathbf{j}} \langle (GA)^k G \rangle_{\mathbf{x}\mathbf{y}}|, \tag{5.76}$$

where $\sum J := \sum_{j \in J} |j|$, $|(j_1, j_2)| := j_1 + j_2$ and $\partial^{(j_1, j_2)} := \partial_{ab}^{j_1} \partial_{ba}^{j_2}$. Here, for notational simplicity, we do not carry the dependence on the spectral parameters of the resolvents but assume that implicitly each resolvent has its own spectral parameter and that each A is correctly regularised with respect to its neighbouring resolvents. In particular compared to (5.12), (5.28), (5.61), and (5.69), it is not necessary to distinguish the sets J, J^* .

Proof of Lemma 5.10. Throughout the proof, we denote $\phi_k := \psi_k^{\text{iso}}/\sqrt{N\eta}$. The naive estimate for the derivatives simply is

$$\begin{aligned} |\partial^{\mathbf{l}}((GA)^{k-1}GA')_{ba}| &\prec \eta^{-(k-1)/2} \left(1 + \phi_{k-1} \right), \\ |\partial^{\mathbf{j}} \langle (GA)^k \rangle| &\prec \frac{1}{N\eta^{k/2}} \sum_{k_1+k_2+\dots+k_i=k} \prod_i (1 + \phi_{k_i}) \end{aligned} \tag{5.77}$$

due to (4.10) and recalling (4.16). Here a single derivative just splits the chain with k resolvents into two chains with a total of $k + 1$ resolvents, leaving the estimates on the main and the error term unchanged. Using (5.77) in (5.75) we obtain

$$\begin{aligned}
 |\Xi_1^{\text{av}}| &\prec (N\eta^{1/2})^{-1-|J|} N^{(2-|l|-\sum J)} \sqrt{N\eta} \left(1 + \phi_1\right)^{|J|}, \\
 |\Xi_2^{\text{av}}| &\prec (N\eta)^{-1-|J|} N^{(2-|l|-\sum J)} \sqrt{N\eta} \left(1 + \phi_1\right) \left(1 + \phi_2 + \phi_1^2\right)^{|J|}, \\
 |\Xi_1^{\text{iso}}| &\prec (\sqrt{N\eta})^{-1-|J|} \eta^{1+|J|/2} N^{(4-|l|+|J|-\sum J)/2} \left(1 + \phi_1\right)^{|J|}, \\
 |\Xi_2^{\text{iso}}| &\prec (\sqrt{N\eta}^{3/2})^{-1-|J|} \eta^{1+|J|/2} N^{(4-|l|+|J|-\sum J)/2} \left(1 + \phi_1\right) \left(1 + \phi_2 + \phi_1^2\right)^{|J|},
 \end{aligned}
 \tag{5.78}$$

and therefore have proved (5.74a) and (5.74c) in all cases except $|l| + \sum J = 2$ and (5.74b) and (5.74d) in all cases except $|l| + \sum J - |J| < 4$. For the remaining cases we need a more refined estimate using the following *Ward lemma*:

Lemma 5.11. *Let \mathbf{x} be any deterministic vector of bounded norm, let $w_1, \dots, w_k \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ be spectral parameters and A_1, \dots, A_k deterministic matrices of bounded norm. Then for $G_i = G(w_i)$ it holds that*

$$\frac{1}{N} \sum_a \left| (G_1 \mathring{A}_1^{w_1, w_2} \dots \mathring{A}_{k-1}^{w_{k-1}, w_k} G_k A_k)_{\mathbf{x}a} \right| \prec \frac{1}{\sqrt{N\eta}} \frac{1}{\eta^{(k-1)/2}} \left(1 + \phi_1 + \dots + \phi_{2k}\right)^{1/2},$$

which improves upon the term-wise bound by a factor of $(N\eta)^{-1/2}$ at the expense of replacing $1 + \phi_k$ by $1 + \sqrt{\phi_1 + \dots + \phi_{2k}}$.

The proof of the above *Ward lemma* is largely based on yet another more general estimate.

Lemma 5.12. *Let \mathbf{x}, \mathbf{y} be normalised vectors, let $w_1, \dots, w_{k+1} \in \mathbf{D}_{\ell+1}^{(\epsilon_0, \kappa_0)}$ be spectral parameters and A_1, \dots, A_k be deterministic matrices of bounded norm such that a of them are regular, i.e. $\mathring{A}_i^{w_i, w_{i+1}} = A_i$ for all $i \in \mathcal{I}$ for some $\mathcal{I} \subset [k]$ of cardinality a . Then with $G_i = G(w_i)$ it holds that*

$$|(G_1 A_1 G_2 \dots A_k G_{k+1})_{\mathbf{x}\mathbf{y}}| \prec \frac{1}{\eta^{k-a/2}} \left(1 + \phi_1 + \dots + \phi_a\right).
 \tag{5.79}$$

We defer the proof of Lemma 5.12 to the end of this section.

Proof of Lemma 5.11. By Cauchy-Schwarz and the norm bound on the middle A_k we have

$$\begin{aligned}
 &\left(\frac{1}{N} \sum_a \left| (G_1 \mathring{A}_1^{w_1, w_2} \dots \mathring{A}_{k-1}^{w_{k-1}, w_k} G_k A_k)_{\mathbf{x}a} \right| \right)^2 \\
 &\lesssim \frac{1}{N} \left(G_1 \mathring{A}_1^{w_1, w_2} \dots \mathring{A}_{k-1}^{w_{k-1}, w_k} G_k G_k^* \mathring{A}_{k-1}^{\bar{w}_k, \bar{w}_{k-1}} \dots \mathring{A}_1^{\bar{w}_2, \bar{w}_1} G_1 \right)_{\mathbf{x}\mathbf{x}} \\
 &\prec \frac{1}{N\eta^k} \left(1 + \phi_1 + \dots + \phi_{2k}\right)
 \end{aligned}$$

due to Lemma 5.12 for $2k$ resolvents and $a = 2k - 2$ regularised A -matrices. \square

The rest of the proof is split into several cases.

Treatment of (5.74a) and (5.74c) for $|\mathbf{l}| + \sum J = 2$: For the case $|\mathbf{l}| + \sum J = 2$ we either have $|\mathbf{l}| \in \{0, 2\}$ or $\sum J = 1 = |J|$. In the former case an off-diagonal resolvent is guaranteed to be present in the first factor of (5.75) (by parity) and in the latter case the second factor consists of a single off-diagonal resolvent chain. In either case we may use Lemma 5.11 to gain a factor of $1/\sqrt{N\eta}$ compared to (5.77) and obtain

$$\begin{aligned} |\Xi_1^{\text{av}}| &\prec (N\eta^{1/2})^{-1-|J|}(1 + \phi_1)^{(|J|-1)+} \left(1 + \phi_1 + \mathbf{1}(|J| \geq 1)\phi_2^{1/2}\right), \\ |\Xi_2^{\text{av}}| &\prec (N\eta)^{-1-|J|}(1 + \phi_1^2 + \phi_2)^{(|J|-1)+} \left(1 + \phi_1^3 + \phi_2^{3/2} + \mathbf{1}(|J| \geq 1)(\phi_3 + \phi_4)^{3/4}\right), \end{aligned} \tag{5.80}$$

where we used the fact that for $|J| = 0$ only a single factor of $(1 + \phi_1)$ needs to be replaced by a factor of $(1 + (\phi_1 + \phi_2)^{1/2})$ for Ξ_2^{av} and no factor needs to be replaced for Ξ_1^{av} . Moreover, we used $\phi_1(\phi_3 + \phi_4)^{1/2} + \phi_1^2\phi_2^{1/2} \lesssim \phi_1^3 + \phi_2^{3/2} + (\phi_3 + \phi_4)^{3/4}$ by a simple Young inequality. Now (5.80) implies (5.74a) and (5.74c) by another simple Young inequality.

Treatment of (5.74b) and (5.74d) for $|\mathbf{l}| + \sum J - |J| \in \{2, 3\}$: In this case we can simply use Lemma 5.11 for the two resolvent chains in the first factor of (5.76) involving \mathbf{x}, \mathbf{y} to gain a factor of $(N\eta)^{-1}$ compared to (5.77) at the expense of replacing $1 + \phi_1$ by $1 + \phi_1^{1/2} + \phi_2^{1/2}$ in case of Ξ_2^{iso} which proves (5.74b) and (5.74d) in this case.

Treatment of (5.74b) and (5.74d) for $|\mathbf{l}| + \sum J - |J| = 0$: In this case we necessarily have $|\mathbf{l}| = 0$ and $|J| \geq 2$ and $|\mathbf{j}| = 1$ for all $\mathbf{j} \in J$. In particular all factors of (5.76) consist of two resolvent chains evaluated in $(\mathbf{x}, a), (\mathbf{y}, b)$ or $(\mathbf{x}, b), (\mathbf{y}, a)$, respectively. This allows to use Lemma 5.11 four times (twice for the a - and twice for the b -summation) to gain a factor of $(N\eta)^{-2}$ compared to (5.77) at the expense of replacing

$$\text{one factor of } (1 + \phi_1) \quad \text{by} \quad (1 + (\phi_1 + \phi_2)^{1/2})$$

in case of Ξ_1^{iso} and

$$\text{one factor of } (1 + \phi_1)(1 + \phi_1^2 + \phi_2) \quad \text{by} \quad (1 + (\phi_1 + \phi_2)^{1/2})(1 + \phi_1 + \phi_2 + (\phi_3 + \phi_4)^{1/2}) \tag{5.81}$$

in case of Ξ_2^{iso} . This concludes the proof in case of Ξ_1^{iso} and together with

$$(1 + (\phi_1 + \phi_2)^{1/2})(1 + \phi_1 + \phi_2 + (\phi_3 + \phi_4)^{1/2}) \lesssim 1 + (\phi_1 + \phi_2)^{3/2} + (\phi_3 + \phi_4)^{3/4}$$

also in case of Ξ_2^{iso} .

Treatment of (5.74b) and (5.74d) for $|\mathbf{l}| + \sum J - |J| = 1$: In this case we necessarily have $|J| \geq 1$ and either $|\mathbf{l}| = 0$ or $|\mathbf{j}| = 1$ for all $\mathbf{j} \in J$. In either case we can use Lemma 5.11

twice for the first factor and once for some other factor in (5.76) to gain a factor of $(N\eta)^{-3/2}$ compared to (5.77) at the expense of replacing (5.81) in case of Ξ_1^{iso} and

$$\begin{aligned} &\text{one factor of } (1 + \phi_1)(1 + \phi_1^2 + \phi_2) \text{ by} \\ &(1 + (\phi_1 + \phi_2)^{1/2})((1 + \phi_1)(1 + \phi_1 + \phi_2)^{1/2} + (\phi_3 + \phi_4)^{1/2}) \end{aligned}$$

in case of Ξ_2^{iso} . Together with

$$(1 + (\phi_1 + \phi_2)^{1/2})((1 + \phi_1)(1 + \phi_1 + \phi_2)^{1/2} + (\phi_3 + \phi_4)^{1/2}) \lesssim 1 + (\phi_3 + \phi_4)^{3/4} + \phi_2^{3/2} + \phi_1^2$$

this concludes the proof also in this case. \square

It remains to give the proof of Lemma 5.12.

Proof of Lemma 5.12. The proof is via induction, i.e. we assume that (5.79) has been established for resolvent chains of up to k resolvents. For $k + 1$ resolvents and $a = k$, i.e. in case when all deterministic matrices are regular, the claim follows by definition of ψ_k^{iso} . Therefore we may assume that some A_j is not regular which we decompose into its regular component $\mathring{A}_j^{w_j, w_{j+1}}$ and a linear combination of E_{\pm} . By linearity it thus suffices to check (5.79) for the cases $A_j = E_{\pm}$, and moreover, by chiral symmetry $G_j E_- G_{j+1} = -E_- G(-w_j) E_+ G_{j+1}$ and $\mathring{A}_j^{w_{j-1}, w_j} E_- = \mathring{A}_j^{w_{j-1}, -w_j}$ (recall Lemma 3.3) the estimate for E_- follows from the estimate for E_+ upon replacing w_j by $-w_j$. Therefore it suffices to check (5.79) in case $A_j = E_+$.

If $\mathfrak{s}_j = -\text{sgn}(\text{Im } w_j \text{Im } w_{j+1}) = +$, i.e. the adjacent spectral parameters lie in opposite half-planes, then we use the resolvent identity (3.21) to write

$$A_{j-1} G_j E_+ G_{j+1} A_{j+1} G_{j+2} = A_{j-1} \frac{G_j - G_{j+1}}{w_j - w_{j+1}} A_{j+1} G_{j+2}.$$

We discuss each of the two resulting summands separately. For the summand involving G_{j+1} , if A_{j-1} was not counted as regularised, i.e. $j - 1 \notin \mathcal{I}$, then the claim follows by induction and the trivial estimate $|w_j - w_{j+1}| \geq \eta$ since k has been reduced by one, while a has been preserved. On the other hand, if A_{j-1} was correctly regularised, then we use Lemma 3.3 to write

$$\mathring{A}_{j-1}^{w_{j-1}, w_j} = \mathring{A}_{j-1}^{w_{j-1}, \bar{w}_j} = \mathring{A}_{j-1}^{w_{j-1}, w_{j+1}} + \mathcal{O}(|\bar{w}_j - w_{j+1}|) E_+ + \mathcal{O}(|\bar{w}_j - w_{j+1}|) E_- . \tag{5.82}$$

Inserting (5.82) into $A_{j-1} G_{j+1} A_{j+1} G_{j+2} / (w_j - w_{j+1})$ the claimed bound follows from induction since for the $\mathring{A}_{j-1}^{w_{j-1}, w_{j+1}}$ -term a has been preserved and k has been reduced by one compensating for $|w_j - w_{j+1}| \geq \eta$, while for E_{\pm} both k, a have been reduced by one and $|\bar{w}_j - w_{j+1}| / |w_j - w_{j+1}| \leq 1$. Next, for the summand involving G_j , the argument is completely analogous, apart from the two error terms in

$$\begin{aligned} \mathring{A}_{j+1}^{w_j, w_{j+1}} &= \mathring{A}_{j+1}^{w_j, w_{j+2}} + \mathcal{O}(|w_j - \bar{w}_{j+1}| + |w_j - \mathfrak{s}_{j+1}w_{j+2}|)E_{\mathfrak{s}_{j+1}} \\ &\quad + \mathcal{O}(|w_j - \bar{w}_{j+1}| + |w_j + \mathfrak{s}_{j+1}\bar{w}_{j+2}|)E_{-\mathfrak{s}_{j+1}}, \end{aligned} \tag{5.83}$$

appearing for an $A_{j+1} = \mathring{A}_{j+1}^{w_{j+1}, w_{j+2}}$, which has been correctly regularised. Here, we applied Lemma 3.3 and denoted, as usual, $\mathfrak{s}_{j+1} = -\text{sgn}(\text{Im } w_{j+1} \text{Im } w_{j+2})$. Now, for the error terms, we assume that the second summand in each $\mathcal{O}(\dots)$ is non-zero (otherwise we are back to (5.82)) and argue by induction: Indeed, using (2.16) and applying a resolvent identity (3.21), we find

$$\begin{aligned} &\frac{|w_j - \bar{w}_{j+1}| + |w_j - \mathfrak{s}_{j+1}w_{j+2}|}{w_j - w_{j+1}} G_j E_{\mathfrak{s}_{j+1}} G_{j+2} \\ &= \frac{|w_j - \bar{w}_{j+1}| + |w_j - \mathfrak{s}_{j+1}w_{j+2}|}{(w_j - w_{j+1})(w_j - \mathfrak{s}_{j+1}w_{j+2})} \mathfrak{s}_{j+1} (G(w_j) - G(\mathfrak{s}_{j+1}w_{j+2})) E_{\mathfrak{s}_{j+1}}, \end{aligned} \tag{5.84}$$

such that, in the resulting chain we have reduced k by *two* and a by one, and the prefactor in (5.84) is bounded by $1/\eta$. The argument for the second error in (5.83) is completely analogous, after realizing that $(|w_j - \bar{w}_{j+1}| + |w_j + \mathfrak{s}_{j+1}\bar{w}_{j+2}|)/(|w_j - w_{j+1}| |w_j + \mathfrak{s}_{j+1}w_{j+2}|) \leq 1/\eta$.

On the contrary, if $\mathfrak{s}_j = -\text{sgn}(\text{Im } w_j \text{Im } w_{j+1}) = -$, i.e. the adjacent spectral parameters lie the same half-plane (without loss of generality the upper one), then we use the integral representation from Lemma 5.1 to write

$$A_{j-1} G_j E_+ G_{j+1} A_{j+1} = \frac{1}{2\pi i} \int_{\Gamma} \frac{A_{j-1} G(z) A_{j+1}}{(z - w_j)(z - w_{j+1})} dz, \tag{5.85}$$

where Γ is an appropriately chosen contour. If $j - 1, j + 1 \notin \mathcal{I}$, i.e. both A_{j-1}, A_{j+1} were not counted as regularised, then the claim follows by induction and estimating the integral by η^{-1} (up to log factors) since k has been reduced by one, and a has been preserved. On the other hand, if both A_{j-1}, A_{j+1} were counted as regularised, then we use Lemma 3.3 to write them as

$$\begin{aligned} \mathring{A}_{j-1}^{w_{j-1}, w_j} &= \mathring{A}_{j-1}^{w_{j-1}, z} + \mathcal{O}(|w_j - z|)E_+ + \mathcal{O}(|w_j - z|)E_-, \\ \mathring{A}_{j+1}^{w_{j+1}, w_{j+2}} &= \mathring{A}_{j+1}^{z, w_{j+2}} + \mathcal{O}(|w_{j+1} - z|)E_+ + \mathcal{O}(|w_{j+1} - z|)E_-. \end{aligned} \tag{5.86}$$

The resulting term with $\mathring{A}_{j-1}^{w_{j-1}, z}, \mathring{A}_j^{z, w_{j+2}}$ can be estimated by induction since k has been reduced by one, a has been preserved and the integral may be estimated by η^{-1} . The other terms with either one or two E_{\pm} can also be estimated by induction since the integral is at most logarithmically divergent, k has been reduced by one and a by at most two. Finally, if in (5.85) one of A_{j-1}, A_{j+1} were counted as regularised, then we use the relevant expansion from (5.86), so that for the resulting term with \mathring{A} , k has been reduced by one, and a has been preserved, so that the η^{-1} estimate on the integral is affordable. The other term with E_{\pm} can also be estimated by induction with both a, k

reduced by one, and the integral being at most logarithmically divergent. This concludes the proof. \square

6. Proof of the reduction inequalities, Lemma 4.10

During the proof of Lemma 4.10, we will heavily rely on the following integral representation for the absolute value $|G|$ of a resolvent (see also [26, Lemma 5.1]).

Lemma 6.1. (Integral representation for the absolute value of a resolvent)

Let $w = e + i\eta \in \mathbf{C} \setminus \mathbf{R}$. Then the absolute value of the resolvent $G(w)$ can be represented as

$$|G(e + i\eta)| = \frac{2}{\pi} \int_0^\infty \operatorname{Im} G(e + i\sqrt{\eta^2 + s^2}) \frac{ds}{\sqrt{\eta^2 + s^2}}. \tag{6.1}$$

Proof. This immediately follows from the functional calculus for H and the identity

$$\frac{1}{|x - i\eta|} = \frac{1}{i\pi} \int_0^\infty \left(\frac{1}{(x - i(\eta^2 + s^2)^{1/2})} - \frac{1}{x + i(\eta^2 + s^2)^{1/2}} \right) \frac{ds}{\sqrt{\eta^2 + s^2}}. \quad \square$$

Proof of Lemma 4.10. To keep the notation simpler within this proof we may often denote

$$A_i = \mathring{A}_i = \mathring{A}_i^{w_i, w_{i+1}},$$

i.e. sometimes we drop the spectral parameters $w_i = e_i + i\eta_i$.

We start with the proof of (4.27), for which, similarly to [26, Lemma 3.6], we get

$$\Psi_4^{\text{av}} \lesssim N\eta + N^2\eta^2 \left(\langle |G_1|A_1|G_2|A_1^* \rangle \langle |G_2|A_2|G_3|A_2^* \rangle \langle |G_3|A_3|G_4|A_3^* \rangle \langle |G_4|A_4|G_1|A_4^* \rangle \right)^{1/2}, \tag{6.2}$$

by Lemma 4.3, spectral decomposition, and a Schwarz inequality. Next, we use (6.1) to write

$$\langle |G_1|A_1|G_2|A_1^* \rangle = \frac{4}{\pi^2} \iint_0^\infty \langle \operatorname{Im} G(w_{1,s}) \mathring{A}_1^{w_1, w_2} \operatorname{Im} G(w_{2,t}) (\mathring{A}_1^{w_1, w_2})^* \rangle \frac{dsdt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}}, \tag{6.3}$$

where we defined $w_{i,s} := e_i + i\sqrt{\eta_i^2 + s^2}$. The very large s, t -regimes in (6.3) can be easily shown to be negligible (e.g. see [26, Proof of Lemma 5.1]), i.e. even if not stated explicitly we assume that the upper integration limit can be replaced by N^{100} . Additionally, we can restrict to the case when $\eta := \min_j |\operatorname{Im} w_j| \leq 1$, when this is not the case we use the local law in the regime $\eta > 1$ from Theorems 4.4–4.5 (see [26, Proof of Lemma 5.1] for

a detailed argument). We remark that this argument is not circular since in the proof of the local law for $\eta > 1$ sketched below Remark 4.6 one does not use the reduction inequalities in (4.27)–(4.28).

In order to estimate the rhs. of (6.3) we write $\text{Im } G = \frac{1}{2i}(G - G^*)$ for both $\text{Im } G$ to obtain four terms with two resolvents; to keep the presentation concise we only present the estimate for one of them. From now on we consider only the term $\langle |G_1|A_1|G_2|A_1^* \rangle$, the bound for all the other terms in the last line of (6.2) is completely analogous and so omitted. In the following we will often use the approximations from Lemma 3.3 (omitting the trivial $\wedge 1$ in the errors for notational simplicity):

$$\begin{aligned} \mathring{A}^{w_1, w_2} &= \mathring{A}^{w_1, s, w_2, t} + \mathcal{O}(|\sqrt{\eta_1^2 + s^2} - \eta_1| + |\sqrt{\eta_2^2 + t^2} - \eta_2|)E_+ \\ &\quad + \mathcal{O}(|\sqrt{\eta_1^2 + s^2} - \eta_1| + |\sqrt{\eta_2^2 + t^2} - \eta_2|)E_-, \\ (\mathring{A}^{w_1, w_2})^* &= (\mathring{A}^*)^{w_2, t, w_1, s} + \mathcal{O}(|e_1 - e_2| + \sqrt{\eta_1^2 + s^2} + \sqrt{\eta_2^2 + t^2})E_+ \\ &\quad + \mathcal{O}(|e_1 + e_2| + \sqrt{\eta_1^2 + s^2} + \sqrt{\eta_2^2 + t^2})E_-. \end{aligned} \tag{6.4}$$

We point out that when taking the adjoint of the first formula to arrive at the second we used that for any w_1, w_2 it holds $(\mathring{A}^{w_1, w_2})^* = (\mathring{A}^*)^{\overline{w_2}, \overline{w_1}}$, see Lemma 3.3. Recall that within this proof we always assume that $\eta \leq 1$. From now on for the error terms we will always use the bounds

$$|\sqrt{\eta_1^2 + s^2} - \eta_1| \lesssim s, \quad \sqrt{\eta_1^2 + s^2} \leq \eta_1 + s, \tag{6.5}$$

and a similar bound with η_1, s replaced with η_2, t . The first bound is not optimal for small η_1 , but good enough for our estimates. Then using (6.4) we write [J: I removed the σ 's from the \mathcal{O} 's]

$$\begin{aligned} &\iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_1, w_2} G(w_{2,t})(\mathring{A}_1^{w_1, w_2})^* \rangle \frac{dsdt}{\sqrt{\eta_1^2 + s^2}\sqrt{\eta_2^2 + t^2}} \\ &= \iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_1, s, w_2, t} G(w_{2,t})(\mathring{A}_1^*)^{w_2, t, w_1, s} \rangle \frac{dsdt}{\sqrt{\eta_1^2 + s^2}\sqrt{\eta_2^2 + t^2}} \\ &\quad + \sum_{\sigma \in \{+, -\}} \iint_0^\infty \langle G(w_{1,s})E_\sigma G(w_{2,t})(\mathring{A}_1^*)^{w_2, t, w_1, s} \rangle \mathcal{O}(\eta_1 + \eta_2 + s + t) \frac{dsdt}{\sqrt{\eta_1^2 + s^2}\sqrt{\eta_2^2 + t^2}} \\ &\quad + \sum_{\sigma \in \{+, -\}} \iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_1, s, w_2, t} G(w_{2,t})E_\sigma \rangle \mathcal{O}(\eta_1 + \eta_2 + s + t) \frac{dsdt}{\sqrt{\eta_1^2 + s^2}\sqrt{\eta_2^2 + t^2}} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\sigma, \tau \in \{+, -\}} \iint_0^\infty \langle G(w_{1,s}) E_\sigma G(w_{2,t}) E_\tau \rangle \mathcal{O}(\eta_1^2 + \eta_2^2 + s^2 + t^2) \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}} \\
 & + \iint_0^\infty \langle G(w_{1,s}) \left[\sum_\sigma \mathcal{O}(|e_1 - \sigma e_2|) E_\sigma \right] G(w_{2,t}) (\mathring{A}_1^*)^{w_{2,t}, w_{1,s}} \rangle \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}} \\
 & + \iint_0^\infty \langle G(w_{1,s}) \mathring{A}_1^{w_{1,s}, w_{2,t}} G(w_{2,t}) [\mathcal{O}(|e_1 - e_2|) E_+ + \mathcal{O}(|e_1 + e_2|) E_-] \rangle \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}} \\
 & + \iint_0^\infty \langle G(w_{1,s}) \left[\sum_\sigma \mathcal{O}(|e_1 - \sigma e_2|) E_\sigma \right] G(w_{2,t}) \left[\sum_\tau \mathcal{O}(|e_1 - \tau e_2|) E_\tau \right] \rangle \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}}.
 \end{aligned} \tag{6.6}$$

We now estimate the terms in the rhs. of (6.6) one by one. In the following estimates we will always omit $\log N$ -factors. We start with

$$\left| \iint_0^\infty \langle G(w_{1,s}) \mathring{A}_1^{w_{1,s}, w_{2,t}} G(w_{2,t}) (\mathring{A}_1^*)^{w_{2,t}, w_{1,s}} \rangle \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}} \right| < 1 + \frac{\psi_2^{\text{av}}}{N\eta},$$

which readily follows by the definition of Ψ_2^{av} in (4.15) and from the assumption $\Psi_2^{\text{av}} < \psi_2^{\text{av}}$. For the third to the fifth line in (6.6) we use the bound

$$\begin{aligned}
 & \left| \iint_0^\infty \langle G(w_{1,s}) E_\sigma G(w_{2,t}) B \rangle \mathcal{O}(\eta_1 + \eta_2 + s + t) \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}} \right| \\
 & < \iint_0^\infty \left(\frac{1}{\sqrt{\eta_1^2 + s^2}} \wedge \frac{1}{\sqrt{\eta_2^2 + t^2}} \right) [\eta_1 + \eta_2 + s + t] \frac{ds dt}{\sqrt{\eta_1^2 + s^2} \sqrt{\eta_2^2 + t^2}} \lesssim 1,
 \end{aligned} \tag{6.7}$$

for any deterministic norm bounded matrices B and for $\sigma \in \{+, -\}$. For the fifth line of (6.6) we used the bound $(s^2 + t^2) \wedge 1 \leq (s + t) \wedge 1$ (recall that $\wedge 1$ is omitted in the error terms in (6.6) for notational simplicity). Note that here we used:

$$|\langle G(w_{1,s}) E_\sigma G(w_{2,t}) B \rangle| < \frac{1}{\sqrt{\eta_1^2 + s^2}} \wedge \frac{1}{\sqrt{\eta_2^2 + t^2}}, \tag{6.8}$$

which holds uniformly in matrices with $\|B\| \lesssim 1$. We point out that to obtain the bound (6.8) we used spectral decomposition of the resolvents and that $\langle \mathbf{w}_i, E_\sigma \mathbf{w}_j \rangle = \delta_{i,\sigma j}$ to bound¹⁸

¹⁸ We point out that here \mathbf{w}_i denotes an eigenvector of H^Λ , and it should not be confused with the spectral parameters $w_{1,s}, w_{2,t}$.

$$\begin{aligned} |\langle G(w_{1,s})E_\sigma G(w_{2,t})B \rangle| &= \left| \frac{1}{2N} \sum_i \frac{\langle \mathbf{w}_i, B\mathbf{w}_{\sigma i} \rangle}{(\lambda_i - w_{1,s})(\lambda_i - \sigma w_{2,t})} \right| \\ &\lesssim \frac{1}{N} \sum_i \frac{1}{|\lambda_i - w_{1,s}| |\lambda_i - \sigma w_{2,t}|} \\ &\prec \frac{1}{|\operatorname{Im} w_{1,s}| \vee |\operatorname{Im} w_{2,t}|}, \end{aligned}$$

where in the last inequality we used the single resolvent local law.

Finally, for the last three lines in (6.6) we use that for any norm bounded matrix B , by resolvent identity (3.21), we have (recall that $E_+ = I$)

$$|\langle G(w_{1,s})BG(w_{2,t}) \rangle| \prec \frac{1}{|w_{1,s} - w_{2,t}|}, \quad |\langle G(w_{1,s})BG(w_{2,t})E_- \rangle| \prec \frac{1}{|w_{1,s} + w_{2,t}|}, \tag{6.9}$$

which after the integration in (6.6) gives a bound of order one, as a consequence of

$$\frac{|e_1 \pm e_2|}{|w_{1,s} \pm w_{2,t}|} \lesssim 1.$$

Note that here it is important that the error terms in (6.6) involving $|e_1 - e_2|$ are always multiplied with the matrix E_+ , while errors of order $|e_1 + e_2|$ are in the direction of E_- .

Combining the computations in (6.6)–(6.9) we conclude that

$$|\langle |G_1|A_1|G_2|A_1^* \rangle| \prec 1 + \frac{\psi_2^{\text{av}}}{N\eta}, \tag{6.10}$$

which, after plugging it in the rhs. of (6.2), clearly implies (4.27).

For (4.28) for Ψ_3^{iso} , we find

$$\Psi_3^{\text{iso}} \lesssim \sqrt{N\eta} + N\eta^2 \left((G_1A_1|G_2|A_1^*G_1^*)_{\mathbf{xx}} (G_4^*A_3^*|G_3|A_3G_4)_{\mathbf{yy}} \langle |G_2|A_2|G_3|A_2^* \rangle \right)^{1/2}, \tag{6.11}$$

again by Lemma 4.3, spectral decomposition, and a Schwarz inequality. Then, using again the integral representation (6.1), we find that

$$(G_1A_1|G_2|A_1^*G_1^*)_{\mathbf{xx}} = \frac{2}{\pi} \int_0^\infty (G_1A_1 \operatorname{Im} G(w_{2,s})A_1^*G_1^*)_{\mathbf{xx}} \frac{ds}{\sqrt{\eta_2^2 + s^2}},$$

recalling the notation $w_{2,s} = e_2 + i\sqrt{\eta_2^2 + s^2}$. The estimate for this term is fairly similar to the one in (6.3), hence we present only the main differences and skip the details; actually the current case is easier since we now have only one $|G|$.

After splitting $\operatorname{Im} G = \frac{1}{2i}(G - G^*)$ and handling both terms separately, we can write, similarly to (6.6) and using (6.4)–(6.5), the following approximation:

$$\begin{aligned} & \int_0^\infty (G_1 A_1 G(w_{2,s}) A_1^* G_1^*)_{\mathbf{x}\mathbf{x}} \frac{ds}{\sqrt{\eta_2^2 + s^2}} \\ &= \int_0^\infty (G_1 \mathring{A}_1^{w_1, w_2, s} G(w_{2,s}) (\mathring{A}_1^*)^{w_2, s, w_1} G_1^*)_{\mathbf{x}\mathbf{x}} \frac{ds}{\sqrt{\eta_2^2 + s^2}} + \mathcal{E}. \end{aligned} \tag{6.12}$$

Here \mathcal{E} is an error coming from all the errors in (6.4). For the first term in the second line of (6.12) we use the bound

$$\left| \int_0^\infty (G_1 \mathring{A}_1^{w_1, w_2, s} G(w_{2,s}) (\mathring{A}_1^*)^{w_2, s, w_1} G_1^*)_{\mathbf{x}\mathbf{x}} \frac{ds}{\sqrt{\eta_2^2 + s^2}} \right| \prec \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right), \tag{6.13}$$

which follows by the definition of Ψ_2^{iso} . For the error term we do not write the details, since once we replace (6.8)–(6.9) with (here B, B_1, B_2 are deterministic norm bounded matrices)

$$\begin{aligned} |(G_1 B_1 G(w_{2,s}) B_2 G_1^*)_{\mathbf{x}\mathbf{x}}| &\leq (G_1 B_1 B_1^* G_1^*)_{\mathbf{x}\mathbf{x}}^{1/2} (G_1 B_2^* G(w_{2,s}) G(w_{2,s})^* B_2 G_1^*)_{\mathbf{x}\mathbf{x}}^{1/2} \\ &\prec \frac{1}{\eta \sqrt{\eta_2^2 + s^2}} \\ |(G_1 E_\sigma G(w_{2,s}) B G_1^*)_{\mathbf{x}\mathbf{x}}| &\prec \frac{1}{\eta |w_1 - w_{2,s}|}, \end{aligned} \tag{6.14}$$

respectively, the estimate

$$|\mathcal{E}| \prec \frac{1}{\eta} \tag{6.15}$$

follows completely analogously. The estimates (6.14) follow by repeated applications of the resolvent identity (3.21) (after commuting E_σ with G in case of the second formula), the trivial bound $\|G\| \leq 1/\eta$ and the single resolvent local law. Combining, (6.13)–(6.15) we conclude

$$|(G_1 A_1 |G_2| A_1^* G_1^*)_{\mathbf{x}\mathbf{x}}| \prec \frac{1}{\eta} \left(1 + \frac{\psi_2^{\text{iso}}}{\sqrt{N\eta}} \right). \tag{6.16}$$

The bound in (6.16), together with (6.10) to estimate the averaged term in (6.11), concludes the proof (4.28) for Ψ_3^{iso} .

Analogously to (6.11), for Ψ_4^{iso} we find that

$$\begin{aligned} \Psi_4^{\text{iso}} &\lesssim \sqrt{N\eta} + N\eta^{5/2} \left((G_1 A_1 | G_2 | A_1^* G_1^*)_{\mathbf{x}\mathbf{x}} (G_5^* A_4^* | G_4 | A_4 G_5)_{\mathbf{y}\mathbf{y}} \right. \\ &\quad \left. \times \langle |G_2 | A_2 G_3 A_3 | G_4 | A_3^* G_3^* A_2^* \rangle \right)^{1/2} \\ &\lesssim \sqrt{N\eta} + N^{3/2} \eta^{5/2} \left((G_1 A_1 | G_2 | A_1^* G_1^*)_{\mathbf{x}\mathbf{x}} (G_5^* A_4^* | G_4 | A_4 G_5)_{\mathbf{y}\mathbf{y}} \right)^{1/2} \\ &\quad \times \left(\langle |G_2 | A_2 | G_3 | A_2^* \rangle \langle |G_3 | A_3 | G_4 | A_3^* \rangle \langle |G_4 | A_3^* | G_3 | A_3 \rangle \langle |G_3 | A_2^* | G_2 | A_2 \rangle \right)^{1/4} \end{aligned}$$

where in the last inequality we used spectral decomposition and a bound as in [26, Proof of Lemma 3.6] to bound the trace with four G 's and four A 's in terms of a product of traces containing only two G 's and two A 's. Finally, using the bounds (6.10), (6.16), we conclude the proof of (4.28) for Ψ_4^{iso} as well. \square

Data availability

No data was used for the research described in the article.

Appendix A. Motivating derivations of the regularisation

In this appendix, we shall motivate and derive the regularisation (3.2) introduced in Definition 3.1 by considering two basic examples. We also use these examples to present two different approaches to guess the right regularisation. Before the details, we give an informal summary of these two model calculations.

First, in Section A.1, we compute

$$\mathbf{E} \left| \langle \underline{WG(i\eta)A} \rangle \right|^2, \tag{A.1}$$

which is the leading contribution to $\langle (G - M)B \rangle$ in the single-resolvent local law, with $A = \mathcal{X}[B]M$, see (5.18). We will show that, in order to be able to reduce its naive size $1/(N\eta)^2$ to the target $1/(N^2\eta)$, we need that $\langle A, V_{\pm} \rangle = 0$, i.e. we need $A \in \mathbf{C}^{2N \times 2N}$ to be orthogonal to two certain directions V_{\pm} in $\mathbf{C}^{2N \times 2N}$. For simplicity, we chose the spectral parameter $w = i\eta$ to be on the imaginary axis, assuming that $0 \in \mathbf{B}_{\kappa}$ for some $\kappa > 0$. In this case, both cutoff functions (4.7) in the actual definition of the regularisation satisfy $\mathbf{1}_{\delta}^{\pm}(i\eta, i\eta) = 0$ for $\eta > 0$ small enough. Hence, at least *a posteriori*, we really catch both directions V_{\pm} and not only one. This calculation is rather *foundational* and unambiguously reveals two directions V_{\pm} , for which we need that $\langle A, V_{\pm} \rangle = 0$, in order to reduce the naive size of (A.1).

Second, in Section A.2, we consider the averaged chain with two resolvents

$$\langle G^{\Lambda_1}(w_1) A_1 G^{\Lambda_2}(w_2) A_2 \rangle, \tag{A.2}$$

where the resolvents are even allowed to have generally *different*¹⁹ deformations, Λ_1 and Λ_2 . Let $M_1 := M^{\Lambda_1}(w_1)$ and $M_2 := M^{\Lambda_2}(w_2)$. For simplicity, we will assume that the stability operators

$$\mathcal{B}_{m^{(*)}n^{(*)}} := 1 - M_m^{(*)} \mathcal{S}[\cdot] M_n^{(*)}, \quad m, n \in [2], \tag{A.3}$$

for all constellations of adjoints, have at most one *critical* eigenvalue $\beta_{m^{(*)}n^{(*)}}$ which is *not* of order one (with associated right and left eigenvectors $R_{m^{(*)}n^{(*)}}$ and $L_{m^{(*)}n^{(*)}}$, respectively, cf. (B.9) later). As we will show in Lemma B.2 (c), this is the case, e.g., if $\Lambda \equiv \Lambda_1 = \Lambda_2$ and $\text{Re } w_1, \text{Re } w_2 \in \mathbf{B}_\kappa^\Lambda$, and actually remains true for other more general random matrix models with a *flat* [2] self-energy operator $\mathcal{S}[\cdot]$. Recall that $\mathcal{S}[\cdot]$ is flat if

$$c\langle R \rangle \leq \mathcal{S}[R] \leq C\langle R \rangle \tag{A.4}$$

for some constants $c, C > 0$ and any positive semi-definite matrix $R \geq 0$.

Again, the main question is what special property A_1, A_2 must have so that (A.2) be smaller than its naive size of order $1/\eta$ obtained from a simple Schwarz inequality. Similarly to (A.1), we could directly compute the second moment of the corresponding underline term (see Lemma 5.8), but for pedagogical reason we present an alternative argument. Quite *pragmatically*, we start the usual proof via cumulant expansion for a bound on (A.2) and find that certain deterministic terms are too big for general A_1, A_2 . We shall see that there exist two matrices $\tilde{V}_\pm \in \mathbf{C}^{2N \times 2N}$ (which turn out to be certain right eigenvectors $R_{m^{(*)}n^{(*)}}$ of (A.3), see (A.15) and (A.18) later), such that, if $\langle A_i, \tilde{V}_\pm \rangle = 0$, these critical terms are smaller. This suggests a *pragmatic ansatz* of the form (3.2) on the regularisation. We will observe that, for the situation $\Lambda_1 = \Lambda_2$ and $w_1 = w_2 = i\eta$, the expressions for \tilde{V}_\pm in fact *coincide* with those for V_\pm obtained in Section A.1. Notice that the imaginary part of the single-resolvent setup leading to (A.1) is a special case of the two-resolvent setup (A.2) since

$$\text{Im} \langle GB \rangle = \langle (\text{Im } G)B \rangle = \eta \langle GBG^* E_+ \rangle$$

for self-adjoint B . Hence the regularity of B tested against $\text{Im } G(i\eta)$ is the same as the regularity of B between $G(i\eta)$ and $G^*(i\eta) = G(-i\eta)$. This shows, at least in this special case, that the *foundational* and the *pragmatic* approaches lead to the same regularisation. Similar conclusion about the equivalence of both approaches holds in general.

Finally, in Section A.3, motivated by the previous tandem of *foundational* and *pragmatic* computations in Sections A.1 and A.2, respectively, we list generally valid (i.e. for arbitrary w_1, w_2 also away from the imaginary axis) explicit formulas for the directions V_\pm regularising (A.2) in case that $\Lambda_1 = \Lambda_2$. These explicit formulas are identical to those used in the regularisation introduced in Definition 3.1.

¹⁹ All results in the current paper concern the $\Lambda_1 = \Lambda_2$ case; the generalisation $\Lambda_1 \neq \Lambda_2$ is mentioned only to stress that our method is also valid beyond the scope of the current paper.

A.1. Variance calculation of (A.1)

In the following, we simply write $G = G(i\eta)$ for ease of notation. Then, using a cumulant expansion and neglecting cumulants of order at least three (or assuming that X is Ginibre), one gets

$$\begin{aligned}
 \mathbf{E}|\langle \underline{WGA} \rangle|^2 &= \frac{1}{N} \sum_{ab} R_{ab} \mathbf{E} \langle \Delta^{ab} GA \rangle \partial_{ba} \langle \underline{A^* G^* W} \rangle \\
 &= \frac{1}{N} \sum_{ab} R_{ab} \mathbf{E} \langle \Delta^{ab} GA \rangle \langle GA^* G^* \Delta^{ba} \rangle \\
 &\quad + \frac{1}{N^2} \sum_{abcd} R_{ab} R_{cd} \mathbf{E} \langle \Delta^{ab} G \Delta^{dc} GA \rangle \langle A^* G^* \Delta^{ba} G^* \Delta^{cd} \rangle \\
 &= \frac{1}{N^2} \sum_{\sigma} \sigma \mathbf{E} \langle E_{\sigma} G A E_{\sigma} A^* G^* \rangle + \frac{1}{N^2} \sum_{\sigma\tau} \sigma\tau \mathbf{E} \langle E_{\sigma} G^* E_{\tau} GA \rangle \langle E_{\sigma} G E_{\tau} (GA)^* \rangle.
 \end{aligned}
 \tag{A.5}$$

The rescaled cumulant $R_{ab} := N\kappa(ab, ba)$ has been introduced above (5.11) and $\Delta^{ab} \in \mathbf{C}^{2N \times 2N}$ contains only one non-zero entry at position (a, b) , i.e. $(\Delta^{ab})_{cd} = \delta_{ac} \delta_{bd}$.

As we will show, the cumulant expansion (A.5) yields that (up to a constant)

$$\mathbf{E}|\langle \underline{WGA} \rangle|^2 \approx \frac{\mathbf{E}|\langle \text{Im } GA \rangle|^2}{(N\eta)^2} + \frac{\mathbf{E}|\langle \text{Im } GA E_{-} \rangle|^2}{(N\eta)^2} + \mathcal{O}\left(\frac{1}{N^2\eta}\right).
 \tag{A.6}$$

Indeed, the first summand in the last line of (A.5) is estimated by $1/(N^2\eta)$, the target size, with the aid of a trivial Schwarz inequality and a Ward identity using Theorem 2.6. By writing out the summation in the last summand, we get in total four terms. Since their treatment is very similar, we focus on two exemplary terms with $\sigma = \tau = +$ (analogous to $\sigma = \tau = -$) and $\sigma = -\tau = -$ (analogous to $\sigma = -\tau = +$).

For the former, we apply a Ward identity and find it to be given by

$$\frac{\mathbf{E}|\langle \text{Im } GA \rangle|^2}{(N\eta)^2},
 \tag{A.7}$$

which, without any further information on A , using that $\langle GA \rangle \sim 1$ from Theorem 2.6, is too big, compared to the targeted $1/(N^2\eta)$ -size. However, this drastically improves if $\langle \text{Im } M, A \rangle = 0$ (recall that $\text{Im } M$ is self adjoint): Since $\langle (G - M)A \rangle$ and $\langle \underline{WGA} \rangle$ are roughly of the same size (see (5.18)), the contribution (A.7) basically becomes a lower-order correction. We have thus identified the first of the two directions V_{\pm} , to which A has to be orthogonal to in order to reduce the naive size of (A.1), namely

$$V_{+} = \alpha_{+} \text{Im } M \quad \text{for some non-zero } \alpha_{+} \in \mathbf{C}.
 \tag{A.8}$$

The latter case, $\sigma = -\tau = -$, is slightly more involved due to the asymmetry of the two factors in the last summand in the last line of (A.5): For the first factor, again a Ward identity is sufficient. In the second factor, we use (2.16) together with the integral representation [26, Eq. (3.14)]

$$G^*G^* = \int_{\mathbf{R}} \frac{\text{Im } G(x + i\eta/2)}{(x + i\eta/2)^2} dx,$$

similar to Lemma 5.1, in the approximate form $G^*G^* \sim \text{Im } G/\eta$. This follows (at least as an effective upper bound) by replacing the Cauchy kernel in the integral

$$|\langle G^*G^*E_-A^* \rangle| \leq \int_{\mathbf{R}} \frac{|\langle \text{Im } G(x + i\eta/2)E_-A^* \rangle|}{x^2 + (\eta/2)^2} dx \sim \frac{\text{Im } G(i\eta)}{\eta}$$

by a δ -distribution. Overall, this leaves us (roughly) with

$$\frac{\mathbf{E} |\langle \text{Im } GAE_- \rangle|^2}{(N\eta)^2} \tag{A.9}$$

for the second case. Hence, arguing for (A.9) completely analogous as done for (A.7), we find the second direction V_- , to which A has to be orthogonal to, in order to reduce the naive size of (A.1), namely

$$V_- = \alpha_- \text{Im } ME_- \quad \text{for some non-zero } \alpha_- \in \mathbf{C}. \tag{A.10}$$

We point out that the first term in (A.6) would have worked in the exact same way also for spectral parameters $w = e + i\eta$ with $e \neq 0$. However, the second direction V_- would *not* have been visible in this scenario, since the second term in (A.6) would have been replaced by (at least for an upper bound)

$$\frac{\mathbf{E} |\langle \text{Im } G(e + i\eta)AE_- \rangle|^2}{N^2\eta(|e| + \eta)} + \frac{\mathbf{E} |\langle \text{Im } G(e + i\eta)AE_- \rangle \langle \text{Im } G(-e + i\eta)E_-A^* \rangle|}{N^2\eta(|e| + \eta)}.$$

A.2. General structural regularisation in (A.2)

We begin with the general rather *structural* regularizing decomposition of a matrix A (recall (3.2)), which shall be conducted as (dropping the tilde, which has been temporarily introduced below (A.3))

$$\boxed{A^\circ \equiv \mathring{A} := A - \langle V_+, A \rangle U_+ - \langle V_-, A \rangle U_-} \tag{A.11}$$

for some $U_\sigma, V_\sigma \in \mathbf{C}^{2N \times 2N}$ to be determined but subject to the conditions $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma,\tau}$ and $\langle U_\sigma, U_\sigma \rangle = 1$. We point out, that the following calculations are largely insensitive to

the form of the self-energy operator $\mathcal{S}[\cdot]$ (but see Footnote 20) and hence the conclusions for U_σ and V_σ derived in this section are valid beyond our concrete model (up to the fact that, due to the chiral symmetry (2.16), the regularisation involves a *two*-dimensional projection).

The goal of the present subsection is to *show* that V_\pm must be chosen as certain right eigenvectors $R_{m^{(*)}n^{(*)}}$ of (A.3). This follows by expanding (A.2) and identifying several terms, whose size is too big for general deterministic matrices. Now, these terms can be neutralised, if $\langle A_i, R_{m^{(*)}n^{(*)}} \rangle = 0$ for certain right eigenvectors. However, as already mentioned in Section 3, for the directions U_\pm there are *a priori* no further constraints or conditions (apart from orthogonality and normalisation). Hence, as it turns out to be convenient for our proofs, we will choose the matrices U_σ in such a way, that a resolvent identity, i.e. the transformation of a product into a difference,

$$G^{\Lambda_1}(w_1)U_\sigma G^{\Lambda_2}(w_2) \approx (G^{\Lambda_1}(w_1) - G^{\Lambda_2}(\sigma w_2))U_\sigma,$$

can be applied (here, the symbol ‘ \approx ’ neglects lower order terms). Finally, the condition $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma,\tau}$ will guarantee that the regularisation is idempotent, i.e. $(\mathring{A})^\circ = \mathring{A}$. Note that our general ansatz (A.11) is restricted to the non-degenerate situation, where U_σ and V_σ are non-orthogonal, $\langle V_\sigma, U_\sigma \rangle \sim 1$. This is guaranteed for our concrete model with deformations $\Lambda_1 = \Lambda_2$ (see Section A.3) but requires some non-trivial arguments in more general cases.

Although the regularisation is inherently two-dimensional (at least for our model), we also define

$$\mathring{A}^\sigma = A^{\circ\sigma} := A - \langle V_\sigma, A \rangle U_\sigma, \quad \sigma \in \{+, -\},$$

and refer to $A^{\circ\sigma}$ as the σ -regular component (or σ -regularisation) of A and to $\langle V_\sigma, A \rangle U_\sigma$ as its σ -singular component. Note that $(A^{\circ+})^{\circ-} = (A^{\circ-})^{\circ+} = \mathring{A}$, since $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma,\tau}$.

As usual, we use the common notation $\eta_i := |\text{Im } w_i|$ for $i \in [2]$ and abbreviate (see (3.8))

$$\mathfrak{s}_i := -\text{sgn}(\text{Im } w_i \text{Im } w_{i+1}), \quad i \in [2], \tag{A.12}$$

where the indices are understood cyclically modulo 2 (cf. Definition 4.2). This means that, in particular, $\mathfrak{s}_1 = \mathfrak{s}_2$ due to the short length of the chain (A.2). In the following, we will drop the arguments by writing, e.g., $M_1 = M^{\Lambda_1}(w_1)$ and $G_2 = G^{\Lambda_2}(w_2)$. Moreover, we take $A_1 = \mathring{A}_1$ and $A_2 = \mathring{A}_2$ to be regular, i.e. orthogonal to some yet to be specified V_\pm .

Now, by means of

$$G_1 = M_1 - M_1 \underline{W} G_1 + M_1 \mathcal{S}[G_1 - M_1] G_1,$$

we immediately find

$$G_1 A_1 G_2 = M_1 A_1 G_2 - M_1 \underline{W G_1} A_1 G_2 + M_1 \mathcal{S}[G_1 - M_1] G_1 A_1 G_2,$$

from which we conclude that

$$\begin{aligned} \mathcal{B}_{12}[G_1 A_1 G_2] &= M_1 A_1 M_2 + M_1 A_1 (G_2 - M_2) - M_1 \underline{W G_1} A_1 G_2 \\ &\quad + M_1 \mathcal{S}[G_1 - M_1] G_1 A_1 G_2 + M_1 \mathcal{S}[G_1 A_1 G_2] (G_2 - M_2). \end{aligned}$$

This implies

$$\begin{aligned} \langle (G_1 A_1 G_2 - M_{12}^{A_1}) A_2 \rangle &= \langle M_1 A_1 (G_2 - M_2) \mathcal{X}_{21}[A_2] \rangle - \langle M_1 \underline{W G_1} A_1 G_2 \mathcal{X}_{21}[A_2] \rangle \\ &\quad + \langle M_1 \mathcal{S}[G_1 - M_1] G_1 A_1 G_2 \mathcal{X}_{21}[A_2] \rangle \\ &\quad + \langle M_1 \mathcal{S}[G_1 A_1 G_2] (G_2 - M_2) \mathcal{X}_{21}[A_2] \rangle \end{aligned}$$

where we defined

$$M_{12}^{A_1} := \mathcal{B}_{12}^{-1}[M_1 A_1 M_2] = M_1 \mathcal{X}_{12}[A_1] M_2 = M(w_1, A_1, w_2) \tag{A.13}$$

(recall (4.4) and Definition 4.1) and used the shorthand notation

$$\mathcal{X}_{mn}[B] = ((\mathcal{B}_{nm}^*)^{-1}[B^*])^* = (\mathcal{B}_{m^*n^*}^{-1})^*[B], \quad B \in \mathbf{C}^{2N \times 2N}.$$

The adjoint of \mathcal{B}_{nm} is understood with respect to the standard (normalised) inner product $\langle S, T \rangle := \langle S^* T \rangle$ for $S, T \in \mathbf{C}^{2N \times 2N}$, which is given by

$$\mathcal{B}^* \equiv \mathcal{B}^*(w_1, w_2)[\cdot] := 1 - \mathcal{S}[(M(w_1))^* \cdot (M(w_2))^*]. \tag{A.14}$$

So far, the regularisation of A_1 and A_2 has been rather *structural*. To make it more concrete, we must allow V_σ and U_σ to be potentially different depending on which of the A_i is regularised. In order to do so, we also temporarily introduce the additional index i , referring to the considered A_i . That is, we will write $V_{\sigma,i}$ instead of V_σ .

The matrices $V_{\mathfrak{s}_i,i}$ (recall (A.12) for the definition of \mathfrak{s}_i) shall be determined by requiring that

$$\|M_{12}^{A_1}\| = \|M_1 \mathcal{X}_{12}[A_1] M_2\| \lesssim \|A_1\| \quad \text{for } i = 1 \quad \text{and} \quad \|\mathcal{X}_{21}[A_2]\| \lesssim \|A_2\| \quad \text{for } i = 2,$$

meaning that the (adjoint of the) stability operator has a bounded inverse on regular observables (i.e. subtracting the \mathfrak{s}_i -singular component amounts to removing the ‘bad direction’ of the stability operators \mathcal{X}_{12} and \mathcal{X}_{12} , respectively). From this condition, we find the characterisation of $V_{\mathfrak{s}_{1,1}}$ and $V_{\mathfrak{s}_{2,2}}$, namely

$$V_{\mathfrak{s}_{1,1}} = R_{1^*2^*} = (R_{21})^* \quad \text{and} \quad V_{\mathfrak{s}_{2,2}} = R_{2^*1^*} = (R_{12})^*, \tag{A.15}$$

up to a normalisation constant, which can be specified only after determining U_σ (recall that $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma,\tau}$ and $\langle U_\sigma, U_\sigma \rangle = 1$). Recall from (A.3), that we denote by $R_{m^{(*)}n^{(*)}}$ and $L_{m^{(*)}n^{(*)}}$ the (normalised) right and left eigenvectors of $\mathcal{B}_{m^{(*)}n^{(*)}}$ corresponding to the (potentially) *critical eigenvalue* $\beta_{m^{(*)}n^{(*)}}$.

Indeed, in order to verify that (A.15) is the right choice for $V_{s_i,i}$, we use the decomposition

$$\mathcal{X}_{mn} = (\mathcal{B}_{m^*n^*}^{-1})^* = \frac{1}{\beta_{m^*n^*}} |L_{m^*n^*}\rangle \langle R_{m^*n^*}| + \mathcal{O}(1), \tag{A.16}$$

where $\mathcal{O}(1)$ is a shorthand notation for a linear operator $\mathcal{E} : \mathbf{C}^{2N \times 2N} \rightarrow \mathbf{C}^{2N \times 2N}$ satisfying $\|\mathcal{E}[B]\| \lesssim \|B\|$. This linear operator is represented by a contour integration of the form

$$\frac{1}{2\pi i} \oint \frac{dz}{z - \mathcal{B}_{m^*n^*}^*}$$

where the contour encircles all non-critical eigenvalues of $\mathcal{B}_{m^*n^*}^*$ and remains at an order one distance from the entire spectrum. Note that for general non-Hermitian operators the resolvent $(z - \mathcal{B}_{m^*n^*}^*)^{-1}$ would not necessarily be bounded (independently of N) just because z is well away from the eigenvalues. However, the explicit form of \mathcal{S} (see (2.21)) implies²⁰ that $\mathcal{B}_{m^*n^*}^* = 1 + T$ where T is a rank-two operator. For such operators elementary linear algebra shows that

$$\left\| \frac{1}{z - \mathcal{B}_{m^*n^*}^*} \right\| \lesssim [\text{dist}(z, \text{Spec}(\mathcal{B}_{m^*n^*}^*))]^{-2},$$

i.e. the non-Hermitian instability only affects a two-dimensional subspace.

Using (A.16) we find

$$\mathcal{X}_{12}[\mathring{A}_1^{s_1}] = \frac{1}{\beta_{1^*2^*}} (\langle R_{1^*2^*}, A_1 \rangle - \langle V_{s_1,1}, A_1 \rangle \langle R_{1^*2^*}, U_{s_1,1} \rangle) L_{1^*2^*} + \mathcal{O}(1)[A_1]$$

for the decomposition of A_1 and

$$\mathcal{X}_{21}[\mathring{A}_2^{s_2}] = \frac{1}{\beta_{2^*1^*}} (\langle R_{2^*1^*}, A_2 \rangle - \langle V_{s_2,2}, A_2 \rangle \langle R_{2^*1^*}, U_{s_2,2} \rangle) L_{2^*1^*} + \mathcal{O}(1)[A_2],$$

for the decomposition of A_2 . This implies that for (\dots) to be vanishing for every $\mathring{A}_i^{s_i}$, the matrix $V_{s_i,i}$ has to be chosen according to (A.15) (recall $\langle V_{\sigma,i}, U_{\tau,i} \rangle = \delta_{\sigma,\tau}$).²¹ Overall,

²⁰ This is the only place in Section A.2 where the special form of \mathcal{S} is currently used. For more general \mathcal{S} operator an appropriate generalisation of the symmetrised (saturated) self-energy operator [2, Def. 4.5] to two different spectral parameters is needed, see [47, Eq. (2.30)] in the commutative case.

²¹ In case that $\Lambda_1 = \Lambda_2$, by the lower bound (B.8), the choices in (A.15) not necessarily have to be made *exact*, but tolerate an error of the order given in the rhs. of (B.8). Having such a tolerance might be important if one treats the $\Lambda_1 \neq \Lambda_2$ case (contrary to $\Lambda_1 = \Lambda_2$ as done in this paper) and still has to satisfy the constraints $\langle V_\sigma, U_\tau \rangle = \delta_{\sigma,\tau}$ and $\langle U_\sigma, U_\sigma \rangle = 1$.

subtracting the \mathfrak{s}_i -singular component already accounts for removing the ‘bad direction’ of a involved stability operator and thus – in particular – reduces the naive size of the deterministic approximation (A.13).

However, removing the \mathfrak{s}_i -singular component is not sufficient: Although $\langle V_{\mathfrak{s}_i,i}, U_{-\mathfrak{s}_i,i} \rangle = 0$ and thus $U_{-\mathfrak{s}_i,i}$ is \mathfrak{s}_i -regular, we observe that

$$\langle G_1 U_{-\mathfrak{s}_1,1} G_2 U_{-\mathfrak{s}_2,2} \rangle \tag{A.17}$$

still (potentially) has large fluctuations: In our concrete i.i.d. model, take $z \equiv z_1 = z_2$ (to be suppressed from the notation) and $w \equiv w_1 = -w_2$ with $e = \text{Re } w_1$ and $\eta = \text{Im } w_1 > 0$ w.l.o.g., which implies that $\mathfrak{s}_1 = \mathfrak{s}_2 = +$ and $U_\sigma = E_\sigma$ for $\sigma = \pm$ (see the discussion below (3.4)). In this situation, we use (2.16) and thus (A.17) takes the form

$$\langle G(e + i\eta)E_- G(-e - i\eta)E_- \rangle = -\langle G(e + i\eta)G(e + i\eta) \rangle.$$

By construction of $V_{\mathfrak{s}_i,i}$, the corresponding deterministic approximation (A.13) is bounded by one, but this is dominated by the fluctuation of order $1/(N\eta^2)$ in the relevant small regime $\eta \sim N^{-1+\epsilon}$. This example shows again, what we have already established in Section A.1: For our concrete model, at least close to the imaginary axis, the regularisation (3.2) is *necessarily a two-dimensional operation*.

For determining the other directions $V_{-\mathfrak{s}_i,i}$, we note that the regularisation should be designed in such a way, that it covers also the cases where one (or both) of the resolvents G_1, G_2 are taken as an adjoint (see, e.g., (5.13) and (6.10)). Hence, requiring that the same arguments leading to (A.15) should also be followed for (i) $\langle G_1 A_1 G_2^* A_2 \rangle$ and (ii) $\langle G_1^* A_1 G_2 A_2 \rangle$ (considering $\langle G_1^* A_1 G_2^* A_2 \rangle$ would again lead to a conclusion for $V_{\mathfrak{s}_i,i}$ as the relative sign of imaginary parts is preserved), we find that $V_{-\mathfrak{s}_1,1} = (R_{2^*1})^*$ and $V_{-\mathfrak{s}_2,2} = (R_{12^*})^*$ in case (i), and $V_{-\mathfrak{s}_1,1} = (R_{21^*})^*$ and $V_{-\mathfrak{s}_2,2} = (R_{1^*2})^*$ in case (ii). In general, the right eigenvectors for these two cases are not the same. However, as pointed out in Footnote 21, there is a certain *tolerance* in choosing the V_\pm . Therefore, within this tolerance and in order to have a consistent and conceptually simple choice, we take $V_{-\mathfrak{s}_1,1}$ from case (i) and $V_{-\mathfrak{s}_2,2}$ from case (ii), i.e.

$$\boxed{V_{-\mathfrak{s}_1,1} = R_{1^*2} = (R_{2^*1})^* \quad \text{and} \quad V_{-\mathfrak{s}_2,2} = R_{2^*1} = (R_{1^*2})^* .} \tag{A.18}$$

Here, in both situations the spectral parameter being the *right* neighbour of A_i receives a complex conjugate. In comparison, if we took $V_{-\mathfrak{s}_1,1}$ from case (ii) and $V_{-\mathfrak{s}_2,2}$ from case (i), we would have ended up with the alternative regularisation from Footnote 12, where the *left* neighbour of A_i received a complex conjugate. Again, the relations in (A.18) are understood up to a normalizing constant, which can be specified only after determining U_σ .

Now, it is very important to observe that, for our concrete model with $\Lambda_1 = \Lambda_2$ and $w_1 = w_2 = i\eta$ (in particular, $\mathfrak{s}_1 = \mathfrak{s}_2 = -$), our choices for V_\pm in (A.15) and (A.18) *agree*

with those in (A.8) and (A.10) obtained from a variance calculation with only a single resolvent. This follows from the explicit formulas for the critical right eigenvector given later in (B.9), Lemma B.1 (a), and (2.19)

A.3. *Explicit formulas for our concrete model and $\Lambda_1 = \Lambda_2$*

In this subsection, we will give explicit formulas for V_{\pm} and U_{\pm} for our concrete model with one fixed deformation Λ . In fact, for $\Lambda_1 = \Lambda_2$, the so far unspecified matrices U_{σ} can be characterised by requiring that, jointly with the symmetry relation $E_- G^z(-w) E_- = -G^z(w)$, a resolvent identity (see (3.21) for the standard resolvent identity) can be applied to $G_2 U_{\sigma} G_1$. This yields, together with the normalisation $\langle U_{\sigma}, U_{\sigma} \rangle = 1$, that²²

$$U_+ = E_+ \quad \text{and} \quad U_- = E_- .$$

The singular (or critical) eigenvectors of the stability operators characterizing $V_{s_i,i}$ can also be explicitly calculated. Using (A.15) and (A.18), we infer, by means of (B.9) and the normalisation/orthogonality condition $\langle V_{\sigma,i}, U_{\tau,i} \rangle = \delta_{\sigma,\tau}$, that

$$\begin{aligned} V_{s_1,1} &= \frac{M_2 E_{s_1} M_1}{\langle M_2 E_{s_1} M_1 E_{s_1} \rangle}, & V_{-s_1,1} &= \frac{M_2^* E_{-s_1} M_1}{\langle M_2^* E_{-s_1} M_1 E_{-s_1} \rangle}, \\ V_{s_2,2} &= \frac{M_1 E_{s_2} M_2}{\langle M_1 E_{s_2} M_2 E_{s_2} \rangle}, & V_{-s_2,2} &= \frac{M_1^* E_{-s_2} M_2}{\langle M_1^* E_{-s_2} M_2 E_{-s_2} \rangle}, \end{aligned} \tag{A.19}$$

matching the definition of the regularisation given in (4.8) and (3.7). The normalisation is obvious and the orthogonality readily follows from (2.19) in combination with Lemma B.1.

Finally, we remark that in order to define the regularisation (3.7) and work with (A.15) and (A.18), it is *not* necessary to have the explicit forms for $V_{\sigma,i}$ at hand. Instead, the single instance of relevant *explicit* formulas is the proof of Theorem 2.7, more precisely, the bound in Proposition 3.4, where one needs that for $|\text{Im } w_1| \sim N^{-1+\epsilon}$, e.g., $(R_{1*1})^*$ is close to $\text{Im } M_1$ (up to a normalisation). But this is true beyond our model, as easily follows after taking the imaginary part of the general *matrix Dyson equation* (see [36])

$$-\frac{1}{M} = w - A + \mathcal{S}[M], \quad \text{Im } w \cdot \text{Im } M > 0$$

with self-adjoint *matrix of expectations* $A = A^*$ and (flat) *self-energy operator* $\mathcal{S}[\cdot]$. In fact, this yields

$$(1 - M\mathcal{S}[\cdot]M^*)(\text{Im } M) = (\text{Im } w) M M^* ,$$

²² Note that the assignment of \pm is *a priori* not determined, but we chose it in that way. This is also reflected in (A.15) and (A.18).

i.e. for $|\operatorname{Im} w| \ll 1$ very small, $\operatorname{Im} M$ is an approximate right eigenvector of the stability operator $1 - M\mathcal{S}[\cdot]M^*$ corresponding to the *critical* eigenvalue (recall the discussion below (A.3)).

Appendix B. The stability operator and its spectral properties: proof of Lemma 3.3

In this appendix, based on several elementary properties of the MDE

$$-\frac{1}{M} = w - \hat{\Lambda} + \mathcal{S}[M], \quad w \in \mathbf{C} \setminus \mathbf{R}, \tag{B.1}$$

(recall (2.20)) and its unique solution M , which are briefly collected in Section 2 (see also Appendix B.1 of the arXiv:2301.03549 version of this article), we will discuss the associated two-body stability operator

$$\mathcal{B} \equiv \mathcal{B}(w_1, w_2)[\cdot] := 1 - M(w_1)\mathcal{S}[\cdot]M(w_2) \tag{B.2}$$

and its adjoint \mathcal{B}^* (see (A.14)). Moreover, we also explain the relation between the regularisation from Definition 3.1 and the stability operator. In fact, after proving and combining Lemma B.1 with Lemma B.3, we will complete the proof of Lemma 3.3.

Due to its connection to the regularisation, the two-body stability operator (B.2) and its adjoint (A.14) play a crucial role in the entire paper. These operators depend on two (a priori) different spectral parameters w_1, w_2 via the solutions $M_1 = M(w_1)$ and $M_2 = M(w_2)$ of the MDE (B.1). For these solutions, we have the following basic lemma.

Lemma B.1. *Let $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ be two spectral parameters and $M_1 = M(w_1), M_2 = M(w_2)$ the corresponding solutions to (B.1).*

(a) *Then we have the M -Ward identity,*

$$M_1 - M_2 = [(w_1 - w_2) + (\langle M_1 \rangle - \langle M_2 \rangle)] M_2 M_1. \tag{B.3}$$

In particular, M_1 and M_2 commute and it holds that

$$(1 - \langle MM^* \rangle) \langle \operatorname{Im} M \rangle = \operatorname{Im} w \langle MM^* \rangle. \tag{B.4}$$

(b) *Fix $\kappa > 0$ and let $\operatorname{Re} w_1, \operatorname{Re} w_2 \in \mathbf{B}_\kappa$. Then, for $\operatorname{Im} w_1 \operatorname{Im} w_2 > 0$, we have the perturbative estimate*

$$\|M(w_1) - M(w_2)\| = \mathcal{O}(|w_1 - w_2| \wedge 1).$$

Proof. Part (a) is an immediate consequence of the MDE (B.1) using the fact that

$$M = (\hat{\Lambda} - (w + m))^{-1}$$

is a resolvent of $\hat{\Lambda}$. The special case (B.4) follows from (B.3) with $w_1 = w$ and $w_2 = \bar{w}$, and taking a trace.

For part (b), we focus on the case of small imaginary parts for the spectral parameters (the complementary regime being trivial) and use that M is analytic away from the real axis and differentiate (B.1) w.r.t. w , such that we find

$$\partial_w M = \frac{1}{1 - \langle M^2 \rangle} M^2$$

by means of $\mathcal{S}[M^2] = \langle M^2 \rangle$ as follows from the explicit form of M in (2.17)–(2.18). Next, using (B.4), the denominator is lower bounded as

$$|1 - \langle M^2 \rangle| = |(1 - \langle MM^* \rangle) - 2i\langle M \text{Im } M \rangle| \geq 2|\langle (\text{Im } M)^2 \rangle| \geq 2\langle \text{Im } M \rangle^2, \tag{B.5}$$

which shows that $\|\partial_w M\| \lesssim 1$ in the bulk. Now the claim follows from the fundamental theorem of calculus together with the boundedness of M , see (2.22). \square

Armed with this information, we can now turn to the following lemma, collecting several basic spectral properties stability operator \mathcal{B} . Its proof will be given at the end of this section.

Lemma B.2. *Let $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ and M_1, M_2 be the respective solutions of (B.1).*

(a) *The associated two-body stability operator*

$$\mathcal{B} = 1 - M_1 \mathcal{S}[\cdot] M_2$$

has two non-trivial eigenvalues β_{\pm} (the other $(2N)^2 - 2$ are equal to one), given by

$$\beta_{\pm} = 1 \mp \langle M_1 E_{\pm} M_2 E_{\pm} \rangle. \tag{B.6}$$

The corresponding right- and left-eigenvectors

$$\mathcal{B}[R_{\pm}] = \beta_{\pm} R_{\pm}, \quad \mathcal{B}^*[L_{\pm}^*] = \bar{\beta}_{\pm} L_{\pm}^*,$$

take the explicit form

$$R_{\pm} = M_1 E_{\pm} M_2, \quad L_{\pm} = E_{\pm}, \tag{B.7}$$

up to a normalisation ensuring that $\langle L_{\pm}, R_{\pm} \rangle = 1$.

(b) *The eigenvalues (B.6) can be lower bounded as*

$$|\beta_{\pm}| \gtrsim (|\text{Re } w_1 \mp \text{Re } w_2| + |\text{Im } w_1| + |\text{Im } w_2|) \wedge 1. \tag{B.8}$$

In particular, the inverse stability operator \mathcal{B}^{-1} exists.

- (c) Fix $\kappa > 0$ and denote $\mathfrak{s} := -\text{sgn}(\text{Im } w_1 \text{Im } w_2)$. Then, for $\text{Re } w_1, \text{Re } w_2 \in \mathbf{B}_\kappa$, we have that $|\beta_{-\mathfrak{s}}| \gtrsim 1$.

By the last item, given $\mathfrak{s} := -\text{sgn}(\text{Im } w_1 \text{Im } w_2)$, we will always refer to

$$(\beta := 1 - \mathfrak{s}\langle M_1 E_\mathfrak{s} M_2 E_\mathfrak{s} \rangle, R := M_1 E_\mathfrak{s} M_2, L := E_\mathfrak{s}) \tag{B.9}$$

as the *critical eigentriple* (and accordingly β as the *critical eigenvalue* etc.), consisting of the eigenvalue and the corresponding right- and left-eigenvector. Moreover, the estimate (B.8) shows that, if we have (recall (3.6))

$$\mathbf{1}_\delta^\pm(w_1, w_2) := \phi_\delta(\text{Re } w_1 \mp \text{Re } w_2) \phi_\delta(\text{Im } w_1) \phi_\delta(\text{Im } w_2) = 0$$

for some $\delta > 0$, then the inverse stability operator \mathcal{B}^{-1} is bounded and none of the eigenvalues β_\pm is really critical. In the complementary regime, $\mathbf{1}_\delta^\pm(w_1, w_2) = 1$, and $\text{Re } w_1, \text{Re } w_2 \in \mathbf{B}_\kappa$, we shall now explain the interplay between the critical eigentriple (B.9) and the regularisation (3.7).

Lemma B.3. *Let $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ with $\text{Re } w_1, \text{Re } w_2 \in \mathbf{B}_\kappa$ for some fixed $\kappa > 0$ and denote the relative sign of imaginary parts by $\mathfrak{s} := -\text{sgn}(\text{Im } w_1 \text{Im } w_2)$. Moreover, let $M_1 = M(w_1), M_2 = M(w_2)$ be the respective solutions of (B.1) and $A \in \mathbf{C}^{2N \times 2N}$ a bounded deterministic matrix.*

- (a) If $\mathbf{1}_\delta^\mathfrak{s}(w_1, w_2) = 1$ for some $\delta > 0$ small enough, the critical left- and right-eigenvectors (B.9) are normalised as $\langle L, R \rangle \sim 1$. In particular, if $\mathbf{1}_\delta^\pm(w_1, w_2) = 1$, the respective denominator in the regularisation \mathring{A}^{w_1, w_2} (see (3.7)) is bounded away from zero.
- (b) The operator \mathcal{X}_{12} , acting as

$$\mathcal{X}_{12}[B] := ((\mathcal{B}_{12}^*)^{-1}[B^*])^* = (1 - \mathcal{S}[M_1 \cdot M_2])^{-1}[B], \quad B \in \mathbf{C}^{2N \times 2N},$$

where $\mathcal{B}_{12} := 1 - M_1 \mathcal{S}[\cdot] M_2$, is well defined and bounded on the \mathfrak{s} -regular component $\mathring{A}^\mathfrak{s}$ (w.r.t. the pair of spectral parameters (w_1, w_2)) of any bounded A . This means, for

$$\mathring{A}^\mathfrak{s} := A - \mathbf{1}_\delta^\mathfrak{s}(w_1, w_2) \frac{\langle M_1 A M_2 E_\mathfrak{s} \rangle}{\langle M_1 E_\mathfrak{s} M_2 E_\mathfrak{s} \rangle} E_\mathfrak{s} \tag{B.10}$$

it holds that $\|\mathcal{X}_{12}[\mathring{A}^\mathfrak{s}]\| \lesssim 1$.

In particular, combining Lemma B.1 (b) with Lemma B.3 (a), (2.19), and Lemma B.1 (a), we conclude the perturbative statements from Lemma 3.3.

Proof of Lemma B.3. For part (a), similarly to the proof of Lemma B.2 (c) given below, we focus on the extreme case $w_2 = \mathfrak{s}\bar{w}_1$, where the critical eigentriple is given by

$$(\beta = 1 - \mathfrak{s}\langle M(w_1)E_{\mathfrak{s}}M(\mathfrak{s}\bar{w}_1)E_{\mathfrak{s}} \rangle, R = M(w_1)E_{\mathfrak{s}}M(\mathfrak{s}\bar{w}_1), L = E_{\mathfrak{s}}). \tag{B.11}$$

Now by means of the chiral symmetry (2.19), we readily obtain

$$\langle L, R \rangle = \mathfrak{s}\langle M_1M_1^* \rangle = \mathfrak{s}\frac{\langle \text{Im } M_1 \rangle}{\text{Im } w_1 + \langle \text{Im } M_1 \rangle} \sim 1,$$

where we used (B.4) in the second step. This principal normalisation of order persists after small perturbation of w_2 around the extreme case, but as long as $\mathbf{1}_{\delta}^{\mathfrak{s}}(w_1, w_2) = 1$. Our claim for the denominators in the regularisation (3.7) follows immediately from the representation in (B.11).

For part (b), we first note that, by means of Lemma B.2, the statement is trivial for constellations of spectral parameters w_1, w_2 satisfying $\mathbf{1}_{\delta}^{\mathfrak{s}}(w_1, w_2) = 0$ and we can hence focus on the complementary extreme case $\mathbf{1}_{\delta}^{\mathfrak{s}}(w_1, w_2) = 1$. Then it follows from the explicit form

$$\mathcal{X}_{12}[B] = B + \sum_{\sigma} \sigma \frac{\langle M_1BM_2E_{\sigma} \rangle}{1 - \sigma \langle M_1E_{\sigma}M_2E_{\sigma} \rangle} E_{\sigma}$$

and Lemma B.2 that

$$\mathcal{X}_{12}[B] = \mathfrak{s}\frac{1}{\beta} \langle M_1BM_2E_{\mathfrak{s}} \rangle E_{\mathfrak{s}} + \mathcal{O}(1)[B], \tag{B.12}$$

where $\mathcal{O}(1)$ is a shorthand notation for a linear operator $\mathcal{E} : \mathbf{C}^{2N \times 2N} \rightarrow \mathbf{C}^{2N \times 2N}$ satisfying $\|\mathcal{E}[B]\| \lesssim \|B\|$. Now, plugging $\hat{A}^{\mathfrak{s}}$ from (B.10) into (B.12) yields the desired. \square

It remains to give the proof of Lemma B.2.

Proof of Lemma B.2. For (a), we first observe that, due to the simple structure of $\mathcal{S}[\cdot]$, indeed $(2N)^2 - 2$ of the $(2N)^2$ eigenvalues of \mathcal{B} are equal to one. The expressions (B.6) and (B.7) can be verified by direct computation, invoking Lemma B.1 in combination with the chiral symmetry (2.19).

For (b) with $w_1 \neq \pm w_2$, we first find that

$$\frac{1}{\beta_{\pm}} = \frac{1}{1 \mp \langle M_1E_{\pm}M_2E_{\pm} \rangle} = 1 + \frac{\langle M_1 \rangle \mp \langle M_2 \rangle}{w_1 \mp w_2} \tag{B.13}$$

as a consequence of Lemma B.1 (a) and the chiral symmetry. Now, using that $|\langle M \rangle| \leq \langle MM^* \rangle^{1/2} < 1$, which follows from $MM^* = \text{Im } M / (\text{Im } w + \langle \text{Im } M \rangle)$ (see Lemma B.1 (a)), we conclude that

$$|\beta_{\pm}| \gtrsim |\operatorname{Re} w_1 \mp \operatorname{Re} w_2| \wedge 1 \quad (\text{B.14})$$

by application of a triangle inequality in (B.13). Next, we estimate

$$\min \{|\beta_+|, |\beta_-|\} \geq |1 - \langle M_1 M_1^* \rangle^{1/2} \langle M_2 M_2^* \rangle^{1/2}| \gtrsim (|\operatorname{Im} w_1| + |\operatorname{Im} w_2|) \wedge 1, \quad (\text{B.15})$$

where in the first step we used $\langle MM^* \rangle < 1$ together with a Schwarz inequality, and (B.4) in the second step. Combining (B.14) and (B.15) yields the claim.

Finally, for (c), we consider the case of small imaginary parts for the spectral parameters (the complementary regime being trivial) and focus on the extreme case $w_1 = -\mathfrak{s}w_2$. Then, using (2.19) and (B.5), we obtain

$$|\beta_{-\mathfrak{s}}| = |1 - \langle M_1^2 \rangle| \geq 2\langle \operatorname{Im} M_1 \rangle^2 \gtrsim 1. \quad (\text{B.16})$$

This principal lower bound persists after small perturbations of w_2 , and the complementary regime can be dealt with by (B.8). \square

References

- [1] O.H. Ajanki, L. Erdős, T. Krüger, Quadratic vector equations on complex upper half-plane, *Am. Math. Soc.* 261 (1261) (2019).
- [2] O.H. Ajanki, L. Erdős, T. Krüger, Stability of the matrix Dyson equation and random matrices with correlations, *Probab. Theory Relat. Fields* 173 (2019) 293–373.
- [3] G. Akemann, R. Tribe, A. Tsareas, O. Zeitner, On the determinantal structure of conditional overlaps for the complex Ginibre ensemble, *Random Matrices: Theory Appl.* 09 (04) (2020) 2050015.
- [4] J. Alt, L. Erdős, T. Krüger, The Dyson equation with linear self-energy: spectral bands, edges and cusps, *Doc. Math.* 25 (2020) 1421–1539.
- [5] N. Anantharaman, E. Le Masson, Quantum ergodicity on large regular graphs, *Duke Math. J.* 164 (2015) 723–765.
- [6] Z.D. Bai, Circular law, *Ann. Probab.* 25 (1997) 494–529.
- [7] J. Banks, J. Garza-Vargas, A. Kulkarni, N. Srivastava, Overlaps, eigenvalue gaps, and pseudospectrum under real Ginibre and absolutely continuous perturbations, arXiv:2005.08930, 2020.
- [8] S. Belinschi, M.A. Nowak, R. Speicher, W. Tarnowski, Squared eigenvalue condition numbers and eigenvector correlations from the single ring theorem, *J. Phys. A, Math. Theor.* 50 (2007) 105204.
- [9] L. Benigni, Eigenvectors distribution and quantum unique ergodicity for deformed Wigner matrices, *Ann. Inst. Henri Poincaré Probab. Stat.* 56 (2020) 2822–2867.
- [10] L. Benigni, G. Cipolloni, Fluctuations of eigenvector overlaps and the Berry conjecture for Wigner matrices, arXiv:2212.10694, 2022.
- [11] L. Benigni, P. Lopatto, Fluctuations in local quantum unique ergodicity for generalized Wigner matrices, *Commun. Math. Phys.* 391 (2022) 401–454.
- [12] P. Biane, On the free convolution with a semi-circular distribution, *Indiana Univ. Math. J.* 46 (1997) 705–718.
- [13] P. Bourgade, G. Dubach, The distribution of overlaps between eigenvectors of Ginibre matrices, *Probab. Theory Relat. Fields* 177 (2020) 397–464.
- [14] P. Bourgade, H.-T. Yau, The eigenvector moment flow and local quantum unique ergodicity, *Commun. Math. Phys.* 350 (2017) 231–278.
- [15] P. Bourgade, H.-T. Yau, J. Yin, Local circular law for random matrices, *Probab. Theory Relat. Fields* 159 (2014) 545–595.
- [16] P. Bourgade, H.-T. Yau, J. Yin, The local circular law II: the edge case, *Probab. Theory Relat. Fields* 159 (2014) 619–660.
- [17] P. Bourgade, H.-T. Yau, J. Yin, Random band matrices in the delocalized phase I: quantum unique ergodicity and universality, *Commun. Pure Appl. Math.* 73 (2020) 1526–1596.

- [18] S. Brooks, E. Lindenstrauss, Joint quasimodes, positive entropy, and quantum unique ergodicity, *Invent. Math.* 198 (2014) 219–259.
- [19] J.T. Chalker, B. Mehlig, Eigenvector statistics in non-Hermitian random matrix ensembles, *Phys. Rev. Lett.* 81 (1998) 3367–3370.
- [20] J.T. Chalker, B. Mehlig, Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles, *J. Math. Phys.* 41 (2000) 3233–3256.
- [21] G. Cipolloni, L. Erdős, D. Schröder, Edge universality for non-Hermitian random matrices, *Probab. Theory Relat. Fields* 179 (2021) 1–28.
- [22] G. Cipolloni, L. Erdős, D. Schröder, Fluctuation around the circular law for random matrices with real entries, *Electron. J. Probab.* 26 (24) (2021) 1–61.
- [23] G. Cipolloni, L. Erdős, D. Schröder, Eigenstate thermalisation hypothesis for Wigner matrices, *Commun. Math. Phys.* 388 (2021) 1005–1048.
- [24] G. Cipolloni, L. Erdős, D. Schröder, Thermalisation for Wigner matrices, *J. Funct. Anal.* 282 (2022) 109394.
- [25] G. Cipolloni, L. Erdős, D. Schröder, Normal fluctuation in quantum ergodicity for Wigner matrices, *Ann. Probab.* 50 (2022) 984–1012.
- [26] G. Cipolloni, L. Erdős, D. Schröder, Optimal multi-resolvent local laws for Wigner matrices, *Electron. J. Probab.* 27 (2022) 1–38.
- [27] G. Cipolloni, L. Erdős, D. Schröder, Rank-uniform local law for Wigner matrices, *Forum Math. Sigma* 10 (2022) E96.
- [28] G. Cipolloni, L. Erdős, D. Schröder, Central limit theorem for linear eigenvalue statistics of non-Hermitian random matrices, *Commun. Pure Appl. Math.* 76 (2023) 946–1034.
- [29] G. Cipolloni, L. Erdős, D. Schröder, Functional central limit theorems for Wigner matrices, *Ann. Appl. Probab.* 33 (2023) 447–489.
- [30] Y. Colin de Verdière, Ergodicité et fonctions propres du laplacien, *Commun. Math. Phys.* 102 (1985) 497–502.
- [31] N. Cook, W. Hachem, J. Najim, D. Renfrew, Non-Hermitian random matrices with a variance profile (I): deterministic equivalents and limiting ESDs, *Electron. J. Probab.* 23 (110) (2018) 1–61.
- [32] L. D’Alessio, Y. Kafri, A. Polkovnikov, M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, *Adv. Phys.* 65 (2016) 239–362.
- [33] K.R. Davidson, D.A. Herrero, N. Salinas, Quasidiagonal operators, approximation, and C^* -algebras, *Indiana Univ. Math. J.* 38 (1989) 973–998.
- [34] J.M. Deutsch, Quantum statistical mechanics in a closed system, *Phys. Rev. A* 43 (1991) 2046–2049.
- [35] J.M. Deutsch, Eigenstate thermalization hypothesis, *Rep. Prog. Phys.* 81 (2018) 082001.
- [36] L. Erdős, The matrix Dyson equation and its applications for random matrices, in: Alexei Borodin, Ivan Corwin (Eds.), *IAS/Park City Math. Ser.* 26, Alice Guionnet (2019) 75–158.
- [37] L. Erdős, H.C. Ji, Wegner estimate and upper bound on the eigenvalue condition number of non-Hermitian random matrices, arXiv:2301.04981, 2023.
- [38] L. Erdős, A. Knowles, H.-T. Yau, J. Yin, The local semicircle law for a general class of random matrices, *Electron. J. Probab.* 18 (59) (2013) 1–58.
- [39] L. Erdős, T. Krüger, D. Schröder, Random matrices with slow correlation decay, *Forum Math. Sigma* 7 (2019) E8.
- [40] Y.V. Fyodorov, On statistics of bi-orthogonal eigenvectors in real and complex Ginibre ensembles: combining partial Schur decomposition with supersymmetry, *Commun. Math. Phys.* 363 (2018) 579–603.
- [41] V.L. Girko, The circular law, *Teor. Veroyatnosti Primenen* 29 (1984) 669–679.
- [42] J. Grela, P. Warchol, Full dysonian dynamics of the complex Ginibre ensemble, *J. Phys. A, Math. Theor.* 51 (2018) 425203.
- [43] Y. He, A. Knowles, Mesoscopic eigenvalue statistics of Wigner matrices, *Ann. Appl. Probab.* 27 (2017) 1510–1550.
- [44] J.W. Helton, R.R. Far, R. Speicher, Operator-valued semicircular elements: solving a quadratic matrix equation with positivity constraints, *Int. Math. Res. Not.* 2007 (9) (2007) rnm086.
- [45] V. Jain, A. Sah, M. Sawhney, On the real Davies’ conjecture, *Ann. Probab.* 49 (2021) 3011–3031.
- [46] A. Knowles, J. Yin, Eigenvector distribution of Wigner matrices, *Probab. Theory Relat. Fields* 155 (2013) 543–582.
- [47] B. Landon, P. Lopatto, P. Sosoe, Single eigenvalue fluctuations of general Wigner type matrices, arXiv:2105.01178, 2021.
- [48] E. Lindenstrauss, Invariant measures and arithmetic quantum unique ergodicity, *Ann. Math.* 163 (2006) 165–219.

- [49] J. Marcinek, H.-T. Yau, High dimensional normality of noisy eigenvectors, *Commun. Math. Phys.* 395 (2022) 1007–1096.
- [50] Z. Rudnick, P. Sarnak, The behaviour of eigenstates of arithmetic hyperbolic manifolds, *Commun. Math. Phys.* 161 (1994) 195–213.
- [51] A.I. Šnirel'man, Ergodic properties of eigenfunctions, *Usp. Mat. Nauk* 29 (1974) 181–182.
- [52] K. Soundararajan, Quantum unique ergodicity for $SL_2(\mathbf{Z}) \backslash \mathbf{H}$, *Ann. Math.* 172 (2010) 1529–1538.
- [53] M. Srednicki, Chaos and quantum thermalization, *Phys. Rev. E* 50 (1994) 888–901.
- [54] T. Tao, V. Vu, Random matrices: universal properties of eigenvectors, *Random Matrices: Theory Appl.* 1 (2012) 1150001.
- [55] T. Tao, V. Vu, Random matrices: universality of local spectral statistics of non-Hermitian matrices, *Ann. Probab.* 43 (2015) 782–874.
- [56] T. Tao, V. Vu, M. Krishnapur, Random matrices: universality of ESDs and the circular law, *Ann. Probab.* 38 (2010) 2023–2065.
- [57] J. Yin, The local circular law III: general case, *Probab. Theory Relat. Fields* 160 (2014) 679–732.
- [58] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, *Duke Math. J.* 55 (1987) 919–941.