



Rational Singularities for Moment Maps of Totally Negative Quivers

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Abstract

We prove that the zero-fiber of the moment map of a totally negative quiver has rational singularities. Our proof consists in generalizing dimension bounds on jet spaces of this fiber, which were introduced by Budur. We also transfer the rational singularities property to other moduli spaces of objects in 2-Calabi-Yau categories, based on recent work of Davison. This has interesting arithmetic applications on quiver moment maps and moduli spaces of objects in 2-Calabi-Yau categories. First, we generalize results of Wyss on the asymptotic behaviour of counts of jets of quiver moment maps over finite fields. Moreover, we interpret the limit of counts of jets on a given moduli space as its p -adic volume under a canonical measure analogous to the measure built by Carocci, Orecchia and Wyss on certain moduli spaces of coherent sheaves.

Keywords Quiver moment maps · Jet schemes · Rational singularities · p -adic integration

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1 Introduction

Let Q be a quiver and $\mathbf{d}$ a dimension vector. The moment map $\mu_{Q,\mathbf{d}}$ is a central object in the geometric representation theory of quivers. It is a building block of Lusztig's nilpotent variety [50, 56, 57] and Nakajima's quiver varieties [61, 62], which were used in geometric realizations of Kac-Moody Lie algebras and associated quantum groups. A systematic study of geometric properties of $\mu_{Q,\mathbf{d}}$ and of quiver varieties was then undertaken by Crawley-Boevey in the early 2000s [22, 24, 25]. Since then, many other aspects of quiver moment maps have been investigated, such as the cohomology and singularities of quiver varieties [6, 41–43] or counts of $\mathbb{F}_q$ -points of $\mu_{Q,\mathbf{d}}^{-1}(0)$ [10, 30, 31, 58]. The latter have proved a useful technique to study cohomological Hall

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algebras and other counts of quiver representations such as Kac's polynomials (see [65] for a survey).

More recently, Wyss observed that the count of jets of $\mu_{Q,\mathbf{d}}^{-1}(0)$ over finite fields has an interesting asymptotic behaviour [70]. Consider the sequence $q^{-n \dim \mu_{Q,\mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q,\mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$. When $\mathbf{d} = \mathbf{1}$, Wyss showed that this sequence converges when n goes to infinity if, and only if, the graph underlying Q is 2-connected. Moreover, by computing the local Igusa zeta function of $\mu_{Q,\mathbf{1}}^{-1}(0)$, Wyss found an explicit rational fraction $W_Q \in \mathbb{Q}(T)$ in terms of the graphical hyperplane arrangement associated to Q , such that:

$$W_Q(q) = \lim_{n \rightarrow +\infty} q^{-n \dim \mu_{Q,\mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q,\mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n)).$$

As discussed in [70], W_Q enjoys nice numerical properties: it has a palindromic numerator W'_Q ; W'_Q conjecturally has non-negative coefficients and is related to the asymptotic analog of Kac's polynomials over $\mathbb{F}_q[t]/(t^n)$. This is reminiscent of similar results on Kac's polynomials (i.e. when $n = 1$) [26, 28, 58] and raises the following questions: can we generalize these asymptotic counts to higher dimension vectors ($\mathbf{d} > \mathbf{1}$)? and can we find a geometric interpretation of these counts?

Convergence Results for Counts of Jets We can reformulate the first question as follows: for which pairs $(Q, \mathbf{d})$ does this sequence converge, when n goes to infinity? By work of Mustață, Aizenbud-Avni and Glazer [1, 38, 60], if $\mu_{Q,\mathbf{d}}^{-1}(0)$ is a local complete intersection, the sequence $q^{-n \dim \mu_{Q,\mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q,\mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ is bounded precisely when $\mu_{Q,\mathbf{d}}^{-1}(0)$ has rational singularities. Using the local Igusa zeta function of $\mu_{Q,\mathbf{d}}^{-1}(0)$, we further show that the counts *converge* when $\mu_{Q,\mathbf{d}}^{-1}(0)$ has rational singularities.

Rational singularities of $\mu_{Q,\mathbf{d}}$ were brought into focus by Aizenbud-Avni [2] and Budur [13], when $Q = S_g$ is the g -loop quiver, in connection with representation growth of arithmetic groups. They established that $\mu_{S_g,\mathbf{d}}^{-1}(0)$ has rational singularities for g large enough. Budur's proof (which was later generalized to more general moment maps by Herbig, Schwarz and Seaton [44]) relies on a stratification of $\mu_{Q,\mathbf{d}}^{-1}(0)$. Using étale slices, the problem can be reduced to the study of the deepest stratum of $\mu_{Q,\mathbf{d}}^{-1}(0)$ for various auxiliary quivers. Budur introduced an explicit bound on the dimension of jet spaces over the deepest stratum, by applying earlier results of Crawley-Boevey [25]. Using this bound, one proves that $\mu_{Q,\mathbf{d}}^{-1}(0)$ has rational singularities thanks to a criterion by Mustață [60].

We identify a large class of pairs $(Q, \mathbf{d})$ for which the above dimension bounds hold in most cases, and which is closed under taking auxiliary quivers. This is the class of pairs $(Q, \mathbf{d})$ where Q is totally negative i.e. the graph underlying Q is complete with at least two loops at each vertex. Actually, we also require a mild condition on the dimension vector $\mathbf{d}$, to ensure that $\mu_{Q,\mathbf{d}}^{-1}(0)$ is a complete intersection. In that case, we say that the pair $(Q, \mathbf{d})$ has property (P) (see Definition 2.2.1). Our main result is the following:

Theorem 1.0.1 *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ such that $(Q, \mathbf{d})$ has property (P). Then $\mu_{Q, \mathbf{d}}^{-1}(0)$ has rational singularities.*

As mentioned above, the dimension bounds introduced by Budur can be established for most, but not all pairs $(Q, \mathbf{d})$. Actually the bound fails for some totally negative quivers. This failure occurs for precisely one pair $(Q, \mathbf{d})$ among those considered in [13], due to a computational gap in the proof (see Remark 3.0.4). For general pairs $(Q, \mathbf{d})$ with property (P), we show that the bound only fails when $\mathbf{d} = \underline{1}$, in which case we can exploit Wyss' results on counts of jets to prove that $\mu_{Q, \mathbf{d}}^{-1}(0)$ has rational singularities regardless. As a corollary, we obtain a partial answer to our initial question:

Corollary 1.0.2 *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ such that $(Q, \mathbf{d})$ has property (P). Then the sequence of rational numbers*

$$q^{-n \dim \mu_{Q, \mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q, \mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n)), \quad n \geq 1$$

converges when n goes to infinity.

To the best of our knowledge, classifying pairs $(Q, \mathbf{d})$ such that $\mu_{Q, \mathbf{d}}^{-1}(0)$ has rational singularities remains an open problem.

Asymptotic Counts of Jets and p -adic Integrals We also interpret the limit of our counts of jets as a p -adic volume. We take a broader perspective and consider moduli stacks of objects in 2-Calabi-Yau categories. These moduli stacks are locally modelled on $[\mu_{Q, \mathbf{d}}^{-1}(0)/\mathrm{GL}(\mathbf{d})]$, by work of Davison [29]. This makes our total negativity assumption on quivers natural from a moduli-theoretic point of view: it translates to a homological assumption on the category we consider, which we call total negativity as well (see also [32]). We also require that the locus of simple objects in the associated moduli space be dense, to ensure that the local models $\mu_{Q, \mathbf{d}}^{-1}(0)$ are built from quivers satisfying property (P).

As before, for counts of jets on moduli stacks to converge, we need to analyze singularities of those stacks (or their atlases, see Remark 4.2.7). Our main result implies rational singularities statements for moduli stacks of totally negative 2-Calabi-Yau categories. The moduli stacks we consider are quotient stacks of the form $[X/G]$, where X is a scheme acted on by a linear algebraic group G . The associated moduli spaces are good categorical quotients $X//G$ in the sense of Geometric Invariant Theory [34, Ch. 6.]. It was shown for several moduli stacks of this form that $X//G$ has rational singularities (more specifically symplectic singularities, as introduced in [5]). In fewer cases, it was also shown that X has rational singularities, which implies by a theorem of Boutot (see [9, Corollaire]) that $X//G$ also has rational singularities. Examples of such moduli spaces include quiver varieties [6, 13], character varieties [13, 15], multiplicative quiver varieties [49, 64] and moduli of sheaves on K3 surfaces [4, 14, 48]. These moduli spaces all parametrize objects of a 2-Calabi-Yau category, in a precise (differential-graded) sense, which was formalized by Davison in [29]. Davison shows, using a formality argument, that these moduli spaces are étale-locally modelled

on moment maps of quivers. Our theorem goes as follows (see Theorem 4.1.2 for a more precise statement):

Theorem 1.0.3 *Let $[X/G]$ be a quotient stack parametrizing objects in a totally negative subcategory of a 2-Calabi-Yau category. Suppose that $X//G$ contains a dense open subset parametrizing simple objects. Then X is locally complete intersection and has rational singularities.*

Let us give some examples. For representation varieties of fundamental groups of compact Riemann surfaces, a rational singularities result was already proved in [13], as well as for their De Rham and Dolbeault analogs. We discuss in details rational singularities for moduli of sheaves on K3 surfaces, as mentioned in the introduction of [14]. Finally, we prove the following result for representations spaces of multiplicative preprojective algebras $R(\Lambda^q(Q), \mathbf{d})$, thereby dealing with the last example in Davison's list (see [29, §1.1.1.]):

Corollary 1.0.4 *Let Q be a quiver, $q \in (\mathbb{C}^\times)^{Q_0}$ and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ such that $(Q, \mathbf{d})$ has property (P). Then $R(\Lambda^q(Q), \mathbf{d})$ has rational singularities.*

With these results at hand, we can extend our results on counts of jets to these moduli stacks. If $[X/G]$ is a quotient stack parametrizing objects in a totally negative 2-Calabi-Yau category, we show that the sequence $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ also converges when n goes to infinity. We interpret the limit as the p -adic volume of an analytic manifold $X^{\natural}$ associated to X , following recent work of Carocci, Orecchia and Wyss [16] (see Theorem 4.2.6 for a more precise statement). The proof relies on a local description of X as a fiber of a flat map φ , whose fibers all have rational singularities, and a Fubini theorem by Aizenbud and Avni (see [2]), applied to φ .

Theorem 1.0.5 *Let $[X/G]$ be a quotient stack parametrizing objects in a totally negative 2-Calabi-Yau category. Suppose that $X//G$ contains a dense open subset parametrizing simple objects.*

Then for p large enough, the canonical measure μ_{can} on $X^{\natural}$ - introduced in [16] - is well-defined. Moreover, the sequence $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/t^n)$, $n \geq 1$ converges and its limit is given by:

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathbb{F}_q[t]/(t^n))}{q^{n \dim X_{\mathbb{Q}}}} = \mu_{\text{can}}(X^{\natural}).$$

In [16], the authors recover BPS invariants of some local surfaces as p -adic integrals on a good moduli space of coherent sheaves. This moduli space is analogous to $X//G$ in our notation, instead of X . As Kac's polynomials can be interpreted as BPS invariants of a certain 3-Calabi-Yau category [58], we expect a connection between the counts of jets studied in this paper and BPS invariants of quivers.

In Sections 2.1 and 2.2, we recall well-known facts on quiver representations and the geometry of moment maps. In Section 2.3, we introduce l.c.i. singularities, rational singularities and the criterion by Mustață that we use to prove our main result. In Section 2.4, we discuss the stratification of $\mu_{Q, \mathbf{d}}^{-1}(0)$ mentioned above and how it

interacts with rational singularities. In Section 2.5, we briefly introduce the other moduli spaces that we deal with in this paper, following [29]. In Section 2.6, we discuss arithmetic consequences of rational singularities, which initially motivated our work in the context of quiver representations.

Section 3 is the heart of the paper, where we prove Theorem 1.0.1. Finally, we apply our main result in Section 4 to prove Theorem 1.0.3, Corollary 1.0.2, and Theorem 1.0.5.

2 Preliminaries

2.1 Quiver Representations and their Moduli

We collect here elementary notions on quiver representations and their moduli (see for instance [63]). A quiver is the datum $Q = (Q_0, Q_1, s, t)$ of a set of vertices Q_0 and a set of arrows Q_1 , along with source and target maps $s, t : Q_1 \rightarrow Q_0$. Note that a quiver may have loops and parallel arrows connecting the same vertices.

Let us fix $\mathbb{K}$ a base field. A representation V of Q is the datum of $\mathbb{K}$ -vector spaces $(V_i)_{i \in Q_0}$ and $\mathbb{K}$ -linear maps $(V_a : V_{s(a)} \rightarrow V_{t(a)})_{a \in Q_1}$. A morphism between two representations V and W is the datum of $\mathbb{K}$ -linear maps $(\varphi_i : V_i \rightarrow W_i)_{i \in Q_0}$ such that, for all arrows $a \in Q_1$, $W_a \circ \varphi_{s(a)} = \varphi_{t(a)} \circ V_a$. We will only consider finite-dimensional representations of Q and define the dimension vector of V as the tuple $\dim(V) = (\dim V_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$. Finite-dimensional representations of Q over $\mathbb{K}$ form an abelian category $\text{Rep}_{\mathbb{K}}(Q)$.

Given a dimension $\mathbf{d} \in \mathbb{N}^{Q_0}$, we can fix bases of the vector spaces V_i , $i \in Q_0$ and obtain from the linear maps V_a , $a \in Q_1$ a tuple of matrices:

$$(x_a)_{a \in Q_1} \in R(Q, \mathbf{d}) := \prod_{a \in Q_1} \text{Mat}(d_{t(a)} \times d_{s(a)}, \mathbb{K}).$$

Conversely, a point $x \in R(Q, \mathbf{d})$ yields a $\mathbf{d}$ -dimensional representation of Q . Two points $x, y \in R(Q, \mathbf{d})$ correspond to isomorphic representations if, and only if, they lie in the same orbit of the following action of $\text{GL}(\mathbf{d}) = \prod_i \text{GL}(d_i, \mathbb{K})$:

$$\begin{aligned} \text{GL}(\mathbf{d}) \times R(Q, \mathbf{d}) &\rightarrow R(Q, \mathbf{d}) \\ (g_i)_{i \in Q_0} ; (x_a)_{a \in Q_1} &\mapsto (g_{t(a)} x_a g_{s(a)}^{-1})_{a \in Q_1}. \end{aligned}$$

The above $\text{GL}(\mathbf{d})$ -variety is the starting point of quiver moduli. Assume for simplicity that $\mathbb{K}$ is algebraically closed, of characteristic zero. Moduli spaces of quiver representations were constructed by King [53] using techniques of Geometric Invariant Theory. The GIT quotient $R(Q, \mathbf{d}) // \text{GL}(\mathbf{d})$ parametrizes closed orbits of $R(Q, \mathbf{d})$ under the action of $\text{GL}(\mathbf{d})$. King further shows that closed orbits correspond precisely to semisimple representations of Q . Recall that a representation V is called simple if it contains no non-zero, proper subrepresentation; more generally, V is called semisimple if it decomposes into a direct sum of simple representations.

We introduce a few more notions which we will use below to define the class of pairs $(Q, \mathbf{d})$ we focus on in this article.

Definition 2.1.1 The Euler form of Q is the following bilinear form:

$$\begin{aligned} \langle \bullet, \bullet \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} &\rightarrow \mathbb{Z} \\ (\mathbf{d}, \mathbf{e}) &\mapsto \sum_{i \in Q_0} d_i e_i - \sum_{a \in Q_1} d_{s(a)} e_{t(a)}. \end{aligned}$$

The symmetrized Euler form is defined by: $(\mathbf{d}, \mathbf{e}) = \langle \mathbf{d}, \mathbf{e} \rangle + \langle \mathbf{e}, \mathbf{d} \rangle$.

Definition 2.1.2 A quiver is called totally negative if $(\mathbf{d}, \mathbf{e}) < 0$ for all $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0} \setminus \{0\}$.

One can check that Q is totally negative if, and only if, (i) Q has at least two loops at each vertex and (ii) any pair of vertices of Q is joined by at least one arrow. We will use the following definition in Section 2.2 to describe the geometry of moment maps of totally negative quivers. We denote by $(\epsilon_i)_{i \in Q_0}$ the canonical basis of $\mathbb{Z}^{Q_0}$.

Definition 2.1.3 The fundamental domain of Q is the following set of dimension vectors:

$$F_Q := \left\{ \mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\} \mid \begin{array}{l} \forall i \in Q_0, (\mathbf{d}, \epsilon_i) \leq 0 \\ \text{supp}(\mathbf{d}) \text{ is connected} \end{array} \right\},$$

where $\text{supp}(\mathbf{d})$ is the full subquiver of Q with set of vertices $\{i \in Q_0 \mid d_i \neq 0\}$.

Note that, for a totally negative quiver, $F_Q = \mathbb{N}^{Q_0} \setminus \{0\}$.

2.2 Geometry of Quiver Moment Maps

In this section, we introduce moment maps associated to quivers and recall some of their geometric properties, assuming $\mathbb{K}$ is algebraically closed of characteristic zero (see also [24]). Given a quiver Q , we define its double quiver $\overline{Q}$ by adjoining one arrow $a^* : t(a) \rightarrow s(a)$ to Q for each $a \in Q_1$. Thus $\overline{Q}_1 = Q_1 \sqcup Q_1^*$ and we call $Q^* = (Q_0, Q_1^*, s, t)$ the opposite quiver of Q .

The moment map of Q is the following morphism of algebraic varieties:

$$\begin{aligned} \mu_{Q, \mathbf{d}} : R(Q, \mathbf{d}) \times R(Q^*, \mathbf{d}) &\rightarrow \mathfrak{gl}(\mathbf{d}) \\ (x_a)_{a \in Q_1} ; (y_a)_{a \in Q_1} &\mapsto \left(\sum_{\substack{a \in Q_1 \\ t(a)=i}} x_a y_a - \sum_{\substack{a \in Q_1 \\ s(a)=i}} y_a x_a \right)_{i \in Q_0}. \end{aligned}$$

We will be interested in the fiber of $\mu_{Q, \mathbf{d}}$ over $0 \in \mathfrak{gl}(\mathbf{d})$. The subscheme $\mu_{Q, \mathbf{d}}^{-1}(0)$ is invariant under the action of $\text{GL}(\mathbf{d})$ and its orbits correspond to isomorphism classes of modules over the preprojective algebra Π_Q . We refer to [23] for a definition of Π_Q , as we will only work with moduli of Π_Q -modules in this paper, as described above.

Just as for $R(Q, \mathbf{d})$, one can form a GIT quotient $\mu_{Q, \mathbf{d}}^{-1}(0) // \text{GL}(\mathbf{d})$, whose points parametrize semisimple representations of $\overline{Q}$ whose orbit lie in $\mu_{Q, \mathbf{d}}^{-1}(0)$, i.e. semisimple Π_Q -modules. We will be interested in pairs $(Q, \mathbf{d})$ such that there exists a simple Π_Q -module of dimension $\mathbf{d}$. These were characterized combinatorially by Crawley-Boevey in [24]. Let us now introduce the class of pairs we focus on in this article.

Definition 2.2.1 Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ a dimension vector. The pair $(Q, \mathbf{d})$ has property (P) if:

1. the quiver Q is totally negative;
2. if $\text{supp}(\mathbf{d})$ has two vertices joined by only one edge, then $\mathbf{d}_{|\text{supp}(\mathbf{d})} \neq (1, 1)$.

We now show the existence of a simple Π_Q -module for all these pairs:

Proposition 2.2.2 Suppose that $(Q, \mathbf{d})$ has property (P). Then there exists a simple Π_Q -module of dimension $\mathbf{d}$.

Proof Since Q is totally negative, $\mathbf{d}$ lies in the fundamental region of Q . By contradiction, assume that there exist no simple Π_Q -modules of dimension $\mathbf{d}$. By [24, Thm. 8.1.], one of the following holds:

1. the subquiver $\text{supp}(\mathbf{d})$ is an extended Dynkin quiver with minimal imaginary root δ and $\mathbf{d} = m\delta$ for some $m \geq 2$;
2. the subquiver $\text{supp}(\mathbf{d})$ splits as a disjoint union $(Q_0)' \sqcup (Q_0)''$ such that there is a unique arrow joining $(Q_0)'$ and $(Q_0)''$, say at vertices i' and i'' , and $d_{i'} = d_{i''} = 1$;
3. the subquiver $\text{supp}(\mathbf{d})$ splits as a disjoint union $(Q_0)' \sqcup (Q_0)''$ such that there is a unique arrow joining $(Q_0)'$ and $(Q_0)''$, say at vertices i' and i'' , $d_{i'} = 1$, the restriction of $\text{supp}(\mathbf{d})$ to $(Q_0)''$ is an extended Dynkin quiver with minimal imaginary root δ and $\mathbf{d}_{|(Q_0)''} = m\delta$ for some $m \geq 2$.

Cases 1. and 3. cannot happen, as Q has at least two vertices at each vertex. Since the graph underlying Q is complete, case 2. can only happen if $\text{supp}(\mathbf{d})$ has two vertices joined by a single edge and $\mathbf{d}_{|\text{supp}(\mathbf{d})} = (1, 1)$. This is ruled out by definition of property (P). Thus we have reached a contradiction and there exists a simple Π_Q -module of dimension $\mathbf{d}$. $\square$

From Crawley-Boevey's study of the geometric properties of $\mu_{Q, \mathbf{d}}^{-1}(0)$ [24, Thm. 1.2. - Cor. 1.4.], we deduce:

Corollary 2.2.3 If $(Q, \mathbf{d})$ has property (P), then $\mu_{Q, \mathbf{d}}^{-1}(0)$ is a reduced, irreducible complete intersection of dimension $\mathbf{d} \cdot \mathbf{d} - 1 + 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle)$. Moreover, $\mu_{Q, \mathbf{d}}^{-1}(0) // \text{GL}(\mathbf{d})$ has dimension $2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle)$.

We prove our main result using dimension estimates on constructible subsets of $\mu_{Q, \mathbf{d}}^{-1}(0)$. These rely on older estimates proved by Crawley-Boevey in [25, §6]. for which we need some additional notations:

Definition 2.2.4 [24, §1] Let M be a semisimple Π_Q -module (Q arbitrary), which decomposes into non-isomorphic direct summands as follows:

$$M \simeq \bigoplus_{i=1}^r M_i^{\oplus e_i}.$$

The semisimple type of M is the multiset $(\dim(M_i), e_i ; 1 \leq i \leq r)$.

We call τ simple (resp. strictly semisimple) if it corresponds to a simple (resp. semisimple, but not simple) representation. Similarly, we call $x \in \mu_{Q,\mathbf{d}}^{-1}(0)$ simple (resp. strictly semisimple) if the corresponding Π_Q -module is. Given a dimension vector $\mathbf{d}$, we define $\tau_{\min,\mathbf{d}} := (\epsilon_i, d_i ; i \in \text{supp}(\mathbf{d}))$. It is the semisimple type of the $\mathbf{d}$ -dimensional Π_Q -module corresponding to $0 \in \mu_{Q,\mathbf{d}}^{-1}(0)$.

Definition 2.2.5 [25, §6] Let N_i , $1 \leq i \leq r$ be a collection of non-isomorphic simple Π_Q -modules. A Π_Q -module M has top-type $(j_s, m_s, 1 \leq s \leq h)$ if it admits a filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_h = M$, such that $M_s/M_{s-1} \simeq N_{j_s}^{\oplus m_s}$ and $\text{hom}(M_s, N_{j_s}) = m_s > 0$.

Note that, if M is semisimple with top-type $(j_s, m_s, 1 \leq s \leq h)$ w.r.t. N_i , $1 \leq i \leq r$ and all N_i appear as subquotients of M^1 , then the semisimple type of M is:

$$\tau = \left(\dim(N_i), \sum_{j_s=i} m_s ; 1 \leq i \leq r \right).$$

We can now state Crawley-Boevey's dimension bound:

Proposition 2.2.6 [25, Lem. 6.2.] Let N_i , $1 \leq i \leq r$ be a collection of non-isomorphic simple Π_Q -modules and $(j_s, m_s, 1 \leq s \leq h)$ a top-type. Define:

$$z_s = \begin{cases} 0 & \text{if } \langle \dim(N_{j_s}), \dim(N_{j_s}) \rangle = 1 \text{ or if there is no } t < s \text{ such that } j_t = j_s, \\ m_t & \text{otherwise, for the largest } t < s \text{ such that } j_t = j_s. \end{cases}$$

Then the subset of $\mu_{Q,\mathbf{d}}^{-1}(0)$ corresponding to Π_Q -modules of top-type $(j_s, m_s, 1 \leq s \leq h)$ is constructible, of dimension at most:

$$\mathbf{d} \cdot \mathbf{d} - 1 + (1 - \langle \mathbf{d}, \mathbf{d} \rangle) + \sum_{s=1}^h m_s z_s - \sum_{s=1}^h m_s^2 \cdot (1 - \langle \dim(N_{j_s}), \dim(N_{j_s}) \rangle).$$

2.3 Local Complete Intersection and Rational Singularities

In this section, we recall results on local complete intersection and rational singularities, assuming $\mathbb{K}$ is algebraically closed. In particular, we state a criterion of Mustața [60] for a local complete intersection to have rational singularities, in terms of jet spaces.

We begin with some definitions. Let $X/\mathbb{K}$ be a scheme of finite type.

Definition 2.3.1 The scheme X is locally complete intersection (or l.c.i. for short) if it can be covered by affine open subsets which are complete intersections in some affine space. In that case, we say that X has l.c.i. singularities.

¹ This can always be assumed, by possibly forgetting some N_i .

Having l.c.i. singularities is a property of local rings: X is locally complete intersection if, and only if, for all $x \in X$, $\mathcal{O}_{X,x}$ is a complete intersection ring. Moreover, this is a local property for the smooth topology:

Proposition 2.3.2 [67, Tag 069P] *Let $f : X \rightarrow Y$ be a smooth morphism between schemes of finite type. If Y has l.c.i. singularities, then X has l.c.i. singularities. The converse holds if f is surjective.*

We now turn to rational singularities. We further assume that $\mathbb{K}$ is of characteristic zero.

Definition 2.3.3 The scheme X has rational singularities if for some (hence for all) resolution of singularities $p : \tilde{X} \rightarrow X$, the natural morphism $\mathcal{O}_X \rightarrow \mathbf{R}p_*\mathcal{O}_{\tilde{X}}$ is an isomorphism. In other words, the canonical morphism $\mathcal{O}_X \rightarrow p_*\mathcal{O}_{\tilde{X}}$ is an isomorphism and $\mathbf{R}^i p_*\mathcal{O}_{\tilde{X}} = 0$ for all $i > 0$. A point $x \in X$ is called a rational singularity if there exists a Zariski-open neighborhood $U \ni x$ which has rational singularities.

We will also use a relative notion of rational singularities, introduced by Aizenbud and Avni [2], in our proof of Theorem 1.0.5.

Definition 2.3.4 [2, Def. II.] Let $f : X \rightarrow Y$ be a morphism between smooth, irreducible varieties over $\mathbb{K}$. The morphism f is called FRS (flat with rational singularities) if it is flat and for every $y \in Y(\mathbb{K})$, the fibre $X \times_Y y$ has rational singularities.

If X has rational singularities, then it is normal (hence reduced) and Cohen-Macaulay - see [36, II.1.]. While the above definition might seem quite abstract, one can show directly that having rational singularities is a local property with respect to smooth morphisms. This is surely common knowledge for the experts, but we include a proof for the reader's convenience, as we could not find one in the literature.

Lemma 2.3.5 *Let $f : X \rightarrow Y$ be a smooth morphism between schemes of finite type. If Y has rational singularities, then X has rational singularities. The converse holds if f is surjective.*

Proof Consider $p_Y : \tilde{Y} \rightarrow Y$ a resolution of Y and $\tilde{X} := X \times_Y \tilde{Y}$. Then, since f is smooth, the canonical morphism $p_X : \tilde{X} \rightarrow X$ is also a resolution of singularities, so that we get the following cartesian diagram, where horizontal maps are resolutions of singularities and the vertical maps are smooth (hence flat):

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p_X} & X \\ \downarrow \tilde{f} & & \downarrow f \\ \tilde{Y} & \xrightarrow{p_Y} & Y \end{array}$$

Suppose that Y has rational singularities. Then flat base change yields:

$$f^*\mathbf{R}(p_Y)_*\mathcal{O}_{\tilde{Y}} \simeq \mathbf{R}(p_X)_*\tilde{f}^*\mathcal{O}_{\tilde{Y}} \simeq \mathbf{R}(p_X)_*\mathcal{O}_{\tilde{X}}.$$

Since by assumption $\mathbf{R}(p_Y)_*\mathcal{O}_{\tilde{Y}} \simeq \mathcal{O}_Y$, we obtain $\mathcal{O}_X \simeq f^*\mathcal{O}_Y \simeq \mathbf{R}(p_X)_*\mathcal{O}_{\tilde{X}}$, hence X has rational singularities.

Conversely, suppose that X has rational singularities and f is surjective. Then flat base change and the rational singularities assumption for X yield:

$$f^*\mathbf{R}(p_Y)_*\mathcal{O}_{\tilde{Y}} \simeq \mathbf{R}(p_X)_*f^*\mathcal{O}_{\tilde{Y}} \simeq \mathbf{R}(p_X)_*\mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X \simeq f^*\mathcal{O}_Y.$$

Since f^* is exact, we obtain taking cohomology sheaves:

$$f^*\mathbf{R}^i(p_Y)_*\mathcal{O}_{\tilde{Y}} \simeq \begin{cases} f^*\mathcal{O}_Y, & \text{if } i = 0; \\ 0, & \text{else.} \end{cases}$$

By fpqc descent (f is surjective), we finally obtain:

$$\mathbf{R}^i(p_Y)_*\mathcal{O}_{\tilde{Y}} \simeq \begin{cases} \mathcal{O}_Y, & \text{if } i = 0; \\ 0, & \text{else.} \end{cases}$$

Thus Y has rational singularities, which finishes the proof. $\square$

In particular, the above lemma shows that having rational singularities is an étale-local property. We will use this fact many times in Section 2.4 in order to transfer rational singularities from $\mu_{Q,d}^{-1}(0)$ to its étale slices.

Finally, we recall the definition of jet spaces. These spaces are strongly related to singularities of algebraic varieties, via motivic integration (and as we will see in Section 2.6, p-adic integration). See for example [35, 59, 60].

Definition 2.3.6 [17, §3.2.] Let $\mathbb{K}$ be a field (of any characteristic) and $m \geq 0$. Let X be a finite-type $\mathbb{K}$ -scheme. The m -th jet scheme of X is the $\mathbb{K}$ -scheme X_m representing the following functor of points:

$$\begin{aligned} \mathbb{K} - \text{CAlg} &\rightarrow \text{Sets} \\ R &\mapsto X(R[t]/(t^{m+1})). \end{aligned}$$

We now state a criterion by Mustață, which characterizes rational singularities for locally complete intersection varieties. This criterion is key to Budur's proof of rational singularities for moment maps of g-loop quivers.

Proposition 2.3.7 [60, Thm. 0.1. & Prop. 1.4.] *Let X be a locally complete intersection variety. Let X_{sg} be its singular locus and $\pi_m : X_m \rightarrow X$ its m -th jet space. Then X has rational singularities if, and only if, for all $m \geq 1$, $\dim \pi_m^{-1}(X_{\text{sg}}) < (m+1) \cdot \dim(X)$.*

2.4 Etale Slices

In this section, we describe étale slices of $\mu_{Q,d}^{-1}(0)$ in terms of semisimple type, following Crawley-Boevey [25] and Budur [13]. We also explain how one can transfer the rational singularities property from the étale slices to $\mu_{Q,d}^{-1}(0)$. This technique is

at the heart of the inductive reasoning in [13], which we extend to a larger class of quivers in this paper.

Let $x \in \mu_{Q,\mathbf{d}}^{-1}(0)$ be a semisimple point of type $\tau = (\mathbf{d}_i, e_i; 1 \leq i \leq r)$. Then its stabilizer satisfies $\mathrm{GL}(\mathbf{d})_x \simeq \mathrm{GL}(\mathbf{e})$. Luna's étale slice theorem [55] then yields a $\mathrm{GL}(\mathbf{e})$ -invariant, locally closed subvariety $S \subseteq \mu_{Q,\mathbf{d}}^{-1}(0)$ such that the commutative square below is cartesian, with étale horizontal maps (the upper horizontal map is given by the action):

$$\begin{array}{ccc} (S \times^{\mathrm{GL}(\mathbf{d})_x} \mathrm{GL}(\mathbf{d}), [x, \mathrm{Id}]) & \longrightarrow & (\mu_{Q,\mathbf{d}}^{-1}(0), x) \\ \downarrow & & \downarrow \\ (S // \mathrm{GL}(\mathbf{d})_x, x) & \longrightarrow & (\mu_{Q,\mathbf{d}}^{-1}(0) // \mathrm{GL}(\mathbf{d}), x). \end{array}$$

Moreover, by work of Crawley-Boevey [25] and Budur [13], the étale slice has an étale-local description in terms of a certain pair $(Q', \mathbf{e})$, which satisfies:

- the set of vertices is $(Q')_0 = \{1, \dots, r\}$ and $\mathbf{e}_i := e_i$,
- the double quiver Q' has $2 \cdot (1 - \langle \mathbf{d}_i, \mathbf{d}_i \rangle)$ loops at each vertex i and $-(\mathbf{d}_i, \mathbf{d}_j)$ arrows from vertex i to vertex j .

We call such a quiver an auxiliary quiver attached to semisimple type τ . Note that Q' is only determined by τ up to orientation. In what follows, we will abuse notations and denote by Q_τ any choice of Q' , as both Q_τ and property (P) for Q_τ do not depend on orientation.

Recall that for a G -variety X with quotient map $q : X \rightarrow X // G$ (G is a reductive group), an open subset $U \subseteq X$ is called G -saturated if $q^{-1}(q(U)) = U$. Then the following holds:

Proposition 2.4.1 [25, §4.] [13, Thm. 2.9.] *There exists a $\mathrm{GL}(\mathbf{d})_x$ -saturated open neighborhood $W \subseteq S$ of x and a $\mathrm{GL}(\mathbf{e})$ -equivariant² morphism $f : (W, x) \rightarrow (\mu_{Q_\tau, \mathbf{e}}^{-1}(0), 0)$ such that the commutative diagram below has cartesian squares and étale horizontal maps:*

$$\begin{array}{ccccc} (\mu_{Q_\tau, \mathbf{e}}^{-1}(0) \times^{\mathrm{GL}(\mathbf{e})} \mathrm{GL}(\mathbf{d}), [0, \mathrm{Id}]) & \longleftarrow & (W \times^{\mathrm{GL}(\mathbf{d})_x} \mathrm{GL}(\mathbf{d}), [x, \mathrm{Id}]) & \longrightarrow & (\mu_{Q,\mathbf{d}}^{-1}(0), x) \\ \downarrow & & \downarrow & & \downarrow \\ (\mu_{Q_\tau, \mathbf{e}}^{-1}(0) // \mathrm{GL}(\mathbf{e}), 0) & \longleftarrow & (W // \mathrm{GL}(\mathbf{e}), x) & \longrightarrow & (\mu_{Q,\mathbf{d}}^{-1}(0) // \mathrm{GL}(\mathbf{d}), x). \end{array}$$

Therefore analyzing singularities of $\mu_{Q,\mathbf{d}}^{-1}(0)$ at closed orbits boils down to analyzing $0 \in \mu_{Q_\tau, \mathbf{e}}^{-1}(0)$. The following result of Le Bruyn, Procesi [54] tells us that there are finitely many semisimple types τ to consider and that they come with a partial order (see also [13, §2.] for a detailed exposition). For simplicity, we call $M(Q, \mathbf{d}) := \mu_{Q,\mathbf{d}}^{-1}(0) // \mathrm{GL}(\mathbf{d})$.

² Using $\mathrm{GL}(\mathbf{d})_x \simeq \mathrm{GL}(\mathbf{e})$.

Proposition 2.4.2 [13, Thm. 2.2.] *Let τ be a semisimple type and $M(Q, \mathbf{d})_\tau$ the subset of semisimple Π_Q -modules of type τ . Then $M(Q, \mathbf{d})_\tau$ is locally closed and there are finitely many types τ such that $M(Q, \mathbf{d})_\tau \neq \emptyset$. Moreover:*

$$\overline{M(Q, \mathbf{d})_\tau} = \bigcup_{\tau' \leq \tau} M(Q, \mathbf{d})_{\tau'},$$

where $\tau' \leq \tau$ if, and only if, there exist semisimple points $x', x \in \mu_{Q, \mathbf{d}}^{-1}(0)$ of types τ', τ such that $\mathrm{GL}(\mathbf{d})_x \subseteq \mathrm{GL}(\mathbf{d})_{x'}$.

Let $q : \mu_{Q, \mathbf{d}}^{-1}(0) \rightarrow M(Q, \mathbf{d})$ be the quotient map. Then we define $\left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_\tau := q^{-1}(M(Q, \mathbf{d})_\tau)$. We now show how to prove that $\mu_{Q, \mathbf{d}}^{-1}(0)$ has rational singularities, also at non-closed orbits.

Proposition 2.4.3 *Let τ be a semisimple type arising from $(Q, \mathbf{d})$. Then $\mu_{Q, \mathbf{e}}^{-1}(0)$ has rational singularities if, and only if, $\bigcup_{\tau' \geq \tau} \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'}$ has rational singularities.*

Proof We first make the following observation: any semisimple point $x \in \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_\tau$ has a $\mathrm{GL}(\mathbf{d})$ -saturated open neighborhood U_x contained in the open subset $\bigcup_{\tau' \geq \tau} \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'} = q^{-1}\left(\bigcup_{\tau' \geq \tau} M(Q, \mathbf{d})_{\tau'}\right)$. By possibly shrinking it, one may assume that U_x is contained in the image of the étale morphism $W \times^{\mathrm{GL}(\mathbf{d})_x} \mathrm{GL}(\mathbf{d}) \rightarrow \mu_{Q, \mathbf{d}}^{-1}(0)$ from Proposition 2.4.1. Moreover, for all $\tau' \geq \tau$, $U_x \cap \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'} \neq \emptyset$. Indeed, $q(U_x)$ is an open neighborhood of $q(x) \in M(Q, \mathbf{d})_\tau$ and $M(Q, \mathbf{d})_\tau \subseteq \overline{M(Q, \mathbf{d})_\tau}$, so $U_x \cap \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'} = q^{-1}(q(U_x) \cap M(Q, \mathbf{d})_{\tau'}) \neq \emptyset$.

Suppose that $\mu_{Q, \mathbf{e}}^{-1}(0)$ has rational singularities. Then by Proposition 2.4.1 and Lemma 2.3.5, for any semisimple point $x \in \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_\tau$, U_x has rational singularities. Since for $\tau' \geq \tau$, $U_x \cap \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'} \neq \emptyset$, there exists some semisimple point $x' \in \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'}$ whose neighborhood $U_{x'}$ has rational singularities (this may require shrinking $U_{x'}$). Since all semisimple points in $\left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'}$ are étale-locally modelled on the same auxiliary quiver $(Q_{\tau'}, \mathbf{e})$, we deduce that for all $x' \in \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'}$, $U_{x'}$ has rational singularities. Finally, the open subset $\bigcup_{\tau' \geq \tau} \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'} \subseteq \mu_{Q, \mathbf{d}}^{-1}(0)$ is covered by the open neighborhoods U_x , $x \in \bigcup_{\tau' \geq \tau} \left(\mu_{Q, \mathbf{d}}^{-1}(0)\right)_{\tau'}$, so it has rational singularities. The converse follows by applying the same reasoning to $\left(\mu_{Q, \mathbf{e}}^{-1}(0)\right)_{\tau_{\min}}$, where $\tau_{\min}$ is the semisimple type of $0 \in \mu_{Q, \mathbf{e}}^{-1}(0)$. $\square$

2.5 Local models for moduli of 2-Calabi-Yau categories

In this section, we gather local models of some moduli stacks of objects of 2-Calabi-Yau categories, following [29]. Such local models were obtained separately in [13, 25], [8, 49] and [4, 14] and united in the more general framework of [29]. In each case, local models are constructed from so-called Ext-quivers of semisimple (or polystable) objects. The moduli stacks we consider are also disjoint unions of global quotient stacks $[X_\alpha/G_\alpha]$, where X_α is a scheme and G_α is a reductive group. This gives us local models for the schemes X_α , which generalize the étale slices described above.

Let us first give a definition of 2-Calabi-Yau category which covers our examples below. For simplicity, we use the definition for triangulated categories given in [51, §2.6.], although Davison works with a refined notion of Calabi-Yau structures suited to differential-graded enhancements of the triangulated categories we consider. As we will see below, the coarser definition is sufficient for the computations of Ext-quivers covered in this article. In what follows, $\mathbb{K}$ denotes a base field.

Definition 2.5.1 Let $d \in \mathbb{Z}$ and $\mathcal{T}$ a $\mathbb{K}$ -linear, Hom-finite, triangulated category admitting a Serre functor as defined in [51, §2.6.]. We say that $\mathcal{T}$ is d -Calabi-Yau if there exists a family of $\mathbb{K}$ -linear forms $t_X : \text{Hom}(X, X[d]) \rightarrow \mathbb{K}$, $X \in \mathcal{T}$ such that, for all $p, q \in \mathbb{Z}$ satisfying $p+q = d$ and for all $f \in \text{Hom}(X, Y[p])$ and $g \in \text{Hom}(Y, X[q])$, the pairing:

$$\begin{aligned} \text{Hom}(X, Y[p]) \times \text{Hom}(Y, X[q]) &\rightarrow \mathbb{K} \\ (f, g) &\mapsto t_X(g[p] \circ f) \end{aligned}$$

is non-degenerate and $t_X(g[p] \circ f) = (-1)^{pq} \cdot t_Y(f[q] \circ g)$.

The triangulated categories considered in the articles are subcategories of $D^b(\mathcal{C})$, where $\mathcal{C}$ is an abelian category e.g. modules over an algebra or quasi-coherent sheaves over an algebraic variety. In particular, if $D^b(\mathcal{C})$ is 2-Calabi-Yau, then for all $X \in \mathcal{C}$, $\text{Ext}_{\mathcal{C}}^1(X, X)$ inherits a symplectic form. Thus $\text{Ext}_{\mathcal{C}}^1(X, X)$ must be even-dimensional.

One can also define a notion of Calabi-Yau algebras as in [37, §3.2.] and [7, §7]. Set A an algebra. Consider the category $\mathcal{C}$ of right A -modules and the triangulated subcategory $D^b(A)$ of $D^b(\mathcal{C})$ formed by complexes whose total cohomology is finite-dimensional. By [51, §4.1-2.], if A is a d -Calabi-Yau algebra, then $D^b(A)$ is d -Calabi-Yau as a triangulated category. Here is the definition:

Definition 2.5.2 Let A be a $\mathbb{K}$ -algebra and $d \in \mathbb{Z}$. The algebra A is called d -Calabi-Yau if:

1. As an $A - A$ -bimodule, A admits a projective resolution of finite length by finite-dimensional projective $A - A$ -bimodules,
2. There is a quasi-isomorphism $\text{RHom}(A, A^{\text{op}} \otimes A) \simeq A[-d]$ of complexes of $A - A$ -bimodules.

We now concretely describe the moduli stacks that we deal with in this paper. Here $\mathbb{K}$ is algebraically closed, of characteristic zero.

2.5.1 Π_Q -modules

Given a quiver Q , we consider the moduli stack $\mathfrak{M}_{\Pi_Q}$ of representations of the preprojective algebra Π_Q . It is the union of the following quotient stacks:

$$\mathfrak{M}_{\Pi_Q} = \bigsqcup_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathfrak{M}_{\Pi_Q}(\mathbf{d}) = \bigsqcup_{\mathbf{d} \in \mathbb{N}^{Q_0}} \left[\mu_{Q, \mathbf{d}}^{-1}(0) / \mathrm{GL}(\mathbf{d}) \right].$$

When the underlying graph of Q has no Dynkin diagram of type A, D, or E among its connected components (in particular, when Q is totally negative), the algebra Π_Q is 2-Calabi-Yau [51, §4.2.]. Moreover:

$$\mathrm{hom}_{\Pi_Q}(M, N) - \mathrm{ext}_{\Pi_Q}^1(M, N) + \mathrm{ext}_{\Pi_Q}^2(M, N) = (\dim(M), \dim(N)).$$

We already saw étale-local models of $\mu_{Q, \mathbf{d}}^{-1}(0)$ and GIT quotients $M_{\Pi_Q}(\mathbf{d}) := \mu_{Q, \mathbf{d}}^{-1}(0) // \mathrm{GL}(\mathbf{d})$ in Sections 2.2 and 2.4. Let us just mention how the auxiliary quivers $(\bar{Q}_\tau, \mathbf{e})$ are related to Ext-groups when Π_Q is 2-Calabi-Yau. Given a semisimple Π_Q -module $M = \bigoplus_{i=1}^r M_i^{\oplus e_i}$ of type $\tau = (\mathbf{d}_i, e_i; 1 \leq i \leq r)$, the Ext-quiver of M is the quiver with set of vertices $\{1, \dots, r\}$ and $\mathrm{ext}_{\Pi_Q}^1(M_i, M_j)$ arrows from vertex i to vertex j . The identity above shows that the Ext-quiver of M is $\overline{Q}_\tau$.

2.5.2 $\Lambda^q(Q)$ -modules

Let Q be a quiver and $q \in (\mathbb{K}^\times)^{Q_0}$. The multiplicative preprojective algebra $\Lambda^q(Q)$ was introduced by Crawley-Boevey and Shaw in [27]. We refer to their article for a precise definition and directly describe moduli of $\Lambda^q(Q)$ -modules. Fix a total ordering $<$ of Q_1 . A $\Lambda^q(Q)$ -module of dimension $\mathbf{d}$ is given by a collection of matrices M_a, M_a^* , $a \in Q_1$ of size $d_{t(a)} \times d_{s(a)}$ (resp. $d_{s(a)} \times d_{t(a)}$) such that, for all $a \in Q_1$, $I_{d_{t(a)}} + M_a M_a^*$ and $I_{d_{s(a)}} + M_a^* M_a$ are invertible and:

$$\prod_{\substack{a \in (Q_1, <) \\ t(a)=i}} (1 + M_a M_a^*) \times \prod_{\substack{a \in (Q_1, <) \\ s(a)=i}} (1 + M_a^* M_a)^{-1} = q_i \cdot I_{d_i}.$$

As in the case of additive preprojective algebras, two collections of matrices correspond to isomorphic modules if they are conjugated by an element of $\mathrm{GL}(\mathbf{d})$. Therefore, we obtain the moduli stack:

$$\mathfrak{M}_{\Lambda^q(Q)} = \bigsqcup_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathfrak{M}_{\Lambda^q(Q)}(\mathbf{d}) = \bigsqcup_{\mathbf{d} \in \mathbb{N}^{Q_0}} [R(\Lambda^q(Q), \mathbf{d}) / \mathrm{GL}(\mathbf{d})].$$

where $R(\Lambda^q(Q), \mathbf{d}) \subseteq R(\overline{Q}, \mathbf{d})$ is the affine, locally closed subvariety defined by the above conditions. Note that $\Lambda^q(Q)$ and $R(\Lambda^q(Q), \mathbf{d})$ do not depend (up to isomorphism) on the orientation of Q , nor on the choice of the ordering on Q_1 - see [27, Thm. 1.4.]. Moreover, since $R(\Lambda^q(Q), \mathbf{d})$ is affine, we obtain a GIT quotient $M_{\Lambda^q(Q)}(\mathbf{d}) =$

$R(\Lambda^q(Q), \mathbf{d}) // \mathrm{GL}(\mathbf{d})$, which parametrizes semisimple $\Lambda^q(Q)$ -modules of dimension $\mathbf{d}$ similarly to $M_{\Pi_Q}(\mathbf{d})$.

When Q is connected and contains at least one (not necessarily oriented) cycle, Kaplan and Schedler proved that $\Lambda^q(Q)$ is a 2-Calabi-Yau algebra ([49, Thm. 1.2.]). This is the case, for instance, when Q is totally negative. Moreover, for any pair of $\Lambda^q(Q)$ -modules (M, N) , the following identity holds as in the additive case (see [27, Thm. 1.6.]):

$$\mathrm{hom}_{\Lambda^q(Q)}(M, N) - \mathrm{ext}_{\Lambda^q(Q)}^1(M, N) + \mathrm{ext}_{\Lambda^q(Q)}^2(M, N) = (\dim(M), \dim(N)).$$

Given a semisimple $\Lambda^q(Q)$ -module $M = \bigoplus_{i=1}^r M_i^{\oplus e_i}$ of type $\tau = (\mathbf{d}_i, e_i; 1 \leq i \leq r)$, one constructs an auxiliary quiver and a dimension vector $(Q_\tau, \mathbf{e})$ as in the additive case (see Section 2.4). Likewise, the identity above shows that the Ext-quiver of M is Q_τ .

2.5.3 Semistable Coherent Sheaves on K3 Surfaces

Let (S, H) be a complex, projective polarized K3 surface. We consider the moduli stack $\mathfrak{M}_{S,H}$ of Gieseker H -semistable, coherent sheaves on S , as described in [46]. It is the union of the following quotient stacks:

$$\mathfrak{M}_{S,H} = \bigsqcup_{\mathbf{v} \in H^*(S, \mathbb{Z})} \mathfrak{M}_{S,H}(\mathbf{v}) = \bigsqcup_{\mathbf{v} \in H^*(S, \mathbb{Z})} [U_{\mathbf{v}, m_{\mathbf{v}}}^{\mathrm{ss}} / \mathrm{GL}(P(m_{\mathbf{v}}))].$$

where $\mathfrak{M}_{S,H}(\mathbf{v})$ is the substack of semistable sheaves with Mukai vector $\mathbf{v}$. Given a coherent sheaf $\mathcal{F}$ on S , its Mukai vector is by definition $\mathbf{v}(\mathcal{F}) := (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), \frac{1}{2}c_1(\mathcal{F})^2 - c_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F}))$ and determines the Hilbert polynomial of $\mathcal{F}$. For a given Mukai vector $\mathbf{v}$ and associated Hilbert polynomial $P \in \mathbb{Q}[t]$, there exists an integer $m_{\mathbf{v}} > 0$ such that, for any Gieseker H -semistable, coherent sheaf $\mathcal{F}$ on S with Hilbert polynomial P , the sheaf $\mathcal{F}(m_{\mathbf{v}})$ is generated by global sections (see [46, §4.3.] for details). Such a sheaf $\mathcal{F}$, together with a choice of a basis of $\Gamma(S, \mathcal{F}(m_{\mathbf{v}}))$, corresponds to a point in $\mathrm{Quot}(\mathcal{O}_S(-m_{\mathbf{v}})^{\oplus P(m_{\mathbf{v}})}, P)$. The locus $U_{\mathbf{v}, m_{\mathbf{v}}} \subseteq \mathrm{Quot}(\mathcal{O}_S(-m_{\mathbf{v}})^{\oplus P(m_{\mathbf{v}})}, P)$ of quotient sheaves with Mukai vector $\mathbf{v}$ is an open and closed subset (see [45, Ch. 10.2.]), which is preserved under the action of $\mathrm{GL}(P(m_{\mathbf{v}}))$ on $\mathrm{Quot}(\mathcal{O}_S(-m_{\mathbf{v}})^{\oplus P(m_{\mathbf{v}})}, P)$. This action admits a linearisation such that the semistable locus $U_{\mathbf{v}, m_{\mathbf{v}}}^{\mathrm{ss}} \subseteq \mathrm{Quot}(\mathcal{O}_S(-m_{\mathbf{v}})^{\oplus P(m_{\mathbf{v}})}, P)$ parametrizes globally generated quotients $\mathcal{O}_S(-m_{\mathbf{v}})^{\oplus P(m_{\mathbf{v}})} \twoheadrightarrow \mathcal{F}$ inducing an isomorphism on global sections. Thus $\mathfrak{M}_{S,H}(\mathbf{v}) \simeq [U_{\mathbf{v}, m_{\mathbf{v}}}^{\mathrm{ss}} / \mathrm{GL}(P(m_{\mathbf{v}}))]$. Taking the associated GIT quotient, we obtain the moduli space $M(\mathbf{v}) = M_{S,H}(\mathbf{v})$, which parametrizes H -polystable sheaves on S . We denote by $M^s(\mathbf{v}) \subseteq M(\mathbf{v})$ the open locus of H -stable sheaves.

Let us denote Mukai vectors by $\mathbf{v} = (r, \mathbf{c}, a) \in \mathbb{Z}_{\geq 0} \oplus \mathrm{NS}(S) \oplus H^4(S, \mathbb{Z})$. Recall that the Mukai pairing is defined by: $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{c}_1 \cdot \mathbf{c}_2 - r_1 a_2 - r_2 a_1$. Note that $\mathrm{NS}(S)$ is a lattice (see [45, Ch. 1.]). Consider $D^b(S)$ the derived category formed by complexes of quasi-coherent sheaves on S with bounded coherent cohomology. Since S is a K3 surface, the category $D^b(S)$ is 2-Calabi-Yau, by Serre duality. Moreover, for any pair

$\mathcal{F}_1, \mathcal{F}_2$ of semistable coherent sheaves, the following identity holds [46, Cor. 6.1.5.]:

$$\mathrm{hom}(\mathcal{F}_1, \mathcal{F}_2) - \mathrm{ext}^1(\mathcal{F}_1, \mathcal{F}_2) + \mathrm{ext}^2(\mathcal{F}_1, \mathcal{F}_2) = -\mathbf{v}_1 \cdot \mathbf{v}_2 = -(\mathbf{c}_1 \cdot \mathbf{c}_2 - r_1 a_2 - r_2 a_1).$$

Given a polystable sheaf $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{F}_i^{\oplus e_i}$, with distinct stable summands $\mathcal{F}_i$ of Mukai vectors $\mathbf{v}_i$, we construct an auxiliary quiver $\mathcal{Q}'$ satisfying the following conditions: $\mathcal{Q}'_0 = \{1, \dots, r\}$ and the number of arrows from i to j in $\mathcal{Q}'$ is:

$$\mathrm{ext}^1(\mathcal{F}_i, \mathcal{F}_j) = \begin{cases} 2 + \mathbf{v}_i \cdot \mathbf{v}_i, & \text{if } i = j, \\ \mathbf{v}_i \cdot \mathbf{v}_j, & \text{if } i \neq j. \end{cases}$$

Thus, $\overline{\mathcal{Q}'}$ is the Ext-quiver associated to $\mathcal{F}$. Note that $\mathrm{ext}^1(\mathcal{F}_i, \mathcal{F}_i)$ is indeed even, thanks to the 2-Calabi-Yau property.

Local Models

We now describe local models of the above moduli stacks in a unified manner, following [29]. Let us first stress common features of the aforementioned stacks. In what follows, $\mathfrak{M}$ denotes either $\mathfrak{M}_{\Pi_Q}$, $\mathfrak{M}_{\Lambda^q(Q)}$, $\mathfrak{M}_{S,H}$ and $\mathcal{A}$ the corresponding category of objects (the groupoid associated to $\mathcal{A}$ is isomorphic to $\mathfrak{M}(\mathbb{K})$). Then $\mathfrak{M}$ is a union of quotient stacks:

$$\mathfrak{M} = \bigsqcup_{\alpha} \mathfrak{M}_{\alpha} = \bigsqcup_{\alpha} [X_{\alpha}/G_{\alpha}],$$

where X_{α} is a finite-type $\mathbb{K}$ -scheme and G_{α} is a reductive linear algebraic group. An object $F \in \mathcal{A}$ corresponds to a point $x \in \mathfrak{M}$ and we call $\alpha(F)$ the dimension vector (resp. the Mukai vector) of F . If $x \in \mathfrak{M}$ is closed (i.e. the orbit of $x \in X_{\alpha}$ is closed), then x corresponds to a semisimple (or polystable) object $F = \bigoplus_{i=1}^r F_i^{\oplus e_i} \in \mathcal{A}$. We call $\tau(F) = (\alpha(F_i), e_i; 1 \leq i \leq r)$ the type of F (or x). Note that $\mathrm{Aut}(F) \simeq \mathrm{GL}(\mathbf{e})$. As explained above, one can build from $\tau(F)$ an auxiliary quiver $\mathcal{Q}_{\tau}$ such that $\overline{\mathcal{Q}_{\tau}}$ is the Ext-quiver of F .

Another common feature of quotient stacks $\mathfrak{M}_{\alpha} = [X_{\alpha}/G_{\alpha}] \subseteq \mathfrak{M}$ is the existence of good categorical quotients $M_{\alpha} = X_{\alpha} // G_{\alpha}$ in the sense of Geometric Invariant Theory [34, Ch. 6.]. We can stratify M_{α} according to the conjugacy class of stabilizers $G_x \subseteq G_{\alpha}$, $x \in X_{\alpha}$ (where x has a closed orbit). For a reductive subgroup $H \subseteq G_{\alpha}$, the stratum $(M_{\alpha})_{(H)}$ is locally closed and corresponds to closed orbits in X_{α} with stabilizer subgroups in the conjugacy class of H . Moreover:

$$\overline{(M_{\alpha})_{(H)}} = \bigcup_{(H') \leq (H)} (M_{\alpha})_{(H')},$$

where $(H') \leq (H)$ if H is conjugated to a subgroup of H' . Note that the type of x determines the conjugacy class of G_x .

When X_{α} is an affine space with a linear G_{α} -action, properties of this stratification were proved in [66, Lem. 5.5.]. If X_{α} is an affine, finite-type G_{α} -scheme, then there exists a closed G_{α} -equivariant embedding of X_{α} into a G_{α} -representation (see [12,

Prop. 2.3.5.]), so we obtain the stratification by restriction. In general, $X_\alpha // G_\alpha$ is locally isomorphic to an affine GIT quotient, so we obtain the stratification by gluing.

Remark 2.5.3 In [54], Le Bruyn and Procesi showed that, for a quiver Q and a dimension vector $\mathbf{d}$, conjugacy classes of stabilizers of closed orbits in $R(Q, \mathbf{d})$ are in bijection with semisimple types appearing in $R(Q, \mathbf{d})$. However, given a polystable sheaf $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{F}_i^{\oplus e_i}$ on a K3 surface, one can only recover the Hilbert polynomials $P_{\mathcal{F}_i}$, $1 \leq i \leq r$ from the conjugacy class of its stabilizer and not the Mukai vectors $\mathbf{v}(\mathcal{F}_i)$, $1 \leq i \leq r$.

The above moduli stacks all arise from a 2-Calabi-Yau category, as shown in [29, §7.]. Davison works with a dg-enhancement $\mathcal{T}^{\text{dg}}$ of a triangulated category $\mathcal{T}$ containing $\mathcal{A}$. Then $\mathcal{T}^{\text{dg}}$ carries a left 2-Calabi-Yau structure, as defined by Brav and Dyckerhoff [11]. In the examples above, $\mathcal{T}$ is a category of complexes (dg-modules over the dg-version of Π_Q or $\Lambda^q(Q)$, complexes of sheaves on a K3 surface) and $\mathcal{A} \subset \mathcal{T}$ is an abelian subcategory of the category $\mathcal{B} \subset \mathcal{T}$ of complexes concentrated in degree zero. Then $\mathfrak{M}$ sits as an open substack in the truncation of the derived moduli stack of objects of $\mathcal{T}^{\text{dg}}$, defined by Toën and Vaquié [68].

Although under certain assumptions, the left 2-Calabi-Yau structure on $\mathcal{T}^{\text{dg}}$ does make $\mathcal{T}$ a 2-Calabi-Yau category in the sense of Definition 2.5.1 (see for instance [52, §10.1.]), in the examples involving quivers, $\mathcal{T}$ may differ from the derived categories of Π_Q (resp. $\Lambda^q(Q)$). Indeed, in those cases, $\mathcal{T}$ is built from the dg-versions of Π_Q (resp. $\Lambda^q(Q)$). Consequently, in the theorem below, the Ext-quivers associated to semisimple Π_Q -modules (resp. $\Lambda^q(Q)$ -modules) should a priori be computed using Hom-spaces of $\mathcal{T}$.

However, when Q is totally negative, the dg-version of Π_Q (resp. $\Lambda^q(Q)$) is quasi-isomorphic to Π_Q itself (resp. $\Lambda^q(Q)$) - see [51, §4.2.] and [49, Prop. 4.4.] - so $\mathcal{T}$ is equivalent to the derived category of Π_Q (resp. $\Lambda^q(Q)$). In the case of coherent sheaves on a K3 surface S , the dg-category $\mathcal{T}^{\text{dg}}$ under consideration is a dg-enhancement of $D^b(S)$ - see [11, §5.2.], [29, §7.2.5.]. Therefore, in our examples, Ext-quivers can be computed from Ext-groups in $\mathcal{A}$. For this reason, we do not give further details and refer to [29] for a more thorough discussion of the 2-Calabi-Yau structures at play. We collect the common features described above in the following definition:

Definition 2.5.4 We call $[X/G]$ a quotient stack coming from a 2-Calabi-Yau category if:

1. The stack $[X/G]$ is an open substack of the truncation of the derived moduli stack of objects of a dg-category $\mathcal{T}^{\text{dg}}$ endowed with a left 2-Calabi-Yau structure,
2. Closed points of $[X/G]$ correspond to (a subclass of) semisimple objects of an abelian, finite-length, $\mathbb{K}$ -linear subcategory $\mathcal{A} \subset \mathcal{T}$; moreover, if $x \in [X/G](\mathbb{K})$ corresponds to the semisimple object $F = \bigoplus_{i=1}^r F_i^{\oplus e_i} \in \mathcal{A}$, then $(F_1, \dots, F_r)$ is a Σ -collection in $\mathcal{T}$ (as defined in [29, §1.3.]),
3. The group G is reductive and X has a good categorical quotient $X \rightarrow M := X // G$.

It follows from the assumptions above that, if $x \in [X/G](\mathbb{K})$ corresponds to the semisimple object $F = \bigoplus_{i=1}^r F_i^{\oplus e_i} \in \mathcal{A}$, then $\text{Aut}(x) \simeq \text{GL}(\mathbf{e})$.

We now describe local models of X and M . In the case of $\mathfrak{M}_{S,H}(\mathbf{v})$, the theorem below follows from the formality result in [14] (see also [4, §3–4.] and the references therein). In the case of $\mathfrak{M}_{\Lambda_Q^q}$, when $\Lambda^q(Q)$ is 2-Calabi-Yau, a local description can be obtained from [49, Thm. 5.12 - Thm. 5.16.], using a G -equivariant version of Artin's approximation theorem (see, for instance, [3]).

Theorem 2.5.5 [29, Thm. 5.11.] *Let $[X/G]$ be a quotient stack coming from a 2-Calabi-Yau category and $x \in [X/G](\mathbb{K})$ a closed point corresponding to $F \simeq \bigoplus_{i=1}^r F_i^{\oplus e_i}$. Let Q' be an auxiliary quiver associated to F (i.e. Q' is the Ext-quiver of F). Then there exists an affine $\mathrm{GL}(\mathbf{e})$ -variety W , with a fixed point w and a commutative diagram:*

$$\begin{array}{ccccc} ([\mu_{Q',\mathbf{e}}^{-1}(0)/\mathrm{GL}(\mathbf{e})], 0) & \longleftarrow & ([W/\mathrm{GL}(\mathbf{e})], w) & \longrightarrow & ([X/G], x) \\ \downarrow & & \downarrow & & \downarrow \\ (M_{\Pi_{Q',\mathbf{e}}}, 0) & \longleftarrow & (W//\mathrm{GL}(\mathbf{e}), w) & \longrightarrow & (M, x). \end{array}$$

such that the horizontal maps are étale and the squares are cartesian.

Similarly to Section 2.4, this also gives us $\mathrm{GL}(\mathbf{e})$ -equivariant étale morphisms $(W, w) \rightarrow (\mu_{Q',\mathbf{e}}^{-1}(0), 0)$ and $(W, w) \rightarrow (X, x)$ which induce:

$$\begin{array}{ccccc} (\mu_{Q',\mathbf{e}}^{-1}(0) \times^{\mathrm{GL}(\mathbf{e})} G, [0, \mathrm{Id}]) & \longleftarrow & (W \times^{\mathrm{GL}(\mathbf{e})} G, [w, \mathrm{Id}]) & \longrightarrow & (X, x) \\ \downarrow & & \downarrow & & \downarrow \\ (M_{\Pi_{Q',\mathbf{e}}}, 0) & \longleftarrow & (W//\mathrm{GL}(\mathbf{e}), w) & \longrightarrow & (M, x). \end{array}$$

where horizontal maps are étale and squares are cartesian.

2.6 Rational Singularities and Counts of Jets

In this section, we shift gears and introduce arithmetic characterizations of rational singularities, based on works of Aizenbud-Avni [1] and Glazer [38]. In order to count $\mathbb{F}_q$ -rational points of jet spaces, we consider a finite-type scheme $X/\mathbb{Z}$. When X is locally complete intersection, Aizenbud-Avni and Glazer showed that the sequences $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ are bounded for all finite fields precisely when $X_{\mathbb{Q}}$ has rational singularities (see [38, Thm. 4.1.] for the detailed assumptions). There is more: by studying the poles of the local Igusa zeta function associated to an affine scheme, Wyss showed that the sequence $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$, when bounded, has a limit when n goes to infinity [70, Lem. 4.7.]. We will identify this limit with a p -adic integral on an analytic manifold associated to X in Section 4.2.

Suppose $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}_{\mathbb{Z}}^r$. The local Igusa zeta function $Z_f(s)$ is defined as a parametric p -adic integral over $\mathbb{A}_{\mathbb{Z}}^r$ (see [47] for an introduction). Consider F a

finite extension of $\mathbb{Q}_p$, with valuation ring $\mathcal{O} \subset F$, maximal ideal $\mathfrak{m} \subseteq \mathcal{O}$ and residue field $\mathcal{O}/\mathfrak{m} \simeq \mathbb{F}_q$. By definition:

$$Z_f(s) := \int_{\mathcal{O}^r} \|(f_1(x), \dots, f_m(x))\|^s dx,$$

where $s \in \mathbb{C}$ and for $(y_1, \dots, y_m) \in F^m$, $\|(y_1, \dots, y_m)\| := \max_{1 \leq j \leq m} |y_j|$ and $|\bullet|$ is the usual non-archimedean norm on F .

The local Igusa zeta function encodes the sequence $\sharp X(\mathcal{O}/\mathfrak{m}^n)$, $n \geq 1$ into a generating series:

$$\sum_{n \geq 1} \sharp X(\mathcal{O}/\mathfrak{m}^n) \cdot q^{-n(r+s)} = \frac{Z_f(s) - 1}{1 - q^s}.$$

Of course, in this paper, we are interested in the counts $\sharp X(\mathbb{F}_q[t]/(t^n))$ instead of $\sharp X(\mathcal{O}/\mathfrak{m}^n)$. The following result by Aizenbud and Avni³ tells us that, up to working in large enough characteristic, the two counts actually coincide.

Proposition 2.6.1 [1, Prop. 3.0.2.] *Let X be a $\mathbb{Z}$ -scheme of finite type. There is a finite set of primes S such that, for any $p \notin S$, for any q power of p and for any $n \geq 1$, $\sharp X(\mathbb{F}_q[t]/(t^n)) = \sharp X(\mathcal{O}/\mathfrak{m}^n)$.*

The local Igusa zeta function also enjoys an explicit formula as a rational function in $T = q^{-s}$. This was proved by Denef when $r = 1$ [33] and by Veys and Zuniga-Galindo [69] in the general case. The formula relies on a principalization $\pi : W \rightarrow \mathbb{A}_{\mathbb{Q}}^r$ of the ideal $I = (f_1, \dots, f_m) \subseteq \mathbb{Q}[x_1, \dots, x_r]$ - see [69] for details. Then the ideal π^*I defines a divisor $\sum_{i \in T} N_i E_i$ with simple normal crossings, where E_i , $i \in T$ are the irreducible components of $\pi^{-1}(X_{\mathbb{Q}})$. We will also need the following numerical data: $\text{div}(\pi^*(dx_1 \wedge \dots \wedge dx_r)) = \sum_{i \in T} (v_i - 1)E_i$. The quantity $\text{lct}(\mathbb{A}_{\mathbb{Q}}^r, X_{\mathbb{Q}}) = \min_{i \in T} \left\{ \frac{v_i}{N_i} \right\}$ is called the log-canonical threshold of the pair $(\mathbb{A}_{\mathbb{Q}}^r, X_{\mathbb{Q}})$. This is an important invariant in singularity theory and it is related to jet spaces of $X_{\mathbb{Q}}$ (see for instance [59]). Finally, we refer to [33, §2.] for the notion of good reduction of π_F modulo $\mathfrak{m}$ and denote by $\overline{\pi}$, $\overline{E_i}$ and $\overline{X}$ the reductions of π_F , $(E_i)_F$ and X_F . These are schemes and morphisms defined over $\mathbb{F}_q$.

Proposition 2.6.2 [33, Thm. 2.4. - Thm. 3.1.] [69, Thm. 2.10.] *There is a finite set of primes S such that for $p \notin S$, q a prime power of p and F any finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, π_F has good reduction modulo $\mathfrak{m}$. In that case,*

$$Z_f(s) = q^{-r} \cdot \sum_{J \subseteq T} c_J \prod_{j \in J} \frac{(q-1)q^{-N_j s - v_j}}{1 - q^{-N_j s - v_j}},$$

where $c_J = \sharp \{a \in \overline{X}(\mathbb{F}_q) \mid a \in \overline{E_j}(\mathbb{F}_q) \Leftrightarrow j \in J\}$.

³ This result follows from transfer results for p-adic integrals, see [19, 20]. Therefore, it holds for any choice of F , as opposed to [1], where the authors only work with unramified extensions of $\mathbb{Q}_p$.

In [70], Wyss interprets the limit when n goes to infinity of $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$ as a residue of $Z_f(s)$, seen as a rational fraction in T . Summing up results of [1, 38, 70], we obtain:

Proposition 2.6.3 *Set $X = V(f_1, \dots, f_m) \subseteq \mathbb{A}_{\mathbb{Z}}^r$ as above and assume that $X_{\mathbb{Q}}$ is an equidimensional local complete intersection, of codimension c in $\mathbb{A}_{\mathbb{Q}}^r$. Then the following are equivalent:*

1. *the scheme $X_{\mathbb{Q}}$ has rational singularities,*
2. *for almost all primes p , $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ converges for any finite field $\mathbb{F}_q$ of characteristic p .*

If 1. and 2. hold, there is a finite set of primes S such that for $p \notin S$, q a large enough prime power of p and F any finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, $Z_f(s)$ has its poles of largest real part at $\operatorname{Re}(s) = -c$ and:

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathbb{F}_q[t]/(t^n))}{q^{n \dim X_{\mathbb{Q}}}} = \frac{-1}{q^c - 1} \cdot \operatorname{Res}_{T=q^c} Z_f.$$

Proof The implication 2. $\Rightarrow$ 1. is a consequence of the implication the implication (v) $\Rightarrow$ (iii) from [38, Thm. 4.1.]. Glazer's result involves $\sharp X(\mathcal{O}/\mathfrak{m}^n)$ instead of $\sharp X(\mathbb{F}_q[t]/(t^n))$, where $\mathcal{O}$ is the valuation ring of the unramified extension of $\mathbb{Q}_p$ with residue field $\mathcal{O}/\mathfrak{m} \simeq \mathbb{F}_q$. However, for a given X , these counts are equal for almost all characteristics by Proposition 2.6.1.

Conversely, suppose 1. Then by implication (iii) $\Rightarrow$ (iv') from [38, Thm. 4.1.], for almost all primes p , $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ is bounded for any finite field $\mathbb{F}_q$ of characteristic p . Note that we used again Proposition 2.6.1 in order to replace $\sharp X(\mathcal{O}/\mathfrak{m}^n)$ by $\sharp X(\mathbb{F}_q[t]/(t^n))$ in the conclusion of Glazer's theorem.

Moreover, since $X_{\mathbb{Q}}$ has rational singularities, Z_f has its poles of largest real part at $\operatorname{Re}(s) = -\operatorname{lct}(\mathbb{A}_{\mathbb{Q}}^r, X_{\mathbb{Q}}) = -c$. The first equality is due to [69, Thm. 2.7.], while the second follows from [60, Prop. 1.4.] and [59, Cor. 0.2.]. Note that the proof of [69, Thm. 2.7.] requires the existence of a generic F -point on a divisor $(E_i)_F$ satisfying $\operatorname{lct}(\mathbb{A}_{\mathbb{Q}}^r, X_{\mathbb{Q}}) = \frac{v_i}{N_i}$ (see also [69, Rmk. 2.8.2.]). We can assume this is true by taking a finite extension of F , which is why we require that q be a large enough power of p . Then we can apply [70, Lem. 4.7.] and conclude that $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ converges and that its limit is given by the residue formula above. $\square$

Remark 2.6.4 The proof by Aizenbud and Avni [1, Thm. 3.0.3.] of the implication 1. $\Rightarrow$ 2. in Proposition 2.6.3 relies on the Lang-Weil estimates, Denef's explicit formula for $Z_f(s)$ and a characterisation by Mustață of top-dimensional irreducible components of jet schemes of $X_{\mathbb{Q}}$ in terms of the quantities (N_i, v_i) [60, Thm. 3.2.] (when $X_{\mathbb{Q}}$ is irreducible). None of these require that $X_{\mathbb{Q}}$ be l.c.i. For l.c.i. varieties, the irreducibility of jet schemes is implied by the fact that X has rational singularities [60, Thm. 3.3.]. But irreducibility of jet schemes also follows directly from the assumption of Proposition 2.3.7 on the dimension of jet schemes over the singular locus of $X_{\mathbb{Q}}$. Hence, even if $X_{\mathbb{Q}}$ is not l.c.i., if $\dim \pi_m^{-1}(X_{\operatorname{sg}}) < (m+1) \cdot \dim(X)$ for all $m \geq 1$, then the sequence $q^{-n \dim X_{\mathbb{Q}}} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ converges.

Instead, our motivation for working with locally complete intersection varieties has to do with the fact that the above properties of jet schemes become étale-local under this assumption (by Proposition 2.3.5, since they correspond to rational singularities). Moreover, locally complete intersection varieties are the appropriate setting to compute counts of jets as p -adic integrals (see the proof of Theorem 1.0.5 below).

As a consequence of the counts of jets carried out in [70], we obtain rational singularities for certain fibers of quiver moment maps. This result is necessary to complete the proof of Theorem 1.0.1, as explained in Remark 3.0.4. When $X = \mu_{Q,\mathbf{d}}^{-1}(0)$ and $\mathbf{d} = \underline{1}$, Wyss gave a criterion for the existence of this limit and an explicit formula in terms of the graphical hyperplane arrangement associated to Q [70, Cor. 4.27.]. Our initial motivation for this work was to extend this result to higher dimension vectors.

Proposition 2.6.5 [70, Cor. 4.27.] *Let Q be a quiver and $\mathbf{d} = \underline{1}$. Let p be a large enough prime and $\mathbb{F}_q$ be any finite field of characteristic p . Then $q^{-n \dim \mu_{Q,\mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q,\mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ converges if, and only if, the underlying graph of Q is 2-connected i.e. removing one edge does not disconnect the graph. In that case, there exists an explicit rational fraction $W \in \mathbb{Q}(T)$ depending only on the underlying graph of Q such that:*

$$\lim_{n \rightarrow +\infty} \frac{\sharp \mu_{Q,\mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))}{q^{n \dim \mu_{Q,\mathbf{d}}^{-1}(0)}} = W(q).$$

When $\mathbf{d} = \underline{1}$, there exists a simple Π_Q -module of dimension $\mathbf{d}$ exactly when Q is 2-connected. Indeed, for a connected quiver Q , $b(Q) := 1 - \sharp Q_0 + \sharp Q_1 = 1 - \langle \mathbf{d}, \mathbf{d} \rangle$ and Q is 2-connected if, and only if, for any decomposition $\mathbf{d} = \mathbf{d}_1 + \dots + \mathbf{d}_r$, $b(Q) > b(\text{supp}(\mathbf{d}_1)) + \dots + b(\text{supp}(\mathbf{d}_r))$. The statement then follows from [24, Thm. 1.2.]. From this we deduce:

Corollary 2.6.6 *Let Q be a 2-connected quiver and $\mathbf{d} = \underline{1}$. Then $\mu_{Q,\mathbf{d}}^{-1}(0)$ has rational singularities.*

3 Rational Singularities and Totally Negative Quivers

In this section we prove Theorem 1.0.1. Let us recall the statement:

Theorem 3.0.1 *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ such that $(Q, \mathbf{d})$ has property (P). Then $\mu_{Q,\mathbf{d}}^{-1}(0)$ has rational singularities.*

We follow the inductive strategy developed by Budur in [13]. Let us use notations from [13]: we write, for short, $X(Q, \mathbf{d}) = \mu_{Q,\mathbf{d}}^{-1}(0) \subseteq R(\overline{Q}, \mathbf{d})$, $M(Q, \mathbf{d}) = X(Q, \mathbf{d}) // \text{GL}(\mathbf{d})$ and $Z(Q, \mathbf{d}) = \left(\mu_{Q,\mathbf{d}}^{-1}(0) \right)_{\tau_{\min}}$, where $\tau_{\min} = \tau_{\min, \mathbf{d}}$ is the semisimple type of $0 \in X(Q, \mathbf{d})$ - see Section 2.2. We denote by $q : X(Q, \mathbf{d}) \rightarrow M(Q, \mathbf{d})$ the quotient morphism. Recall that we defined (P) the following property of $(Q, \mathbf{d})$: Q is totally negative and if $\text{supp}(\mathbf{d})$ has two vertices joined by exactly one arrow, then $\mathbf{d} \neq \underline{1}$. Budur's reasoning is summed up in the following result.

Theorem 3.0.2 [13, Thm. 3.6.] *Let $\mathcal{M}$ be a class of pairs $(Q, \mathbf{d})$, where Q is a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ is a dimension vector. Suppose that:*

1. *The class $\mathcal{M}$ is stable under the operation of building pairs $(Q_\tau, \mathbf{e})$ as in Section 2.4, for τ a semisimple type occurring in $X(Q, \mathbf{d})$ and $(Q, \mathbf{d}) \in \mathcal{M}$,*
2. *For every $(Q, \mathbf{d}) \in \mathcal{M}$, the variety $X(Q, \mathbf{d})$ contains a simple point and $\langle \mathbf{d}, \mathbf{d} \rangle < 1$,*
3. *For every $(Q, \mathbf{d}) \in \mathcal{M}$ such that $X(Q, \mathbf{d})$ contains strictly semisimple points⁴,*

$$\dim q^{-1}(q(0)) < 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \#\{\text{loops in } Q_0\}).$$

Then for every $(Q, \mathbf{d}) \in \mathcal{M}$, $X(Q, \mathbf{d})$ has rational singularities.

Let us consider $\mathcal{M}$ the class of pairs $(Q, \mathbf{d})$ satisfying property (P) and $\text{supp}(\mathbf{d}) = Q$. $\mathcal{M}$ is preserved under taking auxiliary quivers (this is an easy consequence of [13, Prop. 2.11.]), so in the above theorem, Assumption 1 is verified. Assumption 2 follows from total negativity and Proposition 2.2.2. However, Assumption 3 is satisfied for most, but not all dimension vectors. This is the content of the following Lemma and Remark.

Lemma 3.0.3 *Let $\tau = \tau_{\min}$. Suppose that $(Q, \mathbf{d})$ has property (P). Then either $\mathbf{d} = \underline{1}$ or:*

$$\dim q^{-1}(q(0)) < 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \#\{\text{loops in } Q_0\}).$$

Remark 3.0.4 When $\mathbf{d} = \underline{1}$, we may have:

$$\dim q^{-1}(q(0)) = 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \#\{\text{loops in } Q_0\}).$$

This is the case, for instance, when Q has two vertices with two loops each and joined by two arrows (regardless of their orientation). This quiver arises as an auxiliary quiver for the quiver with one vertex and two loops. Actually, within the class of quivers with dimension vectors considered in [13, Prop. 2.26.], Assumption 3 fails exactly for that pair. This is due to a computational gap in the proof of [13, Prop. 2.23.]. More precisely, on the first line of the proof of [13, Prop. 2.23.], the right-hand side should be $2(g-1)(n^2 - \sum_i \beta_i) - 2r + 2$ instead of $2(g-1)(n^2 - \sum_i \beta_i)$.

Therefore, we show by other means that $X(Q, \mathbf{d})$ has rational singularities when $\mathbf{d} = \underline{1}$. In order to incorporate those cases into Budur's inductive argument, we prove the following modified version of Theorem 3.0.2:

Theorem 3.0.5 *Let $\mathcal{M}$ be a class of pairs $(Q, \mathbf{d})$, where Q is a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ is a dimension vector. Let us make the following assumptions:*

1. *The class $\mathcal{M}$ is stable under the operation of building pairs $(Q_\tau, \mathbf{e})$ as in Section 2.4, for τ a semisimple type occurring in $X(Q, \mathbf{d})$ and $(Q, \mathbf{d}) \in \mathcal{M}$,*
2. *For every $(Q, \mathbf{d}) \in \mathcal{M}$, the variety $X(Q, \mathbf{d})$ contains a simple point,*

⁴ In other words, $\mathbf{d} \neq \epsilon_i$ for all $i \in Q_0$, as $\tau_{\min} = (\epsilon_i, d_i, i \in \text{supp}(\mathbf{d}))$.

3. For every $(Q, \mathbf{d}) \in \mathcal{M}$, suppose either that $X(Q, \mathbf{d})$ has rational singularities or that the following inequality holds:

$$\dim q^{-1}(q(0)) < 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \sharp\{\text{loops in } Q_0\}).$$

Then for every $(Q, \mathbf{d}) \in \mathcal{M}$, $X(Q, \mathbf{d})$ has rational singularities.

Note that we removed the assumption $\langle \mathbf{d}, \mathbf{d} \rangle < 1$ from Assumption 2. This is harmless, for when $X(Q, \mathbf{d})$ contains a simple point, $\langle \mathbf{d}, \mathbf{d} \rangle \leq 1$ and equality only occurs when $\mathbf{d} = \epsilon_i$, for $i \in Q_0$ a vertex without loops, in which case $X(Q, \mathbf{d})$ is smooth. Indeed, if $X(Q, \mathbf{d})$ contains a simple point and $\langle \mathbf{d}, \mathbf{d} \rangle = 1$, then $M(Q, \mathbf{d})$ is reduced to a point by Proposition 2.2.3. Thus $X(Q, \mathbf{d})$ does not contain strictly semisimple points, which can only happen if $\mathbf{d} = \epsilon_i$ for some $i \in Q_0$. The number of loops at i is then $1 - \langle \epsilon_i, \epsilon_i \rangle = 1 - \langle \mathbf{d}, \mathbf{d} \rangle = 0$. At any rate, since Theorem 3.0.1 only concerns totally negative quivers, the assumption $\langle \mathbf{d}, \mathbf{d} \rangle < 1$ is automatically satisfied.

Before proving Theorem 3.0.5, we recall the following results from [13]:

Lemma 3.0.6 [13, Lem. 2.16.] *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ a dimension vector. Assume that:*

$$\dim q^{-1}(q(0)) < 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \sharp\{\text{loops in } Q_0\}).$$

Then $\dim Z(Q, \mathbf{d}) < 2(1 - \langle \mathbf{d}, \mathbf{d} \rangle)$.

Lemma 3.0.7 [13, Lem. 3.3. - Proof of Lem. 3.4.] *Let $\pi_m : X(Q, \mathbf{d})_m \rightarrow X(Q, \mathbf{d})$ be the truncation of m -jets:*

$$\pi_m^{-1}(0) \simeq \begin{cases} R(\overline{Q}, \mathbf{d}) \times X(Q, \mathbf{d})_{m-2}, & m \geq 2, \\ R(\overline{Q}, \mathbf{d}), & m = 1. \end{cases}$$

Moreover, $\dim \pi_m^{-1}(Z(Q, \mathbf{d}) \cap X(Q, \mathbf{d})_{\text{sg}}) \leq \dim Z(Q, \mathbf{d}) + \dim \pi_m^{-1}(0)$ (see [13, Proof of Lem. 3.4. - Eqn. (10)]).

Proof of Theorem 3.0.5 Consider a pair $(Q, \mathbf{d}) \in \mathcal{M}$ for which it is not already assumed that $X(Q, \mathbf{d})$ has rational singularities. We proceed by descending induction on semisimple types occurring in $X(Q, \mathbf{d})$ i.e. we show that $X(Q_\tau, \mathbf{e})$ has rational singularities for all semisimple types τ occurring in $X(Q, \mathbf{d})$. When τ is maximal (i.e. simple, by Assumption 2), Q_τ has one vertex and $\mathbf{e} = \underline{1}$, so $X(Q_\tau, \mathbf{e})$ is smooth.

Let us prove the induction step. We apply Proposition 2.3.7 to $X(Q, \mathbf{d})$, which is a complete intersection by Assumption 2 and Proposition 2.2.3. We show that:

$$\dim \pi_m^{-1}(X(Q, \mathbf{d})_{\text{sg}}) < (m + 1) \cdot \dim X(Q, \mathbf{d}).$$

By induction, we assume that $X(Q_\tau, \mathbf{e})$ has rational singularities for $\tau > \tau_{\min}$. By Proposition 2.4.3, we get that the open subset $X(Q, \mathbf{d}) \setminus Z(Q, \mathbf{d})$ has l.c.i. rational

singularities and we obtain by Proposition 2.3.7:

$$\dim \pi_m^{-1}(X(Q, \mathbf{d})_{\text{sg}} \setminus Z(Q, \mathbf{d})) < (m+1) \cdot \dim X(Q, \mathbf{d}).$$

Now, consider $\tau = \tau_{\min}$, i.e. $(Q_\tau, \mathbf{e}) = (Q, \mathbf{d})$. From Proposition 2.2.3 and Lemmas 3.0.3, 3.0.6, 3.0.7, we obtain:

$$\dim \pi_m^{-1}(Z(Q, \mathbf{d}) \cap X(Q, \mathbf{d})_{\text{sg}}) \leq \dim(Z(Q, \mathbf{d})) + \dim \pi_m^{-1}(0) < \dim M(Q, \mathbf{d}) + \dim \pi_m^{-1}(0).$$

We prove that $\dim \pi_m^{-1}(Z(Q, \mathbf{d}) \cap X(Q, \mathbf{d})_{\text{sg}}) < (m+1) \cdot \dim X(Q, \mathbf{d})$ by induction on m . For $m = 1$, Lemma 3.0.7 gives:

$$\begin{aligned} \dim M(Q, \mathbf{d}) + \dim \pi_m^{-1}(0) &= \dim M(Q, \mathbf{d}) + \dim R(\overline{Q}, \mathbf{d}) \\ &= 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle + \mathbf{d} \cdot \mathbf{d} - \langle \mathbf{d}, \mathbf{d} \rangle) = 2 \dim X(Q, \mathbf{d}). \end{aligned}$$

For $m \geq 2$, we obtain:

$$\begin{aligned} \dim M(Q, \mathbf{d}) + \dim \pi_m^{-1}(0) &= \dim M(Q, \mathbf{d}) + \dim R(\overline{Q}, \mathbf{d}) + \dim X(Q, \mathbf{d})_{m-2}, \\ &\leq 2 \dim X(Q, \mathbf{d}) + (m-1) \cdot \dim X(Q, \mathbf{d}) \\ &= (m+1) \cdot \dim X(Q, \mathbf{d}). \end{aligned}$$

where the second inequality holds by induction on m and using the following inequalities at step $m-2$:

$$\begin{cases} \dim \pi_{m-2}^{-1}(X(Q, \mathbf{d})_{\text{sm}}) = (m-1) \cdot \dim X(Q, \mathbf{d}), \\ \dim \pi_{m-2}^{-1}(Z(Q, \mathbf{d}) \cap X(Q, \mathbf{d})_{\text{sg}}) < (m-1) \cdot \dim X(Q, \mathbf{d}), \\ \dim \pi_{m-2}^{-1}(X(Q, \mathbf{d})_{\text{sg}} \setminus Z(Q, \mathbf{d})) < (m-1) \cdot \dim X(Q, \mathbf{d}). \end{cases}$$

Therefore, at step m , we obtain the following inequalities:

$$\begin{cases} \dim \pi_m^{-1}(Z(Q, \mathbf{d}) \cap X(Q, \mathbf{d})_{\text{sg}}) < (m+1) \cdot \dim X(Q, \mathbf{d}), \\ \dim \pi_m^{-1}(X(Q, \mathbf{d})_{\text{sg}} \setminus Z(Q, \mathbf{d})) < (m+1) \cdot \dim X(Q, \mathbf{d}). \end{cases}$$

So we obtain $\dim \pi_m^{-1}(X(Q, \mathbf{d})_{\text{sg}}) < (m+1) \cdot \dim X(Q, \mathbf{d})$ for all $m \geq 1$, which proves, by Mustașă's criterion, that $X(Q, \mathbf{d})$ has rational singularities. $\square$

To complete the proof, we need to prove Lemma 3.0.3 and show that $X(Q, \mathbf{d})$ has rational singularities when $\mathbf{d} = \underline{1}$. The latter fact can easily be deduced from the results in Section 2.6.

Lemma 3.0.8 *If $(Q, \mathbf{d})$ has property (P) and $\mathbf{d} = \underline{1}$, then $X(Q, \mathbf{d})$ has rational singularities.*

Proof This follows from Corollary 2.6.6 since $\mathbf{d}$ has all entries equal to 1 and the graph underlying Q is 2-connected. We could also use [44, Thm. 2.11.]. $\square$

We now turn to the proof of Lemma 3.0.3. We first prove the following intermediate inequality:

Lemma 3.0.9 *Let $Q = S_g$ be the quiver with one vertex and $g \geq 2$ loops and $d \geq 2$. Let $(j_s, m_s, 1 \leq s \leq h)$ be a top type compatible with $\tau_{\min}$. Then:*

$$2 \cdot (1 - \langle d, d \rangle - g) - (d^2 - 1 + 1 - \langle d, d \rangle + m_2 m_1 + \dots m_h m_{h-1} - g \cdot (m_1^2 + \dots m_h^2)) \geq d - 1.$$

Proof Rearranging terms, the left-hand side reads:

$$\begin{aligned} & g \cdot \left(d^2 + \sum_s m_s^2 - 2 \right) - 2(d^2 - 1) - (m_2 m_1 + \dots m_h m_{h-1}) \\ & \geq 2 \cdot \left(d^2 + \sum_s m_s^2 - 2 \right) - 2(d^2 - 1) - (m_2 m_1 + \dots m_h m_{h-1}) \\ & \geq 2 \left(\sum_s m_s^2 - 1 \right) - (m_2 m_1 + \dots m_h m_{h-1}) \\ & \geq \sum_s m_s^2 - 1 + \frac{1}{2} \cdot (m_1^2 + m_h^2 + (m_1 - m_2)^2 + \dots + (m_{h-1} - m_h)^2 - 2) \\ & \geq d - 1. \end{aligned}$$

Note that both sides are equal when $g = 2$ and $m_1 = \dots = m_s = 1$. $\square$

Proof of Lemma 3.0.3 Without loss of generality, we assume that $\text{supp}(\mathbf{d}) = Q$; otherwise we restrict to $\text{supp}(\mathbf{d})$, which also has property (P). Let g_i be the number of loops of Q at vertex i , r_{ij} be the number of arrows between vertices $i \neq j$ and $(j_s, m_s, 1 \leq s \leq h)$ be a top type compatible with τ . Then, by Proposition 2.2.6, the inequality above holds if the following holds for arbitrary top-type:

$$\sum_i d_i^2 - 1 + 1 - \langle \mathbf{d}, \mathbf{d} \rangle + \sum_s m_s z_s - \sum_s m_s^2 g_s < 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \sum_i g_i).$$

We now want to split this inequality along vertices of Q . Let us first rearrange the indices $1 \leq s \leq h$ so that $j_1 = \dots = j_{s_1} = 1$, and so on until $j_{s_{r-1}+1} = \dots = j_{s_r} = r$. Then $z_{s_{i-1}+1} = 0$ and $z_{s_{i-1}+j+1} = m_{s_{i-1}+j}$ for $j > 0$, $1 \leq i \leq r$, with the convention that $s_0 = 0$. We obtain:

$$\sum_i d_i^2 - 1 + 1 - \langle \mathbf{d}, \mathbf{d} \rangle + \sum_i (m_{s_{i-1}+2} m_{s_{i-1}+1} + \dots m_{s_i} m_{s_{i-1}}) - \sum_i g_i \cdot (m_{s_{i-1}+1}^2 + \dots m_{s_i}^2) < 2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle - \sum_i g_i).$$

We know from Lemma 3.0.9 that for all $1 \leq i \leq r$:

$$\begin{aligned} & 2 \cdot (1 - \langle d_i \epsilon_i, d_i \epsilon_i \rangle - g_i) \\ & - \left(d_i^2 - 1 + 1 - \langle d_i \epsilon_i, d_i \epsilon_i \rangle + m_{s_{i-1}+2} m_{s_{i-1}+1} + \dots + m_{s_i} m_{s_{i-1}} - g_i \cdot (m_{s_{i-1}+1}^2 + \dots m_{s_i}^2) \right) \\ & \geq d_i - 1, \end{aligned}$$

where equality holds for top type ($m_s = 1$, $1 \leq s \leq h$). Taking the remaining terms in the inequality (right-hand side minus left-hand side) gives:

$$1 - \langle \mathbf{d}, \mathbf{d} \rangle - \sum_i (1 - \langle d_i \epsilon_i, d_i \epsilon_i \rangle) - (r - 1) = \sum_{i \neq j} r_{ij} d_i d_j - 2(r - 1).$$

Set r_1 (resp. r_2) the number of vertices $i \in Q_0$ such that $d_i = 1$ (resp. $d_i \geq 2$). Then $r = r_1 + r_2$ (recall that we assume that $\text{supp}(\mathbf{d}) = Q$). Then:

$$\begin{aligned} \sum_{i \neq j} r_{ij} d_i d_j - 2(r - 1) &\geq 4 \cdot \frac{r_2(r_2 - 1)}{2} + 2r_1 r_2 + \frac{r_1(r_1 - 1)}{2} - 2(r - 1) \\ &\geq 2(r_2 - 1)(r - 1) + \frac{r_1(r_1 - 1)}{2}. \end{aligned}$$

Summing everything, we obtain:

$$\begin{aligned} &2 \cdot (1 - \langle \mathbf{d}, \mathbf{d} \rangle) - \sum_i g_i) \\ &- \left(\sum_i d_i^2 - 1 + 1 - \langle \mathbf{d}, \mathbf{d} \rangle + \sum_i (m_{s_{i-1}+2} m_{s_{i-1}+1} + \dots m_{s_i} m_{s_i-1}) \right. \\ &\quad \left. - \sum_i g_i \cdot (m_{s_{i-1}+1}^2 + \dots m_{s_i}^2) \right) \\ &\geq \sum_i (d_i - 1) + 2(r_2 - 1)(r - 1) + \frac{r_1(r_1 - 1)}{2}. \end{aligned}$$

If $\mathbf{d} > \underline{1}$, the right-hand side is positive, so we get the desired inequality. $\square$

Theorem 3.0.1 now follows from Lemmas 3.0.3, 3.0.8 and Theorem 3.0.5.

Remark 3.0.10 Unfortunately, it may happen that $X(Q, \mathbf{d})$ has rational singularities, while the dimension bound from [13, Lem. 2.16.] fails. This is the case, for instance, with the following quivers: $\bullet \rightrightarrows \bullet$ and $\bullet \rightrightarrows \bullet \rightrightarrows \bullet$, when $\mathbf{d} = \underline{1}$ (both are 2-connected).

4 Applications to Moduli of Totally Negative 2-Calabi-Yau Categories

4.1 Rational Singularities

Let $\mathfrak{M} = [X/G]$ be a quotient stack coming from a 2-Calabi-Yau category $\mathcal{T}$ and whose closed points parametrize objects in an abelian subcategory $\mathcal{A} \subseteq \mathcal{T}$ (see Definition 2.5.4). As a consequence of Theorem 3.0.1, if all auxiliary quivers arising from $\mathfrak{M}$ have property (P), then X has l.c.i. and rational singularities. This leads us to the notion of a totally negative category, as introduced in [32]. We define it using Hom-spaces in triangulated categories, similarly to Definition 2.5.1, since in

the example we consider, Ext-quivers can be computed from such Hom-spaces (see Section 2.5).

Definition 4.1.1 Let $\mathcal{C}$ be a full subcategory of a $\mathbb{K}$ -linear, Hom-finite, triangulated category $\mathcal{T}$. Suppose that for all $F_1, F_2 \in \mathcal{T}$, $\text{Hom}(F_1, F_2[i]) = 0$ for all but finitely many $i \in \mathbb{Z}$. $\mathcal{C}$ is called totally negative if, for all $F_1, F_2 \in \mathcal{C}$:

$$\sum_{i \geq 0} (-1)^i \cdot \text{hom}(F_1, F_2[i]) < 0.$$

For example, when Π_Q or $\Lambda^q(Q)$ are 2-Calabi-Yau, the categories of Π_Q -modules or $\Lambda^q(Q)$ -modules are totally negative if, and only if, Q is totally negative (see Section 2.5). Here, we see the category $\mathcal{C}$ of Π_Q -modules (or $\Lambda^q(Q)$ -modules) as a subcategory of its bounded derived category $\mathcal{T}$, which is 2-Calabi-Yau in the sense of Definition 2.5.1.

Note that, in order to prove that X has l.c.i., rational singularities, we only need to show that the auxiliary quivers arising from X are totally negative. In other words, we only need to show that the subcategory $\mathcal{C} \subseteq \mathcal{A}$ generated by simple summands occurring at closed points of $\mathfrak{M}$ is totally negative. We also require an additional property on the subset of simple objects so that auxiliary quivers also satisfy property (P).

Theorem 4.1.2 Let $\mathfrak{M} = [X/G]$ be a quotient stack coming from a 2-Calabi-Yau category and $M := X // G$. Assume that:

1. For any closed point $x \in \mathfrak{M}(\mathbb{K})$, the Ext-quiver associated to x is the double of a totally negative quiver;
2. Simple objects form a dense subset of M .

Then X is locally complete intersection and has rational singularities.

Proof Let $x \in X(\mathbb{K})$ be a point with closed orbit and $(\overline{Q'}, \mathbf{e})$ the associated Ext-quiver. Then by Theorem 2.5.5, there exists an affine $\text{GL}(\mathbf{e})$ -variety W , with a fixed point w and a commutative diagram:

$$\begin{array}{ccccc} (\mu_{Q', \mathbf{e}}^{-1}(0) \times^{\text{GL}(\mathbf{e})} G, [0, \text{Id}]) & \longleftarrow & (W \times^{\text{GL}(\mathbf{e})} G, [w, \text{Id}]) & \longrightarrow & (X, x) \\ \downarrow & & \downarrow & & \downarrow \\ (M_{\Pi_{Q', \mathbf{e}}}, 0) & \longleftarrow & (W // \text{GL}(\mathbf{e}), w) & \longrightarrow & (M, x) \end{array}$$

such that the horizontal maps are étale and the squares are cartesian. Since the set of simple objects is dense in M , one can find a simple object in the image of $(W // \text{GL}(\mathbf{e}), w) \rightarrow (M, x)$. Moreover, strongly étale morphisms preserve stabilizers (see [13, Lem. 2.6.]), so we can find a semisimple point in $\mu_{Q', \mathbf{e}}^{-1}(0)$ whose stabilizer is $\mathbb{K}^\times$ i.e. there is a simple $\Pi_{Q'}$ -module of dimension $\mathbf{e}$. From this and assumption 1., we deduce that Q' satisfies property (P).

Therefore, by Theorem 3.0.1 and Lemmas 2.3.2, 2.3.5, any semisimple point of X has a G -saturated neighborhood which is locally complete intersection and has rational singularities. Since X is covered by such neighborhoods, the conclusion follows. $\square$

We now give a few examples and apply our result to moduli of coherent sheaves on K3 surfaces and representations of multiplicative preprojective algebras.

$\Lambda^q(Q)$ -modules

In the case of the multiplicative preprojective algebra $\Lambda^q(Q)$, the auxiliary quivers are computed as in the additive case from Q itself. Therefore, if Q is totally negative, then all auxiliary quivers are. The variety $R(\Lambda^q(Q), \mathbf{d})$ is also known to be an affine complete intersection, by [27, Thm. 1.11.]. Thus, we obtain:

Corollary 4.1.3 *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus 0$ such that $(Q, \mathbf{d})$ has property (P). Then $R(\Lambda^q(Q), \mathbf{d})$ is a complete intersection and has rational singularities.*

Proof We apply Theorem 4.1.2. It remains to check assumption 2. Since $(Q, \mathbf{d})$ has property (P), there exists a simple Π_Q -module of dimension $\mathbf{d}$. Thus by [24, Thm. 1.2.] and [27, Thm. 1.11.], simple points form an open dense subset of $R(\Lambda^q(Q), \mathbf{d})$, hence a dense subset of $M_{\Lambda^q(Q), \mathbf{d}}$. $\square$

Semistable Coherent Sheaves on a K3 Surface

In the case of sheaves on a K3 surface, controlling Ext-quivers of all polystable sheaves with given Mukai vector is more delicate. In [14], it was already mentioned that for many polystable sheaves, one obtains the auxiliary quivers of the g -loop quiver. We spell out conditions for this to be true. However, one can also obtain étale-local models which are much more difficult to study. We illustrate this with the case of one dimensional sheaves on an elliptic K3 surface.

Let (S, H) be a complex, projective, polarized K3 surface. Given a polystable sheaf $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{F}_i^{\oplus e_i}$, with distinct stable summands $\mathcal{F}_i$ of Mukai vectors $\mathbf{v}_i$, recall that the Ext-quiver Q' has the following number of arrows from i to j :

$$\text{ext}^1(\mathcal{F}_i, \mathcal{F}_j) = \begin{cases} 2 + \mathbf{v}_i \cdot \mathbf{v}_i, & \text{if } i = j, \\ \mathbf{v}_i \cdot \mathbf{v}_j, & \text{if } i \neq j. \end{cases}$$

Q' is totally negative if, and only if, $\mathbf{v}_i \cdot \mathbf{v}_j > 0$ for all i, j . To see this, one can simply check when Q' has at least two loops at each vertex and one arrow joining any pair of vertices (recall from Section 2.5 that $\mathbf{v}_i \cdot \mathbf{v}_i$ is even). Otherwise, one can also notice that:

$$(\mathbf{d}, \mathbf{e})_{Q'} = - \left(\sum_i d_i \mathbf{v}_i \right) \cdot \left(\sum_i e_i \mathbf{v}_i \right).$$

We now give examples of auxiliary quivers arising from $\mathfrak{M}_{S,H}(\mathbf{v})$. We rely on criteria for the existence of (semi)stable sheaves of a given Mukai vector, due to Yoshioka [72, Thm. 8.1.] (see also [48, §2.4.]).

Definition 4.1.4 A Mukai vector $\mathbf{v}$ is called primitive if it cannot be written $\mathbf{v} = m\mathbf{v}_0$, for some other Mukai vector $\mathbf{v}_0$ and $m \geq 2$.

A primitive vector $\mathbf{v} = (r, \mathbf{c}, a)$ is called positive if $\mathbf{v} \cdot \mathbf{v} \geq -2$ and one of the following holds:

- $r > 0$ and $\mathbf{c} \in \text{NS}(S)$,
- $r = 0$, $\mathbf{c} \in \text{NS}(S)$ is effective and $a \neq 0$,
- $r = 0$, $\mathbf{c} = 0$ and $a > 0$.

Let $\text{Amp}(S)$ be the space of polarizations of S and $\mathbf{v}$ a primitive, positive Mukai vector. One can associate walls in $\text{Amp}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ to $\mathbf{v}$ i.e. subspaces of real codimension one, such that, for H lying outside these walls, there are no strictly semistable sheaves with Mukai vector $\mathbf{v}$ (see [72, §1.4.] for $\text{rk}(\mathbf{v}) = 0$ and [46, Ch. 4.C.] for $\text{rk}(\mathbf{v}) > 0$). Those polarizations are called $\mathbf{v}$ -generic. The existence results for semistable sheaves are summed up in the following:

Proposition 4.1.5 [4, Thm. 2.2-4.] *If $\mathbf{v}$ is a primitive, positive Mukai vector, then $M(\mathbf{v}) \neq \emptyset$. If moreover H is $\mathbf{v}$ -generic, then $M(\mathbf{v}) = M^s(\mathbf{v})$.*

In general, there is no reason for auxiliary quivers of all polystable sheaves in $M(\mathbf{v})$ to be totally negative. However, if we choose a moduli space $M(\mathbf{v})$ of high enough dimension and a generic polarization, then the auxiliary quiver of any polystable sheaf in $M(\mathbf{v})$ is totally negative (and it arises from a g -loop quiver). This was already observed in [14].

Proposition 4.1.6 *Let $\mathbf{v} = m\mathbf{v}_0$, where $\mathbf{v}_0$ is a primitive, positive Mukai vector and assume that H is $\mathbf{v}_0$ -generic. Suppose also that $\mathbf{v} \cdot \mathbf{v} > 0$. Then for any polystable sheaf $\mathcal{F} = \bigoplus_i \mathcal{F}_i^{\oplus e_i}$ with Mukai vector $\mathbf{v}$, the auxiliary quiver of $\mathcal{F}$ is an auxiliary quiver of some g -loop quiver, where $g \geq 2$.*

Proof Call $\mathbf{v}_i$ the Mukai vector of $\mathcal{F}_i$. We leave out the obvious case where $\mathcal{F}$ is stable. Since H is $\mathbf{v}_0$ -generic and by construction of walls, there exist $r_i \in \mathbb{Q}$ such that $\mathbf{v}_i = r_i \mathbf{v}$ (see [48, §2.4.]). Since $\mathbf{v}_0$ is primitive, $\mathbf{v}_i = m_i \mathbf{v}_0$ for some $m_i \in \mathbb{Z}$. Moreover, $\mathbf{v}_0 \cdot \mathbf{v}_0$ is a positive even integer, hence $\mathbf{v}_0 \cdot \mathbf{v}_0 = 2g - 2$, with $g \geq 2$. Consequently, the auxiliary quiver of $\mathcal{F}$ has $1 + m_i^2(g - 1)$ loops at each vertex and $2m_i m_j(g - 1)$ arrows joining vertices $i \neq j$. One recognizes the auxiliary quiver of the g -loop quiver for the semisimple type $(m_i, e_i; 1 \leq i \leq r)$ (see [13, Prop. 2.26.]). $\square$

In that case, [13, Thm. 1.1.] yields the following corollary, already observed in [14]:

Corollary 4.1.7 *Under the hypotheses of Proposition 4.1.6, $U_{\mathbf{v}, m_{\mathbf{v}}}^{\text{ss}}$ is locally complete intersection and has rational singularities.*

In [4, §6.], Arbarello and Saccá analyzed Ext-quivers of a pure, one-dimensional torsion sheaf $\mathcal{F}$ with primitive, positive Mukai vector $\mathbf{v} = (0, [\mathbf{D}], \chi)$ (D effective, $\chi \neq 0$). Write $D = \sum_{i \in I} n_i D_i$ where $n_i \geq 0$ and D_i is integral. Then the auxiliary quiver of $\mathcal{F}$ has vertex set I , $g(D_i)$ loops at vertex i and $D_i \cdot D_j$ arrows joining vertices i and j . The following example illustrates how difficult the geometry of étale-local models of $\mathfrak{M}_{S, H}(\mathbf{v})$ gets.

Example 4.1.8 Let $\mathbf{v} = (0, [\mathbf{D}], 1)$, where D is an elliptic curve in S (for instance, suppose S is an elliptic K3 surface and D is a generic fibre). Then $\mathbf{v}$ is primitive, positive, $\mathbf{v} \cdot \mathbf{v} = 2g(D) - 2 = 0$ and taking a $\mathbf{v}$ -generic polarization, we obtain by Proposition 4.1.5 a stable sheaf $\mathcal{F}$ with Mukai vector $\mathbf{v}$. For any $n \geq 1$, the auxiliary quiver of $\mathcal{F}^{\oplus n}$ is the Jordan quiver (one vertex with one loop), with dimension vector n . The associated local model is the commuting scheme:

$$C_2 := V([x, y]_{i,j}, 1 \leq i, j \leq n) \subset \mathrm{Mat}(n, \mathbb{K})_x \times \mathrm{Mat}(n, \mathbb{K})_y$$

Commuting schemes have been studied by many authors and were only recently proved to be normal and Cohen-Macaulay (see [18] and the references therein). Whether commuting schemes have rational singularities is a more difficult question and, as far as we know, this problem is still open.

4.2 Counts of Jets and p-adic Integrals

In this section, we go back to our initial arithmetic problem on counts of jets. We deduce two consequences of Theorem 4.1. First, we give a partial answer to our original question on counts of quiver representations over $\mathbb{F}_q[t]/(t^n)$. We then consider a general quotient stack $[X/G]$ coming from a totally negative 2-Calabi-Yau category and show that the counts of jets on X converge to the p-adic volume of a certain analytic manifold associated to X , following [2] and [16, 71].

Convergence for Counts of Jets Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. If $(Q, \mathbf{d})$ has property (P), we already know from Proposition 2.2.3 that $\mu_{Q, \mathbf{d}}^{-1}(0)$ is a complete intersection. Thus we obtain from Proposition 2.6.3:

Corollary 4.2.1 *Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ such that $(Q, \mathbf{d})$ has property (P). Then for almost all primes p and for all finite fields $\mathbb{F}_q$ of characteristic p , $q^{-n \dim \mu_{Q, \mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q, \mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))$ converges when n goes to infinity.*

Now that we have established the existence of $\lim_{n \rightarrow +\infty} q^{-n \dim \mu_{Q, \mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q, \mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))$ for all totally negative quivers, it is natural to ask whether this limit is uniform over all finite fields $\mathbb{F}_q$, as shown by Wyss in the case $\mathbf{d} = \underline{1}$.

Conjecture 4.2.2 Let Q be a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0} \setminus \{0\}$ such that $(Q, \mathbf{d})$ has property (P). There exists a rational fraction $W \in \mathbb{Q}(T)$ such that, for almost all primes p and for all finite fields $\mathbb{F}_q$ of characteristic p :

$$W(q) = \lim_{n \rightarrow +\infty} q^{-n \dim \mu_{Q, \mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q, \mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n)).$$

Counts of Jets and p-adic Volumes of Mildly Singular Schemes Another natural question is whether the quantity $\lim_{n \rightarrow +\infty} q^{-n \dim \mu_{Q, \mathbf{d}}^{-1}(0)} \cdot \sharp \mu_{Q, \mathbf{d}}^{-1}(0)(\mathbb{F}_q[t]/(t^n))$ has a geometric interpretation. We address this by showing that the limit can be interpreted

as a p -adic integral, using earlier work of Aizenbud-Avni [2]. Our result holds for a wider class of schemes X , including atlases of quotient stacks $[X/G]$ parametrizing objects of totally negative 2-Calabi-Yau categories. We then identify this p -adic integral to the canonical volume of X built, for instance, in [16, 71].

Consider F some finite extension of $\mathbb{Q}_p$, with residue field $\mathbb{F}_q$ and valuation ring $\mathcal{O} \subset F$. Given a scheme X which is Gorenstein, of pure dimension d over $\mathcal{O}$, one can construct a canonical measure μ_{can} on $X^{\natural} := X^{\text{sm}}(F) \cap X(\mathcal{O})$. This canonical measure is built by glueing measures obtained from gauge forms (i.e. nowhere vanishing differential forms of top degree) on X^{sm} (the smooth locus of X relative to $\mathcal{O}$) - see [71, §4] or [16, §3]. Let us briefly recall how this construction works.

Suppose that the structure morphism $X \rightarrow \text{Spec}(\mathcal{O})$ is Gorenstein, of pure dimension d (see [67, Tag 0C02]). Then X admits a canonical invertible sheaf $\Omega_{X/\mathcal{O}}$, which restricts to $\Omega_{X^{\text{sm}}/\mathcal{O}}^d$ on the smooth locus X^{sm} (see [21, §3.5]). Consider trivializing open subsets $V_k \subseteq X$ for $\Omega_{X/\mathcal{O}}$ and non-vanishing sections $\omega_k \in \Gamma(V_k, \Omega_{X/\mathcal{O}})$. Note that $X^{\natural} = \bigcup_k V_k^{\natural}$, since for $x \in X(\mathcal{O})$, if $x_{\mathbb{F}_q} \in V_k(\mathbb{F}_q)$, then $x \in V_k(\mathcal{O})$. The sections ω_k induce measures ν_k on $V_k^{\natural}$ - see [2, §3.1.] or [16, §3] for the construction of these measures. Moreover, for $x \in V_k^{\natural} \cap V_l^{\natural}$, $\frac{\omega_k}{\omega_l}(x) \in \mathcal{O}^{\times}$. The change of variables formula for p -adic integrals then implies that the measures ν_k glue to a measure μ_{can} which does not depend on the choices of ω_k .

Given a $\mathbb{Z}$ -scheme X such that $X_{\mathbb{Q}}$ has l.c.i. and rational singularities, we show that the counts of jets we consider converge to $\mu_{\text{can}}(X^{\natural})$, when the residual characteristic of F is large enough. Let us first state an intermediate result for appropriate $\mathcal{O}$ -schemes:

Lemma 4.2.3 *Let X be a finite type $\mathcal{O}$ -scheme. Assume that X is flat over $\mathcal{O}$, with l.c.i. geometric fibers of pure dimension d . Assume also that $X_{\bar{F}}$ has rational singularities.*

Then there is a well-defined canonical measure μ_{can} on $X^{\natural} = X^{\text{sm}}(F) \cap X(\mathcal{O})$. Moreover, the sequence $q^{-nd} \cdot \sharp X(\mathcal{O}/(\varpi^n))$, $n \geq 1$ converges and its limit is given by:

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathcal{O}/(\varpi^n))}{q^{nd}} = \mu_{\text{can}}(X^{\natural}).$$

Proof Since $X \rightarrow \text{Spec}(\mathcal{O})$ is flat, with l.c.i. geometric fibers, it is a Gorenstein morphism (of pure dimension d , by assumption - see [67, Tag 0C02]). Then the construction above applies and $X^{\natural}$ can be endowed with a canonical measure μ_{can} .

We now describe X locally as a fiber of an appropriate morphism $\varphi : \mathbb{A}_{\mathcal{O}}^r \rightarrow \mathbb{A}_{\mathcal{O}}^m$. On the one hand, this allows us to rewrite counts of $\mathcal{O}/(\varpi^n)$ -points as measures of p -adic balls with respect to the pushforward by φ of the Haar measure on $\mathcal{O}^r$, following [70, §4.2.]. On the other hand, if φ is a FRS morphism (see [2, Def. II.]), we can apply a Fubini theorem by Aizenbud and Avni [2, Thm. 3.16.] to compute this pushforward measure and relate it to the canonical measure on X .

As $X \rightarrow \text{Spec}(\mathcal{O})$ is flat, with l.c.i. geometric fibers, we can cover X with affine open subsets of the form:

$$U_i = \text{Spec}(\mathcal{O}[T_1, \dots, T_{r_i}]/(f_{1,i}, \dots, f_{m,i})),$$

whose nonempty fibers over $\mathcal{O}$ have dimension $d = r_i - m_i$ (see [67, Tag 01UB]). Observe that U_i is of pure dimension d over $\mathcal{O}$ if, and only if, $(U_i)_{\mathbb{F}_q} \neq \emptyset$, as U_i is flat over $\mathcal{O}$. Moreover, $X^\natural = \bigcup_i U_i^\natural$ as noted above, and $\sharp X(\mathcal{O}/(\varpi^n))$ can be obtained from $\sharp U_i(\mathcal{O}/(\varpi^n))$ by inclusion-exclusion (see [1, Lem. 3.1.1.]). Therefore, the U_i whose contribution to $\mu_{\text{can}}(X^\natural)$ and $\sharp X(\mathcal{O}/(\varpi^n))$ is non-zero are of pure dimension d over $\mathcal{O}$ and we may restrict to those for the rest of the proof.

Let U_i be such an open subset. We temporarily drop the subscript i to ease notations. Consider $\varphi := (f_1, \dots, f_m) : \mathbb{A}_{\mathcal{O}}^r \rightarrow \mathbb{A}_{\mathcal{O}}^m$. The locus $M = \{x \in \mathbb{A}_{\mathcal{O}}^r \mid \dim_x(\varphi^{-1}(\varphi(x))) \leq d\}$ is open, by Chevalley's semicontinuity theorem [40, Thm. 13.1.3.], and contains $U = \varphi^{-1}(0)$. Moreover, $\dim_x(\varphi^{-1}(\varphi(x))) \geq r - m = d$ by construction, so $\dim_x(\varphi^{-1}(\varphi(x))) = d$ for all $x \in M$. Therefore, $\varphi|_M$ is flat (see [39, Prop. 6.1.5.]) and $N := \varphi(M) \subseteq \mathbb{A}_{\mathcal{O}}^m$ is open. We have thus obtained a morphism $\varphi : M \rightarrow N$ which is flat, with l.c.i. geometric fibers of pure dimension d , and such that $U = \varphi^{-1}(0)$.

Let us now relate counts of $\mathcal{O}/(\varpi^n)$ -points on U to the pushforward by φ of the Haar measure on $\mathcal{O}^r$. Let $\omega_M \in \Gamma(M, \Omega_{M/\mathcal{O}}^r)$ (resp. $\omega_N \in \Gamma(N, \Omega_{N/\mathcal{O}}^m)$) be a gauge form and μ_M (resp. μ_N) the associated measure on $M^\natural$ (resp. $N^\natural$). We also call $\varphi : M^\natural \rightarrow N^\natural$ the map of F -analytic manifolds induced by φ . Then $\varpi^n \mathcal{O}^m \subseteq N^\natural$ and

$$\frac{\sharp U(\mathcal{O}/(\varpi^n))}{q^{nd}} = \sharp \{x \in M(\mathcal{O}/(\varpi^n)) \mid \varphi(x) = 0 \in (\mathcal{O}/(\varpi^n))^m\} \cdot \frac{q^{-nr}}{q^{-nm}} = \frac{\varphi_* \mu_M(\varpi^n \mathcal{O}^m)}{\mu_N(\varpi^n \mathcal{O}^m)}.$$

As shown in [2, §3.3.], the measure $\varphi_*(\mu_M)$ is absolutely continuous with respect to μ_N and its density at $y \in N^\natural$ is given by a p -adic integral on $\varphi^{-1}(y)^\natural$. We now build explicit gauge forms inducing the corresponding measures on $\varphi^{-1}(y)^\natural$. As before, φ is a Gorenstein morphism of pure dimension d , so there exists a canonical invertible sheaf $\Omega_{M/N}$, which restricts to $\Omega_{M^{\text{sm}}/N}^d$ on the smooth locus M^{sm} of φ . Moreover, there is an isomorphism of invertible sheaves $\Omega_{M/\mathcal{O}}^r \simeq \Omega_{M/N} \otimes \varphi^* \Omega_{N/\mathcal{O}}^m$, by [21, Thm. 4.3.3.]. From this we obtain a nowhere-vanishing section $\eta \in \Gamma(M, \Omega_{M/N})$ such that $\omega_M = \eta \otimes \varphi^* \omega_N$ and which restricts to a nowhere-vanishing section $\eta_y \in \Gamma(\varphi^{-1}(y), \Omega_{\varphi^{-1}(y)/\mathcal{O}})$ for any $y \in N^\natural$, by [21, Thm. 3.6.1.]. Let us call μ_y the measure induced by the gauge form $(\eta_y)|_{\varphi^{-1}(y)^{\text{sm}}}$ on $\varphi^{-1}(y)^\natural$. When $y = 0$, we simply write $\eta := \eta_0$ and $\mu := \mu_0$ for the induced measure on $U^\natural = \varphi^{-1}(0)^\natural$. By construction, μ is the restriction of μ_{can} to $U^\natural$.

The next step is to apply [2, Thm. 3.16.]. We first need to restrict μ_M to a locus where φ_F is FRS. Since φ_F is flat, the locus where geometric fibers of φ_F have rational singularities is open - see [36, Thm. 4] - and contains U_F . Denote by $Z_F \subseteq M_F$ the complement of this locus. Then $Z_F(F) \cap U(F) = \emptyset$. Since $M^\natural$ and $N^\natural$ are compact, φ is closed and there exists some $n_0 \geq 0$ such that $Z_F(F) \cap \varphi^{-1}(\varpi^{n_0} \mathcal{O}^m) = \emptyset$. Consequently, φ_F is FRS on $M_F \setminus Z_F$ and $A := \varphi^{-1}(\varpi^{n_0} \mathcal{O}^m)$ is a compact subset of $(M_F \setminus Z_F)(F)$. By [2, Thm. 3.16.], the density of $\varphi_*((\mu_M)|_A)$ with respect to μ_N is given by the function $y \mapsto \mathbf{1}_{\varpi^{n_0} \mathcal{O}^m}(y) \cdot \mu_y(\varphi^{-1}(y)^\natural)$, which is continuous. Therefore,

for $n \geq n_0$:

$$\begin{aligned} \frac{\sharp U(\mathcal{O}/(\varpi^n))}{q^{nd}} &= \frac{\varphi_* \mu_M(\varpi^n \mathcal{O}^m)}{\mu_N(\varpi^n \mathcal{O}^m)} = \frac{\varphi_* ((\mu_M)|_A)(\varpi^n \mathcal{O}^m)}{\mu_N(\varpi^n \mathcal{O}^m)} \\ &= \frac{\int_{\varpi^n \mathcal{O}^m} \mu_y(\varphi^{-1}(y)^{\natural}) d\mu_N}{\int_{\varpi^n \mathcal{O}^m} d\mu_N} \xrightarrow{n \rightarrow +\infty} \mu_0(\varphi^{-1}(0)^{\natural}) = \mu(U^{\natural}). \end{aligned}$$

Finally, we compute $\mu_{\text{can}}(X^{\natural})$ from $\mu_i(U_i^{\natural})$ by inclusion-exclusion. Counts of $\mathcal{O}/(\varpi^n)$ -points on X can also be computed by inclusion-exclusion, as mentioned above. Adding up the contributions of all U_i , we obtain:

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathcal{O}/(\varpi^n))}{q^{nd}} = \mu_{\text{can}}(X^{\natural}).$$

□

We are now in a position to prove our main technical result for a large class of $\mathbb{Z}$ -schemes of finite type. We show how to relate the counts $\sharp X(\mathbb{F}_q[t]/(t^n))$ to the canonical measure on $X^{\natural} = X^{\text{sm}}(F) \cap X(\mathcal{O})$ when the residual characteristic is large enough.

Theorem 4.2.4 *Let X be a $\mathbb{Z}$ -scheme of finite type and assume that $X_{\mathbb{Q}}$ is locally complete intersection, of pure dimension d and has rational singularities. Then for p large enough, the $\mathcal{O}$ -scheme $X_{\mathcal{O}}$ satisfies the assumptions of Lemma 4.2.3. Moreover, the sequence $q^{-nd} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ converges and its limit is given by:*

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathbb{F}_q[t]/(t^n))}{q^{nd}} = \mu_{\text{can}}(X^{\natural}).$$

Proof Assume that $X_{\mathcal{O}}$ satisfies the hypotheses of Lemma 4.2.3. From [1, Prop. 3.0.2.], we obtain that, for p large enough:

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathbb{F}_q[t]/(t^n))}{q^{nd}} = \lim_{n \rightarrow +\infty} \frac{\sharp X(\mathcal{O}/(\varpi^n))}{q^{nd}} = \mu_{\text{can}}(X^{\natural}).$$

Let us now check that for p large enough: (i) X is flat over $\mathcal{O}$, with l.c.i. geometric fibers of pure dimension d and (ii) $X_{\bar{F}}$ has rational singularities. (ii) follows straightforwardly from the assumption on $X_{\mathbb{Q}}$ by base change, so let us check (i). By generic flatness, $X_{\mathbb{Z}[\frac{1}{N}]}$ is flat over $\mathbb{Z}[\frac{1}{N}]$ for some $N \geq 1$. Thus by [67, Tag 01UE, Tag 01UF] and [40, Thm. 13.1.3.], the locus $U \subseteq X_{\mathbb{Z}[\frac{1}{N}]}$ where the structure morphism $X_{\mathbb{Z}[\frac{1}{N}]} \rightarrow \text{Spec}(\mathbb{Z}[\frac{1}{N}])$ has l.c.i. geometric fibers of pure dimension d is open and contains $X_{\mathbb{Q}}$, by assumption. By [40, Cor. 9.5.2.], we obtain that $\{s \in \text{Spec}(\mathbb{Z}) \mid U_s = X_s\} \subseteq \text{Spec}(\mathbb{Z})$ is open i.e. for N large enough, the structure morphism is flat, with l.c.i. geometric fibers of pure dimension d . So the same holds for $X_{\mathcal{O}}$ for $p \geq N$.

Remark 4.2.5 Note that a similar result holds for a scheme X defined over the ring of integers of some number field, with the same proof.

Application to Moduli of Totally Negative 2-Calabi-Yau Categories Let us now consider a quotient stack $[X/G]$ coming from a 2-Calabi-Yau category. If $[X/G]$ is locally modelled on moment maps of totally negative quivers, then Theorem 4.2.4 applies and we obtain the desired interpretation of limits of jet counts for a large class of moduli spaces.

Theorem 4.2.6 *Suppose that X is of finite type, of pure dimension d and defined over $\mathbb{Q}$. Assume also that $[X/G]$ satisfies the hypotheses of Theorem 4.1.2.*

Then for p large enough, the canonical measure μ_{can} on $X^{\natural} = X^{\text{sm}}(F) \cap X(\mathcal{O})$ is well-defined. Moreover, the sequence $q^{-nd} \cdot \sharp X(\mathbb{F}_q[t]/(t^n))$, $n \geq 1$ converges and its limit is given by:

$$\lim_{n \rightarrow +\infty} \frac{\sharp X(\mathbb{F}_q[t]/(t^n))}{q^{nd}} = \mu_{\text{can}}(X^{\natural}).$$

Note that the assumption that X is of pure dimension d may be dropped. Since X is locally complete intersection, its connected components are equidimensional and the counts of jets may be carried out separately on each connected component with the appropriate value of d .

Remark 4.2.7 When G is connected, counts of jets on X are equivalent to counts of jets on the moduli stack $[X/G]$, weighted by their number of automorphisms:

$$\sum_{z \in [X/G](\mathbb{F}_q[t]/(t^n)) / \sim} \frac{1}{\sharp \text{Aut}(z)} = \frac{\sharp X(\mathbb{F}_q[t]/(t^n))}{\sharp G(\mathbb{F}_q[t]/(t^n))}.$$

Indeed, the datum of a principal G -bundle $P \rightarrow \text{Spec}(\mathbb{F}_q[t]/(t^n))$ and a G -equivariant morphism $P \rightarrow X$ is equivalent to the datum of an orbit of $X(\mathbb{F}_q[t]/(t^n))$ under the action of $G(\mathbb{F}_q[t]/(t^n))$. This is due to the fact that the restricted morphism $P_{\mathbb{F}_q} \rightarrow \text{Spec}(\mathbb{F}_q)$ has a section, by Lang's theorem. By Hensel's lemma, this section extends to $\text{Spec}(\mathbb{F}_q[t]/(t^n))$ and so $P \simeq G \times \text{Spec}(\mathbb{F}_q[t]/(t^n))$. As a consequence, when $[X/G]$ satisfies the assumptions of Theorem 4.1.2, the weighted count of jets on $[X/G]$, once normalized, converges to $\frac{\mu_{\text{can}}(X^{\natural})}{\mu_{\text{can}}(G^{\natural})}$ as n goes to infinity.

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References

1. Aizenbud, A., Avni, N.: Counting points of schemes over finite rings and counting representations of arithmetic lattices. *Duke Math. J.* **167**(14), 2721–2743 (2018). <https://doi.org/10.1215/00127094-2018-0021>
2. Aizenbud, A., Avni, N.: Representation growth and rational singularities of the moduli space of local systems. *Invent. Math.* **204**(1), 245–316 (2016). <https://doi.org/10.1007/s00222-015-0614-8>
3. Alper, J., Hall, J., Rydh, D.: A Luna étale slice theorem for algebraic stacks. *Ann. Math.* **191**(3), 675–738 (2020). <https://doi.org/10.4007/annals.2020.191.3.1>
4. Arbarello, E., Saccà, G.: Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties. *Adv. Math.* **329**, 649–703 (2018). <https://doi.org/10.1016/j.aim.2018.02.003>
5. Beauville, A.: Symplectic singularities. *Invent. Math.* **139**(3), 541–549 (2000). <https://doi.org/10.1007/s00229900043>
6. Bellamy, G., Schedler, T.: Symplectic resolutions of quiver varieties. *Selecta Math.* **27**(36) (2021). <https://doi.org/10.1007/s00029-021-00647-0>
7. van den Bergh, M.: Calabi-Yau algebras and superpotentials. *Selecta Math. (N.S.)* (2015). <https://doi.org/10.1007/s00029-014-0166-6>
8. Bocklandt, R., Galluzzi, F., Vaccarino, F.: The Nori-Hilbert scheme is not smooth for 2-Calabi-Yau algebras. *J. Noncommut. Geom.* **10**, 745–774 (2016). <https://doi.org/10.4171/JNCG/247>
9. Boutot, J.-F.: Singularités rationnelles et quotients par les groupes réductifs. *Invent. Math.* **88**(1), 65–68 (1987). <https://doi.org/10.1007/BF01405091>
10. Bozec, T., Schiffmann, O., Vasserot, E.: On the number of points of nilpotent quiver varieties over finite fields. *Annales Scientifiques de l'École Normale Supérieure* **53**(6), 1501–1544 (2020). <https://doi.org/10.24033/asens.2452>
11. Brav, C., Dyckerhoff, T.: Relative Calabi-Yau structures. *Compos. Math.* **155**, 372–412 (2019). <https://doi.org/10.1112/S0010437X19007024>
12. Brion, M.: Some structure theorems for algebraic groups. In: *Algebraic Groups: Structures and Actions*. Ed. by Can, M.B. vol. 94, pp. 53–126 (2017). <https://doi.org/10.1090/pspum/094>
13. Budur, N.: Rational singularities, quiver moment maps and representations of surface groups. *Int. Math. Res. Notices* **2021**(15), 11782–11817 (2021). <https://doi.org/10.1093/imrn/rnz236>
14. Budur, N., Zhang, Z.: Formality conjecture for K3 surfaces. *Compos. Math.* **155**(5), 902–911 (2019). <https://doi.org/10.1112/S0010437X19007206>
15. Budur, N., Zordan, M.: On representation zeta functions for special linear groups. *Int. Math. Res. Notices* **2020**(3), 868–882 (2020). <https://doi.org/10.1093/imrn/rny057>
16. Carocci, F., Orecchia, G., Wyss, D.: BPS-invariants from p-adic integrals. *Compos. Math.* **160**(7), 1525–1550 (2024). <https://doi.org/10.1112/S0010437X24007176>
17. Chambert-Loir, A., Nicaise, J., Sebag J.: *Motivic integration*. Springer (2018). <https://doi.org/10.1007/978-1-4939-7887-8>

18. Charbonnel, J.-Y.: Projective dimension and commuting variety of a reductive Lie algebra. In: [arXiv:2006.12942](https://arxiv.org/abs/2006.12942) (2020)
19. Cluckers, R., Loeser, F.: Ax-Kochen-Eršov theorems for p-adic integrals and motivic integration. In: Geometric methods in algebra and number theory. Springer (2005). https://doi.org/10.1007/0-8176-4417-2_5
20. Cluckers, R., Loeser, F.: Constructible exponential functions, motivic Fourier transform and transfer principle. *Ann. Math.* **171**(2), 1011–1065 (2010). <https://doi.org/10.4007/annals.2010.171.1011>
21. Conrad, B.: Grothendieck duality and base change. Springer (2000). <https://doi.org/10.1007/b75857>
22. Crawley-Boevey, W.: Decomposition of Marsden-Weinstein reductions for representations of quivers. In: *Compos. Math.* (2002). <https://doi.org/10.1023/A:1013793632709>.
23. Crawley-Boevey, W.: DMV lectures on representations of quivers, preprojective algebras and deformations of quotient singularities. Available at <https://www.math.uni-bielefeld.de/~wcrawley/dmvlecs.pdf> (1999)
24. Crawley-Boevey, W.: Geometry of the moment map for representations of quivers. *Compos. Math.* **126**(3), 257–293 (2001). <https://doi.org/10.1023/A:1017558904030>
25. Crawley-Boevey, W.: Normality of Marsden-Weinstein reductions for representations of quivers. *Math. Ann.* **325**(1), 55–79 (2003). <https://doi.org/10.1007/s00208-002-0367-8>
26. Crawley-Boevey, W., van den Bergh, M.: Absolutely indecomposable representations and Kac-Moody Lie algebras. *Invent. Math.* **155**(3), 537–559 (2004). <https://doi.org/10.1007/s00222-003-0329-0>
27. Crawley-Boevey, W., Shaw, P.: Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem. *Adv. Math.* **201**, 180–208 (2006). <https://doi.org/10.1016/j.aim.2005.02.003>
28. Davison, B.: A boson-fermion correspondence in cohomological Donaldson-Thomas theory. *Glasgow Math. J.* **65**, S28–S52 (2023). <https://doi.org/10.1017/S001708952200009X>
29. Davison, B.: Purity and 2-Calabi-Yau categories. In: [arXiv:2106.07692](https://arxiv.org/abs/2106.07692) (2021)
30. Davison, B.: Purity of critical cohomology and Kac’s conjecture. *Math. Res. Lett.* **25**(2), 469–488 (2018). <https://doi.org/10.4310/MRL.2018.v25.n2.a6>
31. Davison, B.: The integrality conjecture and the cohomology of preprojective stacks. *J für die reine und angewandte Mathematik* **804**, 105–154 (2023). <https://doi.org/10.1515/crelle-2023-0065>
32. Davison, B., Hennecart, L., Schlegel-Mejia, S.: BPS Lie algebras for totally negative 2-Calabi-Yau categories and nonabelian Hodge theory for stacks. In: [arXiv:2212.07668](https://arxiv.org/abs/2212.07668) (2022)
33. Denef, J.: On the degree of Igusa’s local zeta function. *Am. J. Math.* **109**(6), 991–1008 (1987). <https://doi.org/10.2307/2374583>
34. Dolgachev, I.: *Lectures on Invariant Theory*. Cambridge University Press (2003). <https://doi.org/10.1017/CBO9780511615436>
35. Ein, L., Lazarsfeld, R., Mustață, M.: Contact loci in arc spaces. *Compos. Math.* **140**, 1229–1244 (2004). <https://doi.org/10.1112/S0010437X04000429>
36. Elkik, R.: Singularités rationnelles et déformations. *Invent. Math.* **47**, 139–147 (1978). <https://doi.org/10.1007/BF01578068>
37. Ginzburg, V.: Calabi-Yau algebras. In: [arXiv:0612139](https://arxiv.org/abs/0612139) (2006)
38. Glazer, I.: On rational singularities and counting points of schemes over finite rings. *Algebra Number Theory* **13**(2), 485–500 (2019). <https://doi.org/10.2140/ant.2019.13.485>
39. Grothendieck, A.: Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie. In: *Publications mathématiques de l’IHES* 24, pp. 5–231 (1965)
40. Grothendieck, A.: Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie. In: *Publications mathématiques de l’IHES* 28, pp. 5–255 (1966)
41. Hausel, T.: Kac’s conjecture from Nakajima quiver varieties. *Invent. Math.* **181**(1), 21–37 (2010). <https://doi.org/10.1007/s00222-010-0241-3>
42. Hausel, T., Letellier, E., Rodríguez-Villegas, F.: Arithmetic harmonic analysis on character and quiver varieties. *Duke Math. J.* **160**(2), 323–400 (2011). <https://doi.org/10.1215/00127094-1444258>
43. Hausel, T., Letellier, E., Rodríguez-Villegas, F.: Arithmetic harmonic analysis on character and quiver varieties II. *Adv. Math.* **234**, 85–128 (2013). <https://doi.org/10.1016/j.aim.2012.10.009>
44. Herbig, H.-C., Schwarz, G., Seaton, C.: When does the zero fiber of the moment map have rational singularities? In: [arXiv:2108.07306](https://arxiv.org/abs/2108.07306) (2021)
45. Huybrechts, D.: *Lectures on K3 surfaces*. Cambridge University Press (2016). <https://doi.org/10.1017/CBO9781316594193>

46. Huybrechts, D., Lehn, M.: The geometry of moduli spaces of sheaves. Cambridge University Press (2010). <https://doi.org/10.1017/CBO9780511711985>
47. Igusa, J.: An Introduction to the Theory of Local Zeta Functions. American Mathematical Society (2000). <https://doi.org/10.1090/amsip/014>
48. Kaledin, D., Lehn, M., Sorger, C.: Singular symplectic moduli spaces. *Invent. Math.* **164**(3), 591–614 (2006). <https://doi.org/10.1007/s00222-005-0484-6>
49. Kaplan, D., Schedler, T.: Multiplicative preprojective algebras are 2-Calabi-Yau. *Lect. Notes. Pure. Appl.* **17**(4), 831–883 (2023). <https://doi.org/10.2140/ant.2023.17.831>
50. Kashiwara, M., Saito, Y.: Geometric construction of crystal bases. *Duke Math. J.* **89**(1), 9–36 (1997). <https://doi.org/10.1215/S0012-7094-97-08902-X>
51. Keller, B.: Calabi-Yau triangulated categories. In: Trends in representation theory of algebras and related topics, pp. 467–489 (2008). <https://doi.org/10.4171/062>
52. Keller, B., Wang, Y.: An introduction to relative Calabi-Yau structures. In: [arXiv:2111.10771](https://arxiv.org/abs/2111.10771) (2021)
53. King, A.D.: Moduli of representations of finite-dimensional algebras. *Q. J. Math.* **45**(2), 515–530 (1994). <https://doi.org/10.1093/qmath/45.4.515>
54. Le Bruyn, L., Procesi, C.: Semisimple representations of quivers. *Trans. Amer. Math. Soc.* **317**(2), 585–598 (1990). <https://doi.org/10.1090/S0002-9947-1990-0958897-0>
55. Luna, D.: Slices étales. *Bull. Soc. Math. France* **33**, 81–105 (1973). <https://doi.org/10.24033/msmf.110>
56. Lusztig, G.: Quivers, perverse sheaves and quantized enveloping algebras. *J. Amer. Math. Soc.* **4**(2), 365–421 (1991). <https://doi.org/10.2307/2939279>
57. Lusztig, G.: Semicanonical bases arising from enveloping algebras. *Adv. Math.* **151**(2), 129–139 (2000). <https://doi.org/10.1006/aima.1999.1873>
58. Mozgovoy, S.: Motivic Donaldson-Thomas invariants and McKay correspondence. In: [arXiv:1107.6044](https://arxiv.org/abs/1107.6044) (2011)
59. Mustăţă, M.: Singularities of pairs via jet schemes. *J. Amer. Math. Soc.* **15**(3), 599–615 (2002). <https://doi.org/10.1090/S0894-0347-02-00391-0>
60. Mustăţă, M.: Jet schemes of locally complete intersection canonical singularities. *Invent. Math.* **145**(3), 397–424 (2001). <https://doi.org/10.1007/s002220100152>
61. Nakajima, H.: Instantons on ALE spaces, quiver varieties and Kac-Moody algebras. *Duke Math. J.* **76**(2), 365–416 (1994). <https://doi.org/10.1215/S0012-7094-94-07613-8>
62. Nakajima, H.: Quiver varieties and Kac-Moody algebras. *Duke Math. J.* **91**(3), 515–560 (1998). <https://doi.org/10.1215/S0012-7094-98-09120-7>
63. Reineke, M.: Moduli of representations of quivers. In: [arXiv:0802.2147](https://arxiv.org/abs/0802.2147) (2008)
64. Schedler, T., Tirelli, A.: Symplectic resolutions for multiplicative quiver varieties and character varieties for punctured surfaces. In: Representation Theory and Algebraic Geometry. Springer, pp. 393–459 (2022). https://doi.org/10.1007/978-3-030-82007-7_9
65. Schiffmann, O.: Kac polynomials and Lie algebras associated to quivers and curves. In: Proceedings of the International Congress of Mathematicians, pp. 1393–1424 (2018). https://doi.org/10.1142/9789813272880_0102
66. Schwarz, G.: Lifting smooth homotopies of orbit spaces. *Publications Mathématiques de l’IHÉS* **51**, 37–135 (1980)
67. The Stacks Project Authors. Stacks Project. <https://stacks.math.columbia.edu> (2024)
68. Toën, B., Vaquié, M.: Moduli of objects in dg-categories. *Annales scientifiques de l’Ecole normale supérieure* **40**(3), 387–444 (2007). <https://doi.org/10.1016/j.ansens.2007.05.001>
69. Veys, W., Zuniga-Galindo, W.A.: Zeta functions for analytic mappings, log-principalization of ideals and Newton polyhedra. *Trans. Amer. Math. Soc.* **360**(4), 2205–2227 (2008). <https://doi.org/10.1090/S0002-9947-07-04422-4>
70. Wyss, D.: Motivic and p-adic Localisation Phenomena. PhD thesis. EPFL (2017)
71. Yasuda, T.: The wild McKay correspondence and p-adic measures. *J. European Math. Soc.* **19**(12), 3709–3743 (2017). <https://doi.org/10.4171/JEMS/751>
72. Yoshioka, K.: Moduli spaces of sheaves on abelian surfaces. *Math. Ann.* **321**(4), 817–884 (2001). <https://doi.org/10.1007/s002080100255>