Algebra & Number Theory

Volume 18
2024
No. 2

Partial sums of typical multiplicative functions over short moving intervals

Mayank Pandey, Victor Y. Wang and Max Wenqiang Xu

msp

ALGEBRA AND NUMBER THEORY 18:2 (2024)
https://doi.org/10.2140/ant.2024.18.389

Partial sums of typical multiplicative functions over short moving intervals

Mayank Pandey, Victor Y. Wang and Max Wenqiang Xu

We prove that the k-th positive integer moment of partial sums of Steinhaus random multiplicative functions over the interval $(x, x+H]$ matches the corresponding Gaussian moment, as long as $H \ll x/(\log x)^{2k^2+2+o(1)}$ and H tends to infinity with x. We show that properly normalized partial sums of typical multiplicative functions arising from realizations of random multiplicative functions have Gaussian limiting distribution in short moving intervals $(x, x+H]$ with $H \ll X/(\log X)^{W(X)}$ tending to infinity with X, where x is uniformly chosen from $\{1, 2, \dots, X\}$, and $W(X)$ tends to infinity with X arbitrarily slowly. This makes some initial progress on a recent question of Harper.

1. Introduction

We are interested in the partial sums behavior of a family of completely multiplicative functions f supported on moving short intervals. Formally, for positive integers X, let $[X] := \{1, 2, \dots, X\}$ and

$$\mathcal{F}_X := \{f : [X] \to \{|z| = 1\} : f \text{ is completely multiplicative}\}.$$

For $f \in \mathcal{F}_X$, the function values $f(n)$ for all $n \leqslant X$ are uniquely determined by $(f(p))_{p \leqslant X}$. The Steinhaus random multiplicative function is defined by selecting $f(p)$ uniformly at random from the complex unit circle and defining $f(n)$ completely multiplicatively. One may view $\mathcal{F}_X$ as the family of all Steinhaus random multiplicative functions.

Let H be another positive integer. We are interested in for a typical $f \in \mathcal{F}_{X+H}$, whether the random partial sums

$$A_H(f, x) := \frac{1}{\sqrt{H}} \sum_{x<n\leqslant x+H} f(n), \tag{1-1}$$

where x is uniformly chosen from $[X]$, behave like a complex standard Gaussian. In this note, we provide a positive answer (Theorem 1.2) when $H \ll_A X/\log^A X$ holds for all $A > 0$. As we explain in Section 4, the answer is negative for $H \gg X \exp(-(\log\log X)^{1/2-\varepsilon})$, but the question remains open between these two thresholds.

MSC2020: primary 11K65; secondary 11D45, 11D57, 11D79, 11N37.
Keywords: random multiplicative function, short moving intervals, multiplicative Diophantine equations, paucity, Gaussian behavior, correlations of divisor functions.

We formalize the question by explaining how to measure the elements in $\mathcal{F}_X$. Via complete multiplicativity of $f \in \mathcal{F}_X$, define on $\mathcal{F}_X$ the product measure

$$\nu_X := \prod_{p \leqslant X} \mu_p,$$

where for any given prime p, we let μ_p denote the uniform distribution on the set $\{f(p)\} = \{|z| = 1\}$. For example, $\nu_X(\mathcal{F}_X) = 1$.

Question 1.1 [Harper 2022, open question (iv)]. What is the distribution of the normalized random sum defined in (1-1) (for most f) as x is uniformly chosen from $[X]$?

1A. *Main results.* In this note, we make some progress on Question 1.1. We use the notation $\xrightarrow{d}$ to denote convergence in distribution.

Theorem 1.2. *Let integer X be large and $W(X)$ tend to infinity arbitrarily slowly as X tends to infinity. Let $H := H(X) \ll X(\log X)^{-W(X)}$ and $H \to +\infty$ as $X \to +\infty$. Then, for almost all $f \in \mathcal{F}_{X+H}$, as $X \to +\infty$,*

$$\frac{1}{\sqrt{H}} \sum_{x<n\leqslant x+H} f(n) \xrightarrow{d} \mathcal{CN}(0, 1), \tag{1-2}$$

where x is chosen uniformly from $[X]$.

Here "almost all" means the total measure of such f is $1 - o_{X\to+\infty}(1)$ under ν_{X+H}.[1] Also, $\mathcal{CN}(0, 1)$ denotes the standard complex normal distribution; a standard complex normal random variable Z (with mean 0 and variance 1) can be written as $Z = X + iY$, where X and Y are independent real normal random variables with mean 0 and variance $\frac{1}{2}$. Recall that a real normal random variable W with mean 0 and variance σ^2 satisfies

$$\mathbb{P}(W \leqslant t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/(2\sigma^2)}\, dx.$$

To prove Theorem 1.2, we establish moment statistics in several situations. We first show that the integer moments of random multiplicative functions f supported on suitable short intervals match the corresponding Gaussian moments. We write $\mathbb{E}_f$ to mean "average over $f \in \mathcal{F}_X$ with respect to ν_X" (where $\mathcal{F}_X$ is always clear from context).

Theorem 1.3. *Let $x, H, k \geqslant 1$ be integers. Let $f \in \mathcal{F}_{x+H}$. Let $E(k) = 2k^2 + 2$. Then*

$$\mathbb{E}_f \left| \frac{1}{\sqrt{H}} \sum_{x<n\leqslant x+H} f(n) \right|^{2k} = k! + O_k\left(H^{-1} + \frac{H^{1/2}}{\max(x, H)^{1/2}} + \frac{H \cdot (\log x + \log H)^{E(k)}}{\max(x, H)} \right),$$

with an implied constant depending only on k.

[1] More precisely, there exist nonempty measurable sets $\mathcal{G}_{X,H} \subseteq \mathcal{F}_{X+H}$ of measure $1 - o_{X\to+\infty}(1)$ (under ν_{X+H}) such that for every sequence of functions $f_X \in \mathcal{G}_{X,H}$ ($X \geqslant 1$), the random variable on the left-hand side of (1-2) (with $f = f_X$) converges in distribution to $\mathcal{CN}(0, 1)$ as $X \to +\infty$.

Notice that $k!$ is the $2k$-th moment of the standard complex Gaussian distribution. Given an integer $k \geqslant 1$, let $E'(k)$ be the smallest real number $r \geqslant 0$ such that for every $\varepsilon > 0$, we have $\mathbb{E}_f |A_H(f, x)|^{2k} \to k!$ whenever

$$x \to +\infty \quad \text{and} \quad (\log x)^{\varepsilon} \leqslant H \leqslant x/(\log x)^{r+\varepsilon}.$$

Theorem 1.3 shows that $E'(k) \leqslant E(k)$.[2] The paper [Chatterjee and Soundararajan 2012] studies the case $k = 2$, showing in particular that $E'(2) \leqslant 1$. In the case that f is supported on $\{1, 2 \ldots, x\}$, the $2k$-th moments for general k were studied in [Batyrev and Tschinkel 1998; de la Bretèche 2001a; 2001b; Granville and Soundararajan 2001; Heap and Lindqvist 2016; Harper 2019; Harper et al. 2015] and it is known that the moments there do not match Gaussian moments: for instance, by [Harper 2019, Theorem 1.1], there exists some constant $c > 0$ such that for all positive integers $k \leqslant c(\log x/\log\log x)$ (assuming x is large),

$$\mathbb{E}_f \left| \frac{1}{\sqrt{x}} \sum_{n \leqslant x} f(n) \right|^{2k} = e^{-k^2 \log(k \log(2k)) + O(k^2)} (\log x)^{(k-1)^2}. \tag{1-3}$$

While (1-3) is quite uniform over k, it is unclear how uniform in k one could make our Theorem 1.3. (See Remark 2.3 for some discussion of the k-aspect in our work.)

Remark 1.4. The powers of $\log x$ above are significant. For instance, Theorem 1.3 in the range $H \gg x$ follows directly from (1-3), since $(k-1)^2 \leqslant E(k)$. One may also wonder how far our bound $E'(k) \leqslant E(k)$ is from the truth. Based on a circle method heuristic for (1-4) along the lines of [Hooley 1986, Conjecture 2], with a Hardy–Littlewood contribution on the order of $(H^{2k}/Hx^{k-1})(\log x)^{(k-1)^2}$, and an additional contribution of roughly $k!H^k$ from trivial solutions, it is plausible that one could improve the right-hand side in Theorem 1.3 to $k! + O_k((H^{k-1}/x^{k-1})(\log x)^{(k-1)^2})$ for $H \in [x^{1-\delta}, x]$. If true, this would suggest that $E'(k) \leqslant k-1$ and we believe this might be the true order. For a discussion of how one might improve on Theorem 1.3, see the beginning of Section 4.

By orthogonality, Theorem 1.3 is a statement about the Diophantine point count

$$\#\{(n_1, n_2, \ldots, n_{2k}) \in (x, x+H]^{2k} : n_1 n_2 \cdots n_k = n_{k+1} n_{k+2} \cdots n_{2k}\}. \tag{1-4}$$

The circle method, or modern versions thereof such as [Duke et al. 1993; Heath-Brown 1996], might lead to an asymptotic for (1-4) uniformly over $H \in [x^{1-\delta}, x]$ for $k = 2$, unconditionally (compare [Heath-Brown 1996, Theorem 6]), or for $k = 3$, conditionally on standard number-theoretic hypotheses (compare [Wang 2021]). Alternatively, "multiplicative" harmonic analysis along the lines of [de la Bretèche 2001b; Harper et al. 2015; Heap and Lindqvist 2016] may in fact lead to an unconditional asymptotic over $H \in [x^{1-\delta}, x]$ for all k, with many main terms involving different powers of $\log x$, $\log H$. Nonetheless, for all k, we obtain an unconditional asymptotic for (1-4) uniformly over $H \ll x/(\log x)^{Ck^2}$, by replacing

[2] After writing the paper, the authors learned that for $H \leqslant x/\exp(C_k \log x/\log\log x)$, the Diophantine statement underlying Theorem 1.3 has essentially appeared before in the literature; see [Bourgain et al. 2014, proof of Theorem 34]. However, we handle a more delicate range of the form $H \leqslant x/(\log x)^{Ck^2}$.

the complicated "off-diagonal" contribution to (1-4) with a *larger but simpler* quantity; see Section 2 for details.

Remark 1.5. An analog of (1-4) for polynomial values $P(n_i)$ is studied in [Klurman et al. 2023; Wang and Xu 2022], and a similar flavor counting question to (1-4) is studied in [Fu et al. 2021] using the decoupling method.

After Theorem 1.3, our next step towards Theorem 1.2 is to establish concentration estimates for the moments of the random sums (1-1). We write $\mathbb{E}_x$ to denote "expectation over x uniformly chosen from $[X]$" (where X is always clear from context).

Theorem 1.6. *Let $X, k \geqslant 1$ be integers with X large. Suppose that $H := H(X) \to +\infty$ as $X \to +\infty$. There exists a large absolute constant $A > 0$ such that the following holds as long as $H \ll X(\log X)^{-C_k}$ with $C_k = Ak^{Ak^{Ak}}$. Let $f \in \mathcal{F}_{X+H}$; then*

$$\mathbb{E}_f\left(\mathbb{E}_x\left|\frac{1}{\sqrt{H}}\sum_{x<n\leqslant x+H} f(n)\right|^{2k} - k!\right)^2 = o_{X\to+\infty}(1). \tag{1-5}$$

Furthermore, for any fixed positive integer $\ell < k$, we have

$$\mathbb{E}_f\left|\mathbb{E}_x\left(\frac{1}{\sqrt{H}}\sum_{x<n\leqslant x+H} f(n)\right)^k\left(\frac{1}{\sqrt{H}}\sum_{x<n\leqslant x+H} \overline{f(n)}\right)^\ell\right|^2 = o_{X\to+\infty}(1). \tag{1-6}$$

We prove Theorem 1.3 in Section 2, and then we prove Theorem 1.6 in Section 3.

Proof of Theorem 1.2, assuming Theorem 1.6. We use the notation $A_H(f,x)$ from (1-1). By Markov's inequality, Theorem 1.6 implies that there exists a set of the form

$$\mathcal{G}_{X,H} := \Big\{f \in \mathcal{F}_{X+H} : \mathbb{E}_x|A_H(f,x)|^{2k} - k! = o_{X\to+\infty}(1) \text{ for all } k \leqslant V(X),$$
$$\mathbb{E}_x[A_H(f,x)^k\overline{A_H(f,x)^\ell}] = o_{X\to+\infty}(1) \text{ for all distinct } k,\ell \leqslant V(X)\Big\}$$

for some $V(X) \to +\infty$ (making a choice of $V(X)$ based on $W(X)$) such that

$$\nu_{X+H}(\mathcal{G}_{X,H}) = 1 - o_{X\to+\infty}(1).$$

Since the distribution $\mathcal{CN}(0,1)$ is uniquely determined by its moments (see e.g., [Billingsley 2012, Theorem 30.1 and Example 30.1]), Theorem 1.2 follows from the method of moments [Gut 2005, Chapter 5, Theorem 8.6] (applied to sequences of random variables $A_H(f,x)$ indexed by $f \in \mathcal{G}_{X,H}$ as $X \to +\infty$). □

We believe results similar to our theorems above should also hold in the (extended) Rademacher case, though we do not pursue that case in this paper.

1B. ***Notation.*** For any two functions $f, g : \mathbb{R} \to \mathbb{R}$, we write $f \ll g$, $g \gg f$, $g = \Omega(f)$ or $f = O(g)$ if there exists a positive constant C such that $|f| \leqslant Cg$, and we write $f \asymp g$ or $f = \Theta(g)$ if $f \ll g$ and $g \gg f$. We write O_k to indicate that the implicit constant depends on k. We write $o_{X\to+\infty}(g)$ to denote a quantity f such that f/g tends to zero as X tends to infinity.

2. Moments of random multiplicative functions in short intervals

In this section, we prove Theorem 1.3. For integers $k, n \geqslant 1$, let $\tau_k(n)$ denote the number of positive integer solutions $(d_1, \dots, d_k)$ to the equation $d_1 \cdots d_k = n$. It is known that (see [Norton 1992, Theorem 1.29 and Corollary 1.36])

$$\tau_k(n) \ll n^{O(\log k/\log\log n)} \quad \text{as } n \to +\infty, \text{ provided } k = o_{n\to+\infty}(\log n). \tag{2-1}$$

As we mentioned before, when $H \geqslant x$, Theorem 1.3 is implied by (1-3). From now on, we focus on the case $H \leqslant x$. We split the proof into two cases: small H and large H. For small H, we illustrate the general strategy and carelessly use divisor bounds; for large H, we take advantage of bounds of Shiu [1980] and Henriot [2012] on mean values and correlations of multiplicative functions over short intervals, together with a decomposition idea.

2A. ***Case 1:*** $\boldsymbol{H \leqslant x^{1-\varepsilon k^{-1}}}$**.** Here we take ε to be a small absolute constant, e.g., $\varepsilon = \frac{1}{100}$.

We begin with the following proposition.

Proposition 2.1. *Let $k, y, H \geqslant 1$ be integers. Suppose y is large and $k \leqslant \log\log y$. Then $N_k(H; y)$, the number of integer tuples $(h_1, h_2, \dots, h_k) \in [-H, H]^k$ with $y \mid h_1h_2\cdots h_k$ and $h_1h_2\cdots h_k \neq 0$, is at most $(2H)^k \cdot O(H^{O(k\log k/\log\log y)}/y)$.*

Proof. The case $k = 1$ is trivial; one has $N_1(H; y) \leqslant 2H/y$. Suppose $k \geqslant 2$. Whenever $y \mid h_1h_2\cdots h_k \neq 0$, there exists a factorization $y = u_1u_2\cdots u_k$ where u_i are positive integers such that $u_i \mid h_i \neq 0$ for all $1 \leqslant i \leqslant k$. (Explicitly, one can take $u_1 = \gcd(h_1, y)$ and $u_i = \gcd(h_i, y/\gcd(y, h_1h_2\cdots h_{i-1}))$.) It follows that $N_k(H; y) = 0$ if $y > H^k$, and

$$N_k(H; y) \leqslant \sum_{u_1u_2\cdots u_k = y} N_1(H; u_1)N_1(H; u_2)\cdots N_1(H; u_k) \leqslant \tau_k(y)\cdot(2H)^k/y \tag{2-2}$$

if $y \leqslant H^k$. By the divisor bound (2-1), Proposition 2.1 follows. □

Corollary 2.2. *Let $k, H, x \geqslant 1$ be integers. Suppose x is large and $k \leqslant \log\log x$. Then $S_k(x, H)$, the set of integer tuples $(h_1, h_2, \dots, h_k, y) \in [-H, H]^k \times (x, x+H]$ with $y \mid h_1h_2\cdots h_k$ and $h_1h_2\cdots h_k \neq 0$, has size at most $(2H)^k \cdot O(H^{1+O(k\log k/\log\log x)}/x)$.*

Proof. $\#S_k(x, H) = \sum_{x<y\leqslant x+H} N_k(H; y)$. But here $N_k(H; y) \ll (2H)^k \cdot H^{O(k\log k/\log\log x)}/x$. □

The $2k$-th moment in Theorem 1.3 is H^{-k} times the point count (1-4) for the Diophantine equation

$$n_1n_2\cdots n_k = n_{k+1}n_{k+2}\cdots n_{2k}. \tag{2-3}$$

There are $k!H^k(1+O(k^2/H)) = k!H^k + O_k(H^{k-1})$ trivial solutions. (We call a solution to (2-3) *trivial* if the tuple $(n_{k+1}, \dots n_{2k})$ equals a permutation of $(n_1, \dots n_k)$.) The number of trivial solutions is clearly $\geqslant k!H(H-1)\cdots(H-k+1)$, and $\leqslant k!H^k$.) It remains to bound $N_k(x, H)$, the number of nontrivial solutions $(n_1, \dots, n_{2k}) \in (x, x+H]^{2k}$ to (2-3).

We will show that $N_k(x, H) \ll H^k \cdot (H/x)^{1/2}$. To this end, let $N'_k(x, H)$ denote the number of nontrivial solutions in $(x, x+H]^{2k}$ with the further constraint that

$$n_{2k} \notin \{n_1, n_2, \dots, n_k\}. \tag{2-4}$$

Then for any $k \geqslant 2$, we have

$$N_k(x, H) \leqslant N'_k(x, H) + k \cdot (H+1) \cdot N_{k-1}(x, H), \tag{2-5}$$

since for each $(n_1, \dots, n_{2k}) \in (x, x+H]^{2k}$, either (2-4) holds or there exists $i \in [k]$ satisfying $n_i = n_{2k} \in (x, x+H]$.

A key observation is that for nontrivial solutions to (2-3) with constraint (2-4),[3]

$$n_{2k} \mid (n_1 - n_{2k})(n_2 - n_{2k}) \cdots (n_k - n_{2k}),$$

and if we write $h_i := n_i - n_{2k}$ then $h_i \in [-H, H]$ are nonzero. Given $h_1, h_2, \dots, h_k, y$, let

$$C_{h_1,\dots,h_k,y} := \prod_{1 \leqslant i \leqslant k} (h_i + y).$$

Then $N'_k(x, H)$ is (upon changing variables from $n_1, \dots, n_k$ to $h_1, \dots, h_k$) at most

$$\sum_{\substack{(h_1,\dots,h_k,n_{2k}) \in S_k(x,H) \\ h_i + n_{2k} > 0}} \#\biggl\{(n_{k+1}, \dots, n_{2k-1}) \in (x, x+H]^{k-1} : \biggl(\prod_{i=1}^{k-1} n_{k+i}\biggr) \biggm| C_{h_1,\dots,h_k,n_{2k}}\biggr\}. \tag{2-6}$$

If x is large and k is fixed (or $k \leqslant \log\log x$, say), then by the divisor bound (2-1), the quantity (2-6) is at most

$$\ll (H+x)^{O(k \log k/\log\log x)} \cdot \#S_k(x, H) \ll O(H)^k \cdot O(H \cdot x^{-1+O(k \log k/\log\log x)}),$$

where in the last step we used Corollary 2.2.

By (2-5), it follows that x is large and k is fixed (or $k \leqslant \log\log x$, say), then

$$N_k(x, H) \leqslant k \cdot \max_{1 \leqslant j \leqslant k} (O(kH)^{k-j} \cdot N'_j(x, H)) \ll k \cdot O(kH)^k \cdot O(H \cdot x^{-1+O(k \log k/\log\log x)}). \tag{2-7}$$

(Note that $N_1(x, H) = 0$.) So in particular, $N_k(x, H) \ll H^k \cdot (H/x)^{1/2}$ for fixed k (or for x large and $k \leqslant (\log\log x)^{1/2-\delta}$, say), since $H \leqslant x^{1-\varepsilon k^{-1}}$. This suffices for Theorem 1.3.

[3]After writing the paper, the authors learned that this observation has appeared before in the literature (see [Bourgain et al. 2014, proof of Lemma 22]); however, we take the idea further, both in Section 2 and in Section 3.

Remark 2.3. The argument above in fact gives, in Case 1, a version of Theorem 1.3 with an implied constant of $O(k!k^2)$, uniformly over $k \leqslant (\log\log x)^{1/2-\delta}$, say. However, in Case 2 below, our proof relies on a larger body of knowledge for which the k-dependence does not seem easy to work out; this is why we essentially keep k fixed in Theorem 1.3.

2B. *Case 2:* $x^{1-2\varepsilon k^{-1}} \leqslant H \leqslant x$. Again, one can assume $\varepsilon = \frac{1}{100}$. In this case, we employ the following tool due to Henriot [2012, Theorem 3]. For the multiplicative functions f in (2-8) (and in similar places below), we let $f(m) := 0$ if $m \leqslant 0$.

Definition 2.4. Given a real $A_1 \geqslant 1$ and a function $A_2 = A_2(\epsilon) \geqslant 1$ (defined for reals $\epsilon > 0$), let $\mathcal{M}(A_1, A_2)$ denote the set of nonnegative multiplicative functions $f(n)$ such that $f(p^\ell) \leqslant A_1^\ell$ (for all primes p and integers $\ell \geqslant 1$) and $f(n) \leqslant A_2 n^\epsilon$ (for all $n \geqslant 1$).

Lemma 2.5. *Let $f_1, f_2 \in \mathcal{M}(A_1, A_2)$ and $\beta \in (0,1)$. Let $a, q \in \mathbb{Z}$ with $|a|, q \geqslant 1$ and $\gcd(a,q) = 1$. If $x, y \geqslant 2$ are reals with $x^\beta \leqslant y \leqslant x$ and $x \geqslant \max(q, |a|)^\beta$, then*

$$\sum_{x \leqslant n \leqslant x+y} f_1(n) f_2(qn+a) \ll_{\beta, A_1, A_2} \Delta_D \cdot y \cdot \sum_{n_1 n_2 \leqslant x} \frac{f_1(n_1) f_2(n_2)}{n_1 n_2}, \tag{2-8}$$

where $\Delta_D \leqslant \prod_{p \mid a^2} (1 + (2A_1 + A_1^2) p^{-1})$. Furthermore,

$$\Delta_D \leqslant \left(\frac{|a|}{\phi(|a|)}\right)^{2A_1 + A_1^2} \quad (\textit{where } \phi \textit{ denotes Euler's totient function}). \tag{2-9}$$

Proof. Everything but (2-9) follows from [Henriot 2012, Theorem 3] and the unraveling of definitions done in [Matomäki et al. 2019, proof of Lemma 2.3(ii)]; in the notation of [Henriot 2012, Theorem 3], we take

$$(k, Q_1(n), Q_2(n), \alpha, \delta, A, B, F(n_1, n_2)) = \big(2, n, qn+a, \tfrac{9}{10}\beta, \tfrac{9}{10}\beta, A_1, A_2(\epsilon)^2, f_1(n_1) f_2(n_2)\big),$$

where $\epsilon = \alpha/(100(2+\delta^{-1}))$.[4] The inequality (2-9) then follows from the fact that $1 + rp^{-1} \leqslant (1+p^{-1})^r \leqslant (1-p^{-1})^{-r}$ for every prime p and real $r \geqslant 1$. □

Also useful to us will be the following immediate consequence of Shiu [1980, Theorem 1].

Lemma 2.6. *Let $f \in \mathcal{M}(A_1, A_2)$ and $\beta \in (0,1)$. If $x, y \geqslant 2$ are reals with $x^\beta \leqslant y \leqslant x$, then*

$$\sum_{x \leqslant n \leqslant x+y} f(n) \ll_{\beta, A_1, A_2} \frac{y}{\log x} \exp\left(\sum_{p \leqslant x} \frac{f(p)}{p}\right).$$

We will apply the above results to $f = \tau_k$ over intervals of the form $[x, x+y]$ with $y \gg x^{1/2k}$, say. Here $\tau_k \in \mathcal{M}(k, O_{k,\epsilon}(1))$, by (2-1) and the fact that $\tau_k(p) = k$ and

$$\tau_k(mn) \leqslant \tau_k(m)\tau_k(n) \quad \text{for arbitrary integers } m, n \geqslant 1. \tag{2-10}$$

[4] In fact, one could extract a more complicated version of (2-8) from [Henriot 2012, Theorem 3], which in some cases (e.g., if $f_1 = f_2 = \tau_k$) would improve the right-hand side of (2-8) by roughly a factor of $\log x$.

Also, recall, for integers $k \geqslant 1$ and reals $x \geqslant 2$, the standard bound

$$\sum_{n \leqslant x} \tau_k(n) \ll_k \frac{x}{\log x} \exp\biggl(\sum_{p \leqslant x} \frac{k}{p}\biggr) \ll_k x(\log x)^{k-1} \tag{2-11}$$

(see e.g., [Matomäki et al. 2019, Section 2.2]) and the consequence

$$\sum_{n_1 n_2 \leqslant x} \tau_k(n_1)\tau_k(n_2) = \sum_{n \leqslant x} \tau_{2k}(n) \ll_k x(\log x)^{2k-1}. \tag{2-12}$$

(See [Norton 1992] for a version of (2-11) with an explicit dependence on k. For Lemmas 2.5 and 2.6, we are not aware of any explicit dependence on β, A_1, A_2 in the literature.)

Lemma 2.7. *Let $V, U, q \geqslant 1$ be integers with $q \leqslant U^{k-2}$, where $k \geqslant 2$. Let $\rho \in \{-1, 1\}$. Then*

$$\sum_{\substack{u \in [U, 2U) \\ 1 \leqslant v \leqslant V}} \tau_k(u)\tau_k(\rho v + uq) \ll_k VU(1 + \log VU)^{3k}.$$

Proof. First suppose $V \geqslant U$. If $u \in [U, 2U)$, then $I := \{\rho v + uq : 1 \leqslant v \leqslant V\}$ is an interval of length $V \geqslant \max(V, U)$ contained in $[-V, V + 2U^{k-1}]$, so by Lemma 2.6 and (2-11), we obtain the bound

$$\sum_{1 \leqslant v \leqslant V} \tau_k(\rho v + uq) \ll_k V(1 + \log V)^{k-1}.$$

(We consider the cases $0 \in I$ and $0 \notin I$ separately. The former case follows directly from (2-11); the latter case requires Lemma 2.6.) Then sum over u using (2-11). Since $(1 + \log V)^{k-1}(1 + \log U)^{k-1} \leqslant (1 + \log VU)^{2k-2}$, Lemma 2.7 follows.

Now suppose $V \leqslant U$. By casework on $d := \gcd(v, q) \leqslant q$, we have

$$\sum_{\substack{u \in [U, 2U) \\ 1 \leqslant v \leqslant V}} \tau_k(u)\tau_k(\rho v + uq) \leqslant \sum_{d \mid q} \tau_k(d) \sum_{\substack{u \in [U, 2U) \\ 1 \leqslant a \leqslant V/d \\ \gcd(a, q/d) = 1}} \tau_k(u)\tau_k(\rho a + uq/d).$$

Since $d \mid q$ and $1 \leqslant a \leqslant V/d$, we have $U \geqslant \max(a, q^{1/k})$. Now for any fixed $1 \leqslant a \leqslant V/d$,

$$\sum_{u \in [U, 2U)} \tau_k(u)\tau_k(\rho a + uq/d) \ll_k \biggl(\frac{a}{\phi(a)}\biggr)^{2k+k^2} \cdot U \cdot (1 + \log U)^{2k}$$

by Lemma 2.5 and (2-12), provided $\gcd(a, q/d) = 1$. Upon summing over $1 \leqslant a \leqslant V/d$ using [Montgomery and Vaughan 2007, page 61, (2.32)], it follows that

$$\sum_{\substack{u \in [U, 2U) \\ 1 \leqslant v \leqslant V}} \tau_k(u)\tau_k(\rho v + uq) \ll_k \sum_{d \mid q} \tau_k(d) \cdot \tfrac{V}{d} \cdot U \cdot (1 + \log U)^{2k}.$$

Since $\sum_{d \leqslant q} (\tau_k(d)/d) \ll_k (1 + \log q)^k$ (by (2-11)) and $q \leqslant U^{k-2}$, Lemma 2.7 follows. □

Lemma 2.8. *Let $V_1, U_2, \dots, U_k \geqslant 1$ be integers, where $k \geqslant 2$. Let $\varepsilon_1 \in \{-1, 1\}$. Then*

$$\sum_{\substack{v_1,u_2,\dots,u_k\geqslant 1\\ u_i\in[U_i,2U_i)\\ v_1\leqslant V_1}} \tau_k(u_2)\cdots\tau_k(u_k)\tau_k(\varepsilon_1 v_1 + u_2\cdots u_k) \ll_k L_k(V_1U_2\cdots U_k),$$

where $L_k(r) := r\cdot(1+\log r)^{3k+(k-2)(k-1)} = r\cdot(1+\log r)^{k^2+2}$ for $r \geqslant 1$.

Proof. We may assume $U_2 \geqslant \cdots \geqslant U_k$. Let $q := u_3\cdots u_k \leqslant U_2^{k-2}$ and apply Lemma 2.7 (with $(V, U) = (V_1, U_2)$) to sum over u_2, v_1. Then sum over the $k-2$ variables $u_3, \dots, u_k$ using (2-11). □

With the lemmas above in hand, we now build on the strategy from Case 1 to attack Case 2. As before, we let $N_k'(x, H)$ denote the number of nontrivial solutions $(n_1, \dots, n_k, n_{k+1}, \dots, n_{2k}) \in (x, x+H]^{2k}$ to (2-3) with constraint (2-4). Again, for such solutions we write $h_i = n_i - n_{2k} \in [-H, H]\setminus\{0\}$, and there exist positive integers u_i $(1 \leqslant i \leqslant k)$ such that $u_i \mid h_i$ with $u_1u_2\cdots u_k = n_{2k} \in (x, x+H]$; so $u_i \leqslant H$, and there exist signs $\varepsilon_i \in \{-1, 1\}$ and positive integers $v_i \leqslant H/U_i$ with $h_i = \varepsilon_i u_i v_i$, whence

$$C_{h_1,\dots,h_k,n_{2k}} := \prod_{i=1}^{k}(h_i + n_{2k}) = \prod_{1\leqslant i\leqslant k}(\varepsilon_i u_i v_i + u_1u_2\cdots u_k).$$

As before, the quantity $N_k'(x, H)$ is at most (2-6). Upon splitting the range $[H]$ for each u_i into $\leqslant 1 + \log_2 H \ll 1 + \log x$ dyadic intervals, we conclude that

$$N_k'(x, H) \leqslant \sum_{\varepsilon_i, U_i} \sum_{\substack{u_i\in[U_i,2U_i)\\ v_i\leqslant H/U_i\\ x<n_{2k}\leqslant x+H\\ h_i+n_{2k}>0}} \tau_k(C_{h_1,\dots,h_k,n_{2k}}) \leqslant 2^k\cdot O(1+\log x)^k\cdot \mathcal{S}(x, H), \tag{2-13}$$

where we let $n_{2k} := u_1u_2\cdots u_k$ and $h_i := \varepsilon_i u_i v_i$ in the sum over u_i, v_i (for notational brevity), and where $\mathcal{S}(x, H)$ denotes the maximum of the quantity

$$S(\vec{\varepsilon}, \vec{U}) := \sum_{\substack{u_i\in[U_i,2U_i)\\ v_i\leqslant H/U_i\\ x<n_{2k}\leqslant x+H\\ h_i+n_{2k}>0}} \tau_k(C_{h_1,\dots,h_k,n_{2k}}) = \sum_{\substack{u_i\in[U_i,2U_i)\\ v_i\leqslant H/U_i\\ x<n_{2k}\leqslant x+H\\ h_i+n_{2k}>0}} \tau_k\Bigl(\prod_{1\leqslant i\leqslant k}(\varepsilon_i u_i v_i + u_1u_2\cdots u_k)\Bigr)$$

over all tuples $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$ and $\vec{U} = (U_1, \dots, U_k)$ where each $U_i \in [H]\cap\{1, 2, 4, 8, \dots\}$ with $2^{-k}x < U_1\cdots U_k \leqslant x+H$. Now, for the rest of Section 2, fix a choice of $\varepsilon_1, \dots, \varepsilon_k, U_1, \dots, U_k$ with

$$\mathcal{S}(x, H) = S(\vec{\varepsilon}, \vec{U}).$$

By symmetry, we may assume that $U_1 \geqslant U_2 \geqslant \cdots \geqslant U_k$.

We now bound $S(\vec{\varepsilon}, \vec{U})$, assuming $k \geqslant 2$. (For $k = 1$, we can directly note that $N_1'(x, H) = 0$.) A key observation is that since $U_1U_2\cdots U_k \leqslant x+H \leqslant 2x$ and $U_1 \geqslant U_2 \geqslant \cdots \geqslant U_k \geqslant 1$, we have (since

$H \geqslant x^{1-2\varepsilon}$ and $k \geqslant 2$)

$$\frac{H}{U_k} \geqslant \frac{H}{U_{k-1}} \geqslant \cdots \geqslant \frac{H}{U_2} \geqslant \frac{H}{(U_1U_2)^{1/2}} \geqslant \frac{x^{1-2\varepsilon}}{(2x)^{1/2}} \gg x^{1/3}.$$

By the submultiplicativity property (2-10), we have that $S(\vec{\varepsilon}, \vec{U})$ is at most

$$\sum_{\substack{u_i \in [U_i, 2U_i) \\ x < u_1u_2\cdots u_k \leqslant x+H}} \sum_{v_i \leqslant H/u_i} \tau_k(u_1)\tau_k(u_2)\cdots\tau_k(u_k) \prod_{1 \leqslant i \leqslant k} \tau_k(\varepsilon_i v_i + u_1u_2\cdots u_{-i}\cdots u_k), \tag{2-14}$$

where u_{-i} means that the factor u_i is not included. But for each $i \geqslant 2$ and $u_i \in [U_i, 2U_i)$, Lemma 2.6 and (2-11) imply (since $u_1u_2\cdots u_{-i}\cdots u_k \leqslant u_1\cdots u_k \ll x$ and $H/u_i \gg x^{1/3}$)

$$\sum_{v_i \leqslant H/u_i} \tau_k(\varepsilon_i v_i + u_1u_2\cdots u_{-i}\cdots u_k) \ll_k (H/U_i)\cdot(1+\log x)^{k-1}; \tag{2-15}$$

compare the use of Lemma 2.6 and (2-11) in the proof of Lemma 2.7. By (2-15) (multiplied over $2 \leqslant i \leqslant k$) and Lemma 2.8 (with $V_1 = H/U_1$), we conclude that the quantity (2-14) (and thus $S(\vec{\varepsilon}, \vec{U})$) is at most

$$\ll_k \frac{H^{k-1}(1+\log x)^{(k-1)^2}}{U_2\cdots U_k} \cdot L_k((H/U_1)\cdot U_2\cdots U_k) \cdot \max_{\substack{u_2,\ldots,u_k \geqslant 1 \\ u_i \in [U_i, 2U_i)}} \sum_{\substack{u_1 \in [U_1, 2U_1) \\ x < u_1u_2\cdots u_k \leqslant x+H}} \tau_k(u_1).$$

For the innermost sum, first note that $(U_2\cdots U_k)^{1/(k-1)} \leqslant (U_1\cdots U_k)^{1/k} \leqslant (2x)^{1/k}$ which implies that

$$H/(u_2\cdots u_k) \gg_k H/(U_2\cdots U_k) \gg_k x^{1-2\varepsilon k^{-1}}/x^{(k-1)/k} \geqslant x^{1/2k}$$

(since $H \geqslant x^{1-2\varepsilon k^{-1}}$); then by Lemma 2.6 and (2-11), we have (for any given $u_2, \ldots, u_k$)

$$\sum_{\substack{u_1 \geqslant 1 \\ x < u_1u_2\cdots u_k \leqslant x+H}} \tau_k(u_1) \ll_k \frac{H}{U_2\cdots U_k} \cdot (1+\log x)^{k-1}.$$

It follows that $S(\vec{\varepsilon}, \vec{U})$ is at most

$$\ll_k \frac{H^{k-1}(1+\log x)^{(k-1)^2}}{U_2\cdots U_k} \cdot \frac{H}{U_1} \cdot U_2\cdots U_k(1+\log x)^{k^2+2} \cdot \frac{H}{U_2\cdots U_k} \cdot (1+\log x)^{k-1},$$

which simplifies to $O_k(1)\cdot H^k \cdot (H/x)\cdot(1+\log x)^{2k^2-k+2}$.

Plugging the above estimate into (2-13), we have (assuming $k \geqslant 2$)

$$N'_k(x, H) \ll_k O(1+\log x)^k \cdot \mathcal{S}(x, H) \ll_k H^k \cdot (H/x)\cdot(1+\log x)^{2k^2+2}, \tag{2-16}$$

in the given range of H. Then by using the first part of (2-7) (and noting that $N_1(x, H) = N'_1(x, H) = 0$) as before, we have (for arbitrary $k \geqslant 1$)

$$N_k(x, H) \leqslant k \cdot \max_{1 \leqslant j \leqslant k} (O(kH)^{k-j} \cdot N'_j(x, H)) \ll_k H^k \cdot (H/x)\cdot(1+\log x)^{2k^2+2},$$

which suffices for Theorem 1.3.

3. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. Let rad_k be the multiplicative function

$$\mathrm{rad}_k(n) = \min_{n_1 \cdots n_k = n} [n_1, \dots, n_k],$$

where $[n_1, \dots, n_k]$ denotes the least common multiple of $n_1, \dots, n_k$. In particular, for prime powers p^ℓ we have

$$\mathrm{rad}_k(p^\ell) = p^{\lceil \ell/k \rceil}. \tag{3-1}$$

Recall that we use $\tau_k(n)$ to denote the k-folder divisor function as defined in (2-6). We begin with the following sequence of lemmas.

Lemma 3.1. *Let $k, y, X, H \geqslant 1$ be integers. Then $M_k(X, H; y) := \{(x, t_1, t_2, \dots, t_k) \in [X] \times [H]^k : y \mid (x + t_1)(x + t_2) \cdots (x + t_k)\}$ has size at most $H^k \tau_k(y) \cdot (1 + X/\mathrm{rad}_k(y))$.*

Proof. Suppose that $y \mid (x + t_1) \dots (x + t_k)$. Then there exist integers $y_1, \dots, y_k \geqslant 1$ with $y_1 \cdots y_k = y$ and $y_i \mid x + t_i$ $(1 \leqslant i \leqslant k)$.

For any given choice of $y_1, \dots, y_k, t_1, \dots, t_k$, the conditions $y_i \mid x + t_i$, when satisfiable, impose on x a congruence condition modulo $[y_1, \dots, y_k]$. It follows that for any given $t_1, \dots, t_k$, the number of values of $x \in [X]$ with $(x, t_1, \dots, t_k) \in M_k(X, H; y)$ is at most

$$\sum_{y_1 \cdots y_k = y} (1 + X/[y_1, \dots, y_k]) \leqslant \tau_k(y) \cdot (1 + X/\mathrm{rad}_k(y)).$$

Lemma 3.1 follows upon summing over $t_1, \dots, t_k \in [H]$. □

Remark 3.2. For a typical value of $y \leqslant X$, Lemma 3.1 saves a factor of roughly y over the trivial bound $H^k X$, even if $H \leqslant X^{1-\delta}$, say. Lemma 3.1 is close to optimal on average over $y \leqslant X$, as one can prove by considering prime values of y, for instance. In some regimes, one can do better by other arguments: one can first fix a choice of y_i (then select x and choose $t_i \equiv -x \mod y_i$) to get

$$|M_k(X, H; y)| \leqslant \sum_{y_1 \cdots y_k = y} X \prod_i (1 + H/y_i) \leqslant \tau_k(y) X \max_{y_1 \cdots y_k = y} \prod_i (1 + H/y_i),$$

which beats Lemma 3.1 when $H \geqslant y$ and $y/\mathrm{rad}_k(y)$ is large, but not in general.

Lemma 3.3. *Let $k, y, X, H \geqslant 1$ be integers. Then $B_k(X, H; y)$, which denotes the set of integer tuples $(x, t_1, \dots, t_k, h_1, \dots, h_k) \in [X] \times [H]^k \times [-H, H]^k$ with $y \mid (x + t_1)(x + t_2) \cdots (x + t_k) h_1 h_2 \cdots h_k$ and $h_1 h_2 \cdots h_k \neq 0$, has size at most $O(H)^{2k} \cdot \tau_2(y) \tau_k(y)^2 \cdot O(1 + X/\mathrm{rad}_k(y))$.*

Proof. We write $y = uv$ with $u \mid (x + t_1)(x + t_2) \cdots (x + t_k)$ and $v \mid h_1 h_2 \cdots h_k$ (where $u, v \geqslant 1$). The number of choices of (u, v) is $\leqslant \tau_2(y)$. Using the notation in Lemma 3.1 and Proposition 2.1, we then find that

$$|B_k(X, H; y)| \leqslant \sum_{uv = y} |M_k(X, H; u)| \cdot N_k(H; v) \leqslant \tau_2(y) \max_{uv = y} |M_k(X, H; u)| \cdot N_k(H; v).$$

Now for any fixed u, v, we apply Lemma 3.1 to bound $|M_k(X, H; u)|$ and (2-2) to bound $N_k(H; v)$, getting

$$|M_k(X, H; u)| \leqslant H^k \tau_k(u) \cdot (1 + X/\operatorname{rad}_k(u)) \quad \text{and} \quad N_k(H; v) \leqslant (2H)^k \tau_k(v)/v,$$

respectively. This leads to the total bound

$$|B_k(X, H; y)| \ll \tau_2(y) H^{2k} \tau_k(y)^2 \cdot \left(1 + \frac{X}{v \operatorname{rad}_k(u)}\right).$$

For any $uv = y$, we have

$$v \operatorname{rad}_k(u) \geqslant \operatorname{rad}_k(y),$$

by the multiplicativity of rad_k, the formula (3-1), and the inequality $p^{\ell_2} p^{\lceil \ell_1/k \rceil} \geqslant p^{\lceil (\ell_1+\ell_2)/k \rceil}$ (valid for all primes p and integers $\ell_1, \ell_2 \geqslant 0$). Thus we complete the proof. □

If we allowed $h_1 h_2 \cdots h_k = 0$, we would have $X \cdot O(H)^{2k-1}$ tuples in $B_k(X, H; y)$. Lemma 3.3 gives a relative saving of roughly y/H on average over $y \ll X$; this follows from (the proof of) Lemma 3.5 below, whose proof requires the following lemma.

Lemma 3.4. *Let $K, k \geqslant 2$ be integers. For integers $i \geqslant 1$, let*

$$c_i := \sum_{(i-1)k < j \leqslant ik} \binom{j+K-1}{K-1}.$$

Then $c_i \leqslant K^K (ik)^K$. Furthermore, for all primes p and reals $s > 1$, we have

$$\sum_{j \geqslant 1} \tau_K(p^j) \frac{p^j}{\operatorname{rad}_k(p^j)} p^{-js} \leqslant 1 + \frac{c_1}{p^s} + \frac{c_2}{p^{2s}} + \cdots.$$

Proof. The first part is clear, since $c_i \leqslant \sum_{0 \leqslant j \leqslant ik} \binom{j+K-1}{K-1} = \binom{ik+K}{K} \leqslant (K+ik)^K \leqslant K^K (ik)^K$ (since $K, k \geqslant 2$). The second part follows from the inequality

$$\sum_{(i-1)k < j \leqslant ik} \frac{\tau_K(p^j) p^j}{\operatorname{rad}_k(p^j) p^{js}} = \sum_{(i-1)k < j \leqslant ik} \frac{\binom{j+K-1}{K-1}}{p^{\lceil j/k \rceil} p^{j(s-1)}} \leqslant \sum_{(i-1)k < j \leqslant ik} \frac{\binom{j+K-1}{K-1}}{p^i p^{i(s-1)}} = \frac{c_i}{p^{is}},$$

which holds because we have $\lceil j/k \rceil = i$ and $j \geqslant i$ whenever $(i-1)k < j \leqslant ik$. □

It turns out that to prove the key Lemma 3.7 (below) for Theorem 1.6, we need a bound of the form (3-2).

Lemma 3.5. *Let $k, X, H \geqslant 1$ be integers with X large and $H \leqslant X$. There exists a positive integer $C_k = O(k^{O(k^{O(k)})})$ (depending only on k) such that the following holds*:

$$\mathbb{E}_{x \in [X]} \sum_{y \in (x, x+H]} \tau_{2k}(y)^{2k} \cdot \tau_2(y) \tau_k(y)^2 \cdot (1 + X/\operatorname{rad}_k(y)) \ll_k H (\log X)^{C_k}. \tag{3-2}$$

Proof. The case $k = 1$ is clear by (2-11) (since $\mathrm{rad}_1(y) = y$), so suppose $k \geqslant 2$ for the remainder of this proof. Let $K := (2k)^{2k} \cdot 2k^2 \leqslant k^{4k+3}$. Then $\tau_{2k}(y)^{2k}\tau_2(y)\tau_k(y)^2 \leqslant \tau_K(y)$, since for all integers $j_1, j_2 \geqslant 1$ we have $\tau_{j_1}(y)\tau_{j_2}(y) \leqslant \tau_{j_1 j_2}(y)$ by [Benatar et al. 2022, (3.2)]. By Rankin's trick, the left-hand side of (3-2) is therefore at most H times

$$\sum_{y \leqslant x+H} \tau_K(y) \cdot (X^{-1} + \mathrm{rad}_k(y)^{-1}) \ll_K (\log X)^{K-1} + \sum_{n \geqslant 1} \tau_K(n) \frac{n}{\mathrm{rad}_k(n)} n^{-1-1/\log X}.$$

By Lemma 3.4 and the multiplicativity of τ_K and rad_k, we find that for $s > 1$, we have

$$\sum_{n \geqslant 1} \tau_K(n) \frac{n}{\mathrm{rad}_k(n)} n^{-s} \leqslant \prod_{p \geqslant 2} \left(1 + \frac{c_1}{p^s} + \frac{c_2}{p^{2s}} + \cdots\right), \tag{3-3}$$

where $c_i \leqslant K^K (ik)^K \leqslant K^{2K}(2K)^K 2^{i/2}$ (since $k \leqslant K$ and $i^K/2^{i/2} \leqslant (2K/\log 2)^K/e^K$, and $e \log 2 \geqslant 1$). But then

$$\frac{c_2}{p^2} + \frac{c_3}{p^3} + \cdots \ll \frac{K^{4K}}{p^2}.$$

Therefore, the right-hand side of (3-3) is at most

$$\prod_{p \geqslant 2} \left(1 + \frac{1}{p^s}\right)^{c_1} \prod_{p \geqslant 2} \left(1 + \frac{1}{p^2}\right)^{O(K^{4K})}.$$

After plugging in $s = 1 + 1/\log X$ and the bound $c_1 \leqslant K^{2K}$, Lemma 3.5 follows. □

We also need a simple but finicky combinatorial estimate.

Lemma 3.6. *Let $k, x, H \geqslant 1$ be integers. Let $\mathcal{A}_{1,2}(x, H)$ be the number of tuples $(a_1, \ldots, a_{2k}) \in (x, x + H]^{2k}$ satisfying both*

(1) $\{a_1, \ldots, a_k\} = \{a_{k+1}, \ldots, a_{2k}\}$ (*in the usual sense, without multiplicities*), *and*

(2) $a_1 \cdots a_k = a_{k+1} \cdots a_{2k}$.

Let $\mathcal{A}_1(x, H)$ be the number of tuples $(a_1, \ldots, a_{2k}) \in (x, x + H]^{2k}$ satisfying (1) (*but not necessarily* (2)). *Then $\mathcal{A}_{1,2}(x, H) \geqslant k!H^k - O_k(H^{k-1})$ and $\mathcal{A}_1(x, H) \leqslant k!H^k + O_k(H^{k-1})$.*

Proof. Call a tuple $(a_1, \ldots, a_{2k}) \in (x, x + H]^{2k}$ good if it satisfies (1). Let $\mathcal{A}_1^\star$ be the number of good tuples where $a_1, \ldots, a_k$ are pairwise distinct. Let $\mathcal{A}_1^\dagger$ be the number of remaining good tuples, namely good tuples where $\prod_{1 \leqslant i < j \leqslant k}(a_i - a_j) = 0$. Then $\mathcal{A}_1 \leqslant \mathcal{A}_1^\star + \mathcal{A}_1^\dagger$.

Clearly $\mathcal{A}_1^\star = k!H(H-1)\cdots(H-k+1)$ (since when the a_i are all different for $1 \leqslant i \leqslant k$, condition (1) implies that $(a_{k+1}, \ldots, a_{2k})$ is a permutation of $(a_1, \ldots, a_k)$; and conversely, when $(a_{k+1}, \ldots, a_{2k})$ is a permutation of $(a_1, \ldots, a_k)$, both (1) and (2) hold). Furthermore, $\mathcal{A}_{1,2} \geqslant \mathcal{A}_1^\star$.

On the other hand, $\mathcal{A}_1^\dagger \leqslant \binom{H}{k-1} \cdot (k-1)^{2k}$ (since if $\prod_{1 \leqslant i < j \leqslant k}(a_i - a_j) = 0$, then $\{a_1, \ldots, a_k\}$ must lie in some $(k-1)$-element subset $S \subseteq (x, x + H]$, and then condition (1) implies that each of $a_1, \ldots, a_{2k}$ is an element of S).

We now know $\mathcal{A}_1^\star = k!H^k + O_k(H^{k-1})$ and $\mathcal{A}_1^\dagger \ll_k H^{k-1}$. So $\mathcal{A}_{1,2} \geqslant \mathcal{A}_1^\star \geqslant k!H^k - O_k(H^{k-1})$, and $\mathcal{A}_1 \leqslant \mathcal{A}_1^\star + \mathcal{A}_1^\dagger \leqslant k!H^k + O_k(H^{k-1})$. □

Given integers $x_1, x_2, H \geqslant 1$, let $I_j = (x_j, x_j + H]$ for $j \in \{1, 2\}$. We are now ready to estimate the size of the set

$$\{(n_1, n_2, \dots, n_{2k}; m_1, m_2, \dots, m_{2k}) \in I_1^{2k} \times I_2^{2k} : n_1 \cdots n_k m_1 \cdots m_k = n_{k+1} \cdots n_{2k} m_{k+1} \cdots m_{2k}\}. \tag{3-4}$$

Lemma 3.7. *Fix an integer $k \geqslant 1$; let $\mathcal{C}_k$ be as in Lemma 3.5. Let X, H be large integers with $H := H(X) \to +\infty$ as $X \to +\infty$. Suppose $H \ll X(\log X)^{-2\mathcal{C}_k}$. Then in expectation over $x_1, x_2 \in [X]$, the size of the set (3-4) is $k!^2 H^{2k} + o_{X\to+\infty}(H^{2k})$.*

Proof. We roughly follow the proof from Section 2 of Theorem 1.3; however, the present situation is more complicated in some aspects, which we address using some new symmetry tricks.

First, let $T_k^\star(I_1, I_2)$ be the subset of (3-4) satisfying the following conditions:

(1) If $u \in \{m_{k+1}, \dots, m_{2k}\}$, then $u \in \{m_1, \dots, m_k\}$.

(2) If $u \in \{m_1, \dots, m_k\}$, then $u \in \{m_{k+1}, \dots, m_{2k}\}$.

(3) If $u \in \{n_{k+1}, \dots, n_{2k}\}$, then $u \in \{n_1, \dots, n_k\}$.

(4) If $u \in \{n_1, \dots, n_k\}$, then $u \in \{n_{k+1}, \dots, n_{2k}\}$.

In the notation of Lemma 3.6, applied with $a = m$ and $a = n$ (separately), we have $\#T_k^\star(I_1, I_2) \geqslant \mathcal{A}_{1,2}(x_1, H)\mathcal{A}_{1,2}(x_2, H)$ and $\#T_k^\star(I_1, I_2) \leqslant \mathcal{A}_1(x_1, H)\mathcal{A}_1(x_2, H)$, so

$$\#T_k^\star(I_1, I_2) = (k!H^k + O_k(H^{k-1}))^2 = k!^2 H^{2k} + O_k(H^{2k-1}). \tag{3-5}$$

In general, given an element $\mathfrak{n} \in I_1^{2k} \times I_2^{2k}$ of (3-4), let $\mathcal{U}$ be the set of integers u that violate at least one of the conditions (1)–(4) above. Then $\mathfrak{n} \in T_k^\star(I_1, I_2)$ if and only if $\mathcal{U} = \varnothing$. This simple observation will help us estimate the size of (3-4).

Let $N_k^\star(I_1, I_2)$ be the subset of (3-4) satisfying the following conditions:

(1) $n_{2k} \notin \{n_1, \dots, n_k\}$. (This implies, but is not equivalent to, $n_{2k} \in \mathcal{U}$.)

(2) If $u \in \mathcal{U}$, then $\tau_{2k}(u) \leqslant \tau_{2k}(n_{2k})$.

Then (3-4) has size at least $\#T_k^\star(I_1, I_2)$ and we claim that (3-4) has size at most

$$\leqslant \#T_k^\star(I_1, I_2) + 2k \cdot \#N_k^\star(I_1, I_2) + 2k \cdot \#N_k^\star(I_2, I_1).$$

First note that for each element $\mathfrak{n}$ of (3-4) lying outside of $T_k^\star(I_1, I_2)$, there exist $v \in \mathcal{U}$ and $(a, b, c) \in \{m, n\} \times \{0, k\} \times [k]$, with $\tau_{2k}(v) = \max_{u\in\mathcal{U}} \tau_{2k}(u)$, such that $a_{b+c} = v$ and $a_{b+c} \notin \{a_{(k-b)+i} : i \in [k]\}$; the existence of v with $\tau_{2k}(v) = \max_{u\in\mathcal{U}} \tau_{2k}(u)$ follows from the fact that $\mathcal{U} \neq \varnothing$, and the existence of (a, b, c) then follows from the definition of $\mathcal{U}$. The claim then follows from the definitions of $N_k^\star(I_1, I_2)$ and $N_k^\star(I_2, I_1)$, upon summing over all possibilities for a, b, c.

It follows that in expectation over $x_1, x_2 \in [X]$, the size of (3-4) is

$$\mathbb{E}_{x_1,x_2} \# T_k^\star(I_1, I_2) + O(2k \cdot \mathbb{E}_{x_1,x_2} \# N_k^\star(I_1, I_2)). \tag{3-6}$$

The projection $I_1^{2k} \times I_2^{2k} \ni (n_1, \dots, n_{2k}; m_1, \dots, m_{2k}) \mapsto (n_1, \dots, n_k; m_1, \dots, m_k; n_{2k}) \in I_1^k \times I_2^k \times I_1$, i.e., "forgetting" $n_{k+1}, \dots, n_{2k-1}, m_{k+1}, \dots, m_{2k}$, defines a map π from $N_k^\star(I_1, I_2)$ to the set

$$D_k^\star(I_1, I_2) := \{(n_1, \dots, n_k; m_1, \dots, m_k; n_{2k}) \in I_1^k \times I_2^k \times I_1 : n_{2k} \mid n_1 \cdots n_k m_1 \cdots m_k, n_{2k} \notin \{n_1, \dots, n_k\}\}.$$

We now bound the fibers of π. Suppose $(n_1, \dots, n_{2k}; m_1, \dots, m_{2k}) \in N_k^\star(I_1, I_2)$. Let $S_1 := \{i \in \{k+1, \dots, 2k\} : n_i \notin \mathcal{U}\}$ and $S_2 := \{j \in \{k+1, \dots, 2k\} : m_j \notin \mathcal{U}\}$, and let

$$z := \prod_{i \in \{k+1,\dots,2k\} \setminus S_1} n_i \prod_{j \in \{k+1,\dots,2k\} \setminus S_2} m_j = \frac{n_1 \cdots n_k m_1 \cdots m_k}{\prod_{i \in S_1} n_i \prod_{j \in S_2} m_j}.$$

Then the following hold:

- $n_i \in \{n_1, \dots, n_k\}$ for all $i \in S_1$, and $m_j \in \{m_1, \dots, m_k\}$ for all $j \in S_2$.
- z depends only on $n_1, \dots, n_k, m_1, \dots, m_k, (n_i)_{i \in S_1}, (m_j)_{j \in S_2}$.
- $\tau_{2k-|S_1|-|S_2|}(z) \leqslant \tau_{2k}(z) \leqslant \tau_{2k}(n_{2k})^{2k-|S_1|-|S_2|}$. (The upper bound on $\tau_{2k}(z)$ arises as follows: since z is the product of $2k - |S_1| - |S_2|$ elements u_l of $\mathcal{U}$, we have an upper bound $\leqslant \prod_{1 \leqslant l \leqslant 2k-|S_1|-|S_2|} \tau_{2k}(u_l)$, which is $\leqslant \prod_{1 \leqslant l \leqslant 2k-|S_1|-|S_2|} \tau_{2k}(n_{2k})$.)

Therefore, the fiber of π over $(n_1, \dots, n_k; m_1, \dots, m_k; n_{2k}) \in D_k^\star(I_1, I_2)$ has size at most

$$\sum_{S_1, S_2 \subseteq \{k+1,\dots,2k\}} k^{|S_1|} \cdot k^{|S_2|} \cdot \tau_{2k}(n_{2k})^{2k-|S_1|-|S_2|} = \sum_{0 \leqslant l \leqslant 2k} \binom{2k}{l} k^l \tau_{2k}(n_{2k})^{2k-l}, \tag{3-7}$$

where each S_t $(1 \leqslant t \leqslant 2)$ runs through all possible subsets of $\{k+1, \dots, 2k\}$.

The right-hand side of (3-7) equals $(k + \tau_{2k}(n_{2k}))^{2k} \leqslant (k+1)^{2k} \tau_{2k}(n_{2k})^{2k}$, so upon summing over $(n_1, \dots, n_k; m_1, \dots, m_k; n_{2k}) \in D_k^\star(I_1, I_2)$, we conclude that

$$\# N_k^\star(I_1, I_2) \leqslant (k+1)^{2k} \sum_{(n_1,\dots,n_k; m_1,\dots,m_k; n_{2k}) \in D_k^\star(I_1, I_2)} \tau_{2k}(n_{2k})^{2k}. \tag{3-8}$$

We use (3-8) to bound $\mathbb{E}_{x_2} \# N_k^\star(I_1, I_2)$. Note that if $(n_1, \dots, n_k; m_1, \dots, m_k; n_{2k}) \in D_k^\star(I_1, I_2)$ and $y := n_{2k}$ (so that in particular, $m_i - x_2 \in [H]$ and $n_i - y \in [-H, H] \setminus \{0\}$ for all $i \in [k]$), then $y \in (x_1, x_1 + H]$ and

$$(x_2, m_1 - x_2, \dots, m_k - x_2, n_1 - y, \dots, n_k - y) \in B_k(X, H; y),$$

in the notation of Lemma 3.3. Therefore, summing (3-8) over $x_2 \in [X]$ gives the inequality

$$X \cdot \mathbb{E}_{x_2} \# N_k^\star(I_1, I_2) = \sum_{x_2 \in [X]} \# N_k^\star(I_1, I_2) \ll_k \sum_{y \in (x_1, x_1+H]} \tau_{2k}(y)^{2k} \cdot |B_k(X, H; y)|.$$

We next apply Lemma 3.3 to give an upper bound on $|B_k(X, H; y)|$, which leads to

$$X \cdot \mathbb{E}_{x_2} \# N_k^\star(I_1, I_2) \ll_k \sum_{y \in (x_1, x_1+H]} \tau_{2k}(y)^{2k} O(H)^{2k} \cdot \tau_2(y)\tau_k(y)^2 \cdot O(1 + X/\operatorname{rad}_k(y)).$$

Average over x_1 by using Lemma 3.5, to get

$$\mathbb{E}_{x_1,x_2} \# N_k^\star(I_1, I_2) \ll_k O(H)^{2k} \cdot H \cdot X^{-1} (\log X)^{\mathcal{C}_k}. \tag{3-9}$$

This is $\ll_k H^{2k}(\log X)^{-\mathcal{C}_k}$ in our range of H. By (3-5) and (3-9), quantity (3-6) is $k!^2 H^{2k} + O_k(H^{2k-1}) + O_k(H^{2k}(\log X)^{-\mathcal{C}_k})$. Lemma 3.7 follows. □

Proof of Theorem 1.6. Assume A is large and $H \ll X(\log X)^{-C_k}$, where $C_k = Ak^{Ak^{Ak}}$. Let $C := 10$, so that the quantity $E(k) = 2k^2 + 2$ in Theorem 1.3 satisfies

$$E(k) \leqslant 4Ck^2, \quad E(k+\ell) \leqslant 5Ck^2 \quad \text{for all } 1 \leqslant \ell \leqslant k-1. \tag{3-10}$$

(This is just for uniform notational convenience.)

(a) We prove (1-5), a bound on the quantity

$$\mathbb{E}_f (\mathbb{E}_x |A_H(f, x)|^{2k} - k!)^2, \tag{3-11}$$

where $A_H(f, x)$ is defined as in (1-1). By expanding the square, we can rewrite (3-11) as

$$\mathbb{E}_f (\mathbb{E}_x |A_H(f, x)|^{2k})^2 - 2k! \mathbb{E}_f \mathbb{E}_x |A_H(f, x)|^{2k} + k!^2. \tag{3-12}$$

The subtracted term in (3-12) can be computed by switching the summation: it equals

$$-2k! \mathbb{E}_x \mathbb{E}_f |A_H(f, x)|^{2k}. \tag{3-13}$$

We estimate (3-13) by a combination of trivial bounds (based on the divisor bound (2-1)) and the moment estimate in Theorem 1.3. We split the sum $\mathbb{E}_x \mathbb{E}_f |A_H(f, x)|^{2k}$ into two ranges, and apply Theorem 1.3 and (3-10), to get that $X \cdot \mathbb{E}_x \mathbb{E}_f |A_H(f, x)|^{2k}$ equals

$$\begin{aligned} &\sum_{1 \leqslant x \leqslant H(\log X)^{5Ck^2}} \mathbb{E}_f |A_H(f, x)|^{2k} + \sum_{H(\log X)^{5Ck^2} \leqslant x \leqslant X} \mathbb{E}_f |A_H(f, x)|^{2k} \\ &\qquad = \sum_{1 \leqslant x \leqslant H(\log X)^{5Ck^2}} O((\log X)^{4Ck^2}) + \sum_{H(\log X)^{5Ck^2} \leqslant x \leqslant X} (k! + O((\log X)^{-Ck^2})). \end{aligned}$$

Upon summing over both ranges of x above, it follows that $\mathbb{E}_x \mathbb{E}_f |A_H(f, x)|^{2k} = k! + o_{X \to +\infty}(1)$ in the given range of H (provided A is large enough that $C_k \geqslant 10Ck^2$).

We next focus on the first term in (3-12). We expand out the expression and switch the expectations to get that the first term in (3-12) is

$$\mathbb{E}_{x_1} \mathbb{E}_{x_2} \mathbb{E}_f |A_H(f, x_1)|^{2k} |A_H(f, x_2)|^{2k}. \tag{3-14}$$

Now we use orthogonality and apply Lemma 3.7 to see that (3-14) is $k!^2 + o_{X\to+\infty}(1)$ in the given range of H (if A is sufficiently large). Combining the above together, (1-5) follows.

(b) We prove (1-6), a bound on the quantity (in the notation $A_H(f,x)$ from (1-1))

$$\mathbb{E}_f|\mathbb{E}_x[A_H(f,x)^k\overline{A_H(f,x)^\ell}]|^2 = X^{-2}\sum_{x_1,x_2\in[X]}\mathcal{B}_H(x_1,x_2), \tag{3-15}$$

where $1\leqslant\ell\leqslant k-1$ and $\mathcal{B}_H(x_1,x_2) := \mathbb{E}_f A_H(f,x_1)^k\overline{A_H(f,x_1)^\ell A_H(f,x_2)^k}A_H(f,x_2)^\ell$. This is the same as counting solutions to

$$n_1n_2\cdots n_k\cdot m_1m_2\cdots m_\ell = n_{k+1}n_{k+2}\cdots n_{k+\ell}\cdot m_{\ell+1}m_{\ell+2}\cdots m_{\ell+k}, \tag{3-16}$$

where $x_1\leqslant n_i\leqslant x_1+H$ and $x_2\leqslant m_i\leqslant x_2+H$ for all $1\leqslant i\leqslant k+\ell$. Suppose that $x_1\geqslant x_2$. The left-hand side in (3-16) is

$$n_1n_2\cdots n_k\cdot m_1m_2\cdots m_\ell \geqslant x_1^k x_2^\ell,$$

while the right-hand side in (3-16) is

$$n_{k+1}n_{k+2}\cdots n_{k+\ell}\cdot m_{\ell+1}m_{\ell+2}\cdots m_{\ell+k} \leqslant (x_1+H)^\ell(x_2+H)^k \leqslant x_1^\ell x_2^k\bigl(1+\tfrac{H}{x_2}\bigr)^{k+\ell}.$$

To make them equal, we must have

$$x_1/x_2 \leqslant (x_1/x_2)^{k-\ell} \leqslant \bigl(1+\tfrac{H}{x_2}\bigr)^{2k},$$

which implies that (under the assumption $Hk = o(x_2)$)

$$x_2\leqslant x_1\leqslant x_2+O(kH).$$

From now on, we only need to consider two cases:

(1) $\min(x_1,x_2)\ll kH$.

(2) $|x_1-x_2| = O(kH)$.

We first deal with case (1): $\min(x_1,x_2)\ll kH$. By the Cauchy–Schwarz inequality,

$$|\mathcal{B}_H(x_1,x_2)|^2 \ll_k (\mathbb{E}_f|A_H(f,x_1)|^{2(k+\ell)})\cdot(\mathbb{E}_f|A_H(f,x_2)|^{2(k+\ell)}).$$

Theorem 1.3 and (3-10) imply that $\mathcal{B}_H(x_1,x_2)\ll_k(\log X)^{5Ck^2}$. So the contribution to (3-15) over all pairs (x_1,x_2) with $\min\{x_1,x_2\}\leqslant H$ is at most $\ll 1/(\log X)^{C_k-10Ck^2}$, which is $o_{X\to+\infty}(1)$ by our choice of C_k.

We next deal with case (2): $|x_1-x_2| = O(kH)$. Assume $x_2<x_1$. Then all the variables m_i, n_j are in $[x_2, x_1+H]$, so by Theorem 1.3 and (3-10), the contribution in this case to (3-16) over x_1, x_2 is at most

$$\ll_k XH(\log X)^{10Ck^2}\cdot H^{k+\ell}(\log X)^{5Ck^2} \ll X^2(\log X)^{15Ck^2-C_k}\cdot H^{k+\ell} = X^2\cdot o_{X\to+\infty}(H^{k+\ell}),$$

by our choice of C_k. After dividing by $X^2H^{k+\ell}$, we see that the total contribution to (3-15) in this case is $o_{X\to+\infty}(1)$.

Combining the two cases above, we obtain the desired (1-6). □

4. Concluding remarks

Recall the exponent $E'(k)$ defined after Theorem 1.3. As we mentioned before, Theorem 1.3 implies $E'(k) \leqslant E(k) = 2k^2 + 2$, and the truth may be that $E'(k)$ grows linearly in k. The method used in [de la Bretèche 2001b; Harper et al. 2015; Heap and Lindqvist 2016] may help to extend Theorem 1.3, i.e., to improve on the bound $E'(k) \leqslant E(k)$. Alternatively, one might try to improve on Theorem 1.3 via Hooley's Δ-function technique [1979]; note that $(x, x+H] \subseteq (x, ex]$ if $H \leqslant x$.

The true threshold in the problem studied in Theorem 1.2 is more delicate. A closely related problem is to understand for what range of H, as $X \to +\infty$, the following holds:

$$\frac{1}{\sqrt{H}} \sum_{X<n\leqslant X+H} f(n) \xrightarrow{d} \mathcal{CN}(0,1), \tag{4-1}$$

where f is a Steinhaus random multiplicative function over the short interval $(X, X+H]$. In contrast to the problem we studied in this paper, X is first fixed in (4-1) and the random multiplicative function f varies. For this question, it is known that [Soundararajan and Xu 2022] if $H \to +\infty$ and $H \ll X/(\log X)^{2\log 2-1+\varepsilon}$, then such a central limit theorem holds. In the other direction, by using Harper's remarkable results and methods [2020] one may be able to show that

$$\mathbb{E}_f \left| \frac{1}{\sqrt{H}} \sum_{X<n\leqslant X+H} f(n) \right| = o_{X\to+\infty}(1), \quad \text{if } H \gg \frac{X}{\exp((\log\log X)^{1/2-\varepsilon})}; \tag{4-2}$$

see [Soundararajan and Xu 2022] for more discussions. Thus, in the above range of H, the $\sqrt{H}$-normalized partial sums do not have Gaussian limiting distribution. It would be interesting to know if another choice of normalization would lead to a Gaussian distribution. Now we return to the question we studied in Theorem 1.2. We established "typical Gaussian behavior" over a range of the form $H \ll X/(\log X)^{W(X)} = X/(\exp(W(X)\log\log X))$ (where $H \to +\infty$). It seems that to extend the range of H so that such a Gaussian behavior holds, significant new ideas would be needed. It would be interesting to understand the whole story for all ranges of H, for both the question studied in Theorem 1.2 and that in (4-1).

Acknowledgements

We thank Andrew Granville and the anonymous referee for many detailed comments that led us to significantly improve the results and presentation of our work. We thank and Adam Harper for helpful discussions and useful comments and corrections on earlier versions. We also thank Yuqiu Fu, Larry Guth, Kannan Soundararajan, Katharine Woo, and Liyang Yang for helpful discussions. Finally, we thank Peter Sarnak for introducing us (the authors) to each other during the "50 Years of Number Theory and Random Matrix Theory" Conference at IAS and making the collaboration possible.

References

[Batyrev and Tschinkel 1998] V. V. Batyrev and Y. Tschinkel, "Manin's conjecture for toric varieties", *J. Algebraic Geom.* **7**:1 (1998), 15–53. MR Zbl

[Benatar et al. 2022] J. Benatar, A. Nishry, and B. Rodgers, "Moments of polynomials with random multiplicative coefficients", *Mathematika* **68**:1 (2022), 191–216. MR Zbl

[Billingsley 2012] P. Billingsley, *Probability and measure*, 3rd ed., Wiley, Hoboken, NJ, 2012. MR Zbl

[Bourgain et al. 2014] J. Bourgain, M. Z. Garaev, S. V. Konyagin, and I. E. Shparlinski, "Multiplicative congruences with variables from short intervals", *J. Anal. Math.* **124** (2014), 117–147. MR Zbl

[de la Bretèche 2001a] R. de la Bretèche, "Compter des points d'une variété torique", *J. Number Theory* **87**:2 (2001), 315–331. MR Zbl

[de la Bretèche 2001b] R. de la Bretèche, "Estimation de sommes multiples de fonctions arithmétiques", *Compositio Math.* **128**:3 (2001), 261–298. MR Zbl

[Chatterjee and Soundararajan 2012] S. Chatterjee and K. Soundararajan, "Random multiplicative functions in short intervals", *Int. Math. Res. Not.* **2012**:3 (2012), 479–492. MR Zbl

[Duke et al. 1993] W. Duke, J. Friedlander, and H. Iwaniec, "Bounds for automorphic L-functions", *Invent. Math.* **112**:1 (1993), 1–8. MR Zbl

[Fu et al. 2021] Y. Fu, L. Guth, and D. Maldague, "Decoupling inequalities for short generalized Dirichlet sequences", preprint, 2021. arXiv 2104.00856

[Granville and Soundararajan 2001] A. Granville and K. Soundararajan, "Large character sums", *J. Amer. Math. Soc.* **14**:2 (2001), 365–397. MR Zbl

[Gut 2005] A. Gut, *Probability: a graduate course*, Springer, 2005. MR Zbl

[Harper 2019] A. J. Harper, "Moments of random multiplicative functions, II: High moments", *Algebra Number Theory* **13**:10 (2019), 2277–2321. MR Zbl

[Harper 2020] A. J. Harper, "Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos", *Forum Math. Pi* **8** (2020), art. id. e1. MR Zbl

[Harper 2022] A. J. Harper, "A note on character sums over short moving intervals", preprint, 2022. arXiv 2203.09448

[Harper et al. 2015] A. J. Harper, A. Nikeghbali, and M. Radziwiłł, "A note on Helson's conjecture on moments of random multiplicative functions", pp. 145–169 in *Analytic number theory*, edited by C. Pomerance and M. T. Rassias, Springer, 2015. MR Zbl

[Heap and Lindqvist 2016] W. P. Heap and S. Lindqvist, "Moments of random multiplicative functions and truncated characteristic polynomials", *Q. J. Math.* **67**:4 (2016), 683–714. MR Zbl

[Heath-Brown 1996] D. R. Heath-Brown, "A new form of the circle method, and its application to quadratic forms", *J. Reine Angew. Math.* **481** (1996), 149–206. MR Zbl

[Henriot 2012] K. Henriot, "Nair–Tenenbaum bounds uniform with respect to the discriminant", *Math. Proc. Cambridge Philos. Soc.* **152**:3 (2012), 405–424. MR Zbl

[Hooley 1979] C. Hooley, "On a new technique and its applications to the theory of numbers", *Proc. London Math. Soc.* (3) **38**:1 (1979), 115–151. MR Zbl

[Hooley 1986] C. Hooley, "On some topics connected with Waring's problem", *J. Reine Angew. Math.* **369** (1986), 110–153. MR Zbl

[Klurman et al. 2023] O. Klurman, I. D. Shkredov, and M. W. Xu, "On the random Chowla conjecture", *Geom. Funct. Anal.* **33**:3 (2023), 749–777. MR Zbl

[Matomäki et al. 2019] K. Matomäki, M. Radziwiłł, and T. Tao, "Correlations of the von Mangoldt and higher divisor functions II: Divisor correlations in short ranges", *Math. Ann.* **374**:1-2 (2019), 793–840. MR Zbl

[Montgomery and Vaughan 2007] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory, I: Classical theory*, Cambridge Studies in Advanced Mathematics **97**, Cambridge University Press, 2007. MR Zbl

[Norton 1992] K. K. Norton, "Upper bounds for sums of powers of divisor functions", *J. Number Theory* **40**:1 (1992), 60–85. MR Zbl

[Shiu 1980] P. Shiu, "A Brun–Titchmarsh theorem for multiplicative functions", *J. Reine Angew. Math.* **313** (1980), 161–170. MR Zbl

[Soundararajan and Xu 2022] K. Soundararajan and M. W. Xu, "Central limit theorems for random multiplicative functions", preprint, 2022. arXiv 2212.06098

[Wang 2021] V. Y. Wang, "Approaching cubic diophantine statistics via mean-value L-function conjectures of random matrix theory type", preprint, 2021. arXiv 2108.03398v1

[Wang and Xu 2022] V. Y. Wang and M. W. Xu, "Paucity phenomena for polynomial products", preprint, 2022. arXiv 2211.02908

Communicated by Andrew Granville
Received 2022-07-24 Revised 2023-03-08 Accepted 2023-05-13

mayankpandey9973@gmail.com *Princeton University, Princeton, NJ, United States*

vywang@alum.mit.edu *Courant Institute, New York University, New York, NY, United States*

maxxu@stanford.edu *Department of Mathematics, Stanford University, Stanford, CA, United States*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2024 is US $525/year for the electronic version, and $770/year (+$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 18 No. 2 2024