# Bidding Games with Charging

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## Abstract

*Graph games* lie at the algorithmic core of many automated design problems in computer science. These are games usually played between two players on a given graph, where the players keep moving a token along the edges according to pre-determined rules (turn-based, concurrent, etc.), and the winner is decided based on the infinite path (aka *play*) traversed by the token from a given initial position. In bidding games, the players initially get some monetary budgets which they need to use to bid for the privilege of moving the token at each step. Each round of bidding affects the players' available budgets, which is the only form of update that the budgets experience. We introduce bidding games *with charging* where the players can additionally improve their budgets during the game by collecting vertex-dependent monetary rewards, aka the "charges." Unlike traditional bidding games (where all charges are zero), bidding games with charging allow non-trivial recurrent behaviors. For example, a reachability objective may require multiple detours to vertices with high charges to earn additional budget. We show that, nonetheless, the central property of traditional bidding games generalizes to bidding games with charging: For each vertex there exists a *threshold* ratio, which is the necessary and sufficient fraction of the total budget for winning the game from that vertex. While the thresholds of traditional bidding games correspond to unique fixed points of linear systems of equations, in games with charging, these fixed points are no longer unique. This significantly complicates the proof of existence and the algorithmic computation of thresholds for infinite-duration objectives. We also provide the lower complexity bounds for computing thresholds for Rabin and Streett objectives, which are the first known lower bounds in any form of bidding games (with or without charging), and we solve the following *repair problem* for safety and reachability games that have unsatisfiable objectives: Can we distribute a given amount of charge to the players in a way such that the objective can be satisfied?

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## 1 Introduction

Two-player *graph games* have deep connections to foundations of mathematical logic [26], and constitute a fundamental model of computations with applications in *reactive synthesis* [25] and multi-agent systems [2]. A graph game is played on a graph, called the *arena*, as follows. A token is placed on an initial vertex and the two players move the token throughout the arena to produce an infinite path, called a *play*. The winner is determined based on whether


**(a)** Strategies may depend on the available budget.

**(b)** Nontrivial solution for safety objective of Player 2 when the unsafe vertex (which is $t$) is reachable from every other vertex.

**Figure 1** Examples to demonstrate the distinctive features of bidding games with charging, compared to traditional bidding games (without charging). The double circled vertices are the ones that Player 1 wants to reach (reachability objective), or, dually, the ones that Player 2 wants to avoid (safety objective). When a vertex $v$ has nonzero reward for at least one of the players, the rewards are shown next to $v$ in the vector notation $\begin{bmatrix} R_1(v) \\ R_2(v) \end{bmatrix}$. The threshold budget of Player 1 for each vertex is shown in `blue` next to the vertex.

the play fulfills a given temporal objective (or specification). Traditionally, graph games are *turn-based*, where the players move the token in alternate turns. *Bidding games* are graph games where who moves the token at each step is determined by an auction (a *bidding*). Concretely, both players are allocated initial budgets, and in each turn, they concurrently place bids from their available budgets, the highest bidder moves the token, and pays his bid according to one of the following pre-determined mechanisms. In *Richman* bidding, the bid is paid to the lower bidder, in *poorman* bidding, the bid is paid to an imaginary "bank" and the money is lost, and in *taxman* bidding, a fixed fraction of the bid is paid to the bank (the "tax") and the rest goes to the lower bidder. The outcome of the game is an infinite play and, as usual, the winner is determined based on whether the play fulfills a given objective.

Bidding games model strategic decision-making problems where resources need to be invested dynamically towards the fulfillment of an objective. For example, a taxi driver needs to decide how to "invest" his gas supply in order to collect as many passengers as possible, internet advertisers need to invest their advertising budgets in ongoing auctions for advertising slots with the goal of maximizing visibility [5], or a coach in an NBA tournament needs to decide his roster for each game while "investing" his players' limited energy with the goal of winning the tournament [8]. While in all these scenarios the investment resources can be "charged," e.g., by visiting a gas station, by adding funds, or by allowing the players to rest, respectively, charging budgets cannot be modeled in traditional bidding games.

We study, for the first time, *bidding games with charging*, where the players can increase their available budgets by collecting vertex-dependent charges. Every vertex $v$ in the arena is labeled with a pair of non-negative rational numbers denoted $R_1(v)$ and $R_2(v)$. Suppose the game enters a vertex $v$, where for $i \in \{1, 2\}$, Player $i$'s budget is $B_i$ with $B_1 + B_2 = 1$. First, the budgets are charged to $B'_1 = B_1 + R_1(v)$ and $B'_2 = B_2 + R_2(v)$. Second, we normalize the sum of budgets to 1 by defining $B''_1 = B'_1/(B'_1 + B'_2)$ and $B''_2 = B'_2/(B'_1 + B'_2)$. Finally, the players bid from their new available budgets $B''_1$ and $B''_2$, and the bids are resolved using any of the traditional mechanisms. Note that traditional bidding games are a special case of bidding games with charging in which all charges are 0. The normalization step plays an important role and will be discussed in Ex. 4.

▶ **Example 1.** We illustrate the model and show a distinctive feature that is not present in traditional bidding games. Consider the bidding game in Fig. 1a, where the objective of Player 1, the reachability player, is to reach $d$ and the objective of Player 2, the safety player, is to prevent this. Consider Richman bidding. We show that from vertex $b$, Player 1 can win with a budget of $B_1 = \frac{1}{4} + \epsilon \leq 1$ for every $\epsilon > 0$. Player 1 bids $\frac{1}{4}$ at $b$. We consider two cases. *First*, Player 2 wins the bidding and proceeds to $c$. She pays Player 1 at least $\frac{1}{4}$, and Player 1's budget at $c$ becomes at least $\frac{1}{2} + \epsilon$. Player 1 can now win the bidding by bidding all of his budget (recall that the sum of budgets is 1), and can proceed to $d$ to win the game. *Second*, suppose that Player 2 loses the bidding at $b$. Player 1 proceeds to $a$ with a budget of $\epsilon$. We charge his budget to $B_1' = 2 + \epsilon$ and after re-normalizing his new budget becomes $B_1'' = \frac{B_1'}{3} > \frac{2}{3}$. Player 1 increases his budget by forcing the game to stay in $a$ for three consecutive turns: He first bids $\frac{1}{3}$, and his budget exceeds $(\frac{1}{3} + 2)/3 = \frac{7}{9}$, then he bids $\frac{2}{9}$ and $\frac{4}{27}$ in the following two turns, after which his budget exceeds $\frac{73}{81} > \frac{7}{8}$. Since every budget greater than $\frac{7}{8}$ suffices to guarantee winning three consecutive biddings, he can now force the game to reach $d$, resulting in a win.

We point out a distinction from traditional bidding games (without charging). In games without charging, it is known that if a player wins, he can win using a *budget agnostic* winning strategy: For every vertex $v$, there is a successor $u$ such that upon winning the bidding at $v$, the strategy proceeds to $u$ regardless of the current available budget.[1] However, it is not hard to see that there is no winning budget-agnostic strategy in the game above; indeed, in order to win, Player 1 must eventually go right from $b$, but when his budget is $0.25 < B_1 \leq 0.75$, he needs to go left and going right will make him lose. ⌟

Another distinctive feature of bidding games with charging is that safety games have non-trivial solutions, while in traditional bidding games, the only way to ensure safety is by reaching a vertex with no path to the unsafe vertices [21, 4, 7]. Therefore, charging opens doors to new applications of bidding games for when safety objectives are involved. For example, in *auction-based scheduling* [14], bidding games are used to compose two policies at runtime such that the objectives of both policies are fulfilled. With traditional bidding games, auction-based scheduling cannot support long-run safety due to the aforementioned reasons. Bidding games with charging creates the possibility to extend auction-based scheduling for richer classes of objectives than what can be supported currently.

▶ **Example 2.** We show that Player 2, the safety player, wins the game depicted in Fig. 1b starting from $b$ when Player 1's budget is $B_1 < \frac{3}{8}$. Fulfilling safety requires the game to forever loop over $a$ and $b$; such an outcome is not possible in traditional bidding games since $t$ is reachable from both $a$ and $b$. After charging at $b$, we have $B_1'' < \frac{1}{2}$. Player 2 bids $\frac{1}{2}$, trivially wins the bid and moves the token to $a$. Her budget is charged to at least $\frac{6}{7}$, meaning that Player 1's budget is at most $\frac{1}{7}$. She bids $\frac{3}{16}$, trivially wins the bidding and move the token to $b$. When entering $b$ her budget is at least $\frac{6}{7} - \frac{3}{16} > \frac{5}{8}$, meaning that Player 1's budget is less than $\frac{3}{8}$, and she can keep repeating the same strategy to win the game. ⌟

The central quantity in bidding games is the pair of *thresholds* on the players' budgets which enable them to win. Formally, for $i \in \{1, 2\}$, Player $i$'s threshold at vertex $v$, denoted $Th_i(v)$, is the smallest value in $[0, 1]$ such that for every $\epsilon > 0$, Player $i$ can guarantee winning from $v$ with an initial budget of $Th_i(v) + \epsilon$. The thresholds in the vertices in Figures 1a and 1b are depicted beside them in blue. When $Th_1(v) + Th_2(v) = 1$, we say that a *threshold*

[1] We refrain from calling the strategy *memoryless* since it might bid differently in successive visits to $v$.

**Table 1** Upper complexity bounds for bidding games with charging ("w/ chg.") in comparison with traditional bidding games ("w/o chg.").

| | Reachability | | Safety | | Büchi | | Co-Büchi | |
|---|---|---|---|---|---|---|---|---|
| | w/ chg. | w/o chg. | w/ chg. | w/o chg. | w/ chg. | w/o chg. | w/ chg. | w/o chg. |
| Richman | coNP | NP ∩ coNP | NP | NP ∩ coNP | $\Pi_2^P$ | NP ∩ coNP | $\Sigma_2^P$ | NP ∩ coNP |
| Taxman and poorman | PSPACE | PSPACE | PSPACE | PSPACE | 2-EXP | PSPACE | 2-EXP | PSPACE |

exists and define the threshold to be $Th(v) = Th_1(v)$. Existence of thresholds is a form of *determinacy*: for every Player 1 budget $B_1 \neq Th_1(v)$, one of the players has a winning strategy. We establish that bidding games with charging are also determined for reachability and Büchi objectives, and, dually, for safety and co-Büchi objectives. The proofs of these claims are however significantly more involved than the case of traditional bidding games. For instance, for traditional bidding games, the existence of thresholds for Büchi objectives follows from the existence of thresholds for reachability objectives, with the observation that for every bottom strongly connected component (BSCC), every vertex has a threshold 0 or 1, so that winning the Büchi game boils down to *reaching* one of the BSCCs with thresholds 0 (the "winning" BSCCs). This approach fails for games with charging. First, players may be able to trap the game within an SCC that is not part of any BSCC, and second, the thresholds in a BSCC might not be all 0 or 1 as seen in Ex. 2. In order to show the existence of thresholds in Büchi games, we develop a novel fixed point algorithm that is based on repeated solutions to reachability bidding games.

We study the complexity of finding thresholds. Here too, the techniques differ and are more involved than traditional bidding games. In Richman-bidding games without charging, thresholds correspond to the unique solution of a system of linear equations. In games with charging, however, thresholds correspond to the least and the greatest fixed points, and we present a novel encoding of the problem using mixed-integer linear programming. We summarize our complexity results in Tab. 1 along with a comparison with known results in traditional bidding games. Finally, we show that Richman games with Rabin and Streett objectives are NP-hard and coNP-hard, respectively. This result establishes the first lower complexity bound in any form of bidding games (with or without charging). Upper bounds for Rabin and Streett objectives are left open.

Finally, we introduce and study a *repair* problem in bidding games: Given a bidding game, a target threshold $t$ in a vertex $v$, and a repair budget $C$, decide if it is possible to add charges to the vertices of $\mathcal{G}$ in a total sum that does not exceed $C$ such that $Th(v) \leq t$. Repairing is relevant when the bidding game is not merely given to us as a fixed input, but rather the design of the game is part of the solution itself. For instance, we have already mentioned auction-based scheduling [14], where the strongest guarantees can be provided when in two bidding games that are played on the same arena, the sum of thresholds in the initial vertex is less than 1. When this requirement fails, repairing can be applied to lower the thresholds. We show that the repair problem for safety objectives is in PSPACE and for reachability objectives is in 2EXPTIME.

## Related work

Bidding games (without charging) were introduced by Lazarus et al. [22, 21], and were extended to infinite-duration objectives by Avni et al. [4, 5, 6, 9]. Many variants of bidding games have been studied, including *discrete*-bidding [19, 1, 12], which restricts the granularity of bids, *all-pay* bidding [8, 9], which model allocation of non-refundable resources, partial-information games [10], which restricts the observation power of one of the players, and non-zero-sum bidding games [23], which allow the players' objectives to be non-complimentary. The inspiration behind charging comes from other forms of resource-constrained games that allow to refill depleted resources or accumulate new resources to perform certain tasks [17, 16, 15]. The unique challenge in our case is the additional layer of bidding, which separates resource (budget) accumulation and spending.

## 2 Bidding Games with Charging

A *bidding game with charging* is a two-player game played on an arena $\langle V, E, R_1, R_2\rangle$ between Player 1 and Player 2,[2] where $V$ is a finite set of vertices, $E \subseteq V \times V$ is a set of directed edges, and $R_1, R_2 : V \to \mathbb{R}_{\geq 0}$ are the charging functions of Player 1 and Player 2, respectively. We denote the set of *successors* of the vertex $v$ by $\mathcal{S}(v) = \{u : \langle v, u\rangle \in E\}$. Bidding games with no charging will be referred to as *traditional* bidding games, which is a special case with $R_1 \equiv R_2 \equiv 0$. The default ones in this paper are bidding games with charging and, to avoid clutter, we typically refer to them simply as *bidding games*.

A bidding game proceeds as follows. A *configuration* of a bidding game is a pair $c = \langle v, B_1\rangle \in V \times [0, 1]$, which indicates that the token is placed on the vertex $v$ and Player 1's current budget is $B_1$. We always normalize the sum of budgets to 1, thus, implicitly, Player 2's budget is $B_2 = 1 - B_1$. At configuration $c$, we charge and normalize the budgets. Formally, the game proceeds to an *intermediate configuration* $c' = \langle v, B_1'\rangle$ defined by $B_1' = \frac{B_1+R_1(v)}{1+R_1(v)+R_2(v)}$. Player 2's budget becomes $B_2' = 1 - B_1' = \frac{B_2+R_2(v)}{1+R_1(v)+R_2(v)}$. Then, the players simultaneously bid for the privilege of moving the token. Formally, for $i \in \{1, 2\}$, Player $i$ chooses an *action* $\langle b_i, u_i\rangle$, where $b_i \in [0, B_i']$ and $u_i \in \mathcal{S}(v)$. Given both players' actions, the next configuration is $\langle u, B_1''\rangle$, where $u = u_1$ when $b_1 \geq b_2$ and $u = u_2$ when $b_2 > b_1$, and $B_1''$ is determined based on the *bidding mechanism* defined below. Note that we arbitrarily break ties in favor of Player 1, but it can be shown that all our results remain valid no matter how ties are resolved. In the definitions below we assume that Player 1 is the higher bidder, i.e., $b_1 \geq b_2$, and the case where $b_2 > b_1$ is dual:

**Richman bidding.** The higher bidder pays his bid to the lower bidder. Formally, $B_1'' = B_1' - b_1$, and $B_2'' = B_2' + b_1$.

**Poorman bidding.** The higher bidder pays his bid to the bank and we re-normalize the budget to sum up to 1. Formally, $B_1'' = \frac{B_1'-b_1}{1-b_1}$, and $B_2'' = \frac{B_2'}{1-b_1}$.

**Taxman bidding.** For a predetermined and fixed fraction $\tau \in [0, 1]$, called the *tax rate*, the higher bidder pays fraction $\tau$ of his bid to the bank, and the rest to the lower bidder. Formally, $B_1'' = \frac{B_1'-b_1}{1-\tau\cdot b_1}$, and $B_2'' = \frac{B_2'+(1-\tau)\cdot b_1}{1-\tau\cdot b_1}$. Note that taxman bidding with $\tau = 0$ coincides with Richman bidding and with $\tau = 1$ coincides with poorman bidding.

In a bidding game, a *history* is a finite sequence $\langle v_0, B_0\rangle, \langle v_0, B_0'\rangle, \ldots, \langle v_n, B_n\rangle, \langle v_n, B_n'\rangle$ which alternates between configurations and intermediate configurations. For $i \in \{1, 2\}$, a *strategy* for Player $i$ is a function $\pi_i$ that maps a history to an action $\langle b_i, u_i\rangle$. We typically consider *memoryless* strategies, which are functions from intermediate configurations to actions. An initial configuration $c_0 = \langle v_0, B_0\rangle$ and two strategies $\pi_1$ and $\pi_2$ give rise to an infinite *play*, denoted $\mathsf{play}(c_0, \pi_1, \pi_2)$, and is defined inductively, where the inductive step is based on the definitions above. Let $\mathsf{play}(c_0, \pi_1, \pi_2) = \langle v_0, B_0\rangle, \langle v_0, B_0'\rangle, \ldots$. The *path* that corresponds to $\mathsf{play}(c_0, \pi_1, \pi_2)$ is $v_0, v_1, \ldots \in V^\omega$.

[2] We will use the pronouns "he" and "she" for Player 1 and Player 2, respectively.

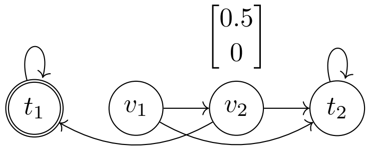



**Figure 2** Example of a poorman bidding game where without normalization thresholds are not uniquely determined.

Each game is equipped with an *objective* $\varphi \subseteq V^\omega$. Each play has a *winner*. Player 1 wins a play if its corresponding path is in $\varphi$, and Player 2 wins otherwise. For an objective $\varphi$, a Player 1 strategy $\pi_1$ is *winning* from a configuration $c$ if for every Player 2 strategy $\pi_2$, the play $\mathsf{play}(c, \pi_1, \pi_2)$ is winning for Player 1, and the definition for Player 2 is dual. For $i \in \{1, 2\}$, we say Player $i$ *wins* from configuration $c$ for $\varphi$ if he has a winning strategy from $c$. We will use $\overline{x}$ to denote the complement of $x$, where $x$ can be either an objective or a set of vertices. We consider the following objectives:

**Reachability.** For a set of vertices $T \subseteq V$, the reachability objective is defined as $\text{Reach}(T) := \{v_0 v_1 \ldots \in V^\omega \mid \exists i \in \mathbb{N} \,.\, v_i \in T\}$. Intuitively, $T$ represents the set of target vertices, and $\text{Reach}(T)$ is satisfied if $T$ is eventually visited by the given path.

**Safety.** For a set of vertices $S \subseteq V$, the safety objective is defined as $\text{Safe}(S) := \{v_0 v_1 \ldots \in V^\omega \mid \forall i \in \mathbb{N} \,.\, v_i \in S\}$. Intuitively, $S$ represents the set of safe vertices, and $\text{Safe}(S)$ is satisfied if $S$ is not left ever during the given path. Safety and reachability are dual to each other, i.e., $\text{Safe}(S) = \overline{\text{Reach}(\overline{S})}$.

**Büchi.** For a set of vertices $B \subseteq V$, the Büchi objective is defined as $\text{Büchi}(B) := \{v_0 v_1 \ldots \in V^\omega \mid \forall i \in \mathbb{N} \,.\, \exists j > i \,.\, v_j \in B\}$. Intuitively, $\text{Büchi}(B)$ is satisfied if $B$ is visited infinitely often during the given path.

**Co-Büchi.** For a set of vertices $C \subseteq V$, the co-Büchi objective is defined as $\text{Co-Büchi}(C) := \{v_0 v_1 \ldots \in V^\omega \mid \exists i \in \mathbb{N} \,.\, \forall j > i \,.\, v_j \in C\}$. Intuitively, $\text{Co-Büchi}(C)$ is satisfied if only $C$ is visited from some point onward during the play. Büchi and co-Büchi objectives are dual to each other, i.e., $\text{Co-Büchi}(C) = \overline{\text{Büchi}(\overline{C})}$.

A central concept in bidding games is the pair of *thresholds* for the two players. Roughly, they are the smallest budgets needed by the respective player for winning the game from a given vertex. We formalize this below.

▶ **Definition 3** (Thresholds). *Let $\mathcal{G}$ be a given arena and $M \in \{Richman, poorman, taxman\}$ be a given bidding mechanism . For an objective $\varphi$, the thresholds $Th_1^{\mathcal{G},M,\varphi}, Th_2^{\mathcal{G},M,\varphi} : V \to [0, 1]$ are functions such that for every $v \in V$ and every $\epsilon > 0$:*

- $Th_1^{\mathcal{G},M,\varphi}(v) := \inf_{B_1 \in [0,1]}\{B_1 :$ *Player 1 wins from* $\langle v, B_1 + \epsilon\rangle$ *for* $\varphi$*, for every* $\epsilon > 0\}$.
- $Th_2^{\mathcal{G},M,\varphi}(v) := \inf_{B_2 \in [0,1]}\{B_2 :$ *Player 2 wins from* $\langle v, 1 - B_2 - \epsilon\rangle$ *for* $\overline{\varphi}$*, for every* $\epsilon > 0\}$.

*When $Th_1^{\mathcal{G},M,\varphi}(v) + Th_2^{\mathcal{G},M,\varphi}(v) = 1$ for every vertex $v$, we say that the* threshold exists *in $\mathcal{G}$, denote it $Th^{\mathcal{G},M,\varphi}(v)$, and define $Th^{\mathcal{G},M,\varphi}(v) = Th_1^{\mathcal{G},M,\varphi}(v)$.*

Whenever the game graph and the bidding mechanism are clear from the context, we simply write $Th_1^\varphi$, $Th_2^\varphi$, and $Th^\varphi$.

▶ **Example 4** (The importance of normalization). Consider the poorman bidding game that is depicted in Fig. 2. Intuitively, Player 1 wins from $v_1$ iff he wins the first two consecutive biddings. Formally, the game starts at $v_1$ and $t_i$ is Player $i$'s target, for $i \in \{1, 2\}$. We first analyze the game with a normalization step. We argue that $Th(v_2) = \frac{1}{4}$; indeed, since

$\frac{1/4+1/2}{3/2} = \frac{1}{2}$, entering $v_2$ with a budget greater than $\frac{1}{4}$ allows Player 1 to secure winning. We argue that $Th(v_1) = \frac{4}{7}$; indeed, Player 1 must bid above Player 2's budget and win the bidding, and note that $\frac{4/7-3/7}{3/7} = \frac{1}{4}$. Note that thresholds are in fact a *ratio*. Stated differently, consider a configuration $\langle v_1, B_1, B_2 \rangle$ with $B_1 + B_2$ not necessarily equals 1, then Player 1 wins iff $\frac{B_1}{B_1+B_2} > \frac{4}{7}$. Crucially, the ratio between $B_1$ and $B_2$ is fixed. We will prove that this is a general phenomenon on which our algorithms depend.

When a normalization step is *not* performed, Player 1's threshold for winning is a non-linear function of Player 2's initial budget. Intuitively, when no normalization is performed, the charge is more meaningful when the budgets are smaller. Consider a configuration $\langle v_1, B_1, B_2 \rangle$ with $B_1 + B_2$ being not necessarily equal to 1. Note that Player 1 must win the first bidding, thus the second configuration must be $\langle v_2, B_1 - B_2, B_2 \rangle$. When no normalization is performed after charging, the intermediate configuration is $\langle v_2, B_1 - B_2 + 0.5, B_2 \rangle$. Clearly, Player 1 wins iff $B_1 - B_2 + 0.5 > B_2$. For example, when $B_2 = 1$, then Player 1's threshold is $\frac{3}{2}$ and when $B_2 = 2$, then Player 1's threshold is $\frac{7}{2}$. These amount to ratios of $\frac{3}{5}$ and $\frac{7}{11}$, respectively, meaning that Player 1's threshold is a non-linear function of Player 2's budget. We point out that this is also the case in poorman discrete-bidding games [11], where thresholds can only be approximated, even in extremely simple games. ⌟

We formulate the decision problem related to the computation of thresholds. We will write that a given objective $\varphi$ *is of type* Reach, Safe, Büchi, or Co-Büchi if $\varphi$ can be expressed as a reachability, safety, Büchi, or co-Büchi objective (on a given arena), respectively.

▶ **Definition 5** (Finding threshold budgets). *Let* $M \in \{Richman, poorman, taxman\}$*, and* $S \in \{\text{Reach}, \text{Safe}, \text{Büchi}, \text{Co-Büchi}\}$*. The problem* $\text{THRESH}_S^M$ *takes as input an arena* $\mathcal{G}$*, an initial vertex* $v$*, and an objective* $\varphi \in S$*, and accepts the input iff* $Th_1^{\mathcal{G},M,\varphi}(v) \le 0.5$*.*

## 3 Reachability Bidding Games with Charging

In this section, we show the existence of thresholds in taxman-bidding games with charging with reachability and, dually, with safety objectives. Throughout this section, we fix an arena $\mathcal{G} = \langle V, E, R_1, R_2 \rangle$. For a given set of vertices $T \subseteq V$, the objective of Player 1, the reachability player, is $\text{Reach}(T)$, and, the objective of Player 2, the safety player, is $\text{Safe}(\overline{T})$.

### 3.1 Bounded-Horizon Reachability and Safety

We start with the simpler case of *bounded-horizon* reachability objectives, and in the next section, we will extend the technique to general games. Let $t \in \mathbb{N}$. The bounded-horizon reachability, denoted $\text{Reach}(T, t)$, intuitively requires Player 1 to reach $T$ within $t$ steps. Formally, $\text{Reach}(T, t) \coloneqq \{v_0 v_1 \ldots \mid \exists i \le t \,.\, v_i \in T\}$. *Bounded-horizon safety* is the dual objective $\text{Safe}(\overline{T}, t) \coloneqq \{v_0 v_1 \ldots \mid \forall i \le t \,.\, v_i \notin T\} = V^\omega \setminus \text{Reach}(T, t)$.

In the following, we characterize the thresholds for $\text{Reach}(T, t)$ and $\text{Safe}(\overline{T}, t)$ by induction on $t$. The induction step relies on the following operator on functions.

▶ **Definition 6.** *Define the function* $\texttt{clamp}_{[0,1]}(x) \coloneqq \min(1, \max(0, x))$*; that is, given* $x$*,* $\texttt{clamp}_{[0,1]}(x) = x$*, when* $0 < x < 1$*, and otherwise it "saturates"* $x$ *at the boundaries* $0$ *or* $1$*. Let* $\tau \in [0,1]$ *be the tax rate. We define two operators on functions* $Av_1, Av_2 \colon [0,1]^V \to [0,1]^V$ *as follows. For* $i \in \{1,2\}$ *and* $f \in [0,1]^V$*:*

$$Av_i(f)(v) \coloneqq \texttt{clamp}_{[0,1]}\left( \frac{(1-\tau)f(v^-) + f(v^+)}{[f(v^+) - f(v^-) - 1]\tau + 2} \cdot (1 + R_1(v) + R_2(v)) - R_i(v) \right)$$

*where* $v^+$ *and* $v^-$ *are the successors of* $v$ *with the largest and the smallest value of* $f(\cdot)$*, respectively, i.e.,* $v^+ = \arg\max_{u \in \mathcal{S}(v)} f(u)$ *and* $v^- = \arg\min_{u \in \mathcal{S}(v)} f(u)$*.*

Note that for Richman bidding, i.e., when $\tau = 0$, for $i \in \{1, 2\}$, we have

$$Av_i(f)(v) \coloneqq \texttt{clamp}_{[0,1]}\left(\frac{f(v^+) + f(v^-)}{2} \cdot \big(1 + R_1(v) + R_2(v)\big) - R_i(v)\right)$$

In this case, the function $Av_i$ computes the average (the name "$Av$" stands for "average") of its argument $f$ on $v^-$ and $v^+$, and then performs an affine transformation followed by the saturation $\texttt{clamp}_{[0,1]}(\cdot)$ on the result. For poorman bidding, i.e., when $\tau = 1$, we have

$$Av_i(f)(v) \coloneqq \texttt{clamp}_{[0,1]}\left(\frac{f(v^+)}{f(v^+) - f(v^-) + 1} \cdot \big(1 + R_1(v) + R_2(v)\big) - R_i(v)\right)$$

We define two functions $f_1$ and $f_2$ which will be shown to coincide with the thresholds.

▶ **Definition 7.** *Define the functions $f_1, f_2 \colon V \times \mathbb{N} \to [0, 1]$ inductively on $t$. For every $v \in T$ and $t \in \mathbb{N}$, define $f_1(v, t) \coloneqq 0$ and $f_2(v, t) \coloneqq 1$. For every $v \notin T$, define $f_1(v, 0) \coloneqq 1$ and $f_2(v, 0) \coloneqq 0$, and for every $t > 0$, define $f_1(v, t) \coloneqq Av_1(f_1(\cdot, t-1))(v)$ and $f_2(v, t) \coloneqq Av_2(f_2(\cdot, t-1))(v)$.*

Lem. 8 shows that $f_1$ and $f_2$ coincide with the thresholds of the (bounded-horizon) reachability and safety players, respectively. Intuitively, for Reach$(T, 0)$, Player 1 wins with even zero budget from vertices that are already in $T$, and loses with even the maximum budget from vertices that are not in $T$. We capture this as $f_1(v, 0) = 0$ if $v \in T$, and $f_1(v, 0) = 1$ otherwise. Furthermore, if Player 1 has a budget more than $f_1(v, t)$ at $v$, then we show that he has a memoryless policy such that no matter which vertex $v'$ the token reaches in the next step, his budget will remain more than $f_1(v', t-1)$. It follows inductively that he will reach $T$ in $t$ steps from $v$. The argument for the safety player is dual. Lem. 8 also establishes the existence of thresholds.

▶ **Lemma 8.** *For every vertex $v \in V$ and $t \geq 0$, we have $Th_1^{\text{Reach}(T,t)}(v) = f_1(v, t)$ and $Th_2^{\text{Reach}(T,t)}(v) = f_2(v, t)$. Moreover, thresholds exist: $Th_1^{\text{Reach}(T,t)}(v) + Th_2^{\text{Reach}(T,t)}(v) = 1$.*

**Proof.** We sketch the proof for Richman bidding and the full proof can be found in the extended version of the paper [13]. We show $f_1(v, t) \geq Th_1^{\text{Reach}(T,t)}(v)$. It is dual to show $f_2(v, t) \geq Th_2^{\text{Reach}(T,t)}(v)$, and the other directions of the inequalities follow from the relationship $f_1(v, t) = 1 - f_2(v, t)$, which is not hard to verify. The proof of $f_1(v, t) \geq Th_1^{\text{Reach}(T,t)}(v)$ proceeds by induction over $t$. The base case, $t = 0$, is not hard to verify. For $t \geq 1$, assume that $f_1(v, t-1) \geq Th_1^{\text{Reach}(T,t-1)}(v)$, and we prove the claim for $t$. Let $v \in V$ and $B_1 > f_1(v, t)$. We describe a Player 1 winning strategy from $\langle v, B_1 \rangle$. Player 1 bids $b_1 = \frac{f_1(v^+, t-1) - f_1(v^-, t-1)}{2}$, and proceeds to $v^-$ upon winning the bidding (recall that $v^-$ is the successor of $v$ that attains the minimal value of $f_1(\cdot, t-1)$). If Player 1 wins the bidding, he pays $b_1$ to Player 2, and it can be verified that his new available budget in the next vertex $v^-$ remains above $f_1(v^-, t-1)$. On the other hand, if Player 1 loses the bidding, he receives at least $b_1$ from Player 2 (because Player 2 must have bid higher than $b_1$), and the token is moved by Player 2 to some successor $v'$ of $v$. It can be verified that even in this case, Player 1's new available budget remains above $f_1(v^+, t-1) > f_1(v', t-1)$. By the induction hypothesis, from the new vertex Player 1 can reach $T$ in at most $t-1$ steps. Therefore, from $v$ Player 1 can reach $T$ in at most $t$ steps. ◀

The following lemma establishes monotonicity of $f_1$ and $f_2$ with respect to $t$, which will play a key role in the proof of existence of thresholds for the unbounded counterparts of the objectives. Intuitively, reaching $T$ within $t$ steps is harder than reaching $T$ within $t' > t$ turns, thus less budget is needed for the latter case. Dually, guaranteeing safety for $t$ turns is easier than guaranteeing safety for $t' > t$ turns.

▶ **Lemma 9.** *For $v \in V$ and $t' > t$, it holds that $f_1(v, t') \leq f_1(v, t)$ and $f_2(v, t') \geq f_2(v, t)$.*

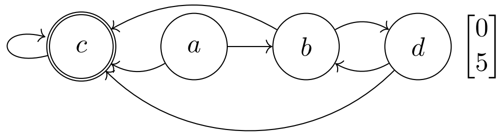



**Figure 3** Non-unique fixed points of the threshold update functions.

| $t \backslash v$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0.5 | 0 | 1 |
| 2 | 1 | 0.75 | 0.5 | 0 | 1 |
| 3 | 0.625 | 0.75 | 0.5 | 0 | 1 |
| 4 | 0.0625 | 0.5625 | 0.5 | 0 | 1 |
| 5 | 0 | 0.28125 | 0.5 | 0 | 1 |
| $\geq 6$ | 0 | 0.25 | 0.5 | 0 | 1 |

**Figure 4** Finite-Horizon reachability thresholds for the game in Fig. 1a. The number at row $i$ and column $v$ is $Th_1^{\mathrm{Reach}(d,i)}(v)$.

## 3.2 Existence of Thresholds (for Reachability and Safety Objectives)

We define two functions $f_1^*$ and $f_2^*$ which will be shown to coincide with the thresholds for the unbounded horizon reachability and safety objectives, respectively.

▶ **Definition 10.** *Define the functions $f_1^*, f_2^* : V \to [0,1]$, such that for every $v \in V$:*

$$f_1^*(v) \coloneqq \lim_{t\to\infty} f_1(v,t) \quad \textit{and} \quad f_2^*(v) \coloneqq \lim_{t\to\infty} f_2(v,t).$$

Since $f_1$ and $f_2$ are bounded in $[0,1]$ and monotonic by Lem. 9, the limits in Def. 10 are well defined. Since $f_1(v,0)$ and $f_2(v,0)$ assign, respectively, the maximum (i.e., 1) and the minimum (i.e., 0) value to every vertex $v \notin T$, hence from the Kleene fixed point theorem, it follows that $f_1^*$ and $f_2^*$ will be, respectively, the greatest and the least fixed points of the operators $Av_1$ and $Av_2$ on the directed-complete partial order $\langle [0,1]^V, \leq \rangle$.

▶ **Proposition 11.** *Consider the directed-complete partial order $L = \langle [0,1]^V, \leq \rangle$, where for every $x, y \in [0,1]^V$, $x \leq y$ iff $x_i \leq y_i$ for every $i \in V$. The functions $f_1^*$ and $f_2^*$ are, respectively, the greatest and the least fixed points of the functions $Av_1$ and $Av_2$ on $L$, subjected to the constraints $f_1^*(v) = 0$ and $f_2^*(v) = 1$ for every $v \in T$.*

The following example demonstrates that, unlike traditional bidding games, the fixed points of the functions $Av_1$ and $Av_2$ on $L$ may not be unique.

▶ **Example 12** (Multiple fixed-points)**.** Consider the bidding game in Fig. 3, where the objective of Player 1 is to reach $c$. It can be easily verified that both $f_1^* = \{a \mapsto 0.25, b \mapsto 0.5, c \mapsto 0, d \mapsto 1\}$ and $f_1^{*\prime} \equiv 0$ are fixed points of the operator $Av_1$ over $L$ in this case. ⌟

The following theorem establishes the existence of thresholds.

▶ **Theorem 13.** *For every vertex $v \in V$, it holds that $Th_1^{\mathrm{Reach}(T)}(v) = f_1^*(v)$ and $Th_2^{\mathrm{Reach}(T)}(v) = f_2^*(v)$. Moreover, thresholds exist: $Th_1^{\mathrm{Reach}(T)}(v) + Th_2^{\mathrm{Reach}(T)}(v) = 1$.*

**Proof.** We prove that $f_1^*(v) \geq Th_1^{\mathrm{Reach}(T)}(v)$, for every $v \in V$. Let $B_1 > f_1^*(v)$. There exists a $t \in \mathbb{N}$ such that $B_1 > f_1(v,t)$. Player 1 uses the strategy from Lem. 8, guaranteeing that $T$ is reached within $t$ steps. Next, we prove that $f_2^*(v) \geq Th_2^{\mathrm{Reach}(T)}(v)$, for every $v \in V$. Let $B_2 > f_2^*(v)$. We know that $f_2^* = Av_2(f_2^*)$ (from Prop. 11), and let $v^+, v^-$ be the successors of $v$ with the greatest and the least value of $f_2$. Player 2 bids $b_2 = \frac{f_2(v^+) - f_2(v^-)}{[f_2(v^+) - f_2(v^-) - 1]\tau + 2}$, which evaluates under Richman bidding to $b_2 = \frac{f_2^*(v^+) - f_2^*(v^-)}{2}$ and under poorman bidding to $b_2 = \frac{f_2(v^+)-f_2(v^-)}{f_2(v^+)-f_2(v^-)+1}$. Player 2 proceeds to $v^-$ upon winning. In the extended version [13], we prove that no matter how Player 1 bids, Player 2's strategy guarantees that in the next vertex $v'$ her new budget $B'_2 > f_2^*(v')$. Recall that by construction, $f_2^*(v'') = 1$, for every $v'' \in T$. Thus, $T$ is never reached since Player 2's budget can never exceed 1. Finally, the other directions, i.e., the inequalities $f_1^*(v) \leq Th_1^{\mathrm{Reach}(T)}(v)$ and $f_2^*(v) \leq Th_2^{\mathrm{Reach}(T)}(v)$, and the existence claim follows from the observation that $f_1^*(v) = 1 - f_2^*(v)$, for every $v \in V$. ◀

It follows that the threshold can be computed using fixed point iterations sketched in Prop. 11; this iterative approach is illustrated in the following example.

▶ **Example 14.** Consider the bidding game in Fig. 1a with Richman bidding. The finite horizon reachability thresholds are depicted in table 4. Suppose the game starts from $\langle a, 0.1\rangle$ which is winning for the reachability player. In this case, $t = 4$ is the smallest integer for which Player 1's budget 0.1 at $a$ is larger than $Th_1^{\mathrm{Reach}(d,t)}(a)$, which is 0.0625. Therefore, according to the strategy of Player 1 as described in the proof of Lem. 8, Player 1 has a strategy for reaching $d$ in 4 steps. First, his budget is charged to $\frac{0.1+2}{3} = 0.7$. His winning strategy (described in the proof of Lem. 8) dictates that he should bid $\frac{Th_1^{\mathrm{Reach}(d,3)}(b) - Th_1^{\mathrm{Reach}(d,3)}(a)}{2} =$ 0.0625. In case of winning, he pays the bid to the safety player and keeps the token at $a$, with budget 0.6375 which is then charged to $0.8791\bar{6}$. This is enough for winning 3 consecutive biddings and moving the token to $d$. In case of losing the first bid, his budget will increase to at least 0.76. This amount is more than both $Th_1^{\mathrm{Reach}(d,3)}(a)$ and $Th_1^{\mathrm{Reach}(d,3)}(b)$, therefore he can guarantee a win in at most 3 steps.

Now suppose the game starts from $\langle b, 0.2\rangle$ meaning that the safety player has 0.8 budget and can win the game. She has a budget-agnostic strategy which dictates her to bid 0.25 in $b$ (see the proof of Lem. 8 for a sketch of Player 2's strategy). She definitely wins this bidding as her opponent has only 0.2 budget. She then moves the token to $c$ and pays 0.25 to the reachability player, leaving her with 0.55 of the total budget. She can then bid 0.5, win the bidding and move the token to $e$, where the token stays indefinitely. ⌟

## 3.3 Complexity Bounds (for Reachability and Safety Objectives)

Since $f_1^*$ and $f_2^*$ are fixed points of the operators $Av_1$ and $Av_2$, respectively, hence $f_1^* = Av_1(f_1^*)$ and $f_2^* = Av_2(f_2^*)$. Moreover, $f_1^*$ is the greatest fixed point, which means that $Th_1^{\mathrm{Reach}(T)} = f_1^*$ can be computed by finding the element-wise maximum function $h$ in $[0,1]^V$ that satisfies $h(v) = 0$ for $v \in T$ and $h(v) = Av_1(h)(v)$ for $v \notin T$. This is formalized below:

$$\max_h \sum_{v \in V} h(v)$$

subjected to constraints:

$$\begin{aligned}
&\forall v \in T \,.\, h(v) = 0,\\
&\forall v \notin T \,.\, h(v) = Av_1\left(h(\cdot)\right)(v)\\
&\qquad = \mathtt{clamp}_{[0,1]}\left(\frac{(1-\tau)h(v^-) + h(v^+)}{[h(v^+) - h(v^-) - 1]\tau + 2} \cdot (1 + R_1(v) + R_2(v)) - R_1(v)\right),\\
&h(v^+) = \max_{u \in \mathcal{S}(v)} h(u), \quad h(v^-) = \max_{u \in \mathcal{S}(v)} h(u).
\end{aligned} \tag{1}$$

▶ **Proposition 15.** *The solution of the optimization problem in* (1) *is equivalent to the threshold function* $Th_1^{\mathrm{Reach}(T)}$ *of the reachability player.*

Due to Thm. 13, for every vertex $v \in V$, we have $Th_2^{\text{Reach}(T)}(v) = 1 - Th_1^{\text{Reach}(T)}(v)$. Consequently, we obtain the following upper complexity bounds.

▶ **Theorem 16.** *The following hold:*
**(i)** $\text{THRESH}^{\text{taxman}}_{\text{Reach}} \in \text{PSPACE}$ *and* $\text{THRESH}^{\text{taxman}}_{\text{Safe}} \in \text{PSPACE}$,
**(ii)** $\text{THRESH}^{\text{Richman}}_{\text{Reach}} \in \text{coNP}$ *and* $\text{THRESH}^{\text{Richman}}_{\text{Safe}} \in \text{NP}$.

**Proof.**

**Proof of (i).** It can be shown that we can construct a polynomially sized (w.r.t. the game) function $\varphi\colon \mathbb{R}^V \to \mathbb{B}$ such that for every $h \in \mathbb{R}^V$, $\varphi(h)$ is true iff $h$ is a fixed point of $Av_1$ (details can be found in the extended version [13]). Hence, if the system has a solution where $h(v) > 0.5$, the greatest fixed point $Th_1^{\text{Reach}(T)}$ satisfies $Th_1^{\text{Reach}(T)}(v) > 0.5$. This is an instance of existential theory of reals which is known to be in PSPACE. Therefore, it is possible to decide $Th_1^{\text{Reach}(T)}(v) > 0.5$ (equivalently $Th_2^{\text{Reach}(T)}(v) < 0.5$) in PSPACE.

**Proof of (ii).** We provide the following reduction from the optimization problem to an instance of MILP; a different proof with a polynomial certificate is provided in the extended version [13]. Let $O$ be the optimization problem as stated in the Section 3 with $\tau = 0$.

Let $M$ be any constant strictly greater than $\max_{u \in V}\{1 + R_1(u) + R_2(u)\}$. For each node $u$ define two new variables $h^-(u), h^+(u)$ and add the following constraints to $O$:

$$
\begin{aligned}
&h^+(w) \geq h(w)\ \forall w \in \mathcal{S}(u) && h^-(w) \leq h(w)\ \forall w \in \mathcal{S}(u)\\
&h^+(w) \leq h(w) + (1 - b_u^w)\cdot M\ \forall w \in \mathcal{S}(u) && h^-(w) \geq h(w) - (1 - c_u^w)\cdot M\ \forall w \in \mathcal{S}(u)\\
&\sum_{w \in \mathcal{S}(u)} b_u^w = 1 && \sum_{w \in \mathcal{S}(u)} c_u^w = 1\\
&b_u^w \in \{0,1\}\ \forall w \in \mathcal{S}(u) && c_u^w \in \{0,1\}\ \forall w \in \mathcal{S}(u)
\end{aligned}
$$

This guarantees that $h^+(u) = h(u^+)$ and $h^-(u) = h(u^-)$, so they can be replaced. Next, replace each $\min(1, x)$ by $\frac{(x-1)-|x-1|}{2} + 1$ and $\max(0, x)$ by $\frac{x+|x|}{2}$. Then replace each $|y|$ with a fresh variable $a_y$ and add the following constraints to $O$:
1. $-a_y \leq y \leq a_y$
2. $y + M \cdot z_y \geq a_y \wedge -y + M \cdot (1 - z_y) \geq a_y \wedge z_y \in \{0,1\}$

The first constraint ensures that $|y| \leq a_y$ and the second one that $|y| \geq a_y$. Therefore, it is guaranteed that $|y| = a_y$. The MILP instance $O$ is equivalent to the optimization problem in section 3. In order to decide whether $Th_1(v) \geq 0.5$ it suffices to decide satisfiability of $O$ with the additional constraint that $h(v) \geq 0.5$ and this decision problem is known to be in NP. ◀

## 4 Büchi Bidding Games with Charging

We proceed to Büchi objectives, for which the proof of existence of thresholds is shown to be significantly more involved than for Büchi games without charging. The key distinction is that thresholds in traditional strongly-connected Büchi games are trivial: If even one of the vertices is a Büchi target vertex, the Büchi player's threshold in *each* vertex is 0 and otherwise is 1 [3]. This property gives us a simple reduction from traditional Büchi bidding games to reachability bidding games. With charging, this property no longer holds. For example, alter the game in Fig. 1b to make it strongly-connected by adding an edge from $t$ to $b$. The thresholds remain above 0, i.e., there are initial budgets with which Player 2 wins.

Our existence proof, which is inspired by an existence proof for discrete-bidding games [12], follows a fixed-point characterization that is based on solutions to *frugal-reachability* games, which are defined below. We note that the proof has a conceptual similarity with Zielonka's algorithm [27] in turn-based Büchi games, which characterizes the set of winning vertices based on repeated calls to an algorithm for turn-based reachability games.

## 4.1 Frugal-Reachability Objectives

We introduce frugal reachability objectives. Consider a taxman-bidding game with charging $\mathcal{G} = \langle V, E, R_1, R_2 \rangle$. Let $T \subseteq V$ be a set of target vertices and $\mathsf{fr} : T \to [0, 1]$ be a function that assigns each target with a *frugal budget*. The *frugal reachability* objective $\mathrm{FrugalReach}(T, \mathsf{fr})$ requires Player 1 to reach $T$ such that the first time a vertex $v \in T$ is reached, Player 1's budget must exceed $\mathsf{fr}(v)$, thus:

$$\mathrm{FrugalReach}(T, \mathsf{fr}) \coloneqq \{\langle v_0, B_1^0 \rangle \langle v_1, B_1^1 \rangle \ldots \mid \exists i \,.\, v_i \in T \wedge B_1^i > \mathsf{fr}(v_i) \wedge \forall j < i \,.\, v_j \notin T\}$$

We stress that $\mathrm{FrugalReach}(T, \mathsf{fr})$ is a set of *plays*, whereas the other objectives we have considered so far (reachability, Büchi, etc.) were sets of *paths*.

Existence of thresholds $Th_1^{\mathrm{FrugalReach}(T,\mathsf{fr})}$ and $Th_2^{\mathrm{FrugalReach}(T,\mathsf{fr})}$ for the frugal-reachability objective and its dual are shown in the following theorem. The proof can be found in the extended version [13] and follows similar arguments as reachability bidding games with the following change in the base case. For $v \in T$ and $t \in \mathbb{N}$, recall that we define $f_1(v, t) = 0$ (Def. 7), which intuitively means that Player 1 wins if he reaches $v$ with *any* budget. Instead, we now define $f_1(v, t) = \mathsf{fr}(v)$, requiring Player 1 to reach $v$ with a budget of $\mathsf{fr}(v)$. Dually, we define $f_2(v, t) = 1 - \mathsf{fr}(v)$.

▶ **Theorem 17.** *The thresholds* $Th_1^{\mathrm{FrugalReach}(T,\mathsf{fr})}$ *and* $Th_2^{\mathrm{FrugalReach}(T,\mathsf{fr})}$ *exist.*

## 4.2 Bounded-Visit Büchi and Co-Büchi

We first prove the existence of thresholds for the simpler case of bounded-visit Büchi and co-Büchi objectives, where we impose, respectively, lower and upper bounds on the number of visits to the Büchi target vertices $B \subseteq V$. Let $k \in \mathbb{N}$ be a given bound. The bounded-visit Büchi, denoted as $\mathrm{Büchi}(B, k)$, intuitively requires Player 1 to visit $B$ at least $k$ times. Formally, $\mathrm{Büchi}(B, k) \coloneqq \{v_0 v_1 \ldots \mid |\{i \in \mathbb{N} \mid v_i \in B\}| \geq k\}$. Bounded-visit co-Büchi is the dual objective $\text{Co-Büchi}(\overline{B}, k) \coloneqq \{v_0 v_1 \ldots \mid |\{i \in \mathbb{N} \mid v_i \in B\}| < k\} = V^\omega \setminus \mathrm{Büchi}(B, k)$.

Like before, we introduce two functions $g_1$ and $g_2$, which will be shown to characterize the thresholds for $\mathrm{Büchi}(B, k)$ and $\text{Co-Büchi}(B, k)$, respectively.

▶ **Definition 18.** *Define the functions* $g_1, g_2 \colon V \times \mathbb{N} \to [0, 1]$ *inductively as follows. For every* $v \in V$*, define* $g_1(v, 0) = 0$*. For every* $v \in B$*, define* $g_1(v, 1) \coloneqq 0$ *and* $g_1(v, k) \coloneqq Av_1(g_1(\cdot, k-1))(v)$ *for* $k > 1$*, and for every* $v \notin B$ *and every* $k > 0$*, define* $g_1(v, k) \coloneqq Th_1^{\mathrm{FrugalReach}(B, g_1(\cdot,k))}(v)$*. We proceed to define* $g_2$*. For every* $v \in V$*, define* $g_2(v, 0) \coloneqq 1$*. For every* $v \in B$*, define* $g_2(v, 1) \coloneqq 1$ *and* $g_2(v, k) \coloneqq Av_2(g_2(\cdot, k-1))(v)$*, for* $k > 1$*, and for every* $v \notin B$ *and every* $k > 0$*, define* $g_2(v, k) \coloneqq Th_2^{\mathrm{FrugalReach}(B, 1-g_2(\cdot,k))}(v)$*.*

We prove the existence of thresholds and their correspondence to $g_1$ and $g_2$.

▶ **Lemma 19.** *For every* $v \in V$ *and* $k \geq 0$*, we have* $g_1(v, k) = Th_1^{\mathrm{Büchi}(B,k)}(v)$ *and* $g_2(v, k) = Th_2^{\mathrm{Büchi}(B,k)}(v)$*. Moreover, the thresholds exist:* $Th_1^{\mathrm{Büchi}(B,k)}(v) + Th_2^{\mathrm{Büchi}(B,k)}(v) = 1$*.*

**Proof.** The proof proceeds by induction over $k$ (see details in the extended version [13]). For the base case, $k = 0$, clearly, Player 1 wins since no further visit to $B$ is required and Player 2 loses, which coincides with the definitions $g_1(v, 0) = 0$ and $g_2(v, 0) = 1$, for all $v \in V$. We describe the inductive step for Player 1. For $v \in B$, the proof follows from the induction hypothesis: similar to Lem. 8, when $B_1 > g_1(v, k)$, Player 1 can bid so that no matter how Player 2 bids and moves the token (upon winning), in the next configuration $\langle v', B_1' \rangle$, we have $B_1' > g_1(v', k-1)$. Finally, for $v \notin B$, recall that $g_1(v, k) := Th_1^{\text{FrugalReach}(B, g_1(\cdot, k))}(v)$. That is, a budget of $B_1 > g_1(v, k)$ means that he can follow a winning strategy in the frugal-reachability game, which forces the game to $B$ and upon reaching $v' \in B$, Player 1's budget exceeds $g_1(v', k)$. The proof then follows from the induction hypothesis. ◀

We establish monotonicity of the thresholds, which confirms that *Player* 1 needs higher budget for forcing larger numbers of visits to $B$.

▶ **Lemma 20.** *For $v \in V$ and $k \in \mathbb{N}$, we have $Th_1^{\text{Büchi}(B,k)}(v) \leq Th_1^{\text{Büchi}(B,k+1)}(v)$ and $Th_2^{\text{Büchi}(B,k)}(v) \geq Th_2^{\text{Büchi}(B,k+1)}(v)$. Moreover, the thresholds are bounded by 0 and 1.*

## 4.3 Existence of Thresholds (for Büchi and Co-Büchi Objectives)

We define two functions $g_1^*$ and $g_2^*$, which will be shown to coincide with the thresholds for the general (unbounded) Büchi and co-Büchi objectives, respectively.

▶ **Definition 21.** *Define the functions $g_1^*, g_2^* \colon V \to \mathbb{R}$ as follows. For every $v \in B$, define $g_1^*(v) := \lim_{k\to\infty} g_1(v, k)$ and $g_2^*(v) := \lim_{k\to\infty} g_2(v, k)$. For every $v \notin B$, define $g_1^*(v) := Th_1^{\text{FrugalReach}(B,\mathsf{fr})}(v)$ where $\mathsf{fr} \colon b \mapsto g_1^*(b)$ for every $b \in B$ and $\mathsf{fr} \colon v \mapsto 0$ (can be arbitrary) for every $v \notin B$. Likewise, for every $v \notin B$, define $g_2^*(v) := Th_2^{\text{FrugalReach}(B,\mathsf{fr})}(v)$ where $\mathsf{fr} \colon b \mapsto 1 - g_2^*(b)$ for every $b \in B$ and $\mathsf{fr} \colon v \mapsto 0$ (can be arbitrary) for every $v \notin B$.*

Monotonicity (Lem. 20) and boundedness of $g_1$ and $g_2$ imply the well-definedness of $g_1^*$ and $g_2^*$. We now establish the existence and the characterization of thresholds.

▶ **Theorem 22.** *For every $v \in V$, we have $Th_1^{\text{Büchi}(B)}(v) = g_1^*(v)$ and $Th_2^{\text{Büchi}(B)}(v) = g_2^*(v)$. Moreover, thresholds exist: $Th_1^{\text{Büchi}(B)}(v) + Th_2^{\text{Büchi}(B)}(v) = 1$.*

**Proof.** First, we show that $g_1^*(v) \geq Th_1^{\text{Büchi}(B)}(v)$. Consider a configuration $\langle v, B_1 \rangle$. When $B_1 > g_1^*(v)$, Player 1 wins as follows. If $v \notin B$, he plays according to a winning strategy in a frugal-reachability game to guarantee reaching some $v' \in B$ with a budget that exceeds $g_1^*(v')$. For $v \in B$, he bids so that in the next configuration $\langle v', B_1' \rangle$, we have $B_1' > g_1^*(v')$. Second, we show that $g_2^*(v) \geq Th_2^{\text{Büchi}(B)}(v)$. When $B_2 = 1 - B_1 > g_2^*(v)$, Player 2 wins as follows. If $v \in B$, then there exists $k$ such that $B_2 > g_2(v, k)$. Lem. 19 shows that she can win the co-Büchi objective by preventing $B$ to be reached more than $k$ times. If $v \notin B$, she has a strategy to make the token either (i) not reach $B$, or (ii) reach $v' \in B$ with a budget at least $g_2^*(v')$. In both cases, she wins by repeating the strategy. Finally, by Lem. 19, we have $g_1(v, k) + g_2(v, k) = 1$, for all $k \in \mathbb{N}$. Thus, in the limit, we have $g_1^*(v) + g_2^*(v) = 1$, for $v \in B$. From this, the other sides of the above inequalities, i.e., $g_1^*(v) \leq Th_1^{\text{Büchi}(B)}(v)$ and $g_2^*(v) \leq Th_2^{\text{Büchi}(B)}(v)$, and the existence claim follow in a straightforward manner. ◀

## 4.4 Complexity Bounds (for Büchi and Co-Büchi Objectives)

The computation of the thresholds $Th_1^{\text{Büchi}(B)} \equiv g_1^*$ and $Th_2^{\text{Büchi}(B)} \equiv g_2^*$ involves a nested fixed point computation. For example, for $g_1^*$, the outer fixed point is the smallest fixed point of the sequence $g_1(\cdot, 0), g_1(\cdot, 1), \ldots$ for vertices in $B$, and for every $k = 0, 1, \ldots$, the inner fixed point is the usual greatest fixed point for frugal reachability thresholds required to reach $B$ with the leftover frugal budget $g_1(\cdot, k)$ from outside $B$. The nested fixed point can be characterized as the solution of the following bilevel optimization problem.

$$\min_{h \in \mathbb{R}^V} \sum_{b \in B} h(b)$$

subjected to constraints:

$$\begin{aligned}
&h \in \arg\max_{h' \in \mathbb{R}^V} \left\{ \sum_{v \in V \setminus B} h'(v) \;\middle|\; \forall b \in B \,.\, h'(b) = h(b) \right\}, \\
&\forall v \in V \,.\, h(v) = Av_1\left(h(\cdot)\right)(v) \\
&\qquad = \mathtt{clamp}_{[0,1]}\left( \frac{(1-\tau)h(v^-) + h(v^+)}{[h(v^+) - h(v^-) - 1]\tau + 2} \cdot (1 + R_1(v) + R_2(v)) - R_1(v) \right), \\
&h(v^+) = \max_{u \in \mathcal{S}(v)} h(u), \quad h(v^-) = \min_{u \in \mathcal{S}(v)} h(u). 
\end{aligned} \tag{2}$$

▶ **Proposition 23.** *The solution of the optimization problem in* (2) *is equivalent to the threshold function* $Th_1^{\text{Büchi}(B)}$ *of the Büchi player.*

▶ **Theorem 24.** *The following hold:*

**(i)** $\text{THRESH}^{\text{taxman}}_{\text{Büchi}} \in \text{2EXPTIME}$ *and* $\text{THRESH}^{\text{taxman}}_{\text{Co-Büchi}} \in \text{2EXPTIME}$*, and*

**(ii)** $\text{THRESH}^{\text{Richman}}_{\text{Büchi}} \in \Pi_2^{\text{P}}$ *and* $\text{THRESH}^{\text{Richman}}_{\text{Co-Büchi}} \in \Sigma_2^{\text{P}}$

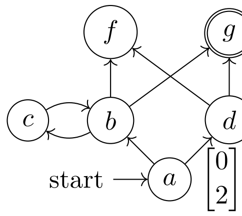
The bounds in (i) follow from a reduction to an equivalent query in the theory of reals. For (ii), we can check if the solution of the optimization problem in (2) is larger than 0.5, and if this is true then we conclude that $0.5 < Th_1^{\text{Büchi}(B)}(v)$ and can output a *negative* answer. Since (2) is a bilevel MILP, hence the check can be done in $\Sigma_2^{\text{P}}$ [20], and the overall complexity is $\Pi_2^{\text{P}}$. The other case is dual. Details of the proof can be found in the extended version [13].

## 5 Lower Complexity Bounds

In this section we show how to simulate a turn-based game using a Richman-bidding game with charging. Thus, solving Richman-bidding games with charging is at least as hard as their turn-based counterparts. Specifically, we obtain that solving Rabin bidding games with charging is NP-hard. This is a distinction from traditional Richman-bidding games, where solving Rabin games is in NP and coNP. Since taxman-bidding games generalize Richman-bidding games, hence it follows that Rabin taxman-bidding games are also NP-hard.

▶ **Lemma 25.** *Given a turn-based game* $\mathcal{G}$*, an initial vertex* $v$*, and an objective* $\varphi$*, there is a bidding game with charging* $\mathcal{G}'$ *of size linear in* $\mathcal{G}$*, with the same objective and initial vertex such that Player 1 can win* $\mathcal{G}$ *from* $v$ *if and only if* $\langle \mathcal{G}, v, \varphi \rangle \in \text{THRESH}^{\text{Richman}}_{\mathcal{S}}$.

**Proof.** The definitions of turn-based games and the detailed proof can be found in the extended version [13]. Intuitively, $\mathcal{G}'$ contains the same set of vertices as $\mathcal{G}$ with two additional sink vertices $s_1$ and $s_2$, where $s_i$ is losing for Player $i$, for $i \in \{1, 2\}$. For every vertex $v$, if $v$ is controlled by Player 1 in $\mathcal{G}$, then in $\mathcal{G}'$, we define Player 1's charge to be $R_1(v) = 2$. Moreover, we add an edge from $v$ to $s_1$, requiring Player 1 to win the bidding in $v$. Note that even if Player 1 starts with a budget of $\epsilon > 0$, at $v$, after charging and




| $R'_1$ | $Th_1(a)$ |
|---|---|
| – | 1 |
| $c, e \mapsto 1$ | 1 |
| $b \mapsto 2$ | 0.75 |
| $d \mapsto 2$ | 0.75 |
| $a, \ldots, e \mapsto 0.4$ | 0.62 |
| $a \mapsto 2$ | 0.5 |
| $b, d \mapsto 1$ | 0 |

**Figure 5** Illustrating the repair problem. LEFT: A reachability game with the objective Reach($\{g\}$). RIGHT: With no repair (first row), $Th_1(a) = 1$. We depict repairs (first col.) and the changes they imply to the thresholds (second col.), for a repair budget of $C = 2$.

normalization, his budget exceeds 2/3. Player 2's vertices are dual. It is not hard to verify that Player 1 can win in $\mathcal{G}$ from $v$ if and only if $Th^{\varphi}(v) = 0$, and Player 2 can win in $\mathcal{G}$ from $v$ if and only if $Th^{\varphi}(v) = 1$. ◀

Since turn-based Rabin games are NP-hard, we obtain the following.

▶ **Theorem 26.** *We have* $\text{THRESH}^{\text{Richman}}_{\text{Rabin}} \in$ NP-hard *and* $\text{THRESH}^{\text{Richman}}_{\text{Streett}} \in$ coNP-hard*.*

## 6 Repairing Bidding Games

In this section, we introduce the *repair* problem for bidding games. Intuitively, the goal is to add minimal charges to the vertices of an arena so as to decrease the threshold in the initial vertex to a target threshold. Formally, we define the following problem.

▶ **Definition 27** (Repairing bidding games). *Consider an arena $\mathcal{G}$ with a vertex $v$, a bidding mechanism $M \in \{Richman, poorman, taxman\}$, a class of objectives $S \in \{\text{Reach}, \text{Safe}, \text{Büchi}, \text{Co-Büchi}\}$, and a repair budget $C \in \mathbb{R}_{\geq 0}$. The set of* repaired arenas*, denoted Repaired$(\mathcal{G}, C)$, are arenas obtained from $\mathcal{G}$ by adding Player 1 charges whose sum does not exceed $C$. Formally, Repaired$(\mathcal{G}, C) \coloneqq \{\langle V, E, R'_1, R_2\rangle$ is an arena $\mid \forall v \in V \,.\, R'_1(v) \geq R_1(v) \wedge \sum_{v \in V}(R'_1(v) - R_1(v)) \leq C\}$. The problem* $\text{R\_THRESH}^M_S$ *takes as input $\langle \mathcal{G}, v, \varphi, C\rangle$, where $\varphi \in S$, and accepts iff there exists $\mathcal{G}' \in$ Repaired$(\mathcal{G}, C)$ with $\langle \mathcal{G}', v, \varphi\rangle \in \text{THRESH}^M_S$.*

▶ **Example 28.** We illustrate the non-triviality of the repair problem in Fig. 5. Observe that neither assigning charges uniformly nor assigning charges to a single vertex, decrease the threshold sufficiently, whereas adding a charge of 1 to both $b$ and $d$ is a successful repair. ⌟

▶ **Theorem 29.** *The following hold:*
**(i)** $\text{R\_THRESH}^{\text{Richman}}_{\text{Reach}} \in$ 2EXPTIME*,*
**(ii)** $\text{R\_THRESH}^{\text{Richman}}_{\text{Safe}} \in$ PSPACE*.*

**Proof.** We introduce notation for the proof. Let $G = \langle V, E, R_1, R_2\rangle$ be a bidding game and $U$ be a set of vertices. Define $A^G_U \colon [0,1]^V \to \{0,1\}$ such that for every $h \in [0,1]^V$, $A^G_U(h) = 1$ iff $h(v) = Av_1(h)(v)$ for every $v \notin U$. Observe that $Th_1^{\text{Reach}(T)}$ is the largest $h$ for which $A^G_T(h) = 1$ and moreover $h(v) = 0$ for every $v \in T$.

**Proof of (i).** Consider a bidding game $G = \langle V, E, R_1, R_2\rangle$ where the objective of Player 1 is Reach($T$) for some $T \subseteq V$. The goal is to check if it is possible to increase $R_1$ by a total of $C$ such that the reachability threshold at $a \in V$ falls below 0.5. This is equivalent to:

$$\exists R_1' \in \mathbb{R}^V \, . \, (R_1' \geq R_1) \wedge (|R_1' - R_1|_1 \leq C) \wedge \Big(\forall Th_1 \in \mathbb{R}^V \, . \, \Big[\big(\forall v \in T \, . \, Th_1(v) = 0\big) \wedge A_T^{\mathcal{G}'}(Th_1)\Big] \Rightarrow Th_1(a) \leq 0.5\Big)$$

where $\mathcal{G}' = \langle V, E, R_1', R_2 \rangle$. The validity of the above formula can be checked by applying a quantifier elimination method. Therefore, the decision problem is in 2EXPTIME.

**Proof of (ii).** Consider a bidding game $G = \langle V, E, R_1, R_2 \rangle$ where the objective of Player 1 is Safe($T$) for some $T \subseteq V$. The goal is to check if it is possible to increase $R_1$ by a total of $C$ such that the safety threshold at $a \in V$ falls below 0.5. This is equivalent to:

$$\exists R_1' \in \mathbb{R}^V \, . \, (R_1' \geq R_1) \wedge (|R_1' - R_1|_1 \leq C) \wedge \Big(\exists Th_1 \in \mathbb{R}^V \; s.t. \; \big[Th_1(a) \leq 0.5 \wedge A_T^{\mathcal{G}'}(Th_1) \wedge \forall u \in T, Th_1(u) = 1\big]\Big)$$

where $\mathcal{G}' = \langle V, E, R_1', R_2 \rangle$. The above formula can be seen as an input instance of existential theory of reals which is known to be in PSPACE. ◀

## 7 Conclusion and Future Work

We introduce and study a generalization of bidding games in which players' budgets are charged throughout the game. One advantage of the model over traditional bidding games is that long-run safety is not trivial. We show that the model maintains the key favorable property of traditional bidding games, namely the existence of thresholds, the proof of which is, however, significantly more challenging due to the non-uniqueness of thresholds. We characterize thresholds in terms of greatest and least fixed points of certain monotonic operators. Finally, we establish the first complexity lower bounds in continuous-bidding games and study, for the first time, a repair problem in this model.

There are plenty of open questions and directions for future research. First, it is important to extend the results to richer classes of $\omega$-regular objectives, like parity, Rabin, and Streett, as well as to quantitative objectives, like mean-payoff. Second, tightening the complexity bounds is an important open question. For example, it might be the case that finding thresholds in Richman-bidding games with charging is in NP and coNP. Third, traditional reachability Richman-bidding games are equivalent to a class of stochastic games [18] called *random-turn games* [24], and the equivalence is general and intricate in infinite-duration games [4, 5, 7, 9]. It is unknown if such a connection exists for games with charging, and if it does, then many of the open questions may be solved via available tools for stochastic games. Finally, there are various possible extensions, like charges disappearing after a vertex is visited, charges that are collectible in multiple installments, etc.

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