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Aharonov–Casher Theorems for Dirac Operators on Manifolds with Boundary and APS Boundary Condition

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Abstract. The Aharonov–Casher theorem is a result on the number of the so-called zero modes of a system described by the magnetic Pauli operator in $\mathbb{R}^2$. In this paper we address the same question for the Dirac operator on a flat two-dimensional manifold with boundary and Atiyah–Patodi–Singer boundary condition. More concretely we are interested in the plane and a disc with a finite number of circular holes cut out. We consider a smooth compactly supported magnetic field on the manifold and an arbitrary magnetic field inside the holes.

Contents

List of Symbols

Ω_k	Open ball in $\mathbb{C}$ with centre at $w_k \in \mathbb{C}$ and radius R_k,
(r_k, φ_k)	Polar coordinates around the point w_k, we set $\varphi_k = 0$ to be the axis parallel with the Cartesian positive x-axis
$Cl(V)$	Clifford algebra on a vector space V
$C_0^\infty(X)$	Smooth functions with compact support in X
$const$	A general constant which can be of different value from one (in)equality sign to another
$int\,\gamma$	Interior of a curve γ
$(\cdot,\cdot)_E$	Inner product on fibres of a bundle E
$\Gamma(M,E)$	Smooth sections of a bundle E over a manifold M
$L^2(M,E)$	Square integrable sections of the bundle E over a Riemannian manifold M
$L^2(M,g;\mathbb{C}^2)$	$\mathbb{C}^2$-valued square integrable functions on a Riemannian manifold M with metric g
M°	Interior of a manifold M
TM	Tangent space of a manifold M
T^*M	Cotangent space of a manifold M
T_p^*M	Fibre of the cotangent space above the point $p \in M$
$(\cdot)^T$	Vector transposition
∂X	Boundary of a region X
$\overline{X}$	Closure of a subset $X \subset \mathbb{C}$
$\lfloor y \rfloor$	The biggest integer strictly less than $y \in \mathbb{R}$
$\|\cdot\|_S$	The norm on a space S

1. Introduction

This paper is inspired by work of Aharonov and Casher (AC), [2], discussing the number of zero modes (*i.e.* of eigenfunctions corresponding to the zero eigenvalue) of the Dirac operator with magnetic field. We extend their result on $\mathbb{R}^2$ to a plane with holes, Theorem 8, a disc with holes, Theorem 10, and finally to a sphere with holes, Theorem 24. All our results are concerning a particular choice of the extension of the Dirac operator, given by the Atiyah–Patodi–Singer (APS) boundary condition. In [2] the zero modes have a definite chirality, which depends on the sign of the flux Φ of the magnetic field. The number of zero modes then depends on the magnitude of the flux, namely it is $\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$. On the plane with holes we reproduce the same result. We would like to point out that considering a self-adjoint realisation D of the Dirac operator, the zero modes of D coincide with the zero modes of its square D^2 which we may refer to as an operator of Pauli type. Such operators describe the non-relativistic limit of Dirac operators, and due to its positivity the zero energy states are also its ground states.

The analytic index of the Dirac operator is a closely related quantity as it computes the difference of the number of zero modes with positive and negative chirality[1]. These are also referred to as modes with spin-up or spin-down. Atiyah and Singer proved in [6] that the analytic index is equal to the topological index of the underlying closed manifold. Using the stereographic projection the AC theorem can be reformulated as a result on a sphere (see *e.g.* [18, Theorem 8.3.]). Since the sphere and the disc with holes are compact manifolds, we also have the index theorem for them. It gives the formula for the difference between the number of the zero modes with positive and negative chirality. Our adaptation of the AC theorem then gives each of these numbers separately and is thus a stronger result. However, one should of course keep in mind that the index theorem is valid for a very general setting.

A continuation of Atiyah and Singer's work resulted in the generalisation of the index formula for manifolds with boundary by Atiyah, Patodi and Singer in the series of papers [3–5]. The authors introduced a boundary condition, nowadays known as the APS boundary condition, which we adopted here for the definition of the domain of our Dirac operator. It is a global boundary condition based on preservation of chirality upon reflection on the boundary. A manifold X that has a product structure near the boundary can be extended by an infinite cylinder. From the analytical point of view the APS boundary condition is tailored so that any zero mode of the Dirac operator on X satisfying this condition can be extended to this infinite cylinder as a square integrable function plus a function constant along the infinite direction of the cylinder. For more details see [10, Sect. 22E].

The literature on zero modes is vast, and we will mention only a couple of works generalising the AC theorem. A proof of the result on a two-sphere is due to Avron and Tomaras (but was not published) and it can be found, for example, in [27] or [18, Appx. A.3]. For generalisation to measure-valued magnetic fields see [19]. Singular Aharonov–Bohm-type fields were considered by Hirokawa and Ogurisu in [25], by Persson in [30] and by Geyler and Štovíček in [22]. Rozenblum and Shirokov, [32], showed that for certain singular magnetic fields there could be an infinite dimensional space of zero modes with having possibly both spin-up and spin-down modes. Results for the case of even dimensional Euclidean spaces were discussed by Persson in [31]. Bony, Espinoza and Raikov investigate almost periodic potentials in [9]. On a bounded domain with Dirichlet boundary condition the related result was studied in [17].

The aim of this paper is to extend the Aharonov–Casher theorem exactly to these cases when a compact boundary is present. Since the APS boundary

[1] A reader that is not familiar with the notion of positive (negative) chirality can, for purposes of the results in this paper, think of the eigenvectors of the third Pauli matrix which is the diagonal matrix $\sigma_3 := \mathrm{diag}(1, -1)$, corresponding to eigenvalue $+1$ (-1). The concept will be more generally introduced in Sect. 1.1.

condition is a condition forged for the index theorem it seems to be a reasonable candidate to start with when studying the AC formula (due to the relation between the AC and index formulas mentioned above) for Dirac operators on manifolds with boundary. Of course, there are many other choices of boundary conditions that would make the Dirac operator self-adjoint (that are not discussed here) and the validity of the AC formula is heavily dependent on the realisation we use. For a detailed classification of self-adjoint realisations of Dirac-type operators on manifolds with boundary we refer an interested reader to works [7,8].

Our main motivation to study this problem is the mathematical interest to see how could the AC theorem be influenced by a presence of a boundary. We also find it a curious problem to explore the zero modes on a non-compact manifold, where the index theorem is not applicable. The particular setting of a plane or a ball with holes is of interest also due to the Aharonov–Bohm (AB) effect. This phenomenon gives a possibility to observe magnetic field in quantum mechanics even if the field is supported in a region inaccessible to the particle. The net observable effect then depends only on the flux of the field in this region. In our setting of magnetic field supported in the holes the AC formula precisely demonstrates such properties. Let us mention that the AB effect is often studied using the model of an infinitesimally thin and infinitely long solenoid. This corresponds to the magnetic field formally given as $\alpha\delta_0$, where α is the magnetic flux and δ_0 the delta distribution with support at $\{0\}$. There are many works considering the Schrödinger-type operators with such a point interaction (also referred to as the AB field). The domain of a realisation is then characterised by the behaviour of the functions at the singularity occurring at the origin. The self-adjoint extensions were classified in [1] and [14] for Schrödinger and Pauli operators with the AB field, respectively. From the recent literature studying such singular interaction let us mention [11,30] for results related to Pauli and Dirac operators, [13,29] studying Schrödinger operators and [15,16] investigating Bessel operators that include Schrödinger operators with the AB field.

Let us sketch the central points of the proofs of our main results in Theorems 8 and 10. The first steps rely on the idea used in the original Aharonov–Casher paper. In particular the equation for the zero modes decouples and we can analyse each component separately. Each of the components then factorises as a product of an (anti)analytic function g and an exponential whose argument depends on the magnetic scalar potential. While AC consider the simply connected manifold $\mathbb{R}^2$, where the function g has a Taylor series, in our case g has only the Laurent series on a neighbourhood of each of the holes. To achieve the starting point of Aharonov and Casher we use the APS boundary condition to extend g (anti)analytically to the interior of the holes. The main difficulty here is to find a suitable way to compare the boundary values to the boundary condition. This is a local analysis and this step requires that the boundaries of our holes are indeed circular. In the unbounded case of $\mathbb{R}^2$ with holes we then complete the analysis by cutting off the Taylor series of g. For that we use the condition that the zero modes need to be in the domain of the operator and therefore have to be square integrable at infinity. This is the same mechanism as in [2]. For the case of a disc the eigenfunctions have to again satisfy the APS boundary condition, which provides us with the cut off on the Taylor series of g. The highest possible power in the series determines the number of the zero modes. Due to the different source of the constraint on this power we arrive at different results. In Remark 11 we, however, show that for those values of fluxes, where the results yield a different number of zero modes, the extra zero mode on the disc is not square integrable at infinity when considered on a disc of a growing radius.

Finally, we briefly outline the content of this paper. In this introduction we give the definition of the Dirac operator on an orientable two-dimensional Riemannian manifold and the APS boundary condition. We further discuss the magnetic field. Introducing our particular setting we find an explicit form of the APS boundary condition and establish the gauge invariance of the problem in Lemma 7. Using Lemma 7 we can without loss of generality study only fluxes mod 2π inside each hole. We refer to this as "gauging away" integer multiples of 2π of the flux.

In Sect. 2 we state and prove the main results. To briefly summarise, we obtain the same result as Aharonov and Casher in the case of the Dirac operator on a plane with holes. On a disc with holes our statement is in accordance with the index theorem.

We extend the Aharonov–Casher theorem to a sphere with holes in Sect. 3. Despite this being a direct consequence of our result on a disc with holes, due to the fact that the two cases are related by stereographic projection, we first need some theoretical preparation in form of treating the Dirac operator with APS boundary condition in a conformal metric. The proof also requires analysis of the spinors under the change of coordinates by the Möbius transform which we discuss in Appx. B.

In Sect. 4 we use the generalised index formula by Grubb [24] and Gilkey [23] of the index theorem on manifolds with boundary, to compute the index of the magnetic Dirac operator and compare it to our result on the disc region. Let us remark that the original result in [3] cannot be applied directly since it was restricted to manifolds that have a product structure near the boundary.

Let us mention that this work is based on the author's PhD thesis [21].

1.1. Dirac Operator and the APS Boundary Condition

Let M be a two-dimensional oriented Riemannian manifold with compact boundary ∂M and metric g. Let further E be a two-dimensional complex vector bundle equipped with an inner product $(\cdot,\cdot)_E$ on the fibres of E. Denote by $\mathrm{End}(E)$ the bundle of endomorphisms of the bundle E. If there is a vector bundle map

$$\sigma : T^*M \to \mathrm{End}(E)$$

which is Hermitian, *i.e.* $\sigma(\zeta) = \sigma(\zeta)^*$ for all $\zeta \in T^*M$, and satisfies the Clifford relations

$$\sigma(\zeta)\sigma(\mu) + \sigma(\mu)\sigma(\zeta) = 2g(\zeta, \mu) \qquad \text{for any } \zeta, \mu \in T_p^*M \text{ at all } p \in M\,, \quad (1)$$

we call E a Spin^c spinor bundle.[2] over M. The mapping σ is called the Clifford multiplication. The Clifford multiplication further extends to a unique mapping from the bundle of Clifford algebras $Cl(T^*M)$. That is the quotient $\otimes T^*M/I_g$ of the bundle of tensor algebras $\otimes T^*M := \oplus_{k\geq 0}(T^*M)^{\otimes k}$ by the bundle of ideals I_g generated by $\{\zeta \otimes \zeta - 2\,g(\zeta,\zeta) \mid \zeta \in T^*M\}$. In even dimensions we call the $\mathrm{i}^{3\dim(M)/2}$-multiple of the Clifford multiplication by the volume form (which is an involution) the chirality operator and refer to its eigenvectors with eigenvalue $+1$ (or -1) as spin-up (or down) vectors. Locally we can always choose the representation of $Cl(\mathbb{R}^2)$ by the constant Pauli matrices

$$\sigma^1 = \sigma(\,\mathrm{d}x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma^2 = \sigma(\,\mathrm{d}y) = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \qquad (2)$$

Note that in $\mathbb{R}^2$ with the standard metric the chirality operator corresponds to the third Pauli matrix $\sigma_3 = \mathrm{diag}(1, -1)$. In what follows we use the standard notation $\Gamma(M, E)$ for smooth sections of the bundle E and $L^2(M, E)$ for the square integrable sections w.r.t. the volume element generated by the Riemannian metric on M. Let us next equip E with a connection ∇. We call ∇ a Spin^c connection if it is

1. metric:
$$X(\zeta, \mu)_E = (\nabla_X \zeta, \mu)_E + (\zeta, \nabla_X \mu)_E\,,$$
for any sections $\zeta, \mu \in \Gamma(M, E)$ and any vector field $X \in TM$, and,
2. compatible with the Clifford multiplication σ:
$$[\nabla_X, \sigma(\mu)] = \sigma(\nabla_X^{LC} \mu)\,,$$
for all vector fields X and one-forms μ on M. Here, ∇^{LC} is the Levi-Civita connection on the cotangent space T^*M of M.

Definition 1. Let E be a Spin^c spinor bundle over M with Clifford multiplication σ and a *Spin*^{c} connection ∇. The Dirac operator $D : \Gamma(M, E) \to \Gamma(M, E)$ is the following composition

$$D = -\mathrm{i} \sum_{j\leq 2} \sigma(e^j) \nabla_{e_j}\,,$$

where $(e_j)_{j\in\{1,2\}}$ is an orthonormal basis on TM and $(e^j)_{j\leq 2}$ is the corresponding dual basis on T^*M.

Note that the definition is independent of a particular choice of $(e_j)_{j\leq 2}$, so D is well defined globally. We remark, that one can use a Spin^c connection on a Spin^c bundle to form a Dirac operator with magnetic field (see further

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[2] One might rather call E a bundle of irreducible complex Clifford modules. These two concepts were, however, shown to be identical, see [12, Sect. 3]

Sect. 1.2). The Dirac operator is a first order operator which is elliptic, symmetric and whose principal symbol is the Clifford multiplication. Furthermore, it can be extended as a closed linear map to the maximal domain of D

$$\mathrm{dom}(D^{\max}) := \{u \in L^2(M,E) \mid Du \in L^2(M,E)\}\,. \tag{3}$$

To introduce the Atiyah–Patodi–Singer (APS) boundary condition we are following the formalism for elliptic boundary conditions used in [7,8]. We, however, diverted with the convention for the Clifford multiplication which in the cited papers is considered to be anti-hermitian and satisfies the Clifford relations (1) with an extra minus sign on the right-hand side.

Notation 2. *Let $\nu^\sharp$ be the normalised inner normal vector field on ∂M. We will denote by $\nu \in T^*M$ the local co-vector field on the boundary ∂M dual to $\nu^\sharp$. The local space of co-vectors tangent to the boundary is defined by*

$$T^*\partial M := \{\xi \in T^*M \mid g(\xi,\nu) = 0\}\,.$$

We further write $\xi^\sharp$ for the normalised tangent vector which is dual to $\xi \in T^*\partial M$.

With this notation we can locally write the Dirac operator in the neighbourhood of the boundary as

$$D = -\mathrm{i}\sigma(\nu)(\nabla_{\nu^\sharp} + A_0) \quad \text{with} \quad A_0 = \sigma(\nu)\sigma(\xi)\nabla_{\xi^\sharp}\,, \tag{4}$$

where we used that by the Clifford relations $\sigma(\nu)^2$ is the identity on fibres of E (restricted to ∂M). As in the Appx. 2 of [8] we then define the canonical boundary operator which anti-commutes with $\sigma(\nu)$.

Definition 3. Let E be a Spin^c spinor bundle over M with Clifford multiplication σ and a *Spin*^{c} connection ∇ and let D be a Dirac operator on E. We define the canonical boundary operator adapted to D by

$$A := \frac{1}{2}(A_0 - \sigma(\xi)\nabla_{\xi^\sharp}\sigma(\nu)) = A_0 - \frac{\kappa}{2}\,,$$

where κ is the eigenvalue of the shape operator of the boundary w.r.t. the normal field ν, *i.e.* $\nabla^{LC}_{\xi^\sharp}\nu = \kappa\xi$. In fact κ is the principal curvature of the boundary.

Note that A was chosen so that the anti-commutator $\{A,\sigma(\nu)\}$ vanishes. In our two-dimensional case it is also not difficult to check by a direct computation that A_0 is symmetric. For a general dimension this is shown in [8, Appx. 1]. By definition the canonical boundary operator A is thus also symmetric. It follows that it is essentially self-adjoint on $L^2(\partial M;\mathbb{C}^2)$, since ∂M is a compact manifold. The importance of boundary operators is that one can use them for a construction of elliptic boundary conditions (see [7, Defs. 1.9, 1.10, and Theorem 1.12]) which give rise to domains that are subsets of $H^1_{loc}(M,E) = \{u \in L^2_{loc}(M,E) \mid \nabla u \in L^2_{loc}(M,E)\}$. Here $L^2_{loc}(M,E)$ denotes sections of E that are square integrable over each compact subset $K \subset M$ and, in particular, we may have $K \cap \partial M \neq \emptyset$.

Since A is a self-adjoint elliptic operator on the compact manifold ∂M, it has purely discrete spectrum. Let us denote by $\{v_k \mid k \in \mathbb{Z} \setminus \{0\}\}$ a set of orthonormal eigenvectors of A corresponding to eigenvalues $\lambda_k \neq 0$. We order these eigenvalues as $\ldots \leq \lambda_{-k} \leq \lambda_{-k+1} \ldots \leq \lambda_{-1} < 0 < \lambda_1 \leq \ldots < \lambda_k \leq \lambda_{k+1} \leq \ldots$. Let us further assume that there is the decomposition into two mutually orthogonal spaces $\ker A = \operatorname{span}\{v_0\} \oplus \operatorname{span}\{\sigma(\nu) v_0\}$. In general v_0 could be a set of vectors but in our case it is only one vector. We define the APS (Atiyah–Patodi–Singer) boundary condition as the following closure of a subset of smooth sections on the boundary

$$BC_{APS} := \overline{\operatorname{span}(\{v_k\}_{\lambda_k<0} \cup v_0)}\,. \tag{5}$$

We point out that v_0 and $\sigma(\nu)v_0$ are exchangeable and that we are making a choice here. The closure in (5) is w.r.t. the norm

$$\left\| \sum_{k\in\mathbb{Z}} c_k v_k \right\|^2_{\check{H}(A)} := \sum_{\lambda_k<0} |c_k|^2 (1+\lambda_k^2)^{1/2} + \sum_{\lambda_k\geq 0} |c_k|^2 (1+\lambda_k^2)^{-1/2}\,, \quad c_k \in \mathbb{C}\,. \tag{6}$$

Further, $\check{H}(A)$ denotes the closure of $C^\infty(\partial M, \mathbb{C}^2)$ in this norm. We call the realisation D^{APS} of D on the domain

$$\operatorname{dom}(D^{\mathrm{APS}}) = \{u \in \operatorname{dom}(D^{\max}) \mid \gamma_0 u \in BC_{APS}\}\,, \tag{7}$$

the Dirac operator with APS boundary condition. The trace map $\gamma_0 u = u\big|_{\partial M}$ is well defined for $u \in C_0^\infty(M, \mathbb{C}^2)$, in particular $\operatorname{supp} u \cap \partial M$ can be non-empty, and by [7, Theorem 1.7.(2)] it extends to a bounded linear map

$$\gamma_0 : \operatorname{dom}(D^{\max}) \to \check{H}(A)\,. \tag{8}$$

Theorem 4.12. in [8] tells us that (7) is a self-adjoint realisation. Recalling that $\sigma(\nu)$ and A anti-commute, the self-adjointness can be also seen directly from the Green's formula (*cf.* [8, Proposition 2.1])

$$\int_M (D\psi, \varphi)_E = \int_M (\psi, D^*\varphi)_E - \int_{\partial M} (-\mathrm{i}\sigma(\nu)\psi, \varphi)_E\,.$$

Here ψ, φ are compactly supported functions on M. In particular their support and the boundary ∂M can have a non-empty intersection. By D^* we denoted the adjoint of D. Let us remark that there are also other choices of more general APS boundary conditions (see [7, Example 1.16(b)]). If M is compact then Theorem 5.3 in [8] implies that the Dirac operator D^{APS} with the APS boundary condition is Fredholm. We recall that Fredholm operator is an operator with closed range and a finite dimensional kernel and cokernel.

In this work we are also interested in M being a plane with holes. In this case zero is an eigenvalue of finite geometrical multiplicity embedded in the essential spectrum (see Theorem 8 and Cor. 9 bellow) and therefore D^{APS} does not have a closed range (*cf.* [26, Theorem 5.2]). Thus, it is not Fredholm. Since we will consider manifolds with several components of boundary, we introduce the notation $BC_{APS}\,|_{\partial\Omega_j}$ for the APS boundary condition on the component $\partial\Omega_j$ of the boundary.

1.2. Magnetic Field and Minimal Coupling

Let us consider the connection $\nabla = \mathrm{d} - \mathrm{i}\alpha$ on a the trivial bundle E over $\mathbb{C}$ with fibre $\mathbb{C}^2$. The term $-\mathrm{i}\alpha$ is called the local connection one-form and it satisfies $-\mathrm{i}\alpha(Y) \in \mathrm{i}\mathbb{R}$ for all vector fields Y on $\mathbb{C}$. Writing $\alpha = \frac{1}{2}(a\,\mathrm{d}\bar{z} + \bar{a}\,\mathrm{d}z)$, $a \in C^\infty(\mathbb{C})$ and using the standard notation $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ we obtain the Dirac operator (*cf.* the representation of the Clifford multiplication (2))

$$D_a = -2\mathrm{i}\begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \bar{a} \\ a & 0 \end{pmatrix},$$

known from physics to be (by the principle of correspondence and the minimal coupling) the Hamiltonian of a relativistic charged mass-less particle in a magnetic field of vector potential α. The field strength is the closed two-form $\beta = \mathrm{d}\alpha$ and two different connection one-forms $\alpha^{(1)}, \alpha^{(2)}$ correspond to the same magnetic field β if they differ by an exact form. This ambiguity is the well-known gauge invariance. To put this in the context of the vector formalism we can write $a = a_x + \mathrm{i}a_y$, for some $a_x, a_y \in C^\infty(\mathbb{R}^2)$. Then defining[3] $a^j = g^{jk}\,a_k$ for $j,k \in \{x,y\}$, the vector potential $\vec{a} = (a^x, a^y)$ corresponds to the magnetic field $\vec{B} = (0,0,B)$ such that $\mathrm{curl}(a^x, a^y, 0) = (0,0,B)$ with $\beta = B(z)\frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$. We will introduce the Aharonov–Casher gauge

$$\partial_z h(z) = -\frac{\mathrm{i}\bar{a}}{2}, \tag{9}$$

using the scalar potential h satisfying $-\Delta h = B$ on $\mathbb{C}$. For B decaying sufficiently fast at infinity (not necessarily smooth) we can write the solution of this Poisson equation

$$h(z) = -\frac{1}{2\pi}\int_{\mathbb{C}} \log|z - z'| B(z')\frac{\mathrm{i}}{2}\,\mathrm{d}z' \wedge \overline{\mathrm{d}z'}. \tag{10}$$

Notice that this gauge is automatically divergence free

$$\partial_x a_x + \partial_y a_y = \mathrm{Re}\,(2\partial_{\bar{z}}\bar{a}) = \mathrm{Re}\,(4\mathrm{i}\partial_{\bar{z}}\partial_z h) = \mathrm{Re}\,(-\mathrm{i}B) = 0\,.$$

Another quantity that describes the magnetic field is called the magnetic flux

$$\Phi := \int_{\mathbb{C}} B\frac{\mathrm{i}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

Remark 4. Let us now consider a smooth magnetic field B with compact support. By elliptic regularity (see *e.g.* [20, Sec. 6.3, Thm. 3]) the potential h is then also a smooth function. Note that the Poisson equation $-\Delta h = B$ determines h up to an addition of a harmonic function and our particular choice corresponds to the unique gauge choice via the relation (9). It yields a divergence free vector potential $a(z)$ that is bounded at infinity. We will refer to the choice (9), (10) of $a(z)$ as the Aharonov–Casher gauge.

[3] g^{jk} here refers to the components of the metric g in the coordinate basis $(\mathrm{d}x, \mathrm{d}y)$ of one-forms.

To see the boundedness, let $R' > 0$ be such that $R' > 2|z'|$ for all $z' \in \operatorname{supp} B$ and taking $|z| > R' > 2|z'|$ we can use the bound

$$\left|\frac{B(z')}{z - z'}\right| \leq \frac{2}{R'}|B(z')| \in L^1(\mathbb{C}), \quad z' \in \operatorname{supp} B,$$

to apply the dominated convergence theorem and obtain

$$|\partial_z h(z)| \leq \frac{const}{|z|} \int_{\mathbb{C}} \frac{|B(z')|}{1 - \frac{|z'|}{|z|}} \frac{\mathrm{i}}{2} \, \mathrm{d}z' \wedge \overline{\mathrm{d}z'} \leq \frac{const}{|z|} \int_{\mathbb{C}} |B(z')| \mathrm{i} \, \mathrm{d}z' \wedge \overline{\mathrm{d}z'} \leq \frac{const}{|z|},$$

for $|z|$ large.

From (10) one also deduces the asymptotic behaviour of the scalar potential

$$h(z) = -\frac{\Phi}{2\pi} \log|z| + \mathcal{O}(|z|^{-1}), \tag{11}$$

as $|z|$ tends to infinity. Moreover, in the case of a spherically symmetric B there is no error term and for z outside of support of B

$$h(z) = -\frac{\Phi}{2\pi} \log|z|, \tag{12}$$

by the Newton's law.

Remark 5. In the flat space $\mathbb{R}^2$ we clearly have $a_x = a^x$ and $a_y = a^y$. Notice that if we consider the polar coordinates $(r, \varphi) = (\sqrt{x^2 + y^2}, \arctan \frac{y}{x})$ and write $\alpha = (a_r, a_\varphi)$ for the components in the normalised basis $(\mathrm{d}r, r\,\mathrm{d}\varphi)$ and $\vec{a} = (a^r, a^\varphi)$ for the components in the dual basis $\left(\partial_r, \frac{\partial_\varphi}{r}\right)$ we also have $a_r = a^r$ and $a_\varphi = a^\varphi$.

1.3. Problem Set-up

We start by establishing some notation:

- Let $\mathscr{M}$ be either the complex plane or a disc $\Omega_{out} \subset \mathbb{C}$ with centre at the origin with radius R_{out}.
- $\Omega_j \subset \mathscr{M}$, $j \in \{1, 2, \ldots, N\}$ refers to a ball with centre at $w_j \in \mathscr{M}$ and radius $R_j > 0$.
- $M = \mathscr{M} \setminus \cup_{k \leq N} \Omega_k$, $N \in \mathbb{N}$ is our two-dimensional manifold of interest.
- (r_j, φ_j) denote the polar coordinates at $w_j \in \Omega_j$.
- B_j, $j \leq N$ denotes the magnetic field with support inside Ω_j, while $B_0 \in C_0^\infty(M^\circ)$.

Complementing the above notation for magnetic field on $\mathscr{M}$ we denote

$$B = B_{sing} + B_0, \tag{13}$$

where $\operatorname{supp} B_{sing} \subset \cup_{k \leq N} \Omega_k$. Later in Lemma 7 we will show that without loss of generality we may assume $B_{sing} = \sum_{k \leq N} \Phi'_k \delta_{w_k}$, where $\Phi'_k \in [-\pi, \pi)$[4]

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[4] we can choose any interval of length 2π, but this choice is the most convenient one for the purposes of our analysis.

differs by an integer multiple of 2π from the flux of B through the k-th hole

$$\Phi_k := \int_{\Omega_k} B(z) \frac{\mathrm{i}}{2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \, .$$

We refer to Φ_k' as a normalised flux through the hole Ω_k. The total flux is then the sum

$$\Phi := \Phi_0 + \sum_{k \le N} \Phi_k' \, ,$$

where $\Phi_0 = \int_M B_0(z) \frac{\mathrm{i}}{2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$ is the flux through the bulk of M.

The Dirac Operator and the APS Boundary Condition Explicitly. For finding more concrete form of the APS boundary condition for the above described setting we will make use of the Dirac operator expressed in polar coordinates (r, φ)

$$D_a = -\mathrm{i} \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(\partial_r - \mathrm{i}\frac{\partial_\varphi}{r}) \\ \mathrm{e}^{\mathrm{i}\varphi}(\partial_r + \mathrm{i}\frac{\partial_\varphi}{r}) & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(a_r - \mathrm{i}a_\varphi) \\ \mathrm{e}^{\mathrm{i}\varphi}(a_r + \mathrm{i}a_\varphi) & 0 \end{pmatrix}, \tag{14}$$

where we use notation from Remark 5.

To gain some intuition for the abstract setting, we first work out an example of finding the boundary condition for the case of one hole.

Example 6. Consider the manifold $M = \mathbb{C} \backslash \Omega$ with Ω being a ball of radius R centred at the origin. We assume there is no magnetic field in the bulk (*i.e.* $B_0 = 0$) and put magnetic field formed by one Aharonov–Bohm flux $B = \Phi\delta$ inside the hole. Later, in Lemma 7 we show that without loss of generality we can always consider that the magnetic field inside a hole is of this form and, moreover, that $\Phi \in [-\pi, \pi)$. The key simplifying point is that for this field we can choose the gauge so that $a_r = 0$ and $a_\varphi = \frac{\Phi}{2\pi r}$. Note, that in this setting the inward normal one-form on the boundary is simply $\nu = \mathrm{d}r$. In accordance we will denote $\sigma(\nu) = \sigma^r$. The Clifford connection along the radial field ∂_r is given by $\nabla_{\partial_r} = \partial_r - \mathrm{i}a_r$. To find the boundary operator A_0 we compare (4) to the expression (14) for D in polar coordinates near the boundary $\partial\Omega$ and conclude

$$\sigma^r = \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} = (\sigma^r)^* = (\sigma^r)^{-1}.$$

With (4) this further yields $A_0 = \sigma^3 \left(\frac{\mathrm{i}\partial_\varphi}{R} + a_\varphi \right)$, where $\sigma^3 = \mathrm{diag}(1, -1)$ is the third Pauli matrix. Finally, using that the principal curvature of the circular boundary $\partial\Omega$ is $\frac{1}{R}$ we obtain by Definition 3 the canonical boundary operator on $\partial\Omega$

$$A = \sigma^3 R^{-1} \left(\mathrm{i}\partial_\varphi + \Phi/2\pi \right) - (2R)^{-1} \, .$$

A simple analysis of the eigenvalueproblem $Au = \lambda u$ then reveals the eigenfunctions

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}k\varphi} \\ 0 \end{pmatrix} \quad \text{with eigenvalue } \lambda = R^{-1} \left(-k - \frac{\Phi}{2\pi} - \frac{1}{2} \right) \quad \text{and} \quad \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}k\varphi} \end{pmatrix}$$

$$\text{with eigenvalue } \lambda = R^{-1} \left(k - \frac{\Phi}{2\pi} - \frac{1}{2} \right) .$$

The APS boundary condition is therefore given by the closure of the set

$$BC_{APS}\mid_{\partial\Omega} = \overline{\operatorname{span}\left\{ \left[\begin{pmatrix} \mathrm{e}^{\mathrm{i}k\varphi} \\ 0 \end{pmatrix}\right]_{k>\frac{\Phi}{2\pi}-\frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}k\varphi} \end{pmatrix}\right]_{k\le\frac{\Phi}{2\pi}+\frac{1}{2}} \right\}} \tag{15}$$

in the $\check{H}$ norm (6). For completion we write down the domain of the operator D_a^{APS} in this setting, acting as (14) on

$$\operatorname{dom}(D_a^{\mathrm{APS}}) = \{\psi \in \operatorname{dom}(D_a^{\max}) \mid \gamma_0\psi \in BC_{APS}\mid_{\partial\Omega}\}.$$

Notice that the sign of the eigenvalues of the boundary operator is independent of the radius of the hole and the boundary condition is invariant under the scaling. However, it does not directly make sense to consider $R \to 0$ which might otherwise be interesting to investigate in view of the many extensions of the Dirac operator with Aharonov–Bohm field (*cf.* [11,30]).

Now, let us fix an index $j \le N$ and work out the boundary condition on the boundary of a j-th hole in the general setting. We choose the polar coordinates (r_j, φ_j) around the centre of the hole Ω_j. Then noting that in our case $\nabla_{\nu^\sharp} = \partial_{r_j} - \mathrm{i}a_{r_j}$ we obtain by (4) $A_0\mid_{\partial\Omega_j} = \sigma^3\left(\frac{\mathrm{i}\partial_{\varphi_j}}{R_j} + a_{\varphi_j}\right)$ and consequently, Definition 3 yields

$$A\mid_{\partial\Omega_j} = \sigma^3\left(\frac{\mathrm{i}\partial_{\varphi_j}}{R_j} + a_{\varphi_j}\right) - \frac{1}{2R_j}. \tag{16}$$

Solving the eigenvalue problem for $A\mid_{\partial\Omega_j}$ on the circle $\partial\Omega_j$ we find

$$A\mid_{\partial\Omega_j}\begin{pmatrix}\psi_\ell^j \\ 0\end{pmatrix} = \lambda_\ell^j\begin{pmatrix}\psi_\ell^j \\ 0\end{pmatrix} \quad \text{and}$$
$$A\mid_{\partial\Omega_j}\begin{pmatrix}0 \\ \psi_\ell^j\end{pmatrix} = -\lambda_{\ell-1}^j\begin{pmatrix}0 \\ \psi_\ell^j\end{pmatrix} \quad \text{with} \begin{cases} R_j\lambda_\ell^j = \Phi_j/2\pi - 1/2 - \ell \\ \psi_\ell^j = \mathrm{e}^{\mathrm{i}\varphi_j\ell}\mathrm{e}^{\mathrm{i}\int_{\gamma_j}\vec{a}\cdot\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi_j}{2\pi}\varphi_j} \end{cases}, \tag{17}$$

where ℓ is an integer and the path $\gamma_j \subset \partial\Omega_j$ connects the points $(R_j, 0)$ and $(R_j, \varphi_j) \in \partial\Omega_j$. This further leads to the APS boundary condition (5) on our chosen component of the boundary

$$BC_{APS}\mid_{\partial\Omega_j} = \overline{\operatorname{span}\left\{ \left[\begin{pmatrix} \psi_\ell^j \\ 0 \end{pmatrix}\right]_{\ell>\frac{\Phi_j}{2\pi}-\frac{1}{2}}, \left[\begin{pmatrix} 0 \\ \psi_\ell^j \end{pmatrix}\right]_{\ell\le\frac{\Phi_j}{2\pi}+\frac{1}{2}} \right\}}, \tag{18}$$

where the closure is in the norm of $\check{H}(A|_{\partial\Omega_j})$ (see (6)). Denoting (r, φ) the polar coordinates at the origin we notice that the inner normal vector on the boundary $\partial\Omega_{out}$ corresponds to $-\partial_r$. Taking this into account means that formally $A\mid_{\partial\Omega_{out}}$ looks like (16) with a minus sign, $A\mid_{\partial\Omega_{out}} = -\sigma^3\left(\frac{\mathrm{i}\partial_\varphi}{R_{out}} + a_\varphi\right) + \frac{1}{2R_{out}}$. Therefore, we infer the solution of the eigenvalue problem immediately from (17)

$$A\mid_{\partial\Omega_{out}}\begin{pmatrix}\psi_\ell \\ 0\end{pmatrix} = -\lambda_\ell\begin{pmatrix}\psi_\ell \\ 0\end{pmatrix} \text{ and}$$
$$A\mid_{\partial\Omega_{out}}\begin{pmatrix}0 \\ \psi_\ell\end{pmatrix} = \lambda_{\ell-1}\begin{pmatrix}0 \\ \psi_\ell\end{pmatrix} \quad \text{with} \begin{cases} R_{out}\lambda_\ell = \Phi/2\pi - 1/2 - \ell \\ \psi_\ell = \mathrm{e}^{\mathrm{i}\varphi\ell}\mathrm{e}^{\mathrm{i}\int_{\gamma_{out}}\vec{a}\cdot\mathrm{d}\vec{s} - \mathrm{i}\frac{\Phi}{2\pi}\varphi} \end{cases}, \tag{19}$$

where $\gamma_{out} \subset \partial\Omega_{out}$ connects the points $(R_{out}, 0)$ and (R_{out}, φ). The APS boundary condition on the outer component of the boundary thus reads

$$BC_{APS}\,|_{\partial\Omega_{out}} = \overline{\operatorname{span}\left\{ \left[\begin{pmatrix}\psi_\ell \\ 0\end{pmatrix}\right]_{\ell < \frac{\Phi}{2\pi} - \frac{1}{2}}, \left[\begin{pmatrix}0 \\ \psi_\ell\end{pmatrix}\right]_{\ell \geq \frac{\Phi}{2\pi} + \frac{1}{2}} \right\}}, \tag{20}$$

where the closure is in the norm of $\check{H}(A|_{\partial\Omega_{out}})$ (see (6)).

The canonical APS boundary condition is gauge invariant in the sense of the following lemma.

Lemma 7. *Let D_a^{APS} and $D_{\widetilde{a}}^{\mathrm{APS}}$ be two Dirac operators with the APS boundary condition on M corresponding to magnetic fields with fluxes Φ and $\widetilde{\Phi}$, respectively, such that*

$$\Phi = \sum_{j \leq N} \Phi_j + \Phi_0 \qquad \textit{and} \qquad \widetilde{\Phi} = \sum_{j \leq N} \widetilde{\Phi}_j + \Phi_0\,,$$

where Φ_j and $\widetilde{\Phi}_j$ are the fluxes through the hole Ω_j, $j \leq N$ of a and $\widetilde{a}$, respectively, and Φ_0 is the flux of a smooth magnetic field supported inside the interior of M. If for all $j \leq N$

$$\widetilde{\Phi}_j = \Phi_j + m_j 2\pi\,,$$

for some $m_j \in \mathbb{Z}$, then D_a^{APS} and $D_{\widetilde{a}}^{\mathrm{APS}}$ are unitarily equivalent

$$\mathscr{U}^* D_a^{\mathrm{APS}} \mathscr{U} = D_{\widetilde{a}}^{\mathrm{APS}}\,,$$

with the unitary operator

$$\mathscr{U} : L^2(M, \mathbb{C}^2) \to L^2(M, \mathbb{C}^2)\,,$$
$$\mathscr{U} : u \mapsto \exp\left[\mathrm{i} \int_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right] u\,,$$

where γ connects a fixed point $z_0 \in M$ and the point $z \in M$.

Proof. First we notice that $\mathscr{U}$ is independent of a particular choice of the path γ in its definition as for an arbitrary loop $\gamma \subset M$ it is an identity operator $\exp\left[\mathrm{i} \oint_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right] = 1$. Thus, we can see immediately from the equalities

$$\oint_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s} = \int_{int\,\gamma} B - \widetilde{B} = -2\pi \sum_{\{j | \Omega_j \subset int\,\gamma\}} m_j\,,$$

where in the first equality we used the relation $\operatorname{curl} \vec{a} = (0, 0, B)$ and Stokes theorem. Notice also that choosing another point $z_1 \in M$ as a starting point of γ instead of z_0 amounts to multiplication by a constant $K = \mathrm{e}^{-\mathrm{i} \int_{z_1}^{z_0} (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}}$. Since $\overline{K} = K^{-1}$ the map $K^{-1}\mathscr{U}$ is also unitary. An explicit computation of the partial derivatives

$$\partial_z \mathscr{U} = \frac{\mathrm{i}}{2} \overline{(a - \widetilde{a})}(z) \mathscr{U}\,, \qquad \text{and} \qquad \partial_{\bar{z}} \mathscr{U} = \frac{\mathrm{i}}{2} (a - \widetilde{a})(z) \mathscr{U}$$

shows that the action of the unitarily transformed Dirac operator is indeed the one with the potential $\widetilde{a}$, as

$$\mathscr{U}^* D_a \mathscr{U} = D_a - 2\mathrm{i}\mathscr{U}^* \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \mathscr{U} = D_a - \mathrm{i} \begin{pmatrix} 0 & \mathrm{i}(\overline{a - \widetilde{a}}) \\ \mathrm{i}(a - \widetilde{a}) & 0 \end{pmatrix}.$$

Moreover, taking our observation that $\overline{K} = K^{-1}$ into account we see that the relation $\mathscr{U}^* D_a \mathscr{U} = D_{\widetilde{a}}$ holds independently of the starting point of γ.

Finally we need to check that the boundary condition is preserved by $\mathscr{U}$. To do so, we show that the boundary operators are unitarily equivalent

$$A(\widetilde{a}) = \mathscr{U}^* A(a) \mathscr{U}\,. \tag{21}$$

Here $A(a)$ denotes the canonical boundary operator adapted to D_a and γ in the definition of $\mathscr{U}$ ends at a point on the boundary $z \in \partial\Omega$. From this we see that the restriction to the boundary of a spinor $u\big|_{\partial\Omega_j}$ is in the negative spectral subspace of $A(\widetilde{a})$ if and only if $(\mathscr{U}u)\big|_{\partial\Omega_j}$ is in the negative spectral subspace of $A(a)$. To see that (21) holds, we write $\alpha = a_\rho \nu + a_s \xi$ with some smooth functions a_ρ, a_s on a neighbourhood of the boundary and consider $z = \gamma(s) \in \partial\Omega_j$. By path independence we have $\mathscr{U} = K \mathrm{e}^{\mathrm{i}\int_0^s (a_s - \widetilde{a}_s)\,\mathrm{d}s}$ with s the arc parameter of $\gamma \subset \partial\Omega_j$. In this notation the boundary operator from Definition 3 reads

$$A(a) = \sigma(\nu)\sigma(\xi)\left(\partial_s - \mathrm{i}a_s\right) - \kappa/2\,,$$

and we can easily compute its commutator with $\mathscr{U}$ using the fundamental theorem of calculus

$$[A(a), \mathscr{U}] = \sigma(\nu)\sigma(\xi)\partial_s(\mathscr{U}) = \sigma(\nu)\sigma(\xi)\mathscr{U}(z)(\mathrm{i}a_s(z) - \mathrm{i}\widetilde{a}_s(z))\,,$$

which yields $A(a)\mathscr{U}u = \mathscr{U}A(a)u + [A(a), \mathscr{U}]u = \mathscr{U}A(\widetilde{a})u.$ ☐

Note that Lemma 7 holds also in case of a smooth non-circular boundary.

Our results are stated in the Aharonov–Casher gauge. It is important to stress out that this is just for the convenience in the computations. Of course, the spectral results are independent of this gauge choice as established by Lemma 7.

2. Main Results

Using the set-up introduced in Sect. 1.3 we are in a position to state the main theorems of this paper. The first result concerns an Aharonov–Casher-type theorem for Dirac operators on $\mathbb{R}^2$ with circular holes. More precisely we will prove

Theorem 8. *Let $M = \mathbb{C} \setminus \cup_{k \le N} \Omega_k$ and let D_a^{APS} be the Dirac operator with the magnetic field (13) in the gauge (9), (10). If $\Phi \neq 0$ then there are*

$$\left\lfloor \frac{|\Phi|}{2\pi} \right\rfloor$$

zero modes of the operator D_a^{APS} *with the APS boundary conditions* (18) *on the inner components of the boundary. These states have spin-up (*i.e. *are eigenvectors of* $\sigma_3 = \mathrm{diag}(1, -1)$ *with eigenvalue* $+1$*) if* $\Phi > 0$ *and spin-down (*i.e. *are eigenvectors of* σ_3 *with eigenvalue* -1*) if* $\Phi < 0$*. If* $\Phi = 0$ *then there are no zero modes.*

Corollary 9. *Under the assumptions of Theorem 8, zero is an eigenvalue embedded in the essential spectrum of the operator* D_a^{APS}*.*

Proof. Due to the unboundedness of M and the existence of the zero modes we can construct a Weyl sequence as follows. Let $\mathbb{B}$ denote a ball centred at the origin such that $\cup_{k \le N}\Omega_k \subset \mathbb{B}$. Consider a compactly supported radial smooth function φ on $\mathbb{C} \setminus \mathbb{B}$. If $\Phi \neq 0$, choose a zero mode $u \in \mathrm{dom}(D_a^{\mathrm{APS}})$. Denote $\varphi_n(z) := \frac{1}{n}\varphi(\frac{z}{n})$. Then $u_n(z) := u(z)\varphi_n(z) \subset \mathrm{dom}(D_a^{\mathrm{APS}})$ tends weekly to zero, while

$$D_a u_n = (D_a u)\varphi_n + u D_0 \varphi_n = u D_0 \varphi_n \,.$$

Using that in polar coordinates D_0 acts as $D_0 = -\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\vartheta} \\ \mathrm{e}^{\mathrm{i}\vartheta} & 0 \end{pmatrix}\partial_r$ on radial functions, one concludes $\|D_a u_n\| \to 0$ as $n \to \infty$.

If $\Phi = 0$, there are no zero modes. In view of Remark 4 the vector potential corresponding to the smooth compactly supported magnetic field in the bulk is bounded. Moreover, thanks to Lemma 7 we can, without loss of generality, consider that the magnetic field inside the holes is formed by the Aharonov Bohm fluxes. The vector potential of such field is bounded outside of $\mathbb{B}$. Consequently the proof works with sequence $u_n = \varphi_n$. □

For the case when the underlying manifold is a disc with holes, the number of zero modes differs. In particular for the total flux in the range $(\pi, 2\pi] \bmod 2\pi k$, for $0 \le k \in \mathbb{Z}$ or $[-2\pi, -\pi] \bmod 2\pi k'$, for $0 \ge k' \in \mathbb{Z}$, we obtain an extra zero mode as opposed to the unbounded case, see Remark 11.

Theorem 10. *Let* $M = \Omega_{out} \setminus \cup_{k \le N}\Omega_k$ *and let* D_a^{APS} *be the Dirac operator with the magnetic field* (13) *in the gauge* (9), (10)*. Then there are*

$$\left| \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor \right|$$

zero modes of the operator D_a^{APS} *with the APS boundary conditions* (18) *on the inner components and* (20) *on the outer component of the boundary. In particular, there are no zero modes in the case* $\Phi \in (-\pi, \pi]$*. If* $\Phi > 0$ *then all the zero modes have spin- up. If* $\Phi < 0$ *then they have spin-down.*

Remark 11.
- The particular form of the (non-normalised) zero modes of the Dirac operator in the Aharonov–Casher gauge with the APS boundary condition is also known from the proof. Depending on the sign of the total flux Φ, they are purely spin-up or purely spin-down

$$\begin{pmatrix} u^+ \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u^- \end{pmatrix},$$

where

$$u^+(z) = \mathrm{e}^{h(z)} \sum_{0\le n<\frac{\Phi}{2\pi}-1} a_n z^n\,, \quad u^-(z) = \mathrm{e}^{-h(z)} \sum_{0\le n<-\frac{\Phi}{2\pi}-1} b_n \overline{z}^n\,, \quad \text{if } \mathscr{M}=\mathbb{C}\,,$$

and,

$$u^+(z) = \mathrm{e}^{h(z)} \sum_{0\le n<\frac{\Phi}{2\pi}-\frac{1}{2}} a'_n z^n\,, \quad u^-(z) = \mathrm{e}^{-h(z)} \sum_{0\le n\le-\frac{\Phi}{2\pi}-\frac{1}{2}} b'_n \overline{z}^n\,, \quad \text{if } \mathscr{M}=\Omega_{out}\,, \tag{22}$$

with some coefficients $a_n, b_n, a'_n, b'_n \in \mathbb{C}$.

- Notice that for certain values of fluxes there is an extra zero mode for the bounded region. This happens in particular if there exists an integer k such that

$$\frac{\Phi}{2\pi} = k+\epsilon \quad \text{with } \epsilon \in \begin{cases} (1/2, 1] & \text{if } k\ge 0 \\ [-1, -1/2] & \text{if } k \le 0 \end{cases}. \tag{23}$$

If we let the radius of the disc Ω_{out} grow to infinity we can compare the asymptotics of the zero modes (22) as $|z|\to\infty$ and easily check that the zero mode with the highest power of z in g^+ (or $\overline{z}$ in g^- in case of spin-down zero modes) satisfying the APS boundary condition would not be square integrable at infinity exactly for the values of fluxes in (23). They exhibit the behaviour $u^\pm(z) \sim |z|^{\mp\epsilon_\pm}$ as $|z|\to\infty$ with $\epsilon_+ \in (1/2, 1]$ and $\epsilon_- \in [-1, -1/2]$. Therefore, they are decaying at infinity and thus are resonances of the Dirac operator in the unbounded case when $\mathscr{M}=\mathbb{C}$, *i.e.* the operator from Theorem 8. To complete the picture we recall that for values $\Phi/2\pi \in (-1/2, 1/2]$ there are no zero modes on either plane or a disc with holes.

The standard argument of Aharonov and Casher from [2] will be very useful for showing Theorems 10 and 8. We will first use it to prove the following

Proposition 12. *Let $D_a^{\max}$ be the maximal extension* (3) *of the Dirac operator on M with the magnetic field* (13) *in the gauge* (9), (10) *and let $D_a^{\max}u=0$ for some $u\in \operatorname{dom}(D_a^{\max})$. Then u is of the form $u=(u^+,u^-)$ with*

$$u^\pm = \mathrm{e}^{\pm h} g^\pm\,,$$

where g^+ is analytic on M and g^- is anti-analytic on M. Vice versa, any function of this form such that $u\in\operatorname{dom}(D_a^{\max})$ satisfies $D_a^{\max}u=0$.

Proof. To find solutions of $D_a^{\max}u=0$, $u=(u^+,u^-)\in L^2(M;\mathbb{C}^2)$ we rewrite the problem using the Aharonov–Casher gauge (9) as

$$\mathrm{e}^h\partial_{\overline{z}}(\mathrm{e}^{-h}u^+) = \left[\partial_{\overline{z}} - \frac{\mathrm{i}a}{2}\right]u^+ = 0\,, \quad \mathrm{e}^{-h}\partial_z(\mathrm{e}^{h}u^-) = \left[\partial_z - \frac{\mathrm{i}\overline{a}}{2}\right]u^- = 0\,.$$

This is satisfied if and only if the function $g^+ := \mathrm{e}^{-h}u^+$ is analytic and $g^- := \mathrm{e}^h u^-$ is anti-analytic on M. □

Remark 13. To avoid any confusions, we would like emphasise that by taking $u \in L^2(M;\mathbb{C}^2)$ in the previous proof, we are not claiming that this is the maximal domain $\mathrm{dom}(D_a^{\max})$. If $u \in L^2(M;\mathbb{C}^2)$ such that $D_a^{\max}u = 0$ it is by definition in the maximal domain.

2.1. Proof for Unbounded Region with Holes

The main idea of the proofs of Theorems 10 and 8 is to show that if $u \in \mathrm{dom}(D_a^{\mathrm{APS}})$ and hence, $(u^+, u^-)^T$ satisfies the APS boundary condition (18), then the functions g^+ and g^- from Proposition 12 can be extended analytically in z and $\bar{z}$ respectively inside the holes of M. In the following example we demonstrate that for the case of a single hole with a magnetic field inside the hole this extension is a straightforward process.

Example 14. For one hole in the plane we worked out the APS boundary condition in Example 6. Here we use the set up and notation from that example. For the functions $g^\pm$ from Proposition 12 we can write the Laurent series

$$g^+ = \sum_{n\in\mathbb{Z}} a_n z^n \qquad \text{and} \qquad g^- = \sum_{n\in\mathbb{Z}} b_n \bar{z}^n\,,$$

with some complex coefficients a_n, b_n. Taking into account (12), the formal restriction of $u^\pm = \mathrm{e}^{\pm h} g^\pm$ to the boundary $\partial\Omega$ then reads

$$\gamma_0 u^+ = R^{-\Phi/2\pi} \sum_{n\in\mathbb{Z}} a_n R^n \mathrm{e}^{\mathrm{i}n\varphi} \qquad \text{and} \qquad \gamma_0 u^- = R^{\Phi/2\pi} \sum_{n\in\mathbb{Z}} b_n R^n \mathrm{e}^{-\mathrm{i}n\varphi}\,.$$

Here γ_0 is the trace map as in (8). By Lemma 7 we can restrict ourselves to $\Phi \in [-\pi,\pi)$. The boundary condition (15) then yields that $a_n = 0 = b_n$ for $n < 0$. Consequently, g^+ has to be analytic and g^- anti-analytic on $\mathbb{C}$.

We leave the argument that the trace γ_0 is indeed given by the formal restriction to the boundary to the general case, which is worked out in Sect. 2.1.1

While the argument in our example is rather simple, extending $g^\pm$ (anti-)analytically inside the holes requires a new approach if we have several holes. Let us fix an index j and denote $\mathcal{A}$ an open annulus co-centred with the hole Ω_j of inner radius R_j and outer radius R such that $\mathcal{A} \cap \operatorname{supp} B = \emptyset$ with B as in (13). In particular the scalar potential $h(z)$ is bounded on $\mathcal{A}$.

Recall that g^+ is analytical and g^- is anti-analytical on M. Thus, on $\mathcal{A}$ they have the Laurent series

$$g^+(z) = \sum_{n\in\mathbb{Z}} a_n (z-w_j)^n\,, \qquad g^-(z) = \sum_{n\in\mathbb{Z}} b_n \overline{(z-w_j)}^n\,, \tag{24}$$

with some $a_n, b_n \in \mathbb{C}$. To find the boundary values of $u^\pm$ and compare them to the boundary condition (18), it is convenient to introduce the following functions for $z \in \mathcal{A} \cup \overline{\Omega_j} \backslash \{w_j\}$

$$\begin{aligned} G_j^+(z) &:= -\mathrm{i}\int_{\gamma(z_{0j},z)} \vec{a}\,\mathrm{d}\vec{s} + \int_{\gamma(z_{0j},z)} \frac{\Phi_j'}{2\pi(z'-w_j)}\,\mathrm{d}z'\,,\\ G_j^-(z) &:= -\mathrm{i}\int_{\gamma(z_{0j},z)} \vec{a}\,\mathrm{d}\vec{s} - \int_{\gamma(z_{0j},z)} \frac{\Phi_j'}{2\pi\overline{(z'-w_j)}}\,\mathrm{d}\bar{z}'\,, \end{aligned} \tag{25}$$

where by $\gamma(z_{0j}, z) \subset \mathcal{A} \cup \overline{\Omega_j} \backslash \{w_j\}$ we denoted the path of integration with the endpoints $z_{0j} = w_j + R_j$ and $z \in \mathcal{A} \cup \overline{\Omega_j} \backslash \{w_j\}$. We stress that throughout the whole section we are, owing to Lemma 7, assuming $B\big|_{\Omega_j} = B_j = \Phi'_j \delta_{w_j}$, with the normalised flux $\Phi'_j \in [-\pi, \pi)$. This, further, allows us to extend the definition of the vector potential a that is given by (9) inside the region $\Omega_j \setminus \{w_j\}$.

The definition (25) is motivated by the fact that the restrictions of $G_j^{\pm}(z)$ to the boundary $\partial\Omega_j$ satisfy

$$G_j^{\pm}(z)\,|_{z\in\partial\Omega_j} = -\mathrm{i}\int_{\gamma_j} \vec{a}\,\mathrm{d}\vec{s} + \mathrm{i}\frac{\Phi'_j}{2\pi}\varphi_j\,,$$

where $\gamma_j \subset \partial\Omega_j$ is the curve connecting the points $z_{0j} = w_j + R_j$ and z counter-clockwise. The lemma below further shows that $G_j^{\pm}(z)$ are well defined on $\mathcal{A} \cup \overline{\Omega_j} \setminus \{w_j\}$.

Lemma 15. *$G_j^{\pm}(z)$ are independent of the choice of the path $\gamma(z_{0j}, z)$ contained in $\mathcal{A} \cup \overline{\Omega_j} \backslash \{w_j\}$.*

Proof. We show the equivalent statement that $G_j^{\pm}(z) = 0$ for any loop $\gamma = \gamma(z_{0j}, z = z_{0j}) \subset \mathcal{A} \cup \overline{\Omega_j} \backslash \{w_j\}$. By definition of the flux the first summands on the right-hand sides of (25) read

$$-\mathrm{i}\int_{\gamma} \vec{a}\,\mathrm{d}\vec{s} = -\mathrm{i}\ell\Phi'_j\,,$$

where $\ell \in \mathbb{Z}$ is the winding number of the loop γ around the point w_j. The result then follows from the definition of the winding number

$$\ell := \frac{1}{2\pi\mathrm{i}}\int_{\gamma} \frac{1}{z' - w_j}\,\mathrm{d}z' = \frac{-1}{2\pi\mathrm{i}}\int_{\gamma} \frac{1}{\overline{z' - w_j}}\,\mathrm{d}\bar{z}'\,.$$

□

Now we show that $G_j^{\pm}$ satisfy another important property.

Proposition 16. *Define*

$$F_j^{\pm}(z) := \pm h(z) + G_j^{\pm}(z)\,, \tag{26}$$

with h given by (10). The functions $F_j^{+}(z)$ and $F_j^{-}(z)$ are analytic on $\mathcal{A} \cup \overline{\Omega_j}$ in z and $\bar{z}$, respectively. And, in particular, there are the following series of the exponentials

$$\mathrm{e}^{F_j^+} = \sum_{k\geq 0} c_k^{+}(z - w_j)^k\,, \quad \mathrm{e}^{F_j^-} = \sum_{k\geq 0} c_k^{-}\overline{(z - w_j)}^k\,,$$

with $c_0^{\pm} \neq 0$.

Before we prove this statement let us present a lemma.

Lemma 17. *Let g be defined on a domain in $\mathbb{C}$. If either*

$$g_{an}(z) := \int_\gamma g(w)\,\mathrm{d}w = \int_\gamma (g_1 + \mathrm{i}g_2)(\,\mathrm{d}t_1 + \mathrm{i}\,\mathrm{d}t_2)\,,$$

or

$$g_{anti}(z) := \int_\gamma g(w)\,\mathrm{d}\bar{w} = \int_\gamma (g_1 + \mathrm{i}g_2)(\,\mathrm{d}t_1 - \mathrm{i}\,\mathrm{d}t_2)\,,$$

are independent of the path γ connecting a fixed point $z_0 \in \mathbb{C}$ and a point $z \in \mathbb{C}$, then g_{an} or $g_{anti}(z)$ are analytic in z or $\bar{z}$, respectively.

Proof. Due to the independence on path γ it holds $\partial_x \int_\gamma (g_1 + \mathrm{i}g_2)\,\mathrm{d}t_2 = 0$ and $\partial_y \int_\gamma (g_1 + \mathrm{i}g_2)\,\mathrm{d}t_1 = 0$. So we have

$$\partial_x \int_\gamma g(w)\,\mathrm{d}w = \partial_x \int_\gamma (g_1 + \mathrm{i}g_2)\,\mathrm{d}t_1 = g(z)$$
$$\partial_y \int_\gamma g(w)\,\mathrm{d}w = \partial_y \int_\gamma \mathrm{i}(g_1 + \mathrm{i}g_2)\,\mathrm{d}t_2 = \mathrm{i}g(z)\,,$$

where in the second equalities we used the fundamental theorem of calculus. Hence, $\frac{1}{2}(\partial_x \pm \mathrm{i}\partial_y)\int_\gamma g(w)(\,\mathrm{d}t_1 \pm \mathrm{i}\,\mathrm{d}t_2) = 0$. □

Proof of Proposition 16. The analyticity follows from the fact, which will be proved below, stating that $F_j^\pm$ have the following forms on $\mathcal{A} \cup \overline{\Omega_j} \setminus \{w_j\}$

$$\begin{aligned} F_j^+(z) &= h(z_{0j}) + \int_{\gamma(z_{0j},z)} \sum_{\substack{k\le N\\ k\neq j}} \left(2\partial_{z'} h_{[k]}\right)\,\mathrm{d}z'\,, \\ F_j^-(z) &= -h(z_{0j}) - \int_{\gamma(z_{0j},z)} \sum_{\substack{k\le N\\ k\neq j}} \left(2\partial_{\bar{z}'} h_{[k]}\right)\,\mathrm{d}\bar{z}'\,, \end{aligned} \tag{27}$$

where $h_{[k]} = \frac{-\Phi_k'}{2\pi}\log|z - w_k|$ for $z \neq w_k$ is the scalar potential of the field B_k inside the hole Ω_k. A direct computation (or the "defining" Poisson equation $-\Delta h_{[k]} = B_k$ for $h_{[k]}$) yields that the integrands $\sum_{k\neq j} 2\partial_z h_{[k]}$ and $\sum_{k\neq j} 2\partial_{\bar{z}} h_{[k]}$ are analytic on $\mathcal{A} \cup \overline{\Omega_j}$ in z and $\bar{z}$, respectively. It follows from Lemma 17 that this indeed implies analyticity of $F_j^\pm$.

Now we will show that the equalities (27) hold. To that end we use the relation (9), *i.e.*

$$a_x = \partial_y h\,, \quad a_y = -\partial_x h\,,$$

and write with the aid of the fundamental theorem of calculus

$$h(z) = h(z_{0j}) + \int_{\gamma(z_{0j},z)} \partial_x h\,\mathrm{d}x + \partial_y h\,\mathrm{d}y\,,$$

where $\gamma(z_{0j}, z) \subset \mathcal{A}\cup\overline{\Omega_j}\backslash\{w_j\}$ is an arbitrary path connecting z_{0j} and z. Thus, we get

$$\begin{aligned} h - \mathrm{i}\int_{\gamma(z_{0j},z)} \vec{a}\,\mathrm{d}\vec{s} &= h(z_{0j}) + \int_{\gamma(z_{0j},z)} \partial_x h\,\mathrm{d}x + \partial_y h\,\mathrm{d}y - \mathrm{i}\partial_y h\,\mathrm{d}x + \mathrm{i}\partial_x h\,\mathrm{d}y \\ &= h(z_{0j}) + 2\int_{\gamma(z_{0j},z)} \partial_{z'} h\,\mathrm{d}z'\,, \end{aligned}$$

and similarly (or by complex conjugation), $-h - \mathrm{i}\int_{\gamma(z_{0j},z)} \vec{a}\,\mathrm{d}\vec{s} = -h(z_{0j}) - 2\int_{\gamma(z_{0j},z)} \partial_{\bar{z}'} h\,\mathrm{d}\bar{z}'$.

Finally, we recall that the concrete form (10) of the potential function for $B_j = \Phi'_j\delta_{w_j}$ is $h_{[j]} = \frac{-\Phi'_j}{2\pi}\log|z-w_j|$ and compute the derivatives

$$\begin{aligned} \partial_z h_{[j]}(z) &= \frac{-\Phi'_j}{4\pi}\partial_z \log|z-w_j|^2 = -\frac{1}{4\pi}\frac{\Phi'_j}{z-w_j}\,, \\ \text{and} \qquad \partial_{\bar{z}} h_{[j]}(z) &= -\frac{1}{4\pi}\frac{\Phi'_j}{\overline{z-w_j}}\,, \end{aligned}$$

which together with the definitions (26) and (25) give (27). □

2.1.1. Trace of the Eigenfunctions. Using the properties of $G_j^{\pm}$ we are able to find the trace γ_0 of functions that have a form of the zero modes on $\mathcal{A}$. For brevity we will denote by $\check{H}_j$ the space $\check{H}(A|_{\partial\Omega_j})$, recall (6) for the definition of the norm on $\check{H}(A)$.

Lemma 18. *Consider the function $u = (u^+, u^-)^T$ which has a convergent Laurent series on $\mathcal{A}$*

$$u = \sum_{n\in\mathbb{Z}} (\mathrm{e}^h a_n (z-w_j)^n, \mathrm{e}^{-h} b_n (\overline{z-w_j})^n)^T \in L^2(\mathcal{A}, \mathbb{C}^2)\,, \qquad a_n, b_n \in \mathbb{C}\,.$$

Then we have the convergence

$$u^Q = \begin{pmatrix} u^{+,Q} \\ u^{-,Q} \end{pmatrix} := \sum_{|n|\le Q} \begin{pmatrix} \mathrm{e}^h a_n (z-w_j)^n \\ \mathrm{e}^{-h} b_n (\overline{z-w_j})^n \end{pmatrix} \to \begin{pmatrix} u^+ \\ u^- \end{pmatrix}\,, \qquad \text{as } Q \to \infty \tag{28}$$

in the operator graph norm $\|\cdot\|_{D_a,\mathcal{A}} := \|\cdot\|^2_{L^2(\mathcal{A},\mathbb{C}^2)} + \|D_a(\cdot)\|^2_{L^2(\mathcal{A},\mathbb{C}^2)}$.

Proof. By Proposition 12 it holds that $D_a u^Q = D_a u = 0$ on $\mathcal{A}$. Hence, the operator graph norm of u has contribution only from $\|u\|^2_{L^2(\mathcal{A},\mathbb{C}^2)} = \|u^+\|^2_{L^2(\mathcal{A})} + \|u^-\|^2_{L^2(\mathcal{A})}$. We compute the first summand.

$$\begin{aligned} \|u^+\|^2_{L^2(\mathcal{A})} &= 2\pi\int_{R_j}^{R} r\,\mathrm{d}r \left(\mathrm{e}^{2h}\sum_{n\in\mathbb{Z}} |a_n|^2 r^{2n}\right) \\ &\ge C\left(|a_{-1}|^2 \ln\frac{R}{R_j} + \sum_{n\neq -1} |a_n|^2 \frac{R^{2n+2} - R_j^{2n+2}}{2n+2}\right)\,, \end{aligned}$$

where the constant $C = 2\pi \min_{z\in\mathcal{A}} \mathrm{e}^{2h(z)}$ is non-zero. This shows that the sum on the right-hand side is convergent and therefore

$$\|u^+ - u^{+,Q}\|^2_{L^2(\mathcal{A})} \le 2\pi \max_{z\in\mathcal{A}} \mathrm{e}^{2h(z)} \sum_{n>Q} |a_n|^2 \frac{R^{2n+2} - R_j^{2n+2}}{2n+2} \to 0\,, \qquad \text{as } Q \to \infty\,.$$

The proof for u^- is a matter of substituting e^{2h} by e^{-2h} and the coefficients a_n by b_n in the computation above. □

As a corollary we obtain from [7, Theorem 1.7] the convergence in $\check{H}_j$ of the traces of u^Q to the trace of the zero mode u.

Lemma 19. *Let u^Q be as in Lemma 18. In $\check{H}_j$ we have the following convergence of the traces $\gamma_0(u^Q)$*

$$\lim_{Q\to\infty} \gamma_0 \begin{pmatrix} u^{+,Q} \\ u^{-,Q} \end{pmatrix} = \sum_{n\in\mathbb{Z}} \sum_{k\ge 0} R_j^{n+k} \begin{pmatrix} a_n c_k^+ \psi^j_{n+k} \\ b_n c_k^- \psi^j_{-(n+k)} \end{pmatrix} =: \begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix}. \tag{29}$$

The vectors ψ^j_ℓ were introduced in (17) and $c_k^\pm$ are the coefficients from Proposition 16.

Proof. First we show that the sum in (29) is an element of $\check{H}_j$. Note that due to the definite chirality of eigenvectors of $A|_{\partial\Omega_j}$ we have $\|(v^+, v^-)\|^2_{\check{H}_j} = \|(v^+, 0)\|^2_{\check{H}_j} + \|(0, v^-)\|^2_{\check{H}_j}$ for any $(v^+, v^-)^T \in \check{H}_j$. Therefore, we work out the proof only for the spin-up part and leave out the spin-down part which is analogous. Let u^+ be as in Lemma 18. Then boundedness of $\mathrm{e}^{G_j^+}$ on $\mathcal{A}$ implies $\mathrm{e}^{G_j^+} u^+ \in L^2(\mathcal{A})$. We compute its norm explicitly

$$\begin{aligned}
\|\mathrm{e}^{G_j^+} u^+\|^2_{L^2(\mathcal{A})} &= \int_{R_j}^R \mathrm{d}r \int_0^{2\pi} r\,\mathrm{d}\varphi \left| \mathrm{e}^{G_j^+ + h} g^+ \right|^2 \\
&= \int_{R_j}^R \mathrm{d}r \int_0^{2\pi} r\,\mathrm{d}\varphi \left| \sum_{\substack{k\ge 0\\ n\in\mathbb{Z}}} a_n c_k^+ r^{k+n} \mathrm{e}^{\mathrm{i}\varphi(k+n)} \right|^2 \\
&= 2\pi \left| \sum_{\substack{k+n=-1\\ k\ge 0}} a_n c_k^+ \right|^2 \ln \frac{R}{R_j} + 2\pi \sum_{\ell\ne -1} \left| \sum_{\substack{k+n=\ell\\ k\ge 0}} a_n c_k^+ \right|^2 \frac{1}{2\ell+2} (R^{2\ell+2} - R_j^{2\ell+2})\,,
\end{aligned}$$

where in the second equality we used that $\mathrm{e}^{G_j^+(z)+h(z)} = \mathrm{e}^{F_j^+}$ is the analytic function on $\mathcal{A} \cup \overline{\Omega_j}$ from Proposition 16. This expression can be used to bound the $\check{H}_j$ norm of the spin-up component of (29)

$$\left\|\left(\sum_{\ell\in\mathbb{Z}}\sum_{\substack{n,k\\ n+k=\ell}} R_j^\ell a_n c_k^+ \psi_\ell^j\,,0\right)^T\right\|^2_{\check{H}_j} = \sum_{\ell\in\mathbb{Z}}\left|\sum_{\substack{n+k=\ell\\ k\geq 0}} R_j^\ell a_n c_k^+ \psi_\ell^j\right|(1+|\lambda_\ell^j|^2)^s\,,$$

$$\begin{cases} s=1/2 & \text{if } \lambda_\ell^j<0\\ s=-1/2 & \text{if } \lambda_\ell^j\geq 0\end{cases}, \tag{30}$$

with λ_ℓ^j from (17). Indeed, observing that

1. If $K\in[-1,0)$, then for any $n\geq 0$ and $C<1/\sqrt{5}$ it holds
$$n+1\geq C\sqrt{(n-K)^2+1}\,,\qquad \frac{1}{n+1}\geq C\frac{1}{\sqrt{(n+K)^2+1}}\,.$$
2. There exist constants $C_{<,>,0}$ such that $\ln\frac{R}{R_j}\geq\frac{C_0}{R_j^2}$, and
$$R^n-R_j^n\geq\left(RR_j^{-1}\right)^n R_j^n\left(1-\left(R_jR^{-1}\right)^n\right)\geq C_>n^2R_j^n\,,\qquad \text{if } n>0\,,$$
$$R_j^n-R^n=R_j^n\left(1-\left(R_jR^{-1}\right)^{|n|}\right)\geq C_<R_j^n\,,\qquad \text{if } n<0\,,$$

we deduce $\|\mathrm{e}^{G_j^+}u^+\|^2_{L^2(\mathcal{A})}\geq const\|(u_0^+,0)^T\|^2_{\check{H}_j}$.

Applying Proposition 16 and using the definition of ψ_ℓ^j we obtain

$$\gamma_0[(u^{+,Q},0)^T]=\sum_{|n|\leq Q}R_j^n a_n \mathrm{e}^{\mathrm{i}n\varphi_j-G_j^+(z)}\mathrm{e}^{(h+G_j^+)(z)}\big|_{z\in\partial\Omega_j}=\sum_{|n|\leq Q}R_j^{n+k}a_nc_k^+\psi_{n+k}^j$$

which with the convergence of the sum in (29) in $\check{H}_j$ norm finishes the proof for the spin-up component. Similarly, one shows $\|\mathrm{e}^{G_j^-}u^-\|^2_{L^2(\mathcal{A})}\geq const\|(0,u_0^-)^T\|^2_{\check{H}_j}$ and the convergence in the spin-down component of (29). □

With these technical preliminaries we are ready to present a key statement for the proof of Theorem 8.

Proposition 20. *Let (u^+,u^-) be a zero mode of the Dirac operator D_a^{APS} with the magnetic field (13) that satisfies the APS boundary condition (18) on $\partial\Omega_j$. Then the functions g^+ and g^- from Proposition 12 can be analytically extended inside the region Ω_j in z and $\overline{z}$, respectively.*

Proof. On $\partial\Omega_j$ we compare the trace given by Lemma 19 with the APS boundary condition (18). We remind that we are using the normalised fluxes through the holes (see Lemma 7) which satisfy $\Phi_j'\in[-\pi,\pi)$ and therefore

$$\gamma_0(u^+,u^-)=\sum_{n\in\mathbb{Z}}\sum_{k\geq 0}R_j^{n+k}\begin{pmatrix}a_nc_k^+\psi_{n+k}^j\\ b_nc_k^-\psi_{-(n+k)}^j\end{pmatrix}=\begin{pmatrix}\sum_{\ell\geq 0}\beta_\ell^+\psi_\ell^j\\ \sum_{\ell\leq 0}\beta_\ell^-\psi_\ell^j\end{pmatrix}.$$

Here $\beta_\ell^\pm\in\mathbb{C}$ are some constant coefficients. Hence, $a_nc_k^+=b_nc_k^-=0$ whenever $n+k<0$. In particular, since $c_0^\pm\neq 0$, it holds $a_n=b_n=0$ for all $n<0$.

This in turn means that g^+ and g^- can be analytically extended inside Ω_j in variables z and $\overline{z}$ respectively. □

Applying now the L^2 integrability condition at infinity to $u^\pm$ the Aharonov–Casher-type result for our setting $\mathscr{M} = \mathbb{C}$ follows.

Proof of Theorem. 8. By Proposition 20 the zero modes are of the form

$$u = \left(\mathrm{e}^h \sum_{n=0}^{n^+} a_n z^n \,, \mathrm{e}^{-h} \sum_{n=0}^{n^-} b_n \overline{z}^n \right)^T$$

with $a_n, b_n \in \mathbb{C}$ and some integers $n^\pm$. Since the requirement $u \in \mathrm{dom}(D_a)$ in particular implies square integrability at infinity, we obtain from the asymptotics (11) of the potential function h the conditions $n^+ - \frac{\Phi}{2\pi} < -1$ and $n^- + \frac{\Phi}{2\pi} < -1$, where $\Phi = \Phi_0 + \sum_{k \le N} \Phi'_j$. From this we infer that there are $\lfloor \frac{\Phi}{2\pi} \rfloor$ zero modes of spin-up and $\lfloor -\frac{\Phi}{2\pi} \rfloor$ zero modes of spin-down provided that $|\Phi| > 2\pi$. □

2.2. Proof for the Bounded Region with Holes

In the case of the bounded domain the condition of the square integrability, responsible for cutting off the infinite series in the final step in proof of Theorem 8, is substituted by the APS boundary condition (20) on the outer boundary $\partial\Omega_{out}$. We denote by $\mathcal{A}^{out} \subset M$ an open annulus whose outer boundary is $\partial\Omega_{out}$ such that it satisfies $\mathcal{A}^{out} \cap \mathrm{supp}\, B = \emptyset$ and by $\widetilde{\Omega}$ the union $(\Omega_{out})^C \cup \mathcal{A}^{out}$ where $(\cdot)^C$ stands for the complement in $\mathbb{C}$. To apply the boundary condition on $\partial\Omega_{out}$ we follow a similar process as in the case of checking the boundary conditions on the inner components of the boundary. Let $\gamma(z_0^{out}, z)$ be a path connecting $z_0^{out} = R_{out}$ and a point $z \in \widetilde{\Omega}$. We define

$$G^+(z) := -\mathrm{i} \int_{\gamma(z_0^{out}, z)} \vec{a}\, \mathrm{d}\vec{s} + \int_{\gamma(z_0^{out}, z)} \frac{\Phi}{2\pi z'}\, \mathrm{d}z' \,,$$

$$G^-(z) := -\mathrm{i} \int_{\gamma(z_0^{out}, z)} \vec{a}\, \mathrm{d}\vec{s} - \int_{\gamma(z_0^{out}, z)} \frac{\Phi}{2\pi \overline{z}'}\, \mathrm{d}\overline{z}' \,.$$

By definition the restrictions to the boundary $\partial\Omega_{out}$ satisfy

$$G^\pm(z) \,|_{z \in \partial\Omega_{out}} = -\mathrm{i} \int_{\gamma_{out}} \vec{a}\, \mathrm{d}\vec{s} + \mathrm{i} \frac{\Phi}{2\pi} \varphi \,,$$

where $\gamma_{out} \subset \partial\Omega_{out}$ connects the points $z_0^{out} = R_{out}$ and $z \in \partial\Omega_{out}$. A direct adaptation of the proof of Lemma 15 shows that $G^\pm(z)$ are independent of the choice of path $\gamma(z_0^{out}, z)$ contained in $\widetilde{\Omega}$.

We prove another key property of these functions.

Lemma 21. *Let us define*

$$F^\pm(z) := \pm h(z) + G^\pm(z) \,.$$

Then $F^+(z)$ and $F^-(z)$ are analytic in z and $\overline{z}$, respectively, on $\widetilde{\Omega}$. Moreover, $F^\pm(z) \to const$ as $|z| \to \infty$.

Proof. Similarly as in the proof of Proposition 16, it can be shown that it holds

$$F^+(z) = h(z_0^{out}) + \int_{\gamma(z_0^{out},z)} \left(2\partial_{z'} h + \frac{\Phi}{2\pi z'}\right) \mathrm{d}z'$$
$$F^-(z) = -h(z_0^{out}) - \int_{\gamma(z_0^{out},z)} \left(2\partial_{\bar{z}'} h + \frac{\Phi}{2\pi \bar{z}'}\right) \mathrm{d}\bar{z}' \,, \tag{31}$$

where $\gamma(z_0^{out}, z) \subset \widetilde{\Omega}$ is an arbitrary path connecting z_0^{out} and z. Then F^+ is analytic on $\widetilde{\Omega}$ as $2\partial_z h + \frac{\Phi}{2\pi z}$ is analytic, and F^- is anti-analytic as $2\partial_{\bar{z}} h + \frac{\Phi}{2\pi \bar{z}}$ is anti-analytic on that region (recall the Poisson equation $-\Delta h = B$ and Remark 17).

Since we are further interested in the limit $|z| \to \infty$, let us assume that $|z| > R'$ for some $R' > 2R_{out}$. We will show that the absolute value of the integrand in (31) decays like $|z|^{-2}$ when $|z|$ tends to infinity. First for the singular parts of the magnetic field $B_j = \Phi'_j \delta_{w_j}$ we have for any $z \in \widetilde{\Omega}$

$$2\partial_z h_{[j]} + \frac{\Phi'_j}{2\pi z} = \frac{-\Phi'_j}{2\pi} \frac{w_j}{z(z - w_j)} \,, \tag{32}$$

with $h_{[j]} = \frac{-\Phi'_j}{2\pi} \log |z - w_j|$ as before, in the proof of Proposition 16. In particular, the absolute value of the right-hand side is indeed bounded by a constant multiple of $|z|^{-2}$ for $|z| > R'$. For the bulk part of the magnetic field $B_0 \in C_0^\infty(M)$ with the scalar potential $h_0(z) = -\frac{1}{2\pi} \int_M B_0(z') \log |z - z'| \frac{\mathrm{i}}{2} \mathrm{d}z' \wedge \overline{\mathrm{d}z'}$ on $\widetilde{\Omega}$ we compute the derivative $\partial_z h_0$ using the dominated convergence theorem similarly as in Remark 4. Then using the definition of the flux Φ_0 we obtain the following estimate

$$\begin{aligned} \left| 2\partial_z h_0 + \frac{\Phi_0}{2\pi z} \right| &= \left| -\frac{1}{2\pi} \int_{\mathbb{C}} \left(\frac{B_0(z')}{z - z'} - \frac{B_0(z')}{z} \right) \frac{\mathrm{i}}{2} \mathrm{d}z' \wedge \overline{\mathrm{d}z'} \right| \\ &\leq \frac{1}{2\pi} \int_{\mathbb{C}} \left| \frac{B_0(z') z'}{z(z - z')} \right| \frac{\mathrm{i}}{2} \mathrm{d}z' \wedge \overline{\mathrm{d}z'} \leq const |z|^{-2} \,, \end{aligned} \tag{33}$$

for all $|z| > R'$. In the last inequality we used that $\left| \frac{B_0(z') z'}{z(z - z')/|z|^2} \right| \leq 2|B_0(z') z'| \in L^1(\mathbb{C})$. Let us define

$$C_0 := \int_0^\infty \left(2\partial_z h(z_0^{out} + t) + \frac{\Phi}{2\pi (z_0^{out} + t)} \right) \mathrm{d}t \,.$$

Then this is a well-defined constant. Indeed, since an integral of analytic function along a bounded interval is bounded we have with use of (33) and (32)

$$|C_0| \leq C_1 + \int_{R'}^\infty \frac{C_2}{t^2} \mathrm{d}t < \infty \,,$$

with some constants $C_{1,2} > 0$. Further by independence on the path $\gamma \subset \widetilde{\Omega}$ (we can choose γ along the real axis and then along the arc corresponding to

$|z| = const$) and by (33), (32) we estimate

$$\left|\int_\gamma 2\partial_{z'} h + \frac{\Phi}{2\pi z'}\,\mathrm{d}z' - C_0\right| \leq const\left| -\int_{|z|}^\infty \frac{\mathrm{d}t}{t^2} + \frac{1}{|z|}\int_0^{\arg(z)} \mathrm{d}\varphi\right| ,$$

which is arbitrarily small as $|z| \to \infty$ and hence concludes the proof for F^+. The proof of asymptotics for F^- at infinity is analogous. □

Corollary 22. *The exponentials of $F^\pm$ have the following series on $\widetilde{\Omega}$*

$$\mathrm{e}^{F^+(z)} = \sum_{n\leq 0} d_n^+ z^n \quad \textit{and} \quad \mathrm{e}^{F^-(z)} = \sum_{n\leq 0} d_n^- \overline{z}^n ,$$

for some $d_n^\pm \in \mathbb{C}$ with $d_0^\pm \neq 0$.

Proof. By the previous lemma and by analyticity of $\exp(z)$ on $\mathbb{C}$ the function $\mathrm{e}^{F^+(w^{-1})}$ is analytic and $\mathrm{e}^{F^-(w^{-1})}$ is anti-analytic on the interior of $\mathbb{C}\backslash\widetilde{\Omega}$ and converge to a non-zero constant as $w \to 0$. This implies existence of the Taylor series $\mathrm{e}^{F^+(w^{-1})} = \sum_{k\geq 0} d_k^+ w^k$ and $\mathrm{e}^{F^-(w^{-1})} = \sum_{k\geq 0} d_k^- \overline{w}^k$ with $d_0^\pm \neq 0$. Thus, on the complement $\widetilde{\Omega}$ we have

$$\mathrm{e}^{F^+(z)} = \sum_{n\leq 0} d_n^+ z^n \quad \text{and} \quad \mathrm{e}^{F^-(z)} = \sum_{n\leq 0} d_n^- \overline{z}^n .$$

□

To apply the boundary condition on the outer boundary we find the boundary values of a function that has the form of our zero modes when restricted to $\mathcal{A}^{out} \subset M$. For conciseness we denote by $\check{H}_{out}$ the space $\check{H}(A|_{\partial\Omega_{out}})$.

Lemma 23. *Let $u = (u^+, u^-)^T = \sum_{n\geq 0}(\mathrm{e}^h a_n z^n, \mathrm{e}^{-h} b_n \overline{z}^n)^T \in L^2(\mathcal{A}^{out}, \mathbb{C}^2)$. Then its trace on $\partial\Omega^{out}$ is*

$$\gamma_0(u^+, u^-)^T = \sum_{n\geq 0}\sum_{k\leq 0} R_{out}^{n+k} \begin{pmatrix} a_n d_k^+ \psi_{n+k} \\ b_n d_k^- \psi_{-(n+k)} \end{pmatrix} =: \begin{pmatrix} v_0^+ \\ v_0^- \end{pmatrix} . \tag{34}$$

The vectors ψ_ℓ were introduced in (19), and the coefficients $d_k^\pm$ are from Corollary 22.

Proof. It is not difficult to see that Lemma 18 holds also with $\mathcal{A}$ substituted by $\mathcal{A}^{out}$ and $z - w_j$ substituted by z. Hence, $\gamma_0(u) = \lim_{Q\to\infty} \sum_{0\leq n\leq Q} R_{out}^n$ $(\mathrm{e}^h a_n \mathrm{e}^{\mathrm{i}\varphi n}, \mathrm{e}^{-h} b_n \mathrm{e}^{-\mathrm{i}\varphi n})^T$ in $\check{H}_{out}$. Mildly alternating the steps of the proof of Lemma 19 we find the bound

$$\|(\mathrm{e}^{G^+}u^+, \mathrm{e}^{G^-}u^-)\|_{L^2(\mathcal{A}^{out},\mathbb{C}^2)} \geq const\|(v_0^+, v_0^-)^T\|_{\check{H}_{out}} ,$$

showing that the sum in (34) converges in $\check{H}_{out}$. The statement then follows from this convergence and by applying Corrolary 22 to the expression

$$\sum_{0\leq n\leq Q} R_{out}^n \left(a_n \mathrm{e}^{\mathrm{i}\varphi n}\mathrm{e}^{-G^+(z)}\mathrm{e}^{(h+G^+)(z)}, b_n \mathrm{e}^{-\mathrm{i}\varphi n}\mathrm{e}^{-G^-(z)}\mathrm{e}^{(h+G^-)(z)}\right)^T \Big|_{z\in\partial\Omega_{out}} .$$

□

The proof of the Aharonov–Casher result in the case of the bounded domain with holes and a circular outer boundary now comes out along the lines of the proof of Proposition 20.

Proof of Theorem 10. Since the zero modes need to satisfy the APS boundary condition on the inner components of the boundary, $\partial\Omega_j$, $j \le N$, they have by Proposition 20 and Proposition 12 the form

$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \sum_{n\ge 0} \begin{pmatrix} \mathrm{e}^h a_n z^n \\ \mathrm{e}^{-h} b_n \bar{z}^n \end{pmatrix}$$

on the interior of Ω^{out}. Using Lemma 23 and the boundary condition (20) we have

$$\gamma_0 \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \sum_{n\ge 0}\sum_{k\le 0} R_{out}^{n+k} \begin{pmatrix} a_n d_k^+ \psi_{n+k} \\ b_n d_k^- \psi_{-(n+k)} \end{pmatrix} = \begin{pmatrix} \sum_{\ell < \frac{\Phi}{2\pi} - \frac{1}{2}} \beta_\ell^+ \psi_\ell \\ \sum_{\ell \ge \frac{\Phi}{2\pi} + \frac{1}{2}} \beta_\ell^- \psi_\ell \end{pmatrix},$$

with some $\beta_\ell^\pm \in \mathbb{C}$, which imposes the restrictions $a_n d_k^+ = 0$ if $n + k \ge \frac{\Phi}{2\pi} - \frac{1}{2}$ and $b_n d_k^- = 0$ if $-(n+k) < \frac{\Phi}{2\pi} + \frac{1}{2}$. We recall that $d_0^\pm \ne 0$ to deduce that there are $\lfloor \frac{\Phi}{2\pi} - \frac{1}{2} \rfloor + 1 = \lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \rfloor$ spin-up and $\{ -\frac{\Phi}{2\pi} - \frac{1}{2} \} + 1 = \{ \frac{-\Phi}{2\pi} + \frac{1}{2} \}$ spin-down zero modes. The symbol $\{y\}$ denotes the biggest integer smaller or equal to $y \ge 0$. The proof is now concluded by noticing that the equality $\lfloor y + \frac{1}{2} \rfloor = -\{ -y + \frac{1}{2} \}$ holds for any $y \le \frac{1}{2}$. □

3. Aharonov–Casher on a Sphere with Holes

In this section we will prove a version of the Aharonov–Casher theorem for the magnetic Dirac operator on a sphere with holes whose boundaries are equipped with APS boundary conditions. This corresponds to our set-up from Sect. 1.3 putting $\mathscr{M} = \mathbb{S}^2$. In particular, let us consider the manifold $M = \mathbb{S}^2 \setminus \cup_{k\le N} \Omega_k$, where $\cup_{k\le N}\Omega_k$ is a union of mutually disjoint open discs on $\mathbb{S}^2$. We again consider the magnetic field (13) on M for which we additionally pose a requirement that the overall flux on the sphere sums to zero

$$\int_{\mathbb{S}^2} B_0 + B_{sing} = 0\,. \tag{35}$$

To motivate the condition (35), recall that the vector potential one-form α is globally defined on M and therefore the flux through the N-th hole is $\Phi_N = -\Psi$, where Ψ is the total flux minus Φ_N. This is so, since $\int_{\partial\Omega_N} \alpha$ can be integrated either as $-\Psi$ or as Φ_N as $\partial\Omega_N$ is boundary of both Ω_N and Ω_N^C which are both bounded regions. Here $(\cdot)^C$ denotes the complement in $\mathbb{S}^2$. We will consider a semi-total flux which we define as the bulk contribution Φ_0 plus the normalised fluxes (*cf.* Sectt1.3) through all the holes but one and we choose to omit the flux of the N-th hole

$$\widehat{\Phi} = \Phi_0 + \sum_{j\le N-1} \Phi_j'\,, \qquad \Phi_j' \in [-\pi, \pi)\,.$$

The reasoning behind this comes from Lemma 28 establishing the gauge invariance of this problem which we state later. It turns out that the problem of finding the zero modes is again gauge invariant and one can gauge away integer multiples of 2π of the flux inside each of the holes apart from exactly one. The number of zero modes then depends on the semi-total flux. Moreover, the result does not depend on which hole was left out with non-normalised flux. More precisely the following theorem holds.

Theorem 24. *Let D be the Dirac operator on M with magnetic field* (13) *in the Aharonov–Casher gauge that satisfies the condition* (35)*. Then there are*

$$\left|\left\lfloor \frac{\widehat{\Phi}}{2\pi} + \frac{1}{2} \right\rfloor\right|$$

zero modes of the operator D with the domain given by the APS boundary conditions. If $\widehat{\Phi} > 0$ then all the zero modes have spin-up. If $\widehat{\Phi} < 0$ then they have spin-down.

The definition of the Dirac operator on a sphere is covered in Appx. A which also includes a proof of the statement that it is unitarily equivalent to the Dirac operator on a disc with holes with a conformal metric.

Proof. We can rotate the sphere so that the centre of the hole Ω_N becomes the north pole N'. Then we perform a transformation P which is the stereographic projection from N' composed with a reflection (see also (41)), to obtain a bounded region $P(M) \subset \mathbb{C} \simeq \mathbb{R}^2$ whose all components of the boundary are circles. This way we get the Dirac operator D^W on the region $P(M)$ with metric

$$g^W = W^2(\,\mathrm{d}x^2 + \mathrm{d}y^2)\,, \quad \text{where} \quad W = \left(1 + \frac{x^2+y^2}{4}\right)^{-1}, \tag{36}$$

which is unitarily equivalent to the Dirac operator on M, by Cor. 35 in Appx. A. The statement is then a direct consequence of Proposition 29 proved below. □

Remark 25. 1. Notice that in particular, there are no zero modes in the case $\widehat{\Phi} \in (-\pi, \pi]$.

2. Let us point out that the number $\left|\left\lfloor \frac{\widehat{\Phi}}{2\pi} + \frac{1}{2} \right\rfloor\right|$ where $\widehat{\Phi} = \sum_{j \le N-1} \Phi'_j$ does not depend on the numbering of the holes. This is because we sum only over the normalised values of the fluxes and due to the condition that the global flux is zero, expressed by (35). Hence, if we fix an index $j_0 \le N-1$ and put

$$\Phi^I = \Phi'_{j_0} + \Phi_{rest}\,,$$

where $\Phi_{rest} = \sum_{j \le N-1, j \ne j_0} \Phi'_j$, we have by (35) the flux $-\Phi^I$ through the hole Ω_N. To normalise this value we note that for any $y \in \mathbb{R}$ it holds $y - \lfloor y + \frac{1}{2} \rfloor \in (-\frac{1}{2}, \frac{1}{2}]$. Thus,

$$\frac{\Phi'_N}{2\pi} = -\left(\frac{\Phi^I}{2\pi} - \left\lfloor \frac{\Phi^I}{2\pi} + \frac{1}{2} \right\rfloor\right) \in \left[-\frac{1}{2}, \frac{1}{2}\right),$$

is the ($\frac{1}{2\pi}$ multiple of the) normalised flux through the N-th hole. The total flux $\Phi^{II} = \Phi_{rest} + \Phi'_N$, *i.e.* omitting the contribution from j_0, then satisfies

$$\left\lfloor \frac{\Phi^{II}}{2\pi} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\Phi_{rest}}{2\pi} + \frac{1}{2} - \frac{\Phi^{I}}{2\pi} + \left\lfloor \frac{\Phi^{I}}{2\pi} + \frac{1}{2} \right\rfloor \right\rfloor$$
$$= \left\lfloor -\frac{\Phi'_{j_0}}{2\pi} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{\Phi^{I}}{2\pi} + \frac{1}{2} \right\rfloor$$
$$= \left\lfloor \frac{\Phi^{I}}{2\pi} + \frac{1}{2} \right\rfloor ,$$

where in the last equality we used that $\frac{\Phi'_{j_0}}{2\pi} \in [-\frac{1}{2}, \frac{1}{2})$.

To see that this result is a direct consequence of the bounded case we need to investigate the Dirac operator with the APS boundary condition under Möbius transform, which is a particular case of a conformal transform.

3.1. The Dirac Operator with APS Boundary Condition in the Conformal Metric g^W

Let M be a two-dimensional manifold with metric g and let E be a Spin^c spinor bundle over M with Clifford multiplication σ and *Spin*^{c} connection ∇. In [18, Sect. 4] the authors showed how σ, ∇ and the Levi-Civita connection ∇^{LC} are modified under a general conformal transformation taking the metric g to a metric $g^W = W^2 g$ for some $W : M \to \mathbb{R} \setminus \{0\}$. We summarise their results in the following proposition.

Proposition 26. *In the conformal metric $g^W = W^2 g$ we have*

$$\sigma^W(\mu) = W^{-1}\sigma(\mu) ,$$
$$\nabla^W_{\mu^\sharp} u = \nabla_{\mu^\sharp} u + \frac{1}{4} W^{-1}[\sigma(\mu), \sigma(\mathrm{d}W)]u ,$$
$$\nabla^{LC,W}_{\mu^\sharp}(\zeta) = \nabla^{LC}_{\mu^\sharp}\zeta - W^{-1}\mu^\sharp(W)\zeta + W^{-1}(\zeta, \mathrm{d}W)\mu - W^{-1}\zeta(\mu^\sharp)\,\mathrm{d}W ,$$

for any spinor u, vector field $\mu^\sharp$ and a one form ζ. We denoted by μ the one-form dual to $\mu^\sharp$ with respect to the metric g.

We point out that for any $\zeta \in T^*M$ it holds $\sigma^W(W\zeta) = \sigma(\zeta)$ and that if ζ is normalised in the metric g then $W\zeta$ is normalised in the conformal metric g^W. As a consequence of Proposition 26 we then obtain the relations of the Dirac operators and their boundary operators under a conformal transform. In what follows, we will use the earlier introduced Notation 2 on page 5, where the normalisation refers to normalisation in metric g.

Corollary 27. *Consider a two-dimensional manifold M with the metric g which is conformally transformed to a manifold M^W with metric $g^W = W^2 g$. The Dirac operators D on M and D^W on M^W and their respective adapted boundary operators are related by*

$$D^W = W^{-3/2} D W^{1/2} \text{ and}$$
$$A^W = W^{-1} A .$$

In particular we see that the APS boundary condition is not *conformally invariant.*

Proof. The proof for D^W is presented in [18, Theorem 4.3], so we show only the relation for A^W. Writing locally on the boundary $D = \sigma(\nu)\nabla_{\nu^\sharp} + \sigma(\xi)\nabla_{\xi^\sharp}$ and using $\sigma(\nu)^2 = 1$ we recall that by Definition 3 the canonical boundary operator A adapted to D in the metric g reads

$$2A = \sigma(\nu)\sigma(\xi)\nabla_{\xi^\sharp} - \sigma(\xi)\nabla_{\xi^\sharp}\sigma(\nu)\,.$$

Changing the metric from g to $g^W = W^2 g$ in this formula, Proposition 26 further gives

$$\begin{aligned}
2A^W &= \sigma(\nu)\sigma(\xi)W^{-1}\nabla^W_{\xi^\sharp} - \sigma(\xi)W^{-1}\nabla^W_{\xi^\sharp}\sigma(\nu)\\
&= W^{-1}\Big(\sigma(\nu)\sigma(\xi)\big(\nabla_{\xi^\sharp} + \frac{1}{4}W^{-1}[\sigma(\xi),\sigma(\,\mathrm{d}W)]\big)\\
&\quad - \sigma(\xi)\big(\nabla_{\xi^\sharp} + \frac{1}{4}W^{-1}[\sigma(\xi),\sigma(\,\mathrm{d}W)]\big)\sigma(\nu)\Big)\\
&= W^{-1}\left(\sigma(\nu)\sigma(\xi)\nabla_{\xi^\sharp} - \sigma(\xi)\nabla_{\xi^\sharp}\sigma(\nu)\right) + \frac{W^{-2}}{4}R = W^{-1}2A + \frac{W^{-2}}{4}R\,,
\end{aligned}$$

where

$$\begin{aligned}
R &:= \sigma(\nu)\sigma(\xi)[\sigma(\xi),\sigma(\,\mathrm{d}W)] - \sigma(\xi)[\sigma(\xi),\sigma(\,\mathrm{d}W)]\sigma(\nu)\\
&= -\sigma(\xi)\{[\sigma(\xi),\sigma(\,\mathrm{d}W)],\sigma(\nu)\}\,.
\end{aligned}$$

Since the pair (ν,ξ) forms a local orthonormal basis of the one-forms, we can write $\mathrm{d}W = (\,\mathrm{d}W,\xi)\xi + (\,\mathrm{d}W,\nu)\nu$ to obtain

$$[\sigma(\xi),\sigma(\,\mathrm{d}W)] = (\,\mathrm{d}W,\nu)\left(\{\sigma(\xi),\sigma(\nu)\} - 2\sigma(\nu)\sigma(\xi)\right) = -2(\,\mathrm{d}W,\nu)\sigma(\nu)\sigma(\xi)\,.$$

Therefore, using the anti-commutation identity $\{EF,G\} = E\{F,G\} - [E,G]F$ holding for any operators E, F, G, we infer $R = 0$, which concludes the proof of $A^W = W^{-1}A$. □

Let us restrict to the specific case of the Dirac operator on $P(M)$ with $M = \mathbb{S}^2\setminus \cup_{j\le N}\,\Omega_j$ and P the stereographic projection composed with a reflection defined in Appx. A. Note that $P(M)$ is a conformal transformation of $\Omega_{out}\setminus\cup_{j\le N-1}\Omega_j$ and the new metric g^W is given by (36). As in the case of the standard metric we can use the arguments for gauge invariance from Lemma 7 for the holes $P(\Omega_j)$, $j\le N-1$ to find the following.

Lemma 28. *Let a and $\widetilde{a}$ be two magnetic vector potentials whose fluxes differ by an integer multiple m_j of 2π on the inner hole Ω_j, for all $j\le N-1$. Then we have the unitary equivalence between the Dirac operators on $P(M)$ in the metric g^W with APS boundary condition, corresponding to the magnetic fields a and $\widetilde{a}$*

$$\mathscr{U}^* D^W_a \mathscr{U} = D^W_{\widetilde{a}}\,,$$

with the unitary

$$\mathscr{U} : L^2(\mathbb{C}, g^W; \mathbb{C}^2) \to L^2(\mathbb{C}, g^W; \mathbb{C}^2)$$
$$\mathscr{U} : u \mapsto \exp\left[\mathrm{i}\int_\gamma (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right] u\,,$$

where $\gamma \subset P(M)$ is a curve connecting a fixed point $z_0 \in P(M)$ and the point z.

We recall from the proof of Lemma 7 that replacing the starting point z_0 of γ by a different point $z_1 \in P(M)$ amounts merely to a multiplication by the constant $K = \exp\left[\mathrm{i}\int_{z_0}^{z_1} (\vec{a} - \widetilde{\vec{a}})\, \mathrm{d}\vec{s}\right]$ satisfying $\overline{K} = K^{-1}$.

Proof. Taking the commutativity of $\mathscr{U}$ and W into account, the statement follows directly from Lemma 7 and Cor. 27. □

By this lemma we can without loss of generality work with the normalised fluxes $\Phi'_j \in [-\pi, \pi)$ inside the holes $\Omega_j \subset \mathbb{S}^2$ for $j \leq N-1$ as well as assume that the magnetic field inside the holes is modelled by such a normalised flux multiple of the Dirac delta function δ_{w_j}, at the centre w_j of Ω_j. Harvesting all this preparation we are able to find the zero modes of the conformal Dirac operator on $\mathbb{C}$ and prove the following proposition whose immediate consequence is Theorem 24.

Proposition 29. *The zero modes of the Dirac operator D^W on $P(M)$ in the metric g^W and magnetic field satisfying the condition 35 in the Aharonov–Casher gauge with the APS boundary condition are of the form*

$$\begin{pmatrix} u^+ \\ 0 \end{pmatrix}, \quad u^+(z) = W^{-1/2}(z)\mathrm{e}^{h(z)} \sum_{0 \leq n < \frac{\hat{\Phi}}{2\pi} - \frac{1}{2}} a_n z^n\,,$$
$$\begin{pmatrix} 0 \\ u^- \end{pmatrix}, \quad u^-(z) = W^{-1/2}(z)\mathrm{e}^{-h(z)} \sum_{0 \leq n \leq -\frac{\hat{\Phi}}{2\pi} - \frac{1}{2}} b_n \overline{z}^n$$

for some coefficients $a_n, b_n \in \mathbb{C}$.

Proof. Consider a zero mode $u \in \ker(D^W)$. Then by Cor. 27 we know that for $v(z) = W(z)^{1/2}u$ it holds $Dv(z) = 0$ on $P(M)$ with $D = W^{3/2}D^W W^{-1/2}$ being the Dirac operator on $P(M)$ in the standard metric on $\mathbb{C}$. We choose coordinates $\tilde{z}$ on $P(\mathbb{S}^2 \setminus \{N'\})$ with origin at $P\big((0,0,-1)^T\big)$ and mark with tilde functions on $P(M)$ expressed in these coordinates. Let us fix an arbitrary index $j \leq N-1$. We write similarly $f_j(z_j)$ for a function f on $P(M)$ in the coordinates z_j obtained by the Möbius transform $Y_{t_j} : \tilde{z} \mapsto z_j$ (see Appx. B and Lemma 36) with t_j being the antipodal point of the centre w_j of the hole $\Omega_j \subset \mathbb{S}^2$. An important observation is that $W_j(z_j)$ is a positive constant on $(Y_{t_j} \circ P)(\partial\Omega_j)$ and therefore u satisfies the APS boundary condition on $(Y_{t_j} \circ P)(\partial\Omega_j)$ for D^W if and only if v satisfies the boundary condition (18) on $(Y_{t_j} \circ P)(\partial\Omega_j)$. By Proposition 12 the spin-up component v^+ takes the form

$$v_j^+(z) = \mathrm{e}^{h_j(z)} g_j^+(z)\,, \quad j \leq N-1\,, \tag{37}$$

where $g_j^+(z)$ is analytic on $(Y_{t_j} \circ P)(M)$ and can be analytically extended to the hole $(Y_{t_j} \circ P)(\Omega_j)$ by Proposition 20. In Appx. B we argue that under the change of coordinates given by the Möbius transform

$$Y_{t_j} : \tilde{z} \mapsto z_j = \frac{a\tilde{z} + b}{c\tilde{z} + d},$$

for some a, b, c, d complex numbers dependent on t_j (the number a here should not be confused with the vector potential a), the spinor u needs to satisfy the relation (51), and therefore,

$$W_j^{-1/2}(Y_{t_j}(\tilde{z}))v_j^+(Y_{t_j}(\tilde{z})) = \tilde{W}^{-1/2}(\tilde{z})\tilde{\mathscr{G}}(\tilde{z})\tilde{v}^+(\tilde{z}), \quad \mathscr{G}(z) = \frac{cz + d}{|cz + d|},$$

for all $j \leq N - 1$. Employing (49) and (37) this now leads to analyticity of $\tilde{g}^+(\tilde{z})$ on $P(\Omega_j)$ as

$$g_j^+(Y_{t_j}(\tilde{z})) = \mathrm{e}^{\tilde{h}(\tilde{z}) - h_j(Y_{t_j}(\tilde{z}))}|c\tilde{z} + d|\frac{c\tilde{z} + d}{|c\tilde{z} + d|}\tilde{g}^+(\tilde{z}) = (c\tilde{z} + d)\tilde{g}^+(\tilde{z}),$$

where we used that the functions $h_j(Y_{t_j}(\tilde{z}))$ and $\tilde{h}(\tilde{z})$ are in fact the same function h expressed in different sets of coordinates. Hence, using that $(c\tilde{z} + d)^{-1}$ is analytic on $P(\Omega_j)$[5] and that $j \leq N - 1$ was arbitrary we conclude that g^+ is analytic on $P(\mathbb{S}^2 \setminus \Omega_N)$. Similarly as above thanks to $W(\tilde{z}) = const > 0$ on $P(\partial\Omega_N)$ the boundary conditions on $P(\partial\Omega_N)$ for $u \in \mathrm{dom}(D^W)$ and $v \in \mathrm{dom}(D)$ coincide (see Cor. 27). Therefore, we may apply the same steps as in the proof of Theorem 10 and obtain

$$u^+(z) = W^{-1/2}(z)\mathrm{e}^{h(z)} \sum_{n < \frac{\hat{\Phi}}{2\pi} - \frac{1}{2}} a_n z^n$$

on $P(M)$. The form of the modes u^- on $P(M)$ is shown by an adaptation of the previous to be

$$u^-(z) = W^{-1/2}(z)\mathrm{e}^{-h(z)} \sum_{n \leq -\frac{\hat{\Phi}}{2\pi} - \frac{1}{2}} b_n \bar{z}^n.$$

Here both a_n and b_n are some complex coefficients. □

4. Relation to the Index Theorem

Here we assume for a moment that the dimension n of the base manifold M is even (not necessarily two). In that case a Spin^c spinor bundle E with Clifford multiplication σ and a *Spin*^{c} connection can be defined, provided that a certain topological condition[6] is imposed on M. We refer to *e.g.* [28, Appx. D] or [33, Sec. 10.8]) for the precise definitions of the above terms in dimension $n > 2$, for $n = 2$ recall Sec. 1.1. Due to the Clifford relations the chirality operator then anti-commutes with $\sigma(\zeta)$ for all $\zeta \in T^*M \subset Cl(\mathbb{R}^n)$ with $Cl(\mathbb{R}^n)$ denoting the

[5] Note that the point $\tilde{z} = -d/c \notin P(\Omega_j)$ is in fact the image of the antipodal point t_j of w_j under the mapping P.

[6] For further details on this condition see [28, Theorem D.2].

Clifford algebra on $\mathbb{R}^n$ and induces thus a $\mathbb{Z}_2$ grading of the bundle E. This means that we can write $E = E_+ \oplus E_-$ where $E_\pm$ are the ± 1 eigenspaces or the chirality operator. If D is the Dirac operator on E, it can be then written in the following form

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

where $D_\pm : \Gamma(E_\pm) \to \Gamma(E_\mp)$ are mutual formal adjoints. We remark that the Dirac operator is defined by the same formula as in two dimensions with the difference that now the index j in Definition 1 runs up to n. We wish to introduce the quantity index, which is well defined for Fredholm operators. Therefore, in the following we assume that D is Fredholm. That is, for example, true when M is a compact manifold and D satisfies the APS boundary condition on ∂M, *cf.* [8, Definition 5.1, Ex. 5.2, Theorem 5.3][7].

Definition 30. We define the analytical index (or index) of the Dirac operator D by

$$\mathrm{ind}(D) = \dim\ker(D_+) - \dim\ker(D_-)\,. \tag{38}$$

Atiyah and Singer showed in [6] that if the manifold M is compact and has no boundary, then the analytical index is equal to the topological index

$$\mathrm{ind}(D) = \int_M AS\,. \tag{39}$$

The integrand AS depends both on the Riemannian curvature R_M of M and the magnetic two-form[8] β on the bundle E. For flat manifolds, *i.e.* $R_M = 0$, it corresponds to the Chern character of the bundle $AS = \mathrm{Ch}(E)_{[n]} = \left(\exp\frac{\beta}{2\pi}\right)_{[n]}$, where the subscript $[n]$ refers to the n-th degree part of the form. The expression exp is to be understood as the series expansions.

The index theorem was extended to manifolds with boundary in [3], where Atiyah, Patodi and Singer proved the formula for the index assuming, that M has a product structure near the boundary. Neglecting this assumption one obtains an additional boundary term that vanishes in the case of a product structure. The extended formula was proven by Grubb in [24, Cor. 5.3]. More explicit expression of the boundary term was given by Gilkey in [23]. From Gilkey's formula it follows that in case of the APS boundary condition given by the canonical boundary operator (*cf.* Definition 3) the above-mentioned additional boundary term vanishes. In particular in our two-dimensional case we obtain by Theorem 8.4.d and Theorem 1.4 in [23]

$$\mathrm{ind}(D) = \int_M AS - \frac{1}{2}(\eta([A]_{11}) + \dim\,\ker[A]_{11})\,, \tag{40}$$

---

[7] [8] shows that it is true even for all D-elliptic boundary conditions, see [8, Definition 4.7] for the definition.

[8] In a general dimension the magnetic two-form is the trace $2^{-n/2}\mathrm{Tr}(\mathrm{i}R)$ of the $\mathrm{End}(E)$-valued curvature $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ of the *Spin*^{c} connection ∇. Here X, Y are arbitrary vector fields on E.

and $[A]_{11}$ is its top left component of the canonical boundary operator.

We will consider the Dirac operator D_a with magnetic field (13). Recall that A was computed in Sec. 1.3. The first term in the integral is the bulk contribution as in (39). Since in our case M is flat, and, since we are in two dimensions, we have $\int_M AS = \int_M \frac{\beta}{2\pi} = \frac{\Phi_0}{2\pi}$. The η-invariant $\eta(A)$ is defined as the analytic extension of the function

$$\eta_s(A) = \sum_{\lambda \in spec(A)\setminus\{0\}} |\lambda|^{-s} sgn(\lambda)\,,$$

at the value $s = 0$ and is well defined for Dirac operators as was shown in [3]. The sum runs over the non-zero eigenvalues of the boundary operator A. For the simple case $T = -\mathrm{i}\partial_t - c$, $c \in \mathbb{R}$, on the first Sobolev space $H^1([0, 2\pi])$ with periodic boundary condition, it is shown in a greater detail for example in [21, Appx. D] that the analytic continuation yields

$$\eta(-\mathrm{i}\partial_t - c) = \begin{cases} -1 + 2\langle c\rangle & \text{if } c \in \mathbb{R}\setminus\mathbb{Z} \\ 0 & \text{if } c \in \mathbb{Z} \end{cases}\,,$$

where $\langle c\rangle$ is the unique number $\widetilde{c} \in (0, 1)$ such that $c - \widetilde{c} \in \mathbb{Z}$. Note that the eta-invariant $\eta(T)$ depends only on the eigenvalues of T and hence the formula for $\eta(T)$ yields directly a result for the η-invariant of any operator whose spectrum is of the form $\{n + c\}_{n\in\mathbb{Z}}$. Solving the eigenvalue problem for the top left component of the boundary operator $[A_{11}]$ restricted to the inner and outer components of the boundary gives the spectra

$$spec\big([A|_{\partial\Omega_j}]_{11}\big) = \left\{-R_j^{-1}\left(n - \frac{\Phi_j'}{2\pi} + \frac{1}{2}\right) \mid n \in \mathbb{Z}\right\}$$
$$spec\big([A|_{\partial\Omega_{out}}]_{11}\big) = \left\{R_{out}^{-1}\left(n - \frac{\Phi}{2\pi} + \frac{1}{2}\right) \mid n \in \mathbb{Z}\right\}.$$

Employing then the property $\eta(cL) = \mathrm{sgn}(c)\eta(L)$ for a constant c and an elliptic operator L we deduce form (17) and (19)

$$\eta\big([A|_{\partial\Omega_j}]_{11}\big) = 1 - 2\left\langle \frac{\Phi_j'}{2\pi} - \frac{1}{2}\right\rangle \quad \text{and} \quad \eta\big([A|_{\partial\Omega_{out}}]_{11}\big) = -1 + 2\left\langle \frac{\Phi}{2\pi} - \frac{1}{2}\right\rangle$$

for all $j \le N$. Let us denote by I_1 the set of indices j such that $1 = \dim\ker([A|_{\partial\Omega_j}]_{11}) \in \{0, 1\}$, by $|I_1|$ the number of elements in I_1 and let $I_0 = \dim([A|_{\partial\Omega_{out}}]_{11}) \in \{0, 1\}$. We make the following observations

1. By definition of the normalised fluxes (recall Sec. 1.3) if $j \notin I_1$ we have $\left\langle \frac{\Phi_j'}{2\pi} - \frac{1}{2}\right\rangle = \frac{\Phi_j'}{2\pi} + \frac{1}{2}$.
2. For $j \in I_1$ it holds $\frac{\Phi_j'}{2\pi} - \frac{1}{2} = -1$ and thus, $\sum_{j\in I_1}\left(\frac{\Phi_j'}{2\pi}\right) = -\frac{|I_1|}{2}$.
3. $\eta([A|_{\partial\Omega_{out}}]_{11}) + I_0 = \begin{cases} 1 & \text{if } I_0 = 1 \\ -1 + 2\left\langle \frac{\Phi}{2\pi} - \frac{1}{2}\right\rangle & \text{if } I_0 = 0 \end{cases}.$

Omitting the outer boundary contribution in the index formula (40) for now, we arrive at the expression

$$\int_M AS - \frac{1}{2}\sum_{j\le N}(\eta([A|_{\partial\Omega_j}]_{11}) + \dim\ker[A|_{\partial\Omega_j}]_{11}) = \frac{\Phi_0}{2\pi} - \frac{1}{2}\sum_{j\notin I_1} -2\left(\frac{\Phi_j'}{2\pi}\right) - \frac{|I_1|}{2} = \frac{\Phi_0}{2\pi} + \sum_{j\le N}\left(\frac{\Phi_j'}{2\pi}\right) = \frac{\Phi}{2\pi}.$$

Finally, for the index of the Dirac operator D_a with magnetic field (13) in the gauge (9), (10) and with domain (7) we have

$$\operatorname{ind}(D_a) = \frac{\Phi}{2\pi} - \begin{Bmatrix} \frac{1}{2} & \text{if } I_0 = 1 \\ -\frac{1}{2} + \left\langle \frac{\Phi}{2\pi} - \frac{1}{2}\right\rangle & \text{if } I_0 = 0 \end{Bmatrix} = \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor,$$

where in the last equality we used that $\frac{\Phi}{2\pi} + \frac{1}{2} \in \mathbb{Z}$ if $I_0 = 1$. Note that this formula is in agreement with our result Theorem 10, by which we immediately infer:

Corollary 31. *Under the assumptions of Theorem 10 we obtain the index for D_a (defined by (38)),*

$$\operatorname{ind}(D_a) = \left\lfloor \frac{\Phi}{2\pi} + \frac{1}{2} \right\rfloor.$$

5. Conclusion

We showed a version of the Aharonov–Casher theorem on some two-dimensional manifolds with boundary. In particular our manifolds are a plane with holes (Theorem 8), a disc with holes (Theorem 10) and a sphere with holes (Theorem 24). We consider the APS boundary condition on the boundary and show that the number of zero modes depends only on the sum of the flux corresponding to the smooth magnetic field on the manifold and the rational part of the fluxes through the holes. In particular our results imply the index theorem for these special choices of the manifolds. Moreover, since the index is a topological invariant, the index theorem is implied also for arbitrarily shaped holes. To prove the Aharonov–Casher theorem, *i.e.* that all zero modes have a definite chirality, is, for such domains still an open problem.

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Appendix A: The Dirac Operator Under the Stereographic Projection

For conciseness we will write in the following $\widetilde{M} = \mathbb{S}^2 \backslash \{N'\}$ for the sphere without the north pole $N' = (0, 0, 1)^T \in \mathbb{S}^2$ and $M = \mathbb{S}^2 \backslash \cup_{j \leq N} \Omega_j$ for the sphere without the holes $\Omega_j \subset \mathbb{S}^2$, $j \leq N$. It is convenient to map the Dirac operator on the sphere to the plane by the stereographic projection. Here we will argue that due to this mapping we can perform the analysis for finding the zero modes of the Dirac operator on $\mathbb{S}^2$ by investigating the problem on $\mathbb{C}$ with a metric that is conformal to the standard metric on $\mathbb{C}$. We will denote by $P : \widetilde{M} \to \mathbb{C}$ the stereographic projection from the north pole composed with reflection across the x axis. In particular a point

$$\omega = \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}, \quad \vartheta \in (0, \pi], \varphi \in (0, 2\pi],$$

is mapped by P to the point $P(\omega) = 2\cot\frac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \in \mathbb{C}$, *i.e.*

$$x := (P(\omega))_x = 2\cot(\vartheta/2)\cos\varphi\,, \quad y := (P(\omega))_y = -2\cot(\vartheta/2)\sin\varphi\,. \tag{41}$$

Lemma 32. *The tangent map $P_* : (T\widetilde{M}, g^{\mathbb{S}^2}) \to (T\mathbb{R}^2, g^W)$, where $g^{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$ and*

$$g^W = W^2(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \quad W = \left(1 + \frac{x^2+y^2}{4}\right)^{-1}\,, \tag{42}$$

is an isometry.

Proof. Using the definition of the push-forward map $(\cdot)_*$ the statement follows by a direct computation from (41). □

We further obtain a unitary between the square integrable functions over $\mathbb{C}$ with the metric g^W and the square integrable functions over the pre-image $P^{-1}(\mathbb{C})$ with the standard metric on $\mathbb{S}^2$.

Lemma 33. *The pullback of the stereographic projection composed with reflection across the x axis $P^* : L^2(\mathbb{C}, g^W; \mathbb{C}^2) \to L^2(P^{-1}(\mathbb{C}), g^{\mathbb{S}^2}; \mathbb{C}^2)$ acting as $(P^*u)(\omega) := u(P(\omega))$ is a unitary operator.*

Proof. Finding the differentials $\mathrm{d}x$ and $\mathrm{d}y$ from (41) one easily verifies that the volume form changes as

$$\mathrm{d}\vartheta \wedge \sin\vartheta\,\mathrm{d}\varphi = \sin^4(\vartheta/2)\,\mathrm{d}x \wedge \mathrm{d}y = \left(1 + \frac{x^2+y^2}{4}\right)^{-2} \mathrm{d}x \wedge \mathrm{d}y\,.$$

With the notation $(\cdot,\cdot)_{\mathbb{S}^2}$ for the inner product on $L^2(\mathbb{S}^2, g^{\mathbb{S}^2}; \mathbb{C}^2)$ and $(\cdot,\cdot)_W$ for the inner product on $L^2(\mathbb{C}, g^W; \mathbb{C}^2)$ we then obtain

$$\begin{aligned}(P^*f_1, P^*f_2)_{\mathbb{S}^2} &= \int_0^\pi \int_0^{2\pi} f_1 \circ P(\vartheta,\varphi)\,\overline{f_2 \circ P(\vartheta,\varphi)}\,\mathrm{d}\vartheta \wedge \sin\vartheta\,\mathrm{d}\varphi \\ &= \int_{\mathbb{R}^2} f_1(x,y)\overline{f_2(x,y)}\left(1 + \frac{x^2+y^2}{4}\right)^{-2} \mathrm{d}x \wedge \mathrm{d}y = (f_1, f_2)_W\,,\end{aligned}$$

for any $f_{1,2} \in L^2(\mathbb{C}, g^W; \mathbb{C}^2)$ and for W given by (42). □

Definition 34. We define the Spin^c spinor bundle over $\widetilde{M}$ as the pullback of the Spin^c spinor bundle $\mathcal{S}$ over $\mathbb{C} \sim \mathbb{R}^2$ by the stereographic projection composed with reflection P

$$P^*\mathcal{S} = \{(\omega, u) \in \mathbb{S}^2 \times \mathcal{S} \mid \pi(u) = P(\omega)\}\,,$$

where π is the bundle projection of $\mathcal{S}$. We have as in Lemma 33 the map $P^* : \Gamma(\mathbb{R}^2, \mathcal{S}) \to \Gamma(\widetilde{M}, P^*\mathcal{S})$ given by $(P^*u)(\omega) = (u \circ P)(\omega)$. The corresponding Clifford multiplication and the Clifford connection on such bundle are given by

$$\begin{aligned}\sigma^{\widetilde{M}}(P^*\zeta)P^* &:= P^*\sigma^W(\zeta), \quad \zeta \in T^*\mathbb{R}^2 \\ \nabla^{\widetilde{M}}_X P^* &:= P^*\nabla^W_{P_*X}, \quad X \in T\widetilde{M},\end{aligned} \tag{43}$$

where σ^W and ∇^W refer to the Clifford multiplication and Clifford connection on $\mathcal{S}$.

Now we are ready to state a corollary which will reduce our analysis of the Dirac operator on the sphere with holes to the investigation of the corresponding Dirac operator on a disc with holes in a metric conformal to the standard metric on $\mathbb{R}^2 \simeq \mathbb{C}$.

Corollary 35. *The Dirac operator D^M on M is unitarily equivalent to the Dirac operator D^W on $P(M) \subset (\mathbb{C}, g^W)$,*

$$D^M P^* = P^* D^W .$$

Proof. We denote by s^j, $j = 1, 2$ an orthonormal (in $g^{\mathbb{S}^2}$) basis on T^*M, by s_j the dual basis and by e^j its counterpart on $T^*\mathbb{R}^2$ such that $P^*e^j = s^j$. Note, that by Lemma 32 the last relation defines e^j that form an orthonormal frame on $T^*\mathbb{R}^2$ in the metric g^W. Using the definitions (43) we obtain for any section u on $\mathbb{R}^2$

$$D^M P^* u = \sum_{j\le 2} \sigma^M(s^j)\nabla^M_{s_j} P^* u = \sum_{j\le 2} P^*(\sigma^W(e^j)\nabla^W_{e_j} u) = P^*(D^W u) .$$

For the canonical boundary operators A^M on ∂M and A^W on $P(\partial M)$ adapted to D^M and D^W respectively it holds again by (43)

$$\begin{aligned} 2A^M P^* &= \sigma^M(P^*\nu)\sigma^M(P^*\xi)\nabla^M_X P^* - \sigma^M(P^*\xi)\nabla^M_X \sigma^M(P^*\nu)P^* \\ &= P^*(\sigma^W(\nu)\sigma^W(\xi)\nabla^W_{P_*X}) - P^*(\sigma^W(\xi)\nabla^W_{P_*X}\sigma^W(\nu)) = 2P^*A^W , \end{aligned}$$

where ν and ξ are the normal and tangent co-vector fields on the boundary $P(\partial M)$ and X is the dual vector field to $P^*\xi$. We see that λ is an eigenvalue of A^W with eigenfunction v if and only if it is an eigenvalue of A^M with an eigenfunction P^*v. Hence, $\mathrm{dom}(D^M) = P^*\mathrm{dom}(D^W)$. □

Appendix B: Remarks on Möbius Transform

Möbius transform is a mapping $Y : \mathbb{C} \to \mathbb{C}$ of the form $Y(z) = \frac{az+b}{cz+d}$ such that $ad - bc = 1$. Notice that it is an analytic mapping on $\mathbb{C} \setminus \{-\frac{d}{c}\}$ whose z derivative reads

$$\partial_z Y(z) = \frac{ad - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2} . \tag{44}$$

Such transforms can be obtained by the composition of the inverse stereographic projection from the plane to a sphere, a rotation on the sphere and stereographic projecting back to the plane.

Lemma 36. *The Möbius transform $Y_\omega = PRP^{-1}$, where P is the stereographic projection from the north pole N' followed by the reflection across the x axis* *(see (41)) and R is the rotation on $\mathbb{S}^2$ along $\varphi = const$ (*i.e. *along a certain meridian) which maps a point $\omega \in \mathbb{S}^2 \backslash \{N'\}$ to the north pole N'*

$$\omega = \begin{pmatrix} \cos\varphi_0 \sin\vartheta_0 \\ \sin\varphi_0 \sin\vartheta_0 \\ \cos\vartheta_0 \end{pmatrix} \mapsto R(\omega) = N' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vartheta_0 \in (0,\pi]\,, \varphi_0 \in (0, 2\pi]\,,$$

has the form $Y_\omega(z) = \frac{az+b}{cz+d}$ with the matrix of coefficients

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\frac{\vartheta_0}{2} & 2\mathrm{e}^{-\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2} \\ -\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2} & \cos\frac{\vartheta_0}{2} \end{pmatrix}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1\,.$$

Moreover, for the composition $Y_{\omega_1} \circ Y_{\omega_2}$ for any $\omega_{1,2} \in \mathbb{S}^2 \backslash \{N'\}$, the coefficients satisfy the following relations

$$a = \overline{d}\,, \quad b = -4\overline{c}\,, \quad |a|^2 + 4|c|^2 = |d|^2 + \frac{1}{4}|b|^2 = 1\,. \tag{45}$$

Proof. One can easily check that $\pm 2\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi_0}$ are the two fixed points of Y_ω, which with the additional conditions

$$ad - bc = 1 \quad \text{and} \quad Y_\omega : P(\omega) = 2\mathrm{e}^{-\mathrm{i}\varphi_0}\cot\frac{\vartheta_0}{2} \mapsto \infty\,,$$

leads to the result

$$Y_\omega(z) = \frac{\cos\frac{\vartheta_0}{2} z + 2\mathrm{e}^{-\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2}}{-\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_0}\sin\frac{\vartheta_0}{2} z + \cos\frac{\vartheta_0}{2}}\,. \tag{46}$$

In particular, let us point out that the relations between the coefficients of the Möbius transform (46) satisfy (45). For a composition of two such Möbius transforms $Y_{\omega_1} \circ Y_{\omega_2}$ for

$$\omega_j = \begin{pmatrix} \cos\varphi_j \sin\vartheta_j \\ \sin\varphi_j \sin\vartheta_j \\ \cos\vartheta_j \end{pmatrix}, j = 1, 2\,,$$

we compute using (46)

$$\begin{aligned} Y_{\omega_1} \circ Y_{\omega_2}(z) = \frac{az+b}{cz+d}\,, \quad &\text{with} \\ a &= \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - \mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)}\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} \\ d &= \cos\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} - \mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}\sin\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2} = \overline{a} \\ b &= 2\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\mathrm{e}^{-\mathrm{i}\varphi_2} + 2\mathrm{e}^{-\mathrm{i}\varphi_1}\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} \\ c &= -\frac{1}{2}\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_2}{2}\mathrm{e}^{\mathrm{i}\varphi_2} - \frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_1}\sin\frac{\vartheta_1}{2}\cos\frac{\vartheta_2}{2} = -\frac{\overline{b}}{4}\,. \end{aligned}$$

□

In what follows the particular choice of the point ω is not important so we will generally use the notation Y instead of Y_ω. Notice that Lemma 32 implies that the tangent mapping Y_*, "pushforwarding" vectors at a point z

to vectors at $Y(z)$, is an isometry on the tangent space of $(\mathbb{C}, g^W)$ with the conformal metric $g^W = W^2 g = (1+|z|^2/4)^{-1} g$ where g is the standard metric on $\mathbb{C}$.

In the last part of this section we will find the relation between the spinor u expressed in a set of coordinates on $\mathbb{C}$ and in the coordinates which are their Möbius transform. Let u be a section of the trivial $Spin^c$ spinor bundle over $\mathbb{C}$ and denote by $u_j(z_j)$ this section in coordinates z_j, $j \in \{1,2\}$. Then we have the relation

$$u_1(z_1) = \mathcal{G}(z_2) u_2(z_2) \tag{47}$$

for some $\mathcal{G} \in GL(2)$.

Remark 37. In fact the structure group of a $Spin^c$-spinor bundle over M is the group $Spin^c(2) := Spin(2) \times U(1)/\{\pm(1,1)\}$, where $/\{\pm(1,1)\}$ refers to the identification of classes $[(1,1)]$ and $[(-1,-1)]$, and, $Spin(2) \simeq SO(2)$ is the spin group of $\mathbb{R}^2$, so more precisely $\mathcal{G} \in Spin^c(2) \subset GL(2)$. More details on Spin and $Spin^c$ groups can be found *e.g.* in [28,33].

Assume further that the coordinates are related by the Möbius transform $Y : z_2 \mapsto z_1 = \frac{az_2+b}{cz_2+d}$. Since we know how the one-forms on $\mathbb{C}$ transform under a change of coordinates, we can find $\mathcal{G}$ by applying relation (47) on a spinor $\sigma^W(\mathcal{T})u$. Here $\sigma^W(\mathcal{T})$ is the Clifford multiplication in metric g^W (see Proposition 26) by a real one form

$$\mathcal{T} = \frac{1}{2}(\overline{\tau}\,\widehat{\mathrm{d}z} + \tau\,\widehat{\mathrm{d}\overline{z}}),$$

where $\widehat{\mathrm{d}z} = W(z)\,\mathrm{d}z$ and similarly $\widehat{\mathrm{d}\overline{z}} = W(z)\,\mathrm{d}\overline{z}$ denote the orthonormal basis of one forms on $\mathbb{C}$ in metric g^W. We denote by $\mathcal{T}_j = \mathrm{Re}\,(\overline{\tau_j}\,\widehat{\mathrm{d}z_j})$ the one form $\mathcal{T}$ in the bases $(\widehat{\mathrm{d}z_j}, \widehat{\mathrm{d}\overline{z}_j})$, $j \in \{1,2\}$ and note that

$$\tau_1(Y(z)) = \frac{W(z)}{W(Y(z))\overline{\partial_z Y(z)}}\tau_2(z) = \frac{|cz+d|^2}{(cz+d)^2}\tau_2(z)\,. \tag{48}$$

The second equality is a result of (44) and the relations (45) for the coefficients of a Möbius transform as

$$\frac{W(z)}{W(Y(z))} = \frac{4|cz+d|^2 + |az+b|^2}{4+|z|^2}|cz+d|^{-2} = |cz+d|^{-2}\,, \quad \text{since}$$
$$|az+b|^2 = |az|^2 + |b|^2 + 2\,\mathrm{Re}\,(az\overline{b}) = 4 + |z|^2 - 4|cz+d|^2\,. \tag{49}$$

By (47) (taking $\sigma^W(\mathcal{T})u$ instead of u) we now obtain

$$\sigma^W(\mathcal{T}_1)u_1(Y(z)) = \mathcal{G}(z)\sigma^W(\mathcal{T}_2)u_2(z) = \mathcal{G}(z)\sigma^W(\mathcal{T}_2)\mathcal{G}^{-1}(z)u_1(Y(z))\,.$$

Therefore, we require

$$\mathcal{G}(z)^{-1}\sigma^W(\mathcal{T}_1)\mathcal{G}(z) = \sigma^W(\mathcal{T}_2)\,. \tag{50}$$

Proposition 26 implies $\sigma^W(\widehat{\mathrm{d}z}) = \sigma(\mathrm{d}z)\,, \sigma^W(\widehat{\mathrm{d}\overline{z}}) = \sigma(\mathrm{d}\overline{z})$ and hence by (2)

$$\sigma^W(\mathcal{T}) = \begin{pmatrix} 0 & \overline{\tau} \\ \tau & 0 \end{pmatrix}.$$

We can check that setting

$$\mathcal{G}(z) = |cz+d|^{-1}\begin{pmatrix}(cz+d) & 0\\ 0 & \overline{(cz+d)}\end{pmatrix} \in SO(2)\,,$$

it indeed solves (50), as

$$\mathcal{G}(z)^{-1}\sigma^{W}(\mathcal{T}_1)\mathcal{G}(z) = \begin{pmatrix}0 & \frac{(\overline{cz+d})^2}{|cz+d|^2}\overline{\tau_1}\\ \frac{cz+d^2}{|cz+d|^2}\tau_1 & 0\end{pmatrix}$$

corresponds to the correct transformation (48) of the components of the one form $\mathcal{T}$ establishing the equality between the right-hand side and $\sigma^{W}(\mathcal{T}_2)$.

For a reference we write the transformation relation for spinors on $\mathbb{C}$ under the Möbius transform once more with the particular form of $\mathcal{G}(z)$

$$u_1(z_1) = |cz_2+d|^{-1}\begin{pmatrix}(cz_2+d) & 0\\ 0 & \overline{(cz_2+d)}\end{pmatrix} u_2(z_2)\,. \tag{51}$$

References

[1] Adami, R., Teta, A.: On the Aharonov–Bohm Hamiltonian. Lett. Math. Phys. **43**, 43–54 (1998)

[2] Aharonov, Y., Casher, A.: The ground state of a spin 1/2 charged particle in a two-dimensional magnetic field. Phys. Rev. A **19**, 2461–2462 (1979)

[3] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. I. In: Math. Proc. Camb. Philos. Soc., vol. 77, pp. 43–69. Cambridge University Press (1975)

[4] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. II. In: Mathematical Proceedings of the Cambridge Philosophical Society, vol. 78, p. 405 (1975)

[5] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. III. In: Mathematical Proceedings of the Cambridge Philosophical Society, vol. 79, pp. 71–99. Cambridge University Press (1976)

[6] Atiyah, M.F., Singer, I.M.: The index of elliptic operators on compact manifolds. Bull. Am. Math. Soc. **69**(3), 422–433 (1963)

[7] Bär, C., Ballmann, W.: Boundary Value Problems for Elliptic Differential Operators of First Order. vol. 17 of Surv. Differ. Geom., pp. 1–78. Int. Press, Boston, MA (2012)

[8] Bär, C., Ballmann, W.: Guide to elliptic boundary value problems for Dirac-type operators. In: Arbeitstagung Bonn 2013, vol. 319 of Progress Mathematics, pp. 43–80. Birkhäuser/Springer, Cham (2016)

[9] Bony, J.F., Espinoza, N., Raikov, G.: Spectral properties of 2D Pauli operators with almost-periodic electromagnetic fields. Publ. Res. Inst. Math. Sci. **55**(3), 453–487 (2019)

[10] Booß Bavnbek, B., Wojciechowski, K.P.: Elliptic boundary problems for Dirac operators. Math.: Theory Appl. Birkhäuser Boston, Inc., Boston, MA (1993)

[11] Borrelli, W., Correggi, M., Fermi, D.: Pauli Hamiltonians with an Aharonov–Bohm flux. J. Spectr. Theory, online first (2024)

[12] Chen, X.: Bundles of Irreducible Clifford Modules and the Existence of Spin Structures. Ph.D. thesis, Stony Brook University, New York (2017)

[13] Correggi, M., Fermi, D.: Schrödinger Operators with Multiple Aharonov–Bohm Fluxes. Ann. Henri Poincaré (2024)

[14] Dabrowski, L., Šťovíček, P.: Aharonov–Bohm effect with delta-type interaction. J. Math. Phys. **39**(1), 47–62 (1998)

[15] Dereziński, J., Faupin, J.: Perturbed Bessel operators. Boundary conditions and closed realizations. J. Funct. Anal., **284**, 109728 (2023)

[16] Dereziński, J., Georgescu, V.: On the domains of Bessel operators. Ann. H. Poincaré **22**, 3291–3309

[17] Elton, D.M.: Approximate zero modes for the Pauli operator on a region. J. Spectr. Theory **6**(2), 373–413 (2016)

[18] Erdős, L., Solovej, J.P.: The kernel of Dirac operators on S3 and R3. Rev. Math. Phys. **13**, 1247–1280 (2001)

[19] Erdős, L., Vougalter, V.: Pauli operator and Aharonov–Casher theorem for measure valued magnetic fields. Commun. Math. Phys. **225**(2), 399–421 (2002)

[20] Evans, L.C.: Partial Differential Equations, vol. 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition (2010)

[21] Fialová, M.: Aharonov–Casher theorem for manifolds with boundary. Ph.D. thesis, University of Copenhagen (2022)

[22] Geyler, V.A., Šťovíček, P.: Zero modes in a system of Aharonov–Bohm fluxes. Rev. Math. Phys. **16**(7), 851–907 (2004)

[23] Gilkey, P.B.: On the index of geometrical operators for Riemannian manifolds with boundary. Adv. Math. **102**(2), 129–183 (1993)

[24] Grubb, G.: Heat operator trace expansions and index for general Atiyah–Patodi–Singer boundary problems. Commun. Partial. Differ. Equ. **17**(11–12), 2031–2077 (1992)

[25] Hirokawa, M., Ogurisu, O.: Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field. J. Math. Phys. **42**(8), 3334–3343 (2001)

[26] Kato, T.: Perturbation Theory for Linear Operators. Springer, Grundlehren der mathematischen Wissenschaften, 1995. Reprint of the 1980 edition

[27] Kirsch, W., Cycon, H.L., Froese, R.G., Simon, B.: Schrödinger Operators. Springer, Berlin (1987). Corrected and extended 2nd printing

[28] Lawson, H.B., Jr., Michelsohn, M.-L.: Spin Geometry. Princeton Mathematical Series, vol. 38. Princeton University Press, Princeton, NJ (1989)

[29] Pankrashkin, K., Richard, S.: Spectral and scattering theory for the Aharonov–Bohm operators. Rev. Math. Phys. **23**(01), 53–81 (2011)

[30] Persson, M.: On the Dirac and Pauli operators with several Aharonov–Bohm solenoids. Lett. Math. Phys. **78**(2), 139–156 (2006)

[31] Persson, M.: Zero modes for the magnetic Pauli operator in even-dimensional Euclidean space. Lett. Math. Phys. **85**(2–3), 111–128 (2008)

[32] Rozenblum, G., Shirokov, N.: Infiniteness of zero modes for the Pauli operator with singular magnetic field. J. Funct. Anal. **233**(1), 135–172 (2006)

[33] Taylor, M.E.: Partial Differential Equations II. Qualitative Studies of Linear Equations, vol. 116 of Applied Mathematical Sciences, second edition. Springer, New York (2011)

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