Journal of Algebra 663 (2025) 81–118

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Research Paper

On modulo ℓ cohomology of p-adic Deligne–Lusztig varieties for GL_n

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A R T I C L E   I N F O

Article history:
Received 30 April 2024
Available online 17 September 2024
Communicated by Gunter Malle

Keywords:
p-adic Deligne-Lusztig spaces
Modular representation theory
Local Langlands correspondence

A B S T R A C T

In 1976, Deligne and Lusztig realized the representation theory of finite groups of Lie type inside étale cohomology of certain algebraic varieties. Recently, a p-adic version of this theory started to emerge: there are p-adic Deligne–Lusztig spaces, whose cohomology encodes representation theoretic information for p-adic groups – for instance, it partially realizes the local Langlands correspondence with characteristic zero coefficients. However, the parallel case of coefficients of positive characteristic $\ell \neq p$ has not been inspected so far. The purpose of this article is to initiate such an inspection. In particular, we relate cohomology of certain p-adic Deligne–Lusztig spaces to Vignéras's modular local Langlands correspondence for $\mathbf{GL}_n$.

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https://doi.org/10.1016/j.jalgebra.2024.08.033

0. Introduction

0.1. Context and history

The realization of representations of finite groups of Lie type – groups of points of reductive groups $\mathbf{G}$ over $\mathbb{F}_q$ – in the cohomology of the so-called Deligne–Lusztig varieties dates back to [10]. At that time, the desire was to describe representations of the finite group $G = \mathbf{G}(\mathbb{F}_q)$ over an algebraically closed field of characteristic zero.[1] After certain choices, one explicitly produces a variety $\dot{X}$ with an action of the finite group $G \times T = \mathbf{G}(\mathbb{F}_q) \times \mathbf{T}(\mathbb{F}_q)$ where $\mathbf{T}$ is an $\mathbb{F}_q$-rational torus of $\mathbf{G}$. The ℓ-adic cohomology $H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)$ for $\ell \neq p$ inherits this action – one can regard its Euler characteristic $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)]$ as an element of the Grothendieck group $\mathrm{G}_0(G \times T, \overline{\mathbb{Q}}_\ell)$ of finite-dimensional representations. This decomposes into isotypic components labelled by characters of T, and the procedure of picking such a weight space relates characters of T with representations of G. The categories $\mathrm{rep}(G, \overline{\mathbb{Q}}_\ell)$ and $\mathrm{rep}(T, \overline{\mathbb{Q}}_\ell)$ of finite-dimensional representations are semisimple, and the results of classical Deligne–Lusztig theory make the above relation meaningful and computable.

There are two interesting directions in which one can try to generalize this theory. Firstly, we can consider reductive groups $\mathbf{G}$ over a non-archimedean local field K instead of $\mathbb{F}_q$, and seek a description of smooth representations of the locally profinite group $\mathbf{G}(K)$ with coefficients in an algebraically closed field of characteristic zero. The existence of a parallel theory was conjectured already by Lusztig, and its development is still in progress. The consequent relation between smooth characters of $\mathbf{T}(K)$ and smooth representations of $\mathbf{G}(K)$ should be tightly connected to the local Langlands correspondence. See [16,15,3,5,6,8,7,13].

Secondly, one can consider different coefficient rings Λ, for instance an algebraically closed field of positive characteristic $\ell \neq p$. This becomes interesting already in the classical case – when $\ell \mid |G|$, the category $\mathrm{rep}(G, \overline{\mathbb{F}}_\ell)$ ceases to be semisimple. Suddenly, the Grothendieck group does not contain enough information to fully describe the representation theory of G. A natural way to circumvent these issues is to work in the finer setting of the bounded derived category $\mathrm{D}^b(G, \Lambda)$, as in [4,1,11]. Nevertheless, the semisimplified information is interesting.

In this paper, we are interested in the combination of the two directions above: we consider the coefficients $\Lambda = \overline{\mathbb{F}}_\ell$ of positive characteristic $\ell \neq p$ for the p-adic Deligne–Lusztig theory. In doing so, we necessarily end up phrasing certain arguments on the level of derived categories.

In particular, we carry forward the results of [8] concerning the partial realization of the local Langlands correspondence for $\mathbf{GL}_n$ to the modular setting with coefficients $\Lambda = \overline{\mathbb{F}}_\ell$ on the level of Grothendieck groups. This has two main ingredients – the results

[1] Such coefficient fields are indistinguishable from the point of view of first order logic, so the representation theory is independent of the specific choice.

of [8] and the comparison to modular local Langlands correspondence constructed by Vignéras [28,27,26].

0.2. Our approach and results

Let K be a nonarchimedean local field with residue field $\mathbb{F}_q$ of characteristic p. In other words, K is either a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, or the function field $\mathbb{F}_q(\!(\varpi)\!)$. Let $\ell \neq p$ be another prime. Consider the group $\mathbf{G} = \mathbf{GL}_n$ over K with $G = \mathbf{G}(K)$ and its maximal torus $\mathbf{T}$, whose K-points $T = \mathbf{T}(K)$ correspond to the multiplicative group $L^\times$ of the degree n unramified extension L/K.

To sketch our approach and results, consider the following diagram.

$$\begin{array}{ccccc}
\pm[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta] & \mathscr{A}_n^0(K, \overline{\mathbb{Q}}_\ell) & \overset{r_\ell}{\dashrightarrow} & \mathscr{A}_n^0(K, \overline{\mathbb{F}}_\ell) & \pm[H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}] \\
\updownarrow & \uparrow & & \uparrow & \updownarrow \\
\theta & \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg}} & \overset{r_\ell}{\dashrightarrow} & \mathscr{X}(L, \overline{\mathbb{F}}_\ell)^{\mathrm{sg}} & \psi \\
\updownarrow & \downarrow & & \downarrow & \updownarrow \\
\mathrm{cInd}_{\mathcal{W}_L}^{\mathcal{W}_K}(\mu \cdot \theta) & \mathscr{G}_n^0(K, \overline{\mathbb{Q}}_\ell) & \overset{r_\ell}{\dashrightarrow} & \mathscr{G}_n^0(K, \overline{\mathbb{F}}_\ell) & \mathrm{cInd}_{\mathcal{W}_L}^{\mathcal{W}_K}(\mu \cdot \psi)
\end{array}$$

Diagram 1. The reduction diagram.

The left-hand side of Diagram 1 takes values in the characteristic zero field $\overline{\mathbb{Q}}_\ell$, while the parallel right-hand side lives over the field $\overline{\mathbb{F}}_\ell$ of positive characteristic. The middle row parametrizes smooth characters of the multiplicative group $L^\times$ in strongly general position[2] in each of the characteristics. From here, we can go to the bottom row, which parametrizes n-dimensional smooth irreducible representations of the Weil group $\mathcal{W}_K$. This goes by adjusting the given character by the so-called rectifier μ, inflating and inducing – it is explicit and well understood.

The passage to the irreducible supercuspidal representations of G parametrized by the upper row is the interesting part. Here we consider a suitable Deligne–Lusztig space $\dot{X}$ equipped with commuting actions of G and T, take its cohomology, look at the isotypic component[3] of the character of T we started with, and consider its Euler characteristic in the corresponding Grothendieck group. For characteristic zero coefficients, this is done in [6,8,7], leading to the following.

Theorem 0.1 *([8, Theorem A]). Assume $p > n$. Let $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg}}$ be a character in strongly general position. Then $\pm[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ is up to sign an irreducible supercuspidal representation of G and the left-hand side of Diagram 1 partially realizes the local Langlands correspondence.*

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[2] The notion of *strongly general position* is a strengthening of the notion of *general position*; see Definition 2.6.

[3] In characteristic zero, this is given by picking the weight space of θ. In positive characteristic, one needs to do this on the derived level – the naive definition of isotypic part yields and extra multiplicity for small ℓ.

Now consider the horizontal *reduction mod ℓ* maps r_ℓ in Diagram 1. These make sense after additional technical assumptions and are well-defined only on the level of Grothendieck groups. At this cost, we can prove clean compatibility statements about the parallel sides of the diagram. The results of Vignéras [28,27,26] further show that for $\mathbf{GL}_n$, the local Langlands correspondence can be partially reduced via r_ℓ to the modular setting. Putting these compatibilities together with the above theorem, we obtain our main result.

Theorem 0.2 *(Theorem 5.25). Assume $p > n$. Let $\psi \in \mathscr{X}(L, \overline{\mathbb{F}}_\ell)^{\mathrm{sg}}$ be a character in strongly general position. Then $\pm[H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}]$ is up to sign an irreducible supercuspidal representation of G, and the right-hand side of Diagram 1 partially realizes the modular local Langlands correspondence.*

Taking the isotypic part of ψ naively results in an extra multiplicity for those ℓ which are small with respect to G. We make this explicit in Theorem 5.29.

0.3. Consequences and further questions

Our Theorem 0.2 shows that the cohomology of p-adic Deligne–Lusztig spaces for $\mathbf{GL}_n$ encodes a big part of modular representation theory of the locally profinite group $G = \mathbf{GL}_n(K)$.

The employed compatibilities between the sides of Diagram 1 are quite general, so one could hope to apply them for other reductive groups $\mathbf{G}$ where the theory is less developed. In such a setting, there is a good notion of p-adic Deligne–Lusztig spaces [13]. However, the modular local Langlands correspondence – let alone its relation to the characteristic zero case – is not understood. A better understanding of $H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)$ may be fruitful.

This leads to another question: can we understand the cohomology $H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)$ already on the level of the bounded derived category $\mathrm{D}^b(G, \overline{\mathbb{F}}_\ell)$ of G-representations? Indeed, such an understanding is necessary for mimicking the geometric arguments of characteristic zero Deligne–Lusztig theory in the modular setting. Already on the level of classical Deligne–Lusztig theory with modular coefficients, this carries important information [4,1,11].

The geometric methods are close in spirit to the recent construction [12], while having more explicit features. One could hope to use them to get a better handle on this approach.

0.4. Structure of this article

After the current §0, this article is structured as follows. In §1 we discuss the smooth representation theory of locally profinite groups. We recall the notion of reduction modulo ℓ, relating characteristic zero and characteristic ℓ coefficients. In §2 we review the smooth character theory of the $T = L^\times$ and discuss the notion of strongly general position of characters. In §3 we recall the modular local Langlands correspondence of Vignéras and its relationship with the usual local Langlands correspondence via reduction modulo ℓ. In §4 we review certain homotopical complexes representing étale cohomology with group actions. These were constructed by Rickard [20] and give an important tool for our comparison of characteristic zero and characteristic ℓ cohomology.

In §5 we turn towards p-adic Deligne–Lusztig theory, concentrating on the hyperspecial Deligne–Lusztig space of Coxeter type for $\mathbf{GL}_n$. We generalize the construction of the isotypic parts in the étale cohomology groups $H_c^\bullet(\dot{X}, \Lambda)$ of [6,8,7,13] to the finer setting of the bounded derived category $\mathrm{D}^b(G, \Lambda)$ for Λ such as $\overline{\mathbb{Z}}_\ell$ and $\overline{\mathbb{F}}_\ell$. At the same time, we record a few statements from [6,8,7,13] about the characteristic zero case and employ them to get control over its characteristic ℓ counterpart. We then use the body of this paper to establish the necessary compatibilities between the vertical maps and the horizontal reduction modulo ℓ in Diagram 1, deducing Theorem 5.25. We finally comment on a non-derived version of our result in Theorem 5.29, leading to extra multiplicities for those ℓ which are small with respect to $\mathbf{G}$. This yields extra information about the structure of the cohomology.

0.5. Notation

Throughout, K will be a local non-archimedean field with ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{p}_K$ and residue field $k = \mathbb{F}_q$ of size q and characteristic p. A uniformizer of $\mathcal{O}_K$ is denoted $\varpi = \varpi_K$. Such K is either a finite extension of the p-adic numbers $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, or the function field $\mathbb{F}_q(\!(\varpi)\!)$.

We fix an integer $n \in \mathbb{N}$ and denote by L the degree n unramified extension of K. Thus the residue field of L is $\mathbb{F}_{q^n}$. We fix a different prime $\ell \neq p$ and let $m = v_\ell(q^n - 1)$ be the ℓ-adic valuation of $q^n - 1 = |\,\mathbb{F}_{q^n}^\times\,|$. In other words, ℓ^m is the size of the ℓ-torsion part of $\mathbb{F}_{q^n}^\times$.

0.6. Acknowledgements

This paper originated from my master thesis written at the University of Bonn during the academic year 2020/2021. I want to thank Alexander Ivanov for mathematical discussions and feedback on earlier drafts. I further want to thank Peter Scholze for suggestions about local systems, leading to a clarification of the result for small ℓ.

During my stay at the University of Bonn, I was supported by the DAAD study scholarship for graduates of all disciplines (57440925). Some final improvements of this paper were done at the Institute of Science and Technology Austria.

1. Grothendieck groups of smooth representations

We review the representation theory of locally profinite groups in sufficient generality valid also for characteristic ℓ coefficients. A very good reference is the book [28]. See also [2] for coefficients of characteristic zero.

1.1. Coefficients rings

We will be considering representations with coefficients in a commutative ring Λ. The interest lies in the following choices.

Setup 1.1. Fix a prime $\ell \neq p$. Let $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell \supseteq \mathfrak{m}$, $\mathbb{F}_\ell$ denote respectively the ℓ-adic numbers, their ring of integers with its maximal ideal, and their residue field of size ℓ. By $\overline{\mathbb{Q}}_\ell$, $\overline{\mathbb{Z}}_\ell \supseteq \overline{\mathfrak{m}}$, $\overline{\mathbb{F}}_\ell$ we mean the algebraic closure of $\mathbb{Q}_\ell$ with its ring of integers, its maximal ideal, and residue field. In general, we write E, $\mathcal{O}_E \supseteq \mathfrak{m}_E$, F for a field extension $\mathbb{Q}_\ell \subseteq E \subseteq \overline{\mathbb{Q}}_\ell$ with ring of integers $\mathcal{O}_E$, its maximal ideal $\mathfrak{m}_E$, and residue field F. For finite extensions, a uniformizer of $\mathcal{O}_E$ is denoted ϖ_E.

Denote respectively μ_r, μ_∞, μ_{ℓ^∞}, $\mu_{\ell^\infty}^\perp$ the functors of r-th roots of unity, roots of unity, roots of unity of order a power of ℓ, roots of unity of order coprime to ℓ. The canonical surjection $\mathcal{O}_E \to F$ induces a multiplicative map $\mathcal{O}_E^\times \to F^\times$. The *Teichmüller map* gives a canonical multiplicative section $\mathcal{O}_E^\times \leftarrow F^\times$. It identifies $F^\times = \mu_\infty(F)$ with the direct summand $\mu_{\ell^\infty}^\perp(E)$ of the multiplicative group $E^\times$ whose torsion part is $\mu_\infty(E) = \mu_{\ell^\infty}(E) \times \mu_{\ell^\infty}^\perp(E)$.

1.2. Categories of smooth representations and their Grothendieck groups

Given a locally profinite topological group G and a coefficient ring Λ, we have the category of *smooth* representations $\mathrm{Rep}(G,\Lambda)$ of G on Λ-modules, i.e. continuous representations with respect to the discrete topology on Λ-modules. We denote by $\mathrm{K}^b(G,\Lambda)$ and $\mathrm{D}^b(G,\Lambda)$ its bounded homotopy category and bounded derived category.

A closed subgroup $H \leq G$ is again locally profinite and we have the functors of compact induction cInd_H^G, restriction Res_H^G and induction Ind_H^G. When G possesses a compact open subgroup whose pro-order is invertible in Λ, all three of the above functors are exact [28, §I.5.10].

Remark 1.2. Any open subgroup $H \leq G$ is automatically closed (hence clopen). Indeed, H is the complement of its other cosets gH, which are open as images of H under the homeomorphisms $g\cdot : G \to G$ for varying $g \in G$.

If $H \leq G$ is clopen, cInd_H^G, Res_H^G and Ind_H^G form two adjoint pairs in this ordering. Moreover, cInd_H^G is exact: for each $V \in \mathrm{Rep}(H,\Lambda)$, the underlying vector space of $\mathrm{cInd}_H^G V$ is given by $\bigoplus_{H\backslash G} V$. See [28, §I.5] for details.

Definition 1.3. A representation $V \in \mathrm{Rep}(G,\Lambda)$ is called

- of *finite length*, if it has a finite composition series,
- of *finite type*, if it is finitely generated as G-representation,

- *admissible*, if V^H is a finitely generated Λ-module for each compact open subgroup $H \leq G$.

In order to have well-behaved Grothendieck groups, we will further restrict to the subcategory

$$\mathrm{rep}(G, \Lambda) \subseteq \mathrm{Rep}(G, \Lambda)$$

of representations of finite length.[4] We denote by $\mathrm{Rep}(G, \Lambda)^{\mathrm{ft}} \subseteq \mathrm{Rep}(G, \Lambda)$ the subcategory of representations of finite type. A representation of finite length is automatically of finite type.

Example 1.4. Assume Λ is artinian. Note that if G is finite, $\mathrm{rep}(G, \Lambda) \cong \mathrm{mod}(\Lambda[G])$ is the category of finitely generated representations of G over Λ. Similarly when G is profinite, the conditions "finite type" and "finite length" are equivalent for smooth representations, so $\mathrm{rep}(G, \Lambda)$ is the subcategory of representations of finite type.

Lemma 1.5. *Let Λ be a coefficient ring and $H \leq G$ a clopen subgroup. Then cInd_H^G preserves the property of being of finite type.*

Notation 1.6. Given a coefficient ring Λ and a locally profinite group G, we denote by $\mathrm{G}_0(G, \Lambda)$ the Grothendieck group of the category $\mathrm{rep}(G, \Lambda)$ of smooth representations of G over Λ of finite length discussed above.

We denote by $\mathrm{Irr}(G, \Lambda)$ a set of representatives of isomorphism classes of irreducibles in $\mathrm{rep}(G, \Lambda)$. The imposed finite length condition forces $\mathrm{G}_0(G, \Lambda)$ to be the free abelian group on $\mathrm{Irr}(G, \Lambda)$.

When G is abelian and Λ is an algebraically closed field (and G/G' is countable for any compact open subgroup G'), the Schur lemma holds: all elements of $\mathrm{Irr}(G, \Lambda)$ are one-dimensional. For more general Λ, we write $\mathrm{Hom}_{\mathrm{Grp}}(G, \Lambda^\times)$ for the character group.

1.3. Integral representations

Consider a coefficient ring Λ among $E, \mathcal{O}_E, F$ as in Setup 1.1. In order to relate these different choices of coefficients, we recall the subcategory $\mathrm{rep}(G, E)^{\mathrm{int}} \subseteq \mathrm{rep}(G, E)$ of *integral representations*.

Definition 1.7. A representation $V \in \mathrm{Rep}(G, E)$ is called *integral* if it lies in the essential image of the functor $(E \otimes_{\mathcal{O}_E} -) : \mathrm{Rep}(G, \mathcal{O}_E) \to \mathrm{Rep}(G, E)$.

[4] This is necessary so that the Grothendieck group $\mathrm{G}_0(\mathrm{rep}(G, \Lambda))$ does not degenerate by Eilenberg swindle.

In other words, V is integral if it contains a full G-stable $\mathcal{O}_E$-lattice M. A choice of such isomorphism is referred to as an *integral structure* of V.

Definition 1.8. Let $\mathrm{rep}(G,E)^{\mathrm{int}} \subseteq \mathrm{rep}(G,E)$ denote the full abelian subcategory of integral representations. We denote by $\mathrm{G}_0(G,E)^{\mathrm{int}} \leq \mathrm{G}_0(G,E)$ the subgroup given by the Grothendieck group of $\mathrm{rep}(G,E)^{\mathrm{int}}$.

Notice that there is a small statement to check, namely that integral representations cut out an abelian subcategory closed on subobjects, quotients and extensions – we then really have a map on Grothendieck groups, which is injective by Jordan–Hölder theorem.

Lemma 1.9. *Let $0 \to V' \to V \to V'' \to 0$ be a short exact sequence in $\mathrm{rep}(G,E)$. Then V is integral $\iff$ both V' and V'' are integral.*

Proof. Given an integral structure M on a representation V, we obtain an integral structure on any subrepresentation resp. quotient by intersecting resp. quotienting M. On the other hand, if V', V'' have integral structures M', M'', we obtain an integral structure M on their extension $0 \to V' \to V \to V'' \to 0$ by lifting M'' to some $\widetilde{M}''$, rescaling M' by some $c \in E^\times$ so that G maps some finite generating set of $\widetilde{M}'' \in \mathrm{rep}(G,\mathcal{O}_E)$ into $cM' + \widetilde{M}''$, and finally putting $M := cM' + \widetilde{M}''$. □

Example 1.10. If G is profinite, an integral structure for $V \in \mathrm{rep}(G,E)$ always exists: one takes any finite generating set of V and closes it on translations by G. The resulting set is still finite, hence generates an $\mathcal{O}_E$-lattice M inside V.

Remark 1.11. Note that for locally profinite G an integral structure need not exist. The basic counterexample appears for characters: A character $\theta : G \to E^\times$ is integral if and only if it factors through the inclusion $\mathcal{O}_E^\times \subseteq E^\times$.

Indeed – up to a scalar, there is only one candidate for an integral structure M of the one-dimensional representation given by θ. This M is stable if and only if θ doesn't hit any element with negative ℓ-adic valuation, i.e. if and only if θ hits only elements of $E^\times$ with zero ℓ-adic valuation.

Example 1.12. A relevant example of the above failure appears for the multiplicative group $G := K^\times$ of a non-archimedean local field K. Any smooth character of G is then uniquely determined by its restriction to a smooth character of the profinite group $\mathcal{O}_K^\times$ and its value at some uniformizer ϖ_K. This gives an identification of abelian groups

$$\mathrm{Hom}_{\mathrm{Grp}}(K^\times, E^\times) \cong \mathrm{Hom}_{\mathrm{Grp}}(\mathcal{O}_K^\times, E^\times) \times E^\times.$$

Since each smooth character θ of $\mathcal{O}_K^\times$ factors through a finite quotient, its image lies in the torsion subgroup $\mu(E)$ whose elements have zero ℓ-adic valuation. The integrality of θ thus depends only on the ℓ-adic valuation of the image of the uniformizer. The subgroup of integral characters thus identifies as $\mathrm{Hom}_{\mathrm{Grp}}(K^\times, E^\times)^{\mathrm{int}} = \mathrm{Hom}_{\mathrm{Grp}}(\mathcal{O}_K^\times, E^\times) \times \mathcal{O}_E^\times \subseteq \mathrm{Hom}_{\mathrm{Grp}}(\mathcal{O}_K^\times, E^\times) \times E^\times$.

Example 1.13. For p-adic reductive groups, the above examples describe the essence of the failure of integrality – by [28, p. II.4.13], a representation of such G over $\overline{\mathbb{Q}}_\ell$ is integral if and only if the central character of its cuspidal support is integral.

For $G = GL_n(K)$, we have $Z(GL_n(K)) = K^\times$ via the diagonal embedding. The integrality of a cuspidal representation of $GL_n(K)$ is thus equivalent to the integrality of the corresponding character of $K^\times$, discussed in the previous example.

Lemma 1.14. *Let G be a locally profinite group and H an open subgroup. If a smooth representation V of H is integral, then so is $\mathrm{cInd}_H^G V$.*

Proof. Let $V \in \mathrm{Rep}(H, E)$, and let M be an integral structure of V. We claim that $\mathrm{cInd}_H^G M$ gives an integral structure of $\mathrm{cInd}_H^G V$. The natural map $E \otimes_{\mathcal{O}_E} \mathrm{cInd}_H^G M \to \mathrm{cInd}_H^G V$ is injective because M is a lattice in V.

On the other hand, any $f \in \mathrm{cInd}_H^G V$ is compactly supported modulo H. Hence it is supported on finitely many points of the discrete quotient space $H\backslash G$, this being the case by clopeness of H. Because M is a full lattice, we can now multiply f by a suitable scalar so that its values for each of these finitely many cosets land in M – indeed, M is stable under the H-action, so all values of f on a given coset lie in M whenever one of them does. This shows surjectivity, proving the lemma. □

In other words, cInd_H^G canonically preserves integral structures on smooth representations.

1.4. Reduction modulo ℓ

Consider a coefficient ring Λ among $E, \mathcal{O}_E, F$ as in Setup 1.1. For a locally profinite group G, we will discuss the existence of *reduction modulo ℓ* map

$$\mathrm{G}_0(G, E) \geq \mathrm{G}_0(G, E)^{\mathrm{int}} \xrightarrow{r_\ell} \mathrm{G}_0(G, F).$$

This r_ℓ is only well-defined on the level of Grothendieck groups of finite length representations. The domain $\mathrm{G}_0(G, E)^{\mathrm{int}}$ of r_ℓ was introduced in Definition 1.8.

When G is finite, r_ℓ forms one side of the so-called Cartan–Brauer triangle; it is usually called the *decomposition map* [22]. With certain technical restrictions, r_ℓ is defined also in the locally profinite setting [28]. We now describe the construction of r_ℓ; the well-definedness in cases of interest is discussed immediately afterwards.

Construction 1.15. Start with $V \in \mathrm{rep}(G, E)^{\mathrm{int}}$ and choose an integral structure M. Then $M/\mathfrak{m}M$ is naturally a G-representation with coefficients in the residue field F. If $M/\mathfrak{m}M$ has finite length, it lies in $\mathrm{rep}(G, F)$ and we may pass to its class $r_\ell(V) := [V] \in \mathrm{G}_0(G, F)$.

Remark 1.16. Whenever Construction 1.15 determines a well-defined map $\mathcal{A} \to \mathrm{G}_0(G, F)$ independent of the choices of integral structures on some abelian subcategory $\mathcal{A}$ of $\mathrm{rep}(G, E)^{\mathrm{int}}$, it automatically gives a map $\mathrm{G}_0(\mathcal{A}) \to \mathrm{G}_0(G, F)$ on Grothendieck groups. We will denote this map by r_ℓ as well.

To see this, one only needs to note that relations given by short exact sequences are sent to such relations again. Given a short exact sequence $0 \to V' \to V \to V'' \to 0$ in $\mathcal{A}$, one can choose integral structures fitting into a short exact sequence $0 \to M' \to M \to M'' \to 0$ as in the proof of Lemma 1.9 and reduce via these, obtaining the desired relation.

Remark 1.17. Note that r_ℓ preserves vector space dimension.

The following *Brauer-Nesbitt principle* addresses the independence of r_ℓ on the choice of integral structure in reasonable generality.

Proposition 1.18 *([28, §I.9.6]). Let G be a locally profinite group. Assume $V \in \mathrm{rep}(G, E)$ is admissible and contains an integral structure M of finite type such that $M/\mathfrak{m}M$ has finite length. Then $r_\ell(V) \in \mathrm{G}_0(G, F)$ is well-defined and independent of the choice of M.*

Discussion 1.19. Let us now record three important instances when Construction 1.15 works. These include all cases relevant later.

(i) If G is profinite, the above construction yields a well-defined map

$$r_\ell : \mathrm{G}_0(G, E) \to \mathrm{G}_0(G, F).$$

Indeed, all representations V in question are automatically finite-dimensional by compactness of G. We have already observed this implies $\mathrm{rep}(G, E)^{\mathrm{int}} = \mathrm{rep}(G, E)$ for profinite G. The assumptions on admissibility and finite length of Proposition 1.18 are clearly satisfied for finite dimensional representations, so the reduction $r_\ell(V) \in \mathrm{G}_0(G, F)$ is independent of the choices made. This is a priori true for irreducible V, but immediately extends to all representations by looking at their composition series. By Remark 1.16, we get a map $\mathrm{G}_0(G, E) \to \mathrm{G}_0(G, F)$.

(ii) If G is a locally profinite group and $\mathrm{G}_0(G, E)^{\mathrm{fd,\,int}}$ is the Grothendieck group of the category $\mathrm{rep}(G, E)^{\mathrm{fd,\,int}}$ of finite-dimensional integral representations, the above construction yields a well-defined map

$$r_\ell : \mathrm{G}_0(G, E)^{\mathrm{fd,\,int}} \to \mathrm{G}_0(G, F)^{\mathrm{fd}}.$$

The argument is the same as in (i), the only difference being that the finite-dimensionality comes as an assumption.

(iii) If G is a p-adic reductive group, the above construction yields a well-defined map

$$r_\ell : \mathrm{G}_0(G, E)^{\mathrm{int}} \to \mathrm{G}_0(G, F).$$

Such G is locally profinite, but we are now dealing with infinite-dimensional representations. Luckily, the assumptions of Proposition 1.18 (for irreducible V) hold by [28, II.5.11.b]. Note that the admissibility condition in this proposition is automatic, since any irreducible representation (and hence any finite-length representation) of G is admissible. The extension to all finite length representation and to a map of Grothendieck groups is again immediate.

Example 1.20. Take G finite abelian and $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$. Then each irreducible representation is a character and the reduction $r_\ell : \mathrm{G}_0(G, \overline{\mathbb{Q}}_\ell) \to \mathrm{G}_0(G, \overline{\mathbb{F}}_\ell)$ is induced by postcomposing characters with $\mu_\infty(\overline{\mathbb{Q}}_\ell) = \mu_\infty(\overline{\mathbb{Z}}_\ell) \to \mu_\infty(\overline{\mathbb{F}}_\ell)$. Since $\mu_\infty(\overline{\mathbb{F}}_\ell)$ sits inside $\mu_\infty(\overline{\mathbb{Q}}_\ell)$ as the direct summand $\mu^{\perp}_{\ell^\infty}(\overline{\mathbb{Q}}_\ell)$, the fibers of the surjection $\mathrm{Irr}(G, \overline{\mathbb{Q}}_\ell) \to \mathrm{Irr}(G, \overline{\mathbb{F}}_\ell)$ are given by $\mathrm{Hom}_{\mathrm{Grp}}(G, \mu_{\ell^\infty}(\overline{\mathbb{Q}}_\ell))$. In particular, the size of these fibers is given by the number of elements of G whose order is a power of ℓ.

1.5. Compatibility of r_ℓ and cInd

Let G be a locally profinite group and $H \le G$ a clopen subgroup. The functor $\mathrm{cInd}_H^G : \mathrm{Rep}(H, \Lambda) \to \mathrm{Rep}(G, \Lambda)$ is exact by §1.2, but it need not restrict to a map $\mathrm{rep}(H, \Lambda) \to \mathrm{rep}(G, \Lambda)$ between finite length representations nor the associated Grothendieck groups.

Notation 1.21. We denote by $\mathrm{rep}(G, \Lambda)^{\mathrm{sub}} \subseteq \mathrm{rep}(G, \Lambda)$ any Serre subcategory: a nonempty strictly full (abelian) subcategory closed on subobjects, quotients and extensions [25, Tag 02MP]. By looking at Jordan-Hölder series, such $\mathrm{rep}(G, \Lambda)^{\mathrm{sub}}$ corresponds to a subset of irreducible objects in $\mathrm{rep}(G, \Lambda)$. Its Grothendieck group $\mathrm{G}_0(G, \Lambda)^{\mathrm{sub}} \le \mathrm{G}_0(G, \Lambda)$ is the subgroup spanned by these irreducibles.

If cInd_H^G restricts to a functor $\mathrm{rep}(H, \Lambda)^{\mathrm{sub}} \to \mathrm{rep}(G, \Lambda)^{\mathrm{sub}}$, we get an induced map $\mathrm{cInd}_H^G : \mathrm{G}_0(H, \Lambda)^{\mathrm{sub}} \to \mathrm{G}_0(G, \Lambda)^{\mathrm{sub}}$. On classes of genuine representations, this sends $[V] \mapsto [\mathrm{cInd}_H^G(V)]$. We now discuss its behaviour with respect to r_ℓ.

Lemma 1.22. *Let G be a locally profinite group and $H \le G$ a clopen subgroup. Consider subcategories as in Notation 1.21 such that the following two reduction maps r_ℓ are well-defined:*

$$\mathrm{G}_0(G, E)^{\mathrm{sub,int}} \xrightarrow{r_\ell} \mathrm{G}_0(G, F), \qquad \mathrm{G}_0(H, E)^{\mathrm{sub,int}} \xrightarrow{r_\ell} \mathrm{G}_0(H, F)$$

Let $[V] \in \mathrm{G}_0(H, E)^{\mathrm{sub,int}}$ be a class of a genuine representation V such that $\mathrm{cInd}_H^G[V]$ lies in $\mathrm{G}_0(G, E)^{\mathrm{sub}}$. Then it is integral and

$$r_\ell \circ \mathrm{cInd}_H^G[V] = \mathrm{cInd}_H^G \circ r_\ell[V] \in \mathrm{G}_0(G, F).$$

Proof. First note that $\mathrm{cInd}_H^G[V] \in \mathrm{G}_0(G,E)^{\mathrm{sub,int}}$. Indeed, it has finite length by assumption and it is integral by Lemma 1.14 – given $V \in \mathrm{rep}(H,E)$ with an integral structure M, the compact induction $\mathrm{cInd}_H^G M$ gives an integral structure of $\mathrm{cInd}_H^G V$.

Reducing by r_ℓ with respect to these two integral structures shows that

$$r_\ell \circ \mathrm{cInd}_H^G[V] = \mathrm{cInd}_H^G \circ r_\ell[V],$$

because cInd_H^G over $\mathcal{O}_E$-coefficients commutes with modding out the image of the maximal ideal $\mathfrak{m}_E \subseteq \mathcal{O}_E$ by explicit inspection. □

Remark 1.23. In particular, the setup of Lemma 1.22 implies that $\mathrm{cInd}_H^G \circ r_\ell[V]$ is a well-defined class in the Grothendieck group of finite length representations $\mathrm{G}_0(G,F)$.

Corollary 1.24. *Suppose we are given subcategories of finite length representations as in Notation 1.21 so that the following square is well-defined. Then it commutes.*

$$\begin{array}{ccc} \mathrm{G}_0(G,E)^{\mathrm{sub,int}} & \xrightarrow{r_\ell} & \mathrm{G}_0(G,F)^{\mathrm{sub}} \\ {\scriptstyle \mathrm{cInd}_H^G}\uparrow & & {\scriptstyle \mathrm{cInd}_H^G}\uparrow \\ \mathrm{G}_0(H,E)^{\mathrm{sub,int}} & \xrightarrow{r_\ell} & \mathrm{G}_0(H,F)^{\mathrm{sub}} \end{array}$$

Proof. Use Lemma 1.22 for classes of irreducible representations $[V] \in \mathrm{G}_0(H,E)^{\mathrm{sub,int}}$. □

1.6. Finite groups and permutation modules

For a finite group G and a commutative coefficient ring Λ, we have the subclass of objects in $\mathrm{Rep}(G,\Lambda)^{\mathrm{ft}}$ of *permutation modules*. By definition, these are cut out by the essential image of the free Λ-module functor

$$\Lambda[-] : \mathrm{FinSet}_G \to \mathrm{Rep}(G,\Lambda)^{\mathrm{ft}} \qquad \text{sending} \qquad S \mapsto \Lambda[S].$$

For each $S \in \mathrm{FinSet}_G$, denote $\varepsilon : \Lambda[S] \to \Lambda$ the natural augmentation map given by summing up the coefficients in the canonical basis of $\Lambda[S]$.

We recall the following standard characterization of projective permutation modules.

Lemma 1.25. *Let $H \le G$ be a subgroup. Then the permutation module $\Lambda[G/H] \in \mathrm{Rep}(G,\Lambda)^{\mathrm{ft}}$ is projective if and only if $|H| \in \Lambda^\times$.*

Proof. The projectivity of $\Lambda[G/H]$ is equivalent to the splitting of the projection $\pi : \Lambda[G] \twoheadrightarrow \Lambda[G/H]$. To give a splitting of π is the same thing as to lift the element $1 \cdot H \in \Lambda[G/H]$ to some $h \in \Lambda[G]$ invariant under the left action of H. Writing such h in the natural basis labelled by elements of G, it must have constant coefficients along the left cosets of H.

If $|H| \notin \Lambda^\times$, such lift h does not exist as $\varepsilon(h) \notin \Lambda^\times$ but $\varepsilon(1 \cdot H) = 1$. On the other hand if $|H| \in \Lambda^\times$, the element $h := |H|^{-1} \cdot \sum_{g \in H} g$ gives such lift. $\square$

Setup 1.26. We will use the following (possibly non-commutative) rings A as coefficients. All modules are implicitly left modules.

(i) We denote by A any torsion Artin algebra. In particular, this covers the case $A = \Lambda[G]$ with $\Lambda = F$ or its finite self-extensions.
(ii) We further allow $A = \Lambda[G]$ for any Λ from Setup 1.1. Such A is an inverse limit of a flat system of torsion Artin algebras, or its subsequent flat base change.

For a subclass $\mathcal{M}$ of objects of an abelian category $\mathcal{A}$, we write $\mathrm{add}(\mathcal{M})$ for the smallest additive idempotent-complete full subcategory of $\mathcal{A}$ containing $\mathcal{M}$. In particular, starting from the class $\mathcal{M} = \{\Lambda[G/H] \mid \ell \nmid |H|\}$ inside $\mathrm{rep}(G, \Lambda)$, we see that $\mathrm{add}(\mathcal{M}) = \mathrm{add}(\Lambda[G])$ consists precisely of finite projective $\Lambda[G]$-modules. Indeed, since $\mathcal{M}$ contains only projectives by Lemma 1.25, the same is true for $\mathrm{add}(\mathcal{M})$; the other direction is obvious as $\Lambda[G] \in \mathcal{M}$.

2. The structure of T and its characters

In this section we discuss the characters of $T = L^\times$ with values in $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$. Apart from their general structure, we consider the question of their integrality and discuss the reduction map r_ℓ. We keep track of the Galois action on them.

2.1. Characters of T

Let $T = L^\times$ be the locally profinite group of units in the degree n unramified extension L/K as in §0.5. We denote by $T_\mathcal{O} = T^0 = \mathcal{O}_L^\times$ the units in the ring of integers $\mathcal{O}_L$ of L; this is a profinite subgroup of T. Note that $L^\times$ can be written as the pushout of abelian groups

$$\begin{array}{ccc} K^\times & \hookrightarrow & L^\times \\ \uparrow & & \uparrow \\ \mathcal{O}_K^\times & \hookrightarrow & \mathcal{O}_L^\times \end{array} \tag{2.1}$$

Thus for $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$ we can write

$$\mathrm{Irr}(T, \Lambda) = \mathrm{Irr}(K^\times, \Lambda) \times_{\mathrm{Irr}(\mathcal{O}_K^\times, \Lambda)} \mathrm{Irr}(\mathcal{O}_L^\times, \Lambda) = \Lambda^\times \times \mathrm{Irr}(T_\mathcal{O}, \Lambda), \tag{2.2}$$

the second equality given by fixing a uniformizer ϖ_K. For $\Lambda = \overline{\mathbb{Q}}_\ell$, the question of integrality is answered as in Example 1.12 – a character is integral if and only if the image ϖ_K lies in $\overline{\mathbb{Z}}_\ell^\times$.

Lemma 2.1. *The reduction map* $r_\ell : \mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}} \to \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)$ *is surjective.*

Proof. Postcomposition with the Teichmüller lift gives a section of $\mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell) \to \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)$. Since the image of the Teichmüller map lands in $\mu_\infty(\overline{\mathbb{Q}}_\ell) \subseteq \overline{\mathbb{Z}}_\ell^\times$, this section factors though $\mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}}$. □

Now we return to the square (2.1) to address the Galois action. The group $\mathrm{Gal}(L/K)$ acts on $\mathrm{Irr}(T, \Lambda)$ by precomposition with its defining action. Under the identification (2.2) this action decomposes into the trivial action (induced by precomposition on $K^\times$) times the natural action on $\mathcal{O}_L^\times$. Altogether, there are two independent contributions to $\mathrm{Irr}(T, \Lambda)$ – the "profinite" contribution of $T_\mathcal{O}$ and the "free" contribution of the uniformizer.

2.2. *Characters of the profinite* $T_\mathcal{O}$

We have the locally profinite group $T = L^\times$ with its profinite subgroup $T_\mathcal{O} = T^0 = \mathcal{O}_L^\times$. For $h \geq 1$ set $T^h := 1 + \mathfrak{p}_L^h$. For any $0 \leq h \leq h'$, we obtain the finite group $T^h_{h'} := T^h / T^{h'}$. In particular, we get the truncations $T_h := T_h^0 = \mathcal{O}_L^\times / T_L^h$.

The profinite group $\mathcal{O}_L^\times$ has a decreasing filtration

$$\mathcal{O}_L^\times = T^0 \supset T^1 \supset \cdots \supset T^h \supset \ldots$$

by clopen subgroups which form a basis of neighbourhoods of 1; this filtration is also $\mathrm{Gal}(L/K)$-stable. The truncation T_h fits into a $\mathrm{Gal}(L/K)$-equivariant short exact sequence of abelian groups

$$1 \to T_h^1 \to T_h \to T_1 \to 1. \tag{2.3}$$

The cokernel T_1 is isomorphic to $\mathbb{F}_{q^n}^\times$. On the other hand, the kernel T_h^1 has size $q^{n(h-1)}$. So the short exact sequence (2.3) is canonically split, giving the decomposition of T_h into p-torsion part and p-torsion-free part. As $\ell \neq p$, the size of the kernel T_h^1 is coprime to ℓ.

Applying $\mathrm{Hom}_{\mathrm{Grp}}(-, \Lambda^\times)$ to the split sequence (2.3) gives a split short exact sequence of character groups with coefficients in Λ, naturally in Λ. In particular, we apply this for $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Z}}_\ell, \overline{\mathbb{F}}_\ell$. Comparing characters with values in $\overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$ via the natural maps from characters with values in $\overline{\mathbb{Z}}_\ell$ implies that this sequence is compatible with reduction modulo ℓ – the following diagram is commutative and $\mathrm{Gal}(L/K)$-equivariant.

$$\begin{array}{ccccccccc}
1 & \longleftarrow & \mathrm{Irr}(T_h^1, \overline{\mathbb{Q}}_\ell) & \longleftarrow & \mathrm{Irr}(T_h, \overline{\mathbb{Q}}_\ell) & \longleftarrow & \mathrm{Irr}(T_1, \overline{\mathbb{Q}}_\ell) & \longleftarrow & 1 \\
 & & \downarrow{\scriptstyle r_\ell} & & \downarrow{\scriptstyle r_\ell} & & \downarrow{\scriptstyle r_\ell} & & \\
1 & \longleftarrow & \mathrm{Irr}(T_h^1, \overline{\mathbb{F}}_\ell) & \longleftarrow & \mathrm{Irr}(T_h, \overline{\mathbb{F}}_\ell) & \longleftarrow & \mathrm{Irr}(T_1, \overline{\mathbb{F}}_\ell) & \longleftarrow & 1
\end{array}$$

Since the order of T_h^1 is coprime to ℓ, the left-hand side reduction map is an isomorphism by Example 1.20. On the other hand, the order of T_1 need not be coprime to

ℓ. If we denote by m its ℓ-adic valuation, it also follows from Example 1.20 that the right-hand map is a surjection with fibers of size ℓ^m. We may record the following.

Lemma 2.2. *The reduction* $r_\ell : \mathrm{Irr}(T_h, \overline{\mathbb{Q}}_\ell) \to \mathrm{Irr}(T_h, \overline{\mathbb{F}}_\ell)$ *is a* $\mathrm{Gal}(L/K)$*-equivariant surjection with fibers of size* ℓ^m*.*

The smoothness of $\theta \in \mathrm{Irr}(T_{\mathcal{O}}, \Lambda)$ is equivalent to the existence of an integer $h \in \mathbb{N}$ such that θ factors over T_h. The smallest such h is called the *level* of θ and denoted $\mathrm{lvl}(\theta)$. We will often confuse $\theta \in \mathrm{Irr}(T_{\mathcal{O}}, \Lambda)$ with such factorization in $\mathrm{Irr}(T_h, \Lambda)$ for some $h \geq \mathrm{lvl}(\theta)$. The statement of Lemma 2.2 carries word for word to a statement about smooth characters of the profinite group $T_{\mathcal{O}}$.

Remark 2.3. Altogether, given a character $\psi \in \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)$, we can choose lifts $\theta_i \in \mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}}$ for $i = 1, \ldots, \ell^m$ of ψ under r_ℓ whose restrictions to $T_{\mathcal{O}}$ are precisely the unique ℓ^m lifts of $\psi|_{T_{\mathcal{O}}}$. Indeed, one simply needs to lift the image of the uniformizer $\varpi \in \mathcal{O}_K$ from $\overline{\mathbb{F}}_\ell^\times$ to $\overline{\mathbb{Z}}_\ell^\times$ as discussed in the previous section – the Teichmüller map provides a canonical choice.

2.3. General position of characters

We now discuss the action of $\mathrm{Gal}(L/K)$ on $\mathrm{Irr}(T, \Lambda)$ by precomposition more closely. There is the following standard vocabulary.

Definition 2.4. A smooth character $\theta \in \mathrm{Irr}(T, \Lambda)$ lies in *general position* if it has trivial $\mathrm{Gal}(L/K)$-stabilizer. We denote the set of characters in general position by $\mathrm{Irr}(T, \Lambda)^{\mathrm{gen}}$. When referring to the local Langlands correspondence, it is also denoted $\mathscr{X}(L, \Lambda)$.

Remark 2.5. Since $T = L^\times$ is the pushout (2.1), the $\mathrm{Gal}(L/K)$-action on T is uniquely determined by its restriction to $T_{\mathcal{O}}$. Therefore, a character $\theta \in \mathrm{Irr}(T, \Lambda)$ is in general position if and only if its restriction $\theta|_{T_{\mathcal{O}}}$ has trivial $\mathrm{Gal}(L/K)$-stabilizer.

The discussion of the preceding section shows that for our purposes, $\theta|_{T^1}$ is the better behaved part of θ. We consider the following stronger condition.

Definition 2.6. A smooth character $\theta \in \mathrm{Irr}(T, \Lambda)$ lies in *strongly general position* if $\theta|_{T^1}$ has trivial stabilizer in $\mathrm{Gal}(L/K)$. We denote the subset of characters in strongly general position by $\mathrm{Irr}(T, \Lambda)^{\mathrm{sg}}$ or by $\mathscr{X}(L, \Lambda)^{\mathrm{sg}}$.

One reason for this definition is the appearance of the same condition in [8, Theorem A], which we use crucially. Another reason is that it behaves well with respect to r_ℓ.

Lemma 2.7. *A smooth character* $\theta \in \mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}}$ *lies in strongly general position if and only if its reduction* $r_\ell(\theta) \in \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)$ *lies in strongly general position.*

Proof. By smoothness, $\theta|_{T_{\mathcal{O}}}$ is a character of T_h for some big enough h. Lying in strongly general position means that the restriction to T_h^1 has trivial $\mathrm{Gal}(L/K)$-stabilizer. But the reduction map $r_\ell : \mathrm{Irr}(T_h^1, \overline{\mathbb{Q}}_\ell) \to \mathrm{Irr}(T_h^1, \overline{\mathbb{F}}_\ell)$ is a $\mathrm{Gal}(L/K)$-equivariant isomorphism, implying the result. □

Remark 2.8. Note that Lemma 2.7 would not be true if we replaced *strongly general position* by *general position* everywhere.

3. Local Langlands correspondence

In this section we recall certain facts about the local Langlands correspondence for $G = \mathbf{GL}_n(K)$ with coefficients $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$. See [28,27,26] for the modular setting and [2] for the characteristic zero setting.

3.1. Notation and the statement

Consider the coefficients $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$. In both cases we use the following standard notation. Let $\mathscr{A}_n^0(K,\Lambda)$ be the set of isomorphism classes of irreducible supercuspidal representations of $\mathbf{GL}_n(K)$ over Λ. On the other side, let $\mathscr{G}_n^0(K,\Lambda)$ be the set of isomorphism classes of irreducible n-dimensional representations of the Weil group $\mathcal{W}_K$ over Λ.

For us, the local Langlands correspondence is a unique bijection

$$\mathscr{G}_n^0(K,\Lambda) \xrightarrow{\mathrm{LL}} \mathscr{A}_n^0(K,\Lambda),$$

satisfying certain naturality properties. Its unique existence is known for $\mathbf{GL}_n(K)$ for both $\Lambda = \overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$. When $\Lambda = \overline{\mathbb{Q}}_\ell$ we call this simply the *local Langlands correspondence*; for the case $\Lambda = \overline{\mathbb{F}}_\ell$ we use the name *modular local Langlands correspondence.*

3.2. The modular local Langlands correspondence and reduction mod ℓ

The modular Local Langlands correspondence for $\mathbf{GL}_n(K)$ has been deduced by reduction modulo ℓ from the zero characteristic case by Vignéras [28,27,26]. A brief summary can be also found in [23].

In addition to the notation of the previous section, we write

$$\mathscr{A}_n^0(K,\overline{\mathbb{Q}}_\ell) \supseteq \mathscr{A}_n^0(K,\overline{\mathbb{Q}}_\ell)^{\mathrm{int}} \supseteq \mathscr{A}_n^0(K,\overline{\mathbb{Q}}_\ell)^{\ell-\mathrm{scu}}$$

for the consecutive subsets of $\mathscr{A}_n^0(K,\overline{\mathbb{Q}}_\ell)$ given by integral representations resp. integral representations with supercuspidal reduction modulo ℓ.

Analogously let

$$\mathscr{G}_n^0(K,\overline{\mathbb{Q}}_\ell) \supseteq \mathscr{G}_n^0(K,\overline{\mathbb{Q}}_\ell)^{\mathrm{int}} \supseteq \mathscr{G}_n^0(K,\overline{\mathbb{Q}}_\ell)^{\ell-\mathrm{irr}}$$

be the consecutive subsets of $\mathscr{G}_n^0(K, \overline{\mathbb{Q}}_\ell)$ given by integral representations resp. integral representations with irreducible reduction modulo ℓ.

Theorem 3.1 *(Vignéras). The following diagram commutes*

$$\begin{array}{ccccc} \mathscr{A}_n^0(K,\overline{\mathbb{Q}}_\ell) & \hookleftarrow & \mathscr{A}_n^0(K,\overline{\mathbb{Q}}_\ell)^{\ell-\mathrm{scu}} & \xrightarrow{r_\ell} & \mathscr{A}_n^0(K,\overline{\mathbb{F}}_\ell) \\ \mathrm{LL}\uparrow\cong & & \mathrm{LL}\uparrow\cong & & \mathrm{LL}\uparrow\cong \\ \mathscr{G}_n^0(K,\overline{\mathbb{Q}}_\ell) & \hookleftarrow & \mathscr{G}_n^0(K,\overline{\mathbb{Q}}_\ell)^{\ell-\mathrm{irr}} & \xrightarrow{r_\ell} & \mathscr{G}_n^0(K,\overline{\mathbb{F}}_\ell) \end{array}$$

Proof. Up to reformulation, this is [23, Theorem 4.6.]. For details, see [27,26]. □

To phrase this in words: the local Langlands correspondence preserves integral representations and identifies ℓ-supercuspidal representations of G with ℓ-irreducible representations of $\mathcal{W}_K$. After the reduction r_ℓ, it descends to a bijection realizing the modular local Langlands correspondence.

3.3. Weil induction of characters

We need the following partial parametrization of $\mathscr{G}_n^0(K, \Lambda)$. Let L be the degree n unramified extension of K and $\mathscr{X}(L, \Lambda) \subseteq \mathrm{Irr}(T, \Lambda)$ the subset of characters in general position. There is a map

$$\begin{gathered} \sigma : \mathscr{X}(L,\Lambda)/\operatorname{Gal}(L/K) \to \mathscr{G}_n^0(K,\Lambda) \\ \theta \mapsto \sigma(\theta) := \mathrm{cInd}_{\mathcal{W}_L}^{\mathcal{W}_K}\left(\mathcal{W}_L \to \mathcal{W}_L^{\mathrm{ab}} \xrightarrow{\mathrm{rec}_L} L^\times \xrightarrow{\mu\cdot\theta} \Lambda\right). \end{gathered} \tag{3.1}$$

Here, the *rectifying character* μ is determined by sending a uniformizer ϖ to $(-1)^{n-1}$ and being trivial on the compact part under the identification $\mathrm{Irr}(T, \Lambda) = \Lambda^\times \times \mathrm{Irr}(T_\mathcal{O}, \Lambda)$. The map rec_L is the Artin reciprocity map of class field theory.

The following seems well-known, but we lack a reference in the modular case. For completeness, we sketch a proof.

Proposition 3.2. *The map* (3.1) *is a well-defined injection.*

Proof. Since μ is stabilized by the $\mathrm{Gal}(L/K)$-action, the adjustment $\theta \mapsto \mu \cdot \theta$ only permutes $\mathscr{X}(L, \Lambda)$ resp. $\mathscr{X}(L, \Lambda)/\operatorname{Gal}(L/K)$, and thus is irrelevant for the statement – hence we ignore it.

Given a character $\theta \in \mathrm{Irr}(L^\times, \Lambda)$, denote V the corresponding one-dimensional representation of $\mathcal{W}_L$ obtained via the reciprocity isomorphism of class field theory.

We have a short exact sequence $1 \to \mathcal{W}_L \to \mathcal{W}_K \to \mathrm{Gal}(L/K) \to 1$. In particular, $\mathcal{W}_L$ is normal in $\mathcal{W}_K$. For $g \in \mathrm{Gal}(L/K)$, we denote ${}^gV$ the twisted representation obtained from V by precomposing with the conjugation $\mathcal{W}_L \to \mathcal{W}_L$, $w \mapsto \tilde{g}w\tilde{g}^{-1}$ by any lift $\tilde{g} \in \mathcal{W}_K$ of g; the isomorphism class of the resulting representation is independent of the choice of $\tilde{g}$. Therefore, $\mathrm{Gal}(L/K)$ acts on the character group of $\mathcal{W}_L$.

The Galois group $\mathrm{Gal}(L/K)$ also acts on the character group $\mathrm{Irr}(L^\times, \Lambda)$ via precomposition with its defining action on L. It is a standard result of local class field theory that the identification of the character groups via the reciprocity map is $\mathrm{Gal}(L/K)$-equivariant with respect to the above described actions, the relevant fact being that the reciprocity map comes from a morphism of modulations [19, Definition 1.5.10.].

Using the normality of $\mathcal{W}_L$ in $\mathcal{W}_K$, we obtain a Mackey decomposition

$$\mathrm{Res}^{\mathcal{W}_K}_{\mathcal{W}_L} \mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V \cong \bigoplus_{g \in \mathrm{Gal}(L/K)} {}^g V.$$

Note this has dimension $n = |\mathrm{Gal}(L/K)|$. The terms ${}^gV$ on the right-hand side are one-dimensional, so in particular irreducible – it follows that this $\mathcal{W}_L$-representation is semisimple.

From now on, assume θ is in general position as in the statement. By the above discussion, the terms ${}^gV$ on the right hand-side are pairwise non-isomorphic.

Consider any nonzero subrepresentation $U \subseteq \mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V$. By the Krull–Schmidt theorem, $\mathrm{Res}^{\mathcal{W}_K}_{\mathcal{W}_L} U$ contains a $\mathcal{W}_L$-representation isomorphic to ${}^gV$ for some $g \in \mathrm{Gal}(L/K)$. For any $h \in \mathrm{Gal}(L/K)$, the translate of this by a lift of hg^{-1} gives a representation of $\mathcal{W}_L$ isomorphic to ${}^hV$. Using Krull–Schmidt, this forces $\dim_\Lambda U \geq n$, showing $U = \mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V$. Altogether, $\mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V$ is irreducible.

We thus have a map $\mathscr{X}(L, \Lambda) \to \mathscr{G}^0_n(K, \Lambda)$ and it remains to see that it identifies precisely the Galois orbits. This is now immediate – given two characters $\theta, \theta' \in \mathscr{X}(L, \Lambda)$ with corresponding $\mathcal{W}_L$-representations V, V', Frobenius reciprocity and the Mackey formula above yield

$$\mathrm{Hom}_{\mathcal{W}_K}\left(\mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V, \mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V'\right) \cong \mathrm{Hom}_{\mathcal{W}_L}\left(V, \mathrm{Res}^{\mathcal{W}_K}_{\mathcal{W}_L} \mathrm{cInd}^{\mathcal{W}_K}_{\mathcal{W}_L} V'\right).$$

By the already proven irreducibility, the left-hand side is nonzero if and only if the inductions are isomorphic, whereas the right-hand side is nonzero if and only if V, V' (hence the original characters θ, θ') lie in the same $\mathrm{Gal}(L/K)$-orbit. □

We will also need a compatibility of the map (3.1) for $\Lambda = \overline{\mathbb{Q}}_\ell, \overline{\mathbb{F}}_\ell$ with the reduction r_ℓ.

Lemma 3.3. *The following well-defined diagram commutes*

$$\begin{array}{ccc} \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg,int}} & \xrightarrow{r_\ell} & \mathscr{X}(L, \overline{\mathbb{F}}_\ell)^{\mathrm{sg}} \\ \downarrow & & \downarrow \\ \mathscr{G}^0_n(K, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}} & \xrightarrow{r_\ell} & \mathscr{G}^0_n(K, \overline{\mathbb{F}}_\ell) \end{array}$$

Proof. Take $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg,\,int}}$, and $\psi \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg}}$ be its image under r_ℓ. We have already seen that all of this makes sense – the smooth character θ has an integral structure by design; r_ℓ preserves its strongly general position by Lemma 2.7. The adjustment by μ does not affect integrality; the formation of the inflated representation to $\mathcal{W}_L$ commutes with r_ℓ. Compact induction carries forward the integral structure and commutes with r_ℓ by Lemma 1.22, whose assumptions are satisfied by Discussion 1.19 (ii) – indeed, $|\,\mathcal{W}_K\,/\,\mathcal{W}_L\,| = n$ is finite, so all representations in question are finite-dimensional. The diagram thus commutes.

Finally, we should point out that the vertical maps land in irreducible representations of $\mathcal{W}_K$ by Proposition 3.2. □

4. Group actions on étale cohomology

For an algebraic variety X, we have its compactly supported étale cohomology groups $H^\bullet_c(X, \Lambda)$; these are the cohomology groups of the complex $\mathrm{R}\Gamma_c(X, \Lambda) \in \mathrm{D}^b(\Lambda)$. When X carries an action of a finite group G, we get an induced action on its étale cohomology: $\mathrm{R}\Gamma_c(X, \Lambda) \in \mathrm{D}^b(G, \Lambda)$.

Rickard [20] noticed that this complex has a canonical representative $\mathrm{G}\Gamma_c(X, \Lambda)$ in the homotopy category $\mathrm{K}^b(G, \Lambda)$ of a specific shape. Many constructions on $\mathrm{R}\Gamma_c(X, \Lambda)$ – which are necessarily derived in nature – can be performed directly on this representative. This makes $\mathrm{G}\Gamma_c(X, \Lambda)$ an important tool for understanding the behaviour of representations coming from cohomology under the change of coefficients. We briefly review the properties of Rickard's complexes [20]. The relevance for classical modular Deligne–Lusztig theory is considered in [4,1,11].

4.1. Conventions

Let X be a separated scheme of finite type over a separably closed field k. Let Λ be as in Setup 1.1. All cohomology groups are étale cohomology groups with compact support or their ℓ-adic version [24,17]. We denote them $H^\bullet_c(X, \Lambda)$.

Given a coefficient ring A as in Setup 1.26-(i), we denote $\mathrm{Sh}(X, A)$ the category of constructible étale sheaves of A-modules on X. We denote its bounded homotopy category by $\mathrm{K}^b(X, A)$ and its bounded derived category by $\mathrm{D}^b(X, A)$. By $\mathrm{R}\Gamma_c(X, -) : \mathrm{D}^b(X, A) \to \mathrm{D}^b(A)$, we denote the derived functor of compactly supported global sections. For a coefficient ring A as in Setup 1.1-(ii), we use $\mathrm{Sh}(X, A)$ for the corresponding category of ℓ-adic sheaves.

If $\mathcal{M} \subseteq \mathrm{Mod}(A)$ is an idempotent complete abelian subcategory, we denote $\mathrm{Sh}(X, \mathcal{M}) \subseteq \mathrm{Sh}(X, A)$ the full subcategory of sheaves with stalks in $\mathcal{M}$. We further denote $\mathrm{K}^b(X, \mathcal{M})$ and $\mathrm{D}^b(X, \mathcal{M})$ the subcategories of objects which can be represented by a complex with entries in $\mathrm{Sh}(X, \mathcal{M})$.

We write $\iota : \mathrm{K}^b(-) \to \mathrm{D}^b(-)$ for the canonical localization functor.

4.2. Rickard's complex

In order to effectively manipulate $\mathrm{R}\Gamma_c(X, \Lambda)$, we need to work with explicit representatives; in order to perform certain operation on $\mathrm{R}\Gamma_c(X, \Lambda)$, such a representative has to be sufficiently acyclic. Rickard [20] constructed a useful canonical representative $\mathrm{K}\Gamma_c(X, \Lambda)$ for $\mathrm{R}\Gamma_c(X, \Lambda)$ on the level of the homotopy category. See also [11].

Let X be a separated scheme of finite type over a field k and A a coefficient ring as in Setup 1.26. Rickard's construction gives a functor

$$\mathrm{K}\Gamma_c(X, -) : \mathrm{Sh}(X, A) \to \mathrm{K}^b(A)$$

with the following properties.

Properties 4.1 *(Rickard's complex).*

(1) *Rickard's functor* $\mathrm{K}\Gamma_c(X, -)$ *is additive and gives a functorial representative for* $\mathrm{R}\Gamma_c(X, -)$ *on the level of* $\mathrm{K}^b(A)$*. In other words,* $\mathrm{R}\Gamma_c(X, -) \cong \iota(\mathrm{K}\Gamma_c(X, -))$*.*
(2) *For any* $\mathscr{F} \in \mathrm{Sh}(X, A)$*, the complex* $\mathrm{K}\Gamma_c(X, \mathscr{F})$ *has finite type terms and is concentrated in degrees* $0, \ldots, 2\dim X$*. More precisely, it is concentrated in the interval of degrees where* $H^i_c(X, \mathscr{F}) \neq 0$*.*
(3) *If all stalks of* $\mathscr{F} \in \mathrm{Sh}(X, A)$ *lie in* $\mathrm{add}(M)$ *for some* $M \in \mathrm{mod}(A)$*, then also* $\mathrm{K}\Gamma_c(X, \mathscr{F})$ *lands in* $\mathrm{K}^b(\mathrm{add}(M))$*. Put in different words, if* $\mathcal{M}$ *is an additive idempotent complete subcategory of* $\mathrm{mod}(A)$*, then* $\mathrm{K}\Gamma_c(X, -)$ *restrict to a functor* $\mathrm{Sh}(X, \mathcal{M}) \to \mathrm{K}^b(\mathcal{M})$*.*
(4) *Rickard's complex* $\mathrm{K}\Gamma_c(-, -)$ *is functorial in the first variable in the following sense: a finite morphism of pairs* $(X, \mathscr{F}) \to (Y, \mathscr{G})$ *induces a map* $\mathrm{K}\Gamma_c(Y, \mathscr{G}) \to \mathrm{K}\Gamma_c(X, \mathscr{F})$*.*
(5) *If* $F : \mathrm{mod}(A) \to \mathrm{mod}(B)$ *is an additive functor, we have a canonical isomorphism*

$$F(\mathrm{K}\Gamma_c(X, \mathscr{F})) = \mathrm{K}\Gamma_c(X, F^{\mathrm{sh}}(\mathscr{F}))$$

inside $\mathrm{K}^b(B)$*. Here,* $F^{\mathrm{sh}}(\mathscr{F})$ *stands for the sheafification of the presheaf given by applying* F *sectionwise.*

Proof. See [20, Theorem 2.7. and Lemma 2.8.] for coefficients from Setup 1.26-(i). The extension to coefficients from Setup 1.26-(ii) is done as in [20, Theorem 3.5.] and further base change. □

Remark 4.2. Property (5) is quite strong. For instance – together with property (1) – it reveals that $\mathrm{R}\Gamma_c(X, F^{\mathrm{sh}}(\mathscr{F}))$ may be computed by naively applying F to the representing complex $\mathrm{K}\Gamma_c(X, \mathscr{F})$. Also, (5) shows that the construction of $\mathrm{K}\Gamma_c(X, -)$ is compatible with forgetful functors between module categories. Consequently, we can dismiss these forgetful maps from the notation.

Remark 4.3. Without any difficulty, one can extend $\mathrm{K}\Gamma_c(X,-)$ to a functor $\mathrm{K}^b(X,A)\to\mathrm{K}^b(A)$.

4.3. Finite group actions

Let X be a separated scheme of finite type over a separably closed field k. Consider an action of a finite group G on X.

Notation 4.4 *(Stack quotients).* We denote $\langle X/G\rangle$ the stack quotient of X by G. The conventions from §4.1 extend to algebraic stacks.

Notation 4.5 *(Coarse quotients).* We denote X/G the quotient of X by G in the sense of algebraic spaces. This gives a course moduli space for $\langle X/G\rangle$ by [14,21].

When X is a quasi-projective scheme over k, so is X/G. It can be constructed as follows: find a G-invariant affine covering of X (which exists by quasi-projectivity), take invariants $(-)^G$ on functions and then glue back [18, Appendix A]. This is the approach taken in [20,10]; it is sufficient for this paper.

A *G-equivariant sheaf* is an object $\mathscr{F}\in\mathrm{Sh}(X,\Lambda)$ equipped with isomorphisms $\alpha_g:\mathscr{F}\to(g^{-1})^*\mathscr{F}$ for each $g\in G$ satisfying the cocycle condition. More conceptually, this amounts to an object $\mathscr{F}\in\mathrm{Sh}(\langle X/G\rangle,\Lambda)$. In particular, $\mathrm{Sh}(\langle \mathrm{pt}/G\rangle,\Lambda)$ is the category of finite type G-representations; similarly for $\mathrm{D}^b(\langle \mathrm{pt}/G\rangle,\Lambda)$.

The cohomology of an equivariant sheaf carries a natural G-action: denoting $f:\langle X/G\rangle\to\langle \mathrm{pt}/G\rangle$ the canonical map, base change allows to identify

$$\mathrm{R}\Gamma_c(X,\mathscr{F})=Rf_!\,\mathscr{F}\in\mathrm{D}^b(G,\Lambda).$$

Taking the constant sheaf Λ with the trivial G-equivariant structure, we get $\mathrm{R}\Gamma_c(X,\Lambda)\in\mathrm{D}^b(G,\Lambda)$.

Here is an alternative way of computing this complex via the coarse quotient. Let $\pi:X\to X/G$ be the quotient map. Given a G-equivariant sheaf $\mathscr{F}$ in $\mathrm{Sh}(X,\Lambda)$, the pushforward $\pi_*\mathscr{F}$ naturally lives in $\mathrm{Sh}(X/G,\Lambda[G])$. Since π is finite, $R\pi_*=\pi_*=\pi_!=R\pi_!$ has no higher cohomology. Given any further map $h:X/G\to S$ to some scheme S, we deduce

$$R(h\circ\pi)_!=Rh_!R\pi_!$$

In particular,

$$\mathrm{R}\Gamma_c(X,\mathscr{F})=\mathrm{R}\Gamma_c(X/G,\pi_*\mathscr{F})\in\mathrm{D}^b(G,\Lambda).\tag{4.1}$$

Notation 4.6. Given a point $x\in X(k)$, we write $C_G(x)$ for its stabilizer under the G-action. Denoting $y=\pi(x)$ its image under the quotient map $\pi:X\to X/G$, the stalk of the pushforward of Λ identifies as $(\pi_*\Lambda)_y\cong\Lambda[G/C_G(x)]$.

Definition 4.7 *(Rickard's complex of a group action).* For a finite group G acting on X and a G-equivariant sheaf $\mathscr{F}$, denote

$$\mathrm{G}\Gamma_c(X, \mathscr{F}) := \mathrm{K}\Gamma_c(X/G, \pi_* \mathscr{F}) \in \mathrm{K}^b(G, \Lambda)$$

By §4.2, this is a complex of finite type $\Lambda[G]$-modules representing $\mathrm{R}\Gamma_c(X, \mathscr{F}) \in \mathrm{D}^b(G, \Lambda)$.

Lemma 4.8. *For a finite group G acting on X, the complex $\mathrm{G}\Gamma_c(X, \Lambda) \in \mathrm{K}^b(G, \Lambda)$ has terms in $\mathcal{M} = \mathrm{add}\{\Lambda[G/C_G(x)] \mid x \in X(k)\}$.*

Proof. This follows from the Property 4.1-(3), because the stalks of the relevant sheaf $\pi_*\Lambda$ on X/G are precisely of the form $\Lambda[G/C_G(x)]$. See [20, Theorems 3.2. and 3.5.]. □

Lemma 4.9. *Let Λ be as in Setup 1.1. Then on the level of bounded homotopy categories*

$$\mathrm{G}\Gamma_c(X, \Lambda) \cong \mathrm{G}\Gamma_c(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \Lambda.$$

Proof. This is an instance of Property 4.1-(5) for $F = (- \otimes_{\mathbb{Z}_\ell} \Lambda)$. □

Corollary 4.10 *(Euler characteristic and reduction). The map $r_\ell : \mathrm{G}_0(G, \overline{\mathbb{Q}}_\ell) \to \mathrm{G}_0(G, \overline{\mathbb{F}}_\ell)$ sends*

$$r_\ell : [H^\bullet_c(X, \overline{\mathbb{Q}}_\ell)] \mapsto [H^\bullet_c(X, \overline{\mathbb{F}}_\ell)]. \tag{4.2}$$

Proof. Indeed, this can be rewritten as $r_\ell : [\mathrm{G}\Gamma_c(X, \overline{\mathbb{Q}}_\ell)] \mapsto [\mathrm{G}\Gamma_c(X, \overline{\mathbb{F}}_\ell)]$. By Lemma 4.8, $\mathrm{G}\Gamma_c(X, \overline{\mathbb{Z}}_\ell)$ is a complex of torsionfree $\overline{\mathbb{Z}}_\ell$-modules; by Lemma 4.9 it thus gives an integral structure of $\mathrm{G}\Gamma_c(X, \overline{\mathbb{Q}}_\ell)$ which reduces to $\mathrm{G}\Gamma_c(X, \overline{\mathbb{F}}_\ell)$ modulo ℓ and the statement follows. □

Corollary 4.10 can be also checked directly without Rickard's complexes. In particular if G is trivial, it amounts to the standard equality of Betti numbers of X with $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$ coefficients [17, p. 166]. We will employ similar reasoning in Corollary 5.9 for derived isotypic parts.

4.4. Derived isotypic parts

Let Y be a separated scheme of finite type over a separably closed field k. Let G and H be finite groups. Suppose Y carries commuting actions of G and H. In other words, $G \times H$ act on Y and we get the complexes

$$\mathrm{G}\Gamma_c(Y, \Lambda) \in \mathrm{K}^b(G \times H, \Lambda) \qquad \text{and} \qquad \mathrm{R}\Gamma_c(Y, \Lambda) \in \mathrm{D}^b(G \times H, \Lambda).$$

We now want to make sense of the isotypic parts for the H-action on them.

Notation 4.11. For $\theta \in \mathrm{Rep}(H, \Lambda)$, we denote

$$(-)_\theta := (- \otimes_{\Lambda[H]} \theta) : \mathrm{Rep}(G \times H, \Lambda) \to \mathrm{Rep}(G, \Lambda)$$

the usual tensor product with θ. We use the same notation for the induced functor on homotopy categories

$$(-)_\theta := (- \otimes_{\Lambda[H]} \theta) : \mathrm{K}^b(G \times H, \Lambda) \to \mathrm{K}^b(G, \Lambda).$$

We call this $(-)_\theta$ the *isotypic part* of θ.

Notation 4.12. For $\theta \in \mathrm{Rep}(H, \Lambda)$, we write

$$(-)_{\mathrm{L}\,\theta} := (- \otimes^{\mathrm{L}}_{\Lambda[H]} \theta) : \mathrm{D}^b(G \times H, \Lambda) \to \mathrm{D}^b(G, \Lambda)$$

for the derived tensor product with θ. We call $(-)_{\mathrm{L}\,\theta}$ the *derived isotypic part* of θ.

It turns out that when H acts freely, the derived isotypic parts of $\mathrm{R}\Gamma_c(Y, \Lambda)$ may be computed as the non-derived isotypic parts of $\mathrm{G}\Gamma_c(Y, \Lambda)$.

Lemma 4.13. *Suppose Y carries commuting actions of finite groups G and H. Assume that H acts freely on Y. Let $\theta \in \mathrm{Rep}(G, \Lambda)^{\mathrm{ft}}$. Then on the level of $\mathrm{D}^b(G, \Lambda)$, we have*

$$\mathrm{R}\Gamma_c(Y, \Lambda)_{\mathrm{L}\,\theta} \cong \iota(\mathrm{G}\Gamma_c(Y, \Lambda)_\theta).$$

Proof. Denote $\rho : Y \to Y/(G \times H)$ the quotient map. We compute:

$$\begin{aligned}
\mathrm{R}\Gamma_c(Y, \Lambda)_{\mathrm{L}\,\theta} &= \mathrm{R}\Gamma_c(Y/(G \times H), \rho_*\Lambda) \otimes^{\mathrm{L}}_{\Lambda[H]} \theta \\
&= \mathrm{R}\Gamma_c(Y/(G \times H), \rho_*\Lambda \otimes^{\mathrm{L}}_{\Lambda[H]} \theta) \\
&= \mathrm{R}\Gamma_c(Y/(G \times H), \rho_*\Lambda \otimes_{\Lambda[H]} \theta) \\
&= \iota(\mathrm{K}\Gamma_c(Y/(G \times H), \rho_*\Lambda \otimes_{\Lambda[H]} \theta)) \\
&= \iota(\mathrm{K}\Gamma_c(Y/(G \times H), \rho_*\Lambda) \otimes_{\Lambda[H]} \theta) \\
&= \iota(\mathrm{G}\Gamma_c(Y, \Lambda)_\theta)
\end{aligned}$$

Indeed, the first equality is just the definition of $(-)_{\mathrm{L}\,\theta}$ together with (4.1). The second line is a general property of étale cohomology [24, pp. XVII, 5.2.9]. The third equality uses the freeness of the H-action on Y: each stalk of $\rho_*\Lambda$ is the permutation module on the corresponding $(G \times H)$-orbit inside Y, so in particular free as $\Lambda[H]$-module – we thus do not need to derive the tensor product. The fourth line holds by Property 4.1-(1), whereas the fifth one is Property 4.1-(5). The last line is again just the definition of $(-)_\theta$. □

Remark 4.14. Denoting $\pi : Y \to Y/H$ be the obvious quotient map, we can alternatively think of $\mathrm{R}\Gamma(Y, \Lambda)_{\mathrm{L}\,\theta}$ as the cohomology $\mathrm{R}\Gamma_c(Y/H, \Lambda_\theta)$ of the G-equivariant local system

$$\Lambda_\theta := \pi_* \Lambda \otimes_{\Lambda[T]} \theta$$

on Y/H.

Proof of Remark 4.14. This is clear say from the third or fourth line of the above computation: to conclude, one only needs to factor ρ as

$$Y \xrightarrow{\pi} Y/H \xrightarrow{\rho'} Y/(G \times H)$$

and identify the coefficient sheaves

$$\rho_* \Lambda \otimes_{\Lambda[H]} \theta = \rho'_* \pi_* \Lambda \otimes_{\Lambda[H]} \theta = \rho'_*(\pi_* \Lambda \otimes_{\Lambda[H]} \theta)$$

by commuting $(-)_\theta$ with the complementary pushforward on stalks. The isotypic part from Lemma 4.13 thus indeed identifies with $\mathrm{R}\Gamma_c(Y/H, \Lambda_\theta) = \mathrm{G}\Gamma_c(Y/H, \Lambda_\theta)$. □

5. p-adic Deligne–Lusztig spaces and their cohomology

In §5.1 we provide a brief account of the p-adic Deligne–Lusztig spaces of [13]. In §5.2 we specialize to the case of our interest for GL_n following [8,6]. In §5.3 we discuss étale cohomology of the truncated spaces $\dot{X}_h$ with coefficients in any Λ from Setup 1.1 and its derived isotypic parts. We use Rickard's complexes to control their behaviour with respect to reduction r_ℓ. In §5.4 we consider the isotypic parts of the cohomology of whole space $\dot{X}$. In §5.5 we show that this definition is independent of h up to an explicit even cohomological shift. In §5.6 we prove that it gives a well-defined class in the Grothendieck group of finite length representations for both $\Lambda = \overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$. During the process, we prove that these classes are compatible under the reduction r_ℓ. In §5.7 we review the partial realization of the local Langlands correspondence in characteristic zero via p-adic Deligne–Lusztig spaces of [8]. In §5.8 we deduce our main result – Theorem 5.25. In the complementary §5.9 we present a more naive version of this theorem, resulting in an extra multiplicity for small ℓ.

5.1. Overview of p-adic Deligne–Lusztig spaces

Let K be a local non-archimedean field as in §0.5. For a scheme X over K, the *loop space* LX is the presheaf on the opposite category of perfect k-algebras given by $LX : R \mapsto X(\mathbb{W}(R)[\varpi^{-1}])$. When X actually lives over $\mathcal{O}_K$, the *positive loop space* L^+X resp. the *truncated loop spaces* L_h^+X for $h \in \mathbb{N}$ defined as $L^+X : R \mapsto X(\mathbb{W}(R))$ resp. $L_h^+X : R \mapsto X(\mathbb{W}_h(R))$. See [29,13].

Let $\mathbf{G}$ be an unramified reductive group over K. Fix a maximal K-rational maximally split torus $\mathbf{T}$, which splits after an unramified extension, and a K-rational Borel $\mathbf{B} = \mathbf{UT}$ with unipotent radical $\mathbf{U}$. Let $\breve{K}$ be the maximal unramified extension of K and $\sigma \in \mathrm{Aut}(\breve{K}/K)$ the Frobenius lift. Then $\mathbf{G}$ and $\mathbf{T}$ split over $\breve{K}$ and we denote N resp. W the normalizer resp. the Weyl group of $\mathbf{T}_{\breve{K}}$ in $\mathbf{G}_{\breve{K}}$. The Bruhat decomposition yields a stratification of $\mathbf{G}/\mathbf{B} \times \mathbf{G}/\mathbf{B}$ into locally closed subspaces $\mathcal{O}(w)$, $w \in W$. The bigger space $\mathbf{G}/\mathbf{U} \times \mathbf{G}/\mathbf{U}$ can be decomposed into $\mathcal{O}(\dot{w})$, $\dot{w} \in N_G(T)$.

Fix $b \in \mathbf{G}(\breve{K})$, and $\dot{w} \in N$ resp. its image $w \in W$. The p-adic Deligne–Lusztig spaces are defined in [13, Definition 8.3.] as the following fiber products in presheaves on the opposite category of perfect algebras:

$$\begin{array}{ccc} \dot{X}_{\dot{w}}(b) & \longrightarrow & L(\mathcal{O}(\dot{w})) \\ \downarrow & & \downarrow \\ L(\mathbf{G}/\mathbf{U}) & \xrightarrow{\mathrm{id}\times b\sigma} & L(\mathbf{G}/\mathbf{U}) \times L(\mathbf{G}/\mathbf{U}) \end{array} \qquad \begin{array}{ccc} X_w(b) & \longrightarrow & L(\mathcal{O}(w)) \\ \downarrow & & \downarrow \\ L(\mathbf{G}/\mathbf{B}) & \xrightarrow{\mathrm{id}\times b\sigma} & L(\mathbf{G}/\mathbf{B}) \times L(\mathbf{G}/\mathbf{B}) \end{array} \tag{5.1}$$

They depend up to isomorphism only on the σ-conjugacy class $[b] := \{g^{-1}b\sigma(g) \mid g \in \mathbf{G}(\breve{K})\}$ of b. There is a natural map

$$\dot{X}_{\dot{w}}(b) \to X_w(b). \tag{5.2}$$

The spaces (5.2) carry actions of the profinite groups

$$G_b := \mathbf{G}_b(K) = \{g \in \mathbf{G}(\breve{K}) \mid g^{-1}b\sigma(g) = b\},$$
$$T_w := \mathbf{T}_w(K) = \{t \in \mathbf{T}(\breve{K}) \mid t^{-1}\dot{w}\sigma(t) = \dot{w}\}.$$

More precisely, both $\dot{X}_{\dot{w}}(b)$ and $X_w(b)$ carry an action of G_b such that (5.2) is G_b-equivariant [13, §8.2.1.]. Furthermore, $\dot{X}_{\dot{w}}(b)$ carries a T_w-action, commuting with the above G_b-action [13, §8.2.2.]. The map (5.2) is a T_w-torsor [13, Proposition 11.9.].

Given a reductive $\mathcal{O}_K$-model of $\mathbf{G}$, we can replace the loop space $L(-)$ in (5.1) by the integral and truncated loop spaces $L^+(-)$ and $L_h^+(-)$, $h \in \mathbb{N}$. This yields the presheaves

$$\dot{X}_{\dot{w}}(b)^+ \quad \text{and} \quad \dot{X}_{\dot{w}}(b)_h^+ \quad \text{resp.} \quad X_w(b)^+ \quad \text{and} \quad X_w(b)_h^+.$$

When $h = 1$, these truncated spaces are representable by the (perfections of) usual Deligne–Lusztig varieties. In this sense, the above definition generalizes the standard setting.

In many cases, p-adic Deligne–Lusztig spaces are ind-representable by perfect pfp schemes [13, Theorem 9.1., Corollary 9.2.]. Moreover, pfp perfect schemes have a well-behaved theory of étale sheaves equipped with six operations [29, Appendix A, §A.3.]; it agrees with the theory of étale sheaves on finite type models. In particular, compactly supported cohomology of such spaces is well-defined.

5.2. The case of our interest

From now on, we specialize to the case of split $\mathbf{G} = \mathbf{GL}_n$ over K and its hyperspecial parahoric model $\mathbf{G}_{\mathcal{O}} = \mathbf{GL}_n$ over $\mathcal{O}_K$. We put $b = 1$ and consider a Coxeter element $w \in W$, we take $\dot{w} \in N$ to be a lift of w with the same Kottwitz invariant as b. This is the most elementary case among those studied in [8,6,13]. For brevity, we denote

$$\dot{X} := \dot{X}_{\dot{w}}(b) \qquad \text{resp.} \qquad \dot{X}_{\mathcal{O}} := \dot{X}_{\dot{w}}(b)^{+} \qquad \text{resp.} \qquad \dot{X}_h := \dot{X}_{\dot{w}}(b)^{+}_{h},$$
$$X := X_w(b) \qquad \text{resp.} \qquad X_{\mathcal{O}} := X_w(b)^{+} \qquad \text{resp.} \qquad X_h := X_w(b)^{+}_{h}.$$

These spaces satisfy [8, §2.6.]

$$\dot{X} \cong \coprod_{G/G_{\mathcal{O}}} \dot{X}_{\mathcal{O}} \qquad \text{and} \qquad \dot{X}_{\mathcal{O}} \cong \lim_{h} \dot{X}_h. \tag{5.3}$$

In accordance with §5.1 they bear commuting actions of the following groups:

- $\dot{X}$ has commuting actions of the locally profinite groups $G = \mathbf{GL}_n(K)$ and $T = \mathbf{T}_w(K) = L^{\times}$;
- $\dot{X}_{\mathcal{O}}$ has commuting action of the profinite subgroups $G_{\mathcal{O}} = \mathbf{GL}_n(\mathcal{O}_K)$ and $T_{\mathcal{O}} = \mathcal{O}_L^{\times}$;
- $\dot{X}_h$, $h \in \mathbb{N}$ have commuting actions of the finite quotients

$$G_h = \mathbf{GL}_n(\mathcal{O}_K)/(1 + M_{n\times n}(\mathfrak{p}_K^h)) \qquad \text{and} \qquad T_h = \mathcal{O}_L^{\times}/T^h = \mathcal{O}_L^{\times}/(1 + \mathfrak{p}_L^h).$$

These actions are compatible with the isomorphisms (5.3) in the obvious way and the morphism (5.2) from $\dot{X}$ to X is equivariant.

Proposition 5.1 *([6, Proposition 7.4.]). The truncated spaces $\dot{X}_h$ for $h \in \mathbb{N}$ are representable by perfections of smooth affine $\overline{\mathbb{F}}_q$-schemes of finite type.*

Remark 5.2. In particular, there is a good notion of compactly supported étale cohomology for these finite level $\dot{X}_h$ and the discussion of §4 applies. The compactly supported cohomology of $\dot{X}_h$ is concentrated only in the range $[\dim \dot{X}_h, 2\dim \dot{X}_h]$ by affineness. The Poincaré duality between $H^{\bullet}$ and $H_c^{\bullet}$ holds by smoothness.

The discussion in §5.1 implies the following.

Proposition 5.3. *The T_h-action on $\dot{X}_h$ is free and the quotient map $\pi : \dot{X}_h \to X_h$ is a T_h-torsor.*

We now remind the reader about the notation $T_h^{h-1} = \ker(T_h \to T_{h-1})$ from §2.2. The maps $\dot{X}_h \to \dot{X}_{h-1}$ factor as follows.

Proposition 5.4 *([6, Proposition 7.7.]). The morphism $\dot{X}_h \to \dot{X}_{h-1}$ factors through a perfectly smooth map $\dot{X}_h/T_h^{h-1} \to \dot{X}_{h-1}$ whose fibers are isomorphic to the perfection of $\mathbb{A}^{n-1}$.*

5.3. Isotypic parts at finite level

Let us now discuss the equivariant cohomology of the truncated spaces $\dot{X}_h$, $h \in \mathbb{N}$ and their derived isotypic parts. These finite level phenomena play a crucial for the discussion of the cohomology of the whole space $\dot{X}$ in §5.4.

The spaces $\dot{X}_h$ and X_h are equipped with commuting actions of G_h and T_h. Their compactly supported cohomology is well-defined and inherits an action of $G_h \times T_h$. Appealing to §4.3 for some finite type models, we get the complexes

$$\mathrm{G}\Gamma_c(\dot{X}_h, \Lambda) \in \mathrm{K}^b(G_h \times T_h, \Lambda) \qquad \text{and} \qquad \mathrm{R}\Gamma_c(\dot{X}_h, \Lambda) \in \mathrm{D}^b(G_h \times T_h, \Lambda).$$

We now want to consider the isotypic parts for the T_h-action on them. In the semisimple case, this takes the form of a direct sum decomposition indexed by characters. In the non-semisimple case, one needs to be careful in choosing the right definition – see Remark 5.10. To not distract the reader, we simply present the correct notions: for any $\theta \in \mathrm{Hom}_{\mathrm{Grp}}(T_h, \Lambda^\times)$, the discussion from §4.4 gives the complexes

$$\mathrm{G}\Gamma_c(\dot{X}_h, \Lambda)_\theta \in \mathrm{K}^b(G_h, \Lambda) \qquad \text{and} \qquad \mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G_h, \Lambda).$$

In our situation, the derived isotypic parts of $\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)$ may be computed as the non-derived isotypic parts of $\mathrm{G}\Gamma_c(\dot{X}_h, \Lambda)$. This plays an important role in our comparison.

Lemma 5.5. *On the level of $\mathrm{D}^b(G_h, \Lambda)$, we have*

$$\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \cong \iota(\mathrm{G}\Gamma_c(\dot{X}_h, \Lambda)_\theta).$$

Proof. Since the action of T_h on $\dot{X}_h$ is free by Proposition 5.3, the claim follows from Lemma 4.13 applied to the action of $G_h \times T_h$ on $\dot{X}_h$. □

Remark 5.6. Note that this argument applies to other truncated Deligne–Lusztig spaces of §5.1.

Remark 5.7 *(Description via local systems).* Denoting $\pi : \dot{X}_h \to X_h$ the obvious quotient map, we can alternatively think of $\mathrm{R}\Gamma(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}$ as the cohomology $\mathrm{R}\Gamma_c(X_h, \Lambda_\theta)$ of the G_h-equivariant local system

$$\Lambda_\theta := \pi_*\Lambda \otimes_{\Lambda[T_h]} \theta$$

on $X_h = \dot{X}_h/T_h$. Indeed, this follows from Remark 4.14.

Lemma 5.5 allows to compute $\mathrm{R}\Gamma(\dot{X}_h,\Lambda)_{\mathrm{L}\,\theta}$ by applying the additive functor $(-)_\theta$ to the concrete representative $\mathrm{G}\Gamma(\dot{X}_h,\Lambda)$ in the homotopy category. This procedure is compatible with base changes of the coefficient ring Λ, so in particular with reduction modulo ℓ.

Notation 5.8. We abuse notation and write $[H_c^\bullet(\dot{X}_h,\Lambda)_{\mathrm{L}\,\theta}] := [\mathrm{R}\Gamma_c(\dot{X}_h,\Lambda)_{\mathrm{L}\,\theta}] \in \mathrm{G}_0(G_h,\Lambda)$.

Corollary 5.9. *The reduction map* $r_\ell : \mathrm{G}_0(G_h,\overline{\mathbb{Q}}_\ell) \to \mathrm{G}_0(G_h,\overline{\mathbb{F}}_\ell)$ *sends*

$$[H_c^\bullet(\dot{X}_h,\overline{\mathbb{Q}}_\ell)_\theta] \mapsto [H_c^\bullet(\dot{X}_h,\overline{\mathbb{F}}_\ell)_{\mathrm{L}\,r_\ell(\theta)}].$$

Proof. Let $\theta \in \mathrm{Irr}(T_h,\overline{\mathbb{Q}}_\ell)$ regarded as an element in $\mathrm{Rep}(T_h,\overline{\mathbb{Z}}_\ell)^{\mathrm{ft}}$; then $(\theta \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell) = r_\ell(\theta)$ by design. By Lemma 4.9, the complex $\mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Z}}_\ell)$ specializes to analogous complexes with $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{F}}_\ell$ coefficients under the obvious base changes. Tensoring with θ and using the associativity of the tensor product, we obtain the following identification.

$$\begin{array}{ccccc} \mathrm{K}^b(G_h,\overline{\mathbb{Q}}_\ell) & \xleftarrow{(-\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell)} & \mathrm{K}^b(G_h,\overline{\mathbb{Z}}_\ell) & \xrightarrow{(-\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell)} & \mathrm{K}^b(G_h,\overline{\mathbb{F}}_\ell) \\ \mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Q}}_\ell)_\theta & \longleftarrow\!\shortmid & \mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Z}}_\ell)_\theta & \longmapsto & \mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{F}}_\ell)_{r_\ell(\theta)} \end{array}$$

Since T_h acts freely on $\dot{X}_h$, the complex $\mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Z}}_\ell)$ actually consists of projective $\overline{\mathbb{Z}}_\ell[T_h]$-modules by Lemma 4.8. The map $\mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Z}}_\ell)_\theta \to \mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Q}}_\ell)_\theta$ is thus injective, so $\mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Z}}_\ell)_\theta$ defines an integral structure of $\mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{Q}}_\ell)_\theta$ which reduces to $\mathrm{G}\Gamma_c(\dot{X}_h,\overline{\mathbb{F}}_\ell)_{r_\ell(\theta)}$ modulo ℓ. □

Remark 5.10. When the category $\mathrm{rep}(T_h,\Lambda)$ is semisimple, the complexes $\mathrm{G}\Gamma(\dot{X}_h,\Lambda)$ decompose into a direct sum of isotypic components for the T_h-action and everything can be computed on the graded vector space $H_c^\bullet(\dot{X}_h,\Lambda)$. When $\mathrm{rep}(T_h,\Lambda)$ is not semisimple, this is not the case. Although one can still consider naive definitions of isotypic components on the graded vector space $H_c^\bullet(\dot{X}_h,\Lambda)$, they do not behave well with respect to geometric arguments. We briefly return to such naive definition in §5.9; this reveals a pathology in terms of an explicit multiplicity ℓ^m – see Theorem 5.29.

Remark 5.11. In the case $h = 1$, the space $\dot{X}_1$ is the perfection of the usual Deligne–Lusztig variety and $G_1 = \mathbf{GL}_n(\mathbb{F}_q)$. The complexes $\mathrm{R}\Gamma_c(\dot{X}_1,\Lambda)_{\mathrm{L}\,\theta}$ were studied in [4,1]; they play an important role in the structure of $\mathrm{D}^b(G_1,\overline{\mathbb{F}}_\ell)$.

5.4. Isotypic parts for whole Deligne–Lusztig spaces

The goal of this section is to introduce the isotypic parts of cohomology of the whole Deligne–Lusztig varieties $\dot{X}$. More precisely, we provide a definition of the derived isotypic components

$$\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G, \Lambda),$$

where Λ is a coefficient ring and $\theta \in \mathrm{Hom}_{\mathrm{Grp}}(T, \Lambda^\times)$.

Since $\dot{X}$ is infinite-dimensional, the compactly supported cohomology does not a priori make sense. The actual construction proceeds as in [8] – we take the cohomology at a finite level $\dot{X}_h$ for sufficiently big $h \geq 1$, and construct the cohomology of $\dot{X}$ out of it by an explicit representation-theoretic procedure.

Take $\dot{X}$, $\dot{X}_{\mathcal{O}}$, $\dot{X}_h$ resp. their quotients X, $X_{\mathcal{O}}$, X_h as in §5.2 and Λ as in Setup 1.1.

Construction 5.12. Let $\theta \in \mathrm{Hom}_{\mathrm{Grp}}(T, \Lambda^\times)$ and $h \geq \mathrm{lvl}(\theta)$. Then we define

$$\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta} := \mathrm{cInd}_{ZG_{\mathcal{O}}}^{G} \mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G, \Lambda)$$

where $\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}$ is regarded as an element of $\mathrm{D}^b(ZG_{\mathcal{O}}, \Lambda)$ by inflation to $G_{\mathcal{O}}$ and by letting the center Z of G act via θ. This is well-defined up to an even cohomological shift.

Discussion 5.13. Let us expand the construction in words and point out what needs to be checked. The restriction of the smooth character θ to $T_{\mathcal{O}}$ factors through any T_h with $h \geq \mathrm{lvl}(\theta)$. By §5.3, we get the derived isotypic part

$$\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G_h, \Lambda).$$

We will show in Lemma 5.15 that the equivariant tower of the finite level Deligne–Lusztig spaces $\dot{X}_h$ is *representation stable*; the derived complex above is thus independent of the choice of $h \geq \mathrm{lvl}(\theta)$ up to an even cohomological shift.

Having a good notion of derived isotypic parts at finite level, we regard $\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}$ as an object of $\mathrm{D}^b(G_{\mathcal{O}}, \Lambda)$ by inflating along the quotient map $G_{\mathcal{O}} \to G_h$. This makes good sense, because the inflation is exact. Furthermore, we upgrade this to an object of $\mathrm{D}^b(ZG_{\mathcal{O}}, \Lambda)$ by letting the center Z act via the desired character θ of T.

Finally, we obtain the sought-for complex

$$\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G, \Lambda)$$

by applying the functor $\mathrm{cInd}_{ZG_{\mathcal{O}}}^{G}$ of compact induction from the clopen subgroup $ZG_{\mathcal{O}}$. This makes good sense, because $\mathrm{cInd}_{ZG_{\mathcal{O}}}^{G}$ is exact by §1.2.

We have completely ignored the question of finiteness throughout the discussion. However, we will eventually need some sort of finiteness to have a good notion of Euler characteristic. We will discuss this thoroughly in §5.6. For the moment, we record one elementary lemma in this direction.

Lemma 5.14. *Let $\theta \in \mathrm{Hom}_{\mathrm{Grp}}(T, \Lambda^\times)$. Then $\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G, \Lambda)$ is of finite type.*

Proof. At the finite level, $\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta_h}$ is of finite type in $\mathrm{D}^b(G_h, \Lambda)$ by Lemma 4.8. This clearly remains true when we inflate to $G_{\mathcal{O}}$ and when we let Z act via θ, simply because we add more structure to the same complex of Λ-modules. Finally, compact induction from the clopen subgroup $ZG_{\mathcal{O}}$ preserves finite type by Lemma 1.5, so that

$$\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta} = \mathrm{cInd}_{ZG_{\mathcal{O}}}^{G} \mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(G, \Lambda)$$

is of finite type: it can be represented by a bounded complex with finite type terms. □

5.5. Representation stability of the truncated Deligne–Lusztig spaces

To provide the missing input for Construction 5.12, we discuss the representation stability of the sequence of spaces $\dot{X}_h$. This works up to an explicit even cohomological shift.

Lemma 5.15. *Let $\theta \in \mathrm{Hom}_{\mathrm{Grp}}(T, \Lambda^\times)$ and $h \geq h' \geq \mathrm{lvl}(\theta)$. Then we have a canonical identification*

$$\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta} \cong \mathrm{R}\Gamma_c(\dot{X}_{h'}, \Lambda)_{\mathrm{L}\,\theta}[2(n-1)(h'-h)] \in \mathrm{D}^b(G_h, \Lambda).$$

Proof. From Propositions 5.3 and 5.4 we obtain a commutative diagram of schemes

$$\begin{array}{ccccc} \dot{X}_h & \xrightarrow{g} & \dot{X}_h/T_h^{h-1} & \xrightarrow{f} & \dot{X}_{h-1} \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi''} \\ & & X_h & \xrightarrow{\overline{f}} & X_{h-1} \end{array}$$

The fibers of f are perfections of affine spaces $\mathbb{A}^{n-1}$ over the base field $\overline{\mathbb{F}}_q$. At the same time, the maps g and π, resp. π', π'' are finite quotient maps for the group actions of T_h^{h-1} and T_h resp. T_{h-1}. For any $\theta \in \mathrm{Rep}(T_{h'}, \Lambda)^{\mathrm{ft}} \subseteq \mathrm{Rep}(T_h, \Lambda)^{\mathrm{ft}}$, we can compute inside $\mathrm{D}^b(G_h, \Lambda)$ as follows.

$$\begin{aligned} \mathrm{R}\Gamma_c(\dot{X}_h, \Lambda) \otimes^{\mathrm{L}}_{\Lambda[T_h]} \theta &\cong \mathrm{R}\Gamma_c(X_h, \pi_*\Lambda) \otimes^{\mathrm{L}}_{\Lambda[T_h]} \theta \\ &\cong \mathrm{R}\Gamma_c(X_h, \pi_*\Lambda) \otimes^{\mathrm{L}}_{\Lambda[T_h]} \Lambda[T_{h-1}] \otimes^{\mathrm{L}}_{\Lambda[T_{h-1}]} \theta \\ &\cong \mathrm{R}\Gamma_c(X_h, \pi_*\Lambda \otimes^{\mathrm{L}}_{\Lambda[T_h]} \Lambda[T_{h-1}]) \otimes^{\mathrm{L}}_{\Lambda[T_{h-1}]} \theta \end{aligned}$$

$$\cong \mathrm{R}\Gamma_c(X_h, \pi'_*\Lambda) \otimes^{\mathrm{L}}_{\Lambda[T_{h-1}]} \theta$$
$$\cong \mathrm{R}\Gamma_c(\dot{X}_h/T_h^{h-1}, \Lambda) \otimes^{\mathrm{L}}_{\Lambda[T_{h-1}]} \theta$$
$$\cong (\mathrm{R}\Gamma_c(\dot{X}_{h-1}, \Lambda) \otimes^{\mathrm{L}}_{\Lambda[T_{h-1}]} \theta)[-2(n-1)]$$

The first equality holds by (4.1). The second is just the abusive identification of θ as representation of either T_h or T_{h-1}; the second tensor product is actually nonderived. The third line holds by the projection formula for étale cohomology.

For the fourth line, we need to see that $\pi_*\Lambda \otimes^{\mathrm{L}}_{\Lambda[T_h]} \Lambda[T_{h-1}] \cong \pi'_*\Lambda$. Here, we note that $\Lambda[T_{h-1}] \cong \Lambda[T_h/T_h^{h-1}]$ is projective as $\Lambda[T_h]$-module by Lemma 1.25 as T_h^{h-1} is a p-group for $h \geq 2$. The desired isomorphism can be checked on stalks; by the above projectivity it reduces to the nonderived formula

$$\Lambda[T_h/C_{T_h}(x)] \otimes_{\Lambda[T_h]} \Lambda[T_{h-1}] \cong \Lambda[T_{h-1}/C_{T_{h-1}}(x)],$$

which is clear. (Here, x is any geometric point of X_h and $C_{T_h}(x)$ is its stabilizer with respect to our character θ.) The fifth line is then obtained in the same way as the first.

The sixth line holds because f is a smooth map with fibers perfections of $\mathbb{A}^{n-1}$. Indeed, one can use smooth base change together with the canonical local identifications of the form $\mathrm{R}\Gamma_c(V \times \mathbb{A}^{n-1}, \Lambda) \cong \mathrm{R}\Gamma_c(V, \Lambda) \otimes^{\mathrm{L}}_{\Lambda} \mathrm{R}\Gamma_c(\mathbb{A}^{n-1}, \Lambda) \cong \mathrm{R}\Gamma_c(V, \Lambda)[-2(n-1)]$. One should observe that this is compatible with the T_h-actions. □

Remark 5.16. In the fourth step, we employed the projectivity of $\Lambda[T_{h-1}]$. Alternatively, we can use that the T_h-action on $\dot{X}_h$ is free, showing the stalkwise projectivity of the other factor $\pi_*\Lambda$.

Remark 5.17. The above stabilization holds only up to an even degree shift – as remarked in [8, p. 10], one can formally get rid of this shift. In the current context when all $\dot{X}_h$ are perfectly smooth, this is simply achieved by taking cohomology $\mathrm{R}\Gamma(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}$ without the compact support condition.

5.6. Finiteness and Euler characteristic

Construction 5.12 is given by a composition of several exact functors. It follows that whenever it preserves finite length of Euler characteristic, it specializes to the level of Grothendieck groups.

Construction 5.18. Let $h \geq \mathrm{lvl}(\theta)$ and assume that $\mathrm{cInd}^G_{ZG_{\mathcal{O}}}[\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}]$ is a formal linear combination of finite length representations. Then it gives a well-defined element of $\mathrm{G}_0(G, \Lambda)$ which we denote

$$[\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta}] := \mathrm{cInd}^G_{ZG_{\mathcal{O}}}[\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}] \in \mathrm{G}_0(G, \Lambda).$$

We alternatively denote this element by $[H_c^\bullet(\dot{X}, \Lambda)_{\mathrm{L}\,\theta}]$. It is independent of the choice of $h \geq \mathrm{lvl}(\theta)$ by Lemma 5.15.

Beware that by our conventions, $\mathrm{G}_0(G, \Lambda)$ is the Grothendieck group of smooth *finite length* representations. A priori, we only know that $\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta}$ is of *finite type* by Lemma 5.14. This necessitates in the extra assumption that $\mathrm{cInd}_{ZG_{\mathcal{O}}}^G[\mathrm{R}\Gamma_c(\dot{X}_h, \Lambda)_{\mathrm{L}\,\theta}]$ is a formal linear combination of finite length representations. Whenever $\mathrm{R}\Gamma_c(\dot{X}, \Lambda)_{\mathrm{L}\,\theta} \in \mathrm{D}^b(\dot{X}, \Lambda)$ can be represented by a complex with finite length entries, its Euler characteristic gives rightaway an element of $\mathrm{G}_0(G, \Lambda)$ which agrees with Construction 5.18.

We now explain why Construction 5.18 applies to the cases considered in this paper – see §5.6.1 for $\Lambda = \overline{\mathbb{Q}}_\ell$ and §5.6.2 for $\Lambda = \overline{\mathbb{F}}_\ell$.

5.6.1. Characteristic zero coefficients

Employing results of [28] and [8], we now show that Construction 5.18 applies for $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_{\mathrm{L}\,\theta}]$ when θ lies in general position. At the same time we keep track of integral structures. This will imply the same finiteness for $[\mathrm{R}\Gamma_c(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\theta}]$ via reduction r_ℓ in §5.6.2.

Since we work in characteristic zero, we can work non-derived. The main input is the following.

Proposition 5.19 *([8, Theorem 5.1.]). Let $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)$. Then $\pm[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ is admissible.*

We now make the promised discussion; we also address the question of integrality.

Proposition 5.20. *For any $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}}$ we have a well-defined element*

$$[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta] \in \mathrm{G}_0(G, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}}.$$

Proof. Denote $h = \mathrm{lvl}(\theta)$, so that $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta] = \mathrm{cInd}_{ZG_{\mathcal{O}}}^G[H_c^\bullet(\dot{X}_h, \overline{\mathbb{Q}}_\ell)_\theta]$ holds by definition. We want to prove that this lands in finite length representations, and that it is integral.

Let us first discuss why $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ is of finite length. By Lemma 5.14 we already know that $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ is of finite type. We now use results of [8], which are valid under the assumption that θ lies in general position. By [8, Corollary 3.3], the finite level class $[H_c^\bullet(\dot{X}_h, \overline{\mathbb{Q}}_\ell)_\theta]$ is up to sign an irreducible representation, which we call V for the moment. By Proposition 5.19, $\mathrm{cInd}_{ZG_O}^G V$ is a finite direct sum of irreducible supercuspidal representations, i.e. it is admissible. Now by [28, p. II.5.10] admissibility and finite type together imply finite length, as desired.

Secondly, we address the integrality of $H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta$. Again, $H_c^\bullet(\dot{X}_h, \overline{\mathbb{Q}}_\ell)_\theta$ is obviously an integral G_h-representation. The inflation to $G_{\mathcal{O}}$ clearly preserves integrality. Also prolonging to $ZG_{\mathcal{O}}$ by θ doesn't hurt the choice of integral structure by the integrality assumption on θ. Finally, $\mathrm{cInd}_{ZG_{\mathcal{O}}}^{G}$ sends an integral structure of the original representation to an integral structure of the induced representation by the clopeness of $ZG_{\mathcal{O}}$ and Lemma 1.14. □

5.6.2. Characteristic ℓ coefficients

We now show that Construction 5.18 applies for $[H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}]$ with ψ in general position. At the same time, we check that this is given by $r_\ell[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ for any lift θ of ψ.

Lemma 5.21. *Let $\psi \in \mathscr{X}(L, \overline{\mathbb{F}}_\ell)$ and $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}}$ one of its lifts under r_ℓ. Then*

$$r_\ell[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta] = [H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}] \in \mathrm{G}_0(G, \overline{\mathbb{F}}_\ell).$$

Proof. By Lemma 2.2 and subsequent discussion, $\psi|_{T_{\mathcal{O}}}$ has precisely ℓ^m lifts under r_ℓ, each of which can be non-uniquely extended to a desired lift θ from the statement. Such a lift has the same level $h = \mathrm{lvl}(\psi) = \mathrm{lvl}(\theta)$. At this finite level, Corollary 5.9 yields an equality

$$r_\ell[H_c^\bullet(\dot{X}_h, \overline{\mathbb{Q}}_\ell)_\theta] = [H_c^\bullet(\dot{X}_h, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}].$$

Now we go through Construction 5.18 of the Euler characteristic of $\dot{X}$. First we inflate to $G_{\mathcal{O}}$; this operation is additive and commutes with r_ℓ. Then we extend to $ZG_{\mathcal{O}}$-representations by letting Z act via ψ; this is again additive and can be commuted through r_ℓ to the action of θ on the left-hand side.

Finally, we apply $\mathrm{cInd}_{ZG_{\mathcal{O}}}^{G}$ to both sides; this is additive and commutes with r_ℓ. More precisely, Lemma 1.22 simultaneously shows that Construction 5.18 applies to $[H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}]$ and that

$$r_\ell[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta] = [H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}].$$

Indeed, during the entire computation, we stayed in the Grothendieck groups of finite length integral representations. Up to the final step, this is clear by the choice of θ and finite dimensionality of the representations. Finally, $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ is integral of finite length by Proposition 5.20. In particular, the application of Lemma 1.22 is correct for the $ZG_{\mathcal{O}}$-representation constructed above; the reduction maps r_ℓ indeed land in the correct targets by Discussion 1.19 (iii) resp. (ii). □

5.7. Realization of the characteristic zero local Langlands correspondence

We briefly summarize the main results of [6,8,7,13] regarding the relationship of the isotypic parts $[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ to the local Langlands correspondence with $\overline{\mathbb{Q}}_\ell$-coefficients.

Let $\theta \in \mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)$. The Howe decomposition asserts the existence of a unique tower of subfields $K = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{t-1} \subsetneq L_t = L$ such that

$$\theta = (\theta_0 \circ N_{L/L_0}) \cdot (\theta_1 \circ N_{L/L_1}) \cdots (\theta_0 \circ N_{L/L_{t-1}}) \cdot (\theta_t)$$

for some characters $\theta_i : L_i^\times \to \overline{\mathbb{Q}}_\ell$ of unique levels. From this, we get numerical invariants of θ. Regarding θ as an element of $\mathrm{Irr}(T_h, \overline{\mathbb{Q}}_\ell)$ for a fixed $h \geq \mathrm{lvl}(\theta)$, [7, Theorem 6.1.1.] defines an integer $\mathrm{cd}(\theta)$ (called r_χ there) by an explicit formula dependent only on the group G and $\theta|_{T^1}$.

Proposition 5.22. *The function* $\mathrm{cd} : \mathrm{Irr}(T_h, \overline{\mathbb{Q}}_\ell) \to \mathbb{Z}$ *factors through the reduction* r_ℓ : $\mathrm{Irr}(T_h, \overline{\mathbb{Q}}_\ell) \to \mathrm{Irr}(T_h, \overline{\mathbb{F}}_\ell)$*. We use the same notation* $\mathrm{cd} : \mathrm{Irr}(T_h, \overline{\mathbb{F}}_\ell) \to \mathbb{Z}$ *for this induced map.*

Proof. By the discussion in §2.2, the lifts of given $\psi \in \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)$ along r_ℓ differ at most by a character of $T_1 = \mathcal{O}_L^\times / T^1$, i.e. the restrictions of all these lifts to T^1 are equal. The statement now follows from the definition of cd discussed above. □

Theorem 5.23 *([8]). Assume $p > n$. Let $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg}}$. Then $M(\theta) := (-1)^{\mathrm{cd}(\theta)} \cdot [H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$ is an irreducible supercuspidal representation of G and the association $\sigma(\theta) \leftrightarrow M(\theta)$ is a partial realization of the local Langlands correspondence.*

Proof. Up to the explicit determination of the sign $\pm[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$, this is [8, Theorem A].

The sign is discussed in the course of [8, §7.1.]. By [7, Theorem 6.1.1.], we get the definition of cd, which returns the single nonzero degree of the $\theta|_{T^1}$-isotypic component of the cohomology of certain closed subscheme X_h^1 of our $\dot{X}_h$. Therefore it gives also the single nonzero degree of the θ-isotypic component of another closed subscheme $X_{h,n'}$, which is given by a finite disjoint union of copies of X_h^1, see [7, §2]. By [8, Corollary 4.2.], there is an identification $[H_c^\bullet(X_{h,n'}, \overline{\mathbb{Q}}_\ell)_\theta] = [H_c^\bullet(\dot{X}_h, \overline{\mathbb{Q}}_\ell)_\theta]$ inside $\mathrm{G}_0(G_h, \overline{\mathbb{Q}}_\ell)$ and consequently $(-1)^{\mathrm{cd}(\theta)} \cdot [H_c^\bullet(\dot{X}_h, \overline{\mathbb{Q}}_\ell)_\theta]$ is indeed an (irreducible) representation; the same sign then appears for cohomology of the whole $\dot{X}$ by design. □

Remark 5.24. It is expected that this holds for any $\theta \in \mathscr{X}(L, \overline{\mathbb{Q}}_\ell)$, i.e. after replacing "strongly general position" by "general position".

5.8. Main result

We are now ready to piece everything together and deduce our main result.

Theorem 5.25. *Assume $p > n$. Let $\psi \in \mathscr{X}(L, \overline{\mathbb{F}}_\ell)^{\mathrm{sg}}$. Then:*

(1) the Weil representation $\sigma(\psi)$ is irreducible,
(2) the representation $M(\psi) := (-1)^{\mathrm{cd}(\psi)} \cdot [H_c^\bullet(X, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}]$ is irreducible supercuspidal,
(3) the association $\sigma(\psi) \longleftrightarrow M(\psi)$ partially realizes the modular local Langlands correspondence.

Proof. Fix an integral lift $\theta \in \mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)^{\mathrm{sg,int}}$ of $\psi \in \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)^{\mathrm{sg}}$ as in Lemma 2.1; such θ lies in strongly general position by Lemma 2.7. As ψ in particular lies in general position, Proposition 3.2 shows that $\sigma(\psi)$ is irreducible. By Lemma 3.3 we have $r_\ell(\sigma(\theta)) = \sigma(\psi)$.

Using the assumption $p > n$, Theorem 5.23 applies to the strongly general character θ. Thus $M(\theta) := \pm[H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)_\theta]$ is an irreducible supercuspidal representation of G and the sign identifies as $(-1)^{\mathrm{cd}(\theta)}$. By Lemma 5.21, reduction modulo ℓ returns the modular representation in question:

$$r_\ell\left(M(\theta)\right) = (-1)^{\mathrm{cd}(\theta)} \cdot r_\ell[H_c^\bullet(X, \overline{\mathbb{Q}}_\ell)_\theta] = (-1)^{\mathrm{cd}(\psi)} \cdot [H_c^\bullet(X, \overline{\mathbb{F}}_\ell)_{\mathrm{L}\,\psi}] = M(\psi). \tag{5.4}$$

Moreover, still by Theorem 5.23, the association

$$\sigma(\theta) \leftarrow\!\shortmid \theta \mapsto M(\theta) = (-1)^{\mathrm{cd}(\theta)} \cdot [H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)_\theta]$$

realizes the local Langlands correspondence over $\overline{\mathbb{Q}}_\ell$.

Now Theorem 3.1 applies: since $\sigma(\theta)$ is sent by the local Langlands correspondence to $M(\theta)$, its irreducible reduction $\sigma(\psi)$ is sent by the modular local Langlands correspondence to $r_\ell(M(\theta))$, which is thus irreducible supercuspidal. Since $r_\ell(M(\theta)) = M(\psi)$ by (5.4), the proof is finished. □

5.9. A version with multiplicities

In this complementary section, we present a variation on Theorem 5.25 coming from a naive definition of isotypic parts. The proof of this version goes along the same lines as the proof of Theorem 5.25.

When $\ell \nmid |T_1|$, the category $\mathrm{rep}(T_h, \Lambda)$ is semisimple and these naive isotypic parts coincide with the derived ones. When $\ell \mid |T_1|$, an extra multiplicity ℓ^m appears in our theorem. This discrepancy reveals that the complexes $\mathrm{G}\Gamma_c(\dot{X}_h, \overline{\mathbb{F}}_\ell)$ contain interesting extensions. The reader may compare this with the results of [4,1].

Notation 5.26. Let Λ be an algebraically closed field and G, G' two finite groups. Then there is a bijection [9, Theorem 10.33]

$$\mathrm{Irr}(G, \Lambda) \times \mathrm{Irr}(G', \Lambda) \xrightarrow{\cong} \mathrm{Irr}(G \times G', \Lambda), \qquad (V, V') \mapsto V \otimes_\Lambda V'$$

inducing canonical isomorphism of free abelian groups $\mathrm{G}_0(G \times G', \Lambda) \cong \mathrm{G}_0(G, \Lambda) \otimes_{\mathbb{Z}} \mathrm{G}_0(G', \Lambda)$.

We equip each G_0 with the scalar product making the classes of irreducible representations into an orthonormal basis. For any $V' \in \mathrm{Irr}(G', \Lambda)$ we then have the orthonormal projection

$$(-)[V'] : \mathrm{G}_0(G \times G', \Lambda) \to \mathrm{G}_0(G, \Lambda), \qquad [M] \mapsto [M][V']$$

We call it the *naive V'-isotypic part.*

Considering this for the groups $G_h \times T_h$ of our interest, we now address the compatibility of naive isotypic parts with the reduction map r_ℓ.

Lemma 5.27. *Let $M \in \mathrm{G}_0(G_h \times T_h, \overline{\mathbb{Q}}_\ell)$. Take $\psi \in \mathrm{Irr}(T_h, \overline{\mathbb{F}}_\ell)$ and let $\theta_i \in \mathrm{Irr}(T_h, \overline{\mathbb{Q}}_\ell)$ for $i \in I = \{1, \ldots, \ell^m\}$ be its finitely many lifts. Then*

$$\sum_{i \in I} r_\ell\left(M[\theta_i]\right) = r_\ell(M)[\psi].$$

Proof. The map r_ℓ is additive, as well as the formation of an isotypic part. Hence both sides of the equality are additive in M. So we only need to check the equality on the generators $[V \otimes_\Lambda V']$ from above, where it obviously holds. □

Remark 5.28. Altogether, one can define the naive isotypic parts of the cohomology of $\dot{X}$ as in Construction 5.18: for $\theta \in \mathrm{Irr}(T, \Lambda)$ and $h \geq \mathrm{lvl}(\theta)$, this naive isotypic part is given by

$$[H_c^\bullet(\dot{X}, \Lambda)][\theta] := \mathrm{cInd}_{ZG_\mathcal{O}}^{G} [H_c^\bullet(\dot{X}_h, \Lambda)][\theta] \in \mathrm{G}_0(G, \Lambda).$$

The well-definedness of this variant can be bootstrapped from what we've already seen. For instance, consider the independence on h. We know this holds for $\Lambda = \overline{\mathbb{Q}}_\ell$, because the naive and derived isotypic parts coincide. The case of $\Lambda = \overline{\mathbb{F}}_\ell$ then follows by reduction modulo ℓ through Corollary 4.10. Similar reasoning together with Lemma 5.27 shows that $[H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)][\theta]$ lands in the correct Grothendieck group $\mathrm{G}_0(G, \overline{\mathbb{F}}_\ell)$.

Replacing derived isotypic parts by the naive ones, our main theorem transforms as follows.

Theorem 5.29. *In the setting of Theorem 5.25, the irreducible supercuspidal representation $M(\psi)$ is given by*

$$[M(\psi)] = (-1)^{\mathrm{cd}(\psi)} \cdot \frac{1}{\ell^m} \cdot [H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)][\psi].$$

In other words, $[H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)][\psi] \in \mathrm{G}_0(G, \overline{\mathbb{F}}_\ell)$ is an ℓ^m multiple of the class of $M(\psi)$. When ℓ doesn't divide $|T_1| = q^n - 1$, the multiplicity is 1; this is the case when $\mathrm{rep}(T_h, \overline{\mathbb{F}}_\ell)$ is semisimple. When ℓ divides $|T_1|$, this multiplicity becomes nontrivial. This discrepancy shows that the complexes $\mathrm{R\Gamma}(\dot{X}_h, \overline{\mathbb{F}}_\ell)$ and $\mathrm{G\Gamma}(\dot{X}_h, \overline{\mathbb{F}}_\ell)$ contain interesting information for small ℓ.

Proof. Denote by θ_i for $i = 1, \ldots, \ell^m$ lifts of ψ under $r_\ell :\ \mathrm{Irr}(T, \overline{\mathbb{Q}}_\ell)^{\mathrm{int}} \to \mathrm{Irr}(T, \overline{\mathbb{F}}_\ell)$, whose restrictions to $T_\mathcal{O}$ are pairwise distinct as in Remark 2.3.

The proof of Theorem 5.25 applies for each of these θ_i separately, giving irreducible supercuspidal representations $M(\theta_i) := (-1)^{\mathrm{cd}(\psi)} \cdot [H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)][\theta_i]$ of G – indeed, the naive and derived isotypic parts coincide here. By the arguments in that proof, each of these representations reduces to $M(\psi)$ modulo ℓ, which is independent of $i = 1, \ldots, \ell^m$. By Lemma 5.27 and additivity of r_ℓ we infer

$$(-1)^{\mathrm{cd}(\psi)} \cdot [H_c^\bullet(\dot{X}, \overline{\mathbb{F}}_\ell)][\psi] = \sum_{i=1}^{\ell^m} r_\ell((-1)^{\mathrm{cd}(\psi)}[H_c^\bullet(\dot{X}, \overline{\mathbb{Q}}_\ell)][\theta_i]) = \sum_{i=1}^{\ell^m} r_\ell[M(\theta_i)]$$
$$= \ell^m \cdot [M(\psi)]. \quad \square$$

Remark 5.30. This naive version can be deduced without Rickard's complexes. Indeed, only Corollary 4.10 is needed and this can be proved directly.

Data availability

No data was used for the research described in the article.

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