Books, Hallways, and Social Butterflies: A Note on Sliding Block Puzzles

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Interest in sliding block puzzles dates back to the 15-puzzle, seemingly invented by Noyes Chapman in 1874 (see [23] for an account of the fascinating history of the puzzle). The game consists of fifteen movable square blocks numbered 1, 2, … , 15 and arranged within a 4×4 square box, leaving one empty space (see Figure 1). The task at hand is to start from a given configuration of the numbered blocks and reach the desired target configuration, where the only allowed move is to slide a numbered block into an adjacent empty space. This task seemed to be unpredictably either very easy to accomplish, or completely impossible, and the puzzle turned into a worldwide sensation in the spring of 1880. A particularly challenging instance, known as the 13-15-14 puzzle, consisted of initial and target configurations that differed by a single swap (historically this swap involved the blocks labeled 14 and 15). The craze of this puzzle was such that it consistently made newspaper headlines in 1880, with an article in the *New York Times* lamenting that it was "threatening our free institutions" [23, p. 9]. Various prizes were offered for anyone who could solve this challenge, beginning with a $25 set of teeth and culminating with Sam Loyd's famous $1,000 cash prize.

The notorious 13-15-14 challenge is now known to be impossible. The earliest mathematical accounts were published shortly after the puzzle became a sensation. Johnson and Storey (in tandem) [14] and Schubert [22] independently realized that every configuration can be assigned a parity, and only configurations of the same parity can be reached from each other.

It is natural to wonder how much of this analysis is dependent on the particular setup of the 15-puzzle. Indeed, over the years, many versions and extensions of the 15-puzzle have been proposed and studied, drawing the attention of those in both the recreational and academic communities (including Knuth and Coxeter). See, for example, [2, 3, 6, 10, 11, 20, 24, 27, 28], [23, Chapter 9 and pp. 114–115 of Chapter 7], and [12, B60, E19, E20]. In Figure 2, we single out a particular variation on the 15-puzzle, which we shall reuse later on in a few examples.

Perhaps most famously, in 1974, Wilson [27] generalized the 15-puzzle to the setting in which blocks may be slid around on an arbitrary graph. Specifically, we can consider a graph $\mathcal{G}$ and place a numbered block on each but one of the vertices. Then the only legal move is to move a block onto an empty adjacent vertex. Wilson gave a complete characterization of the graphs $\mathcal{G}$ for which the corresponding configuration space is connected (in the language of graph theory, $\mathcal{G}$ should be biconnected and nonbipartite, and not equal to a particular 7-vertex graph called θ_0; see Figure 7).

Wilson's theorem is not entirely the end of the story. As we have discussed, the 15-puzzle does not have a connected configuration space (its corresponding graph G is bipartite). However, if we remove an additional block, leaving two empty spaces, then it is easy to show that the configuration space becomes connected. In general, for a given graph it is unclear exactly how much more power is granted by the addition of more empty spaces. In this paper, we set out to understand the full picture for sliding block puzzles in general graphs with an arbitrary number of empty spaces (see, for example, Figure 3). In particular, we provide a complete answer to the following question.

Question 1. *Given an arbitrary graph $\mathcal{G}$, how many empty vertices are necessary to connect all the configurations of its associated sliding block puzzle?*

Remark 2. Unsurprisingly, connectedness of the configuration space is a property that is monotone in k: if k empty vertices are enough to connect all configurations, any number $k' > k$ of empty vertices is certainly also enough to achieve connectivity. Although intuitively clear, this remark can be seen as a particular case of [8, Proposition 2.1].

Remark 3. Since making this paper publicly available, we were made aware that the answer to the question above has previously appeared in the literature. Specifically, this result (in different language) is claimed without proof in a paper of Kornhauser, Miller, and Spirakis [17] appearing in the conference proceedings of FOCS'84 (which is a conference in theoretical computer science; there are connections to memory management in distributed systems). The proof can be found in Kornhauser's thesis [16]. Nonetheless, we believe our new proof provides a different outlook on this theorem, and we hope that our reframing within the language of friends and strangers graphs might help bring attention to this result from different communities. We also provide a different decision algorithm to check whether two puzzles can be reconfigured into each other (see the proof of our main theorem, simplifying, in our view, the work of [16]), together with a new result giving an exact count of the number of connected components of the space of all configurations.

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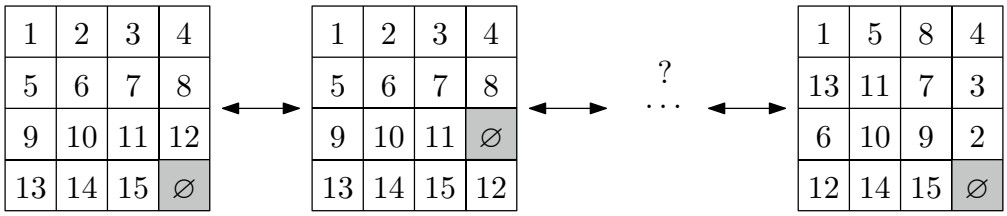

Figure 1: Two configurations of the 15-puzzle are not necessarily connected by elementary moves, depending on the parity of the initial and target configurations.

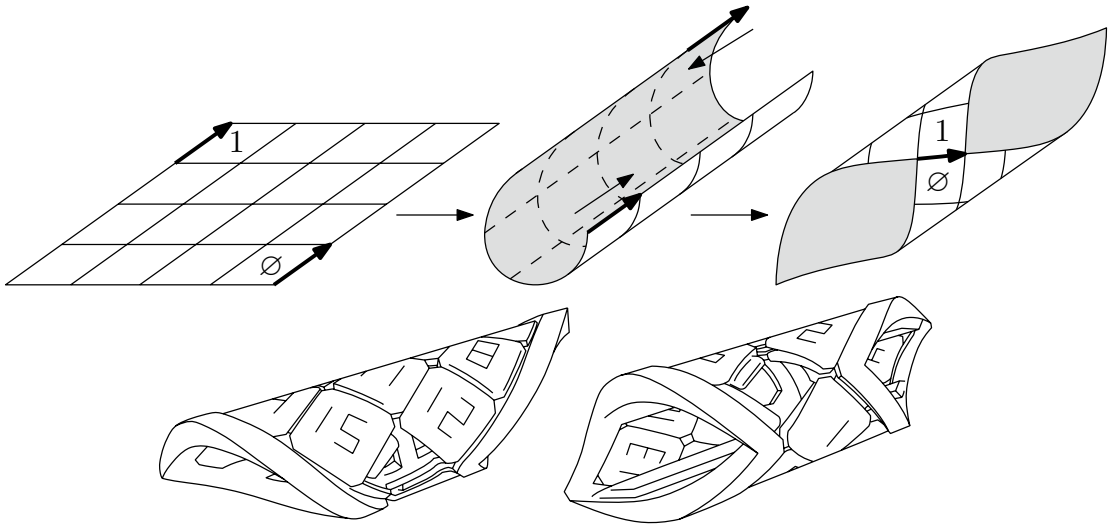

Figure 2: Henry Segerman's coiled 15 wraps around the standard 15-puzzle on a cylinder, connecting the top left square with the bottom right square.

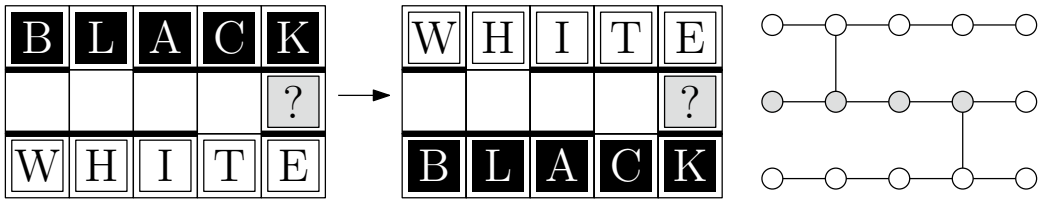

Figure 3: Nob Yoshigahara's Black and White puzzle (1982) is an example of a sliding block puzzle with more than one empty space. The player must interchange the black and white tiles, respecting the order of the letters. Note the immovable barriers between the rows in bold. The corresponding underlying graph $\mathcal{G}$ is depicted on the right, with the four empty spaces highlighted in gray.

Friends and Strangers Graphs

Recently, Defant and Kravitz situated sliding block puzzles as a special case of the broader framework of friends and strangers graphs [8] (see also [1, 4, 7, 13, 18, 19, 25, 26]), which we shall recall and adopt in the rest of this article.

Informally, in this new framework, one may think of sliding blocks as people walking on an underlying simple graph $\mathcal{G}$ (e.g., the 4×4 grid graph in the case of the 15-puzzle). Whenever two people find themselves at the endpoints of a common edge of $\mathcal{G}$, they may swap their position/vertex in $\mathcal{G}$ if and only if they are friends with

each other. In this context, an empty space can be seen as a social butterfly, i.e., a person who is friends with everyone else. The friendships are specified using a second graph $\mathcal{F}$, in which each vertex corresponds to a fixed person, and edges indicate friendships (see Figure 4). In our notation, we make use of capital letters to denote the positions of the people (i.e., the vertices of $\mathcal{G}$) and lowercase letters to refer to the people themselves (i.e., the vertices of $\mathcal{F}$), and we say that a person $x \in \mathtt{V}(\mathcal{F})$ lies on a vertex $X \in \mathtt{V}(\mathcal{G})$.

Formally, given two simple graphs $\mathcal{G}$ and $\mathcal{F}$ on n vertices, the friends and strangers graph $\mathrm{FS}(\mathcal{G}, \mathcal{F})$ is the graph whose vertex set is the set of bijections $\sigma : \mathtt{V}(\mathcal{G}) \to \mathtt{V}(\mathcal{F})$, where $\mathtt{V}(\mathcal{G})$ denotes the vertex set of a graph $\mathcal{G}$. We call such a bijection a configuration. Two configurations $\sigma, \sigma' : \mathtt{V}(\mathcal{G}) \to \mathtt{V}(\mathcal{F})$ are then adjacent in $\mathrm{FS}(\mathcal{G}, \mathcal{F})$ if and only if there exists an edge $\{A, B\}$ in $\mathcal{G}$ such that:

- $\{\sigma(A), \sigma(B)\}$ is an edge in $\mathcal{F}$,
- $\sigma(A) = \sigma'(B)$ and $\sigma(B) = \sigma'(A)$,
- $\sigma(C) = \sigma'(C)$, for all $C \in \mathtt{V}(\mathcal{G}) - \{A, B\}$.

When this is the case, the operation that transforms σ into σ' is referred to as a $(\mathcal{G}, \mathcal{F})$-friendly swap across $\{A, B\}$ in [8], or simply a swap across $\{A, B\}$.

A vertex $v \in \mathtt{V}(\mathcal{F})$ with degree $n - 1$ is called a social butterfly. On the other hand, a vertex $v \in \mathtt{V}(\mathcal{F})$ who is friends with nobody except social butterflies is called asocial.

Example 4. The 15-puzzle can be encoded by setting $\mathcal{G} = \mathrm{Path}_4 \square \mathrm{Path}_4$ and setting $\mathcal{F}$ to be the complete bipartite graph $\mathcal{K}_{1,15}$ (i.e., the star graph with $n - 1$ edges). In this example, the center of the star is the unique social butterfly, and everyone else is asocial. See Figure 4.

Example 5. The coiled 15-puzzle can be encoded using the same friendship graph $\mathcal{F} = \mathcal{K}_{1,15}$ and setting $\mathcal{G}$ to be the 4×4 grid graph $\mathrm{Path}_4 \square \mathrm{Path}_4$ to which a single edge joining the top-left and bottom-right vertices is added (for example). See Figure 4.

The central question and results of this article may then be understood under two different guises: either as the reconfiguration of sliding block puzzles using empty spaces or as the reconfiguration of an asocial crowd walking on a graph with the help of social butterflies.

In the case of the 15-puzzle—and more generally, in Wilson's paradigm—the friendship graph $\mathcal{F}$ turns out to be a star graph (as seen in Examples 4 and 5). When we are considering multiple social butterflies, the right generalization of star graphs is that of generalized book graphs, as defined, for example, in [5]. Given integers $n > k$, we consider the k-book graph on n vertices $\mathcal{B}^{(k)}_{n-k}$, consisting of $n - k$ copies of the complete graph $\mathcal{K}_{k+1}$ on $k + 1$ vertices joined along a common $\mathcal{K}_k$ (see Figure 5).

Remark 6. Using our terminology, book graphs can be seen as the family of graphs in which every person is either asocial or a social butterfly.

We may now reformulate our main question in the language of the friends and strangers graphs.

Question 7 (Friends and strangers reformulation). *Given an arbitrary graph $\mathcal{G}$ on n vertices, for which values of k is $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ connected? In other words, what is the minimal number of social butterflies required to freely reconfigure all the people on $\mathcal{G}$ if everyone else is asocial?*

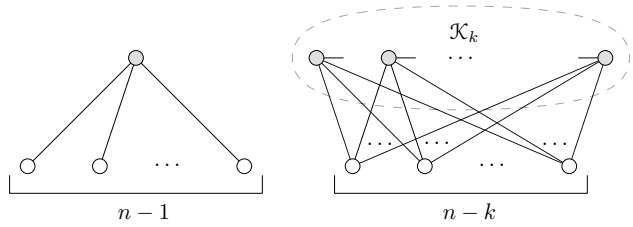

Figure 5: On the left, the star graph $\mathcal{K}_{1,n-1}$, alternatively seen as the book graph $\mathcal{B}^{(1)}_{n-1}$, with its single common vertex $\mathcal{K}_1$ highlighted in gray. On the right, a general k-book graph on n vertices, with its common $\mathcal{K}_k$ indicated in gray.

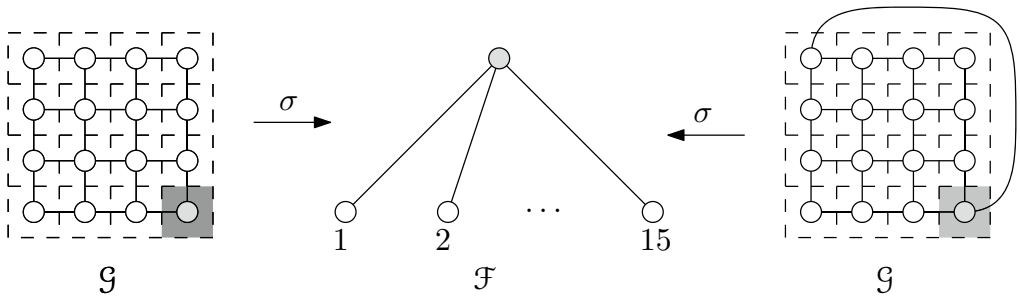

Figure 4: The friendship graph $\mathcal{F}$ associated with the 15-puzzle (left) and the coiled 15-puzzle (right) is the same star graph $\mathcal{K}_{1,15}$ on 16 vertices (middle). For each of these puzzles, a configuration is encoded as a bijection from the vertices of $\mathcal{G}$ to those of $\mathcal{F} = \mathcal{K}_{1,15}$.

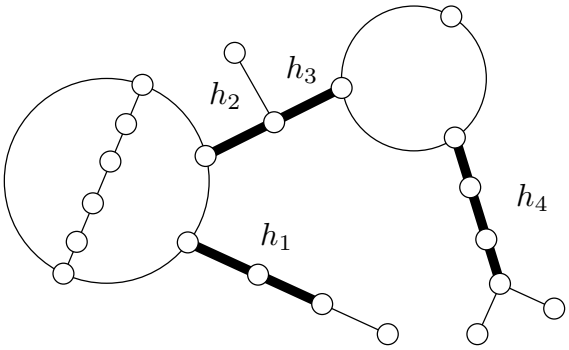

Figure 6: A graph with four maximal hallways h_1, h_2, h_3, and h_4 (highlighted in bold) of order 3, 2, 2, and 4 respectively.

To formulate our main theorem, which precisely answers this last question, we first need to define the notion of a hallway. Recall first that a path $\mathcal{P}$ in the graph $\mathcal{G}$ is a sequence $\mathcal{P} = (X_0, X_1, \dots, X_n)$ of distinct vertices of $\mathcal{G}$ such that for all $1 \leq i \leq n$, X_{i-1} and X_i are adjacent in $\mathcal{G}$. The path $\mathcal{P}$ is said to be bare if all its inner vertices have degree 2 in $\mathcal{G}$. A vertex $V \in \mathbb{V}(\mathcal{G})$ is a cut-vertex of $\mathcal{G}$ if removing it increases the number of connected components of $\mathcal{G}$. A hallway of order k in $\mathcal{G}$ is a bare path $\mathcal{H} = (X_1, X_2, \dots, X_k)$ in $\mathcal{G}$ between cut-vertices X_1 and X_k that is not part of any cycle (in particular, a single cut-vertex is a hallway of order 1). See Figure 6. A hallway is called dark if its order is greater than or equal to k (the relevant value of k will usually be clear from the context).

Let us denote by $\kappa(\mathcal{G})$ the maximal order of a hallway in $\mathcal{G}$. We then define the function $\kappa^* : \mathcal{G} \to \{1, 2, \dots, n\} \cup \{\infty\}$ to be equal to $\kappa(\mathcal{G}) + 1$, except in the following cases:

- If $\mathcal{G}$ is not connected, then $\kappa^*(\mathcal{G}) = \infty$.
- If $\mathcal{G}$ is a cycle, then $\kappa^*(\mathcal{G}) = n - 1$.
- If $\mathcal{G}$ is either bipartite or the θ_0 graph (defined in Figure 7), then $\kappa^*(\mathcal{G}) = 2$.

Theorem 8 (Main theorem). *Let $\mathcal{G}$ be an arbitrary graph on n vertices and let $\mathcal{B}^{(k)}_{n-k}$ be the k-book graph on n vertices. The smallest k such that $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ is connected is exactly $\kappa^*(\mathcal{G})$.*

In other words, our main theorem states that the only obstructions to a connected configuration space are cycles, bipartite graphs, the theta graph θ_0, and dark hallways.

Our main theorem can actually also be used to derive corollaries about the connected components of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ for general $\mathcal{G}$ and k (i.e., when $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ may not be connected). In particular, we describe an efficient algorithm for determining whether two configurations can be reached from each other, and we provide a general formula for the number of connected components of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$.

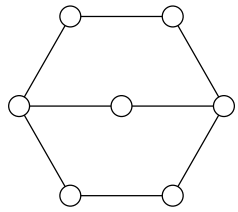

Figure 7: The θ_0 graph.

Overview of the Proof

Before giving an informal overview of the proof of Theorem 8, we begin by recalling the situation for sliding block puzzles with exactly one empty vertex. Wilson's theorem [27] (together with some observations due to Defant and Kravitz [8]) gives a description of the component structure of $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$, as follows. Recall that a graph is biconnected if it has no cut-vertex.

Theorem 9 (Wilson's theorem). *Let $\mathcal{G}$ be a graph on n vertices. We have the following cases:*

1. *$n \leq 2$. In this case, $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$ is trivially connected.*
2. *$\mathcal{G}$ is biconnected, not bipartite, not a cycle, and not θ_0. In this case, $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$ is connected.*
3. *$\mathcal{G}$ is biconnected and bipartite, has at least three vertices, and is not a cycle. In this case, $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$ has exactly two connected components of equal size.*
4. *$\mathcal{G}$ is not biconnected. In this case, $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$ is disconnected.*
5. *$\mathcal{G}$ is a cycle. In this case, $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$ has as many connected components as cyclic orders of $\{1, \dots, n\}$ (i.e., $n!/2$ components of size $n(n-1)$).*
6. *$\mathcal{G}$ is the θ_0 graph. $\mathrm{FS}(\theta_0, \mathcal{K}_{1,n-1})$ is not connected, and one can obtain a full description of the six connected components and their structure (see [27] for the details).*

Moreover, in the setting of item 2 above, we have the following characterization of the components of $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$. Under some chosen identification of both $\mathbb{V}(\mathcal{G})$ and $\mathbb{V}(\mathcal{K}_{1,n-1})$ with $\{1, 2, \dots, n\}$, configurations can be seen as permutations of $\{1, 2, \dots, n\}$. Denoting by $S \in \mathbb{V}(\mathcal{G})$ the vertex supporting the unique social butterfly in the identity permutation/configuration, the two components of $\mathrm{FS}(\mathcal{G}, \mathcal{K}_{1,n-1})$ correspond exactly to the permutations whose parities are the same (respectively the opposite) as the parity of the distance in $\mathcal{G}$ between the vertex supporting their social butterfly and S.

To be precise about attribution, Wilson's 1974 paper [27] proves cases 1, 2, and 5, together with the characterization for case 2 described at the end of the theorem statement (these are all part of [8, Theorem 1]). Defant and Kravitz [8] observe case 4 (see the discussion preceding [8, Theorem 2.5]), and in the setting of case 3, they prove a lower bound for the number of connected components in terms

of the number of cut-vertices (see [8, Proposition 2.6]). For the exact number of connected components in the setting of case 3, we refer the reader to our more general Corollary 22, which relies on definitions not yet introduced.

Example 10. The first two configurations of the 15-puzzle shown in Figure 1 are trivially connected, since they differ by a single flip. Alternatively, this can be checked using Theorem 9: if we work under the identification in which the first configuration corresponds to the identity permutation (where 16 is mapped to the empty vertex), the second configuration then corresponds to the transposition (12 16); see Figure 8. This permutation is odd, but so is the distance between the empty spaces of the first and second configurations, and they are thus in the same connected component. Likewise, the last configuration shown in Figure 1 is (2 12 13 5)(3 9)(6 9 11), which is even. Since the empty spaces of the first and last configurations are in the same position (and 0 is even), these configurations are in fact also connected.

Example 11. Because the underlying graph of the coiled 15-puzzle (see Figure 2) is no longer bipartite, case 1 of Theorem 9 applies, and all configurations become connected.

In order to understand the situation in which more social butterflies are involved, we first recall that a block in a graph is a maximal biconnected subgraph, and we recall that every connected graph $\mathcal{G}$ admits a canonical treelike decomposition into its blocks (see, for example, [9, Section 3.1]). More formally, let us denote the set of cut-vertices of $\mathcal{G}$ by $\mathcal{A}$ and the set of blocks of $\mathcal{G}$ by $\mathfrak{B}$. The block graph is defined to be the graph with vertex set $\mathcal{A} \cup \mathfrak{B}$ and edge set $\{A\mathcal{B} \mid \mathcal{B} \in \mathfrak{B}, A \in \mathcal{B}\}$. The block graph of a connected graph $\mathcal{G}$ is a tree. We will further distinguish between blocks consisting of a single edge and biconnected blocks on at least three vertices, the latter of which we shall refer to as libraries (see Figure 9).

In order to prove our main theorem, we provide a variation of Theorem 9 that allows us to fully understand the situation on each library (see the balloon lemma in the next section). We then show how one can use the block decomposition of a graph to inductively bootstrap this strengthening and obtain an analysis of the whole graph from the individual analysis of each of its libraries. The intuition for this latter step stems from the observation that hallways play a crucial role in ferrying people across the block graph and between libraries. Namely, if a hallway is longer than the number of social butterflies available (i.e., if the hallway is dark), we cannot ferry any asocial people across it, as follows from the following lemma.

Lemma 12. *Fix $k \geq 1$. If $\mathcal{G}$ is a graph with a dark hallway, then* $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ *is disconnected.*

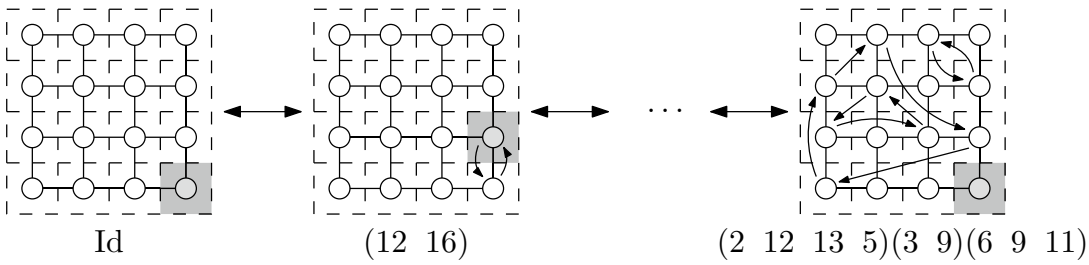

Figure 8: Theorem 9 gives a simple criterion for checking whether two configurations are connected when only one empty vertex is involved.

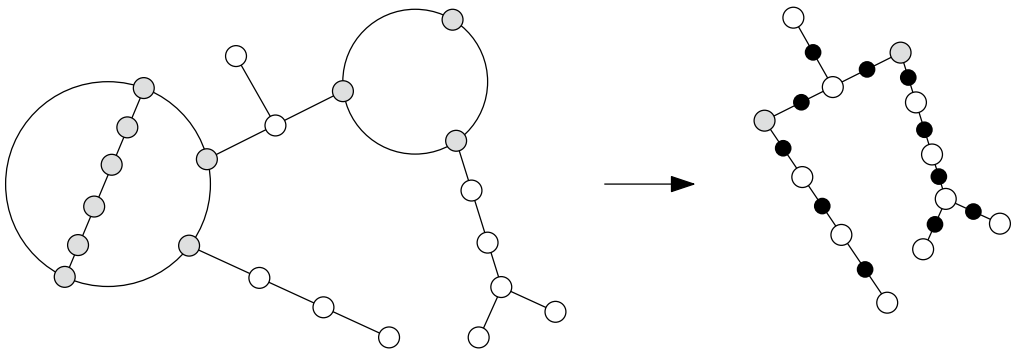

Figure 9: A graph (left) and its associated block graph (right), with its cut-vertices in black, two libraries in gray, and all other blocks in white.

Lemma 12 is easy to see directly, but it can also be seen as a corollary of [8, Theorem 6.1], which more generally states that $\mathrm{FS}(\mathcal{G}, \mathcal{F})$ is disconnected whenever $\mathcal{G}$ has a dark hallway and $\mathcal{F}$ has minimum degree at most k.

To establish our results, we shall rely heavily on the connectedness of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ for a few special types of graphs $\mathcal{G}$ that we analyze in the following section in a series of lemmas of increasing generality.

The Snake Tongue, the Stopwatch, the Hourglass, and the Balloon

In this section, we establish the connectivity of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ for some special graphs $\mathcal{G}$. First we prove the snake tongue lemma, then a series of lemmas culminating in the balloon lemma, taking a short detour for the hourglass lemma.

To give a quick summary of how everything comes together, the proof of our main theorem will be by induction; the snake tongue lemma and the balloon lemma are used in the inductive step (in particular, the snake tongue lemma will be used to ferry people across hallways), while the base cases of the induction are provided by the stringless balloon lemma (proved on the way to the balloon lemma) and the hourglass lemma.

We point out to the reader that in the proofs of all the following lemmas, we implicitly make use of Remark 2, which states that each proof has only to be done for the smallest relevant k (usually $k = 2$).

Our first special graph is the snake tongue graph $\mathcal{D}_n$, $n \geq 4$, defined as the Dynkin diagram of type $\mathcal{D}_n$, i.e., the graph on n vertices obtained from a path on $n-1$ vertices by adding a leaf to the penultimate vertex of the path (see Figure 10).

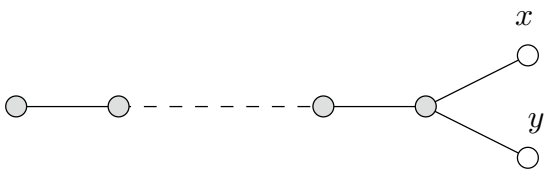

Figure 10: The snake tongue graph $\mathcal{D}_n$.

Remark 13. From now on, in all our figures, we shall indicate vertices occupied by social butterflies in gray.

Lemma 14 (Snake tongue lemma). *For all $n \in \mathbb{N}$, $n-2 \leq k \leq n$, $\mathrm{FS}(\mathcal{D}_n, \mathcal{B}^{(k)}_{n-k})$ is connected.*

In words, the snake tongue lemma says that $\mathcal{D}_n$ can be reconfigured with $n-2$ social butterflies.

Proof. The lemma is trivial if $k \geq n-1$, because then $B^{(k)}_{n-k}$ is a clique and everyone can move around freely. If $k = n-2$, there are only two asocial people, who can exchange their positions by moving to the "tip of the snake tongue" and then moving back in the opposite order. (Formally, this is a special case of [8, Theorem 6.5], since $\mathcal{B}^{(k)}_{n-k}$ has minimum degree at least $n-2$). □

We next define the stopwatch graph $\mathcal{SW}_n$, $n \geq 2$, as a cycle $\mathcal{C}_{n-1}$ of length $n-1$ (the dial of the stopwatch) with an added leaf (the crown of the stopwatch); see Figure 11. We denote by O and P the unique vertices of degree 1 and 3, respectively, and write $V_1, \ldots, V_{n-2}$ for the other vertices (ordered around the dial, so that V_1 and V_{n-2} are neighbors of P).

Lemma 15 (Stopwatch lemma). *For all $n \in \mathbb{N}$, $2 \leq k \leq n$, $\mathrm{FS}(\mathcal{SW}_n, \mathcal{B}^{(k)}_{n-k})$ is connected.*

Proof. It is enough to show that all configurations with a social butterfly on O can be reached from each other, seeing as we can always bring a social butterfly to O.

Observe first that there is a sequence of swaps π that fixes the social butterfly at O and cyclically permutes the people on the dial vertices $V_1, V_2, \ldots, V_{n-2}$ (including the social butterfly lying on P) by one position. This permutation π can

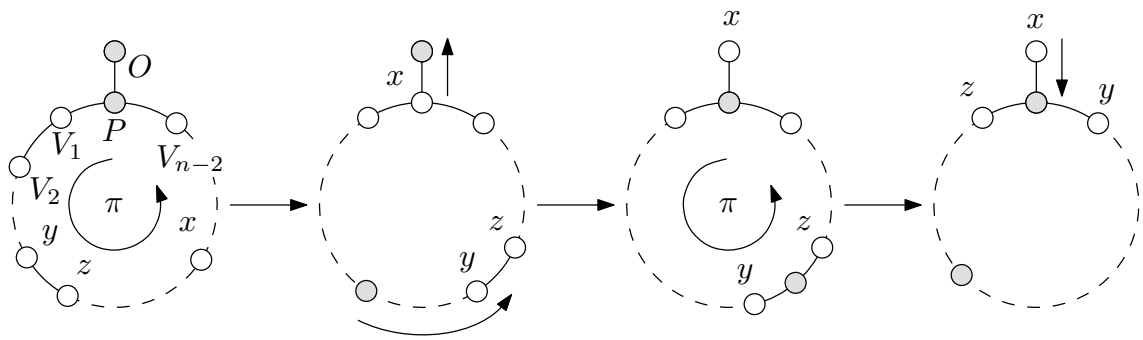

Figure 11: The stopwatch lemma: any person x in the dial of a stopwatch can be removed and inserted back in the cycle between any other two persons y and z lying on successive vertices of the dial. To do so requires "ticking" the stopwatch the appropriate number of times, storing x in the crown of the stopwatch, then ticking it some more before inserting the stored person x back between y and z.

perhaps be seen more intuitively in the language of sliding blocks: the empty space allows us to slide all the nonempty blocks by one position along the dial (counterclockwise, for example). Note that in this scenario, the empty vertex itself is seen as moving clockwise $n-2$ positions around the dial. This motion can be repeated as many times as needed to obtain any power of π; we call this ticking the stopwatch.

For all other reconfigurations, the key observation is that we can store any person x in the crown: after ticking the stopwatch to bring x to P, we may swap it with the social butterfly at O. While doing this, we can position the other social butterfly left on the dial between any two people y and z lying on two chosen consecutive vertices of the dial. We then tick the stopwatch until this last social butterfly lies on P, before swapping it with x to insert it back between z and y. Since there is once again a social butterfly on O, we can repeat this entire operation as many times as required, and all permutations can be reached by a sequence of such insertions via the insertion sort algorithm (see, for example, [15, 5.2.1]). □

Corollary 16. *For all* $k \in \{2, 3, \dots, 7\}$, $\mathrm{FS}(\theta_0, \mathcal{B}^{(k)}_{7-k})$ *is connected.*

Proof of Corollary 16. Notice that the θ_0 graph can be seen as a stopwatch graph $\mathcal{SW}_6$ with a single additional edge drawn between its crown and the unique vertex at distance 4 from the crown. This additional edge only makes reconfiguration easier. □

Next, we say that a graph $\mathcal{H}$ is an hourglass if it can be obtained by joining two cycles at a single vertex, called its throat.

Lemma 17 (Hourglass lemma). *For all* $n \in \mathbb{N}$, $2 \le k \le n$, *if* $\mathcal{H}$ *is an hourglass on* n *vertices, then* $\mathrm{FS}(\mathcal{H}, \mathcal{B}^{(k)}_{n-k})$ *is connected.*

Proof. Let $\mathcal{H}$ be an hourglass graph on n vertices and denote by $\mathcal{X}$ and $\mathcal{Y}$ its two cycles joined at its throat vertex T. Denote by X (respectively Y) one of the two neighbors of T in $\mathcal{X}$ (respectively $\mathcal{Y}$). We first note that there is always a snake tongue $\mathcal{D}_4$ in the throat of the hourglass, with its unique degree-3 vertex coinciding with the throat vertex. Moreover, we may select it so that the (two-pointed) tip of the tongue lies in $\mathcal{X}$, and the rest of the tongue in $\mathcal{Y}$; see Figure 12. By bringing two social butterflies to either T and X or T and Y, we can operate the stopwatch lemma in either of the stopwatches induced by $\mathcal{X} \cup Y$ or $\mathcal{Y} \cup X$. This allows us to bring any person x in $\mathcal{X}$ (respectively y in $\mathcal{Y}$) to a non-throat vertex of the snake tongue lying in $\mathcal{X}$ (respectively $\mathcal{Y}$). Using the snake tongue lemma, we may then exchange x and y across T between $\mathcal{X}$ and $\mathcal{Y}$. By repeatedly applying this procedure, we can reach any desired partition of the people across $\mathcal{X}$ and $\mathcal{Y}$. An application of the stopwatch lemma in both $\mathcal{X}$ and $\mathcal{Y}$ is then enough to reconfigure the people in each of the two cycles of the hourglass. □

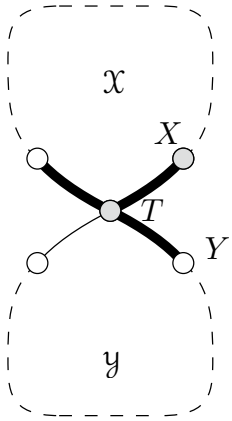

Figure 12: An hourglass graph consisting of two cycles joined at a single throat vertex T. Its $\mathcal{D}_4$ snake tongue subgraph is highlighted in bold.

We are now in a position to prove our last lemma, which establishes connectivity of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ for a much more general class of graphs called balloons. A graph $\mathcal{Q}$ is a balloon if it can be obtained by adding a single leaf to a biconnected graph. Balloons can thus be seen as a generalization of stopwatches.

Lemma 18 (Balloon lemma). *Let* $\mathcal{Q}$ *be a balloon, obtained by adding a leaf* O *to a vertex* P *of a biconnected graph* $\mathcal{G}$ *on* $(n-1)$ *vertices. For all* $k \ge 2$, $\mathrm{FS}(\mathcal{Q}, \mathcal{B}^{(k)}_{n-k})$ *is connected.*

On our way to proving the balloon lemma, we will also prove the following variation of it, which will be used in the base case of the induction of the proof of our Theorem 8. This following lemma confirms our first impression that the addition of a second empty vertex/social butterfly grants us significantly more power. Recall that it allows us to connect all configurations of the 15-puzzle. We show that more generally, two social butterflies are in fact enough to guarantee connectivity for all biconnected graphs except cycles (i.e., in contrast to Theorem 9, neither bipartite graphs nor the θ_0 graph is problematic).

Lemma 19 (Stringless balloon lemma). *Let* $2 \le k \le n$ *and consider a biconnected graph* $\mathcal{G}$ *on* n *vertices that is not a cycle of length strictly greater than* $k+1$. *The friends and strangers graph* $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ *is connected.*

Proof of the stringless balloon lemma. If $\mathcal{G}$ happens to be the θ_0 graph, the lemma is already given by Corollary 16. If $\mathcal{G}$ is a cycle on fewer than $k+1$ vertices, or if $\mathcal{G}$ consists of a single edge, the result is trivial. If $\mathcal{G} \neq \theta_0$ is not bipartite, then $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ is connected for every positive k and there is nothing further to prove. So let $\mathcal{G}$ be a biconnected bipartite graph on at least three vertices that is not a cycle.

Recall the characterization of the components of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(1)}_{n-1})$ in case 2 of Theorem 9 (there are two connected

components with an explicit description in terms of parity).

Identifying once again $\mathbb{V}(\mathcal{G})$ and $\mathbb{V}(\mathcal{B}^{(k)}_{n-k})$ with $\{1, 2, \dots, n\}$, it is enough to show that there exist configurations (seen as permutations in this context) σ and σ', reachable from each other, that fix a social butterfly on some arbitrary vertex X of $\mathcal{G}$ but have opposite parities. Because the stopwatch lemma allows full reconfiguration of the stopwatch and thus allows one to obtain a permutation of arbitrary signature of its dial, it is enough to see that $\mathcal{G}$ contains a stopwatch. Indeed, $\mathcal{G}$ is biconnected with at least three vertices and thus contains a cycle. Consider a shortest such cycle $\mathcal{C}$. Since $\mathcal{G} \neq \mathcal{C}$ is connected, there is an additional edge incident to $\mathcal{C}$, with which we may form a stopwatch (Figure 13). □

Proof of the balloon lemma. Let $\mathcal{B}$ be a balloon, obtained by adding a leaf O to a vertex P of a biconnected graph $\mathcal{G}$ on $(n-1)$ vertices. The case that $\mathcal{G}$ is a cycle is already treated by the stopwatch lemma, and the case that $\mathcal{G}$ is an edge is trivial. Suppose then that $\mathcal{G}$ is a biconnected graph that is not a cycle. We consider two configurations σ and σ' of $\mathcal{B}$ with social butterflies positioned at O and P, and show that these configurations can be reached from each other (it is always easy to bring two social butterflies to O and P).

Since $\mathcal{G}$ is biconnected, it has a cycle $\mathcal{C}$ containing P. Let us call X one of the two neighbors of P on $\mathcal{C}$ and call x the person lying on X in the configuration σ. We begin by swapping x successively with the two social butterflies lying on P and O, so that x is now lying on O and the two social butterflies at P and X. We may then apply the stringless balloon lemma to reconfigure $\mathcal{G}$ as we wish, leaving O and P fixed. In particular, we may choose to reach a configuration σ_1 such that σ_1 and σ' agree on everything but $\mathcal{C}$ and share the same set of people lying on $\mathcal{C}$. We then swap x back onto X, bringing back the two social butterflies to rest on O and P. To finish the proof, we then apply the stopwatch lemma to reconfigure the stopwatch obtained by adding the leaf O at P to the cycle $\mathcal{C}$ to match σ' on $\mathcal{C}$. □

Proof of the Main Theorem

Rough Sketch of the Proof

We first recall that $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ is disconnected if there exist dark hallways in $\mathcal{G}$ (Lemma 12). This observation combined with Theorem 9 yields a lower bound of $\kappa^*(\mathcal{G})$ on the smallest k such that $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ is connected. We may then focus on proving the upper bound. Note also that if $\mathcal{G}$ is not connected, then neither is $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ (hence the infinite value of κ^*), and we may thus restrict our attention to connected graphs.

As outlined in the overview of the proof above, we wish to bootstrap our analysis of each library. We do this by induction on the number of blocks.

The structure of our proof is as follows. We first isolate a block of $\mathcal{G}$ corresponding to a leaf of its block graph. Let us call this block $\mathcal{L}$. Since blocks intersect in unique cut-vertices, $\mathcal{L}$ intersects its neighboring block in a single cut-vertex, which we denote by V. Let us call $\mathcal{A}$ the subgraph of $\mathcal{G}$ induced by the union of the complement of $\mathbb{V}(\mathcal{L})$ with V. (So $\mathcal{A}$ and $\mathcal{L}$ partition the edges of $\mathcal{G}$, and they share a single vertex V). Our proof then consists of two main ingredients. First, we need to show that we can ferry people across V between $\mathcal{A}$ and $\mathcal{L}$ to reach any desired "people-partition" between $\mathcal{L}$ and $\mathcal{A}$ (the ferrying lemma). More specifically, we will single out a special type of partition that we call balanced; such partitions roughly speaking have the social butterflies balanced between the two sides $\mathcal{A}$ and $\mathcal{L}$ in a way that facilitates induction (the exact definition will be given shortly thereafter). Second, we need to show that all the configurations that feature the same balanced partition can be reached from one another (the island lemma).

Geometrically, our two steps correspond to showing the existence of subgraphs of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ such that each member of a pair of vertices in these subgraphs can be reached from the other member; note that this might involve leaving the subgraph and spelling out a path from any configuration σ to any of these subgraphs. We think of our subgraphs as "islands," one for each balanced partition, and we think of our paths between islands as "ferries"; see Figure 14.

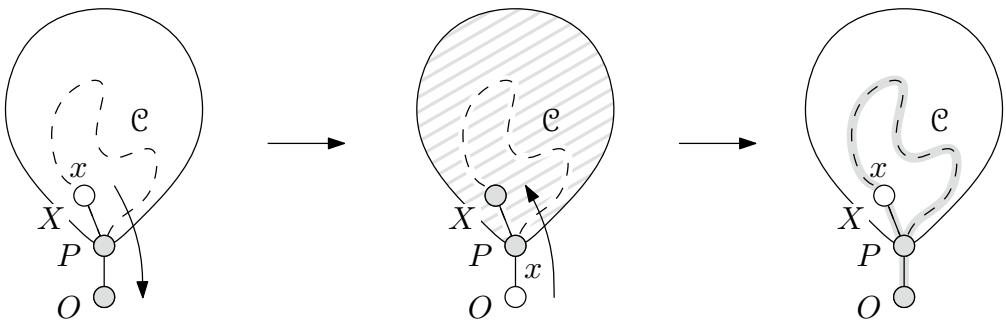

Figure 13: The sequence of reconfigurations from σ to σ'. The gray hatchings indicate the completion of the same configuration as the target configuration σ', restricted to the hatched area.

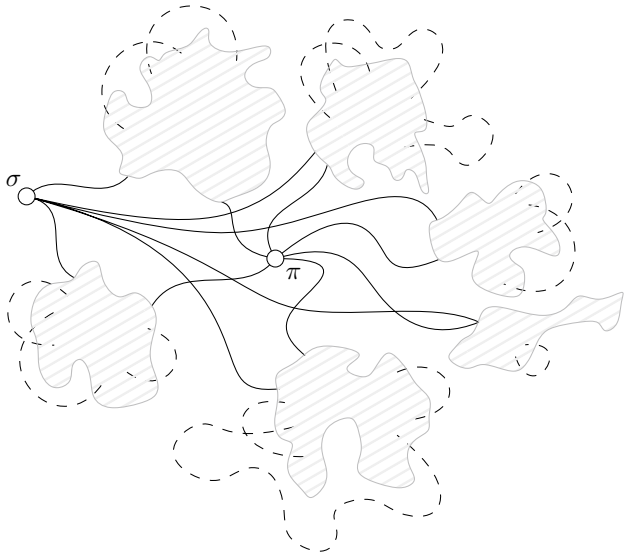

Figure 14: A pictorial overview of the proof of the main theorem: configurations sharing the same balanced partition form "islands" that are connected to one another in $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ (hatched in gray). Every nonbalanced configuration (such as σ or π in the figure) has a path (a "ferry") to all these connected configurations.

The Base Cases

Because of the peculiar role played by cycles in our theorem, we need two different base cases for our induction:

Case 1: $\mathcal{G}$ is an hourglass (which has two blocks). In this case, $\kappa(\mathcal{G}) = 1$, so $\kappa^*(\mathcal{G}) = 2$, and the theorem follows from the hourglass lemma.

Case 2: $\mathcal{G}$ consists of a single block (i.e., it is biconnected). In this case, $\kappa(\mathcal{G}) = 1$, but $\kappa^*(\mathcal{G})$ depends on the structure of $\mathcal{G}$.

- If $\mathcal{G} \neq \theta_0$ is not bipartite, then $\kappa^*(\mathcal{G}) = 1$, and the desired result follows from Theorem 9.
- If $\mathcal{G} = \theta_0$ or $\mathcal{G}$ is bipartite, then $\kappa^*(\mathcal{G}) = 2$. At least two social butterflies are necessary for connectedness by Theorem 9, and two social butterflies suffice by the stringless balloon lemma.
- If $\mathcal{G}$ is a cycle, then $\kappa^*(\mathcal{G}) = n - 1$. It is impossible to change the relative order of the asocial people around the cycle, so $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ is connected if and only if $k \geq n - 1$ (i.e., there is at most one asocial person).

The Induction Step

Now fix a connected graph $\mathcal{G}$ on n vertices that is not an hourglass and has at least two blocks. Suppose that the theorem holds for all subgraphs of $\mathcal{G}$ with fewer blocks than $\mathcal{G}$. We will prove the theorem for $\mathcal{G}$.

Let us fix $k \leq n$ and assume that $\mathcal{G}$ does not contain any dark hallways. Our goal is to prove that $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ is connected as in the sketch of the proof. Fix a leaf $\mathcal{L}$ of the block graph of $\mathcal{G}$, and let $\mathcal{A}$ be the subgraph induced by the union of the complement of $\mathcal{L}$ with V, where V is the unique cut-vertex separating $\mathcal{A}$ from $\mathcal{L}$. Since $\mathcal{G}$ is not an hourglass, we may choose the leaf $\mathcal{L}$ in such a way that $\mathcal{A}$ is not a cycle.

We define a block party to be a partition $(\mathcal{X}, \mathcal{Y})$ of $\mathrm{V}(\mathcal{B}^{(k)}_{n-k})$ into two sets of size $|\mathcal{X}| = |\mathcal{A}|$ and $|\mathcal{Y}| = |\mathcal{L}| - 1$. For a block party $(\mathcal{X}, \mathcal{Y})$, we then say that a configuration σ is $(\mathcal{X}, \mathcal{Y})$-balanced if the following conditions hold:

1. The people of $\mathcal{X}$ are those lying in $\mathcal{A}$, i.e., $\sigma(\mathrm{V}(\mathcal{A})) = \mathcal{X}$.
2. The people of $\mathcal{Y}$ are those lying in $\mathcal{L} - \{V\}$, i.e., $\sigma(\mathrm{V}(\mathcal{L}) - \{V\}) = \mathcal{Y}$.
3. There is a social butterfly lying on the cut-vertex V and on a vertex of $\mathcal{A}$ adjacent to V.
4. Either there are no social butterflies in $\mathcal{Y}$, or $\mathcal{X}$ consists entirely of social butterflies (roughly speaking, we want all the social butterflies to be in $\mathcal{A}$, but if there are more than $|\mathcal{A}|$ social butterflies, we allow some of them to overflow into $\mathcal{Y}$).

If $(\mathcal{X}, \mathcal{Y})$ has the right number of social butterflies in $\mathcal{X}$ and in $\mathcal{Y}$ for an $(\mathcal{X}, \mathcal{Y})$-balanced configuration to exist, then we say that $(\mathcal{X}, \mathcal{Y})$ is a balanced block party. Furthermore, we can associate to every configuration σ a unique block party $(\mathcal{X}_\sigma, \mathcal{Y}_\sigma) := (\sigma(\mathrm{V}(\mathcal{A})), \sigma(\mathrm{V}(\mathcal{L}) - \{V\}))$. For each balanced block party $(\mathcal{X}, \mathcal{Y})$, we define the $(\mathcal{X}, \mathcal{Y})$-cluster to be the set of configurations whose associated block party is $(\mathcal{X}, \mathcal{Y})$.

As outlined in the sketch of the proof at the start of this section, there are two main ingredients to our proof.

Lemma 20 (Island lemma). *For every balanced block party $(\mathcal{X}, \mathcal{Y})$, all the configurations in the $(\mathcal{X}, \mathcal{Y})$-cluster are part of the same connected component of $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$.*

Lemma 21 (Ferrying lemma). *For every configuration σ and every balanced block party $(\mathcal{X}, \mathcal{Y})$, there exists a path in $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ between σ and the $(\mathcal{X}, \mathcal{Y})$-cluster.*

Proof of the island lemma. Let σ and σ' be two $(\mathcal{X}, \mathcal{Y})$-balanced configurations. Our aim is to construct a sequence of swaps from σ to σ' (Figure 15).

By assumption, $\mathcal{G}$ does not have any dark hallways, and thus neither does $\mathcal{A}$. Since $\mathcal{A}$ is not a cycle, we may use our induction assumption to reconfigure $\mathcal{A}$ in isolation to reach a $(\mathcal{X}, \mathcal{Y})$-balanced configuration σ_1 that agrees with σ' on $\mathcal{A}$ (i.e., $\sigma_1|_{\mathcal{A}} = \sigma'|_{\mathcal{A}}$). Since σ_1 is balanced, we can select a neighbor P of V in $\mathcal{A}$ supporting a social butterfly. The subgraph induced by the union of P with $\mathcal{L}$ forms a balloon, and two social butterflies lie on P and V. Thus, we may apply the balloon lemma to reconfigure $\mathcal{L} - \{V\}$ as needed and reach σ'. □

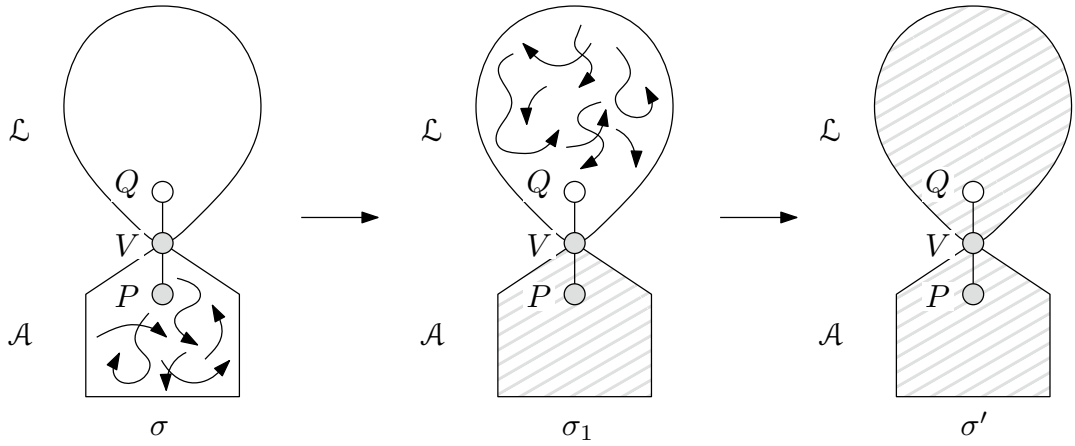

Figure 15: The sequence of reconfigurations from σ to σ'. The hatchings once again indicate the completion of the same configuration as the target configuration σ', restricted to the hatched area.

We have now verified that all configurations lying inside a cluster of balanced configurations corresponding to a fixed balanced block party are connected to one another. To complete the proof, we exhibit a path from any configuration to any balanced cluster.

Proof of the ferrying lemma. Let us fix a configuration σ and its associated block party $(\mathcal{X}_\sigma, \mathcal{Y}_\sigma)$, together with the target balanced block party $(\mathcal{X}, \mathcal{Y})$. We begin by successively bringing as many as possible of k available social butterflies to $\mathcal{A}$, thereby reaching an $(\mathcal{X}', \mathcal{Y}')$-balanced configuration σ' for some balanced block party $(\mathcal{X}', \mathcal{Y}')$. We want to show that for every collection of asocial people $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, there is always a sequence of swaps that allows us to alter $(\mathcal{X}', \mathcal{Y}')$ by "trading x with y," as follows.

Claim. *For all asocial $x \in \mathcal{X}'$ and $y \in \mathcal{Y}'$, there exists a sequence of swaps that takes σ' to some $(\mathcal{X}' \cup \{y\} - \{x\}, \mathcal{Y}' \cup \{x\} - \{y\})$-balanced configuration.*

The above claim suffices to prove the island lemma. Indeed, we can reach $(\mathcal{X}, \mathcal{Y})$ from $(\mathcal{X}', \mathcal{Y}')$ by repeatedly trading people between the two parts.

Proof of the claim. First, note that if $\mathcal{A}$ is a path, then the assumption that there do not exist any dark hallways forces $\mathcal{A}$ to be a path of length at most k, and the desired result is trivial (in the $(\mathcal{X}', \mathcal{Y}')$-balanced configuration σ', only social butterflies lie in $\mathcal{A}$).

So we may assume that $\mathcal{A}$ is not a path. Similarly to what we did in the proof of the hourglass lemma, we find an adequately short snake tongue subgraph in $\mathcal{G}$ straddling V and extending into both $\mathcal{A}$ and $\mathcal{L}$ (see Figure 16), so that we may use the snake tongue lemma to ferry people across V. To be precise, the fact we need is as follows.

Fact. There always exists a snake tongue subgraph $\mathcal{D}_m$, $4 \le m \le k+1$, such that the entirety of $\mathcal{D}_m$ is contained in $\mathcal{A}$, save for the very last edge of the "back" of the tongue, which lies in $\mathcal{L}$.

This snake tongue can be found by exploring from V into $\mathcal{A}$. If V has two neighbors in $\mathcal{A}$, we have immediately found a suitable snake tongue $\mathcal{D}_4$, and otherwise, the desired snake tongue amounts to a vertex of degree $d \ge 3$ in $\mathbb{V}(\mathcal{A}) \cup \{V\}$ within distance at most $k-1$ of V. If we cannot find such a snake tongue, then the depth-$(k-1)$ neighborhood of $\mathcal{V}$ in $\mathcal{A}$ must be a path. Since $\mathcal{A}$ has no dark hallways, this means that $\mathcal{A}$ itself must be a path, contradicting the assumption we made at the start of the proof of the claim.

Proceeding with the proof of the claim, let us write X and X' for the two vertices at the tip of the snake tongue graph D_m, and Y for its back vertex (see Figure 16). Since $\mathcal{A}$ is not a cycle, our induction assumption guarantees that we may reconfigure $\mathcal{A}$ as desired. In particular, we may bring any person $x \in \mathcal{X}'$ to X and saturate the rest of $D_m \cap \mathcal{A}$ with $m-1 \le k$ social butterflies. Note that V has a neighbor $X'' \ne X$ in D_m (if $m = 4$, we take $X'' = X'$, and otherwise, we let X' be a vertex on the path section of the snake tongue). The subgraph induced by X'' and $\mathcal{L}$ is a balloon, and two social butterflies now lie on X'' and V. We may thus apply the balloon lemma in $\mathcal{L}$ to bring any person $y \in \mathcal{Y}'$ to Y. The snake tongue lemma then allows us to reconfigure D_m so that y now lies on X and x on Y. This final configuration $\sigma_{x,y}$ satisfies the conclusion of the claim. □

The proof of the claim completes the proof of the ferrying lemma. □

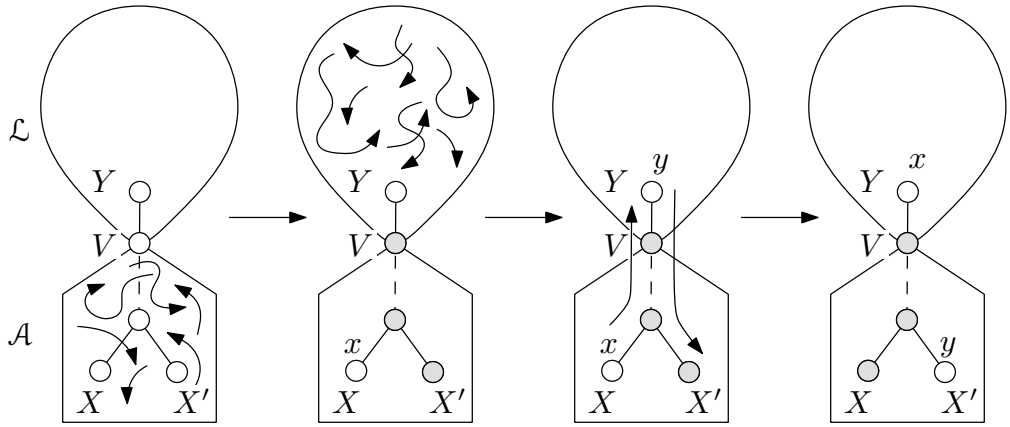

Figure 16: The sequence of swaps to alter $(\mathcal{X}', \mathcal{Y}')$ by trading x with y.

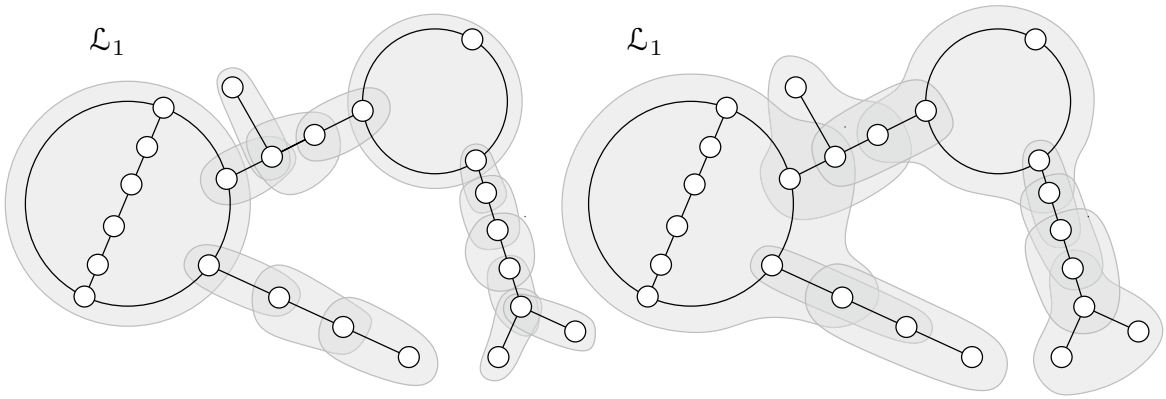

Figure 17: The social circles of the graph $\mathcal{G}$ for $k = 1$ (left diagram) and $k = 2$ (right diagram).

A Simple Algorithm to Determine When Two Configurations Are Connected

The ideas in the proof of our main theorem can be straightforwardly adapted to provide an algorithm that determines whether two configurations can be reached from each other.

Given a connected graph $\mathcal{G}$ (and some k), a connected subgraph $\mathcal{G}'$ of $\mathcal{G}$ is called well-lit if it does not contain any dark hallways (for the subgraph). A maximal well-lit subgraph is called a social circle (see Figure 17). By construction, the social circles of a graph $\mathcal{G}$ cover all of its vertices. It is easy to see that all social circles contain at least k vertices (in fact, unless $k = n$, each social circle contains at least $k + 1$ vertices, but we will not need this fact).

Setting aside issues related to the exceptions in Theorem 9, our main theorem allows us to freely reconfigure inside each social circle. Thus, to check that two configurations can be reached from each other, we need only verify that each social circle contains the same set of people in both configurations and further consider the conditions of Theorem 9 within each social circle.

To be more precise, consider the social circles $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$ of $\mathcal{G}$. For all $i \in [m]$, we arbitrarily fix a set $\mathcal{O}_i$ of k vertices in the social circle $\mathcal{L}_i$. The key observation is that if all k social butterflies lie on the vertices in $\mathcal{O}_i$, then none of the other asocial people who are in $\mathcal{L}_i$ at that point can ever leave $\mathcal{L}_i$ (since asocial people cannot traverse dark hallways). That is to say, if we have two configurations σ_i, τ_i for which the k social butterflies lie in $\mathcal{O}_i$, then we have no hope of reaching τ_i from σ_i unless $\mathcal{L}_i$ contains the same set of people in σ_i and τ_i. But note that for any configurations σ, τ, we can always reach some configuration σ_i from σ, and some configuration τ_i from τ, such that σ_i, τ_i have the k social butterflies lying in $\mathcal{O}_i$ (indeed, social butterflies can move freely through the graph, displacing asocial people as they move).

Below, we present an algorithm (actually the pair of Algorithms 1 and 2) for applying these observations to check whether two configurations can be reached from each other. For ease of notation, given a social circle $\mathcal{L}$ in a graph $\mathcal{G}$ and some $\sigma \in \mathbb{V}(\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k}))$, we write $\sigma|_{\mathcal{L}}$ for the restriction of σ to the vertices in $\mathcal{L}$. If we write $n_{\mathcal{L}}$ and $k_{\sigma,\mathcal{L}}$ for the number of vertices in $\mathcal{L}$ and the number of social

Algorithm 1*

Connect($\sigma, \tau, \mathcal{G}, k$) Checking whether two configurations σ and τ are connected

Consider the social circles $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$ of $\mathcal{G}$.
Fix a set $\mathcal{O}_i$ of k vertices in each social circle $\mathcal{L}_i$.
for $i = 1, 2, \ldots, m$ **do**
  In both σ and τ, move all social butterflies to $\mathcal{O}_i$ (arbitrarily) and consider the resulting configurations σ_i, τ_i.
  Run **SocialCircle($\sigma_i, \tau_i, \mathcal{L}_i, k$)**
end for
Return "CONNECTED"

Algorithm 2*

SocialCircle($\sigma_i, \tau_i, \mathcal{L}_i, k$) Checking connectivity in the social circle $\mathcal{L}_i$

if $\sigma_i(\mathtt{V}(\mathcal{L}_i)) \neq \tau_i(\mathtt{V}(\mathcal{L}_i))$ **then**
  Return "NOT CONNECTED"
end if
if $k < |\mathcal{L}_i| - 1$ and $\mathcal{L}_i$ is a cycle **then**
  if the cyclic ordering of σ_i and τ_i differ **then**
    Return "NOT CONNECTED"
  end if
end if
if $k = 1$ **then**
  if $\mathcal{L}_i$ is bipartite **then**
    if σ_i and τ_i do not share the same parity **then**
      Return "NOT CONNECTED"
    end if
  else if $\mathcal{L}_i \cong \theta_0$ **then**
    If we have reached this point then $\sigma_i(\mathtt{V}(\mathcal{L}_i)) = \tau_i(\mathtt{V}(\mathcal{L}_i))$. Note that $\sigma_i|_{\mathcal{L}_i}$ and $\tau_i|_{\mathcal{L}_i}$ can then be viewed as vertices of $\mathrm{FS}(\mathcal{L}_i, B_{7-1}^{(1)}) \cong \mathrm{FS}(\theta_0, B_{7-1}^{(1)})$, which is a specific graph on 7! vertices with 6 connected components (characterised in [27]).
    if $\sigma_i|_{\mathcal{L}_i}$ and $\tau_i|_{\mathcal{L}_i}$ are in different components of $\mathrm{FS}(\mathcal{L}_i, B_{7-1}^{(1)})$ **then**
      Return "NOT CONNECTED"
    end if
  end if
end if

butterflies in $\mathcal{L}$ with respect to σ, then $\sigma(\mathtt{V}(\mathcal{L}))$ is a set of $n_{\mathcal{L}}$ people including $k_{\sigma,\mathcal{L}}$ social butterflies. We can therefore identify $\sigma(\mathtt{V}(\mathcal{L}))$ with the vertex set of $\mathcal{B}_{n_{\mathcal{L}}-k_{\sigma,\mathcal{L}}}^{(k_{\sigma,\mathcal{L}})}$ and view $\sigma|_{\mathcal{L}}$ as a vertex of $\mathrm{FS}(\mathcal{L}, \mathcal{B}_{n_{\mathcal{L}}-k_{\sigma,\mathcal{L}}}^{(k_{\sigma,\mathcal{L}})})$.

We note that the algorithm is efficient (i.e., it terminates in polynomial time). Let n, e, m be the numbers of vertices, edges, and social circles of $\mathcal{G}$. First, we note that the blocks and cut-vertices of $\mathcal{G}$ can be found in time $O(n + e)$ [21], after which the social circles can be returned in time $O(kn)$ (remember that the block graph is always a forest). The operation of "moving" the social butterflies to $\mathcal{O}_i$ requires computing paths between all social butterflies and social circles, which can be done in time $O(\min\{m, k\}(n + e))$ by either conflating k points inside each social circle to a single node and running a breadth-first search rooted at that node, or running such a search directly from each social butterfly. Regarding the algorithm SocialCircle, all the criteria take $O(|\mathcal{L}_i|)$ time to compute. Since $\sum_i |\mathcal{L}_i| < kn$, the overall running time of our algorithm is $O(\min\{m, k\}(n + e) + 2kn)$. Note that this is $O(n^3)$ in the

worst case (and the cubic lower bound of [16] still holds), but it is much faster on many graphs.

The Number of Components of the Friends and Strangers Graph

The previous algorithm not only informs us about the connectivity between two particular configurations, but it also shows the following:

Corollary 22. *The number of connected components of* $\mathrm{FS}(\mathcal{G}, \mathcal{B}^{(k)}_{n-k})$ *is precisely*

$$\binom{n-k}{|\mathcal{L}_1|-k, |\mathcal{L}_2|-k, \ldots, |\mathcal{L}_m|-k} \cdot \prod_{i=1}^{m} \lambda_i \,,$$

where λ_i *is defined to be equal to* 1*, except in the following cases*:

- If $k < |\mathcal{L}_i| - 1$ and $\mathcal{L}_i$ is a cycle, then $\lambda_i := \frac{|\mathcal{L}_i|!}{2}$.
- If $k = 1$, then
  - if $\mathcal{L}_i$ is bipartite, but not a cycle, then $\lambda_i := 2$.
  - if $\mathcal{L}_i \equiv \theta_0$, then $\lambda_i := 6$.

Indeed, the collection of asocial people lying in $\mathcal{L}_i$ in the configuration σ_i, as i ranges over $[m]$, forms a partition of all asocial people. As shown by our previous algorithm, two configurations can be reached from each other if and only if they agree on this partition and satisfy the additional constraints of Theorem 9. This confirms that the previous formula is both a lower and an upper bound for the number of connected components.

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