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On the chromatic number of powers of subdivisions of graphs

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ABSTRACT

For a given graph $G = (V, E)$, we define its *nth subdivision* as the graph obtained from G by replacing every edge by a path of length n. We also define the *mth power* of G as the graph on vertex set V where we connect every pair of vertices at distance at most m in G. In this paper, we study the chromatic number of powers of subdivisions of graphs and resolve the case $m = n$ asymptotically. In particular, our result confirms a conjecture of Mozafari-Nia and Iradmusa in the case $m = n = 3$ in a strong sense.

1. Introduction

Let $G = (V, E)$ be a simple graph. A *total colouring* of G is an assignment of colours to its vertices and edges so that no pair of adjacent vertices or edges has the same colour, and no edge has the same colour as either of its endpoints. We denote by $\chi''(G)$ (called the *total chromatic number* of G) the minimum number of colours needed in a total colouring of G.

The *total colouring conjecture*, posed independently by Vizing in 1964 [15] and by Behzad [3] in his Ph.D. dissertation in 1965, states that, for every simple graph G with maximum degree $\Delta(G)$, we have $\chi''(G) \leq \Delta(G) + 2$. Nowadays, there are partial advances towards this conjecture. For example, Reed and Molloy proved in [9] that, if $\Delta(G)$ is sufficiently large, then $\chi''(G) \leq \Delta(G)+C$, where C can be taken to be 10^{26} (however, the authors state that the constant is not optimised and a detailed analysis could yield a much better constant). Hind, Reed, and Molloy proved in [6] that, if $\Delta(G)$ is sufficiently large, then $\chi''(G) \leq \Delta(G) + 8(\log \Delta(G))^8$. We refer the reader to [8,10,17] for some history and further results in this line of research.

In this paper, we study generalisations of the total colouring conjecture. For a given graph $G = (V, E)$, we define its *nth subdivision* as the graph obtained from G by replacing every edge with a path of length n. We also define the mth power of G as the graph on vertex set V where we connect every pair of vertices at distance[1] at most m in G. We denote by $G^{\frac{1}{n}}$ and G^m the nth subdivision and the mth power of G, respectively. Finally, we define $G^{\frac{m}{n}}$ to be $\left(G^{\frac{1}{n}}\right)^m$. For instance, $G^{\frac{1}{1}} = G$. In $G^{\frac{m}{n}}$, the vertices that were already in G are called *branch vertices*, whereas the vertices that were added because of the subdivision are called *inner vertices*.

Note that $\chi(G^{\frac{1}{n}}) \leq 3$ holds for all $n > 1$ and all graphs G because we can always assign colour 1 to the branch vertices and then alternatingly colour the inner vertices of the subdivision with the colours 2 and 3. Observe also that

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[1] The *length* of a path P is the number of edges of P. The *distance* between two vertices u and v is the minimum length of a u-v-path.

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$\chi''(G) = \chi(G^{\frac{2}{2}})$; hence, the total colouring conjecture states that $\chi(G^{\frac{2}{2}}) \leq \Delta(G) + 2$. For other powers of subdivisions, it was shown by Iradmusa [7, Lemma 3], and later also by Hartke, Liu, and Petříčková [5, Lemma 2.6], that $\chi(G^{\frac{2}{3}}) = \Delta(G)+1$ whenever $\Delta(G) \geq 3$. More generally, the latter group of authors studied the chromatic number of $G^{\frac{m}{n}}$ when $1 < m < n$ and determined it up to an additive constant of 2 (see [5, Theorems 1 and 2] and [7, Theorem 1]). The case $m = n$ is much less understood. Wang and Liu proved in [16] that, if $\Delta(G) \leq 3$, then $\chi(G^{\frac{3}{3}}) \leq 7$. This was extended by Mozafari-Nia and Iradmusa in [11], who showed that, if $\Delta(G) \leq 4$, then $\chi(G^{\frac{3}{3}}) \leq 9$, and conjectured that $\chi(G^{\frac{3}{3}}) \leq 2\Delta(G) + 1$. In [12–14], the same authors checked the validity of this conjecture for some classes of graphs such as k-degenerate graphs, cycles, forests, complete graphs, hypercubes, outerplanar graphs, and regular bipartite graphs.

Our main result proves this conjecture in asymptotic form (when $\Delta \to \infty$), and shows that the multiplicative constant can in fact be taken to be 1.

Theorem 1.1. *There is a constant $C > 0$ such that, for all graphs G with maximum degree Δ,*

$$\chi(G^{\frac{3}{3}}) \leq \Delta + C \log \Delta.$$

The constant C in Theorem 1.1 can be taken to be 28 when Δ is sufficiently large, although in our argument we have not attempted to optimise it. The proof uses some ideas from Guiduli [4] as well as Alon, McDiarmid, and Reed [2]. In the first paper, the author considers the incidence colouring number $\iota(G)$, which can be seen as the colouring number where one only has to colour the inner vertices of $G^{\frac{3}{3}}$. We show that the proof can be extended by probabilistic means to colour the branch vertices as well.

Concerning lower bounds for this problem, Guiduli [4] also notes that, if G is a Paley graph, then $\iota(G) \geq \Delta + \Omega(\log \Delta)$. This implies that, for infinitely many Δ, there are graphs G with maximum degree Δ such that

$$\chi(G^{\frac{3}{3}}) \geq \Delta + \Omega(\log \Delta).$$

As a byproduct of the ideas developed to prove Theorem 1.1, we are also able to obtain the following generalisation for the chromatic number of $G^{\frac{k}{k}}$ when $k \geq 2$:

Theorem 1.2. *For every integer $k \geq 2$, there exists a constant C_k such that, for every graph G, we have:*

$$\left\lfloor \frac{k}{2} \right\rfloor \Delta(G) \leq \chi(G^{\frac{k}{k}}) \leq \left\lfloor \frac{k}{2} \right\rfloor \Delta(G) + C_k \log \Delta(G).$$

This short paper is organised as follows. In Section 2, we recall basic notions concerning the directed linear arboricity of a graph and prove Theorem 1.1. We sketch the proof of Theorem 1.2 and discuss the difficulties arising when trying to generalise our method to fractions $\frac{r}{s}$ where $r > s$ in Section 3.

2. The chromatic number of $G^{\frac{2}{3}}$: Proof of Theorem 1.1

The proof of Theorem 1.1 follows the arguments from the proof of Theorem 3.1 in [4]. There, the author considers the *directed star arboricity* $\mathrm{dst}(D)$ of a directed graph D, defined as the smallest number of directed star forests needed to cover D, where the edges of the star are directed away from the centre. The directed star arboricity is closely connected to the *incidence colouring number* $\iota(G)$ of a graph G. This is the smallest number of colours needed to colour the vertex–edge pairs (v, e), with $v \in e$, of G in such a way that (v, e) and (w, f) receive different colours if $v = w$ or $vw = e$ or $vw = f$. By viewing the pair (v, e) as an orientation of e towards v, we observe the following connection between these two notions: If $S(G)$ is the directed graph where each edge of G is replaced by both directed edges, then $\iota(G) = \mathrm{dst}(S(G))$. In [4], Guiduli showed that $\mathrm{dst}(D) \leq k + 20 \log k + 84$, where k is the larger of the maximum indegree and the maximum outdegree of D.

For an edge $e = vw$ in G, we write e^v for the neighbour of v in $G^{\frac{1}{3}}$ on the subdivision of e. In other words, the edge $e = vw$ in G defines the path v, e^v, e^w, w in $G^{\frac{1}{3}}$. In $G^{\frac{2}{3}}$, the vertices e^v, e^w are inner vertices, whereas the vertices v and w are branch vertices. For a branch vertex v, let $I_v := \{e^v : e \in E(G)\}$ be the set of all inner vertices which are neighbours of v in $G^{\frac{1}{3}}$. Note that $\{I_v\}_{v \in V(G)}$ partitions the set of inner vertices.

If we identify the inner vertex e^v of $G^{\frac{2}{3}}$ with the incidence pair (v, e), we can quickly see that $\iota(G)$ is the colouring number of $G^{\frac{2}{3}}$ when we only have to colour the inner vertices. In particular, $\chi(G^{\frac{2}{3}}) \geq \iota(G)$. We can easily complete a colouring of the inner vertices to a colouring of all of $G^{\frac{2}{3}}$ by using $\chi(G) \leq \Delta(G) + 1$ additional colours. Our result shows that we can in fact accomplish this task with only logarithmically many (in Δ) additional colours. Hence, Theorem 1.1 is a slight generalisation of Theorem 3.1 in [4].

We will follow the proof of [4] and adjust it in such a way that it becomes apparent that we can also colour the branch vertices of $G^{\frac{2}{3}}$. The proof uses a version of the Lovász Local Lemma (Lemma 3.4 in [4]), which we restate here for completeness.

Lemma 2.1. *Let H be a simple graph on vertex set $V = [n]$ with maximum degree $\Delta(H) \leq d$. A probability event A_i is associated with each vertex $i \in V$ such that $\Pr(A_i) \leq 1/(4d)$ and the event A_i is independent of all A_j for which j is not adjacent to i in H. Then $\Pr(\bar{A_i} \cap \cdots \cap \bar{A_n}) > 0$.*

Given a graph G with maximum degree Δ, our general strategy for colouring $\chi(G^{\frac{2}{3}})$ will be the following:

(1) Find a proper colouring c of the branch vertices of $G^{\frac{2}{3}}$ using colours in $[\Delta + 1]$.
(2) For each branch vertex v of $G^{\frac{2}{3}}$, find a list $L_v \subseteq [\Delta + C \log \Delta]$ of colours (C will be defined explicitly later) of size $\Theta(\log \Delta)$. For a branch vertex w and an edge $e = wv$ of G, we will colour the inner vertex e^w with a colour from the list L_v. For that, we will require the family $\{L_v\}_{v \in V(G)}$ to have the following property: for each branch vertex w, $\{L_v \setminus \{c(v), c(w)\}\}_{v \in N_G(w)}$ has a transversal[2] T_w.
(3) Colour the inner vertices of $G^{\frac{2}{3}}$ around w according to the transversal T_w obtained in Step (2).
At this point of the colouring process, the only monochromatic edges that can occur are of the form $e^v f^w$, where $e = vw$ and we allow $f = e$. But this can only happen if L_w contains the colour of f^w (note that the colour of e^v comes from L_w). Thus, for each branch vertex v, there are at most $\Theta(\log \Delta)$ many inner vertices of the form e^v that have to be recoloured. As a final step, we use a small number of additional colours to resolve the conflicts:
(4) Use $\Theta(\log \Delta)$ new colours to recolour every e^v for which there exists a monochromatic edge of the form $e^v f^w$ without creating new monochromatic edges.

Step (1) is implemented by invoking the `Greedy Colouring Algorithm`. Therefore, we only need to justify the existence of transversals in Step (2), as well as the fact that $\Theta(\log \Delta)$ of new colours suffice for the recolouring in Step (4).

Before going into the proof of Step (2) (which is based on the Lovász Local Lemma) we need to prove Lemma 2.2, stated below. A version of Lemma 2.2 where r is taken to be at least $5 \log k + 20$ and all $F_1, \ldots, F_k$ equal the empty set was proven in [2] (see Lemma 2.5 therein). The two proofs are very similar and are based on verifying Hall's condition.

Lemma 2.2. *There exists an integer k_0 such that, for all $k \geq k_0$ and every integer r satisfying $7 \log k \leq r \leq k$, the following holds: Let $S_1, \ldots, S_k$ be independent random subsets of $[k+r]$, each of which is generated by sampling r elements from $[k+r]$ uniformly and independently at random with replacement. Furthermore, let $F_1, \ldots, F_k$ be arbitrary fixed subsets of $[k+r]$ of size two. Then the probability that the family of sets $\{S_1 \setminus F_1, \ldots, S_k \setminus F_k\}$ does not have a transversal is at most $k^{1-\frac{r}{5}}$.*

Note that the number 5 in the exponent is not optimal. A more careful analysis can lead to a better constant.

Proof. Our goal is to show that the family $\{S_1 \setminus F_1, \ldots, S_k \setminus F_k\}$ violates Hall's condition, and therefore has no transversal, with probability at most $k^{1-\frac{r}{5}}$. For $j \in [k]$, let P_j be the probability that there is a set $J \subseteq [k]$ of size j with $\left|\bigcup_{i \in J} S_i \setminus F_i\right| < |J|$. Our aim is to show that $P_j \leq k^{-\frac{r}{5}}$ for each j, which implies that Hall's Theorem is violated with probability at most $\sum_{j=1}^{k} P_j \leq k^{1-\frac{r}{5}}$. We have

$$P_j \leq \binom{k}{j}\binom{k+r}{j}\left(\frac{j+2}{k+r}\right)^{rj} \leq \binom{k+r}{j}^2\left(\frac{j+2}{k+r}\right)^{rj},$$

since there are $\binom{k}{j}$ ways to choose $J \subseteq [k]$ with $|J| = j$, $\binom{k+r}{j}$ ways to pick a subset $S \subseteq [k+r]$ of size j, and $\left(\frac{j+2}{k+r}\right)^{rj}$ is an upper bound for the probability that $\bigcup_{i \in J} S_i \setminus F_i \subseteq S$ (this implies that $\left|\bigcup_{i \in J} S_i \setminus F_i\right| \leq |J|$, which contains the event $\left|\bigcup_{i \in J} S_i \setminus F_i\right| < |J|$).

In order to study this quantity we need to distinguish three cases. In all cases, k will be sufficiently large.

Case 1: $\frac{k+r}{2} \leq j \leq k$. Then,

$$\begin{aligned}
P_j &\leq \binom{k+r}{k+r-j}^2\left(1 - \frac{k+r-j-2}{k+r}\right)^{rj} \\
&\leq (k+r)^{2(k+r-j)} \exp\left(-\frac{rj(k+r-j-2)}{k+r}\right) \\
&= \exp\Big((k+r-j)\Big(2\log(k+r) - \frac{k+r-j-2}{k+r-j} \cdot \frac{rj}{k+r}\Big)\Big) \\
&\leq \exp\Big((k+r-j)\Big(2\log(k+k) - \frac{2}{3} \cdot \frac{r}{2}\Big)\Big) \\
&\leq \exp\Big((k+r-j)\Big(2\log k + 2 - \frac{r}{3}\Big)\Big).
\end{aligned}$$

[2] A *transversal* of a family $S_1, \ldots, S_m$ of sets consists of m distinct elements $x_1, \ldots, x_m$ such that $x_i \in S_i$.

For the second inequality we used the inequalities $\binom{a}{b} \le a^b$ and $1-x \le e^{-x}$. Furthermore, we used the assumptions that $\frac{k+r}{2} \le j \le k$ and $k \ge k_0$ is sufficiently large to deduce that $\frac{j}{k+r} \ge \frac{1}{2}$ and $\frac{k+r-j-2}{k+r-j} \ge \frac{2}{3}$, used in the fourth inequality. Since $r \ge 7\log k$ and $k \ge k_0$ is sufficiently large, the above quantity is bounded above by

$$\exp\left(-(k+r-j)\left(\frac{1}{4}\log k - 2\right)\right) \le \exp\Big(-\frac{r}{5}\log k\Big).$$

Case 2: $\log k \le j \le \frac{k+r}{2}$. In this case we have

$$P_j \le \binom{k+r}{j}^2 \left(\frac{j+2}{k+r}\right)^{rj} \le \left(\frac{e(k+r)}{j}\right)^{2j} \left(\frac{1.1\cdot j}{k+r}\right)^{rj}$$
$$= \left(e^2\cdot 1.1^r\Big(\frac{j}{k+r}\Big)^{r-2}\right)^j \le \left(e^2\cdot 1.1^r\Big(\frac{1}{2}\Big)^{\frac{r}{2}}\right)^{\log k} \le \exp\Big(-\frac{r}{5}\log k\Big).$$

For the second inequality, we used that $\binom{n}{x} \le \left(\frac{en}{x}\right)^x$ for every integer $1 \le x \le n$. We used the assumptions that $\log k \le j \le \frac{k+r}{2}$ and $k \ge k_0$ is sufficiently large to deduce that $j+2 \le 1.1j$ and $\frac{j}{k+r} \le \frac{1}{2}$ in the second and third inequalities respectively. For the last inequality, we used that $1.1^r\left(\frac{1}{2}\right)^{\frac{r}{2}} \le \exp\left(-\frac{r}{4}\right)$ holds for all positive r.

Case 3: $1 \le j \le \log k$. Then

$$\frac{3j}{k+r} \le \frac{3\log k}{k} \le \frac{1}{\sqrt{k}}.$$

Hence,

$$P_j \le \binom{k+r}{j}^2 \left(\frac{j+2}{k+r}\right)^{rj} \le \left(\frac{e(k+r)}{j}\right)^{2j} \left(\frac{3\cdot j}{k+r}\right)^{rj}$$
$$= \left(e^2 3^2\left(\frac{3j}{k+r}\right)^{r-2}\right)^j \le \left(100\left(\frac{1}{\sqrt{k}}\right)^{r-2}\right)^j \le \left(\frac{1}{\sqrt{k}}\right)^{r-3}$$
$$= \exp\left(-\frac{1}{2}(r-3)\log k\right) \le \exp\Big(-\frac{r}{5}\log k\Big).$$

In the second inequality, we again used that $\binom{n}{x} \le \left(\frac{en}{x}\right)^x$.

This shows that, for each $j \in [k]$, we have $P_j \le k^{-\frac{r}{5}}$, as needed. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let Δ be large enough. We will start by defining a (not necessarily proper) colouring of $G^{\frac{2}{3}}$ using at most $\Delta + 7\log\Delta$ colours. Let $V := V(G)$ be the set of branch vertices of $G^{\frac{2}{3}}$. We start by taking a proper colouring of G with $\Delta+1$ colours, $c : V(G) \to [\Delta+1]$, which exists by the `Greedy Colouring Algorithm`. Such a colouring defines a colouring on the branch vertices of $G^{\frac{2}{3}}$ in which two incident branch vertices have different colours.

We continue with Step (2), which is the content of the following claim:

Claim 2.3. *There exists an assignment of a list $L_v \subseteq [\Delta + 7\log\Delta]$ of size $7\log\Delta$ to each vertex $v \in V$ such that, for each $w \in V$, the set family $\{L_v \setminus \{c(v), c(w)\} : v \in N_G(w)\}$ has a transversal T_w.*

Proof of Claim 2.3. For each $v \in V$, generate L_v randomly by performing $7\log\Delta$ independent uniform samplings from $[\Delta + 7\log\Delta]$. Let B_w be the bad event that the family $\{L_v \setminus \{c(v), c(w)\} : v \in N_G(w)\}$ does not have a transversal. By Lemma 2.2, we have $\Pr(B_w) \le \Delta^{1-\frac{7\log\Delta}{5}}$. Furthermore, B_w is independent of all but at most Δ^2 other B_v's, namely those corresponding to vertices at distance at most two from w in G. Hence, the dependency graph has degree at most Δ^2 and $\Pr(B_w) \le 1/(4\Delta^2)$ for Δ large enough. Hence, applying the Lovász Local Lemma (Lemma 2.1) gives that the probability that no bad event happens is positive. In particular, the required list assignment exists. □

Let $\{L_v\}_{v\in V}$ be a collection of lists such that all the transversals T_w exist, guaranteed by Claim 2.3. Now we extend the colouring c to the inner vertices: For each edge $e = vw$, let $c(e^v)$ be the transversal element of T_v corresponding to the set $L_w \setminus \{c(v), c(w)\}$. This colouring is not necessarily proper, but note that there cannot be a monochromatic edge between a branch vertex and an inner vertex because we excluded $\{c(v), c(w)\}$ from L_w in T_v. Furthermore, an edge of the form $e_1^v e_2^v$ can also not be monochromatic since these two colours come from the same transversal T_v.

The only conflicts that remain are those between two inner vertices of the form e^v and f^w with $v \ne w$. Since they are connected in $G^{\frac{2}{3}}$, they must be of distance at most three in $G^{\frac{1}{3}}$. Unless $f = e = vw$, this shortest path must pass through either v or w. We call this branch vertex the *corresponding branch vertex* for the conflict (e^v, f^w). If $f = e = vw$, we choose v or w arbitrarily and call it the corresponding branch vertex.

Now consider a conflicting pair (e^v, f^w) of inner vertices and assume, without loss of generality, that v is the corresponding branch vertex. It follows that $f = vw$ and $c(e^v) = c(f^w) \in L_v$. Therefore, if for each $v \in V$ we properly recolour all e^v with $c(e^v) \in L_v$ with new colours, then we have found a proper colouring of $G^{\frac{2}{3}}$. Let I_v be the set of all inner vertices e^v satisfying $c(e^v) \in L_v$. Observe that, as there are no conflicts of the form (e^v, f^w) with $v = w$, all the colours $c(e^v)$ for $e^v \in I_v$ are distinct and contained in L_v. Thus $|I_v| \le |L_v|$.

Claim 2.4. *There is a proper colouring of $\bigcup_{v \in V} I_v$ using at most $21 \log \Delta$ new colours.*

In the proof of Claim 2.4 we use the following auxiliary lemma, taken from [4].

Lemma 2.5 (*Lemma 3.2 in [4]*). *Let D be a directed graph (with possible multiple edges) such that every vertex has indegree at most c. Then $dst(D) \le 3c$.*

Proof of Claim 2.4. Let $S = \bigcup_{x \in V(G)} I_x$ be the set of vertices in $G^{\frac{2}{3}}$ to be recoloured with new colours. Consider the directed graph D with $V(D) = V(G)$ and $E(D) = \{(w, v) : e = vw, e^v \in S\}$. Now each vertex in $e^v \in S$ corresponds to a directed edge in D. Observe also that, for each $w \in V(D)$, the set of edges directed away from w corresponds to a subset of S that is an independent set in $G^{\frac{2}{3}}$. Similarly, a directed star forest where the edges of each star are directed away from its centre in D corresponds to an independent set in $G^{\frac{2}{3}}$. Furthermore, the maximum indegree in D is at most $\max_{v \in V} |I_v| \le \max_{v \in V} |L_v| \le 7 \log \Delta$. By Lemma 2.5 , this implies that D can be partitioned into at most $21 \log \Delta$ directed star forests where the edges of each star are directed away from its centre. By using a different new colour for each of the corresponding independent sets in $G^{\frac{2}{3}}$, we obtain the required colouring. □

After introducing these $21 \log \Delta$ new colours, we are using $\Delta + 28 \log \Delta$ colours in total to properly colour$G^{\frac{2}{3}}$, as claimed. □

3. Proof of Theorem 1.2 and further comments

In this paper, we obtained asymptotically tight bounds for $\chi(G^{\frac{2}{3}})$ as the maximum degree of G grows. The method used to obtain upper bounds for $\chi(G^{\frac{2}{3}})$ can be extended to yield upper bounds for $\chi(G^{\frac{k}{k}})$. Specifically, for every integer $k \ge 2$ there exists a constant C_k such that for every graph G the following holds:

$$\left\lfloor \frac{k}{2} \right\rfloor \Delta(G) \le \chi(G^{\frac{k}{k}}) \le \left\lfloor \frac{k}{2} \right\rfloor \Delta(G) + C_k \log \Delta(G).$$

The lower bound of $\lfloor k/2 \rfloor \Delta(G)$ comes from the cliques that are 'centred' around branch vertices of maximum degree. For the upper bound on $\chi(G^{\frac{k}{k}})$, we follow the same four-step strategy as for $\chi(G^{\frac{2}{3}})$, which slightly differs depending on the parity of k in the first two steps. If k is even, then in Step (1) we use a proper colouring with $\Delta(G) + 1$ colours to colour the branch vertices of $G^{\frac{k}{k}}$. Otherwise, if k is odd, then in Step (1), we use a total colouring of G with $\Delta(G) + O(1)$ colours, which is known to exist by [9], to colour the branch vertices along with the middle vertices on the subdivided edges.

In Step (2), we assign a list $L_v \subseteq [\lfloor \frac{k}{2} \rfloor \Delta(G) + C'_k \log \Delta(G)]$ of size $\Theta(\log \Delta(G))$ to every vertex v. For a branch vertex w and an edge $e = wv$ of G, we will colour the inner vertices of e that are strictly closer to w than to v with a colour from the list L_v. For that, we will require the family $\{L_v\}_{v \in V(G)}$ to have the following property: for each branch vertex w, there exist (simple) subsets $L_{v,w} \subseteq L_v \setminus F_{v,w}$ of size $\lfloor k/2 \rfloor$ for each $v \in N_G(w)$ such that no element appears in two different sets $L_{v,w}$ and $L_{v',w}$. Here $F_{v,w}$ has size at most three and contains $c(v)$, $c(w)$, and, when k is odd, the colour of the middle vertex of the subdivided edge vw. To study the likelihood of the existence of these subsets of $\{L_v\}_{v \in V(G)}$ we appeal to the generalised Hall's condition in place of Hall's condition, that is, $|N(S)| \ge \lfloor k/2 \rfloor |S|$ (see [1, Corollary 1.2]), for every set S. Steps (3) and (4) are identical.

A more complicated problem arises when dealing with fractions $\frac{r}{s}$ when r is greater than s. In this situation, we must use more colours than the ones used by Brooks' Theorem, as the colouring of a specific branch vertex may influence not only their neighbours in G, but vertices at a higher distance.

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Data availability

No data was used for the research described in the article.

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