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# BIRKHOFF CONJECTURE FOR NEARLY CENTRALLY SYMMETRIC DOMAINS

V. Kaloshin, C.E. Koudjinan, and Ke Zhang

**Abstract.** In this paper we prove a perturbative version of a remarkable Bialy–Mironov (Ann. Math. 196(1):389–413, 2022) result. They prove non perturbative Birkhoff conjecture for centrally-symmetric convex domains, namely, a centrally-symmetric convex domain with integrable billiard is ellipse. We combine techniques from Bialy–Mironov (Ann. Math. 196(1):389–413, 2022) with a local result by Kaloshin–Sorrentino (Ann. Math. 188(1):315–380, 2018) and show that a domain close enough to a centrally symmetric one with integrable billiard is ellipse. To combine these results we derive a slight extension of Bialy–Mironov (Ann. Math. 196(1):389–413, 2022) by proving that a notion of rational integrability is equivalent to the $C^0$-integrability condition used in their paper.

## 1 Introduction

A mathematical billiard is a system describing the inertial motion of a point mass inside a domain, with elastic reflections at the boundary (which is assumed to have infinite mass). This simple model was first proposed by G. D. Birkhoff as a mathematical playground where "the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting qualitative questions need to be considered," [Bir20, pp. 155–156].

This dynamical system associated to billiards has simple local dynamics, however, its study turns out to be really complex and has many important open questions, see e.g. [Gut12]. In this paper we study integrable billiards and Birkhoff conjecture. Let us first recall some properties of the billiard map. We refer e.g. to [Tab05, Sib04], for a more comprehensive introduction to the study of billiards.

Let $\Omega$ be a bounded strictly convex domain in $\mathbb{R}^2$ (for short *billiard table*) with $C^r$ boundary $\partial\Omega$, with $r \geq 3$.[1] The phase space $\mathscr{M}$ of the billiard map consists of unit vectors $(x, v)$ whose foot points $x$ are on $\partial\Omega$ and that have inward directions. The billiard ball map $T : \mathscr{M} \to \mathscr{M}$ takes $(x, v)$ to $(x', v')$, where $x'$ represents the point where the trajectory starting at $x$ with velocity $v$ hits the boundary $\partial\Omega$ again, and $v'$ is the reflected velocity, according to the standard reflection law: angle of incidence

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[1] Observe that if $\Omega$ is not convex, then the billiard map is not continuous; in this article we will be interested only in strictly convex domains. Moreover, as pointed out by Halpern [Hal77], if the boundary is not at least $C^3$, then the flow might not be complete.

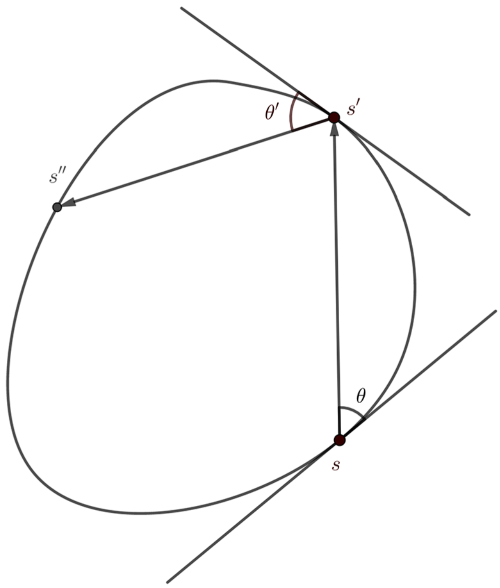


Figure 1: Billiard map: $T(s,\theta) = (s',\theta')$.

is equal to the angle of reflection. Assume that the boundary $\partial\Omega$ is parametrized by arc–length $s$ and let $\gamma : \mathbb{T}_{|\partial\Omega|} \to \mathbb{R}^2$ denote such a parametrization, where $\mathbb{T}_{|\partial\Omega|} := \mathbb{R}/\mathbb{Z}\cdot|\partial\Omega|$ and $|\partial\Omega|$ is the length of $\partial\Omega$. Let $\theta$ be the angle between $v$ and the positive tangent to $\partial\Omega$ at $s$. Hence, $\mathscr{M}$ can be identified with the annulus $\mathbb{A}_\Omega := \mathbb{T}_{|\partial\Omega|} \times (0,\pi)$ and the billiard map $T$ can be described as (see *fig.* 1)

$$T\colon (s,\theta) \longmapsto (s',\theta'),$$

where $(s,\theta),(s',\theta') \in \mathbb{A}_\Omega$.

A billiard trajectory is called $(p,q)$–periodic if the trajectory is $q$–periodic and the orbit winds around the boundary $p$ times within one period. Equivalently, after lifing the trajectory $\{(s_n,\theta_n)\}_{n\in\mathbb{Z}}$ to the universal cover $\mathbb{R}\times(0,\pi)$, we have $s_q = s_0 + p|\partial\Omega|$. For such orbits, the rotation number is given by $p/q$. Furthermore, there exist at least two periodic orbits of rotation number $p/q$,[2] for any rational $p/q$ by a Theorem by Birkhoff [Bir13].

The central objects of this paper are the notions of *caustics* and integrability. To state a version of Birkhoff conjecture, we introduce some notions.

Definition 1. **(i)** A curve $\mathscr{C} \subseteq \Omega$ is called a caustic for the billiard on the table $\Omega$ if any billiard orbit having one segment tangent to $\mathscr{C}$ has all its segments tangent to it.

A caustic is called $(p,q)$–rationally integrable ($p,q \in \mathbb{N}$ with $p \le q/2$) if all its tangential orbits are $(p,q)$–periodic; whenever $p=1$, we shall simply call such a caustic $q$–rationally integrable.

[2] In the present paper, rationals are considered in reduced form.

**(ii)** A billiard table $\Omega$ is called $q_0$–rationally integrable (*resp.* weakly integrable) if it admits a $(p,q)$–rationally (*resp.* a $q$–rationally) integrable caustic for each $0 < p/q \leq 1/q_0$ (*resp.* $q \geq q_0$).

We start with a classical result by M. Bialy [Bia93] who proved the following theorem concerning global integrability: if the phase space of the billiard ball map is globally foliated by continuous invariant curves that are not null–homotopic, then it is a circular billiard.

A strong version of Birkhoff conjecture states that for any $q_0 > 1$, a $q_0$–rationally weakly integrable billiard is a billiard in an ellipse. For domains close to ellipses this conjectures was settled for $q_0 = 3$ in [ASK16, KS18] and for $q_0 = 4, 5$ in [HKS18]. For $q_0 > 3$ in a recent work of Koval [Kov21]. Recently Bialy-Mironov [BM17] proved a remarkable global result stating that any centrally symmetric domain that is $C^0$-integrable in the phase space between the boundary and a 4–rationally integrable caustic is an ellipse [BM22].

Our main result is a perturbative version of this result and is a combination of [KS18] and [BM22]: *a 4-rationally integrable domain with a 3–rationally integrable caustic sufficiently close to any centrally symmetric domain is an ellipse.*

**1.1 Main result (a technical formulation).** Let $\Omega$ be a strictly convex subset of $\mathbb{R}^2$ containing the origin $O$, with boundary $\partial\Omega$. It is more convenient to use the support function to represent the boundary, defined by

$$h_\Omega(\psi) = \sup\{x\cos\psi + y\sin\psi : \quad (x,y) \in \Omega\}, \qquad \psi \in [0, 2\pi].$$

(see *fig.* 3). Denoting by $\gamma_\Omega(\psi) = (x_\Omega(\psi), y_\Omega(\psi))$ the Cartesean coordinates of the point on $\partial\Omega$ corresponding to $(\psi, h_\Omega(\psi))$, we have[3]

$$\begin{cases} x_\Omega(\psi) = h_\Omega(\psi)\cos\psi - h'_\Omega(\psi)\sin\psi \\ y_\Omega(\psi) = h_\Omega(\psi)\sin\psi + h'_\Omega(\psi)\cos\psi, \end{cases} \tag{1}$$

where $h'_\Omega$ denotes the derivative of $h_\Omega$. In particular, if the support functions of two boundaries are $C^m$-close, then their corresponding parametrization (1) by $\psi$ are $C^{m-1}$-close.

Moreover, let $h_\Omega \in C^m$. Then, denoting by $s(\psi)$ the arc–length parameter corresponding to $\psi$, we have

$$s'(\psi) = \sqrt{x'_\Omega(\psi)^2 + y'_\Omega(\psi)^2} \overset{(1)}{=} h_\Omega(\psi) + h''_\Omega(\psi) = 1/k_\Omega(\psi) > 0, \tag{2}$$

where $k_\Omega(\psi)$ is the curvature of $\partial\Omega$ *w.r.t.* the parameter $\psi$ and for the last equality we refer the reader to [Res15]. The change of parametrization $\psi \mapsto s(\psi)$ is a $C^{m-1}$–diffeomorphism.

[3] We refer the reader to [Res15] for more details.

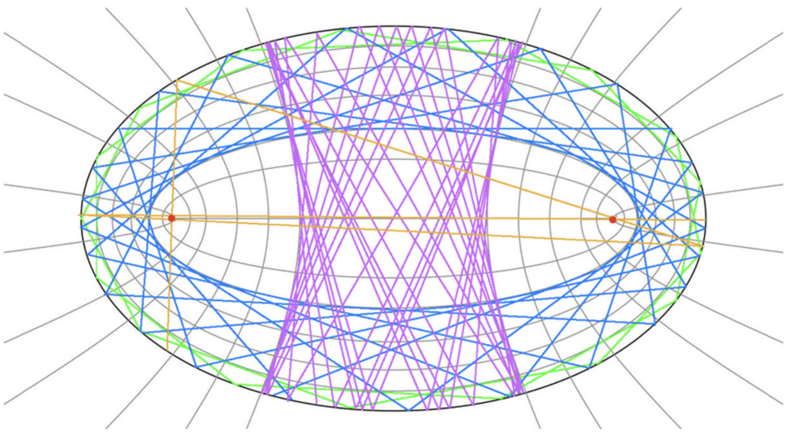

Figure 2: Billiard in an ellipse.

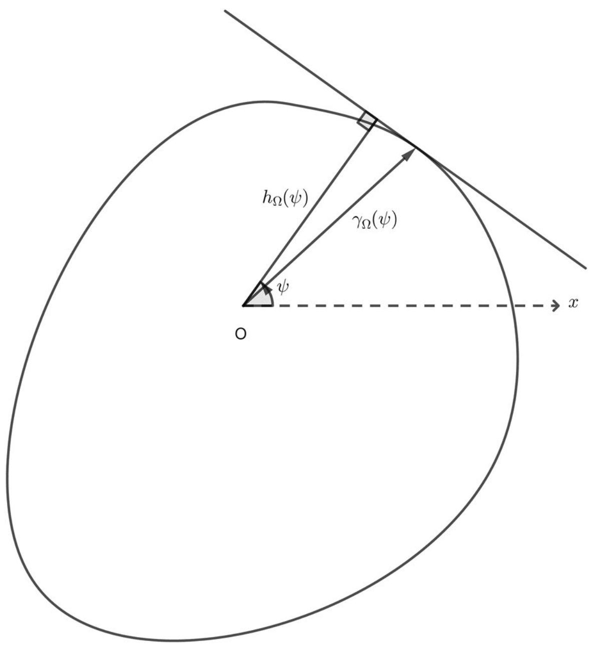


Figure 3: Support function of $\Omega$.

Example The support function of the ellipse $\mathcal{E}_{a,b} \coloneqq \{(x,y) \in \mathbb{R}^2 : \ x^2/a^2 + y^2/b^2 = 1\}$, $a \geq b > 0$, is given by

$$h_{\mathcal{E}_{a,b}}(\psi) = \sqrt{a^2 \cos^2 \psi + b^2 \sin^2 \psi} = a\sqrt{1 - e^2 \sin^2 \psi}, \qquad e \coloneqq \sqrt{1 - (b/a)^2}. \tag{3}$$

Moreover, the billiard dynamic in a non–degenerate ellipse (*i.e.* $a > b$) is illustrated in *fig.* 2.

**Theorem 2.** *Let $\Omega_0$ be a centrally symmetric, strictly convex domain with support function $h_0 \in C^m(\mathbb{R}/(2\pi\mathbb{Z}))$, with $m \geq 40$. Assume that the curvature satisfies*

$$k_{\Omega_0}(\psi) > \kappa_0 > 0,$$

*and $\|h_0\|_{C^m} < M$. Then there exists $\varepsilon_0 > 0$ depending only on $\kappa_0$, $|\partial\Omega_0|$, and $M$, such that the following holds.*

*Assume that the domain $\Omega_\varepsilon$ with support function $h_\varepsilon$ satisfies*

$$\|h_\varepsilon\|_{C^m} < 2M, \quad \|h_\varepsilon - h\|_{C^5} < \varepsilon_0,$$

*and $\Omega_\varepsilon$ admits a 3–rationally integrable caustic, and satisfies either of the following integrability conditions*:

(1) *The phase space between the 4–rationally integrable caustic and the boundary is foliated by continuous invariant graphs over the $\mathbb{T}$ component.*
(2) *The billiard is 4-rationally integrable.*

*Then $\Omega_\varepsilon$ is an ellipse.*

Remark 3. (i) Condition (1) in Theorem 2, called $C^0$-integrability, is identical to the one in [BM22]. We will show that condition (2) is equivalent to condition (1), following arguments of [A+15].

(ii) There are two different notions of "rational integrability" in the literature. In [KS18], it corresponds to our "rational weak integrability". Our definition of "rational integrability" agrees with the one in [Kov21] and is strictly stronger (since we also require $(p, q)$-caustics to exist).

If a billiard is 4-rationally integrable, and in addition admits a 3-rationally integrable caustic, then the billiard is 3-rationally weakly integrable, whence verifies the condition in [KS18].

The reminder of the paper is organized as follows. In § 2, after recalling the classical (§ 2.1) and the non–standard (§ 2.2) symplectic coordinates systems of the billiard map, we provide quantitative estimates on the change between the corresponding parameters (§ 2.3). In § 3.1, we provide some properties of the 4–periodic orbits in a billiard table which is close to a centrally symmetric one (Theorem 10). In § 3.2, we prove that a nearly–centrally–symmetric and 4–rationally integrable billiard table is necessarily close to some ellipse $\mathcal{E}_0$. In § 3.3.1, we prove the closeness of the billiard table to the ellipse $\mathcal{E}_0$ in the elliptic polar coordinates associated to $\mathcal{E}_0$ (Lemma 17). In § 3.3.3, we complete the proof of the main Theorem 2. We collect in § Appendix A some technical facts. In § Appendix B, we prove an interpolation–type result. In § Appendix C, we prove the uniqueness of the $(p, q)$-loop orbit, which is a more general version of $q$-loop orbits needed for our proof. See Definition 5. In § Appendix D, we prove the equivalence between the rational and the $C^0$–integrability.

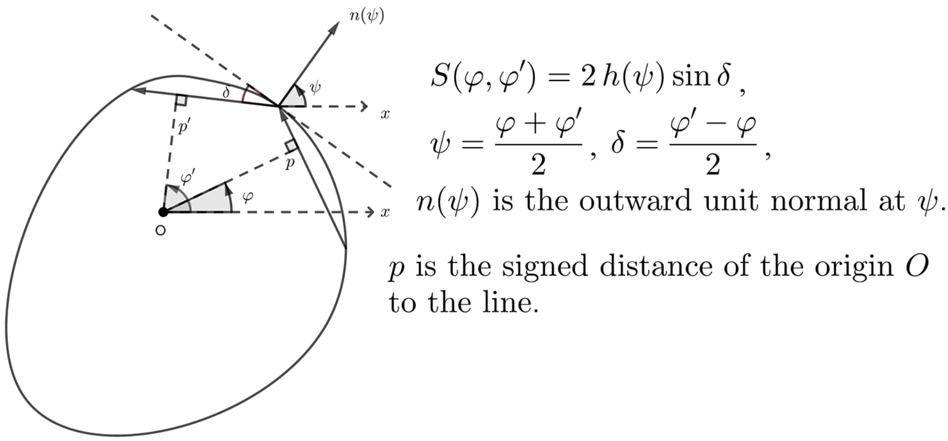


Figure 4: Non standard Bialy–Mironov generating function.

## 2 Generating functions and some auxiliary facts

**2.1 The traditional billiard generating function.** Denote the Euclidean distance between two points of $\partial\Omega$ by

$$l(s, s') := |\gamma(s) - \gamma(s')|.$$

Then, one checks easily

$$\partial_s l(s, s') = -\cos\theta, \qquad \partial_{s'} l(s, s') = \cos\theta', \tag{4}$$

where $T(s, \theta) = (s', \theta')$. In particular, if we lift everything to the universal cover and introduce the new coordinates $(x, y) = (s, \cos\theta) \in \mathbb{R} \times [-1, 1]$, then the billiard map is a twist map with $l$ as generating function, and it preserves the area form $dx \wedge dy$.[4]

For the main text of this paper, we will only use the Bialy–Mironov generating function introduced in the next section. However, we do use the traditional generating function in the Appendix.

**2.2 Bialy–Mironov generating function.** The following non–standard generating function for the billiard map has been discovered by Bialy and Mironov [BM17] (see *fig.* 4)

$$S(\varphi, \varphi') = 2\, h\left(\frac{\varphi + \varphi'}{2}\right) \sin\left(\frac{\varphi' - \varphi}{2}\right).$$

In the $(\varphi, p)$–coordinates, the billiard map $T$ reads: $T(\varphi, p) = (\varphi', p')$ *iff*

$$\begin{cases} p = -S_1(\varphi, \varphi') = h(\psi)\cos\delta - h'(\psi)\sin\delta \\ p' = S_2(\varphi, \varphi') = h(\psi)\cos\delta + h'(\psi)\sin\delta \end{cases} \tag{5}$$

[4] See [Tab05, Sib04] for details.

where $\psi = \frac{1}{2}(\varphi + \varphi')$, $\delta = \frac{1}{2}(\varphi' - \varphi)$, and $S_1 := \partial_\varphi S$, $S_2 := \partial_{\varphi'} S$, (for later use) $S_{12} := \partial^2_{\varphi\varphi'} S$, *etc.*

Denote by $\gamma_\varepsilon$ the $\psi$–parametrization of $\partial\Omega$ given by (1), by $S_\varepsilon$ its Bialy–Mironov generating function and by $T_\varepsilon$ the billiard map in the coordinate $(\varphi, p)$.

**2.3 Some estimates using the support function parametrization.** Given $\varepsilon_1 > 0$, define the space

$$\mathcal{V} = \{h \in C^m(\mathbb{T}) : \quad \|h\|_{C^m} < 2M, \quad \|h - h_0\|_{C^5} < \varepsilon_1\}. \tag{6}$$

We assuem that $\varepsilon_1$ is small enough so that the curvature function associated to $h \in \mathcal{V}$ satsifies

$$k_\Omega(\psi) > \kappa_0/2, \quad \forall \psi \in \mathbb{T}.$$

Moreover, $\varepsilon_1$ will be chosen so that Proposition 7 applies, which depends only on $\kappa_0$, $|\partial\Omega_0|$, $M$.

Throughout the proof we assume $h_\varepsilon \in \mathcal{V}$ and set

$$\varepsilon = \|h_\varepsilon - h_0\|_{C^5}.$$

We shall write $f = O_n(\varepsilon)$ if $\|f\|_{C^n} \le C_1\varepsilon$ for a constant $C_1$ depending only on $\kappa_0$, $|\partial\Omega_0|$, $M$. In particular, $f = O_n(1)$ means $f$ has bounded $C^n$ norm.

Denote by $s_\varepsilon$ the arc–length parameter of $\partial\Omega$. Then,

LEMMA 4. *The change of parametrization $\psi \mapsto s_\varepsilon(\psi)$ is a $C^{m-1}$–diffeomorphism with inverse $s \mapsto \psi_\varepsilon(s)$, and we have*

$$\psi_\varepsilon = O_{m-1}(1), \quad s_\varepsilon = O_{m-1}(1), \quad \widetilde{\psi}_\varepsilon = O_4(\varepsilon), \quad \tilde{s}_\varepsilon = O_4(\varepsilon), \tag{7}$$

*where $\widetilde{\psi}_\varepsilon := \psi_\varepsilon - \psi_0$ and $\tilde{s}_\varepsilon := s_\varepsilon - s_0$.*

*Proof.* Indeed, (2) implies $s_\varepsilon$ is a $C^{m-1}$–diffeomorphism and $s'_\varepsilon = h_\varepsilon + h''_\varepsilon = h_0 + h''_0 + O_3(\varepsilon) = s'_0 + O_3(\varepsilon)$. Thus, $\widetilde{s}_\varepsilon = O_4(\varepsilon)$. Therefore, by the implicit function theorem, its inverse $\psi_\varepsilon$ satisfies $\widetilde{\psi}_\varepsilon = O_4(\varepsilon)$. □

## 3 Proof of Theorem 2

**3.1 Properties of 4–periodic orbits in nearly–centrally–symmetric billiard tables.** In this section, we prove that the properties of the one parameter family of 4–gons “persists” up to small corrections on tables which are close to a centrally–symmetric one and admits a 4–rationally integrable caustic.

DEFINITION 5. Given $q \ge 2$, an orbit segment of the billiard is called a *q-loop orbit* if the segment bounces $q$ times back to its starting point, winding around the table exactly once.

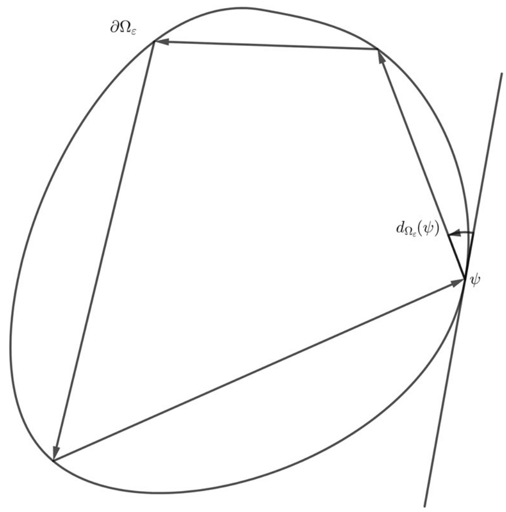


Figure 5: The 4–loop angle function $d_{\Omega_\varepsilon}$ in $\partial\Omega_\varepsilon$.

Remark 6. A q-loop orbit is not necessarily a periodic orbit, as the final angle of incidence is not required to equal to the inital angle of reflection.

For a billiard $\Omega$ with parametrized boundary $\gamma$, the $q$-loop orbit starting at any $\gamma(\psi)$ always exists, by taking the maximum perimeter $q$-gon that starts and ends at $\gamma(\psi)$. It is unique if the billiard is nearly circular, see [HZ22]. In Proposition 7, we show that the 4-loop orbit exists for both the billiards $\Omega_0$ and $\Omega_\varepsilon$.

Proposition 7. *There exists $\varepsilon_2 > 0$ and $C > 0$ depending only on $\kappa_0$ and $\|\gamma_0\|_{C^4}$, such that the following hold.*

*Consider the space of closed curves $\gamma : \mathbb{T} \to \mathbb{R}^2$*

$$\mathcal{V}_\gamma = \{\gamma : \|\gamma - \gamma_0\|_{C^2} < \varepsilon_2, \quad \|\gamma\|_{C^4} < C\},$$

*then each $\gamma$ is the boundary of a convex billiard (since $\varepsilon_2$ is small enough). Suppose there exists at least one $\gamma_\varepsilon \in \mathcal{V}_\gamma$ such that $\gamma_\varepsilon$ admits a 4–rationally integrable caustic, then for all $\gamma \in \mathcal{V}_\gamma$, the corresponding billiard admits unique 4-loop orbit starting at every point on the boundary.*

Proposition 7 follows from a more general result for $(p, q)$-loop orbit, for both twist map and billiards. The proof is presented in Appendix C.

Definition 8. Assume the billiard $\Omega$ admits a unique 4-loop orbit starting at any point on the boundary. We define $d_\Omega(\psi) \in (0, \pi)$ to be the unique initial angle for the unique 4-loop orbit of the billiard in $\Omega$ starting at the point $\gamma(\psi)$.(see *fig.* 5).

We can choose $\varepsilon_1$ in (6) small enough so that $h_\varepsilon \in \mathcal{V}$ implies $\gamma_\varepsilon \in \mathcal{V}_\gamma$. Therefore the 4-loop angle functions $d_\varepsilon$ and $d_0$ are both defined.

The following lemma is derived from the proof of Proposition 7, whose proof is in Appendix C.

LEMMA 9. (i) *$d_\varepsilon$ is $C^{m-1}$ with $d_\varepsilon = O_{m-1}(1)$, and $\tilde{d}_\varepsilon = d_\varepsilon - d_0 = O_3(\varepsilon)$.*
(ii) *There exists $0 < \underline{d} < \overline{d} < \pi$ such that $\underline{d} < d_0,\ d_\varepsilon < \overline{d}$.*

We have the following.

**Theorem 10** (Perturbative Theorem 4.1 [BM22]). *(i) $d_\varepsilon(\psi+\pi) = d_\varepsilon(\psi) + O_3(\varepsilon)$ and each 4–periodic billiard trajectory is $O_2(\varepsilon)$–close to a parallelogram.*

*(ii) The tangent line to $\Omega$ at the vertices of any 4–periodic billiard trajectory form a quadrilateral which is $O_2(\varepsilon)$–close to a rectangle.*

*(iii) $d_\varepsilon(\psi+\frac{\pi}{2}) = \frac{\pi}{2} - d_\varepsilon(\psi) + O_2(\varepsilon)$.*

*(iv) There exists $R_0 > 0$ such that*

$$h_\varepsilon^2(\psi) + h_\varepsilon^2(\psi + \frac{\pi}{2}) = R_0^2 + O_3(\varepsilon) =: R_\varepsilon^2\,, \tag{8}$$

$$h_\varepsilon(\psi) = R_0 \sin d_\varepsilon(\psi) + O_2(\varepsilon), \qquad h_\varepsilon(\psi+\frac{\pi}{2}) = R_0 \cos d_\varepsilon(\psi) + O_2(\varepsilon)\,. \tag{9}$$

*Proof.* (i) Observe that, by Lemma A.1, the central symmetry of $\Omega_0$ implies $d_0(\psi + \pi) = d_0(\psi)$. Then, by Lemma 9, we have

$$d_\varepsilon(\psi+\pi) = d_0(\psi+\pi) + \tilde{d}_\varepsilon(\psi+\pi) = d_0(\psi) + \tilde{d}_\varepsilon(\psi+\pi) =$$
$$d_\varepsilon(\psi) + \tilde{d}_\varepsilon(\psi+\pi) - \tilde{d}_\varepsilon(\psi) = d_\varepsilon(\psi) + O_3(\varepsilon).$$

Next, we show the last part of (i): writing

$$\Psi_\varepsilon^2(\psi) = \psi + \pi + f(\psi), \tag{10}$$

it is enough to show

$$f = O_2(\varepsilon). \tag{11}$$

Indeed, by Lemma A.3

$$\begin{aligned}
\Psi_\varepsilon^3(\psi) &= \Psi_\varepsilon(\psi + \pi + f(\psi)) \\
&= \Psi_\varepsilon(\psi+\pi) + \int_0^1 \Psi_\varepsilon'(\psi + \pi + tf(\psi))dt \cdot f(\psi) \\
&= \Psi_\varepsilon(\psi) + \pi + \widetilde{\Psi}_\varepsilon(\psi+\pi) - \widetilde{\Psi}_\varepsilon(\psi) + \int_0^1 \Psi_\varepsilon'(\psi + \pi + tf(\psi))dt \cdot f(\psi) \\
&=: \Psi_\varepsilon(\psi) + \pi + f_1(\psi),
\end{aligned}$$

and

$$\begin{aligned}
\psi + 2\pi &= \Psi_\varepsilon^4(\psi) \\
&= \Psi_\varepsilon(\Psi_\varepsilon(\psi) + \pi + f_1(\psi)) \\
&= \Psi_\varepsilon(\Psi_\varepsilon(\psi) + \pi) + \int_0^1 \Psi_\varepsilon'(\Psi_\varepsilon(\psi) + \pi + tf_1(\psi))dt \cdot f_1(\psi)
\end{aligned}$$

$$
\begin{aligned}
&= \Psi_\varepsilon^2(\psi) + \pi + \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)+\pi) - \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)) \\
&\quad + \int_0^1 \Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi+tf_1(\psi))dt \cdot f_1(\psi) \\
&= \psi + \pi + f(\psi) + \pi + \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)+\pi) - \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)) \\
&\quad + \int_0^1 \Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi+tf_1(\psi))dt \cdot f_1(\psi) \\
&=: \psi + 2\pi + f_2(\psi),
\end{aligned}
$$

*i.e.*

$$f_2(\psi) \equiv 0,$$

where

$$f_1(\psi) := \widetilde{\Psi}_\varepsilon(\psi+\pi) - \widetilde{\Psi}_\varepsilon(\psi) + \int_0^1 \Psi_\varepsilon'(\psi+\pi+tf(\psi))dt \cdot f(\psi), \tag{12}$$

$$
\begin{aligned}
f_2(\psi) &:= f(\psi) + \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)+\pi) - \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)) + \int_0^1 \Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi+tf_1(\psi))dt \cdot f_1(\psi) \\
&= \Big(1 + \Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi+t_1f_1(\psi)) \cdot \Psi_\varepsilon'(\psi+\pi+t_2f(\psi))\Big) f(\psi) + \\
&\quad + \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)+\pi) - \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)) + (\widetilde{\Psi}_\varepsilon(\psi+\pi) - \widetilde{\Psi}_\varepsilon(\psi))\Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi \\
&\quad + t_3f_1(\psi)),
\end{aligned} \tag{13}
$$

for some $t_1, t_2, t_3 \in (0,1)$.

Hence, as $0 \equiv f_2(\psi)$, it follows

$$
\begin{aligned}
-\Big(1 + \Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi+t_1f_1(\psi)) \cdot \Psi_\varepsilon'(\psi+\pi+t_2f(\psi))\Big) f(\psi) = \\
\widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)+\pi) - \widetilde{\Psi}_\varepsilon(\Psi_\varepsilon(\psi)) + \\
+ (\widetilde{\Psi}_\varepsilon(\psi+\pi) - \widetilde{\Psi}_\varepsilon(\psi)) \cdot \Psi_\varepsilon'(\Psi_\varepsilon(\psi)+\pi+t_3f_1(\psi)).
\end{aligned}
$$

By Lemma A.3, $\widetilde{\Psi}_\varepsilon = O_3(\varepsilon)$. Now, observe that $\Psi_\varepsilon$ is strictly increasing by the strict convexity of the caustic $\mathscr{C}$. Then, $\Psi_\varepsilon' > 0$ and

$$1 + \Psi_\varepsilon'(\psi_1) \cdot \Psi_\varepsilon'(\psi_2) > 1, \qquad \forall\, \psi_1, \psi_2 \in \mathbb{T}, \tag{14}$$

Thus, the Faà di Bruno's formula yields (11) by induction.

(ii) Indeed,

$$d_\varepsilon(\Psi_\varepsilon^2(\psi)) \overset{(10)}{=} d_\varepsilon(\psi+\pi+f_1(\psi)) \overset{(12),(11),(A.2)}{=} d_\varepsilon(\psi+\pi) + O_2(\varepsilon) \overset{(i)}{=} d_\varepsilon(\psi) + O_2(\varepsilon), \tag{15}$$

so that, as in [BM22],

$$2\pi = 2\left(d_\varepsilon(\psi) + d_\varepsilon(\Psi_\varepsilon(\psi)) + d_\varepsilon(\Psi_\varepsilon^2(\psi)) + d_\varepsilon(\Psi_\varepsilon^3(\psi))\right)$$

$$\overset{(15)}{=} 4\left(d_\varepsilon(\psi) + d_\varepsilon(\Psi_\varepsilon(\psi))\right) + O_2(\varepsilon)$$

*i.e.*

$$d_\varepsilon(\psi) + d_\varepsilon(\Psi_\varepsilon(\psi)) = \frac{\pi}{2} + O_2(\varepsilon), \tag{16}$$

proving (ii).

(iii) Recalling that $\psi$ is the angle formed by the outer unit normal of $\partial\Omega_\varepsilon$ at $\psi$ with the $x$–axis, (16) then implies

$$\Psi_\varepsilon(\psi) = \psi + \frac{\pi}{2} + O_2(\varepsilon), \tag{17}$$

which, together with (16) imply the last assertion in (iii).

Next, we prove $0 < \underline{d} \le d_\varepsilon(\psi) \le \overline{d} < \pi/2$. Indeed, $d_\varepsilon$ being continuous on the compact $\mathbb{T}$ then, it admits a minimum and a maximum *i.e.* there are $\psi_{min}, \psi_{max} \in \mathbb{T}$ *s.t.* $d_\varepsilon(\psi_{min}) \le d_\varepsilon(\psi) \le d_\varepsilon(\psi_{max})$, forall $\psi \in \mathbb{T}$. Obviously, $d_\varepsilon(\psi) > 0$ forall $\psi \in \mathbb{T}$. Thus, $d_\varepsilon(\psi_{min}) > 0$, and $0 < d_\varepsilon(\psi_{max} + \frac{\pi}{2}) = \frac{\pi}{2} - d_\varepsilon(\psi_{max}) + O_3(\varepsilon)$ which implies that $\frac{\pi}{2} - d_\varepsilon(\psi_{max}) > 0$, concluding the proof of (iii).

(iv) First of all, by Lemma A.1,

$$\begin{aligned} h_\varepsilon(\psi + \pi) = h_0(\psi + \pi) + \tilde{h}_\varepsilon(\psi + \pi) = h_0(\psi) + \tilde{h}_\varepsilon(\psi + \pi) = h_\varepsilon(\psi) \\ + \tilde{h}_\varepsilon(\psi + \pi) - \tilde{h}_\varepsilon(\psi). \end{aligned} \tag{18}$$

Then, just as in [BM22], we obtain on one hand

$$\begin{aligned} &h_\varepsilon(\psi)\cos d_\varepsilon(\psi) + h'_\varepsilon(\psi)\sin d_\varepsilon(\psi) \\ &\quad \overset{(5)}{=} h_\varepsilon(\Psi_\varepsilon(\psi))\cos d_\varepsilon(\Psi_\varepsilon(\psi)) - h'_\varepsilon(\Psi_\varepsilon(\psi))\sin d_\varepsilon(\Psi_\varepsilon(\psi)) \\ &\quad \overset{(17)}{=} h_\varepsilon(\psi + \frac{\pi}{2})\cos d_\varepsilon(\psi + \frac{\pi}{2}) - h'_\varepsilon(\psi + \frac{\pi}{2})\sin d_\varepsilon(\psi + \frac{\pi}{2}) + O_2(\varepsilon)\,, \end{aligned}$$

and, on the other hand

$$\begin{aligned} &h_\varepsilon(\psi + \frac{\pi}{2})\cos d_\varepsilon(\psi + \frac{\pi}{2}) - h'_\varepsilon(\psi + \frac{\pi}{2})\sin d_\varepsilon(\psi + \frac{\pi}{2}) + O_2(\varepsilon) \overset{(17)}{=} \\ &\quad \overset{(17)}{=} h_\varepsilon(\Psi_\varepsilon(\psi))\cos d_\varepsilon(\Psi_\varepsilon(\psi)) - h'_\varepsilon(\Psi_\varepsilon(\psi))\sin d_\varepsilon(\Psi_\varepsilon(\psi)) \\ &\quad \overset{(5)}{=} h_\varepsilon(\Psi^2_\varepsilon(\psi))\cos d_\varepsilon(\Psi^2_\varepsilon(\psi)) + h'_\varepsilon(\Psi^2_\varepsilon(\psi))\sin d_\varepsilon(\Psi^2_\varepsilon(\psi)) \\ &\quad \overset{(10),(11)}{=} h_\varepsilon(\psi + \pi)\cos d_\varepsilon(\psi + \pi) - h'_\varepsilon(\psi + \pi)\sin d_\varepsilon(\psi + \pi) + O_2(\varepsilon) \\ &\quad \overset{(18)}{=} h_\varepsilon(\psi)\cos d_\varepsilon(\psi + \pi) - h'_\varepsilon(\psi)\sin d_\varepsilon(\psi + \pi) + O_2(\varepsilon) \\ &\quad \overset{(i)}{=} h_\varepsilon(\psi)\cos d_\varepsilon(\psi) - h'_\varepsilon(\psi)\sin d_\varepsilon(\psi) + O_2(\varepsilon)\,, \end{aligned}$$

and, summing and subtracting up yields

$$\begin{cases} h_\varepsilon(\psi+\frac{\pi}{2})\cos d_\varepsilon(\psi+\frac{\pi}{2}) = h_\varepsilon(\psi)\cos d_\varepsilon(\psi) + O_2(\varepsilon)\,, \\ h'_\varepsilon(\psi+\frac{\pi}{2})\sin d_\varepsilon(\psi+\frac{\pi}{2}) = -h'_\varepsilon(\psi)\sin d_\varepsilon(\psi) + O_2(\varepsilon)\,, \end{cases}$$

which, combined with $d_\varepsilon(\psi+\frac{\pi}{2}) = \frac{\pi}{2} - d_\varepsilon(\psi) + O_2(\varepsilon)$ (*cf.* (iii)) yields

$$\begin{cases} h_\varepsilon(\psi+\frac{\pi}{2})\sin d_\varepsilon(\psi) = h_\varepsilon(\psi)\cos d_\varepsilon(\psi) + O_2(\varepsilon)\,, \\ h'_\varepsilon(\psi+\frac{\pi}{2})\cos d_\varepsilon(\psi) = -h'_\varepsilon(\psi)\sin d_\varepsilon(\psi) + O_2(\varepsilon)\,. \end{cases} \tag{19}$$

Now, multiplying the two equations in (19) side–wise, we obtain

$$\left(h_\varepsilon(\psi+\frac{\pi}{2})h'_\varepsilon(\psi+\frac{\pi}{2}) + h_\varepsilon(\psi)h'_\varepsilon(\psi)\right)\sin 2d_\varepsilon(\psi) = O_2(\varepsilon)\,,$$

which together with

$$\begin{aligned}\sin 2d_\varepsilon(\psi) &= \sin 2d_0(\psi) + 2\tilde{d}_\varepsilon(\psi)\int_0^1 \cos(2d_0(\psi) + t2\tilde{d}_\varepsilon(\psi))dt \\ &\overset{Lemma\ 9}{=} \sin 2d_0(\psi) + O_3(\varepsilon)\,,\end{aligned}$$

imply

$$\frac{1}{2}\left(h_\varepsilon^2(\psi) + h_\varepsilon^2(\psi+\frac{\pi}{2})\right)' = h'_\varepsilon(\psi)h_\varepsilon(\psi) + h'_\varepsilon(\psi+\frac{\pi}{2})h_\varepsilon(\psi+\frac{\pi}{2}) = O_2(\varepsilon),$$

proving (8), $R_0$ being the constant of integration from 0 to $\psi$.

From (8), it follows

$$h_\varepsilon(\psi) = R_\varepsilon \sin \mathbf{d}_\varepsilon(\psi), \qquad h_\varepsilon(\psi+\frac{\pi}{2}) = R_\varepsilon\cos\mathbf{d}_\varepsilon(\psi), \tag{20}$$

for some $\mathbf{d}_\varepsilon(\psi) \in [0,\pi/2]$ since $h_\varepsilon > 0$. Since $h_\varepsilon > 0$, (20) yields $\mathbf{d}_\varepsilon(\psi) = -i\log\frac{1}{R_\varepsilon}(h_\varepsilon \times (\psi+\frac{\pi}{2}) + ih_\varepsilon(\psi))$ and, hence, $\mathbf{d}_\varepsilon(\psi)$ is $C^{m+1}$–smooth. Now, plugging (20) in the first equation in (19), we obtain $R_\varepsilon\sin(\mathbf{d}_\varepsilon - d_\varepsilon) = O_2(\varepsilon)$ and, by definition of $R_\varepsilon$ in (8), yields $\sin(\mathbf{d}_\varepsilon - d_\varepsilon) = O_2(\varepsilon)$. Then, by continuity, $\mathbf{d}_\varepsilon(\psi) - d_\varepsilon(\psi) \overset{(8)}{=} n_\varepsilon\pi + O_2(\varepsilon)$, for some $n_\varepsilon \in \mathbb{Z}$. It turns out that $n_\varepsilon = 0$ since $d_\varepsilon(\psi), \mathbf{d}_\varepsilon(\psi) \in [0,\pi/2]$. Hence, plugging $\mathbf{d}_\varepsilon(\psi) = d_\varepsilon(\psi) + O_2(\varepsilon)$ in (20) and recalling the definition of $R_\varepsilon$ as in (8), we obtain (9). □

**3.2 Nearly–centrally–symmetry and 3–rational integrability imply closeness to an ellipse.** Denote

$$U_\varepsilon := -h_\varepsilon(h'_\varepsilon)^2(h_\varepsilon + h''_\varepsilon)\left(\frac{1}{2}d_\varepsilon - \frac{1}{4}\sin 2d_\varepsilon\right)$$

$$+ (h_\varepsilon'' h_\varepsilon^2 + 3h_\varepsilon (h_\varepsilon')^2)(h_\varepsilon + h_\varepsilon'') \left( \frac{1}{8} d_\varepsilon - \frac{1}{32} \sin 4d_\varepsilon \right).$$

We have the following statement.

PROPOSITION 11 (see [BM22]). *Suppose for the billiard $\Omega_\varepsilon$, the part of the phase space between the 4–rationally integrable caustic and the boundary is foliated by $C^0$–rotational invariant curves, then*

$$\int_0^{2\pi} U_\varepsilon \, d\psi \leq 0. \tag{21}$$

*Proof.* We refer to *cf.* [BM22, Eq. (19)], noting that the proof in [BM22, Sect. 5.1] there up to this formula does not use the central symmetry assumption. □

The assumption of $C^0$-integrability can be replaced by rational integrability.

PROPOSITION 12 (See Proposition D.1). *The billiard $\Omega$ is $q_0$-rationally integrable if and only if the part of the phase space between the $q_0$–rationally integrable caustic and the boundary is foliated by $C^0$–rotational invariant curves.*

We now proceed with (21).

LEMMA 13 (Perturbative Lemma 5.1 [BM22]).

$$\int_0^{2\pi} U_\varepsilon(\psi) d\psi = \frac{\pi R_0^4}{512} \int_0^{2\pi} ((\mu_\varepsilon'')^2 - 4(\mu_\varepsilon')^2) d\psi + O(\varepsilon), \tag{22}$$

*with* $\mu_\varepsilon(\psi) = \cos(2d_\varepsilon(\psi))$.

*Proof.* In our setting, [BM22, Eq. (20)] reads

$$\begin{cases} h_\varepsilon = R_0 \sin d_\varepsilon + O_2(\varepsilon)\,, \\ h_\varepsilon' = R_0 \cos d_\varepsilon \, d_\varepsilon' + O_1(\varepsilon)\,, \\ h_\varepsilon'' = R_0 \cos d_\varepsilon \, d_\varepsilon'' - R_0 \sin d_\varepsilon (d_\varepsilon')^2 + O(\varepsilon)\,, \\ d_\varepsilon(\psi + \frac{\pi}{2}) = \frac{\pi}{2} - d_\varepsilon(\psi) + O_2(\varepsilon)\,. \end{cases} \tag{23}$$

Thus, mimicking the proof of [BM22, Lemma 5.1], we obtain (22). □

Writing $\mu_\varepsilon$, $\mu_0$ in Fourier series as $\mu_\varepsilon =: \sum_{j\in\mathbb{Z}} \hat\mu_{\varepsilon,j}\, e^{ij\psi}$ and $\mu_0 =: \sum_{j\in\mathbb{Z}} \hat\mu_{0,j}\, e^{ij\psi}$, we have:

LEMMA 14. *We have* $\mu_\varepsilon - (\hat\mu_{0,-2}\, e^{-i2\psi} + \hat\mu_{0,2}\, e^{i2\psi}) = O_1(\sqrt{\varepsilon})$.

*Proof.* Indeed, by (22) and (21), we have, for some $C > 0$

$$\int_0^{2\pi} ((\mu_\varepsilon'')^2 - 4(\mu_\varepsilon')^2) d\psi \leq C\, \varepsilon. \tag{24}$$

Since $\mu_0 = \cos(2d_0)$ is $\pi$–periodic,[5] we have $\hat{\mu}_{0,2k+1} = 0$ for all $k \in \mathbb{Z}$ so that

$$|\hat{\mu}_{\varepsilon,\pm 1}| = |\hat{\mu}_{\varepsilon,\pm 1} - \hat{\mu}_{0,\pm 1}| = |\int_0^{2\pi} (\mu_\varepsilon - \mu_0)(\psi)\, e^{\pm i\psi} d\psi| \leq 2\pi \|\mu_\varepsilon - \mu_0\|_{C^0} =$$
$$= 4\pi \|\sin(d_\varepsilon - d_0)\sin(d_\varepsilon + d_0)\|_{C^0} \leq 4\pi \|\sin(d_\varepsilon - d_0)\|_{C^0} \overset{Lemma\ 9}{=} O(\varepsilon). \quad (25)$$

Thus,

$$\begin{aligned}\sum_{k\in\mathbb{Z}\setminus\{0,\pm 2\}} k^4 |\hat{\mu}_{\varepsilon,k}|^2 &= (|\hat{\mu}_{\varepsilon,-1}|^2 + |\hat{\mu}_{\varepsilon,1}|^2) + \sum_{|k|\geq 3} k^4 |\hat{\mu}_{\varepsilon,k}|^2 \\ &\leq (|\hat{\mu}_{\varepsilon,-1}|^2 + |\hat{\mu}_{\varepsilon,1}|^2) + \sum_{|k|\geq 3} (k^2-4)k^2 |\hat{\mu}_{\varepsilon,k}|^2 \\ &= 4(|\hat{\mu}_{\varepsilon,-1}|^2 + |\hat{\mu}_{\varepsilon,1}|^2) + \sum_{k\in\mathbb{Z}} (k^2-4)k^2 |\hat{\mu}_{\varepsilon,k}|^2 \\ &= 4(|\hat{\mu}_{\varepsilon,-1}|^2 + |\hat{\mu}_{\varepsilon,1}|^2) + \frac{1}{2\pi}\int_0^{2\pi} ((\mu''_\varepsilon)^2 - 4(\mu'_\varepsilon)^2) d\psi \\ &\qquad \text{(by Parseval's identity)} \\ &\overset{(24),(25)}{\leq} C\,\varepsilon,\end{aligned}$$

*i.e.* $\mu_\varepsilon - (\hat{\mu}_{\varepsilon,0} + \hat{\mu}_{\varepsilon,-2}\, e^{-i2\psi} + \hat{\mu}_{\varepsilon,2}\, e^{i2\psi}) \in H^2(\mathbb{T})$ and $\|\mu_\varepsilon - (\hat{\mu}_{\varepsilon,0} + \hat{\mu}_{\varepsilon,-2}\, e^{-i2\psi} + \hat{\mu}_{\varepsilon,2}\, e^{i2\psi})\|_{H^2} \leq C\sqrt{\varepsilon}$. Therefore, Sobolev's embedding theorem yields $\mu_\varepsilon - (\hat{\mu}_{\varepsilon,0} + \hat{\mu}_{\varepsilon,-2}\, e^{-i2\psi} + \hat{\mu}_{\varepsilon,2}\, e^{i2\psi}) = O_1(\sqrt{\varepsilon})$. To conclude the proof, we show $\hat{\mu}_{\varepsilon,0} = O(\varepsilon)$ and $\hat{\mu}_{\varepsilon,\pm 2} = \hat{\mu}_{0,\pm 2} + O(\varepsilon)$. Indeed, since (recall (9))

$$R_0^2 \mu_\varepsilon = R_0^2 \cos 2d_\varepsilon(\psi) = R_0^2(\cos^2 d_\varepsilon - \sin^2 d_\varepsilon) = h_\varepsilon^2(\psi + \frac{\pi}{2}) - h_\varepsilon^2(\psi) + O_2(\varepsilon),$$

We obtain, taking the average over $[0, 2\pi]$, $\hat{\mu}_{\varepsilon,0} = O(\varepsilon)$.

Finally, for any $k \in \mathbb{Z}$, $|\hat{\mu}_{\varepsilon,k} - \hat{\mu}_{0,k}| \leq \|\mu_\varepsilon - \mu_0\|_{C^0} = O(\varepsilon)$. □

Denote by $\mathcal{E}_0$ the ellipse given by the support function

$$\check{h}_0 := \frac{1}{\sqrt{2}} R_0 \sqrt{1 - (\hat{\mu}_{0,-2}\, e^{-i2\psi} + \hat{\mu}_{0,2}\, e^{i2\psi})},$$

by $\check{d}_0$ its 4–loop angle function. Let $\check{\mu}_0 = \cos(2\check{d}_0)$ and $\check{h}_\varepsilon := h_\varepsilon$. Let $0 \leq e_0 < 1$ be the eccentricity and $\mathfrak{c}_0$ the semi–focal distance of the ellipse $\mathcal{E}_0$, and by $\check{\gamma}_0$ its $\psi$–parametrization. We are in position to complete the present section by proving that $\Omega_\varepsilon$ is close to the ellipse $\mathcal{E}_0$.

[5] Recall that $d_0$ is $\pi$–periodic.

Observe that (*cf.* [BM22, Eq. (20)])

$$\begin{cases} \check{h}_0 = \check{R}_0 \sin \check{d}_0 \,, \\ \check{h}_0' = \check{R}_0 \cos \check{d}_0 \, \check{d}_0' \,, \\ \check{h}_0'' = \check{R}_0 \cos \check{d}_0 \, \check{d}_0'' - \check{R}_0 \sin \check{d}_0 (\check{d}_0')^2 \,, \\ \check{d}_0(\psi + \frac{\pi}{2}) = \frac{\pi}{2} - \check{d}_0(\psi) \,. \end{cases} \tag{26}$$

for some $\check{R}_0 > 0$.

LEMMA 15. **(i)** $\check{h}_\varepsilon = \check{h}_0 + O_1(\sqrt{\varepsilon})$.
**(ii)** $h_\varepsilon - \check{h}_0 = O_2(\varepsilon^{\frac{1}{8}})$.

*Proof.* **(i)** Indeed, recalling Lemma 14, we have

$$\begin{aligned} h_\varepsilon^2 &\overset{(23)}{=} R_0^2 \sin^2 d_\varepsilon + O_2(\varepsilon) \\ &= \frac{R_0^2}{2}(1 - \cos 2d_\varepsilon) + O_2(\varepsilon) \\ &= \frac{R_0^2}{2}(1 - \mu_\varepsilon) + O_2(\varepsilon) \\ &= \check{h}_0^2 + O_1(\sqrt{\varepsilon}), \end{aligned}$$

hence

$$h_\varepsilon = \check{h}_0 + O_1(\sqrt{\varepsilon}). \tag{27}$$

**(ii)** We apply Lemma B.1 with $f = \check{h}_\varepsilon$, $\delta = \sqrt{\varepsilon}$, $l = m - 1$ to get the claim. □

LEMMA 16. $\gamma_\varepsilon \in C^{m-1}(\mathbb{T})$ *with* $\gamma_\varepsilon = O_{m-1}(1)$ *and* $\gamma_\varepsilon - \check{\gamma}_0 = O_1(\varepsilon^{\frac{1}{8}})$.

*Proof.* Since

$$\gamma_\varepsilon(\psi) = h_\varepsilon(\psi) \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} + h_\varepsilon'(\psi) \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix},$$

For some uniform constant $C$

$$\|\gamma_\varepsilon\|_{C^{m-1}} \le C \|h_\varepsilon\|_{C^m},$$

and

$$\|\gamma_\varepsilon - \check{\gamma}_0\|_{C^1} \le C \|h_\varepsilon - \check{h}_0\|_{C^2} = O(\varepsilon^{\frac{1}{8}}). \tag{28}$$

□

**3.3 Proof of Theorem 2 using [KS18].** For arbitrary eccentricity $0 \le e_0 < 1$ of the ellipse $\mathcal{E}_0$, we shall apply the following local Birkhoff conjecture result [KS18, Main Theorem]. For, we first introduce the so–called elliptic polar coordinates.

3.3.1 *Elliptic polar coordinates.* Assume that the ellipse $\mathcal{E}_0$ has semi-focal distance $\mathfrak{c}_0$ and eccentricity $e_0$. We consider the elliptic polar coordinates adapted to this family, given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Phi(\lambda, \theta) = \mathfrak{c}_0 \begin{bmatrix} \cosh\lambda\cos\theta \\ \sinh\lambda\sin\theta \end{bmatrix} : \quad [0,\infty)\times\mathbb{T} \to \mathbb{R}^2.$$

$\Phi$ is a diffeomorphism from $(0,\infty)\times\mathbb{T}$ onto its image. In these coordinates, the ellipse $\mathcal{E}_0$ is given by

$$\{(\lambda_0, \theta) : \theta \in \mathbb{T}\}$$

where $e_0 = 1/\cosh(\lambda_0)$.

Let $\check{\gamma}_0(\psi)$ be the parametrization for $\mathcal{E}_0$ in the support function parameter, and as before $\gamma_\varepsilon = \gamma_\varepsilon(\psi)$ is the paremetrization of $\Omega_\varepsilon$, and recall $\gamma_\varepsilon = O_{m-1}(1)$, $\gamma_\varepsilon - \check{\gamma} = O_1(\varepsilon)$.

LEMMA 17. *There exists $\varepsilon_2 > 0$ depending only on $M$ and $\check{\gamma}_0$ such that if*

$$\|\gamma_\varepsilon - \check{\gamma}_0\|_{C^1} < \varepsilon_2,$$

*there exists a $C^{m-1}$ function $\lambda_\varepsilon : \mathbb{T} \to \mathbb{R}$ such that $\gamma_\varepsilon$ is represented as the graph of $\lambda_\varepsilon$ in elliptic polar coordinates, i.e.*

$$\{(\lambda_\varepsilon(\theta), \theta) : \theta \in \mathbb{T}\} = \{\Phi^{-1}\circ\gamma_\varepsilon(\psi) : \psi \in \mathbb{T}\}.$$

*Moreover,*

$$\lambda_\varepsilon = O_{m-1}(1), \quad \lambda_\varepsilon - \lambda_0 = O_1(\varepsilon^{\frac{1}{8}}).$$

*Proof.* Assume that $\varepsilon_2$ is chosen such that the curve $\gamma_\varepsilon$ is contained in a compact tubular neighborhood of $\mathcal{E}_0$. Throughout the proof, $C$ denotes a constant that depends only on $\mathcal{E}_0$ and $\varepsilon_1$, but may change meaning from line to line.

Denote

$$(\bar{\lambda}_\varepsilon(\psi), \bar{\theta}_\varepsilon(\psi)) = \Phi^{-1}\circ\gamma_\varepsilon(\psi),$$

and define $\bar{\lambda}_0$, $\bar{\theta}_0$ similarly for $\check{\gamma}_0$. Then $\bar{\lambda}_0(\psi) = \lambda_0$ and $\bar{\theta}_0(\psi)$ is a diffeomorphism $\mathbb{T}\to\mathbb{T}$ with $C^m$ norm depending only on $\check{\gamma}_0$. Indeed, on $\check{\gamma}_0$, the change of parameter from $\psi$ to arclength parameter $s$, and the one from $s$ to $\theta$ are both $C^\infty$ smooth.

Using smoothness of $\Phi^{-1}$, we have

$$\|\bar{\lambda}_\varepsilon\|_{C^{m-1}}, \|\bar{\theta}_\varepsilon\|_{C^{m-1}} \le C, \quad \|\bar{\lambda}_\varepsilon - \lambda_0\|_{C^1}, \|\bar{\theta}_\varepsilon - \bar{\theta}_0\|_{C^1} \le C\|\gamma_\varepsilon - \check{\gamma}_0\|_{C^1}.$$

If $\varepsilon_1$ is small enough, $\bar{\theta}_\varepsilon(\psi)$ is also a diffeomorphism $\mathbb{T}\to\mathbb{T}$ with $\|\bar{\theta}_\varepsilon^{-1} - \bar{\theta}_0^{-1}\|_{C^1} \le C\|\gamma_\varepsilon - \check{\gamma}_0\|_{C^1} \le C\varepsilon^{\frac{1}{8}}$, $\|\bar{\theta}_\varepsilon^{-1} - \bar{\theta}_0^{-1}\|_{C^{m-1}} \le C\|\gamma_\varepsilon - \check{\gamma}_0\|_{C^{m-1}} = O(1)$, where we have applied Lemma 16. Then

$$\lambda_\varepsilon(\theta) = \bar{\lambda}_\varepsilon \circ \bar{\theta}_\varepsilon^{-1}(\theta)$$

verifies the conclusion of the lemma. □

3.3.2 *Statement of the local Birkhoff conjecture result in* [KS18].

**Theorem 18** ([KS18]). *Let $\ell \geq 39$. For every $M > 0$, there exists $\delta = \delta(M) > 0$ such that the following holds: if $\Omega$ is a 3–rationally weakly integrable $C^\ell$–smooth billiard table so that $\partial\Omega = \mathcal{E}_0 + \lambda_\Omega$ with $\|\lambda_\Omega\|_{C^\ell} \leq M$ and $\|\lambda_\Omega\|_{C^1} \leq \delta$ then $\partial\Omega$ is an ellipse.*

3.3.3 *Completion of the proof of Theorem 2.* By Lemma 17, $\lambda_\varepsilon = O_\ell(1)$ with $\ell := m - 1 \geq 39$, and $\lambda_\varepsilon - \lambda_0 = O_1(\varepsilon^{\frac{1}{8}})$. Choose $\varepsilon_0$ small enough so that $\varepsilon \in (0, \varepsilon_0)$ implies $\|\lambda_\varepsilon - \lambda_0\|_{C^1} < \delta$, where $\delta$ is from Theorem 18, we conclude that $\partial\Omega_\varepsilon$ is an ellipse. □

## Appendix A: Some technical facts

Lemma A.1. *Let $\Omega$ be a strictly convex planar domain and denote by $h$ its support function. Then $\Omega$ is centrally symmetric with center at the origin iff $h(\psi + \pi) = h(\psi)$, for all $\psi \in [0, 2\pi]$.*

*Proof.* Recall (1), that

$$\gamma(\psi) = h(\psi) \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} + h'(\psi) \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix}.$$

Then, $\Omega$ is centrally symmetric *iff* $\gamma(\psi + \pi) + \gamma(\psi) \equiv 0$ which, in turn, is equivalent to $h(\psi + \pi) - h(\psi) \equiv 0$. □

Lemma A.2. *We have $T_0, T_\varepsilon \in C^{m-1}$, $T_0, T_\varepsilon = O_{m-1}(1)$, and $\tilde{T}_\varepsilon = T_\varepsilon - T_0 = O_4(\varepsilon)$.*

*Proof.* The formula (1) and $h \in C^m$ implies $\gamma \in C^{m-1}$. By [Dou82], the billiard map is $C^{m-2}$. To upgrade regularity, we note that (*cf.* [Gol01, Proposition 25.5])

$$T'(\varphi, p) = \begin{pmatrix} -S_{11} \cdot (S_{12})^{-1} & -(S_{12})^{-1} \\ S_{21} - S_{22} \cdot S_{11} \cdot (S_{12})^{-1} & -S_{22} \cdot (S_{12})^{-1} \end{pmatrix}, \tag{A.1}$$

we get $T_0', T_\varepsilon' \in C^{m-2}$ and $T_0, T_\varepsilon \in C^{m-1}$. Moreover, $S_\varepsilon - S_0 = O_5(\varepsilon)$ and (A.1) implies $\tilde{T}_\varepsilon = O_4(\varepsilon)$. □

Consider the map $\Psi_\varepsilon : \mathbb{T} \ni \psi \mapsto \psi' \in \mathbb{T}$, where $\psi'$ is the $\psi$–parameter corresponding to the next bouncing point of the billiard trajectory in $\Omega_\varepsilon$ starting at $\psi$ with the angle $d_\varepsilon(\psi)$; we shall denote its lift to $\mathbb{R}$ by $\Psi_\varepsilon$ as well. Then,

Lemma A.3. *(i)* $\Psi_0(\psi + \pi) = \Psi_0(\psi) + \pi$.
*(ii)* $\Psi_\varepsilon = \Psi_0 + \widetilde{\Psi}_\varepsilon$ *and* $\Psi_\varepsilon(\psi + \pi) = \Psi_\varepsilon(\psi) + \pi + \widetilde{\Psi}_\varepsilon(\psi + \pi) - \widetilde{\Psi}_\varepsilon(\psi)$, *with*

$$\widetilde{\Psi}_\varepsilon = O_3(\varepsilon). \tag{A.2}$$

*Proof.* (i) This is obvious by the symmetry of $\Omega_0$ and the uniqueness of the direction which comes back at the starting point after 3 reflections.

(ii) Observe that

$$\Psi_\varepsilon = \psi_\varepsilon \circ \pi_1 \circ T_\varepsilon \circ (s_\varepsilon, \cos d_\varepsilon).$$

Then, Taylor's expansion yields, for some $t_0, t_1, t_2 \in (0,1)$,

$$\widetilde{\Psi}_\varepsilon = \widetilde{\psi}_\varepsilon \circ \pi_1 T_\varepsilon(s_\varepsilon, \cos d_\varepsilon) + \psi_0' \left(\pi_1 T_0(s_0, \cos d_0) + R_1\right) \cdot [\pi_1 T_0'((s_0, \cos d_0) + R_2)]$$

where

$$R_1 = t_2 \Big[\pi_1 T_0'((s_0, \cos d_0) + t_1(\tilde{s}_\varepsilon, -\sin(d_0 + t_0 \tilde{d}_\varepsilon) \cdot \tilde{d}_\varepsilon)$$
$$\cdot (\tilde{s}_\varepsilon, -\sin(d_0 + t_0 \tilde{d}_\varepsilon) \cdot \tilde{d}_\varepsilon) + \pi_1 \widetilde{T}_\varepsilon(s_\varepsilon, d_\varepsilon)\Big],$$
$$R_2 = \Big[\pi_1 T_0'((s_0, \cos d_0) + t_1(\tilde{s}_\varepsilon, -\sin(d_0 + t_0 \tilde{d}_\varepsilon) \cdot \tilde{d}_\varepsilon)$$
$$\cdot (\tilde{s}_\varepsilon, -\sin(d_0 + t_0 \tilde{d}_\varepsilon) \cdot \tilde{d}_\varepsilon) + \pi_1 \widetilde{T}_\varepsilon(s_\varepsilon, \cos d_\varepsilon)\Big].$$

Therefore, (A.2) holds. Moreover,

$$\begin{aligned} \Psi_\varepsilon(\psi + \pi) &= \Psi_0(\psi + \pi) + \widetilde{\Psi}_\varepsilon(\psi + \pi) \\ &\overset{(i)}{=} \Psi_0(\psi) + \pi + \widetilde{\Psi}_\varepsilon(\psi + \pi) \\ &= \Psi_\varepsilon(\psi) + \pi + \widetilde{\Psi}_\varepsilon(\psi + \pi) - \widetilde{\Psi}_\varepsilon(\psi). \end{aligned}$$

□

## Appendix B: Interpolation–type result

LEMMA B.1. *Suppose $f \in C^l(\mathbb{T})$ where $l \geq 6$, $\|f\|_{C^l} = C > 0$ and $\|f\|_{C^1} = \delta$. Then*

$$f = O_2(\delta^{\frac{1}{4}}).$$

*Proof.* Choose $\sigma = \frac{7}{2}$, we estimate the Sobolev norm $\|f\|_{H^\sigma}$ which bounds $\|f\|_{C^2}$. Let $\hat{f}_k$ be the Fourier series of $f$, then by the Hausdorf-Young inequality

$$|k|^l |\hat{f}_k| \leq \|(f^{(l)})\hat{}\|_{L^\infty} \leq \|f^{(l)}\|_{L^1} \leq \|f\|_{C^l},$$

we obtain (write $\langle f \rangle = \max\{1, |k|\}$)

$$|\hat{f}_k| \leq C \langle k \rangle^{-l}$$

and similarly

$$|\hat{f}_k| \leq \delta \langle k \rangle^{-1}.$$

For $a > 0$ to be determined,

$$\sum_{\langle k \rangle \geq \delta^{-a}} \langle k \rangle^{2\sigma} |\hat{f}_k|^2 \leq C^2 \sum_{\langle k \rangle \geq \delta^{-a}} \langle k \rangle^{2(\sigma - l)} \leq C^2 \delta^{2a} \sum_{\langle k \rangle \geq \delta^{-a}} \langle k \rangle^{2(\sigma - l + 1)} \leq C_1 \delta^{2a}$$

for some $C_1 > 0$, since $\sigma - l + 1 = \frac{9}{2} - l \le -\frac{3}{2}$. On the other hand,

$$\sum_{\langle k\rangle < \delta^{-a}} \langle k\rangle^{2\sigma} |\hat{f}_k|^2 \le \delta^2 \sum_{\langle k\rangle < \delta^{-a}} \langle k\rangle^{2(\sigma-1)} \le 2\delta^2 \delta^{-a} \cdot \delta^{-2a(\sigma-1)} = 2\delta^{2+a-2a\sigma}.$$

Set $2a = 2 + a - 2a\sigma$, we get $a = \frac{2}{1+2\sigma} = \frac{1}{4}$. It follows that

$$\|f\|_{C^2} \le \|f\|_{H^\sigma} = \left(\sum_k \langle k\rangle^{2\sigma} |\hat{f}_k|^2\right)^{\frac{1}{2}} = O(\delta^a) = O(\delta^{\frac{1}{4}}). \qquad \square$$

## Appendix C: Uniqueness of the $(p, q)$-loop orbits

We will begin with the general setting of twist maps and specialize to billiards later in this section.

Consider a $C^2$ generating function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies the following conditions:

(1) $H(x + 1, x' + 1) = H(x, x')$.
(2) There exists $M > 0$ such that $\sup_{x,x'} |\partial_{ij} H(x, x')| < M$, $i, j \in 1, 2$.
(3) There exists $\rho > 0$ such that $\partial_{12} H(x, x') < -\rho$ for all $x$, $x'$.

We remark that condition (3) implies the traditional twist ($x' \mapsto \partial_2 H(x, x')$ is monotone) and superlinearity ($\lim_{|x'-x|\to\infty} H(x, x')/|x' - x| = \infty$) conditions. In fact $H$ satisfies the conditions defined by Mather (see [MF94]), which implies all the classical results of Aubry-Mather theory. Then $H$ defines an exact area preserving twist map $F = F(x, r)$ on $\mathbb{R} \times \mathbb{R}$ via

$$F(x, r) = (x', r') \iff r = -\partial_1 H(x, x'), \quad r' = \partial_2 H(x, x'), \tag{C.1}$$

which also projects to a map of $\mathbb{T} \times \mathbb{R}$. Note that

$$\partial_{12} H(x, x') = -\frac{\partial r}{\partial x'} < -\rho < 0,$$

which implies the mapping $r \mapsto x' = \pi_1 F(x, r)$ is an increasing global diffeomorphism. Moreover, $\|DF\|$, $\|DF^{-1}\|$ are bounded by a constant depending only on $\|D^2 H\|_{C^0}$ and $\rho$.

An orbit $(x_k, r_k)$ of $F$ on $\mathbb{R} \times \mathbb{R}$ is uniquely determined by the sequence $(x_k)$, and any orbit $(x_k, r_k) \in \mathbb{T} \times \mathbb{R}$ admits a unique lift $(\tilde{x}_k)$ once a lift of $x_0$ is chosen.

We say $(x, r)$ is $(p, q)$-periodic if for the lifted map $F$,

$$F^q(x, r) = (x + p, r).$$

We say the orbit of $(x, r)$ is a $(p, q)$-*loop* orbit if for the lifted map $F$,

$$\pi_1 F^q(x, r) = x + p.$$

$(p,q)$-loop orbits aren't necessarily periodic on $\mathbb{T}\times\mathbb{R}$.

An invariant curve $\gamma$ of $F$ is called *essential* if it is not homotopic to a point. $\gamma$ is said to be $(p,q)$–rationally integrable if every orbit on $\gamma$ is $(p,q)$-periodic. We have the following classical theorem by Birkhoff ([Bir22]), and the proof of the Lipschitz constant can be found in [HF83].

**Theorem C.1** ([Bir22, HF83]). *Any essential invariant curve of a twist map is a Lipschitz graph over $\mathbb{T}$. Moreover, the Lipschitz constant depends only on $\|D^2H\|_{C^0}$ and $\rho$.*

PROPOSITION C.2. *Suppose the twist map $F$ admits a $(p,q)$–rationally integrable curve $\gamma$. Then there is $\varepsilon_3>0$, $c>0$, depending only on $q$, $\|D^2H\|_{C^0}$, $\|D^3H\|_{C^0}$, and $\rho$ such that the following hold.*

*If $F_1$ is a twist map whose generating function $H_1$ satisfies $\|H_1-H\|_{C^2}<\varepsilon_3$, then (after lifting to $\mathbb{R}\times\mathbb{R}$) the equation*

$$F_1^q(x,r)=x+p$$

*has a unique solution $\bar r(x)$ for every $x$. Moreover,*

$$|\partial_r F_1^q(x,\bar r(x))|>c. \tag{C.2}$$

LEMMA C.3. *Let $\gamma$ be a continuous invariant graph of the twist map $F:\mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R}$, and suppose $\mathbb{T}\times\mathbb{R}\setminus\gamma=U^+\cup U^-$, where $U^\pm$ are the components above and below $\gamma$. Then both $U^\pm$ are invariant.*

*There exists $c>0$ depending only on $\|D^2H\|_{C^0}$, such that if $(x_k,r_k)_{k\in\mathbb{Z}}$ is an orbit in either $U^+$ or $U^-$, for all $j<k$, we have*

$$\begin{aligned}\pi_1F^{k-j}(V(x_j)\cap\gamma)-x_k &> c^{k-j}\rho\,\mathrm{vdist}((x_j,r_j),\gamma), \quad \text{if } \{(x_j,r_j)\}\subseteq U^-,\\ \pi_1F^{k-j}(V(x_j)\cap\gamma)-x_k &< -c^{k-j}\rho\,\mathrm{vdist}((x_j,r_j),\gamma), \quad \text{if } \{(x_j,r_j)\}\subseteq U^+,\end{aligned} \tag{C.3}$$

*where*

$$\mathrm{vdist}((x,r),\gamma)=\mathrm{dist}((x,r),V(x)\cap\gamma).$$

*the comparison is done after lifting $F$ to $\mathbb{R}\times\mathbb{R}$.*

*Proof.* The twist property implies that for all $x\in\mathbb{R}$, the curve $F(V(x))$ is a monotonically increasing curve. This implies for any $(x,r)$ above $\gamma$, the image $F(x,r)$ will still be above $\gamma$. The invariance of $U^\pm$ follows.

We only prove the lemma in the case that the orbit is contained in $U^-$, as the other case is similar. Let $(x_k,\delta_k)=V(x_k)\cap\gamma$, then by assumption, $\delta_k>r_k$. By the twist propery,

$$\pi_1F(x_k,\delta_k)-x_{k+1}\ge\int_{r_k}^{\delta_k}\partial_r\pi_1F(x,r)dr>\rho(\delta_k-r_k)>0,\quad \forall k\in\mathbb{Z}.$$

In particular, $\pi_1F(x_j,\delta_j)>x_{j+1}$.

Since $\gamma$ is an invariant graph, the mapping

$$g_\gamma : x \mapsto \pi_1 F(V(x) \cap \gamma)$$

is an orientation preserving diffeomorphism on $\mathbb{T}$. Moreover, since $\gamma$ is Lipschitz with constant depending only on $\|D^2H\|_{C^0}$ and $\rho$, the Lipschitz constant of $g_\gamma$ and $g_\gamma^{-1}$ depends only on the same constants.

For the rest of the proof, $F$ stands for the lifted map to $\mathbb{R} \times \mathbb{R}$. Let $c = 1/\|Dg_\gamma^{-1}\|_{C^0} \in (0,1]$, then for all $y > x \in \mathbb{R}$

$$\pi_1 F(V(y) \cap \gamma) - \pi_1 F(V(x) \cap \gamma) > c(y-x) > 0. \tag{C.4}$$

We first prove by induction that

$$\pi_1 F^{k-j}(x_j, \delta_j) - x_k > 0. \tag{C.5}$$

Indeed, assume by induction that $\pi_1 F^{k-j-1}(x_j, \delta_j) - x_{k-1} > 0$, we have

$$\pi_1 F^{k-j}(x_j, \delta_j) = \pi_1 F(V(\pi_1 F^{k-j-1}(x_j, \delta_j)) \cap \gamma) > \pi_1 F(V(x_{k-1}) \cap \gamma) > x_k.$$

We now have

$$\begin{aligned} \pi_1 F^{k-j}(x_j, \delta_j) &> \pi_1 F^{k-j-1}(V(x_{j+1} + \rho(\delta_j - r_j)) \cap \gamma) \\ &> c^{k-j-1}\rho(\delta_j - r_j) + \pi_1 F^{k-j-1}(x_{j+1}, \delta_{j+1}) > c^{k-j}\rho(\delta_j - r_j) + x_k, \end{aligned}$$

where the second inequality is by applying (C.4) $k-j-1$ times, and the third is due to (C.5). □

*Proof of Proposition C.2.* Lemma C.3 implies that if $(x,r) \in U^-$ (resp $U^+$), then for the lifted map $F$, $\pi_1 F^q(x,r) < x + p$ (resp. $\pi_1 F^q(x,r) > x + p$). In other words, for any given $x$, the unique solution to

$$\pi_1 F^q(x,r) = x + p$$

is given by the condition $(x,r) \in \gamma$.

For the "moreover" part, we note that $\pi_1 F^q(x,r) = x + p$ for $(x,r) \in \gamma$, and (C.3) implies

$$\partial_r \pi_1 F^q(x,r) > c^q \rho. \tag{C.6}$$

There exists $r_0 > 0$ depending only on $q$ and $\|D^2F\|_{C^0}$ (hence $\|D^3H\|_{C^0}$), such that

$$\partial_r \pi_1 F^q(x,r) > c^q\rho/2, \text{ if } \operatorname{vdist}((x,r),\gamma) < r_0. \tag{C.7}$$

Moreover, from (C.3),

$$\begin{aligned} &F^q(V(x) \cap \gamma + (0,r)) - (x+p) > c^q \rho r_0, \quad r \ge r_0, \\ &F^q(V(x) \cap \gamma - (0,r)) - (x+p) < -c^q \rho r_0, \quad r \ge r_0. \end{aligned} \tag{C.8}$$

For any $F_1$ that is $\varepsilon_0$-$C^1$-close to $F$ with $\varepsilon_0$ depending only on $r_0$ and $c^q\rho$, (C.7) and (C.8) holds for $F_1$, as long as we replace $c^q\rho$ with $c^q\rho/2$. These equations imply uniqueness of the solution

$$\pi_1 F^q(x,r) - (x+p) = 0.$$

Finally, (C.2) follows from (C.7). □

We now specialize to the billiard case. Given a billiard boundary $\Omega$, define the generating function

$$H_\Omega(s_1,s_2) = -l(s_1,s_2)$$

for all $s_1, s_2 \in \mathbb{R}$ such that $0 \le s_2 - s_1 \le 1$. Then the billiard map can be written as

$$F(s_1,r_1) = (s_2,r_2) \iff r_1 = -\partial_1 H_\Omega(s_1,s_2), \quad r_2 = \partial_2 H_\Omega(s_1,s_2),$$

where the relation between the $r$ variable and the reflection angle $\theta$ is $r = -\cos\theta$ (see Fig. 1). One checks that the billiard map is defined on $\mathbb{T}\times(-1,1)$ extensible continuously to $[-1,1]$.

While the billiard map is a twist map in this sense, it has vanishing twist at the boundary. Indeed, if

$$F(s_1,r_1) = (s_2,r_2), \quad r_1 = -\cos\theta_1, \quad r_2 = -\cos\theta_2,$$

where $\theta_1(s_1,s_2) = \angle(\dot\gamma(s_1), \gamma(s_2)-\gamma(s_1))$, $\theta_2(s_1,s_2) = \angle(\gamma(s_2)-\gamma(s_1), \dot\gamma(s_2)) \in [0,\pi]$, then

$$\partial_{12}H(s_1,s_2) = \frac{\partial r_2}{\partial s_1} = -\frac{\partial r_1}{\partial s_2} = \sin\theta_2 \frac{\partial\theta_2}{\partial s_1} = -\frac{\sin\theta_2}{\sin\theta_1} l(s_1,s_2), \tag{C.9}$$

where the last identity $\frac{\partial\theta_2}{\partial s_1} = -\frac{l}{\sin\theta_1}$ can be found in [KS06], (V.4.9). As $s_2 - s_1 \to 0$, (C.9) converges to 0, as seen from the next lemma.

LEMMA C.4. *Suppose the boundary $\gamma$ is normalized to arclength 1 with arclength parametrization, and assume the curvature bound*

$$0 < \kappa_0 \le \kappa(s) \le \kappa_1.$$

*Then there exists $C > 1$ depending only on $\kappa_0$, $\kappa_1$ such that*

$$\begin{aligned} C^{-1}\operatorname{dist}_{\mathbb{T}}(s_1,s_2) &\le l(s_1,s_2) \le C\operatorname{dist}_{\mathbb{T}}(s_1,s_2),\\ C^{-1}\operatorname{dist}_{\mathbb{T}}(s_1,s_2) &\le \theta_1,\theta_2 \le C\operatorname{dist}_{\mathbb{T}}(s_1,s_2),\\ C^{-1}\operatorname{dist}_{\mathbb{T}}(s_1,s_2) &\le |\partial_{12}H(s_1,s_2)| \le C\operatorname{dist}_{\mathbb{T}}(s_1,s_2). \end{aligned}$$

*for all $s \in \mathbb{T}$.*

*Proof.* Thoughout the proof, we write $f \lesssim g$ if $f \le Cg$ for a constant depending only on $\kappa_0$, $\kappa_1$, and $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

For a fixed $s_1 \in \mathbb{R}$, consider the function

$$\alpha(s) = \angle(\dot\gamma(s), \dot\gamma(s_1))$$

defined which maps $[s_1, s_1+1]$ diffeomorphically to $[0, 2\pi]$.

Perform a rotation so that $\dot\gamma(s_1) = (1,0)$, then $\gamma(s_2) - \gamma(s_1) = \int_{s_1}^{s_2} \begin{bmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{bmatrix} ds.$ This mean in general

$$l(s_1, s_2) = |\gamma(s_2) - \gamma(s_1)| = \left| \int_{s_1}^{s_2} \begin{bmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{bmatrix} ds \right|.$$

Let

$$S_1 = \{s \in [s_1, s_1+1] : \alpha(s) \in [0, \frac{\pi}{4}] \cup [2\pi - \frac{\pi}{4}, 2\pi]\}, \quad S_2 = \overline{[s_1, s_1+1] \setminus S_1}.$$

If $s \in S_1$, notting $\cos\alpha(s) \ge 0$, we have

$$l(s_1, s_2) \ge \int_{s_1}^{s_2} \cos\alpha(s) ds \ge \min_{s \in [s_1, s_2]} \cos\alpha(s) \operatorname{dist}_{\mathbb{T}}(s_1, s_2) \gtrsim (s_2 - s_1).$$

For any $t_1 < t_2$

$$\alpha(t_2) - \alpha(t_1) = \int_{t_1}^{t_2} \kappa(s) ds \approx t_2 - t_1,$$

we have

$$\frac{\pi}{4} \le \min\{\alpha(s_*), 2\pi - \alpha(s_*)\} \lesssim \min\{s_* - s_1, s_1 + 1 - s_*\} = \operatorname{dist}_{\mathbb{T}}(s_1, s_*).$$

Let $s_*$ denote the $s \in [s_1, s_1+1]$ which minimimizes $l(s_1, s)$ on the set over all $s \in S_2$. Then either $\alpha(s_*) \in [\frac{\pi}{4}, \pi]$ or $\alpha(s_*) \in [\pi, 2\pi - \frac{\pi}{4}]$. Assume the former as the other case is similar. For $s_2 \in S$,

$$\begin{aligned} l(s_1, s_2) &\ge l(s_1, s_*) \ge \int_{s_1}^{s_*} \sin\alpha(s) ds \ge \int_{s_1+\pi/8}^{s_1+\pi/4} \sin\alpha(s) ds \gtrsim \operatorname{dist}_{\mathbb{T}}(s_1, s_*) \\ &\gtrsim \frac{\pi}{4} \gtrsim \operatorname{dist}_{\mathbb{T}}(s_1, s_2) \end{aligned}$$

since $\operatorname{dist}_{\mathbb{T}}(s_2 - s_1) \le \frac{1}{2}$. We have proven the bound $l(s_1, s_2) \gtrsim \operatorname{dist}_{\mathbb{T}}(s_1, s_2)$. The other bound is trivial since chord length is always smaller than arclength: $l(s_1, s_2) \le \operatorname{dist}_{\mathbb{T}}(s_1, s_2)$. The first inequality follows.

For the second inequality, let $\theta_1$, $\theta_2$ be as in (C.9). Rotate the axis so that $\gamma'(s_1) = (1,0)$, we have

$$\sin\theta_1 = \frac{\int_{s_1}^{s_2} \sin\alpha(s) ds}{l(s_1, s_2)} = -\frac{\int_{s_2}^{s_1+1} \sin\alpha(s) ds}{l(s_1, s_2)}.$$

If $s_2 \in S_1$, then

$$|\sin\theta_1| \approx \frac{\mathrm{dist}_{\mathbb{T}}(s_1,s_2)^2}{l(s_1,s_2)} \approx \mathrm{dist}_{\mathbb{T}}(s_1,s_2).$$

If $s_2 \in S_2$, we let $s_*$ minimize $\sin\theta_1$ over $S_2$, argue similarly as before to get

$$|\sin\theta_1| \gtrsim \frac{\mathrm{dist}_{\mathbb{T}}(s_1,s_*)^2}{l(s_1,s_2)} \gtrsim \mathrm{dist}_{\mathbb{T}}(s_1,s_2).$$

The upper bound $|\sin\theta_1| \lesssim \mathrm{dist}_{\mathbb{T}}(s_1,s_2)$ is trivial since $\mathrm{dist}_{\mathbb{T}}(s_1,s_2) \gtrsim 1$ when $s_2 \in S_2$. Moreover, the same estimate works for $\theta_2$ by symmetry. By (C.9), we get

$$|\partial_{12}H(s_1,s_2)| \approx \mathrm{dist}_{\mathbb{T}}(s_1,s_2)$$

as required. □

PROPOSITION C.5. *Let $\Omega$ be a strictly convex billiard table and let $\gamma(s)$ be the arc-length parametrized boundary, and $\kappa(s)$ the curvature function, we assume that there is $\kappa_0 > 0$ and $C > 0$ such that*

$$\kappa(s) > \kappa_0, \quad |\partial\Omega| < C, \quad \|\gamma\|_{C^3} < C.$$

*Suppose the billiard map $T$ admits a $(p,q)$-rationally integrable caustic.*

*Then there is $\varepsilon_4 > 0$, $c > 0$, depending only on $q$, $C$, and $\kappa_0$, such that if billiard table $\Omega_1$ whose boundary $\gamma_1$ satisfies $\|\gamma_1 - \gamma\|_{C^2} < \varepsilon_4$, for the lifted billiard map $T_1$, the equation*

$$\pi_1 F_1^q(s,r) = s + p$$

*has a unique solution $\bar{r}(s)$, with*

$$|\partial_r \pi_1 F_1^q(s,\bar{r}(s))| > c, \quad |\bar{r}(s) \pm 1| > c.$$

*Proof.* Without loss of generality, assume $|\partial\Omega|$ is normalized to 1. We continue the use of notations $\lesssim$ and $\approx$, where the constant factor depends on $q$, $C$, and $\kappa_0$.

Suppose $F(s_1,r_1) = (s_2,r_2)$, then Lemma C.4 implies

$$|\partial_r \pi_1 F(s_1,r_1)| \approx \frac{1}{|\partial_{12}H(s_1,s_2)|} \approx \mathrm{dist}_{\mathbb{T}}(s_1,s_2)^{-1}.$$

Assume that $s_2 - s_1 \in (0,\frac{1}{2})$ so that $\mathrm{dist}_{\mathbb{T}}(s_1,s_2) = s_2 - s_1$, then

$$\pi_1 F(s_1,r_1) - s_1 = \pi_1 F(s_1,r_1) - \pi_1 F(s_1,-1) = \int_{-1}^{r} \partial_r \pi_1 F(s_1,\sigma)d\sigma$$

$$\lesssim |r+1|(s_2-s_1)^{-1} = 2\sin^2\frac{\theta_1}{2}(s_2-s_1)^{-1} \approx s_2 - s_1,$$

noting that $\theta_1 \approx s_2 - s_1$. If $(s_k,r_k)$, $0 \le k \le q$ is a $(p,q)$-loop orbit, then

$$p = s_q - s_0 \approx s_k - s_{k-1}$$

for any $1 \le k \le q$, where the constant factor may depend on $q$. Therefore, there exists a constant $\delta_0 > 0$ such that

$$s_k - s_{k-1} > \delta_0, \quad 1 \le k \le q$$

for any $q$-loop orbit. Since $|\bar{r}(s)+1| = |r_0+1| \approx (s_1 - s_0)^2$, there exists $c > 0$ such that $|\bar{r}(s)+1| > c$. When $s_2 - s_1$ is close to 1, a symmetric argument yields $|\bar{r}(s) - 1| > c$.

The billiard map $F$ is uniformly twist on the part of the phase space where $1 - \delta_0 > s_2 - s_1 > \delta_0$. Modifying the generating function $H_\Omega(s_1, s_2)$ on the set $s_2 - s_1 \le \delta_0$ and $s_2 - s_1 \ge p - \delta_0$ so that the modified generating function to $H$ is uniformly twist and defined on $\mathbb{R} \times \mathbb{R}$, with the map denoted $\tilde{F}$. We now apply Proposition C.2 to get uniqueness of $(p,q)$-loop orbit for the map $\tilde{F}$ and any small $C^3$ perturbation $H_1$ to the generating function $H$. Our proposition now follows since any $q$-loop orbit of $F$ must be a $(1,q)$-loop orbit of $\tilde{F}$, and a $C^3$-small perturbation to $\gamma$ results in a $C^3$-small perturbation to the generating function $H$. □

Finally, we are ready to prove Propisition 7.

*Proof of Proposition 7.* Choose $\varepsilon_5 > 0$ such that all the curves in

$$\mathcal{V}_1 = \{\gamma : \|\gamma - \gamma_0\|_{C^2} < \varepsilon_1, \quad \|\gamma\|_{C^4} < C\},$$

all admits the bounds

$$\kappa(s) > \underline{\kappa}/2.$$

For a curve $\gamma = \gamma(\psi)$ parametrized using the normal angle $\psi$, let $\psi = g_\gamma(s)$ be the coordinate change to the arclength coordinates. The coordinate changes $g_\gamma$ and $g_\gamma^{-1}$ admits uniform $C^3$ bounds depending only on $\kappa_0$ and $C$. Therefore there exists $C_1 > 1$ such that

$$C_1^{-1}\|\gamma - \gamma_0\|_{C^2} \le \|\gamma \circ g_\gamma - \gamma_0 \circ g_{\gamma_0}\|_{C^2} < C_1\|\gamma - \gamma_0\|_{C^2}$$

and

$$\|\gamma \circ g_\gamma\|_{C^3} < C_1$$

for all $\gamma \in \mathcal{V}_1$.

Let $\varepsilon_4$ be the small parameter given in Proposition C.5, which depends only on $\underline{\kappa}$, $q$, $C$ and $C_1$. In particular, if $\gamma_\varepsilon \in \mathcal{V}_1$ admits 4-rationally-integrable caustic, then any arclength parametrized curve $\eta$ satisfying

$$\|\eta - \gamma_\varepsilon \circ g_{\gamma_\varepsilon}\|_{C^2} < \varepsilon_4$$

admits unique 4-loop orbits. Define $\varepsilon_1 = \min\{\varepsilon_4, \varepsilon_5\}$, and

$$\mathcal{V} = \{\gamma : \|\gamma - \gamma_0\| < C_1^{-1}\varepsilon_0,\ \|\gamma\|_{C^4} < C\},$$

then if $\gamma_\varepsilon \in \mathcal{V}$ admits a 4-caustic, for any other $\gamma \in \mathcal{V}$, we have

$$\|\gamma \circ g_\gamma - \gamma_\varepsilon \circ g_{\gamma_\varepsilon}\|_{C^2} \le C_1 \|\gamma - \gamma_\varepsilon\|_{C^2} < \varepsilon_4,$$

hence $\gamma$ admits unique 4-loop orbits. □

*Proof of Lemma 9.* Let $F_0(s,r)$, $F_\varepsilon(s,r)$ denote the billiard map for $\Omega_0$, $\Omega_\varepsilon$ in the traditional coordinates, and let $\psi = g_0(s)$, $\psi = g_\varepsilon(s)$ the coordinate change between $\psi$ and $s$. In the notation of Proposition C.5,

$$-\cos d_0(\psi) = \bar{r}_0 \circ g_0^{-1}(\psi), \quad -\cos d_\varepsilon(\psi) = \bar{r}_\varepsilon \circ g_\varepsilon^{-1}(\psi).$$

Then there exists $c > 0$ such that $|\bar{r}_0 \pm 1| > c$, $|\bar{r}_\varepsilon \pm 1| > c$, or equivalently, $0 < \underline{d} < d_0$, $d_\varepsilon < \overline{d} < \pi$ for some $\underline{d}$, $\overline{d}$.

By Proposition C.5, $\bar{r}$ is the unique solution to

$$F^4(s,r) = s + 1,$$

and since $|\partial_r \pi_1 F^4(s,r)| > c > 0$, the implicit function applies. When $F$ is $C^{m-1}$, so is $\bar{r}$ with $\bar{r} = O_m(1)$. The implicit function theorem also implies

$$\|\bar{r}_\varepsilon - \bar{r}_0\|_{C^3} \lesssim \|F_\varepsilon - F_0\|_{C^3} \lesssim \|\gamma_0 - \gamma_\varepsilon\|_{C^4} \lesssim \|h_0 - h_\varepsilon\|_{C^5}.$$ □

## Appendix D: Relation between rational and $C^0$ integrability

Let $p_1/q_1 < p_2/q_2 \in \mathbb{Q} \cap (0, \infty)$, we say the billiard map in the domain $\Omega$ is rationally integrable in the interval $[p_1/q_1, p_2/q_2]$ if for every rational number $\rho \in [p_1/q_1, p_2/q_2] \cap \mathbb{Q}$, there exists a smooth, strictly convex caustic on which the dynamics is conjugate to a rigid rotation with rotation number $\rho$. Same as in the previous section, we use the generating function $H_\Omega(s_1, s_2) = -l(s_1, s_2)$, and the billiard map is given by

$$F(s,r) = (s_1, r_1), \quad \Longleftrightarrow \quad r = -\partial_1 H_\Omega(s, s_1), \quad r_1 = \partial_2 H_\Omega(s, s_1),$$

with $r = -\cos\theta$. The map is defined on $\mathbb{T} \times (-1,1)$ extending continously to $\mathbb{T} \times \{-1, 1\}$. In this point of view, rotation numbers of invariant curve can range from 0 to 1 (with the counter clockwise rotation counts as rotation number $[1/2, 1]$).

A homotopically non-trivial invariant curve shall be called an essential invariant curve.

We will prove the following statement.

Proposition D.1. *Suppose the billiard admit two essential invariant curves $\gamma_1$, $\gamma_2$ on which the dynamics are conjugate to rigid rotations of rotation numbers $\rho_1 < \rho_2 \in \mathbb{Q}$. Then the billiard is rationally integrable on $[\rho_1, \rho_2]$ if and only if it is $C^0$-integrable on the phase space between $\gamma_1$ and $\gamma_2$.*

Proposition D.1 follows from a more general result about twist maps. The assumption for the generating function is the same as in the previous section.

Definition D.2. We say an orbit $\{(x_k, r_k)\}_{k\in\mathbb{Z}}$ of $F$ has no conjugate points if for all $j < k$, we have

$$\frac{\partial(\pi_1 F^{k-j}(x,r))}{\partial r}\Big|_{(x_j,r_j)} \neq 0,$$

where $\pi_1(x,r) = x$.

Proposition D.3. *Suppose $H$ satisfies the conditions* (1) - (3) *stated above. Let $\gamma_1$ and $\gamma_2$ be two essential invariant curves of $F$, on which the dynamics are conjugate to rigid rotations of rotation numbers $\rho_1 < \rho_2 \in \mathbb{Q}$. Then the following are equivalent.*

(a) *$F$ is rationally integrable on $[\rho_1, \rho_2]$.*
(b) *$F$ has no conjugate points for all the orbits in the phase space between $\gamma_1$ and $\gamma_2$.*
(c) *$F$ is $C^0$-integrable on the phase space between $\gamma_1$ and $\gamma_2$.*

*Proof of Proposition D.1 using Proposition D.3.* Suppose the billiard table is normalized to perimeter 1. The generating function $H_\Omega(s_1, s_2)$ can be extended to the set $s_2' - s_1' \in [0,1]$ in $\mathbb{R}\times\mathbb{R}$, and for any $\epsilon > 0$, $\partial_{12}H_\omega < -\rho(\epsilon) < 0$ over $s_2' - s_1' \in [\epsilon, 1-\epsilon]$. The generating function admits a smooth, periodic extension to $\mathbb{R}\times\mathbb{R}$ keeping the estimate $\partial_{12}H < -\rho(\epsilon)$ and uniform bound on the second derivatives (see [MF94], Sect. 8). There exists $\delta(\epsilon) > 0$ satisfying $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$, such that extended map coincide with the billiard map over the set $\mathbb{T}\times[-1+\delta(\epsilon), 1-\delta(\epsilon)]$. Finally, our proposition follows by choosing $\epsilon$ small enough such that the phase space between the two rational invariant curves are contained in the set $\mathbb{T}\times[-1+\delta(\epsilon), 1-\delta(\epsilon)]$, and applying Proposition D.3 to the extension system. □

For the case of Tonelli Hamiltonians on $\mathbb{T}^n\times\mathbb{R}^n$ and under the assumption that there is a totally periodic Lagrangian invariant graph for every rational vector, Proposition D.3 is essentially proven in Sect. 2.2 and 2.3 of [A+15]. We provide a proof here because the twist map case cannot be directly reduced to the Tonelli case (although the proofs are very similar), and also the notion of rationally integrable on an interval is purely one-dimensional. We mostly follow the arguments of [A+15], and also mention [AMS22] where some analogous statements in higher dimensional twist maps are proven.

We will start with the implication (a) ⇒ (c).

For $c \in \mathbb{R}$, define $A_c : \mathbb{T}\times\mathbb{T}\to\mathbb{R}$ by

$$A_c(x,y) = \min\{H(x',y') - c(y'-x') : \quad x' = x,\, y' = y \mod 1\}.$$

Given $x_0', \dots, x_n' \in \mathbb{R}$, let us denote

$$H\left((x_k')_{k=0}^n\right) = \sum_{k=0}^{n-1} H(x_k', x_{k+1}'),$$

and define $A_c^n : \mathbb{T}\times\mathbb{T}\to R$ by

$$A_c^n(x,y) = \min\left\{\sum_{k=0}^{n-1} A_c(x_k,x_{k+1}) : \quad x_0 = x,\, x_n = y\right\}$$
$$= \min\left\{H((x'_k)_{k=0}^n) : \quad x'_0 = x,\, x'_n = y \mod 1\right\}.$$

The *Lax-Oleinik* operator $T : C(\mathbb{T}) \to C(\mathbb{T})$ is defined as

$$T_c u(x) = \min_{y\in\mathbb{T}} \{u(y) + A_c(y,x)\}.$$

We have the following standard results from weak KAM theory.

PROPOSITION D.4 (see [Fat05, Zav10]). (1) *All $A_c^n(x,y)$ are equi-Lipschitz in both variables.*

(2) *There exists unique $\alpha(c)\in\mathbb{R}$ called Mather's alpha function such that all $A_c^n(x,y)+n\alpha(c)$ are equibounded. In particular, $-\alpha(c) = \lim_{n\to\infty}\frac{1}{n}A_c^n(x,y)$.*
(3) *$\alpha(c)$ is a convex function in $c\in\mathbb{R}$.*
(4) *(Weak KAM Theorem) There exists $u\in C(\mathbb{T})$ called a weak KAM solution such that $T_c u + \alpha(c) = u$.*
(5) *If $u$ is a weak KAM solution, then for every $x\in\mathbb{T}$, there exists an $F$-orbit $(x_k,r_k)_{k=-\infty}^0$ with $x_0 = x$, such that*

$$A_c^n(x_{-n},x_0) + u(x_{-n}) = \sum_{k=-n}^{0} A_c(x_k,x_k+1) + u(x_{-n}) = u(x_0).$$

*This orbit is called a calibrated orbit for $u$.*

Birkhoff's Theorem says any continuous invariant graph $\gamma$ of $F$ must be Lipschitz. Then there exists a $C^{1,1}$ function $u$ and $c\in\mathbb{R}$ such that $\gamma = \{(x, c+du(x)) : x\in\mathbb{T}\}$.

LEMMA D.5. *Let $u\in C^1(\mathbb{T})$ and $\gamma = \{(x,du(x)) : x\in\mathbb{T}\}$ is invariant under $T$. Then $u$ is a weak KAM solution*:

$$T_c u + \alpha(c) = u.$$

*Suppose in addition that $\gamma$ is a $p/q$ periodic invariant curve. Let $(x'_k,r'_k)_{k=0}^q$ the lift of any orbit on $\gamma$ to $\mathbb{R}\times\mathbb{R}$. Then*

$$H\left((x'_k)_{k=0}^q\right) = \min\left\{H\left((y'_k)_{k=0}^q\right) : y'_0 = x_0,\, y'_q = x'_q = x'_0 + p\right\}. \tag{D.1}$$

*Conversely, any sequence satisfying (D.1) is the projection of a lifted orbit from $\gamma$. If $(x_k)$ denote the projection of $x'_k$ to $\mathbb{T}$, we have*

$$A_c^q(x_0,x_0) + q\alpha(c) = 0. \tag{D.2}$$

*Proof.* Let $x'\in\mathbb{R}$ be a lift of $x\in\mathbb{T}$, then

$$T_c u(x) = \min_{y'\in\mathbb{R}} \{u(y') + H(y',x') - c(x'-y')\},$$

where $u$ is treated as a periodic function on $\mathbb{R}$. If $y_0'$ reaches the minimum, then

$$du(y_0') + \partial_1 H(y_0', x') + c = 0.$$

This means $F(y_0', c + du(y_0')) = (x', p)$ where $p = \partial_2 H(y_0', x')$. Since $\gamma$ is invariant, we must have $p = c + du(x')$. Since $F$ is a diffeomorphism, $y_0'$ is unique. This implies $T_c u$ is differentiable at $x$ and

$$d(T_c u)(x') = \partial_2 H(y_0', x') - c = du(x').$$

It follows that $T_c u - u$ is a constant, which necessarily equals $-\alpha(c)$. Note that this argument proves that the minimum in

$$\min_y \{u(y) + A_c(y, x)\} = T_c u(x) = u(x) - \alpha(c)$$

is necessarily achieved at $x_{-1}$, where $F(x_{-1}, c + du(x_{-1}) = (x, du(x))$.

Suppose $\gamma$ is $p/q$ periodic. For every $x \in \mathbb{T}$, let $x_0 = x$, $(x_k, c + du(x_k)) = T^k(x_0, c + du(x_0)) \in \mathbb{T} \times \mathbb{R}$, and let $x_k'$ denote the projection of the lift of $(x_k, c + du(x_k))$. The argument in the first half of this proof implies $(x_k')_{0 \le k \le q-1}$ is the unique minimizer over $y_k'$ for

$$u(y_0') + \sum_{k=0} \left( H(y_k', y_{k+1}') - c(y_{k+1}' - y_k') \right), \quad \text{such that } y_q' = x_q'.$$

Therefore it is also the unique minimizer over $(y_k')_{k=1}^{q-1}$ which satisfies $y_0' = x_0'$ and $y_q' = x_q'$. Under this constraint, $u(y_0')$ and $\sum_{k=0}^{q-1} c(y_{k+1}' - y_k')$ are constants, therefore $(x_k')$ is the unique minimizer of $\sum_{k=0}^{p-1} H(y_k', y_{k+1}')$ over the same constraints. This proves both (D.1) and its converse.

Moreover, we know that

$$u(x_0) - q\alpha(c) = T_c^q u(x_0) = u(x_0) + A_c^n(x_0, x_0),$$

this proves (D.2). □

We define the Peierl's barrier $h_c : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ by

$$h_c(x, y) = \liminf_{n \to \infty} \left( A_c^n(x, y) + n\alpha(c) \right).$$

The *projected Aubry set* is

$$\mathcal{A}(c) = \{x : \quad h_c(x, x) = 0\}.$$

The following properties hold for the Peierls barrier.

PROPOSITION D.6 (See [Fat05, Zav10]). (1) *$h_c(x, y)$ is Lipschitz in both variables, $h_c(x, x) \ge 0$.*

(2) *For every $x \in \mathbb{T}$, $h_c(x, \cdot)$ is a weak KAM solution.*

(3) *If $x \in \mathcal{A}(c)$, then there is a unique $h_c(x, \cdot)$ calibrated orbit ending at $(x, r(x))$.*

(4) (*Mather's graph theorem*) $r(x)$ *is a Lipchitz function over* $\mathcal{A}(c)$.

For $x \in \mathcal{A}(c)$, let $(x, r(x))$ be given by Proposition D.6. Then

$$\tilde{\mathcal{A}}(c) = \{(x, r(x)) : x \in \mathcal{A}(c)\}$$

is well defined and is called the Aubry set. It is a compact invariant set under $F$.

LEMMA D.7. *For every* $x \in \mathbb{T}$, *let* $(x_k, r_k)_{k=-\infty}^{0}$ *be any orbit calibrated by* $h_c(x,\cdot)$ *and* $x'_k$ *be the lift of the orbit to* $\mathbb{R}$. *Then the limit*

$$\lim_{k\to-\infty} \frac{x'_0 - x'_k}{|k|}$$

*exists and depends only on* $c$, *denoted* $\rho(c)$. *The map* $c \mapsto \rho(c)$ *is monotone over all* $c$*'s for which* $\tilde{\mathcal{A}}(c)$ *is a graph over* $\mathbb{T}$.

*Proof.* Let $(x'_k)$ corresponds to a lifted orbit, we call $(k, x'_k)$ the graph of $(x'_k)$. We claim that the graph of any two lifted calibrated orbits $(x'_k)$ and $(y'_k)$ cannot cross each other in $\mathbb{R}^2$.

Suppose the contrary holds. By the Aubry Crossing Lemma (see [MF94], also note that it only holds in dimension 1), if any two graphs $(k, x'_k)_{k=-n}^{0}$, $(k, y'_k)_{k=-n}^{0}$ crosses each other in $\mathbb{R}^2$, then there exists two other sequences $z'_k$, $w'_k$ such that $z'_{-k} = x'_{-k}$, $z'_0 = y_0$, $w_{-k} = y'_{-k}$, $w_0 = x'_0$ such that

$$H((z'_k)) + H((w'_k)) < H((x'_k)) + H((y'_k)).$$

It follows that

$$A^n_c(z_{-n}, z_0) + A^n_c(w_{-n}, w_0) < A^n_c(x_{-k}, x_0) + A^n_c(y_{-k}, y_0),$$

where removing $'$ stands for projection to $\mathbb{T}$. This contradicts the fact that $x_{-k}$ and $y_{-k}$ mninimizes $h_c(y,\cdot) + A^n_c(\cdot, x_0)$ and $h_c(y,\cdot) + A^n_c(\cdot, y_0)$ respectively.

If a calibrated orbit does not have a well defined rotation number, then there exists two lifts of it that intersect each other. Similarly, any two orbits with different rotation numbers admit intersecting lifts. This proves $\rho$ is well defined. Moreover, the twist property implies that if $r_2 > r_1$, then the rotation number of the orbit of $(x, r_2)$ is larger than that of $(x, r_1)$. This implies $c \mapsto \rho(c)$ is monotone. □

The following proposition proves (a) ⇒ (c) in Proposition D.3.

PROPOSITION D.8. *Suppose* $F$ *is rationally integrable in* $[\rho_1, \rho_2]$. *Then for any* $c \in \mathbb{R}$ *such that* $\rho(c) \in [\rho_1, \rho_2]$, *the Aubry set* $\tilde{\mathcal{A}}(c)$ *projects onto* $\mathbb{T}$. *For every* $c \in \mathbb{T}$, *the map* $G_x : c \mapsto \tilde{\mathcal{A}}(c) \cap \pi^{-1}(x)$ *is a homeomorphism onto its image.*

*As a corollary, the set in* $\mathbb{T} \times \mathbb{R}$ *bounded by the invariant curves of rotation numbers* $\rho_1 < \rho_2 \in \mathbb{Q}$ *are foliated by the Aubry sets* $\tilde{\mathcal{A}}(c)$, $\rho(c) \in [\rho_1, \rho_2]$.

*Proof.* Suppose $c \in (\rho_1, \rho_2)$, we first show that $\mathcal{A}(c) = \mathbb{T}$. Given $x \in \mathbb{T}$, let $n_j \to \infty$ such that $A^{n_j}_c(x, x) \to h_c(x, x)$, and let $(x^j_k, r^j_k)_{k=-n_j}^{0}$ be minimizers for $A^{n_j}_c(x, x)$. let

$(x_k^\infty, r_k^\infty)_{k=-\infty}^0$ be any limit point of the sequence $(x_k^j, r_k^j)$ in $j$ in term wise convergence. We have

$$A_c^{n_j}(x, x_k^j) + (n_j - k)\alpha(c) + A^k(x_k^j, x) + k\alpha(c) = A_c^{n_j}(x, x) + n_j\alpha(c)$$

Taking lim inf to both sides, we get

$$h_c(x, x_k^\infty) + A^k(x_k^\infty, x) + k\alpha(c) \le h_c(x, x).$$

Since the opposite inequality follows directly from definition, we get $x_k^\infty$ is calibrated by $h_c(x, \cdot)$. It follows from Lemma D.7 that $\rho((x_k^\infty)) = \rho(c)$. Since $(x_k^j)$ converges to $(x_k^\infty)$, then for any lift $(\xi_k^j)$ of $(x_k^j)$, we have

$$\frac{\xi_{n_j}^j - \xi_0^j}{n_j} \to \rho(c) \in (\rho_1, \rho_2).$$

Since the orbit $(x_k^j)$ is minimizing, $(\xi_k^j)$ is a minimizer of (D.1) with $p = \xi_{n_j}^j - \xi_0^j$ and $q = n_j$. Lemma D.5 then implies the orbit $(x_k^j)$ is contained in some periodic invariant curve $\gamma$ and $A_c^{n_j}(x, x) + n_j\alpha(c) = 0$. Taking lim inf, we get

$$h_c(x, x) \le 0.$$

Since $h_c(x, x) \ge 0$, we have $x \in \mathcal{A}(c)$. Moreover, the argument also proves that for any $(x, r) \in \mathcal{A}(c)$, there exists a sequence $\gamma_k$ of periodic invariant curves such that $\gamma_k \cap \pi^{-1}(x) \to (x, r)$, where $\pi(x, r) = x$ is the projection. Since all those invariant curves are Aubry sets, this means there exists $c_k$ such that $\rho(c_k) \to \rho(c)$ and $\tilde{\mathcal{A}}(c_k) \cap \pi^{-1}(x) \to \tilde{\mathcal{A}}(c) \cap \pi^{-1}(x)$.

Suppose $c_1, c_2 \in \mathbb{R}$ is such that $\tilde{\mathcal{A}}(c_1) \cap \tilde{\mathcal{A}}(c_2) \neq \emptyset$. Proposition D.6 implies that there must exists $x \in \mathcal{A}(c_1) \cap \mathcal{A}(c_2)$ such that $u_1 = h_{c_1}(x, \cdot)$ and $u_2 = h_{c_2}(x, \cdot)$ have a common calibrated orbit at $x$. Let $(x'_k)_{k \le 0}$ denote the lift of this orbit, we have

$$\alpha(c_1) = \lim_{n\to\infty} \frac{1}{n} H((x_k)_{k=-n}^0) - \frac{c_1 \cdot (x'_0 - x'_{-n})}{n} = \lim_{n\to\infty} \frac{1}{n} H((x_k)_{k=-n}^0) - c_1 \cdot \rho((x'_k)),$$

where the limit exists due to Lemma D.7. Apply the same calculation to $c'$ over the same orbit, we get

$$\alpha(c_1) - \alpha(c_2) = (c_1 - c_2) \cdot \rho((x_k)).$$

The above equality combined with the convexity of $\alpha$ implies $\alpha$ is a linear function on $[c_1, c_2]$. Suppose $(y_k^{(n)}, r_k^{(n)})_{k=0}^n$ is the lift of a minimizer for $A^n_{(c_1+c_2)/2}(x,x)$, then

$$\begin{aligned}
&A^n_{(c_1+c_2)/2}(x,x) + n\alpha\left(\frac{c_1+c_2}{2}\right)\\
&= H\left((y_k)_{k=0}^n\right) - \frac{c_1+c_2}{2}\cdot(y_n - y_0) + \alpha\left(\frac{c_1+c_2}{2}\right)\\
&= \frac{1}{2}\left(H\left((y_k)\right) - c_1\cdot(y_n-y_0) + \alpha(c_1)\right) + \frac{1}{2}\left(H\left((y_k)\right) - c_2\cdot(y_n-y_0) + \alpha(c_2)\right)\\
&\geq \frac{1}{2}\left(A^n_{c_1}(x,x) + n\alpha(c_1) + A^n_{c_2}(x,x) + n\alpha(c_2)\right) \geq 0.
\end{aligned} \tag{D.3}$$

Suppose $x \in \mathcal{A}((c_1+c_2)/2)$, then there exists $n_k \to \infty$ such that $A^{n_k}_{(c_1+c_2)/2)}(x,x) \to 0$. Then (D.3) implies that the $c_1$ and $c_2$ action of the same orbits also converges to 0. This implies any limit points of $(x, r_0^{(n_k)})$ converges to a point in $\tilde{\mathcal{A}}(c_1) \cap \tilde{\mathcal{A}}(c_2)$. As a result, $\tilde{\mathcal{A}}((c_1+c_2)/2) \subseteq \tilde{\mathcal{A}}(c_1) \cap \tilde{\mathcal{A}}(c_2)$. Since all three Aubry set are graphs over $\mathbb{T}$ as we just proved, we have $\tilde{\mathcal{A}}(c_1) = \tilde{\mathcal{A}}(c_2)$. Finally, $c_1 = c_2$ since $c$ is uniquely determined by the invariant graph $\gamma$ via the relation $\gamma = (x, c + du(x))$.

Since $\psi : c \mapsto \rho(c)$ is monotone, the set $\{c : \rho(c) \in [\rho_1, \rho_2]\}$ is an interval $[c_1, c_2]$. The set $\psi([c_1, c_2])$ contains all rational numbers in $[\rho_1, \rho_2]$. We have proven that the map $G_x : c \mapsto \tilde{\mathcal{A}}(c) \cap \pi^{-1}(x)$ is one-to-one over $[c_1, c_2]$. Moreover, the disjointedness of different $\tilde{\mathcal{A}}(c)$ implies the map $c \mapsto G_x(c)$ is monotone, and we have proved that $G_x$ is the continuous extension of $G_x|\{c : \rho(c) \in \mathbb{Q}\}$, hence continuous. The above argument shows that $G_x$ is a homeomorphism. □

*Proof of Proposition D.3.* We have already proven (a) ⇒ (c).

To prove (c) ⇒ (b), note that by Lemma D.5, every invariant curve of the system is the graph of the gradient of a weak KAM solution, and every orbit on the curve is calibrated by this solution. This implies, in particular, that every orbit is minimizing. It is well known that minimizing orbits don't have conjugate points, see for example [Arn10, CI99]. This proves (c) ⇒ (b).

We now prove (b) ⇒ (a). We follow the proof in [A+15], Sect. 2.2. Let $U$ denote the part of the phase space between two invariant curves. For $x \in \mathbb{T}$, let $V(x) = \{x\} \times [-1, 1]$ be the vertical fiber at $x$. Then the no conjugate point condition imply that the map $F^k$ (lifted to $\mathbb{R} \times [-1, 1]$) is a global diffeomorphism from $V(x) \cap U$ to its image. In particular, the set $F^k(V(x) \cap U) \cap V(x)$ has at most one point.

Given $x' \in \mathbb{R}$ and $p/q \in [\rho_1, \rho_2]$, define the $(p, q)$-minimal action function

$$M_{p,q}(x') = \min\{H((x_k)_{k=0}^q : \quad x_0 = x',\, x_q = x' + p\}.$$

It's known since Birkhoff that the critical points of $M_{p,q}$ corresponds to periodic orbit. In particular, $\min M_{p,q}$ and $\max M_{p,q}$ corresponds to periodic orbits. Before proceeding, let's first prove the following:

Claim *If $(x_k)_{k=1}^q$ is the lift of a periodic orbit outside of $U$, then $(x_q - x_0)/q \notin [\rho_1, \rho_2]$.*

*Proof of claim.* There are two cases, either $(x_0, r_0)$ is below $\gamma_1$, or it is above $\gamma_2$ (in terms of the $r$ coordinate). We will only prove assuming the former as the other case is similar. Let $(x_k, \delta_k) = V(x_k) \cap \gamma_1$, then by Lemma C.3,

$$\pi_1 F^k(x_0, \delta_0) > x_k$$

for all $k \geq 0$. It follows that $\frac{x_q - x_0}{q} \leq \lim_{k\to\infty} \frac{\pi_1 F^{kq}(x_0,\delta_0)}{kq} = \rho_1$. The inequality is strict since $\gamma_1$ already contain all periodic orbits of rotation number $\rho_1$. This concludes the proof of the Claim.

Continuing with the proof, let $(y_k)_{k=0}^q$ be the maximizer for $M_{p,q}$. Since $(y_k)$ is periodic, we have $\pi_1 F^{nq}(y_0) = y_0 + np$ for all $n \geq 1$. Our claim implies that the periodic orbit associated with $(y_k)$ is contained in $U$, hence $y_0$ is the unique solution to the equation

$$\pi_1 F^{nq}(y_0) = y_0 + np.$$

Moreover, since the orbit realizing $M_{np,nq}$ satisfies the same equation, they must coincide. Let

$$C = \max_{p-1 < x_q - x_0 < p+1} \min_{x_1,\dots,x_{q-1}} H((x_k)_{k=0}^q),$$

we have

$$nH((y_k)_{k=0}^q) = M_{np,nq}(y_0) \leq 2C + (n-2) \min M_{p,q},$$

because we can move the inner $(n-2)$ cycles to the minimum cycles of $M_{p,q}$, and pay $C$ action connecting $y_0$ to the cycle in the first $q$ steps, and another $C$ action connecting the cycle back to $y_{nq}$. Divide by $n$ and take limit, we get $\max M_{p,q} = H((y_k)_{k=0}^q) = \min M_{p,q}$, implying $M_{p,q}$ is constant. Since every point $x$ is a critical point of $M_{p,q}$, there exists a invariant curve consisting entirely of $p/q$ periodic orbits. This concludes the proof (b) $\Rightarrow$ (a). □

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V. Kaloshin, C.E. Koudjinan
Institute of Science and Technology Austria (ISTA), Klosterneuburg, Austria.
vadim.kaloshin@gmail.com
koudjinanedmond@gmail.com

Ke Zhang
University of Toronto, Toronto, Canada.
kzhang@math.toronto.edu

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