Morse Predecomposition of an Invariant Set

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Abstract

Motivated by the study of recurrent orbits and dynamics within a Morse set of a Morse decomposition we introduce the concept of Morse predecomposition of an isolated invariant set within the setting of both combinatorial and classical dynamical systems. While Morse decomposition summarizes solely the gradient part of a dynamical system, the developed generalization extends to the recurrent component as well. In particular, a chain recurrent set, which is indecomposable in terms of Morse decomposition, can be represented more finely in the Morse predecomposition framework. This generalization is achieved by forgoing the poset structure inherent to Morse decomposition and relaxing the notion of connection between Morse sets (elements of Morse decomposition) in favor of what we term 'links'. We prove that a Morse decomposition is a special case of Morse predecomposition indexed by a poset. Additionally, we show how a Morse predecomposition may be condensed back to retrieve a Morse decomposition.

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1 Introduction

A Morse decomposition of a flow on a compact metric space is a finite collection of disjoint, compact invariant sets, referred to as *Morse sets*, such that all the recurrent dynamics of the flow is contained in the union of the Morse sets and the Morse sets can be indexed by a partially ordered set (poset). Morse decompositions provide a rigorous, computable approximation to Ch. Conley's fundamental result on the decomposition of a compact invariant set into its chain recurrent part and gradient-like part [1, 2]. Identification of nontriviality of Morse sets can be done via the Conley index [1], but more detailed understanding of the structure of the dynamics within a Morse sets requires additional information. With this in mind, in this paper we seek to extend the concept of a Morse decomposition to a Morse predecomposition that provides a more refined representation of the internal structure of Morse sets. In essence, we give up the partial order condition and we weaken the concept of a connection between two isolated invariant sets. We present the definition both in the combinatorial setting of multivector fields [3, 4] (see Definition 4.3) and in the classical setting of flows (see Definition 5.2). We prove that a Morse predecomposition with a partial order as the preorder is always a Morse decomposition (see Corollary 4.15 and Theorem 5.13) and show how, in the combinatorial case, one can reconstruct a Morse decomposition from a given Morse predecomposition (see Theorem 4.13).

Our presentation is motivated by the fact that algorithms for constructing Morse decompositions are based on a combinatorial representation of dynamics in the form of a multivalued map acting on a collection of cells decomposing the phase space. The multivalued map sends a cell Q into a collection of cells that captures the forward evolution of Q under the dynamics. The action of the multivalued map may be interpreted as a directed graph with the collection of cells as vertices. The strongly connected components of this graph identify isolating blocks and the associated isolated invariant sets form the constructed Morse decomposition.

In the context of discrete dynamics, i.e., dynamics generated by a continuous function $f: X \to X$, there is extensive work and broadly applicable software for identification of Morse decompositions [5–8] and analysis of the structure of dynamics within the Morse sets [9–13]. Comparable tools are much less developed in the context of flows, e.g., dynamics generated by a differential equation [14]. For the purposes of this paper we focus on the theory of combinatorial multivector fields [3, 4, 15, 16] (an extension of the methods of combinatorial topological dynamics introduced by R. Forman [17, 18]) for which the above mentioned graph theoretic algorithms produce Morse decompositions and for which identification of finer structure of dynamics within Morse sets is possible, e.g., the identification of periodic orbits and chaotic dynamics [19]. We expect that effective extension of these latter results will require combinatorial decompositions of strongly connected components and a deeper understanding of the dynamic interpretations of these decompositions, which are subjects of this paper.

To understand the idea of a Morse predecomposition on a simple example, consider the flow visualized in Fig. 1 (top) and the isolated invariant set S highlighted in pink. We recall that a *Morse decomposition* of S is a finite indexed family $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of mutually disjoint isolated invariant sets (called Morse sets) with a partial order
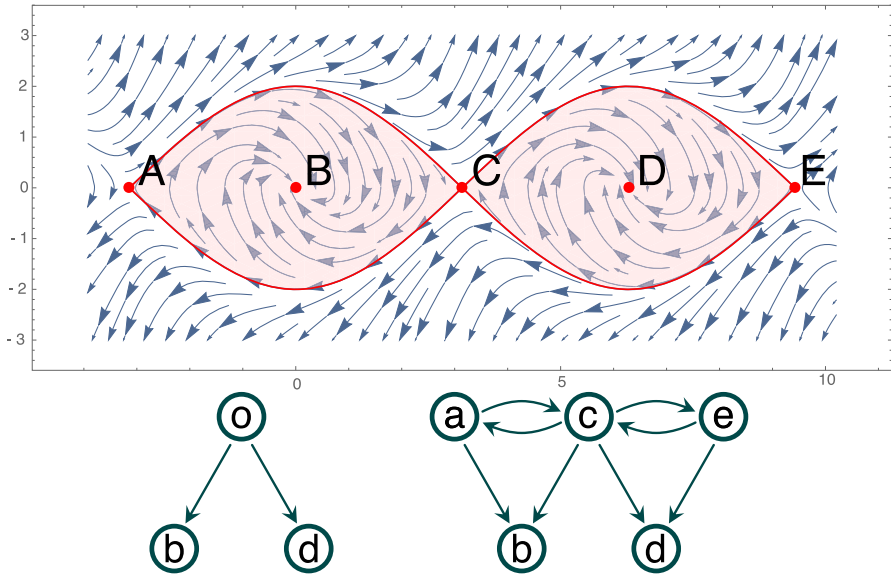
Fig. 1 Top: An isolated invariant set S marked in pink. Bottom left: The flow-defined Conley model for the minimal Morse decomposition of S. Bottom right: The flow-defined Conley model for the minimal Morse predecomposition of S

$(\mathbb{P}, \leq)$ such that for every point $x \in S \setminus \bigcup \mathcal{M}$ we can find $p, q \in \mathbb{P}$ such that $q < p$, $\alpha(x) \subset M_p$ and $\omega(x) \subset M_q$. The finest Morse decomposition of S consists of three Morse sets, $M_b = \{B\}$, $M_d = \{D\}$ and set M_o composed of the stationary points A, C, E, together with the heteroclinic orbits connecting them. The connections between the Morse sets are presented in Fig. 1 (bottom left) in the form of an acyclic directed graph (digraph). The Morse decomposition cannot capture the stationary points A, C, E and the heteroclinic connections between them.

We define a *Morse predecomposition* of S as a finite indexed family $\mathcal{M}' = \{M_p \mid p \in \mathbb{P}'\}$ of mutually disjoint, closed invariant sets such that for all $x \in S$ we have $\alpha(x) \cap M_p \neq \emptyset$ and $\omega(x) \cap M_q \neq \emptyset$ for some $p, q \in \mathbb{P}'$ where $\mathbb{P}'$ is an arbitrary finite set. With such a definition we can split M_o into three sets M_a, M_c and M_e corresponding to the stationary points A, C, and E as independent elements of $\mathcal{M}'$. Thus, the resulting Morse predecomposition is $\mathcal{M}' = \{M_b, M_d, M_a, M_b, M_c\}$. The corresponding digraph representing the connections is presented in Fig. 1 (bottom right). We call it the *Conley model* of $\mathcal{M}'$ (see Sect. 5.1).

The directed graph presented in Fig. 2 has four strongly connected components marked in light green. There are also three strongly connected sets which are not strongly connected components. They are marked in orange. Every strongly connected set is contained in a strongly connected component. However, a strongly connected component may have many non-empty, proper, disjoint, strongly connected subsets. Hence, even on a purely combinatorial level the internal structure of a strongly connected component may vary.

Fig. 2 A directed graph with four strongly connected components marked in light green. The three sets marked in orange are strongly connected but not strongly connected components
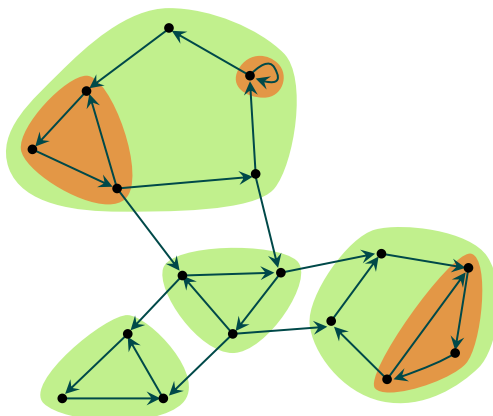
Under weak conditions, every strongly connected component of a digraph obtained from a cellular decomposition of the phase space of a flow identifies an isolating neighborhood and a Morse set inside. This raises the question whether a strongly connected subset of a strongly connected component also identifies an isolating neighborhood and an isolated invariant set. This is not true in general, but in the context of the combinatorial multivector field models discussed above we can provide conditions under which the answer is positive. Thus, we first introduce Morse predecompositions for multivector fields where the concept arises naturally, is relatively simple, and suggests how to set up the analogous definition for flows. Finally, we introduce a concept of *Conley predecomposition* (Definition 5.15) – a construction analogous to Morse predecomposition but with isolating neighborhoods taking over the role of Morse sets (that is isolated invariant sets). Conley predecomposition plays multiple roles. First, in computations one usually works with isolating neighborhoods instead of Morse sets as they are easier to handle [5, 6]. Secondly, it forms a bridge between the combinatorial and classical dynamics, because, as already mentioned, under certain transversality conditions [19], a combinatorial Morse set corresponds to an isolating neighborhood in the original phase space. Thus, a combinatorial Morse predecomposition can be translated into a Conley predecomposition which, in turn, leads to a continuous Morse (pre)decomposition. Finally, Conley predecomposition helps to formalize the idea of stability of a Morse predecomposition.

This paper is organized as follows. Section 2 contains preliminaries. In Sect. 3 we introduce basic concepts and facts concerning dynamics, both classical and combinatorial. The combinatorial formulation of Morse predecomposition, as well as its properties are presented in Sect. 4. Section 5 contains the definition of the Morse predecomposition with the related results for classical dynamical systems. Concluding remarks together with an outline of the further direction of the research are discussed in the final Sect. 6.

2 Preliminaries

This section gathers notation, terminology and basic results needed in the sequel.

2.1 Sets, Families and Maps

We denote the sets of real numbers, non-negative real numbers, integers, non-negative integers and natural numbers respectively by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{N}$.

By a *family* we mean a set whose elements are sets. Given a set X, we denote the family of all subsets of X by $\mathcal{P}(X)$.

An *indexed family* is, formally speaking, a function assigning to every $\iota \in J$ the set A_ι (see [20, Section I.1]). Informally, we will write an indexed family as $\{A_\iota \mid \iota \in J\}$. Every family $\mathcal{A}$ may be consider an indexed family $\{A \mid A \in \mathcal{A}\}$. The converse is not true. Although an indexed family is often identified with the family of its values, there is a subtle difference. The family of values may be smaller, because A_ι and A_κ may be equal as sets, but they are different in the indexed family if $\iota \neq \kappa$. In the case when the elements of a family are mutually disjoint, the difference between families and indexed families matters only in the case of the empty set, which may appear only once in a family but many times in an indexed family.

A *partition* of a set X is a family $\mathcal{E} \subset \mathcal{P}(X)$ of mutually disjoint, non-empty subsets of X such that $\bigcup \mathcal{E} = X$. Given two families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ we say that $\mathcal{A}$ is *inscribed* in $\mathcal{B}$ or $\mathcal{A}$ is a refinement of $\mathcal{B}$ if for each $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ such that $A \subset B$.

We write $f : X \nrightarrow Y$ for a *partial map* from X to Y, that is, a map defined on a subset dom $f \subset X$, called the *domain* of f, and such that the set of values of f, denoted by im f, is contained in Y.

We write $F : X \multimap Y$ to denote a *multivalued map*, that is a map defined on X with subsets of Y as values. We identify such an F with the relation

$$\{\, (x, y) \in X \times Y \mid y \in F(x) \,\}.$$

Such an identification lets us define the *composition* $G \circ F$ of multivalued maps $F : X \multimap Y$ and $G : Y \multimap Z$ as the composition of relations. In particular, we are interested in the iterates of a multivalued map $F : X \multimap X$ considered as a multivalued dynamical system.

2.2 Relations

Let X be a set and let $R \subset X \times X$ be a binary relation in X. A relation $R' \subset X \times X$ is an *extension* of the relation R if $R \subset R'$. A set X with a reflexive and transitive relation R is called a *preordered set*; the relation R is called a *preorder*. If R is additionally antisymmetric we call it a *partial order*, and X a *partially ordered set* (or a poset). The total relation $X \times X$ is clearly a preorder containing every binary relation R in X. Hence, the family of preorders extending a given R is not empty. The intersection of this family is easily seen to be the smallest preorder in X extending R. We call it the *preorder induced by R* and denote it $\leq_R$. The following proposition is straightforward.

Proposition 2.1 *If the preorder $\leq_R$ induced by a relation $R \subset X \times X$ admits an extension which is a partial order, then $\leq_R$ itself is a partial order.*

2.3 Digraphs

Note that the pair (X, R) may be considered a *directed graph* (a *digraph*) with X as its set of *vertices* and R as its set of *edges*. Vice versa, given a digraph (V, E), the collection of edges E is a relation on the set of vertices V. This lets us freely mix the terminology typically used for relations with the terminology used for digraphs.

Given a digraph $G = (V, E)$, by a *walk in G from v to w of length $k > 0$* we mean a sequence $v = v_0, v_1, \ldots v_k = w$ of vertices in V such that $(v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots k$. Such a walk is a *cycle* if $v_0 = v_k$. A cycle of length one is a *loop*. A digraph is *acyclic* if it has no cycles other then loops. A subset $A \subset V$ of vertices is *strongly connected* if for any $v, w \in A$ there is a walk $v = v_0, v_1, \ldots v_k = w$ such that $v_i \in A$ for $i \in \{0, 1, \ldots, k\}$. A vertex $v \in V$ is strongly connected if $\{v\}$ is strongly connected which happens if and only if there is a loop at v. A *strongly connected component* of G is a strongly connected subset of vertices which is maximal with respect to inclusion. Note that two strongly connected components of G are either disjoint or equal but, in general, the family of all strongly connected components need not be a partition of V.

By mixing the terminology of relations and digraphs, we say that $G = (V, E)$ is *reflexive* if E considered as a relation in V is reflexive or, equivalently, if there is a loop at every vertex v. Note that the family of strongly connected components of a reflexive digraph $G = (V, E)$ is a partition of the vertex set V. Given a reflexive digraph $G = (V, E)$, we write $\leq_G$ for the *preorder in V induced by E*. We note that the preorder $\leq_G$ of an acyclic digraph G is a always a partial order.

2.4 Topological Spaces

For the basic terminology concerning topological spaces we refer the reader to [20, 21]. In terms of notation, given a topological space X, we denote by int A the *interior* of $A \subset X$ and by cl A the closure of A. A subset $A \subset X$ is *locally closed* if mo $A :=$ cl $A \backslash A$ is closed (see [20, Problem 2.7.1] for several equivalent conditions for local closedness).

A topological space X is a *finite topological space* if the underlying set X is finite. In this paper we consider finite topological spaces satisfying T_0 separation axiom. By Alexandrov Theorem [22] every finite, T_0 topological space $(X, \mathcal{T})$ can be uniquely identified with a finite poset $(X, \leq_{\mathcal{T}})$ defined by

$$x \leq_{\mathcal{T}} y \iff x \in \text{cl } y. \tag{1}$$

In particular, a subset A of a finite topological space is locally closed if and only if it is *convex* with respect to the poset, that is if $x \leq y \leq z$ and $x, z \in A$ imply $y \in A$.

3 Dynamical Systems

In this section we gather basic concepts and results in dynamics needed in the paper. We first recall classical definitions and statements for flows and then we introduce the concept of a combinatorial multivector field and the associated combinatorial dynamical system. Note that our notation and terminology for combinatorial systems shadows that for flows. We do so to emphasize the similarities.

3.1 Flows

Assume that (X, d) is a locally compact metric space and $\varphi : \mathbb{R} \times X \to X$ is a flow on X.

We recall that a partial map $\gamma : \mathbb{R} \nrightarrow X$ is a *solution* of φ if $\gamma(s + t) = \varphi(t, \gamma(s))$ for all $s, s + t \in \operatorname{dom} \gamma$. A *solution through* $x \in X$ is a solution such that $x \in \operatorname{im} \gamma$. A *solution in* $S \subset X$ is a solution such that $\operatorname{im} \gamma \subset S$. A solution γ is *full* if $\operatorname{dom} \gamma = \mathbb{R}$; otherwise it is *partial*.

We recall that the *maximal invariant set* of $N \subset X$ is given by

$$\operatorname{Inv}(N) = \operatorname{Inv}(N, \varphi) := \{ x \in N \mid \varphi(\mathbb{R}, x) \subset N \}.$$

We say that a set $S \subset X$ is *invariant* if $\operatorname{Inv} S = S$. A compact set $N \subset X$ is an *isolating neighborhood* if $\operatorname{Inv} N \subset \operatorname{int} N$. An invariant set $S \subset X$ is *an isolated invariant set* if there exists an isolating neighborhood N such that $\operatorname{Inv} N = S$.

A fundamental feature of isolating neighborhoods is that they persist under a small perturbation. More precisely, we have the following proposition.

Proposition 3.1 [23, Proposition 1.1] *Given a continuously parametrized family of dynamical systems*

$$\varphi_\lambda : \mathbb{R} \times X \to X, \quad \lambda \in [-1, 1]$$

and an isolating neighborhood for φ_0, *there is an* $\varepsilon > 0$ *such that* N *is also an isolating neighborhood for* φ_λ *with* $\lambda \in (-\varepsilon, \varepsilon)$. $\qquad \square$

The *omega limit set of a point* $x \in X$, denoted by $\omega(x)$, is the maximal invariant set contained in the closure of the forward solution passing through x:

$$\omega(x) := \bigcap_{t>0} \operatorname{cl} \varphi([t, +\infty), x).$$

Symmetrically, we have the *alpha limit set of* x, denoted by $\alpha(x)$ defined as

$$\alpha(x) := \bigcap_{t<0} \operatorname{cl} \varphi((-\infty, t], x).$$

As a point x uniquely determines the corresponding solution $\gamma(t) := \varphi(t, x)$ we use also notation $\alpha(\gamma)$ and $\omega(\gamma)$. This will allow us to be more consistent with the notation used for multivalued combinatorial dynamical systems.

Let $S \subset X$ be a compact invariant set. An invariant set $A \subset S$ is an *attractor in S* if there exists a neighborhood U of A such that $\omega(U \cap S) = A$. A compact invariant set $R \subset S$ is a *repeller in S* if there exists a neighborhood V of R such that $\alpha(V \cap S) = R$. If $S = X$ we just say that A is an attractor (repeller).

3.2 Chain Recurrence

Let S be a compact metric space. Fix an open cover $\mathcal{U}$ of S and a $T > 0$. A $(\mathcal{U}, T)$-*chain from x to y of length n* is a sequence $x = x_0, x_1, ..., x_n = y$ such that there exist sequences $t_1, t_2, \ldots, t_n$ of reals numbers and $U_1, U_1, \ldots, U_n \in \mathcal{U}$ satisfying $t_i \geq T$ and $x_i, \varphi(t_i, x_{i-1}) \in U_i$ for each $i = 1, 2 \ldots n$. Let us denote by $R(S)$ the set of points $x \in S$ such that for every open cover $\mathcal{U}$ of S, and every $T > 0$ there exists a non-constant $(\mathcal{U}, T)$-chain in S from x to itself. A set $A \subset S$ is *chain recurrent* if $R(A) = A$.

As a consequence of [1, 4.1.D] and [1, 6.3.C] and dual results for α limit sets we obtain the following proposition.

Proposition 3.2 *For every point $x \in S$, the limit sets $\alpha(x)$ and $\omega(x)$ are non-empty, compact, invariant, connected and chain recurrent.*

By [24, Theorem 4.12(5)] we immediately get the following property.

Proposition 3.3 *If $C \subset S$ is a connected, invariant, chain recurrent set and there exists a non-empty isolated invariant set A such that $A \subsetneq C$, then A is neither an attractor nor a repeller in C.*

The following lemma may be proved analogously to [25, Theorem 6.2].

Lemma 3.4 *Assume S is connected and $A \subsetneq S$ is a non-empty, isolated invariant set. If A is not an attractor (respectively not a repeller), then there exists an $x \in S \setminus A$ such that $\alpha(x) \subset A$ (respectively $\omega(x) \subset A$).*

Theorem 3.5 *Assume $C \subset S$ is a connected, invariant, chain recurrent set and A is a non-empty isolated invariant set in C. If $A \subsetneq C$, then there exist points $x, y \in C \setminus A$ such that $\alpha(x) \subset A$ and $\omega(y) \subset A$.*

Proof It follows from Proposition 3.3 that A is neither an attractor nor a repeller in C. Hence, the conclusion follows immediately from Lemma 3.4.   □

3.3 Combinatorial Dynamical Systems

Let X be a finite T_0 topological space. A *combinatorial dynamical system* on X induced by a multivalued map $\Pi : X \multimap X$ is a multivalued map $\Pi : \mathbb{Z}^+ \times X \multimap X$ defined recursively for $x \in X$ and $n \in \mathbb{Z}^+$ by $\Pi(0, x) := \{x\}$ and $\Pi(n+1, x) := \Pi(\Pi(n, x))$. Formally, the map $\Pi : X \multimap X$ is the *generator* of the combinatorial dynamical system

but, to simplify the terminology, we often refer to $\Pi : X \multimap X$ as the combinatorial dynamical system.

A *solution* of Π is a partial map $\gamma : \mathbb{Z} \nrightarrow X$ such that dom γ is an interval in $\mathbb{Z}$ and $\gamma(t+1) \in \Pi(\gamma(t))$ for $t, t+1 \in$ dom γ. We recall that Π may be considered as a binary relation in X, hence, also as a directed graph G_Π. In terms of the digraph G_Π, a solution may be identified with a walk in G_Π. If dom γ is finite, we call γ a *path*. If dom $\gamma = \mathbb{Z}$, we call γ a *full solution*. We denote the restriction of a full solution γ to the intervals $[p, q]$, $[0, \infty)$, and $(-\infty, 0]$ respectively by $\gamma_{[p,q]}$, γ^+, and γ^-.

A *shift* is a map $\tau_s : \mathbb{Z} \ni t \mapsto t+s$. Let γ and ψ be solutions such that min dom $\gamma = p$ and max dom $\psi = q$ with $p, q \in \mathbb{Z}$. Moreover, assume that $\gamma(p) \in \Pi(\psi(q))$. Then, we define the *composition* of γ and ψ by $\gamma \cdot \psi := \gamma \cup (\psi \circ \tau_{q-p})$ (the union on the right-hand side is taken over partial maps interpreted as relations). It is straightforward to observe that the composition of solutions is also a solution.

3.4 Combinatorial Dynamics Induced by Multivector Fields

In this subsection we briefly recall basic concepts of combinatorial dynamics generated by multivector fields. For a comprehensive introduction into multivector fields see [4].

A *multivector field* on X is a partition $\mathcal{V}$ of X into locally closed subsets of X (see Sec. 2.4 for the definition of local closedness). We refer to the elements of the partition as *multivectors*. Note that mo $V = \text{cl } V \setminus V$ is always closed for a multivector V, because a multivector is locally closed. We say that a multivector V is *critical* if the relative singular homology $H(\text{cl } V, \text{mo } V)$ is non-trivial. Otherwise, we say that V is *regular*.

For $A \subset X$ we set

$$\mathcal{V}_A := \{V \in \mathcal{V} \mid V \cap A \neq \emptyset\} \quad \text{and} \quad [A]_\mathcal{V} := \bigcup \mathcal{V}_A.$$

We say that A is $\mathcal{V}$-*compatible* if $A = [A]_\mathcal{V}$. In particular, for every $x \in X$ set $[x]_\mathcal{V} := [\{x\}]_\mathcal{V}$ is the unique multivector $V \in \mathcal{V}$ containing x.

A multivector field $\mathcal{V}$ induces a combinatorial dynamical system generated by $\Pi_\mathcal{V} : X \multimap X$ given for an $x \in X$ by

$$\Pi_\mathcal{V}(x) := \text{cl } x \cup [x]_\mathcal{V}.$$

As mentioned $\Pi_\mathcal{V}$ induces a directed graph $G_{\Pi_\mathcal{V}}$. We will denote it for short as $G_\mathcal{V}$.

By a solution of $\mathcal{V}$ we mean a solution of the combinatorial dynamical system $\Pi_\mathcal{V}$. For a full solution γ we define combinatorial *limit sets* as

$$\alpha(\gamma) := \left\langle \bigcap_{t \in \mathbb{Z}^-} \gamma((-\infty, t]) \right\rangle_\mathcal{V} \quad \text{and} \quad \omega(\gamma) := \left\langle \bigcap_{t \in \mathbb{Z}^+} \gamma([t, \infty)) \right\rangle_\mathcal{V}, \qquad (2)$$

where $\langle A \rangle_\mathcal{V}$ denotes the $\mathcal{V}$-*hull of* A, that is the smallest locally closed and $\mathcal{V}$-compatible set containing A.

We say that a full solution γ is *non-essential* if either $\alpha(\gamma)$ or $\omega(\gamma)$ is contained in a single regular multivector; otherwise we say that γ is *essential*. These definitions of non-essential and essential solutions are easily seen to be equivalent to the definitions given in [4]. We denote by $\mathrm{eSol}(A)$ the set of all essential solutions in A. Additionally we set

$$\mathrm{eSol}(x, A) := \{\gamma \in \mathrm{eSol}(A) \mid \gamma(0) = x\}$$

and we define the *maximal invariant subset of* $S \subset X$ as

$$\mathrm{Inv}\, S := \{x \in S \mid \mathrm{eSol}(x, S) \neq \emptyset\}.$$

We say that a set S is *invariant* if $\mathrm{Inv}\, S = S$. A closed set N is an *isolating set* for an invariant set S if $\Pi_{\mathcal{V}}(S) \subset N$ and every path in N with endpoints in S is contained in S. If an invariant set admits an isolating set we call it an *isolated invariant set*. Note that the combinatorial notion of isolation is weaker than its classical counterpart. In particular, a closed set N can be an isolating set for two different isolated invariant sets. Moreover, a set isolated by N does not need to be contained in the interior of N (N might no have an interior at all). The need for a modification of isolation in finite topological spaces comes from the tightness of such spaces caused by the lack of the Hausdorff axiom.

Theorem 3.6 [4, Propositions 4.10, 4.12 and 4.13] *Assume $S \subset X$ is invariant. Then S is an isolated invariant set if and only if S is locally closed and $\mathcal{V}$-compatible.*

As an immediate consequence of Theorem 3.6 we obtain the following corollary.

Corollary 3.7 *Assume S is an isolated invariant set of $\mathcal{V}$. Then $\mathcal{V}_S := \{V \in \mathcal{V} \mid V \subset S\}$ is a multivector field on S. We call it the* induced *multivector field.* □

Theorem 3.8 [4, Theorem 6.15] *Let $\gamma \in \mathrm{eSol}(X)$. Then both $\alpha(\gamma)$ and $\omega(\gamma)$ are non-empty, connected, isolated invariant sets. Moreover, the sets $\alpha(\gamma)$ and $\omega(\gamma)$ are strongly connected with respect to $G_{\mathcal{V}}$.*

The following proposition follows immediately from the definition of a strongly connected set.

Proposition 3.9 *Assume $C \subset X$ is strongly connected with respect to $G_{\mathcal{V}}$. Then, there exists a path $\rho : [0, n]_{\mathbb{Z}} \to C$ such that $\mathrm{im}\, \rho = C$ and $\rho(0) = \rho(n)$.*

Proposition 3.10 *Let $\gamma \in \mathrm{eSol}(X)$. Then there exist $\rho, \rho' \in \mathrm{eSol}(\alpha(\gamma))$ such that $\mathrm{im}\, \rho = \alpha(\gamma)$ and $\mathrm{im}\, \rho' = \omega(\gamma)$.*

Proof By Theorem 3.8, set $\alpha(\gamma)$ is strongly connected with respect to $G_{\mathcal{V}}$. Hence, by Proposition 3.9, we can construct a path $\theta : [0, n]_{\mathbb{Z}} \to X$ such that $\mathrm{im}\, \theta = \alpha(\gamma)$ and $\theta(0) = \theta(n)$. Define $\rho := \ldots \cdot \theta \cdot \theta \cdot \theta \cdot \ldots$. Note that ρ is essential, because otherwise $\alpha(\gamma)$ would consist of only a single regular multivector which contradicts the fact that $\alpha(\gamma)$ is an isolated invariant set. Moreover, one easily verifies that $\alpha(\rho) = \omega(\rho) = \mathrm{im}\, \rho = \alpha(\gamma)$. The argument for $\omega(\gamma)$ is analogous. □

3.5 Morse Decompositions in Classical and Combinatorial Dynamics

We now recall the classical concept of Morse decomposition for flows and its counterpart for multivector fields. Later in the paper, after introducing Morse predecompositions, we will give different but equivalent definitions of Morse decomposition, formulated in terms of Morse predecompositions.

Consider an isolated invariant set S of a flow φ on a locally compact metric space X. The following definition of Morse decomposition, in the formulation by J. Reineck [26, Definition 1.3], goes back to R. Franzosa [27].

Definition 3.11 An indexed family $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of mutually disjoint isolated invariant subsets of S indexed by a finite poset $\mathbb{P}$ is a *Morse decomposition* of S if for every $x \in S$ either $x \in M_p$ for some $p \in \mathbb{P}$ or there exist $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(x) \subset M_p$ and $\omega(x) \subset M_q$.

Consider now an isolated invariant set S of a combinatorial multivector field $\mathcal{V}$ on a finite topological space X. The following definition of Morse decomposition for multivector fields was introduced in [4, Definition 7.1]

Definition 3.12 An indexed family $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of mutually disjoint isolated invariant subsets of S indexed by a poset $\mathbb{P}$ is a *Morse decomposition* of S if for every essential solution γ in S either im $\gamma \subset M_p$ for some $p \in \mathbb{P}$ or there exist $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$.

As pointed out in [28], Morse decomposition is an extremely general tool, but this generality has led to subtle variances in its definition. Some approaches exclude the empty set as a Morse set, some allow for many empty Morse sets. The latter approach requires a properly set up definition but has many benefits, in particular in the context of the stability of various invariants associated with Morse decomposition. In [28], the issue is resolved by differentiating Morse representation from Morse decomposition. This paper is based on the original definition by Franzosa [29] in which Morse decomposition is considered as a properly understood indexed family. An indexed family also allows for many empty Morse sets, as pointed out in Sect. 2.1.

4 Morse Predecomposition in Combinatorial Dynamics

In this section we introduce Morse predecompositions in the setting of combinatorial multivector fields. We assume in this section that $\mathcal{V}$ is a multivector field on a finite topological space X and X is invariant with respect to $\mathcal{V}$. Such an assumption is not restrictive, because if X is not invariant, we can replace X by its invariant part and $\mathcal{V}$ by its restriction to the maximal invariant subset.

4.1 Combinatorial Morse Predecomposition

In order to generalize the concept of Morse decomposition we first introduce the following definition classifying full solutions which keep visiting an isolated invariant set S_1 backward in time and an isolated invariant set S_2 forward in time.

Definition 4.1 Consider invariant sets S_1, S_2 of a multivector field $\mathcal{V}$ in X. A full solution $\gamma : \mathbb{Z} \to X$ is a *link from S_1 to S_2* if $\alpha(\gamma) \cap S_1 \neq \emptyset \neq \omega(\gamma) \cap S_2$. The link γ is *homoclinic* if $S_1 = S_2$. Otherwise, it is *heteroclinic*. A homoclinic link γ is *trivial* if $\operatorname{im} \gamma \subset S_1 = S_2$. Otherwise, it is *non-trivial*.

Proposition 4.2 *Let S_1, $S_2 \subset X$ be isolated invariant sets. There exists a link γ from S_1 to S_2 if and only if there exists a path $\rho : [a, b] \to X$ such that $\rho(a) \in S_1$ and $\rho(b) \in S_2$.*

Proof Let γ be a link from S_1 to S_2. Let $a, b \in \mathbb{Z}$ such that $\gamma(a) \in \alpha(\gamma)$ and $\gamma(b) \in \omega(\gamma)$. By Theorem 3.8 for any $x_1 \in S_1 \cap \alpha(\gamma)$ there exists a path ρ_1 from x_1 to $\gamma(a)$. Similarly we can find $x_2 \in S_2 \cap \omega(\gamma)$ a path ρ_2 from $\gamma(b)$ to x_2. Hence, $\rho := \rho_1 \cdot \gamma_{[a,b]} \cdot \rho_2$ is a path from S_1 to S_2.

To prove the opposite implication consider a path $\rho : [a, b] \to X$ such that $\rho(a) \in S_1$ and $\rho(b) \in S_2$. Take $\varphi_1 \in \operatorname{eSol}(\rho(a), S_1)$ and $\varphi_2 \in \operatorname{eSol}(\rho(b), S_2)$ and define essential solution $\gamma := \varphi_1^- \cdot \rho \cdot \varphi_2^+$. We have $\alpha(\gamma) = \alpha(\varphi_1) \subset S_1$ and $\omega(\gamma) = \omega(\varphi_2) \subset S_2$. Hence, γ is a link from S_1 to S_2. □

Definition 4.3 An indexed family $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$ of mutually disjoint isolated invariant subsets in X is a *Morse predecomposition* of X if every essential solution γ in X is a link from M_p to M_q for some $p, q \in \mathbb{P}$. We refer to the sets in $\mathcal{M}$ as *pre-Morse sets*. We also define a Morse predecomposition of an isolated invariant set $S \subset X$ as a Morse predecomposition with respect to the induced multivector field $\mathcal{V}_S$.

Proposition 4.4 *Let $\mathcal{M}$ be a Morse predecomposition of X. Then, for every non-empty isolated invariant set S we have $S \cap \bigcup \mathcal{M} \neq \emptyset$. In particular, if $V \in \mathcal{V}$ is a critical multivector then $V \subset \bigcup \mathcal{M}$.*

Proof Assume to the contrary that S is a non-empty isolated invariant set such that $S \cap \bigcup \mathcal{M} = \emptyset$. Since S is non-empty, we can find a $\varphi \in \operatorname{eSol}(S)$. Clearly, $\alpha(\varphi) \subset S$ and $\omega(\varphi) \subset S$ because of Theorem 3.6 and (2). Thus, φ is an essential solution while not being a link which contradicts the assumption that $\mathcal{M}$ is a Morse predecomposition. □

Links of a system form a relation L on an indexing set $\mathbb{P}$ of a Morse predecomposition $\mathcal{M}$ defined as

$$L := \{(p, q) \in \mathbb{P} \times \mathbb{P} \mid \text{ there exists a non-trivial link from } M_p \text{ to } M_q\}.$$

The digraph $G_{\mathcal{M}} := (\mathbb{P}, L)$ is referred to as *the digraph induced by Morse predecomposition $\mathcal{M}$*.

We say that a preorder $\leq$ in $\mathbb{P}$ is *admissible* if the existence of a link from M_p to M_q implies $q \leq p$. Clearly, the preorder induced by $G_{\mathcal{M}}$ is the minimal admissible preorder. We denote it by $\leq_{\mathcal{M}}$ and we call it the *flow induced preorder*.

A *Conley model* of $\mathcal{M}$ is a digraph $C = (N, E)$ with a map $\nu_{G_{\mathcal{M}}, C} : \mathbb{P} \to N$ such that it preserves the graph induced preorders. Clearly, $G_{\mathcal{M}}$ itself is a Conley model. We call $G_{\mathcal{M}}$ the *flow-defined Conley model of $\mathcal{M}$*.
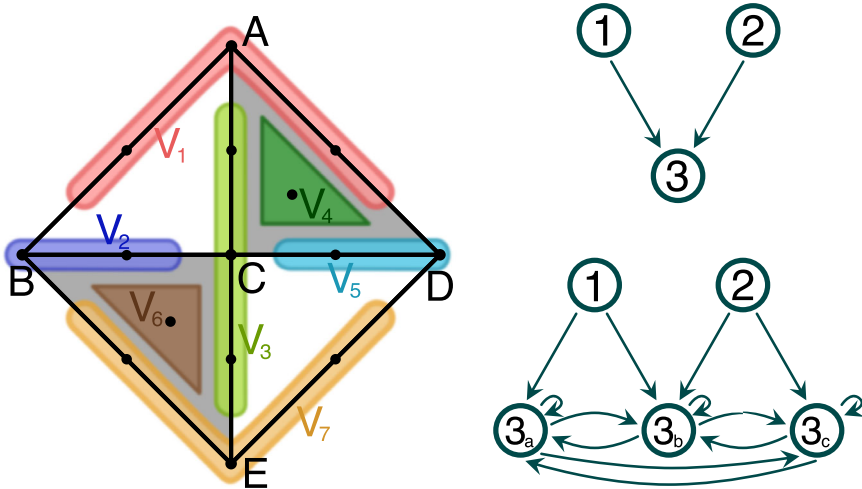
Fig. 3 Left: A simplicial complex K with a multivector field consisting of two regular multivectors, V_2 and V_5, and five critical multivectors, V_1, V_3, V_4, V_6, V_7. Top right: the digraph induced by Morse decomposition $\{M_1, M_2, M_3\}$ where $M_1 := V_4$, $M_2 := V_6$, and $M_3 =: K \backslash V_4 \backslash V_6$. Bottom right: the digraph induced by Morse predecomposition $\mathcal{M}$ consisting of: $M_1 = V_4$, $M_2 = V_6$, $M_3^A = V_1$, $M_3^C = V_3$, and $M_3^E = V_7$

4.2 Combinatorial Morse Decomposition

In this section we present a definition of Morse decomposition formulated as a special case of Morse predecomposition. We later prove (see Theorem 4.11) that this definition is equivalent to Definition 3.12. We begin with the definition of a connection, a special case of a link.

Definition 4.5 Given a Morse predecomposition $\mathcal{M}$, a link γ from M_p to M_q is a *connection* if $\operatorname{im} \gamma \cap \bigcup \mathcal{M} \subset M_p \cup M_q$ and $\alpha(\gamma) \subset M_p$, $\omega(\gamma) \subset M_q$.

Clearly, a trivial link is a homoclinic connection. However, note that not every homoclinic connection is a trivial link.

We say that an invariant subset T of an invariant set S is *saturated in S* if every homoclinic connection γ from T to T in S is trivial. We say that a Morse predecomposition $\mathcal{M}$ is *saturated* if every $M \in \mathcal{M}$ is saturated.

Definition 4.6 A Morse predecomposition $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ is called a *Morse decomposition* if it is saturated and among admissible preorders there is a partial order. Note that then, by Proposition 2.1, the flow induced preorder is a partial order.

Example 4.7 Consider a simplicial complex K in Fig. 3 consisting of 5 vertices, 8 edges, and 2 triangles. We refer to a simplex by the list of vertices in its boundary. Let $\mathcal{V}$ be a multivector field on K consisting of seven multivectors:

$$V_1 := \{A, AB, AD\}, \quad V_2 := \{B, BC\}, \quad V_3 := \{C, AC, CE\},$$
$$V_4 := \{ACD\}, \quad V_5 := \{D, CD\}, \quad V_6 := \{BCE\},$$
$$V_7 := \{E, BE, DE\}.$$

Fig. 4 A multivector field on a cellular complex in which we assume that the edge AC is identified with the edge $A'C'$ and the edge AB is identified with the edge $A''B'$ The multivector field consists of four regular multivectors, V_1, V_2, V_3, V_4, and critical multivector V_5
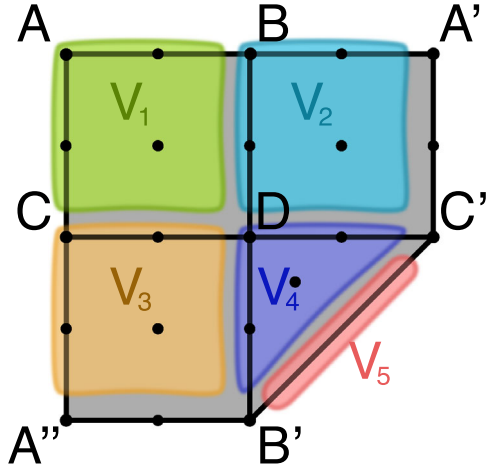

Multivectors V_2 and V_5 are regular, all others are critical. The finest possible Morse decomposition consists of three isolated invariant sets $M_1 = V_4$, $M_2 = V_6$, and $M_3 = K \backslash V_4 \cup V_6$. The Hasse diagram of the admissible partial order coincides with the induced digraph $G_\mathcal{M}$ and is presented in Fig. 3 (top right). On the other hand, Morse predecomposition allows us to treat every critical multivector contained in M_3 as a separate pre-Morse set. Hence, we obtain a Morse predecomposition $\mathcal{M}$ consisting of five sets: $M_1 = V_4$, $M_2 = V_6$, $M_3^A = V_1$, $M_3^C = V_3$, and $M_3^E = V_7$. The induced digraph is presented in Fig. 3 (bottom right). ◇

However, the following example shows that such a minimal Morse predecomposition need not be unique.

Example 4.8 Consider a cellular complex L in Fig. 4 where we assume that the edge AC is identified with the edge $A'C'$ and the edge AB with the edge $A''B'$. Consider a multivector field $\mathcal{V}$ on L consisting of five multivectors:

$$V_1 := \{A, AB, AC, ABCD\}, \quad V_2 := \{B, A'B, BD, BA'DC'\},$$
$$V_3 := \{C, CD, CA'', CDA''B'\}, \quad V_4 := \{D, DC', DB', DC'B'\},$$
$$V_5 := \{B'C'\}.$$

Note that the only critical multivector is V_5. The multivector field $\mathcal{V}$ on L is recurrent in the sense that for any two cells in L we can construct a path connecting them. Thus, the only possible Morse decomposition of L with respect to $\mathcal{V}$ is the trivial one, with L as the only Morse set. However, L admits two non-trivial Morse predecompositions. Let $M_1 := V_1 \cup V_2$, $M_2 := V_1 \cup V_3$, and $M_3 := V_5$. Observe that M_1 is a non-empty isolated invariant set, because for every point in M_1 we can find a periodic solution, for instance, $\ldots, A, AB, B, BA', A, \ldots$. The same observation applies to M_2. The set M_3 is an isolated invariant set as a critical multivector. One can verify that $\mathcal{M}_1 := \{M_1, M_3\}$ and $\mathcal{M}_2 := \{M_2, M_3\}$ are both non-trivial Morse predecompositions of L. We note
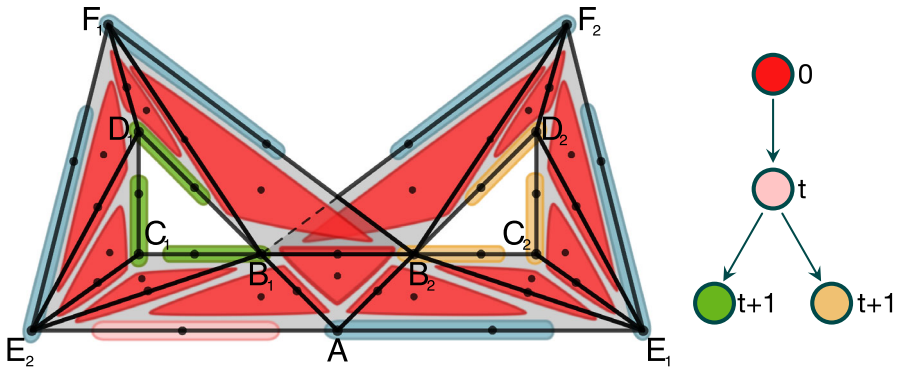
Fig. 5 Left: A combinatorial model of Lorenz type dynamics with a Morse decomposition consisting of four Morse sets: a repeller marked with red multivectors, a saddle marked with a pink multivector and two attracting periodic orbits marked respectively with green and yellow multivectors. Right: The associated Conley-Morse digraph

that the digraph induced by $\mathcal{M}_1$ as well as $\mathcal{M}_2$ is a clique. Clearly, neither of these Morse predecompositions is a refinement of the other.   ◊

The following example presents a combinatorial version of the Williams model [30] of the Lorenz attractor. A similar combinatorial model was introduced in [31, Fig. 3] and studied in [19, Fig. 4].

Example 4.9 Consider the combinatorial multivector field presented in Fig. 5 (left). It has exactly one critical multivector, a singleton edge AE_2 at bottom left. All other multivectors are doubletons. The minimal Morse decomposition consists of four Morse sets: a repeller marked with red multivectors, a saddle marked with a pink multivector and two attracting periodic orbits marked respectively with green and yellow multivectors. The associated Conley-Morse digraph is presented in Fig. 5 (right). In particular, the Conley index of the repeller is zero. Such a Conley index says nothing about the invariant set inside. However, the repeller contains two repelling periodic orbits sharing the middle bottom triangle AB_1B_2. Each of them is an isolated invariant set with the cells of the other on a non-trivial homoclinic connection. It is straightforward to verify that one can obtain a Morse predecomposition by replacing the repeller in the Morse decomposition by one of the repelling orbits. The resulting two minimal Morse predecompositions together with the Conley-Morse graphs are presented in Fig. 6. ◊

Given a Morse predecomposition $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of X and $C \subset X$ define

$$\mathbb{P}_C = \mathbb{P}_C(\mathcal{M}) := \{\, p \in \mathbb{P} \mid M_p \cap C \neq \emptyset \,\}.$$

Lemma 4.10 *Let $\mathcal{M}$ be a Morse predecomposition of X and let $C \subset X$ be an isolated invariant subset of X which is strongly connected in $G_\mathcal{V}$. If there is an $r \in \mathbb{P}_C(\mathcal{M})$ such that M_r is saturated in X then $C \subset M_r$. In particular, $\mathbb{P}_C = \{r\}$.*

Proof To see that $C \subset M_r$ consider an $x \in C$. By Proposition 3.9 we can construct a path $\rho : [0, n]_{\mathbb{Z}} \to C$ through x such that $\operatorname{im} \rho = C$ and $\rho(0) = \rho(n)$. Since
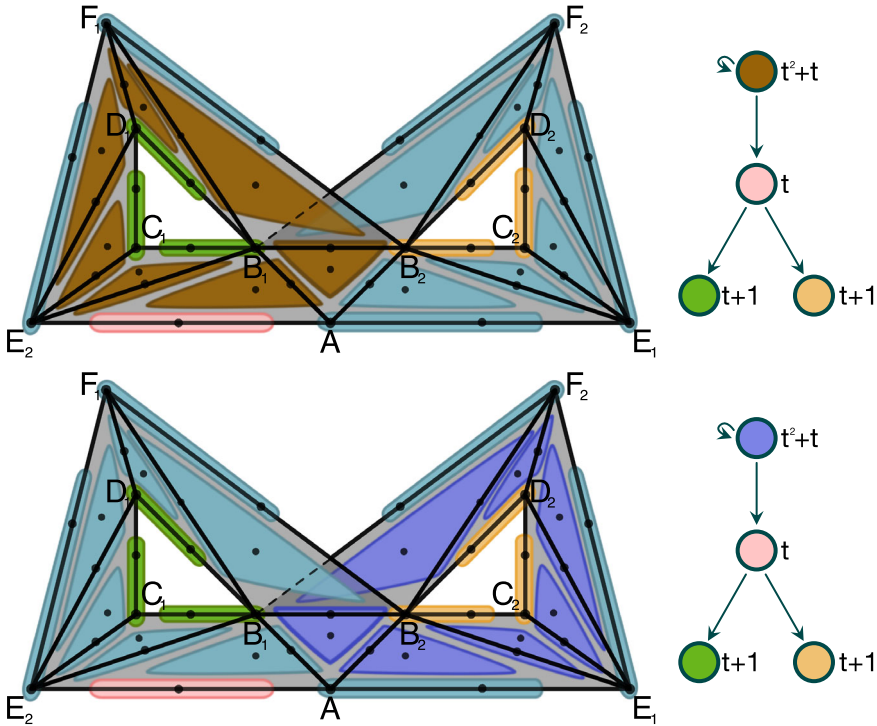
Fig. 6 Top left: A Morse predecomposition with Conley-Morse digraph consisting of four pre-Morse sets: a repelling periodic orbit on the left marked with brown multivectors, a saddle marked with a pink multivector and two attracting periodic orbits marked respectively with green and yellow multivectors. Bottom left: A Morse predecomposition with Conley-Morse digraph symmetric to the top one with the brown repelling orbit replaced by the violet repelling orbit on the right. Top and bottom right: The digraphs induced by the Morse predecompositions on the left

$r \in \mathbb{P}_C(\mathcal{M})$, we can find a $y \in M_r \cap C$. Then $y = \rho(k)$ for some $k \in [0, n]_{\mathbb{Z}}$. Without loss of generality we may assume that $k = 0$. Let γ be an essential solution through y in M_r and let $\gamma' := \gamma^- \cdot \rho \cdot \gamma^+$. Then γ' is an essential solution through x and $\alpha(\gamma') = \alpha(\gamma) \subset M_r$ and $\omega(\gamma') = \omega(\gamma) \subset M_r$. Therefore, γ' is a homoclinic connection of M_r. Since M_r is saturated we have $x \in \operatorname{im} \gamma' \subset M_r$. Consequently, $C \subset M_r$. Since the elements of $\mathcal{M}$ are mutually disjoint, we also get $\mathbb{P}_C = \{r\}$. $\qquad\square$

The following theorem gives characterizations of Morse decomposition. In particular, condition (iii) shows that Definition 4.6 is equivalent to Definition 3.12 which is the definition of Morse decomposition assumed in [4, Definition 7.1].

Theorem 4.11 *Let $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ be an indexed family of mutually disjoint subsets of X. Then, the following conditions are mutually equivalent.*

(i) The family $\mathcal{M}$ is a Morse decomposition of $\mathcal{S}$.

(ii) Each M_p is a saturated, isolated invariant subset of X and $\mathbb{P}$ admits a partial order $\leq$ such that for every essential solution γ in X there exist $p, q \in \mathbb{P}$ satisfying $p \geq q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$.

(iii) *Each M_p is an isolated invariant subset of X and $\mathbb{P}$ admits a partial order $\leq$ such that for every essential solution γ in X either* im $\gamma \subset M_p$ *for some $p \in \mathbb{P}$ or there exist $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$.*

In particular, every Morse set in a Morse decomposition is saturated and isolated.

Proof Assume (i). Then $\mathcal{M}$ is a saturated Morse predecomposition and the flow induced preorder $\leq_\mathcal{M}$ is a partial order. In particular, each M_p is a saturated, isolated invariant subset of X. We claim that the partial order $\leq_\mathcal{M}$ fulfills the requirements in condition (ii). To see this consider an essential solution γ in X. By the definition of Morse predecomposition there exist $p, q \in \mathbb{P}$ such that $\alpha(\gamma) \cap M_p \neq \emptyset$ and $\omega(\gamma) \cap M_q \neq \emptyset$. By the definition of flow induced preorder we obtain $p \leq_\mathcal{M} q$ and by Theorem 3.8 and Lemma 4.10 we conclude that $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. Hence, (ii) follows from (i).

Assume in turn (ii). Then, each M_p is an isolated invariant subset of X. We claim that the partial order $\leq$ given in (ii) fulfills the requirements of condition (iii). To see this, consider an essential solution γ in X and assume there is no $p \in \mathbb{P}$ such that im $\gamma \subset M_p$. By (ii) there exist $p, q \in \mathbb{P}$ satisfying $p \geq q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. We claim that $p > q$. To see this assume to the contrary that $p = q$. We first prove that under this assumption we have $\mathbb{P}_{\text{im}\,\gamma} \subset \{p\}$. To prove this consider an $r \in \mathbb{P}_{\text{im}\,\gamma}$ such that $r \neq p$. Then, by Proposition 4.2, one can construct a link from M_p to M_r and a link from M_r to M_q, which gives $q \leq r \leq p = q$, which contradicts with the assumption that $\leq$ is a partial order. It proves that $\mathbb{P}_{\text{im}\,\gamma} \subset \{p\}$. However, this means that γ is a homoclinic connection to M_p. Since M_p is saturated, we have im $\gamma \subset M_p$, a contradiction. Hence, $p > q$ which completes the proof of (iii).

Finally, assume (iii) and consider an essential solution γ in X. In order to prove that $\mathcal{M}$ is a Morse predecomposition, we have to show that γ is a link from M_p to M_q for some $p, q \in \mathbb{P}$. If there is a $p \in \mathbb{P}$ such that im $\gamma \subset M_p$, then also $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_p$, which shows that γ is a link from M_p to M_p. Otherwise, we conclude from (iii) that there exist $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$, which also proves that γ is a link. This completes the proof that $\mathcal{M}$ is a Morse predecomposition.

Clearly, $\leq$ is then an admissible partial order for $\mathcal{M}$. We still need to prove that each $M_r \in \mathcal{M}$ is saturated. To see this consider a full solution γ such that $\alpha(\gamma) \cup \omega(\gamma) \subset M_r$. If im $\gamma \not\subset M_r$, then we conclude from (iii) the existence of $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. However, Morse sets are mutually disjoint and limit sets are non-empty. Therefore $p = r = q$, a contradiction proving that M_r is saturated. □

4.3 Consolidation Theorem

Assume that $\mathcal{V}$ is a combinatorial multivector field on a finite topological space X and consider a fixed Morse predecomposition $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$. Given $\mathbb{Q} \subset \mathbb{P}$ we denote the family of links in X between pre-Morse sets with indexes in $\mathbb{Q}$ by

$$\text{eSol}_\mathbb{Q}(X) := \{\gamma \in \text{eSol}(X) \mid \mathbb{P}_{\alpha(\gamma)} \cap \mathbb{Q} \neq \emptyset \neq \mathbb{P}_{\omega(\gamma)} \cap \mathbb{Q}\},$$

and we define the *connection set* of $\mathbb{Q}$ by

$$M_{\mathbb{Q}} := \bigcup \{ \operatorname{im} \gamma \mid \gamma \in \operatorname{eSol}_{\mathbb{Q}}(X) \}. \tag{3}$$

Lemma 4.12 *Assume $\leq$ is an admissible preorder with respect to Morse predecomposition $\mathcal{M}$. Then for every $\mathbb{Q} \subset \mathbb{P}$ the set $M_{\mathbb{Q}}$ is a saturated isolated invariant set. Moreover, if $\mathbb{Q}$ is convex with respect to $\leq$ then $M_{\mathbb{Q}} \cap M_r = \emptyset$ for all $r \in \mathbb{P} \setminus \mathbb{Q}$.*

Proof First observe that $\operatorname{eSol}(x, M_{\mathbb{Q}})$ is non-empty for every $x \in M_{\mathbb{Q}}$ directly by definition (3). Thus, $M_{\mathbb{Q}}$ is invariant. We will show that $M_{\mathbb{Q}}$ is $\mathcal{V}$-compatible and locally closed. To prove $\mathcal{V}$-compatibility take an $x \in M_{\mathbb{Q}}$ and a $\gamma \in \operatorname{eSol}(x, M_{\mathbb{Q}})$. Consider a $y \in [x]_{\mathcal{V}}$ and an essential solution $\gamma' := \gamma^- \cdot y \cdot \gamma^+$. Clearly, $\gamma' \in \operatorname{eSol}_{\mathbb{Q}}$. It follows that $y \in M_{\mathbb{Q}}$. Therefore, if $x \in M_{\mathbb{Q}}$ then $[x]_{\mathcal{V}} \subset M_{\mathbb{Q}}$, which proves the $\mathcal{V}$-compatibility of $M_{\mathbb{Q}}$.

To show that $M_{\mathbb{Q}}$ is locally closed it suffices to prove that $M_{\mathbb{Q}}$ is convex with respect to the order $\leq_{\mathcal{T}}$ on X given by the topology (see (1) in Sect. 2.4). Hence, consider $x, z \in M_{\mathbb{Q}}$ and a $y \in X$ such that $x \leq_{\mathcal{T}} y \leq_{\mathcal{T}} z$. Since there exist $\gamma_x \in \operatorname{eSol}(x, M_{\mathbb{Q}})$ and $\gamma_z \in \operatorname{eSol}(z, M_{\mathbb{Q}})$, we can construct an essential solution $\psi := \gamma_z^- \cdot y \cdot \gamma_x^+$. We have $\alpha(\psi) \cap M_p = \alpha(\gamma_z) \cap M_p \neq \emptyset$ and $\omega(\psi) \cap M_q = \omega(\gamma_x) \cap M_q \neq \emptyset$ for some $p, q \in \mathbb{Q}$. Hence, $\psi \in \operatorname{eSol}_{\mathbb{Q}}$ and $y \in \operatorname{im} \psi \subset M_{\mathbb{Q}}$. This proves that $M_{\mathbb{Q}}$ is convex with respect to $\leq_{\mathbb{Q}}$, that is locally closed. Hence, we obtain from Theorem 3.6 that $M_{\mathbb{Q}}$ is an isolated invariant set. It follows directly from the definition of $M_{\mathbb{Q}}$ that $M_{\mathbb{Q}}$ is saturated.

To prove the remaining assertion assume that $\mathbb{Q} \subset \mathbb{P}$ is convex with respect to the admissible preorder $\leq$ and $M_{\mathbb{Q}} \cap M_r \neq \emptyset$ for some $r \in \mathbb{P} \setminus \mathbb{Q}$. Then we can select an $x \in \operatorname{im} \gamma \cap M_r$ for some $\gamma \in \operatorname{eSol}(x, M_{\mathbb{Q}})$ and, by Definition (3), we can find $p, q \in \mathbb{Q}$ such that $\alpha(\gamma) \cap M_p \neq \emptyset$ and $\omega(\gamma) \cap M_q \neq \emptyset$. Since M_r is invariant there exists a $\psi \in \operatorname{eSol}(x, M_r)$. Consider essential solutions $\gamma_q := \psi^- \cdot \gamma^+$ and $\gamma_p := \gamma^- \cdot \psi^+$. We have

$$\alpha(\gamma_q) \subset M_r, \qquad\qquad \omega(\gamma_q) \cap M_q \neq \emptyset,$$
$$\omega(\gamma_p) \subset M_r, \qquad\qquad \alpha(\gamma_p) \cap M_p \neq \emptyset.$$

By the admissibility of $\leq$ we have $q \leq r \leq p$. Since $p, q \in \mathbb{Q}$ and $r \notin \mathbb{Q}$, this contradicts the assumption that $\mathbb{Q}$ is convex. $\qquad\square$

Theorem 4.13 (Consolidation theorem) *Let $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$ be a Morse predecomposition of X and let $\leq$ be an admissible preorder. Assume $\mathcal{Q}$ is a partition of $\mathbb{P}$ consisting of non-empty, convex with respect to $\leq$ subsets of $\mathbb{P}$ such that the induced relation $\leq_{\mathcal{Q}}$ is a partial order, where $\leq_{\mathcal{Q}}$ is defined for $Q, Q' \in \mathcal{Q}$ by $Q \leq_{\mathcal{Q}} Q'$ if and only if there exist a $q \in Q$ and a $q' \in Q'$ such that $q \leq q'$. Then, $\mathcal{M}' := \{ M_Q \mid Q \in \mathcal{Q} \}$ is a Morse decomposition of X.*

Proof We see from Lemma 4.12 that M_Q is a saturated isolated invariant set for every $Q \in \mathcal{Q}$. Consider $M_Q, M_{Q'} \in \mathcal{M}'$ for some $Q, Q' \in \mathcal{Q}$ such that $Q \neq Q'$. To see that M_Q and $M_{Q'}$ are disjoint, suppose to the contrary that there exists an $x \in M_Q \cap M_{Q'}$. Select a $\gamma \in \operatorname{eSol}(x, M_Q)$ and a $\gamma' \in \operatorname{eSol}(x, M_{Q'})$. Then $\gamma_{QQ'} := \gamma^- \cdot (\gamma')^+$ is an

essential solution satisfying $\alpha(\gamma_{QQ'}) = \alpha(\gamma) \subset M_Q$ and $\omega(\gamma_{QQ'}) = \omega(\gamma') \subset M_{Q'}$. It follows that $\gamma_{QQ'}$ is a link from M_q to $M_{q'}$ for some $q \in Q$ and $q' \in Q'$. Since $\leq$ is admissible, we have $q' \leq q$ and in consequence, also $Q' \leq_{\mathcal{Q}} Q$. Similarly, considering $\gamma_{Q'Q} := \gamma^- \cdot (\gamma')^+$ we have $Q \leq_{\mathcal{Q}} Q'$. Since $\leq_{\mathcal{Q}}$ is a partial order, we obtain $Q = Q'$, a contradiction. Hence, $\mathcal{M}'$ is a Morse predecomposition.

Consider now a $\gamma \in \mathrm{eSol}(X)$. By the definition of Morse predecomposition we can find $q, q' \in \mathbb{P}$ such that $\alpha(\gamma) \cap M_q \neq \emptyset$ and $\omega(\gamma) \cap M_{q'} \neq \emptyset$. Since $\leq$ is an admissible preorder, we have $q' \leq q$. Since $\mathcal{Q}$ is a partition, there are some $Q, Q' \in \mathcal{Q}$ such that $q \in Q$ and $q' \in Q'$. By Proposition 3.10 we can find a $\psi \in \mathrm{eSol}(X)$ such that $\alpha(\psi) = \omega(\psi) = \mathrm{im}\,\psi = \alpha(\gamma)$. Therefore, $\psi \in \mathrm{eSol}_Q$ and $\alpha(\gamma) = \mathrm{im}\,\psi \subset M_Q$. Similarly, we argue that $\omega(\gamma) \subset M_{Q'}$. Since $q' \leq q$ we have $Q' \leq_{\mathcal{Q}} Q$. Therefore, $\mathcal{M}$ satisfies condition (ii) of Theorem 4.11 which means that $\mathcal{M}$ is a Morse decomposition. $\square$

Corollary 4.14 *Let $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ be a Morse predecomposition of X. If $\mathcal{Q}$ is a partition of $\mathbb{P}$ into strongly connected components of graph $G_{\mathcal{M}}$ then $\mathcal{M}' := \{M_Q \mid Q \in \mathcal{Q}\}$ is a Morse decomposition of X.*

Corollary 4.15 *Let $\mathcal{M}$ be a Morse predecomposition of X. If there is a partial order among admissible preorders in $\mathbb{P}$ then $\mathcal{M}' := \{M_{\{p\}} \mid p \in \mathbb{P}\}$ is a Morse decomposition of X.*

We note that $M_{\{p\}}$ in general may be bigger than M_p, because it may contain points on solutions homoclinic to M_p. However, $M_{\{p\}} = M_p$ if M_p is saturated. Therefore, we also have the following corollary.

Corollary 4.16 *Let $\mathcal{M}$ be a saturated Morse predecomposition of X. If there is a partial order among admissible preorders in $\mathbb{P}$ then $\mathcal{M}$ is a Morse decomposition of X.*

5 Predecompositions and Decompositions for Flows

In this section we introduce the concept of Morse predecomposition for flows. We show that, as in the combinatorial case, it generalizes Morse decomposition. Moreover, we introduce the notion of Conley predecomposition which allows us to formulate a stability result. However, the analysis of Morse predecompositions for flows becomes more complicated. In particular, it is an open question if a counterpart of the consolidation theorem (Theorem 4.13) holds for flows.

5.1 Morse Predecompositions

Let $\varphi : \mathbb{R} \times X \to X$ be a fixed flow on a locally compact metric space X and let S be a compact, non-empty invariant set with respect to φ. Consider closed invariant sets S_1, S_2 contained in S.

Definition 5.1 A full solution $\gamma : \mathbb{R} \to S$ is a *link from S_1 to S_2 in S* if $\alpha(\gamma) \cap S_1 \neq \emptyset \neq \omega(\gamma) \cap S_2$. A link γ is *homoclinic* if $S_1 = S_2$. Otherwise it is *heteroclinic*. Link

γ is a *connection from S_1 to S_2* if $\alpha(\gamma) \subset S_1$ and $\omega(\gamma) \subset S_2$. A homoclinic link γ is *trivial* if im $\gamma \subset S_1 = S_2$. Otherwise, it is *non-trivial*.

Clearly, a trivial link is a homoclinic connection. However, note that not every homoclinic connection is a trivial link. Observe also that a link need not be a connection. However, since S is compact, limit sets of full solutions in S are always non-empty and, consequently, every connection is a link.

Definition 5.2 A *Morse predecomposition of S* is a finite indexed family $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of mutually disjoint closed invariant subsets of S such that every full solution $\gamma : \mathbb{R} \to S$ is a link from M_p to M_q for some $p, q \in \mathbb{P}$. We refer to the sets in $\mathcal{M}$ as *pre-Morse* sets.

Proposition 5.3 *A finite indexed family $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of mutually disjoint closed invariant subsets of S is a Morse predecomposition of S if and only if we have $T \cap \bigcup \mathcal{M} \neq \emptyset$ for every non-empty closed invariant set $T \subset S$.*

Proof Assume $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ is a Morse predecomposition of S and $T \subset S$ is a non-empty closed invariant set. Select an $x \in T$. Since T is invariant, there is a full solution $\gamma : \mathbb{R} \to T$ through x. Let $q \in \mathbb{P}$ be such that $\omega(\gamma) \cap M_q \neq \emptyset$. Since $\omega(\gamma) \subset \mathrm{cl}\,\mathrm{im}\,\gamma \subset \mathrm{cl}\,T = T$, we conclude that $\emptyset \neq \omega(\gamma) \cap M_q \subset T \cap \bigcup \mathcal{M}$.

Assume in turn that $T \cap \bigcup \mathcal{M} \neq \emptyset$ for every non-empty closed invariant set $T \subset S$ and consider a full solution $\gamma : \mathbb{R} \to S$. By Proposition 3.2, the sets $\alpha(\gamma)$ and $\omega(\gamma)$ are non-empty, closed and invariant. Hence, we there exist $p, q \in \mathbb{P}$ such that $\alpha(\gamma) \cap M_p \neq \emptyset$ and $\omega(\gamma) \cap M_q \neq \emptyset$, which proves that γ is a link and, therefore, $\mathcal{M}$ is a Morse predecomposition of S. □

The *digraph induced by Morse predecomposition $\mathcal{M}$* has $\mathbb{P}$ as its vertex set and an edge from p to q if there exists a non-trivial link from M_p to M_q in S. We denote it $G_{\mathcal{M}}$.

We say that a preorder $\leq$ in $\mathbb{P}$ is *admissible* if the existence of a link from M_p to M_q implies $q \leq p$. Clearly, the preorder induced by the transitive closure of $G_{\mathcal{M}}$ is admissible. We denote it by $\leq_{\mathcal{M}}$ and we call it the *flow induced preorder*.

A *Conley model of $\mathcal{M}$* is a digraph $C = (V, E)$ with a map $v_{G_{\mathcal{M}}, C} : \mathbb{P} \to V$ such that it preserves the graph induced preorders. Clearly, $G_{\mathcal{M}}$ itself is a Conley model. We call $G_{\mathcal{M}}$ the *flow-defined Conley model* of $\mathcal{M}$.

Example 5.4 Consider the flow in Fig. 7 (top left). The equilibrium M_1 is an isolated invariant set. However, the only Morse predecomposition of D is

$$\mathcal{S} := \{D\}.$$

The flow-defined Conley model consist of a single vertex and no edges, $G = (\{v\}, \emptyset)$. Another possible Conley model consists of a single vertex and a self edge, $G = (\{v\}, \{(v, v)\})$.   ◇
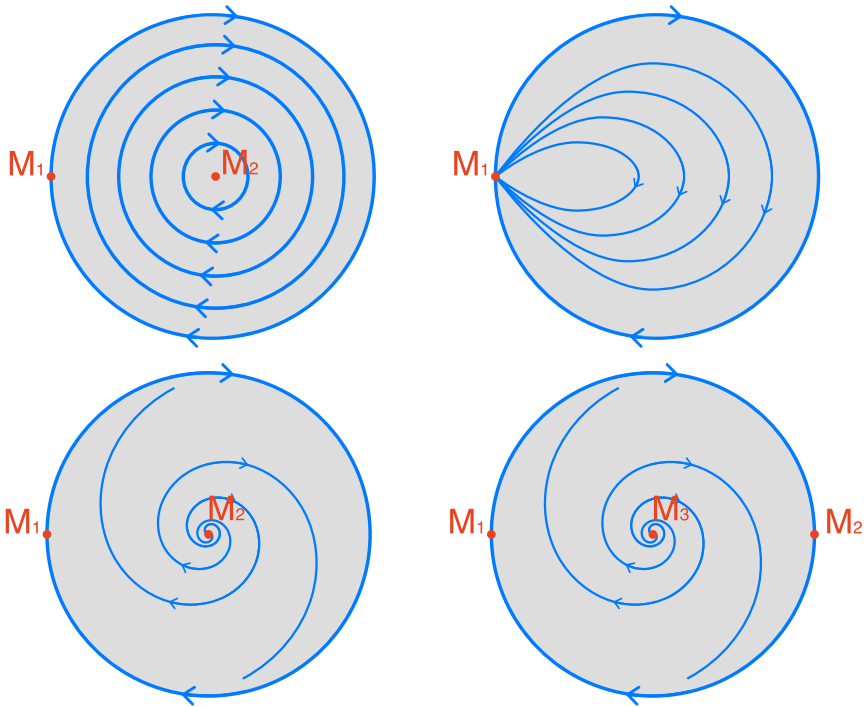
Fig. 7 Four examples of flows on the unit disk D in $\mathbb{R}^2$. Stationary points are indicated in red. In all examples $\{D\}$ is a Morse decomposition, a trivial one. Top left: The only Morse predecomposition is the trivial one $\{D\}$. It is saturated. Top right: $\{M_1\}$ is also a Morse predecomposition. It is not saturated and M_1 is not isolated. Bottom left: $\{M_1, M_2\}$ is also a Morse predecomposition. It is not saturated but both M_1 and M_2 are isolated. Bottom right: $\{M_1, M_2, M_3\}$ is also a Morse predecomposition. It is saturated, hence M_1, M_2 and M_3 are isolated

5.2 Saturated Invariant Sets and Morse Predecompositions

We say that an invariant subset T of an invariant set S is *saturated in S* if $\operatorname{im}\gamma \subset T$ for every homoclinic connection γ from T to T in S. We say that a Morse predecomposition $\mathcal{M}$ is *saturated* if every $M_p \in \mathcal{M}$ is saturated.

Example 5.5 Consider the flow indicated in Fig. 7 (top right). It admits Morse predecomposition $\{M_1\}$ consisting of the equilibrium M_1. Clearly, M_1 is not saturated. Therefore, this Morse predecomposition is not saturated. Unlike Example 5.4, the flow-defined Conley model consists of a single vertex and necessarily also a loop, $G = (\{v\}, \{(v, v)\})$. ◇

Proposition 5.6 *Assume that $S = X$. Given a pre-Morse set M_p in a saturated Morse predecomposition $\mathcal{M}$ of S, every compact neighborhood N of M_p disjoint from all other pre-Morse sets and such that $M_p \subset \operatorname{int} N$ is an isolating neighborhood for M_p. In particular, every pre-Morse set in a saturated Morse predecomposition is an*

isolated invariant set. If $S \subsetneq X$ then every $M_p \in \mathcal{M}$ is an isolated invariant set relatively to S.

Proof Fix a pre-Morse set M_p in $\mathcal{M}$. Let N be a compact neighborhood of M_p which is disjoint from all other pre-Morse sets in $\mathcal{M}$. We claim that Inv $N = M_p$. Clearly, $M_p \subset$ Inv N. To prove the opposite inclusion, assume to the contrary that there exists an $x \in$ Inv $N \setminus M_p$. Then $x \in S \setminus \bigcup \mathcal{M}$ and by the definition of Morse predecomposition there exist $M_q, M_r \in \mathcal{M}$ such that $\alpha(x) \cap M_q \neq \emptyset$, and $\omega(x) \cap M_r \neq \emptyset$. Since $x \in$ Inv N, we have $\varphi(\mathbb{R}, x) \subset N$ and in consequence, $\alpha(x) \cup \omega(x) \subset N$. This implies $M_q = M_p = M_r$, because N is disjoint from pre-Morse sets other than M_p. If M_p is empty then by Proposition 3.2 we have a contradiction. Otherwise, since every pre-Morse set in $\mathcal{M}$ is saturated, we conclude that $x \in M_p$. This is a contradiction proving that N is an isolating neighborhood for M_p.

The case when $S \subsetneq X$ is a straightforward consequence of the main statement. $\quad\square$

Example 5.7 Consider the flow in Fig. 7 (bottom left). It admits a Morse predecomposition $\{M_1, M_2\}$ consisting of the equilibriums M_1 and M_2 shown in red. Here, M_2 is saturated but M_1 is not saturated. Therefore, this Morse predecomposition is not saturated but both pre-Morse sets are isolated invariant sets. The flow-defined Conley model

$$G = (\{v_1, v_2\}, \{(v_1, v_1), (v_2, v_1)\})$$

consists of two vertices and two edges. Exactly one of these edges is a loop. $\quad\diamond$

Example 5.8 Consider the flow in Fig. 7 (bottom right). It admits a Morse predecomposition $\{M_1, M_2, M_3\}$ consisting of the equilibria M_1, M_2 and M_3. This Morse predecomposition is saturated. In particular, all pre-Morse sets are isolated invariant sets. The flow-defined Conley model

$$G = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_1), (v_3, v_1), (v_3, v_2)\})$$

consists of three vertices and four edges. None of these edges is a loop. $\quad\diamond$

Example 5.9 Take as the phase space X a torus represented as the quotient of the square $[0, 1] \times [0, 1]$ in the complex plane through the relation identifying its left edge with the right edge and its bottom edge with the top edge. We can identify the equivalence classes with the points in $[0, 1) \times [0, 1)$. Note, that the identification preserves orientation. Consider the flow φ on X (see Fig. 8, left) induced by the vector field

$$v : X \ni z \rightarrow k(z)w(z) \in X$$

where $w(z) := 1 + \theta i$ with θ being a non-zero irrational number and $k(z) := \min\{|z' - z| \mid z \in \{A, B, C, D\}\}$ where $A = \frac{i}{8}$, $B = \frac{3i}{8}$, $C = \frac{5i}{8}$, and $D = \frac{7i}{8}$. Observe that v is a continuous vector field with four stationary points A, B, C, and D. To understand the features of full solutions of φ it is convenient to look into the flow $\bar{\varphi}$ on X induced by w. This flow may be considered a suspension of an irrational rotation
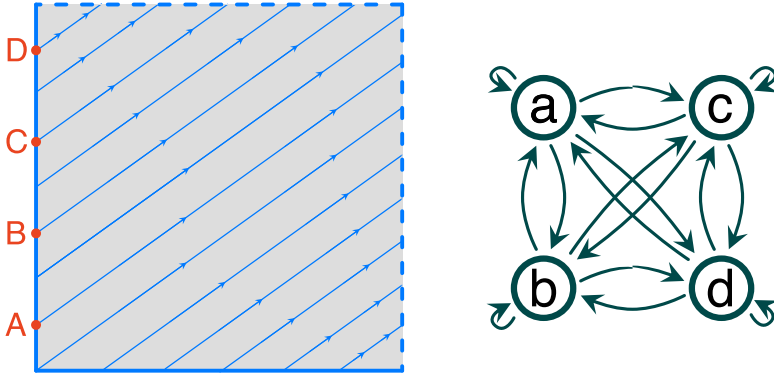
Fig. 8 Left: an irrational rotation on a torus slowing down around four stationary points A, B, C, and D. Note that there are no heteroclinic nor homoclinic connections between the points. The torus has a Morse predecomposition $\mathcal{M} := \{\{A\}, \{B\}, \{C\}, \{D\}\}$. There is a link between any two of these points. Right: flow-defined Conley model of $\mathcal{M}$

on a circle. It is well known (see, for instance, [32, Theorem 3.2.3]) that trajectories of an irrational rotation are dense. In consequence, the alpha and omega limit sets of a full solution of $\bar{\varphi}$ coincide with the torus. Since the vector fields v and w differ only by a scalar factor, a full solution γ of φ is a rescaled full solution of $\bar{\varphi}$. It follows that the alpha or the omega limit set of γ is either the whole torus X or one of the four stationary points. Therefore, the only Morse decomposition is the trivial one, consisting of a single Morse set X; while the family $\mathcal{M} := \{\{A\}, \{B\}, \{C\}, \{D\}\}$ is a Morse predecomposition. However, it follows that we can find a link between every pair of sets in $\mathcal{M}$. Thus, the digraph induced by $\mathcal{M}$ is a clique (see Fig. 8, right). Moreover, notice that $\mathcal{M}$ is saturated.   ◇

Given a Morse predecomposition $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ of S and $C \subset S$ define

$$\mathbb{P}_C := \mathbb{P}_C(\mathcal{M}) := \{p \in \mathbb{P} \mid C \cap M_p \neq \emptyset\}.$$

Lemma 5.10 *Let $\mathcal{M}$ be a Morse predecomposition of an invariant set S and let $C \subset S$ be a closed, connected and chain recurrent subset of S. If $\mathbb{P}_C$ admits a minimal element r with respect to an admissible preorder $\leq$ and M_r is saturated in S, then $C \subset M_r$. In particular, $\mathbb{P}_C = \{r\}$. Moreover, the conclusion follows if $\mathcal{M}$ allows for an admissible partial order, because every finite set has a minimal element with respect to a poset.*

Proof Assume to the contrary that the inclusion $C \subset M_r$ does not hold. Then $A := C \cap M_r \subsetneq C$. We claim that A is an isolated invariant subset of C. Clearly, $A = C \cap M_r$ is invariant as an intersection of invariant sets. To see that A is isolated in C take N, an isolating neighborhood for M_r. We have

$$\mathrm{Inv}(C \cap N) \subset \mathrm{Inv}\, C \cap \mathrm{Inv}\, N = C \cap M_r = A$$

and

$$A = C \cap M_r \subset C \cap \operatorname{int} N \subset \operatorname{int}_C (C \cap N),$$

which shows that $C \cap N$ is an isolating neighborhood for A in C. This proves that A is an isolated invariant set in C. By Theorem 3.5, we can find an $x \in C \backslash A = C \backslash M_r$ such that $\alpha(x) \subset A \subset M_r$. We have $\omega(x) \subset C$, because C is invariant and closed. By the definition of Morse predecomposition there exist $p, q \in \mathbb{P}$, such that $\alpha(x) \cap M_p \neq \emptyset$ and $\omega(x) \cap M_q \neq \emptyset$. Since $\alpha(x) \subset M_r$ and pre-Morse sets are mutually disjoint, we must have $p = r$. We also get $C \cap M_q \supset \omega(x) \cap M_q \neq \emptyset$ which implies $q \in \mathbb{P}_C$. Since $q \leq p = r$ and r is minimal in $\mathbb{P}$, we conclude that $q = r = p$. Hence, $x \in M_r$, because M_r is saturated, a contradiction. Thus, $C \subset M_r$ is proved. Since the elements of $\mathcal{M}$ are mutually disjoint, we also get $\mathbb{P}_C = \{r\}$. $\qquad \square$

Proposition 5.11 *Let $\mathcal{M}$ be a Morse predecomposition. $\mathcal{M}$ is saturated if and only if for every non-trivial link γ there are $p, q \in \mathbb{P}$ such that $p \neq q$ and γ is a link from M_p to M_q.*

Proof Assume $\mathcal{M}$ is a saturated Morse predecomposition and γ is a non-trivial link. Then, by the definition of Morse predecomposition, we know that $\mathbb{P}_{\alpha(\gamma)} \neq \emptyset \neq \mathbb{P}_{\omega(\gamma)}$. We claim that there is no $r \in \mathbb{P}$ such that $\mathbb{P}_{\alpha(\gamma)} = \mathbb{P}_{\omega(\gamma)} = \{r\}$. Indeed, if such an $r \in \mathbb{P}$ exists, then by Lemma 5.10 we have $\alpha(\gamma) \subset M_r$ and $\omega(\gamma) \subset M_r$. Since M_r is saturated we conclude that $\operatorname{im} \gamma \subset M_r$, which contradicts the non-triviality of γ. Therefore, $\mathbb{P}_{\alpha(\gamma)} \cup \mathbb{P}_{\omega(\gamma)}$ contains at least two different elements, among which we can find p and q such that γ is a link from p to q and $p \neq q$.

To prove the opposite implication suppose that $\mathcal{M}$ is not saturated. It means that we can find $r \in \mathbb{P}$ with a full solution γ such that $\alpha(\gamma) \cup \omega(\gamma) \subset M_r$ and $\operatorname{im} \gamma \not\subset M_r$. It follows that $\mathbb{P}_{\alpha(\gamma)} \cup \mathbb{P}_{\omega(\gamma)} = \{r\}$. On the other hand, the assumptions implies that γ is a link from M_p to M_q for some $p \neq q$ in $\mathbb{P}$. Therefore, $\mathbb{P}_{\alpha(\gamma)} \cup \mathbb{P}_{\omega(\gamma)}$ contains at least two elements, which gives a contradiction. $\qquad \square$

5.3 Morse Decompositions

We now present a definition of Morse decomposition for flows formulated in terms of Morse predecomposition. However, Theorem 5.13 shows that Definition 5.12 is equivalent to the classical one, that is Definition 3.11 (compare it with condition (iii) in Theorem 5.13).

Definition 5.12 A Morse predecomposition $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ is called a *Morse decomposition* if it is saturated and among the admissible preorders there is a partial order.

Then, by Proposition 2.1, the flow induced preorder is a partial order.

Theorem 5.13 *Let $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ be an indexed family of mutually disjoint subsets of S. Then, the following conditions are mutually equivalent.*

(i) The family $\mathcal{M}$ is a Morse decomposition of S.

(ii) Each M_p is a closed, saturated invariant subset of S and $\mathbb{P}$ admits a partial order $\leq$ such that for every full solution γ in S there exist $p, q \in \mathbb{P}$ satisfying $p \geq q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$.

(iii) Each M_p is a closed, isolated invariant subset of S and $\mathbb{P}$ admits a partial order $\leq$ such that for every full solution γ in S through an $x \notin \bigcup\{ M_p \mid p \in \mathbb{P} \}$ there exist $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$.

In particular, every Morse set in a Morse decomposition is saturated and isolated.

Proof Assume (i). Then $\mathcal{M}$ is a saturated Morse predecomposition and the flow induced preorder $\leq_{\mathcal{M}}$ is a partial order. In particular, each M_p is closed, invariant and saturated. We claim that the partial order $\leq_{\mathcal{M}}$ fulfills the requirements in condition (ii). To see this, consider a full solution γ in S. By the definition of Morse predecomposition there exist $p, q \in \mathbb{P}$ such that $\alpha(\gamma) \cap M_p \neq \emptyset$ and $\omega(\gamma) \cap M_q \neq \emptyset$. By the definition of flow induced preorder we have $p \leq_{\mathcal{M}} q$ and from Lemma 5.10 we conclude that $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. Hence, (ii) follows from (i).

Assume in turn (ii). Then, obviously, each M_p is a closed invariant subset of S. By Proposition 5.6 each M_p is also isolated. We claim that the partial order $\leq$ given in (ii) fulfills the requirements in condition (iii). To see this, consider a full solution γ in S through an $x \notin \bigcup\{ M_p \mid p \in \mathbb{P} \}$. By (ii) there exist $p, q \in \mathbb{P}$ satisfying $p \geq q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. We cannot have $p = q$, because then $M_p = M_q$ is not saturated, contradicting (ii). Therefore, $p > q$, proving (iii).

Finally, assume (iii) and consider a full solution γ in S. In order to prove that $\mathcal{M}$ is a Morse predecomposition, we have to show that γ is a link from M_p to M_q for some $p, q \in \mathbb{P}$. Consider first the case when $\operatorname{im} \gamma \subset \bigcup\{ M_r \mid r \in \mathbb{P} \}$. Since $\operatorname{im} \gamma$ is connected and the sets M_r are compact and mutually disjoint, we must have $\operatorname{im} \gamma \subset M_r$ for some $r \in \mathbb{P}$. Then $\alpha(\gamma) \cup \omega(\gamma) \subset \operatorname{cl} \operatorname{im} \gamma \subset M_r$. Therefore, γ is a link from M_r to M_r. Consider now the case when $\operatorname{im} \gamma \not\subset \bigcup\{ M_r \mid r \in \mathbb{P} \}$. Then there is an $x \in \operatorname{im} \gamma \setminus \bigcup\{ M_r \mid r \in \mathbb{P} \}$ and we conclude from (iii) the existence of $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. In particular, γ is a link from M_p to M_q. This completes the proof that $\mathcal{M}$ is a Morse predecomposition. Clearly, $\leq$ is then an admissible partial order for $\mathcal{M}$. We still need to prove that each $M_r \in \mathcal{M}$ is saturated. To see this, consider a full solution γ such that $\alpha(\gamma) \cup \omega(\gamma) \subset M_r$. If $\operatorname{im} \gamma \not\subset M_r$, then also $\operatorname{im} \gamma \not\subset \bigcup\{ M_r \mid r \in \mathbb{P} \}$ and, from (iii), we conclude the existence of $p, q \in \mathbb{P}$ satisfying $p > q$, $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. However, Morse sets are mutually disjoint and limit sets are non-empty. Therefore $p = r = q$, a contradiction proving that M_r is saturated. $\qquad\square$

Theorem 5.14 *Assume $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ is a saturated Morse predecomposition of S. If there is a partial order among the admissible preorders for $\mathcal{M}$, then $\mathcal{M}$ is a Morse decomposition.*

Proof Assume $\leq$ is an admissible partial order for $\mathcal{M}$. Consider a non-trivial link γ. By Proposition 5.11, there are $p, q \in \mathbb{P}$ such that $q < p$ and $\alpha(\gamma) \cap M_p \neq \emptyset \neq \omega(\gamma) \cap M_q$. By Proposition 3.2, the limit sets $\alpha(\gamma)$ and $\omega(\gamma)$ are invariant, chain recurrent and connected. It follows that $\mathbb{P}_{\alpha(\gamma)} \neq \emptyset \neq \mathbb{P}_{\omega(\gamma)}$. Therefore, both these sets admit minimal elements. From Lemma 5.10 we obtain $\alpha(\gamma) \subset M_p$ and $\omega(\gamma) \subset M_q$. This proves that $\mathcal{M}$ is actually a Morse decomposition. $\qquad\square$

5.4 Conley Predecomposition

A set $A \subset \mathbb{R}$ is *right infinite* if

$$\sup\{\, b - a \mid a, b \in \mathbb{R}^+, \ [a, b] \subset A \,\} = \infty.$$

Similarly, $A \subset \mathbb{R}$ is *left infinite* if

$$\sup\{\, b - a \mid a, b \in \mathbb{R}^-, \ [a, b] \subset A \,\} = \infty.$$

We say that $A \subset \mathbb{R}$ is *bi-infinite* if it is both left and right infinite.

Definition 5.15 A finite collection $\mathcal{N} = \{N_v \mid v \in \mathbb{V}\}$ of mutually disjoint isolating neighborhoods in X is called a *Conley predecomposition of S* if for every $x \in X$, $\{\, t \in \mathbb{R} \mid \varphi(t, x) \in \bigcup\{\text{int } N \mid N \in \mathcal{N}\} \,\}$ is bi-infinite.

The *associated digraph* $G_\mathcal{N}$ has $\mathbb{V}$ as its set of vertices and an edge from v to w if there is a partial solution $\rho : [a, b] \to S$ satisfying $\gamma(a) \in \text{int } N_v$ and $\gamma(b) \in \text{int } N_w$.

A Conley predecomposition $\mathcal{N} = \{N_v \mid v \in \mathbb{V}\}$ is *inscribed* in a Conley predecomposition $\mathcal{N}' = \{N'_w \mid w \in \mathbb{W}\}$ if for every $v \in \mathbb{V}$ there exists a $\nu(v) := w \in W$ such that $N_v \subset N'_w$. Then, we have a well defined map preserving the digraph induced preorders $\nu : \mathbb{V} \to \mathbb{W}$.

We say that a Conley predecomposition $\mathcal{N}$ is an *approximation* of a Morse predecomposition $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$ if for every $p \in \mathbb{P}$ there is a $\mu(p) := v \in \mathbb{V}$ such that $\text{Inv } N_v = M_p$. Clearly, such a v is uniquely determined by p, because if $M_p = \text{Inv } N_{v_i}$ for $i = 1, 2$ then $\emptyset \neq M_p = \text{Inv } N_{v_i} \subset \text{int } N_{v_i}$, which gives $v_1 = v_2$. Therefore, we have a well defined map $\mu : \mathbb{P} \to \mathbb{V}$.

Proposition 5.16 *Let $\mathcal{N} = \{N_v \mid v \in \mathbb{V}\}$ be a Conley predecomposition approximating a Morse predecomposition $\mathcal{M} = \{M_p \mid p \in \mathbb{P}\}$. Then $G_\mathcal{N}$ is a Conley model for $\mathcal{M}$ with map $\mu : \mathbb{P} \to \mathbb{V}$ preserving the digraph induced orders given by $\mu(p) := v$, where $\text{Inv } N_v = M_p$.*

Proof To see that μ is injective, observe that $\mu(p) = \mu(q)$ for some $p, q \in \mathbb{P}$ implies $M_p = \text{Inv } N_{\mu(p)} = \text{Inv } N_{\mu(q)} = M_q$ which gives $p = q$. If (p, q) is an edge in $G_\mathcal{M}$, then there is a full solution $\gamma : \mathbb{R} \to S$ such that $\alpha(\gamma) \cap M_p \neq \emptyset$ and $\omega(\gamma) \cap M_q \neq \emptyset$. Hence, we can find $t, s \in \mathbb{R}$, $t < s$ such that $\gamma(t) \in \text{int } N_{\mu(p)}$ and $\gamma(s) \in \text{int } N_{\mu(q)}$. This shows that there is also an edge in $G_\mathcal{N}$ from $\mu(p)$ to $\mu(q)$, which proves that μ induces a relation preserving map. $\qquad\square$

Example 5.17 Consider the flow presented in Fig. 9 (top) and the isolated invariant set S consisting of a planar disk. It admits a minimal Morse predecomposition $\mathcal{M} := \{M_0, M_1, M_2, M_3, M_4\}$ where each pre-Morse set is a stationary point. Note that points M_2 and M_4 are created simply by locally slowing down the flow. The flow induced Conley model is presented in the middle left. Taking a small disk around each stationary point, we obtain a Conley decomposition $\mathcal{N} := \{N_0, N_1, N_2, N_3, N_4\}$ approximating $\mathcal{M}$. The associated digraph $G_\mathcal{N}$ is presented in Fig. 9 (middle right).
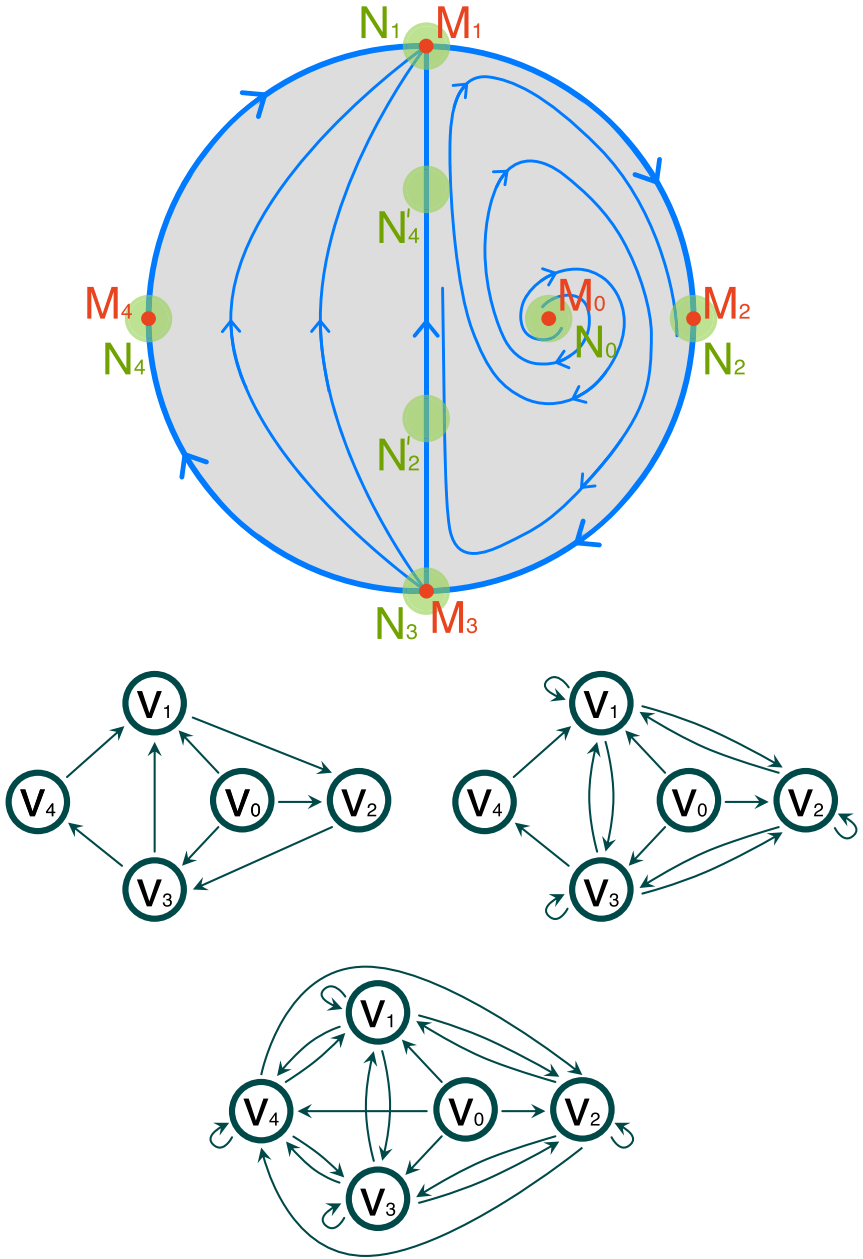
Fig. 9 Top: An isolated invariant set S consisting of a planar disk with a minimal Morse predecomposition $\mathcal{M} = \{M_0, M_1, M_2, M_3, M_4\}$ comprising five stationary points, and Conley predecompositions $\mathcal{N} := \{N_0, N_1, N_2, N_3, N_4\}$ and $\mathcal{N}' := \{N_0, N_1, N_2 \cup N_2', N_3, N_4 \cup N_4'\}$. Middle left: The flow-defined Conley model for $\mathcal{M}$. Middle right: The Conley model derived from $\mathcal{N}$. Bottom: The Conley model derived from $\mathcal{N}'$

We note that not every Conley decomposition approximating $\mathcal{M}$ will give the same digraph. For instance, for $\mathcal{N}' := \{N_0, N_1, N_2 \cup N_2', N_3, N_4 \cup N_4'\}$ we obtain another Conley decomposition approximating $\mathcal{M}$ with the associated Conley model presented in Fig. 9 (bottom). ◇

Proposition 5.18 *Let $\mathcal{N}$ be a Conley predecomposition of S. Then*

$$\mathcal{N}^\bullet := \mathcal{N}_\varphi^\bullet := \{ \operatorname{Inv} N \mid N \in \mathcal{N},\ \operatorname{Inv}(N, \varphi) \neq \emptyset \}$$

is a Morse predecomposition of S and $\mathcal{N}$ is an approximation of it.

Proof Let $\rho : \mathbb{R} \to S$ be a full solution. Since $\rho^{-1}\left(\bigcup\{\operatorname{int} N \mid N \in \mathcal{N}\}\right)$ is bi-infinite and $\mathcal{N}$ is finite, there are $N_-, N_+ \in \mathcal{N}$ such that $\rho^{-1}(N_-)$ is left infinite and $\rho^{-1}(N_+)$ is right infinite. We claim that $\alpha(\rho) \cap \operatorname{Inv} N_- \neq \emptyset$ and $\omega(\rho) \cap \operatorname{Inv} N_+ \neq \emptyset$. Since $\rho^{-1}(N_+)$ is right infinite, we can find a sequence t_n of real numbers such that $\rho([t_n - n, t_n + n]) \subset N_+$ and $\lim_{n\to\infty} t_n = \infty$ for $n \in \mathbb{N}$. Let $x_n := \rho(t_n) \in N_+$. Without loss of generality we may assume that $\lim x_n = x_* \in N_+$. Then, there is a full solution $\gamma : \mathbb{R} \to N_+$ through x_*. This shows that $x_* \in \operatorname{Inv} N_+ \in \mathcal{N}^\bullet$. Clearly, also $x_* \in \omega(\rho)$. Therefore, $\omega(\rho) \cap \operatorname{Inv} N_+ \neq \emptyset$. Similarly we prove that $\alpha(\rho) \cap \operatorname{Inv} N_- \neq \emptyset$. This proves that $\mathcal{N}^\bullet$ is a Morse predecomposition and $\mathcal{N}$ is its approximation. □

Consider a continuously parametrized family of dynamical systems

$$\varphi_\lambda : \mathbb{R} \times X \to X, \quad \lambda \in [-1, 1]$$

on a compact metric space X.

Theorem 5.19 *(Stability of Conley predecomposition) Let $\mathcal{N}$ be a Conley predecomposition of X with respect to φ_0. Then, there exists an $\varepsilon > 0$ such that $\mathcal{N}$ is an approximation of $\mathcal{N}_{\varphi_\lambda}^\bullet$ for every $\lambda \in (-\varepsilon, \varepsilon)$. In particular, $G_\mathcal{N}$ is a Conley model of Morse predecomposition $\mathcal{N}_{\varphi_\lambda}^\bullet$ for $\lambda \in (-\varepsilon, \varepsilon)$.*

Proof Since $\mathcal{N}$ consists of a finite number of isolating neighborhoods, the conclusion follows immediately from Proposition 3.1. □

5.5 Asymptotic Characterization of Morse Predecompositions

In this section we study Morse predecompositions whose pre-Morse sets are isolated invariant sets. In such a case we provide theorems which characterize a Morse predecomposition in terms of the asymptotic behavior of trajectories in isolating neighborhoods surrounding the isolated invariant sets.

Proposition 5.20 *A subset $A \subset \mathbb{R}$ is right infinite if and only if*

$$\forall K > 0\ \exists a \geq K\ [a, a + K] \subset A. \tag{4}$$

A subset $A \subset \mathbb{R}$ is left infinite if and only if

$$\forall K > 0\ \exists a \leq -K\ [a - K, a] \subset A.$$

Proof Obviously condition (4) implies that A is right infinite. To see the opposite implication assume $A \subset \mathbb{R}$ is right infinite. Fix a $K > 0$. We can find $c, d \in \mathbb{R}^+$ such that $[c, d] \subset A$ and $d - c \geq 2K$. Let $a := \max(K, c)$. Then $K \leq a \leq K + c$ and $a + K \leq c + 2K \leq d$. Therefore, $[a, a + K] \subset [c, d] \subset A$, which proves (4). The proof of the remaining assertion is analogous. □

Let N be an isolating neighborhood and let γ be a full solution. We say that γ *wades through N in plus (minus) infinity* if there exist sequences $t^-, t^+ : \mathbb{N} \to \mathbb{R}^+$ such that t^-, t^+ as well as $t^+ - t^-$ tend to plus (minus) infinity and for all $n \in \mathbb{N}$

$$\gamma([t_n^-, t_n^+]) \subset N. \tag{5}$$

If γ wades through N in plus (minus) infinity we write $\gamma \overset{+}{\to} N$ ($\gamma \overset{-}{\to} N$).

Proposition 5.21 *Assume N is an isolating neighborhood and γ is a full solution. Then the following conditions are mutually equivalent.*

(i) The preimage $\gamma^{-1}(N)$ is right (respectively left) infinite.
(ii) The solution γ wades through N in plus (respectively minus) infinity.
(iii) The intersection of Inv N *and $\omega(\gamma)$ (respectively $\alpha(\gamma)$) is non-empty.*

Proof We give the proof that (i) is equivalent to (ii) in the case when $\gamma^{-1}(N)$ is right infinite. The proof in the other case is analogous. Assume (i) and set

$$t^+(0) := t^-(0) := \inf\{ s \geq 0 \mid \gamma(s) \in N \}.$$

Proceeding recursively, assume t_{n-1}^+ and t_{n-1}^- are already defined for some $n \in \mathbb{N}$. Using Proposition 5.20 for $K := \max(n, t_{n-1}^+)$ select an $a \geq K$ such that $[a, a+K] \subset \gamma^{-1}(N)$ and set $t_n^- := a$, $t_n^+ := a + K$. Then $t_n^+ > t_n^- \geq n$, $t_n^+ - t_n^- \geq n$ and $[t_n^-, t_n^+] \subset \gamma^{-1}(N)$. In particular, the sequences t^-, t^+, $t^+ - t^-$ tend to infinity. Hence, γ wades through N in plus infinity, which proves (ii). Obviously, (ii) implies (i). Hence the proof of the equivalence of (i) and (ii) is completed.

We now give the proof that (ii) is equivalent to (iii) in the case when γ wades through N in plus infinity. The proof in the other case is analogous. Assume (ii). Then we can choose sequences $t^-, t^+ : \mathbb{N} \to \mathbb{R}^+$ such that t^-, t^+ as well as $t^+ - t^-$ tend to plus infinity and (5) is satisfied for all $n \in \mathbb{N}$. Set $t_n := \frac{t_n^- + t_n^+}{2}$, $x_n := \gamma(t_n)$ and $\gamma_n(t) := \gamma(t_n + t)$. By taking a subsequence, if necessary, we may assume that $\lim_{n \to \infty} x_n = x_*$ for some $x_* \in \omega(\gamma)$. Since $\gamma_n([t_n^- - t_n, t_n^+ - t_n]) = \gamma([t_n^-, t_n^+]) \subset N$, we obtain $x_* \in$ Inv N. Hence, Inv $N \cap \omega(\gamma) \neq \emptyset$, which proves (iii).

Finally assume (iii). As a consequence of [25, Theorem 2.1], for each $n \in \mathbb{N}$ we can choose U_n, an open neighborhood of M, such that

$$u \in U_n \implies \varphi([-n, n], u) \subset N. \tag{6}$$

Since $\omega(\gamma) \cap M \neq \emptyset$, we can choose a $T_n > 2n$ such $\varphi(T_n, x) \in U_n$. Set $t_n^- := T_n - n$ and $t_n^+ := T_n + n$. Then, $t_n^+ > t_n^- > n$ and $t_n^+ - t_n^- = 2n$ proving that t^+, t^- and

$t^+ - t^-$ tend to infinity. Moreover, since $u_n := \varphi(T_n, x) \in U_n$, we have from (6) that

$$\varphi([t_n^-, t_n^+], x) = \varphi([-n, n], \varphi(T_n, x)) = \varphi([-n, n], u_n) \subset N,$$

which proves (ii). □

Theorem 5.22 *Assume* $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$ *is a finite indexed family of mutually disjoint isolated invariant sets in* S. *Then, the following conditions are equivalent.*

(i) *The family* $\mathcal{M}$ *is a Morse predecomposition of* S.
(ii) *There exists a family* $\mathcal{N} = \{ N_p \mid p \in \mathbb{P} \}$ *of mutually disjoint isolating neighborhoods in* S *such that* $M_p = \mathrm{Inv}\, N_p$ *and for every full solution* γ *in* S *there exist* $p, q \in \mathbb{P}$ *such that* $\gamma \xrightarrow{-} N_p$ *and* $\gamma \xrightarrow{+} N_q$.

Proof Consider a finite family $\mathcal{M} = \{ M_p \mid p \in \mathbb{P} \}$ of mutually disjoint isolated invariant sets in S and assume (i). Since $\mathcal{M}$ is finite and its elements are mutually disjoint, we can choose a family $\mathcal{N} = \{ N_p \mid p \in \mathbb{P} \}$ of mutually disjoint isolating neighborhoods such that $M_p = \mathrm{Inv}\, N_p$ for every $p \in \mathbb{P}$. Let γ be a full solution in S. Then, by the definition of Morse predecomposition, there exist $p, q \in \mathbb{P}$ such that γ is a link from M_p to M_q. It follows from the definition of link and Proposition 5.21 that $\gamma \xrightarrow{-} N_p$ and $\gamma \xrightarrow{+} N_q$. This proves (ii). Assume in turn that there exists a family $\mathcal{N} = \{ N_p \mid p \in \mathbb{P} \}$ which meets the requirements of (ii). To prove that $\mathcal{M}$ is a Morse predecomposition of S we only need to verify that every full solution in S is a link between elements of $\mathcal{M}$, because, by the main assumption, the family $\mathcal{M}$ consists of mutually disjoint isolated invariant sets in S. Hence, let $\gamma : \mathbb{R} \to S$ be a full solution. Then, by (ii) there exist $p, q \in \mathbb{P}$ such that $\gamma \xrightarrow{-} N_p$ and $\gamma \xrightarrow{+} N_q$ and it follows from Proposition 5.21 that γ is a link from M_p to M_q. □

As an immediate consequence of Theorem 5.22 and Proposition 5.21 we conclude the following corollary.

Corollary 5.23 *Assume* $\mathcal{N} = \{ N_p \mid p \in \mathbb{P} \}$ *is a family of mutually disjoint isolating neighborhoods in* S *such that* $\mathrm{Inv}\, N_p \neq \emptyset$ *and for every full solution* γ *in* S *there exist* $p, q \in \mathbb{P}$ *such that* $\gamma \xrightarrow{-} N_p$ *and* $\gamma \xrightarrow{+} N_q$. *Then* $\mathcal{N}$ *is a Conley predecomposition of* S. *Moreover, there is an edge from* p *to* q *in the flow-defined Conley model of* $\mathcal{N}^\bullet$ *if and only if there is a full solution* γ *in* S *which wades through* N_p *in minus infinity and through* N_q *in plus infinity.*

6 Discussion

The proposed concept of Morse predecomposition is a natural generalization of Morse decomposition. It provides insight into the internal structure of Morse sets. It is defined in the classical setting of flows as well as in the combinatorial setting of multivector fields. Therefore, it may be a useful tool in the algorithmic analysis of recurrent dynamics. In particular, Morse predecomposition can give a refined insight of a parameter space of a numerically studied dynamical systems [5, 6]. Moreover, the

possibility of decomposing a recurrent invariant set might be crucial for development of a computer assisted proof for chaos based on the techniques developed in [19].

Several other questions remain to be answered. A particularly interesting problem concerns the counterpart of Theorem 4.13 for flows, that is, whether every Morse pre-decomposition naturally induces a Morse decomposition via consolidation of cliques in the flow-defined Conley model. This problem is currently open. Furthermore, there is a natural question for an extension of the predecomposition to the setting of continuous discrete-time dynamical systems. We speculate that the development of such a generalization might be even more demanding. However, it would potentially allow for the study of more complex examples compared to the ones presented in the paper. That theory would also open additional questions. For instance, we suspect that the U-horseshoe – a 2-dimensional chaotic discrete-time dynamical system, might not admit a non-trivial Morse predecomposition at all, but this require further investigation. Or a more general one, which chaotic isolated invariant sets admit a non-trivial Morse predecomposition?

Another important issue which needs to be addressed is the algorithm for constructing Morse predecompositions in the combinatorial setting. Theorem 4.1 in [33] shows that the algorithm for minimal Morse decompositions of combinatorial multi-vector fields reduces to the algorithm for strongly connected components in a directed graph. In the case of predecomposition one would search for possibly small strongly connected subsets of strongly connected components. This may be achieved by computing the simple cycles in the graph. However, unlike the decomposition case where every strongly connected component is automatically an isolated invariant set (see [33, Theorem 4.1]), in the predecomposition case one has to take the smallest locally closed, $\mathcal{V}$-compatible superset of the strongly connected set to guarantee isolation. Moreover, contrary to a Morse decomposition, there may be many different minimal Morse predecompositions in the combinatorial setting (see Example 4.8). Therefore, to fully understand the structure of an isolated invariant set an algorithm should be able to retrieve all minimal predecompositions.

Finally, it would be interesting to investigate the relation between Morse predecomposition in the combinatorial case and Morse predecomposition in the classical case in the spirit of the result for Morse decomposition [34].

Declarations

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