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Density amplifiers of cooperation for spatial games

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Spatial games provide a simple and elegant mathematical model to study the evolution of cooperation in networks. In spatial games, individuals reside in vertices, adopt simple strategies, and interact with neighbors to receive a payoff. Depending on their own and neighbors' payoffs, individuals can change their strategy. The payoff is determined by the Prisoners' Dilemma, a classical matrix game, where players cooperate or defect. While cooperation is the desired behavior, defection provides a higher payoff for a selfish individual. There are many theoretical and empirical studies related to the role of the network in the evolution of cooperation. However, the fundamental question of whether there exist networks that for low initial cooperation rate ensure a high chance of fixation, i.e., cooperation spreads across the whole population, has remained elusive for spatial games with strong selection. In this work, we answer this fundamental question in the affirmative by presenting network structures that ensure high fixation probability for cooperators in the strong selection regime. Besides, our structures have many desirable properties: (a) they ensure the spread of cooperation even for a low initial density of cooperation and high temptation of defection, (b) they have constant degrees, and (c) the number of steps, until cooperation spreads, is at most quadratic in the size of the network.

cooperation | spatial games | evolutionary dynamics

Significance

Spatial games provide a broad framework to study the spread of strategies over networks. Individuals residing in vertices play a matrix game with neighbors and update strategies based on the payoff. We address a long-standing problem: Do there exist networks that promote the spread of cooperation for Prisoners' dilemma? We present networks that ensure the spread of cooperation with high probability. We also establish several robustness properties of our networks: They promote cooperation even with a low initial cooperation rate and high temptation and ensure that cooperation spreads quickly. Due to the broad connection of spatial games in several applications ranging from biology to physics, our structures are significant and relevant in all these domains.

Game theory is a broad field that provides the mathematical foundations related to decision-making, which has many applications in economics (1, 2), computer science and artificial intelligence (3), evolutionary dynamics (4–8), and physics (9–11). For example: One-shot or matrix games have been studied in refs. 12 and 13; their generalization as normal and extensive form games have wide applications in economics (1, 14); and evolutionary games study competitive and collaborative behavior in biology (15–24).

An important class of games that arise in many different contexts is games on graphs. The graph represents a network or a population structure. Individuals reside in vertices, and the interactions happen over the edges of the graph. Well-known examples of games over graphs are game of life (25), games with cellular automata (26); evolutionary games on graphs (27–29); and spatial games (30–35).

One of the classical examples in game theory to represent social tension by a matrix game is Prisoners' Dilemma (PD) (13), which has been studied both theoretically (3) as well as experimentally (36, 37). In PD, individuals can choose to cooperate or defect. If they mutually cooperate, they both receive payoff R. If they mutually defect, they receive payoff P. A cooperator facing a defector obtains payoff S, while a defector facing a cooperator gets T. The payoffs follow inequality $T > R > P > S$, which represents the dilemma between altruistic behavior and self-interest. Mutual cooperation is the desired behavior; however, defection is the dominant strategy, which achieves the best payoff for a selfish individual no matter the strategy of the other player. Hence mutual defection is the only Nash equilibrium. The population structure can help the cooperation to overcome this trap (38–44).

Spatial games provide the framework to study PD games over structured populations (graphs) played over multiple rounds. Individuals residing in vertices can adopt only simple strategies, always cooperate C, or always defect D in every round. In one round, the payoff of an individual is obtained by interacting with all neighbors by playing pairwise PD. Based on the individual's payoff and the payoffs of neighbors, the strategy is updated by a variant of a replicator dynamic (5, 15, 30). The strategy can be updated synchronously, where every individual can update its strategy; or asynchronously, where only one individual from a random pair of neighbors can update its strategy. The strategy update can be either deterministic: The individual with a lower payoff adopts the strategy of a more successful individual; or randomized: With some probability (given by the Fermi function), even the strategy with a lower payoff can replace a higher payoff strategy. This represents the specific regime of spatial games with strong selection, i.e., in the deterministic setting we have the best-response dynamics, and

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in the randomized setting replacement probability depends exponentially on payoff difference.

The study of spatial games was initiated in the seminal work of Nowak and May (30) and since then, it received a lot of attention from several research perspectives. Ohtsuki et al. (45) theoretically examine the graphs conducive to cooperation. Traulsen et al. (46) examine how humans update their strategies in the real world. Many works (47–49) simulate the process on grids or natural graphs that approximate the social networks (scale-free graphs). On scale-free graphs, cooperation is promoted in comparison to highly regular graphs. The reason is that scale-free graphs have large differences between the degrees of the vertices. Highly connected cooperators can tolerate a lot of defection in the neighborhood while spreading cooperation.

A fundamental question in spatial games is the role of the graph structure in boosting or amplifying the desired cooperative behavior. There are two notions of cooperation amplification: 1) an increase of cooperation in the stationary state during coexistence; and 2) an increase in the probability that the cooperators replace all defectors, this event is called cooperator fixation. Most works on spatial games (9–11) study mainly the aspect that cooperation increases in the stationary state with results on scale-free networks (47–49). Moreover, these works focus on empirical results and do not provide theoretical guarantees with analytical results. Our work is distinguished in two aspects. First, as compared to increasing cooperation in the stationary state, we consider fixation, which is more desirable as it ensures complete cooperation in the stationary state. However, note that ensuring fixation is more challenging and usually takes a long time to achieve in simulations. Second, our goal is to establish analytical results that can provide theoretical guarantees on fixation probability. By amplifying cooperation, we mean increasing the fixation probability. This type of amplifying has been studied in many contexts: such as amplification for Moran processes (27, 50, 51) where the mutant has constant advantage, or amplification for evolutionary games for weak selection regimes (52). Despite the importance of amplifying cooperation, in the context of spatial games in strong selection, the existence of graphs that amplify cooperation has remained elusive even after two decades of active research, which is the question we address.

In this work, we present a classification of density amplifiers in spatial games. A weak density amplifier ensures fixation with a positive probability for a low initial cooperation rate. A strong density amplifier guarantees the fixation probability close to 1 for a low initial cooperation rate. We show that previously studied structures, such as grids and regular graphs are not even weak density amplifiers. In contrast, we present graph structures that are not only weak but even strong density amplifiers. We demonstrate this amplifying effect across three replicator dynamics: deterministic synchronous, deterministic asynchronous, and randomized asynchronous. Thus, our construction answers the open question in the affirmative for the existence of density amplifiers in spatial games.

Besides answering the open question, we present several other results related to our construction. First, we present two robustness results: Our structure ensures high fixation probability even with: (a) a low rate of initial cooperation; and (b) high temptation-reward ratio T/R above 2. Previous literature in the context of spatial games (38, 48, 49, 52) requires a very low temptation-reward ratio T/R and shows that with a high initial cooperation rate (around 50%) structures can ensure a steady-state of cooperation rate (around 50%). These results neither start with a low cooperation rate (e.g., 5%) and ensure a steady-state cooperation rate of 50%, nor start with a high initial cooperation rate and ensure fixation. In contrast, our structures with a low initial cooperation rate of 5% ensure fixation, which is a significant advancement in the study of spatial games. Second, we show that our structure ensures the fixation quickly. In contrast to many structures in the literature where fixation requires exponentially many steps, see refs. 30 and 53, we show that for our structures the fixation happens within quadratic steps. Third, our construction has degree variations between neighbors, similar to scale-free networks, however, we ensure that the maximal degree is constant (proportional to T/R). Finally, we supplement our theoretical guarantees for large population limit by simulation results to show the effectiveness of our structures even for small population sizes. The simulation results consider small population sizes with various ratios of T/R and show that the fixation probability is high even for a small initial density of cooperation and large T/R.

The framework of spatial games is quite general, with several regimes, e.g., weak and strong selection and deterministic and stochastic dynamics. Our work provides density amplification in the strong selection regime. The extension of the density amplification for all selection regimes and dynamics are an interesting direction for future work.

Results

Model. As in the classical literature (30, 48), we focus on PD with normalized payoffs, where $R = 1$, $P = \varepsilon$, $S = 0$, and $T = b > 1$. The individuals follow the replicator dynamics. The detailed explanation is in Fig. 1. The individuals play PD with all neighbors and collect the payoff, then change their strategies based on their and the neighbor's payoffs. We examine the three main variants in replicator dynamics: 1) Synchronous deterministic: Every individual that has a neighbor with a higher payoff adopts a strategy of the neighbor with the highest payoff; 2) Asynchronous deterministic: One edge is selected randomly, and the individual with a smaller payoff adopts its neighbor's strategy; and 3) Asynchronous randomized: One edge is selected randomly. Let x and y be the selected neighbors. If the payoffs are P_x and P_y then y changes its strategy to x's strategy with probability $\frac{1}{1+e^{(P_y-P_x)/K}}$, where K (selection intensity) is the noise parameter, otherwise y adopts x's strategy. See Fig. 1 *D–F* for details on replicator dynamics.

The dynamic starts with some cooperators already present in a graph, we denote the initial density of cooperation by p. Usually, simulation studies start with large p (around $\frac{1}{2}$). However, ensuring initial configurations with such a large fraction of cooperators is unrealistic. Here, we consider p to be small, which goes to zero in the limit of a large population. This requires the spread of cooperation even from a disadvantageous initial position. Spatial games have rich and complex dynamics. Deciding whether the cooperation survives on a general graph is computationally hard (53). They give rise to beautiful patterns even for grids; see Fig. 2. Given these complicated dynamics, analytical results are challenging to achieve. The landscape of results is dominated by simulations and case studies. The main challenge is to establish results with provable guarantees, which have been elusive.

We say that cooperators fixate if they replace all defectors and the probability of this event is called the fixation probability. Traditionally, previous works on spatial games consider the density of cooperators in the steady-state of coexistence of

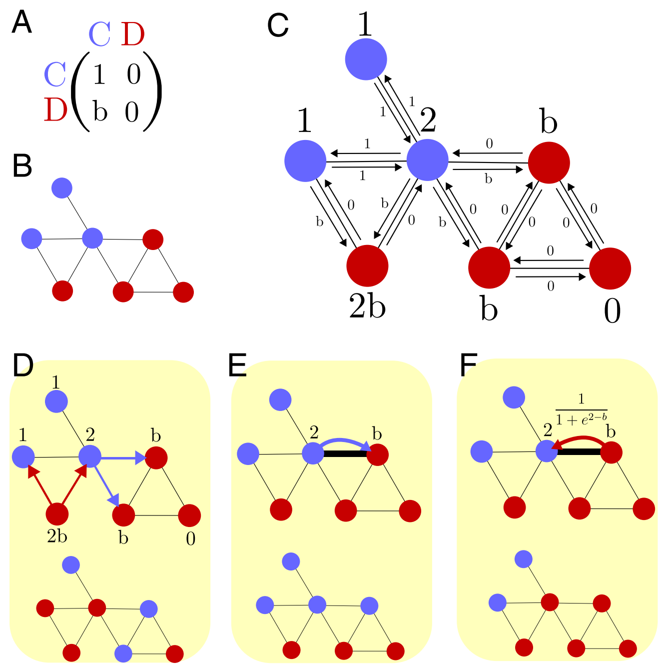

Fig. 1. The description of the Prisoner's dilemma on graphs. (*A*) The payoff matrix of Prisoner's dilemma, we have $b > 1$ and $\varepsilon = 0$. (*B*) The initial configuration on a graph, with cooperators being (pale) blue and defectors (dark) red. (*C*) Payoffs of all players in one turn. The payoff of every cooperator is equal to the number of cooperating neighbors, and the payoff of every defector is *b* times the number of cooperating neighbors. (*D*) Synchronous deterministic updating: Every individual updates its strategy to the strategy of a neighbor with the highest payoff. The arrows denote the spread of cooperation/defection. (*E*) Asynchronous deterministic updating: One random edge is selected (highlighted in the figure), and the individual with a higher payoff replaces the individual with a lower payoff. (*F*) Asynchronous randomized updating: One random edge is selected, and one individual replaces the other with the probability given the Fermi function of the payoffs. In the example, the defector replaces the cooperator with probability $\frac{1}{1+e^{2-b}}$.

cooperators and defectors. The fixation probability is much more desirable as it ensures only cooperators in the steady state.

Now, we define the notion of density amplifiers. Given temptation $b > 1$, the initial density of cooperators $p > 0$ and number of vertices n, we call a family of graphs: (a) weak density amplifiers if the fixation probability for all graphs in the family is above 0, even in the limit of large n; (b) mild density amplifiers if the fixation probability for all graphs in the family is higher than the probability that cooperators will become extinct; (c) strong density amplifiers if the fixation probability tends to 1 in the limit of large n.

Note that complete graphs (unstructured populations) are not even weak density amplifiers. By design of the Prisoner's dilemma, every defector in the population has a bigger payoff than any cooperator. Moreover, results of ref. 45 imply that regular graphs are also not even weak density amplifiers for all b. Some graphs, for example, stars, are weak density amplifiers: If the central vertex of the star starts as a cooperator, then the cooperation has a chance to replace all defectors. However, if the central vertex starts as a defector, then cooperation disappears. Hence stars are not mild density amplifiers for $p \leq \frac{1}{2}$. The two fundamental open questions are (a) The existence of mild density amplifiers; and (b) the existence of strong density amplifiers.

Analytical Results. To answer positively to both questions about the existence of density amplifiers, we describe two graph families parameterized by n, the number of nodes, and b, the temptation. Later, we show that graphs $\mathcal{A}^d$ are strong density amplifiers for deterministic (synchronous and asynchronous) settings and graphs $\mathcal{A}^r$ are strong density amplifiers for randomized settings. To simplify the construction, we first increase the defectors' payoff against the cooperator by the highest degree times ε and then round up the payoff to the nearest higher integer. It allows us to treat the payoff of a defector interacting with a defector as 0. This change gives the defector an advantage by giving them a higher payoff. Finally, if there is a tie in the accumulated payoff, then cooperators win ties. However, note that since we have increased the defectors' payoff, a tie implies a higher payoff for cooperators in the original setting. Thus this procedure still only provides an advantage to defectors.

The structure of $\mathcal{A}^d(n, b)$ is as follows: There are four types of vertices: big, small, bridge, and leaf. The graph consists of a path on which big and small vertices alternate. Between two neighboring big and small vertices there are $b - 1$ bridge vertices connected to both of them. Moreover, to every small vertex, b leaf vertices connect, and to every big vertex, $10b^2 - 2b$ leaf vertices connect. The number n bounds the number of big vertices that can be in the graph (around $\frac{n}{10b^2}$). Fig. 3 shows the structure of $\mathcal{A}^d$ together with the spread of cooperation.

The structure $\mathcal{A}^r(n, b)$ is similar. The graph contains big, small, and leaf vertices. Again, big and small vertices alternate on a path. Every big vertex connects to $10b^2 - 2$ leaf vertices. This time, small vertices neighbor only two big vertices.

We describe the spread of cooperation on $\mathcal{A}^d$ in a deterministic (synchronous or asynchronous) setting. We suppose that b is arbitrary and n sufficiently large. Since the initial density p is above 0, we know that in the graph, there are a lot of seeded

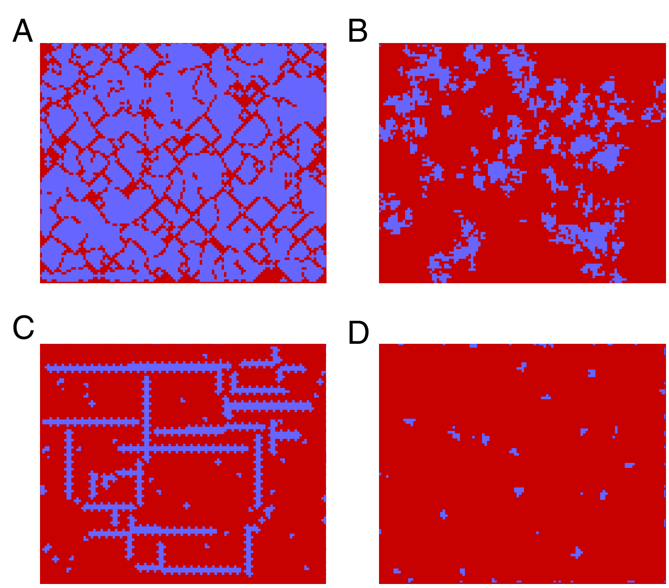

Fig. 2. Simulations of the spatial games on the lattice with unchangeable boundary and size 100×120. Cooperators are denoted blue, and defectors are red. The initial density of cooperators is $p = \frac{1}{2}$, and the position is recorded after reaching equilibria. (*A*) Process for $b = 1.3$ and synchronous deterministic updating. (*B*) Process for $b = 1.99$ and synchronous deterministic updating. Synchronicity and determinism ensure that the cooperators create stable structures that are impossible to invade. (*C*) Process for 1.03 and deterministic asynchronous updating. (*D*) Process for $b = 1.03$ and randomized asynchronous updating. In the randomized updating, cooperators cannot create structures, therefore defectors slowly erode the cooperation.

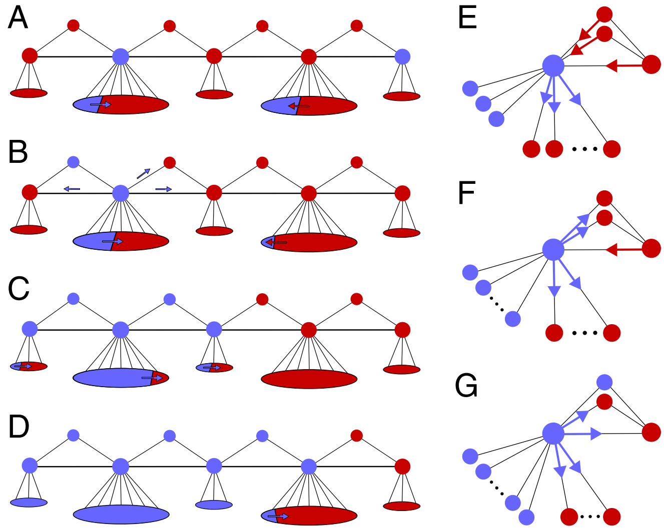

Fig. 3. The spread of cooperation in $\mathcal{A}^d$. (*A*) Big and small vertices alternate on a line. One bridge vertex is connected to neighboring small and big vertex, leaf vertices only connect to small and big vertices. More leaf vertices connect to the big vertex. A big vertex that started as a cooperator with a lot of cooperating neighbors spreads the cooperation while the cooperation recedes everywhere else. (*B*) The cooperating big vertex becomes invincible. Cooperation still recedes everywhere else. (*C*) The invincible vertex spreads the cooperation to a small vertex, which in turn starts spreading cooperation among the bridge neighbors. (*D*) The small vertex can convert the big neighbor, and the spread of cooperation continues. (*E*) The neighborhood of a seeded vertex for $b = 3$. The seeded vertex can convert only leaf neighbors; bridge and small vertices can have a higher payoff. (*F*) After a big vertex has more than $2b$ cooperating neighbors, it can convert bridge vertices. (*G*) Invincible vertex has at least $3b^2$ neighbors and can spread the cooperation to any neighbor.

vertices. Seeded vertices are big vertices that are cooperators and have at least b cooperating neighbors. Seeded vertices can convert other leaf vertices to cooperation; see Fig. 3 *E–G* for a detailed description of the neighborhood of the seeded vertex. With substantial probability, the seeded vertex converts at least $3b^2$ of leaf vertices to cooperators. This cooperator cannot be converted: No neighbor can have a higher payoff than $3b^2$ since no neighbor has more than $3b$ neighbors. We call such a vertex invincible. After an invincible vertex appears, the cooperation cannot die out. Moreover, from every position, the probability that the cooperation fixates is nonzero and happens quickly.

Observe that the probability that a big vertex becomes invincible depends only on the neighborhood of the big vertex and is also independent from all events happening at higher distances in the graph. That means increasing the size of the graph from n vertices to $2n$ vertices increases the probability of invincible vertex appearing (and thus fixation probability) roughly from ρ to $1 - (1 - \rho)(1 - \rho) = 2\rho - \rho^2$. Also, for setting $p = \frac{1}{2}$, the expected number of cooperating neighbors of a big vertex is around $5b^2$, which means that a big vertex initialized as a cooperator is invincible from the start, so the probability of an invincible vertex not appearing is exponentially small.

Similar reasoning holds for randomized asynchronous settings. Some big vertices have a cooperator in a neighborhood. This cooperator has a small but positive chance to spread to the big vertex and then convert all leaf vertices to cooperators. This vertex, so-called stronghold, is more likely to convert another neighboring vertex to a stronghold than to turn to a defector. The spread of cooperators is a walk on a biased Markov Chain. Since the bias is significant, the cooperators spread over the whole graph with a large probability.

In summary, we show that the family of graphs $\mathcal{A}^d(n, b)$ ensures high fixation probability for synchronous deterministic and asynchronous deterministic processes, and family of graphs $\mathcal{A}^r(n, b)$ ensures the same for asynchronous randomized setting; see *SI Appendix*, Theorems 1 and 2 for details. In other words, we establish the existence of strong density amplifiers for all three replicator dynamics. Along with the main result proving the existence of strong density amplifiers, our construction has several other desirable properties. First, (a) in contrast to existing simulation studies that consider high $p = 1/2$ (11, 48, 54), our structure ensures high fixation probability even for low p, for instance, for p of order of $n^{-\frac{1}{3(b+1)}}$, that goes to 0 as n goes to ∞, the fixation probability is at least $1 - e^{-0.2n^{\frac{1}{3}}}$ in both density amplifier families; (b) in contrast to previous studies that consider low b, our structure ensures high fixation probability for high b, for instance for b that is of order of $\sqrt{\log n}$ that goes to ∞ as n goes to ∞, the fixation probability is again at least $1 - e^{-0.2n^{\frac{1}{3}}}$ in both density amplifier families. It shows the robustness of our result with respect to low initial cooperation and high temptation. Second, the number of steps to achieve fixation is asymptotically quadratic in the size of the graph for both $\mathcal{A}^d$ and $\mathcal{A}^r$ (*SI Appendix*, Lemmas 3 and 6). It means that the fixation is achieved quickly with respect to the network size. Third, our graph structures have constant degree, i.e., the degree is bounded by $10b^2$ even when the population size increases to ∞. The constant-degree property is desirable as it ensures that every individual finishes all interactions in constant time in every round.

Significance of results. We further emphasize the strength and significance of our results in the following ways:

Initial cooperation rate. First, while previous literature for constant or weak selection, considers the probability that a single cooperator fixates, we show in *SI Appendix*, section 5 that for spatial games a lower bound on the initial cooperation rate that is required for the deterministic setting. This complements our initial cooperation density requirement.

Extinction of defectors. Our strong density amplifiers ensure a high fixation probability of cooperation even with a low initial cooperation rate. This also ensures that even if there is a relatively high initial density of defectors (say $\frac{1}{2}$) still the defectors become extinct. In other words, our strong density amplifiers ensure high fixation for cooperators and low fixation for defectors.

Payoff matrix. While we focus our results on the classic matrix from refs. 5 and 30, we also show the condition under which we obtain strong density amplifiers for the general Prisoners' dilemma payoff matrix. We show that our results about the existence of strong density amplifiers hold in the randomized update for all parameters, and for the deterministic update when $R^2 > T \cdot P$ (*SI Appendix*, section 6).

Initialization. We focus on the randomized initialization of cooperators as this is one of the most difficult conditions where cooperators are not clustered. We argue (in *SI Appendix*, section 7) that our results hold for other initialization, e.g., temperature and correlated initialization.

Mutation rate. We present simulation results that consider the cooperation rate over time for various mutation rates (*SI Appendix*, Fig. S3), and our structure significantly outperforms star and grid.

Table 1. Fixation probabilities for a single random mutant in graph occupied by individuals of the other type: The first column describes the graph type, and the second and third columns represent the fixation probability of a single cooperator and defector, respectively

	ρ_C^n	ρ_D^n	$\rho_C^n \bowtie \rho_D^n$	$A_n = \frac{\rho_C^n}{\rho_D^n}$	$\lim_{n\to\infty} A_n$	$B_n = \frac{\rho_C^n}{\rho_n}$	$\lim_{n\to\infty} B_n$
Complete graph	$2^{-c_1 n}$	c_2	$<<$	$2^{-c_1 n}/c_2$	0	$n2^{-c_1 n}$	0
Grid	$2^{-c_3 n}$	c_4	$<<$	$2^{-c_3 n}/c_4$	0	$n2^{-c_3 n}$	0
Star	$c_5 n^{-1}$	$c_6 n^{-1}$	$\approx$	c_5/c_6	c_5/c_6	c_5	c_5
$\mathcal{A}^r$	c_7	$2^{-c_8 n}$	$>>$	$c_7 2^{c_8 n}$	∞	$c_7 n$	∞

In the fourth column, we compare the two fixation probabilities where $\bowtie$ is the comparison operator, which can be very small $<<$, comparable $\approx$, or very large $>>$. The fifth and sixth columns consider the ratio A_n of the two fixation probabilities and the respective large-population limit. The seventh and eighth columns consider the ratio B_n of the fixation probability of a cooperator to the fixation probability of a neutral mutant, denoted ρ_n, which is $1/n$, and the respective large-population limit. In the table, $c_1, c_2, \ldots, c_8$ denote constants that are independent of n. The table summarizes the following: (a) for complete graph and grid, the fixation probability of cooperators is exponentially small in n, whereas for defectors it is constant; (b) for star, the fixation probabilities are proportional to $1/n$; and (c) for $\mathcal{A}^r$ the fixation probability of cooperators is constant, whereas for defectors it is exponentially small. The large-population limit of the desired comparison ratios vanishes for the complete graph and grid, is constant for the star, and goes to ∞ for $\mathcal{A}^r$. Observe that only $\mathcal{A}^r$ yields that the desired ratio goes to ∞ in the large-population limit.

Fixation probability bounds for single mutant. As mentioned above, in the deterministic setting we show that the initial density of cooperators is required. We also consider the bounds on fixation probability of a single mutant: a single cooperator among defectors (fixation probability denoted as ρ_C^n in population of size n) and a single defector among cooperators (fixation probability of defector denoted as ρ_D^n). For $\mathcal{A}^r$ we show that ρ_C^n is constant and ρ_D^n is exponentially small in n. In contrast, for complete graph and grid, ρ_C^n is exponentially small in n, whereas ρ_D^n is constant, and for star both ρ_C^n and ρ_D^n are proportional to $1/n$. The results are summarized in Table 1 (*SI Appendix*, section 8) and the asymptotic fixation probabilities are highlighted in Fig. 4.

Simulation Results. Finally, we supplement our theoretical findings with simulation results. While the theory provides guarantees for large population sizes, the simulation results demonstrate the effectiveness of our structure even for small population sizes, with the code available at ref. 55. We examine three baselines: complete graph, grid, and star. No matter the process and the temptation (b), the fixation probability on these graphs does not change much. On a complete graph, every defector has a higher payoff than every cooperator, which means the fixation probability tends to 0. Similarly, on a grid, the last defector has a higher payoff than all neighbors, so it cannot be converted in a deterministic setting. In a randomized setting, the fixation probability is also 0. On a star, the probability that the center is a cooperator is proportional to p. If at the same time $p \geq \frac{b}{n-1}$, the central cooperator has a higher payoff than neighbors and can spread. This means the fixation probability is proportional to p.

We run the asynchronous deterministic setting for three temptations and graphs: $\mathcal{A}^d(10^4, 2)$, $\mathcal{A}^d(10^4, 4)$, and $\mathcal{A}^d(10^4, 6)$. We examine how the initial cooperator density p influences the fixation of cooperation in $\mathcal{A}^d$ against the baseline graphs. For every graph and value of p, we report the fixation probability by averaging over $5 \cdot 10^4$ runs. Every run is simulated until one of the two things happens: Cooperators become extinct, or cooperators create an invincible vertex (this is a big vertex with $3b^2$ cooperating neighbors). Since the invincible vertex cannot be converted and cooperators eventually fixate, these two conditions are equivalent to simulating until cooperators or defectors spread over the whole graph. The first row of Fig. 5 shows the results. We see that for all settings, the fixation probability increases with increasing initial probability. At $p = 0.04$, the fixation probability on $\mathcal{A}^d(10^4, 2)$ is already almost 1. The fixation probability for other graphs rises more slowly since the temptation to defect is larger, and the graphs $\mathcal{A}^d(10^4, 4)$ and $\mathcal{A}^d(10^4, 6)$ have fewer big vertices than the graph $\mathcal{A}^d(10^4, 2)$.

For the asynchronous randomized setting, we again examine three graphs: $\mathcal{A}^r(10^4, 2)$, $\mathcal{A}^r(10^4, 4)$, and $\mathcal{A}^r(10^4, 6)$. We

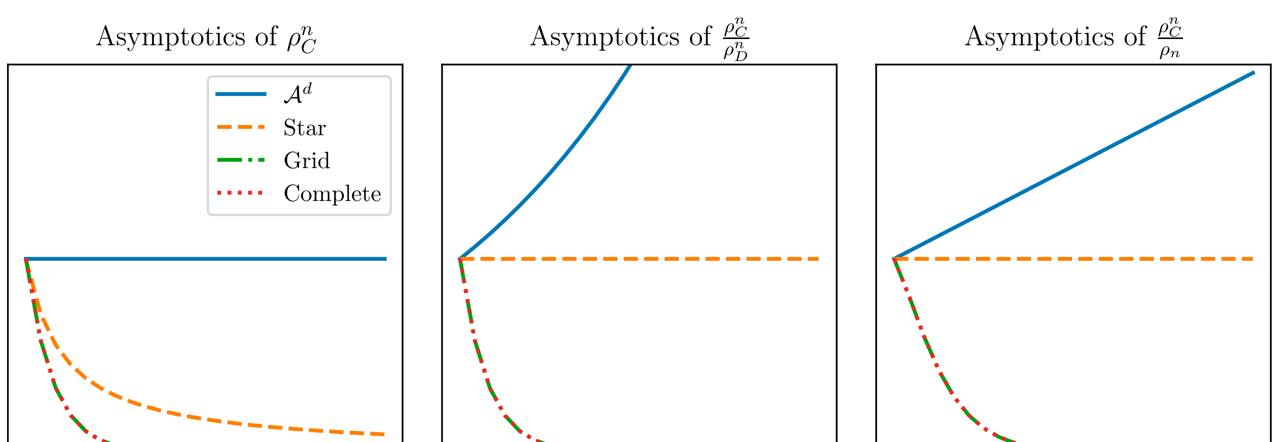

Fig. 4. Asymptotics of fixation probabilities. We examine the relationship between the fixation probability of one cooperator in asynchronous randomized setting and the graph size as described in Table 1. We first show the asymptotic of the fixation probability and then compare it to two baselines. We see that $\mathcal{A}^r$ is the only graph that grows to infinity while compared to the baselines.

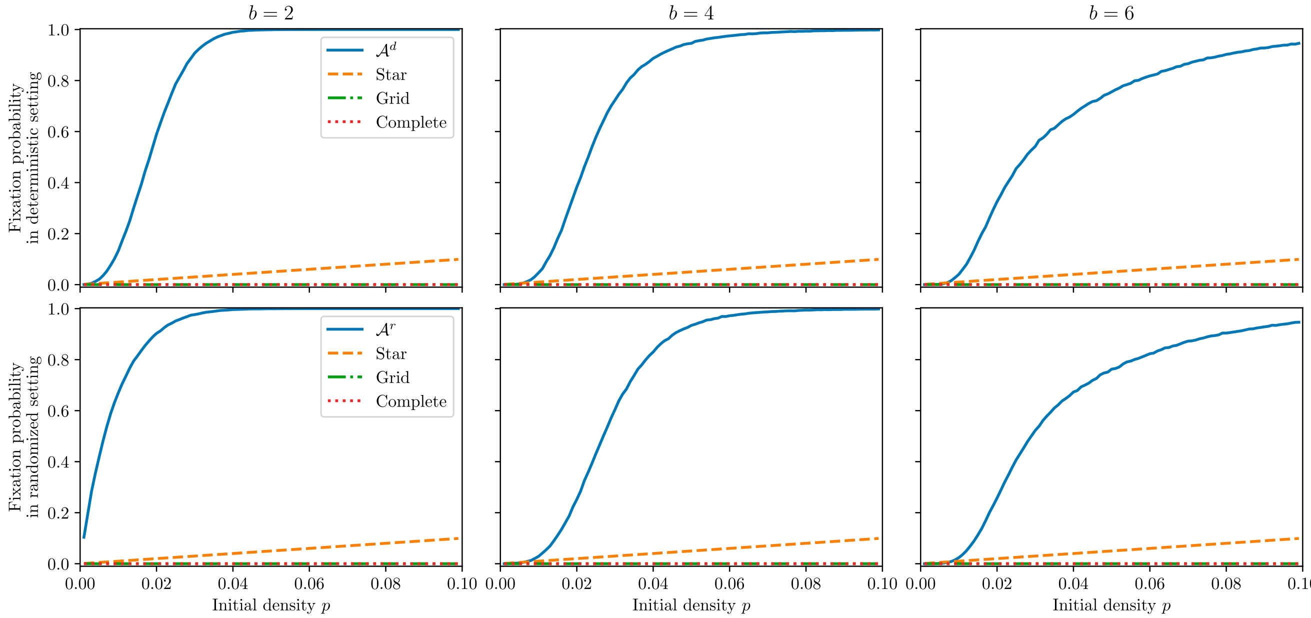

Fig. 5. Simulation results. We examine the fixation probability (y-axis) with respect to the starting density of cooperation p (x-axis) for four graphs: a density amplifier, a star, a grid, and a complete graph. The rows examine asynchronous deterministic and asynchronous randomized setting, the columns examine different temptation $b \in \{2, 4, 6\}$. All graphs have 10^4 vertices and the results are averaged over $5 \cdot 10^4$ runs. Cooperators on the grid and complete graph have fixation probability 0. On star, the fixation probability is proportional p. The fixation probability on both density amplifiers quickly reaches 1 for $p = 0.04$ already.

examine how the initial cooperator density p influences the fixation of cooperation in $\mathcal{A}^r$ against the baseline graphs. We run the process for every combination of parameters $5 \cdot 10^4$ times until either cooperation or defection fixate and report the fixation probability. In the second row of Fig. 5, as our result suggests, we see that cooperation increases even more steeply than in the deterministic setting. At $p = 0.02$, the fixation probability on $\mathcal{A}^r(10^4, 2)$ is already almost 1.

Discussion

In this work, we present $\mathcal{A}^d$ and $\mathcal{A}^r$, the first graph families that increase the fixation probability of cooperation in spatial games with Prisoners' dilemma with strong selection. Moreover, the graph families ensure the fixation with several desirable properties: (a) they are robust with respect to a low initial density of cooperation and high temptation; (b) they have a constant degree; and (c) they ensure a fast spread of cooperation within quadratic steps in the size of the network.

First, the study of evolutionary graph theory for constant (or frequency-independent) fitness has been widely studied in the context of the Moran process (27). The role of graphs that can amplify the fixation probability has been a key topic of interest (56, 57). Moreover, the time to fixation is another important aspect (58), and there is a very interesting tradeoff (59). The existence of strong density amplifiers with fast fixation time has been established in ref. 50. However, the techniques of these works do not extend to evolutionary games, which represent frequency-dependent selection.

Second, the study of evolutionary games on graphs, which represent frequency-dependent selection, also received broad attention (30, 45). The computational hardness of such games has been established in refs. 53 and 60. In the regime of weak selection, the role of graphs that help in increasing cooperative behavior has been considered by Allen et al. (52), and by Fotouhi et al. (61). While Allen et al. give the theoretical algorithms and guarantee, Fotouhi et al. present natural graphs and empirical results. However, these results do not extend to strong selection, where the existence of density amplifiers is an interesting open question. Spatial games intuitively represent the strong selection limit, and our results complement the existing results in the literature establishing the existence of strong density amplifiers in this regime.

Finally, we believe that the structures we present have wider applicability. The reasoning in our proof can be extended to other two-player matrix games (dilemmas), such as snowdrift and stag-hunt games. Even in these cases, prosocial behavior creates invincible parts of the graph. Our structures will be useful in problems where there is a hard-to-reach absorbing state that is desired. Exploring the role of our structures in specific applications is an interesting direction for future work.

Materials and Methods

In this section, we describe the main ingredients for both proofs.

Proof Sketch for Asynchronous Deterministic Updating. The graph family $\mathcal{A}^d$ are strong density amplifiers. The proof has three main ingredients. First, we show that with substantial probability there are at least $n^{\frac{1}{3}}$ big vertices that are cooperators and have at least b cooperating neighbors (we call them seeded vertices). Second, we prove that one seeded vertex becomes invincible with a large probability. Third, we describe how one invincible vertex can convert the whole graph to cooperation.

A vertex starts as a cooperator with probability p and we suppose that

$$p \geq \frac{1}{n^{1/(3b+3)}}. \qquad \textbf{[1]}$$

Let us examine the probability that a big vertex is a cooperator and has at least b cooperating leaf neighbors. With probability p, the big vertex is a cooperator. It has $10b^2 - 2b$ leaf neighbors and the number of cooperating neighbors follows a binomial distribution, which gives the probability

$$p \cdot \sum_{x \geq b} p^x (1-p)^{10b^2-2b-x} \binom{10b^2-2b}{x}. \quad [2]$$

We split the graph to $n^{\frac{1}{3}}$ parts. In one part, using $b \geq 2$, Eqs. **1** and **3**, we have that there is no seeded vertex with probability at most

$$\left(1 - \frac{10b^2+b+2}{n^{1/3}}\right)^{\frac{n^{2/3}}{10b^2+b+2}} < e^{-2n^{1/3}}. \quad [3]$$

From union bound and Eq. **3**, we get that in every part of the graph, there is at least one seeded vertex (that means at least $n^{1/3}$ seeded vertices in the whole graph) with probability at least

$$1 - e^{-n^{1/3}}. \quad [4]$$

The seeded vertex has a payoff big enough to convert any leaf to cooperation. In the worst case, the neighboring small and bridge vertices can revert the seeded vertex to defection. In one active step, the probability of the big vertex being turned into defection is at most $\frac{2b}{10b^2-4b} < \frac{1}{4b}$. When the seeded vertex has at least $2b$ cooperating leaf neighbors, the bridge vertices stop being threatening. The failure probability in one active step decreases to $\frac{2}{9b^2-2(b-1)} < \frac{1}{4b^2}$ until the big vertex has b^2 cooperating neighbors. Finally, the failure probability is at most $\frac{2}{7b^2-2(b-1)} \leq \frac{1}{3b^2}$ in one step until the big vertex has at least $3b^2$ cooperating neighbors. At that point, it becomes invincible, no neighbor can have a higher payoff, so it cannot be converted. From union bound, the seeded vertex becomes invincible with probability at least

$$\left(1 - b \cdot \frac{1}{4b}\right)\left(1 - b^2 \cdot \frac{1}{4b^2}\right)\left(1 - 2b^2 \cdot \frac{1}{3b^2}\right) \geq \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{3} \geq \frac{3}{16}. \quad [5]$$

One invincible vertex can convert the rest of the graph to cooperation. It first converts all bridge vertices and neighboring small vertex. This small vertex converts its leaf neighbors, then the neighboring big vertex and remaining bridge vertices. By this, the big vertex has at least b cooperating neighbors and can start converting other vertices and becomes invincible. The probability of everything happening is nonzero and the invincible vertex cannot be converted, which means it happens eventually. To increase the number of invincible vertices by one, we need the number of steps around the newly invincible vertex to be around $\mathcal{O}((20b)^{2b})$. This means the number of steps until all vertices are converted to cooperators is

$$\mathcal{O}\left((20b)^{2b} n^2\right). \quad [6]$$

That means, for initial rate of cooperation at least $p \geq \frac{1}{n^{1/(3b+3)}}$ (Eq. **1**), from Eqs. **4** and **5**, the fixation probability is at least

$$\left(1 - e^{-n^{1/3}}\right) \cdot \left(1 - \left(\frac{3}{16}\right)^{n^{1/3}}\right) \geq 1 - e^{-0.2n^{1/3}}. \quad [7]$$

Note that for b around $\sqrt{\log n}$ (which is unbounded if n grows), the initial density of cooperation is around $e^{-\sqrt{\log n}}$, which tends to 0 as n grows. However, the theorem still holds and guarantees fixation probability that goes to 1 with growing n.

Proof Sketch for Asynchronous Randomized Updating. The graph family $\mathcal{A}^r$ are strong density amplifiers. The proof consists of two important steps. First, we show that a big vertex that is seeded (has one cooperator in the neighborhood) becomes a stronghold (all its leaf neighbors are cooperators) with a large probability. Second, we estimate the probability that a stronghold vertex converts the rest of the graph.

We split the spread of cooperation into several stages. In the first stage, the big vertex v becomes a cooperator. Let i denote the number of cooperating neighbors of v. In the second stage, we have $i \leq b$, in the third $b < i \leq 3b$, and in the fourth $3b < i$.

The first stage succeeds with probability

$$\frac{1}{1+e^{b-0}}. \quad [8]$$

During the second and third stages, we suppose that the edge between the small and big vertex was not selected. This happens with probability at most

$$\left(1 - \frac{2}{10b^2-3b}\right)^{3b} > 1 - \frac{6b}{10b^2-3b} > 1 - \frac{1}{\frac{17}{12}b}. \quad [9]$$

With probability $\frac{1}{1+e^{b-i}}$ after selecting the edge between leaf and big vertex, the cooperation spreads, that means the second phase succeeds with the probability

$$\prod_{i=0}^{b} \frac{1}{1+e^{b-i}}. \quad [10]$$

By the same reasoning, the third phase succeeds with probability

$$\prod_{i=b+1}^{3b} \frac{1}{1+e^{b-i}} > \prod_{i=1}^{2b} \frac{1}{1+e^{-i}}, \quad [11]$$

and fourth with

$$\prod_{i=3b+1}^{10b^2} \frac{1}{1+e^{2b-i}} = \prod_{i=0}^{10b^2} \frac{1}{1+e^{-b-1-i}} > \prod_{i=0}^{\infty} \frac{e^{b+1+i}}{1+e^{b+1+i}}. \quad [12]$$

Combining Eqs. **9**–**12**, we get the success probability

$$2^{-5}e^{-b^2}. \quad [13]$$

Having a stronghold in the graph and selecting an edge between the stronghold and neighboring small vertex means that either stronghold becomes a defector (with probability roughly $\frac{1}{1+e^{10b^2-2}}$) or the neighboring big vertex becomes seeded and it might become a stronghold with probability $2^{-5}e^{-b^2}$. We track the number of strongholds using one dimensional Markov Chain. Observe that the ratio between increasing and decreasing the number of strongholds is

$$e^{6b^2}. \quad [14]$$

Suppose that $p \geq \frac{e^{b^2} 2^3 \cdot 10b^2}{n^2 3}$, then a big vertex is seeded with probability at least $2^5 p$ and from Eq. **13**, we have that a big vertex becomes a stronghold with probability at least $e^{-b^2}p$. With probability at least $1 - e^{\frac{49}{16}n^{1/3}}$, at least $n^{1/3}$ seeded vertices turns into stronghold. In the Markov Chain tracking the number of strongholds (with the ratio Eq. **14**), the strongholds disappear with probability at most $e^{-6b^2 n^{1/3}}$. Therefore the fixation probability is at least

$$1 - e^{-3n^{1/3}}. \quad [15]$$

Again, note that we can set $b = (\log n)^{1/3}$ which grows with n, and then p tends to 0 with large n and the fixation probability tends to 1.

Data, Materials, and Software Availability. Code data have been deposited in Code: Amplifiers of Cooperation (DOI: 10.5281/zenodo.10832535) (55).

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