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Diagonal cubic forms and the large sieve

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Abstract

Let $N(X)$ be the number of integral zeros $(x_1, \dots, x_6) \in [-X, X]^6$ of $\sum_{1 \leqslant i \leqslant 6} x_i^3$. Works of Hooley and Heath-Brown imply $N(X) \ll_\epsilon X^{3+\epsilon}$, if one assumes automorphy and grand Riemann hypothesis for certain Hasse–Weil L-functions. Assuming instead a natural large sieve inequality, we recover the same bound on $N(X)$. This is part of a more general statement, for diagonal cubic forms in $\geqslant 4$ variables, where we allow approximations to Hasse–Weil L-functions.

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1 | INTRODUCTION

Fix an integer $m \geqslant 4$. Fix integers $F_1, \dots, F_m \in \mathbb{Z} \setminus \{0\}$ and let

$$F(\boldsymbol{x}) := \sum_{1 \leqslant i \leqslant m} F_i x_i^3,$$

where $\boldsymbol{x} = (x_1, \dots, x_m)$. We are interested in the behavior, as $X \to \infty$, of the point count

$$N_F(X) := |\{\boldsymbol{x} \in \mathbb{Z}^m \cap [-X, X]^m : F(\boldsymbol{x}) = 0\}|.$$

Certain varieties, $V_{\boldsymbol{c},k}$, play a key role. For each $\boldsymbol{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ and field k, let

$$V_{\boldsymbol{c},k} := \left\{ (\xi_1, \dots, \xi_m) \in \mathbb{P}_k^{m-1} : \sum_{1 \leqslant i \leqslant m} F_i \xi_i^3 = \sum_{1 \leqslant i \leqslant m} c_i \xi_i = 0 \right\},$$

where $\mathbb{P}_k^{m-1}$ is the projective space with coordinates $\xi_1, \dots, \xi_m$ over k.

In the special case $F = \sum_{1 \leqslant i \leqslant 6} x_i^3$, with $m = 6$, we abbreviate $N_F(X)$ to $N(X)$. In this case, building on [9], the papers [10] and [8] each proved

$$N(X) \ll_\epsilon X^{3+\epsilon}, \tag{1.1}$$

assuming Hypothesis HW of [9, Section 6]; [8, Section 4] for the Hasse–Weil L-function of each smooth variety $V_{\boldsymbol{c},\mathbb{Q}}$ with $\boldsymbol{c} \neq \boldsymbol{0}$. Unconditionally, by [17, Theorem 1.2],

$$N(X) \ll_\epsilon X^{7/2}/(\log X)^{5/2-\epsilon}$$

for $X \geqslant 2$, via methods stemming from work such as [2, 4, 11, 18].

Hypothesis HW practically amounts to automorphy, plus the grand Riemann hypothesis (GRH). Automorphy remains open [21, Appendix A]. Hooley suggested that a zero-density hypothesis would suffice in place of GRH [9, p. 51]. Following the usual paths laid out in [12, Theorem 10.4], a general such density hypothesis is provable assuming automorphy, a large sieve inequality, and progress on the grand Lindelöf hypothesis (GLH).

In this paper, we show that a large sieve inequality *by itself* would imply Equation (1.1). The precise large sieve inequality we need will be stated in Section 2, as Hypothesis 2.1.

Theorem 1.1. *Suppose $m \in \{5, 6\}$. Assume Hypothesis 2.1. Then*

$$N_F(X) \ll_\epsilon X^{3(m-2)/4+\epsilon}, \tag{1.2}$$

for all reals $X \geqslant 1$ and $\epsilon > 0$.

For $m = 6$, the exponent in Equation (1.2) matches Equation (1.1). In Section 2, we state a more general result, Theorem 2.7, valid for all $m \geqslant 4$. Our methods might also apply elsewhere [21, Section 9.1]. For instance, Wang [20] explained how one may hope to use the modularity of elliptic curves over $\mathbb{Q}$ to *unconditionally* produce an absolute constant $\delta > 0$ such that

$$|\{a \in \mathbb{Z} : 1 \leqslant a \leqslant A\} \setminus \{x^2 + y^3 + z^3 : x, y, z \in \mathbb{Z}_{\geqslant 0}\}| \ll A^{6/7-\delta}.$$

This would then improve on the existing bound $O_\epsilon(A^{6/7+\epsilon})$ due to Brüdern [3].

Conventions

We let $\mathbb{Z}_{\geqslant c} := \{n \in \mathbb{Z} : n \geqslant c\}$. We let $\mathbf{1}_E := 1$ if a statement E holds, and $\mathbf{1}_E := 0$ otherwise. For integers $n \geqslant 1$, we let $\mu(n)$ denote the Möbius function.

We write $f \ll_S g$, or $g \gg_S f$, to mean $|f| \leqslant Cg$ for some $C = C(S) > 0$. The implied constant C is always allowed to depend on m and F, in addition to S. We let $O_S(g)$ denote a quantity that is $\ll_S g$. We write $f \asymp_S g$ if $f \ll_S g \ll_S f$.

2 | FRAMEWORK AND RESULTS

Let $\mathfrak{D} := 3(\prod_{1 \leqslant i \leqslant m} F_i)^{2^{m-2}} \in \mathbb{Z}$. For each $\boldsymbol{c} \in \mathbb{Z}^m$, let

$$\Delta(\boldsymbol{c}) := \mathfrak{D} \prod_{(v_2,\dots,v_m) \in \{1,-1\}^{m-1}} \left((c_1^3/F_1)^{1/2} + \sum_{2 \leqslant i \leqslant m} v_i (c_i^3/F_i)^{1/2} \right) \in \mathbb{Z}. \tag{2.1}$$

For each field k in which $\Delta(\boldsymbol{c})$ is invertible, the variety $V_{\boldsymbol{c},k}$ is a smooth complete intersection, by the Jacobian criterion for smoothness. Let

$$\mathcal{S} := \{\boldsymbol{c} \in \mathbb{Z}^m : \Delta(\boldsymbol{c}) \neq 0\}, \qquad \mathcal{S}(C) := \mathcal{S} \cap [-C, C]^m. \tag{2.2}$$

For each $\boldsymbol{c} \in \mathcal{S}$ and prime p, we define a local Euler factor $L_p(s, \boldsymbol{c})$, following Serre [16] and Kahn [13, Section 5.6]. First, choose a prime $\ell \neq p$, and let

$$M(\boldsymbol{c}, \ell) := H^{m-3}(V_{\boldsymbol{c},\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)/H^{m-3}(\mathbb{P}^{m-1}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

where $H^i(W, \mathbb{Q}_\ell)$ denotes the ith ℓ-adic cohomology group of W. Let $M(\boldsymbol{c}, \ell)^{I_p} \subseteq M(\boldsymbol{c}, \ell)$ denote the group of inertia invariants of $M(\boldsymbol{c}, \ell)$. Let $\alpha_{\boldsymbol{c},j}(p) \in \mathbb{C}$, for $1 \leqslant j \leqslant \dim M(\boldsymbol{c}, \ell)^{I_p}$, be the geometric Frobenius eigenvalues on $M(\boldsymbol{c}, \ell)^{I_p}$. Finally, let

$$\tilde{\alpha}_{\boldsymbol{c},j}(p) := \frac{\alpha_{\boldsymbol{c},j}(p)}{p^{(m-3)/2}}, \qquad L_p(s, \boldsymbol{c}) := \prod_{1 \leqslant j \leqslant \dim M(\boldsymbol{c},\ell)^{I_p}} (1 - \tilde{\alpha}_{\boldsymbol{c},j}(p) p^{-s})^{-1}. \tag{2.3}$$

On multiplying over p, we obtain for each $\boldsymbol{c} \in \mathcal{S}$ a global *Hasse–Weil L-function*

$$L(s, \boldsymbol{c}) := \prod_p L_p(s, \boldsymbol{c}) = \sum_{n \geqslant 1} \lambda_{\boldsymbol{c}}(n) n^{-s}, \tag{2.4}$$

for some coefficients $\lambda_{\boldsymbol{c}}(n) \in \mathbb{C}$ defined by expanding the product over p. We now state Hypothesis 2.1. It asserts a large sieve inequality, Equation (2.5), in a certain range.

Hypothesis 2.1. For all reals $C, N, \epsilon > 0$ with $N \leqslant C^3$, we have

$$\sum_{\boldsymbol{c} \in \mathcal{S}(C)} \left| \sum_{n \leqslant N} v_n \lambda_{\boldsymbol{c}}(n) \right|^2 \ll_\epsilon C^\epsilon \max(C^m, N) \sum_{n \leqslant N} |v_n|^2 \tag{2.5}$$

for all vectors $(v_n)_{1 \leqslant n \leqslant N} \in \mathbb{C}^{\lfloor N \rfloor}$.

We now make some general comments on $L(s, \boldsymbol{c})$. By [13, Sections 5.6.3 and 5.6.4] and [15, Corollary 1.2], the factors $L_p(s, \boldsymbol{c})$ are independent of the choice of ℓ, and we have

$$|\tilde{\alpha}_{\boldsymbol{c},j}(p)| \leqslant 1. \tag{2.6}$$

By Equation (2.6), the product and series in Equation (2.4) converge absolutely for $\Re(s) > 1$.

We have $\dim M(\boldsymbol{c}, \ell)^{I_p} \leqslant \dim M(\boldsymbol{c}, \ell) \ll_m 1$ by [14, Corollary of Theorem 3]. Therefore, by Equation (2.6), we have $\lambda_{\boldsymbol{c}}(n) \ll_\epsilon n^\epsilon$ for all $n \geqslant 1$. Thus, Equation (2.5) is the large sieve inequality that one would naturally expect to hold. In fact, Equation (2.5) could potentially hold in the range $N \leqslant C^A$ for any constant $A > 0$. However, we will only need it in the range $N \leqslant C^3$.

The coefficients $\lambda_{\boldsymbol{c}}(n)$ can be interpreted geometrically, but it would take us too far afield to detail anything but the simplest case. For each $\boldsymbol{c} \in \mathbb{Z}^m$ and prime p, let

$$E_{\boldsymbol{c}}(p) := \frac{|\{\boldsymbol{x} \in \mathbb{F}_p^m : F(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} = 0\}| - p^{m-2}}{p - 1}, \qquad E^\natural_{\boldsymbol{c}}(p) := \frac{E_{\boldsymbol{c}}(p)}{p^{(m-3)/2}},$$

where $\boldsymbol{c}\cdot\boldsymbol{x} := \sum_{1\leqslant i\leqslant m} c_i x_i$. If $p \nmid \Delta(\boldsymbol{c})$, then $M(\boldsymbol{c},\ell)^{I_p} = M(\boldsymbol{c},\ell)$ and

$$\lambda_{\boldsymbol{c}}(p) = \sum_{1\leqslant j\leqslant \dim M(\boldsymbol{c},\ell)} \tilde{\alpha}_{\boldsymbol{c},j}(p) = (-1)^{m-3}E^{\natural}_{\boldsymbol{c}}(p), \tag{2.7}$$

by Equation (2.3) and the Grothendieck–Lefschetz trace formula.

We emphasize that our L-functions are normalized differently than in [8, 9]. If $H(s,\boldsymbol{c})$ is the L-function associated with $V_{\boldsymbol{c},\mathbb{Q}}$ in [8, Section 4], then

$$H(s+\frac{m-3}{2},\boldsymbol{c}) = L(s,\boldsymbol{c}).$$

Proof framework

We will analyze $N_F(X)$ using the *delta method*, due to [5, 7]. This method features some complete exponential sums that we now recall. Let

$$S_{\boldsymbol{c}}(n) := \sum_{\substack{1\leqslant a\leqslant n:\\ \gcd(a,n)=1}} \sum_{1\leqslant x_1,\dots,x_m\leqslant n} e^{2\pi i(aF(\boldsymbol{x})+\boldsymbol{c}\cdot\boldsymbol{x})/n}, \qquad S^{\natural}_{\boldsymbol{c}}(n) := \frac{S_{\boldsymbol{c}}(n)}{n^{(m+1)/2}}, \tag{2.8}$$

for all $\boldsymbol{c}\in\mathbb{Z}^m$ and integers $n\geqslant 1$. It is known that $S_{\boldsymbol{c}}(n)$ is *multiplicative* in n, meaning that $S_{\boldsymbol{c}}(1)=1$ and $S_{\boldsymbol{c}}(n_1n_2) = S_{\boldsymbol{c}}(n_1)S_{\boldsymbol{c}}(n_2)$ whenever $\gcd(n_1,n_2)=1$ [8, Lemma 4.1]. Thus, $S^{\natural}_{\boldsymbol{c}}(n)$ is also multiplicative in n. For each $\boldsymbol{c}\in\mathbb{Z}^m$, let

$$\Phi(\boldsymbol{c},s) := \sum_{n\geqslant 1} S^{\natural}_{\boldsymbol{c}}(n)n^{-s} = \prod_p \Phi_p(\boldsymbol{c},s),$$

where $\Phi_p(\boldsymbol{c},s) := \sum_{l\geqslant 0} S^{\natural}_{\boldsymbol{c}}(p^l)p^{-ls}$. Ultimately, we will see that $S^{\natural}_{\boldsymbol{c}}(n)$ is related to $\lambda_{\boldsymbol{c}}(n)$ in a way that allows us to apply a large sieve inequality, like Equation (2.5), to the delta method.

Before proceeding, we recall two basic definitions from the theory of Dirichlet series. For $f,g:\mathbb{Z}_{\geqslant 1}\to\mathbb{C}$, the *Dirichlet convolution* $f*g:\mathbb{Z}_{\geqslant 1}\to\mathbb{C}$ is defined by the formula

$$(f*g)(n) := \sum_{ab=n} f(a)g(b).$$

A Dirichlet series $\sum_{n\geqslant 1} f(n)n^{-s}$ is said to be *invertible* if $f(1)\neq 0$, or equivalently, if there exists $g:\mathbb{Z}_{\geqslant 1}\to\mathbb{C}$ with $(f*g)(n) = \mathbf{1}_{n=1}$.

Our work is based on approximations of Dirichlet series. For each $\boldsymbol{c}\in\mathcal{S}$, let $\Psi(\boldsymbol{c},s)$ be an invertible Dirichlet series. The function $\boldsymbol{c}\mapsto\Psi(\boldsymbol{c},s)$, from $\mathcal{S}$ to the set of Dirichlet series, will be denoted simply by Ψ. For each $\boldsymbol{c}\in\mathcal{S}$ and integer $n\geqslant 1$, let

$$b_{\boldsymbol{c}}(n),\ a_{\boldsymbol{c}}(n),\ a'_{\boldsymbol{c}}(n)$$

be the n^{-s} coefficients of the Dirichlet series

$$\Psi(\boldsymbol{c},s),\ \Psi(\boldsymbol{c},s)^{-1},\ \Phi(\boldsymbol{c},s)/\Psi(\boldsymbol{c},s),$$

respectively. In terms of Dirichlet convolution, this means that

$$(a_{\boldsymbol{c}} * b_{\boldsymbol{c}})(n) = \mathbf{1}_{n=1}, \qquad a'_{\boldsymbol{c}} = S^{\natural}_{\boldsymbol{c}} * a_{\boldsymbol{c}}, \qquad S^{\natural}_{\boldsymbol{c}} = a'_{\boldsymbol{c}} * b_{\boldsymbol{c}}. \tag{2.9}$$

For us, the following particular definition of *approximation* will be convenient.

Definition 2.2. Call Ψ an *approximation of* Φ if the following three conditions hold:

(1) If $\boldsymbol{c} \in \mathcal{S}$, then $b_{\boldsymbol{c}}(n)$ is multiplicative in n.
(2) For all $\boldsymbol{c} \in \mathcal{S}$, integers $n \geqslant 1$, and reals $\epsilon > 0$, we have

$$\max(|b_{\boldsymbol{c}}(n)|, |a'_{\boldsymbol{c}}(n)|) \ll_\epsilon n^\epsilon \sum_{d \mid n} |S^{\natural}_{\boldsymbol{c}}(d)|.$$

(3) For all $\boldsymbol{c} \in \mathcal{S}$ and primes $p \nmid \Delta(\boldsymbol{c})$, we have $a'_{\boldsymbol{c}}(p) \ll p^{-1/2}$.

Theorem 2.3. *Suppose that for each* $\boldsymbol{c} \in \mathcal{S}$*, we have*

$$\Psi(\boldsymbol{c}, s) \in \left\{ \Phi(\boldsymbol{c}, s), \quad \prod_{p \nmid \Delta(\boldsymbol{c})} \Phi_p(\boldsymbol{c}, s), \quad \prod_{p \nmid \Delta(\boldsymbol{c})} L_p(s, \boldsymbol{c})^{(-1)^{m-3}}, \quad L(s, \boldsymbol{c})^{(-1)^{m-3}} \right\}.$$

Then, Ψ *is an approximation of* Φ.

Theorem 2.3 provides natural examples of approximations. It will not be used until Section 8, so we defer the proof to that section. For the rest of Section 2, fix an approximation Ψ of Φ.

Hypotheses

Our main general result, Theorem 2.7, will assume that *either* of two specific hypotheses holds. Our first hypothesis is the following:

Hypothesis 2.4. For all reals $C, N, \epsilon > 0$ with $N \leqslant C^3$, we have

$$\sum_{\boldsymbol{c} \in \mathcal{S}(C)} \left| \sum_{n \in I} b_{\boldsymbol{c}}(n) \right|^2 \ll_\epsilon C^\epsilon \max(C^m, N)\, N \tag{2.10}$$

for all real intervals $I \subseteq (0, N]$.

The following two remarks may help to clarify the nature of this hypothesis.

(1) If $\Psi = L(s, \boldsymbol{c})^{-1}$, then Hypothesis 2.4 would easily follow from GRH. On the other hand, if $\Psi = L(s, \boldsymbol{c})$, then Hypothesis 2.4 would follow from GLH plus a technical bound on $|\{\boldsymbol{c} \in \mathcal{S}(C) : L(s, \boldsymbol{c}) \text{ has a pole at } s = 1\}|$.
(2) A *density bound*, namely $|\{\boldsymbol{c} \in \mathcal{S}(C) : |\sum_{n \in I} b_{\boldsymbol{c}}(n)| \geqslant N^\sigma\}| \ll_\epsilon C^{m+\epsilon}/N^{2\sigma - 1}$ for $N \leqslant C^3$ and $\sigma \geqslant 1/2$, would follow from Hypothesis 2.4. But $C^{m+\epsilon}/N^{2\sigma-1}$ could be quite large even if $N = C^3$ and $\sigma = 1$. This is unlike in some density applications, for example, [12, Theorem 10.5], where further input may be needed near $\sigma = 1$.

If $\Psi = L(s, \boldsymbol{c})^{-1}$, then Hypothesis 2.4 is perhaps unattractive in that $b_{\boldsymbol{c}}(n)$ involves the Möbius function $\mu(n)$. We might thus wish to pass from $b_{\boldsymbol{c}}(n)$ to $a_{\boldsymbol{c}}(n)$. This is possible, to some extent, in the situation of the following definition:

Definition 2.5. Call Ψ *standard* if for all $\boldsymbol{c} \in \mathcal{S}$, integers $n \geqslant 1$, and reals $\epsilon > 0$, we have

$$\max(|b_{\boldsymbol{c}}(n)|, |a_{\boldsymbol{c}}(n)|) \ll_\epsilon n^\epsilon.$$

Let $\vartheta \in \{0, 1\}$ if Ψ is standard, and let $\vartheta := 0$ if Ψ is non-standard. Let

$$\gamma_{\boldsymbol{c}}(n) := (1 - \vartheta) \cdot b_{\boldsymbol{c}}(n) + \vartheta \cdot \mu(n)^2 a_{\boldsymbol{c}}(n). \tag{2.11}$$

We now come to our main hypothesis: a large sieve inequality for $\gamma_{\boldsymbol{c}}$, in a certain range.

Hypothesis 2.6. For all reals $C, N, \epsilon > 0$ with $N \leqslant C^3$, we have

$$\sum_{\boldsymbol{c} \in \mathcal{S}(C)} \left| \sum_{n \leqslant N} v_n \gamma_{\boldsymbol{c}}(n) \right|^2 \ll_\epsilon C^\epsilon \max(C^m, N) \sum_{n \leqslant N} |v_n|^2 \tag{2.12}$$

for all vectors $(v_n)_{1 \leqslant n \leqslant N} \in \mathbb{C}^{\lfloor N \rfloor}$.

Again, some brief remarks may be helpful.

(1) When $\vartheta = 1$, the factor $\mu(n)^2$ in Equation (2.11) simply restricts us to square-free moduli n.
(2) Hypothesis 2.6 remains open in general [21, Remark 4.1.10].

Results

Fix a smooth, compactly supported function $w : \mathbb{R}^m \to \mathbb{R}$. Assume that

$$\boldsymbol{0} \notin \overline{\{\boldsymbol{x} \in \mathbb{R}^m : w(\boldsymbol{x}) \neq 0\}}. \tag{2.13}$$

For reals $X \geqslant 1$, let

$$N_{F,w}(X) := \sum_{\boldsymbol{x} \in \mathbb{Z}^m} w(\boldsymbol{x}/X)\, \mathbf{1}_{F(\boldsymbol{x})=0}. \tag{2.14}$$

If $m \geqslant 5$, then let $N'_{F,w}(X) := N_{F,w}(X)$. If $m = 4$, then let Υ denote the set of two-dimensional rational vector spaces L with $F|_L = 0$, and let

$$N'_{F,w}(X) := \sum_{\boldsymbol{x} \in \mathbb{Z}^m \setminus (\bigcup_{L \in \Upsilon} L)} w(\boldsymbol{x}/X)\, \mathbf{1}_{F(\boldsymbol{x})=0}. \tag{2.15}$$

Theorem 2.7. *Assume Hypothesis 2.6 or Hypothesis 2.4. Then for some constant $\mathfrak{c}(F, w) \in \mathbb{R}$, we have*

$$N'_{F,w}(X) - \mathfrak{c}(F, w) X^{m-3} \ll_\epsilon X^{3(m-2)/4+\epsilon}, \tag{2.16}$$

for all reals $X \geqslant 1$ and $\epsilon > 0$.

Note that m, F, w are fixed. In other words, the implied constant in Equation (2.16) is allowed to depend on m, F, w in addition to ϵ. Also, for numerical reference,

$$3(m-2)/4 = 1.5 \cdot \mathbf{1}_{m=4} + 2.25 \cdot \mathbf{1}_{m=5} + 3 \cdot \mathbf{1}_{m=6} + \cdots.$$

In particular, if $5 \leqslant m \leqslant 6$, then $m-3 \leqslant 3(m-2)/4$, and Equation (2.16) simply says

$$N_{F,w}(X) \ll_\epsilon X^{3(m-2)/4+\epsilon}.$$

The rest of the paper is devoted to the proof of Theorems 1.1, 2.3, and 2.7. In Section 3, we reduce Hypothesis 2.4 to Hypothesis 2.6. In Sections 4–7, we recall the delta method for $N_{F,w}(X)$, then analyze parts of it unconditionally and parts of it using Hypothesis 2.4. In Section 8, we tie together the previous sections to complete the proofs.

3 | A CONVERSION BETWEEN STANDARD COEFFICIENTS

In this section, we prove a useful consequence of Hypothesis 2.6. First, we record some standard lemmas that will be repeatedly used throughout the paper.

Lemma 3.1. *Let $N, h \in \mathbb{Z}_{\geqslant 1}$. Then, there are at most $O_h(N^{1/h})$ integers $n \in [N, 2N)$ such that $v_p(n) \geqslant h$ holds for all primes $p \mid n$.*

Proof. This is classical; see, for example, [1]. □

To proceed, we need to introduce some notation. We write $u \mid v^\infty$ if there exists $k \in \mathbb{Z}_{\geqslant 1}$ with $u \mid v^k$. For an integer $c \neq 0$, we let $\mathrm{sq}(c)$ (resp. $\mathrm{cub}(c)$) denote the largest square-full (resp. cube-full) positive integer divisor of c. We also let $\mathrm{sq}(0) := 0$.

Lemma 3.2. *Let $N, R \in \mathbb{Z}_{\geqslant 1}$. Then, there are at most $O_\epsilon(N^\epsilon R^\epsilon)$ positive integers $n \leqslant N$ with $n \mid R^\infty$.*

Proof. We have $\sum_{n \mid R^\infty} \mathbf{1}_{n \leqslant N} \leqslant \sum_{n \mid R^\infty} (N/n)^\epsilon = N^\epsilon \prod_{p \mid R} (1-p^{-\epsilon})^{-1} \ll_\epsilon N^\epsilon R^\epsilon$. □

Lemma 3.3. *Let $N \in \mathbb{Z}_{\geqslant 1}$. Then, the following hold:*

(1) *We have*

$$\sum_{n \leqslant N:\, n=\mathrm{sq}(n)} n^{-1/2} \ll_\epsilon N^\epsilon.$$

(2) *We have*

$$\sum_{|c| \leqslant N} \mathrm{sq}(c)^{1/2} \ll_\epsilon N^{1+\epsilon}.$$

(3) *For any $t \in \mathbb{R}$, we have*

$$\sum_{1 \leqslant n \leqslant N} \mathrm{cub}(n)^t \ll_{t,\epsilon} N^\epsilon \max(N, N^{1/3+t}).$$

Proof.

(1): By the $h = 2$ case of Lemma 3.1 in dyadic intervals $n \in [2^k, 2^{k+1})$, we have

$$\sum_{n \leqslant N:\, n=\mathrm{sq}(n)} n^{-1/2} \ll \sum_{0 \leqslant k \leqslant \log_2 N} (2^k)^{1/2}(2^k)^{-1/2} \ll_\epsilon N^\epsilon.$$

(2): There are at most N/d positive integers $n \leqslant N$ with $\mathrm{sq}(n) = d$. Therefore,

$$\sum_{|c| \leqslant N} \mathrm{sq}(c)^{1/2} = 2 \sum_{1 \leqslant n \leqslant N} \mathrm{sq}(n)^{1/2} \leqslant 2 \sum_{d \leqslant N:\, d=\mathrm{sq}(d)} \frac{N}{d} \cdot d^{1/2} \ll_\epsilon N^{1+\epsilon},$$

where the last inequality follows from Equation (1).

(3): There are at most N/n_3 positive integers $n \leqslant N$ with $\mathrm{cub}(n) = n_3$. Thus

$$\sum_{1 \leqslant n \leqslant N} \mathrm{cub}(n)^t \leqslant \sum_{n_3 \leqslant N:\, n_3=\mathrm{cub}(n_3)} \frac{N}{n_3} \cdot n_3^t$$

$$\ll_t \sum_{0 \leqslant k \leqslant \log_2 N} (2^k)^{1/3}(2^k)^{t-1} N \ll_{t,\epsilon} N^\epsilon \max(N, N^{1/3+t}),$$

by the $h = 3$ case of Lemma 3.1 in dyadic intervals $n_3 \in [2^k, 2^{k+1})$. □

Proposition 3.4. *Fix an approximation Ψ of Φ. Assume Hypothesis 2.6. Then, Hypothesis 2.4 holds.*

Proof. First, suppose $\vartheta = 0$. Then, $\gamma_{\boldsymbol{c}} = b_{\boldsymbol{c}}$ by Equation (2.11). For C, N, I as in Hypothesis 2.4, the bound (2.12) with $v_n := \mathbf{1}_{n \in I}$ thus trivially implies Equation (2.10), as desired.

Now, suppose $\vartheta = 1$. Then in particular, Ψ is standard. For the rest of the proof, let C, d, N denote *positive* variables. For integers d and intervals I, let

$$A_{\boldsymbol{c}}(d, I) := \sum_{n \in I} \mathbf{1}_{\gcd(d,n)=1}\, \mu(n) a_{\boldsymbol{c}}(n).$$

We have $\gamma_{\boldsymbol{c}}(n) = \mu(n)^2 a_{\boldsymbol{c}}(n)$ by Equation (2.11). Taking $v_n := \mathbf{1}_{n\in I}\mathbf{1}_{\gcd(d,n)=1}\, \mu(n)$ in Equation (2.12), and observing that $\mu(n)^3 = \mu(n)$, we find that Hypothesis 2.6 implies

$$\sum_{\boldsymbol{c} \in S(C)} |A_{\boldsymbol{c}}(d, I)|^2 \ll_\epsilon C^\epsilon \max(C^m, N) N \tag{3.1}$$

uniformly over reals C, integers d, reals $N \leqslant C^3$, and real intervals $I \subseteq (0, N]$.

To proceed, we rewrite $b_{\boldsymbol{c}}(n)$ using multiplicativity. First, by Equation (2.9), for primes p we have

$$b_{\boldsymbol{c}}(p) = -a_{\boldsymbol{c}}(p).$$

Furthermore, an integer $n \geqslant 1$ can be *uniquely* expressed in the form $n_1 d$, where d is square-full, n_1 is coprime to d, and n_1 is square-free. Therefore, for all $n \geqslant 1$, we have

$$b_{\boldsymbol{c}}(n) = \sum_{n_1 d = n} \mathbf{1}_{\gcd(d,n_1)=1}\, \mu(n_1) a_{\boldsymbol{c}}(n_1) \cdot \mathbf{1}_{d=\mathrm{sq}(d)}\, b_{\boldsymbol{c}}(d). \tag{3.2}$$

We note here that $\mu(n_1)$ is supported on square-free integers n_1.

Consider a real C, a real $N \leqslant C^3$, and a real interval $I \subseteq (0, N]$. Let $B_{\boldsymbol{c}}(I) := \sum_{n \in I} b_{\boldsymbol{c}}(n)$. By Equation (3.2), we have

$$\begin{aligned} B_{\boldsymbol{c}}(I) &= \sum_{n_1 d \in I} \mathbf{1}_{\gcd(d,n_1)=1}\, \mu(n_1) a_{\boldsymbol{c}}(n_1) \cdot \mathbf{1}_{d=\mathrm{sq}(d)}\, b_{\boldsymbol{c}}(d) \\ &= \sum_{d \leqslant N:\, d=\mathrm{sq}(d)} b_{\boldsymbol{c}}(d) \cdot A_{\boldsymbol{c}}(d, I/d). \end{aligned}$$

By the Cauchy–Schwarz inequality over d, it follows that

$$\begin{aligned} \sum_{\boldsymbol{c} \in \mathcal{S}(C)} |B_{\boldsymbol{c}}(I)|^2 &\leqslant \sum_{\boldsymbol{c} \in \mathcal{S}(C)} \left(\sum_{d \leqslant N:\, d=\mathrm{sq}(d)} |b_{\boldsymbol{c}}(d)|^2 d^{-1/2} \right) \left(\sum_{d \leqslant N:\, d=\mathrm{sq}(d)} d^{1/2} |A_{\boldsymbol{c}}(d, I/d)|^2 \right) \\ &\ll_\epsilon N^\epsilon \sum_{\boldsymbol{c} \in \mathcal{S}(C)} \sum_{d \leqslant N:\, d=\mathrm{sq}(d)} d^{1/2} |A_{\boldsymbol{c}}(d, I/d)|^2, \end{aligned} \tag{3.3}$$

by Lemma 3.3(1), since $b_{\boldsymbol{c}}(d) \ll_\epsilon d^\epsilon$ by Definition 2.5. Yet for all integers d, we have

$$\sum_{\boldsymbol{c} \in \mathcal{S}(C)} |A_{\boldsymbol{c}}(d, I/d)|^2 \ll_\epsilon C^\epsilon \max(C^m, N/d)\,(N/d)$$

by Equation (3.1), since $N/d \leqslant N \leqslant C^3$ and $I/d \subseteq (0, N/d]$. Plugging this into Equation (3.3), we get

$$\begin{aligned} \sum_{\boldsymbol{c} \in \mathcal{S}(C)} |B_{\boldsymbol{c}}(I)|^2 &\ll_\epsilon N^\epsilon \sum_{d \leqslant N:\, d=\mathrm{sq}(d)} d^{1/2} [C^\epsilon \max(C^m, N/d)\,(N/d)] \\ &\ll_\epsilon N^{2\epsilon} C^\epsilon \max(C^m, N)\, N, \end{aligned}$$

where the second inequality follows from Lemma 3.3(1) and the trivial bound $\max(C^m, N/d) \leqslant \max(C^m, N)$. Thus, Equation (2.10) holds, uniformly over C, N, I. □

4 | DELTA METHOD INGREDIENTS

Let $X \geqslant 1$. Assume Equation (2.13), that is, that w is supported away from $\mathbf{0} \in \mathbb{R}^m$. Such an assumption is implicit in some of the integral estimates in [7, 8]. Set

$$Y := X^{(\deg F)/2} = X^{3/2}. \tag{4.1}$$

Fix $\epsilon_0 \in (0, 10^{-10}]$ and set

$$Z := Y/X^{1-\epsilon_0} = X^{1/2+\epsilon_0}. \tag{4.2}$$

Let $\varrho_0(x) := \exp(-(1-x^2)^{-1})$ for $|x| < 1$, and $\varrho_0(x) := 0$ for $|x| \geqslant 1$. Let

$$\varrho(x) := \frac{4\varrho_0(4x-3)}{\int_{y \in \mathbb{R}} \varrho_0(y)\, dy}.$$

For $x > 0$ and $y \in \mathbb{R}$, let

$$h(x,y) := \sum_{j \geqslant 1} \frac{1}{xj}\left(\varrho(xj) - \varrho\left(\frac{|y|}{xj}\right)\right).$$

This is precisely the function $h(x,y)$ defined in [7, Section 3]. For $\boldsymbol{c} \in \mathbb{Z}^m$ and $n > 0$, let

$$I_{\boldsymbol{c}}(n) := \int_{\boldsymbol{x} \in \mathbb{R}^m} w(\boldsymbol{x}/X)\, h(n/Y, F(\boldsymbol{x})/Y^2)\, \mathrm{e}^{-2\pi \mathrm{i}(\boldsymbol{c}\cdot\boldsymbol{x}/n)}\, d\boldsymbol{x}.$$

Let $\|\boldsymbol{c}\| := \max_{1 \leqslant i \leqslant m}(|c_i|)$. We now recall two standard results on the integral $I_{\boldsymbol{c}}(n)$.

Proposition 4.1 [7, par. 1 of Section 7]. *The functions $n \mapsto I_{\boldsymbol{c}}(n)$ are supported on a range of the form $n \leqslant M_0(F,w)Y$, uniformly over $\boldsymbol{c} \in \mathbb{Z}^m$, for some constant $M_0(F,w) > 0$.*

Lemma 4.2 [8, (3.9)]. *If $\|\boldsymbol{c}\| \geqslant Z$ and $n \geqslant 1$, then $I_{\boldsymbol{c}}(n) \ll_{\epsilon_0,A} \|\boldsymbol{c}\|^{-A}$, for all $A > 0$.*

Proposition 4.1 and Lemma 4.2, together with the trivial bound $|S_{\boldsymbol{c}}(n)| \leqslant n^{1+m}$, imply

$$Y^{-2} \sum_{n \geqslant 1} \sum_{\|\boldsymbol{c}\| > Z} n^{-m} |S_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n)| \ll_{\epsilon_0,A} X^{-A}, \tag{4.3}$$

for all $A > 0$. Here, $S_{\boldsymbol{c}}(n)$ is defined as in Equation (2.8). By [7, Theorem 2, (1.2)], we have

$$(1 + O_A(Y^{-A}))\, N_{F,w}(X) = Y^{-2} \sum_{n \geqslant 1} \sum_{\boldsymbol{c} \in \mathbb{Z}^m} n^{-m} S_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n). \tag{4.4}$$

Equivalently, in terms of $S^{\natural}_{\boldsymbol{c}}(n)$, we have

$$(1 + O_A(X^{-A}))\, N_{F,w}(X) = X^{-3} \sum_{n \geqslant 1} \sum_{\boldsymbol{c} \in \mathbb{Z}^m} n^{(1-m)/2} S^{\natural}_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n). \tag{4.5}$$

In view of Equation (4.3), analyzing $N_{F,w}(X)$ reduces to understanding the quantity

$$\Sigma_0 := X^{-3} \sum_{n \geqslant 1} \sum_{\boldsymbol{c} \in [-Z,Z]^m} n^{(1-m)/2} S^{\natural}_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n). \tag{4.6}$$

(Here, $I_{\boldsymbol{c}}(n) = I_{\boldsymbol{c}}(n)\, \mathbf{1}_{n \leqslant M_0(F,w)Y}$. But it is more convenient to keep the factor $\mathbf{1}_{n \leqslant M_0(F,w)Y}$ implicit, in order to allow for more flexible technique later on.)

We now recall some standard formulas for $S_{\boldsymbol{c}}$ at primes p and prime powers p^l.

Proposition 4.3. *Say $p \nmid \boldsymbol{c}$. Then $S^{\natural}_{\boldsymbol{c}}(p) = E^{\natural}_{\boldsymbol{c}}(p) + O(p^{-1/2})$.*

Proof. Let

$$E(p) := \frac{|\{\boldsymbol{x} \in \mathbb{F}_p^m : F(\boldsymbol{x}) = 0\}| - p^{m-1}}{p-1}.$$

By [8, p. 680], we have $S_{\boldsymbol{c}}(p) = p^2 E_{\boldsymbol{c}}(p) - pE(p)$ and $E(p) \ll p^{(m-2)/2}$. Thus,

$$S_{\boldsymbol{c}}(p) = p^2 E_{\boldsymbol{c}}(p) + O(p^{m/2}).$$

Now divide by $p^{(m+1)/2}$. □

Proposition 4.4. *Say $p \nmid \Delta(\boldsymbol{c})$. Then, $S_{\boldsymbol{c}}(p^l) = 0$ for all integers $l \geqslant 2$.*

Proof. This follows immediately from [8, Lemma 4.4]. □

Fix an approximation Ψ of Φ. Recall the definition of $\mathcal{S}$ from Equation (2.2). For each $\boldsymbol{c} \in \mathcal{S}$, we have $S^{\natural}_{\boldsymbol{c}} = a'_{\boldsymbol{c}} * b_{\boldsymbol{c}}$ by Equation (2.9). The following result controls the coefficients $a'_{\boldsymbol{c}}$ and $b_{\boldsymbol{c}}$.

Proposition 4.5. *Let $\boldsymbol{c} \in \mathcal{S}$. Then $a'_{\boldsymbol{c}}(n)$ is multiplicative in n. Moreover, for all primes p and integers $k \geqslant 1$, we have*

$$a'_{\boldsymbol{c}}(p) \cdot \mathbf{1}_{p \nmid \Delta(\boldsymbol{c})} \ll p^{-1/2},$$
$$\max(|a'_{\boldsymbol{c}}(p^k)|, |b_{\boldsymbol{c}}(p^k)|) \ll_{\epsilon} p^{k\epsilon} + p^{k\epsilon} \sum_{d \mid p^k} |S^{\natural}_{\boldsymbol{c}}(d)| \cdot \mathbf{1}_{p \mid \Delta(\boldsymbol{c})}.$$

Proof. By Equation (2.9), we have $(a_{\boldsymbol{c}} * b_{\boldsymbol{c}})(n) = \mathbf{1}_{n=1}$ and $a'_{\boldsymbol{c}} = S^{\natural}_{\boldsymbol{c}} * a_{\boldsymbol{c}}$. Since $b_{\boldsymbol{c}}, S^{\natural}_{\boldsymbol{c}}$ are multiplicative, it follows that $a_{\boldsymbol{c}}, a'_{\boldsymbol{c}}$ are too. It remains to bound $a'_{\boldsymbol{c}}(p^k), b_{\boldsymbol{c}}(p^k)$. When $p \mid \Delta(\boldsymbol{c})$, there is nothing to prove, since condition (2) in Definition 2.2 already gives what we want. Now assume $p \nmid \Delta(\boldsymbol{c})$. Then condition (3) in Definition 2.2 gives $a'_{\boldsymbol{c}}(p) \ll p^{-1/2}$. On the other hand, $E^{\natural}_{\boldsymbol{c}}(p) \ll 1$ by Equations (2.7) and (2.6). Therefore, condition (2) in Definition 2.2 gives

$$b_{\boldsymbol{c}}(p^k), a'_{\boldsymbol{c}}(p^k) \ll_{\epsilon} p^{k\epsilon} \sum_{d \mid p^k} |S^{\natural}_{\boldsymbol{c}}(d)| \ll p^{k\epsilon},$$

because $S^{\natural}_{\boldsymbol{c}}(p) = E^{\natural}_{\boldsymbol{c}}(p) + O(p^{-1/2}) \ll 1$ by Proposition 4.3 and $S^{\natural}_{\boldsymbol{c}}(p^l) \cdot \mathbf{1}_{l \geqslant 2} = 0$ by Proposition 4.4. This completes the proof. □

Let $\omega(n)$ denote the number of distinct prime factors of n. The following result, which is due to [8, 9], gives a general pointwise bound on $S^{\natural}_{\boldsymbol{c}}(n)$.

Proposition 4.6. *For some constant $A_F > 0$, we have*

$$n^{-1/2} |S^{\natural}_{\boldsymbol{c}}(n)| \leqslant A_F^{\omega(n)} \prod_{1 \leqslant i \leqslant m} \gcd\left(\mathrm{cub}(n)^2, \gcd(\mathrm{cub}(n), \mathrm{sq}(c_i))^3\right)^{1/12}$$

for all $\boldsymbol{c} \in \mathbb{Z}^m$ and integers $n \geqslant 1$.

Proof. By definition, $S^{\natural}_{\boldsymbol{c}}(n) = n^{-(m+1)/2} S_{\boldsymbol{c}}(n)$. Moreover, since F is diagonal, we have

$$S_{\boldsymbol{c}}(p^l) \ll_F p^{l(1+m/2)} \prod_{1 \leqslant i \leqslant m} \gcd\left(\mathrm{cub}(p^l)^2, \gcd(\mathrm{cub}(p^l), \mathrm{sq}(c_i))^3\right)^{1/12},$$

by [8, (5.1) and (5.2)] for $l \geqslant 2$ and [6, Lemma 11] for $l = 1$. The desired result follows immediately from the multiplicativity of $S_{\boldsymbol{c}}$. □

We have stated Proposition 4.6 uniformly over $\boldsymbol{c} \in \mathbb{Z}^m$. We proceed to analyze the vectors $\boldsymbol{c}$ in sets based on which coordinates c_i are nonzero. For the rest of Section 4, we fix a set

$$\mathcal{I} \subseteq \{1, 2, \dots, m\}. \tag{4.7}$$

Let

$$\mathcal{R} := \{\boldsymbol{c} \in \mathbb{Z}^m \cap [-Z, Z]^m : \mathbf{1}_{c_i \neq 0} = \mathbf{1}_{i \in \mathcal{I}} \text{ for all } i \in \{1, 2, \dots, m\}\}. \tag{4.8}$$

By definition, if $\boldsymbol{c} \in \mathcal{R}$, then $c_i \neq 0$ if and only if $i \in \mathcal{I}$.

Proposition 4.6 implies that for all $\boldsymbol{c} \in \mathcal{R}$ and integers $n \geqslant 1$, we have

$$n^{-1/2} S^{\natural}_{\boldsymbol{c}}(n) \ll_\epsilon n^\epsilon \operatorname{cub}(n)^{(m-|\mathcal{I}|)/6} \prod_{i \in \mathcal{I}} \gcd(\operatorname{cub}(n), \operatorname{sq}(c_i))^{1/4}. \tag{4.9}$$

We will repeatedly use Equation (4.9) later in the present paper. We now turn to $I_{\boldsymbol{c}}(n)$.

Lemma 4.7 [7, 8]. *Assume* $|\mathcal{I}| \geqslant 1$. *Then uniformly over* $\boldsymbol{c} \in \mathcal{R}$, *reals* $n \geqslant 1$, *and integers* $k \in \{0, 1\}$, *we have*

$$n^k (\partial/\partial n)^k I_{\boldsymbol{c}}(n) \ll_{k,\epsilon} X^{m+\epsilon} \left(\frac{X\|\boldsymbol{c}\|}{n}\right)^{1-(m+|\mathcal{I}|)/4} \prod_{i \in \mathcal{I}} \left(\frac{\|\boldsymbol{c}\|}{|c_i|}\right)^{1/2}.$$

Proof. By [8, Lemma 3.2], since F is diagonal, we have

$$\begin{aligned} n^k (\partial/\partial n)^k I_{\boldsymbol{c}}(n) &\ll_{k,\epsilon} \left(\frac{X\|\boldsymbol{c}\|}{n}\right) X^{m+\epsilon} \prod_{1 \leqslant i \leqslant m} \min\left[\left(\frac{n}{X|c_i|}\right)^{1/2}, \left(\frac{n}{X\|\boldsymbol{c}\|}\right)^{1/4}\right] \\ &\leqslant \left(\frac{X\|\boldsymbol{c}\|}{n}\right) X^{m+\epsilon} \prod_{i \in \mathcal{I}} \left(\frac{n}{X|c_i|}\right)^{1/2} \prod_{i \notin \mathcal{I}} \left(\frac{n}{X\|\boldsymbol{c}\|}\right)^{1/4}. \end{aligned} \tag{4.10}$$

After writing $(\frac{n}{X|c_i|})^{1/2} = (\frac{n}{X\|\boldsymbol{c}\|})^{1/2} (\frac{\|\boldsymbol{c}\|}{|c_i|})^{1/2}$ in the final line of Equation (4.10), the desired inequality follows from the fact that $1 - |\mathcal{I}|/2 - (m - |\mathcal{I}|)/4 = 1 - (m + |\mathcal{I}|)/4$. □

For later convenience, we now make a definition: for $\boldsymbol{c} \in \mathbb{Z}^m$ and integers $N \geqslant 1$, let

$$\|I_{\boldsymbol{c}}\|_{1,\infty;N} := \sup_{n \in \mathbb{R}:\, N \leqslant n \leqslant 4N} (|I_{\boldsymbol{c}}(n)| + |n(\partial/\partial n) I_{\boldsymbol{c}}(n)|). \tag{4.11}$$

In the rest of Section 4, we will concern ourselves only with $\boldsymbol{c} \in \mathcal{R}$ such that $\Delta(\boldsymbol{c}) \neq 0$. If $|\mathcal{I}| = 0$, then no such $\boldsymbol{c}$ exist, because $\mathcal{R} = \{\mathbf{0}\}$ by Equation (4.8). Therefore, we may and do assume $|\mathcal{I}| \geqslant 1$ for the rest of Section 4. To proceed further, we break $\mathcal{R}$ into dyadic pieces. For each $i \in \mathcal{I}$, let

$C_i \in \{2^t : t \in \mathbb{Z}_{\geqslant 0}\}$ with $1 \leqslant C_i \leqslant Z$. Write

$$\mathcal{C} := \{\mathbf{c} \in \mathcal{R} : |c_i| \in [C_i, 2C_i) \text{ for all } i \in \mathcal{I}\}, \qquad C := \max_{i \in \mathcal{I}}(C_i). \tag{4.12}$$

Proposition 4.8. *Suppose $N_0 \in \mathbb{Z}_{\geqslant 1}$ and $N_0 \ll X^{O(1)}$. Then*

$$\sum_{\mathbf{c} \in \mathcal{C} : \Delta(\mathbf{c}) \neq 0} \left(\sum_{n_0 \in [N_0, 2N_0)} |a'_{\mathbf{c}}(n_0)| \right)^2 \ll_\epsilon X^\epsilon N_0^{1+(m-|\mathcal{I}|)/3} \prod_{i \in \mathcal{I}} C_i.$$

Proof. Consider an integer $n_0 \in [N_0, 2N_0)$. If $n_{\mathbf{c}} := \prod_{p | \Delta(\mathbf{c})} p^{v_p(n_0)}$ and $n_2 := \mathrm{sq}(n_0/n_{\mathbf{c}})$, then Proposition 4.5 implies

$$\begin{aligned} a'_{\mathbf{c}}(n_0) &= a'_{\mathbf{c}}\left(\frac{n_0}{n_{\mathbf{c}} n_2}\right) \cdot a'_{\mathbf{c}}(n_2) \cdot a'_{\mathbf{c}}(n_{\mathbf{c}}) \\ &\ll_\epsilon \left(\frac{n_0}{n_{\mathbf{c}} n_2}\right)^{-1/2+\epsilon} \cdot n_2^\epsilon \cdot |a'_{\mathbf{c}}(n_{\mathbf{c}})| \\ &\leqslant n_0^{-1/2+\epsilon} (n_{\mathbf{c}} n_2)^{1/2} |a'_{\mathbf{c}}(n_{\mathbf{c}})|. \end{aligned}$$

Since $n_{\mathbf{c}} \mid \Delta(\mathbf{c})^\infty$ and n_2 is square-full, we find, upon summing over n_0, that

$$\begin{aligned} \sum_{n_0 \in [N_0, 2N_0)} |a'_{\mathbf{c}}(n_0)| &\ll_\epsilon \sum_{\substack{n_{\mathbf{c}} n_2 \leqslant 2N_0 : \\ n_{\mathbf{c}} | \Delta(\mathbf{c})^\infty,\ n_2 = \mathrm{sq}(n_2)}} \frac{N_0}{n_{\mathbf{c}} n_2} \cdot N_0^{-1/2+\epsilon} (n_{\mathbf{c}} n_2)^{1/2} |a'_{\mathbf{c}}(n_{\mathbf{c}})| \\ &\ll_\epsilon N_0^{1/2+2\epsilon} \sum_{\substack{n_{\mathbf{c}} \leqslant 2N_0 : \\ n_{\mathbf{c}} | \Delta(\mathbf{c})^\infty}} n_{\mathbf{c}}^{-1/2} |a'_{\mathbf{c}}(n_{\mathbf{c}})| \\ &\ll_\epsilon N_0^{1/2+2\epsilon} (N_0 C)^\epsilon \max_{\substack{n_{\mathbf{c}} \leqslant 2N_0 : \\ n_{\mathbf{c}} | \Delta(\mathbf{c})^\infty}} n_{\mathbf{c}}^{-1/2} |a'_{\mathbf{c}}(n_{\mathbf{c}})|, \end{aligned}$$

where we have used Lemma 3.3(1) to sum over $n_2 \leqslant 2N_0/n_{\mathbf{c}}$, and then used Lemma 3.2 to bound the sum over $n_{\mathbf{c}}$ by a maximum. Furthermore,

$$\max_{\substack{n_{\mathbf{c}} \leqslant 2N_0 : \\ n_{\mathbf{c}} | \Delta(\mathbf{c})^\infty}} n_{\mathbf{c}}^{-1/2} |a'_{\mathbf{c}}(n_{\mathbf{c}})| \ll_\epsilon N_0^{2\epsilon} \max_{\substack{d \leqslant 2N_0 : \\ d | \Delta(\mathbf{c})^\infty}} d^{-1/2} |S^\natural_{\mathbf{c}}(d)|,$$

since $a'_{\mathbf{c}}(n_{\mathbf{c}}) \ll_\epsilon n_{\mathbf{c}}^\epsilon \sum_{d | n_{\mathbf{c}}} |S^\natural_{\mathbf{c}}(d)|$ by condition (2) in Definition 2.2. But

$$\begin{aligned} \sum_{\mathbf{c} \in \mathcal{C} : \Delta(\mathbf{c}) \neq 0} \max_{\substack{d \leqslant 2N_0 : \\ d | \Delta(\mathbf{c})^\infty}} d^{-1} |S^\natural_{\mathbf{c}}(d)|^2 &\leqslant \sum_{\mathbf{c} \in \mathcal{C}} \max_{d \leqslant 2N_0} d^{-1} |S^\natural_{\mathbf{c}}(d)|^2 \\ &\ll_\epsilon N_0^{(m-|\mathcal{I}|)/3+2\epsilon} \sum_{\mathbf{c} \in \mathcal{C}} \prod_{i \in \mathcal{I}} \mathrm{sq}(c_i)^{1/2} \end{aligned}$$

by Equation (4.9), since $\gcd(\mathrm{cub}(d), \mathrm{sq}(c_i))^{1/4} \leqslant \mathrm{sq}(c_i)^{1/4}$. Yet

$$\sum_{\boldsymbol{c}\in\mathcal{C}} \prod_{i\in\mathcal{I}} \mathrm{sq}(c_i)^{1/2} \ll_\epsilon \prod_{i\in\mathcal{I}} C_i^{1+\epsilon}, \tag{4.13}$$

by Lemma 3.3(2). Proposition 4.8 follows upon combining the previous four displays. □

We are now prepared to prove a crucial bound for Section 5.

Lemma 4.9. *Suppose $N_0, N \in \mathbb{Z}_{\geqslant 1}$ and $N_0, N \ll X^{O(1)}$. Let*

$$Q_{\boldsymbol{c}} = \|I_{\boldsymbol{c}}\|_{1,\infty;N} \sum_{n_0\in[N_0,2N_0)} |a'_{\boldsymbol{c}}(n_0)|.$$

Then

$$\left(\sum_{\boldsymbol{c}\in\mathcal{R}:\,\Delta(\boldsymbol{c})\neq 0} Q_{\boldsymbol{c}}^2\right)^{1/2} \ll_\epsilon X^{m+\epsilon} N_0^{1/2+(m-|\mathcal{I}|)/6} (X/N)^{1-(m+|\mathcal{I}|)/4} \max[Z^{1+(|\mathcal{I}|-m)/4}, 1].$$

Proof. With notation as in Proposition 4.8, consider an element $\boldsymbol{c} \in \mathcal{C}$. Then by Equation (4.12), we have $|c_i| \asymp C_i$ for all $i \in \mathcal{I}$, whence $\|\boldsymbol{c}\| \asymp C$. Now Equation (4.11) and Lemma 4.7 imply

$$\|I_{\boldsymbol{c}}\|_{1,\infty;N} \ll_\epsilon X^{m+\epsilon} (XC/N)^{1-(m+|\mathcal{I}|)/4} \prod_{i\in\mathcal{I}} (C/C_i)^{1/2},$$

since $|\mathcal{I}| \geqslant 1$. By Proposition 4.8, it follows that

$$\sum_{\boldsymbol{c}\in\mathcal{C}:\,\Delta(\boldsymbol{c})\neq 0} Q_{\boldsymbol{c}}^2 \ll_\epsilon X^{2m+3\epsilon} N_0^{1+(m-|\mathcal{I}|)/3} (XC/N)^{2-(m+|\mathcal{I}|)/2} \prod_{i\in\mathcal{I}} C.$$

By Equation (4.12) we have $1 \leqslant C \leqslant Z$, since $1 \leqslant C_i \leqslant Z$ for all i. The quantity $C^{2-(m+|\mathcal{I}|)/2} \prod_{i\in\mathcal{I}} C = C^{2+(|\mathcal{I}|-m)/2}$ is maximized either at $C = Z$ or $C = 1$, so we conclude that

$$\sum_{\boldsymbol{c}\in\mathcal{C}:\,\Delta(\boldsymbol{c})\neq 0} Q_{\boldsymbol{c}}^2 \ll_\epsilon X^{2m+3\epsilon} N_0^{1+(m-|\mathcal{I}|)/3} (X/N)^{2-(m+|\mathcal{I}|)/2} \max[Z^{2+(|\mathcal{I}|-m)/2}, 1].$$

Summing over all possibilities for $\mathcal{C}$, we get

$$\sum_{\boldsymbol{c}\in\mathcal{R}:\,\Delta(\boldsymbol{c})\neq 0} Q_{\boldsymbol{c}}^2 \ll_\epsilon X^{2m+4\epsilon} N_0^{1+(m-|\mathcal{I}|)/3} (X/N)^{2-(m+|\mathcal{I}|)/2} \max[Z^{2+(|\mathcal{I}|-m)/2}, 1].$$

Lemma 4.9 follows upon taking a square root. □

Having analyzed $I_{\boldsymbol{c}}$ and $a'_{\boldsymbol{c}}$ above, we now concentrate on $b_{\boldsymbol{c}}$ for the rest of Section 4.

Proposition 4.10. *Let the C_i, as well as C and $\mathcal{C}$, be as specified before Proposition 4.8. Suppose $N_1 \in \mathbb{Z}_{\geqslant 1}$ and $N_1 \ll X^{O(1)}$. Then*

$$\sum_{\boldsymbol{c}\in\mathcal{C}:\,\Delta(\boldsymbol{c})\neq 0}\left(\sum_{n_1\in[N_1,2N_1]}|b_{\boldsymbol{c}}(n_1)|\right)^2 \ll_\epsilon X^\epsilon N_1^{\max(2,1+(m-|\mathcal{I}|)/3)}\prod_{i\in\mathcal{I}}C_i.$$

Proof. We mimic the proof of Proposition 4.8. Consider an integer $n_1 \in [N_1, 2N_1]$. If $n_{\boldsymbol{c}} := \prod_{p|\Delta(\boldsymbol{c})} p^{v_p(n_1)}$, then by Proposition 4.5 and the multiplicativity of $b_{\boldsymbol{c}}$, we have

$$b_{\boldsymbol{c}}(n_1) = b_{\boldsymbol{c}}(n_1/n_{\boldsymbol{c}})\, b_{\boldsymbol{c}}(n_{\boldsymbol{c}}) \ll_\epsilon (n_1/n_{\boldsymbol{c}})^\epsilon\, |b_{\boldsymbol{c}}(n_{\boldsymbol{c}})| \leqslant n_1^\epsilon\, |b_{\boldsymbol{c}}(n_{\boldsymbol{c}})|.$$

Upon summing over n_1, then,

$$\begin{aligned}\sum_{n_1\in[N_1,2N_1]}|b_{\boldsymbol{c}}(n_1)| &\ll_\epsilon \sum_{n_{\boldsymbol{c}}\leqslant 2N_1:\,n_{\boldsymbol{c}}|\Delta(\boldsymbol{c})^\infty}\frac{N_1}{n_{\boldsymbol{c}}}\cdot N_1^\epsilon\,|b_{\boldsymbol{c}}(n_{\boldsymbol{c}})|\\ &\ll_\epsilon N_1^{1+2\epsilon}C^\epsilon \max_{n\leqslant 2N_1} n^{-1}|b_{\boldsymbol{c}}(n)|\end{aligned}$$

by Lemma 3.2. Condition (2) in Definition 2.2 implies

$$\max_{n\leqslant 2N_1} n^{-1}|b_{\boldsymbol{c}}(n)| \ll_\epsilon N_1^{2\epsilon}\max_{n\leqslant 2N_1} n^{-1}|S^\natural_{\boldsymbol{c}}(n)|.$$

But by Equation (4.9), we have

$$\sum_{\boldsymbol{c}\in\mathcal{C}}\max_{n\leqslant 2N_1} n^{-2}|S^\natural_{\boldsymbol{c}}(n)|^2 \ll_\epsilon N_1^{2\epsilon}\max(1, N_1^{-1+(m-|\mathcal{I}|)/3})\sum_{\boldsymbol{c}\in\mathcal{C}}\prod_{i\in\mathcal{I}}\operatorname{sq}(c_i)^{1/2}.$$

The desired result follows upon combining the last three displays with Equation (4.13). □

Lemma 4.11. *Suppose $N_1 \in \mathbb{Z}_{\geqslant 1}$ and $N_1 \ll X^{O(1)}$. Then*

$$\sum_{\boldsymbol{c}\in\mathcal{R}:\,\Delta(\boldsymbol{c})\neq 0}\left(\sum_{n_1\in[N_1,2N_1]}|b_{\boldsymbol{c}}(n_1)|\right)^2 \ll_\epsilon X^\epsilon N_1^{\max(2,1+(m-|\mathcal{I}|)/3)}Z^{|\mathcal{I}|}.$$

Proof. This follows from Proposition 4.10 upon summing over all possibilities for $\mathcal{C}$. □

We need the following lemma in Section 5. Let

$$\beta := 1 + 10\cdot M_0(F,w) \ll 1. \tag{4.14}$$

Lemma 4.12. *Assume Hypothesis 2.4. Then*

$$\sum_{\boldsymbol{c}\in\mathcal{R}:\,\Delta(\boldsymbol{c})\neq 0}\left|\sum_{n_1\in I} b_{\boldsymbol{c}}(n_1)\right|^2 \ll_\epsilon \min\left(X^\epsilon Z^m N_1, X^\epsilon Z^{|\mathcal{I}|}N_1^{\max(2,1+(m-|\mathcal{I}|)/3)}\right), \tag{4.15}$$

for all positive integers $N_1 \leqslant \beta Y$ and real intervals $I \subseteq [N_1, 2N_1]$.

Proof. The bound $X^\epsilon Z^m N_1$ in Equation (4.15) follows upon applying Equation (2.10) with $C = (2\beta)^{1/3} Z$ and $N = 2N_1$. Meanwhile, $X^\epsilon Z^{|\mathcal{I}|} N_1^{\max(2,1+(m-|\mathcal{I}|)/3)}$ comes from Lemma 4.11. □

5 | CONTRIBUTION FROM SMOOTH HYPERPLANE SECTIONS

Recall the key quantity Σ_0 from Equation (4.6), involving a sum over $\boldsymbol{c} \in [-Z, Z]^m$. In this section, we concentrate on vectors $\boldsymbol{c} \in \mathcal{S}(Z) = \mathcal{S} \cap [-Z, Z]^m$, in the notation of Equation (2.2). Let

$$\Sigma_1 := X^{-3} \sum_{\boldsymbol{c} \in \mathcal{S}(Z)} \sum_{n \geqslant 1} n^{(1-m)/2} S^\natural_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n).$$

We will prove the following result:

Theorem 5.1. *Assume Hypothesis 2.4. Then*

$$\Sigma_1 \ll_{\epsilon_0} X^{3(m-2)/4 + O(\epsilon_0)}. \tag{5.1}$$

For each $n \geqslant 1$, we have $S^\natural_{\boldsymbol{c}}(n) = \sum_{n_0 n_1 = n} a'_{\boldsymbol{c}}(n_0) b_{\boldsymbol{c}}(n_1)$, since $S^\natural_{\boldsymbol{c}} = a'_{\boldsymbol{c}} * b_{\boldsymbol{c}}$ by Equation (2.9). Thus

$$\Sigma_1 = X^{-3} \sum_{\boldsymbol{c} \in \mathcal{S}(Z)} \sum_{n_0 \geqslant 1} a'_{\boldsymbol{c}}(n_0) \sum_{n_1 \geqslant 1} (n_0 n_1)^{(1-m)/2} I_{\boldsymbol{c}}(n_0 n_1) b_{\boldsymbol{c}}(n_1). \tag{5.2}$$

By Proposition 4.1, we have $I_{\boldsymbol{c}}(n) = 0$ when $n > \beta Y/10$, where β is as in Equation (4.14). Thus

$$\Sigma_1 = X^{-3} \sum_{\boldsymbol{c} \in \mathcal{S}(Z)} \sum_{(N_0, N_1) \in \mathcal{A}} \lozenge_{\boldsymbol{c}, N_0, N_1}, \tag{5.3}$$

where

$$\mathcal{A} := \{(N_0, N_1) \in \{2^t : t \in \mathbb{Z}_{\geqslant 0}\}^2 : N_0 N_1 \leqslant \beta Y/10\},$$

$$\lozenge_{\boldsymbol{c}, N_0, N_1} := \sum_{n_0 \in [N_0, 2N_0)} a'_{\boldsymbol{c}}(n_0) \sum_{n_1 \in [N_1, 2N_1)} (n_0 n_1)^{(1-m)/2} I_{\boldsymbol{c}}(n_0 n_1) b_{\boldsymbol{c}}(n_1).$$

For convenience, let $N := N_0 N_1$, let $B_{\boldsymbol{c}}(J) := \sum_{n_1 \in J} b_{\boldsymbol{c}}(n_1)$ for intervals J, and let

$$\heartsuit_{\boldsymbol{c}, n_0, N_1} := \sum_{n_1 \in [N_1, 2N_1)} (n_0 n_1)^{(1-m)/2} I_{\boldsymbol{c}}(n_0 n_1) b_{\boldsymbol{c}}(n_1).$$

Recall $\|I_{\boldsymbol{c}}\|_{1,\infty;N}$ from Equation (4.11). We now have enough notation to state a key lemma:

Lemma 5.2. *Let $(N_0, N_1) \in \mathcal{A}$. Then, there exists a probability measure $\nu = \nu_{N_0, N_1}$, supported on the real interval $[N_1, 2N_1]$, such that for all $\boldsymbol{c} \in \mathcal{S}$ and $n_0 \in \mathbb{Z} \cap [N_0, 2N_0)$, we have*

$$\heartsuit_{\boldsymbol{c}, n_0, N_1} \ll N^{(1-m)/2} \|I_{\boldsymbol{c}}\|_{1,\infty;N} \int_{x \in [N_1, 2N_1]} |B_{\boldsymbol{c}}([N_1, x))| \, d\nu(x). \tag{5.4}$$

Proof. Let $\boldsymbol{c} \in \mathcal{S}$ and $n_0 \in \mathbb{Z} \cap [N_0, 2N_0)$. For brevity, let $I(n) = n^{(1-m)/2} I_{\boldsymbol{c}}(n)$. Then

$$\heartsuit_{\boldsymbol{c},n_0,N_1} = \sum_{n_1 \in [N_1, 2N_1)} I(n_0 n_1) \cdot b_{\boldsymbol{c}}(n_1).$$

By partial summation over n_1, it follows that

$$\begin{aligned} |\heartsuit_{\boldsymbol{c},n_0,N_1}| &\leqslant \|I(r)\|_{L^\infty([N,4N])} \, |B_{\boldsymbol{c}}([N_1, 2N_1))| + n_0 \, \|I'(r)\|_{L^\infty([N,4N])} \sum_{k \in [N_1, 2N_1)} |B_{\boldsymbol{c}}([N_1, k))| \\ &\ll \|I(r)\|_{L^\infty([N,4N])} \, |B_{\boldsymbol{c}}([N_1, 2N_1))| + \frac{N}{N_1} \, \|I'(r)\|_{L^\infty([N,4N])} \sum_{k \in [N_1, 2N_1)} |B_{\boldsymbol{c}}([N_1, k))|, \end{aligned}$$

where $\|f(r)\|_{L^\infty([N,4N])} := \sup_{r \in [N,4N]} |f(r)|$ for continuous functions $f : [N, 4N] \to \mathbb{C}$. Here

$$\max(\|I(r)\|_{L^\infty([N,4N])}, N \, \|I'(r)\|_{L^\infty([N,4N])}) \ll N^{(1-m)/2} \, \|I_{\boldsymbol{c}}\|_{1,\infty;N}$$

by Equation (4.11). Finally, let

$$\nu := \frac{1}{2} \left(\delta_{2N_1} + \frac{1}{N_1} \sum_{k \in [N_1, 2N_1)} \delta_k \right),$$

where δ_k is the Dirac measure supported on the singleton set $\{k\}$. Then, ν is a probability measure supported on $[N_1, 2N_1]$. Also, the last three displays imply Equation (5.4). □

Let $(N_0, N_1) \in \mathcal{A}$. Let $\mathcal{I}$ and $\mathcal{R}$ be as in Equations (4.7) and (4.8), respectively. Since we are presently only interested in $\boldsymbol{c} \in \mathcal{S}$, we may and do assume $|\mathcal{I}| \geqslant 1$. For each $\boldsymbol{c} \in \mathcal{S}$, we have

$$\begin{aligned} |\diamondsuit_{\boldsymbol{c},N_0,N_1}| &\leqslant \sum_{n_0 \in [N_0, 2N_0)} |a'_{\boldsymbol{c}}(n_0) \, \heartsuit_{\boldsymbol{c},n_0,N_1}| \\ &\ll N^{(1-m)/2} \, \|I_{\boldsymbol{c}}\|_{1,\infty;N} \sum_{n_0 \in [N_0, 2N_0)} |a'_{\boldsymbol{c}}(n_0)| \int_{x \in [N_1, 2N_1]} |B_{\boldsymbol{c}}([N_1, x))| \, d\nu(x), \end{aligned}$$

where the first and second inequality are justified by the triangle inequality and Lemma 5.2, respectively. Abbreviating $B_{\boldsymbol{c}}([N_1, x))$ to $B_{\boldsymbol{c}}(x)$ for convenience, we deduce that

$$\sum_{\boldsymbol{c} \in \mathcal{R} : \, \Delta(\boldsymbol{c}) \neq 0} |\diamondsuit_{\boldsymbol{c},N_0,N_1}| \ll_\epsilon X^{m+\epsilon} Q_1 \left(\sum_{\boldsymbol{c} \in \mathcal{R} : \, \Delta(\boldsymbol{c}) \neq 0} \left(\int_{x \in [N_1, 2N_1]} |B_{\boldsymbol{c}}(x)| \, d\nu \right)^2 \right)^{1/2} \tag{5.5}$$

by the Cauchy–Schwarz inequality and Lemma 4.9, where

$$Q_1 := N^{(1-m)/2} N_0^{1/2+(m-|\mathcal{I}|)/6} (X/N)^{1-(m+|\mathcal{I}|)/4} \max[Z^{1+(|\mathcal{I}|-m)/4}, 1]. \tag{5.6}$$

Now, for the rest of Section 5, we assume Hypothesis 2.4. We have

$$\left(\int_{x \in [N_1, 2N_1]} |B_{\boldsymbol{c}}(x)| \, d\nu \right)^2 \ll \int_{x \in [N_1, 2N_1]} |B_{\boldsymbol{c}}(x)|^2 \, d\nu$$

by the Cauchy–Schwarz inequality, so

$$\left(\sum_{\boldsymbol{c}\in\mathcal{R}\,:\,\Delta(\boldsymbol{c})\neq 0}\left(\int_{x\in[N_1,2N_1]}|B_{\boldsymbol{c}}(x)|\,d\nu\right)^2\right)^{1/2} \ll \left(\int_{x\in[N_1,2N_1]}\sum_{\boldsymbol{c}\in\mathcal{R}\,:\,\Delta(\boldsymbol{c})\neq 0}|B_{\boldsymbol{c}}(x)|^2\,d\nu\right)^{1/2}$$
$$\ll_\epsilon X^\epsilon Q_2$$

by Equation (4.15), where

$$Q_2 := \min\left(Z^m N_1, Z^{|\mathcal{I}|}N_1^{\max(2,1+(m-|\mathcal{I}|)/3)}\right)^{1/2}. \tag{5.7}$$

Lemma 5.3. *We have* $Q_1Q_2 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}$.

Proof. We split the proof into four cases.

Case 1: $|\mathcal{I}| = m$. Then, $Q_2 = (Z^m N_1)^{1/2}$, since $|\mathcal{I}| = m$ and $N_1 \geqslant 1$. Therefore, $Q_1Q_2 = Q_3$, where

$$Q_3 := Z^{m/2}N_1^{1/2}\cdot N^{(1-m)/2}N_0^{1/2}(X/N)^{1-m/2}\max[Z,1]. \tag{5.8}$$

But $Q_3 = Z^{m/2}X^{1-m/2}\max[Z,1]$, since $N_1N_0 = N$. By Equation (4.2), we have $Z = X^{1/2+\epsilon_0} \geqslant 1$, so

$$Q_3 = X^{1-m/2}Z^{1+m/2} = X^{3/2-m/4+(1+m/2)\epsilon_0}.$$

Thus, $Q_1Q_2 = Q_3 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}$.

Case 2: $|\mathcal{I}| = m-1$ *and* $N_1 \geqslant Z$. Then $Q_2 = (Z^m N_1)^{1/2}$, by Equation (5.7). Therefore, $Q_1Q_2 = Q_4$, where

$$Q_4 := Z^{m/2}N^{1-m/2}N_0^{(m-|\mathcal{I}|)/6}(X/N)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1],$$

since $N_1N_0 = N$. Since $(m-|\mathcal{I}|)/6 \geqslant 0$ and $N_0 = N/N_1 \leqslant N/Z$, we have

$$Q_4 \leqslant Z^{m/2}N^{1-m/2}(N/Z)^{(m-|\mathcal{I}|)/6}(X/N)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1]. \tag{5.9}$$

The right-hand side of Equation (5.9) is *decreasing* as a function of N, because

$$1-m/2+(m-|\mathcal{I}|)/6-1+(m+|\mathcal{I}|)/4 = (|\mathcal{I}|-m)/12 < 0. \tag{5.10}$$

Since $N \geqslant N_1 \geqslant Z$, it follows that

$$Q_4 \leqslant Z^{m/2}Z^{1-m/2}(Z/Z)^{(m-|\mathcal{I}|)/6}(X/Z)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1]$$
$$\ll_{\epsilon_0} X^{O(\epsilon_0)}X^{1-(m+|\mathcal{I}|)/8}\max[X^{1/2+(|\mathcal{I}|-m)/8},1],$$

since $Z = X^{1/2+\epsilon_0}$. But $|\mathcal{I}| = m-1$, so

$$X^{1-(m+|\mathcal{I}|)/8}\max[X^{1/2+(|\mathcal{I}|-m)/8},1] = \max[X^{3/2-m/4}, X^{9/8-m/4}] = X^{3/2-m/4}. \tag{5.11}$$

Thus, $Q_1Q_2 = Q_4 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}$.

Case 3: $1 \leqslant |\mathcal{I}| \leqslant m-2$. By Equation (5.7), we have $Q_2 \leqslant (Z^{|\mathcal{I}|}N_1^{\max(2,1+(m-|\mathcal{I}|)/3)})^{1/2}$. Since $N_1N_0 = N$, it follows that $Q_1Q_2 \leqslant Q_5$, where

$$Q_5 := Z^{|\mathcal{I}|/2}N_1^{\max(1/2,(m-|\mathcal{I}|)/6)}N^{1-m/2}N_0^{(m-|\mathcal{I}|)/6}(X/N)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1].$$

Since $N_0 \geqslant 1$ and $N_1N_0 = N$, we have $N_1^{\max(1/2,(m-|\mathcal{I}|)/6)}N_0^{(m-|\mathcal{I}|)/6} \leqslant N^{\max(1/2,(m-|\mathcal{I}|)/6)}$. Thus,

$$Q_5 \leqslant Z^{|\mathcal{I}|/2}N^{\max(1/2,(m-|\mathcal{I}|)/6)}N^{1-m/2}(X/N)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1]. \tag{5.12}$$

The right-hand side of Equation (5.12) is *weakly decreasing* in N, because

$$\begin{aligned}\max(1/2,(m-|\mathcal{I}|)/6) + 1 - m/2 - 1 + (m+|\mathcal{I}|)/4 &= \max(1/2,(m-|\mathcal{I}|)/6) + (|\mathcal{I}|-m)/4 \\ &\leqslant 0,\end{aligned}$$

in view of the inequality $|\mathcal{I}| - m \leqslant -2$. Since $N \geqslant 1$ and $|\mathcal{I}| \leqslant m$, it follows that

$$\begin{aligned}Q_5 &\leqslant Z^{|\mathcal{I}|/2}X^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1] \\ &\leqslant Z^{|\mathcal{I}|/2}X^{1-(m+|\mathcal{I}|)/4}Z \\ &\ll_{\epsilon_0} X^{O(\epsilon_0)}X^{3/2-m/4},\end{aligned}$$

since $Z = X^{1/2+\epsilon_0}$. Thus, $Q_1Q_2 \leqslant Q_5 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}$.

Case 4: $|\mathcal{I}| = m-1$ *and* $N_1 \leqslant Z$. Arguing as in Case 3, we have $Q_1Q_2 \leqslant Q_5$. But if we hold N_1 constant, and plug $N_0 = N/N_1$ into Q_5, then Q_5 is *decreasing* in N, by Equation (5.10). Since $N \geqslant N_1$, it follows that $Q_5 \leqslant Q_6$, where

$$Q_6 := Z^{|\mathcal{I}|/2}N_1^{\max(1/2,(m-|\mathcal{I}|)/6)}N_1^{1-m/2}(X/N_1)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1].$$

But Q_6 is *increasing* in N_1, because

$$\max(1/2,(m-|\mathcal{I}|)/6) + 1 - m/2 - 1 + (m+|\mathcal{I}|)/4 = 1/4 > 0,$$

in view of the equality $|\mathcal{I}| = m-1$. Since $N_1 \leqslant Z$ and $|\mathcal{I}| = m-1$, it follows that

$$\begin{aligned}Q_6 &\leqslant Z^{|\mathcal{I}|/2}Z^{1/2}Z^{1-m/2}(X/Z)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1] \\ &= Z\,(X/Z)^{1-(m+|\mathcal{I}|)/4}\max[Z^{1+(|\mathcal{I}|-m)/4},1] \\ &\ll_{\epsilon_0} X^{O(\epsilon_0)}X^{1-(m+|\mathcal{I}|)/8}\max[X^{1/2+(|\mathcal{I}|-m)/8},1],\end{aligned}$$

since $Z = X^{1/2+\epsilon_0}$. But $|\mathcal{I}| = m-1$, so it follows from Equation (5.11) that $Q_6 \ll_{\epsilon_0} X^{O(\epsilon_0)} X^{3/2-m/4}$. Thus, $Q_1 Q_2 \leqslant Q_5 \leqslant Q_6 \ll_{\epsilon_0} X^{3/2-m/4+O(\epsilon_0)}$. □

Remark 5.4. Interestingly, the quantity Q_3 in Equation (5.8) is constant over $(N_0, N_1) \in \mathcal{A}$.

By Lemma 5.3, the left-hand side of Equation (5.5) is $\ll_{\epsilon_0} X^{m+O(\epsilon_0)} X^{3/2-m/4}$. Upon summing over $(N_0, N_1) \in \mathcal{A}$ and the set of $2^m - 1$ possible sets $\mathcal{R}$, it follows from (5.3) that

$$\Sigma_1 \ll_{\epsilon_0} X^{-3} X^{m+O(\epsilon_0)} X^{3/2-m/4} = X^{3(m-2)/4+O(\epsilon_0)}.$$

This yields the desired inequality, Equation (5.1).

6 | CONTRIBUTION FROM THE CENTRAL TERMS

Here, we address the $\boldsymbol{c} = \boldsymbol{0}$ contribution to Equation (4.6), using the theory of $I_{\boldsymbol{0}}(n)$ developed in [7]. We roughly follow [7, Section 12, par. 2]. Let

$$\Sigma_2 := X^{-3} \sum_{n \geqslant 1} n^{(1-m)/2} S^{\natural}_{\boldsymbol{0}}(n) I_{\boldsymbol{0}}(n). \tag{6.1}$$

We begin with a slight extension of [19, Lemma 4.9].

Lemma 6.1. *If* $N \geqslant 1$, *then* $\sum_{n \in [N,2N)} n^{-m} |S_{\boldsymbol{0}}(n)| \ll_\epsilon N^{(4-m)/3+\epsilon}$.

Proof. We have $S^{\natural}_{\boldsymbol{0}}(n) \ll_\epsilon n^{1/2+\epsilon} \operatorname{cub}(n)^{m/6}$ by Proposition 4.6. Thus

$$n^{-m} S_{\boldsymbol{0}}(n) \ll_\epsilon n^{1-m/2+\epsilon} \operatorname{cub}(n)^{m/6}.$$

Taking $t = m/6$ in Lemma 3.3(3), we get

$$\sum_{n \in [N,2N)} n^{-m} |S_{\boldsymbol{0}}(n)| \ll_\epsilon N^{1-m/2+\epsilon} \max(N, N^{1/3+m/6}) = N^{(4-m)/3+\epsilon},$$

where we note that $\max(N, N^{1/3+m/6}) = N^{1/3+m/6}$ because $N \geqslant 1$ and $m \geqslant 4$. □

Lemma 6.1 implies, in particular, the familiar fact that the singular series

$$\mathfrak{S} := \sum_{n \geqslant 1} n^{-m} S_{\boldsymbol{0}}(n) \tag{6.2}$$

converges absolutely for $m \geqslant 5$. It is also known that the real density

$$\sigma_{\infty,w} := \lim_{\epsilon \to 0} (2\epsilon)^{-1} \int_{|F(\boldsymbol{x})| \leqslant \epsilon} w(\boldsymbol{x})\, d\boldsymbol{x} \tag{6.3}$$

exists; see, for example, [7, Theorem 3]. Yet for all $n \ll Y$, [7, Lemma 13] implies

$$X^{-m} I_{\mathbf{0}}(n) = \sigma_{\infty,w} + O_A((n/Y)^A), \tag{6.4}$$

for all $A > 0$. If $m \geqslant 5$, then via Equation (6.4) with $A = (m-4)/3$, we get

$$\begin{aligned} &\sum_{n \leqslant M_0(F,w)Y} n^{-m} S_{\mathbf{0}}(n) X^{-m} I_{\mathbf{0}}(n) \\ &= \sigma_{\infty,w} \sum_{n \leqslant M_0(F,w)Y} n^{-m} S_{\mathbf{0}}(n) + \sum_{n \leqslant M_0(F,w)Y} O((n/Y)^{(m-4)/3} n^{-m} |S_{\mathbf{0}}(n)|) \\ &= \sigma_{\infty,w} \mathfrak{S} + O_\epsilon(Y^{(4-m)/3+\epsilon}), \end{aligned}$$

by Lemma 6.1 and Equation (6.2). Also, by Proposition 4.1, we have $I_{\mathbf{0}}(n) = 0$ for all $n > M_0(F,w)Y$. Since $n^{-m} S_{\mathbf{0}}(n) = n^{(1-m)/2} S^\natural_{\mathbf{0}}(n)$ and $Y = X^{3/2}$, it follows that if $m \geqslant 5$, then

$$\Sigma_2 = X^{m-3} [\sigma_{\infty,w} \mathfrak{S} + O_\epsilon(X^{(4-m)/2+\epsilon})] = \sigma_{\infty,w} \mathfrak{S} X^{m-3} + O_\epsilon(X^{(m-2)/2+\epsilon}), \tag{6.5}$$

where Σ_2 is the quantity defined in Equation (6.1). On the other hand, for all $m \geqslant 4$,

$$\Sigma_2 \ll X^{m-3} \sum_{n \leqslant M_0(F,w)Y} n^{-m} |S_{\mathbf{0}}(n)| \ll_\epsilon X^{m-3+\epsilon} \tag{6.6}$$

by Proposition 4.1 and Lemma 6.1, since $I_{\mathbf{0}}(n) \ll X^m$ by [7, Lemma 16].

7 | CONTRIBUTION FROM SINGULAR HYPERPLANE SECTIONS

In this section, we study the quantity

$$\Sigma_3 := X^{-3} \sum_{n \geqslant 1} \sum_{\boldsymbol{c} \in [-Z,Z]^m :\, \Delta(\boldsymbol{c})=0,\, \boldsymbol{c} \neq \mathbf{0}} n^{(1-m)/2} S^\natural_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n). \tag{7.1}$$

We will prove the following result, extending work of Heath-Brown. Recall the definitions of $N_{F,w}(X)$ and $N'_{F,w}(X)$ from Equations (2.14) and (2.15), respectively.

Theorem 7.1. *If $m \geqslant 5$, then*

$$\Sigma_3 \ll_{\epsilon_0} X^{3(m-2)/4+O(\epsilon_0)}. \tag{7.2}$$

If $m = 4$, then

$$\Sigma_3 = N_{F,w}(X) - N'_{F,w}(X) + O_{\epsilon_0}(X^{3(m-2)/4+O(\epsilon_0)}). \tag{7.3}$$

The cases $m = 4$ and $m = 6$ of this result are due to Heath-Brown. For instance, the estimate (7.3) for $m = 4$ follows directly from [8, Lemmas 7.2 and 8.1], in view of the tail estimate (4.3). Therefore, we may and do assume $m \geqslant 5$, for the rest of Section 7.

We combine ideas from [8, 9]. Let $\mathcal{I}$ and $\mathcal{R}$ be as in Equations (4.7) and (4.8), respectively. Since we are only interested in $\boldsymbol{c} \neq \mathbf{0}$, we may and do assume $|\mathcal{I}| \geqslant 1$. Let C and $\mathcal{C}$ be as in Equation (4.12), for some $C_i \in \{2^t : t \in \mathbb{Z}_{\geqslant 0}\}$ with $1 \leqslant C_i \leqslant Z$.

By Proposition 4.1, the sum Σ_3 from Equation (7.1) is supported on $n \leqslant M_0(F, w)Y$. Let

$$\Sigma_4 := X^{-3} \sum_{n \leqslant M_0(F,w)Y} \sum_{\boldsymbol{c} \in \mathcal{C} :\, \Delta(\boldsymbol{c})=0} n^{(1-m)/2} S^{\natural}_{\boldsymbol{c}}(n) I_{\boldsymbol{c}}(n). \tag{7.4}$$

Now, consider an element $\boldsymbol{c} \in \mathcal{C}$ with $\Delta(\boldsymbol{c}) = 0$, assuming such a $\boldsymbol{c}$ exists. Denote the nonempty fibers of the map $\mathcal{I} \to \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, $i \mapsto F_i c_i \bmod (\mathbb{Q}^\times)^2$ by

$$\mathcal{I}(k) := \{i \in \mathcal{I} : F_i c_i \equiv g_k \bmod (\mathbb{Q}^\times)^2\},$$

for $1 \leqslant k \leqslant K$, say, where the g_k are signed, nonzero square-free integers. Trivially, we have $\sum_{1\leqslant k \leqslant K} |\mathcal{I}(k)| = |\mathcal{I}|$. For each $i \in \mathcal{I}(k)$, we may write

$$c_i = g_k F_i^{-1} e_i^2 \tag{7.5}$$

with $e_i \in \mathbb{Z}$. Moreover, by Equation (2.1) and the $\mathbb{Q}$-linear independence of square roots of distinct square-free integers, we may choose the signs of the integers e_i so that

$$\sum_{i \in \mathcal{I}(k)} F_i (e_i/F_i)^3 = 0. \tag{7.6}$$

Since $c_i \neq 0$ implies $e_i \neq 0$ for all $i \in \mathcal{I}(k)$, we immediately deduce from Equation (7.6) that

$$|\mathcal{I}(k)| \geqslant 2. \tag{7.7}$$

We now prove a general lemma that will allow us, in Lemma 7.3, to exploit the structure uncovered in the previous paragraph.

Lemma 7.2. *Let $J \in \mathbb{Z}_{\geqslant 2}$, let $d_1, \dots, d_J \in \mathbb{Z}_{\geqslant 1}$, and let $G, E_1, \dots, E_J \in \mathbb{R}_{>0}$. Then*

$$\sum_{\substack{1 \leqslant g \leqslant G : \\ \mu(g)^2 = 1}} \prod_{1 \leqslant i \leqslant J} \sum_{\substack{1 \leqslant e_i \leqslant E_i : \\ d_i \mid \mathrm{sq}(g e_i^2)}} d_i^{1/2} \leqslant \prod_{1 \leqslant i \leqslant J} (2^{\omega(d_i)} G^{1/2} E_i).$$

Proof. By Hölder's inequality over g, we may assume that $E_1 = \cdots = E_J = E$ and $d_1 = \cdots = d_J = d$, say. Let $S := \{h \mid d : \mu(h)^2 = 1\}$. Now consider integers $g, e \geqslant 1$ with g square-free. Then, $\mathrm{sq}(ge^2) = \gcd(g, e)e^2$. Therefore, if $d \mid \mathrm{sq}(ge^2)$, and we let $h := \gcd(g, e, d)$, then

$$h \in S, \qquad (d/h) \mid e^2,$$

whence e is divisible by the integer $\prod_{p \mid (d/h)} p^{\lceil v_p(d/h)/2 \rceil} \geqslant (d/h)^{1/2}$. Thus, given $h \in S$, the number of possible $e \in [1, E]$ is at most $E/(d/h)^{1/2}$. It follows that

$$\sum_{\substack{1 \leqslant e \leqslant E : \\ d \mid \mathrm{sq}(ge^2)}} d^{1/2} \leqslant \sum_{h \in S} (d^{1/2} \cdot \mathbf{1}_{h \mid g} \cdot E/(d/h)^{1/2}) = \sum_{h \in S} (\mathbf{1}_{h \mid g} \cdot h^{1/2} E), \tag{7.8}$$

for every square-free $g \geqslant 1$. By Equation (7.8), and Hölder's inequality over h, we get

$$\begin{aligned}\sum_{\substack{1\leqslant g\leqslant G:\\ \mu(g)^2=1}} \left(\sum_{\substack{1\leqslant e\leqslant E:\\ d\mid \mathrm{sq}(ge^2)}} d^{1/2}\right)^J &\leqslant \sum_{\substack{1\leqslant g\leqslant G:\\ \mu(g)^2=1}} \left(\sum_{h\in S}(\mathbf{1}_{h\mid g}\cdot h^{1/2}E)\right)^J \\ &\leqslant \sum_{\substack{1\leqslant g\leqslant G:\\ \mu(g)^2=1}} |S|^{J-1}\sum_{h\in S}(\mathbf{1}_{h\mid g}\cdot h^{1/2}E)^J \\ &\leqslant |S|^{J-1}\sum_{\substack{h\in S:\\ h\leqslant G}}(G/h)(h^{1/2}E)^J \\ &\leqslant |S|^J G^{J/2}E^J,\end{aligned}$$

where in the last step we note that $h^{J/2-1} \leqslant G^{J/2-1}$. This suffices, since $|S| = 2^{\omega(d)}$. □

Lemma 7.3. *Let $n \geqslant 1$ be an integer. Then*

$$\sum_{\boldsymbol{c}\in C:\,\Delta(\boldsymbol{c})\neq 0} n^{-1}|S^{\natural}_{\boldsymbol{c}}(n)|^2 \ll_\epsilon n^\epsilon\,\mathrm{cub}(n)^{(m-|\mathcal{I}|)/3}\prod_{i\in\mathcal{I}} C_i^{1/2+\epsilon}.$$

Proof. Let $n_3 := \mathrm{cub}(n)$. Fix a set $\mathcal{J} \subseteq \mathcal{I}$ with $|\mathcal{J}| \geqslant 2$. Let $G \in \{2^t : t \in \mathbb{Z}_{\geqslant 0}\}$, and let $E_i := (2F_iC_i/G)^{1/2}$ for each $i \in \mathcal{J}$. Let $\tau(\cdot)$ be the divisor function. Then

$$\begin{aligned}&\sum_{\substack{|g|\in[G,2G):\\ \mu(|g|)^2=1}}\prod_{i\in\mathcal{J}}\sum_{\substack{|e_i|\leqslant(2F_iC_i/|g|)^{1/2}:\\ gF_i^{-1}e_i^2\in\mathbb{Z}\setminus\{0\}}}\gcd(n_3,\mathrm{sq}(gF_i^{-1}e_i^2))^{1/2}\\ &\leqslant 2^{1+|\mathcal{J}|}\sum_{\substack{g\in[G,2G):\\ \mu(g)^2=1}}\prod_{i\in\mathcal{J}}\sum_{1\leqslant e_i\leqslant E_i}\gcd(n_3,\mathrm{sq}(ge_i^2))^{1/2}\\ &\leqslant 2^{1+|\mathcal{J}|}\sum_{\substack{g\in[G,2G):\\ \mu(g)^2=1}}\prod_{i\in\mathcal{J}}\sum_{d_i\mid n_3}\sum_{\substack{1\leqslant e_i\leqslant E_i:\\ d_i\mid \mathrm{sq}(ge_i^2)}} d_i^{1/2}\\ &\leqslant 2^{1+|\mathcal{J}|}\prod_{i\in\mathcal{J}}(\tau(n_3)^2G^{1/2}E_i)\\ &= 2^{1+|\mathcal{J}|}\tau(n_3)^{2|\mathcal{J}|}\prod_{i\in\mathcal{J}}(2F_iC_i)^{1/2},\end{aligned}$$

where in the penultimate step we use Lemma 7.2 for each possible choice of divisors $d_i \mid n_3$, and we note that $2^{\omega(d_i)} \leqslant 2^{\omega(n_3)} \leqslant \tau(n_3)$. Moreover, if $G > \min_{i\in\mathcal{J}}(2F_iC_i)$, then

$$\sum_{\substack{|g|\in[G,2G):\\ \mu(|g|)^2=1}}\prod_{i\in\mathcal{J}}\sum_{\substack{|e_i|\leqslant(2F_iC_i/|g|)^{1/2}:\\ gF_i^{-1}e_i^2\in\mathbb{Z}\setminus\{0\}}}\gcd(n_3,\mathrm{sq}(gF_i^{-1}e_i^2))^{1/2} = 0,$$

since the sum over one of the variables $e_i \in \mathbb{Z} \setminus \{0\}$ is empty. On summing the penultimate display over all $G \in \{2^t : t \in \mathbb{Z}_{\geqslant 0}\}$ with $G \leqslant \min_{i\in\mathcal{J}}(2F_iC_i)$, we conclude that

$$\sum_{\substack{g\in\mathbb{Z}\setminus\{0\}:\\ \mu(|g|)^2=1}} \prod_{i\in\mathcal{J}} \sum_{\substack{|e_i|\leqslant(2F_iC_i/|g|)^{1/2}:\\ gF_i^{-1}e_i^2\in\mathbb{Z}\setminus\{0\}}} \gcd(n_3, \mathrm{sq}(gF_i^{-1}e_i^2))^{1/2} \ll_{|\mathcal{J}|,\epsilon} n_3^\epsilon \prod_{i\in\mathcal{J}}(2F_iC_i)^{1/2+\epsilon}. \tag{7.9}$$

Recall the constraints (7.5) and (7.7) on $\{\boldsymbol{c} \in C : \Delta(\boldsymbol{c}) = 0\}$. Applying Equation (7.9) with $\mathcal{J} = \mathcal{I}(k)$, for each $k \in [1, K]$, and multiplying the resulting K inequalities, we get

$$\prod_{1\leqslant k\leqslant K} \sum_{\substack{g_k\in\mathbb{Z}\setminus\{0\}:\\ \mu(|g_k|)^2=1}} \prod_{i\in\mathcal{I}(k)} \sum_{\substack{|e_i|\leqslant(2F_iC_i/|g_k|)^{1/2}:\\ g_kF_i^{-1}e_i^2\in\mathbb{Z}\setminus\{0\}}} \gcd(n_3, \mathrm{sq}(g_kF_i^{-1}e_i^2))^{1/2} \ll_\epsilon n_3^\epsilon \prod_{i\in\mathcal{I}} C_i^{1/2+\epsilon}, \tag{7.10}$$

since $K \leqslant |\mathcal{I}| \leqslant m$, and the variables m, F_i are fixed. On summing Equation (7.10) over all possible choices for the sets $\mathcal{I}(k) \subseteq \mathcal{I}$, we deduce that

$$\sum_{\boldsymbol{c}\in C:\,\Delta(\boldsymbol{c})=0} \prod_{i\in\mathcal{I}} \gcd(n_3, \mathrm{sq}(c_i))^{1/2} \ll_\epsilon n_3^\epsilon \prod_{i\in\mathcal{I}} C_i^{1/2+\epsilon}. \tag{7.11}$$

Lemma 7.3 follows immediately from Equations (4.9) and (7.11). □

Remark 7.4. Interestingly, the proof of Equation (7.11) uses the constraint (7.6) only through Equation (7.7).

Taking $n_3 = 1$ in Equation(7.11) implies

$$|\{\boldsymbol{c} \in C : \Delta(\boldsymbol{c}) = 0\}| \ll_\epsilon \prod_{i\in\mathcal{I}} C_i^{1/2+\epsilon}.$$

Therefore, Lemma 7.3 implies

$$\sum_{\boldsymbol{c}\in C:\,\Delta(\boldsymbol{c})=0} n^{-1/2}|S^\natural_{\boldsymbol{c}}(n)| \ll_\epsilon n^\epsilon \,\mathrm{cub}(n)^{(m-|\mathcal{I}|)/6} \prod_{i\in\mathcal{I}} C_i^{1/2+\epsilon}, \tag{7.12}$$

by the Cauchy–Schwarz inequality over $\boldsymbol{c}$.

Let $N \in \{2^t : t \in \mathbb{Z}_{\geqslant 0}\}$ with $1 \leqslant N \leqslant M_0(F, w)Y$. By Lemma 4.7, Equation (7.12), and the $t = (m - |\mathcal{I}|)/6$ case of Lemma 3.3(3), the sum

$$\Sigma_5 := X^{-3} \sum_{n\in[N,2N)} \sum_{\boldsymbol{c}\in C:\,\Delta(\boldsymbol{c})=0} n^{(1-m)/2}|S^\natural_{\boldsymbol{c}}(n)I_{\boldsymbol{c}}(n)|$$

satisfies the bound $\Sigma_5 \ll_\epsilon X^{m-3+\epsilon}Q_7$, where

$$\begin{aligned} Q_7 &:= N^{1-m/2}(XC/N)^{1-(m+|\mathcal{I}|)/4} \max(N, N^{1/3+(m-|\mathcal{I}|)/6})\, C^{|\mathcal{I}|/2} \\ &= X^{1-(|\mathcal{I}|+m)/4} \max(N^{1+(|\mathcal{I}|-m)/4}, N^{1/3+(|\mathcal{I}|-m)/12})\, C^{1+(|\mathcal{I}|-m)/4}. \end{aligned}$$

Since $N^{1+(|\mathcal{I}|-m)/4} = (N^{1/3+(|\mathcal{I}|-m)/12})^3$, we will analyze Q_7 according to the sign of

$$\mathfrak{e} := 1 + (|\mathcal{I}| - m)/4.$$

Case 1: $\mathfrak{e} \leqslant 0$. Then, since $N, C \gg 1$, we have

$$Q_7 \ll X^{1-(|\mathcal{I}|+m)/4} \leqslant X^{(6-m)/4},$$

where the final inequality holds because $X \geqslant 1$ and $|\mathcal{I}| \geqslant -2$.

Case 2: $\mathfrak{e} \geqslant 0$. Then, since $N \ll Y$ and $C \ll Z$, we have

$$Q_7 \ll X^{1-(|\mathcal{I}|+m)/4} Y^{1+(|\mathcal{I}|-m)/4} Z^{1+(|\mathcal{I}|-m)/4}.$$

Plugging in Equations (4.1) and (4.2), we get

$$Q_7 \ll_{\epsilon_0} X^{1-(|\mathcal{I}|+m)/4+O(\epsilon_0)} (X^2)^{1+(|\mathcal{I}|-m)/4} = X^{3+(|\mathcal{I}|-3m)/4+O(\epsilon_0)}.$$

Moreover, if $|\mathcal{I}| \leqslant 2m - 6$, then $3 + (|\mathcal{I}| - 3m)/4 \leqslant (6 - m)/4$.

If $m \geqslant 6$, then $1 \leqslant |\mathcal{I}| \leqslant m \leqslant 2m - 6$, so regardless of what $|\mathcal{I}|$ is, it follows that

$$\begin{aligned} \Sigma_5 &\ll_{\epsilon_0} X^{m-3+\epsilon_0} Q_7 \\ &\ll_{\epsilon_0} X^{m-3+\epsilon_0} X^{(6-m)/4+O(\epsilon_0)} \\ &= X^{3(m-2)/4+O(\epsilon_0)}, \end{aligned}$$

whence by summing over all possibilities for N and C we get

$$\Sigma_3, \Sigma_4 \ll_{\epsilon_0} X^{3(m-2)/4+O(\epsilon_0)},$$

where Σ_3, Σ_4 are as defined in Equations (7.1) and (7.4), respectively. This completes the proof of Equation (7.2) for $m \geqslant 6$. For the rest of Section 7, we relinquish the previous definitions of $\mathcal{C}$ and C.

For $m = 5$, we first show that a natural extension of [8, Lemma 7.1] holds.

Lemma 7.5. *If* $5 \leqslant m \leqslant 6$ *and* $C \gg 1$, *then* $|\{\mathbf{c} \in \mathbb{Z}^m \cap [-C, C]^m : \Delta(\mathbf{c}) = 0\}| \ll_\epsilon C^{m-3+\epsilon}$.

Proof. For $m = 6$, this follows directly from [8, Lemma 7.1]. Now, suppose $m = 5$. A *partition* of m is an infinite, weakly decreasing sequence of nonnegative integers $\lambda_1, \lambda_2, \ldots$, such that $\sum_{k \geqslant 1} \lambda_k = m$. For any partition of m, let

$$e_k := 2 \cdot \mathbf{1}_{2 \leqslant \lambda_k \leqslant 4} + (\lambda_k - 2) \cdot \mathbf{1}_{\lambda_k \geqslant 5}$$

for $k \geqslant 1$. Let θ denote the maximum value of $\frac{1}{2} \sum_{k \geqslant 1} e_k$ over all partitions of m. By [8, p. 687], we have $|\{\mathbf{c} \in \mathbb{Z}^m \cap [-C, C]^m : \Delta(\mathbf{c}) = 0\}| \ll_\epsilon C^{\theta+\epsilon}$.

Clearly $\lambda_3 \leqslant \lfloor m/3 \rfloor = 1$, so $e_k = 0$ for all $k \geqslant 3$. If $\lambda_2 \leqslant 1$, then $e_k = 0$ for all $k \geqslant 2$, so $\sum_{k\geqslant 1} e_k = e_1 \leqslant m-2$. If $\lambda_2 \geqslant 2$, then $\lambda_1 \leqslant m - \lambda_2 \leqslant 3$, so $e_k \leqslant 2$ for all $k \geqslant 1$, whence $\sum_{k\geqslant 1} e_k = e_1 + e_2 \leqslant 4$. In either case, $\sum_{k\geqslant 1} e_k \leqslant 4$. Therefore, $\theta \leqslant 2 = m-3$. □

We now recall a bound from [8] that is valid for all $m \geqslant 4$.

Lemma 7.6. *Fix $\varepsilon > 0$. Suppose $1 \ll N \ll X^{3/2}$ and $1 \ll C \ll X^{1/2+\varepsilon}$. Let*

$$A = \sum_{N<q\leqslant 2N} \sum_{C<\|\boldsymbol{c}\|\leqslant 2C:\, \Delta(\boldsymbol{c})=0} q^{-m} S_{\boldsymbol{c}}(q) I_{\boldsymbol{c}}(q).$$

Then, there exist reals $X_1, X_2, X_3 \gg 1$ and an integer $H \geqslant 1$ such that $X_1X_2X_3 \asymp N$ and

$$A \ll_\varepsilon X^{m+4\varepsilon} N^{-m} X_1^{1+m/2} X_2^{2/3+2m/3} X_3^{1+2m/3} H^{1/2} \left(\frac{N}{XC}\right)^{(m-2)/4} \mathcal{N}_1 \mathcal{N}_2(H),$$

where in terms of the quantity $\mathfrak{D} = 3(\prod_{1\leqslant i\leqslant m} F_i)^{2^{m-2}}$ from Section 2, we let

$$\mathcal{N}_1 := \sum_{(q_1,q_2,q_3):\, X_i<q_i\leqslant 2X_i} \mathbf{1}_{\mathrm{cub}(q_1)=1} \mathbf{1}_{q_2=\mathrm{cub}(q_2)} \mathbf{1}_{q_3 \mid \mathfrak{D}^\infty},$$

$$\mathcal{N}_2(H) := \sum_{C<\|\boldsymbol{c}\|\leqslant 2C} \mathbf{1}_{H\mid \boldsymbol{c}} \mathbf{1}_{\Delta(\boldsymbol{c})=0}.$$

Proof. This is immediate from [8, pp. 688–689, from the definition of A on p. 688 to the definition of $\mathcal{N}_2(H)$ on p. 689]. What Heath-Brown calls P (resp. X), we call X (resp. N). Moreover, in terms of Heath-Brown's notation n and G, our m and Δ satisfy $m = n$ and $\Delta(\boldsymbol{c}) = 3G(\boldsymbol{c})$. However, our $C, q, \boldsymbol{c}, X_1, X_2, X_3, H$ match Heath-Brown's notation. □

Applying Lemma 3.1 to q_2 and Lemma 3.2 to q_3, it is clear that

$$\mathcal{N}_1 \ll_\varepsilon X_1 X_2^{1/3} X_3^{\varepsilon}.$$

Now, assume $5 \leqslant m \leqslant 6$. Then, $\mathcal{N}_2(H) = 0$ unless $H \leqslant 2C$, in which case

$$\mathcal{N}_2(H) \ll_\varepsilon (C/H)^{m-3+\varepsilon}$$

by Lemma 7.5. Plugging the last two displays into Lemma 7.6, with $\varepsilon := \epsilon_0$, we get

$$A \ll_{\epsilon_0} X^{m+O(\epsilon_0)} N^{-m} X_1^{2+m/2} X_2^{1+2m/3} X_3^{1+2m/3} H^{1/2} \left(\frac{N}{XC}\right)^{(m-2)/4} \left(\frac{C}{H}\right)^{m-3}.$$

Since $m-3 \geqslant 1/2$, we have $H^{1/2}(C/H)^{m-3} \leqslant C^{m-3}$. Moreover, $m \leqslant 6$ implies $2+m/2 \geqslant 1+2m/3$, so $X_1^{2+m/2} X_2^{1+2m/3} X_3^{1+2m/3} \ll (X_1X_2X_3)^{2+m/2} \asymp N^{2+m/2}$. Thus

$$A \ll_{\epsilon_0} X^{m+O(\epsilon_0)} N^{2-m/2} \left(\frac{N}{XC}\right)^{(m-2)/4} C^{m-3}. \tag{7.13}$$

Since $2 - m/2 + (m-2)/4 = (6-m)/4 \geqslant 0$ (resp. since $m - 3 \geqslant (m-2)/4$), the right-hand side of Equation (7.13) is weakly increasing in N (resp. in C). Therefore

$$A \ll_{\epsilon_0} X^{m+O(\epsilon_0)}(X^{3/2})^{2-m/2}(X^{1/2})^{m-3} = X^{3m/4+3/2+O(\epsilon_0)}.$$

Summing over $1 \ll N = M_0(F,w)Y/2^{k_1}$ and $1 \ll C = Z/2^{k_2}$ with $k_1, k_2 \in \mathbb{Z}_{\geqslant 1}$, we get

$$\Sigma_3 \ll X^{-3}X^{3m/4+3/2+O(\epsilon_0)} = X^{3(m-2)/4+O(\epsilon_0)},$$

where Σ_3 is the quantity defined in Equation (7.1). This completes the proof of Equation (7.2).

8 | PROOF OF MAIN RESULTS

In this section, we first prove Theorem 2.7, because it builds directly on our work in Sections 4–7 on the delta method. We then prove Theorem 2.3 using Equations (2.6), (2.7), and Proposition 4.3. Finally, we combine Theorems 2.3 and 2.7 to prove Theorem 1.1.

Proof of Theorem 2.7. By Proposition 3.4, we see that Hypothesis 2.6 implies Hypothesis 2.4. Therefore, we may and do assume Hypothesis 2.4. Now recall the quantity Σ_0 from Equation (4.6). By Equation (4.5) and the tail estimate (4.3), we have

$$N_{F,w}(X) - \Sigma_0 \ll_{A,\epsilon_0} X^{-A}.$$

Case 1: $m = 4$. Then, adding Equations (5.1), (6.6), and (7.3) together, we get

$$\Sigma_0 = \Sigma_1 + \Sigma_2 + \Sigma_3 = N_{F,w}(X) - N'_{F,w}(X) + O_{\epsilon_0}(X^{3(m-2)/4+O(\epsilon_0)}) + O_{\epsilon_0}(X^{m-3+\epsilon_0}).$$

It follows that $N'_{F,w}(X) \ll_{\epsilon_0} X^{3(m-2)/4+O(\epsilon_0)}$. Let $\mathfrak{c}(F,w) := 0$.

Case 2: $m \geqslant 5$. Then adding Equations (5.1), (6.5), and (7.2) together, we get

$$\Sigma_0 = \Sigma_1 + \Sigma_2 + \Sigma_3 = \mathfrak{c}(F,w)X^{m-3} + O_{\epsilon_0}(X^{3(m-2)/4+O(\epsilon_0)}) + O_{\epsilon_0}(X^{(m-2)/2+\epsilon_0}),$$

where $\mathfrak{c}(F,w) := \sigma_{\infty,w}\mathfrak{S}$. It follows that $N_{F,w}(X) - \mathfrak{c}(F,w)X^{m-3} \ll_{\epsilon_0} X^{3(m-2)/4+O(\epsilon_0)}$.

In each case, taking $\epsilon_0 \to 0$ gives the desired result, Equation (2.16). □

Proof of Theorem 2.3. Let $\boldsymbol{c} \in \mathcal{S}$. Since $\Psi(\boldsymbol{c}, s)$ has an Euler product, condition (1) in Definition 2.2 clearly holds. It remains to prove that conditions (2) and (3) hold.

Case 1: $\Psi(\boldsymbol{c}, s) = \Phi(\boldsymbol{c}, s)$. Then, conditions (2) and (3) are trivial, since

$$(b_{\boldsymbol{c}}(n), a'_{\boldsymbol{c}}(n)) = (S^{\natural}_{\boldsymbol{c}}(n), \mathbf{1}_{n=1}).$$

Case 2: $\Psi(\boldsymbol{c}, s) = \prod_{p \nmid \Delta(\boldsymbol{c})} \Phi_p(\boldsymbol{c}, s)$. Then, conditions (2) and (3) are trivial, since

$$(b_{\boldsymbol{c}}(n), a'_{\boldsymbol{c}}(n)) = (S^{\natural}_{\boldsymbol{c}}(n) \cdot \mathbf{1}_{\gcd(n,\Delta(\boldsymbol{c}))=1}, S^{\natural}_{\boldsymbol{c}}(n) \cdot \mathbf{1}_{n \mid \Delta(\boldsymbol{c})^\infty}).$$

Case 3: $\Psi(\boldsymbol{c}, s) \in \{\prod_{p\nmid\Delta(\boldsymbol{c})} L_p(s,\boldsymbol{c})^{(-1)^{m-3}}, L(s,\boldsymbol{c})^{(-1)^{m-3}}\}$. Then by Equation (2.6), we have

$$b_{\boldsymbol{c}}(n), a_{\boldsymbol{c}}(n) \ll_\epsilon n^\epsilon. \tag{8.1}$$

But $a'_{\boldsymbol{c}} = S^\natural_{\boldsymbol{c}} * a_{\boldsymbol{c}}$, by Equation (2.9). Therefore, condition (2) holds. Furthermore, if $p \nmid \Delta(\boldsymbol{c})$, then $a_{\boldsymbol{c}}(p) = -b_{\boldsymbol{c}}(p)$ by Equation (2.9) and $b_{\boldsymbol{c}}(p) = (-1)^{m-3}\lambda_{\boldsymbol{c}}(p) = E^\natural_{\boldsymbol{c}}(p)$ by Equation (2.7), so

$$a'_{\boldsymbol{c}}(p) = S^\natural_{\boldsymbol{c}}(p) + a_{\boldsymbol{c}}(p) = S^\natural_{\boldsymbol{c}}(p) - E^\natural_{\boldsymbol{c}}(p) \ll p^{-1/2}$$

by Proposition 4.3. Therefore, condition (3) also holds. □

Proof of Theorem 1.1. Let $\Psi := L(s,\boldsymbol{c})^{(-1)^{m-3}}$. Then, Ψ is an approximation of Φ, by Theorem 2.3. Moreover, Ψ is standard by Equation (8.1) and Definition 2.5. Now, let $\vartheta := 1$. Then, $\gamma_{\boldsymbol{c}}(n) = \mu(n)^m\lambda_{\boldsymbol{c}}(n)$ by Equation (2.11), since for all primes p we have $a_{\boldsymbol{c}}(p) = (-1)^{m-2}\lambda_{\boldsymbol{c}}(p)$ by the definition of $a_{\boldsymbol{c}}$. Upon plugging in $\mu(n)^m v_n$ for v_n in Hypothesis 2.1, we immediately find that Hypothesis 2.6 holds. Let $\varsigma : \mathbb{R} \to \mathbb{R}$ be a nonnegative, smooth, compactly supported function such that $\varsigma(t) = 1$ for all $t \in [1,4]$, and $\varsigma(t) = 0$ for all $t \notin [\frac{1}{2}, 8]$. Let

$$w(\boldsymbol{x}) := \varsigma\left(\sum_{1\leqslant i\leqslant m} x_i^2\right).$$

Then, Theorem 2.7 implies $N_{F,w}(X) \ll_\epsilon X^{3(m-2)/4+\epsilon}$ for all $X \geqslant 1$. Since $w(\boldsymbol{x}/2^k) = 1$ for all $\boldsymbol{x} \in \mathbb{Z}^m$ in the annulus $4^k \leqslant \sum_{1\leqslant i\leqslant m} x_i^2 \leqslant 4^{k+1}$, it follows that

$$\begin{aligned} N_F(X) - 1 &= |\{\boldsymbol{x} \in \mathbb{Z}^m \cap [-X,X]^m : F(\boldsymbol{x}) = 0,\ \boldsymbol{x} \neq \boldsymbol{0}\}| \\ &\leqslant \sum_{0\leqslant k\leqslant \log_4(4mX^2)} N_{F,w}(2^k) \\ &\ll_\epsilon \sum_{0\leqslant k\leqslant \log_4(4mX^2)} (2^k)^{3(m-2)/4+\epsilon} \\ &\ll ((4mX^2)^{1/2})^{3(m-2)/4+\epsilon} \\ &\ll_\epsilon X^{3(m-2)/4+\epsilon}, \end{aligned}$$

for all $X \geqslant 1$. This implies Theorem 1.1. □

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