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Loschmidt echo for deformed Wigner matrices

László Erdős[1] · Joscha Henheik[1] · Oleksii Kolupaiev[1]

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Abstract
We consider two Hamiltonians that are close to each other, $H_1 \approx H_2$, and analyze the time decay of the corresponding *Loschmidt echo* $\mathfrak{M}(t) := |\langle \psi_0, \mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1}\psi_0\rangle|^2$ that expresses the effect of an imperfect time reversal on the initial state ψ_0. Our model Hamiltonians are deformed Wigner matrices that do not share a common eigenbasis. The main tools are new two-resolvent laws for such H_1 and H_2.

Keywords Quantum dynamics · Loschmidt echo · Matrix Dyson equation

Mathematics Subject Classification 60B20 · 82C10

1 Introduction

Recent quantum technological advances put quantum mechanical time reversal procedures in the focus of both experimental [25, 32, 37, 38, 40, 42] and theoretical [16, 17, 30, 31, 33, 43–45] research (see also the review [29] for a concise overview). The basic physical setup consists of an initial (normalized) quantum state ψ_0 and two self-adjoint Hamiltonians close to each other, $H_1 \approx H_2$, each governing the evolution of the system during a time span t. First, the initial state ψ_0 evolves under the Hamiltonian H_1 from time zero to t, resulting in the state $\psi_t = \exp(-\mathrm{i}H_1 t)\psi_0$. Then, during a second evolution between t and $2t$, one applies the Hamiltonian H_2 backward in time, equivalently the Hamiltonian $-H_2$ in forward time, aiming to recover the initial

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✉ Oleksii Kolupaiev
oleksii.kolupaiev@ist.ac.at

László Erdős
lerdos@ist.ac.at

Joscha Henheik
joscha.henheik@ist.ac.at

1 Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria

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state ψ_0. A schematic summary of this process is given by

$$\psi_0 \xrightarrow[H_1]{t} \psi_t \xrightarrow[-H_2]{t} \psi_0' . \tag{1.1}$$

Note that if $H_2 = H_1$, the restoration of ψ_0 would be perfect, $\psi_0' = \psi_0$ for any time t. However, in realistic setup the second Hamiltonian is never a perfect copy of the first one: the nonzero difference between H_1 and H_2 regularly leads to an imperfect recovery ψ_0' of ψ_0 and the discrepancy also depends on time.

This imperfection in the time reversal is captured in the scalar *overlap function* [28, 45, 49] (sometimes also called *fidelity amplitude* [27, 28, 50])

$$\mathfrak{S}(t) = \mathfrak{S}^{(E_0)}_{H_1,H_2}(t) := \left\langle \psi_0, \mathrm{e}^{\mathrm{i}H_2 t}\mathrm{e}^{-\mathrm{i}H_1 t}\psi_0 \right\rangle \tag{1.2}$$

where it is assumed that the initial state is supported[1] around its energy $\langle \psi_0, H_1\psi_0\rangle \approx \langle \psi_0, H_2\psi_0\rangle \approx E_0$. The central object of our paper is the absolute value square of the overlap function

$$\mathfrak{M}(t) = \mathfrak{M}^{(E_0)}_{H_1,H_2}(t) := \left| \mathfrak{S}^{(E_0)}_{H_1,H_2}(t) \right|^2 . \tag{1.3}$$

This was coined the *fidelity*, e.g., by Gorin et al. [28], or the *Loschmidt echo* by Peres [41] and Jalabert–Pastawski in [34] owing to its connection to the classical Loschmidt's paradox of time reversibility [6, 39].

In addition to (1.2)–(1.3), we will also consider an *averaged overlap function* and an *averaged Loschmidt echo*, defined as

$$\overline{\mathfrak{S}}(t) = \overline{\mathfrak{S}}^{(E_0,\eta_0)}_{H_1,H_2}(t) := \mathrm{Av}\left[\mathfrak{S}^{(E)}_{H_1,H_2}(t)\right] \quad \text{and} \quad \overline{\mathfrak{M}}(t) = \overline{\mathfrak{M}}^{(E_0,\eta_0)}_{H_1,H_2}(t) := \left|\overline{\mathfrak{S}}^{(E_0,\eta_0)}_{H_1,H_2}(t)\right|^2, \tag{1.4}$$

respectively. In (1.4), by Av[...], we denoted an averaging over initial states with energies E in a small energy window of size η_0 around E_0 (see (2.5) below for a precise implementation of this concept).

The Loschmidt echo is a basic object in the study of complex quantum system and has attracted considerable attention in different areas of research, e.g., quantum chaos [30, 31, 33, 34, 41, 45], quantum information theory [24, 26], and statistical mechanics [16, 17, 43, 44]. The Loschmidt echo, as a measurable physical quantity, is observed and predicted to follow a quite universal behavior as a function of time (cf. the discussion of our main results around (1.6)–(1.7) below). On a high level (see [29]), the reason for the robust universal features is that the subsequent forward and backward evolutions act as a "filter" for irrelevant details. The typical behavior of the Loschmidt echo can be structured in three consecutive phases (see Fig. 1, cf. also [29, Figure 4]): After an initial short-time parabolic decay, $\mathfrak{M}(t) \approx 1 - \gamma t^2$, the Loschmidt echo exhibits an intermediate-time asymptotic exponential decay[2], $\mathfrak{M}(t) \approx \mathrm{e}^{-\Gamma t}$. Finally,

[1] This means that when writing $\psi_0 = \sum_n c_n^{(i)} \phi_n^{(i)}$ in the eigenbasis $\{\phi_n^{(i)}\}_n$ of H_i, only coefficients $c_n^{(i)}$ corresponding to an eigenvalue close to E_0 are non-vanishing.

[2] In the very extreme case, when the difference $H_1 - H_2$ is small compared to the local eigenvalue spacing one observes Gaussian instead of exponential decay, $\mathfrak{M}(t) \approx \mathrm{e}^{-\gamma t^2}$ (see, e.g., [29, Section 2.3.1])

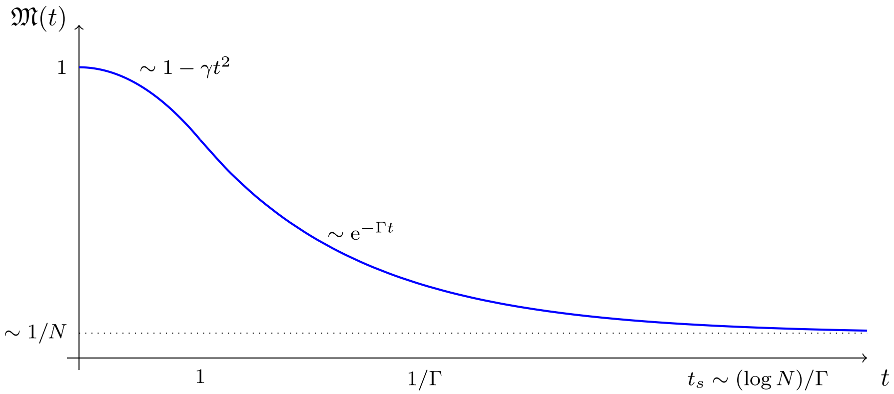

Fig. 1 Illustrated is the typical behavior of the Loschmidt echo in its three consecutive phases: Short-time parabolic decay, intermediate-time asymptotic decay, and long-time saturation. In both of our main results (1.6)–(1.7), the decay parameters γ and Γ generally satisfy $\gamma \sim \Gamma \sim N^{-1}\mathrm{Tr}(H_1 - H_2)^2$; cf. (1.8)

at times t beyond the so-called *saturation time* $t_s \sim (\log N)/\Gamma$, where N is the (effective) Hilbert space dimension, it saturates at a value inversely proportional to N, i.e., $\mathfrak{M}(t) \sim 1/N$. We restrict our study to the first two regimes. For the explanation of technical issues related to the third regime, we refer to the discussion below Theorem 2.4.

There are several ways to determine the behavior of the Loschmidt echo in a given system (see the review [29]): One standard option is to employ semi-classical approximations [34, 49, 52], another one is numerical evaluation [19, 47, 48]. Here, following E. Wigner's original vision of describing chaotic quantum systems by large random matrices [51] and the Bohigas–Giannoni–Schmit (BGS) conjecture [5] (see also further extensive physics literature [7, 8, 15, 17, 18, 27, 35]), we model (part of) the Hamiltonian(s) H_1, H_2 by Wigner random matrices with independent entries. In this setup, we can give a mathematically rigorous and quite precise analysis of certain features of the Loschmidt echo; some of them have been predicted in the physics literature.

Before defining the precise model, we first discuss where the name *echo* for $\mathfrak{M}(t)$ comes from. Fix any time $t > 0$ and consider the two-step process (1.1). For $s \in [0, 2t]$ denote the state at the intermediate time s by ψ_s, namely, $\psi_s = \mathrm{e}^{-\mathrm{i}sH_1}\psi_0$ for $s \in [0, t]$ and $\psi_s = \mathrm{e}^{\mathrm{i}(t-s)H_2}\mathrm{e}^{-\mathrm{i}tH_1}\psi_0$ for $s \in [t, 2t]$. Comparing this notation to (1.1) we see that $\psi_{2t} = \psi_0'$. Denote further the (squared) overlap of ψ_0 and ψ_s by

$$\mathfrak{P}_t(s) := |\langle \psi_0, \psi_s \rangle|^2. \tag{1.5}$$

This quantity depends also on ψ_0 and H_1, H_2, but we suppress this dependence in notations for simplicity. We call $\mathfrak{P}_t(s)$, $s \in [0, 2t]$, the *Loschmidt echo process*. Clearly, $\mathfrak{P}_t(0) = 1$ and $\mathfrak{P}_t(2t) = \mathfrak{M}(t)$. Later in Corollary 2.5 we show that $\overline{\mathfrak{P}}_t(t) \ll \overline{\mathfrak{P}}_t(2t)$ under suitable assumptions, where $\overline{\mathfrak{P}}_t$ is an averaged version of $\mathfrak{P}_t$ defined in (2.9a)-(2.9b). This result means that typically the original complete overlap $\mathfrak{P}_t(0) = 1$

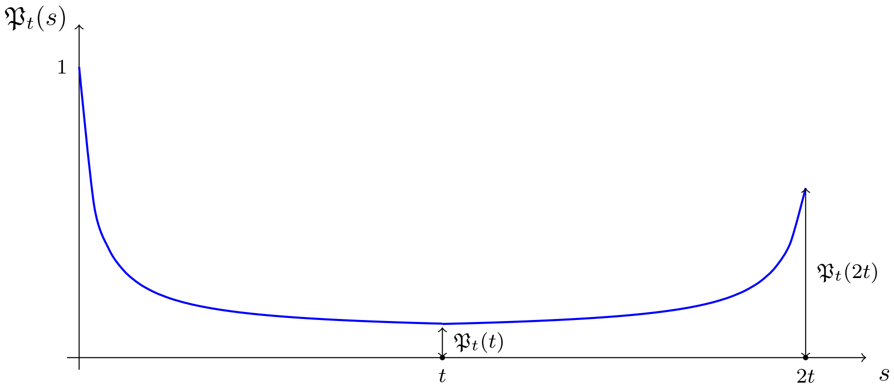

Fig. 2 [*Echo feature*] Schematic behavior of the overlap $\mathfrak{P}_t(s)$ from (1.5) for $s \in [0, 2t]$. At the midpoint, $s = t$, typically $\mathfrak{P}_t(t) \ll \mathfrak{P}_t(2t)$, which indicates a partial recovery between time t and $2t$ of the original complete overlap at time $s = 0$

is partially recovered at the final moment of time $2t$, though at the intermediate time t it is much smaller than $\mathfrak{P}_t(2t)$ (see Fig. 2).

As our main result, we rigorously prove the decay of the Loschmidt echo for two different physical settings (called *Scenario I* and *Scenario II*), which we now describe somewhat informally (see Sect. 2 for more precise statements containing all the technical details).

For our first result (Scenario I, Theorem 2.4), we consider two *deformed* $N \times N$ *Wigner matrices* $H_j = D_j + W$, $j = 1, 2$, with bounded deterministic D_j, satisfying $D_1 \approx D_2$, and W a (common) random Wigner matrix. This setup corresponds to an arbitrary deterministic system modeled by the Hamiltonian D_1 and the time reversed Hamiltonian D_2 nearby, which are both subject to an overall mean-field noise described by the same Wigner matrix W throughout the whole echo process. In this setting, for an energy E_0 in the bulk of the density of states of both H_1 and H_2, we consider the averaged Loschmidt echo (1.4). Our result in Theorem 2.4 then shows (i) *short-time parabolic decay* and (ii) *intermediate-time asymptotic decay* of the form

$$\overline{\mathfrak{M}}(t) \approx \begin{cases} 1 - \gamma t^2 & \text{for} \quad t \ll 1 \\ \mathrm{e}^{-\Gamma t} & \text{for} \quad 1 \ll t \lesssim \Delta^{-2}. \end{cases} \tag{1.6}$$

Both *decay parameters* satisfy $\gamma \sim \Delta^2$ and $\Gamma \sim \Delta^2$, where $\Delta := \langle (D_1 - D_2)^2 \rangle^{1/2}$, and depend on E_0 and the density of states at E_0. Here, we introduced the notation $\langle A \rangle := \frac{1}{N} \mathrm{Tr}\, A$ for any $N \times N$ matrix A. We point out that the quadratic relation $\Gamma \sim \Delta^2$ is in perfect agreement with *Fermi's golden rule*.

For our second main result (Scenario II, Theorem 2.10), we consider a physically different situation: Now the two Hamiltonians[3] are $H_1 = D$ and $H_2 = D + \lambda W$ with the same deterministic D, a standard Wigner matrix W and a small parameter

[3] Within Theorem 2.10, the Hamiltonians $H_1 = D$ and $H_2 = D + \lambda W$ will be denoted by H_0 and H_λ, respectively.

$|\lambda| \ll 1$. The normalization is chosen such that $\mathbf{E}\langle W^2\rangle = 1$. Hence, the imperfection along the backward evolution is modeled by a small Wigner matrix λW indicating an additive noise (see, e.g., [16, Eq. (31)] or [17, Eq. (1)]). For a normalized initial state $\psi_0 \in \mathbf{C}^N$ supported in the bulk of the density of states of both H_1 and H_2 with energy $\langle \psi_0, H_1\psi_0\rangle \approx \langle \psi_0, H_2\psi_0\rangle \approx E_0$, we now consider the usual Loschmidt echo (1.3) without averaging. Similarly to (1.6), our result in Theorem 2.10 then shows (i) *short-time parabolic decay* and (ii) *intermediate-time asymptotic decay* of the form

$$\mathfrak{M}(t) \approx \begin{cases} 1-\gamma t^2 & \text{for} \quad t \ll 1 \\ \mathrm{e}^{-\Gamma t} & \text{for} \quad 1 \ll t \lesssim \lambda^{-2}. \end{cases} \tag{1.7}$$

Here the *decay parameters* satisfy $\gamma = \lambda^2$ and $\Gamma = 2\pi\rho_0(E_0)\lambda^2$, where ρ_0 is the (limiting, as $N \to \infty$) density of states of D. Finally, we note that since $\mathbf{E}\langle W^2\rangle = 1$, in both of our scenarios (1.6)–(1.7) the decay parameters γ and Γ satisfy the general relation

$$\gamma \sim \Gamma \sim \mathbf{E}\langle (H_1 - H_2)^2\rangle\,. \tag{1.8}$$

As corollaries to our main results (1.6)–(1.7) in Theorems 2.4 and 2.10, we also consider the *scrambled Loschmidt echo* [16, 36, 44] $\mathfrak{M}_\delta^{sc}(t)$ and its averaged analog $\overline{\mathfrak{M}}_\delta^{sc}(t)$. They are defined from

$$\mathfrak{S}_\delta^{\mathrm{sc}}(t) := \left\langle \psi_0, \mathrm{e}^{\mathrm{i}H_2 t}\mathrm{e}^{-\mathrm{i}\delta V}\mathrm{e}^{-\mathrm{i}H_1 t}\psi_0\right\rangle \tag{1.9}$$

and its averaged analog $\overline{\mathfrak{S}}_\delta^{\mathrm{sc}}(t)$ as

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) := \left|\mathfrak{S}_\delta^{\mathrm{sc}}(t)\right|^2 \quad \text{and} \quad \overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) := \left|\overline{\mathfrak{S}}_\delta^{\mathrm{sc}}(t)\right|^2,$$

exactly as in (1.3)–(1.4), respectively. In (1.9), H_1 and H_2 are the two Hamiltonians either from Scenario I or Scenario II. The idea behind the quantity in (1.9) is that, between the forward and backward evolution, there is a (short) *scrambling time* δ, in which the system is uncontrolled and governed by another self-adjoint *scrambling Hamiltonian* V [16]. Similarly to (1.1), a schematic summary of this process is given by

$$\psi_0 \xrightarrow[H_1]{t} \psi_t \xrightarrow[V]{\delta} \psi_t' \xrightarrow[-H_2]{t} \psi_0'\,. \tag{1.10}$$

In Corollaries 2.6 and 2.11 (of Theorems 2.4 and 2.10, respectively), we model the scrambling Hamiltonian by another Wigner matrix, $V := \widetilde{W}$, that is *independent* of W; see [16]. As a result, we find that

$$\overline{\mathfrak{M}}_\delta^{sc}(t) \approx (\varphi(\delta))^2\,\overline{\mathfrak{M}}(t) \quad \text{and} \quad \mathfrak{M}_\delta^{sc}(t) \approx (\varphi(\delta))^2\,\mathfrak{M}(t) \tag{1.11}$$

in the setting of Scenario I and Scenario II, respectively, where we denoted $\varphi(\delta) := J_1(2\delta)/\delta$ and J_1 is the first order Bessel function of the first kind. Note that in (1.11) we see the effects of the scrambling Hamiltonian V and the imperfect time reversal of H_1 and H_2 to completely decouple (cf. [16, Eq. (35)]).

We point out that Scenario II, discussed around (1.7), and the corollaries described in (1.11) are primarily given to provide a more comprehensive view of Loschmidt echoes modeled with Wigner matrices. Technically, these are obtained by simple modifications of earlier results and techniques [10, 22] (see the proof in Sect. 7 for details). The mathematically novel principal part of this work therefore consists of Theorem 2.4 analyzing Scenario I.

The proof of Theorem 2.4 relies on a new *two-resolvent global law*, i.e., a concentration estimate for products of resolvents $G_i(z_i) := (H_i - z_i)^{-1}$ for $z_i \in \mathbf{C}\backslash\mathbf{R}$ as the dimension N of the matrix becomes large. By functional calculus, this can then be used for computing more complicated functions of H_i, like the exponential, and thus connecting to the time evolutions above. A typical global law computes, e.g.,

$$\langle \psi_0, G_2(z_2)G_1(z_1)\psi_0 \rangle \tag{1.12}$$

to leading order in N with error terms vanishing like $N^{-1/2+\epsilon}$ with very high probability. The main novelty of this paper is a precise estimate on the deterministic leading term to (1.12). While it is well known that $G_i(z_i) \approx M_i(z_i)$, where the deterministic matrix $M_i(z_i)$ is the solution of the *Matrix Dyson equation* (2.1), it does *not* hold that $G_2(z_2)G_1(z_1) \approx M_2(z_2)M_1(z_1)$ owing to correlations between G_1 and G_2. The correct approximation is

$$G_2(z_2)G_1(z_1) \approx \frac{M_2(z_2)M_1(z_1)}{1 - \langle M_1(z_1)M_2(z_2)\rangle} . \tag{1.13}$$

To control (1.13), we hence need to estimate the denominator of (1.13), which is well known in case of $H_1 = H_2$, i.e., $D_1 = D_2$ [9, 13, 22]. Here, however, the analysis of (1.13) is much more intricate, since for general D_1, D_2 the deterministic approximations $M_1(z_1), M_2(z_2)$ do *not* commute. In our main Proposition 4.2, we optimally track the dependence of (1.13) on the difference $D_1 - D_2$ of the two deformations and on $z_1 - z_2$.

Notations

For positive quantities f, g we write $f \lesssim g$ (or $f = \mathcal{O}(g)$) and $f \sim g$ if $f \le Cg$ or $cg \le f \le Cg$, respectively, for some constants $c, C > 0$ which only depend on the constants appearing in the moment condition (see Assumption 2.1), the bound on M in Assumption 2.2, the constants from Assumption 2.8, or the bulk parameter κ from (2.3). In informal explanations, we frequently use the notation $f \ll g$, which indicates that f is "much smaller" than g. Moreover, we shall also write $w \approx z$ to indicate the closeness of $w, z \in \mathbf{C}$ with a not precisely specified error.

For any natural number n, we set $[n] := \{1, 2, \dots, n\}$. Matrix entries are indexed by lowercase Roman letters a, b, c, ... from the beginning of the alphabet. We denote vectors by bold-faced lowercase Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, or lower case Greek letters $\psi, \phi \in \mathbf{C}^N$, for some $N \in \mathbf{N}$. Vector and matrix norms, $\|\boldsymbol{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $N \times N$ matrix A we use the notation $\langle A \rangle := N^{-1}\mathrm{Tr} A$ for its normalized trace and denote the spectrum of A by $\sigma(A)$. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ we denote their scalar product by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_i \overline{x}_i y_i$. The support of a function f is denoted by $\mathrm{supp}(f)$.

Finally, we use the concept of "with very high probability" *(w.v.h.p.)* meaning that for any fixed $C > 0$, the probability of an N-dependent event is bigger than $1 - N^{-C}$ for $N \geq N_0(C)$. We also introduce the notion of *stochastic domination* (see, e.g., [20]): given two families of non-negative random variables

$$X = \left(X^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)} \right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)} \right)$$

indexed by N (and possibly some parameter u in some parameter space $U^{(N)}$), we say that X is stochastically dominated by Y, if for all $\xi, C > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P} \left[X^{(N)}(u) > N^{\xi} Y^{(N)}(u) \right] \leq N^{-C} \tag{1.14}$$

for large enough $N \geq N_0(\xi, C)$. In this case we use the notation $X \prec Y$ or $X = \mathcal{O}_\prec(Y)$.

2 Main results

The key players of our paper are *deformed Wigner matrices*, i.e., matrices of the form $H = D + W$, where $D = D^* \in \mathbf{C}^{N \times N}$ is a bounded deterministic matrix (called *deformation*), $\|D\| \leq L$ for some N-independent $L > 0$ and $W = W^* \in \mathbf{C}^{N \times N}$ is a real symmetric or complex Hermitian Wigner matrices. This means, its entries are independently distributed random variables according to the laws[4] $w_{ij} \stackrel{\mathrm{d}}{=} N^{-1/2} \chi_{\mathrm{od}}$ for $i < j$ and $w_{jj} \stackrel{\mathrm{d}}{=} N^{-1/2} \chi_{\mathrm{d}}$. For the single entry distributions χ_{od} and χ_{d} we assume the following.

Assumption 2.1 (*Wigner matrix*) We assume that χ_{d} is a centered real random variable, and χ_{od} is a real or complex random variable with $\mathbf{E}\chi_{\mathrm{od}} = 0$ and $\mathbf{E}|\chi_{\mathrm{od}}|^2 = 1$. Furthermore, we assume the existence of higher moments, namely $\mathbf{E}|\chi_{\mathrm{d}}|^p + \mathbf{E}|\chi_{\mathrm{od}}|^p \leq C_p$ for all $p \in \mathbf{N}$, where C_p are positive constants.

It is well known [2, 21] that the resolvent of H, denoted by $G(z) := (H - z)^{-1}$ for $z \in \mathbf{C} \backslash \mathbf{R}$, becomes approximately deterministic in the large N limit. Its deterministic approximation (as a matrix) is given by $M(z)$, the unique solution of the Matrix Dyson equation (MDE)

$$-\frac{1}{M(z)} = z - D + \langle M(z) \rangle \quad \text{for} \quad z \in \mathbf{C} \setminus \mathbf{R} \quad \text{under the constraint} \quad \Im z \, \Im M(z) > 0 \,, \tag{2.1}$$

4 A careful examination of our proof reveals that the entries of W need not be distributed identically. Indeed, only the matching of the second moments is necessary, but higher moments can differ.

where $\Im M(z) := [M(z) - M(z)^*]/2\mathrm{i}$ and positivity is understood as a matrix. The corresponding (N-dependent) *self-consistent density of states (scDos)* is defined as

$$\rho(e) := \frac{1}{\pi} \lim_{\eta \downarrow 0} \langle \Im M(e + \mathrm{i}\eta) \rangle . \tag{2.2}$$

This is a compactly supported Hölder-1/3 continuous function on $\mathbf{R}$ which is in fact real-analytic on the set $\{\rho > 0\}$,[5]. The positive harmonic extension of ρ is denoted by $\rho(z) := \pi^{-1}|\langle \Im M(z)\rangle|$ for $z \in \mathbf{C}\backslash\mathbf{R}$. We point out that not only the tracial quantity $\langle \Im M(e + \mathrm{i}\eta)\rangle$ has an extension to the real axis, but the whole matrix $M(e) := \lim_{\eta\downarrow 0} M(e + \mathrm{i}\eta)$ is well defined (see Lemma B.1 (b) of the arXiv: 2301.03549 version of [14]). Moreover, for any small $\kappa > 0$ (independent of N) we define the κ*-bulk* of the scDos (2.2) as

$$\mathbf{B}_\kappa(\rho) = \{x \in \mathbf{R} \ : \ \rho(x) \geq \kappa\} . \tag{2.3}$$

It is a finite union of disjoint compact intervals, cf. Lemma B.2 in the arXiv: 2301.03549 version of [14]. Note that, for $\Re z \in \mathbf{B}_\kappa$ it holds that $\|M(z)\| \lesssim 1$, as easily follows by taking the imaginary part of (2.1).

Now, the resolvent G is close to M from (2.1) in the following *averaged* and *isotropic* sense:

$$|\langle (G(z) - M(z))B\rangle| \prec \frac{1}{N|\Im z|}, \qquad |\langle \boldsymbol{x} \, , (G(z) - M(z))\boldsymbol{y}\rangle| \prec \frac{1}{\sqrt{N|\Im z|}} , \tag{2.4}$$

uniformly in deterministic vectors $\|\boldsymbol{x}\| + \|\boldsymbol{y}\| \lesssim 1$ and deterministic matrices $\|B\| \lesssim 1$. These estimates are called *local laws* when $|\Im z| \ll 1$ and *global laws* when $|\Im z| \gtrsim 1$. To be precise about their validity, we recall that while (2.4) holds for $\Re z \in \mathbf{B}_\kappa$ and $\mathrm{dist}(\Re z, \mathrm{supp}(\rho)) \gtrsim 1$ for *arbitrary* bounded self-adjoint deformations $D = D^*$ (see [21, Theorem 2.1]), the complementary regime requires the additional Assumption 2.2 on D stated below (see [4, Theorem 2.6] and [23, Theorem 2.8]). A sufficient condition for Assumption 2.2 is discussed in Remark 2.3; see also [3].

In the remainder of this section, we formulate our main results on the two different Loschmidt echo scenarios described in Sect. 1.

2.1 Scenario I: Two deformations of a Wigner matrix

For the first echo scenario, we consider two deformed Wigner matrices, $H_j = D_j + W$, $j \in [2]$, and denote their resolvents and corresponding deterministic approximation

[5] In [1–3], the scDos has been thoroughly analyzed in increasing generality of the ensemble. It is supported on finitely many finite intervals and, roughly speaking, there are three different regimes for the behavior or ρ: In the *bulk* ρ is strictly positive; at the *edge*, ρ vanishes like a square root at the edges of every supporting interval which are well separated; at the *cusp*, where two intervals of support (almost) meet, ρ behaves (almost) as a cubic root. Correspondingly, ρ is locally real analytic, Hölder-1/2, or Hölder-1/3 continuous, respectively. Near the singularities, it has an approximately universal shape (see (A.3a)–(A.3d) in the proof of Lemma A.1).

(2.1) by G_j and M_j, respectively. A natural definition of the averaged Loschmidt echo is

$$\overline{\mathfrak{M}}(t) = \overline{\mathfrak{M}}^{(E_0,\eta_0)}_{H_1,H_2}(t) := \left| \frac{\langle e^{itH_1} \Im G_1(E_0 + i\eta_0) e^{-itH_2} \rangle}{\langle \Im M_1(E_0 + i\eta_0) \rangle} \right|^2 , \tag{2.5}$$

since $\Im G / \langle \Im M \rangle$ in (2.5) effectively localizes around E_0 and *averages* in a window of size $\eta_0 > 0$ assumed to be independent of N. In Remark 2.7 below we comment on the averaging implemented by (2.5). Note that in order to match (1.2)-(1.3) from the introduction we need to replace t by $-t$ in (2.5). However, this replacement does not change the quantity (2.5) since

$$\left|\left\langle e^{itH_1} \Im G_1(E_0 + i\eta_0) e^{-itH_2} \right\rangle\right| = \left|\left\langle e^{itH_2} \Im G_1(E_0 + i\eta_0) e^{-itH_1} \right\rangle\right| = \left|\left\langle e^{-itH_1} \Im G_1(E_0 + i\eta_0) e^{itH_2} \right\rangle\right| ,$$

where in the last step we used that e^{itH_1} and $\Im G_1(E_0 + i\eta_0)$ commute. Using this observation, we will work with (2.5) in the rest of the paper. The same comment applies also to the other versions of the averaged Loschmidt echo defined in Sect. 2.1, namely to (2.9a), (2.9b) and (2.12).

We will henceforth assume that the deformations D_1, D_2 are such that the corresponding solutions M_1, M_2 to (2.1) are bounded.

Assumption 2.2 (*Boundedness of M*) Let D be an $N \times N$ Hermitian matrix and M the solution to (2.1). We assume that there exists an N-independent positive constant L such that $\sup_{z \in \mathbf{C} \setminus \mathbf{R}} \| M(z) \| < L$.

Assumption 2.2 is the basis for the *shape theory* of the scDos, which we briefly described in Footnote 5. We now give a sufficient condition on D for Assumption 2.2 to hold. It basically requires that its ordered eigenvalue sequence has to be piecewise Hölder-$1/2$ continuous as a function of the label.

Remark 2.3 (Sufficient condition for Assumption 2.2) Denote the eigenvalues of any self-adjoint deformation D by $\{d_j\}_{j=1}^N$ labeled in increasing order, $d_j \le d_k$ for $j < k$. Fix a (large) positive constant $L > 0$. The set $\mathcal{M}_L$ of admissible self-adjoint deformations D is defined as follows: we say that $D \in \mathcal{M}_L$ if $\|D\| \le L$ and there exists an N-independent partition $\{I_s\}_{s=1}^m$ of $[0, 1]$ in at most L segments such that for any $s \in [1, m]$ and any $j, k \in [1, N]$ with $j/N, k/N \in I_s$ we have $|d_j - d_k| \le L|j/N - k/N|^{1/2}$. Since the operator $\mathcal{S} = \langle \cdot \rangle$ is flat, condition $D \in \mathcal{M}_L$ implies that D satisfies Assumption 2.2 for some $L' < \infty$ by means of [3, Lemma 9.3].

We can now formulate our first main result.

Theorem 2.4 (Averaged Loschmidt echo with two deformations) *Let W be a Wigner matrix satisfying Assumption 2.1, and $D_1, D_2 \in \mathbf{C}^{N \times N}$ be bounded, traceless*[6] *Hermitian matrices, i.e., $\|D_j\| \le L$ for some $L > 0$ and $\langle D_1 \rangle = \langle D_2 \rangle = 0$, additionally satisfying Assumption 2.2. Fix $\eta_0 \le 1$ and let E_0 be an energy in the bulk of the scDos of H_1 and H_2, i.e., assume that there exist $\delta, \kappa > 0$ such that*

[6] If D_1 or D_2 had a nonzero trace, it could be absorbed by a simple (scalar) energy shift.

$[E_0-\delta, E_0+\delta] \subset \mathbf{B}_\kappa(\rho_1) \cap \mathbf{B}_\kappa(\rho_2)$. *We also assume that parameters* η_0, κ *and* δ *are* N*-independent.*

Consider the deformed Wigner matrices $H_j := D_j + W$ *for* $j \in [2]$ *and the corresponding averaged (at energy* E_0 *in a window of size* $\eta_0 > 0$*) Loschmidt echo* $\overline{\mathfrak{M}}(t)$ *for times* $t \geq 0$ *defined in* (2.5). *Then, we have the following:*

(i) [Short-time parabolic decay] *As* $t \to 0$, *it holds that*

$$\overline{\mathfrak{M}}(t) = 1 - \gamma t^2 + \mathcal{O}\big(\langle D^2\rangle t^3\big) + \mathcal{O}_\prec((N\eta_0)^{-1}) \tag{2.6}$$

where the decay parameter *is given by* $\gamma := \langle (D - \langle PD\rangle)^2 P\rangle$, *where we abbreviated* $D := D_2 - D_1$ *and* $P := \Im M_1(E_0+\mathrm{i}\eta_0)/\langle \Im M_1(E_0+\mathrm{i}\eta_0)\rangle$. *It satisfies* $\gamma \sim \Delta^2 := \langle D^2\rangle$ *and the implicit constant in* $\sim$ *depends only on* κ *and* L.
The implicit constants in the error terms in (2.6) *depend only on* L, δ, κ *and the* C_p*'s from Assumption 2.1.*

(ii) [Intermediate-time asymptotic decay] *Take a (large) positive* K *and consider times* $1 \leq t \leq K/\Delta^2$. *Then, there exists a positive constant* c *such that whenever* $\Delta < c$ *and* $\eta_0 < \Delta/|\log\Delta|$ *it holds that*

$$\overline{\mathfrak{M}}(t) = \mathrm{e}^{-\Gamma t} + \mathcal{O}(\mathcal{E}) + \mathcal{O}_\prec\big(C(t,\eta_0)/N\big), \tag{2.7}$$

where the rate Γ *(explicitly given in* (4.27)*) satisfies* $\Gamma \sim \Delta^2$ *with the implicit constant depending only on* κ *and* L. *Moreover, we denoted*

$$\mathcal{E} = \mathcal{E}(t,\Delta,\eta_0) := \frac{1+\log t}{t} + \Delta|\log\Delta| + \frac{\eta_0|\log\Delta|}{\Delta} \tag{2.8}$$

and $C(t) > 0$ *is a positive constant depending only on* t.
The implicit constants in the error terms in (2.7) *depend only on* L, δ, κ, K *and the* C_p*'s from Assumption 2.1.*

Note that Theorem 2.4 addresses only times t which do not depend on N. Reaching times of order $\log N$, i.e., accessing the saturation time would require a different proof strategy. Since $t \leq K/\Delta^2$, we find that the leading term $\mathrm{e}^{-\Gamma t}$ in (2.7) remains of order one throughout the whole time regime. The error term $\mathcal{E}$ is small compared to this leading term if $t \gg 1$, $\Delta \ll 1$, and $\eta_0 \ll \Delta/|\log\Delta|$; hence, these relations define the regime of the parameters where our theorem is meaningful.

The following corollary to Theorem 2.4 reveals the key property of the Loschmidt echo, the partial recovery of the initial overlap, as discussed in the introduction; see Fig. 2.

Corollary 2.5 (Averaged Loschmidt echo process) *Assume the set-up and the conditions of Theorem 2.4. For time* $t > 0$ *define the* averaged Loschmidt echo process $\overline{\mathfrak{P}}_t(s)$, $s \in [0, 2t]$, *as follows:*

$$\overline{\mathfrak{P}}_t(s) := \left|\frac{\big\langle \mathrm{e}^{\mathrm{i}sH_1}\Im G_1(E_0+\mathrm{i}\eta_0)\big\rangle}{\langle \Im M_1(E_0+\mathrm{i}\eta_0)\rangle}\right|^2, \qquad s \in [0,t], \tag{2.9a}$$

$$\overline{\mathfrak{P}}_t(s) := \left|\frac{\langle e^{itH_1}\Im G_1(E_0+i\eta_0)e^{-i(s-t)H_2}\rangle}{\langle \Im M_1(E_0+i\eta_0)\rangle}\right|^2, \quad s\in(t,2t]. \tag{2.9b}$$

Let $\lim^*$ *be the simultaneous limit in* Δ, η_0, t *such that* $\Delta, \eta_0 \to 0$ *and* $t\to\infty$ *under constraints* $\Delta^2 \ll \eta_0 \ll \Delta/|\log\Delta|$ *and* $1/\eta_0 \ll t \lesssim 1/\Delta^2$. *Here* $a \ll b$ *means that* $a/b \to 0$ *in this limit. Then, almost surely we have*

$$\lim^* \limsup_{N\to\infty} \frac{\overline{\mathfrak{P}}_t(t)}{\overline{\mathfrak{P}}_t(2t)} = \lim^* \limsup_{N\to\infty} \frac{\overline{\mathfrak{P}}_t(t)}{e^{-\Gamma t}} = 0, \tag{2.10}$$

where Γ *is the same as in Theorem 2.4.*

Proof of Corollary 2.5 Firstly take the limit $N\to\infty$ in the denominator $\overline{\mathfrak{P}}_t(2t) = \overline{\mathfrak{M}}(t)$ of (2.10). Recall the definition of $\mathcal{E}$ from (2.8). By means of Theorem 2.4 we have

$$\liminf_{N\to\infty} \overline{\mathfrak{P}}_t(2t) = \liminf_{N\to\infty}\left(e^{-\Gamma t} + \mathcal{O}(\mathcal{E}(t,\Delta,\eta_0))\right) = \liminf_{N\to\infty}\left(e^{-\Gamma t}(1+o(1))\right) \sim 1$$

in the limit $\lim^*$. Here, we used that $\Gamma \sim \Delta^2$ and $t \lesssim \Delta^{-2}$, so $e^{-\Gamma t}\sim 1$. Thus, in order to verify (2.10) it is sufficient to show that

$$\lim^* \limsup_{N\to\infty} \overline{\mathfrak{P}}_t(t) = 0.$$

From the average single resolvent global law for H_1, see (2.4) or [21, Theorem 2.1], we get that

$$\lim_{N\to\infty}\left|\left\langle e^{itH_1}\Im G_1(E_0+i\eta_0)\right\rangle - \int_{\mathbf{R}} e^{itx}\frac{\eta_0}{(x-E_0)^2+\eta_0^2}\rho_1(x)\mathrm{d}x\right| = 0.$$

Recall that $E_0 \in \boldsymbol{B}_\kappa(\rho_1)$. Thus $\langle \Im M(E_0+i\eta_0)\rangle \sim 1$ for $\eta_0\to 0$ and

$$\limsup_{N\to\infty}\overline{\mathfrak{P}}_t(t) \lesssim \limsup_{N\to\infty}\left|\int_{\mathbf{R}} e^{itx}\frac{\eta_0}{(x-E_0)^2+\eta_0^2}\rho_1(x)\mathrm{d}x\right|^2 \lesssim \left(\frac{1}{\eta_0 t}\right)^2. \tag{2.11}$$

In the last inequality, we employed integration by parts. Additionally we used that $\rho_1(x)$ is a bounded function of x which is guaranteed by Assumption 2.2 and that $\rho_1(x)$ has bounded derivative for $|x-E_0|\le\delta$ (see also Footnote 5), where δ was fixed in Theorem 2.4. Both of these bounds (on $\rho_1(x)$ and $\mathrm{d}\rho_1(x)/\mathrm{d}x$) are uniform in N. In the limit $\lim^*$ we have $\eta_0 t\to\infty$, so (2.11) finishes the proof of Corollary 2.5. □

As mentioned in the introduction, we also have the following corollary to Theorem 2.4.

Corollary 2.6 (Scrambled averaged Loschmidt echo with two deformations) *Assume the conditions of Theorem 2.4 and consider (as a variant of (2.5)) the* scrambled *averaged Loschmidt echo*

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) := \left| \frac{\left\langle \mathrm{e}^{-\mathrm{i}\delta \widetilde{W}} \mathrm{e}^{\mathrm{i}tH_1} \Im G_1(E_0+\mathrm{i}\eta_0)\mathrm{e}^{-\mathrm{i}tH_2}\right\rangle}{\langle \Im M_1(E_0+\mathrm{i}\eta_0)\rangle}\right|^2, \tag{2.12}$$

where $\widetilde{W}$ is a Wigner matrix satisfying Assumption 2.1, independent *of W and $0 \le \delta \le N^{2/3-\varepsilon}$ for some fixed $\varepsilon > 0$. Moreover, let φ be the Fourier transform of the semi-circular density of states $\rho_{\mathrm{sc}}(x) := (2\pi)^{-1}\sqrt{[4-x^2]_+}$, which is explicitly given as*

$$\varphi(\delta) := \widehat{\rho_{\mathrm{sc}}}(\delta) = \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}\delta x}\rho_{\mathrm{sc}}(x)\mathrm{d}x = \frac{J_1(2\delta)}{\delta} \tag{2.13}$$

where J_1 is the first-order Bessel function of the first kind.

Then, instead of (2.6)–(2.7), we have that

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2\left[1-\gamma t^2+\mathcal{O}\big(\langle D^2\rangle t^3\big)+\mathcal{O}_\prec((N\eta_0)^{-1})\right]+\mathcal{O}_\prec\big(\delta/(N\eta_0)\big) \quad \textit{as} \quad t\to 0$$

and

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2\left[\mathrm{e}^{-\Gamma t}+\mathcal{O}(\mathcal{E})+\mathcal{O}_\prec\big(C(t)/N\big)\right]+\mathcal{O}_\prec\big(\delta/(N\eta_0)\big) \quad \textit{for} \quad 1\le t\le K/\Delta^2$$

in the short and intermediate time regimes, respectively.

Proof of Corollary 2.6 Denote $A := \mathrm{e}^{\mathrm{i}tH_1}\Im G_1(E_0+\mathrm{i}\eta_0)\mathrm{e}^{-\mathrm{i}tH_2}$ and observe that $\|A\| \le 1/\eta_0$. Then, by residue calculus with the contour $C_\delta := \{z \in \mathbf{C} : \mathrm{dist}(z,[-2,2]) = \delta^{-1}\}$ and a single resolvent law[7] as in (2.4), using only the randomness of $\widetilde{W}$, we find

$$\begin{aligned}
\langle \mathrm{e}^{-\mathrm{i}\delta\widetilde{W}}A\rangle &= \frac{1}{2\pi\mathrm{i}}\oint_{C_\delta}\mathrm{e}^{-\mathrm{i}\delta z}\langle A(W-z)^{-1}\rangle\mathrm{d}z\\
&= \frac{\langle A\rangle}{2\pi\mathrm{i}}\oint_{C_\delta}\mathrm{e}^{-\mathrm{i}\delta z}m_{\mathrm{sc}}(z)\mathrm{d}z + \mathcal{O}_\prec(\delta/(N\eta_0))\\
&= \langle A\rangle\int_{\mathbf{R}}\mathrm{e}^{-\mathrm{i}\delta x}\rho_{\mathrm{sc}}(x)\mathrm{d}x + \mathcal{O}_\prec(\delta/\sqrt{N}) = \langle A\rangle\varphi(\delta)+\mathcal{O}_\prec(\delta/(N\eta_0))\,.
\end{aligned}$$

The rest of the proof follows from Theorem 2.4. □

We close this section by commenting on the effect of the small averaging of the Loschmidt echo over several energy states implemented in (2.5). This is a necessary technical step for our proof in Scenario I that relies on a two-resolvent global law. Note that averaging will not be necessary for Scenario II since it uses only single resolvent global law. This is because randomness is present only in the second Hamiltonian, while the first is modeled by a deterministic matrix.

[7] To be precise, when $\delta < 1$, we use the slightly improved average *global* law $|\langle A(W-z)^{-1}\rangle - m(z)\langle A\rangle| \prec \delta^2\|A\|/N$ (see, e.g., [21, Theorem 2.1]).

Remark 2.7 (Averaging of the Loschmidt echo) We provide two independent non-rigorous arguments for the averaged Loschmidt echo $\overline{\mathfrak{M}}$ and the original Loschmidt echo $\mathfrak{M}$ being close to each other.

(1) First, by means of the Eigenstate Thermalization Hypothesis (ETH) for a deformed Wigner matrix $H = D + W$, see [13, Theorem 2.7], and a single resolvent local law (2.4), it holds that

$$\langle \boldsymbol{u}_j, A\boldsymbol{u}_j \rangle \approx \frac{\langle \Im M(E_0 + \mathrm{i}\eta_0)A \rangle}{\langle \Im M(E_0 + \mathrm{i}\eta_0) \rangle} \approx \frac{\langle \Im G(E_0 + \mathrm{i}\eta_0)A \rangle}{\langle \Im M(E_0 + \mathrm{i}\eta_0) \rangle} . \tag{2.14}$$

Here, A is an arbitrary deterministic matrix, $\boldsymbol{u}_j$ is a (normalized) eigenvector of H with eigenvalue $\approx E_0$, and η_0 a small regularization. In this sense, the pure state $|\boldsymbol{u}_j\rangle \langle \boldsymbol{u}_j|$ is *weakly* close to $\Im G / \langle \Im M \rangle$ (i.e., if tested against a deterministic A), which heuristically supports the implementation of the averaged Loschmidt echo in (2.5). However, the rigorous ETH statements do not allow to choose A depending on the underlying randomness like $A = e^{-\mathrm{i}tH_2} e^{\mathrm{i}tH_1}$.

(2) Another supporting argument uses the fact that the *averaged overlap function* $\overline{\mathfrak{S}}^{(E,\eta_0)}(t)$ (in particular its phase) is approximately constant as long as E varies in a range $|E - E_0| \lesssim \eta_0$. Hence, it is irrelevant if one (a) first averages and then takes absolute value square, or (b) does it the other way around. The fact that $\overline{\mathfrak{S}}^{(E,\eta_0)}(t)$ is slowly varying in E follows by a simple computation using that (i) $\overline{\mathfrak{S}}^{(E_0,\eta_0)}(t) \approx I_{E_0,\eta_0}(t) / \langle \Im M_1(E_0 + \mathrm{i}\eta_0) \rangle$ (see (4.1) and (4.7)), (ii) I_{E_0,η_0} is given by $\mathrm{e}^{\mathrm{i}t\mathfrak{s}_0} \langle \Im M_1(E_0 + \mathrm{i}\eta_0) \rangle$ (see (4.26)), (iii) the exponent $\mathfrak{s}_0$ is Lipschitz continuous on scale Δ (see the last relation of (4.16)), and (iv) we have $t \lesssim \Delta^{-2}$ and $\eta_0 \ll \Delta$ by assumption.

Both, the ETH argument (2.14) and the fact that $\overline{\mathfrak{S}}^{(E,\eta_0)}(t)$ is approximately constant as long as $|E - E_0| \lesssim \eta_0$, independently indicate that the averaged Loschmidt echo $\overline{\mathfrak{M}}$ and the non-averaged Loschmidt echo $\mathfrak{M}$ should practically agree with each other. However, neither of them constitutes a rigorous proof, since (1) the observable A in (2.14) cannot be chosen to depend on the randomness, and (2) we cannot exclude that for some initial fixed energy state ψ_0, $\mathfrak{S}$ in (1.2) behaves very differently from its typical value computed by local averaging.

2.2 Scenario II: Perturbation by a Wigner matrix

For the second echo scenario, we consider a single deformed Wigner matrix $H_\lambda = H_0 + \lambda W$ and the Loschmidt echo

$$\mathfrak{M}(t) = \mathfrak{M}_{H_\lambda, H_0}^{(E_0,\Delta)}(t) := \left| \langle \psi_0, \mathrm{e}^{\mathrm{i}tH_\lambda} \mathrm{e}^{-\mathrm{i}tH_0} \psi_0 \rangle \right|^2 \tag{2.15}$$

for some normalized initial state $\psi_0 \in \mathbf{C}^N$ with energy $E_0 = \langle \psi_0, H_0 \psi_0 \rangle$ and localized in an interval of size Δ around E_0 (see Assumption 2.9 below for a precise statement). The localization parameter Δ plays the same role as η_0 in Sect. 2.1, but here we work with a sharp cutoff in the energy.

The unperturbed Hamiltonian H_0 is assumed to satisfy the following.

Assumption 2.8 (*H_0 and its limiting density of states*) The Hamiltonian H_0 is deterministic, self-adjoint $H_0 = H_0^*$, and uniformly bounded, $\|H_0\| \le C_{H_0}$ for some $C_{H_0} > 0$. We denote the resolvent of H_0 at any spectral parameter $z \in \mathbf{C} \setminus \mathbf{R}$ by $M_0(z) := (H_0 - z)^{-1}$. Moreover, we assume the following:

(i) There exists a compactly supported measurable function $\rho_0 : \mathbf{R} \to [0, +\infty)$ with $\int_{\mathbf{R}} \rho_0(x)\mathrm{d}x = 1$ and two positive sequences $\epsilon_0(N)$ and $\eta_0(N)$, both converging to zero as $N \to \infty$, such that, uniformly in $z \in \mathbf{C}\backslash\mathbf{R}$ with $\eta := |\Im z| \ge \eta_0 \equiv \eta_0(N)$, we have

$$\langle M_0(z)\rangle = m_0(z) + \mathcal{O}(\epsilon_0) \quad \text{with} \quad \epsilon_0 \equiv \epsilon_0(N)\,. \tag{2.16}$$

Here,

$$m_0(z) := \int_{\mathbf{R}} \frac{\rho_0(x)}{x - z}\mathrm{d}x \tag{2.17}$$

is the Stieltjes transform of ρ_0. We refer to ρ_0 as the *limiting density of states*, and to $\mathrm{supp}(\rho_0)$ as the *limiting spectrum* of H_0.

(ii) For small positive constants $\kappa, c > 0$, we define the set of *admissible energies* $\sigma_{\mathrm{adm}}^{(\kappa,c)}$ in the limiting spectrum of H_0 by[8]

$$\sigma_{\mathrm{adm}}^{(\kappa,c)} := \left\{x \in \mathrm{supp}(\rho_0) : \inf_{|y-x|\le\kappa} \rho_0(y) > c,\ \|\rho_0\|_{C^{1,1}([x-\kappa,x+\kappa])} \le 1/c\right\}. \tag{2.18}$$

We assume that for some positive N-independent $\kappa, c > 0$, $\sigma_{\mathrm{adm}}^{(\kappa,c)}$ is not empty.

Assuming that the set of admissible energies in (2.18) is non-empty guarantees the limiting spectrum $\mathrm{supp}(\rho_0)$ has a part, where the limiting density of states behaves regularly, i.e., it is sufficiently smooth and strictly positive (in the *bulk*).

Assumption 2.9 (*Locality of the initial state*) Given Assumption 2.8, we first pick a *reference energy*

$$E_0 \in \sigma_{\mathrm{adm}}^{(\kappa_0,c_0)} \quad \text{for some} \quad \kappa_0, c_0 > 0, \tag{2.19}$$

and further introduce $I_\delta := [E_0 - \delta, E_0 + \delta]$ for any $0 < \delta < \kappa_0$. Moreover, take an N-independent *energy width* $\Delta \in (0, \kappa_0/2)$ and let $\Pi_\Delta := \mathbf{1}_{I_\Delta}(H_0)$ be the spectral projection of H_0 onto the interval I_Δ. Then, we assume that the initial state $\psi_0 \in \mathbf{C}^N$ is normalized, $\|\psi_0\| = 1$, has energy $E_0 = \langle\psi_0, H_0\psi_0\rangle$, and satisfies $\Pi_\Delta\psi_0 = \psi_0$, i.e., ψ_0 is localized in I_Δ.

Theorem 2.10 (Loschmidt echo with a single deformation) *Consider the Loschmidt echo (2.15) for times $t \ge 0$ and assume that its constituents satisfy Assumptions 2.1 and 2.8–2.9. Then, we have the following:*

[8] Here, $C^{1,1}(J)$ denotes the set of continuously differentiable functions with a Lipschitz-continuous derivative on an interval J, equipped with the norm $\|f\|_{C^{1,1}(J)} := \|f\|_{C^1(J)} + \sup_{x,y\in J: x\neq y} \frac{|f'(x)-f'(y)|}{|x-y|}$.

(i) [Short-time parabolic decay] *As* $t \to 0$ *it holds that*

$$\mathfrak{M}(t) = 1 - \lambda^2 t^2 + \mathcal{O}(\lambda^2 t^3) + \mathcal{O}_\prec\big(1/\sqrt{N}\big)\,. \tag{2.20}$$

The implicit constants in the error terms in (2.20) *only depend on* C_{H_0} *and the* C_p*'s from Assumption 2.1.*

(ii) [Intermediate-time asymptotic decay] *For all times* $t \geq 0$*, it holds that*

$$\mathfrak{M}(t) = \mathrm{e}^{-2\pi\rho_0(E_0)\lambda^2 t} + \mathcal{O}(\mathcal{E}) + \mathcal{O}_\prec\big(C(t,\lambda)/\sqrt{N}\big)\,, \tag{2.21}$$

where for any fixed $T > 0$ *the error term* $\mathcal{E}$*, explicitly given in* (7.8)*, satisfies*

$$\lim_{\Delta\to 0}\ \lim_{\substack{t\to\infty,\lambda\to 0\\ \lambda^2 t\leq T}}\ \lim_{N\to\infty} \mathcal{E} = 0$$

and the constant $C(t,\lambda) > 0$ *depends only on its arguments. The implicit constants in the error terms in* (2.21) *depend only on* C_{H_0} *from Assumption 2.8,* κ_0, c_0 *from Assumption 2.9, and the* C_p*'s from Assumption 2.1.*

In the small time regime, $t \to 0$, (2.20) is surely more precise than (2.21), but the latter is more relevant to describe the exponential decay for times of order $t \sim \lambda^{-2}$.

The proof of Corollary 2.11 is completely analogous to the proof of Corollary 2.6 (only using an isotropic law instead of an averaged law) and so omitted.

Corollary 2.11 (Scrambled Loschmidt echo with a single deformation) *Assume the conditions of Theorem 2.10 and consider (as a variant of* (2.15)*) the* scrambled *Loschmidt echo*

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) := \left|\left\langle \psi_0, \mathrm{e}^{\mathrm{i}tH_\lambda}\mathrm{e}^{-\mathrm{i}\delta\widetilde{W}}\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\right\rangle\right|^2\,, \tag{2.22}$$

where $\widetilde{W}$ *is a Wigner matrix satisfying Assumption 2.1,* independent *of* W *and* $0 \leq \delta \leq N^{2/3-\varepsilon}$ *for some fixed* $\varepsilon > 0$*. Moreover, let* φ *be given by* (2.13)*.*

Then, instead of (2.20)–(2.21)*, we have that*

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2\left[1 - \lambda^2 t^2 + \mathcal{O}(\lambda^2 t^3) + \mathcal{O}_\prec\big(1/\sqrt{N}\big)\right] + \mathcal{O}_\prec\big(\delta/\sqrt{N}\big) \quad as \quad t \to 0$$

and

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2\left[\mathrm{e}^{-2\pi\rho_0(E_0)\lambda^2 t} + \mathcal{O}(\mathcal{E}) + \mathcal{O}_\prec\big(C(t,\lambda)/\sqrt{N}\big)\right] + \mathcal{O}_\prec\big(\delta/\sqrt{N}\big) \quad for \quad \lambda^2 t \leq T\,,$$

respectively.

The rest of the paper is devoted to proving Theorems 2.4 and 2.10. The proof of Theorem 2.4 is conducted in Sects. 3–4. In Sect. 7, we prove Theorem 2.10. The proof of several technical results from Sect. 4 is deferred to Sects. 5 and 6, and Appendix A.

3 Short-time parabolic decay in Scenario I: Proof of Theorem 2.4 (i)

In the following, we abbreviate $\widetilde{P} = \Im G_1(E_0 + \mathrm{i}\eta_0)/\langle \Im M_1(E_0 + \mathrm{i}\eta_0)\rangle$, such that $\mathfrak{M}(t)$ can be written as

$$\overline{\mathfrak{M}}(t) = \left|\left\langle \mathrm{e}^{\mathrm{i}tH_1}\widetilde{P}\mathrm{e}^{-\mathrm{i}tH_2}\right\rangle\right|^2 = \left|\left\langle \widetilde{P}\mathrm{e}^{\mathrm{i}tH_1}\mathrm{e}^{-\mathrm{i}tH_2}\right\rangle\right|^2 . \tag{3.1}$$

Next, we trivially Taylor expand $\mathrm{e}^{\mathrm{i}tH_1}$ and $\mathrm{e}^{-\mathrm{i}tH_2}$ to second order, leaving us with

$$\mathrm{e}^{\mathrm{i}tH_1}\mathrm{e}^{-\mathrm{i}tH_2} = 1 + \mathrm{i}t(H_1 - H_2) - \frac{t^2}{2}\big((H_1 - H_2)^2 - [H_1, H_2]\big) + \mathcal{O}(t^3) . \tag{3.2}$$

Plugging this in (3.1), we find

$$\begin{aligned} \overline{\mathfrak{M}}(t) =& \left\langle \widetilde{P}\left(1 - \tfrac{t^2}{2}((H_1 - H_2)^2 - [H_1, H_2])\right)\right\rangle^2 \\ &+ t^2\langle \widetilde{P}(H_1 - H_2)\rangle^2 + \mathcal{O}(t^3) + \mathcal{O}_\prec((N\eta_0)^{-1}) \\ =& 1 - \left\langle (D - \langle PD\rangle)^2 P\right\rangle t^2 + \mathcal{O}(t^3) + \mathcal{O}_\prec((N\eta_0)^{-1}) . \end{aligned} \tag{3.3}$$

Here, we additionally used that $\langle \widetilde{P}[H_1, H_2]\rangle = 0$ since $\widetilde{P}$ is a function of H_1, $D = H_2 - H_1$, and a single resolvent law in the form $\langle \widetilde{P}A\rangle = \langle PA\rangle + \mathcal{O}_\prec((N\eta_0)^{-1})$ for any A with $\|A\| \lesssim 1$. The fact that the decay parameter $\gamma = \langle (D - \langle PD\rangle)^2 P\rangle$ satisfies $\gamma \sim \Delta^2$ is a simple consequence of the *flatness* of the stability operator for a deformed Wigner matrix (see, e.g., [3, Proposition 3.5]).

In order to conclude (2.6), it remains to show that the error term $\mathcal{O}(t^3)$ in (3.3) is actually improvable to $\mathcal{O}(\langle D^2\rangle t^3)$. To see this, we (formally)[9] employ the Baker–Campbell–Hausdorff (BCH) formula, to write the exponentials as

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tH_1}\mathrm{e}^{-\mathrm{i}tH_2} =& \mathrm{e}^K \quad \text{with} \\ K =& \mathrm{i}t(H_1 - H_2) + \frac{t^2}{2}[H_1, H_2] + \frac{\mathrm{i}t^3}{12}\big([H_1, [H_1, H_2]] - [H_2, [H_1, H_2]]\big) \\ &- \frac{t^4}{24}[H_2, [H_1, [H_1, H_2]]] + \ldots \end{aligned} \tag{3.4}$$

and note that every summand in the expression for K in (3.4) can be written as a linear combination of nested commutators of D with $H \equiv H_1$ with one D always being in the innermost commutator. Hence, to conclude the desired, we need to show that (i) all the terms in e^K containing only a single D vanish, when evaluated in $\langle \widetilde{P}\ldots\rangle$, and (ii) all the terms in e^K containing at least two D's lead to an additional $\langle D^2\rangle$-factor in the error term.

[9] In order to guarantee convergence of the BCH expansion (3.4), we need the time t to be small enough such that $|t|(\|H_1\| + \|H_2\|) < \log 2$ [46], which can be achieved in an open interval around zero, since $\|D_i\| \lesssim 1$ and $\|W\| \leq 2 + \epsilon$ with very high probability.

For (i), note that the only way to have just a single D in a nested commutator is precisely $\mathrm{ad}_H^n(D)$ with $\mathrm{ad}_H(D) := [H, D]$. Evaluated in $\langle \widetilde{P} ... \rangle$, this vanishes, $\langle \widetilde{P} \mathrm{ad}_H^n(D) \rangle = 0$, since $[\widetilde{P}, H] = 0$ and hence

$$\langle \widetilde{P} \mathrm{ad}_H^n(D) \rangle = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \langle \widetilde{P} H^{n-k} D H^k \rangle = \langle \widetilde{P} H^n D \rangle \sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0 \,. \tag{3.5}$$

For (ii), we take a product, say, T, of H's and at least two D's, resulting from resolving (a product of) nested commutators, and estimate

$$\big| \langle \widetilde{P} T \rangle \big| \lesssim \langle \widetilde{P} D^2 \rangle = \langle P D^2 \rangle + \mathcal{O}_\prec\big((N\eta_0)^{-1} \big) \lesssim \langle D^2 \rangle + \mathcal{O}_\prec\big((N\eta_0)^{-1} \big) \,. \tag{3.6}$$

In the first step, we estimated all H's and all but two D's in T by their operator norm, additionally using that $\widetilde{P} \geq 0$ and $[H, \widetilde{P}] = 0$. In the second step, we employed the single resolvent law (2.4), while in the last step we used $\|P\| \lesssim 1$.

We have hence shown that all the terms of e^K in (3.4) carrying at least a third power of t, can in fact be bounded with an additional $\langle D^2 \rangle$-factor compared to (3.3). This concludes the proof. □

4 Asymptotic decay in Scenario I: Proof of Theorem 2.4 (ii)

The principal goal of this section is to prove (2.7) in Theorem 2.4 (ii), i.e., study the behavior of $\overline{\mathfrak{M}}(t)$ defined in (2.5) for times $1 \leq t \lesssim \Delta^{-2}$. In order to do so, we compute the random quantity $\big\langle \mathrm{e}^{\mathrm{i}tH_1} \Im G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_2} \big\rangle$ by residue calculus as

$$\begin{aligned} &\Big\langle \mathrm{e}^{\mathrm{i}tH_1} \Im G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_2} \Big\rangle \\ &\quad = \left(\frac{1}{2\pi \mathrm{i}} \right)^2 \oint_{\gamma_1} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1 - z_2)} \frac{\eta_0}{(z_1 - E_0)^2 + \eta_0^2} \langle G_1(z_1) G_2(z_2) \rangle \, \mathrm{d}z_1 \mathrm{d}z_2 \\ &\qquad + \frac{1}{4\pi} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(E_0 + i\eta_0 - z_2)} \langle G_1(E_0 + \mathrm{i}\eta_0) G_2(z_2) \rangle \, \mathrm{d}z_2 . \end{aligned} \tag{4.1}$$

Here, the contours γ_1, γ_2 are chosen to be two semicircles as indicated in Fig. 3. More precisely, we take a (large) constant $R > 0$ such that $\mathrm{supp}\rho_1$ and $\mathrm{supp}\rho_2$ are contained in $[-(R-1), R-1]$. The distance of the flat pieces from the real axis are denoted by $\eta_1 := \min\{1/t, \eta_0/2\}$ and $0 < \eta_2 \lesssim 1/t$. The latter will explicitly be chosen later in Sect. 4.3, where we conclude the proof of Theorem 2.4 (ii). We decompose both contours into their flat in semicircular parts, $\gamma_j = \gamma_j^{(1)} \dot{+} \gamma_j^{(2)}$, $j \in [2]$, and parametrize them as follows:

$$\gamma_1^{(1)} : \; z_1 = E_1 - \mathrm{i}\eta_1 \quad \text{with} \quad E_1 \in [-2R, 2R] \,, \quad \gamma_1^{(2)} : \; z_1 = 2R\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{i}\eta_1 \quad \text{with} \quad \varphi \in [0, \pi] \tag{4.2}$$

$$\gamma_2^{(1)} : \; z_2 = E_2 + \mathrm{i}\eta_2 \quad \text{with} \quad E_2 \in [-R, R] \,, \quad \gamma_2^{(2)} : \; z_2 = R\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{i}\eta_2 \quad \text{with} \quad \varphi \in [0, \pi] \tag{4.3}$$

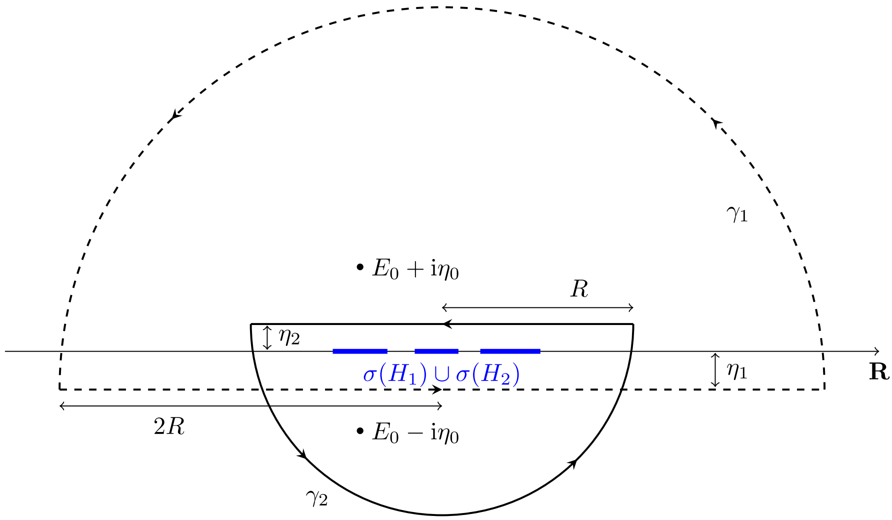

Fig. 3 Sketch of the contours γ_1 (dashed) and γ_2 (full) from (4.2)–(4.3). The union of the spectra of H_1 and H_2 is indicated in blue

Finally, we point out that, in order to (4.1) being valid, γ_1 is chosen in such a way that it encircles $E_0 + \mathrm{i}\eta_0$, but *not* $E_0 - \mathrm{i}\eta_0$.

The following argument leading toward the proof of Theorem 2.4 (ii) is split in three parts. First, in Sect. 4.1, we approximate the random contour integrals (4.1) by their deterministic counterparts by using an appropriate *two resolvent global law* for two different deformations (Proposition 4.1). Afterward, in Sect. 4.2, we collect some preliminary stability bounds (Proposition 4.2) and information on the *shift*, which is the key parameter in our analysis of the Loschmidt echo; see Lemmas 4.4–4.6. Finally, in Sect. 4.3, we summarize the evaluation of the deterministic contour integrals from Sect. 4.1 in five Lemmas 4.7–4.11. Combining these with estimates on the shift from Sect. 4.2, we conclude the proof of Theorem 2.4 (ii) at the end of Sect. 4.3.

4.1 Step (i): Global law with two deformations

The following two resolvent global law will be used to approximate (4.1) by its deterministic counterpart.

Proposition 4.1 (Average two resolvent global law) *Let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be a bounded Hermitian matrices, i.e., $\|D_j\| \leq L$ for some $L > 0$, and W a Wigner matrix satisfying Assumption 2.1. Moreover, let $z_1, z_2 \in \mathbf{C}$ be spectral parameters satisfying $\kappa := \min_{i\in[2]} \mathrm{dist}(z_i, [-(L+2), L+2]) \geq \delta > 0$ and denote $G_j(z_j) := (D_j + W - z_j)^{-1}$ for $j \in [2]$. Then, it holds that*

$$|\langle G_1(z_1)G_2(z_2)\rangle - \langle M(z_1, z_2)\rangle| \prec \frac{C(\delta)}{N}, \tag{4.4}$$

where $C(\delta) > 0$ *is a constant depending*[10] *only on its argument (apart from* L *and the constants from Assumption 2.1). In (4.4), we abbreviated*

$$M(z_1, z_2) = M_{12}(z_1, z_2) := \frac{M_1(z_1)M_2(z_2)}{1 - \langle M_1(z_1)M_2(z_2)\rangle} \tag{4.5}$$

and $M_j = M_j(z_j)$*, for* $j \in [2]$*, is the unique solution to the Matrix Dyson equation (MDE)*

$$-\frac{1}{M_j} = z_j - D_j + \langle M_j\rangle \quad \text{with} \quad \Im M_j(z_j)\Im z_j > 0 \quad \text{for} \quad z_j \in \mathbf{C}\setminus\mathbf{R}\,. \tag{4.6}$$

Proof Using that $\|D_j + W\| \le L + 2 + \epsilon$, $j \in [2]$ with very high probability and the stability bound $|1 - \langle M_1(z_1)M_2(z_2)\rangle|^{-1} \lesssim 1$ for $\kappa := \min_{i\in[2]} \mathrm{dist}(z_i, [-(L+2), L+2]) \gtrsim 1$ from Proposition 4.2 below,[11] the proof works in the same way as [22, Proposition 3.1], [12, Appendix B], [14, Section 5.2], or [13, Section 6.2]. We omit the details for brevity. □

Hence, by means of Proposition 4.1, we find that the random contour integral (4.1) can be approximated by the deterministic quantity

$$\begin{aligned} I_{E_0,\eta_0}(t) :=& \left(\frac{1}{2\pi \mathrm{i}}\right)^2 \oint_{\gamma_1}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1-z_2)} \frac{\eta_0}{(z_1-E_0)^2+\eta_0^2} \langle M(z_1,z_2)\rangle\, \mathrm{d}z_1\mathrm{d}z_2 \\ &+ \frac{1}{4\pi}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(E_0+i\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle\, \mathrm{d}z_2 . \end{aligned} \tag{4.7}$$

up to an error of size $\mathcal{O}_\prec\big(C(\eta_1)C(\eta_2)/N\big)$, where we additionally used that the lengths of the contours are bounded, $\ell(\gamma_j) \lesssim 1$ for $j \in [2]$.

4.2 Step (ii): Preliminary bounds on the stability operator and the shift

As usual in random matrix theory, local/global laws are governed by a *stability operator*, which, in our case is given by

$$\mathcal{B}_{12}(z_1, z_2)[\cdot] := \mathbf{1} - M_1\langle\cdot\rangle M_2 \quad \text{with} \quad M_j \equiv M_j(z_j)\,. \tag{4.8}$$

One can easily see that $\mathcal{B}_{12}(z_1, z_2)$ has a highly degenerate eigenvalue one, and its only non-trivial eigenvalue is given by $1 - \langle M_1 M_2\rangle$ with corresponding eigen“vector" $M_1 M_2$.

The following proposition, whose proof is given in Sect. 5, states an upper bound on the inverse of this non-trivial eigenvalue. A simplified form this stability bound

[10] By carefully tracking δ throughout the proof, one can see that the dependence is inverse polynomially, $C(\delta) \lesssim \delta^{-n}$ for some $n \in \mathbf{N}$. This will, however, be completely irrelevant for our purposes.

[11] In view of (4.10), note that the supports $\mathrm{supp}(\rho_1)$, $\mathrm{supp}(\rho_2)$ of the scDos of H_1 and H_2 are contained in $[-(L+2), (L+2)]$.

already appeared in [11, Lemma 5.2] for the very special case that $D_1 = \alpha D_2$ for some $\alpha \in \mathbf{R}$.

Proposition 4.2 (Stability bound) *Fix a (large) $L > 0$. Uniformly in $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ and traceless Hermitian D_1, D_2 with $|z_j| \le L$, $\|D_j\| \le L$, $j = 1, 2$, it holds that*

$$\left| \frac{1}{1 - \langle M_1 M_2 \rangle} \right| \lesssim \frac{1}{\Delta^2 + (\Re z_1 - \Re z_2)^2 + (\Im \langle M_1 \rangle + \Im \langle M_2 \rangle)^2 + \left| \frac{\Im z_1}{\langle \Im M_1 \rangle} \right| + \left| \frac{\Im z_2}{\langle \Im M_2 \rangle} \right|} \vee 1, \tag{4.9}$$

where we denoted $\Delta^2 := \langle (D_1 - D_2)^2 \rangle$.

In the current Sect. 4, more precisely, the proof of Proposition 4.1 above, only the special case

$$|1 - \langle M_1 M_2 \rangle|^{-1} \lesssim 1 \quad \text{for} \quad \max_{j \in [2]} \mathrm{dist}(z_j, \mathrm{supp}(\rho_j)) \gtrsim 1 \tag{4.10}$$

of Proposition 4.2 is relevant. However, for later reference, we also point out that, in particular, $|1 - \langle M_1 M_2 \rangle|^{-1} \lesssim |z_1 - z_2|^{-2}$ and that the lhs. of (4.9) is bounded by one, whenever z_1, z_2 are in the same half-plane and $\rho_1(z_1) + \rho_2(z_2) \gtrsim 1$ (e.g., if one of them is in the bulk, $\Re z_j \in \mathbf{B}_\kappa(\rho_j)$).

In addition to these bounds, Proposition 4.2 also plays an important role in the analysis of the *shift* $\mathfrak{s}(z_1, z_2)$ of the spectral parameters z_1, z_2 in the *(generalized) M-resolvent identity*

$$\langle M_{12} \rangle = \frac{\langle M_1 M_2 \rangle}{1 - \langle M_1 M_2 \rangle} = \frac{\langle M_1 \rangle - \langle M_2 \rangle}{z_1 - z_2 - \mathfrak{s}(z_1, z_2)}, \tag{4.11}$$

which can easily be obtained by subtracting MDEs (4.6) for M_2 and M_1 from each other. In (4.11), the shift is defined as follows.

Definition 4.3 (The *shift*) Let D_1, D_2 be Hermitian traceless matrices and let $M_j(z_j)$ for $j \in [2]$ be the solution of the MDE (4.6). Then, we define the *shift* (depending on D_1, D_2 and $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$) as

$$\mathfrak{s}(z_1, z_2) := \frac{\langle M_1(z_1)(D_1 - D_2) M_2(z_2) \rangle}{\langle M_1(z_1) M_2(z_2) \rangle}, \tag{4.12}$$

whenever the denominator does not vanish.

As already mentioned above, the shift $\mathfrak{s}$ is the key parameter in our analysis of the Loschmidt echo. We now collect several estimates on $\mathfrak{s}$ in Lemmas 4.4–4.6. The proofs, which are based on the stability bound in Proposition 4.2, are given in Sect. 5.

Lemma 4.4 (Properties of $\mathfrak{s}(z_1, z_2)$) *Fix a (small) $\kappa > 0$ and a (large) $L > 0$. Consider spectral parameters $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ such that $\Im z_1 \Im z_2 < 0$ and $|z_j| \le L$, $\|D_j\| \le L$, for $j \in [2]$. Assume that at least one of these parameters is such that the (positive)* *harmonic extension of the scDos is positive, i.e.,* $\rho_1(z_1) + \rho_2(z_2) \geq \kappa$. *Then, there exists a positive constant* $\mathfrak{c}$ *which depends only on* κ, L *such that for any Hermitian traceless* D_1, D_2 *with* $\Delta := \langle (D_1 - D_2)^2 \rangle^{1/2} \leq \mathfrak{c}$ *we have the following:*

(1) The denominator of the shift (4.12) *is of order one,* $|\langle M_1(z_1) M_2(z_2) \rangle| \sim 1$. *In particular,*

$$|\mathfrak{s}(z_1, z_2)| \lesssim \Delta. \tag{4.13}$$

(2) If $\rho_j(z_j) \geq \kappa/2$, *then*

$$|\partial_{z_j} \mathfrak{s}(z_1, z_2)| \lesssim \Delta. \tag{4.14}$$

Here all implicit constants depend only on κ *and* L.

We now introduce an auxiliary function f, which exactly detects the influence of the shift on the real part of a spectral parameter.

Lemma 4.5 (Definition of f and $\mathfrak{s}_0$) *Fix a (small)* $\kappa > 0$ *and a (large)* $L > 0$. *Consider* $0 < \eta_1, \eta_2 < L$ *and a spectral parameter* $z_2 = E_2 + \mathrm{i}\eta_2$ *such that* $\rho_2(z_2) \geq \kappa$, *and satisfying* $|z_2| \leq L$. *Let* D_1, D_2 *be Hermitian traceless matrices with* $\|D_j\| \leq L$, $j \in [2]$. *Assume that* $\Delta := \langle (D_1 - D_2)^2 \rangle^{1/2} \leq \mathfrak{c}$, *where* $\mathfrak{c}$ *is the constant from Lemma 4.4.*

Then, there exists a unique energy renormalization $f^{\eta_1,\eta_2}(E_2) = f(E_2) \in \mathbb{R}$ *with* $|f(E_2)| \leq L$ *such that*

$$\Re\left(f(E_2) - E_2 - \mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2)\right) = 0.$$

Moreover, denoting the renormalized (one point) shift *by*

$$\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) := \mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2), \tag{4.15}$$

the functions $f^{\eta_1,\eta_2}(E_2)$ *and* $\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)$ *are differentiable in* η_1, η_2 *and for* $E_2 \in \mathbf{B}_\kappa(\rho_2)$ *in the bulk, and the derivatives satisfy*

$$|\partial_{E_2} f^{\eta_1,\eta_2}(E_2) - 1| \lesssim \Delta, \quad \left|\partial_{\eta_j} f^{\eta_1,\eta_2}(E_2)\right| \lesssim \Delta, \; j \in [2], \quad \textit{and} \quad |\partial_{E_2} \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)| \lesssim \Delta. \tag{4.16}$$

Whenever it does not lead to confusion our ambiguities, we will omit the superscripts η_1, η_2 of f^{η_1,η_2} and $\mathfrak{s}_0^{\eta_1,\eta_2}$. Next, we show that the *imaginary part* of the renormalized shift is in fact much smaller than indicated by the upper bounds of order Δ in (4.13)–(4.14) and (4.16).

Lemma 4.6 (Behavior of $\Im\mathfrak{s}_0$) *Fix a (small)* $\kappa > 0$ *and a (large)* $L > 0$. *Let* $E \in \mathbf{B}_\kappa(\rho_2)$ *be in the bulk of* ρ_2. *Then, there exist positive constants* $c_1, c_2 > 0$ *such that for any Hermitian traceless* D_1, D_2 *with* $\|D_j\| \leq L$, $j = 1, 2$, $\Delta < c_1$ *and for any* $0 < \eta_j \leq c_2\Delta$, *for* $j \in [2]$, *it holds that*

$$\Im\mathfrak{s}_0^{\eta_1,\eta_2}(E) \sim \Delta^2. \tag{4.17}$$

Here, c_1, c_2 *and the implicit constants in* (4.17) *depend only on* κ *and* L.

In the following section, armed with the preliminary bounds from Proposition 4.2 and Lemmas 4.4–4.6, we carry out the evaluation of the contour integrals in (4.7).

4.3 Step (iii): Contour integration of the deterministic approximation

Throughout this section, let $[a, b]$ be an interval with length of order one satisfying $\mathrm{dist}(E_0, [a, b]^c) \gtrsim 1$ and $\mathrm{dist}([a, b], (\mathrm{supp}(\rho_1) \cap \mathrm{supp}(\rho_2))^c) \gtrsim 1$. That is, the energy E_0 from Theorem 2.4 is order one away from the boundary of $[a, b]$ and $[a, b]$ is simultaneously in the bulk of ρ_1 and ρ_2. The existence of such an interval is always guaranteed.

As already mentioned above, we now dissect the evaluation of (4.7) in several parts. As the first step, we show that the second line of (4.7) is in fact negligible. The proofs of Lemma 4.7 and all the other Lemmas 4.8–4.11 are given in Sect. 6.

Lemma 4.7 (The second line is negligible) *Under the assumptions of Theorem 2.4 (ii) it holds that*

$$I_{E_0}^{(2)} := \frac{1}{4\pi} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(E_0 + \mathrm{i}\eta_0 - z_2)} \langle M(E_0 + i\eta_0, z_2)\rangle \, \mathrm{d}z_2 = \mathcal{O}\left(\frac{1}{t}\right).$$

For the remaining first line of (4.7), we then find that the main contribution of the γ_2 integral comes from the interval $[a, b] + \mathrm{i}\eta_2$, i.e., we can cut away the tails.

Lemma 4.8 (Cutting tails) *Under the assumptions of Theorem 2.4 (ii) it holds that*

$$\begin{aligned} I_{E_0}^{(1)} &:= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1 - z_2)} \frac{\eta_0}{(z_1 - E_0)^2 + \eta_0^2} \langle M(z_1, z_2)\rangle \mathrm{d}z_1 \mathrm{d}z_2 \\ &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1} \int_a^b \mathrm{e}^{\mathrm{i}t(z_1 - E_2 - i\eta_2)} \frac{\eta_0}{(z_1 - E_0)^2 + \eta_0^2} \langle M(z_1, E_2 + \mathrm{i}\eta_2)\rangle \mathrm{d}z_1 \mathrm{d}E_2 \\ &\quad + \mathcal{O}\left(\frac{1}{t} + \frac{\eta_0}{\Delta}\right). \end{aligned}$$

The following lemma formally implements inside the integral from Lemma 4.8 the approximation

$$\langle M(z_1, E_2 + \mathrm{i}\eta_2)\rangle = \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2 + \mathrm{i}\eta_2)\rangle}{z_1 - (E_2 + \mathrm{i}\eta_2) - \mathfrak{s}(z_1, E_2 + \mathrm{i}\eta_2)} \approx \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2 + \mathrm{i}\eta_2)\rangle}{z_1 - (E_2 + \mathrm{i}\eta_2) - \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)},$$

which is valid in the main contributing regime $E_1 \approx E_2$. This is our *first replacement* $\mathfrak{s}(z_1, E_2 + \mathrm{i}\eta_2) \to \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)$.

Lemma 4.9 (First replacement) *Denote* $\mathfrak{d} := \min_{E_2 \in [a,b]} |\eta_1 + \eta_2 + \Im \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|$. *Then, under the assumptions of Theorem 2.4 (ii), it holds that*

$$\begin{aligned} &\left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1} \mathrm{d}z_1 \int_a^b \mathrm{d}E_2 \, \mathrm{e}^{\mathrm{i}t(z_1 - E_2 - \mathrm{i}\eta_2)} \frac{\eta_0}{(z_1 - E_0)^2 + \eta_0^2} \langle M(z_1, E_2 + \mathrm{i}\eta_2)\rangle \\ &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1} \mathrm{d}z_1 \int_a^b \mathrm{d}E_2 \, \mathrm{e}^{\mathrm{i}t(z_1 - E_2 - \mathrm{i}\eta_2)} \frac{\eta_0}{(z_1 - E_0)^2 + \eta_0^2} \cdot \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2 + \mathrm{i}\eta_2)\rangle}{z_1 - (E_2 + \mathrm{i}\eta_2) - \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)} \\ &\quad + \mathcal{O}\left(\eta_0 + \Delta|\log \Delta| + \Delta|\log \mathfrak{d}|\right). \end{aligned} \tag{4.18}$$

Next, plugging in the Stieltjes representation $\langle M_1(z_1)\rangle = \int_{\mathbf{R}} \rho_1(x)(x-z_1)^{-1}\mathrm{d}x$, the γ_1 integral in Lemma 4.9 can be explicitly computed using residue calculus. The “unwanted" residue contributions arising in this way can be estimated using the oscillatory factor and integration by parts (see the proof of Lemma 4.10 in Sect. 6).

Lemma 4.10 (Residue computation after the first replacement) *Denote* $\mathfrak{a} := \min_{E_2\in[a,b]} |\eta_0 - \eta_2 - \Im \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|$ *and suppose that*

$$\eta_1 + \eta_2 + \Im \mathfrak{s}_0(E_2) > 0, \quad \forall E_2 \in [a,b]\,. \tag{4.19}$$

Then, again under the assumptions of Theorem 2.4 (ii), it holds that

$$\begin{aligned}
&\left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1} \mathrm{d}z_1 \int_a^b \mathrm{d}E_2 \mathrm{e}^{\mathrm{i}t(z_1-E_2-i\eta_2)} \frac{\eta_0}{(z_1-E_0)^2+\eta_0^2} \cdot \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{z_1-(E_2+\mathrm{i}\eta_2)-\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)} \\
&= -\frac{1}{2\pi\mathrm{i}} \int_{\mathbf{R}} \mathrm{d}x \int_a^b \mathrm{d}E_2 \mathrm{e}^{\mathrm{i}t(x-E_2-i\eta_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)}{x-(E_2+\mathrm{i}\eta_2)-\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)} \\
&\quad + \mathcal{O}\left(\frac{|\log\mathfrak{a}|}{t} + \frac{\Delta+t^{-1}}{t\mathfrak{a}} + \eta_0|\log\mathfrak{a}| + \frac{\eta_0(\Delta+t^{-1})}{\mathfrak{a}}\right).
\end{aligned} \tag{4.20}$$

In the following lemma, we (i) complete the integral $\int_a^b$ to a full contour integral $\oint_{\gamma_2}$, i.e., put back the tails that were cut away in Lemma 4.8, and (ii) implement the *second replacement*

$$\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) \to \mathfrak{s}_0 := \mathfrak{s}_0^{\eta_1,\eta_2}\left(\left(f^{\eta_1,\eta_2}\right)^{-1}(E_0)\right) \tag{4.21}$$

inside the integral from Lemma 4.10. This replacement leads to a small error comparing to the leading term since $\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) \approx \mathfrak{s}_0$ in the relevant regime $E_2 \approx E_0$.

Lemma 4.11 (Second replacement) *Let* $\mathfrak{b} := \min_{E_2\in[a,b]} |\eta_2 + \Im\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|$ *and* $\mathfrak{s}_0$ *as in* (4.21)*. Then, again under the assumptions of Theorem 2.4 (ii), it holds that*

$$\begin{aligned}
&-\frac{1}{2\pi\mathrm{i}} \int_{\mathbf{R}} \mathrm{d}x \int_a^b \mathrm{d}E_2\, \mathrm{e}^{\mathrm{i}t(x-E_2-\mathrm{i}\eta_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)}{x-(E_2+i\eta_2+\mathfrak{s}_0^{\eta_1,\eta_2}(E_2))} \\
&= -\frac{1}{2\pi\mathrm{i}} \int_{\mathbf{R}} \mathrm{d}x \oint_{\gamma_2} \mathrm{d}z_2 \mathrm{e}^{\mathrm{i}t(x-z_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)}{x-(z_2+\mathfrak{s}_0)} \\
&\quad + \mathcal{O}\left(\frac{\eta_0+\mathfrak{b}}{\mathfrak{b}}\Delta|\log(\eta_0+\mathfrak{b})| + \eta_0|\log\mathfrak{b}| + \frac{1}{t}\right).
\end{aligned} \tag{4.22}$$

Armed with Lemmas 4.7–4.11, we can finally give the proof of Theorem 2.4 (ii).

Proof of Theorem 2.4 (ii) Combining Lemmas 4.7 - 4.11 we find that

$$I_{E_0,\eta_0}(t) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(x-z_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)\mathrm{d}x}{x-(z_2+\mathfrak{s}_0)} \mathrm{d}z_2 + \mathcal{O}\big(\widehat{\mathcal{E}}(t)\big), \tag{4.23}$$

where we collected all the error terms in

$$\widehat{\mathcal{E}}(t) := \frac{\eta_0}{\Delta} + \Delta|\log\Delta| + \Delta|\log\mathfrak{d}| + \frac{|\log\mathfrak{a}|}{t} + \frac{\Delta+t^{-1}}{t\mathfrak{a}} + \eta_0|\log\mathfrak{a}| + \frac{\eta_0(\Delta+t^{-1})}{\mathfrak{a}} + \frac{\eta_0+\mathfrak{b}}{\mathfrak{b}}\Delta|\log(\eta_0+\mathfrak{b})| + \eta_0|\log\mathfrak{b}|\,.$$

We shall now estimate $\widehat{\mathcal{E}}(t)$ in different time regimes. First note that Lemmas 4.4 and 4.6 imply the existence of positive constants $\{c_j\}_{j=1}^4$ such that

$$\begin{aligned} |\mathfrak{s}(z_1,z_2)| \le c_1\Delta, &\quad \text{for all} \quad |z_1|\le 2R,\ E_2\in[a,b],\ \eta_2\in[0,1], \quad \text{and} \\ c_2\Delta^2 \le \Im\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) \le c_3\Delta^2, &\quad \text{for all} \quad E_2\in[a,b],\ \eta_j\in[0,c_4\Delta],\ j=1,2. \end{aligned} \tag{4.24}$$

First regime: For $1 \le t \le 4Kc_3/(c_4\Delta)$ we take $\eta_2 := 8Kc_1c_3/(c_4t)$. Then, for any $E_2\in[a,b]$ it holds that

$$\eta_2 + \Im\mathfrak{s}_0(E_2) \ge 8Kc_1c_3/(c_1t) - c_1\Delta \ge 4Kc_1c_3/(c_1t) > 0.$$

In particular, the parameters $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{d}$ from Lemmas 4.10, 4.11, and 4.9, respectively, are all of order $1/t$ and $\widehat{\mathcal{E}}(t)$ is bounded as

$$\widehat{\mathcal{E}}(t) \lesssim \frac{1+\log t}{t} + \frac{\eta_0}{\Delta} + \Delta|\log\Delta| + \Delta\log t, \quad \text{for} \quad 1\le t\le \frac{4Kc_3}{c_4}\cdot\frac{1}{\Delta}. \tag{4.25}$$

Second regime: For $4Kc_3/(c_4\Delta) \le t \le 2Kc_3/\eta_0$, we take $\eta_2 := \frac{4Kc_3}{t}$. In this regime, $\eta_2 \le c_4\Delta$, so the positivity of $\eta_2 + \Im\mathfrak{s}_0(E_2)$ follows from (4.24). We also have that $\eta_2 \ge 2\eta_0$ and again $\mathfrak{a}\sim\mathfrak{b}\sim\mathfrak{d}\sim 1/t$. Therefore, (4.25) holds in the whole regime $1\le t\le 2Kc_3/\eta_0$.

Third regime: It remains to study the regime $2Kc_3/\eta_0 \le t \le K/\Delta^2$. If $\eta_0 \le 2c_3\Delta^2$, it is in fact empty; hence, we may assume $\eta_0 \ge 2c_3\Delta^2$. In this case, we take $\eta_2 := \min\{\eta_0/4, c_4\Delta, 1/t\}$ and find that $\mathfrak{a}\sim\eta_0$, $\mathfrak{b}\gtrsim\Delta^2$, $\mathfrak{d}\gtrsim\Delta^2$. Moreover, the error term $\widehat{\mathcal{E}}(t)$ is bounded as

$$\widehat{\mathcal{E}}(t) \lesssim \frac{1+\log t}{t} + \Delta|\log\Delta| + \frac{\eta_0|\log\Delta|}{\Delta}, \quad \text{for} \quad 2Kc_3/\eta_0 \le t\le K/\Delta^2.$$

After having chosen η_2 in all time regimes explicitly, we can perform z_2-integration in (4.23). Note that in all time regimes η_2 was chosen in such a way that $\eta_2 + \Im\mathfrak{s}_0 > 0$, which guarantees that γ_2 encircles the point $x - \mathfrak{s}_0$ for $x \in \operatorname{supp}(\rho_1)$. So, (4.23) evaluates to

$$\begin{aligned} I_{E_0,\eta_0}(t) &= \mathrm{e}^{\mathrm{i}t\mathfrak{s}_0} \int_{\mathbf{R}} \frac{\eta_0}{(x-E_0)^2+\eta_0^2}\rho_1(x)\mathrm{d}x + \mathcal{O}\big(\widehat{\mathcal{E}}(t)\big) \\ &= \mathrm{e}^{\mathrm{i}t\mathfrak{s}_0}\Im\langle M_1(E_0+\mathrm{i}\eta_0)\rangle + \mathcal{O}\big(\widehat{\mathcal{E}}(t)\big). \end{aligned} \tag{4.26}$$

After dividing by $\Im\langle M_1(E_0+\mathrm{i}\eta_0)\rangle$ and taking the absolute value square, it is left to notice that, setting

$$\Gamma := 2\,\Im\mathfrak{s}_0^{0,0}\left(\left(f^{0,0}\right)^{-1}(E_0)\right), \tag{4.27}$$

it holds that

$$\Im\mathfrak{s}_0 = \Im\mathfrak{s}_0^{\eta_1,\eta_2}\left(\left(f^{\eta_1,\eta_2}\right)^{-1}(E_0)\right) = \Gamma/2 + \mathcal{O}(\Delta(\eta_1+\eta_2)) = \Gamma/2 + \mathcal{O}(\Delta/t).$$

Here, we used (4.16) from Lemma 4.5 and (4.14) from Lemma 4.4 together with the bound $\eta_j \lesssim 1/t$, $j = 1, 2$. By Lemma 4.6, we finally see that the implicit constants in $\Gamma \sim \Delta^2$ only depend on κ and L. This finishes the proof of Theorem 2.4 (ii). □

5 Stability operator and shift: Proofs for Sect. 4.2

5.1 Bound on the stability operator: Proof of Proposition 4.2

Throughout the proof, we will use the shorthand notations $E_j := \Re z_j$, $\eta_j := |\Im z_j|$, $\rho_j := \frac{1}{\pi}\left|\langle \Im M_j(z_j)\rangle\right|$ and $\omega_j := z_j + \langle M_j(z_j)\rangle$, for $j \in [2]$.

We will conclude Proposition 4.2 from the following lemma.

Lemma 5.1 *Under the assumptions of Proposition 4.2 and using the notations from above, we have that:*

$$|1-\langle M_1M_2\rangle|^{-1} \lesssim (\eta_1/\rho_1+\eta_1/\rho_2)^{-1} \vee 1. \tag{5.1}$$

$$|1-\langle M_1M_2\rangle|^{-1} \lesssim (\Delta^2+|\omega_1-\overline{\omega}_2|^2)^{-1}. \tag{5.2}$$

$$|1-\langle M_1M_2\rangle|^{-1} \lesssim |z_1-z_2|^{-2} \tag{5.3}$$

Combining (5.1)–(5.3) with the simple observation $|\omega_1 - \overline{\omega}_2| \geq |\langle\Im M_1\rangle + \langle \Im M_2\rangle|$, we conclude (4.9), i.e., the proof of Proposition 4.2. □

Proof of Lemma 5.1 For (5.1), it is sufficient to check that for some $c \in (0, 1)$ we have $|\langle M_1M_2\rangle| \leq (1 - c(\eta_1/\rho_1 + \eta_2/\rho_2)) \vee (1-c)$. This follows from a simple Cauchy–Schwarz inequality $|\langle M_1M_2\rangle| \leq \langle |M_1|^2\rangle^{1/2}\langle|M_2|^2\rangle^{1/2}$ together with the estimate

$$\langle|M_j|^2\rangle^{1/2} = \left(\frac{\langle\Im M_j\rangle}{\Im z_j+\langle\Im M_j\rangle}\right)^{1/2} \lesssim \left(\frac{\rho_j}{\eta_j+\rho_j}\right)^{1/2} \leq \left(1-\frac{1}{2}\cdot\frac{\eta_j}{\rho_j}\right) \vee (1-c), \quad j \in [2]$$

where the first step follows by taking the imaginary part of the MDE (4.6).

For (5.2), we note that it is sufficient to show

$$\Re\langle M_1 M_2\rangle \le 1 - c\left(\left\langle (D_1 - D_2)^2\right\rangle + |\omega_1 - \bar{\omega}_2|^2\right) \quad \text{for some} \quad c > 0\,. \tag{5.4}$$

The idea for proving (5.4) is to translate it to a question for the spectral measures of D_1 and D_2.

In order to do so, for $j \in [2]$, denote the eigenvalues and eigenvectors of D_j by $\{\lambda_k^{(j)}\}_{k=1}^N$ and $\{\boldsymbol{u}_k^{(j)}\}_{k=1}^N$, respectively, and the normalized spectral measure by $\mu_j := N^{-1}\sum_{k=1}^N \delta_{\lambda_k^{(j)}}$. By the MDE (4.6), we immediately see that ω_j solves the equation $\omega_j - z_j = m_{\mu_j}(\omega_j)$, where $m_\mu(z) := \int_{\mathbf{R}} \mathrm{d}\mu(x)(x-z)^{-1}$ is the Stieltjes transform of the probability measure μ. By taking the imaginary part and estimating $|\Im\omega_j| > |\Im\omega_j - \Im z_j|$ we hence find

$$\int \frac{\mathrm{d}\mu_j(x)}{|x-\omega_j|^2} < 1\,. \tag{5.5}$$

Using the above notations, we further see that M_j can be written as $M_j = \sum_{k=1}^N (\lambda_k^{(j)} - \omega_j)^{-1} |\boldsymbol{u}_k^{(j)}\rangle\langle \boldsymbol{u}_k^{(j)}|$ and thus

$$\langle M_1 M_2\rangle = \frac{1}{N^2}\sum_{a,b=1}^N \frac{1}{\lambda_a^{(1)} - \omega_1}\cdot\frac{1}{\lambda_b^{(2)} - \omega_2} f\big(\lambda_a^{(1)}, \lambda_b^{(2)}\big), \quad \text{with} \quad f\big(\lambda_a^{(1)}, \lambda_b^{(2)}\big) := N\big|\langle \boldsymbol{u}_a^{(1)}, \boldsymbol{u}_b^{(2)}\rangle\big|^2\,.$$

Extending $f(x,y)$ to $\mathbf{R}^2$ by zero, we immediately see the following properties of f:

(1) $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$.
(2) $\int f(x,y)\mathrm{d}\mu_2(y) = \mathbf{1}_{\operatorname{supp}\mu_1}(x)$ and $\int f(x,y)\mathrm{d}\mu_1(x) = \mathbf{1}_{\operatorname{supp}\mu_2}(y)$.
(3) On $\mathbf{R}^2$, $\mathrm{d}\nu(x,y) := f(x,y)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)$ is a probability measure with marginals μ_1 and μ_2.

In this way, the desired inequality (5.4) can equivalently be rewritten as

$$\Re\iint \frac{1}{x-\omega_1}\cdot\frac{1}{y-\omega_2}\mathrm{d}\nu(x,y) \le 1 - c\left(\iint (x-y)^2 \mathrm{d}\nu(x,y) + |\omega_1 - \bar{\omega}_2|^2\right)\,. \tag{5.6}$$

In this form, using (5.5), we begin by estimating the lhs. of (5.6) as

$$\Re\iint \frac{1}{x-\omega_1}\cdot\frac{1}{y-\omega_2}\mathrm{d}\nu(x,y) < 1 - \frac{1}{2}\iint \left|\frac{1}{x-\omega_1} - \frac{1}{y-\bar{\omega}_2}\right|^2 \mathrm{d}\nu(x,y),$$

Thus, in order to arrive at (5.6), it suffices to bound

$$\begin{aligned}\iint \left|\frac{1}{x-\omega_1} - \frac{1}{y-\bar{\omega}_2}\right|^2 \mathrm{d}\nu(x,y) &\gtrsim \iint |(x-y)-(\omega_1-\bar{\omega}_2)|^2 \,\mathrm{d}\nu(x,y) \\ &= \iint (x-y)^2 \mathrm{d}\nu(x,y) - 2\Re(\omega_1-\bar{\omega}_2)\iint (x-y)\mathrm{d}\nu(x,y) + |\omega_1-\bar{\omega}_2|^2 \\ &= \iint (x-y)^2 \mathrm{d}\nu(x,y) + |\omega_1-\bar{\omega}_2|^2 .\end{aligned}$$

where in the first step we employed $|x-\omega_1| \lesssim \|D_1\| + |\omega_1| \lesssim 1$ (and analogously for $|y-\bar{\omega}_2|$), while in the last step we used that fact that D_1 and D_2 are traceless. This finishes the proof of (5.2).

Finally, for (5.3), we use (5.2) and (4.11) to get that

$$\begin{aligned}|z_1-z_2|^2 &= |\langle M_2(D_1-D_2)M_1\rangle + (1-\langle M_1M_2\rangle)(z_1-z_2+\langle M_1\rangle - \langle M_2\rangle)|^2 \\ &\lesssim |\langle M_1(D_1-D_2)M_2\rangle|^2 + |1-\langle M_1M_2\rangle| \\ &\lesssim \left\langle (D_1-D_2)^2 \right\rangle + |1-\langle M_1M_2\rangle| \lesssim |1-\langle M_1M_2\rangle| .\end{aligned}$$

□

5.2 Properties of the shift: Proof of Lemmas 4.4–4.6

We finally prove the properties of the shift from Lemmas 4.4–4.6.

Proof of Lemma 4.4 The proof is split in two parts in the statement of the lemma.

Part (1): Given $|\langle M_1M_2\rangle| \sim 1$, note that the bound (4.13) immediately follows since, if, say, z_1 is such that $\rho_1(z_1) \geq \kappa/2$, then $\|M_1\| \lesssim 1$ and $\langle |M_2|^2\rangle^{1/2} \leq 1$. Both of these estimates easily follow by taking the imaginary part of the respective MDEs (4.6).

It is hence left to prove $|\langle M_1M_2\rangle| \sim 1$. The upper bound $|\langle M_1M_2\rangle| \leq 1$ is a consequence of the Cauchy–Schwarz inequality and $\langle |M_j|^2\rangle^{1/2} \leq 1$. In order to prove the lower bound, we may assume w.l.o.g. that $|\langle M_1M_2\rangle| \leq 1/2$, in which case $|1-\langle M_1M_2\rangle| \sim 1$. Now, the numerator in the rhs. of the M-resolvent identity (4.11) is of order one, since $1 \gtrsim |\langle M_1\rangle - \langle M_2\rangle| \gtrsim |\langle \Im M_1\rangle| + |\langle \Im M_2\rangle| \gtrsim 1$. Thus, by (4.11) again, we find that $|(z_1-z_2)\langle M_1M_2\rangle - \langle M_1(D_1-D_2)M_2\rangle| \sim 1$, so, in particular,

$$1 \lesssim |(z_1-z_2)\langle M_1M_2\rangle - \langle M_1(D_1-D_2)M_2\rangle| \lesssim |\langle M_1M_2\rangle| + \Delta .$$

Therefore, for some constant $c > 0$ which depends only on L and κ we have

$$|\langle M_1M_2\rangle| \geq c - \Delta \gtrsim 1,$$

i.e., we get the desired lower bound for $|\langle M_1M_2\rangle|$.

Part (2): Assume w.l.o.g. that $\rho_1(z_1) \geq \kappa/2$. The derivative $\partial_{z_1}\mathfrak{s}(z_1, z_2)$ can be computed explicitly as

$$\partial_{z_1}\mathfrak{s}(z_1, z_2) = \frac{\langle M_1^2(D_1 - D_2)M_2\rangle\langle M_1M_2\rangle - \langle M_1(D_1 - D_2)M_2\rangle\langle M_1^2M_2\rangle}{\langle M_1M_2\rangle^2(1 - \langle M_1^2\rangle)}$$

and we note that, by analogous reasoning as in part (1), the numerator is bounded from above by Δ. Since $|\langle M_1M_2\rangle| \sim 1$, from part (1), it holds that

$$|\partial_{z_1}\mathfrak{s}(z_1, z_2)| \lesssim \frac{\Delta}{|1 - \langle M_1^2\rangle|} \lesssim \Delta,$$

where in the last step we used the bound $|1 - \langle M_1^2\rangle| \gtrsim \rho_1(z_1)^2$ with the aid of Proposition 4.2. □

Proof of Lemma 4.5 The argument is split in two parts: First, we prove existence and uniqueness of the energy renormalization function f. Second, we estimate the partial derivatives (4.16) of f and the renormalized (one point) shift $\mathfrak{s}_0$.

Part (1): Existence and uniqueness of f. First, from Lemma 4.5, we have that, for z_1 with $|z_1| \leq L$ and $\Im z_1 < 0$, it holds that $|\mathfrak{s}(z_1, z_2)| \leq C\Delta$ for some $C > 0$. For fixed $z_2 = E_2 + \mathrm{i}\eta_2$, we introduce the auxiliary (differentiable) function

$$h(E_1) := E_1 - E_2 - \Re\mathfrak{s}(E_1 - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2),$$

which has the property that $h(E_1) < 0$ for $E_1 < E_2 - C\Delta$, and $h(E_2) > 0$ for $E_1 > E_2 + C\Delta$. Hence, $h(E_1) = 0$ has a solution in $\mathcal{I} := [E_2 - C\Delta, E_2 + C\Delta]$. To see uniqueness, we differentiate h and find that $h'(E_1) \geq 1 - c\Delta$ for $E_1 \in \mathcal{I}$ and some $c > 0$ by means of (4.14) from Lemma 4.4. Thus, h has a unique zero on $\mathcal{I}$ (and hence in $(-L, L)$) which we denote by $f(E_2) = f^{\eta_1,\eta_2}(E_2)$—the desired energy renormalization function. Differentiability of f easily follows from the implicit function theorem.

Part (2): Bounds on derivatives. Differentiating the identity $h(f^{\eta_1,\eta_2}(E_2)) = 0$ in E_2, we find that

$$\partial_{E_2} f^{\eta_1,\eta_2}(E_2) = \frac{1 + \Re\partial_2\mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2)}{1 - \Re\partial_1\mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2))} = 1 + \mathcal{O}(\Delta),$$

by means of (4.14) from Lemma 4.4. Here, $\partial_j\mathfrak{s}$ denotes the partial derivative of $\mathfrak{s}$ w.r.t. its j^{th} argument. Similarly,

$$\partial_{\eta_1} f^{\eta_1,\eta_2}(E_2) = -\frac{\Re\,[\mathrm{i}\partial_1\mathfrak{s}]}{1 - \Re\,[\partial_1\mathfrak{s}]}, \quad \partial_{\eta_2} f^{\eta_1,\eta_2}(E_2) = \frac{\Re\,[\mathrm{i}\partial_1\mathfrak{s}]}{1 - \Re\,[\partial_2\mathfrak{s}]},$$

where $\mathfrak{s}$ has arguments $f(E_2) - \mathrm{i}\eta_1$ and $E_2 + \mathrm{i}\eta_2$. This concludes the bound $\left|\partial_{\eta_j} f^{\eta_1,\eta_2}(E_2)\right| \lesssim \Delta$ for $j = 1, 2$. The bound on $|\partial_{E_2}\mathfrak{s}_0(E_2)|$ is obtained in a similar fashion and thus left to the reader. □

Proof of Lemma 4.6 The proof is divided in two parts: In the first part, we prove (4.17) for $\eta_1 = \eta_2 = +0$. In the second part of the argument, we treat the general case as a perturbation thereof.

Part (1): Proof on the real line. Applying the M-resolvent identity (4.11) for $z_1 := f(E) - \mathrm{i}0$ and $z_2 := E + \mathrm{i}0$ and using Proposition 4.2, we find that

$$\left| \frac{\langle M_1(z_1)\rangle - \langle M_2(z_2)\rangle}{f(E) - E - \mathfrak{s}_0(E)} \right| \lesssim \frac{1}{\Delta^2}\,.$$

Since the numerator on the lhs. is of order one and the real part of the denominator vanishes by definition of $f(E)$, we deduce that

$$\Delta^2 \lesssim |\Im\,[f(E) - E - \mathfrak{s}_0(E)]| = |\Im \mathfrak{s}_0(E)|\,,$$

i.e., we have a lower bound on the *modulus* of $\Im\mathfrak{s}_0(E)$. To turn this into a lower bound on $\Im\mathfrak{s}_0(E)$ itself, we need to show that it is positive.

This will be done via a proof by contraction: Suppose that $\Im\mathfrak{s}_0(E) < 0$. By (4.11) for $z_1 := f(E) - \mathrm{i}0$, $z_2 := E + \mathrm{i}0$ we get

$$\frac{\langle M_1 M_2\rangle}{1 - \langle M_1 M_2\rangle} = \frac{\langle M_1\rangle - \langle M_2\rangle}{-\mathrm{i}\Im\mathfrak{s}_0(E)}\,. \tag{5.7}$$

Since $\Im\,[\langle M_1\rangle - \langle M_2\rangle] = -c$ for some $c > 0$ and $|\Re\,[\langle M_1\rangle - \langle M_2\rangle]| \lesssim \Delta$, we obtain, using our assumption $\Im\mathfrak{s}_0(E) < 0$,

$$\langle M_1\rangle - \langle M_2\rangle = |\langle M_1\rangle - \langle M_2\rangle| \mathrm{e}^{-\frac{\mathrm{i}\pi}{2} + \mathrm{i}\mathcal{O}(\Delta)} \quad \text{and} \quad \frac{\langle M_1\rangle - \langle M_2\rangle}{-\mathrm{i}\Im\mathfrak{s}_0(E)} = \left|\frac{\langle M_1\rangle - \langle M_2\rangle}{-\mathrm{i}\Im\mathfrak{s}_0(E)}\right| \mathrm{e}^{\mathrm{i}\pi + \mathrm{i}\mathcal{O}(\Delta)}\,,$$

where here and in the following $\mathcal{O}(\Delta)$ is real-valued. In a similar way, we find that $\langle M_1 M_2\rangle = 1 + \mathcal{O}(\Delta) + \mathrm{i}\mathcal{O}(\Delta)$ and $\langle M_1 M_2\rangle = |\langle M_1 M_2\rangle| \mathrm{e}^{\mathrm{i}\mathcal{O}(\Delta)}$. Hence, (5.7) implies

$$1 - \langle M_1 M_2\rangle = \left| \frac{-\mathrm{i}\Im\mathfrak{s}_0(E)}{\langle M_1\rangle - \langle M_2\rangle} \langle M_1 M_2\rangle \right| \mathrm{e}^{\mathrm{i}\pi + \mathcal{O}(\Delta)}\,,$$

i.e., in particular, $\Re\,[1 - \langle M_1 M_2\rangle] < 0$. On the other hand, it holds that $\Re\,[1 - \langle M_1 M_2\rangle] \geq 1 - |\langle M_1 M_2\rangle| \geq 0$, so we arrived at a *contradiction* and thus $\Im\mathfrak{s}_0(E) > 0$ and $\Im\mathfrak{s}_0(E) \gtrsim \Delta^2$.

For part (1), we are now left to prove $|\Im\mathfrak{s}_0(E)| \lesssim \Delta^2$, which is done via a perturbative argument in Appendix A. This concludes part (1), i.e., $\Im\mathfrak{s}_0^{0,0}(E) \sim \Delta^2$.

Part (2): Extension away from the real line. By (4.16) and the fundamental theorem of calculus, we have

$$\left|\Im\mathfrak{s}_0^{\eta_1,\eta_2}(E) - \Im\mathfrak{s}_0^{0,0}(E)\right| \leq \left|\int_0^{\eta_1} \partial_{\zeta_1}\mathfrak{s}_0^{\zeta_1,\eta_2}(E)\mathrm{d}\zeta_1\right| + \left|\int_0^{\eta_2} \partial_{\zeta_2}\mathfrak{s}_0^{0,\zeta_2}(E)\mathrm{d}\zeta_2\right| \lesssim \Delta(\eta_1 + \eta_2)\,.$$

Hence, if $0 < \eta_j \leq c_2\Delta$ for some $c_2 > 0$ small enough, we obtain $\Im \mathfrak{s}_0^{\eta_1,\eta_2}(E) \sim \Im \mathfrak{s}_0^{0,0}(E) \sim \Delta^2$. □

6 Contour integration: Proof of technical lemmas from Sect. 4.3

The goal of this section is to give the proofs of the technical lemmas from Sect. 4.3, for which we recall the construction of the contours γ_1, γ_2 from Sect. 4, in particular (4.2)–(4.3) and Fig. 1, and the definition of the $[a, b]$ interval from the beginning of Sect. 4.3.

In all of the estimates below, we will frequently use the following simple tools:

- To gain $1/t$-factors from the oscillatory $\mathrm{e}^{\mathrm{i}t(z_1-z_2)}$, we integrate by parts.
- When pulling absolute values inside an integral, we bound $|\mathrm{e}^{\mathrm{i}t(z_1-z_2)}| \lesssim 1$ (recall $|\Im z_j| \lesssim 1/t$).
- The convolution of two Cauchy kernels yields another Cauchy kernel: For $\eta_j > 0$ and $E_j \in \mathbf{R}$, $j \in [2]$ it holds that

$$\int_{\mathbf{R}} \frac{\eta_1}{(x-E_1)^2+\eta_1^2} \frac{\eta_2}{(x-E_2)^2+\eta_2^2} \mathrm{d}x \lesssim \frac{\eta_1+\eta_2}{(E_1-E_2)^2+(\eta_1+\eta_2)^2} . \tag{6.1}$$

We now turn to the proofs of the lemmas from Sect. 4.3.

6.1 The second line of (4.23) is negligible: Proof of Lemma 4.7

We discuss the contributions from the flat and semicircular part of γ_2 separately (recall (4.3)).

First, the smallness of the integral over $\gamma_2^{(2)}$ (the semicircular part) is granted by the factor $\mathrm{e}^{t\Im z_2}$ (note that $\Im z_2 \in [-R+\eta_2, \eta_2]$) and the estimate $|\langle M(E_0+i\eta_0, z_2)\rangle| \lesssim 1$, which follows from (4.9). More precisely, we have that

$$\left| \oint_{\gamma_2^{(2)}} \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle \,\mathrm{d}z_2 \right| \lesssim R \int_{\pi}^{2\pi} \mathrm{e}^{tR\sin\theta} \mathrm{d}\theta \lesssim \frac{1}{t} . \tag{6.2}$$

Next, we bound the integral over $\gamma_2^{(1)}$—the flat part. As a first step, integration by parts yields

$$\begin{aligned} &\left| \int_{\gamma_2^{(1)}} \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle \,\mathrm{d}z_2 \right| \\ &\quad \lesssim \frac{1}{t} + \left| \frac{1}{\mathrm{i}t} \int_{-R}^{R} \mathrm{e}^{-\mathrm{i}tE_2} \partial_{E_2} \langle M(E_0+\mathrm{i}\eta_0, E_2+\mathrm{i}\eta_2)\rangle \,\mathrm{d}E_2 \right| . \end{aligned}$$

The derivative can be explicitly computed as

$$\partial_{z_2}\langle M(z_1,z_2)\rangle = \frac{\langle M_1 M_2^2\rangle}{(1-\langle M_2^2\rangle)(1-\langle M_1M_2\rangle)^2}. \tag{6.3}$$

Since E_0 is in the bulk of ρ_1 and $z_0 := E_0 + \mathrm{i}\eta_0$ and z_2 are in the same half-plane we infer $|1-\langle M_1M_2\rangle| \gtrsim 1$ and thus

$$\left|\oint_{\gamma_2^{(1)}} \mathrm{e}^{\mathrm{i}t(E_0+i\eta_0-z_2)}\langle M(E_0+\mathrm{i}\eta_0,z_2)\rangle\,\mathrm{d}z_2\right| \lesssim \frac{1}{t}+\frac{1}{t}\int_{-R}^{R}\left|\frac{1}{1-\langle M_2(E_2+\mathrm{i}\eta_2)^2\rangle}\right|\mathrm{d}E_2.$$

In order to conclude the proof of Lemma 4.7, we finally use that the one-body stability operator $|1-\langle M_2(E_2+\mathrm{i}\eta_2)^2\rangle|^{-1}$ is locally integrable, see Lemma A.1 in Appendix A. □

6.2 Cutting tails in the first line of (4.23): Proof of Lemma 4.8

For cutting the tails, we focus on the more critical regime, where both parameters are on the horizontal part of the contours, $z_j \in \gamma_j^{(1)}$ for $j \in [2]$ (recall (4.2)–(4.3)). Indeed, if this is not the case, a simple computation using Proposition 4.2 and arguing similarly to (6.2) yields $(1+\eta_0/\Delta)/t \lesssim 1/t$ as an upper bound for the corresponding integrals.

In the critical regime $z_j \in \gamma_j^{(1)}$ for $j \in [2]$ we carry out only the case $E_2 = \Re z_2 \in [b,R]$; for $E_2 \in [-R,a]$ the argument is identical. Let $\delta := (b-E_0)/2$ and split the region of the $E_1 = \Re z_1$-integration into the two parts, $[b-\delta, 2R]$ and $[-2R, b-\delta]$. In the first regime, using $|E_1-E_0| \gtrsim 1$ and, from Proposition 4.2, $|\langle M(E_1-\mathrm{i}\eta_1, E_2+\mathrm{i}\eta_2)\rangle| \lesssim \big((E_1-E_2)^2+\Delta^2\big)^{-1}$, we find that

$$\int_{b-\delta}^{2R}\int_b^R \left|\frac{\eta_0}{(E_1-\mathrm{i}\eta_1-E_0)^2+\eta_0^2}\langle M(E_1-\mathrm{i}\eta_1,E_2+\mathrm{i}\eta_2)\rangle\right|\mathrm{d}E_1\mathrm{d}E_2 \lesssim \frac{\eta_0}{\Delta}.$$

For $E_1 \in [-2R, b-\delta]$, by Proposition 4.2 again, we have $|\langle M(z_1,z_2)\rangle| \lesssim 1$, since $|E_1-E_2| \sim 1$. Using this and integration by parts in E_2, similarly to the proof of Lemma 4.7, in combination with (6.3) and Lemma A.1, we find that

$$\left|\int_{-2R}^{b-\delta}\int_b^R \mathrm{e}^{\mathrm{i}t(z_1-E_2-\mathrm{i}\eta_2)}\frac{\eta_0}{(E_1-\mathrm{i}\eta_1-E_0)^2+\eta_0^2}\langle M(E_1-\mathrm{i}\eta_1,E_2+\mathrm{i}\eta_2)\rangle\mathrm{d}E_1\mathrm{d}E_2\right| \lesssim \frac{1}{t}.$$

This finishes the proof of Lemma 4.8. □

6.3 First replacement: Proof of Lemma 4.9

Let $\delta > 0$ be such that $[a-\delta, b+\delta]$ is in the bulk of ρ_1. We now compare the two integrals on the lhs. and rhs. of (4.18) by taking their difference. Using integration by parts, the contribution from $(z_1, z_2) \in \gamma_1^{(2)} \times [a, b]$ is bounded by η_0/t. Analogously to the proof of Lemma 4.8, we also find that the contribution from $([-R, R]\backslash[a - \delta, b + \delta]) \times [a, b]$ is bounded by η_0, since in this regime $|\langle M(z_1, I, z_2)\rangle| \lesssim 1$ and $|z_1 - z_2 - \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|^{-1} \gtrsim 1$. From now on and until the end of Sect. 6.5 we use the shorthand notation $\mathfrak{s}_0(E_2) := \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)$.

We are hence left to estimate the contribution from the region $[a-\delta, b+\delta]\times[a, b]$. Using that $|\mathfrak{s}(z_1, z_2) - \mathfrak{s}_0(E_2)| \lesssim \Delta|E_1 - \Re f(z_2)|$ by means of Lemma 4.4, we find that this can be bounded by

$$\mathcal{E} := \int_{a-\delta}^{b+\delta}\int_a^b \frac{\eta_0}{(E_1 - E_0)^2 + \eta_0^2} \cdot \frac{\Delta|E_1 - \Re f(z_2)|}{|z_1 - z_2 - \mathfrak{s}(z_1, z_2)| \cdot |z_1 - z_2 - \mathfrak{s}_0(E_2)|} \mathrm{d}E_1 \mathrm{d}E_2 .$$

To have better control on $\mathcal{E}$, we now bound the denominators in the second factor from below. First, using the definition of $\mathfrak{d}$ from the formulation of Lemma 4.9, we get

$$|z_1-z_2-\mathfrak{s}_0(E_2)|^2 = (E_1-E_2-\Re\mathfrak{s}_0(E_2))^2+(\eta_1+\eta_2+\Im\mathfrak{s}_0(E_2))^2 \gtrsim (E_1-f(E_2))^2+\mathfrak{d}^2 . \tag{6.4}$$

Next, using that $|z_1 - z_2 - \mathfrak{s}(z_1, z_2)| \gtrsim \Delta^2$, as simple consequence of the stability bound (4.9), we infer

$$|z_1-z_2-\mathfrak{s}(z_1, z_2)|^2 \sim |z_1-z_2-\mathfrak{s}(z_1, z_2)|^2+\Delta^4 \gtrsim (E_1-E_2-\Re\mathfrak{s}(z_1, z_2))^2+\Delta^4 . \tag{6.5}$$

Finally, using the defining properties of the renormalization function f given in Lemma 4.5, (4.14) from Lemma 4.4, and the triangle inequality, one easily sees that

$$|E_1 - E_2 - \Re\mathfrak{s}(z_1, z_2)| \sim |E_1 - f(E_2)|. \tag{6.6}$$

Hence, combining (6.4) and (6.5)–(6.6) we find that

$$\begin{aligned}\mathcal{E} &\lesssim \int_{a-\delta}^{b+\delta}\int_a^b \frac{\eta_0}{(E_1 - E_0)^2 + \eta_0^2} \cdot \frac{\Delta|E_1 - f(E_2)|}{|E_1 - f(E_2)|^2 + (\min\{\mathfrak{d}, \Delta^2\})^2} \mathrm{d}E_1 \mathrm{d}E_2 \\ &\lesssim \int_{a-\delta}^{b+\delta} \frac{\eta_0\Delta(|\log\Delta| + |\log\mathfrak{d}|)}{(E_1 - E)^2 + \eta_0^2}\mathrm{d}E_1 \lesssim \Delta(|\log\Delta| + |\log\mathfrak{d}|).\end{aligned}$$

where in the second step we changed the integration variable from E_2 to $f(E_2)$ and employed (4.16) from Lemma 4.5. This concludes the proof of Lemma 4.9. □

6.4 Residue computation after the first replacement: Proof of Lemma 4.10

Using the integral representation $\langle M_1(z_1)\rangle = \int_{\mathbf{R}} \rho_1(x)(x - z_1)^{-1}\mathrm{d}x$ and carrying out the residue computation (note that (4.19) ensures $z_2 + \mathfrak{s}_0(E_2)$ is encircled by the contour γ_1), we find the lhs. of (4.20) to equal

$$-\frac{1}{2\pi\mathrm{i}}\int_a^b \mathrm{d}E_2\int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}t(x-E_2-\mathrm{i}\eta_2)}\frac{\eta_0}{(x-E_0)^2+\eta_0^2}\cdot\frac{\rho_1(x)\mathrm{d}x}{x-(E_2+\mathrm{i}\eta_2+\mathfrak{s}_0(E_2))}+\mathcal{E}_1+\mathcal{E}_2\,,$$

where we introduced the shorthand notations

$$\mathcal{E}_1 := -\frac{1}{4\pi}\int_a^b \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-E_2-\mathrm{i}\eta_2)}\frac{\langle M_1(E_0+\mathrm{i}\eta_0)\rangle-\langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{E_0+\mathrm{i}\eta_0-(E_2+\mathrm{i}\eta_2+\mathfrak{s}_0(E_2))}\mathrm{d}E_2,$$
$$\mathcal{E}_2 := \frac{1}{2\pi\mathrm{i}}\int_a^b \mathrm{e}^{\mathrm{i}t\mathfrak{s}_0(E_2)}\frac{\eta_0\left(\langle M_1(E_0+\mathrm{i}\eta_0)\rangle-\langle M_2(E_2+\mathrm{i}\eta_2)\rangle\right)}{(E_2+\mathrm{i}\eta_2+\mathfrak{s}_0(E_2)-E_0)^2+\eta_0^2}\mathrm{d}E_2\,.$$

Moreover, we shall abbreviate $z_0 := E_0+\mathrm{i}\eta_0$, $z_2 := E_2+\mathrm{i}\eta_2$. Then, to estimate $\mathcal{E}_1$, we employ integration by parts and find that since $\left|\partial_{E_2}\langle M_2(z_2)\rangle\right| \lesssim 1$ as $\rho_2(z_2)\gtrsim 1$, using (4.16) from Lemma 4.5, and recalling the definition of $\mathfrak{a}$ from the formulation of Lemma 4.10,

$$\left|\partial_{E_2}\frac{\langle M_1(z_0)\rangle-\langle M_2(z_2)\rangle}{z_0-(z_2+\mathfrak{s}_0(E_2))}\right| \lesssim \frac{1}{|E_0-f(E_2)|+\mathfrak{a}}+\frac{|\langle M_1(z_0)\rangle-\langle M_2(z_2)\rangle|}{|E_0-f(E_2)|^2+\mathfrak{a}^2}.$$

Applying the M-resolvent identity (4.11) to z_0 and z_2 we infer, by application of the stability bound from Proposition 4.2 together with (4.13) and $\eta_2\lesssim 1/t$, $\eta_0\lesssim\Delta$, that $|\langle M_1(z_0)\rangle-\langle M_2(z_2)\rangle|\lesssim|E_0-f(E_2)|+\Delta+1/t$, and hence

$$|\mathcal{E}_1| \lesssim \frac{1}{t}+\frac{1}{t}\int_a^b\left(\frac{1}{|E_0-f(E_2)|+\mathfrak{a}}+\frac{|E_0-f(E_2)|+\Delta+t^{-1}}{|E_0-f(E_2)|^2+\mathfrak{a}^2}\right)\mathrm{d}E_2$$
$$\lesssim \frac{|\log\mathfrak{a}|}{t}+\frac{\Delta+t^{-1}}{t\mathfrak{a}}.$$

Similarly, $\mathcal{E}_2$ admits the bound $|\mathcal{E}_2|\lesssim\eta_0|\log\mathfrak{a}|+\eta_0\mathfrak{a}^{-1}(\Delta+t^{-1})$. This finishes the proof of Lemma 4.10. □

6.5 Second replacement: Proof of Lemma 4.11

The argument is split in two parts. First, we estimate the error of the second replacement within the interval $[a,b]$. Then, we put back the tails to complete the full contour integral.

For the first part, using $|\mathfrak{s}_0(E_2)-\mathfrak{s}_0|\lesssim\Delta|f(E_2)-E_0|$ as a consequence of (4.16), we find the error to be bounded by (a constant times) $\mathcal{E}_1+\mathcal{E}_2$, where

$$\mathcal{E}_1 := \int_{\mathbf{R}}\mathrm{d}x\int_a^b\frac{\eta_0}{(x-E_0)^2+\eta_0^2}\cdot\frac{\Delta|f(E_2)-E_0|}{|x-(z_2+\mathfrak{s}_0(E_2))|^2}\mathrm{d}E_2 \qquad (6.7)$$

and $\mathcal{E}_2$ is the same integral as $\mathcal{E}_1$, but with $\mathfrak{s}_0(E_2)$ being replaced by $\mathfrak{s}_0$. Next, convolving Cauchy kernels (6.1) in the x-variable and using (4.16) together with the definition of

$\mathfrak{b}$ we arrive at

$$\mathcal{E}_1 \lesssim \frac{\eta_0 + \mathfrak{b}}{\mathfrak{b}} \int_a^b \frac{\Delta |f(E_2) - E_0|}{(f(E_2) - E_0)^2 + (\eta_0 + \mathfrak{b})^2} \mathrm{d}E_2 \lesssim \frac{\eta_0 + \mathfrak{b}}{\mathfrak{b}} \Delta |\log(\eta_0 + \mathfrak{b})| \, .$$

For $\mathcal{E}_2$, the argument is similar: We simply replace $f(E_2) - E_0$ in the denominator by $E_0 - E_2 - \Re\mathfrak{s}_0$ and estimate $|E_0 - f(E_2)| \lesssim |E_0 - E_2 - \Re\mathfrak{s}_0|$ in the numerator. This shows that the error for the first bound is bounded by $(\eta_0 + \mathfrak{b})\mathfrak{b}^{-1}\Delta|\log(\eta_0 + \mathfrak{b})|$.

In the second part, we estimate the tails on the rhs. of (4.22). In the regime when $z_2 \in \gamma_2^{(2)}$ we find the bound $1/t$, similarly to (6.2). If instead $z_2 \in \gamma_2^{(1)} \setminus ([a, b] + \mathrm{i}\eta_2)$, say, $E_2 = \Re z_2 \in [b, R]$ for concreteness, we have that $|E_0 - E_2 - \Re\mathfrak{s}_0| \sim 1$, so the singularities in x on the rhs. of (4.22) are separated from each other. Now, pick $\delta \sim 1$ such that $[E_0 - \delta, E_0 + \delta] \subset [a, b]$ and $|x - E_2 - \Re\mathfrak{s}_0| \sim 1$ for any $E_2 \in [b, R]$, $x \in [E_0 - \delta, E_0 + \delta]$. Then, for $|x - E_0| \geq \delta$, it holds that

$$\left| \int_{|x-E_0|\geq\delta} \mathrm{d}x \int_b^R e^{it(x-E_2-i\eta_2)} \frac{\eta_0}{(x-E_0)^2 + \eta_0^2} \cdot \frac{\rho_1(x)\mathrm{d}x}{x - (E_2 + i\eta_2 + \mathfrak{s}_0)} \mathrm{d}E_2 \right| \lesssim \eta_0 |\log \mathfrak{b}| \, ,$$

where, in order to get $\mathfrak{b}$, we employed Lemma 4.6 and (4.16). Finally, for $|x - E_0| \leq \delta$, we employ integration by parts in E_2 and use $|x - E_2 - \Re\mathfrak{s}_0| \sim 1$ for any $E_2 \in [b, R]$, $x \in [E_0 - \delta, E_0 + \delta]$ to get

$$\left| \int_{E_0-\delta}^{E_0+\delta} \mathrm{d}x \mathrm{e}^{\mathrm{i}t(x-\mathrm{i}\eta_2)} \frac{\eta_0 \, \rho_1(x)}{(x - E_0)^2 + \eta_0^2} \int_b^R \mathrm{e}^{-\mathrm{i}tE_2} \frac{\mathrm{d}E_2}{x - (E_2 + i\eta_2 + \mathfrak{s}_0)} \right| \lesssim \frac{1}{t} \, .$$

This finishes the justification of the replacement (4.22) and thus the proof of Lemma 4.11. □

7 Second echo protocol: Proof of Theorem 2.10

The argument for part (i) is very similar to that for the proof of Theorem 2.4 (i). The only two differences are the following: First, the formerly algebraic cancellations $\langle \widetilde{P}[H_1, H_2] \rangle = 0$ below (3.3) and (3.5) are replaced by the estimate $|\langle \psi, W\phi \rangle| \prec \|\psi\| \, \|\phi\| N^{-1/2}$ for deterministic $\phi, \psi \in \mathbf{C}^N$. This follows by residue calculus and using an isotropic global law for the Wigner matrix W together with the fact that the first moment of the semicircular density vanishes, $\int_{\mathbf{R}} x \rho_{\mathrm{sc}}(x) \mathrm{d}x = 0$, by symmetry.

More precisely, using $\|W\| \leq 2+\epsilon$ with very high probability,

$$\begin{aligned}|\langle\psi, W\phi\rangle| &= \left|\frac{1}{2\pi \mathrm{i}}\oint_{|z|=3} z\langle\psi,(W-z)^{-1}\phi\rangle \mathrm{d}z\right| \\ &\lesssim \|\psi\|\,\|\phi\|\left|\frac{1}{2\pi \mathrm{i}}\oint_{|z|=3} z m_{\mathrm{sc}}(z)\mathrm{d}z\right| + \|\psi\|\,\|\phi\|\mathcal{O}_\prec(N^{-1/2}) \\ &\lesssim \|\psi\|\,\|\phi\|\left|\int_{\mathbf{R}} x\rho_{\mathrm{sc}}(x)\mathrm{d}x\right| + \|\psi\|\,\|\phi\|\mathcal{O}_\prec(N^{-1/2}) \prec \|\psi\|\,\|\phi\| N^{-1/2}\end{aligned} \tag{7.1}$$

where, to go to the last line, we used the Stieltjes representation $m_{\mathrm{sc}}(z) = \int_{\mathbf{R}}(x-z)^{-1}\rho_{\mathrm{sc}}(x)\mathrm{d}x$ and simple residue calculus. Second, in the analog of (3.6) it suffices to estimate all the λW simply by operator norm, recalling $\|W\| \leq 2+\epsilon$ with very high probability. The rest of the argument goes along the same lines as in the proof of Theorem 2.4 (i) with straightforward modifications.

Part (ii) may be derived from [22, Theorem 2.4], but here we give a direct proof relying just on the argument given in [22, Section 3.2.1]. First, by means of the single resolvent global law, we have that

$$\begin{aligned}\langle\psi_0, \mathrm{e}^{\mathrm{i}tH_\lambda}\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle &= \frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle\psi_0, G_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle\psi_0, M_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle \mathrm{d}z + \mathcal{O}_\prec\big(C(t,\lambda)/\sqrt{N}\big)\end{aligned} \tag{7.2}$$

for some constant $C(t,\lambda) > 0$ depending only on time t and coupling λ. Next, we approximate $\langle M_\lambda(z)\rangle \approx \overline{m_0(E_0)}$, leading to

$$M_\lambda(z) \approx \frac{1}{H_0 - z - \lambda^2\overline{m_0(E_0)}}\,. \tag{7.3}$$

Plugging the approximation (7.3) into (7.2), we find

$$\frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{e}^{\mathrm{i}tz}\big\langle\psi_0, \big(H_0 - z - \lambda^2\overline{m_0(E_0)}\big)^{-1}\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\big\rangle \mathrm{d}z = \mathrm{e}^{-\mathrm{i}\overline{m_0(E_0)}\lambda^2 t} \tag{7.4}$$

from simple residue calculus for $\lambda > 0$ small enough, using that $|m_0(E_0)| \lesssim 1$ (as follows from ρ_0 being $C^{1,1}$ around E_0; recall (2.18)) and γ encircles the spectrum of H_0. We have thus extracted the main term in (7.2), and it remains to estimate the errors resulting from the replacements in (7.3).

Denoting the spectral decomposition of H_0 by $H_0 = \sum_j \mu_j |\boldsymbol{u}_j\rangle\langle\boldsymbol{u}_j|$ and using Assumption 2.9, we have that

$$\frac{1}{2\pi \mathrm{i}}\oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle\psi_0, M_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle \mathrm{d}z = \sum_{\mu_j\in I_\Delta}\langle\psi_0,\boldsymbol{u}_j\rangle\langle\boldsymbol{u}_j,\psi_t\rangle\widetilde{\vartheta}(j)\,, \tag{7.5}$$

where we denoted $\psi_t := \mathrm{e}^{-\mathrm{i}tH_0}\psi_0$ and

$$\widetilde{\vartheta}(j) := \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{e}^{\mathrm{i}tz}}{\mu_j - z - \lambda^2 \langle M_\lambda(z)\rangle} \mathrm{d}z\,. \tag{7.6}$$

The key to approximating (7.5) is the following lemma, the proof of which is identical to that of [22, Lemma 3.3] and so omitted.

Lemma 7.1 (cf. Lemma 3.3 in [22]) *Under the above assumptions and notations, for every $j \in [N]$ such that $\mu_j \in I_\Delta$, denote*

$$\vartheta(j) := (2\pi\mathrm{i})^{-1} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}(\mu_j - z - \lambda^2\overline{m_0(E_0)})^{-1}\mathrm{d}z.$$

Then, it holds that

$$\sup_{\mu_j \in I_\Delta} \left|\widetilde{\vartheta}(j) - \vartheta(j)\right| \lesssim \mathcal{E} \tag{7.7}$$

for sufficiently small $\lambda > 0$ and N large enough (dependent on λ, cf. [22, Lemma A.1]). Here, recalling (2.16) for the definition of $\epsilon_0 = \epsilon_0(N)$, we denoted

$$\mathcal{E} = \mathcal{E}(\lambda, t, \Delta, N) := \lambda^2 t\,\Delta + \lambda\,(1 + \lambda^2 t) + \frac{\lambda}{\Delta}\left(1 + \frac{\lambda}{\Delta}\right) + \lambda^2 t\,\epsilon_0\,. \tag{7.8}$$

Therefore, by means of Lemma 7.1, employing a Hölder inequality in (7.5), and using (7.4), we find that

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle\psi_0, M_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle\mathrm{d}z = \mathrm{e}^{-\mathrm{i}\overline{m_0(E_0)}\lambda^2 t} + \mathcal{O}(\mathcal{E})\,. \tag{7.9}$$

Combining with (7.2) and taking the absolute value square of (7.9), we arrive at (2.21). This concludes the proof of Theorem 2.10. □

Appendix A Additional proofs

A.1 Upper bound on the renormalized shift: Perturbation argument for Lemma 4.6

The goal of this section is to prove the upper bound $|\Im \mathfrak{s}_0^{0,0}(E)| \lesssim \Delta^2$, as claimed at the end of part (1) of the proof of Lemma 4.6 in Sect. 5.

This is done via a perturbative calculation, which we carry out in a slightly more general setting: Consider two spectral parameters $z_1 = E_1 - \mathrm{i}0$, $z_2 = E_2 + \mathrm{i}0$, such that E_j is in the bulk of ρ_j, $j \in [2]$. Introducing the averaged and relative coordinates

$$D := (D_1 + D_2)/2, \quad z := (E_1 + E_2)/2 + i0, \quad \Theta := (D_2 - D_1)/2 - (E_2 - E_1)/2\,,$$

we find that $D_1 - z_1 = D - z - \Theta$ and $D_2 - z_2 = D - z + \Theta$. Let M be the solution of the MDE with the averaged coordinates, i.e.,

$$-\frac{1}{M} = z - D + \langle M \rangle.$$

Using the identity $MM^* = \Im M / \langle \Im M \rangle$, it is easy to compute by Taylor expansion

$$M_2 M_1 = \frac{1}{\langle \Im M \rangle} \left(\Im M + 2\mathrm{i}\Im\left[\Im M \Theta M\right] + 2\mathrm{i}\Im\left[\frac{\langle \Theta M^2 \rangle}{1 - \langle M^2 \rangle} \Im M \cdot M\right] + \mathcal{O}(|\Theta|^2) \right), \tag{A.1}$$

where $\mathcal{O}(|\Theta|^2)$ indicates terms containing at least two Θ's. Plugging (A.1) in the definition of the shift (4.12), we find that

$$\mathfrak{s}(z_1, z_2) + (z_2 - z_1) = -2 \frac{\langle \Theta \Im M \rangle + 2\mathrm{i}\langle \Theta \Im\left[\Im M \Theta M\right] \rangle + 2\mathrm{i}\left\langle \Theta \Im\left[\frac{\langle \Theta M^2 \rangle}{1 - \langle M^2 \rangle} \Im M \cdot M\right]\right\rangle + \mathcal{O}(\langle |\Theta|^3 \rangle)}{\langle \Im M \rangle + 2\mathrm{i}\langle \Im\left[\Im M \Theta M\right] \rangle + 2\mathrm{i}\left\langle \Im\left[\frac{\langle \Theta M^2 \rangle}{1 - \langle M^2 \rangle} \Im M \cdot M\right]\right\rangle + \mathcal{O}(\langle |\Theta|^2 \rangle)}.$$

which implies

$$\begin{aligned}
&\frac{\langle \Im M \rangle^2}{2}\left[\mathfrak{s}(z_1, z_2) + (z_2 - z_1)\right] \\
&= -\langle \Theta \Im M \rangle \langle \Im M \rangle - 2\mathrm{i}\langle \Im M \rangle \left\langle \left(\Theta - \frac{\langle \Theta \Im M \rangle}{\langle \Im M \rangle} \right) \Im M \Theta \Im M \right\rangle \\
&- 2\mathrm{i}\langle \Im M \rangle \left\langle \left(\Theta - \frac{\langle \Theta \Im M \rangle}{\langle \Im M \rangle} \right) \Im\left[\frac{\langle \Theta M^2 \rangle}{1 - \langle M^2 \rangle} \Im M \cdot M\right] \right\rangle + \mathcal{O}(\langle |\Theta|^3 \rangle).
\end{aligned}$$

Using $\Im[z_2 - z_1] = 0$ and $\Im[\langle \Theta \Im M \rangle \langle \Im M \rangle] = 0$, the imaginary part is given by

$$\frac{\langle \Im M \rangle}{4} \Im \mathfrak{s}(z_1, z_2) = -\left\langle \left(\Theta - \frac{\langle \Theta \Im M \rangle}{\langle \Im M \rangle} \right) \Im M \left(\Theta \Im M + \Im\left[\frac{\langle \Theta M^2 \rangle}{1 - \langle M^2 \rangle} M\right] \right) \right\rangle + \mathcal{O}(\langle |\Theta|^3 \rangle)$$

and hence

$$\begin{aligned}
|\Im \mathfrak{s}(z_1, z_2)| &= \frac{4}{\langle \Im M \rangle} \left| \left\langle \left(\Theta - \frac{\langle \Theta \Im M \rangle}{\langle \Im M \rangle} \right) \Im M \left(\Theta \Im M + \Im\left[\frac{\langle \Theta M^2 \rangle}{1 - \langle M^2 \rangle} M\right] \right) \right\rangle + \mathcal{O}(\langle |\Theta|^3 \rangle) \right| \\
&\lesssim \langle |\Theta|^2 \rangle,
\end{aligned}$$

since $\|\Theta\| \lesssim 1$. Specializing to the setting of Lemma 4.6 this result means that

$$|\Im \mathfrak{s}_0^{0,0}(E)| \lesssim \Delta^2 + |f(E) - E|^2 \lesssim \Delta^2,$$

where in the last step we used (4.13) and Lemma 4.5. This concludes the proof of the upper bound in part (1) of Lemma 4.6.

A.2 The one-body stability operator is locally integrable

In Sect. 6, we frequently use that the one-body stability operator is locally integrable. This is the statement of the following lemma.

Lemma A.1 (Integral of one-body stability operator) *Fix a (large) positive constant L. Uniformly in $\eta \in [0, 1]$ and in D satisfying Assumption 2.2 with constant L we have*

$$\int_{-L}^{L} \frac{\mathrm{d}E}{|1 - \langle M^2(E + i\eta)\rangle|} \lesssim 1. \tag{A.2}$$

Proof With the notation (2.2), we use the classification of local minima of ρ from [3, Theorem 7.1]. This result addresses the case of a *diagonal* deformation, while D in the formulation of Lemma A.1 does not need to be diagonal. Since the deterministic approximation $M(z)$ to the resolvent $(H - z)^{-1}$ of a random matrix H depends only on the first two joint moments of entries of H, we have that $M(z)$ in (A.2) coincides with the deterministic approximation to $(W_{\mathrm{GUE}} + D - z)^{-1}$, where W_{GUE} is a GUE matrix. Let U be a unitary diagonalizing D, i.e., $U^* D U = D_0$, where D_0 is diagonal. Invariance of GUE under unitary conjugations gives that $\widetilde{M}(z) := U^* M(z) U$ is a deterministic approximation to $(W_{\mathrm{GUE}} + D_0 - z)^{-1}$, so $\widetilde{M}(z)$ solves the MDE

$$-\widetilde{M}^{-1}(z) = z - D_0 + \langle \widetilde{M}(z)\rangle, \quad \Im z \Im \widetilde{M}(z) > 0 \quad \text{for} \quad z \in \mathbf{C} \setminus \mathbf{R}\,.$$

We remark that $\widetilde{M}$ satisfies the assumptions of [3, Theorem 7.1], since $\mathcal{S} = \langle \cdot \rangle$ is flat and by means of Assumption 2.2.

Thus, [3, Theorem 7.1] applied to $\widetilde{M}$ together with the observation $\langle \widetilde{M}\rangle = \langle M\rangle$ gives that there exist positive constants $\rho_* > 0$ and $\delta_* > 0$ dependent only on L such that for any local minimum τ_0 of ρ with $\rho(\tau_0) < \rho_*$ one of the following possibilities holds:

$$\rho(\tau_0 + \omega) \sim \min\{\Delta^{-1/6}\omega^{1/2}, \omega^{1/3}\}\,, \qquad \omega \in [0, \delta_*]\,, \qquad \text{(left edge)} \tag{A.3a}$$

$$\rho(\tau_0 + \omega) \sim \min\{\Delta^{-1/6}|\omega|^{1/2}, |\omega|^{1/3}\}, \qquad \omega \in [-\delta_*, 0]\,, \qquad \text{(right edge)} \tag{A.3b}$$

$$\rho(\tau_0 + \omega) \sim |\omega|^{1/3}, \qquad \omega \in [-\delta_*, \delta_*]\,, \qquad \text{(cusp)} \tag{A.3c}$$

$$\rho(\tau_0 + \omega) \sim \tilde{\rho} + \min\{\tilde{\rho}^{-5}\omega^2, |\omega|^{1/3}\}, \qquad \omega \in [-\delta_*, \delta_*]\,, \qquad \text{(internal minimum)} \tag{A.3d}$$

where $\tilde{\rho} \sim \rho(\tau_0)$ in (A.3d). In (A.3a), $\Delta := 1$ if τ_0 is an extreme left edge of suppρ and Δ is the length of the gap between the intervals of support which ends at point τ_0 otherwise (see also [3, Lemma 7.16]),[12] for the right edge (A.3b) Δ is defined similarly.

[12] To be consistent with [3] we use Δ to denote the size of the gap inside of the proof of Lemma A.1. This should not lead to any confusion with the rest of the paper, where Δ is used for the Hilbert-Schmidt norm of $D_1 - D_2$.

As a first preparatory step for (A.2), we give a lower bound for $|1-\langle M^2(E+\mathrm{i}\eta)\rangle|$ in terms of $\rho(E)$. In fact, we will show that uniformly in $E\in[-L,L]$ it holds that

$$|1-\langle M^2(E+\mathrm{i}\eta)\rangle|\gtrsim\rho^2(E)\,. \tag{A.4}$$

By Lemma 5.1 the LHS of (A.4) has a lower bound of order $\rho^2(E+\mathrm{i}\eta)+\eta/\rho(E+\mathrm{i}\eta)$. Recall from [3, Proposition 2.4] that ρ is $1/3$-Hölder regular, i.e., there exists a constant C_0 depending only on L such that $|\rho(z_1)-\rho(z_2)|\le C_0|z_1-z_2|^{1/3}$ uniformly in $z_1,z_2\in\mathbf{C}$ with $\Im z_1\Im z_2>0$ and $|z_j|\le 2L$, $j=1,2$. If $\eta\le\rho(E)^3/(2C_0)^3$, then $\rho(E+\mathrm{i}\eta)\sim\rho(E)$, i.e., (A.4) holds. In the complementary case, $\eta>\rho(E)^3/(2C_0)^3$, we have $\rho(E+\mathrm{i}\eta)<\rho(E)$ and hence $\eta/\rho(E+\mathrm{i}\eta)\gtrsim\rho^2(E)$, i.e., (A.4) again holds.

Now, armed with (A.4), we are ready to prove (A.2). We split the region of integration into several regimes according to the classification of local minima of ρ. For each local minimum τ_0 with $\rho(\tau_0)<\rho_*$ the integration over $\tau_0+[-\delta_*,\delta_*]\cap\mathcal{D}$ will be considered separately. Here, $\mathcal{D}=\mathbf{R}$ for cusps and internal minima, $\mathcal{D}=[0,+\infty)$ for left edges and $(-\infty,0]$ for right edges. The set $\mathcal{D}$ is chosen in such a way that $\tau_0+[-\delta_*,\delta_*]\cap\mathcal{D}$ covers the part, where ρ is positive and small. The complementary regimes, the *bulk regime* (where $\rho\ge\rho_*$) and the *gap regime* (where $\rho=0$), are treated separately.

Bulk regime: It holds that $\rho(E)\ge\rho_*$, hence desired bound on the lhs. of (A.2) in the bulk regime immediately follows from (A.4).

Gap regime: Let $\tau_1<\tau_0$ be two edges of suppρ such that $\rho(E)=0$ for any $E\in[\tau_1,\tau_0]$; the cases when either τ_1 is an extreme right edge or τ_0 is an extreme left edge are treated similarly. Since $\partial_z M=M^2(1-\langle M^2\rangle)^{-1}$, we have $|1-\langle M^2\rangle|^{-1}\le 1+|\langle M'\rangle|$. Together with (A.3a) and (A.3b), this gives that

$$|1-\langle M^2(E+i\eta)\rangle|\gtrsim\Big((\min\{|E-\tau_1|,|E-\tau_0|\})^2+\eta^2\Big)^{1/3},$$

so the integral of $|1-\langle M^2(E+\mathrm{i}\eta)\rangle|^{-1}$ over $E\in[\tau_1,\tau_0]$ is uniformly bounded in $\eta\in[0,1]$.

Internal minimum with $\rho(\tau_0)<\rho_*$: Using (A.4) along with (A.3d), we find that

$$\int_{-\delta_*}^{\delta_*}\frac{\mathrm{d}\omega}{|1-\langle M^2(\tau_0+\omega+i\eta)\rangle|}\lesssim\int_{-\delta_*}^{\delta_*}\frac{\mathrm{d}\omega}{\rho^2(\tau_0+\omega)}\lesssim\int_0^{\tilde\rho^3}\frac{\mathrm{d}\omega}{\tilde\rho^2}+\int_{\tilde\rho^3}^{\delta_*}\frac{\mathrm{d}\omega}{\omega^{2/3}}\lesssim 1\,.$$

Cusp regime: This works in the exact same way as the internal minimum, using (A.3c) instead of (A.3d).

Edge regime: Let τ_0 be a left edge of ρ, for the right edge the argument is the same. First, [3, Corollary 5.3] gives that $|1-\langle M^2(z)\rangle|\gtrsim\rho(z)(|\sigma(z)|+\rho(z))$, where $\sigma(z)$ is a 1/3-Hölder regular function in $\{z\in\mathbf{C}:\Im z>0\}$ (by [3, Lemma 5.5]) and $|\sigma(\tau_0)|\sim\Delta^{1/3}$ (by [3, Theorem 7.7, Lemma 7.16]). Therefore, there exists a (small) positive constant $c\sim 1$ such that for all z with $\Re z\in[\tau_0,\tau_0+c\Delta]$ and $\Im z\in[0,c\Delta]$ it holds that $|\sigma(z)|\sim\Delta^{1/3}$. It is easy to see that the integral of the one-body stability operator over $[\tau_0+c\Delta,\tau_0+\delta_*]$ has an upper bound of order one by means of (A.4) and (A.3a). In the complementary regime $[\tau_0, \tau_0 + c\Delta]$ we distinguish between two cases (i) $\eta \in [0, c\Delta]$ and (ii) $\eta > c\Delta$. In the first case, note that, by the integral representation $\rho(E + i\eta) = \int_{\mathbf{R}} \mathrm{d}x \rho(x)\eta/((x - E)^2 + \eta^2)$ and (A.3a), it holds that $\rho(E + \mathrm{i}\eta) \gtrsim \rho(E)$ for $E \in [\tau_0, \tau_0 + c\Delta]$. Thus,

$$\int_0^{c\Delta} \frac{\mathrm{d}\omega}{|1 - \langle M^2(\tau_0 + \omega + i\eta)\rangle|} \lesssim \Delta^{1/3} \int_0^{c\Delta} \frac{\mathrm{d}\omega}{\omega^{1/2}(\Delta^{1/2} + \omega^{1/2})} \lesssim 1.$$

In the second case, $\eta > c\Delta$, we use (A.4) and the bound $|1 - \langle M^2(z)\rangle| \gtrsim |\Im z|$ to get

$$\int_0^{c\Delta} \frac{\mathrm{d}\omega}{|1 - \langle M^2(\tau_0 + \omega + i\eta)\rangle|} \lesssim \int_0^{c\Delta} \frac{\mathrm{d}\omega}{\rho^2(\tau_0 + \omega) + \Delta} \sim \Delta^{1/3} \int_0^{c\Delta} \frac{\mathrm{d}\omega}{\omega + \Delta^{4/3}} \lesssim 1\,,$$

which concludes the proof for the regular edge.
A careful examination of the proof shows that all implicit constants in the inequalities above depend only on L. □

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Data availability The authors declare that the data supporting the findings of this study are available within the paper.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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References

1. Ajanki, O.H., Erdős, L., Krüger, T.: Quadratic vector equations on complex upper half-plane. Am. Math. Soc. **261**, 1261 (2019)
2. Ajanki, O.H., Erdős, L., Krüger, T.: Stability of the matrix Dyson equation and random matrices with correlations. Probab. Theory Relat. Fields **173**, 293–373 (2019)
3. Alt, J., Erdős, L., Krüger, T.: The Dyson equation with linear self-energy: spectral bands. Edges Cusps. Doc. Math. **25**, 1421–1539 (2020)
4. Alt, J., Erdős, L., Krüger, T., Schröder, D.: Correlated random matrices: band rigidity and edge universality. Ann. Probab. **48**, 963–1001 (2020)
5. Bohigas, O., Giannoni, M.J., Schmit, C.: Characterization of chaotic quantum spectra and universality of level fluctuation laws. Phys. Rev. Lett. **52**(1), 1–4 (1984)

6. Boltzmann, L.: Über die Beziehung eines allgemeine mechanischen Satzes zum zweiten Haupsatze der Wärmetheorie. Sitzungsberichte der Akad. Wiss. Wien **II**(75), 67–73 (1877)
7. Chenu, A., Egusquiza, I.L., Molina-Vilaplana, J., Del Campo, A.: Quantum work statistics, Loschmidt echo and information scrambling. Sci. Rep. **8**(1), 12634 (2018)
8. Chenu, A., Molina-Vilaplana, J., Del Campo, A.: Work statistics, Loschmidt echo and information scrambling in chaotic quantum systems. Quantum **3**, 127 (2019)
9. Cipolloni, G., Erdős, L., Schröder, D.: Eigenstate thermalization hypothesis for Wigner matrices. Commun. Math. Phys. **388**, 1005–1048 (2021)
10. Cipolloni, G., Erdős, L., Schröder, D.: Thermalisation for Wigner matrices. J. Func. Anal. **282**, 109394 (2022)
11. Cipolloni, G., Erdős, L., Schröder, D.: Quenched universality for deformed Wigner matrices. Probab. Theory Relat. Fields **185**(3), 1183–1218 (2023)
12. Cipolloni, G., Erdős, L., Schröder, D.: Optimal multi-resolvent local laws for Wigner matrices. Electron. J. Probab. **27**, 1–38 (2022)
13. Cipolloni, G., Erdős, L., Henheik, J., Kolupaiev, O.: Gaussian fluctuations in the equipartition principle for Wigner matrices. For. Math. Sigma **11** (2023)
14. Cipolloni, G., Erdős, L., Henheik, J., Schröder, D.: Optimal lower bound on eigenvector overlaps for non-Hermitian random matrices. J. Funct. Anal. **287**(4), 110495 (2024)
15. Cucchietti, F.M., Lewenkopf, C.H., Mucciolo, E.R., Pastawski, H.M., Vallejos, R.O.: Measuring the Lyapunov exponent using quantum mechanics. Phys. Rev. E **65**, 046209 (2002)
16. Dabelow, L., Reimann, P.: Persistent many-body quantum echoes. Phys. Rev. Res. **2**(2), 023216 (2020)
17. Dabelow, L., Reimann, P.: Predicting imperfect echo dynamics in many-body quantum systems. Z. Naturforschung A **75**(5), 403–411 (2020)
18. Del Campo, A., Molina-Vilaplana, J., Sonner, J.: Scrambling the spectral form factor: unitarity constraints and exact results. Phys. Rev. D **95**(12), 126008 (2017)
19. De Raedt, H.: Computer simulation of quantum phenomena in nanoscale devices. Annu. Rev. Comput. Phys. **4**, 107–146 (1996)
20. Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: The local semicircle law for a general class of random matrices. Elect. J. Probab. **18**(59), 1–58 (2013)
21. Erdős, L., Krüger, T., Schröder, D.: Random matrices with slow correlation decay. Forum Math. Sigma **7**, E8 (2019)
22. Erdős, L., Henheik, J., Reker, J., Riabov, V.: Prethermalization for deformed Wigner matrices (2023). arXiv:2310.06677
23. Erdős, L., Henheik, J., Riabov, V.: Cusp universality for correlated random matrices (2024). arXiv:2410.06813
24. García-Mata, I., Frahm, K.M., Shepelyansky, D.L.: Shor's factorization algorithm with a single control qubit and imperfections. Phys. Rev. A **78**, 062323 (2008)
25. Gärttner, M., Bohnet, J.G., Safavi-Naini, A., Wall, M.L., Bollinger, J.J., Rey, A.M.: Measuring out-of-time- order correlations and multiple quantum spectra in a trapped ion quantum magnet. Nat. Phys. **13**, 781 (2017)
26. Georgeot, B., Shepelyansky, D.L.: Quantum computer inverting time arrow for macroscopic systems. Eur. Phys. J. D **19**, 263 (2002)
27. Gorin, T., Prosen, T., Seligman, T.H.: A random matrix formulation of fidelity decay. New J. Phys. **6**(1), 20 (2004)
28. Gorin, T., Prosen, T., Seligman, T.H., Žnidarič, M.: Dynamics of Loschmidt echoes and fidelity decay. Phys. Rep. **435**(2–5), 33–156 (2006)
29. Goussev, A., Jalabert, R.A., Pastawski, H.M., Wisniacki, D.: Loschmidt Echo. Scholarpedia **7**(8), 11687 (2012)
30. Gutiérrez, M., Goussev, A.: Long-time saturation of the Loschmidt echo in quantum chaotic billiards. Phys. Rev. E **79**(4), 046211 (2009)
31. Haake, F., Gnutzmann, S., Kuś, M.: Quantum Signatures of Chaos, 4th edn. Springer, Berlin (2018)
32. Hahn, E.L.: Spin echoes. Phys. Rev. **80**, 580–594 (1950)
33. Jacquod, P., Petitjean, C.: Decoherence, entanglement and irreversibility in quantum dynamical systems with few degrees of freedom. Adv. Phys. **58**(2), 67–196 (2009)
34. Jalabert, R.A., Pastawski, H.M.: Environment-independent decoherence rate in classically chaotic systems. Phys. Rev. Lett. **86**, 2490–2493 (2001)

35. Kohler, H., Smolyarenko, I.E., Pineda, C., Guhr, T., Leyvraz, F., Seligman, T.H.: Surprising relations between parametric level correlations and fidelity decay. Phys. Rev. Lett. **100**, 190404 (2008)
36. Kurchan, J.: Quantum bound to chaos and the semiclassical limit. J. Stat. Phys. **171**(6), 965–979 (2018)
37. Levstein, P.R., Usaj, G., Pastawski, H.M.: Attenuation of polarization echoes in nuclear magnetic resonance: a study of the emergence of dynamical irreversibility in many-body quantum systems. J. Chem. Phys. **108**, 2718–2724 (1998)
38. Linnemann, D., Strobel, H., Muessel, W., Schulz, J., Lewis-Swan, R.J., Kheruntsyan, K.V., Oberthaler, M.K.: Quantum-enhanced sensing based on time reversal of nonlinear dynamics. Phys. Rev. Lett. **117**, 013001 (2016)
39. Loschmidt, J.: Über den Zustand des Wärmegleichgewichts eines Systems von Körpern mit Rücksicht auf die Schwerkraft. Sitzungsberichte der Akad. Wiss. Wien **II**(73), 128–142 (1876)
40. Niknam, M., Santos, L.F., Cory, D.G.: Sensitivity of quantum information to environment perturbations measured with a nonlocal out-of-time-order correlation function. Phys. Rev. Res. **2**, 013200 (2020)
41. Peres, A.: Stability of quantum motion in chaotic and regular systems. Phys. Rev. A **30**(4), 1610 (1984)
42. Sánchez, C.M., Levstein, P.R., Buljubasich, L., Pastawski, H.M., Chattah, A.K.: Quantum dynamics of excitations and decoherence in many-spin systems detected with Loschmidt echoes: its relation to their spreading through the Hilbert space. Philos. Trans. R. Soc. A **374**, 20150155 (2016)
43. Schmitt, M., Kehrein, S.: Effective time reversal and echo dynamics in the transverse field Ising model. Eur. Phys. Lett. **115**, 50001 (2016)
44. Schmitt, M., Kehrein, S.: Irreversible dynamics in quantum many-body systems. Phys. Rev. B **98**(18), 180301 (2018)
45. Serbyn, M., Abanin, D.A.: Loschmidt echo in many-body localized phases. Phys. Rev. B **96**(1), 014202 (2017)
46. Suzuki, M.: On the convergence of exponential operators—the Zassenhaus formula, BCH formula and systematic approximants. Commun. Math. Phys. **57**, 193–200 (1977)
47. Tal-Ezer, H., Kosloff, R.: An accurate and efficient scheme for propagating the time dependent Schrödinger equation. J. Chem. Phys. **81**, 3967–3971 (1984)
48. van Dijk, W., Brown, J., Spyksma, K.: Efficiency and accuracy of numerical solutions to the time-dependent Schrödinger equation. Phys. Rev. E **84**, 056703 (2011)
49. Vaníček, J.: Dephasing representation: employing the shadowing theorem to calculate quantum correlation functions. Phys. Rev. E **70**(5), 055201 (2004)
50. Vaníček, J., Cohen, D.: Path integral approach to the quantum fidelity amplitude. Philos. Trans. R. Soc. A Math. Phys. Eng. Sci. **374**(2069), 20150164 (2016)
51. Wigner, E.P.: Characteristic vectors of bordered matrices with infinite dimensions. Ann. Math. **62**(2), 548–564 (1955)
52. Zambrano, E., Ozorio de Almeida, A.M.: Initial-value representation for the Loschmidt echo. Phys. Rev. E **84**, 045201(R) (2011)

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