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Correction

Correction to the paper "The polynomial sieve and equal sums of like polynomials" (IMRN, Vol. 2015, No. 7, 1987–2019)

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This paper corrects an error in an earlier work of the author.

There is an error in Lemma 4.2 of [1] which invalidates the statement of [1, Theorem 1.3]. The first part of [1, Lemma 4.2] states that

$$T_p(0,0) = p^2 + O(p).$$

This is deduced from Lemma 3.6 under the assumption that $H = K = 0$ defines a smooth complete intersection in $\mathbb{P}^4_{\mathbb{F}_p}$. However, this projective surface is easily checked to have (no more than) isolated singularities with $Z_1 = Z_2 = W = 0$, corresponding to the roots of the polynomial $X^4 - Y^4 = 0$. Thus it is only true that

$$T_p(0,0) = p^2 + O(p^{3/2}),$$

where the new error term comes from the analogue of Lemma 3.6 for surfaces with isolated singularities. In summary this means that we have

$$\Sigma_2(p;0,0) = 2p^2 + O(p^{3/2}) \tag{1}$$

in [1, Lemma 4.3] (rather than the version with error term $O(p)$ that is claimed there). The second part of [1, Lemma 4.2] is true as stated.

Tracing through the effect of this, one is led to an estimate for $M_{i,j}$ on [1, p. 2015] in which the term $O(p^{-1} + q^{-1})$ gets replaced by $O(p^{-1/2} + q^{-1/2})$. Following the argument in [1, §6], this leads to a version of [1, Eq. (6.7)] with $B^{2+\varepsilon}/Q$ replaced by $B^{2+\varepsilon}/\sqrt{Q}$ on the right hand side. Ultimately, one is forced to optimise with the choice $Q = B^{1/5}$, rather than $Q = B^{1/6}$, which means that the statement of [1, Theorem 1.3] only holds with the exponent $2 - 1/6$ replaced by $2 - 1/10$.

The author is very grateful to Dr Bonolis for drawing his attention to this flaw and for suggesting an alternative approach that allows us to fully recover the statement of [1, Theorem 1.3]. We begin with a variant of [1, Theorem 1.1], which follows directly from the proof.

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Theorem 1.1. Let $\mathcal{P}$ be a set of primes, with $P = \#\mathcal{P}$. Let $\alpha \in \mathbb{Z}_{>0}$ and let $g \in \mathbb{Z}[\mathbf{y}]$ be a non-zero polynomial. For each $p \in \mathcal{P}$ and $\mathbf{n} \in \mathbb{Z}^m$, let

$$h(\mathbf{n}) = \gcd(c_0(\mathbf{n}), \dots, c_d(\mathbf{n}))$$

and let

$$\tau_p(\mathbf{n}) = \alpha + (\nu_p(\mathbf{n}) - 1)(d - \nu_p(\mathbf{n})),$$

where $\nu_p(\mathbf{n}) = \#\{x \pmod p : f(x;\mathbf{n}) \equiv 0 \pmod p\}$. Suppose that $w(\mathbf{n}) = 0$ if $g(\mathbf{n})h(\mathbf{n}) = 0$ or if $|\mathbf{n}| \geqslant \exp(P)$. Then we have

$$S(\mathcal{A}) \ll \frac{1}{P^2} \sum_{p,q \in \mathcal{P}} \left| S(p,q) \right|,$$

with

$$S(p,q) = \sum_{\substack{\mathbf{n} \in \mathcal{A} \\ \gcd(pq, g(\mathbf{n})h(\mathbf{n}))=1}} w(\mathbf{n}) \tau_p(\mathbf{n}) \tau_q(\mathbf{n}).$$

We now consider the effect of using this result instead of [1, Theorem 1.1] in the proof, which we apply with $\alpha = 1$ and $d = 3$. The first change occurs in [1, Eq. (3.15)], where now

$$N_1(B; H; \mathbf{h}, \varrho) \ll \frac{\log^2 Q}{Q^2} \sum_{p,q \in \mathcal{P}} |S|,$$

with

$$S = \sum_{(r,s) \in \mathcal{A}} \tau_p(r,s) \tau_q(r,s).$$

Here, since $d = 3$, we have

$$\begin{aligned} \tau_p(r,s) &= 1 + (\nu_p(r,s) - 1)(3 - \nu_p(r,s)) \\ &= -\nu_p(r,s)^2 + 4\nu_p(r,s) - 2, \end{aligned} \tag{2}$$

for any prime p, where $\nu_p(r,s)$ is given by [1, Eq. (3.14)].

Continuing with argument, the analogue of [1, Eq. (3.18)] becomes

$$S = \frac{1}{(pq\varrho)^2} \sum_{-pq\varrho/2 < m,n \leqslant pq\varrho/2} \Gamma\left(\frac{2B}{h_1}, m\right) \Gamma\left(\frac{2B}{h_2}, n\right) \Psi(m,n),$$

with

$$\Psi(m,n) = \sum_{\substack{(r,s) \pmod{pq\varrho} \\ \gcd(\varrho, rs)=1 \\ F(r,s) \equiv 0 \pmod{\varrho}}} \tau_p(r,s) \tau_q(r,s) \mathrm{e}_{pq\varrho}(mr + ns).$$

Appealing to the Chinese remainder theorem, we readily obtain the following version of [1, Lemma 3.4].

Lemma 1.2. Suppose that $p \neq q$ and choose $p', q', \overline{pq}, \overline{\varrho} \in \mathbb{Z}$ such that $pq\overline{pq}+\varrho\overline{\varrho} = 1$ and $pp'+qq' = 1$. Then we have

$$\Psi(m,n) = \widetilde{\Sigma}_1(p;\overline{\varrho}q'm,\overline{\varrho}q'n)\widetilde{\Sigma}_1(q;\overline{\varrho}p'm,\overline{\varrho}p'n)\Phi(\varrho;\overline{pq}m,\overline{pq}n),$$

where

$$\widetilde{\Sigma}_i(p;M,N) = \sum_{(r,s) \ (\mathrm{mod}\, p)} \tau_p(r,s)^i \mathrm{e}_p(Mr+Ns),$$

$$\Phi(\varrho;M,N) = \sum_{\substack{(r,s) \ (\mathrm{mod}\, \varrho) \\ \gcd(\varrho,rs)=1 \\ F(r,s)\equiv 0 \ (\mathrm{mod}\, \varrho)}} \mathrm{e}_\varrho(Mr+Ns),$$

for $i \in \{1,2\}$. Suppose that $p = q$ and choose $\overline{p}, \overline{\varrho} \in \mathbb{Z}$ such that $p\overline{p}+\varrho\overline{\varrho} = 1$. Then we have

$$\Psi(m,n) = \begin{cases} p^2\widetilde{\Sigma}_2(p;\overline{\varrho}m',\overline{\varrho}n')\Phi(\varrho;\overline{p}m',\overline{p}n'), & \text{if } (m,n) = p(m',n'), \\ 0, & \text{otherwise.} \end{cases}$$

We now need to prepare a modified version of [1, Lemma 4.3], with the error in the proof of [1, Lemma 4.2] corrected. This is summarised in the following result.

Lemma 1.3. We have $\widetilde{\Sigma}_i(p;M,N) = O(p\,\mathbf{gcd}(p,M,N))$ for $i \in \{1,2\}$, and

$$\widetilde{\Sigma}_1(p;0,0) = O(p^{3/2}).$$

Proof. The first part of [1, Lemma 4.3] is correct as stated, and yields

$$\Sigma_t(p;M,N) = O_t(p\,\mathbf{gcd}(p,M,N)),$$

for $t \geqslant 0$. Recalling (2), this readily implies that $\widetilde{\Sigma}_i(p;M,N) = O(p\,\mathbf{gcd}(p,M,N))$ for $i \in \{1,2\}$.
Next, it follows from (2) that

$$\widetilde{\Sigma}_1(p;0,0) = -\Sigma_2(p;0,0) + 4\Sigma_1(p;0,0) - 2\Sigma_0(p;0,0).$$

To begin with, we have $\Sigma_0(p;0,0) = p^2$, as in [1, Eq. (4.2)]. Next, the first part of [1, Lemma 4.1] yields

$$\Sigma_1(p;0,0) = p^2 + O(p).$$

Combining this with the corrected bound (1), we deduce that

$$\widetilde{\Sigma}_1(p;0,0) = -2p^2 + O(p^{3/2}) + 4p^2 + O(p) - 2p^2 = O(p^{3/2}),$$

as desired. ∎

We now continue through the argument, with the only change arising through our modified Lemmas 1.2 and 1.3. The differences occur in [1, § 6] and the contribution to S from the term $m = n = 0$. This leads to the expression

$$S = M + O\left(\frac{|\Psi(0,0)|B}{\min\{h_1, h_2\}(pq\varrho)^2}\right) + O(E),$$

as an analogue of [1, Eq. (6.2)], where

$$M = \frac{4\Psi(0,0)B^2}{h_1 h_2 (pq\varrho)^2}$$

and

$$E = \sum_{\substack{-pq\varrho/2 < m,n \leqslant pq\varrho/2 \\ (m,n)\neq(0,0)}} \min\left\{\frac{B}{h_1}, \frac{pq\varrho}{|m|}\right\} \min\left\{\frac{B}{h_2}, \frac{pq\varrho}{|n|}\right\} \frac{|\Psi(m,n)|}{(pq\varrho)^2}.$$

The treatment of the two error terms goes through unmodified and so it remains to analyse M.

Appealing to Lemmas 1.2 and 1.3, we obtain

$$\begin{aligned} M = \frac{4\widetilde{\Sigma}_1(p;0,0)\widetilde{\Sigma}_1(q;0,0)\Phi(\varrho;0,0)B^2}{h_1 h_2 (pq\varrho)^2} &\ll \frac{\Phi(\varrho;0,0)B^2}{h_1 h_2 \sqrt{pq}\varrho^2} \\ &\leqslant \frac{\Phi(\varrho;0,0)B^2}{h_1 h_2 \min\{p,q\}\varrho^2}. \end{aligned}$$

Thus we are have recovered the quality of upper bound present in the antepenultimate displayed equation of [1, p. 2015], and the remainder of the argument runs through unchanged.

References

1. Browning, T. D. "The polynomial sieve and equal sums of like polynomials." *IMRN* **2015**, no. 7 (2015): 1987–2019.