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General Section

Averages of multiplicative functions along equidistributed sequences

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For a general family of non-negative functions matching upper and lower bounds are established for their average over the values of any equidistributed sequence.

Contents

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1. Introduction

Averaging multiplicative functions over integer sequences has a long history in number theory. Nair [7] studied the average over the values of an irreducible integer polynomial and this was later greatly generalised by Nair–Tenenbaum [8] and Henriot [4], who brought into focus certain key uniformity issues. When it comes to polynomials in two variables it was later extended to binary forms by La Bretèche–Browning [1] and to principal ideals by Browning–Sofos [2].

Wolke [11] had worked on averages of a multiplicative function $f \geqslant 0$ over the values of an increasing integer sequence, i.e.

$$\sum_{a \in \mathbb{N} \cap [1,T]} f(c_a),$$

under the assumption that the sequence is 'equi-distributed' along arithmetic progressions. With an eye to certain applications to arithmetic statistics and Diophantine equations we aim to study sums that are more general and under weaker assumptions on equidistribution. Omitting certain details for now, we shall work with sums of the form

$$\sum_{a \in \mathcal{A}} f(c_a)\chi(a),$$

where $\mathcal{A}$ is any countable set, $\chi : \mathcal{A} \to [0, \infty)$ is any function of finite support, c_a is an integer sequence, and f is a non-negative arithmetic function with certain multiplicative properties. We will give upper bounds in Theorem 1.9 and matching lower bounds in Theorem 1.13.

1.1. The upper bound

We introduce the necessary notation for the statement of the upper bound.

Definition 1.1 *(Density functions).* Fix $\kappa, \lambda_1, \lambda_2, B, K > 0$. We define $\mathcal{D}(\kappa, \lambda_1, \lambda_2, B, K)$ as the set of multiplicative functions $h : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ having the properties

- for all $B < w < z$ we have

$$\prod_{\substack{p \text{ prime} \\ w \leqslant p < z}} (1 - h(p))^{-1} \leqslant \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + \frac{K}{\log w}\right), \tag{1.1}$$

- for every prime $p > B$ and integers $e \geqslant 1$ we have

$$h(p^e) \leqslant \frac{B}{p}, \tag{1.2}$$

- for every prime p and $e \geqslant 1$ we have

$$h(p^e) \leqslant p^{-e\lambda_1 + \lambda_2}. \tag{1.3}$$

In order to state a result that is sufficiently general but easy to use we use the following set-up from [3, §2.2]. Let $\mathcal{A}$ be an infinite set and for each $T \geqslant 1$ let $\chi_T : \mathcal{A} \to [0, \infty)$ be any function for which

$$\{a \in \mathcal{A} : \chi_T(a) > 0\} \text{ is finite for every } T \geqslant 1. \tag{1.4}$$

We also assume that

$$\lim_{T \to +\infty} \sum_{a \in \mathcal{A}} \chi_T(a) = +\infty. \tag{1.5}$$

Assume that we are given a sequence of strictly positive integers $(c_a)_{a \in \mathcal{A}}$ indexed by $\mathcal{A}$ and denoted by

$$\mathfrak{C} := \{c_a : a \in \mathcal{A}\}.$$

We will be interested in estimating sums of the form

$$\sum_{a \in \mathcal{A}} \chi_T(a) f(c_a), \tag{1.6}$$

where f is an arithmetic function with the following properties:

Definition 1.2 *(A class of functions).* Fix $A \geqslant 1, \epsilon > 0, C > 0$. The set $\mathcal{M}(A, \epsilon, C)$ of functions $f : \mathbb{N} \to [0, \infty)$ is defined by the property that for all coprime m, n one has

$$f(mn) \leqslant f(m) \min\{A^{\Omega(n)}, Cn^{\epsilon}\}.$$

Example 1.3. If c_n is a sequence of positive integers then

$$\sum_{1\leqslant n\leqslant T} f(c_n)$$

is of type (1.6) by taking $\mathcal{A}=\mathbb{N}$ and $\chi_T(n)=\mathbb{1}_{[1,T]}(n)$.

Example 1.4. If $\mathcal{D}\subset\mathbb{R}^n$ is bounded and $Q(x_1,\dots,x_n)$ an integer polynomial then

$$\sum_{\substack{\mathbf{x}\in\mathbb{Z}^n\cap T\mathcal{D}\\ Q(\mathbf{x})\neq 0}} f(|Q(\mathbf{x})|)$$

is of type (1.6) by taking $\mathcal{A}=\{\mathbf{x}\in\mathbb{Z}^n : Q(\mathbf{x})\neq 0\}$ and $\chi_T(\mathbf{x})=\mathbb{1}_{T\mathcal{D}}(\mathbf{x})$.

Example 1.5. If Q_1,Q_2 are integer polynomials in n variables then

$$\sum_{\substack{\mathbf{x}\in(\mathbb{Z}\cap[-T,T])^n\\ Q_1(\mathbf{x})=0,Q_2(\mathbf{x})\neq 0}} f(|Q_2(\mathbf{x})|)$$

is of type (1.6) when $\chi_T(\mathbf{x})=\mathbb{1}_{[0,T]}(\max|x_i|)$, $\mathcal{A}=\{\mathbf{x}\in\mathbb{Z}^n : Q_1(\mathbf{x})=0, Q_2(\mathbf{x})\neq 0\}$.

We will need the following notion of 'regular' distribution of the values of the integer sequence c_a in arithmetic progressions. For a non-zero integer d and any $T\geqslant 1$, let

$$C_d(T)=\sum_{\substack{a\in\mathcal{A}\\ c_a\equiv 0 \pmod d}} \chi_T(a).$$

Definition 1.6 *(Equidistributed sequences).* We say that $\mathfrak{C}$ is equidistributed if there exist positive real numbers $\theta,\xi,\kappa,\lambda_1,\lambda_2,B,K$ with $\max\{\theta,\xi\}<1$, a function $M:\mathbb{R}_{\geqslant 1}\to\mathbb{R}_{\geqslant 1}$ and a function $h_T\in\mathcal{D}(\kappa,\lambda_1,\lambda_2,B,K)$ such that

$$C_d(T)=h_T(d)M(T)\left\{1+O\left(\prod_{\substack{B<p\leqslant M(T)\\ p\nmid d}}(1-h_T(p))^2\right)\right\}+O(M(T)^{1-\xi}) \qquad (1.7)$$

for every $T\geqslant 1$ and every $d\leqslant M(T)^\theta$, where the implied constants are independent of d and T.

It is worth emphasizing that in this definition the constants $\theta,\xi,\kappa,\lambda_1,\lambda_2,B,K$ are all assumed to be independent of T. For example, the bound $h_T(p^e)=O(1/p)$ in (1.2) holds with an implied constant that is independent of e,p as well as T.

From now on we shall often abuse notation by writing M for $M(T)$.

Remark 1.7. The function $M(T)$ can be chosen freely in any way that makes

$$\sum_{a\in\mathcal{A}}\chi_T(a)=M(T)\left\{1+O\left(\prod_{B<p\leqslant M(T)}(1-h_T(p))^2\right)\right\}+O(M(T)^{1-\xi})$$

hold. It particular, it is necessary that it satisfies

$$\lim_{T\to\infty}\frac{1}{M(T)}\sum_{a\in\mathcal{A}}\chi_T(a)=1.$$

One could simply take $M(T):=\sum_{a\in\mathcal{A}}\chi_T(a)$, however, in certain applications it is helpful to have the freedom to choose instead a smooth approximation to $\sum_{a\in\mathcal{A}}\chi_T(a)$ as a function of T.

Example 1.8. In the setting of Example 1.3 define $c_n=n$. Then

$$C_d(T)=\sharp\{1\leqslant n\leqslant T:d\mid n\}=\frac{T}{d}+O(1),$$

thus, one can choose $h_T(d)=1/d$, $M(T)=T$ and $\xi=9/10$. It is important to note that the choice of $M(T)$ and ξ is not unique: one may, for example, alternatively take $M(T)=T+T^{0.4}$ and $\xi=1/2$.

We are now ready to state the main upper bound of this paper.

Theorem 1.9 *(The upper bound). Let $\mathcal{A}$ be an infinite set and for each $T\geqslant 1$ define $\chi_T:\mathcal{A}\to[0,\infty)$ to be any function such that both (1.4) and (1.5) hold. Take a sequence of strictly positive integers $\mathfrak{C}=(c_a)_{a\in\mathcal{A}}$. Assume that $\mathfrak{C}$ is equidistributed with respect to some positive constants $\theta,\xi,\kappa,\lambda_1,\lambda_2,B,K$ and functions $M(T)$ and $h_T\in\mathcal{D}(\kappa,\lambda_1,\lambda_2,B,K)$ as in Definition 1.6. Fix any $A>1$ and assume that f is a function such that for every $\epsilon>0$ there exists $C>0$ for which $f\in\mathcal{M}(A,\epsilon,C)$, which is introduced in Definition 1.2. Assume that there exists $\alpha>0$ and $\widetilde{B}>0$ such that for all $T\geqslant 1$ one has*

$$\sup\{c_a:a\in\mathcal{A},\chi_T(a)>0\}\leqslant\widetilde{B}M^{\alpha},\tag{1.8}$$

where $M=M(T)$ is as in Definition 1.6. Then for all $T\geqslant 1$ we have

$$\sum_{a\in\mathcal{A}}\chi_T(a)f(c_a)\ll M\prod_{B<p\leqslant M}(1-h_T(p))\sum_{a\leqslant M}f(a)h_T(a),$$

where the implied constant is allowed to depend on $\alpha,A,B,\widetilde{B},\theta,\xi,K,\kappa$, λ_i, the function f and the implied constants in (1.7), but is independent of T and M.

Remark 1.10 *(Wolke's density function assumption).* Note that [11, Assumption (A_2)] states that there exist positive constants C_1, C_2 such that for all $e \geqslant 1$ and primes p one has $h_T(p^e) \leqslant C_1 e^{C_2} p^{-e}$. We replace this with (1.3) which is a lighter assumption for large e. This is of high significance in applications where c_a is the sequence of values obtained by a multivariable polynomial, as in this case $h_T(p^e)$ is the density of zeros modulo p^e and one cannot hope for a bound with $\lambda_1 \geqslant 1$.

Remark 1.11 *(Wolke's level of distribution assumption).* Let us comment that [11, Assumption (A_4)] implies that

$$C_1(T) - h_T(1)M \ll \frac{M}{(\log M)^{D_1}}$$

holds for every positive fixed constant D_1, i.e. it demands an arbitrary logarithmic saving. Our assumption in Definition 1.6 is lighter in the sense that it essentially only requires this for a fixed power of $\log M$. To see this, note that when $d = 1$, Definition 1.6 states that

$$C_1(T) - h_T(1)M \ll M \prod_{B<p\leqslant M} (1 - h_T(p))^2 + M^{1-\xi}.$$

In typical applications this is of size $M/(\log M)^\kappa$, where κ is as in (1.1).

Remark 1.12 *(Wolke's growth assumption).* Let us note that Wolke assumes that the function f is multiplicative, which is relaxed in our work by demanding that it is sub-multiplicative as in Definition 1.2. Furthermore, [11, Assumption (F_1)] states that $f(p^e)$ is only allowed to grow polynomially in e for a fixed prime p, whereas, Definition 1.2 relaxes this by assuming that $f(p^e)$ is allowed to grow subexponentially in e.

1.2. The lower bound

We shall see that if f is not too close to 0, then matching lower bounds hold. This is a generalization of the work of Wolke [11, Satz 2], where the main difference lies in the fact that the density functions in Definition 1.1 are now allowed to grow with larger freedom. Furthermore, Wolke's condition that $f(p^m) \geqslant C_0^m$ for some strictly positive real constant C_0 is replaced by the more general condition (1.9).

Theorem 1.13 *(The lower bound). Keep the notation and assumptions of Theorem 1.9. Assume, in addition, that $f : \mathbb{N} \to [0, \infty)$ is a multiplicative function for which*

$$\text{for each } L \geqslant 1 \text{ one has } \inf\{f(m) : \Omega(m) \leqslant L\} > 0 \tag{1.9}$$

and that the error term in Definition 1.6 satisfies

$$C_d(T) = h_T(d)M(T)\left\{1 + o_{T\to\infty}\left(\prod_{\substack{B<p\leqslant M(T)\\ p\nmid d}}(1-h_T(p))^2\right)\right\} + O(M(T)^{1-\xi})$$

whenever $d \leqslant M(T)^\theta$. *Then for all* $T \geqslant 1$ *we have*

$$\sum_{a\in\mathcal{A}} \chi_T(a)f(c_a) \gg M(T) \prod_{p\leqslant M(T)}(1-h_T(p)) \sum_{a\leqslant M(T)} f(a)h_T(a),$$

where the implied constants are independent of T.

We finish the introduction by giving a concrete corollary:

Corollary 1.14. *Let* $Q \in \mathbb{Z}[x_1,\dots,x_n]$ *be irreducible and let* τ *denote the divisor function. Then for all* $T \geqslant 2$ *we have*

$$T^n \log T \ll \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n, |x_i|\leqslant T\\ Q(\mathbf{x})\neq 0}} \tau(|Q(\mathbf{x})|) \ll T^n \log T,$$

where the implied constants depend on Q *and* n.

To prove this, we take $\mathcal{A}$ and χ_T as in Example 1.4. Letting

$$h(d) = d^{-n}\sharp\{\mathbf{x} \in (\mathbb{Z}/d\mathbb{Z})^n : Q(\mathbf{x}) = 0\}, M(T) = (2T)^n$$

and splitting in progressions modulo d we can easily verify (1.7) for some sufficiently small positive θ and ξ. Then Theorem 1.9 gives

$$\begin{aligned} T^{-n} \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n, |x_i|\leqslant T\\ Q(\mathbf{x})\neq 0}} \tau(|Q(\mathbf{x})|) &\ll \prod_{1\ll p\leqslant T}(1-h(p)) \sum_{a\leqslant(2T)^n} \tau(a)h(a) \\ &\ll \prod_{1\ll p\leqslant(2T)^n}(1-h(p))\left(1+2h(p)+\sum_{t\geqslant 2}(t+1)h(p^t)\right). \end{aligned}$$

Furthermore, $h(p^t) \ll p^{-t/\deg(Q)}$ by [6, Lemma 4.10], hence, the sum over all sufficiently large t is $O(p^{-2})$. Since Q is irreducible one can prove that $h(p^t) \leqslant h(p^2) \ll p^{-2}$ for all $t \geqslant 2$, hence, the product is

$$\ll \prod_{1\ll p\leqslant(2T)^n}(1-h(p))(1+2h(p)+O(p^{-2})) \ll \prod_{1\ll p\leqslant(2T)^n}(1+h(p)).$$

Finally, by Chebotarev's density theorem for schemes [9, §9] this is $\ll \log T$. The lower bound of Corollary 1.14 can be proved similarly by Theorem 1.13.

Notation. For a non-zero integer m define

$$\Omega(m) := \sum_{p|m} v_p(m),$$

where v_p is the standard p-adic valuation. Define $P^+(m)$ and $P^-(m)$ respectively to be the largest and the smallest prime factor of a positive integer m and let $P^+(1) = 1$ and $P^-(1) = +\infty$. For a real number x we reserve the notation $[x]$ for the largest integer not exceeding x. Throughout the paper we use the standard convention that empty products are set equal to 1. Throughout the paper we shall also make use of the convention that when iterated logarithm functions $\log t, \log\log t$, etc., are used, the real variable t is assumed to be sufficiently large to make the iterated logarithm well-defined.

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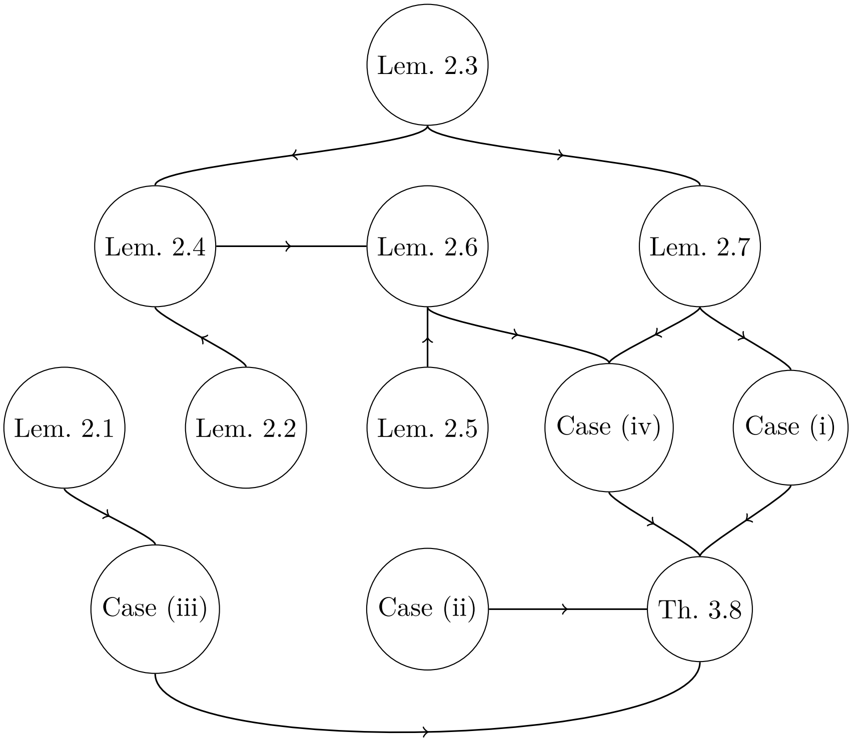
The following constants and functions are recurring throughout the paper:

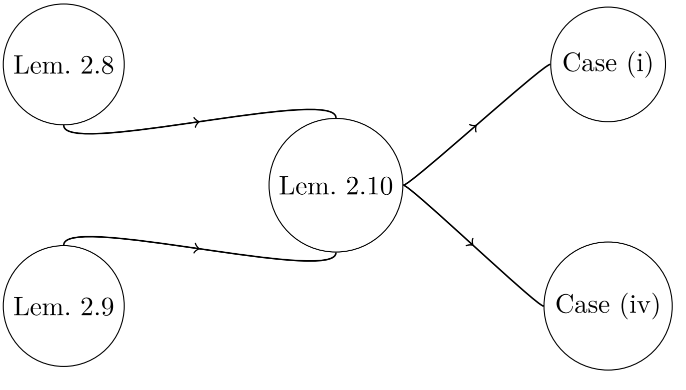
Symbols	First appearance
$F : \mathbb{N} \to [0, \infty)$	Lemma 2.1
c_0, c_1, c_2	Lemma 2.1
C, C'	Lemma 2.3
$G : \mathbb{N} \to [0, \infty)$	Lemma 2.4
$\kappa, \lambda_1, \lambda_2, B, K$	Definition 1.1
$h : \mathbb{N} \to [0, \infty)$	Definition 1.1
$\mathcal{A}, T, \chi_T$	Equations (1.4)-(1.5)
$C_d(T), M, M(T)$	Definition 1.6
θ, ξ	Definition 1.6
$A, \mathcal{M}(A, \epsilon, C)$	Definition 1.2
$\alpha,\ \tilde{B}$	Equation (1.8)
η_1, η_2	Equation (3.1)
Z	Equation (3.2)
b_a, c_a, d_a	Equations (3.3)-(3.5)

2. Preliminary lemmas

The present section consists of a series of preparatory lemmas that will later be used to prove Theorem 1.9. The lemmas that do not rely on sieve theory are structured as follows:



while the following lemmas are independent and rely on sieve theory:



The work of [10, Lemma 1] gives an upper bound on the density of integers all of whose prime factors are relatively small. We shall need a variation of this result where the integers are weighted by a multiplicative function. In the applications it will be important that the bound is of the form $O(x^{o(1)}z^{-c})$ for some positive constant c.

Lemma 2.1. *Fix any positive real numbers c_0, c_1, c_2 and assume that $F : \mathbb{N} \to [0, \infty)$ is a multiplicative function such that*

$$F(p^e) \leqslant \min\left\{\frac{c_0}{p}, \frac{p^{c_1}}{p^{ec_2}}\right\} \tag{2.1}$$

for all primes p and $e \geqslant 1$. Define

$$c := \min\left\{\frac{c_2}{2}, \frac{1}{1 + [2c_1/c_2]}\right\} \quad \text{and} \quad c' := \frac{c + 2(c_0 + c)}{c}.$$

Then for all $x, z \geqslant 2$ we have

$$\sum_{\substack{n \in \mathbb{N} \cap (z, x] \\ p \mid n \Rightarrow p \leqslant (\log x)(\log\log x)}} F(n) \ll z^{-c} \exp\left(\frac{c' \log x}{(\log\log x)^{1/2}}\right),$$

where the implied constant is absolute.

Proof. Let c_4 be a positive constant that will be optimised later. Then the sum over n is

$$\leqslant \frac{1}{z^{c_4}} \sum_{\substack{n \leqslant x \\ p \mid n \Rightarrow p \leqslant y}} F(n) n^{c_4},$$

where $y = (\log x)(\log\log x)$. By Rankin's trick we get the following bound for any $\delta > 0$:

$$\leqslant \frac{x^\delta}{z^{c_4}} \sum_{\substack{n \in \mathbb{N} \\ p \mid n \Rightarrow p \leqslant y}} F(n) \frac{n^{c_4}}{n^\delta} = \frac{x^\delta}{z^{c_4}} \prod_{p \leqslant y} \left(1 + \sum_{e \geqslant 1} F(p^e) p^{e(c_4 - \delta)}\right).$$

For an auxiliary positive integer e_0 we shall control the contribution of the range $e \leqslant e_0$ and $e > e_0$ using the bounds $F(p^e) \leqslant c_0/p$ and $F(p^e) \leqslant p^{c_1 - ec_2}$ respectively. Assume that $c_4 \geqslant \delta$ so that the contribution of the former range contributes

$$\leqslant 1 + \sum_{e=1}^{e_0} \frac{c_0}{p} \frac{p^{ec_4}}{p^{e\delta}} \leqslant 1 + c_0 e_0 p^{e_0(c_4 - \delta) - 1}.$$

Now assume that $c_4 e_0 \leqslant 1$ so that the bound becomes

$$\leqslant 1 + \frac{c_0}{c_4} p^{-\delta e_0} \leqslant 1 + \frac{c_0}{c_4} p^{-\delta} \leqslant 1 + \frac{c_0}{c_4} \frac{1}{p^\delta - 1}.$$

The remaining range contributes

$$\leqslant p^{c_1} \sum_{e \geqslant 1+e_0} p^{e(c_4-\delta-c_2)}.$$

Making the additional assumption that $c_4 \leqslant \frac{1}{2} c_2$ we can bound this by

$$\leqslant p^{c_1} \sum_{e \geqslant 1+e_0} p^{-e(\delta+c_2/2)} \leqslant \frac{p^{c_1}}{p^{e_0(\delta+c_2/2)}} \sum_{j=1}^{\infty} \frac{1}{p^{j(\delta+c_2/2)}} \leqslant \frac{p^{c_1}}{p^{e_0 c_2/2}} \frac{1}{p^\delta - 1}.$$

Further assuming that $2c_1 \leqslant e_0 c_2$ shows that this is $\leqslant \frac{1}{p^\delta - 1}$. Putting the bounds together leads to

$$1 + \sum_{e \geqslant 1} F(p^e) p^{e(c_4-\delta)} \leqslant 1 + \frac{c_0 + c_4}{c_4} \frac{1}{p^\delta - 1},$$

subject to the conditions

$$\delta \leqslant c_4, c_4 e_0 \leqslant 1, c_4 \leqslant \frac{c_2}{2}, \frac{2c_1}{c_2} \leqslant e_0.$$

Putting $e_0 = 1 + [2c_1/c_2]$ shows that these conditions are met for any $\delta \in (0, c_4)$ where

$$c_4 := \min\left\{\frac{c_2}{2}, \frac{1}{1 + [2c_1/c_2]}\right\} = c.$$

Hence, the overall bound becomes

$$\frac{x^\delta}{z^{c_4}} \prod_{p \leqslant y} \left(1 + \frac{c_0 + c_4}{c_4} \frac{1}{p^\delta - 1}\right) \leqslant \frac{x^\delta}{z^{c_4}} \prod_{p \leqslant y} \left(1 + \frac{1}{p^\delta - 1}\right)^{\frac{c_0+c_4}{c_4}}$$
$$\leqslant z^{-c_4} \exp\left(\delta(\log x) + \frac{c_0 + c_4}{c_4} T\right),$$

where

$$T := \sum_{p \leqslant y} \frac{1}{p^\delta - 1} \leqslant \frac{1}{\delta} \sum_{p \leqslant y} \frac{1}{\log p} \leqslant \frac{1}{\delta} \frac{2y}{(\log y)^2},$$

owing to the inequality $\mathrm{e}^t - 1 \geqslant t$ with $t = \delta \log p$ and the prime number theorem. Putting $\delta = (\log\log x)^{-1/2}$ we see that when $y = (\log x)(\log\log x)$ one gets the bound

$$z^{-c_4} \exp\left(\frac{\log x}{(\log\log x)^{1/2}} + \frac{(c_0 + c_4)}{c_4} (\log\log x)^{1/2} \frac{2(\log x)(\log\log x)}{(\log\log x)^2}\right),$$

which is sufficient. □

Lemma 2.2. *Keep the setting of Lemma 2.1 and fix any $\beta_0 > 0$. For all $T \geqslant 2$ with $\log T > 4\beta_0/c_2$, for all $A > 1, c \in \mathbb{N}$, and for any $\beta > 0$ with*

$$\beta \leqslant \min\left\{\frac{c_2}{2}, \frac{\beta_0}{\log T}\right\},$$

the product

$$\prod_{\substack{p \leqslant T \\ p \nmid c}} \Big(1 + \sum_{\substack{i \geqslant 1 \\ j \geqslant 0}} \min\{C' p^{(i+j)c_2/2}, A^{i+j}\} F(p^{i+j})(p^{\beta i} - p^{\beta(i-1)}) \Big(\frac{\mathbb{1}[p > c_0]}{(1 - F(p))} + \mathbb{1}[p \leqslant c_0]\Big)\Big)$$

is $O(\mathrm{e}^{\nu\beta \log T})$, where ν is a positive constant that depends at most on β_0, c_i and A. Furthermore, the implied constant depends at most on A, C', c_0, c_1 and c_2.

Proof. Define p_0 to be the least prime satisfying $2A \leqslant p_0^{c_2/4}$. We will bound the sum over i, j for every individual prime $p \geqslant p_0$ and in the end we shall piece the bounds together for all primes $p \leqslant T$.

Step (1). We start with the contribution of large i, in which case the bound $F(p^e) \leqslant p^{c_1 - ec_2}$ and the crude estimate $p^{\beta i} - p^{\beta(i-1)} \leqslant p^{\beta i}$ will suffice. Define

$$i_1 := 1 + \left[\frac{4(5 + c_1)}{c_2}\right].$$

The contribution of $i \geqslant i_1$ is

$$\leqslant \sum_{i \geqslant i_1} A^i p^{\beta i} \sum_{j \geqslant 0} A^j F(p^{i+j}) \leqslant p^{c_1} \sum_{i \geqslant i_1} A^i p^{(\beta - c_2)i} \sum_{j \geqslant 0} (Ap^{-c_2})^j \leqslant 2p^{c_1} \sum_{i \geqslant i_1} (Ap^{(\beta - c_2)})^i$$

because $Ap^{-c_2} \leqslant 1/2$, a fact that follows from $p \geqslant p_0$. Now we use the assumptions $\beta \leqslant c_2/2$ and $2A \leqslant p_0^{c_2/4} \leqslant p^{c_2/4}$ to see that $Ap^{(\beta - c_2)} \leqslant Ap^{-c_2/2} \leqslant 1/2$. Hence,

$$2p^{c_1} \sum_{i \geqslant i_1} (Ap^{(\beta - c_2)})^i \leqslant 4p^{c_1} (Ap^{(\beta - c_2)})^{i_1} \leqslant 4p^{c_1 - i_1 c_2/4} \leqslant p^{2 + c_1 - i_1 c_2/4}.$$

This is $\leqslant p^{-3}$ because our choice for i_1 makes sure that $5 + c_1 \leqslant i_1 c_2/4$. We have thus shown that for all $p \geqslant p_0$ one has

$$\sum_{\substack{i \geqslant i_1 \\ j \geqslant 0}} A^{i+j} F(p^{i+j})(p^{\beta i} - p^{\beta(i-1)}) \leqslant p^{-3}.$$

Step (2). Let us now bound the contribution of the i, j that satisfy

$$1 \leqslant i < i_1 \quad \text{and} \quad i + j \geqslant i_1.$$

We have

$$\sum_{i=1}^{i_1-1}\sum_{j\geqslant i_1-i}A^{i+j}F(p^{i+j})(p^{\beta i}-p^{\beta(i-1)})\leqslant p^{c_1}\sum_{i=1}^{i_1-1}A^ip^{-ic_2}(p^{\beta i}-p^{\beta(i-1)})\sum_{j\geqslant i_1-i}(Ap^{-c_2})^j.$$

Using the inequality $Ap^{-c_2}\leqslant 1/2$ to bound the sum over j results in the inequality

$$\leqslant 2p^{c_1}\sum_{i=1}^{i_1-1}A^ip^{-ic_2}(p^{\beta i}-p^{\beta(i-1)})(Ap^{-c_2})^{i_1-i}=2p^{c_1}A^{i_1}p^{-c_2i_1}\sum_{i=1}^{i_1-1}(p^{\beta i}-p^{\beta(i-1)}),$$

which is at most $2p^{c_1}(Ap^{(\beta-c_2)})^{i_1}$ that has been previously shown to be at most $\leqslant 1/p^3$.

We have thus proved that for all $p\geqslant p_0$ one has

$$\sum_{\substack{1\leqslant i<i_1\\ j\geqslant i_1-i}}A^{i+j}F(p^{i+j})(p^{\beta i}-p^{\beta(i-1)})\leqslant p^{-3}.$$

Step (3). It remains to study the contribution of cases with $i+j<i_1$. For these we use the assumption $F(p^e)\leqslant c_0/p$ that leads to the bound

$$\leqslant\frac{c_0}{p}\sum_{\substack{i\geqslant 1,j\geqslant 0\\ i+j<i_1}}A^{i+j}(p^{\beta i}-p^{\beta(i-1)})\leqslant\frac{c_0}{p}\sum_{1\leqslant i<i_1}(2A)^i(p^{\beta i}-p^{\beta(i-1)})\sum_{0\leqslant j<i_1-i}(2A)^j.$$

Now since $A>1$ we have $2A>2$. For all $m\geqslant 1$ we have

$$1+(2A)+(2A)^2+\ldots+(2A)^{m-1}\leqslant\frac{(2A)^m}{2A-1}\leqslant(2A)^m.$$

This gives the bound

$$\begin{aligned}&\leqslant\frac{c_0}{p}\sum_{1\leqslant i<i_1}(2A)^i(p^{\beta i}-p^{\beta(i-1)})(2A)^{i_1-i}\\&=\frac{c_0(2A)^{i_1}}{p}\sum_{1\leqslant i<i_1}(p^{\beta i}-p^{\beta(i-1)})\leqslant\frac{c_0(2A)^{i_1}}{p}(p^{\beta i_1}-1).\end{aligned}$$

The assumption $\beta\log T\leqslant\beta_0$ shows that $\beta i_1\log T\leqslant\beta_0(1+\frac{4(5+c_1)}{c_2})$, hence, for primes $p\leqslant T$ we infer $\beta i_1\log p\leqslant\beta_0(1+\frac{4(5+c_1)}{c_2})$. On the other hand, the function $(-1+\mathrm{e}^t)/t$ is bounded in the interval $0\leqslant t\leqslant\beta_0(1+\frac{4(5+c_1)}{c_2})$, thus,

$$p^{\beta i_1}-1=\exp((\log p)\beta i_1)-1\leqslant\beta_1(\log p)\beta i_1,$$

for a positive constant β_1 that depends on β_0 and c_1,c_2. Thus, the contribution of cases with $i+j<i_1$ is

$$\leqslant \frac{c_0(2A)^{i_1}}{p}(p^{\beta i_1}-1) \leqslant \left\{c_0(2A)^{i_1}\beta_1\beta i_1\right\}\frac{\log p}{p}.$$

In conclusion, we saw that for all primes $p \in (p_0, T]$ one has

$$\sum_{\substack{i\geqslant 1\\ j\geqslant 0}} A^{i+j}F(p^{i+j})(p^{\beta i}-p^{\beta(i-1)}) \leqslant 2p^{-3} + \left\{c_0(2A)^{i_1}\beta_1\beta i_1\right\}\frac{\log p}{p}.$$

Step (4). Using the last inequality with the bound $1+x_p \leqslant \exp(x_p)$, valid for all $x_p \in \mathbb{R}$, shows that, once restricted in the range $p > \max\{p_0, c_0\}$, the product in the lemma is

$$\leqslant \exp\Bigg(\sum_{\substack{\max\{p_0,c_0\}<p\leqslant T\\ p\nmid c}} (1-F(p))^{-1}\Big(2p^{-3}+\left\{c_0(2A)^{i_1}\beta_1\beta i_1\right\}\frac{\log p}{p}\Big)\Bigg).$$

Ignoring the condition $p \nmid c$ will produce a larger bound. Using the inequality $F(p) \leqslant c_0/p$ we obtain

$$\ll \exp\Big(c_0(2A)^{i_1}\beta_1\beta i_1 \sum_{\max\{p_0,c_0\}<p\leqslant T}(1-F(p))^{-1}\frac{\log p}{p}\Big),$$

where the implied constant depends at most on c_0. Using the inequality $F(p) \leqslant c_0/p$ and the estimate $\sum_{p\leqslant y}(\log p)/p \ll \log y$ leads to

$$\sum_{\max\{p_0,c_0\}<p\leqslant T}\frac{\log p}{(1-F(p))p} \leqslant \sum_{\max\{p_0,c_0\}<p\leqslant T}\frac{\log p}{p}\left(1+O_{c_0}\left(\frac{c_0}{p}\right)\right) \ll O_{c_0}(1)+\log T,$$

where the implied constant is absolute. The previous bound becomes

$$\ll_{c_0} \exp\Big(c_0(2A)^{i_1}\beta_1\beta i_1\log T\Big).$$

Recall that β_1 depends on β_0 and c_1, c_2. Since i_1 is a function of c_1 and c_2 we can thus write the bound as $\ll_{c_0} \exp(\nu\beta\log T)$ for some $\nu = \nu(\beta_0, c_0, c_1, c_2, A)$. To conclude the proof of the lemma we must deal with the contribution of the primes $p \leqslant \max\{p_0, c_0\}$. Note that for every prime p the corresponding factor in the product of the lemma is

$$\leqslant 1 + \Big(\frac{1}{1-F(p)}+1\Big)\sum_{\substack{i\geqslant 1\\ j\geqslant 0}}\min\{C'p^{(i+j)c_2/2}, A^{i+j}\}F(p^{i+j})p^{\beta i}.$$

Using the bound for F in the assumptions of Lemma 2.1 and the bound $\beta \leqslant \beta_0/\log T$ we see that the sum over i, j is at most

$$C'p^{c_1}\sum_{i\geqslant 1}p^{(-c_2/2+\beta_0/\log T)i}\sum_{j\geqslant 0}p^{-jc_2/2}.$$

Our assumption $4\beta_0/c_2 < \log T$ ensures that $\beta_0/\log T < c_2/4$, hence, we obtain the bound

$$C'p^{c_1}\sum_{i\geqslant 1}p^{-ic_2/4}\sum_{j\geqslant 0}p^{-jc_2/2}\leqslant C'p^{c_1}\sum_{i\geqslant 1}2^{-ic_2/4}\sum_{j\geqslant 0}2^{-jc_2/2}=O_{c_2}(C'p^{c_1}).$$

Taking the product of this quantity over all primes $p\leqslant \max\{p_0,c_0\}$ gives an implied constant that depends on p_0,c_0,c_1,c_2 and C'. Since $p_0=p_0(A,c_2)$ we see that the implied constant also depends on A. □

Lemma 2.3. *Fix any positive constants C,C',ϵ and assume that we are given a function $G:\mathbb{N}\to[0,\infty)$ such that for all coprime positive integers a,b one has*

$$G(ab)\leqslant G(a)\min\{C^{\Omega(b)},C'b^{\epsilon}\}.$$

Then for all coprime positive integers a,b we have $G(ab)\leqslant G(a)H(b)$, where H is the multiplicative function defined as $H(p^e)=\min\{C^e,C'p^{\epsilon e}\}$ for all $e\geqslant 1$ and primes p.

Proof. We will prove this with induction on $\omega(b)$. When $\omega(b)=0$ then $b=1$, hence, the statement clearly holds. Assume that $k\geqslant 0$ and that the statement holds for all $b\in\mathbb{N}$ with $\omega(b)=k$. Now let n,n' be coprime and assume that $\omega(n)=k+1$. We shall show that $G(n'n)\leqslant G(n')H(n)$. Writing $n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}p_{k+1}^{\alpha_{k+1}}$ where each α_i is strictly positive and the p_i are distinct primes, we let $a=n'p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ and $b=p_{k+1}^{\alpha_{k+1}}$ so that

$$G(n'n)=G(ab)\leqslant G(a)H(p_{k+1}^{\alpha_{k+1}})$$

by assumption. Now a can be written as n' multiplied by an integer that is coprime to n' and with exactly k distinct prime factors, thus, our inductive hypothesis shows that

$$G(a)\leqslant G(n')\prod_{i=1}^{k}H(p_i^{\alpha_i}).$$

Combining the two inequalities gives $G(n'n)\leqslant G(n')\prod_{i\leqslant k+1}H(p_i^{\alpha_i})=G(n')H(n)$. □

Lemma 2.4. *Keep the setting of Lemma 2.1, fix any $C>1$ and $C'>0$ and assume that $G:\mathbb{N}\to[0,\infty)$ is a function that satisfies*

$$G(ab)\leqslant G(a)\min\{C^{\Omega(b)},C'b^{c_2/2}\} \tag{2.2}$$

for all coprime positive integers a,b. Fix any positive real number β_0. For any $\Upsilon,\Psi\geqslant 2$ and $\varpi>0$ satisfying

$$\varpi\leqslant\min\left\{\frac{c_2}{2}\log\Psi,\beta_0\right\}$$

we have

$$\sum_{\substack{a>\Upsilon \\ P^+(a)<\Psi}} F(a)G(a) \prod_{\substack{c_0<p \\ p|a}} (1-F(p))^{-1}$$
$$\ll \exp\left(-\varpi \frac{\log \Upsilon}{\log \Psi}\right) \sum_{\substack{n\in\mathbb{N} \\ P^+(n)<\Psi}} F(n)G(n) \prod_{\substack{c_0<p \\ p|n}} (1-F(p))^{-1},$$

where the implied constant depends at most on C, C', β_0 and c_i.

Proof. Define $\beta := \varpi/\log\Psi$. The sum is at most

$$\sum_{P^+(a)<\Psi} F(a)G(a) \prod_{\substack{c_0<p \\ p|a}} (1-F(p))^{-1} \left(\frac{a}{\Upsilon}\right)^{\beta}.$$

Now define the multiplicative function $\psi_\beta : \mathbb{N} \to \mathbb{R}$ via the Dirichet convolution

$$m^\beta = \sum_{\substack{d\in\mathbb{N} \\ d|m}} \psi_\beta(d), \quad m \in \mathbb{N}.$$

Writing $n = a/d$ we obtain

$$\Upsilon^{-\beta} \sum_{\substack{d\in\mathbb{N} \\ P^+(d)<\Psi}} \psi_\beta(d) \sum_{\substack{n\in\mathbb{N} \\ P^+(n)<\Psi}} F(nd)G(nd) \prod_{\substack{c_0<p \\ p|nd}} (1-F(p))^{-1}.$$

Now factor $n = n_0 n_1$, where $\gcd(n_1, d) = 1$ and n_0 is only divisible by primes dividing d. Then the sum over d and n becomes

$$\sum_{\substack{d\in\mathbb{N} \\ P^+(d)<\Psi}} \psi_\beta(d) \sum_{\substack{\mathbf{n}\in\mathbb{N}^2,\ P^+(n_0n_1)<\Psi \\ p|n_0 \Rightarrow p|d \\ \gcd(n_1,d)=1}} F(n_0n_1d)G(n_0n_1d) \prod_{\substack{c_0<p \\ p|n_0n_1d}} (1-F(p))^{-1}.$$

Our assumptions on G together with Lemma 2.3 ensure that $G(n_0n_1d) \leqslant G(n_1)H(n_0d)$, where H is the multiplicative function given by $H(p^e) = \min\{C^e, C'p^{ec_2/2}\}$ for $e \geqslant 1$ and primes p. Together with the multiplicativity of F we obtain the bound

$$\Upsilon^{-\beta} \sum_{\substack{n_1\in\mathbb{N} \\ P^+(n_1)<\Psi}} F(n_1)G(n_1) \prod_{\substack{c_0<p \\ p|n_1}} (1-F(p))^{-1}$$
$$\times \sum_{\substack{n_0,d\in\mathbb{N}, P^+(d)<\Psi \\ p|n_0\Rightarrow p|d \\ \gcd(d,n_1)=1}} F(n_0d)H(n_0d)\psi_\beta(d) \prod_{\substack{c_0<p \\ p|d}} (1-F(p))^{-1}.$$

It is easy to see that $\psi_\beta(p^m) = p^{\beta m} - p^{\beta(m-1)}$ for all $m \geqslant 1$ and primes p. We can use this to write the sum over n_0, d as an Euler product. The Euler product is of the type covered by Lemma 2.2 as can be seen by taking $A = C, c = n_1$ and $T = \Psi$. The assumption of the present lemma on the size of ϖ implies that the assumption of Lemma 2.2 on the size of β. Thus, the sum over a in the lemma is

$$\ll \frac{\mathrm{e}^{\nu\beta\log\Psi}}{\Upsilon^\beta} \sum_{\substack{n\in\mathbb{N} \\ P^+(n)<\Psi}} F(n)G(n) \prod_{\substack{c_0<p \\ p|n}} (1-F(p))^{-1},$$

where $\nu = \nu(\beta_0, c_0, c_1, c_2, C)$ is positive and the implied constant depends at most on C and c_i. Using the fact that $\varpi \leqslant \beta_0$, we can write

$$\mathrm{e}^{\nu\beta\log\Psi} = \mathrm{e}^{\nu\varpi} = O_{\beta_0}(1).$$

Finally, we have $\Upsilon^{-\beta} = \exp(-\varpi \log \Upsilon / \log \Psi)$. $\square$

Lemma 2.5. *Keep the setting of Lemma 2.4 and define for any $V \geqslant 1$ the function*

$$\mathcal{H}(V) := \sum_{\substack{n\in\mathbb{N} \\ P^+(n)<V}} F(n)G(n) \prod_{\substack{c_0<p \\ p|n}} (1-F(p))^{-1}.$$

For $V \geqslant 1$ and $\epsilon > 0$ with $V^{\epsilon c_2/2} > 2C$ and $V^\epsilon > c_0$ we have

$$\mathcal{H}(V) \ll \frac{\mathcal{H}(V^\epsilon)}{\epsilon^{\nu_1}},$$

where $\nu_1 = \nu_1(C, c_0, c_1, c_2)$ is positive and the implied constant depends at most on C, C' and c_i.

Proof. For a prime $p > V^\epsilon$ we have $p^{c_2/2} > 2C$ due to the assumption $V^{\epsilon c_2/2} > 2C$. Now let $j_0 := 1 + [4/c_2 + 2c_1/c_2]$ so that $j_0 c_2 \geqslant 4 + 2c_1$. Then $-j_0 c_2/2 \leqslant -c_1 - 2$, which can be combined with $p^{c_2/2} > 2C$ to show that

$$(Cp^{-c_2})^{j_0} \leqslant p^{-j_0 c_2/2} \leqslant p^{-c_1-2}.$$

By (2.1) and the fact that $C > 1$ we see that

$$\sum_{j=1}^{\infty} C^j F(p^j) \leqslant j_0 C^{j_0} \frac{c_0}{p} + p^{c_1} \sum_{j=1+j_0}^{\infty} (Cp^{-c_2})^j \leqslant j_0 C^{j_0} \frac{c_0}{p} + p^{c_1} 2 (Cp^{-c_2})^{j_0} \leqslant j_0 C^{j_0} \frac{c_0}{p} + \frac{2}{p^2},$$

which is at most ν_1/p, where ν_1 is a positive constant that depends at most on C and c_i. We infer that

$$\prod_{p\in(V^\epsilon,V)}\left(1+\sum_{j=1}^{\infty}\frac{C^jF(p^j)}{1-F(p)}\right)\leqslant\prod_{p\in(V^\epsilon,V)}\left(1+\frac{\nu_1/p}{1-c_0/p}\right)\ll\prod_{p\in(V^\epsilon,V)}\left(1+\frac{1}{p}\right)^{\nu_1}\ll\frac{1}{\epsilon^{\nu_1}},\tag{2.3}$$

with an implied constant that depends at most on C and c_i.

We can now use (2.3) to bound $\mathcal{H}(V)$. Each positive integer n can be written uniquely as $n=n_0n_1$, where $P^+(n_0)\leqslant V^\epsilon$ and $P^-(n_1)>V^\epsilon$. We have $G(n_0n_1)\leqslant G(n_0)C^{\Omega(n_1)}$ by equation (2.2) and together with the multiplicativity of F we obtain

$$\mathcal{H}(V)\leqslant\sum_{\substack{n_0\in\mathbb{N}\\P^+(n_0)\leqslant V^\epsilon}}F(n_0)G(n_0)\prod_{\substack{c_0<p\\p|n_0}}(1-F(p))^{-1}\sum_{\substack{n_1\in\mathbb{N}\\P^-(n_1)>V^\epsilon\\P^+(n_1)<V}}C^{\Omega(n_1)}F(n_1)\prod_{\substack{c_0<p\\p|n_1}}(1-F(p))^{-1}.$$

The assumption $V^\epsilon>c_0$ shows that every prime $p>V^\epsilon$ satisfies $p>c_0$, hence, the sum over n_1 equals

$$\prod_{V^\epsilon<p<V}\left(1+(1-F(p))^{-1}\sum_{j=1}^{\infty}C^jF(p^j)\right).$$

Alluding to (2.3) and noting that the sum over n_0 equals $\mathcal{H}(V^\epsilon)$ concludes the proof. □

Lemma 2.6. *Let F be as in Lemma 2.1 and G be as in Lemma 2.4. Fix any positive real number β_0. For sufficiently large $\Upsilon,\Psi\geqslant 2$ and for all $\varpi>0$ satisfying*

$$\varpi\leqslant\min\left\{\frac{c_2}{2}\log\Psi,\beta_0\right\}$$

we have

$$\sum_{\substack{a>\Upsilon\\P^+(a)<\Psi}}F(a)G(a)\prod_{\substack{c_0<p\\p|a}}(1-F(p))^{-1}\ll\exp\left(-\varpi\frac{\log\Upsilon}{\log\Psi}\right)\sum_{a\leqslant\Psi}F(a)G(a)\prod_{\substack{c_0<p\\p|a}}(1-F(p))^{-1},$$

where the implied constant depends at most on C,β_0 and c_i.

Proof. Taking $\Psi=\Upsilon^\epsilon$ and $\beta_0=\varpi=1$ in Lemma 2.4 shows that

$$\sum_{\substack{a>\Upsilon\\P^+(a)<\Upsilon^\epsilon}}F(a)G(a)\prod_{\substack{c_0<p\\p|a}}(1-F(p))^{-1}\ll\frac{\mathcal{H}(\Upsilon^\epsilon)}{\exp(1/\epsilon)},$$

where the implied constant depends at most on C, C' and c_i. Taking a sufficiently small $\epsilon=\epsilon_0$ in terms of C and c_i makes the right-hand side be $\leqslant\mathcal{H}(\Upsilon^{\epsilon_0})/2$. Furthermore, by the definition of $\mathcal{H}$ we have

$$\mathcal{H}(\Upsilon^{\epsilon_0}) \leqslant \sum_{\substack{a\leqslant \Upsilon\\ P^+(a)<\Upsilon^{\epsilon_0}}} F(a)G(a)\prod_{\substack{c_0<p\\ p|a}}(1-F(p))^{-1} + \sum_{\substack{a> \Upsilon\\ P^+(a)<\Upsilon^{\epsilon_0}}} F(a)G(a)\prod_{\substack{c_0<p\\ p|a}}(1-F(p))^{-1},$$

thus,

$$\mathcal{H}(\Upsilon^{\epsilon_0}) \leqslant \sum_{a\leqslant \Upsilon} F(a)G(a)\prod_{\substack{c_0<p\\ p|a}}(1-F(p))^{-1} + \frac{\mathcal{H}(\Upsilon^{\epsilon_0})}{2}.$$

Hence,

$$\mathcal{H}(\Upsilon^{\epsilon_0}) \leqslant 2\sum_{a\leqslant \Upsilon} F(a)G(a)\prod_{\substack{c_0<p\\ p|a}}(1-F(p))^{-1}.$$

Thus, by Lemma 2.5 we infer that

$$\mathcal{H}(\Psi) \ll \frac{\mathcal{H}(\Psi^{\epsilon_0})}{\epsilon_0^{\nu_1}} \ll \sum_{a\leqslant \Psi} F(a)G(a)\prod_{\substack{c_0<p\\ p|a}}(1-F(p))^{-1}.$$

We conclude the proof by injecting this estimate into Lemma 2.4. □

Lemma 2.7. *Let F be as in Lemma 2.1 and G be as in Lemma 2.4. Fix any positive constant γ and assume that for every prime p we are given a constant $c(p)$ in the interval $[0,\gamma/p]$. Then for all $T \geqslant 1$ we have*

$$\sum_{a\leqslant T} F(a)G(a)\prod_{p|a}(1+c(p)) \leqslant 2^{\gamma\gamma'}\sum_{a\leqslant T}F(a)G(a),$$

where $\gamma' = 1+2(1+c_1)/c_2)c_0C^{1+2(1+c_1)/c_2} + C'(2^{c_2/2}-1)^{-1}$.

Proof. Extending multiplicatively the function c to positive square-free integers we get

$$\prod_{p|a}(1+c(p)) = \sum_{d|a}\mu(d)^2c(d).$$

This turns the sum in the lemma into

$$\sum_{a\leqslant T} F(a)G(a)\sum_{d|a}\mu(d)^2c(d) = \sum_{bd\leqslant T}\mu(d)^2c(d)F(bd)G(bd).$$

By assumption there exists C' such that $G(ab) \leqslant G(a)\min\{C^{\Omega(b)}, C'b^{c_2/2}\}$. By Lemma 2.3 with $\epsilon = c_2/2$ we see that $G(n'n) \leqslant G(n')H(n)$ for all coprime n, n', where H is the multiplicative function given by $H(p^e) = \min\{C^e, C'p^{ec_2/2}\}$ for $e \geqslant 1$ and primes

p. We factor $b = b_0 b_1$, where b_1 is coprime to d and each prime divisor of b_0 divides d. Thus,

$$F(bd)G(bd) = F(b_0 b_1 d)G(b_0 b_1 d) \leqslant F(b_0 d)F(b_1)H(b_0 d)G(b_1),$$

hence, the sum is

$$\leqslant \sum_{b_1 \leqslant T} F(b_1)G(b_1) \sum_{\substack{b_0 d \leqslant T/b_1 \\ p|b_0 \Rightarrow p|d \\ \gcd(b_1,d)=1}} \mu(d)^2 c(d) F(b_0 d) H(b_0 d).$$

We will show that the inner double sum over b_0 and d converges, and we will also upper bound the value that it attains. Dropping the condition $b_0 d \leqslant T/b_1$ we can write it as $\prod_p (1 + \mathcal{E}_p)$, where

$$\mathcal{E}_p = \sum_{\substack{\beta,\delta \in \mathbb{Z} \cap [0,\infty) \\ (\beta,\delta) \neq (0,0) \\ \beta > 0 \Rightarrow \delta > 0}} \mu(p^\delta)^2 c(p^\delta) F(p^{\beta+\delta}) H(p^{\beta+\delta}) = c(p) \sum_{\beta \geqslant 0} F(p^{\beta+1}) H(p^{\beta+1})$$

and the product is taken over all primes $p \nmid b_1$. Let $\mathcal{B}$ be the least integer satisfying $2(1 + c_1) \leqslant (\mathcal{B} + 1)c_2$. To estimate the contribution of $\beta \leqslant \mathcal{B}$ we use $c(p) \leqslant \gamma/p$ to get

$$\sum_{0 \leqslant \beta \leqslant \mathcal{B}} c(p) F(p^{\beta+1}) H(p^{\beta+1}) \leqslant \frac{\gamma}{p} \sum_{0 \leqslant \beta \leqslant \mathcal{B}} F(p^{\beta+1}) C^{\beta+1} \leqslant \frac{\gamma(\mathcal{B}+1) c_0 C^{1+\mathcal{B}}}{p^2}.$$

To bound the contribution of the remaining terms we use $F(p^e) \leqslant p^{c_1 - e c_2}$ to get

$$\sum_{\beta \geqslant 1+\mathcal{B}} c(p) F(p^{\beta+1}) H(p^{\beta+1}) \leqslant C' \gamma p^{-1+c_1} \sum_{\beta \geqslant 1+\mathcal{B}} p^{-(\beta+1)c_2/2}.$$

This is at most

$$C' \gamma p^{-1+c_1} p^{-(\mathcal{B}+2)c_2/2} \sum_{\beta \geqslant 0} 2^{-(\beta+1)c_2/2} = \frac{C' \gamma (2^{c_2/2} - 1)^{-1}}{p^{1-c_1+(\mathcal{B}+2)c_2/2}}.$$

The exponent of p in the right-hand side is strictly larger than 2 owing to our definition of $\mathcal{B}$. We have thus shown that for all primes p one has $0 \leqslant \mathcal{E}_p \leqslant \mathcal{B}' p^{-2}$, where

$$\mathcal{B}' := \gamma(\mathcal{B}+1) c_0 C^{1+\mathcal{B}} + C' \gamma (2^{c_2/2} - 1)^{-1}.$$

By the definition of $\mathcal{B}$ we have $2(1 + c_1) > \mathcal{B} c_2$, hence, $\mathcal{B}' \leqslant \mathcal{D}'$, where

$$\mathcal{D}' := \gamma(1 + 2(1 + c_1)/c_2) c_0 C^{1+2(1+c_1)/c_2} + \gamma C' (2^{c_2/2} - 1)^{-1},$$

hence, $\prod_p (1 + \mathcal{E}_p) \leqslant \prod_p (1 + p^{-2\mathcal{D}'}) \leqslant \prod_p (1 + p^{-2})^{\mathcal{D}'} \leqslant \zeta(2)^{\mathcal{D}'} \leqslant 2^{\mathcal{D}'}$. □

Lemma 2.8. *Fix a positive constant α_1 and let $g : \mathbb{N} \to [0, \infty)$ be a multiplicative function for which $g(p) \leqslant \alpha_1/p$ for all primes p. Then for all $a \in \mathbb{N}, \alpha_2, \alpha_3 > 0$ and $x \geqslant 2$ we have*

$$\sum_{\substack{m\in\mathbb{N},\gcd(m,a)=1\\ p|m\Rightarrow p\in(\alpha_1,x^{\alpha_2})}} \mu(m)^2 g(m) \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_3}\\ p\nmid am}} (1-g(p))^2 \ll \mathcal{C} \prod_{\substack{\alpha_1<p\leqslant x^{\min\{\alpha_2,\alpha_3\}}\\ p\nmid a}} (1-g(p)),$$

where

$$\mathcal{C} := \mathbb{1}[\alpha_2 \leqslant \alpha_3] \prod_{\substack{x^{\alpha_2}<p\leqslant x^{\alpha_3}\\ p\nmid a}} (1-g(p))^2 + \mathbb{1}[\alpha_2 > \alpha_3] \prod_{\substack{x^{\alpha_3}<p\leqslant x^{\alpha_2}\\ p\nmid a}} (1-g(p))^{-1}$$

and the implied constant depends on α_1 but is independent of a, α_2, α_3 and x.

Proof. Let $\mathcal{P}$ be the product of all primes in (α_1, x^{α_2}) that do not divide a. Using that g is multiplicative and $g \geqslant 0$ we see that the sum over m is

$$\begin{aligned}\sum_{m|\mathcal{P}} g(m) \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_3}\\ p\nmid am}} (1-g(p))^2 &= \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_3}\\ p\nmid a}} (1-g(p))^2 \sum_{m|\mathcal{P}} g(m) \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_3}\\ p|m}} (1-g(p))^{-2}\\ &= \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_3}\\ p\nmid a}} (1-g(p))^2 \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_2}\\ p\nmid a, p>x^{\alpha_3}}} (1+g(p)) \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_2}\\ p\nmid a, p\leqslant x^{\alpha_3}}} \left(1+\frac{g(p)}{(1-g(p))^2}\right).\end{aligned}$$

The assumption $g(p) \leqslant \alpha_1/p$ implies that $g(p) < 1$ whenever $p > \alpha_1$, thus, we can use the approximations

$$1+\epsilon = (1-\epsilon)^{-1}(1+O(\epsilon^2)), \quad \left(1+\frac{\epsilon}{(1-\epsilon)^2}\right) = (1-\epsilon)^{-1}(1+O(\epsilon^2))$$

with $\epsilon = g(p)$ respectively in the second and third product. This will produce

$$\ll \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_3}\\ p\nmid a}} (1-g(p))^2 \prod_{\substack{\alpha_1<p\leqslant x^{\alpha_2}\\ p\nmid a}} (1-g(p))^{-1} \ll \mathcal{C} \prod_{\substack{\alpha_1<p\leqslant x^{\min\{\alpha_2,\alpha_3\}}\\ p\nmid a}} (1-g(p))$$

with implied constants that depend at most on α_1. This is because $\prod_p (1+O(g(p)^2))$ converges absolutely due to the assumption $g(p) = O(1/p)$. □

Let us recall a special case of [5, Lemma 6.3] here:

Lemma 2.9 *(Fundamental lemma of Sieve Theory). Let $\kappa > 0, y > 1$. There exist sequences of real numbers $(\lambda_m^{\pm})$ depending only on κ and y with the following properties:*

$$\lambda_1^{\pm} = 1, \tag{2.4}$$

$$|\lambda_m^{\pm}| \leqslant 1 \quad \text{if } 1 < m < y \tag{2.5}$$

$$\lambda_m^{\pm} = 0 \quad \text{if } m \geqslant y, \tag{2.6}$$

and for any integer $n > 1$,

$$\sum_{m|n} \lambda_m^{-} \leqslant 0 \leqslant \sum_{m|n} \lambda_m^{+}. \tag{2.7}$$

Moreover, for any multiplicative function $f(m)$ with $0 \leqslant f(p) < 1$ and satisfying

$$\prod_{w \leqslant p < z} (1 - f(p))^{-1} \leqslant \left(\frac{\log z}{\log w}\right)^{\kappa} \left(1 + \frac{K}{\log w}\right) \tag{2.8}$$

for all $2 \leqslant w < z \leqslant y$ we have

$$\sum_{m|P(z)} \lambda_m^{\pm} f(m) = \left(1 + O\left(\mathrm{e}^{-\sigma}\left(1 + \frac{K}{\log z}\right)^{10}\right)\right) \prod_{p \leqslant z}(1 - f(p)), \tag{2.9}$$

where $P(z)$ is the product of all primes $p \leqslant z$ and $\sigma = \log y / \log z \geqslant 1$, the implied constant depending only on κ.

Lemma 2.10. *Let $g : \mathbb{N} \to [0, 1)$ be as in Lemma 2.8 and assume that there exist constants α_2, α_3 such that*

$$\prod_{w \leqslant p < z} (1 - g(p))^{-1} \leqslant \left(\frac{\log z}{\log w}\right)^{\alpha_2} \left(1 + \frac{\alpha_3}{\log w}\right)$$

for all $2 \leqslant w < z$. Fix any constants $\xi_1, \xi_2 \in (0, 1)$, $\Lambda_1, \Lambda_2 > 0$ and assume that we are given a finite set of non-zero integers $\mathcal{S} = \{s_1, \ldots, s_N\}$ and a set of non-negative real numbers $x, a_1, \ldots, a_N$ such that for all $d \leqslant x^{\xi_1}$ one has

$$\sum_{\substack{1 \leqslant n \leqslant N \\ d | s_n}} a_n = g(d) x (1 + \epsilon_1) + \epsilon_2,$$

where ϵ_i are real numbers that satisfy

$$|\epsilon_1| \leqslant \Lambda_1 \prod_{\substack{\alpha_1 < p \leqslant x \\ p \nmid d}} (1 - g(p))^2 \quad \text{and} \quad |\epsilon_2| \leqslant \Lambda_2 x^{1 - \xi_2}.$$

Fix any constants $\xi_3 \in (0, \xi_1)$ and $\xi_4 > 0$, let $\Gamma = \max\{1/\xi_4, 1/(\xi_1 - \xi_3), 1/\xi_2\}$ and assume that $\log x > 4\alpha_3 \Gamma$.

Then, for all $b \in \mathbb{N}$ satisfying $b \leqslant x^{\xi_3}$ we have

$$\sum_{\substack{1\leqslant n\leqslant N, b|s_n\\ p\leqslant x^{\xi_4} \text{ and } p\nmid b\Rightarrow p\nmid s_n}} a_n \leqslant \mathcal{C}_0\Big(\Gamma^{\alpha_2} x g(b) \prod_{\substack{p\leqslant x\\ p\nmid b}}(1-g(p)) + x^{1-\xi_2/2}\Big),$$

where $\mathcal{C}_0$ is a positive constant that is independent of b, x and ξ_4.

Proof. Let $\gamma = \min\{(\xi_1-\xi_3)/2, \xi_2/4, \xi_4\}$. We employ Lemma 2.9 with

$$\kappa = \alpha_2, K = \alpha_3, y = x^{\min\{\xi_1-\xi_3,\xi_2/2\}}, f(p) = g(p)\mathbb{1}[p > \alpha_1 \ \& \ p \nmid b], z = x^{\gamma},$$

where α_1 is as in Lemma 2.8. To verify (2.8) we note that for all $\alpha_1 < w' < z'$ one has

$$\prod_{w'\leqslant p<z'} \frac{1}{1-f(p)} = \prod_{\substack{w'\leqslant p<z'\\ p\nmid b}} \frac{1}{1-g(p)} \leqslant \prod_{w'\leqslant p<z'} \frac{1}{1-g(p)} \leqslant \left(\frac{\log z'}{\log w'}\right)^{\alpha_2}\left(1+\frac{\alpha_3}{\log w'}\right).$$

Define $\mathcal{P}$ to be the product of all primes $p \in (\alpha_1, z]$ that do not divide b. Then the cardinality in the lemma is bounded by

$$\sum_{\substack{1\leqslant n\leqslant N, b|s_n\\ \gcd(s_n,\mathcal{P})=1}} a_n = \sum_{\substack{1\leqslant n\leqslant N\\ b|s_n}} a_n \sum_{\substack{m|s_n\\ m|\mathcal{P}}} \mu(m) \leqslant \sum_{\substack{1\leqslant n\leqslant N\\ b|s_n}} a_n \sum_{\substack{m|s_n\\ m|\mathcal{P}}} \lambda_m^+ = \sum_{m|\mathcal{P}} \lambda_m^+ \sum_{\substack{1\leqslant n\leqslant N\\ bm|s_n}} a_n,$$

where we used (2.4) and (2.7) in the inequality. By (2.5) the only m that contribute must satisfy $bm \leqslant by \leqslant bx^{\xi_1-\xi_3} \leqslant x^{\xi_1}$. This allows us to use the assumption, thus,

$$\sum_{m|\mathcal{P}} \lambda_m^+ \sum_{\substack{1\leqslant n\leqslant N\\ bm|s_n}} a_n = xg(b)\sum_{m|\mathcal{P}} \lambda_m^+ g(m) + \epsilon_3 + \epsilon_4,$$

where we used (2.5) and the coprimality of b and m, and the ϵ_i are real numbers that satisfy

$$|\epsilon_3| \leqslant \Lambda_2 y x^{1-\xi_2}, \quad |\epsilon_4| \leqslant \Lambda_1 x g(b) \sum_{m|\mathcal{P}} g(m) \prod_{\substack{\alpha_1<p\leqslant x\\ p\nmid bm}} (1-g(p))^2.$$

Our choice of y makes sure that $yx^{1-\xi_2} \leqslant x^{1-\xi_2/2}$, which is acceptable. Note that $\xi_2 < 1$ hence $\gamma < 1$. Thus, when applying Lemma 2.8 with $\alpha_2 = \gamma, \alpha_3 = 1$ one sees that the factor $\mathcal{C}$ appearing in the lemma is at most 1. This leads to the bound

$$\leqslant \Lambda_3\Big(xg(b)\Big|\sum_{m|\mathcal{P}} \lambda_m^+ g(m)\Big| + x^{1-\xi_2/2} + xg(b)\prod_{\substack{\alpha_1<p\leqslant x^{\gamma}\\ p\nmid b}}(1-g(p))\Big),$$

for some positive real number Λ_3 that is independent of b, x and ξ_4. Note that $g(m) = f(m)$ for all $m \mid \mathcal{P}$, thus, by (2.9) we obtain

$$\Big|\sum_{m\mid\mathcal{P}} \lambda_m^+ g(m)\Big| = \Big|\sum_{m\mid\mathcal{P}} \lambda_m^+ f(m)\Big| \leqslant \Lambda_4 \prod_{\alpha_1<p\leqslant z} (1-f(p)) = \Lambda_4 \prod_{\substack{\alpha_1<p\leqslant x^\gamma \\ p\nmid b}} (1-g(p)),$$

for some positive real number Λ_4 that is independent of b, x and ξ_4. We have so far obtained the bound

$$\Lambda_5\Big(xg(b)\prod_{\substack{\alpha_1<p\leqslant x^\gamma \\ p\nmid b}} (1-g(p)) + x^{1-\xi_2/2}\Big)$$

for some positive real number Λ_5 that is independent of b, x and ξ_4. It remains to upper-bound the product over p. For this, we write

$$\prod_{\substack{\alpha_1<p\leqslant x^\gamma \\ p\nmid b}} (1-g(p)) \leqslant \prod_{\substack{\alpha_1<p\leqslant x \\ p\nmid b}} (1-g(p)) \prod_{x^\gamma<p\leqslant x} (1-g(p))^{-1}$$

and use our assumptions to upper-bound it by

$$\leqslant \prod_{\substack{\alpha_1<p\leqslant x \\ p\nmid b}} (1-g(p)) \left(\frac{\log x}{\log x^\gamma}\right)^{\alpha_2} \left(1+\frac{\alpha_3}{\gamma \log x}\right) \leqslant \Lambda_6 \gamma^{-\alpha_2} \prod_{\substack{\alpha_1<p\leqslant x \\ p\nmid b}} (1-g(p)),$$

whenever $\log x > \alpha_3/\gamma$ and where Λ_6 is a positive real number that is independent of b, x and ξ_4. To conclude the proof note that $1/\gamma \leqslant 4\Gamma$, hence, $\gamma^{-\alpha_2} \leqslant (4\Gamma)^{\alpha_2}$ and $\log x > \alpha_3/\gamma$ due to $\log x > 4\Gamma\alpha_3$. $\square$

Lemma 2.11. *Fix any positive c_0, c_1, c_2, assume that F is as in Lemma 2.1 and that there exists $c_3 \geqslant 0$ such that for all primes p and integers $e \geqslant 2$ we have $F(p^e) \leqslant c_3/p^2$. Fix any $C, C' > 0$ and assume that $G : \mathbb{N} \to [0, \infty)$ is a multiplicative function such that for all integers a one has $G(a) \leqslant \min\{C^{\Omega(a)}, C' a^{c_2/2}\}$.*

Then for all $x \geqslant 1$ we have

$$\sum_{\substack{n\leqslant x \\ P^-(n)>c_0}} F(n)G(n) \ll \exp\left(\sum_{c_0<p\leqslant x} F(p)G(p)\right),$$

where the implied constant depends at most on c_i and C, C'.

Proof. We define a multiplicative function H' such that when p is prime and $e \geqslant 2$ one has $H'(p^e) = \min\{C^e, C' p^{c_2 e/2}\}$ while $H'(p) = G(p)$. We claim that $G(a) \leqslant H'(a)$ for all $a \geqslant 1$. Write $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where p_i are distinct primes and $\alpha_i \geqslant 1$ for all i so that

$$G(a) = G((\prod_{i<k} p_i^{\alpha_i})p_k^{\alpha_k}) = G(\prod_{i<k} p_i^{\alpha_i})G(p_k^{\alpha_k}) \leqslant G(\prod_{i<k} p_i^{\alpha_i})H'(p_k^{\alpha_k}).$$

Similarly,

$$G(\prod_{i<k} p_i^{\alpha_i}) = G((\prod_{i<k-1} p_i^{\alpha_i})p_{k-1}^{\alpha_{k-1}}) \leqslant G(\prod_{i<k} p_i^{\alpha_i})H'(p_{k-1}^{\alpha_{k-1}}).$$

Continuing likewise until all factors $p_i^{\alpha_i}$ are exhausted we get $G(a) \leqslant \prod_{i\leqslant k} H'(p_i^{\alpha_i}) = H'(a)$.

Hence, $G(b) \leqslant H'(b)$ for all b and therefore the sum in the lemma is at most

$$\begin{aligned}\sum_{\substack{n\leqslant x\\P^-(n)>c_0}} F(n)H'(n) &\leqslant \prod_{\substack{n\leqslant x\\P^-(n)>c_0}} \left(1+\sum_{e\geqslant 1} F(p^e)H'(p^e)\right)\\ &\leqslant \exp\left(\sum_{c_0<p\leqslant x, e\geqslant 1} F(p^e)H'(p^e)\right)\end{aligned}$$

due to the inequality $1+z \leqslant \mathrm{e}^z$ valid for all $z \in \mathbb{R}$. Let $\mathfrak{E}$ be a positive integer that will be specified later. The contribution of $e > \mathfrak{E}$ is at most

$$p^{c_1}\sum_{e>\mathfrak{E}} p^{-ec_2}H'(p^e) \leqslant C'p^{c_1}\sum_{e>\mathfrak{E}} p^{-ec_2/2} \leqslant C'p^{c_1-\mathfrak{E}c_2/2}(1-2^{-c_2/2})^{-1} \ll p^{c_1-\mathfrak{E}c_2/2}.$$

Taking $\mathfrak{E}$ to be the least positive integer satisfying $2(c_1+2)/c_2 \leqslant \mathfrak{E}$ yields the bound $\ll p^{-2}$. The contribution of the terms in the interval $[2, \mathfrak{E}]$ is

$$\leqslant \sum_{2\leqslant e\leqslant \mathfrak{E}} F(p^e)H'(p^e) \leqslant \sum_{2\leqslant e\leqslant \mathfrak{E}} F(p^e)C^e \leqslant \frac{c_3}{p^2}\sum_{2\leqslant e\leqslant \mathfrak{E}} C^e \ll \frac{1}{p^2}.$$

Thus, the overall bound becomes

$$\exp\left(\sum_{c_0<p\leqslant x, e\geqslant 1} F(p^e)H'(p^e)\right) \leqslant \exp\left(\sum_{c_0<p\leqslant x} F(p)H'(p)\right)\exp\left(\sum_{c_0<p\leqslant x} O(1/p^2)\right),$$

which is sufficient because $H'(p) = G(p)$. □

3. The upper bound

3.1. Start of the proof

Let us define the constants

$$\eta_1 := \frac{1}{\alpha}\min\left\{\frac{\xi}{20}, \frac{\theta}{2}, \frac{1}{2}\right\}, \quad \eta_2 := \min\left\{\frac{\lambda_1}{4(1+\lambda_1+\lambda_2)}, \frac{1}{2}\right\}. \tag{3.1}$$

Define

$$Z := M^{\alpha\eta_1}. \tag{3.2}$$

For $a \in \mathcal{A}$ we factorise $c_a = p_1^{e_1} \cdots p_r^{e_r}$ with primes $p_1 < \cdots < p_r$ and exponents $e_i \geqslant 1$. Let d_a be the unique integer of the form $d_a := p_1^{e_1} \cdots p_i^{e_i}$ satisfying

$$p_1^{e_1} \cdots p_i^{e_i} \leqslant Z < p_1^{e_1} \cdots p_i^{e_i} p_{i+1}^{e_{i+1}} \tag{3.3}$$

and let $b_a := p_{i+1}^{e_{i+1}} \cdots p_r^{e_r}$. By construction we have

$$P^+(d_a) < P^-(b_a), \tag{3.4}$$

$$\gcd(d_a, b_a) = 1, \tag{3.5}$$

$$d_a \leqslant Z. \tag{3.6}$$

The following cases will be considered:

(i) $P^-(b_a) \geqslant Z^{\eta_2}$,
(ii) $P^-(b_a) < Z^{\eta_2}$ and $d_a \leqslant Z^{1/2}$,
(iii) $P^-(b_a) \leqslant (\log Z) \log\log Z$ and $Z^{1/2} < d_a \leqslant Z$,
(iv) $(\log Z) \log\log Z < P^-(b_a) < Z^{\eta_2}$ and $Z^{1/2} < d_a \leqslant Z$.

3.2. Case (i)

The plan in this case is to show that b_a has few prime divisors so that c_a has few prime divisors in a large interval. The density of a with the latter property will be bounded by the Brun sieve.

For the $a \in \mathcal{A}$ in the present case we have

$$M^{\alpha\eta_1\eta_2\Omega(b_a)} = Z^{\eta_2\Omega(b_a)} \leqslant P^-(b_a)^{\Omega(b_a)} \leqslant b_a \leqslant c_a \leqslant \widetilde{B} M^\alpha$$

and therefore $\Omega(b_a) \leqslant \frac{1+\log \widetilde{B}}{\eta_1\eta_2}$ for $M > e^{1/\alpha}$. By (3.5) we have $\gcd(d_a, b_a) = 1$, thus leading via Definition 1.2 to

$$f(c_a) \leqslant f(d_a) A^{\frac{1+\log \widetilde{B}}{\eta_1\eta_2}}.$$

Now let $d := d_a$, so that $d \leqslant Z$ and $d \mid c_a$. Furthermore, c_a is coprime to every prime in the interval $[2, Z^{\eta_2})$ that does not divide d. This is because every prime that divides c_a must necessarily divide d_a or b_a and in our case all prime divisors of b_a are in the interval $[Z^{\eta_2}, \infty)$. In particular, c_a is coprime to every prime in the interval (B, Z^{η_2}) that is coprime to d. Define

$$\mathcal{P} := \prod_{\substack{p\in(B,Z^{\eta_2})\\ p\nmid d}} p.$$

We obtain

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (i)}}} \chi_T(a)f(c_a) \leqslant A^{\frac{1+\log \widetilde{B}}{\eta_1\eta_2}} \sum_{d\leqslant Z} f(d) \sum_{\substack{a\in\mathcal{A},d|c_a\\ \gcd(\mathcal{P},c_a)=1}} \chi_T(a).$$

To deal with the coprimality condition we employ Lemma 2.10 with

$$\mathcal{S} = \{c_a : a\in\mathcal{A}, \chi_T(a)>0\}, \{a_n : 1\leqslant n\leqslant N\} = \{\chi_T(a) : a\in\mathcal{A}, \chi_T(a)>0\}$$

and $x = M, g = h, \alpha_1 = B, \alpha_2 = \kappa, \alpha_3 = K, \xi_1 = \theta, \xi_2 = \xi, b = d, \xi_3 = \alpha\eta_1, \xi_4 = \alpha\eta_1\eta_2$. The assumption $\xi_3 < \xi_1$ is satisfied due to (3.1). Thus,

$$\sum_{\substack{a\in\mathcal{A},d|c_a\\ \gcd(\mathcal{P},c_a)=1}} \chi_T(a) \ll Mh_T(d) \prod_{\substack{B<p\leqslant M\\ p\nmid d}} (1-h_T(p)) + M^{1-\xi/2},$$

where the implied constant is independent of d, M and T but is allowed to depend on $\alpha, \eta_1, \eta_2, K, \kappa, \lambda_i, \theta$ and ξ. This gives the overall bound

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (i)}}} \chi_T(a)f(c_a) \ll A^{\frac{1+\log \widetilde{B}}{\eta_1\eta_2}} \sum_{d\leqslant Z} f(d) \left\{ Mh_T(d) \prod_{\substack{B<p\leqslant M\\ p\nmid d}} (1-h_T(p)) + M^{1-\xi/2} \right\}.$$

Since $f(n) \ll n$, we infer that

$$\sum_{d\leqslant Z} f(d)M^{1-\xi/2} \ll Z^2M^{1-\xi/2} \ll M^{2\alpha\eta_1+1-\xi/2} \leqslant M^{1-\xi/3}$$

due to (3.1). This leads us to

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (i)}}} \chi_T(a)f(c_a) \ll M \sum_{d\leqslant Z} f(d)h_T(d) \prod_{\substack{B<p\leqslant M\\ p\nmid d}} (1-h_T(p)) + M^{1-\xi/3}.$$

We can now extend the sum over d to all $d \leqslant M$ due to (3.1) that guarantees that $Z \leqslant M$. Combining this together with Lemma 2.7 for $F = h_T$, $G = f$ and $c(p) = -1 + (1-h_T(p))^{-1}$ yields

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (i)}}} \chi_T(a)f(c_a) \ll M \prod_{B<p\leqslant M} (1-h_T(p)) \sum_{d\leqslant M} f(d)h_T(d) + M^{1-\xi/3}. \tag{3.7}$$

3.3. Case (ii)

The main idea is to show that the exponent of $P^-(b_a)$ in the prime factorisation of c_a is large and then prove that this cannot happen too often.

Let $q := P^-(b_a)$. Equation (3.3) and the definition of case (ii) respectively show

$$Z < d_a q^{v_q(b_a)}, d_a \leqslant Z^{1/2},$$

thus, $q^{v_q(b_a)} > Z^{1/2}$. For a prime p, we take m_p to be the smallest positive integer such that $p^{m_p} > Z^{1/2}$ and we take n_p to be the largest positive integer such that $p^{n_p} \leqslant M^\theta$. We set $f_p = \min(m_p, n_p)$. Then we always have

$$p^{f_p} > \frac{1}{p} M^{\min\{\alpha\eta_1/2,\theta\}} = \frac{M^{\alpha\eta_1/2}}{p}. \tag{3.8}$$

Also observe that $q^{f_q} \mid c_a$ (by $q^{f_q} \mid q^{m_q}$ and $q^{m_q} \mid c_a$) and $q^{f_q} \leqslant M^\theta$. Thus, we have shown that there exists a prime $q < Z^{\eta_2}$ (due to the definition of case (ii)) that has the properties $q^{f_q} \mid c_a$, $q^{f_q} \leqslant M^\theta$ and (3.8). Hence, by Definition 1.6 we obtain

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (ii)}}} \chi_T(a) \leqslant \sum_{\text{prime } q<Z^{\eta_2}} C_{q^{f_q}}(T) \ll \sum_{\text{prime } q<Z^{\eta_2}} (h_T(q^{f_q})M + M^{1-\xi}) \leqslant M\mathcal{S} + Z^{\eta_2}M^{1-\xi},$$

where $\mathcal{S} := \sum_{q<Z^{\eta_2}} h_T(q^{f_q})$. By (1.3) and (3.8) the sum $\mathcal{S}$ is at most

$$\sum_{q<Z^{\eta_2}} q^{-f_q\lambda_1+\lambda_2} \leqslant M^{-\lambda_1\alpha\eta_1/2} \sum_{q<Z^{\eta_2}} q^{\lambda_1+\lambda_2} \leqslant M^{-\lambda_1\alpha\eta_1/2} Z^{\eta_2(1+\lambda_1+\lambda_2)}.$$

This equals $M^{-\rho}$, where

$$\rho := \lambda_1\alpha\eta_1/2 - \alpha\eta_1\eta_2(1+\lambda_1+\lambda_2) = \alpha\eta_1(\lambda_1/2 - \eta_2(1+\lambda_1+\lambda_2))$$

is strictly positive owing to (3.1). Fix any $\delta > 0$. By Definition 1.2 we have $f(c_a) \leqslant C c_a^{\delta/\alpha}$ for all $a \in \mathcal{A}$, where C is positive and depends on α and δ. Thus, (1.8) shows that for all $a \in \mathcal{A}$ one has $f(c_a) \ll CM^\delta$. We have therefore proved that for every $\delta > 0$ one has

$$\begin{aligned}\sum_{\substack{a\in\mathcal{A}\\ \text{case (ii)}}} \chi_T(a) f(c_a) &\ll C\left(M^{1-\rho+\delta} + Z^{\eta_2}M^{1-\xi+\delta}\right)\\ &= C\left(M^{1-\rho+\delta} + M^{1-\xi+\delta+\alpha\eta_1\eta_2}\right) \ll CM^{1+\delta-\beta_1},\end{aligned} \tag{3.9}$$

where $\beta_1 := \min\{\alpha\eta_1(\lambda_1/2 - \eta_2(1+\lambda_1+\lambda_2)), \xi - \alpha\eta_1\eta_2\}$ is positive due to (3.1) and the fact that $\eta_2 < 1$. Furthermore, the implied constant depends at most on $\alpha, \widetilde{B}, \delta, K, \kappa, \lambda_i, \theta$ and ξ.

3.4. Case (iii)

The key idea in this case is to show that d_a is divisible only by very small primes and then show that this does not happen too often. We have

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (iii)}}} \chi_T(a) \leqslant \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)\leqslant(\log Z)\log\log Z}} \sum_{\substack{a\in\mathcal{A}\\ d\mid c_a}} \chi_T(a) = \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)\leqslant(\log Z)\log\log Z}} C_d(T).$$

Equation (3.1) makes sure that $d \leqslant Z \leqslant M^\theta$, thus, we can employ the estimate in Definition 1.6. It yields the upper bound

$$\ll \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)\leqslant(\log Z)\log\log Z}} (M^{1-\xi} + h_T(d)M) \leqslant ZM^{1-\xi} + M \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)\leqslant(\log Z)\log\log Z}} h_T(d).$$

To bound the sum over d we employ Lemma 2.1 with

$$F = h_T, c_0 = B, c_1 = \lambda_2, c_2 = \lambda_1, x = Z, z = Z^{1/2}.$$

It shows that the sum over d is

$$\ll Z^{-c/2}M^{o(1)} = M^{-\alpha\eta_1 c/2+o(1)} \leqslant M^{-\alpha\eta_1 c/4},$$

where

$$c := \min\left\{\frac{\lambda_1}{2}, \frac{1}{1+[2\lambda_2/\lambda_1]}\right\}.$$

The overall bound becomes

$$\ll ZM^{1-\xi} + M^{1-\alpha\eta_1 c/4} = M^{1-\xi+\alpha\eta_1} + M^{1-\alpha\eta_1 c/4} \ll M^{1-\beta_2},$$

where

$$\beta_2 := \min\{\xi - \alpha\eta_1, \alpha\eta_1 c/4\}$$

is strictly positive by (3.1). Bringing everything together we conclude that for every $\delta > 0$ one has

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (iii)}}} \chi_T(a) f(c_a) \ll M^{1+\delta-\beta_2}. \tag{3.10}$$

3.5. Case (iv)

The main idea is to use the fact that c_a/d_a has no small prime divisors and then apply the Brun sieve to see that this can happen with low probability, even when one counts with the additional weight $A^{\Omega(c_a/d_a)}$.

Recalling (3.5) and Definition 1.2 we see that

$$f(c_a) = f(d_a b_a) \leqslant f(d_a) A^{\Omega(b_a)}.$$

Thus, letting $d = d_a$, we infer that

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (iv)}}} \chi_T(a) f(c_a) \ll \sum_{Z^{1/2}<d\leqslant Z} f(d) \sideset{}{^*}\sum_{\substack{a\in\mathcal{A}\\ d\mid c_a}} \chi_T(a) A^{\Omega(c_a/d)}, \tag{3.11}$$

where $\sum^*$ is subject to the further conditions

$$\gcd(d, c_a/d) = 1 \quad \text{and} \quad (\log Z)\log\log Z < P^-(c_a/d) < Z^{\eta_2}.$$

It would be easier to estimate the sum over a in the right-hand side of (3.11) if the summand $A^{\Omega(c_a/d)}$ was a constant. With this in mind we freeze the value of $P^-(c_a/d)$ as follows: let

$$s := \left[\frac{\log Z}{\log P^-(c_a/d)}\right]$$

so that $Z^{1/(s+1)} < P^-(c_a/d) \leqslant Z^{1/s}$ and $s \in \mathbb{N} \cap [1, s_0]$, where

$$s_0 := \left[\frac{\log Z}{\log\{(\log Z)(\log\log Z)\}}\right] \leqslant \frac{\log Z}{\log\log Z}$$

for Z large enough. By (1.8) we have for a with $\chi_T(a) \neq 0$ that

$$M^{\alpha\eta_1 \frac{\Omega(c_a/d)}{s+1}} = \left(Z^{1/(s+1)}\right)^{\Omega(c_a/d)} < P^-(c_a/d)^{\Omega(c_a/d)} \leqslant c_a/d \leqslant c_a \leqslant \widetilde{B} M^{\alpha}$$

thus, for $M \geqslant \mathrm{e}$ we obtain

$$\Omega(c_a/d) \leqslant (s+1)\left(\frac{1}{\eta_1} + \frac{\log \widetilde{B}}{\alpha\eta_1}\right) \leqslant 2s\left(\frac{1}{\eta_1} + \frac{\log \widetilde{B}}{\alpha\eta_1}\right) = \tau s,$$

where $\tau = \tau(\alpha, \widetilde{B}, \eta_1)$ is a positive constant. Hence the right-hand side of (3.11) is

$$\ll \sum_{1\leqslant s\leqslant s_0} A^{\tau s} \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)<Z^{1/s}}} f(d) \sum_{\substack{a\in\mathcal{A}, d\mid c_a, \gcd(d, c_a/d)=1\\ Z^{1/(s+1)}<P^-(c_a/d)\leqslant Z^{1/s}}} \chi_T(a).$$

The sum over a is at most

$$\sum_{\substack{a\in\mathcal{A},d\mid c_a\\ p\leqslant Z^{1/(s+1)}\text{ and } p\nmid d\Rightarrow p\nmid c_a}}\chi_T(a),$$

which will be bounded by employing Lemma 2.10 with

$$\mathcal{S}=\{c_a: a\in\mathcal{A},\chi_T(a)>0\},\{a_n:1\leqslant n\leqslant N\}=\{\chi_T(a):a\in\mathcal{A},\chi_T(a)>0\},$$
$$g=h_T,\alpha_1=B,\alpha_2=\kappa,\alpha_3=K,x=M,\xi_1=\theta,\xi_2=\xi,\xi_3=\frac{\theta}{2},\xi_4=\frac{\alpha\eta_1}{s+1},b=d,$$

where h_T,B,κ,K,θ,M and ξ are as in Definition 1.6. The assumption $b\leqslant x^{\xi_3}$ of Lemma 2.10 is satisfied due to (3.1). The further assumption $\log x>4\alpha_3\Gamma$ is satisfied for all large enough M compared to $K,\alpha,\eta_1,\theta,\xi$ due to the inequality

$$\Gamma=\max\left\{\frac{1+s}{\alpha\eta_1},\frac{2}{\theta},\frac{1}{\xi}\right\}\ll_{\alpha,\eta_1,\theta,\xi}1+s\leqslant 1+s_0\leqslant 1+\frac{\log Z}{\log\log Z}\ll_{\alpha,\eta_1}\frac{\log M}{\log\log M}.$$

We obtain the upper bound

$$\ll\max\left\{\frac{1+s}{\alpha\eta_1},\frac{2}{\theta},\frac{1}{\xi}\right\}^{\kappa}Mh_T(d)\prod_{\substack{B<p\leqslant M\\ p\nmid d}}(1-h_T(p))+M^{1-\xi/2}$$
$$\ll s^{\kappa}Mh_T(d)\prod_{\substack{B<p\leqslant M\\ p\nmid d}}(1-h_T(p))+M^{1-\xi/2},$$

where the implied constants are independent of s,d and M. Thus, the right-hand side of (3.11) is

$$\ll M\sum_{1\leqslant s\leqslant s_0}A^{\tau s}s^{\kappa}\sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)<Z^{1/s}}}f(d)h_T(d)\prod_{\substack{B<p\leqslant M\\ p\nmid d}}(1-h_T(p))+M^{1-\xi/2}\sum_{1\leqslant s\leqslant s_0}A^{\tau s}\sum_{d\leqslant Z}f(d).$$

We have $\sum_{d\leqslant Z}f(d)\ll Z^2=M^{2\alpha\eta_1}$ by Definition 1.2. Thus,

$$M^{1-\xi/2}\sum_{1\leqslant s\leqslant s_0}A^{\tau s}\sum_{d\leqslant Z}f(d)\ll M^{1-\xi/2+2\alpha\eta_1}s_0A^{\tau s_0}\leqslant M^{1-\xi/3}$$

due to (3.1) and the inequality $s_0\leqslant(\log Z)/(\log\log Z)$ which implies that

$$s_0A^{\tau s_0}\ll A^{2\tau s_0}=Z^{O(1/\log\log Z)}=M^{o(1)}.$$

Thus, the right-hand side of (3.11) is

$$\ll M \sum_{1\leqslant s\leqslant s_0} A^{\tau s} s^{\kappa} \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)<Z^{1/s}}} f(d)h_T(d) \prod_{\substack{B<p\leqslant M\\ p\nmid d}} (1-h_T(p)) + M^{1-\xi/3}.$$

By (3.1) we have $\alpha\eta_1 \leqslant 1$, so that $d \leqslant Z \leqslant M$. Then the product over p is

$$\leqslant \prod_{\substack{B<p\leqslant M\\ p\nmid d}} (1-h_T(p)) = \prod_{B<p\leqslant M} (1-h_T(p)) \prod_{\substack{B<p\\ p\mid d}} (1-h_T(p))^{-1}$$

and we get the bound

$$\ll M \prod_{B<p\leqslant M} (1-h_T(p)) \sum_{1\leqslant s\leqslant s_0} A^{\tau s} s^{\kappa} \sum_{\substack{Z^{1/2}<d\leqslant Z\\ P^+(d)<Z^{1/s}}} f(d)h_T(d) \prod_{\substack{B<p\\ p\mid d}} (1-h_T(p))^{-1} + M^{1-\xi/3}.$$

We now bound the sum over d by alluding to Lemma 2.6 with

$$\Upsilon = Z^{1/2}, \Psi = Z^{1/s}, F = h_T, G = f, c_0 = B, c_1 = \lambda_2, c_2 = \lambda_1, \varpi = \beta_0, C = A,$$

where ϖ is defined via $4A^{\tau} = \mathrm{e}^{\varpi/2}$. This means that ϖ depends on $\alpha, A, \widetilde{B}$, and η_1. Hence, the sum over d is

$$\ll \exp(-\varpi s/2) \sum_{d\leqslant Z^{1/s}} f(d)h_T(d) \prod_{\substack{B<p\\ p\mid d}} (1-h_T(p))^{-1}.$$

We can extend the summation to all $d \leqslant M$ since the summand is non-negative and $Z^{1/s} \leqslant Z \leqslant M$. Thus, the right-hand side of (3.11) is

$$\ll M \prod_{B<p\leqslant M} (1-h_T(p)) \sum_{d\leqslant M} f(d)h_T(d) \prod_{\substack{B<p\\ p\mid d}} (1-h_T(p))^{-1} \sum_{1\leqslant s\leqslant s_0} z^s s^{\kappa} + M^{1-\xi/3},$$

where $z = A^{\tau}\mathrm{e}^{-\varpi/2}$. By the definition of ϖ we have $z = 1/4$, hence, the sum over s is bounded in terms of κ. Thus, we have shown that

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (iv)}}} \chi_T(a) f(c_a) \ll M \prod_{B<p\leqslant M} (1-h_T(p)) \sum_{d\leqslant M} f(d)h_T(d) \prod_{\substack{B<p\\ p\mid d}} (1-h_T(p))^{-1} + M^{1-\xi/3},$$

where the implied constant depends at most on $\alpha, A, \widetilde{B}, B, \lambda_i, \eta_i, \theta, \xi$ and κ. Alluding to Lemma 2.7 with $F = h_T$ and $G = f$ yields

$$\sum_{\substack{a\in\mathcal{A}\\ \text{case (iv)}}} \chi_T(a) f(c_a) \ll M \prod_{B<p\leqslant M} (1-h_T(p)) \sum_{d\leqslant M} f(d)h_T(d) + M^{1-\xi/3}. \tag{3.12}$$

3.6. Proof of Theorem 1.9

The upper bound claimed in Theorem 1.9 derives from (3.7) and (3.12). Taking $\delta = \beta_1/2$ in (3.9) and $\delta = \beta_2/2$ in (3.10) shows that cases (ii) and (iii) contribute $\ll M^{1-\beta_3}$, where β_3 is given by

$$\min\left\{\frac{\alpha\eta_1(\lambda_1/2-\eta_2(1+\lambda_1+\lambda_2))}{2}, \frac{\xi-\alpha\eta_1}{2}, \frac{\alpha\eta_1\lambda_1}{16}, \frac{\alpha\eta_1}{8(1+[2\lambda_2/\lambda_1])}\right\}.$$

The term $M^{1-\xi/3}$ that is present in (3.7) and (3.12) and the term $M^{1-\beta_3}$ may be absorbed in the upper bound from Theorem 1.9, thus concluding the proof.

4. The lower bound

Recall the notation of θ, ξ in Definition 1.6 and let κ, K be as in Definition 1.1. We introduce the constants

$$v = \min\left\{1, \frac{\theta\min\{1/4, \xi/(4\theta)\}}{1+9\kappa+(\log 2)+10(\log K)}\right\}, v_0 := \min\{v/2, \theta/2\}.$$

Let $z := M^v$. For each $c \in \mathbb{N}$ we define

$$c^\flat = \prod_{p \leqslant z} p^{v_p(c)}.$$

Note that for a positive integer d satisfying $P^+(d) \leqslant z$, one has $d = c^\flat$ if and only if d divides c and the smallest prime divisor of c/d strictly exceeds z. Classifying all $a \in \mathcal{A}$ according to the value of $d := c_a^\flat$ we thus obtain

$$\sum_{a\in\mathcal{A}} \chi_T(a) f(c_a) = \sum_{\substack{d\in\mathbb{N}\\ P^+(d)\leqslant z}} \sum_{\substack{a\in\mathcal{A}\\ c_a^\flat = d}} \chi_T(a) f(c_a) = \sum_{\substack{d\in\mathbb{N}\\ P^+(d)\leqslant z}} \sum_{\substack{a\in\mathcal{A}, d|c_a\\ P^-(c_a/d)>z}} \chi_T(a) f(c_a).$$

Note that if $d \leqslant M^{v_0}$ then $P^+(d) \leqslant d \leqslant M^{v_0} \leqslant z$. Thus, since $f \geqslant 0$, we can restrict the sum over d to get the lower bound

$$\sum_{a\in\mathcal{A}} \chi_T(a) f(c_a) \geqslant \sum_{1\leqslant d\leqslant M^{v_0}} \sum_{\substack{a\in\mathcal{A}, d|c_a\\ P^-(c_a/d)>z}} \chi_T(a) f(c_a).$$

Using (1.8) and the inequality $m \geqslant P^-(m)^{\Omega(m)}$ leads to

$$\Omega(c_a/d) \leqslant \frac{\log(c_a/d)}{\log P^-(c_a/d)} \leqslant \frac{\log c_a}{\log P^-(c_a/d)} \leqslant \frac{\log(\widetilde{B}M^\alpha)}{\log(M^{v_0})} \leqslant L_0$$

for some $L_0 = L_0(\widetilde{B}, \alpha, v_0) > 0$. Therefore, by assumption (1.9), $f(c_a/d) \gg 1$, where the implied constant depends at most on L_0. Since $P^+(d) < P^-(c_a/d)$ we see that $d, c_a/d$ are coprime, hence, the multiplicativity of f yields

$$f(c_a) = f(d)f(c_a/d) \gg_{\widetilde{B},\alpha,v_0} f(d).$$

Injecting this into the previous estimate will yield

$$\sum_{a\in\mathcal{A}} \chi_T(a)f(c_a) \gg_{\widetilde{B},\alpha,v} \sum_{1\leqslant d\leqslant M^{v_0}} f(d) \sum_{\substack{a\in\mathcal{A},d|c_a\\ P^-(c_a/d)>z}} \chi_T(a). \tag{4.1}$$

We will now lower bound the sum over $a \in \mathcal{A}$ by arguments similar to the ones in the proof of Lemma 2.10. Using the sequence λ_m^- from Lemma 2.9 we obtain

$$\sum_{\substack{a\in\mathcal{A},d|c_a\\ P^-(c_a/d)>z}} \chi_T(a) \geqslant \sum_{m|\mathcal{P}} \lambda_m^- \sum_{\substack{a\in\mathcal{A}\\ dm|c_a}} \chi_T(a),$$

where $\mathcal{P}$ is the product of all primes $p \leqslant z$. Recall from Lemma 2.9 that λ_m^- is supported on integers $m \leqslant y$. We define $y = M^{\theta\epsilon}$ where $\epsilon = \min\{1/4, \xi/(4\theta)\}$. Then the only m that contribute to the sum satisfy

$$dm \leqslant dy = dM^{\theta\epsilon} \leqslant M^{v_0+\theta\epsilon} \leqslant M^{\theta/2+\theta\epsilon} \leqslant M^{\theta},$$

thus, we can use the assumption on the growth of $C_d(T)$. It yields the estimate

$$Mh_T(d) \sum_{m|\mathcal{P}} \lambda_m^- h_T(m) + o\left(Mh_T(d) \sum_{\substack{m|\mathcal{P}\\ m\leqslant y}} h_T(m) \prod_{\substack{B<p\leqslant M\\ p\nmid dm}} (1 - h_T(p))^2 \right)$$
$$+ O\left(M^{1-\xi} \sum_{m\leqslant y} 1 \right).$$

The last error term is $\ll M^{1-\xi}y = M^{1-\xi+\epsilon\theta}$. Since $\epsilon\theta < \xi/2$, the error term becomes $O(M^{1-\xi/2})$, which is acceptable.

By taking out the largest factor of each $m \mid \mathcal{P}$ that is a product of primes that satisfy $p \leqslant B$ or $p \mid d$, the sum over m in the error term is

$$\leqslant \prod_{p\leqslant B} (1 + h_T(p)) \prod_{B<p|d} (1 + h_T(p)) \sum_{\substack{m\leqslant y,\gcd(m,d)=1\\ p|m\Rightarrow B<p\leqslant z}} \mu(m)^2 h_T(m) \prod_{\substack{B<p\leqslant M\\ p\nmid dm}} (1 - h_T(p))^2.$$

The primes $p \leqslant B$ contribute $O_B(1)$. Using Lemma 2.8 with $\alpha_1 = B$, $x^{\alpha_2} = z = M^v$, $x^{\alpha_3} = M$ and $a = d$, and taking advantage of the fact that $v \leqslant 1$, we infer that

$$\ll \prod_{B<p|d}(1+h_T(p)) \prod_{\substack{M^v<p<M \\ p\nmid d}}(1-h_T(p))^2 \prod_{\substack{B<p<M^v \\ p\nmid d}}(1-h_T(p)),$$

that is at most

$$\ll \prod_{B<p|d}(1-h_T(p))^{-2} \prod_{B<p<M}(1-h_T(p)).$$

To treat the main term sum $\sum_{m|\mathcal{P}} \lambda_m^- h_T(m)$ we use [5, Equation (6.40)], which is a more precise version of (2.8) in the case of λ_m^-. Specifically, [5, Equation (6.40)] states that

$$\sum_{m|\mathcal{P}} \lambda_m^- h_T(m) > (1-\mathrm{e}^{\beta-s}K^{10}) \prod_{p\leqslant z}(1-h_T(p)),$$

where $\beta = 1+9\kappa$ and $s = (\log y)/(\log z)$. In our case one has $s = \epsilon\theta/v$ and a simple calculation shows that our definition of v ensures that $1-\mathrm{e}^{\beta-s}K^{10} \geqslant 1/2$, thus,

$$\sum_{m|\mathcal{P}} \lambda_m^- h_T(m) \gg \prod_{p\leqslant z}(1-h_T(p)).$$

Injecting our estimates in (4.1) gives

$$\sum_{a\in\mathcal{A}} \chi_T(a) f(c_a) \gg M \prod_{p\leqslant M^v}(1-h_T(p)) \sum_{d\leqslant M^{v_0}} f(d)h_T(d) + o(M\mathcal{T}) + O(M^{1-\xi/2}),$$

where

$$\mathcal{T} = \prod_{B<p\leqslant M^v}(1-h_T(p)) \sum_{1\leqslant d\leqslant M^{v_0}} f(d)h_T(d) \prod_{B<p|d}(1-h_T(p))^{-2}.$$

Letting $c(p) = (1-h_T(p))^{-2}-1$ and applying Lemma 2.7 we obtain

$$\sum_{1\leqslant d\leqslant M^{v_0}} f(d)h_T(d) \prod_{B<p|d}(1-h_T(p))^{-2} \ll \sum_{1\leqslant d\leqslant M^{v_0}} f(d)h_T(d).$$

This leads to

$$\sum_{a\in\mathcal{A}} \chi_T(a) f(c_a) \gg M \prod_{p\leqslant M^v}(1-h_T(p)) \sum_{1\leqslant d\leqslant M^{v_0}} f(d)h_T(d) + O(M^{1-\xi/2}).$$

Since $h_T(p) \in [0,1)$ for $p > M^v$ and using that $v \leqslant 1$, the product over $p \leqslant M^v$ is at least $\prod_{p\leqslant M}(1-h_T(p))$. It thus remains to prove

$$\sum_{1\leqslant d\leqslant M^{v_0}} f(d)h_T(d) \gg \sum_{1\leqslant d\leqslant M} f(d)h_T(d).$$

Using the fact that f and h_T are both multiplicative we can write

$$\sum_{1\leqslant d\leqslant M} f(d)h_T(d) = \sum_{\substack{1\leqslant b\leqslant M \\ P^+(b)\leqslant M^{v_0}}} f(b)h_T(b) \sum_{\substack{1\leqslant c\leqslant M/b \\ P^-(c)>M^{v_0}}} f(c)h_T(c)$$

and it suffices to prove that the sum over c is bounded independently of M. We apply Lemma 2.11 to get the upper bound

$$\sum_{\substack{1\leqslant c\leqslant M/b \\ P^-(c)>M^{v_0}}} f(c)h_T(c) \ll \exp\left(\sum_{M^{v_0}<p\leqslant M} f(p)h_T(p)\right).$$

Recall that $f(p) \leqslant A$ and $h_T(p) \leqslant B/p$, so the sum over p is

$$\ll \sum_{M^{v_0}<p\leqslant M} \frac{1}{p} = O(1),$$

thus concluding the proof.

Data availability

No data was used for the research described in the article.

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