Modeling complex quantum systems

Random matrices, BCS theory, and quantum lattice systems

by

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Abstract

This thesis deals with several different models for complex quantum mechanical systems and is structured in three main parts.

In Part I, we study mean field random matrices as models for quantum Hamiltonians. Our focus lies on proving concentration estimates for resolvents of random matrices, so-called *local laws*, mostly in the setting of multiple resolvents. These estimates have profound consequences for eigenvector overlaps and thermalization problems. More concretely, we obtain, e.g., the optimal eigenstate thermalization hypothesis (ETH) uniformly in the spectrum for Wigner matrices, an optimal lower bound on non-Hermitian eigenvector overlaps, and prethermalization for deformed Wigner matrices. In order to prove our novel multi-resolvent local laws, we develop and devise two main methods, the static Ψ*-method* and the dynamical *Zigzag strategy*.

In Part II, we study Bardeen-Cooper-Schrieffer (BCS) theory, the standard mean field microscopic theory of superconductivity. We focus on asymptotic formulas for the characteristic *critical temperature* and *energy gap* of a superconductor and prove universality of their ratio in various physical regimes. Additionally, we investigate multi-band superconductors and show that inter-band coupling effects can only enhance the critical temperature.

In Part III, we study quantum lattice systems. On the one hand, we show a strong version of the *local-perturbations-perturb-locally* (LPPL) principle for the ground state of weakly interacting quantum spin systems with a uniform on-site gap. On the other hand, we introduce a notion of a *local gap* and rigorously justify response theory and the Kubo formula under the weakened assumption of a local gap.

Additionally, we discuss two classes of problems which do not fit into the three main parts of the thesis. These are deformational rigidity of *Liouville metrics* on the torus and relativistic toy models of particle creation via *interior-boundary-conditions* (IBCs).

To Anna and Emma.

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About the Author

Joscha Henheik started his undergraduate studies in Physics at the University of Tübingen and soon later decided to add Mathematics as a second major. After graduating from Tübingen in 2020 with a MSc in Physics and a thesis under supervision of Stefan Teufel, he joined ISTA and became a PhD student of László Erdős. He is interested in a wide range of mathematical problems originating from physics, mostly applying tools from analysis and probability in their solutions.

His previous and current research interests are divided in (but not limited to) three main areas: (i) Random matrix theory, with a focus on multi-resolvent local laws and applications to eigenvector overlaps and thermalization problems; (ii) Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity, with a focus on universality phenomena in different physical regimes; (iii) Quantum lattice systems and adiabatic theory, with a focus on gapped systems at zero temperature and applications to response theory.

List of Collaborators and Publications

This thesis contains joint works with the following collaborators (listed alphabetically):

Giorgio Cipolloni, László Erdős, Oleksii Kolupaiev, Edwin Langmann, Asbjørn Bækgaard Lauritsen, Bipul Poudyal, Jana Reker, Volodymyr Riabov, Barbara Roos, Dominik Schröder, Shoki Sugimoto, Stefan Teufel, Roderich Tumulka, Tom Wessel

This thesis includes the following publications and preprints:

Part I: Random Matrices

[154]; Chap. 1	G. Cipolloni, L. Erdős, J. Henheik, and D. Schröder. Optimal Lower Bound on Eigenvector Overlaps for non-Hermitian Random Matrices. *J. Funct. Anal.*, 287(4), 2024
[152]; Chap. 2	G. Cipolloni, L. Erdős, J. Henheik, and O. Kolupaiev. Gaussian fluctuations in the equipartition principle for Wigner matrices. *Forum Math., Sigma*, 11, 2023
[150]; Chap. 3	G. Cipolloni, L. Erdős, and J. Henheik. Eigenstate thermalisation at the edge for Wigner matrices. *arXiv:2309.05488*, 2023
[151]; Chap. 4	G. Cipolloni, L. Erdős, and J. Henheik. Out-of-time-ordered correlators for Wigner matrices. *Adv. Theor. Math. Phys.*, 2024. Accepted, arXiv: 2402.17609
[153]; Chap. 5	G. Cipolloni, L. Erdős, J. Henheik, and O. Kolupaiev. Eigenvector decorelation for random matrices. *arXiv:2410.10718*, 2024
[239]; Chap. 6	L. Erdős, J. Henheik, and V. Riabov. Cusp universality for correlated random matrices. *arXiv:2410.06813*, 2024
[238]; Chap. 7	L. Erdős, J. Henheik, J. Reker, and V. Riabov. Prethermalization for Deformed Wigner Matrices. *Ann. Henri Poincaré*, pages 1–43, 2024
[237]; Chap. 8	L. Erdős, J. Henheik, and O. Kolupaiev. Loschmidt echo for deformed Wigner matrices. *Lett. Math. Phys.*, 115(1):1–42, 2025
[541]; Chap. 9	S. Sugimoto, J. Henheik, V. Riabov, and L. Erdős. Eigenstate thermalisation hypothesis for translation invariant spin systems. *J. Stat. Phys.*, 190(7):128, 2023

Part II: BCS Theory

[333]; Chap. 10 J. Henheik. The BCS Critical Temperature at High Density. *Math. Phys. Anal. Geom.*, 25(1), 2022

[336]; Chap. 11 J. Henheik and A. B. Lauritsen. The BCS Energy Gap at High Density. *J. Stat. Phys.*, 189(5), 2022

[338]; Chap. 12 J. Henheik, A. B. Lauritsen, and B. Roos. Universality in low-dimensional BCS theory. *Rev. Math. Phys.*, page 2360005, 2023

[337]; Chap. 13 J. Henheik and A. B. Lauritsen. Universal behavior of the BCS energy gap. *J. Spectr. Theory*, 2025. (online first)

[335]; Chap. 14 J. Henheik, E. Langmann, and A. B. Lauritsen. Multi-band superconductors have enhanced critical temperatures. *arXiv:2409.17297*, 2024

Part III: Quantum Lattice Systems

[343]; Chap. 15 J. Henheik, S. Teufel, and T. Wessel. Local stability of ground states in locally gapped and weakly interacting quantum spin systems. *Lett. Math. Phys.*, 112(1):9, 2022

[346]; Chap. 16 J. Henheik and T. Wessel. On adiabatic theory for extended fermionic lattice systems. *J. Math. Phys.*, 63(12):121101, 2022

[347]; Chap. 17 J. Henheik and T. Wessel. Response theory for locally gapped systems. *arXiv:2410.10809*, 2024

Appendix: Miscellaneous Results

[334]; Chap. A J. Henheik. Deformational rigidity of integrable metrics on the torus. *Ergod. Theory Dyn. Syst.*, 45:467–503, 2025

[345]; Chap. B J. Henheik and R. Tumulka. Creation Rate of Dirac Particles at a Point Source. *J. Phys. A: Math. Theor.*, 56(44):445201, 2023

[339]; Chap. C J. Henheik, B. Poudyal, and R. Tumulka. How a Space-Time Singularity Helps Remove the Ultraviolet Divergence Problem. *arXiv:2409.00677*, 2024

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Introduction and Summary of Results

It is a recurrent problem in mathematical physics, to analyze complex quantum systems. These are, in many cases, inevitably intractable due their large number of degrees of freedom, and thus their high-dimensionality. Very often, physicists developed effective models, which are meant to grasp the essence of a system and thereby reducing its complexity.

Although these models are meant to simplify the original complex systems, they are by no means trivial. In this thesis, we deal with some approximating effective models themselves, analyze their properties, and rigorously justify predictions from the physics literature.

This thesis is structured in three main parts:

- In Part I, we study *mean field* random matrices, providing a statistical approach to large disordered quantum systems.
- In Part II, we study Bardeen-Cooper-Schrieffer (BCS) theory, a mean field microscopic theory of superconductivity.
- In Part III, we study quantum lattice systems, which arise by reducing the degrees of freedom of particles to finite dimensional subspaces.

Besides these three main parts of the thesis, we also discuss work on Liouville metrics on the torus and relativistic toy models of particle creation in the appendix.

Part I: Random Matrices

In the first part of this thesis, we consider mean field random matrices, mostly modeling (part of) a quantum Hamiltonian. More precisely, we consider the following random matrix ensembles, for which *mean field* means that the variances have comparable sizes:

(i) **Wigner matrices:** These are real symmetric or complex Hermitian matrices $W \in \mathbf{C}^{N\times N}$, such that the upper-triangular entries $\{w_{ab} : a \le b\}$ are independent, the off-diagonal entries $\{w_{ab} : a < b\}$ are identically distributed with $\mathbf{E}\, w_{ab} = 0$ and $\mathbf{E}\,|w_{ab}|^2 = 1/N$, and the diagonal entries are identically distributed with $\mathbf{E}\, w_{aa} = 0$ and $c/N \le \mathbf{E}\,|w_{aa}|^2 \le C/N$ for some N-independent constants $c, C > 0$.

Wigner matrices include the famous Gaussian Orthogonal/Unitary Ensemble (GOE/GUE), where the entries are suitably normalized real/complex centered Gaussian random variables.

(ii) **Deformed Wigner matrices:** Let $D \in \mathbf{C}^{N\times N}$ be a deterministic self-adjoint (i.e. real symmetric or complex Hermitian) and $W \in \mathbf{C}^{N\times N}$ a Wigner matrix. Then $H = D + W$ is a deformed Wigner matrix.

(iii) **Correlated random matrices:** These are self-adjoint matrices H, for which the cumulants of the entries decay in an integrable manner, and it holds that

$$N\,\mathbf{E}\left[|\mathrm{tr}[(H-\mathbf{E}\,H)R]|^2\right] \geq c\,\mathrm{tr}[R^2]$$

for some N-independent constant $c>0$ and any matrix R in the same symmetry class (i.e. real symmetric or complex Hermitian) as H.

(iv) **Deformed i.i.d. matrices:** A (*non-Hermitian*) i.i.d. matrix $X \in \mathbf{C}^{N\times N}$ has independent and identically distributed entries, which are centered and normalized such that $\mathbf{E}\,|x_{ab}|^2 = 1/N$. Given some deterministic $\Lambda \in \mathbf{C}^{N\times N}$, we call $\Lambda + X$ a deformed i.i.d. matrix.

We point out that the variances in the random matrix ensembles defined above are chosen such that their spectrum remains of order one in the large N limit. Moreover, the self-adjoint models in (i)–(iii) are increasing in generality, in the sense that

$$\text{Wigner matrices} \subsetneq \text{Deformed Wigner matrices} \subsetneq \text{Correlated random matrices}\,.$$

Other mean-field random matrix models include *sample covariance matrices* [587, 433, 88], *Wigner-type matrices* [15], or *invariant ensembles* [54, 109, 108, 304, 353]. These are, however, not studied in this thesis. Further popular random matrix models beyond the scope of this thesis are random band matrices [529, 110, 115, 592], sparse matrices (e.g. as adjacency matrices of random graphs) [250, 242, 51, 52, 50], or free sums of random matrices [47, 46, 359].

Apart from the discussion of our result on non-Hermitian i.i.d. matrices below, we will henceforth focus on Hermitian ones, without further mentioning.

Local laws and applications

Traditionally, random matrix theory aims to analyze the *empirical density of states* (empirical DOS)

$$\mu_N := \frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j}\,,$$

a random probability measure on $\mathbf{R}$ with mass $1/N$ at the eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ of a random matrix, which we temporarily assume to be a Wigner matrix W for concreteness. In [586], Wigner showed that, in this case, the empirical DOS converges weakly, as $N\to\infty$, to the celebrated semi-circular law

$$\rho_{\mathrm{sc}}(x)\mathrm{d}x := \frac{\sqrt{[4-x^2]_+}}{2\pi}\mathrm{d}x\,.$$

His proof proceeded by computing moments of μ_N via a graphical expansion and showing that they converge to the moments of the semi-circular law, given by the Catalan numbers.

It turns out, that, in order to study the empirical DOS at smaller scales (than a *global* weak convergence), it is convenient to analyze the *Stieltjes transform*, uniquely determining the measure. Recall that, given a measure μ, its *Stieltjes transform* at $z\in\mathbf{C}\setminus\mathbf{R}$ is defined as

$$m_\mu(z) := \int_{\mathbf{R}}\frac{\mathrm{d}\mu(z)}{x-z}\,.$$

In case of the empirical DOS, using spectral decomposition, it holds that

$$m_{\mu_N}(z) := \int_{\mathbf{R}}\frac{\mathrm{d}\mu_N(x)}{x-z} = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\lambda_j - z} = \frac{1}{N}\,\mathrm{tr}[G(z)] =: \langle G(z)\rangle$$

where we used the notation $\langle\cdot\rangle := N^{-1}\operatorname{tr}[\cdot]$ for the normalized trace. In this way, one has linked the study of the empirical DOS μ_N to the study of the resolvent $G(z) := (W - z)^{-1}$ of W.

Note that the convergence of $m_{\mu_N}(z) = \langle G(z)\rangle$ to a deterministic value clearly hinges on the spectral parameter $z \in \mathbf{C}\setminus\mathbf{R}$. For Wigner matrices, the fact that $m_{\mu_N}(z)$ converges to the Stieltjes transform $m_{\mathrm{sc}}(z)$ of the semi-circular law for every fixed, i.e. N-independent $z \in \mathbf{C}\setminus\mathbf{R}$, is equivalent to the weak convergence of μ_N to Wigner's semi-circle law. One might thus ask, if, apart from this *global law*, the spectral parameter z can be chosen dependent on N, and convergence of the Stieltjes transforms is still valid as $N \to \infty$. In fact, by spectral decomposition, we see that the convergence of

$$\langle \operatorname{Im} G(z)\rangle = \frac{1}{N}\sum_{j=1}^{N}\frac{\operatorname{Im} z}{(\lambda_j - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \tag{1}$$

to a deterministic value, essentially depends only on the eigenvalues λ_j lying in an interval of size $|\operatorname{Im} z|$ around the energy $\operatorname{Re} z$. This is a consequence of the fact that $x \mapsto \pi^{-1}|\operatorname{Im} z|/[(x-\operatorname{Re} z)^2 + (\operatorname{Im} z)^2]$ is an approximate δ-function around $\operatorname{Re} z$ on scale $|\operatorname{Im} z|$. One thus expects that, as long as $|\operatorname{Im} z|$ is such that the number of eigenvalues effectively contributing to the summation in (1), roughly given by $\sim N|\operatorname{Im} z|\rho_{\mathrm{sc}}(\operatorname{Re} z)$, is $\gg 1$, it still holds that $m_{\mu_N}(z) \to m_{\mathrm{sc}}(z)$ as $N \to \infty$.

Indeed, it is a common phenomenon of rather general random matrices H (including the correlated random matrices introduced above), that their resolvent $G(z) = (H - z)^{-1}$ at a spectral parameter $z \in \mathbf{C}\setminus\mathbf{R}$ tends to concentrate around a deterministic matrix $M(z)$, even for $|\operatorname{Im} z|$ only slightly above the local eigenvalue spacing around the energy $\operatorname{Re} z$. Such estimates are called *local laws* and usually take the following *averaged* and *isotropic* form: Consider a deterministic matrix $B \in \mathbf{C}^{N\times N}$ and deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$. Then it holds that

$$\big|\big\langle\big(G(z) - M(z)\big)B\big\rangle\big| \prec \frac{\|B\|}{N|\operatorname{Im} z|}, \qquad \big|\big\langle \boldsymbol{x}, \big(G(z) - M(z)\big)\boldsymbol{y}\big\rangle\big| \prec \|\boldsymbol{x}\|\,\|\boldsymbol{y}\|\sqrt{\frac{\rho(z)}{N|\operatorname{Im} z|}}, \tag{2}$$

where $\rho(z) := \pi^{-1}|\operatorname{Im}\langle M(z)\rangle|$ is the local self-consistent density of states (scDOS) and the $\prec$-notation is a suitable notion of high-probability bound up to N^{ε}-factors (see, e.g., the definition around (1.1.10) below).

We will now discuss, how the deterministic matrix $M(z)$ in (2) arises, and afterwards explain several (nowadays standard) applications of local laws, namely *eigenvalue rigidity*, *eigenvector delocalization*, and *local eigenvalue universality*.

Matrix Dyson Equation and self-consistent density of states The matrix $M(z)$ in (2) is given as the unique solution [236] to the *matrix Dyson equation* (MDE)

$$-M(z)^{-1} = z - A + \mathcal{S}[M(z)] \quad \text{under the constraint} \quad \operatorname{Im} M(z)\cdot\operatorname{Im} z > 0\,,$$

where $\operatorname{Im} M := (2\mathrm{i})^{-1}(M - M^*)$. The data pair $(A, \mathcal{S})$ of the MDE is given by the first and second moment of the random matrix H. That is, the expectation is given by $A := \mathbf{E}\,H$ and the *self-energy* or *covariance operator* $\mathcal{S}$ is defined as

$$\mathcal{S}[R] := \mathbf{E}\big[(H - \mathbf{E}[H])R(H - \mathbf{E}[H])\big] \quad \text{for} \quad R \in \mathbf{C}^{N\times N}. \tag{3}$$

The limiting self-consistent density of states (scDOS) ρ of H is obtained from the solution M of the MDE as

$$\rho(x) = \lim_{\eta\searrow 0}\rho(x + \mathrm{i}\eta) := \lim_{\eta\searrow 0}\pi^{-1}\langle \operatorname{Im} M(x + \mathrm{i}\eta)\rangle\,.$$

As an example, in the case of Wigner matrices, we have that $\rho = \rho_{\mathrm{sc}}$ and $M(z) = m_{\mathrm{sc}}(z)\mathbf{1}$, where $m_{\mathrm{sc}}(z)$ solves the scalar equation

$$-m_{\mathrm{sc}}(z)^{-1} = z + m_{\mathrm{sc}}(z) \quad \text{with} \quad \operatorname{Im} m_{\mathrm{sc}}(z)\cdot\operatorname{Im} z > 0$$

Moreover, under rather general conditions, it is shown (see [236] for a review) that M is a Hölder-$1/3$ continuous function of z and that the scDOS ρ has the following properties:

(i) the support of ρ consists of finitely many compact intervals;

(ii) wherever ρ is strictly positive (in the *bulk*), it is real analytic;

(iii) there exist two types of singularities, where ρ vanishes, *edges* and *cusps*;

(iv) for an *edge point* $\mathfrak{e} \in \partial \operatorname{supp} \rho$, one has $\rho(\mathfrak{e} \pm x) = c\sqrt{x} + o(\sqrt{x})$ and $\rho(\mathfrak{e} \mp x) = 0$ for $0 < x \ll 1$ and a constant $c = c(\mathfrak{e}) > 0$;

(v) for a *cusp point* $\mathfrak{c} \in \operatorname{supp} \rho$, one has $\rho(\mathfrak{c} + x) = c|x|^{1/3} + o(|x|^{1/3})$ for some constant $c = c(\mathfrak{c}) > 0$.

The spectral regimes corresponding to (ii), (iv), and (v) are called the *bulk*, *edge*, and *cusp regime* of the scDOS, respectively.

Rigidity of eigenvalues Let $\lambda_1 \le \ldots \le \lambda_N$ be the eigenvalues of a correlated random matrix H and ρ the scDOS. Moreover, we define the typical eigenvalue locations of the eigenvalues (*quantiles*) γ_i implicitly via

$$\int_{-\infty}^{\gamma_i} \rho(x)\mathrm{d}x = \frac{i}{N} \quad \text{for} \quad i = 1, \ldots, N\,.$$

Then, as a consequence of the local law (2) and using Helffer-Sjöstrand calculus, one can show that

$$|\lambda_i - \gamma_i| \prec \eta_{\mathfrak{f}}(\gamma_i)\,,$$

i.e. the eigenvalues remain very close to their expected locations and are thus *rigid*. Here, $\eta_{\mathfrak{f}}(E)$ is the local eigenvalue fluctuation scale around an energy E, implicitly defined as

$$\int_{E-\eta_{\mathfrak{f}}(E)}^{E+\eta_{\mathfrak{f}}(E)} \rho(x)\mathrm{d}x = \frac{1}{N}\,.$$

It is of order N^{-1} in the bulk, $N^{-2/3}$ at the edge, and $N^{-3/4}$ at the cusp. It is a consequence of the strong correlation of the eigenvalues, that they fluctuate only on such a small scale.

Delocalization of eigenvectors Another consequence of the local law (2) is that the eigenvectors of H are completely delocalized. This means that, taking $(\boldsymbol{u}_i)_{i=1}^N$ to be the ℓ^2-normalized eigenvectors of H and $\boldsymbol{x} \in \mathbf{C}^N$ a deterministic unit vector, we have $|\langle \boldsymbol{x}, \boldsymbol{u}_i\rangle| \prec N^{-1/2}$ for all $i \in [N] := \{1, \ldots, N\}$. Such so-called *eigenvector delocalization* follows from the local law (2):

$$|\langle \boldsymbol{x}, \boldsymbol{u}_i\rangle|^2 \le C \sum_{j=1}^{N} \frac{\eta^2 |\langle \boldsymbol{x}, \boldsymbol{u}_j\rangle|^2}{(E-\lambda_j)^2 + \eta^2} = C\eta \,\langle \boldsymbol{x}, \operatorname{Im} G(E + \mathrm{i}\eta)\boldsymbol{x}\rangle \prec \eta$$

for some constant $C > 0$ and choosing (in the bulk) $\eta = N^{-1+\varepsilon}$ for some arbitrarily small $\varepsilon > 0$, and $E = \gamma_i$.

Three-step-strategy for universality On the eigenvalue fluctuation scale $\eta_{\mathfrak{f}}$, individual eigenvalues become relevant. On this *microscopic* scale, it has been conjectured by Wigner in the 1950's [586], and later formalized as the *Wigner-Dyson-Mehta (WDM) universality conjecture* [446], roughly stating that the local eigenvalue statistics are insensitive to any specifics of the model apart from the symmetry class. We point out that, in contrast with the local law (2) on *mesoscopic scale* (i.e. above the local eigenvalue spacing), which can be viewed as a law of large numbers (LLN) for the resolvent, the emergence of universal spectral statistics can be viewed as an analog of the central limit theorem (CLT) in the case of weakly correlated random variables.

In order to formulate the universality of local eigenvalue statistics, we (implicitly) define the k-point correlation function $p_N^{(k)}$ via

$$\int_{\mathbf{R}^k} f(\boldsymbol{x}) p_N^{(k)}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \binom{N}{k}^{-1} \sum_{\{i_1,\dots,i_k\}\subset[N]} \mathbf{E}\, f(\lambda_{i_1},\dots,\lambda_{i_k})$$

for any compactly supported smooth test function f, where the summation is running over all k-tuples of distinct elements from $[N]$.

We now formulate the WDM conjecture in the complex Hermitian case.

Conjecture (WDM conjecture in the Hermitian symmetry class)**.** *Assume that $\mathfrak{b}$, $\mathfrak{e}$, and $\mathfrak{c}$ are bulk,*[3] *edge, and cusp points of some scDOS ρ with parameters $\gamma_\mathfrak{e}$ and $\gamma_\mathfrak{c}$ defined in such a way that*

$$\rho(\mathfrak{e}\pm x) = \gamma_\mathfrak{e}^{3/2} x^{1/2}/\pi + o(x^{1/2}) \quad \textit{and} \quad \rho(\mathfrak{c}+x) = \sqrt{3}\gamma_\mathfrak{c}^{4/3}|x|^{1/3}/2\pi + o(|x|^{1/3})\,.$$

Then, for any fixed $k \in \mathbf{N}$, the universal correlation functions are given by

$$\frac{1}{\rho(\mathfrak{b})^k} p_k^{(N)}\left(\mathfrak{b} + \frac{\boldsymbol{x}}{\rho(\mathfrak{b})N}\right) \approx \det\left(\frac{\sin\pi(x_i-x_j)}{\pi(x_i-x_j)}\right)_{i,j\in[k]} \tag{Bulk}$$

$$\frac{N^{k/3}}{\gamma_\mathfrak{e}^k} p_k^{(N)}\left(\mathfrak{e} + \frac{\boldsymbol{x}}{\gamma_\mathfrak{e} N^{2/3}}\right) \approx \det\left(K_{\mathrm{Airy}}(x_i,x_j)\right)_{i,j\in[k]} \tag{Edge}$$

$$\frac{N^{k/4}}{\gamma_\mathfrak{c}^k} p_k^{(N)}\left(\mathfrak{c} + \frac{\boldsymbol{x}}{\gamma_\mathfrak{c} N^{3/4}}\right) \approx \det\left(K_{\mathrm{Pearcey}}(x_i,x_j)\right)_{i,j\in[k]}\,. \tag{Cusp}$$

The approximation is understood up to an error of size $N^{-c(k)}$ when integrated against a smooth test function.

In the bulk and edge case, the universal kernels were computed from the GUE ensemble and are called the *sine kernel* [447] and *Airy kernel* [264], respectively. For the cusp, the *Pearcey kernel* was computed from a GUE matrix with non-zero diagonal expectation $\mathrm{diag}(1,\dots,1,-1,\dots,-1)$ via saddle point analysis of a contour integral obtained via the *Harish-Chandra-Itzykson-Zuber integral* over the unitary group [121]. There is an analogous WDM conjecture for the real symmetry class with the bulk and edge kernels being explicitly known in this case as well. Due to the lack of the *Harish-Chandra-Itzykson-Zuber integral*, however, it is not even known, whether the universal cusp statistics is *determinantal*.

The WDM conjecture remained an open problem for several decades. Using moment computations, similarly to Wigner's original approach, Soshnikov in 1999 [531] showed validity of the WDM conjecture at the edge of Wigner matrices. Only about 15 years ago, it was resolved in the bulk of Wigner matrices in a series of papers by Erdős-Schlein-Yau (and collaborators) [246, 247, 251, 252] and work by Tao-Vu [551]. Subsequently, the WDM conjecture in the bulk and at the edge was proven for more general random matrix ensembles [406, 409, 370, 15, 17, 243, 23], including correlated random matrices [243, 23]. At the cusp, it has been shown only for *Wigner-type matrices* in [244, 155]. For general correlated random matrices, a proof of the WDM conjecture at the cusp was achieved only very recently in [239], which is included as Chapter 6 of this thesis.

The most powerful technique for proving spectral universality is the *three-step-strategy* invented by Erdős-Schlein-Yau (see [248] for a pedagogical introduction):

(i) *Eigenvalue rigidity* as a consequence of the local law (2).

[3] We will call $\mathfrak{b}$ a *bulk point*, if $\rho(\mathfrak{b}) \geq \delta > 0$.

(ii) Addition of a small Gaussian (GOE/GUE) component via Green function comparison (GFT).

(iii) Proof of universality for matrices with a small Gaussian component.

In the second step, one evolves the random matrix H by the Ornstein-Uhlenbeck (OU) flow

$$\mathrm{d}H_t = -\frac{1}{2}\big(H_t - A\big)\mathrm{d}t + \Sigma^{1/2}[\mathrm{d}B_t]\,, \qquad H_0 = H\,, \tag{4}$$

where B_t is a standard real symmetric/complex Hermitian Brownian motion. Moreover, $A = \mathbf{E}\,H$ and the non-negative operator Σ is given by

$$\Sigma[R] := \mathbf{E}\left[(H - \mathbf{E}[H])\,\mathrm{tr}\left(R(H - \mathbf{E}[H])\right)\right].$$

The key feature of the flow (4) is that both the expectation and the covariance operator (3) associated with H are preserved in time. As a consequence, the scDOS remains constant in t, i.e. $\rho_t \equiv \rho$. Moreover, analyzing the (time derivative of the) resolvent $G_t := (H_t - z)^{-1}$ for $|\mathrm{Im}\,z| \ll \eta_{\mathfrak{f}}(\mathrm{Re}\,z)$, one finds that the local eigenvalue statistics are effectively unchanged for running times of the OU flow (4) satisfying

$$t \ll \begin{cases} N^{-1/2} & \text{bulk} \\ N^{-1/6} & \text{edge} \\ N^{-1/4} & \text{cusp}\,. \end{cases} \tag{5}$$

Note that the OU flow (4) has added a Gaussian component of order $\sqrt{t}$, such that we have

$$H_t \overset{\mathrm{d}}{=} \widehat{H}_t + \sqrt{ct}U \tag{6}$$

for some matrix $\widehat{H}_t$, independent of the GOE/GUE matrix U, and a constant $c \sim 1$. By the second step, one hence just has to prove universality for matrices having a Gaussian component of size $\sqrt{t}$.

In the third step, we consider a terminal time $T = N^{\varepsilon}\eta_{\mathfrak{f}}$ for some small $\varepsilon > 0$, and the matrix flow

$$\mathrm{d}\widetilde{H}_t = \frac{\mathrm{d}\widetilde{B}_t}{\sqrt{N}}\,, \qquad \widetilde{H}_0 = \widehat{H}_T \tag{7}$$

where $\widetilde{B}_t$ is an independent standard matrix valued Brownian motion and $\widehat{H}$ is from (6). The solution to (7) is given by $\widetilde{H}_t = \widetilde{H}_0 + \sqrt{t}\widetilde{U}$, with $\widetilde{U}$ being an independent GOE/GUE matrix. In this way, we have $\widetilde{H}_{cT} \overset{\mathrm{d}}{=} H_T$ with c from (6), i.e. the two flows (4) and (7) have a joint target, and thus universality for H_0 can be inferred from universality for $\widetilde{H}_{cT} \overset{\mathrm{d}}{=} H_T$ if T can be chosen sufficiently small.

Indeed, in order to conclude the argument, it remains to check that the eigenvalue statistics along the flow (7) converges to the universal one sufficiently fast. One can easily check that the flow (7) induces the so-called *Dyson Brownian motion* (DBM) for the eigenvalues, the following system of coupled stochastic differential equations (SDEs):

$$\mathrm{d}\lambda_i(t) = \sqrt{\frac{2}{\beta N}}\mathrm{d}b_i(t) + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i(t) - \lambda_j(t)}\mathrm{d}t\,, \qquad \lambda_i(0) = \lambda_i(\widetilde{H}_0)\,, \quad i \in [N]\,, \tag{8}$$

where $(b_i(t))_{i\in[N]}$ is a standard vector valued Brownian motion, and $\beta = 1$ for the real symmetric case, and $\beta = 2$ for the complex Hermitian case.

The most convenient way to study (8) is via the *coupling method*. That is, we consider a comparison process for (8), driven by the *same* Brownian motions, but with initial conditions given by the eigenvalues of an independent GOE/GUE matrix. As a consequence, the difference variables

$(\delta_i(t))_{i\in[N]}$ satisfy a system of *deterministic* parabolic differential equations. Finally, using (i) heat kernel decay estimates for the resulting equations and (ii) the fact that the initial condition is very close to equilibrium (due to *eigenvalue rigidity* from the first step), one finds that $|\delta_i(t)| \ll \eta_{\mathfrak{f}}(\gamma_i)$ for sufficiently long time. This *fast relaxation to equilibrium of the DBM* is achieved if

$$t \gg \begin{cases} N^{-1} & \text{bulk} \\ N^{-1/3} & \text{edge} \\ N^{-1/2} & \text{cusp}\,, \end{cases} \tag{9}$$

leaving quite some room to choose T in (7), such that both (5) and (9) are satisfied. This concludes the three-step-strategy

Multi-resolvent local laws and applications

As discussed above, the local law in (2) has profound applications; most prominently local eigenvalue universality via the three-step-strategy. Due to this huge impact of the *single* resolvent local law, it is natural to study their analog for multiple resolvents, the *multi-resolvent local laws*. The central object of interest are strictly alternating[4] chains of resolvents $G_i = G(z_i) := (H - z_i)^{-1}$ with $z_i \in \mathbf{C} \setminus \mathbf{R}$ and deterministic matrices $B_i \in \mathbf{C}^{N\times N}$, i.e.

$$G_1 B_1 G_2 B_2 ... G_k B_k\,. \tag{10}$$

We now expand the averaged trace of (10) in spectral decomposition

$$\frac{1}{N}\sum_{j_1,...,j_k=1}^{N} \frac{\langle \boldsymbol{u}_{j_1}, B_1 \boldsymbol{u}_{j_2}\rangle\langle \boldsymbol{u}_{j_2}, B_2 \boldsymbol{u}_{j_3}\rangle ... \langle \boldsymbol{u}_{j_k}, B_k \boldsymbol{u}_{j_1}\rangle}{(\lambda_{j_1} - z_1)\,(\lambda_{j_2} - z_2)\,...\,(\lambda_{j_k} - z_k)} \tag{11}$$

where $(\boldsymbol{u}_j)_{j\in[N]}$ are the ℓ^2-normalized eigenvectors with associated eigenvalues $(\lambda_j)_{j\in[N]}$ of H. In this way, one easily sees that resolvent chains like (10) are *multi-point functions* and contain *much more* information than their single resolvent (*one-point function*) analog, since now eigenvector overlaps $\langle \boldsymbol{u}, B\boldsymbol{u}\rangle$ and multi-point eigenvalue correlations are implicitly included.

Now, a multi-resolvent local law is meant to be a concentration estimate for (10), similarly to (2). We emphasize that a multi-resolvent local law is *not* a consequence of a single resolvent law, meaning that, say,

$$G \approx M \implies G^2 \approx M^2$$

as (2) is only true in the above *weak* sense. Instead, the correct heuristic derivation of the deterministic approximation of G^2 is obtained as $G^2 = \partial_z G \approx \partial_z M$. However, this derivative trick breaks down when considering alternating chains of the form (10) involving deterministic B_i's. Hence, establishing a multi-resolvent local law for (10) consists of two tasks: First, finding the right deterministic approximation, second proving concentration around this matrix.

The systematic study of multi-resolvent local laws was initiated by Cipolloni-Erdős-Schröder in [165, 172, 170] for Wigner matrices, on which we focus from now on, for concreteness. The deterministic approximation $M_{[k]}$ for a general resolvent chain as in (10) can be obtained inductively in the length k of the chain (via *recursive Dyson equations*, see, e.g., Definition 1.4.1 and Lemma 1.D.1 below), and (in case of Wigner matrices) expressed via *free probability theory*. The general form of a multi-resolvent local law (in the interesting regime, where $\eta := \min_i |\mathrm{Im}\, z_i| \ll 1$) is then[5]

$$\left|\langle G_1 B_1 G_2 B_2 ... G_k B_k\rangle - \langle M_{[k]}(z_1, ..., z_k; B_1, ..., B_k)\rangle\right| \prec \frac{\prod_{i=1}^k \|B_i\|}{N\eta^k}\,, \tag{12}$$

[4] We can make this assumption w.l.o.g., since any chain containing, say, a G^2, can be reduced to a (linear combination of a) single G by Cauchy's integral formula or a resolvent identity.

[5] The corresponding isotropic multi-resolvent law takes a similar form. From now on, however, we focus on the averaged case for clarity of the presentation.

where the deterministic approximation $M_{[k]} = M_{[k]}(z_1, ..., z_k; B_1, ..., B_k)$ satisfies

$$\left\| M_{[k]} \right\| \lesssim \frac{\prod_{i=1}^{k} \|B_i\|}{\eta^{k-1}} \,. \tag{13}$$

We point out a natural power counting: Every additional resolvent raises the power of η in the denominators in (12) and (13) (compared to (2)), and the fluctuation size on the rhs. of (12) is suppressed by $(N\eta)^{-1} \ll 1$ compared to the size of the deterministic approximation.

Early on [168], Cipolloni-Erdős-Schröder realized a strong improvement in the estimates (12)–(13) stemming from *traceless observables* with $\langle B \rangle = 0$. This stems from the fact, that the inverse of the *two-body stability operator*

$$\mathcal{B}_{ij}[\cdot] := \mathbf{1} - m_{\mathrm{sc}}(z_i)\langle \cdot \rangle m_{\mathrm{sc}}(z_j) \,,$$

which governs the size of the estimates (12)–(13), is *bounded* on traceless B's. The improvement can roughly be captured via the $\sqrt{\eta}$*-rule*, stating that each traceless B reduces both the size of $M_{[k]}$ and the fluctuation bound in (12) by a factor of $\sqrt{\eta} \ll 1$. As an example, it holds that

$$\begin{aligned} &\left| \langle G_1 B_1 G_2 B_2 \rangle - m_1 m_2 \langle \mathring{B}_1 \mathring{B}_2 \rangle - \langle B_1 \rangle \langle B_2 \rangle \frac{m_1 - m_2}{z_1 - z_2} \right| \\ &\qquad < \frac{|\langle B_1 \rangle| \, |\langle B_2 \rangle|}{N\eta^2} + \frac{|\langle B_1 \rangle| \, \|\mathring{B}_2\| + \|\mathring{B}_1\| \, |\langle B_2 \rangle|}{N\eta^{3/2}} + \frac{\|\mathring{B}_1\| \, \|\mathring{B}_2\|}{N\eta} \end{aligned}$$

where we denoted $m_i = m_{\mathrm{sc}}(z_i)$ and $\mathring{B}_i := B_i - \langle B_i \rangle$. Note that, by decomposing the general observables B_i in (12) into their traceless part $\mathring{B}_i$ and tracial part $\langle B_i \rangle$, it is sufficient to study strictly alternating chains of resolvents and traceless matrices (cf. Footnote 4).

The theory of multi-resolvent local laws has rich applications. In this thesis, we focus on two types of physical problems. First, multi-resolvent laws allow to study *eigenvector overlaps* $\langle \boldsymbol{u}_i, B\boldsymbol{u}_j \rangle$ as indicated by (11). More precisely, we establish validity of the *Eigenstate Thermalization Hypothesis* (ETH) introduced by Deutsch [221] and Srednicki [535] in various settings where the quantum Hamiltonian is modeled by a random matrix. The purpose of the ETH is to provide an explanation for thermalization in closed quantum mechanical systems. Roughly speaking, the ETH asserts that the eigenvectors $\boldsymbol{u}_i$ of a chaotic quantum system are uniformly distributed in phase space. In physics terms, this means that the matrix elements of an observable A in the eigenbasis of the Hamiltonian effectively agree with the microcanonical average, and the fluctuations are entropically suppressed, i.e.

$$\langle \boldsymbol{u}_i, A\boldsymbol{u}_j \rangle \sim \delta_{ij} \langle\!\langle A \rangle\!\rangle + \sqrt{\frac{\langle\!\langle A^2 \rangle\!\rangle}{N}} \xi_{i,j} \,.$$

Here, $\langle\!\langle \cdot \rangle\!\rangle$ denotes the microcanonical average and $\xi_{i,j}$ is an order one (pseudo-)random variable. In the following, we will often say that we "showed ETH", if the second term above is established as the correct order in N, and say that we "established the fluctuation around ETH" if $\xi_{i,j}$ is identified as well.

As the second problem, we study *thermalization* questions and other *approach-to-equilibrium phenomena*, where again the random matrix H models a quantum Hamiltonian. In this case, the time evolution generated by H according to the Schrödinger equation, can be related to resolvents by simple contour integration

$$\mathrm{e}^{\mathrm{i}tH} = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma} \mathrm{e}^{\mathrm{i}tz} G(z) \mathrm{d}z$$

for an appropriately chosen $\Gamma \subset \mathbf{C}$ encircling the spectrum of H. The approach to equilibrium can then be determined by analyzing contour integrals of $M_{[k]}$.

Moreover, apart from the physical problems discussed above, we mention that controlling a special two-point function (in a two-resolvent local law), allows to prove CLTs for the linear eigenvalue statistics at macroscopic and mesoscopic scales [329, 393, 157, 156, 500, 500, 416].

Summary of Results

We now briefly summarize our results on random matrices contained in this thesis; afterwards we provide a concise summary of the main ideas for (some of) their proofs.

Chapter 1: Optimal lower bound on eigenvector overlaps for non-Hermitian matrices [154] We consider large deformed i.i.d. matrices $\Lambda + X \in \mathbf{C}^{N\times N}$ and show the following results:

(i) The ℓ^2-normalized bulk right and left singular vectors $(\boldsymbol{u}_i)_{i\in[N]} \subset \mathbf{C}^N$ and $(\boldsymbol{v}_i)_{i\in[N]} \subset \mathbf{C}^N$, respectively, are thermalized. That is, for any bounded deterministic matrix $B \in \mathbf{C}^{N\times N}$ it holds that

$$\max_{i,j\in\text{bulk}} |\langle \boldsymbol{u}_i, B\boldsymbol{u}_j\rangle - \delta_{ij}\langle V_i B\rangle| \prec \frac{1}{\sqrt{N}}$$

for some bounded deterministic matrix $V_i \in \mathbf{C}^N$. Analogous statements hold for the overlaps $\langle \boldsymbol{u}_i, B\boldsymbol{v}_j\rangle$ and $\langle \boldsymbol{v}_i, B\boldsymbol{v}_j\rangle$.

(ii) The diagonal eigenvector overlap $\mathcal{O}_{ii} := \|\boldsymbol{r}_i\|^2 \|\boldsymbol{l}_i\|^2$ is lower bounded as

$$\mathcal{O}_{ii} \succ N\,.$$

Here $(\boldsymbol{r}_i)_{i\in[N]}$ and $(\boldsymbol{l}_i)_{i\in[N]}$ are the bi-orthonormal right- and left-eigenvectors of $\Lambda + X$, respectively, normalized such that $\langle \bar{\boldsymbol{l}}_i, \boldsymbol{r}_j\rangle = \boldsymbol{l}_i^t \boldsymbol{r}_j = \delta_{ij}$

The underlying technical result is a two-resolvent local law for the Hermitization

$$H = \left(\begin{smallmatrix} 0 & \Lambda+X \\ (\Lambda+X)^* & 0 \end{smallmatrix}\right) \in \mathbf{C}^{2N\times 2N}$$

of $\Lambda + X$ with optimal improvements for *regular observables* $A = \mathring{A}^{z_i,z_j}$, generalizing the concept of traceless matrices for Wigner matrices. In fact, the regular observables form an *energy dependent* codimension one subspace, on which the inverse of the stability operator $\mathcal{B}_{ij}[\cdot] := \mathbf{1} - M(z_i)\mathcal{S}[\cdot]M(z_j)$, with $\mathcal{S}$ being the self-energy operator from (3), is bounded.

Chapter 2: Gaussian fluctuations in the equipartition principle for Wigner matrices [152] Let W_1 and W_2 be two independent $N \times N$ Wigner matrices of the same symmetry class and consider the Hamiltonian $H := W_1 + W_2$. As shown in [45], the energy of an eigenstate $\boldsymbol{u}$ of H is equally distributed among the constituents W_1 and W_2. This means that, if $H\boldsymbol{u} = \lambda\boldsymbol{u}$ for a bulk eigenvalue $\lambda \in \mathbf{R}$, it holds that

$$\left|\langle \boldsymbol{u}, W_1\boldsymbol{u}\rangle - \frac{\lambda}{2}\right| \prec \frac{1}{\sqrt{N}}$$

and the same for W_2 instead of W_1. We show Gaussian fluctuations around this *equipartition principle*, i.e.

$$\sqrt{\beta N}\left[\langle \boldsymbol{u}, W_1\boldsymbol{u}\rangle - \frac{\lambda}{2}\right] \Longrightarrow \mathcal{N}(0,1)$$

in the sense of moments. Here, $\beta = 1$ if W_1, W_2 are real symmetric Wigner matrices, and $\beta = 2$ if they are complex Hermitian, and $\mathcal{N}(0,1)$ denotes a real standard Gaussian random variable.

As our main ingredient, we prove ETH and Gaussian fluctuations around ETH for general quadratic forms of the bulk eigenvectors of Wigner matrices with an arbitrary deformation, which are deduced with the aid of appropriate multi-resolvent laws.

Chapter 3: Eigenstate thermalization at the edge for Wigner matrices [150] Let W be a Wigner matrix with normalized eigenvectors $(\boldsymbol{u}_i)_{i\in[N]}$. Then we have validity of the optimal[6] eigenstate thermalization hypothesis (ETH) uniformly in the spectrum

$$\max_{i,j\in[N]} |\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\langle A\rangle| \prec \frac{\langle|\mathring{A}|^2\rangle^{1/2}}{\sqrt{N}} \quad \text{where} \quad \mathring{A} := A - \langle A\rangle\,.$$

Previous results were restricted either to the bulk of the spectrum [169] or to special observables [60]. In fact, the optimal ETH requires to simultaneously and optimally track the Hilbert-Schmidt norm of observables and dependencies on the local density of states in the edge regime. That is, we show

$$\langle \operatorname{Im} G(z_1)\mathring{A}\operatorname{Im} G(z_2)\mathring{A}^*\rangle \prec \rho_{\mathrm{sc}}(z_1)\rho_{\mathrm{sc}}(z_2)\langle|\mathring{A}|^2\rangle\,,$$

noticing the change G to $\operatorname{Im} G$ (compared to the chain in (10) and (10)) and the resulting improvement of having ρ's (which are small at the edge) in the rhs.

Chapter 4: Out-of-time-ordered correlators for Wigner matrices [151] For general observables $A, B \in \mathbf{C}^{N\times N}$, we consider their *out-of-time-ordered correlator* (OTOC)

$$\mathcal{C}_{A,B}(t) := \frac{1}{2}\left\langle \left|\,[A(t), B]\,\right|^2\right\rangle \quad \text{with} \quad A(t) = \mathrm{e}^{-\mathrm{i}tW} A \mathrm{e}^{\mathrm{i}tW}$$

in a mean-field chaotic quantum system described by a Wigner random matrix W. We rigorously identify three time regimes separated by the physically relevant scrambling and relaxation times. The main feature of our analysis is that we express the error terms in the optimal Schatten (tracial) norms of the observables, allowing us to track the exact dependence of the errors on their rank.

This is achieved by establishing multi-resolvent local laws with Schatten norm error terms. For example, for $A \in \mathbf{C}^{N\times N}$ traceless, spectral parameter $z = E + \mathrm{i}\eta$ with $\eta \gg N^{-1}$ in the bulk and $G = G(z) = (W - z)^{-1}$ it holds that (here, $m = m_{\mathrm{sc}}(z)$)

$$\left|\langle GAGA\rangle - m^2\langle A^2\rangle\right| \prec \begin{cases} \dfrac{\|A\|^2}{N\eta} & \text{from [168, Theorem 2.5];} \\ \dfrac{\langle|A|^2\rangle}{\sqrt{N\eta}} & \text{from [169, Theorem 2.2];} \\ \dfrac{\langle|A|^2\rangle}{N\eta} + \dfrac{\langle|A|^4\rangle^{1/2}}{N\sqrt{\eta}} & \text{from Theorem 4.3.3.} \end{cases}$$

The last bound is optimal for observables of any rank and interpolates between the former two bounds, which are optimal only for full rank and bounded rank observables, respectively.

Chapter 5: Eigenvector decorrelation for random matrices [153] Let W be a Wigner matrix and consider two self-adjoint deterministic deformations $D_1, D_2 \in \mathbf{C}^{N\times N}$. Define $H_\ell := W + D_\ell$, $\ell = 1, 2$ and denote their respective eigenvectors by $(\boldsymbol{u}_i^{(\ell)})_{i=1}^N$ which correspond to the eigenvalues $(\lambda_i^{(\ell)})_{i=1}^N$. For eigenvalues in the bulk of the spectra, we prove the *optimal decorrelation estimate*

$$\left|\left\langle \boldsymbol{u}_i^{(1)}, \boldsymbol{u}_j^{(2)}\right\rangle\right|^2 \prec \frac{1}{N}\cdot\frac{1}{\langle(D_1 - D_2)^2\rangle + \left|\lambda_i^{(1)} - \lambda_j^{(2)}\right|^2 + \mathrm{LT}}$$

of the eigenvector overlaps, where the so-called *linear term* LT is (the absolute value of) a specific linear combination of $D_1 - D_2$ and $\lambda_i^{(1)} - \lambda_j^{(2)}$. Our estimate manifests the interplay of two decay effects in three different terms, which can make the eigenvectors $\boldsymbol{u}_i^{(1)}, \boldsymbol{u}_j^{(2)}$ almost orthogonal.

[6] The ETH below is optimal in terms of

Moreover, similarly to the ETH, we identify a one-codimensional subspace of observables A, dependent on λ's and D's, such that the overlap satisfies $\left|\langle \boldsymbol{u}_i^{(1)}, A\boldsymbol{u}_j^{(2)}\rangle\right|^2 \prec N^{-1}$.

On a technical level, our results build on optimal two-resolvent local laws for deformed Wigner matrices not sharing a common eigenbasis.

Chapter 6: Cusp universality for correlated random matrices [239] We consider large Hermitian random matrices H with a general slowly decaying correlation structure among its entries. We prove a local law for the resolvent of H throughout the entire spectrum, in particular including cusp singularities of the density of states. As a consequence, we obtain universality of the local eigenvalue statistics at such cusp points via the three-step-strategy.

Since the density of states typically exhibits only square root edge or cubic root cusp singularities, our result completes the proof of the Wigner-Dyson-Mehta universality conjecture in all spectral regimes for a very general class of random matrices. Previously only the bulk and the edge universality were established in this generality [23], while cusp universality was proven only for Wigner-type matrices with independent entries [244, 155].

Chapter 7: Prethermalization for deformed Wigner matrices [238] We consider a class of weakly perturbed Hamiltonians of the form $H_\lambda = H_0 + \lambda W$, with W being a Wigner matrix, and prove that they exhibit *prethermalization*. That is, the time evolution generated by H_λ relaxes to its ultimate thermal state via an intermediate prethermal state with a lifetime of order λ^{-2}, in agreement with *Fermi's golden rule*.

Additionally, we obtain a general relaxation formula, which expresses the perturbed dynamics via the unperturbed dynamics and the ultimate thermal state $\widetilde{P}_\lambda$. More precisely, under quite general conditions, for an observable $A \in \mathbf{C}^{N\times N}$ and initial state $0 \le P \le \mathbf{1}$, it holds that

$$\operatorname{tr}\left(AP_\lambda(t)\right) \approx \operatorname{tr}\left(A\widetilde{P}_\lambda\right) + \mathrm{e}^{-2\alpha\lambda^2 t}\left[\operatorname{tr}\left(AP_0(t)\right) - \operatorname{tr}\left(A\widetilde{P}_\lambda\right)\right],$$

where we denoted the Heisenberg time evolution of P by $P_\lambda(t) := \mathrm{e}^{-\mathrm{i}tH_\lambda} P \mathrm{e}^{\mathrm{i}tH_\lambda}$.

Chapter 8: Loschmidt echo for deformed Wigner matrices [237] For two Hamiltonians that are close to each other, $H_1 \approx H_2$, we analyze the time-decay of the corresponding *Loschmidt echo*

$$\mathfrak{M}(t) := \left|\left\langle \psi_0, \mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1}\psi_0\right\rangle\right|^2$$

for an energetically localized initial state $\psi_0 \in \mathbf{C}^N$. For our model Hamiltonians we consider deformed Wigner matrices that do not share a common eigenbasis and show that

$$\mathbf{M}(t) \approx \begin{cases} 1-\gamma t^2 & \text{for} \quad t \ll 1 \\ \mathrm{e}^{-\Gamma t} & \text{for} \quad 1 \ll t \lesssim \left(\mathbf{E}\langle (H_1-H_2)^2\rangle\right)^{-1}, \end{cases}$$

where both *decay parameters* satisfy $\gamma \sim \Gamma \sim \mathbf{E}\langle (H_1-H_2)^2\rangle$.

Chapter 9: Eigenstate thermalization hypothesis for translation invariant spin systems [541] We prove ETH for local observables in a typical translation invariant system of quantum spins with mean field interaction. This mathematically verifies observations made in the physics literature [512], that ETH may hold for systems with additional translation symmetries for a naturally restricted class of observables.

Main Ideas of the Proofs

Traditionally, the proof of local laws consists of two largely separated parts. On the one hand, there is a deterministic stability analysis of the MDE, say, for illustrational purpose, $-m^{-1} = z + m$ in the case of Wigner matrices. On the other hand, there is a probabilistic error control when showing that $-G^{-1} \approx z + G$. Combining deterministic stability and probabilistic error control, we find that $G \approx m$, as formally stated in (2).

Both proof parts are, however, highly model dependent and thus considerable effort is needed in establishing a local law. In contrast to that, steps (ii) and (iii) of the three-step-strategy have already reached the status of a *black box* and can be performed essentially model independent. On a technical level, the main goal of this thesis is to elevate the proof of single- and multi-resolvent local laws to this status. In order to do so, we developed two methods.

The Ψ-method In order to prove the results in Chapters 1–2, we developed the Ψ-method, originally invented in [168]. The principal idea is to consider

$$\Psi_k := N\eta^{k/2} \left| \langle G_1 A_1 ... G_k A_k \rangle - \langle M_{[k]} \rangle \right| , \qquad k \in \mathbf{N} , \tag{14}$$

for *regular matrices* $A_i = \mathring{A}_i$ with bounded norm $\|A_i\| \lesssim 1$. In accordance with the $\sqrt{\eta}$-rule, the normalization factor in (14) is chosen in such a way, that the optimal multi-resolvent local law is equivalent to $\Psi_k \prec 1$. Then, by a *minimalistic cumulant expansion* and *recursive moment estimates* we prove that, if $\Psi_k \prec \psi_k$ for some deterministic control quantities $\psi_k \geq 1$, the improved estimates

$$\Psi_k \prec 1 + \sum_{j=1}^{k-1} \psi_j + \sum_{j=1}^{k-1} \frac{\psi_j \psi_{k-j}}{N\eta} + \frac{\sqrt{\psi_{2k}}}{\sqrt{N\eta}} , \qquad k \in \mathbf{N} \tag{15}$$

hold as well. These estimates are called *master inequalities* and are an infinite non-linear coupled hierarchical system of multi-resolvent local laws and their fluctuations. Apart from the target bound of order one on the rhs. of (15), it contains the subsequent *sub-homogeneous*, *non-linear*, and *doubling* term(s).

The way to close the infinite hierarchy (15) of master inequalities a priori induced by the *doubling term*, is the use of *reduction inequalities* taking the form

$$\Psi_{2k} \prec (N\eta)^2 + \psi_k^2 , \qquad k \in \mathbf{N} . \tag{16}$$

Formally, plugging (16) into (15), and using that the sub-homogeneous and non-linear terms in (15) follow a strict *triangular structure* (i.e. involve ψ's with strictly lower indices on the rhs.), we find that $\Psi_k \prec \sqrt{N\eta}$ for all $k \in \mathbf{N}$ as an intermediate bound. Feeding this information back into (15), and using again the triangular structure, we arrive at the target $\Psi_k \prec 1$.

Compared to traditional local law proofs, the Ψ-method avoids tedious cumulant expansions and is thus capable of dealing with general multi-resolvent chains. The key difficulty, however, in the application of the Ψ-method to the problems in Chapters 1–2 is to track the energy dependent regularizations throughout the expansions.

The Zigzag strategy As the main method for proving local laws, in this thesis we develop and apply the Zigzag strategy in Chapters 3–6. As illustrated in these later parts of the thesis, the Zigzag strategy is a powerful, versatile, and fine-tunable tool, which efficiently handles instabilities, such as non-regular observables or singularities in the spectrum (edges and cusps).

We now briefly explain the strategy in the setting of the standard local semicircle law for a Wigner matrix W, consisting of three steps:

(i) **Global law:** For $\eta \sim 1$, prove that $|\langle G - m \rangle| < 1/N$. Since in this regime, the rough bound $\|G\| \lesssim 1$ is affordable, the global law can be obtained by simple independent methods.

(ii) **Characteristic flow (*Zig* step):** Run an Ornstein-Uhlenbeck flow

$$\mathrm{d}W_t = -\frac{W_t}{2}\mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad W_0 = W$$

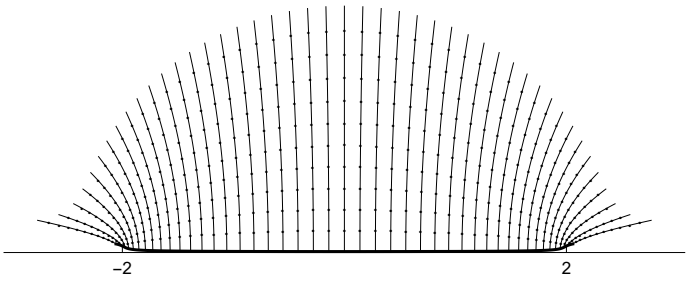
and *simultaneously* evolve the spectral parameter $z = E + \mathrm{i}\eta \in \mathbf{C} \setminus \mathbf{R}$ by

$$\mathrm{d}z_t = -m(z_t)\mathrm{d}t - \frac{z_t}{2}\mathrm{d}t\,. \tag{17}$$

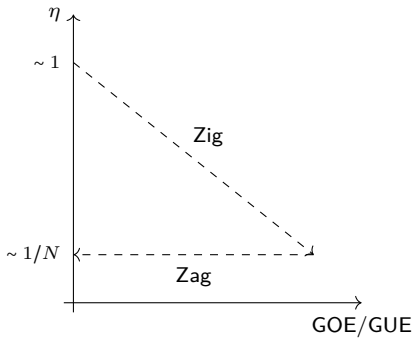
The key effect of running these two flows in tandem is a great cancellation in the differential $\mathrm{d}\langle G_t(z_t)\rangle$ as shown below.

(iii) **Green function comparison (*Zag* step):** The outcome of the second step is the local semicircle law for small η, but with a large Gaussian component. This component is removed by a *self-consistent* GFT argument (via Gronwall inequalities).

We point out that, previously, the characteristic flow idea only appeared in the single-resolvent setting [6, 7, 14, 105, 111, 353], however (mostly) without combining it with a GFT. One main achievement of this thesis is to consolidate the multi-resolvent setting as well. We remark that, in this case, the immediate outcome of the second step are *time-dependent* master inequalities, similarly to (15). The hierarchy can be closed analogously to the static Ψ-method.



(a) Characteristic semicircular flow.



(b) Schematic picture of Zigzag.

The schematic of the Zig and Zag steps is illustrated in Figure (b); often, they are repeated multiple times, reducing η gradually. In Figure (a), we depict trajectories of the deterministic flow (17). As mentioned above, the key effect of combining the OU flow with (17) is a great cancellation in the differential $\mathrm{d}\langle G_t(z_t)\rangle$. To illustrate this, we consider $\mathrm{d}\langle G(z_t)\rangle$, $\mathrm{d}\langle G_t(z)\rangle$, and $\mathrm{d}\langle G_t(z_t)\rangle$, making only the spectral parameter, only the randomness, or both z_t and W_t time dependent, respectively. In fact, by Itô calculus we find that

$$\begin{aligned}
\mathrm{d}\langle G(z_t)\rangle &= (\partial_t z_t)\langle G(z_t)^2\rangle \mathrm{d}t \\
\mathrm{d}\langle G_t(z)\rangle &= \left[\frac{\langle G_t(z)\rangle}{2} + \left(\langle G_t(z)\rangle + \frac{z}{2}\right)\langle G_t(z)^2\rangle\right]\mathrm{d}t + \ldots \mathrm{d}B \\
\mathrm{d}\langle G_t(z_t)\rangle &= \left[\frac{\langle G_t(z_t)\rangle}{2} + \left(\langle G_t(z_t)\rangle + \frac{z_t}{2} + \partial_t z_t\right)\langle G_t(z)^2\rangle\right]\mathrm{d}t + \ldots \mathrm{d}B
\end{aligned}$$

Hence, only for $\mathrm{d}\langle G_t(z_t)\rangle$, the prefactor multiplying the critical $\langle G^2\rangle$ term is small due to the choice of characteristics (17) and the fact that $\langle G\rangle \approx m$. Finally, we point out that, in the above expressions, the *martingale term* $\ldots \mathrm{d}B$ can be handled via the Burkholder-Davis-Gundy (BDG) inequality.

Conclusion

To summarize, the main contributions of Part I of this thesis are as follows:

(1) We generalize the concept of tracelessness for Wigner matrices to *regular observables* and provide a systematic decomposition of general observables, thereby developing the static Ψ-*method* (see Chapters 1–2).

(2) We establish the dynamical *Zigzag strategy* as a powerful method for proving single- and multi-resolvent local laws. Concretely, we prove fine estimates for Wigner matrices (Chapters 3–4), decorrelation estimates for different spectral families of deformed Wigner matrices (Chapter 5), and a cusp local law for correlated matrices (Chapter 6).

(3) We apply the theory of multi-resolvent laws to physical problems, such as prethermalization (Chapter 7), Loschmidt echo (Chapter 8), or ETH in translation invariant systems (Chapter 9).

Part II: BCS Theory

In the second part of this thesis, we study (translation invariant) Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [48]. We consider a system of fermionic particles in $\mathbf{R}^d$, $d = 1, 2, 3$, at temperature $T \geq 0$, chemical potential $\mu > 0$, and interacting via a two-body (effective) interaction denoted by V. The central mathematical object to study is the *BCS free energy functional*

$$\mathcal{F}_{\mathrm{BCS}}(\Gamma) := \int_{\mathbf{R}^d} (p^2 - \mu)\hat{\gamma}(p)\,\mathrm{d}p - TS(\Gamma) + \int_{\mathbf{R}^d} V(x)|\alpha(x)|^2\,\mathrm{d}x \tag{18}$$

on *BCS states* $\Gamma \equiv (\gamma, \alpha)$. These are conveniently represented as 2×2 matrices in momentum space,

$$\hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}}(p) & 1 - \hat{\gamma}(-p) \end{pmatrix} \quad \text{with the constraint} \quad 0 \leq \hat{\Gamma}(p) \leq \mathbf{1}_{\mathbf{C}^2}\,, \tag{19}$$

which encodes the fermionic nature of the particles. The *entropy* term S in (18) is given by

$$S(\Gamma) := -\int_{\mathbf{R}^d} \mathrm{tr}_{\mathbf{C}^2}\left[\hat{\Gamma}(p)\log\hat{\Gamma}(p)\right]\mathrm{d}p\,. \tag{20}$$

The BCS functional (18) can be heuristically derived from a many-body Hamiltonian by making the following simplifying assumptions (see [309, Appendix A] and Section 14.A for details): One restricts the grand canonical pressure functional to *quasi-free states* ρ, drops *direct and exchange terms*, restricts to $SU(2)$-invariant states (ignoring spin), and assumes translation invariance.

The only term kept in the restriction to quasi-free states, is the *pairing term*, amounting to $\int V|\alpha|^2$ in (18). The functions $\hat{\gamma}(p) \sim \langle a_p^\dagger a_p\rangle_\rho$ and $\hat{\alpha}(p) \sim \langle a_p a_{-p}\rangle_\rho$ constituting the BCS state (19), can be interpreted as the *electron density* and *Cooper pair density*, respectively. Here, $a_p^\dagger$ and a_p denote the creation and annihilation operator of an electron with momentum p, respectively.

The Cooper pairs constitute of two electrons with opposite momenta, that are bound together by the effective interaction V. Physically, this arises from the attractive interactions of negatively charged electrons and positively charged ions in the crystal lattice. By "integrating out" the lattice degrees of freedom (*phonons*) then yields an effective attractive interaction V between two electrons. A Cooper pair (formed by two fermions) is then of effective bosonic nature, and hence many Cooper pairs can occupy the same state (form a *condensate*). The condensate of Cooper pairs is then responsible for dispersionless charge transport, i.e. superconductivity.

Hence, the question of whether a material described by the functional (18) is superconducting, can mathematically be rephrased by asking, whether there exists a minimizer of (18) having a non-vanishing Cooper pair density, $\alpha \not\equiv 0$. The minimization problem associated with (18) is, however, highly non-linear. Nevertheless, at least heuristically, one can see a competition in (18) between the entropy and the pairing term: While by simple concavity properties of $x \mapsto x \log x$, the entropy term (20) favors a diagonal state Γ, i.e. with $\alpha \equiv 0$, the attractive interaction V (having at least some negative part) favors non-vanishing α.

The Euler-Lagrange equation associated with (18)–(19) is the celebrated *BCS gap equation*

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{K_T^\Delta(q)} \mathrm{d}q \tag{21}$$

satisfied by the *gap function* $\Delta := -2\widehat{V\alpha}$. In (21) we denoted

$$K_T^\Delta(p) := \frac{E_\Delta(p)}{\tanh\left(\frac{E_\Delta(p)}{2T}\right)} \quad \text{with} \quad E_\Delta(p) := \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}\,. \tag{22}$$

Hence, superconductivity is equivalent[7] to the existence of a non-zero solution to (21). Noting that (i) the gap equation is equivalent to $(K_T^\Delta + V)\alpha = 0$ and (ii) $[0,\infty) \ni \Delta \mapsto K_T^\Delta(p)$ is strictly monotone for every $p \in \mathbf{R}^d$, it turns out that the question of existence of a non-zero solution to the *non-linear* gap equation (21) can be characterized by a *linear* criterion. In fact, it holds that [309]

$$\exists \Delta \not\equiv 0 \quad \text{solving} \quad (21) \quad \Leftrightarrow \quad \inf \mathrm{spec}(K_T + V) < 0\,, \tag{23}$$

i.e. the pseudo-differential operator $K_T + V$ with $K_T(p) \equiv K_T^0(p) := \frac{p^2-\mu}{\tanh((p^2-\mu)/2T)}$ has a negative eigenvalue. The operator $K_T + V$ can be thought of as the Hessian of the functional (18) around the *normal state* $(\gamma_0, 0)$ with $\hat{\gamma}_0(p) = (1 + \mathrm{e}^{(p^2-\mu)/T})^{-1}$ and zero Cooper pair density. Hence existence of a negative eigenvalue of $K_T + V$ shows that one can locally perturb around the normal state by some $\alpha \not\equiv 0$ and reach a strictly lower BCS free energy.

Since $[0,\infty) \ni T \mapsto K_T(p)$ is strictly monotone for every $p \in \mathbf{R}^d$, the linear characterization (23) motivates the definition of the *critical temperature*

$$T_c := \inf\{T \geq 0 : K_T + V \geq 0\}$$

below which ($T < T_c$) superconductivity exists, and above which ($T \geq T_c$) it does not. This threshold is physically measurable and highly important in practical applications.

Another important measurable quantity of a superconductor, measuring stability of the superconducting state, is the (temperature dependent) *energy gap*

$$\Xi(T) := \inf_{p \in \mathbf{R}^d} E_\Delta(p) = \inf_{p \in \mathbf{R}^d} \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$$

associated to a non-zero solution Δ to (21) for temperatures $0 \leq T \leq T_c$. We call $\Xi \approx \Delta(\sqrt{\mu})$ an energy gap due to its interpretation as a spectral gap above the superconducting ground state of a quasi-particle Hamiltonian (see [309, Appendix A]), at least at zero temperature.

The above mathematical formulation of BCS theory has been put forward mostly by Hainzl and Seiringer and their collaborators [309, 316] starting from the late 2000s. Since then, many aspects of BCS theory have been put on rigorous grounds. In particular, for $d = 3$, asymptotic formulas for the critical temperature and energy gap at zero temperature have been established in limits where

[7] With a slight abuse of notation, we will use K_T^Δ to denote both, the pseudo-differential operator and its symbol (i.e. a function).

superconductivity is weak. More precisely, T_c have been studied in the weak coupling (i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$) limit [269, 312] and low-density (i.e. $\mu \to 0$) limit [311, 403]. These results show that, in particular, the *ratio* Ξ/T_c takes a *universal* value, *independent of the microscopic details* of the material, i.e. the interaction V. In Chapters 10–11, for $d = 3$, we obtain similar asymptotic formulas in the high density limit, $\mu \to \infty$; in Chapter 12 we study the weak coupling limit in spatial dimensions $d = 1, 2$, in both cases obtaining universality as a corollary.

BCS theory has also been studied for temperatures close to the critical one. In this regime of the superconducting phase transition, the microscopic BCS theory is well described by the phenomenological Ginzburg-Landau (GL) theory [282]. The fact that GL theory in fact arises from BCS theory as a limiting description for $T \uparrow T_c$ (see [294]), has recently been shown, even including external electromagnetic fields [270, 272, 271, 220, 219]. Finally, we point out that BCS theory has also been investigated in the non-translation-invariant setting [218, 310, 508, 509]. For more detailed introductions to BCS theory, we refer the reader to the Master Thesis of Lauritsen [402], and the PhD Theses of Bräunlich [119], Deuchert [217], Maier [428], and Roos [507].

Summary of Results

We now briefly summarize our results on BCS theory contained in this thesis; afterwards we provide a concise summary of the main ideas for (some of) their proofs.

Chapter 10: The BCS critical temperature at high density [333] In three spatial dimensions, we determine an asymptotic formula for the critical temperature T_c in the limit of high density, modeled by sending the chemical potential to infinity, $\mu \to \infty$, which roughly takes the form

$$T_c \approx \mu \, A \, \mathrm{e}^{B/(\sqrt{\mu} e_\mu)} \tag{24}$$

for some constants $A, B > 0$. Here, $\sqrt{\mu} e_\mu < 0$ (defined precisely in (30) below) measures the strength of the effective interactions for electrons on the Fermi surface $\{p^2 = \mu\}$, which behaves as $|\sqrt{\mu} e_\mu| \sim \mu^{-\delta}$ for some $\delta > 0$. Roughly said, the decay of $|\sqrt{\mu} e_\mu|$ can be obtained from the decay of the Fourier transform of the interaction, $\hat{V}$, in momentum space.

The behavior of T_c in (24) is valid for radially symmetric and (mostly) attractive potentials V, which are well behaved near the origin. In combination with a similar asymptotic formula valid for low densities [311], we thus rigorously prove the existence of *superconducting domes* (i.e. a non-monotonous behavior of T_c as a function of μ) in BCS theory. This has previously been predicted only in the physics literature [400].

Chapter 11: The BCS energy gap at high density [336] Similarly to Chapter 10, we study the BCS energy gap Ξ at zero temperature in the limit of high density, $\mu \to \infty$, and find that

$$\Xi(T = 0) \approx \mu \, C \, \mathrm{e}^{B/(\sqrt{\mu} e_\mu)} \tag{25}$$

for $B > 0$ as in (24) and $C = \pi \mathrm{e}^{-\gamma} A$, where A is from (24) and $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Hence we prove universality of the ratio $\Xi(T = 0)/T_c \approx \pi \mathrm{e}^{-\gamma}$ in the limit of large densities.

Chapter 12: Universality in low-dimensional BCS theory [338] For one and two spatial dimensions, we study the BCS critical temperature and energy gap at zero temperature. As a result, we obtain asymptotic formulas in the weak coupling limit, yielding the *same* universal ratio $\Xi(T = 0)/T_c \approx \pi \mathrm{e}^{-\gamma}$ as in three dimensions.

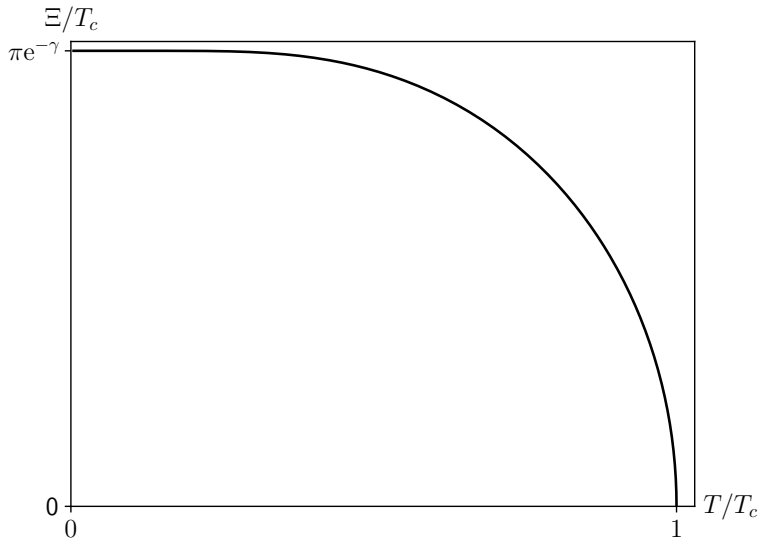
Chapter 13: Universal behavior of the BCS energy gap [337] In this chapter, we focus on the three dimensional setting and prove universality [412, 399] of the ratio $\Xi(T)/T_c$ in the weak coupling limit (i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$) for *all temperatures* $0 \leq T \leq T_c$. More precisely, as $\lambda \to 0$, we show that

$$\frac{\Xi(T)}{T_c} \approx \mathsf{f}_{\mathsf{BCS}}\left(\sqrt{1 - T/T_c}\right)$$

for some universal function $\mathsf{f}_{\mathsf{BCS}}$, taking only the relative temperature T/T_c as an argument. In particular, for temperatures close to the critical one, it holds that

$$\frac{\Xi(T)}{T_c} \approx \sqrt{\frac{8\pi^2}{7\zeta(3)}}\sqrt{1 - T/T_c},$$

where $\zeta(\cdot)$ denotes the Riemann ζ-function. The universal behavior of the ratio $\Xi(T)/T_c$ is illustrated in Figure (c) below.



(c) Universal behavior of the BCS energy gap.

Chapter 14: Multi-band superconductors have enhanced critical temperatures [335] We study an analog of the BCS functional (18) allowing for multiple interacting electronic bands, originally put forward by Suhl-Mathias-Walker [542] and Moskalenko [457] shortly after the original BCS model. As a result, we prove that the critical temperature of a multi-band superconductor can only be increased by inter-band coupling, irrespective of its attractive or repulsive nature and its strength. Moreover, for weak coupling and weaker inter-band coupling, we show the following: The dependence of the increase in critical temperature on the inter-band coupling is (1) linear, if there are two or more equally strongly superconducting bands, or (2) quadratic, if there is only one dominating band.

Main Ideas of the Proofs

The central ingredient for proving asymptotic formulas for T_c and Ξ is the spectral theoretic characterization of these quantities and their translation via the Birman-Schwinger principle.

For a large class of interaction potentials, the gap equation (21) is equivalent to the operator $K_T^{\Delta} + V$ having zero as its lowest eigenvalue with eigenvector α given as the minimizer of the BCS functional.

By the Birman-Schwinger principle, this is equivalent to

$$B_{T,\Delta} := \operatorname{sgn} V |V|^{1/2} \frac{1}{K_T^\Delta} |V|^{1/2} \tag{26}$$

having lowest eigenvalue -1 with eigenvector $\operatorname{sgn} V|V|^{1/2}\alpha$. This step greatly simplifies the analysis, since, in contrast to $K_T^\Delta + V$, we have that $B_{T,\Delta}$ is a *compact* operator.

In order to study the Birman-Schwinger operator, we introduce the integral

$$m(T,\Delta) := \frac{1}{|\mathbb{S}^{d-1}|} \int_{|p|\leq\sqrt{2\mu}} \frac{1}{K_T^\Delta(p)} \mathrm{d}p \tag{27}$$

and decompose $B_{T,\Delta}$ into a dominant first term, and an error term:

$$B_{T,\Delta} = m(T,\Delta) \operatorname{sgn} V|V|^{1/2} \mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2} + \operatorname{sgn} V|V|^{1/2} M_{T,\Delta} |V|^{1/2}, \tag{28}$$

where $\mathfrak{F}_\mu : L^1(\mathbf{R}^d) \to L^2(\mathbb{S}^{d-1})$ is a rescaled Fourier transform restricted to the unit sphere, and $M_{T,\Delta}$ is such that (28) holds. Note that the decomposition (28) effectively approximated $(K_T^\Delta)^{-1}$ in (26) by a multiple of $\mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu$, concentrating around the Fermi sphere $\{p^2 = \mu\}$, where $(K_T^\Delta)^{-1}$ is large (if T and Δ are small). Finally, by simple manipulations, using that $\operatorname{sgn} V|V|^{1/2} M_{T,\Delta}|V|^{1/2}$ is small and that $B_{T,\Delta}$ has lowest eigenvalue -1, we find

$$m(T,\Delta) = -\left[\inf \operatorname{spec}\left(\mathfrak{F}_\mu |V|^{1/2} \frac{1}{1 + \operatorname{sgn} V|V|^{1/2} M_{T,\Delta}|V|^{1/2}} \operatorname{sgn} V|V|^{1/2} \mathfrak{F}_\mu^\dagger\right)\right]^{-1}. \tag{29}$$

Observe that (29) is a highly implicit equation for T and Δ, as they (a) appear both on the lhs. and rhs. of (29), and (b) are coupled through the integral (27). In general, there are thus no explicit formulas available. If, however, superconductivity is weak, i.e. Δ and T_c and thus T are small, one can safely approximate $M_{T,\Delta} \approx M_{0,0}$, thereby solving problem (a).

To overcome problem (b) as well, we note that, in the "extreme cases" $(T = 0, \Delta)$ and $(T_c, \Delta = 0)$, the integral (27) can be (asymptotically) computed explicitly, yielding

$$\begin{aligned} m(T=0,\Delta) &\approx \mu^{d/2-1}\left(\log\frac{\mu}{\Xi(T=0)} + \log(2c_d)\right) \\ m(T_c,\Delta=0) &\approx \mu^{d/2-1}\left(\log\frac{\mu}{T_c} + \log\frac{\mathrm{e}^\gamma}{\pi} + \log(2c_d)\right) \end{aligned}$$

for some explicit dimension dependent constant $c_d > 0$. We thus have that

$$m(T=0,\Delta) \approx -\frac{1}{e_\mu} \quad \text{and} \quad m(T_c,\Delta=0) \approx -\frac{1}{e_\mu} \quad \text{with} \quad e_\mu := \inf \operatorname{spec}(\mathfrak{F}_\mu V \mathfrak{F}_\mu^\dagger) < 0 \tag{30}$$

which yield asymptotic formulas as in (24)–(25) (cf. Chapters 10–11).

For temperatures $T \neq 0$ (see Chapter 13) one has to treat the $\log$-divergences of the integral (27) more carefully. The idea is to consider the *difference*

$$m(T,\Delta) - m(T_c,\Delta=0) \approx 0$$

and prove that this relation is essentially the defining equation for the universal function $\mathsf{f}_{\mathsf{BCS}}$. This approach works for temperatures not too close to the critical one, $T \in [0, (1-\varepsilon)T_c]$.

In addition, for temperatures very close to T_c, we employ the relation of BCS theory to GL theory, showing that α minimizing (18) can very well be approximated by the minimizer of the much simpler GL functional.

Conclusion

To summarize, the main contributions of Part II of this thesis are as follows:

(1) We establish asymptotic formulas for T_c and $\Xi(T=0)$, and hence universality, for high density in dimension $d=3$ (Chapters 10–11), and in the weak coupling limit for $d=1,2$ (Chapter 12).

(2) We prove energy gap universality for all temperatures $0 \leq T \leq T_c$ (Chapter 13), refining previous arguments for universality at zero temperature, and employing GL theory.

(3) We extend the analysis of BCS theory to multi-band systems, and study the effect of inter-band interactions on the critical temperature (Chapter 14).

Part III: Quantum Lattice Systems

In the third part of this thesis, we study many-body quantum lattice systems with short-range interactions. For concreteness, we consider a macroscopic $\Lambda \subseteq \mathbf{Z}^d$. To each $x \in \Lambda$, we attach a local Hilbert space $\mathcal{H}_x$, which, for simplicity, we assume to be of uniformly bounded dimension and can be thought of to model local *spin* degrees of freedom. For a (finite) subset $X \subset \Lambda$, we define the Hilbert space $\mathcal{H}_X := \bigotimes_{x\in X} \mathcal{H}_x$ and the *algebra of local observables* $\mathcal{A}_X := \mathcal{B}(\mathcal{H}_X)$ as the bounded operators on the Hilbert space $\mathcal{H}_X$. Note that one can naturally identify $\mathcal{A}_{X'} \ni A \leftrightarrow A \otimes \mathbf{1}_{X\setminus X'} \in \mathcal{A}_X$ for $X' \subset X$ and one thus has the (formal) embedding $\mathcal{A}_{X'} \subset \mathcal{A}_X$.

In the above setting, we consider self-adjoint Hamiltonians $H = H^\Lambda$ that are *sum-of-local-terms (SLT) operators*

$$H = \sum_{X\subset\Lambda} \Phi(X) \quad \text{with} \quad \Phi(X) \in \mathcal{A}_X\,, \tag{31}$$

where $\|\Phi(X)\|$ decays rapidly as X grows. One instance is the situation of the *interaction* Φ being of finite range, meaning that there exists $R_0 > 0$ such that $\Phi(X) = 0$ whenever $\operatorname{diam}(X) > R_0$. A prototypical example for an SLT operator is the nearest neighbor Heisenberg model with coupling $J \in \mathbf{R}$ and in a magnetic field of strength $h \in \mathbf{R}$ in 3-direction, i.e.

$$H = J \sum_{\substack{x,y\in\Lambda:\\ d(x,y)=1}} \left(\sigma_x^1\sigma_y^1 + \sigma_x^2\sigma_y^2 + \sigma_x^3\sigma_y^3\right) - h\sum_{x\in\Lambda} \sigma_x^3$$

where σ_z^i is the i-th Pauli matrix acting only on site $z \in \Lambda$ (on all the other sites it acts as the identity). We remark that there is a similar setup for fermionic systems, but in this introduction we restrict to quantum spin systems for simplicity of the presentation.

The overarching goal of Part III of this thesis is to study the effect of a (localized) perturbation V on the ground state of a reference Hamiltonian H_0. We do this in two flavors: On the one hand, we aim to prove stability of ground states far away from arbitrarily strong, but localized perturbations. This is an instance of a strong version of the *local-perturbations-perturb-locally (LPPL) principle* [39, 207, 35], studied in Chapter 15. On the other hand, our objective is the justification of (linear) response theory and the Kubo formula (see Chapters 16–17), which we henceforth shall focus on.

In a nutshell, linear response theory provides an answer to the following question: Given a Hamiltonian H_0 with equilibrium state ρ_0, what is the response of the system to a small static perturbation ϵV? Or, more formally, denoting $\langle A\rangle_\rho := \operatorname{tr}(\rho A)$, what is the change

$$\langle A\rangle_{\rho_\epsilon} - \langle A\rangle_{\rho_0} = \epsilon\,\sigma_A + o(\epsilon) \tag{32}$$

in expectation values of an observable A induced by the perturbation ϵV, at least to leading order in the strength $0 < \epsilon \ll 1$? In (32), ρ_ϵ denotes the state, which is reached after the perturbation has

been slowly (*adiabatically*) turned on, and σ_A is the so-called linear response coefficient, which is usually computed by the *Kubo formula*.

Despite the simplicity and empirical success of response theory and the Kubo formula, the problem of justifying it so far has escaped rigorous treatment. Clearly, the central problem lies in determining the state ρ_ϵ. Typically, as prominently advertised by Simon in 1984 in his "Fifteen problems in mathematical physics" [526], the state ρ_ϵ no longer is a non-equilibrium state, and hence the powerful machinery of equilibrium statistical mechanics is not available.

Put briefly, the problem of justifying response theory thus relies on showing that the equilibrium state ρ_0 is driven into a non-equilibrium state $\rho_\epsilon \approx \rho_0$. Note that this clearly goes beyond standard perturbation theory, since the ϵV acts over a long time, and thus an *adiabatic theorem* is needed. However, due to the many-body nature of the problem, the justification of reponse theory also goes beyond the standard adiabatic theorem. In fact, due to the *orthogonality catastrophe*,[8] one cannot expect that ρ_0 and ρ_ϵ are close to each other in any strong topology for macroscopic Λ. Instead, the correct way to measure the distance between ρ_0 and ρ_ϵ is by considering expectation values of *local observables*, allowing to provide error estimates which hold uniformly in the system size.

A first breakthrough in this direction was achieved by Bachmann-de Roeck-Fraas in [36]. On a high level, their main technical idea is to exploit locality estimates in the form of *Lieb-Robinson bounds* (LRBs) [419], allowing a locality preserving treatment of the problem. In fact, standard LRBs roughly take the following form: Consider two observables $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$ with disjoint support $X \cap Y = \varnothing$, ensuring that, in particular, $[A, B] = 0$. Then, for an SLT Hamiltonian H, it holds that

$$\|[A(t), B]\| \lesssim \|A\| \, \|B\| \, \mathrm{e}^{v_{\mathrm{LR}}|t| - \mathrm{dist}(X,Y)} \quad \text{with} \quad A(t) = \mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH} \tag{33}$$

for some constant $v_{\mathrm{LR}} > 0$. This means that, as long as $|t| \ll \mathrm{dist}(X,Y)/v_{\mathrm{LR}}$, the support of $A(t)$ has not reached Y, capturing an effective "light cone" in lattice systems with short-range interactions.

The immense power of LRBs (33) was first realized by Hastings and Wen [326], when proving locality of the generator of the *spectral flow* (see also [39]). Since then, a huge line of research exploiting and proving locality in quantum lattice systems has emerged (see [461, 462, 144] for recent reviews). In regard of the problem of proving response theory, subsequent to [36], Monaco and Teufel [452] extended the result to fermionic lattice systems.

A key assumption in all the above works [326, 39, 36, 452] is the existence of a spectral gap of H_0 separating ρ_0 from the rest of the spectrum. Moreover, as a crucial limitation, the gap is required to remain open for $H_\epsilon = H_0 + \epsilon V$. This apparent restriction in the range of applicability of a many-body adiabatic theorem has then been resolved by Teufel in [556]. Jointly with him, we extended the result to the thermodynamic limit [342] (requiring a spectral gap uniformly in the system size) and proved the adiabatic theorem directly in the thermodynamic limit [341], assuming a gap only for the associated GNS Hamiltonian (see [454] for a construction of the spectral flow in this setting).

We stress that in all these works a *global* spectral gap assumption is crucial for proving adiabatic and response theory for *local* observables. In some sense, the gap "protects" the separated state from far-away influences and thus allows a local treatment around the observable of interest. The foremost goal of Part III is thus to bridge the (conceptual) gap between the *local* nature of the problem (involving *locality* preserving estimates, *local* observables etc.), and the assumption of a *global* spectral gap by introducing a *local dynamical gap condition* (see (35) below).

Summary of Results

We now briefly summarize our results on quantum lattice systems contained in this thesis; afterwards we provide a concise summary of the main ideas for (some of) their proofs.

[8] This means that small errors for single particle wave functions deteriorate when forming a many-particle wave function by taking a large tensor product.

Chapter 15: Local stability of ground states in locally gapped and weakly interacting quantum spin systems [343] We consider weakly interacting quantum spin systems with uniformly gapped on-site terms and prove that localized but otherwise *arbitrary* perturbations change the ground state of such a system only locally, even if the spectral gap gets closed. We call this a *strong version* of the LPPL principle and our result is the first of this kind (followed upon by [35]). In fact, previously, the LPPL principle has already been established in much more general gapped systems, however only for perturbations that do not close the spectral gap of the Hamiltonian (and are thus not *strong*). We also extend our result to Hamiltonians having the appropriate structure of gapped on-site terms and weak interactions only locally in space, and hence some sort of *local gap*.

Although, technically, our results are corollaries of a theorem of Yarotsky [591], the paradigm of a locally gapped Hamiltonian, insensitive to far-away perturbations, has important physical consequences (cf. Chapter 17).

Chapter 16: On adiabatic theory for extended fermionic lattice systems [346] We review recent results on adiabatic theory for ground states of extended gapped fermionic lattice systems under several different assumptions. More precisely, in Chapter 16, we present generalized super-adiabatic theorems for extended but finite as well as infinite systems, assuming either a uniform gap or a gap in the bulk above the unperturbed ground state (cf. [556, 342, 341]). The common idea underlying all these results is the construction of so-called *non-equilibrium almost stationary states* (NEASSs) via locally generated unitaries close to the identity. This approach is known as *space-time adiabatic perturbation theory* [484], or, in very similar contexts, as *Schrieffer-Wolff transformations* [595, 596] or *Lie-Schwinger block diagonalization* [273].

The principal goal of Chapter 16 is to provide an overview of the above mentioned adiabatic theorems and briefly outline the main ideas and techniques required in their proofs. Several of these methods presented in Chapter 16 are then modified upon and advanced for the proofs in Chapter 17.

Chapter 17: Response theory for locally gapped systems [347] In this main chapter of Part III, we introduce a notion of a *local gap* and prove the validity of response theory and the Kubo formula for localized perturbations ϵV. As mentioned above, on a high level, our result shows that the usual spectral gap condition, concerning the interacting many-body system as a whole, is not a necessary condition for understanding local properties of that system.

More concretely, consider an equilibrium state ρ_0 of a Hamiltonian H_0. We say that ρ_0 is *locally dynamically gapped* in a region $\Lambda^{\mathrm{gap}} \subset \Lambda$, whenever the Liouvillian $\mathcal{L}_{H_0}[\cdot] := -\mathrm{i}[H_0, \cdot]$ associated with H_0 is almost invertible on local observables supported well inside Λ^{gap} when tested against the state ρ_0. In order to put our local dynamical gap condition into context, we provide some alternative notions of a local gap and discuss their relations.

Main Ideas of the Proofs

The role of the gap for the construction of NEASSs is that the Liouvillian $\mathcal{L}_{H_0}$ can be *locally* inverted. That is, there exists a locality preserving map[9] $\mathcal{I}_{H_0}$, called the *quasi-local inverse of the Liouvillian*, such that

$$\big[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0}[A] - A, \rho_0\big] = 0 \qquad \forall A \in \mathcal{A}_\Lambda\,. \tag{34}$$

The quasi-local inverse of the Liouvillian is given as a weighted integral of the dynamics generated by H_0, and so fast decay of the weight function (in time) and Lieb-Robinson bounds as in (33) guarantee the locality preserving nature of $\mathcal{I}_{H_0}$.

As mentioned above, the basic idea underlying the proof of the many-body adiabatic theorems and thus response theory, is that the equilibrium state ρ_0 is driven into a non-equilibrium state

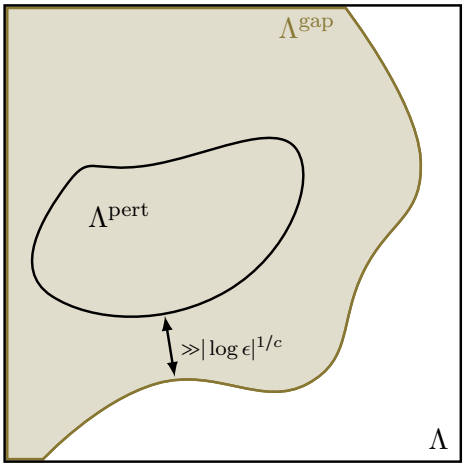
[9] Surely, $\mathcal{I}_{H_0}$ depends on the gap size, but we suppress this dependence for simplicity of the presentation.

$\Pi_\epsilon = \mathrm{e}^{-\mathrm{i}\epsilon S}\rho_0\mathrm{e}^{\mathrm{i}\epsilon S}$. This state, however, will be designed in such a way, that it is almost invariant under the dynamics generated by $H_\epsilon = H_0 + \epsilon V$, thus having engineered a NEASS (non-equilibrium almost-stationary state). In order to achieve this, the generator S of the small unitary transformation relating ρ_0 and Π_ϵ has to be chosen in an appropriate way, in particular preserving locality as an SLT operator.

Indeed, to ensure that $[H_\epsilon, \Pi_\epsilon] = O(\epsilon^2)$ (i.e. invariance up to quadratic terms), we compute

$$\mathrm{e}^{-\mathrm{i}\epsilon S}(H_0 + \epsilon V)\mathrm{e}^{\mathrm{i}\epsilon S} = H_0 + \epsilon\left(V - \mathcal{L}_{H_0}[S]\right) + O(\epsilon^2)\,.$$

From this, we immediately see that choosing $S := \mathcal{I}_{H_0}[V]$ (to leading order), we have created an SLT generator S yielding $[H_\epsilon, \Pi_\epsilon] = O(\epsilon^2)$ by means of (34).



(d) Response theory holds if $\mathrm{dist}(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg |\log\epsilon|^{1/c}$

Our approach in Chapter 17 is to modify the exact invertibility of the Liouvillian (34), which is in fact equivalent to the usual gap condition, in a *local* manner and use it as an *assumption*. In fact, we say that (H_0, ρ_0) is *locally dynamically gapped* in $\Lambda^{\mathrm{gap}} \subset \Lambda$ w.r.t. constants $C, c, \ell > 0$ iff it holds that

$$\begin{aligned}\left|\left\langle\left[\mathcal{L}_{H_0}\circ\mathcal{I}_{H_0}[A] - A, B\right]\right\rangle_{\rho_0}\right| \leq C\|A\|\,\|B\|\,\mathrm{diam}(X)^\ell\,\mathrm{diam}(Y)^\ell \times \\ \times \exp\left(-\max\left\{\mathrm{dist}(X, \Lambda\setminus\Lambda^{\mathrm{gap}}), \mathrm{dist}(Y, \Lambda\setminus\Lambda^{\mathrm{gap}})\right\}^c\right)\end{aligned} \tag{35}$$

for all observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. Under this assumption, by controlling locality properties of the NEASS construction, we show that response theory holds to any order, whenever the perturbation ϵV acts in a region Λ^{pert} which is further than $|\log\epsilon|^{1/c}$ away from the non-gapped region $\Lambda \setminus \Lambda^{\mathrm{gap}}$; cf. Figure (d) above. The linear response coefficient in (32) is given by the Kubo formula, i.e.

$$\sigma_A = -\mathrm{i}\langle[\mathcal{I}_{H_0}[V], A]\rangle_{\rho_0}\,.$$

By using techniques from [35], we will show that the condition (35) is satisfied for ground states of locally (in $\Lambda \setminus \Lambda^{\mathrm{gap}}$) but arbitrarily strongly perturbed Hamiltonians of certain Hamiltonians, having a globally gapped ground state and satisfying *local topological quantum order* (LTQO).

Conclusion

To summarize, the main contributions of Part III of this thesis are as follows:

(1) We provide the first proof of a *strong* version of the LPPL principle for weakly interacting quantum spin systems with uniformly gapped on-site terms (Chapter 15).

(2) We systematically review and provide an exposition of relevant techniques and results required for proving adiabatic theorems in extended fermionic lattice systems (Chapter 16).

(3) We propose a notion of a *local gap*, the local dynamical gap condition (35), prove response theory in that setting, and provide non-trivial examples satisfying (35) (see Chapter 17).

Appendix: Miscellaneous Results

In the appendix, we discuss two more (sets of) problems, which cannot be fit into the three main parts of the thesis.

A *Liouville metrics* on the torus $\mathbf{T}^2$ has line element

$$\mathrm{d}s^2 := \big(f(x) + g(y)\big)\big(\mathrm{d}x^2 + \mathrm{d}y^2\big) \quad \text{with} \quad (x, y) \in \mathbf{T}^2 . \tag{36}$$

It is well known (by constructing an integral of motion), that the geodesic flow associated to (36) is integrable. In Appendix A, we prove that integrable conformal perturbations by trigonometric polynomials are again Liouville metrics. That is, *the class of Liouville metrics is deformationally rigid under a fairly wide class of integrable conformal perturbations*, similarly to rigidity results in the context of the *Birkhoff conjecture* [85, 33, 362, 26]. The proof uses tools from complex analysis, dynamical systems, and Fourier analysis.

In Appendices B–C, we use *interior-boundary conditions* (IBCs) [558, 557, 389] as a method to rigorously implement particle creation and annihilation in relativistic (toy) models. More precisely, in Appendix B, we study Bohmian trajectories of (special) relativistic particles, described by the Dirac equation with a sufficiently strong Coulomb potential. In Appendix C, we study a general relativistic model, where the particle is moving in a super-critical Reissner-Nordström space time. We rigorously construct a Hamiltonian with particle creation and provide asymptotics for the Bohmian trajectories. The proofs use tools from functional analysis, in particular the theory of self-adjoint extension.

Part I

Random Matrices

Chapter 1

Optimal lower bound on eigenvector overlaps for non-Hermitian random matrices

This chapter contains (the extended arXiv:2301.03549 version of) the paper [154]:

G. Cipolloni, L. Erdős, J. Henheik, and D. Schröder. Optimal Lower Bound on Eigenvector Overlaps for non-Hermitian Random Matrices. *J. Funct. Anal.*, 287(4), 2024

Abstract. We consider large non-Hermitian $N \times N$ matrices with an additive independent, identically distributed (i.i.d.) noise for each matrix elements. We show that already a small noise of variance $1/N$ completely thermalises the bulk singular vectors, in particular they satisfy the strong form of Quantum Unique Ergodicity (QUE) with an optimal speed of convergence. In physics terms, we thus extend the Eigenstate Thermalisation Hypothesis, formulated originally by Deutsch [221] and proven for Wigner matrices in [165], to arbitrary non-Hermitian matrices with an i.i.d. noise. As a consequence we obtain an optimal *lower* bound on the diagonal overlaps of the corresponding non-Hermitian eigenvectors. This quantity, also known as the (square of the) eigenvalue condition number measuring the sensitivity of the eigenvalue to small perturbations, has notoriously escaped rigorous treatment beyond the explicitly computable Ginibre ensemble apart from the very recent *upper* bounds given in [44] and [357]. As a key tool, we develop a new systematic decomposition of general observables in random matrix theory that governs the size of products of resolvents with deterministic matrices in between.

Content

1.1 Introduction

Traditional random matrix theory focuses on statistics of eigenvalues, where spectacular universality phenomena arise: the local spectral statistics tend to become universal as the dimension goes to infinity with new distributions arising; most importantly the celebrated *Wigner-Dyson-Mehta bulk statistics* and the *Tracy-Widom edge statistics* in the *Hermitian* spectrum and the *Ginibre statistics* in the *non-Hermitian* spectrum. More recently eigenvectors of *Hermitian* ensembles received considerable attention. They also become universal, albeit in a more conventional way: they tend to be entirely randomised, i.e. Haar distributed [112, 115, 435, 62, 167, 169, 61]. In this paper we study two related questions: how do eigenvectors and singular vectors of a typical *non-Hermitian* random matrix in high dimension look like? To answer them, we introduce a new decomposition of general observables that identifies correlations of the Hermitised resolvents as *entire matrices* at different spectral parameters. This captures correlations of the singular vectors well beyond correlations of traces of resolvents that govern only the singular *values*. Somewhat surprisingly, we are then able to transfer information on singular vectors to the non-Hermitian eigenvectors.

1.1.1 Non-Hermitian eigenvector overlaps

To be specific, we consider non-Hermitian $N \times N$ matrices of the form $\Lambda + X$, where Λ is an arbitrary deterministic matrix and X is random. We assume that the norm of Λ is bounded independently of N and X has independent, identically distributed (i.i.d.) centred matrix elements with variance

$\mathbf{E}|x_{ij}|^2 = \frac{1}{N}$ with some further moment conditions. This normalisation guarantees that $\|X\| \le 2+o(1)$ and the spectrum of X lies essentially in the unit disk (*circular law*) with very high probability, hence Λ and X remain of comparable size as N increases. Note that X perturbs each matrix elements of Λ by a small random amount of order $1/\sqrt{N}$, however the spectra of Λ and $\Lambda + X$ substantially differ.

The analysis of non-Hermitian random matrices is typically much harder than that of the Hermitian ones. Non-Hermitian matrices have two different sets of spectral data: eigenvalues/vectors and singular values/vectors which cannot be directly related. In particular, the study of singular vectors and eigenvectors substantially differ: while singular vectors can still be understood from a Hermitian theory, there is no such route for eigenvectors. Unlike for non-Hermitian *eigenvalues*, where Girko's formula translates their linear statistics into a Hermitian problem, no similar "Hermitisation" relation is known for non-Hermitian *eigenvectors*. Furthermore, left and right eigenvectors differ and their relation is very delicate. Assuming that each eigenvalue μ_i of $\Lambda + X$ is simple, we denote the corresponding left and right eigenvectors by $\boldsymbol{l}_i$, $\boldsymbol{r}_i$, i.e.

$$(\Lambda + X)\boldsymbol{r}_i = \mu_i \boldsymbol{r}_i\,, \qquad \boldsymbol{l}_i^t(\Lambda + X) = \mu_i \boldsymbol{l}_i^t\,,$$

under the standard bi-orthogonality relation $\langle \bar{\boldsymbol{l}}_j, \boldsymbol{r}_i\rangle = \boldsymbol{l}_j^t \boldsymbol{r}_i = \delta_{i,j}$. Note that this relation leaves a large freedom in choosing the normalisation of each eigenvector. The key invariant quantity is the *eigenvector overlap*

$$\mathcal{O}_{ij} := \langle \boldsymbol{r}_j, \boldsymbol{r}_i\rangle\langle \boldsymbol{l}_j, \boldsymbol{l}_i\rangle\,,$$

which emerges in many problems where non-Hermitian eigenvectors are concerned, see e.g. [18, 140, 141, 57, 107, 274]. Two prominent examples are

(i) in numerical linear algebra; where $\sqrt{\mathcal{O}_{ii}}$ is the *eigenvalue condition number* determining how fast μ_i moves under small perturbation in the worst case using the formula

$$\sqrt{\mathcal{O}_{ii}} = \lim_{t\to 0} \sup\left\{\left|\frac{\mu_i(\Lambda + X + tE) - \mu_i(\Lambda + X)}{t}\right| \,:\, E \in \mathbf{C}^{N\times N}, \|E\| = 1\right\} \tag{1.1.1}$$

(see, e.g. [44]);

(ii) in the theory of the *Dyson Brownian motion* for non-Hermitian matrices; where $\mathcal{O}_{ij}$ gives the correlation of the martingale increments for the stochastic evolution of the eigenvalues μ_i and μ_j as the matrix evolves by the natural Ornstein-Uhlenbeck flow (see [299], [107, Appendix A]).

The main result of this paper is an almost optimal *lower* bound of order N on the diagonal overlap $\mathcal{O}_{ii}$, with very high probability. In the context of numerical linear algebra this means that non-Hermitian eigenvalues of $\Lambda + X$ still move at a speed of order $\sqrt{N}$ under the "worst" perturbation E in (1.1.1), despite having added a random smoothing component X to Λ. Note that in numerics one typically views the random smoothing as a tool to reduce the overlap of Λ in order to enhance the stability of its eigenvalues; our result shows a natural limitation for such reduction. It still does not exclude the possibility that a very specially chosen X reduces the eigenvalue condition numbers much more than a typical random one does, in particular it does not disprove the Davidson-Herrero-Salinas conjecture (see [199, Problem 2.11]). However, our N-dependent lower bound on $\mathcal{O}_{ii}$ shows that a naive randomisation argument is not sufficient for resolving this conjecture. Complementary *upper* bounds on $\mathcal{O}_{ii}$ have recently been proven in [44] and [357]. These hold only in expectation sense, as $\mathcal{O}_{ii}$ has a fat-tail, and they are off by a factor N. Very recently this factor was removed in [240]. We remark, however, that N is the most relevant parameter of the problem only from our random matrix theory point of view. Works motivated by numerical analysis, such as [44, 357] and references therein, often focus on tracking the γ-dependence for the problem $\Lambda + \gamma X$ in the small noise regime

$\gamma \ll 1$ in order to reduce the effect of the random perturbation. In this setup the non-optimality of the N-power may be considered less relevant.[1]

In the context of the Dyson Brownian motion, our lower bound on $\mathcal{O}_{ii}$ implies a diffusive lower bound on the eigenvalues of the Ornstein-Uhlenbeck (OU) matrix flow, generalizing the analogous result of Bourgade and Dubach [107, Corollary 1.6] from Ginibre ensemble to arbitrary i.i.d. ensemble (see (1.2.14) later).

1.1.2 Thermalisation of singular vectors

The key step to our lower bound on $\mathcal{O}_{ii}$ is a *thermalisation* result on the singular vectors that is of independent interest. Namely, we show that singular vectors of $\Lambda + X$ are fully randomised in the large N limit in the sense that their quadratic forms with arbitrary test matrices have a deterministic limit with an optimal $N^{-1/2}$ speed of convergence. This holds with very high probability which enables us to make such statement for matrices of the form $(\Lambda - z) + X$ *simultaneously* for any shift parameter z, even for random ones. We will use this for $z = \mu$, an eigenvalue of $\Lambda + X$. This allows us to gain access to eigenvectors of $\Lambda + X$, by noticing that singular vectors and eigenvectors are unrelated in general with an obvious exception: if μ is an eigenvalue of $\Lambda + X$, then any vector in the kernel of $\Lambda + X - \mu$ is an eigenvector of $\Lambda + X$ with eigenvalue μ, and a singular vector of $\Lambda + X - \mu$ with singular value 0. Hence high probability statements for singular vectors can be converted into similar statements for eigenvectors – this key idea may be viewed as the eigenvector version of the transfer principle between eigenvalues and singular values encoded in Girko's formula.

Our thermalisation result for singular vectors may be viewed as the non-Hermitian analogue of the *Quantum Unique Ergodicity* (QUE) for Hermitian Wigner matrices proven in [165]. We now briefly explain the QUE phenomenon and its physics background in the simplest Hermitian context before we consider the singular vectors of $\Lambda + X$. In fact, via a standard Hermitisation procedure we will turn the singular vector problem to a Hermitian eigenvector problem.

For Hermitian random matrices H, that can be considered as the Hamilton operator of a disordered quantum system, a major motivation comes from physics, where the randomisation of the eigenvectors is interpreted as a *thermalisation* effect. The *Eigenstate Thermalisation Hypothesis (ETH)* by Deutsch [221] and Srednicki [535] (see also [233, 222]) asserts that any deterministic Hermitian matrix A (observable), becomes essentially diagonal in the eigenbasis of a "sufficiently chaotic" Hamiltonian, where chaos may come from an additional randomness or from the ergodicity of the underlying classical dynamics. In other words,

$$\langle \boldsymbol{u}_i, A\boldsymbol{u}_j \rangle - \delta_{ij} \langle\langle A \rangle\rangle_i \to 0\,, \qquad \text{as } N \to \infty\,, \tag{1.1.2}$$

where $\{\boldsymbol{u}_i\}$ is a orthonormal eigenbasis of H and the deterministic "averaged" coefficient $\langle\langle A \rangle\rangle_i$ is to be computed from the statistics of H.

In the mathematics literature the same problem is known as the Quantum (Unique) Ergodicity, originally formulated for the Laplace-Beltrami operator on surfaces with ergodic geodesic flow, see [528, 178, 600], on regular graphs [24] and on special arithmetic surfaces [510, 123, 424, 533]. In [165] we proved QUE in the strongest form with an optimal speed of convergence for the eigenvectors of Wigner matrices that, by E. Wigner's vision, can be viewed as the "most random" Hamiltonian. In this case, the diagonal limit $\langle\langle A \rangle\rangle_i$ in (1.1.2) is independent of i and given by the normalised trace $\langle A \rangle := \frac{1}{N} \operatorname{Tr} A$. In fact, in subsequent papers [167, 169] (see also [62]) even the normal fluctuation of $\sqrt{N}\big[\langle \boldsymbol{u}_i, A\boldsymbol{u}_i \rangle - \langle A \rangle\big]$ was proven, followed by the proof of joint Gaussianity of finite many overlaps in [61]. Previously QUE results were proven for rank one observables

[1] As long as γ is N independent, one may set $\gamma = 1$ by a simple rescaling so we refrain from carrying this extra factor in the current paper. We remark that our methods would allow to trace the polynomial γ-dependence in all our main estimates as well, albeit not with an optimal power.

(see [368, 552] under four moment matching and [112] in general) and finite rank observables [435], see also [59] for deformed Wigner matrices and [115] for band matrices. The proofs crucially used that H is Hermitian, heavily relying on sophisticated Hermitian techniques (such as *local laws* and *Dyson Brownian Motion*) developed in the last decade for eigenvalue universality questions.

Back to our non-Hermitian context, we consider the singular vectors $\{\boldsymbol{u}_i, \boldsymbol{v}_i\}_{i=1}^N$ of $\Lambda + X$,

$$(X+\Lambda)(X+\Lambda)^* \boldsymbol{u}_i = \sigma_i^2 \boldsymbol{u}_i\,, \qquad (X+\Lambda)^*(X+\Lambda)\boldsymbol{v}_i = \sigma_i^2 \boldsymbol{v}_i\,,$$

belonging to the singular value σ_i. We view them as the two N-dimensional components of the eigenvectors $\boldsymbol{w}_i = (\boldsymbol{u}_i, \boldsymbol{v}_i)$ of the $2N$-dimensional *Hermitisation* of $\Lambda + X$, defined as

$$H = H^\Lambda := W + \hat{\Lambda}\,, \quad W := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \qquad \hat{\Lambda} := \begin{pmatrix} 0 & \Lambda \\ \Lambda^* & 0 \end{pmatrix}. \tag{1.1.3}$$

In particular, from the overlaps $\langle \boldsymbol{w}_i, A\boldsymbol{w}_j\rangle$ of eigenvectors for the Hermitised problem with a general $(2N)\times(2N)$ matrix A one may read off all the singular vector overlaps of the form $\langle \boldsymbol{u}_i, B\boldsymbol{u}_j\rangle$, $\langle \boldsymbol{v}_i, B\boldsymbol{v}_j\rangle$ and $\langle \boldsymbol{u}_i, B\boldsymbol{v}_j\rangle$ with any $N\times N$ matrix B. Therefore our goal is to show the general thermalisation phenomenon, the convergence of $\langle \boldsymbol{w}_i, A\boldsymbol{w}_j\rangle$ (cf. (1.1.2)), for the Hermitised matrix H^Λ thus generalizing the ETH proven in [165] beyond Wigner matrices and with an additional arbitrary matrix Λ. Unlike in the Wigner case, the limit $\langle\langle A\rangle\rangle_i$ genuinely depends on the index i and part of the task is to determine its precise form. Note that due to the large zero blocks, W has about half as many random degrees of freedom as a Wigner matrix of the same dimension has, moreover the block structure gives rise to potential instabilities, thus the ETH for H^Λ is considerably more involved than for Wigner matrices. In the next section we explain the main new method of this paper that systematically handles all these instabilities.

1.1.3 Structural decomposition of observables

We introduce a new concept for splitting general observables into "regular" and "singular" components; where the singular component gives the leading contribution and the regular component is estimated. In the case of Wigner matrices H in [165, 172] we used the decomposition $A = \langle A\rangle + \mathring{A}$, where the traceless part of A, $\mathring{A} := A - \langle A\rangle$, is the regular component and the projection[2] of A onto the one dimensional space spanned by the identity matrix is the singular component. This gave rise to the following decomposition of resolvent $G = G(w) = (H-w)^{-1}$ for any $w \in \mathbf{C}\setminus\mathbf{R}$:

$$\langle GA\rangle = m\langle A\rangle + \langle A\rangle\langle G - m\rangle + \langle G\mathring{A}\rangle, \tag{1.1.4}$$

where $m = m(w)$ is the Stieltjes transform of the semicircle law. The second term in (1.1.4) is asymptotically Gaussian of size $\langle G-m\rangle \sim (N\eta)^{-1}$ [329] and the last term is also Gaussian, but of much smaller size $\langle G\mathring{A}\rangle \sim \langle \mathring{A}\mathring{A}^*\rangle^{1/2}/(N\eta^{1/2})$ in the interesting regime of small $\eta := |\operatorname{Im} w| \ll 1$ [172].

Similar decomposition governs the traces of longer resolvent chains of Wigner matrices, for example

$$\langle GAG^*B\rangle = \langle GG^*\rangle = \frac{1}{\eta}\langle \operatorname{Im} G\rangle \sim \frac{1}{\eta} \gg 1$$

if $A = B = I$, i.e. both observable matrices are purely singular, while for regular (and bounded) observables $A = \mathring{A}$, $B = \mathring{B}$ we have

$$\langle GAG^*B\rangle \sim 1. \tag{1.1.5}$$

[2] We equip the space of matrices with the usual normalised Hilbert-Schmidt scalar product, $\langle A, B\rangle := \frac{1}{N}\operatorname{Tr} A^*B = \langle A^*B\rangle$.

Both examples indicate the $\sqrt{\eta}$-*rule* (see (1.3.16) and Remark 1.4.6 later), informally asserting that each regular observable renders the size of a resolvent chain smaller by a factor $\sqrt{\eta}$ than its singular counterpart. In [168, 169] we obtained the deterministic leading terms and optimal error estimates on the fluctuation for resolvent chains of arbitrary length

$$\langle G(w_1)A_1G(w_2)A_2\dots\rangle \tag{1.1.6}$$

with arbitrary observables in between. The answer followed the $\sqrt{\eta}$-*rule* hence it heavily depended on the $A_i = \langle A_i\rangle + \mathring{A}_i$ decomposition for each observable.

In particular, in order to estimate $\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\langle A\rangle = \langle \boldsymbol{u}_i, \mathring{A}\boldsymbol{u}_j\rangle$ for ETH in (1.1.2), we had

$$N|\langle \boldsymbol{u}_i, \mathring{A}\boldsymbol{u}_j\rangle|^2 \lesssim \langle \operatorname{Im} G(w_1)\mathring{A}\operatorname{Im} G(w_2)\mathring{A}\rangle \lesssim 1\,,$$

where we first used spectral decomposition of both G's and then used a version of (1.1.5). Here the spectral parameters $w_k = e_k + \mathrm{i}\eta$ are chosen such that e_1 and e_2 be close to the eigenvalues corresponding to $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$, respectively, and $\eta \sim N^{-1}$ in order to resolve the spectrum on the fine scale of the individual eigenvalues.[3]

The key point in all these analyses for Wigner matrices was that the regular/singular concept was *independent* of the spectral parameter: the same universal decomposition into tracial and traceless parts worked in every instance along the proofs. One consequence is the i-independence of the limiting overlap $\langle\langle A\rangle\rangle_i := \langle A\rangle$ in (1.1.2).[4]

For more complicated ensembles, like H^Λ in (1.1.3), especially if an arbitrary matrix Λ is involved, the correct decomposition depends on the location in the spectrum of H where we work. To guess it, first we recall the *single resolvent local law* (Theorem 1.2.6) for the resolvent $G = G^\Lambda(w) = (H^\Lambda - w)^{-1}$, asserting that $\langle GA\rangle \approx \langle MA\rangle$, where $M = M^\Lambda(w)$ solves a nonlinear deterministic equation, the *Matrix Dyson Equation (MDE)*, see (1.2.20) later. Then a heuristic calculation (see Appendix 1.A.1) shows that for $w = e + \mathrm{i}\eta \in \mathbf{C}_+$ we have

$$\mathbf{E}\big|\langle (G-M)A\rangle\big|^2 \approx \frac{\big|\langle \operatorname{Im} MA\rangle\big|^2}{(N\eta)^2} + \frac{\big|\langle \operatorname{Im} MAE_-\rangle\big|^2}{N^2\eta(|e|+\eta)} + \mathcal{O}\Big(\frac{1}{N^2\eta}\Big), \quad E_- := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.1.7}$$

indicating that the singular component of A is *two dimensional*, depends on w, and for any A orthogonal to the two singular directions $\operatorname{Im} M$ and $E_-\operatorname{Im} M$ the size of $\langle (G-M)A\rangle$ is smaller by a factor $\sqrt{\eta}$. The first singular direction is always present. The second singular direction is a consequence of the block structure of H and it is manifested only for w near the imaginary axis. For energies $|e| \sim 1$, only the first singular direction, namely the one involving $\operatorname{Im} M$ plays a role.

What about longer chains (1.1.6)? Each matrix A_i is sandwiched between two resolvents with different spectral parameters w_i, w_{i+1}. We find that the correct decomposition of any A between two resolvents in a chain $\dots G(w)AG(w')\dots$ depends only on w, w' and it has the form

$$A = \langle V_+, A\rangle U_+ + \langle V_-, A\rangle U_- + \mathring{A}\,, \qquad V_\pm = V_\pm^{w,w'}\,, \quad \mathring{A} = \mathring{A}^{w,w'}\,, \tag{1.1.8}$$

where the first two terms form the singular component of A, and $\mathring{A}$, defined by this equation, is the regular component. We will establish that both V_+ and V_- are the right eigenvectors of a certain *stability operator* $\mathcal{B}$ acting on $\mathbf{C}^{2N\times 2N}$ that corresponds to the Dyson equation, and $U_\pm$ will be

[3] Strictly speaking we used $\eta = N^{-1+\xi}$ with any small $\xi > 0$, and all estimates held up to an N^ξ factor but we ignore these technicalities in the introduction.

[4] A quick direct way to see this independence is the special case of Gaussian Wigner matrices (GUE or GOE), where the eigenvectors are Haar distributed, independently of their eigenvalue.

explained later. For example, if $\operatorname{Im} w$ and $\operatorname{Im} w'$ have opposite signs then V_+ is the right eigenvector of

$$\mathcal{B}[\cdot] = 1 - M(\bar{w})\mathcal{S}[\cdot]M(\bar{w}'),$$

where $\mathcal{S}$ is covariance operator for the matrix W in (1.1.3) (see (1.2.21)). $V_\pm$ with other sign combinations are defined very similarly (in Appendix 1.A.3 we present all cases). Actually, the special directions $\operatorname{Im} M$ and $E_-\operatorname{Im} M$ that we found by direct variance calculation in (1.1.7) also emerge *canonically* as eigenvectors of a certain stability operator! Similar variance calculation for longer chains would reveal the same consistency: the variance of the chain (1.1.6) is the smallest if each A_i is regular with respect to the two neighboring spectral parameters w_i, w_{i+1}.

Note that the choice of $V_\pm$ is basically dictated by variance calculations like (1.1.7). However, the matrices $U_\pm$ in (1.1.8) can still be chosen freely up to their linear independence and the normalisation requirement $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$. The latter guarantees that the sum of the singular terms in (1.1.8) is actually a (non-orthogonal) projection $|U_+\rangle\langle V_+| + |U_-\rangle\langle V_-|$ acting on A. Since $V_\pm$ are the right eigenvectors of a stability operator, one may be tempted to choose $U_\pm$ as certain left eigenvectors but we did not find this guiding principle helpful. Instead, we use this freedom to simplify the calculation of the singular terms. Substituting the singular part of A into $\ldots G(w)AG(w')\ldots$, we need to compute $G(w)U_\pm G(w')$ and quite pragmatically we choose $U_\pm$ such that the entity could be applied and thus reduce the length of the chain. Thanks to the spectral symmetry of $H = H^\Lambda$, for its resolvent we have $E_-G(-w)E_- = -G(w)$, and we find that $U_+ = I$, $U_- = E_-$ do the job, which accidentally coincide with the left eigenvectors of the stability operator for the special case of i.i.d. matrices.

In Appendix 1.A.3 we present the canonical choices of $V_\pm$ and $U_\pm$ in a more general situation and explain at which stage of the proof their correct choice emerges. In our current application only $V_\pm$ are nontrivial (in particular energy dependent), while $U_\pm$ are very simple. This is due to the fact that the chain (1.1.6) consists of resolvents of the *same* operator. In more general problems one may take resolvents with two different Λ's in the chain, in which case $U_\pm$ are also nontrivial.

This decomposition scheme is the really novel ingredient of our proofs. Several other tools we use, such as *recursive Dyson equations*, hierarchy of *master inequalities* and *reduction inequalities* have been introduced before (especially in our related works on Wigner matrices [165, 172]), but the dependence of the decomposition on the spectral parameters in the current setup requires quite different new estimates along the arguments. We informally explain the prototype of such an estimate at the beginning of Section 1.4.1.

1.1.4 Notations

We define the $2N \times 2N$ matrices $E_\pm := E_1 \pm E_2$, where

$$E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.1.9}$$

Each entry of the matrix is understood as a multiple of the $N \times N$–identity. By $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$ we denote the upper and lower integer part, respectively, i.e. for $x \in \mathbf{R}$ we define $\lceil x\rceil := \min\{m \in \mathbf{Z} : m \geq x\}$ and $\lfloor x\rfloor := \max\{m \in \mathbf{Z} : m \leq x\}$. We denote $[k] := \{1, \ldots, k\}$ for $k \in \mathbf{N}$ and $\langle A\rangle := d^{-1}\operatorname{Tr}(A)$, $d \in \mathbf{N}$, is the normalised trace of a $d \times d$-matrix. For positive quantities A, B we write $A \lesssim B$ resp. $A \gtrsim B$ and mean that $A \leq CB$ resp. $A \geq cB$ for some N-independent constants $c, C > 0$. We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^{2N}$, for some $N \in \mathbf{N}$, and define

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \sum_i \bar{x}_i y_i, \qquad A_{\boldsymbol{xy}} := \langle \boldsymbol{x}, A\boldsymbol{y}\rangle.$$

Matrix entries are indexed by lower case Roman letters $a, b, c, ...$ from the beginning of the alphabet and unrestricted sums over $a, b, c, ...$ are always understood to be over $\{1, ..., N, N+1, ..., 2N\}$. Analogously, unrestricted sums over lower case Roman letters $i, j, k, ...$ from the middle of the alphabet are always understood to be over $\{-N, ..., -1, 1, ..., N\}$. Finally, the lower case Greek letters σ and τ as indices indicate a sign, i.e. $\sigma, \tau \in \{+, -\}$, and unrestricted sums over σ, τ are understood to be over $\{+, -\}$.

We will use the concept of 'with very high probability', meaning that any fixed $D > 0$, the probability of an N-dependent event is bigger than $1 - N^{-D}$ for all $N \geq N_0(D)$. Also, we will use the convention that $\xi > 0$ denotes an arbitrarily small constant, independent of N. Moreover, we introduce the common notion of *stochastic domination* (see, e.g., [241]): For two families

$$X = \left(X^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right)$$

of non-negative random variables indexed by N, and possibly a parameter u, then we say that X is stochastically dominated by Y, if for all $\varepsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\epsilon} Y^{(N)}(u)\right] \leq N^{-D} \tag{1.1.10}$$

for large enough $N \geq N_0(\epsilon, D)$. In this case we write $X \prec Y$. If for some complex family of random variables we have $|X| \prec Y$, we also write $X = O_{\prec}(Y)$.

Acknowledgment

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1.2 Main results

We consider *real or complex i.i.d. matrices* X , i.e. $N \times N$ matrices whose entries are independent and identically distributed as $x_{ab} \stackrel{\mathrm{d}}{=} N^{-1/2}\chi$ for some real or complex random variable χ satisfying the following assumptions:

Assumption 1.2.1. *We assume that* $\mathbf{E}\,\chi = 0$ *and* $\mathbf{E}\,|\chi|^2 = 1$. *Furthermore, we assume the existence of high moments, i.e., that there exist constants* $C_p > 0$, *for any* $p \in \mathbf{N}$, *such that*

$$\mathbf{E}\,|\chi|^p \leq C_p\,.$$

Additionally, in the complex case, we assume that $\mathbf{E}\,\chi^2 = 0$.

For definiteness, in the sequel we perform our entire analysis for the complex case; the real case being completely analogous and hence omitted.

1.2.1 Non-Hermitian singular vectors and eigenvectors

Fix a deterministic matrix $\Lambda \in \mathbf{C}^{N \times N}$, with N-independent norm bound, $\|\Lambda\| \lesssim 1$. Let $\{\sigma_i\}_{i \in [N]}$ be the singular values of $X + \Lambda$ with corresponding (normalised) left- and right-singular vectors $\{\boldsymbol{u}_i\}_{i \in [N]}$ and $\{\boldsymbol{v}_i\}_{i \in [N]}$, respectively, i.e.

$$(X + \Lambda)\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i \qquad \text{and} \qquad (X + \Lambda)^* \boldsymbol{u}_i = \sigma_i \boldsymbol{v}_i\,. \tag{1.2.1}$$

All these objects naturally depend on Λ, but we will omit this fact from the notation.

Let ν_i, $i \in [N]$, be the increasingly ordered singular values of Λ. Define the *Hermitisation* of Λ as

$$\hat{\Lambda} := \begin{pmatrix} 0 & \Lambda \\ \Lambda^* & 0 \end{pmatrix}. \tag{1.2.2}$$

Due to its block structure, the spectrum of $\hat{\Lambda}$ is symmetric with respect to zero, with eigenvalues $\{\nu_i\}_{0 \neq |i| \leq N}$ such that $\nu_{-i} = -\nu_i$ for all $i \in [N]$. The empirical density of states of $\hat{\Lambda}$ is denoted by

$$\mu_{\hat{\Lambda}} := \frac{1}{2N} \sum_{0 \neq |i| \leq N} \delta_{\nu_i}.$$

Let μ_{sc} be the Wigner semicircle distribution with density $\rho_{\mathrm{sc}}(x) := (2\pi)^{-1}\sqrt{[4-x^2]_+}$, where $[\cdots]_+$ is the positive part of a real number. Define the free additive convolution

$$\mu = \mu^{\Lambda} := \mu_{\mathrm{sc}} \boxplus \mu_{\hat{\Lambda}}, \tag{1.2.3}$$

which is a probability distribution on $\mathbf{R}$. We now briefly recall basic facts about the free convolution with the semicircle density (see, e.g. the classical paper by P. Biane [81]). Most conveniently μ may be defined by inverting its Stieltjes transform

$$m(w) = m^{\Lambda}(w) := \int_{\mathbf{R}} \frac{\mu(\mathrm{d}e)}{e-w}, \qquad w \in \mathbf{C} \setminus \mathbf{R},$$

where m satisfies the implicit equation

$$m(w) = \int_{\mathbf{R}} \frac{\mu_{\hat{\Lambda}}(\mathrm{d}e)}{e - (w + m(w))}. \tag{1.2.4}$$

With the additional constraint $\operatorname{Im} m(w) \cdot \operatorname{Im} w > 0$ this equation has a unique solution that is analytic away from the real axis with $m(\overline{w}) = \overline{m}(w)$. Since $\mu_{\hat{\Lambda}}$ is symmetric to zero with bounded support, μ is also symmetric with support bounded independently of N. Moreover μ is absolutely continuous with respect to Lebesgue measure with density denoted by $\rho = \rho^{\Lambda}$. The density ρ may be obtained[5] as the boundary value of $\operatorname{Im} m$ at any e on the real line, i.e.

$$\rho(e) := \lim_{\eta \downarrow 0} \rho(e + \mathrm{i}\eta), \qquad \rho(w) := \frac{1}{\pi} |\operatorname{Im} m(w)|. \tag{1.2.5}$$

In fact m itself has a continuous extension to the real axis from the upper half plane $m(e) := \lim_{\eta \downarrow 0} m(e + \mathrm{i}\eta)$. Proving the existence of these limits is standard from (1.2.4).

Next, for any (small) $\kappa > 0$, we define the *κ-bulk* of the density ρ as

$$\mathbf{B}_{\kappa} = \mathbf{B}_{\kappa}^{\Lambda} := \{x \in \mathbf{R} : \rho(x) \geq \kappa^{1/3}\} \tag{1.2.6}$$

which is symmetric to the origin. Finally, we denote a (modified) i^{th} quantile of the density ρ by γ_i, i.e.

$$\frac{i+N}{2N} = \int_{-\infty}^{\gamma_i} \rho(e)\,\mathrm{d}e, \qquad |i| \leq N, \tag{1.2.7}$$

and we immediately conclude by symmetry of ρ that $\gamma_i = -\gamma_{-i}$ for every $|i| \leq N$.

Our first main result establishes the *thermalisation of singular vectors* of $X + \Lambda$ in the bulk, i.e. for indices i, j with quantiles γ_i, γ_j uniformly in the *bulk* of the density ρ.

[5] For orientation of the reader we mention that ρ is the deterministic approximation, the so-called *self-consistent density of states (scDos)*, for the empirical eigenvalue density of the Hermitisation of $X + \Lambda$. This connection will be explained in the next Section 1.2.2.

Theorem 1.2.2. (Thermalisation of Singular Vectors)
Fix a bounded $\Lambda \in \mathbf{C}^{N\times N}$ and $\kappa > 0$ independent of N. Let $\{\boldsymbol{u}_i\}_{i\in[N]}$ and $\{\boldsymbol{v}_i\}_{i\in[N]}$ be the (normalised) left- and right-singular vectors of $X+\Lambda$, respectively, where X is an i.i.d. matrix satisfying Assumption 1.2.1. Then, for any deterministic matrix $B \in \mathbf{C}^{N\times N}$ with $\|B\| \lesssim 1$ it holds that[6]

$$\max_{i,j}\left|\langle \boldsymbol{u}_i, B\boldsymbol{u}_j\rangle - \delta_{j,i}\frac{\left\langle \operatorname{Im}\left[\frac{\gamma_j+m(\gamma_j)}{\Lambda\Lambda^*-(\gamma_j+m(\gamma_j))^2}\right]B\right\rangle}{\pi\rho(\gamma_j)}\right| \prec \frac{1}{\sqrt{N}}\,, \tag{1.2.8a}$$

$$\max_{i,j}\left|\langle \boldsymbol{v}_i, B\boldsymbol{v}_j\rangle - \delta_{j,i}\frac{\left\langle \operatorname{Im}\left[\frac{\gamma_j+m(\gamma_j)}{\Lambda^*\Lambda-(\gamma_j+m(\gamma_j))^2}\right]B\right\rangle}{\pi\rho(\gamma_j)}\right| \prec \frac{1}{\sqrt{N}}\,, \tag{1.2.8b}$$

$$\max_{i,j}\left|\langle \boldsymbol{u}_i, B\boldsymbol{v}_j\rangle - \delta_{j,i}\frac{\left\langle \Lambda \operatorname{Im}\left[\left(\Lambda^*\Lambda-(\gamma_j+m(\gamma_j))^2\right)^{-1}\right]B\right\rangle}{\pi\rho(\gamma_j)}\right| \prec \frac{1}{\sqrt{N}}\,, \tag{1.2.8c}$$

where the maximum is taken over all $i,j\in[N]$ such that the quantiles $\gamma_i,\gamma_j \in \mathbf{B}_\kappa$ are in the κ-bulk of the density ρ.

The thermalisation of singular vectors will be a simple corollary to the *Eigenstate Thermalisation Hypothesis (ETH)* for the Hermitisation H^Λ of $X+\Lambda$, which is formulated in Theorem 1.2.7 below. The proof of Theorem 1.2.2 will be given in Section 1.3.

Our second main result concerns the bi-orthonormal left and right eigenvectors $\{\boldsymbol{l}_i\}_{i\in[N]}$ and $\{\boldsymbol{r}_i\}_{i\in[N]}$, respectively, of $X+\Lambda$, with corresponding eigenvalues $\{\mu_i\}_{i\in[N]}$, i.e.

$$(X+\Lambda)\boldsymbol{r}_i = \mu_i\boldsymbol{r}_i\,, \qquad \boldsymbol{l}_i^t(X+\Lambda) = \mu_i\boldsymbol{l}_i^t\,, \tag{1.2.9}$$

where t denotes the transpose of a vector. More precisely, the following theorem provides a lower bound on the diagonal part of the *overlaps matrix*

$$\mathcal{O}_{ij} := \langle \boldsymbol{r}_j, \boldsymbol{r}_i\rangle\langle \boldsymbol{l}_j, \boldsymbol{l}_i\rangle\,, \tag{1.2.10}$$

defined subject to the standard normalisation

$$\langle \bar{\boldsymbol{l}}_j, \boldsymbol{r}_i\rangle = \boldsymbol{l}_j^t\boldsymbol{r}_i = \delta_{i,j}\,. \tag{1.2.11}$$

We restrict our results to eigenvalues μ_i in the *bulk* of $X+\Lambda$ in the following sense.

Definition 1.2.3. *We say that $z\in\mathbf{C}$ is in the* bulk *of $X+\Lambda$ if and only if there exists an N-independent $\kappa>0$ for which*

$$0 \in \mathbf{B}_\kappa^{\Lambda-z} = \{x\in\mathbf{R} : \rho^{\Lambda-z}(x) \geq \kappa^{1/3}\}\,.$$

There is no simple characterisation of the bulk of $X+\Lambda$ in terms of the spectrum of Λ. However, taking the imaginary part of (1.2.4) at $w = 0+\mathrm{i}0$ shows that $0\in\mathbf{B}_\kappa^{\Lambda-z}$ is equivalent to

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\nu_i(\Lambda-z)^2+\kappa^{2/3}} \geq 1\,,$$

where $\nu_i(\Lambda-z)$ are the singular values of $\Lambda-z$.

Theorem 1.2.4. *Consider $X+\Lambda$, with Λ being a deterministic matrix as in* (1.2.2) *and with X being an i.i.d. matrix satisfying Assumption 1.2.1. Then*

$$\mathcal{O}_{ii} > N\,, \tag{1.2.12}$$

where the index $i\in[N]$ is such that μ_i is in the bulk of $X+\Lambda$.

[6] The deterministic terms following the Kronecker symbol $\delta_{j,i}$ in (1.2.8) will be shown to be bounded.

In the introduction we already mentioned the consequence of this result on the sensitivity of an eigenvalue of $X+\Lambda$ under small perturbations. Now we explain its other consequence on the diffusivity of the Dyson-type eigenvalue dynamics. Let each entry of $X = X(t)$ evolve as an independent complex OU process,

$$\mathrm{d}X_{ij} = \frac{\mathrm{d}B_{ij}}{\sqrt{N}} - \frac{1}{2}X_{ij}\mathrm{d}t,$$

where B_{ij} are independent standard complex Brownian motions and the initial condition $X(0)$ satisfies Assumption 1.2.1. A direct calculation [107, Proposition A.1] shows that the eigenvalues $\mu_i = \mu_i(t)$ follow the Dyson-type stochastic dynamics

$$\mathrm{d}\mu_i = \mathrm{d}M_i - \frac{1}{2}\mu_i\mathrm{d}t, \qquad \{\mu_i(0)\} = \mathrm{Spec}X(0), \qquad 1 \le i \le N, \tag{1.2.13}$$

where the martingales M_i have brackets $\langle M_i, M_j\rangle = 0$ and $\mathrm{d}\langle M_i, M_j\rangle_t = \frac{1}{N}\mathcal{O}_{ij}(t)\mathrm{d}t$. In particular, we immediately obtain, for any $\epsilon > 0$ that

$$\mathbf{E}\left[|\mu_i(t) - \mu_i(0)|^2\mathbf{1}(\mu_i(0) \in \mathbf{B}_\kappa)\right] \ge tN^{-\epsilon} \tag{1.2.14}$$

up to some time $t \le T(\kappa)$, where $\mathbf{B}_\kappa$ denotes the κ-bulk of $X(0)$. For Ginibre initial condition $X(0)$ (1.2.14) was established in [107, Corollary 1.6], we now generalise it to i.i.d. initial conditions. We remark that (1.2.13) is similar to its Hermitian counterpart, the standard Dyson Brownian motion (DBM) on the real line, with some notable differences. In particular, in the latter process the eigenvalues cannot cross each other, hence they are quite rigid and confined to an interval of size essential $1/N$, so they are not diffusive beyond a time-scale $1/N$. Along the evolution (1.2.13) the non-Hermitian eigenvalues still repel each other (encoded in the typically negative off-diagonal overlaps, see [107, Theorem 1.3] in the Gaussian case), but they still can pass by each other and not hindering the diffusive behavior (1.2.14).

Example 1.2.5. *The most prominently and extensively studied [283, 40, 555, 113, 114, 597, 554, 164, 171, 166] deformation is $\Lambda = -z$ with $z \in \mathbf{C}$, since it plays a key role in Girko's formula [283] expressing linear statistics of non-Hermitian eigenvalues of X in terms of the Hermitisation of $X - z$. In this case, the self-consistent equation* (1.2.4) *reduces to the well-known cubic relation*

$$-\frac{1}{m} = w + m - \frac{|z|^2}{w+m}\,.$$

As a consequence, the deterministic terms in (1.2.8) *drastically simplify (e.g., the fractions in* (1.2.8a) *and* (1.2.8b) *are simply replaced by $\langle B\rangle$) and one also has explicit formulas for the bulk* (1.2.6) *in terms of solution of a cubic equation. In particular, for $|z| < 1 - \epsilon_\kappa$, the κ-bulk $\mathbf{B}_\kappa$ consists of a single interval, while for $|z| \ge 1 - \epsilon_\kappa$ it consists of two intervals, where $\epsilon_\kappa \sim \kappa^{2/3}$. In the former case $0 \in \mathbf{B}_\kappa$. Consequently, Theorem 1.2.4 gives the lower bound* (1.2.12) *for all the diagonal overlaps $\mathcal{O}_{ii}$ of eigenvectors of X whose eigenvalue μ_i lies in a disk of radius $1-\epsilon$ with some $\epsilon > 0$ independent of N.*

In the next section we explain the key technical result behind our main theorems, the eigenstate thermalisation for the Hermitisation of $X + \Lambda$.

1.2.2 Eigenstate Thermalisation Hypothesis for the Hermitisation of $X + \Lambda$

The key to access the non-Hermitian singular vectors of $X + \Lambda$ is to study its *Hermitisation* [283], which is defined as

$$H = H^\Lambda := \begin{pmatrix} 0 & X+\Lambda \\ (X+\Lambda)^* & 0 \end{pmatrix} =: W + \hat{\Lambda}\,, \tag{1.2.15}$$

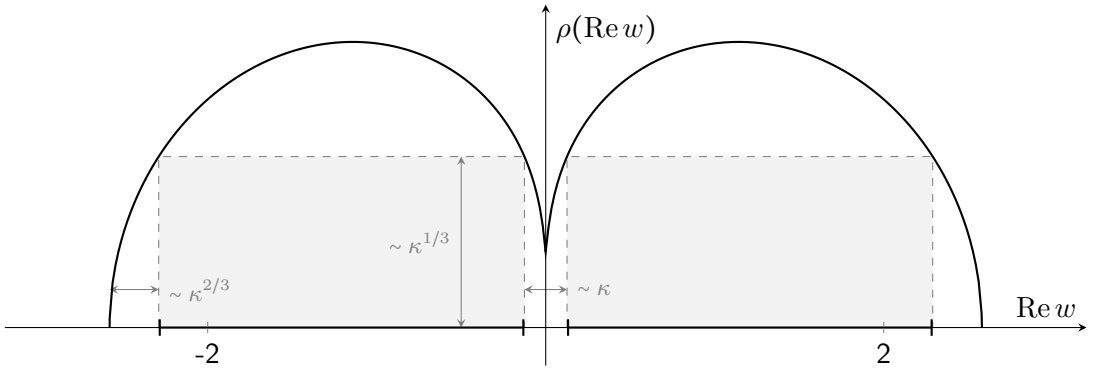

Figure 1.2.1: Depicted is the density ρ for the deformation $\Lambda = -z$ with $|z|$ slightly less than one. On the horizontal axis, we indicated the two components of the bulk $\mathbf{B}_\kappa$. The distance between $\mathbf{B}_\kappa$ and a regular edge scales like $\kappa^{2/3}$, while near an (approximate) cusp the distance between the two components scales linearly (see also (1.2.6) and (1.2.23)).

where $\hat{\Lambda}^* = \hat{\Lambda}$ was defined in (1.2.2) and can also be viewed as the matrix of expectation $\hat{\Lambda} = \mathbf{E}H^\Lambda$.

We denote by $\{\boldsymbol{w}_i\}_{|i|\le N}$ the (normalised) eigenvectors of H and by $\{\lambda_i\}_{|i|\le N}$ the corresponding eigenvalues.[7] By means of the singular value decomposition in (1.2.1), the eigenvalues and eigenvectors of H are related to the singular values and singular vectors of $X + \Lambda$ as follows:

$$\boldsymbol{w}_i = (\boldsymbol{u}_i, \boldsymbol{v}_i)^t \quad \text{and} \quad \lambda_i = \sigma_i \quad \text{for} \quad i \in [N]\,,$$

up to a normalisation, since now $\|\boldsymbol{u}_i\|^2 = \|\boldsymbol{v}_i\|^2 = \frac{1}{2}$. Moreover, the block structure of H induces a symmetric spectrum around zero, i.e. $\lambda_{-i} = -\lambda_i$ for any $i \in [N]$. This symmetry for the eigenvalues is also reflected in the eigenvectors, which satisfy $\boldsymbol{w}_{-i} = E_-\boldsymbol{w}_i$ for any $i \in [N]$. By spectral decomposition, this immediately shows the *chiral symmetry*

$$E_- G(w) = -G(-w)E_-, \qquad \text{with} \quad E_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2.16}$$

for the resolvent $G(w) = G^\Lambda(w) := (H^\Lambda - w)^{-1}$, with spectral parameter $w \in \mathbf{C} \setminus \mathbf{R}$. We also have $\langle G(w)E_- \rangle = 0$ for any w since $\langle \boldsymbol{w}_i, E_-\boldsymbol{w}_i \rangle = \|\boldsymbol{u}_i\|^2 - \|\boldsymbol{v}_i\|^2 = 0$.

A basic feature of a very large class of random matrices is that their resolvent becomes approximately deterministic in the large N limit, often even for any spectral parameter with $|\mathrm{Im}\, w| \ge N^{-1+\epsilon}$; these statements are called *local laws*. In our case the deterministic approximation of the resolvent $G(w)$ is given by

$$M(w) = M^\Lambda(w) := \begin{pmatrix} M_{11}(w) & \frac{\Lambda M_{22}(w)}{w+m(w)} \\ \frac{\Lambda^* M_{11}(w)}{w+m(w)} & M_{22}(w) \end{pmatrix} \in \mathbf{C}^{2N\times 2N}\,, \qquad w \in \mathbf{C} \setminus \mathbf{R}, \tag{1.2.17}$$

with each block being understood as a matrix in $\mathbf{C}^{N\times N}$, where the diagonal entries are defined via

$$M_{11}(w) := \frac{w + m(w)}{\Lambda\Lambda^* - (w + m(w))^2}\,, \qquad M_{22}(w) := \frac{w + m(w)}{\Lambda^*\Lambda - (w + m(w))^2}\,. \tag{1.2.18}$$

Here we require $m(w) = \langle M(w) \rangle$, which is an implicit equation for the function $m(w)$. Simple calculation shows that this implicit equation is exactly (1.2.4). Moreover, one can easily check that $M(w)$ also satisfies the chiral symmetry (1.2.16), i.e.

$$E_- M(w) = -M(-w)E_-\,. \tag{1.2.19}$$

[7] In the definition of the eigenvectors and eigenvalues, we omitted 0 in the set of indices, i.e. $|i| \le N$ really means $i \in \{-N, ..., -1, 1, ..., N\}$.

To derive these formulas systematically, we recall that the deterministic approximation to $G(w)$ is obtained as the unique solution to the *matrix Dyson equation (MDE)* (introduced first in [332] and extensively studied in [16, 17, 22]). The MDE corresponding to the random matrix H is given by

$$-\frac{1}{M(w)} = w - \hat{\Lambda} + \mathcal{S}[M(w)] \tag{1.2.20}$$

under the constraint $\operatorname{Im} M(w) \cdot \operatorname{Im} w > 0$, where $\operatorname{Im} M(w) := \frac{1}{2\mathrm{i}}\big[M(w) - (M(w))^*\big]$. Here $\mathcal{S}[\cdot]$, the *self-energy operator*, is defined via

$$\mathcal{S}[T] := \widetilde{\mathbf{E}}(\widetilde{H} - \mathbf{E}H)T(\widetilde{H} - \mathbf{E}H)$$

for any $T \in \mathbf{C}^{2N\times 2N}$, where $\widetilde{H}$ denotes an independent copy of H. In our case we can write $\mathcal{S}$ in the following two ways

$$\mathcal{S}[T] = 2E_1\langle TE_2\rangle + 2\langle E_1T\rangle E_2 = \sum_{\sigma=\pm}\sigma\langle TE_\sigma\rangle E_\sigma, \tag{1.2.21}$$

with E_1, E_2 defined as in (1.1.9), and $E_\pm = E_1 \pm E_2$. Using $\langle M_{11}(w)\rangle = \langle M_{22}(w)\rangle$ that directly follows from (1.2.18), it is straightforward to check that $M(w)$ as defined in (1.2.17) satisfies the MDE (1.2.20). Since the MDE has a unique solution, we see that the density ρ defined via free convolution in Section 1.2.1 coincides with the *self-consistent density of states (scDos)* corresponding to the MDE, defined as the boundary value of $\frac{1}{\pi}\langle \operatorname{Im} M\rangle$ on the real axis in the theory of MDE [17, 22].

For the reader's convenience in Appendix 1.B.1 we will collect a few facts about M, in particular we will show that it has a continuous extension as a matrix valued function to the real axis, i.e. the limit $M(e) := \lim_{\eta\downarrow 0} M(e+\mathrm{i}\eta)$ exists for any $e \in \mathbf{R}$. This extends the similar result on its trace mentioned in (1.2.5). Moreover, we will also show that for spectral parameters $w \in \mathbf{C}\setminus\mathbf{R}$ with $\operatorname{Re} w \in \mathbf{B}_\kappa$, we have

$$\|M(w)\| \lesssim 1\,. \tag{1.2.22}$$

In particular, together with (1.2.17)–(1.2.18) this implies that the deterministic terms in (1.2.8) are bounded. Finally, we will also prove an important regularity property of the κ-bulk, namely that

$$\operatorname{dist}(\partial\mathbf{B}_{\kappa'}, \mathbf{B}_\kappa) \geq \mathfrak{c}(\kappa - \kappa') \tag{1.2.23}$$

for any small $0 < \kappa' < \kappa$ and some N-independent constant $\mathfrak{c} = \mathfrak{c}(\|\Lambda\|) > 0$. In fact, for our proof it is sufficient if $\mathfrak{c} = \mathfrak{c}(\kappa, \|\Lambda\|)$, i.e. an additional κ dependence is allowed – in Appendix 1.B.1 we will explain that this weaker result is considerably easier to obtain (see Remark 1.B.3). We will also show that $\mathbf{B}_\kappa$ is a finite disjoint union of compact intervals; the number of these *components* depends only on κ and $\|\Lambda\|$.

The above mentioned concentration of G around M is the content of the following single resolvent *local law*, both in *averaged* and *isotropic form*, which we prove in Appendix 1.C.

Theorem 1.2.6. (Single resolvent local law for the Hermitisation H)
Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N\times N}$ and $\kappa > 0$ independent of N. Then, for any $w \in \mathbf{C}\setminus\mathbf{R}$ with $|w| \leq N^{100}$ and $\operatorname{Re} w \in \mathbf{B}_\kappa$, we have

$$|\langle (G(w) - M(w))B\rangle| \prec \frac{1}{N\eta}\,, \qquad |\langle \boldsymbol{x}, (G(w) - M(w))\boldsymbol{y}\rangle| \prec \frac{1}{\sqrt{N\eta}}\,,$$

where $\eta := |\operatorname{Im} w|$, for any bounded deterministic matrix $\|B\| \lesssim 1$ and vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$.

Our main result for the Hermitised random matrix H is the *Eigenstate Thermalisation Hypothesis (ETH)*, that in mathematical terms is the proof of an optimal convergence rate of the strong *Quantum Unique Ergodicity (QUE)* for general observables A, uniformly in the *bulk* (1.2.6) of the spectrum of H, i.e. in the bulk of the scDos ρ.

Theorem 1.2.7. (Eigenstate Thermalisation Hypothesis for the Hermitisation H)
Fix some bounded $\Lambda \in \mathbf{C}^{N\times N}$ and $\kappa > 0$ independent of N. Let $\{\boldsymbol{w}_i\}_{|i|\le N}$ be the orthogonal eigenvectors of the Hermitisation H of $X+\Lambda$, where X is an i.i.d. matrix satisfying Assumption 1.2.1. Then, for any deterministic matrix $A \in \mathbf{C}^{2N\times 2N}$ with $\|A\| \lesssim 1$ it holds that

$$\max_{i,j}\left|\langle \boldsymbol{w}_i, A\boldsymbol{w}_j\rangle - \delta_{j,i}\frac{\langle \operatorname{Im} M(\gamma_j)A\rangle}{\langle \operatorname{Im} M(\gamma_j)\rangle} - \delta_{j,-i}\frac{\langle \operatorname{Im} M(\gamma_j)E_- A\rangle}{\langle \operatorname{Im} M(\gamma_j)\rangle}\right| \prec \frac{1}{\sqrt{N}}\,, \tag{1.2.24}$$

where the maximum is taken over all $|i|,|j| \le N$, such that the quantiles $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$ defined in (1.2.7) *are in the bulk of the scDos ρ.*

The main technical result underlying Theorem 1.2.7 is an averaged local law for *two* resolvents with different spectral parameters, which we will formulate in Theorem 1.4.4 later.

Remark 1.2.8. *Given the optimal bound* (1.2.24)*, following a Dyson Brownian Motion (DBM) analysis similar to [167, 169], it is possible to prove a CLT for single diagonal overlaps $\langle \boldsymbol{w}_i, A\boldsymbol{w}_i\rangle$. However, for the sake of brevity, we do not present this argument here and defer the CLT analysis to future work.*

In the following Section 1.3 we precisely define the regularisation and we will prove our main results formulated above assuming the key technical Proposition 1.3.4. This proposition is obtained from a *local law*, which we prove in Section 1.4. Local laws are proved by a hierarchy of *master and reduction inequalities*, that are derived in Sections 1.5 and 1.6, respectively. Appendix 1.A contains two motivating calculations for the correct regularisation. Several technical and auxiliary results are deferred to the other appendices.

1.3 Proof of the main results

The key to understanding the eigenvector overlaps and showing our main results is an improved bound on the averaged trace of *two* resolvents with *regular* (see Section 1.3.1 below for the precise definition) deterministic matrices A_1, A_2 in between, i.e. for

$$\langle G(w_1)A_1G(w_2)A_2\rangle\,. \tag{1.3.1}$$

Naively, for arbitrary A_1, A_2, estimating (1.3.1) via a trivial Schwarz inequality and Ward identity yields the upper bound $|\langle G(w_1)A_1G(w_2)A_2\rangle| \prec 1/\eta$, where $\eta := \min_j |\operatorname{Im} w_j|$. However, this bound drastically improves, whenever the matrices A_1, A_2 are *regular*, i.e. orthogonal to certain *critical* eigenvectors $V_\pm$ of the associated two-body stability operators (1.B.2), which is denoted as $A_j = \mathring{A}_j$; see (1.3.2) and Definitions 1.3.1 and 1.4.2. In this case, in our key Proposition 1.3.4 we will show that

$$|\langle G(w_1)\mathring{A}_1G(w_2)\mathring{A}_2\rangle| \prec 1$$

even for very small $\eta \sim N^{-1+\epsilon}$ as a consequence of a more precise *local law* for (1.3.1), which we present in Section 1.4. We find that (see Theorem 1.4.4 and Remark 1.4.6) both the size of its deterministic approximation and the fluctuation around this mean heavily depend on whether (one or both of) the matrices A_1, A_2 are regular, i.e. satisfy $\langle V_\pm, A_j\rangle = 0$, or not.

Therefore, the general rather *structural* regularizing decomposition (or *regularisation*) of a matrix A shall be conducted as

$$\boxed{A^\circ \equiv \mathring{A} := A - \langle V_+, A\rangle U_+ - \langle V_-, A\rangle U_-} \tag{1.3.2}$$

for $U_\sigma, V_\sigma \in \mathbf{C}^{2N\times 2N}$ satisfying $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$ and the normalisation $\langle U_\sigma, U_\sigma\rangle = 1$, where recall that $\langle R, T\rangle := \langle R^*T\rangle$ denotes the (normalised) Hilbert-Schmidt scalar product. The *regularisation map*

$$(1-\Pi) : \mathbf{C}^{2N\times 2N} \to \mathbf{C}^{2N\times 2N}\,, \quad A \mapsto \mathring{A}\,, \tag{1.3.3}$$

where Π is a *two-dimensional* (non-orthogonal) projection,[8] is closely related to the built-in chiral symmetry (1.2.16) of our model. Indeed, for other ensembles without this special structure only *one* of the terms $\langle V_\sigma, A\rangle U_\sigma$ in (1.3.2) would be present.

As mentioned above, the matrices $V_\pm$ are determined as *critical* eigenvectors (with corresponding small eigenvalue) of naturally associated two-body stability operators with their precise form worked out in Appendix 1.A and given in (1.A.19). In Appendix 1.A we also give two different calculations that helped us guess these formulas. However, for the directions $U_\pm$ there are *a priori* no further constraints (apart from orthogonality and normalisation). Hence, as it turns out to be convenient for our proofs, we will *choose* the matrices U_σ (in principle, even allowing for two different deformations $\Lambda_1 \neq \Lambda_2$) in such a way, that a resolvent identity

$$G^{\Lambda_1}(w_1)U_\sigma G^{\Lambda_2}(w_2) \approx \big(G^{\Lambda_1}(w_1) - G^{\Lambda_2}(\sigma w_2)\big)U_\sigma\,, \tag{1.3.4}$$

can be applied (here, the symbol '≈' neglects lower order terms). This is used to reduce the number of resolvents in a chain. Note that, again due to the eminent chiral symmetry (1.2.16) for the resolvents, there are in fact *two* matrices U_σ for which a resolvent identity (1.3.4) can be applied.

Although the regularisation (1.3.2) shall be motivated for arbitrary deformations Λ_1, Λ_2 in Appendix 1.A, we will henceforth choose a single bounded deformation $\Lambda \in \mathbf{C}^{N\times N}$, which remains fixed with the just mentioned exception in Appendix 1.A. For a single deformation Λ, this restricts the matrices $U_\pm$ satisfying (1.3.4) to be given by $E_\pm$.

In case that the spectral parameters (w_1, w_2) appearing in (1.3.1) (with a single fixed deformation Λ) are such that *none* of the eigenvectors of the stability operator is *critical* (cf. Appendix 1.B), we consider *every* matrix A as regular. The distinction between these two scenarios is regulated by cutoff functions $\mathbf{1}_\delta^\pm$ introduced in (1.3.6) below.

1.3.1 Regular observables: A bound on (1.3.1)

As already mentioned above, our main result for the Hermitised random matrix, Theorem 1.2.7, shall be derived from a bound on (1.3.1), where we assume the (real parts of the) spectral parameters w_1, w_2 to be in the *bulk* of the scDos ρ (recall (1.2.6)).

We now specify the concept of regularisation (1.3.2) to our setting. The eigenvectors $V_\pm$ will be computed in Appendix 1.A, the matrices $U_\pm$ are simply chosen as $E_\pm$.

Definition 1.3.1. (Regular observables) *Given $\kappa > 0$, let*[9]

$$\delta = \delta(\kappa, \|\Lambda\|) > 0 \tag{1.3.5}$$

be sufficiently small (to be chosen below, see (1.4.22)) and let $w, w' \in \mathbf{C}\setminus\mathbf{R}$ with $\operatorname{Re} w, \operatorname{Re} w' \in \mathbf{B}_\kappa$ be spectral parameters. Furthermore, we introduce the (symmetric) cutoff functions

$$\mathbf{1}_\delta^\pm(w, w') := \phi_\delta(\operatorname{Re} w \mp \operatorname{Re} w')\ \phi_\delta(\operatorname{Im} w)\ \phi_\delta(\operatorname{Im} w')\,, \tag{1.3.6}$$

where $0 \le \phi_\delta \le 1$ is a smooth symmetric bump function on $\mathbf{R}$ satisfying $\phi_\delta(x) = 1$ for $|x| \le \delta/2$ and $\phi_\delta(x) = 0$ for $|x| \ge \delta$.

[8] The condition $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$ guarantees that the regularisation is idempotent, i.e. $(\mathring{A})^\circ = \mathring{A}$ and $\Pi^2 = \Pi$.

[9] Note that the parameter $\delta > 0$ is independent of the matrix size N.

(a) We define the (w,w')-regular component *or* (w,w')-regularisation $\mathring{A}^{w,w'}$ *of a matrix* A *as* [10]

$$\boxed{\mathring{A}^{w,w'} := A - \sum_{\tau=\pm} \mathbf{1}_\delta^{\tau\mathfrak{s}}(w,w') \frac{\langle M(\mathrm{Re}\, w + \mathrm{i}\mathrm{Im}\, w) A M(\mathrm{Re}\, w' + \tau\mathrm{i}\mathrm{Im}\, w') E_{\tau\mathfrak{s}} \rangle}{\langle M(\mathrm{Re}\, w + \mathrm{i}\mathrm{Im}\, w) E_{\tau\mathfrak{s}} M(\mathrm{Re}\, w' + \tau\mathrm{i}\mathrm{Im}\, w') E_{\tau\mathfrak{s}} \rangle} E_{\tau\mathfrak{s}}\,,} \tag{1.3.7}$$

where the relative sign of the imaginary parts is defined as

$$\mathfrak{s} \equiv \mathfrak{s}_{w,w'} := -\operatorname{sgn}(\mathrm{Im}\, w\, \mathrm{Im}\, w')\,. \tag{1.3.8}$$

(b) We say that A *is* (w,w')-regular *if and only if* $A = \mathring{A}^{w,w'}$.

The regularisation shall be revisited in Definition 1.4.2, where we tailor it to certain averaged (1.4.5) or isotropic (1.4.6) resolvent chains.

Remark 1.3.2. *We have several comments concerning the above definition.*

(i) In case that at least one of the spectral parameters is away from the imaginary axis, say $|\mathrm{Re}\, w| > \delta$ *w.l.o.g., then the regularisation in* (1.3.7) *contains at most one summand: If* $\mathbf{1}_\delta^+(w,w') = 1$, *i.e.* $\mathrm{Re}\, w$ *is close to* $\mathrm{Re}\, w'$, *then we have that*

$$\mathring{A}^{w,w'} := A - \frac{\langle M(w) A M(\mathrm{Re}\, w' + \mathfrak{s}\mathrm{i}\mathrm{Im}\, w') \rangle}{\langle M(w) M(\mathrm{Re}\, w' + \mathfrak{s}\mathrm{i}\mathrm{Im}\, w') \rangle} E_+\,,$$

whereas if $\mathbf{1}_\delta^-(w,w') = 1$, *i.e. if* $\mathrm{Re}\, w$ *is close to* $-\mathrm{Re}\, w'$, *then we have that*

$$\mathring{A}^{w,w'} := A - \frac{\langle M(w) A E_- M(-\mathrm{Re}\, w' + \mathfrak{s}\mathrm{i}\mathrm{Im}\, w') \rangle}{\langle M(w) M(-\mathrm{Re}\, w' + \mathfrak{s}\mathrm{i}\mathrm{Im}\, w') \rangle} E_-\,,$$

where we used that $M(w)E_- = -E_- M(-w)$ *(see* (1.2.19)*)*

(ii) The cutoff functions in (1.3.6) *satisfy the basic symmetry properties*

$$\mathbf{1}_\delta^\pm(w,w') = \mathbf{1}_\delta^\pm(\bar{w},w') = \mathbf{1}_\delta^\pm(w,\bar{w}') = \mathbf{1}_\delta^\pm(\bar{w},\bar{w}')\,.$$

However, $\mathring{A}$ *is* not *symmetric in its two spectral parameters, i.e.* $\mathring{A}^{w,w'} \neq \mathring{A}^{w',w}$ *in general*

(iii) For spectral parameters satisfying $\mathbf{1}_\delta^\pm(w,w') > 0$, *it will be shown in Appendix 1.B that the respective denominators in* (1.3.7) *are bounded away from zero. In particular, the linear map* $A \mapsto \mathring{A}$ *is bounded with a bound depending only on* δ *and* $\|\Lambda\|$.

(iv) Whenever it holds that $\mathbf{1}_\delta^\pm(w,w') = 0$ *then also* $\mathbf{1}_{\delta'}^\pm(w,w') = 0$ *for every* $0 < \delta' < \delta$. *Conversely, whenever it holds that* $\mathbf{1}_\delta^\pm(w,w') = 1$ *then also* $\mathbf{1}_{\delta'}^\pm(w,w') = 1$ *for every* $0 < \delta < \delta'$.

(v) We point out that the notion of regularity implicitly depends on κ *and* δ *and hence also on the (norm of the) deformation* Λ.

Moreover, the regularisation defined above satisfies the following elementary properties. The identities in (1.3.10) and (1.3.9) are immediate from the definition, the perturbative statements are proven in Appendix 1.B.

[10] Putting the summation parameter τ at the second spectral parameter w' (and not at w) is simply a free choice, which we made here. More precisely, defining the regularisation as

$$\mathring{A}^{w,w'} := A - \sum_{\tau=\pm} \mathbf{1}_\delta^{\tau\mathfrak{s}}(w,w') \frac{\langle M(\mathrm{Re}\, w + \tau\mathrm{i}\mathrm{Im}\, w) A M(\mathrm{Re}\, w' + \mathrm{i}\mathrm{Im}\, w') E_{\tau\mathfrak{s}} \rangle}{\langle M(\mathrm{Re}\, w + \tau\mathrm{i}\mathrm{Im}\, w) E_{\tau\mathfrak{s}} M(\mathrm{Re}\, w' + \mathrm{i}\mathrm{Im}\, w') E_{\tau\mathfrak{s}} \rangle} E_{\tau\mathfrak{s}}$$

would equally work in our proofs (see Appendices 1.A and 1.B for details).

Lemma 1.3.3. *Fix a bounded deterministic deformation $\Lambda \in \mathbf{C}^{N\times N}$ and let $A \in \mathbf{C}^{2N\times 2N}$ be an arbitrary bounded deterministic matrix.*

(i) Let $w, w' \in \mathbf{C}\setminus\mathbf{R}$ with $\operatorname{Re} w, \operatorname{Re} w' \in \mathbf{B}_\kappa$. Then, we have the identities

$$\big(\mathring{A}^{w,w'}\big)^* = (\mathring{A}^*)^{\bar{w}',\bar{w}}, \quad \mathring{A}^{w,w'}E_- = (A\mathring{E}_-)^{w,-w'}, \quad E_-\mathring{A}^{w,w'} = (\mathring{E_-}A)^{-w,w'}. \tag{1.3.9}$$

(ii) Moreover, by definition it holds that

$$\mathring{A}^{w,\bar{w}'} = \mathring{A}^{w,w'}, \tag{1.3.10}$$

and setting $\mathfrak{s} := -\operatorname{sgn}(\operatorname{Im} w \operatorname{Im} w')$, we have the perturbative estimate[11]

$$\mathring{A}^{\bar{w},w'} = \mathring{A}^{w,w'} + \mathcal{O}(|w-\mathfrak{s}\bar{w}'|\wedge 1)E_{\mathfrak{s}} + \mathcal{O}(|w+\mathfrak{s}w'|\wedge 1)E_{-\mathfrak{s}}. \tag{1.3.11}$$

(iii) Let $w_1, w_1', w_2, w_2' \in \mathbf{C}\setminus\mathbf{R}$ with $\operatorname{Re} w_1, \operatorname{Re} w_1', \operatorname{Re} w_2, \operatorname{Re} w_2' \in \mathbf{B}_\kappa$ as well as $\operatorname{Im} w_1 \cdot \operatorname{Im} w_2 > 0$ and $\operatorname{Im} w_1' \cdot \operatorname{Im} w_2' > 0$ be spectral parameters. Then we have that

$$\mathring{A}^{w_2,w_1'} = \mathring{A}^{w_1,w_1'} + \mathcal{O}\big(|w_1-w_2|\wedge 1\big)E_+ + \mathcal{O}\big(|w_1-w_2|\wedge 1\big)E_-, \tag{1.3.12}$$

$$\mathring{A}^{w_1,w_2'} = \mathring{A}^{w_1,w_1'} + \mathcal{O}\big(|w_1'-w_2'|\wedge 1\big)E_+ + \mathcal{O}\big(|w_1'-w_2'|\wedge 1\big)E_-. \tag{1.3.13}$$

We can now state the bound on (1.3.1) for regular observables, which shall be proven in Section 1.4 as an immediate corollary to a local law for (1.3.1) given in Theorem 1.4.4 and the bound from Lemma 1.4.3.

Proposition 1.3.4. *Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N\times N}$, $\epsilon > 0$, $\kappa > 0$, and let $w_1, w_2 \in \mathbf{C}$ with $|w_1|, |w_2| \le N^{100}$, $\operatorname{Re} w_1, \operatorname{Re} w_2 \in \mathbf{B}_\kappa$, and $|\operatorname{Im} w_1|, |\operatorname{Im} w_2| \ge N^{-1+\epsilon}$. Moreover, let $A_1 \in \mathbf{C}^{2N\times 2N}$ be a (w_1,w_2)*-regular *and $A_2 \in \mathbf{C}^{2N\times 2N}$ a (w_2,w_1)*-regular *deterministic matrix, both satisfying $\|A_1\|, \|A_2\| \lesssim 1$. Then we have*

$$\big|\big\langle G(w_1)\mathring{A}_1^{w_1,w_2}G(w_2)\mathring{A}_2^{w_2,w_1}\big\rangle\big| \prec 1. \tag{1.3.14}$$

1.3.2 Estimating (1.3.1) for general observables

Armed with the correct regularisation, we can now present a systematic analysis of $\langle G(w_1)A_1G(w_2)A_2\rangle$ from (1.3.1) for *arbitrary* bounded deterministic matrices A_1, A_2. Decomposing A_1, A_2 according to Definition 1.3.1 as

$$\begin{aligned} A_1 &= \mathring{A}_1^{w_1,w_2} + \langle\!\langle A_1\rangle\!\rangle^+_{w_1,w_2}E_+ + \langle\!\langle A_1\rangle\!\rangle^-_{w_1,w_2}E_-, \\ A_2 &= \mathring{A}_2^{w_2,w_1} + \langle\!\langle A_2\rangle\!\rangle^+_{w_2,w_1}E_+ + \langle\!\langle A_2\rangle\!\rangle^-_{w_2,w_1}E_-, \end{aligned} \tag{1.3.15}$$

(where $\langle\!\langle\cdot\rangle\!\rangle^\sigma_{w,w'}$ can be read off as the coefficients in (1.3.7)) and plugging (1.3.15) into (1.3.1), we find that

$$\begin{aligned} \big\langle G(w_1)A_1G(w_2)A_2\big\rangle = &\sum_{\sigma,\tau}\langle\!\langle A_1\rangle\!\rangle^\sigma_{w_1,w_2}\langle\!\langle A_2\rangle\!\rangle^\tau_{w_2,w_1}\big\langle G(w_1)E_\sigma G(w_2)E_\tau\big\rangle \\ &+ \sum_\sigma \langle\!\langle A_1\rangle\!\rangle^\sigma_{w_1,w_2}\big\langle G(w_1)E_\sigma G(w_2)\mathring{A}_2^{w_2,w_1}\big\rangle \\ &+ \sum_\tau \langle\!\langle A_2\rangle\!\rangle^\tau_{w_2,w_1}\big\langle G(w_1)\mathring{A}_1^{w_1,w_2}G(w_2)E_\tau\big\rangle \\ &+ \big\langle G(w_1)\mathring{A}_1^{w_1,w_2}G(w_2)\mathring{A}_2^{w_2,w_1}\big\rangle. \end{aligned} \tag{1.3.16}$$

[11] Note that the asymmetry between (1.3.11) and (1.3.10) stems from the asymmetry imposed in the definition of the regularisation, namely by placing the summation index τ in (1.3.7) at the second spectral parameter.

Which terms in (1.3.16) are effectively present depends on the coefficients $\langle\langle\cdot\rangle\rangle^{\sigma}_{w,w'}$, i.e. on the singular components of A_1, A_2. For terms with nonzero coefficients the following rule of thumb applies. Denoting $\eta := \min\big(|\mathrm{Im}\, w_1|, |\mathrm{Im}\, w_2|\big) \geq N^{-1+\epsilon}$, the terms $\langle GEGE\rangle$ in the first line of (1.3.16) are bounded by $1/\eta$, the terms $\langle GEG\mathring{A}\rangle$ in the two middle lines of (1.3.16) are bounded by $1/\sqrt{\eta}$, and $\langle G\mathring{A}G\mathring{A}\rangle$ in the last line is of order one (Proposition 1.3.4). This is in perfect agreement with the $\sqrt{\eta}$-*rule* mentioned in the Introduction (see also Remark 1.4.6 below). Some of these bounds are actually sharp for special values of w_1, w_2, for example

$$\langle G(w)E_+G(\bar{w})E_+\rangle = \frac{\langle \mathrm{Im}\, G(w)\rangle}{\eta} \sim \frac{1}{\eta}, \qquad \text{or} \quad \langle G(w)E_-G(-\bar{w})E_-\rangle = -\frac{\langle \mathrm{Im}\, G(w)\rangle}{\eta},$$

where we used the chiral symmetry (1.2.16). In fact, two terms with $\sigma\tau = -1$ in the first line of (1.3.16) are identically zero by applying the chiral symmetry, followed by the resolvent identity and $\langle GE_-\rangle = 0$. For a middle term in (1.3.16) we have

$$\langle G(w)E_+G(\bar{w})\mathring{A}^{\bar{w},w}\rangle = \frac{1}{\eta}\langle \mathrm{Im}\, G(w)\mathring{A}^{\bar{w},w}\rangle \lesssim 1 + \frac{1}{N\eta}\frac{1}{\sqrt{\eta}}.$$

In the very last relation we treated $\langle G(w)\mathring{A}^{\bar{w},w}\rangle$ and $\langle G(\bar{w})\mathring{A}^{\bar{w},w}\rangle$ separately. In both cases we first used Lemma 1.3.3 to adjust the regularisation to $\mathring{A}^{w,w}$ and $\mathring{A}^{\bar{w},\bar{w}}$, respectively, to match the new single-resolvent setup and then we applied the corresponding single-resolvent local law with regular observable (see Theorem 1.4.5 below).

Note that the most critical estimate concerns the last line of (1.3.16), i.e. the regular part for both observable matrices. The bound (1.3.14) is obtained from a local law with *two* resolvents and *two* regular matrices, while the first and the middle terms in (1.3.16) can be understood already from an improved local law for *one* resolvent and *one* regular matrix (see Theorem 1.4.5 below) after applying resolvent identities and adjusting the regularisation by Lemma 1.3.3. Furthermore, observe that the sizes of the first three lines in (1.3.16) are sensitive to w_1, w_2 via the usual resolvent identity (see (1.3.21) below) and the chiral symmetry (1.2.16), for example

$$|\langle G(w_1)E_+G(w_2)E_+\rangle| = \left|\frac{\langle G(w_1) - G(w_2)\rangle}{w_1 - w_2}\right| \lesssim \frac{1}{|w_1 - w_2|},$$
$$\text{or} \quad |\langle G(w_1)E_-G(w_2)E_-\rangle| = |\langle G(w_1)G(-w_2)\rangle| \lesssim \frac{1}{|w_1 + w_2|},$$

while the last line in (1.3.16) is typically order one.

Summarizing, the singular parts of $\langle G(w_1)A_1G(w_2)A_2\rangle$ can be explicitly computed (using single-resolvent local laws) as explicit functions of w_1, w_2, while the regular part remains of order one. A combination of our decomposition (1.3.7), the perturbation formulas from Lemma 1.3.3, and our single- and two-resolvent local laws together with their explicit deterministic terms from the subsequent Section 1.4 provide an effective recipe to compute $\langle G(w_1)A_1G(w_2)A_2\rangle$ with high precision in all cases. We refrain from formulating it as a comprehensive theorem due to the large number of cases.

1.3.3 Proof of the main results

We will first focus on the proof of Theorem 1.2.7 and turn to the proofs of Theorem 1.2.2 and Theorem 1.2.4 afterwards.

1.3.3.1 Proof of Theorem 1.2.7

As a first step towards the proof of Theorem 1.2.7, we show that (1.2.24) indeed follows from a bound similar to (1.3.14), where G is replaced by $\mathrm{Im}\, G$. The proof of the following simple lemma is given after completion of the proof of Theorem 1.2.7.

Lemma 1.3.5. *Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N\times N}$, $\epsilon > 0$, $\kappa > 0$, and let $B \in \mathbf{C}^{2N\times 2N}$. Then, for any bulk indices $|i|, |j| \le N$, i.e. with $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$, and $\eta \ge N^{-1+\epsilon}$, we have*

$$N\,|\langle \boldsymbol{w}_i, B\boldsymbol{w}_j\rangle|^2 \prec (N\eta)^2 \langle \operatorname{Im} G(\gamma_i + \mathrm{i}\eta) B \operatorname{Im} G(\gamma_j + 2\mathrm{i}\eta) B^*\rangle\,. \tag{1.3.17}$$

The same bound still holds without the factor of two in the argument of the second resolvent in (1.3.17). However, we chose to have it, in order to ensure that the spectral parameters of the two resolvents are always forced to be different.

Proof of Theorem 1.2.7. Having Lemma 1.3.5 at hand, we are left with estimating the rhs. of (1.3.17) for

$$B = A - \frac{\langle \operatorname{Im} M(\gamma_j) A\rangle}{\langle \operatorname{Im} M(\gamma_j)\rangle} E_+ - \frac{\langle \operatorname{Im} M(\gamma_j) E_- A\rangle}{\langle \operatorname{Im} M(\gamma_j)\rangle} E_- \tag{1.3.18}$$

using Proposition 1.3.4. Note that the two terms in (1.2.24) carrying a δ-symbol arise from the orthogonality relations $\langle \boldsymbol{w}_i, E_\pm \boldsymbol{w}_j\rangle = \delta_{j,\pm i}$, following from the spectral symmetry described around (1.2.16).

We now write out $\operatorname{Im} G(w) = (G(w) - G(\bar w))/(2\mathrm{i})$, such that (1.3.17) leaves us with four different terms, each of which can be bounded individually. Since their treatment is completely analogous, we focus on the exemplary term

$$\langle G(\gamma_i + \mathrm{i}\eta) B G(\gamma_j - 2\mathrm{i}\eta) B^*\rangle \tag{1.3.19}$$

with the deterministic matrix B being defined in (1.3.18). We rely on the following simple perturbative lemma, which follows from Lemma 1.3.3 by invoking Lemma 1.B.4.

Lemma 1.3.6. *Using the notation introduced in* (1.3.7)*, the matrix $B \in \mathbf{C}^{2N\times 2N}$ from* (1.3.18) *satisfies*

$$\begin{aligned} B &= \mathring{A}^{\gamma_i+\mathrm{i}\eta, \gamma_j - 2\mathrm{i}\eta} + \mathcal{O}\big(|\gamma_i - \gamma_j| + \eta\big) E_+ + \mathcal{O}\big(|\gamma_i + \gamma_j| + \eta\big) E_-\,, \\ B^* &= (\mathring{A}^*)^{\gamma_j - 2\mathrm{i}\eta, \gamma_i + \mathrm{i}\eta} + \mathcal{O}\big(|\gamma_i - \gamma_j| + \eta\big) E_+ + \mathcal{O}\big(|\gamma_i + \gamma_j| + \eta\big) E_-\,. \end{aligned} \tag{1.3.20}$$

Hence, plugging (1.3.20) into (1.3.19), we get a sum of several terms, which can all be estimated separately. For the 'leading term', we use Proposition 1.3.4 to get that

$$\Big|\big\langle G(\gamma_i + \mathrm{i}\eta) \mathring{A}^{\gamma_i+\mathrm{i}\eta, \gamma_j - 2\mathrm{i}\eta} G(\gamma_j - 2\mathrm{i}\eta) (\mathring{A}^*)^{\gamma_j - 2\mathrm{i}\eta, \gamma_i + \mathrm{i}\eta}\big\rangle\Big| \prec 1\,.$$

Two further representative terms are given by

$$\mathcal{O}\big(|\gamma_i \mp \gamma_j| + \eta\big)\, \langle G(\gamma_i + \mathrm{i}\eta) E_\pm G(\gamma_j - 2\mathrm{i}\eta) C\rangle\,,$$

where $C \in \mathbf{C}^{2N\times 2N}$ is some generic bounded matrix. Now, by using (1.2.16), these terms can be rewritten as

$$\mathcal{O}\big(|\gamma_i \mp \gamma_j| + \eta\big)\, \langle G(\gamma_i + \mathrm{i}\eta) G(\pm(\gamma_j - 2\mathrm{i}\eta)) E_\pm C\rangle\,.$$

For either sign choice (due to the factor two), we can now employ a simple resolvent identity

$$G(w_1) G(w_2) = \frac{G(w_1) - G(w_2)}{w_1 - w_2}\,, \tag{1.3.21}$$

leaving us with

$$\frac{\mathcal{O}\big(|\gamma_i - \gamma_j| + \eta\big)}{(\gamma_i \mp \gamma_j) + (1 \pm 2)\mathrm{i}\eta} \langle [G(\gamma_i + \mathrm{i}\eta) - G(\pm(\gamma_j - 2\mathrm{i}\eta))] C\rangle\,,$$

which is surely stochastically dominated by one by means of Theorem 1.2.6. Thus, collecting all the terms, we find that $|(1.3.19)| \prec 1$.

Finally, we choose $\eta = N^{-1+\xi}$ for an arbitrarily small $\xi > 0$, such that Lemma 1.3.5 with B as in (1.3.18) yields Theorem 1.2.7. □

We conclude with giving a proof of Lemma 1.3.5.

Proof of Lemma 1.3.5. By spectral decomposition we write

$$\begin{aligned}\langle \operatorname{Im} G(\gamma_i + \mathrm{i}\eta) B \operatorname{Im} G(\gamma_j + 2\mathrm{i}\eta) B^* \rangle &= \frac{1}{2N} \sum_{k,l} \frac{2\eta^2 |\langle \boldsymbol{w}_k, B\boldsymbol{w}_l \rangle|^2}{[(\lambda_k - \gamma_i)^2 + \eta^2][(\lambda_l - \gamma_j)^2 + 4\eta^2]} \\ &\succ \frac{|\langle \boldsymbol{w}_i, B\boldsymbol{w}_j \rangle|^2}{N\eta^2},\end{aligned}$$

which proves (1.3.17). We point out that in the last inequality we used rigidity of the eigenvalues [17, 243]:

$$|\lambda_i - \gamma_i| \prec \frac{1}{N}, \tag{1.3.22}$$

which holds for bulk indices as a standard consequence of the single-resolvent local law, Theorem 1.2.6. □

1.3.3.2 Proof of Theorem 1.2.2

The bounds in (1.2.8a), (1.2.8b), and (1.2.8c) follow from Theorem 1.2.7 by choosing

$$A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix},$$

respectively, and invoking (1.2.17)–(1.2.18). □

1.3.3.3 Proof of Theorem 1.2.4

By the definition

$$H^z := \begin{pmatrix} 0 & X + \Lambda - z \\ (X + \Lambda - z)^* & 0 \end{pmatrix}$$

it follows that $\mu \in \operatorname{Spec}(X + \Lambda)$ if and only if $\lambda_1^\mu = 0$. Here by $\{\lambda_i^z\}_{i \in [N]}$ we denoted the increasingly ordered non–negative eigenvalues of H^z. We remark that Λ is omitted by the notation since it is fixed throughout the proof. In particular, using the bound for products of two resolvents and two regular matrices in (1.3.14), we will now prove the lower bound in (1.2.12) for the overlap of left and right eigenvectors corresponding to eigenvalues μ which lies in the bulk of the spectrum of $X + \Lambda$.

Proof of Theorem 1.2.4. Define

$$F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathbf{C}^{2N \times 2N},$$

then by (1.3.14), for $\eta \geq N^{-1}$, we conclude

$$\sup_{z \in \mathrm{bulk}} \langle \operatorname{Im} G^z(\mathrm{i}\eta) F \operatorname{Im} G^z(\mathrm{i}\eta) F^* \rangle \prec 1, \tag{1.3.23}$$

where the supremum is taken over the bulk as given in Definition 1.2.3. The fact that (1.3.23) holds for the supremum over the z's with very high probability follows by a standard grid argument together with the Lipschitz continuity of $z \mapsto \operatorname{Im} G^z$. Here we used that F is regular in the sense of (1.3.7); this immediately follows from the fact that F is (block) off–diagonal and $\operatorname{Im} M(\mathrm{i}\eta)$ is (block) diagonal (see Lemma 1.B.1). We now want to show that if we choose $z = \mu_i$ to be a bulk

eigenvalue of $X + \Lambda$ the upper bound (1.3.23) implies a lower bound on $\mathcal{O}_{ii}$. To make the notation simpler, from now on we denote $\mu = \mu_i$.

Consider the non-Hermitian left/right–eigenvectors $\boldsymbol{l}, \boldsymbol{r}$, with corresponding eigenvalue μ, defined as in (1.2.9) and set

$$\mathcal{P} := \begin{pmatrix} \frac{\bar{\boldsymbol{l}}\bar{\boldsymbol{l}}^*}{\|\boldsymbol{l}\|^2} & 0 \\ 0 & \frac{\boldsymbol{r}\boldsymbol{r}^*}{\|\boldsymbol{r}\|^2} \end{pmatrix}.$$

Clearly $\mathcal{P}$ is a rank two orthogonal projection whose range lies in the kernel of H^μ, recalling that up to scalar multiples the non-Hermitian eigenvectors $\bar{\boldsymbol{l}}, \boldsymbol{r}$ coincide with some singular vectors $\boldsymbol{u}, \boldsymbol{v}$ of $X + \Lambda - \mu$, respectively, forming an eigenvector $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v})$ in the kernel of H^μ. Note that $\mathrm{Ker}(H^\mu)$ has dimension two if μ is a simple eigenvalue, but in general the multiplicity of μ and the multiplicity of $\lambda_1^\mu = 0$ may differ. Let $\mathcal{Q}$ be the orthogonal projection onto the kernel of H^μ, then $\mathcal{P} \leq \mathcal{Q}$. Then, almost surely, by spectral decomposition (and by the spectral symmetry of H^μ)

$$\operatorname{Im} G^\mu(\mathrm{i}\eta) = \frac{\mathcal{Q}}{\eta} + \sum_{i:\lambda_i^\mu \neq 0} \frac{\eta}{(\lambda_i^\mu)^2 + \eta^2} \begin{pmatrix} \boldsymbol{u}_i^\mu \\ \boldsymbol{v}_i^\mu \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_i^\mu \\ \boldsymbol{v}_i^\mu \end{pmatrix}^* \geq \frac{\mathcal{P}}{\eta}.$$

By (1.3.23) we thus obtain

$$1 > \sup_{z \in \mathrm{bulk}} \langle \operatorname{Im} G^z(\mathrm{i}\eta) F \operatorname{Im} G^z(\mathrm{i}\eta) F^* \rangle > \frac{1}{\eta^2} \langle \mathcal{P} F \mathcal{P} F^* \rangle = \frac{\left| \langle \bar{\boldsymbol{l}}, \boldsymbol{r} \rangle \right|^2}{N\eta^2 \|\boldsymbol{r}\|^2 \|\boldsymbol{l}\|^2},$$

which, by (1.2.11), implies

$$\mathcal{O}_{ii} = \|\boldsymbol{r}\|^2 \|\boldsymbol{l}\|^2 > \frac{1}{N\eta^2}.$$

Choosing $\eta = N^{-1+\epsilon/2}$, this concludes the proof. □

1.4 Local laws with regular observables

The goal of the present section is to establish the key Proposition 1.3.4 by proving an averaged local law for a product of two resolvents (of the Hermitisation (1.2.15)) in the bulk of the scDos ρ with *regular* (recall Definition 1.3.1 and see Definition 1.4.2 below) deterministic matrices A_1, A_2 in between. Throughout the rest of this paper, we consider the case of several spectral parameters $w_1, w_2, ...$ and fixed bounded deformations $\Lambda_1 = \Lambda_2 = ... \equiv \Lambda \in \mathbf{C}^{N \times N}$, which we continue to omit from the notation.

Using the abbreviations $G_i := G(w_i) := G^\Lambda(w_i)$ (and analogously for M_i), the deterministic approximation to the resolvent chain

$$G_1 B_1 \cdots B_{k-1} G_k$$

for arbitrary deterministic $B_1, ..., B_k$[12] is denoted by

$$M(w_1, B_1, ..., B_{k-1}, w_k) \tag{1.4.1}$$

and defined recursively in the length k of the chain.

[12]We will use the the notational convention, that the letter B denotes arbitrary (generic) matrices, while A is reserved for *regular* matrices, in the sense of Definition 1.4.2.

Definition 1.4.1. *Fix $k \in \mathbf{N}$ and let $w_1, ..., w_k \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters. As usual, the corresponding solutions to the MDE* (1.2.20) *are denoted by $M(w_j)$, $j \in [k]$. Then, for deterministic matrices $B_1, ..., B_{k-1}$ we recursively define*

$$M(w_1, B_1, ...B_{k-1}, w_k) = \big(\mathcal{B}_{1k}\big)^{-1}\Bigg[M(w_1)B_1M(w_2, ..., w_k) + \sum_{\sigma=\pm}\sum_{l=2}^{k-1}\sigma M(w_1)\langle M(w_1, ..., w_l)E_\sigma\rangle E_\sigma M(w_l, ..., w_k)\Bigg], \tag{1.4.2}$$

where we introduced the shorthand notation

$$\mathcal{B}_{mn} \equiv \mathcal{B}(w_m, w_n) = 1 - M(w_m)\mathcal{S}[\cdot]M(w_n) \tag{1.4.3}$$

for the so-called stability operator, discussed later in Appendix 1.B.

Note that the recursion (1.4.2) is well defined, since on the rhs. of (1.4.2), there are only $M(w_m, ..., w_n)$ appearing for which the number of spectral parameters is strictly smaller than on the lhs. of (1.4.2), i.e. $n - m + 1 < k$. We may call these formulas (1.4.2) *recursive Dyson equations* as they provide us with the correct deterministic quantity for longer resolvent chains. As an example, we have that

$$M(w_1, B_1, w_2) = \mathcal{B}_{12}^{-1}[M_1B_1M_2] = M_1\mathcal{X}_{12}[B_1]M_2\,, \tag{1.4.4}$$

where $\mathcal{B}_{12}^{-1}$ is the inverse stability operator (1.4.3) and $\mathcal{X}_{12} = \big(1 - \mathcal{S}[M_1 \cdot M_2]\big)^{-1}$. We remark that M satisfies several different recursions besides (1.4.2); they are presented in Lemma 1.D.1 (see also [170, Lemma 5.4] for a simpler setup of Wigner matrices). The equivalence of these recursions will be proved via the so-called *meta-argument*, see e.g. [181].

As already mentioned above, we are aiming at local laws for expressions of the form

$$\langle G_1A_1 \cdots G_kA_k\rangle \tag{1.4.5}$$

in the averaged case, or

$$\big(G_1A_1 \ \cdots A_kG_{k+1}\big)_{\boldsymbol{xy}} \tag{1.4.6}$$

in the isotropic case, where the deterministic matrices $A_1, ..., A_k$ are assumed to be *regular*.

The general concept of *regularity* depending on two spectral parameters w and w' has already been introduced in Definition 1.3.1. In the following definition we tailor this concept to observables in chains (1.4.5) and (1.4.6). It basically says that observable A_j, located between $G_j = G(w_j)$ and $G_{j+1} = G(w_{j+1})$ in these chains will naturally be regularised using the spectral parameters w_j and w_{j+1}.

Definition 1.4.2. (Regular observables in chains)
Fix $\kappa > 0$ and let $\delta = \delta(\kappa, \|\Lambda\|) > 0$ be small enough (see (1.3.5) *and* (1.4.22)*). Consider one of the two expressions* (1.4.5) *or* (1.4.6) *for some fixed length $k \in \mathbf{N}$ and bounded matrices $\|A_i\| \lesssim 1$ and let $w_1, ..., w_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters with $\operatorname{Re} w_i \in \mathbf{B}_\kappa$. For any $j \in [k]$, analogously to* (1.3.6)*, we denote*

$$\mathbf{1}_\delta^\pm(w_j, w_{j+1}) := \phi_\delta(\operatorname{Re} w_j \mp \operatorname{Re} w_{j+1})\ \phi_\delta(\operatorname{Im} w_j)\ \phi_\delta(\operatorname{Im} w_{j+1}) \tag{1.4.7}$$

and $\mathfrak{s}_j := -\operatorname{sgn}(\operatorname{Im} w_j \operatorname{Im} w_{j+1})$, where, here and in the following, in case of (1.4.5)*, the indices are understood cyclically modulo k.*

(a) For $i \in [k]$ we define the regular component *or* regularisation *of A_i from* (1.4.5) *or* (1.4.6) (w.r.t. the pair of spectral parameters (w_i, w_{i+1})) *as*

$$\mathring{A}_i := \mathring{A}_i^{w_i, w_{i+1}}\,. \tag{1.4.8}$$

(b) Moreover, we call A_i regular (w.r.t. (w_i, w_{i+1})) *if and only if* $\mathring{A}_i = A_i$.

For example, in case of (1.4.5) for $k = 1$ with spectral parameter $w_1 \in \mathbf{C} \setminus \mathbf{R}$ satisfying $\operatorname{Re} w_1 \in \mathbf{B}_\kappa$, $|\operatorname{Re} w_1| \le \delta/4$ and $|\operatorname{Im} w_1| \le \delta/2$ (recall (1.3.5) and (1.4.7)), the regular component of A_1 is given by

$$\mathring{A}_1 := A_1 - \frac{\langle \operatorname{Im} M_1 A_1 \rangle}{\langle \operatorname{Im} M_1 \rangle} E_+ - \frac{\langle M_1 A_1 M_1 E_- \rangle}{\langle M_1 E_- M_1 E_- \rangle} E_- \,. \tag{1.4.9}$$

Here we used the short–hand notation $M_1 := M(w_1)$.

We emphasise, that our notation $\mathring{\cdot}$ for the regular component of A_i does *not* have an overall fixed meaning but depends on the spectral parameters of the resolvents 'surrounding' the deterministic matrix A_i under consideration, i.e.

$$\langle \cdots G_i A_i G_{i+1} \cdots \rangle \quad \text{or} \quad \big(\cdots G_i A_i G_{i+1} \cdots \big)_{\boldsymbol{xy}} ,$$

or in case of (1.4.5) for $k = 1$ the single spectral parameter involved. However, if we aim at specifying the spectral parameters defining the operation $\mathring{\cdot}$, we add them (or their indices) as a subscript, i.e. write

$$\mathring{A}_i^{w_i, w_{i+1}} \equiv \mathring{A}_i^{i,i+1} \equiv \mathring{A}_i \equiv A_i^\circ \equiv A_i^{\circ_{i,i+1}} \equiv A_i^{\circ_{w_i, w_{i+1}}} ,$$

as done in Definition 1.3.1, and do not use imprecise notation $\mathring{A}_i$.

The just explained caveats are in stark contrast to the case of Wigner matrices [165, 168, 169], where the regular component of a matrix A is simply its traceless part, i.e. $\mathring{A} = A - \langle A \rangle$, irrespective of the spectral parameters involved. Apart from this independence of the location in the spectrum, there is a one further important difference to our case, which we already mentioned in Section 1.3: For Wigner matrices, the condition for A being regular is one-dimensional and hence restricts A to a $(N^2 - 1)$-dimensional subspace of $\mathbf{C}^{N \times N}$ (the traceless matrices), whereas in our case, the regularity condition is two-dimensional (if $\mathbf{1}_\delta^\sigma(\cdot, \cdot) = 1$) and hence restricts a regular matrix A to a $((2N)^2 - 2)$-dimensional subspace of $\mathbf{C}^{2N \times 2N}$, which depends on the 'surrounding' spectral parameters.

We now give bounds on the size of the deterministic term $M(w_1, B_1, ..., B_{k-1}, w_k)$ from (1.4.1), where all B_i are regular in the sense of Definition 1.4.2. The proof of this lemma is presented in Appendix 1.D.

Lemma 1.4.3. (Bounds on M, see [168, Lemma 2.4])
Fix $\kappa > 0$. *Let* $k \in [4]$ *and* $w_1, ..., w_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ *be spectral parameters with* $\operatorname{Re} w_j \in \mathbf{B}_\kappa$. *Then, for bounded* regular *deterministic matrices* $A_1, ..., A_k$ *(in the sense of Definition 1.4.2), we have the bounds*

$$\| M(w_1, A_1, ..., A_k, w_{k+1}) \| \lesssim \begin{cases} \frac{1}{\eta^{\lfloor k/2 \rfloor}} & \text{if } \eta \le 1 \\ \frac{1}{\eta^{k+1}} & \text{if } \eta > 1 \end{cases} , \tag{1.4.10}$$

$$|\langle M(w_1, A_1, ..., A_{k-1}, w_k) A_k \rangle| \lesssim \begin{cases} \frac{1}{\eta^{\lfloor k/2 \rfloor - 1}} \vee 1 & \text{if } \eta \le 1 \\ \frac{1}{\eta^k} & \text{if } \eta > 1 \end{cases} , \tag{1.4.11}$$

for the deterministic approximation (1.4.1) *of a resolvent chain, where* $\eta := \min_j |\operatorname{Im} w_j|$.

For the presentation of our main results, we would only need (1.4.10) and (1.4.11) for $k \in [2]$ from the previous lemma. However, the remaining bounds covered by Lemma 1.4.3 will be instrumental in our proofs of Theorems 1.4.5 and 1.4.4 below (see Sections 1.5 and 1.6).

The main result of the present section and most important input for our proofs in Section 1.3 is the following averaged local law in the bulk of the spectrum for two resolvents and regular matrices.

Theorem 1.4.4. (Local laws with *two* regular matrices)
Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N\times N}$, $\epsilon > 0$ *and* $\kappa > 0$. *Then, for spectral parameters* $w_1, w_2, w_3 \in \mathbf{C}$ *satisfying* $\max_j |w_j| \le N^{100}$, $\operatorname{Re} w_j \in \mathbf{B}_\kappa$ *and* $\eta := \min_j |\operatorname{Im} w_j| \ge N^{-1+\epsilon}$, *deterministic vectors* $\boldsymbol{x}, \boldsymbol{y}$ *with* $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$, *and any* regular *deterministic matrices* $A_1, A_2 \in \mathbf{C}^{2N\times 2N}$ *(cf. Definition 1.4.2), we have the* averaged local law

$$|\langle G_1 A_1 G_2 A_2 - M(w_1, A_1, w_2) A_2\rangle| \prec \begin{cases} \frac{1}{\sqrt{N\eta}} & \text{if } \eta \le 1 \\ \frac{1}{N\eta^3} & \text{if } \eta > 1 \end{cases} \tag{1.4.12a}$$

and the isotropic law

$$\left|\left\langle \boldsymbol{x}, \left(G_1 A_1 G_2 A_2 G_3 - M(w_1, A_1, w_2, A_2, w_3)\right)\boldsymbol{y}\right\rangle\right| \prec \begin{cases} \frac{1}{\eta} & \text{if } \eta \le 1 \\ \frac{1}{\sqrt{N}\eta^4} & \text{if } \eta > 1 \end{cases}. \tag{1.4.12b}$$

Together with (1.4.11) for $k = 2$, this proves Proposition 1.3.4. Moreover, as a byproduct of our proof of Theorem 1.4.4, we obtain the following optimal local laws with a single regular matrix.

Theorem 1.4.5. (Optimal local laws with *one* regular matrix)
Fix a bounded deterministic $\Lambda \in \mathbf{C}^{N\times N}$, $\epsilon > 0$ *and* $\kappa > 0$. *Then, for spectral parameters* $w_1, w_2 \in \mathbf{C}$ *satisfying* $\max_j |w_j| \le N^{100}$, $\operatorname{Re} w_j \in \mathbf{B}_\kappa$ *and* $\eta := \min_j |\operatorname{Im} w_j| \ge N^{-1+\epsilon}$, *deterministic vectors* $\boldsymbol{x}, \boldsymbol{y}$ *with* $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$, *and any* regular *deterministic matrix* A_1 *(cf. Definition 1.4.2), we have the* optimal averaged local law

$$|\langle (G_1 - M_1) A_1\rangle| \prec \begin{cases} \frac{1}{N\eta^{1/2}} & \text{if } \eta \le 1 \\ \frac{1}{N\eta^2} & \text{if } \eta > 1 \end{cases} \tag{1.4.13a}$$

and the optimal isotropic local law

$$\left|\left\langle \boldsymbol{x}, \left(G_1 A_1 G_2 - M(w_1, A_1, w_2)\right)\boldsymbol{y}\right\rangle\right| \prec \begin{cases} \frac{1}{\sqrt{N\eta^2}} & \text{if } \eta \le 1 \\ \frac{1}{\sqrt{N}\eta^3} & \text{if } \eta > 1 \end{cases}. \tag{1.4.13b}$$

Remark 1.4.6. *We have several comments.*

(i) The above local laws are in agreement with the $\sqrt{\eta}$-rule *first established for Wigner matrices for traceless matrices in [168, Theorem 2.5]: Every regular deterministic matrix* A_i *reduces both the size of the deterministic approximation and the error term by a factor* $\sqrt{\eta}$.

(ii) The error terms in Theorem 1.4.4 dealing with two regular matrices can still be improved by a factor $1/\sqrt{N\eta}$, *as shown in [168]. A similar analysis could have been conducted here, but we refrain from doing so, as it is not needed for our main results from Section 1.2. However, the error bounds in* (1.4.13) *with one regular matrix are in fact optimal.*

(iii) Given Theorem 1.2.6, and Theorems 1.4.4–1.4.5, it is possible to deduce similar bounds for averaged and isotropic chains as in (1.4.12), *where not both matrices* A_1, A_2 *are regular (see* (1.3.16)*).*

In the rest of this paper, we give a detailed proof of Theorem 1.4.4 in the much more involved $\eta \le 1$ regime. For $\eta > 1$, the bound simply follows by induction on the number of resolvents in chain by invoking the trivial $\|M(w)\| \lesssim 1/|\operatorname{Im} w|$. The detailed argument has been carried out in [168, Appendix B] for the case of Wigner matrices. However, at a certain technical point (within the proof of the *master inequalities* in Proposition 1.4.9 and the *reduction inequalities* in Lemma 1.4.10), the proof uses Theorems 1.4.4 and 1.4.5 (and even its analogues for longer chains) for the $\eta > 1$ regime. But the master and reduction inequalities are not needed for proving the above estimates in the $\eta > 1$ regime, hence the argument is not circular. With partial exception in Appendix 1.D, where we prove Lemma 1.4.3, throughout the rest of this paper we assume that $\min_j |\operatorname{Im} w_j| =: \eta \le 1$.

1.4.1 Basic control quantities and proof of Theorems 1.4.4 and 1.4.5

Our strategy for proving Theorem 1.4.4 (and thereby Theorem 1.4.5 as a byproduct) is to derive a system of *master inequalities* (Proposition 1.4.9) for the errors in the local laws by cumulant expansion, then use an iterative scheme to gradually improve their estimates. The cumulant expansion introduces longer resolvent chains, potentially leading to an uncontrollable hierarchy, so our master inequalities are complemented by a set of *reduction inequalities* (Lemma 1.4.10) to estimate longer chain in terms of shorter ones. We have used a similar strategy in [168, 169] for Wigner matrices, but now many new error terms due to regularisations need to be handled.

Before entering the detailed proof, we explain the main mechanism of the new type of error terms. Cumulant expansions applied to chains $\dots G_i A_i G_{i+1} \dots$ with regular A_i's introduce more resolvent factors, for example $\dots G_i G_i A_i G_{i+1} \dots$ or $\dots G_i E_- G_i A_i G_{i+1} \dots$ without introducing more A's. Multiple G factors without intermediate A's appear which we wish to reduce to fewer G factors using resolvent identities (1.3.21) or contour integral representations; in the example above we will use

$$G_i G_i = G(w_i)^2 = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{G(z)}{(z-w_i)^2} \mathrm{d}z, \tag{1.4.14}$$

where Γ is an appropriate contour (see Lemma 1.5.1). When this formula is inserted into the chain, we have $\dots G(z) A_i G_{i+1} \dots$, i.e. A_i is not regular any more with respect to the neighboring spectral parameters (z, w_{i+1}) since w_i has been changed to z. We need to regularise A_i to the new situation. Fortunately, the regularisation is Lipschitz continuous by Lemma 1.3.3, so roughly speaking we make an error of order $|z - w_i|$ when we regularise A_i from (w_i, w_{i+1}) to (z, w_{i+1}). This error exactly compensates the higher power of $z - w_i$ in the denominator in (1.4.14), making eventually the adjustment of regularisations harmless in the estimates. We need to meticulously implement this strategy for longer chains and also taking into account the chiral symmetry to reduce $G_i E_- G_i$ in chains like $\dots G_i E_- G_i A_i G_{i+1} \dots$. The precise form of the error terms in Lemma 1.3.3 is essential. It is remarkable that the signs appearing in (1.3.11), (1.3.12), and (1.3.13) exactly match those that arise in the denominators of the contour integral formulas like (1.4.14). We now start the actual proof.

As the basic control quantities in the sequel of the proof, we introduce the normalised differences

$$\Psi_k^{\mathrm{av}}(\boldsymbol{w}_k, \boldsymbol{A}_k) := N\eta^{k/2} |\langle G_1 A_1 \cdots G_k A_k - M(w_1, A_1, ..., w_k) A_k \rangle|, \tag{1.4.15}$$

$$\Psi_k^{\mathrm{iso}}(\boldsymbol{w}_{k+1}, \boldsymbol{A}_k, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta^{k+1}} \left| \big(G_1 A_1 \cdots A_k G_{k+1} - M(w_1, A_1, ..., A_k, w_{k+1}) \big)_{\boldsymbol{xy}} \right| \tag{1.4.16}$$

for $k \in \mathbf{N}$, where we used the short hand notations

$$G_i := G(w_i), \quad \eta := \min_i |\mathrm{Im}\, w_i|, \quad \boldsymbol{w}_k := (w_1, ..., w_k), \quad \boldsymbol{A}_k := (A_1, ..., A_k).$$

The deterministic matrices $\|A_i\| \le 1$, $i \in [k]$, are assumed to be *regular* (i.e., $A_i = \mathring{A}_i$, see Definition 1.4.2) and the deterministic counterparts

$$M(w_1, A_1, ..., A_{k-1}, w_k)$$

used in (1.4.15) and (1.4.16) (see also (1.4.1)) are given recursively in Definition 1.4.1.

For convenience, we extend the above definitions to $k = 0$ by

$$\Psi_0^{\mathrm{av}}(w) := N\eta |\langle G(w) - M(w) \rangle|, \quad \Psi_0^{\mathrm{iso}}(w, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta} \big| \big(G(w) - M(w) \big)_{\boldsymbol{xy}} \big|$$

and observe that

$$\Psi_0^{\mathrm{av}} + \Psi_0^{\mathrm{iso}} \prec 1 \tag{1.4.17}$$

is the usual single-resolvent local law (in the bulk) from Theorem 1.2.6, where here and in the following the arguments of $\Psi_k^{\mathrm{av/iso}}$ shall occasionally be omitted. We remark that the index k counts the number of regular matrices in the sense of Definition 1.4.2.

Throughout the entire argument, let $\epsilon > 0$ and $\kappa > 0$ be *arbitrary* but fixed, and let

$$\mathbf{D}^{(\epsilon,\kappa)} := \left\{ w \in \mathbf{C} : \operatorname{Re} w \in \mathbf{B}_\kappa \,,\ N^{100} \geq |\operatorname{Im} w| \geq N^{-1+\epsilon} \right\} \tag{1.4.18}$$

be the *target spectral domain*, where the κ-bulk $\mathbf{B}_\kappa$ has been introduced in (1.2.6). This target spectral domain $\mathbf{D}^{(\epsilon,\kappa)}$ will be reached by shrinking a larger *initial spectral domain*

$$\mathbf{D}^{(\epsilon_0,\kappa_0)} := \left\{ w \in \mathbf{C} : \operatorname{Re} w \in \mathbf{B}_{\kappa_0} \,,\ N^{100} \geq |\operatorname{Im} w| \geq N^{-1+\epsilon_0} \right\} \tag{1.4.19}$$

many times, say $(L-1)$ times, during our whole argument, where $L = L(\epsilon)$ is an N-independent positive integer to be determined below (see Remark 1.4.12). In (1.4.19), we set $\epsilon_0 := \epsilon/2$ and chose the initial bulk parameter

$$\kappa_0 = \kappa_0(\epsilon,\kappa) = \frac{\kappa}{L(\epsilon)} > 0 \tag{1.4.20}$$

The just mentioned shrinking of domains will be conducted alongside the decreasing family $(\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)})_{\ell \in [L]}$ of spectral domains, defined via

$$\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)} := \left\{ w \in \mathbf{C} : \operatorname{Re} w \in \mathbf{B}_{\ell\kappa_0} \,,\ N^{100} \geq |\operatorname{Im} w| \geq \ell N^{-1+\epsilon_0} \right\} \subset \mathbf{D}^{(\epsilon_0,\kappa_0)} \,. \tag{1.4.21}$$

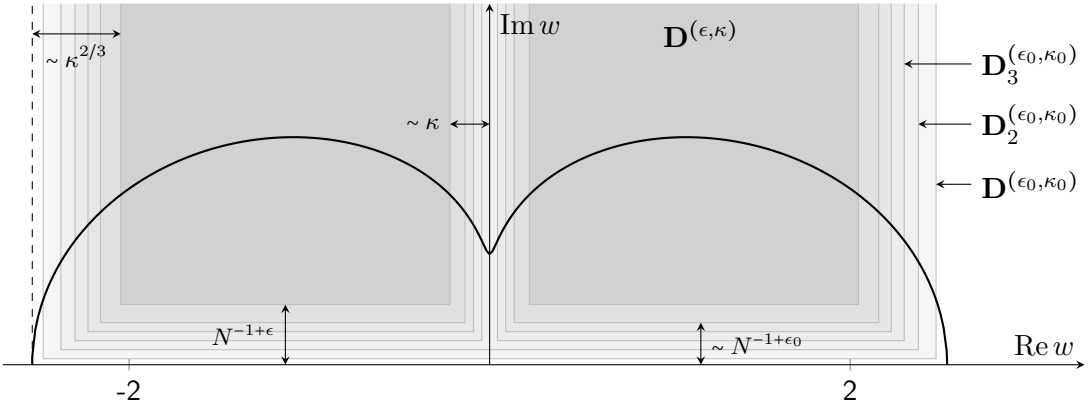

Figure 1.4.1: Depicted are the target spectral domain (1.4.18), the initial spectral domain (1.4.19) and four intermediate domains from the family (1.4.21). The solid black curve represents the symmetric scDos ρ for the perturbation $\Lambda = -z$ with $|z|$ slightly less than one (see Example 1.2.5). Close to a regular edge of the scDos, the horizontal distance between two domains scales like $\kappa^{2/3}$. Near an (approximate) cusp, the scaling agrees with the linear lower bound given in (1.2.23).

Finally, the cut-off parameter $\delta > 0$ used in the definition of the regular component of an observable (see (1.3.5) and (1.4.8) in Definition 1.4.2) shall be chosen by the following two requirements: First, it has to be much smaller than the initial bulk-parameter κ_0 from (1.4.20), i.e.

$$0 < \delta \ll \mathfrak{c}\kappa_0 \,, \tag{1.4.22}$$

where $\mathfrak{c} > 0$ is the same constant as introduced in (1.2.23). Second, it has to be small enough such that the denominators in (1.4.8) (see also Appendix 1.B) as well as in Lemmas 1.5.5, 1.5.7, and 1.E.1 are uniformly bounded away from zero – in case that $\mathbf{1}_\delta^\sigma(w_i, w_{i+1}) = 1$. Note that these requirements also depend on the deformation $\Lambda \in \mathbf{C}^{N\times N}$ (but only via the norm $\|\Lambda\| \lesssim 1$) as it determines the scDos ρ.

Definition 1.4.7. (Uniform bounds in domains)
Let $\epsilon > 0$ and $\kappa > 0$ as above and let $k \in \mathbf{N}$. We say that the bounds

$$\begin{aligned} \big|\langle G(w_1)B_1 \cdots G(w_k)B_k - M(w_1, B_1, ..., w_k)B_k\rangle\big| &\prec \mathcal{E}^{\mathrm{av}}\,, \\ \Big|\big(G(w_1)B_1 \cdots B_k G(w_{k+1}) - M(w_1, B_1, ..., B_k, w_{k+1})\big)_{\boldsymbol{xy}}\Big| &\prec \mathcal{E}^{\mathrm{iso}} \end{aligned} \tag{1.4.23}$$

*hold (ϵ, κ)-*uniformly *for some deterministic control parameters $\mathcal{E}^{\mathrm{av/iso}} = \mathcal{E}^{\mathrm{av/iso}}(N, \eta)$, depending only on N and $\eta := \min_i |\mathrm{Im}\, w_i|$, if the implicit constant in* (1.4.23) *are uniform in bounded deterministic matrices $\|B_j\| \le 1$, deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \le 1$, and* admissible *spectral parameters $w_j \in \mathbf{D}^{(\epsilon,\kappa)}$ satisfying $1 \ge \eta := \min_j |\mathrm{Im}\, w_j|$.*

*Similarly, we use the phrase that a bound holds $(\epsilon_0, \kappa_0, \ell)$-*uniformly *(or simply ℓ-*uniformly*), if the above statement is true with $\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$ in place of $\mathbf{D}^{(\epsilon,\kappa)}$.*

Moreover, we may allow for additional restrictions on the deterministic matrices. For example, we may talk about uniformity under the additional assumption that some (or all) of the matrices are regular *(in the sense of Definition 1.4.2).*

Note that (1.4.23) is stated for a fixed choice of spectral parameters w_j in the left hand side, but it is in fact equivalent to an apparently stronger statement, when the same bound holds with a supremum over the spectral parameters (with the same constraints). More precisely, if $\mathcal{E}^{\mathrm{iso}} \ge N^{-C}$ for some constant $C > 0$, then (1.4.23) implies

$$\sup_{w_1,\ldots,w_{k+1}} \Big|\big(G(w_1)B_1 \cdots B_k G(w_{k+1}) - M(w_1, B_1, ..., B_k, w_{k+1})\big)_{\boldsymbol{xy}}\Big| \prec \mathcal{E}^{\mathrm{iso}} \tag{1.4.24}$$

(and analogously for the averaged bound), where the supremum is taken over all choices of w_j's in the admissible spectral domain $\mathbf{D}^{(\epsilon,\kappa)}$ or $\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$. This bound follows from (1.4.23) by a standard *grid argument* (see, e.g., the discussion after [168, Def. 3.1]). Throughout the entire paper, we will frequently use the equivalence between (1.4.23) and (1.4.24), e.g. when integrating such bounds over some spectral parameters as done in Sections 1.5 and 1.6.

We can now formulate our main results of the present section, Theorem 1.4.4 and Theorem 1.4.5, in the language of our basic control quantities $\Psi_k^{\mathrm{av/iso}}$.

Lemma 1.4.8. (Estimates on $\Psi_1^{\mathrm{av/iso}}$ and $\Psi_2^{\mathrm{av/iso}}$) *For any $\epsilon > 0$ and $\kappa > 0$ we have*

$$\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} \prec 1 \qquad \textit{and} \qquad \Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} \prec \sqrt{N\eta}$$

(ϵ, κ)-uniformly in regular matrices (i.e. for spectral parameters $w_j \in \mathbf{D}^{(\epsilon,\kappa)}$ with $1 \ge \eta := \min_j |\mathrm{Im}\, w_j|$).

Proof of Theorems 1.4.4 and 1.4.5. These immediately follow from Lemma 1.4.8. □

The rest of the proof is structured as follows: First, in Section 1.4.2, we state the *master inequalities* and corresponding *reduction inequalities* on the $\Psi_k^{\mathrm{av/iso}}$ parameters, which we then use in Section 1.4.3 to prove Lemma 1.4.8. Afterwards, in Section 1.5, we prove the master inequalities and, finally, the proof of the reduction inequalities is presented in Section 1.6.

1.4.2 Master inequalities and reduction lemma

We now state the relevant part of a non-linear infinite hierarchy of coupled master inequalities for Ψ_k^{av} and Ψ_k^{iso}. In fact, for our purposes, it is sufficient to have only the inequalities for $k \in [2]$, but with fairly more effort (despite closely following the arguments in Section 1.5) it is possible to obtain analogous estimates for general $k \in \mathbf{N}$.

Proposition 1.4.9. (Master inequalities) *Assume that*

$$\Psi_i^{\mathrm{av/iso}} \prec \psi_i^{\mathrm{av/iso}}, \quad i \in [4], \tag{1.4.25}$$

ℓ-uniformly (i.e. for spectral parameters $w_j \in \mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$ and $1 \geq \min_j |\mathrm{Im}\, w_j|$) in regular matrices, for some deterministic control parameters $\psi_i^{\mathrm{av/iso}}$, which are independent of the spectral parameters w_j. Then it holds that

$$\Psi_1^{\mathrm{av}} \prec 1 + \frac{\psi_1^{\mathrm{av}}}{N\eta} + \frac{\psi_1^{\mathrm{iso}} + (\psi_2^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{1.4.26a}$$

$$\Psi_1^{\mathrm{iso}} \prec 1 + \frac{\psi_1^{\mathrm{iso}} + \psi_1^{\mathrm{av}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{1.4.26b}$$

$$\Psi_2^{\mathrm{av}} \prec 1 + \frac{(\psi_1^{\mathrm{av}})^2 + (\psi_1^{\mathrm{iso}})^2 + \psi_2^{\mathrm{av}}}{N\eta} + \frac{\psi_2^{\mathrm{iso}} + (\psi_4^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\mathrm{iso}})^{1/2} + (\psi_4^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{1.4.26c}$$

$$\Psi_2^{\mathrm{iso}} \prec 1 + \psi_1^{\mathrm{iso}} + \frac{\psi_1^{\mathrm{av}}\psi_1^{\mathrm{iso}} + (\psi_1^{\mathrm{iso}})^2}{N\eta} + \frac{\psi_2^{\mathrm{iso}} + (\psi_1^{\mathrm{iso}}\psi_3^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\mathrm{iso}})^{1/2} + (\psi_4^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{1.4.26d}$$

now $(\ell+1)$-uniformly (i.e. for spectral parameters $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ with $1 \geq \eta := \min_j |\mathrm{Im}\, w_j|$) in regular matrices.

As shown in the above proposition, resolvent chains of length k are estimated by resolvent chains up to length $2k$. In order to avoid the indicated infinite hierarchy of master inequalities with higher and higher k indices, we will need the following *reduction lemma*.

Lemma 1.4.10. (Reduction inequalities) *Assume that $\Psi_n^{\mathrm{av/iso}} \prec \psi_n^{\mathrm{av/iso}}$ holds for $1 \leq n \leq 4$, ℓ-uniformly (i.e. for spectral parameters $w_j \in \mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$ with $1 \geq \eta := \min_j |\mathrm{Im}\, w_j|$) in regular matrices (cf. Definition 1.4.7). Then we have*

$$\Psi_4^{\mathrm{av}} \prec (N\eta)^2 + (\psi_2^{\mathrm{av}})^2, \tag{1.4.27}$$

on the same domain. Furthermore, we have

$$\begin{aligned} \Psi_3^{\mathrm{iso}} &\prec N\eta \left(1 + \frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right)^{1/2}, \\ \Psi_4^{\mathrm{iso}} &\prec (N\eta)^{3/2} \left(1 + \frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right) \end{aligned} \tag{1.4.28}$$

again uniformly in $w_j \in \mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$ and in regular matrices.

1.4.3 Proof of Lemma 1.4.8

Within the proof, we repeatedly use a simple argument, which we call *iteration*.

Lemma 1.4.11. (Iteration) *For every $D > 0$, $\nu > 0$, and $\alpha \in (0,1)$, there exists some $K = K(D,\nu,\alpha)$, such that whenever (i) $X \prec N^D$ on $\mathbf{D}_1^{(\epsilon_0,\kappa_0)}$ and (ii) $X \prec x$ on $\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$ for some $\ell \in \mathbf{N}$, implies that*

$$X \prec A + \frac{x}{B} + x^{1-\alpha}C^\alpha \qquad \text{on} \quad \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$$

for some constants $B \geq N^\nu$ and $A, C > 0$, it also holds that

$$X \prec A + C \qquad \text{on} \quad \mathbf{D}_{\ell+K}^{(\epsilon_0,\kappa_0)}.$$

We can now turn to the proof of Lemma 1.4.8.

Proof of Lemma 1.4.8. Assume that

$$\Psi_j^{\mathrm{av/iso}} \prec \psi_j^{\mathrm{av/iso}}, \quad j \in [4],$$

ℓ-uniformly, for some fixed $\ell > 0$, i.e. it holds on the domain $\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$. Then, by (1.4.26a)–(1.4.26d), using that $N\eta \geq 1$ to remove some lower order terms, we immediately obtain

$$\begin{aligned}
\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} &\prec 1 + \frac{\psi_1^{\mathrm{av}} + \psi_1^{\mathrm{iso}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\mathrm{av}})^{1/2} + (\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}} \\
\Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} &\prec 1 + \psi_1^{\mathrm{iso}} + \frac{(\psi_1^{\mathrm{av}})^2 + (\psi_1^{\mathrm{iso}})^2}{N\eta} + \frac{\psi_2^{\mathrm{av}} + \psi_2^{\mathrm{iso}}}{(N\eta)^{1/2}} \\
&\quad + \frac{(\psi_4^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_1^{\mathrm{iso}}\psi_3^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\mathrm{iso}})^{1/2} + (\psi_4^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}
\end{aligned} \tag{1.4.29}$$

on the domain $\mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$. We point out that to get the second bound in (1.4.29) we also used $\psi_1^{\mathrm{iso}}\psi_1^{\mathrm{av}} \leq (\psi_1^{\mathrm{iso}})^2 + (\psi_1^{\mathrm{av}})^2$. Now, given the first estimate in (1.4.29) we obtain a bound for $\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}}$ which is better than the original a priori bound (1.4.25). We can thus replace the $\psi_1^{\mathrm{av/iso}}$ from (1.4.25) with the rhs. of the first line of (1.4.29). Using iteration in both lines, we thus get

$$\begin{aligned}
\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} &\prec 1 + \frac{(\psi_2^{\mathrm{av}})^{1/2} + (\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \\
\Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} &\prec 1 + \frac{(\psi_4^{\mathrm{av}})^{1/2}}{\sqrt{N\eta}} + \frac{(\psi_2^{\mathrm{av}})^{1/4} + (\psi_2^{\mathrm{iso}})^{1/4}}{(N\eta)^{1/8}} \cdot \frac{(\psi_3^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\mathrm{iso}})^{1/2} + (\psi_4^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}},
\end{aligned} \tag{1.4.30}$$

on the domain $\mathbf{D}_{\ell+K}^{(\epsilon_0,\kappa_0)}$, for some K as in Lemma 1.4.11. We now use the reduction inequalities from Lemma 1.4.10 in the second line of (1.4.30):

$$\begin{aligned}
\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} &\prec 1 + \frac{(\psi_2^{\mathrm{av}})^{1/2} + (\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}} \\
\Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} &\prec (N\eta)^{1/2} + \frac{\psi_2^{\mathrm{av}}}{\sqrt{N\eta}} + (N\eta)^{1/4}(\psi_2^{\mathrm{iso}})^{1/2} + (\psi_2^{\mathrm{av}})^{1/2} + \frac{(\psi_2^{\mathrm{av}}\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \\
&\quad + \left((N\eta)^{1/4} + \frac{(\psi_2^{\mathrm{av}})^{1/4} + (\psi_2^{\mathrm{iso}})^{1/4}}{(N\eta)^{1/8}}\right)\left(1 + \frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}} + \frac{(\psi_2^{\mathrm{av}})^{1/4}}{(N\eta)^{1/8}} + \frac{(\psi_2^{\mathrm{iso}})^{1/2}(\psi_2^{\mathrm{av}})^{1/4}}{(N\eta)^{3/8}}\right),
\end{aligned} \tag{1.4.31}$$

on the domain $\mathbf{D}_{\ell+K}^{(\epsilon_0,\kappa_0)}$. Next, using iteration once again in the second line of (1.4.31), we obtain

$$\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} \prec 1 + \frac{(\psi_2^{\mathrm{av}})^{1/2} + (\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \qquad \Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} \prec (N\eta)^{1/2}$$

on the domain $\mathbf{D}_{\ell+K+K'}^{(\epsilon_0,\kappa_0)}$, for some K' as in Lemma 1.4.11. We point out that here we used Schwarz and Young inequalities for several terms. Finally, using iteration one last time we conclude

$$\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} \prec 1, \qquad \Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} \prec (N\eta)^{1/2}$$

on the domain $\mathbf{D}_{\ell+K+K'+K''}^{(\epsilon_0,\kappa_0)}$, for some K'' as in Lemma 1.4.11. This concludes the proof. □

Remark 1.4.12. *We observe that in every application of Lemma 1.4.11 during the proof of Lemma 1.4.8, the parameter D is uniformly bounded by, say, $D \leq 10$, as follows by estimating every resolvent in $\Psi_k^{\mathrm{av/iso}}$ by norm and using the trivial $1/\eta$-bound on inverse of the stability operator in the iterative definition of $M(w_1, ..., w_k)$ given in Definition 1.4.1. A further quick inspection of the above proof shows, that α can be chosen as fixed $\alpha = 1/2$. Finally, the parameter ν is lower bounded by (some universal positive constant times) ϵ, since $N\eta \geq N^{\epsilon/2}$ by construction of the initial domain* (1.4.19). *Hence, the constants K, K', and K'' only depend on ϵ and therefore also the maximal number $L = L(\epsilon)$ of domain shrinkings.*

1.5 Proof of the master inequalities, Proposition 1.4.9

Before going into the proofs of the master inequalities, we state a simple lemma, which will frequently be used in the following. Recall that the deformation $\Lambda \in \mathbf{C}^{N\times N}$ is fixed and hence omitted from the notation.

Lemma 1.5.1. (Integral representations for products of resolvents)
Let $k \in \mathbf{N}$ and $w_1, ..., w_k \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters, whose imaginary parts have equal sign, i.e. $\operatorname{sgn}(\operatorname{Im} w_1) = ... = \operatorname{sgn}(\operatorname{Im} w_k) =: \tau$. Then, for any $J \subset \mathbf{R}$ being a union of compact intervals such that $\operatorname{Re} w_i \in \mathring{J}$ (the interior) for all $i \in [k]$ and $0 < \tilde{\eta} < \eta := \min_j |\operatorname{Im} w_j|$, we have the integral representation

$$\prod_{j=1}^{k} G(w_j) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma G(z) \prod_{j=1}^{k} \frac{1}{z - w_j} \, \mathrm{d}z \,, \tag{1.5.1}$$

where the contour Γ from (1.5.1) is defined as (see Figure 1.5.1)

$$\Gamma \equiv \Gamma_{\tilde{\eta}}^{\tau}(J) := \begin{cases} \partial\big(J \times [\mathrm{i}\tilde{\eta}, \mathrm{i}\infty)\big) & \text{if} \quad \tau = + \\ \partial\big(J \times (-\mathrm{i}\infty, -\mathrm{i}\tilde{\eta}]\big) & \text{if} \quad \tau = - \end{cases} \tag{1.5.2}$$

and the boundary is parameterised in counter-clockwise orientation.

Proof. This easily follows from residue calculus. For example, for $k = 2$, we have that

$$\frac{1}{\lambda_i - w_1} \frac{1}{\lambda_i - w_2} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{1}{\lambda_i - z} \frac{1}{z - w_1} \frac{1}{z - w_2} \mathrm{d}z$$

for every eigenvalue λ_i of H by definition of the contour. This implies (1.5.1) for $k = 2$ by spectral decomposition for H; the argument for general k is analogous. □

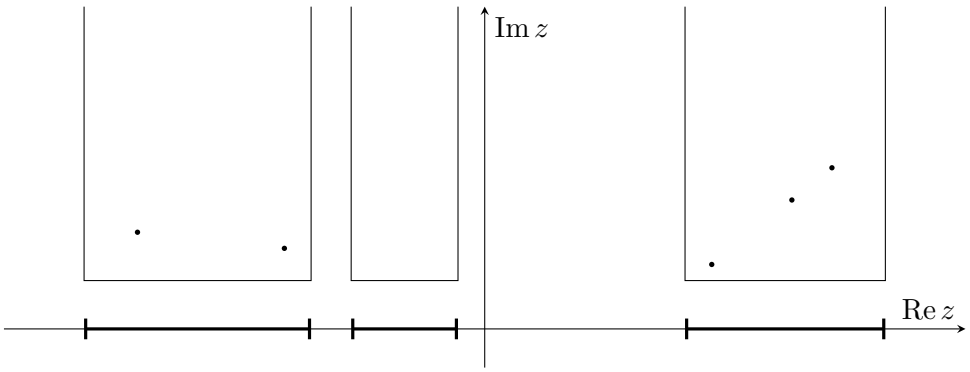

Figure 1.5.1: Depicted is the scenario from Lemma 1.5.1 with five spectral parameters represented as dots in the upper half plane. Moreover, we indicated the union of compact intervals J on the real axis and the contour Γ as described in (1.5.2). Note that one of the three intervals constituting J does not contain any $\operatorname{Re} w_j$.

We recall the definition of the *second order renormalisation*, denoted by underline, from [165]. For functions $f(W), g(W)$ of the random matrix W (see (1.2.15)), we define

$$\underline{f(W)Wg(W)} := f(W)Wg(W) - \widetilde{\mathbb{E}}\big[(\partial_{\widetilde{W}} f)(W)\widetilde{W}g(W) + f(W)\widetilde{W}(\partial_{\widetilde{W}} g)(W)\big] \,, \tag{1.5.3}$$

where $\partial_{\widetilde{W}}$ denotes the directional derivative in the direction of

$$\widetilde{W} := \begin{pmatrix} 0 & \widetilde{X} \\ \widetilde{X}^* & 0 \end{pmatrix} ,$$

where $\widetilde{X}$ is a complex Ginibre matrix that is independent of W. The expectation is taken w.r.t. the matrix $\widetilde{X}$. Note that, if W itself consists of a complex Ginibre matrix X, then $\mathbf{E}\,\underline{f(W)Wg(W)} = 0$, while for X with a general distribution this expectation is independent of the first two moments of X. In other words, the underline renormalises the product $f(W)Wg(W)$ to second order. We remark that underline (1.5.3) is a well-defined notation, if the 'middle' W to which the renormalisation refers is unambiguous. This is always be the case in all our proofs, since the functions f, g will be products of resolvents, never involving explicitly monomials in W.

We note that for any deterministic matrix $R \in \mathbf{C}^{2N\times 2N}$ we have

$$\widetilde{\mathbf{E}}\widetilde{W}R\widetilde{W} = 2\langle RE_2\rangle E_1 + 2\langle RE_1\rangle E_2 = \sum_\sigma \sigma\langle RE_\sigma\rangle E_\sigma = \mathcal{S}[R]$$

and furthermore, that the directional derivative of the resolvent is given by

$$\partial_{\widetilde{W}} G = -G\widetilde{W}G\,.$$

For example, in the special case $f(W) = 1$ and $g(W) = (W + \hat{\Lambda} - w)^{-1} = G$, we thus have

$$\underline{WG} = WG + \mathcal{S}[G]G$$

by definition of the underline in (1.5.3).

Using this underline notation in combination with the identity $G(W + \hat{\Lambda} - w) = E_+$ and the defining equation (1.2.20) for M, we have

$$G = M - M\underline{WG} + M\mathcal{S}[G - M]G = M - \underline{GW}M + G\mathcal{S}[G - M]M\,. \tag{1.5.4}$$

Recall that $\langle GE_-\rangle = 0$ (see below (1.2.16)) which immediately yields that $\mathcal{S}[G] = \sum_\sigma \sigma\langle GE_\sigma\rangle E_\sigma = \langle G\rangle$. Moreover, we have that $\mathcal{S}[M] = \langle M\rangle$, as follows from (1.2.17)–(1.2.18), and hence $\mathcal{S}[\cdot]$ effectively acts like a trace on G and M, i.e.

$$\mathcal{S}[G - M] = \langle G - M\rangle\,. \tag{1.5.5}$$

Now, similarly to [168], the key idea of the proof of Proposition 1.4.9 is using (1.5.4) for some G_j in a chain $G_1A_1\cdots A_kG_{k+1}$ and extending the renormalisation (1.5.3) to the whole product at the expense of adding resolvent products of shorter length. This will be done for each of the four estimates from Proposition 1.4.9 separately and presented in an *underlined lemma* in the beginning of each of the following subsections. Afterwards, the renormalisation of the whole product will be handled by cumulant expansion, exploiting that its expectation vanishes up to second order. While the proofs of the underlined lemmas for $\Psi_1^{\mathrm{av/iso}}$ are presented in detail, we defer the analogous arguments for $\Psi_2^{\mathrm{av/iso}}$ to Appendix 1.E.

1.5.1 Proof of the first master inequality (1.4.26a)

Let $w \equiv w_1$ be a spectral parameter in $\mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ (in particular in the bulk of the scDos, recall (1.4.21)) and $A \equiv A_1$ a (w,w)-*regular* matrix (cf. Definition 1.4.2). We use the notation $w = e + \mathrm{i}\eta$ and we assume without loss of generality (by conjugation with E_-, see (1.2.16)) that $1 \geq \eta > 0$. We also assume that (1.4.25) holds (in this subsection we will need it only for Ψ_1^{av} and Ψ_2^{av}).

Lemma 1.5.2. (Representation as full underlined)
For any regular matrix $A = \mathring{A}$ we have that

$$\langle (G - M)\mathring{A}\rangle = -\langle \underline{WG\mathring{A}'}\rangle + \mathcal{O}_<\big(\mathcal{E}_1^{\mathrm{av}}\big) \tag{1.5.6}$$

for some other regular matrix $A' = \mathring{A}'$, which linearly depends on A (see (1.5.24) *for the precise formula for A'). For the error term in* (1.5.6)*, we used the shorthand notation*

$$\mathcal{E}_1^{\mathrm{av}} := \frac{1}{N\eta^{1/2}}\left(1 + \frac{\psi_1^{\mathrm{av}}}{N\eta}\right). \tag{1.5.7}$$

Having this approximate representation of $\langle (G-M)\mathring{A}\rangle$ as a full underlined term at hand, we turn to the proof of (1.4.26a) via a (minimalistic) cumulant expansion: For fixed indices a, b and a smooth function $f(W)$ we have

$$\mathbf{E}\, w_{ab} f(W) = \sum_{l_1+l_2\geq 1} \frac{1}{l_1! l_2!} \kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2})\, \mathbf{E}\, \partial_{ab}^{l_1} \partial_{ba}^{l_2} f(W), \tag{1.5.8}$$

where $\kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2})$ is the cumulant l_1+1 copies of the random variable w_{ab} and l_2 copies of the random variable w_{ba}, and $\partial_{ab}^{l_1}\partial_{ba}^{l_2}$ denotes the l_1-th derivative in the ab-entry and the l_2-th derivative in the ba-entry.

Proof of (1.4.26a). Let $p \in \mathbf{N}$. Then, starting from (1.5.6), and recalling the second order renormalisation (1.5.3), we have

$$\begin{aligned}
\mathbf{E}\, \langle \underline{WG} A\rangle \langle (G-M)A\rangle^{2p-1} &= \frac{1}{N}\sum_{ab} \mathbf{E}\, \underline{w_{ab}(GA)_{ba}} \langle (G-M)A\rangle^{2p-1} \\
&= \frac{1}{N}\sum_{ab} \mathbf{E}(GA)_{ba}\Big(\kappa(w_{ab}, w_{ba})\partial_{ba} \langle (G-M)A\rangle^{2p-1} + \kappa(w_{ab}, w_{ab})\partial_{ab} \langle (G-M)A\rangle^{2p-1}\Big) \\
&\quad + \frac{1}{N}\sum_{ab}\sum_{l_1+l_2\geq 2} \frac{1}{l_1! l_2!} \kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2})\, \mathbf{E}\, \partial_{ab}^{l_1}\partial_{ba}^{l_2}\big[(GA)_{ba} \langle (G-M)A\rangle^{2p-1}\big].
\end{aligned} \tag{1.5.9}$$

By computing the resolvent derivatives explicitly as

$$\partial_{ab} \langle (G-M)A\rangle = -\frac{1}{N}(GAG)_{ba} \tag{1.5.10}$$

and using that $\kappa(w_{ab}, w_{ba}) = R_{ab}/N$, the first term in the middle line of (1.5.9) simplifies to

$$\frac{1}{N^3}\sum_{ab}(GA)_{ba}(GAG)_{ab} = \frac{\langle E_1 GAE_2 GAG\rangle + \langle E_2 GAE_1 GAG\rangle}{N^2}$$

(up to the factor of $\langle (G-M)A\rangle^{2p-2}$) , the second term being similar up to an additional transposition. Here

$$R_{ab} := \mathbf{1}(a \leq N, b \geq N+1 \text{ or } b \leq N, a \geq N+1)$$

is the indicator function for the off-diagonal blocks of W. For the remaining term in (1.5.9) we simply estimate the cumulants by their size $|\kappa(\{w_{ab}\}^{l_1+1}, \{w_{ba}\}^{l_2})| \leq N^{-(l_1+l_2+1)/2} R_{ab}$ to obtain

$$\begin{aligned}
&\mathbf{E}|\langle (G-M)A\rangle|^{2p} \\
&= \big|-\mathbf{E}\langle \underline{WG} A'\rangle \langle (G-M)A\rangle^{p-1} \langle (G-M)^* A^*\rangle^p\big| + \mathcal{O}_{\prec}\big((\mathcal{E}_1^{\mathrm{av}})^{2p}\big) \\
&\lesssim \mathbf{E}\, \frac{\big|\sum_\sigma \sigma \langle GE_\sigma GA' E_\sigma GA\rangle\big| + \big|\sum_\sigma \sigma \langle G^* E_\sigma GA' E_\sigma G^* A^*\rangle\big|}{N^2} \big|\langle (G-M)A\rangle\big|^{2p-2} \\
&\quad + \sum_{|\boldsymbol{l}|+\sum(J\cup J_*)\geq 2} \mathbf{E}\, \Xi_1^{\mathrm{av}}(\boldsymbol{l}, J, J_*) \big|\langle (G-M)A\rangle\big|^{2p-1-|J\cup J_*|} + \mathcal{O}_{\prec}\big((\mathcal{E}_1^{\mathrm{av}})^{2p}\big),
\end{aligned} \tag{1.5.11}$$

where $\Xi_1^{\mathrm{av}}(\boldsymbol{l}, J, J_*)$ is defined as

$$\Xi_1^{\mathrm{av}} := N^{-(|\boldsymbol{l}|+\sum(J\cup J_*)+3)/2} \sum_{ab} R_{ab} |\partial^{\boldsymbol{l}}(GA')_{ba}| \prod_{\boldsymbol{j}\in J} |\partial^{\boldsymbol{j}} \langle GA\rangle| \prod_{\boldsymbol{j}\in J_*} |\partial^{\boldsymbol{j}} \langle G^* A^*\rangle| \tag{1.5.12}$$

and the summation in the last line of (1.5.11) is taken over tuples[13] $\boldsymbol{l} \in \mathbf{Z}_{\geq 0}^2$ and multisets of tuples $J, J_* \subset \mathbf{Z}_{\geq 0}^2 \setminus \{(0,0)\}$. Moreover, we set $\partial^{\boldsymbol{l}} = \partial^{(l_1,l_2)} := \partial_{ab}^{l_1}\partial_{ba}^{l_2}$, similarly $\partial^{\boldsymbol{j}} = \partial^{(j_1,j_2)} := \partial_{ab}^{j_1}\partial_{ba}^{j_2}$ and we

[13] Note that the role played by (l_1, l_2) here is slightly different than in (1.5.9) above. Here the derivatives are applied to the individual factors according to Leibniz' rule, resulting in $\boldsymbol{l}, J, J_*$, and $\boldsymbol{l}$ encodes only the derivatives hitting the $(GA')_{ab}$ factor.

define $|\boldsymbol{l}| = |(l_1, l_2)| = l_1 + l_2$, $\sum J = \sum_{\boldsymbol{j}\in J} |\boldsymbol{j}|$. In the remainder of the proof, we need to analyze the rhs. of the inequality derived in (1.5.11). We begin with the third line and study the terms involving Ξ_1^{av} from (1.5.12) afterwards.

Before going into the proof, we note that, due to the cumulant expansion in (1.5.11), there are chains of resolvents G and deterministic matrices A appearing, where some of the A's are *not* necessarily regular w.r.t. the spectral parameters of the surrounding G's. The principal idea is to decompose such A with the aid of Lemma 1.3.3 and carefully track the resulting errors. As a rule of thumb, potentially small denominators resulting from resolvent identities (1.3.21) or the integral representation in Lemma 1.5.1 are balanced with the linear perturbative estimates from Lemma 1.3.3. See also Remark 1.5.3 below.

Gaussian contribution: third line of (1.5.11)**.** In order to do so, we need to analyze in total four terms, each of which carries a factor of

$$\langle GE_\sigma GA'E_\sigma GA\rangle \quad \text{or} \quad \langle G^*E_\sigma GA'E_\sigma G^*A^*\rangle\,, \quad \text{for} \quad \sigma = \pm\,.$$

Since their treatment is very similar, we focus on the two exemplary terms

$$\text{(i)} \quad \langle GGA'GA\rangle \qquad \text{and} \qquad \text{(ii)} \quad \langle G^*GA'G^*A^*\rangle\ . \tag{1.5.13}$$

In the analysis of the Gaussian contribution in Section 1.5.2, we will discuss the analogs of the other two terms in more detail.

First term. For the first term in (1.5.13), we apply the integral representation from Lemma 1.5.1 to GG with

$$\tau = +\,, \quad J = \mathbf{B}_{\ell\kappa_0}\,, \quad \text{and} \quad \tilde\eta = \frac{\ell}{\ell+1}\eta\,,$$

for which we recall that $w \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$, i.e. in particular $\eta \geq (\ell+1)N^{-1+\epsilon_0}$ and hence $\tilde\eta \geq \ell N^{-1+\epsilon_0}$. The fact that J is a union of compact intervals follows from the fact the support of the density of H^Λ has finitely many components. In particular, $\Gamma \equiv \Gamma_{\tilde\eta}^{\tau}(J) \subset \mathbf{D}_{\ell}^{(\epsilon_0,\kappa_0)}$. Now, we split the contour Γ in three parts,[14] i.e.

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3\,. \tag{1.5.14}$$

As depicted in Figure 1.5.2, the first part of the contour consists of the entire horizontal part of Γ. The second part, Γ_2, covers the vertical components up to $|\mathrm{Im}\, z| \leq N^{100}$. Finally, Γ_3 consists of the remaining part with $|\mathrm{Im}\, z| > N^{100}$.

Now, the contribution coming from Γ_3 can easily be estimated by one via a trivial norm bound on G. For $z \in \Gamma_2$, we use that $\mathbf{1}_\delta^\pm(z,w) = 0$ for every $w \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ (recall (1.2.23) and (1.4.22)) and hence every matrix is (z,w)-regular. Therefore, after adding and subtracting the corresponding deterministic approximation, we can bound this part by $(1+\psi_2^{\mathrm{av}}/(N\eta))$ with the aid of Lemma 1.4.3. Hence, after splitting the contour integral and bounding each contribution as just described, we find

$$\big|\langle GGA'GA\rangle\big| \prec \left(1+\frac{\psi_2^{\mathrm{av}}}{N\eta}\right) + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{|\langle G(x+\mathrm{i}\tilde\eta)A'G(e+\mathrm{i}\eta)A\rangle|}{(x-e)^2+\eta^2}\,\mathrm{d}x\,. \tag{1.5.15}$$

Next, we decompose $A = \mathring{A} = \mathring{A}^{e+\mathrm{i}\eta,e+\mathrm{i}\eta}$ and $A' = \mathring{A}' = (\mathring{A}')^{e+\mathrm{i}\eta,e+\mathrm{i}\eta}$ according to Lemma 1.3.3 as

$$\mathring{A}^{e+\mathrm{i}\eta,e+\mathrm{i}\eta} = \mathring{A}^{e+\mathrm{i}\eta,x+\mathrm{i}\tilde\eta} + \mathcal{O}\big(|x-e|+\eta\big)E_+ + \mathcal{O}\big(|x-e|+\eta\big)E_-\,,$$

[14] In the case of several $w_1, \ldots, w_k$, the second part might require a further decomposition: If the spectral parameters of the resolvents which are *not* involved in such an integral representation have spectral parameters with imaginary parts of absolute value greater than one, we need to split Γ_2 according to $|\mathrm{Im}\, z| \leq 1$ and $|\mathrm{Im}\, z| > 1$. While the former will be treated exactly as Γ_2 here, the latter shall be estimated by means of the $\eta > 1$-laws, which we discussed after Remark 1.4.6.

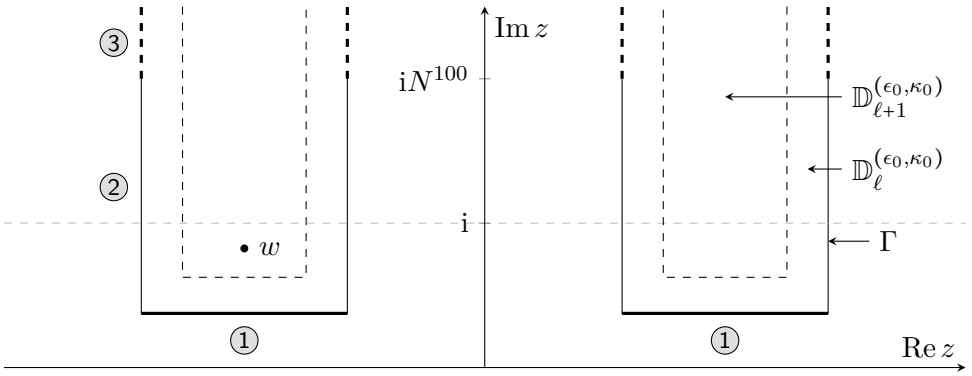

Figure 1.5.2: The contour Γ is split into three parts (see (1.5.14)). In case of multiple spectral parameters, the second part might require a further decomposition at the level indicated by the dashed horizontal line (see Footnote 14). Depicted is the situation, where the bulk $\mathbf{B}_{\ell\kappa_0}$ consists of two components.

$$(\mathring{A}')^{e+\mathrm{i}\eta,e+\mathrm{i}\eta} = (\mathring{A}')^{x+\mathrm{i}\tilde{\eta},e+\mathrm{i}\eta,} + \mathcal{O}\big(|x-e|+\eta\big)E_+ + \mathcal{O}\big(|x-e|+\eta\big)E_- \,.$$

Plugging this into (1.5.15), we obtain several terms contributing to the integral. Adding and subtracting the deterministic approximation, the leading term accounts for

$$\int_{\mathbf{B}_{\ell\kappa_0}} \frac{\big|\langle G(x+\mathrm{i}\tilde{\eta})(\mathring{A}')^{x+\mathrm{i}\tilde{\eta},e+\mathrm{i}\eta} G(e+\mathrm{i}\eta)\mathring{A}^{e+\mathrm{i}\eta,x+\mathrm{i}\tilde{\eta}}\rangle\big|}{(x-e)^2+\eta^2}\,\mathrm{d}x \prec \frac{1}{\eta}\left(1+\frac{\psi_2^{\mathrm{av}}}{N\eta}\right)$$

by means of Lemma 1.4.3. Here the "1" on the right-hand side is due to the contribution of the deterministic approximation $\langle M(x+\mathrm{i}\tilde{\eta},(A')^\circ,e+\mathrm{i}\eta)\mathring{A}^{e+\mathrm{i}\eta,x+\mathrm{i}\tilde{\eta}}\rangle$, while the "$\psi_2^{\mathrm{av}}/(N\eta)$" is due to the definition of Ψ_2^{av} and the bound $\Psi_2^{\mathrm{av}} \prec \psi_2^{\mathrm{av}}$. The error terms can be dealt with by simple resolvent identities (1.3.21) in combination with the usual single-resolvent local law, Theorem 1.2.6, proving them to be bounded by η^{-1}. Indeed, for a generic $B \in \mathbf{C}^{2N\times 2N}$, we consider the exemplary term

$$\begin{aligned}\int_{\mathbf{B}_{\ell\kappa_0}} |\langle G(x+\mathrm{i}\tilde{\eta})E_+G(e+\mathrm{i}\eta)B\rangle| &\frac{|x-e|+\eta}{(x-e)^2+\eta^2}\,\mathrm{d}x \\ &\lesssim \int_{\mathbf{B}_{\ell\kappa_0}} \frac{\big|\big\langle\big(G(x+\mathrm{i}\tilde{\eta})-G(e+\mathrm{i}\eta)\big)B\big\rangle\big|}{(x-e)^2+\eta^2}\,\mathrm{d}x \prec \frac{1}{\eta}\,.\end{aligned}$$

Second term. The second term in (1.5.13) is much simpler than the first. After writing $GG^* = \mathrm{Im}\, G/\eta$, it suffices to realise that, by means of Lemma 1.3.3,

$$A' = (\mathring{A}')^{e+\mathrm{i}\eta,e-\mathrm{i}\eta}, \quad (\mathring{A}')^{e-\mathrm{i}\eta,e-\mathrm{i}\eta} = A' + \mathcal{O}(|e|)E_-, \quad \text{and} \quad A^* = (\mathring{A}^*)^{e-\mathrm{i}\eta,e\pm\mathrm{i}\eta}$$

in order to bound

$$\big|\langle G^*GA'G^*A^*\rangle\big| \prec \frac{1}{\eta}\left(1+\frac{\psi_2^{\mathrm{av}}}{N\eta}\right) + \frac{|e|}{\eta}\,\frac{\big|\langle[G(-e+\mathrm{i}\eta)-G(e-\mathrm{i}\eta)]A^*E_-\rangle\big|}{|e|+\eta} \prec \frac{1}{\eta}\left(1+\frac{\psi_2^{\mathrm{av}}}{N\eta}\right)$$

with the aid of Lemma 1.4.3, the chiral symmetry (1.2.16), a resolvent identity (1.3.21) and Theorem 1.2.6.

This finishes the estimate for the Gaussian contribution from the third line of (1.5.11), for which we have shown that

$$\frac{1}{N^2}\sum_\sigma \big(\big|\langle GE_\sigma GA'E_\sigma GA\rangle\big| + \big|\langle G^*E_\sigma GA'E_\sigma G^*A^*\rangle\big|\big) \prec \frac{1}{N^2\eta}\left(1+\frac{\psi_2^{\mathrm{av}}}{N\eta}\right). \tag{1.5.16}$$

We are now left with the terms from the last line (1.5.11) resulting from higher order cumulants.

Higher order cumulants and conclusion. The terms stemming from higher order cumulants are estimated in Section 1.5.5, the precise bound being given in (1.5.74a). Indeed, plugging (1.5.16) and (1.5.74a) into (1.5.11) we obtain

$$\begin{aligned}\mathbf{E}|\langle(G-M)A\rangle|^{2p} &\prec (\mathcal{E}_1^{\mathrm{av}})^{2p} \\ &+ \sum_{m=1}^{p}\left[\frac{1}{N\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}+(\psi_2^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}}+\frac{(\psi_2^{\mathrm{iso}})^{1/8}}{(N\eta)^{1/8}}\right)\right]^m \left(\mathbf{E}|\langle(G-M)A\rangle|^{2p}\right)^{1-m/2p}\end{aligned}$$

and get the appropriate estimate $\mathbf{E}|\dots|^{2p}$ using Young inequalities. Since p was arbitrary, it follows that

$$\Psi_1^{\mathrm{av}} \prec 1+\frac{\psi_1^{\mathrm{av}}}{N\eta}+\frac{\psi_1^{\mathrm{iso}}+(\psi_2^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}}+\frac{(\psi_2^{\mathrm{iso}})^{1/4}}{(N\eta)^{1/8}}\,.$$

The bound given in Proposition 1.4.9 is an immediate consequence after a further trivial Young inequality. □

Remark 1.5.3. *Although the proof of the first master inequality* (1.4.26a) *is rather short, it already revels a general strategy for dealing with a generic (not strictly) alternating chain*

$$\cdots GGAGAGE\_AGE\_GA\cdots \tag{1.5.17}$$

of resolvents G and deterministic matrices A.

(i) Apply resolvent identites (1.3.21) *and the integral representation from Lemma 1.5.1 in order to reduce a product of resolvents to a linear combination (discrete or continuous, respectively). For terms of the form GE_G instead of GG this additionally requires an application of the chiral symmetry* (1.2.16).

(ii) In the resulting strictly alternating chain, decompose every deterministic A according to the regularisation from Definition 1.4.2 w.r.t. the spectral parameters of its surrounding resolvents by using Lemma 1.3.3.

(iii) Estimate the regular parts coming from this decomposition in terms of $\Psi_k^{\mathrm{av/iso}} \prec \psi_k^{\mathrm{av/iso}}$. Carefully track the resulting errors stemming from the other parts.

These steps shall be applied repeatedly until the entire chain (1.5.17) *has been examined. The first two steps outlined above will be performed mechanically without any complication. However, the third step is non-trivial and requires careful analysis on a case-by-case basis.*

We have already mentioned that, as a rule of thumb, potentially small denominators resulting from Step (i) are balanced with the linear perturbative numerators from Step (ii).

It remains to give a proof of Lemma 1.5.2.

Proof of Lemma 1.5.2. Similarly as in (1.5.6), we suppress the indices of $G \equiv G_1$, $M \equiv M_1$ etc.

We start with the first identity in (1.5.4), such that, after defining the one-body stability operator

$$\mathcal{B} := 1 - M\mathcal{S}[\cdot]M$$

we find

$$\mathcal{B}[G-M] = -M\underline{WG} + M\mathcal{S}[G-M](G-M)$$

and consequently, by inversion, multiplication by $A = \mathring{A}$ (in the sense of (1.4.8), see also (1.4.9)) and taking a trace

$$\langle(G-M)A\rangle = -\langle\underline{WG}\mathcal{X}[A]M\rangle + \langle\mathcal{S}[G-M](G-M)\mathcal{X}[A]M\rangle\,, \tag{1.5.18}$$

where we introduced the linear operator

$$\mathcal{X}[B] := \big((\mathcal{B}^*)^{-1}[B^*]\big)^* = \big(1 - \mathcal{S}[M \cdot M]\big)^{-1}[B] \quad \text{for} \quad B \in \mathbf{C}^{2N\times 2N}.$$

Then, it is important to note that the condition $\mathbf{1}_\delta^+\langle \operatorname{Im} MA\rangle = 0$ (the first of the two imposed via (1.4.9); recall the definition of the cutoff function $\mathbf{1}_\delta^+$ from (1.3.6) and (1.4.7)), is stable under the linear operation $A \mapsto \mathcal{X}[A]M$.

Lemma 1.5.4. *For a generic $B \in \mathbf{C}^{2N\times 2N}$, we find*

$$\langle \mathcal{X}[B]M\operatorname{Im} M\rangle = \langle B\mathcal{B}^{-1}[M\operatorname{Im} M]\rangle = \frac{\mathrm{i}}{2}\frac{\langle B\operatorname{Im} M\rangle}{\langle \operatorname{Im} M\rangle} + \mathcal{O}(\eta). \tag{1.5.19}$$

Proof. Using (1.B.10), we compute

$$\mathcal{B}^{-1}[M\operatorname{Im} M] = \frac{\mathcal{B}^{-1}[M^2 - MM^*]}{2\mathrm{i}} = \frac{\mathrm{i}}{2}\frac{\operatorname{Im} M}{\eta + \langle \operatorname{Im} M\rangle} + \frac{1}{2\mathrm{i}}\frac{1 - \langle MM^*\rangle}{1 - \langle M^2\rangle}M^2.$$

Now, by means of Lemma 1.B.4 and Lemma 1.B.5, we find that

$$|1 - \langle MM^*\rangle| = \mathcal{O}(\eta) \quad \text{and} \quad \big|1 - \langle M^2\rangle\big| \gtrsim 1, \quad \text{respectively.} \qquad \square$$

Recall from (1.5.5) that $\mathcal{S}[G - M] = \langle G - M\rangle$. Therefore, by means of the usual averaged local law, Theorem 1.2.6, which in particular shows that $|\langle \underline{WG}B\rangle| \prec \frac{1}{N\eta}$ for arbitrary $\|B\| \lesssim 1$ (see also Appendix 1.C and [243]), we can write (1.5.18) as

$$\begin{aligned}\langle (G - M)A\rangle = &- \langle \underline{WG}(\mathcal{X}[A]M)^\circ\rangle + \langle G - M\rangle\langle (G - M)(\mathcal{X}[A]M)^\circ\rangle \\ &- \mathbf{1}_\delta^- c_-(\mathcal{X}[A]M)\langle \underline{WG}E_-\rangle + \mathcal{O}_\prec\big(N^{-1}\big),\end{aligned} \tag{1.5.20}$$

where in the underlined term, we used that the E_+ component of the regularisation of $\mathcal{X}[A]M$ is negligible thanks to Lemma 1.5.4 and the regularity of A, and we introduced the short hand notation

$$c_-(\mathcal{X}[A]M) := \frac{\langle M\mathcal{X}[A]MME_-\rangle}{\langle ME_-ME_-\rangle}.$$

Next, with the aid of $WG = I - \hat{\Lambda}G + wG$ and using $\langle GE_-\rangle = 0$ from (1.5.5), we undo the underline in the second to last term, such that we infer

$$\langle \underline{WG}E_-\rangle = -\langle GE_-\hat{\Lambda}\rangle = -\langle (G - M)E_-\hat{\Lambda}\rangle = -\langle (G - M)(E_-\hat{\Lambda})^\circ\rangle.$$

In the second equality, we used that $\langle ME_-\hat{\Lambda}\rangle = 0$, which follows by a simple computation using the explicit form of M given in (1.2.17)–(1.2.18). For the last equality, we note that

$$(E_-\hat{\Lambda})^\circ = E_-\hat{\Lambda} - \mathbf{1}_\delta^+\frac{\langle \operatorname{Im} ME_-\hat{\Lambda}\rangle}{\langle \operatorname{Im} M\rangle}E_+ - \mathbf{1}_\delta^-\frac{\langle ME_-\hat{\Lambda}ME_-\rangle}{\langle ME_-ME_-\rangle}E_- = E_-\hat{\Lambda},$$

which again follows after a simple computation using the fact that $\hat{\Lambda}$ is off-diagonal together with (1.2.17)–(1.2.18).

We can now write (1.5.20) for $A = \mathring{A} = (E_-\hat{\Lambda})^\circ = E_-\hat{\Lambda}$ and solve the resulting equation for $\langle (G - M)E_-\hat{\Lambda}\rangle$. Plugging this back into (1.5.20) yields

$$\begin{aligned}\langle (G - M)A\rangle = &- \langle \underline{WG}(\mathcal{X}[A]M)^\circ\rangle + \langle G - M\rangle\langle (G - M)(\mathcal{X}[A]M)^\circ\rangle + \mathcal{O}_\prec\big(N^{-1}\big) \\ &+ \frac{\mathbf{1}_\delta^- c_-(\mathcal{X}[A]M)}{1 - \mathbf{1}_\delta^- c_-(\mathcal{X}[E_-\hat{\Lambda}]M)}\Big[- \langle \underline{WG}(\mathcal{X}[E_-Z]M)^\circ\rangle\end{aligned} \tag{1.5.21}$$

$$+\langle G-M\rangle\langle (G-M)(\mathcal{X}[E_-Z]M)^\circ\rangle + \mathcal{O}_\prec\big(N^{-1}\big)\Big].$$

Since $\|\mathcal{X}[\mathring{A}]\| \lesssim 1$ (see Lemma 1.B.6), the only thing left to check is, that the denominator in (1.5.21) is bounded away from zero.

Lemma 1.5.5. *For small enough $\delta > 0$, we have that*

$$\big|1 - \mathbf{1}_\delta^-(w,w)\, c_-(\mathcal{X}[E_-\hat{\Lambda}]M)\big| \gtrsim 1\,.$$

Proof. The statement is trivial for $\mathbf{1}_\delta^-(w,w) = 0$ and we hence focus on the case where $\lambda := \mathbf{1}_\delta^-(w,w) \in (0,1]$. First, we note that $\mathcal{X}[E_-\hat{\Lambda}] = E_-\hat{\Lambda}$, which follows from the explicit form of M given in (1.2.17)–(1.2.18) using the fact that $\hat{\Lambda}$ is purely off-diagonal. Next, we use the MDE (1.2.20), the chiral symmetry (1.2.19), and Lemma 1.B.4 (a) to infer

$$1 - c_-(\mathcal{X}[E_-\hat{\Lambda}]M) = 1 - \frac{\langle ME_-\hat{\Lambda}MME_-\rangle}{\langle ME_-ME_-\rangle} = \frac{1}{2}\Big[1 - \frac{w+m}{m}\langle M^2\rangle\Big]\,.$$

Now, specialising to $w = \mathrm{i}\eta$ with sufficiently small η, we find that, to leading order,

$$\mathrm{Re}\,\Big[1 - \frac{\eta + \mathrm{Im}\, m}{\mathrm{Im}\, m}\langle M^2\rangle\Big] \sim \mathrm{Re}\,\big[1 - \langle M^2\rangle\big] = 1 - \langle MM^*\rangle + 2\langle (\mathrm{Im}\, M)^2\rangle \geq 2\langle \mathrm{Im}\, M\rangle^2 \gtrsim 1 \tag{1.5.22}$$

by direct computation. Using Lipschitz continuity of this expression in w, this principal lower bound on $\mathrm{Re}\big[1 - c_-(\mathcal{X}[E_-\hat{\Lambda}]M)\big]$ of order one persists after a small perturbation of w allowing for a non-zero real part, but as long as $\lambda = \mathbf{1}_\delta^-(w,w) > 0$ for some $\delta > 0$ small enough. Hence, we conclude the lower bound

$$\big|1 - \lambda\, c_-(\mathcal{X}[E_-\hat{\Lambda}]M)\big| \geq (1-\lambda)1 + \lambda \mathrm{Re}\big[1 - c_-(\mathcal{X}[E_-\hat{\Lambda}]M)\big] \gtrsim 1 \tag{1.5.23}$$

for the convex combination, by separately considering smaller and larger values of $\lambda \in (0,1]$. □

From the expansion (1.5.21) it is apparent, that the main terms for understanding the size of $\langle (G-M)A\rangle$ are the underlined ones, the rest carrying additional $\langle G-M\rangle$-factors, hence they will become negligible errors. In fact, summarizing our investigations, we have shown that

$$\langle (G-M)\mathring{A}\rangle = -\langle \underline{WG\mathring{A}'}\rangle + \mathcal{O}_\prec\big(\mathcal{E}_1^{\mathrm{av}}\big)\,,$$

where we used the shorthand notation

$$\mathring{A}' := (\mathcal{X}[\mathring{A}]M)^\circ + \frac{\mathbf{1}_\delta^- c_-(\mathcal{X}[\mathring{A}]M)}{1 - \mathbf{1}_\delta^-\, c_-(\mathcal{X}[E_-\hat{\Lambda}]M)}(\mathcal{X}[E_-\hat{\Lambda}]M)^\circ \tag{1.5.24}$$

in the underlined term. Using the usual averaged local law (1.4.17) and (1.4.25), we collected all the error terms from (1.5.21) in $\mathcal{E}_1^{\mathrm{av}}$, defined in (1.5.7). □

1.5.2 Proof of the second master inequality (1.4.26b)

Let $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ for $j \in [2]$ be spectral parameters and A_1 a regular matrix w.r.t. the pair of spectral parameters (w_1, w_2) (see Definition 1.4.2). By conjugation with E_-, we will assume w.l.o.g. that $\mathrm{Im}\, w_1 > 0$ and $\mathrm{Im}\, w_2 < 0$. Moreover, we use the notations $e_j \equiv \mathrm{Re}\, w_j$, $\eta_j := |\mathrm{Im}\, w_j|$ for $j \in [2]$ and define $1 \geq \eta := \min_j |\mathrm{Im}\, w_j|$. We also assume that (1.4.25) holds.

Lemma 1.5.6. (Representation as full underlined)
For $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \le 1$ and any (w_1, w_2)-regular matrix $A_1 = \mathring{A}_1$, we have that

$$\big(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\big)_{\boldsymbol{xy}} = -\big(\underline{G_1 \mathring{A}_1' W G_2}\big)_{\boldsymbol{xy}} + \mathcal{O}_{\prec}\big(\mathcal{E}_1^{\mathrm{iso}}\big) \tag{1.5.25}$$

for some (w_1, w_2)-regular matrix $A_1' = \mathring{A}_1'$, which linearly depends on $A_1 = \mathring{A}_1$ (see (1.5.57)*). For the error term in* (1.5.25)*, we used the shorthand notation*

$$\mathcal{E}_1^{\mathrm{iso}} := \frac{1}{\sqrt{N\eta^2}} \left(1 + \frac{\psi_1^{\mathrm{av}}}{(N\eta)^{1/2}} + \frac{\psi_1^{\mathrm{iso}}}{N\eta}\right). \tag{1.5.26}$$

Note that unlike in Section 1.5.1, now in (1.5.25) the second resolvent G_2 was expanded instead of G_1 rendering the W factor in the middle of the underlined term. This prevents the emergence of resolvent chains in the proof of (1.4.26b), which are 'too long' to be handled within our hierarchical framework of master inequalities (e.g., a chain involving four resolvents would appear in $\widetilde{\Xi}_1^{\mathrm{iso}}$ defined below).

Having this approximate representation of $\big(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\big)_{\boldsymbol{xy}}$ as a full underlined term at hand, we turn to the proof of (1.4.26b) via a (minimalistic) cumulant expansion.

Proof of (1.4.26b). Let $p \in \mathbf{N}$. Then, starting from (1.5.25) and using the same notations as in the proof of (1.4.26a), we obtain

$$\begin{aligned} &\mathbf{E}\big|\big(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\big)_{\boldsymbol{xy}}\big|^{2p} \\ &\lesssim \mathbf{E}\, \widetilde{\Xi}_1^{\mathrm{iso}} \big|\big(G_1 \mathring{A}_1 G_2 - M(\ldots)\big)_{\boldsymbol{xy}}\big|^{2p-2} \\ &\quad + \sum_{|\boldsymbol{l}| + \sum(J \cup J_*) \ge 2} \mathbf{E}\, \Xi_1^{\mathrm{iso}}(\boldsymbol{l}, J, J_*) \big|\big(G_1 \mathring{A}_1 G_2 - M(\ldots)\big)_{\boldsymbol{xy}}\big|^{2p-1-|J \cup J_*|} + \mathcal{O}_{\prec}\big((\mathcal{E}_1^{\mathrm{iso}})^{2p}\big), \end{aligned} \tag{1.5.27}$$

where

$$\begin{aligned} \widetilde{\Xi}_1^{\mathrm{iso}} :=& \frac{\sum_\sigma \Big[\Big|\big(G_1 \mathring{A}_1' E_\sigma G_1 \mathring{A}_1 G_2\big)_{\boldsymbol{xy}} \big(G_1 E_\sigma G_2\big)_{\boldsymbol{xy}}\Big| + \Big|\big(G_1 \mathring{A}_1' E_\sigma G_2\big)_{\boldsymbol{xy}} \big(G_1 \mathring{A}_1 G_2 E_\sigma G_2\big)_{\boldsymbol{xy}}\Big|\Big]}{N} \\ &+ \frac{\sum_\sigma \Big[\Big|\big(G_1 \mathring{A}_1' E_\sigma G_2^* (\mathring{A}_1)^* G_1^*\big)_{\boldsymbol{xx}} \big(G_2^* E_\sigma G_2\big)_{\boldsymbol{yy}}\Big| + \Big|\big(G_1 \mathring{A}_1' E_\sigma G_1^*\big)_{\boldsymbol{xx}} \big(G_2^* (\mathring{A}_1)^* G_1^* E_\sigma G_2\big)_{\boldsymbol{yy}}\Big|\Big]}{N} \end{aligned}$$

and $\Xi_1^{\mathrm{iso}}(\boldsymbol{l}, J, J_*)$ is defined via

$$\begin{aligned} \Xi_1^{\mathrm{iso}} := N^{-(|\boldsymbol{l}| + \sum(J \cup J_*) + 1)/2} \sum_{ab} R_{ab} \big|\partial^{\boldsymbol{l}} \big[(G_1 \mathring{A}_1')_{\boldsymbol{x}a} \big(G_2\big)_{b\boldsymbol{y}}\big]\big| \\ \times \prod_{\boldsymbol{j} \in J} \big|\partial^{\boldsymbol{j}} \big(G_1 \mathring{A}_1 G_2\big)_{\boldsymbol{xy}}\big| \prod_{\boldsymbol{j} \in J_*} \big|\partial^{\boldsymbol{j}} \big(G_2^* (\mathring{A}_1)^* G_2^*\big)_{\boldsymbol{yx}}\big|. \end{aligned} \tag{1.5.28}$$

In the remainder of the proof, we need to analyze the rhs. of the inequality derived in (1.5.27). Following the general strategy outlined in Remark 1.5.3, we begin with the second line and study the terms involving Ξ_1^{iso} from (1.5.28) afterwards.

Gaussian contribution: third line of (1.5.27)**.** In order to do so, following Remark 1.5.3, we need to analyze in total eight terms, each of which carries one of the summands in the definition of $\widetilde{\Xi}_1^{\mathrm{iso}}$ as a factor. Since their treatment is very similar, we focus on the two exemplary terms

$$\text{(i)}\ \big(G_1 \mathring{A}_1' E_- G_1 \mathring{A}_1 G_2\big)_{\boldsymbol{xy}} \big(G_1 E_- G_2\big)_{\boldsymbol{xy}}, \quad \text{(ii)}\ \big(G_1 \mathring{A}_1' E_- G_1^*\big)_{\boldsymbol{xx}} \big(G_2^* (\mathring{A}_1)^* G_1 E_- G_2\big)_{\boldsymbol{yy}}, \tag{1.5.29}$$

showcasing the key difficulties. Recall that, in the analysis of the Gaussian term in Section 1.5.1 we discussed analogs of the above terms with the choice $\sigma = +$.

Term (i) in (1.5.29). For the first term, we decompose, similarly to Lemma 1.3.3,

$$(\mathring{A}_1')^{1,2}E_- = \big((\mathring{A}_1')^{1,2}E_-\big)^{\circ_{1,1}} + \mathcal{O}\big(|e_1+e_2|+|\eta_1-\eta_2|\big)E_+ + \mathcal{O}\big(|e_1+e_2|+|\eta_1-\eta_2|\big)E_- . \tag{1.5.30}$$

Inserting this into the first term in (1.5.29) and using Lemma 1.4.3, we find

$$\Big|\big(G_1\mathring{A}_1'E_-G_1\mathring{A}_1G_2\big)_{\boldsymbol{xy}}\Big| \prec \frac{1}{\eta}\left(1+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right) + \big(|e_1+e_2|+|\eta_1-\eta_2|\big)\sum_\sigma \Big|\big(G_1E_\sigma G_1\mathring{A}_1G_2\big)_{\boldsymbol{xy}}\Big| . \tag{1.5.31}$$

In the last term, we focus on $\sigma = -$, while $\sigma = +$ can be dealt with by Lemma 1.5.1. In fact, using (1.2.16) and a resolvent identity (1.3.21), we obtain

$$\Big|\big(G_1E_-G_1\mathring{A}_1G_2\big)_{\boldsymbol{xy}}\Big| = \left|\frac{1}{w_1}\big([G(-w_1)-G(w_1)]\mathring{A}_1^{w_1,w_2}G(w_2)\big)_{(E_-\boldsymbol{x})\boldsymbol{y}}\right| \prec \frac{1}{\eta^2}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right),$$

where in the last step we used Lemma 1.4.3 and the trivial approximation

$$\mathring{A}_1^{-w_1,w_2} = \mathring{A}_1^{w_1,w_2} + \mathcal{O}(1)E_+ + \mathcal{O}(1)E_- .$$

For the second factor in the first term in (1.5.29), we use (1.2.16) and employ the integral representation from Lemma 1.5.1 with

$$\tau = +, \quad J = \mathbf{B}_{\ell\kappa_0}, \quad \text{and} \quad \tilde{\eta} = \frac{\ell}{\ell+1}\eta,$$

for which we recall that $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$, i.e. in particular $\eta \geq (\ell+1)N^{-1+\epsilon_0}$ and hence $\tilde{\eta} \geq \ell N^{-1+\epsilon_0}$. After splitting the contour integral and estimating the contribution as described around (1.5.14), we find, with the aid of Lemma 1.4.3 and absorbing logarithmic corrections into '$\prec$', that

$$\begin{aligned}\Big|\big(G_1E_-G_2\big)_{\boldsymbol{xy}}\Big| &\prec 1 + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{\big|\big(G(x+\mathrm{i}\tilde{\eta})\big)_{\boldsymbol{x}(E_-\boldsymbol{y})}\big|}{\big|\big(x-e_1-\mathrm{i}(\eta_1-\tilde{\eta})\big)\big(x+e_2-\mathrm{i}(\eta_2-\tilde{\eta})\big)\big|}\mathrm{d}x \\ &\prec 1 + \frac{1}{|e_1+e_2|+\eta_1+\eta_2}\end{aligned} \tag{1.5.32}$$

where in the last step we used the usual single resolvent local law from Theorem 1.2.6. Notice the key cancellation of the $|e_1+e_2|$ factor in (1.5.31) and (1.5.32). Collecting all the estimates, we have shown that

$$\big|(1.5.29)\,(\mathrm{i})\big| \prec \frac{1}{\eta^2}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right). \tag{1.5.33}$$

Term (ii) in (1.5.29). In the first factor in the second term in (1.5.29), we again employ the decomposition (1.5.30) to find

$$\Big|\big(G_1\mathring{A}_1'E_-G_1^*\big)_{\boldsymbol{xx}}\Big| \prec \frac{1}{\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)+\frac{|e_1+e_2|+|\eta_1-\eta_2|}{\eta} \tag{1.5.34}$$

with the aid of Theorem 1.2.6 and Lemma 1.4.3 as well as a resolvent identity (1.3.21) and Lemma 1.5.1 for the E_+ and E_- in (1.5.30), respectively.

In the second factor, similarly to (1.5.32) above, we use Lemma 1.5.1 together with the decomposition[15]

$$(\mathring{A}_1^{w_1,w_2})^* = (\mathring{A}_1^*)^{\bar{w}_2,\bar{w}_1} = (\mathring{A}_1^*)^{\bar{w}_2,w_1} = (\mathring{A}_1^*)^{\bar{w}_2,x+\mathrm{i}\tilde{\eta}} + \sum_\sigma \mathcal{O}_\sigma(|x-e_1|+|\eta_1-\tilde{\eta}|)E_\sigma$$

[15] The notation $\sum_\sigma \mathcal{O}_\sigma(\alpha)E_\sigma$ means that, for each σ, the term $\mathcal{O}_\sigma(\alpha)$ is a scalar function $g_\sigma(\alpha)$ of order $\mathcal{O}(\alpha)$.

from Lemma 1.3.3 for arbitrary x to find

$$\begin{aligned}\left|\left(G_2^*(\mathring{A}_1)^* G_1 E_- G_2\right)_{\boldsymbol{yy}}\right| \prec & \frac{1}{\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right) \\ & + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{\left|\left(G(\bar{w}_2)(\mathring{A}_1^*)^{\bar{w}_2, x+\mathrm{i}\tilde{\eta}} G(x+\mathrm{i}\tilde{\eta})\right)_{\boldsymbol{y}(E_-\boldsymbol{y})}\right|}{\left|\left(x-e_1-\mathrm{i}(\eta_1-\tilde{\eta})\right)\left(x+e_2-\mathrm{i}(\eta_2-\tilde{\eta})\right)\right|}\mathrm{d}x \\ & + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{\sum_\sigma \left|\left(G(\bar{w}_2) E_\sigma G(x+\mathrm{i}\tilde{\eta})\right)_{\boldsymbol{y}(E_-\boldsymbol{y})}\right|}{|x+e_2-\mathrm{i}(\eta_2-\tilde{\eta})|}\mathrm{d}x \\ \prec & \frac{1}{\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1+\frac{1}{|e_1+e_2|+\eta_1+\eta_2}\right)+\frac{1}{\eta}\,.\end{aligned} \tag{1.5.35}$$

Now, combining (1.5.34) and (1.5.35), we obtain

$$\left|(1.5.29)\,(\mathrm{ii})\right| \prec \frac{1}{\eta^2}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)^2\,. \tag{1.5.36}$$

This finishes the estimate for the Gaussian contribution from the third line of (1.5.27), for which we have shown that

$$\widetilde{\Xi}_1^{\mathrm{iso}} \prec \frac{1}{N\eta^2}\left(1+\frac{(\psi_1^{\mathrm{iso}})^2}{N\eta}+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right) \tag{1.5.37}$$

as easily follows by combining (1.5.33) with (1.5.36) and using a Schwarz inequality.

We are now left with the terms from the last line (1.5.27) resulting from higher order cumulants.

Higher order cumulants and conclusion. The estimate stemming from higher order cumulants is given in (1.5.74b). Then, plugging (1.5.37) and (1.5.74b) into (1.5.27), we find, similarly to Section 1.5.1, that

$$\Psi_1^{\mathrm{iso}} \prec 1+\frac{\psi_1^{\mathrm{iso}}}{N\eta}+\frac{\psi_1^{\mathrm{iso}}+\psi_1^{\mathrm{av}}}{(N\eta)^{1/2}}+\frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}+\frac{(\psi_2^{\mathrm{iso}})^{1/4}}{(N\eta)^{1/8}}\,.$$

The bound given in Proposition 1.4.9 is an immediate consequence after a trivial Young inequality. □

It remains to give a proof of Lemma 1.5.6. This is much more involved than for the previous underlined Lemma 1.5.2. The proof of Lemma 1.5.2 crucially used that the orthogonality $\langle \mathrm{Im}\, MA\rangle = 0$ is (almost) preserved under the operation $A \mapsto \mathcal{X}[A]M$ (see Lemma 1.5.4). This is simply not available here, since we deal with two spectral parameters w_1, w_2.

Proof of Lemma 1.5.6. We denote $A_1 \equiv \mathring{A}_1$, except we wish to emphasise A_1 being regular. Just as in Section 1.5.1, we start with

$$G_2 = M_2 - M_2\underline{WG_2} + M_2\mathcal{S}[G_2-M_2]G_2\,,$$

such that we get

$$G_1\tilde{A}_1G_2 = G_1\tilde{A}_1M_2 - G_1\tilde{A}_1M_2\underline{WG_2} + G_1\tilde{A}_1M_2\mathcal{S}[G_2-M_2]G_2$$

for $\tilde{A}_1 = \mathcal{X}_{12}[A_1]$ and $A_1 = \mathring{A}_1$ (note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ by Lemma 1.B.6), where we introduced the linear operator

$$\mathcal{X}_{12}[B] := \big(1-\mathcal{S}[M_1\,\cdot\, M_2]\big)^{-1}[B] \quad \text{for} \quad B \in \mathbf{C}^{2N\times 2N}\,. \tag{1.5.38}$$

Extending the underline to the whole product, we obtain

$$G_1\tilde{A}_1G_2 = M_1\tilde{A}_1M_2 + (G_1-M_1)\tilde{A}_1M_2 - \underline{G_1\tilde{A}_1M_2WG_2}$$

$$+ G_1\tilde{A}_1M_2\mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[G_1\tilde{A}_1M_2]G_2\,,$$

from which we conclude that

$$\begin{aligned} G_1\big(\tilde{A}_1 - \mathcal{S}[M_1\tilde{A}_1M_2]\big)G_2 = {} & M_1\tilde{A}_1M_2 + (G_1 - M_1)\tilde{A}_1M_2 - \underline{G_1\tilde{A}_1M_2WG_2} \\ & + G_1\tilde{A}_1M_2\mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)\tilde{A}_1M_2]G_2 \end{aligned}$$

and thus

$$\begin{aligned} G_1A_1G_2 = {} & M_1\mathcal{X}_{12}[A_1]M_2 + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1\mathcal{X}_{12}[A_1]M_2WG_2} \\ & + G_1\mathcal{X}_{12}[A_1]M_2\mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)\mathcal{X}_{12}[A_1]M_2]G_2\,. \end{aligned} \tag{1.5.39}$$

We note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ by means of Lemma 1.B.6.

Then, we need to further decompose $\mathcal{X}_{12}[A_1]M_2$ in the last three terms in (1.5.39) as

$$\mathcal{X}_{12}[A_1]M_2 = \big(\mathcal{X}_{12}[A_1]M_2\big)^\circ + \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)E_\sigma\,, \tag{1.5.40}$$

where we suppressed the spectral parameters (and the relative sign of their imaginary parts, which has been fixed by $\operatorname{Im} w_1 > 0$ and $\operatorname{Im} w_2 < 0$) in the notation for the linear functionals $c_\sigma(\cdot)$ on $\mathbf{C}^{2N\times 2N}$ defined as

$$c_+(B) := \frac{\langle M_1BM_2\rangle}{\langle M_1M_2\rangle} \qquad \text{and} \qquad c_-(B) := \frac{\langle M_1BM_2^*E_-\rangle}{\langle M_1E_-M_2^*E_-\rangle}\,. \tag{1.5.41}$$

Plugging (1.5.40) into (1.5.39) we find $G_1A_1G_2$ to equal

$$\begin{aligned} & M_1\mathcal{X}_{12}[A_1]M_2 + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ WG_2} \\ & + G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ\mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}\big[(G_1 - M_1)\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ\big]G_2 \\ & + \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)\big[-\underline{G_1E_\sigma WG_2} + G_1E_\sigma\mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}[(G_1 - M_1)E_\sigma]G_2\big]\,. \end{aligned} \tag{1.5.42}$$

Recall that the regular component is defined w.r.t. the pair of spectral parameters (w_1, w_2). In particular, $\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ = \big(\mathcal{X}_{12}[A_1]M_2\big)^{\circ_{1,2}}$ in the last term in the second line of (1.5.42) is *not* regular as defined via the conditions with one resolvent (1.4.9).

In the last line of (1.5.42) we now undo the underline and find the bracket $\big[\cdots\big]$ to equal (the negative of)

$$\begin{aligned} & G_1E_\sigma WG_2 + G_1E_\sigma\mathcal{S}[M_2]G_2 + G_1\mathcal{S}[M_1E_\sigma]G_2 \\ = {} & G_1E_\sigma + G_1\big(E_\sigma(w_2 - \hat{\Lambda} + \mathcal{S}[M_2]) + \mathcal{S}[M_1E_\sigma]\big)G_2 \\ = {} & G_1E_\sigma - G_1\big(E_\sigma M_2^{-1} - \mathcal{S}[M_1E_\sigma]\big)G_2 =: G_1E_\sigma - G_1\Phi_\sigma G_2\,, \end{aligned}$$

where we used $WG_2 = E_+ + w_2G_2 - \hat{\Lambda}G_2$ in the first step and the MDE (1.2.20) in the second step. Moreover, we introduced the shorthand notation

$$\Phi_\sigma := E_\sigma\frac{1}{M_2} - \mathcal{S}[M_1E_\sigma]\,. \tag{1.5.43}$$

From the expansion (1.5.42) it is apparent (and it can also be checked by hand using the explicit form of (1.5.43)) that

$$M_1E_\sigma = M_1\big(E_\sigma M_2^{-1}\big)M_2 = M_1\mathcal{X}_{12}[\Phi_\sigma]M_2 = M(w_1, \Phi_\sigma, w_2)\,,$$

where in the last step we used (1.4.4). This finally yields that $G_1A_1G_2$ equals

$$M(w_1, A_1, w_2) + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ WG_2} \tag{1.5.44}$$

$$+ G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}\big[(G_1 - M_1)\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ\big]G_2$$
$$+ \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)\big[-(G_1 - M_1)E_\sigma + \big(G_1\Phi_\sigma G_2 - M(w_1, \Phi_\sigma, w_2)\big)\big]\,.$$

The last term in the last line of (1.5.44) requires further decomposition of Φ_σ from (1.5.43) (completely analogous to (1.5.40) and (1.5.41)) as

$$\Phi_\sigma = \mathring{\Phi}_\sigma + \sum_\tau \mathbf{1}_\delta^\tau\, c_\tau(\Phi_\sigma)E_\tau\,.$$

Using the explicit form of Φ_σ, we further observe that

$$c_\tau(\Phi_\sigma) \sim \delta_{\sigma,\tau} \quad \text{and} \quad c_\tau(\mathcal{X}_{12}[\Phi_\sigma]M_2) \sim \delta_{\sigma,\tau}\,. \tag{1.5.45}$$

Therefore, by means of the first relation in (1.5.45), the expansion (1.5.44) can be carried out further as

$$\begin{aligned} &M(w_1, A_1, w_2) + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ W G_2} \\ &+ G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}\big[(G_1 - M_1)\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ\big]G_2 \\ &+ \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)\Big[-(G_1 - M_1)E_\sigma + \big(G_1\mathring{\Phi}_\sigma G_2 - M(w_1, \mathring{\Phi}_\sigma, w_2)\big) \\ &\qquad + c_\sigma(\Phi_\sigma)\big(G_1 E_\sigma G_2 - M(w_1, E_\sigma, w_2)\big)\Big]. \end{aligned} \tag{1.5.46}$$

Next, we write (1.5.46) for both, $A_1 = \mathring{A}_1 = \mathring{\Phi}_+$ and $A_1 = \mathring{A}_1 = \mathring{\Phi}_-$, and solve the two resulting linear equations for $G_1\mathring{\Phi}_\pm G_2 - M(w_1, \mathring{\Phi}_\pm, w_2)$. Observe that by means of the second relation in (1.5.45) the original system of linear equations boils down to two separate ones. Thus, plugging the solutions for $G_1\mathring{\Phi}_\pm G_2 - M(w_1, \mathring{\Phi}_\pm, w_2)$ back into (1.5.46) we arrive at

$$\begin{aligned} G_1A_1G_2 =& M(w_1, A_1, w_2) + (G_1 - M_1)\mathcal{X}_{12}[A_1]M_2 - \underline{G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ W G_2} \\ &+ G_1\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}\big[(G_1 - M_1)\big(\mathcal{X}_{12}[A_1]M_2\big)^\circ\big]G_2 \\ &+ \sum_\sigma \frac{\mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)}{1 - \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)}\Big[(G_1 - M_1)\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2 - \underline{G_1\big(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2\big)^\circ W G_2} \\ &+ G_1\big(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2\big)^\circ \mathcal{S}[G_2 - M_2]G_2 + G_1\mathcal{S}\big[(G_1 - M_1)\big(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2\big)^\circ\big]G_2 \\ &- (G_1 - M_1)E_\sigma + c_\sigma(\Phi_\sigma)\big(G_1E_\sigma G_2 - M(w_1, E_\sigma, w_2)\big)\Big]. \end{aligned} \tag{1.5.47}$$

We now need to check that the denominators in (1.5.47) are bounded away from zero.

Lemma 1.5.7. *For small enough $\delta > 0$, we have that*

$$\big|1 - \mathbf{1}_\delta^\sigma(w_1, w_2)\, c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)\big| \gtrsim 1 \qquad \textit{for} \quad \sigma = \pm\,.$$

Proof. The statements are trivial for $\mathbf{1}_\delta^\sigma(w_1, w_2) = 0$ and we hence focus on cases where $\lambda^\sigma := \mathbf{1}_\delta^\sigma(w_1, w_2) \in (0, 1]$. First, we compute

$$\begin{aligned} 1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2) &= \langle M_1\rangle \frac{\langle M_1M_2M_2\rangle}{\langle M_1M_2\rangle^2} \qquad \text{and} \\ 1 - c_-(\mathcal{X}_{12}[\mathring{\Phi}_-]M_2) &= \frac{\langle M_1E_-M_2^*M_2^{-1}E_-\rangle + \langle M_1\rangle\langle M_1E_-M_2^*E_-\rangle}{1 + \langle M_1E_-M_2E_-\rangle}\ \frac{\langle M_1E_-M_2M_2^*E_-\rangle}{\langle M_1E_-M_2^*E_-\rangle^2} \end{aligned} \tag{1.5.48}$$

for arbitrary spectral parameters w_1, w_2. Recall that we assumed the two spectral parameters to be on different halfplanes, i.e. $\mathfrak{s}_1 = -\operatorname{sgn}(\operatorname{Im} w_1 \operatorname{Im} w_2) = +$, hence we shall specialise (i) the first expression in (1.5.48) to $w_2 = \bar{w}_1$ and (ii) the second expression in (1.5.48) to $w_2 = -w_1$.

In this case, for the first expression in (1.5.48), using Lemma 1.B.4 and $\operatorname{Im} M_1 \operatorname{Im} w_1 > 0$, we obtain

$$\big|1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2)\big| = \big|\langle M_1\rangle \frac{\langle \operatorname{Im} M_1 M_1^*\rangle}{\langle \operatorname{Im} M_1\rangle^2}(\langle \operatorname{Im} M_1\rangle + \operatorname{Im} w_1)\big| \geq \langle \operatorname{Im} M_1\rangle^2 \gtrsim 1 \tag{1.5.49}$$

in the bulk of the spectrum. This principal lower bound of order one persists after a small perturbation of w_2 around the special case $w_2 = \bar{w}_1$, but as long as $\lambda^+ = 1$ (for some $\delta > 0$ small enough), which proves the claim for $\sigma = +$ and $\lambda^+ = 1$. A further direct computation by estimating real and imaginary part of $1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2)$ instead of its absolute value in (1.5.49), similarly to (1.5.22) shows that also the convex combination

$$(1-\lambda^+)1 + \lambda^+\big[1 - c_+(\mathcal{X}_{12}[\mathring{\Phi}_+]M_2)\big]$$

is bounded away from zero (in absolute value), by separately considering small and large values of $\lambda^+ \in (0,1)$. For the second expression in (1.5.48), the argument is similar and hence omitted. □

Next, we take the scalar product of (1.5.47) with two deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$ satisfying $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \leq 1$. In the resulting expression,there are two particular terms, namely the ones of the form

$$\big(G_1\mathcal{S}\big[(G_1 - M_1)\mathring{A}_1^{1,2}\big]G_2\big)_{\boldsymbol{xy}} \quad \text{and} \tag{1.5.50}$$

$$c_\sigma(\mathcal{X}_{12}[\mathring{A}_1^{1,2}]M_2)c_\sigma(\Phi_\sigma)\big(G_1E_\sigma G_2 - M(w_1, E_\sigma, w_2)\big)_{\boldsymbol{xy}}, \tag{1.5.51}$$

whose direct (naive) estimates are $1/(N\eta^2)$ and $1/\eta$, respectively, and thus do not match the target size. Hence, they have to be discussed in more detail. In our notation, we emphasised that the regularisation is defined w.r.t. the spectral parameters (w_1, w_2), i.e., in particular, $A_1^\circ = A_1^{\circ_{1,2}}$.

Estimating (1.5.50). For the term (1.5.50), we expand

$$\big(G_1\mathcal{S}\big[(G_1 - M_1)\mathring{A}_1^{1,2}\big]G_2\big)_{\boldsymbol{xy}} = \sum_\sigma \sigma\langle (G_1 - M_1)\mathring{A}_1^{1,2}E_\sigma\rangle\big(G_1E_\sigma G_2\big)_{\boldsymbol{xy}} \tag{1.5.52}$$

and observe that, by definition of $\cdot^\circ$ in (1.4.8), we have, similarly to Lemma 1.3.3 (see also (1.5.30)),

$$\mathring{A}_1^{1,2}E_\sigma = \big(\mathring{A}_1^{1,2}E_\sigma\big)^{\circ_{1,1}} + \mathcal{O}\big(|e_1 - \sigma e_2| + |\eta_1 - \eta_2|\big)E_+ + \mathcal{O}\big(|e_1 - \sigma e_2| + |\eta_1 - \eta_2|\big)E_-. \tag{1.5.53}$$

Now, in the second term in (1.5.52) for $\sigma = +$ and $E_\sigma = E_+$, we use a resolvent identity (1.3.21) and the usual isotropic local law (1.4.17) to estimate it as

$$\big|\big(G_1G_2\big)_{\boldsymbol{xy}}\big| < 1 + \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2}. \tag{1.5.54}$$

Furthermore, in the second term in (1.5.1) for $\sigma = -$ and $E_\sigma = E_-$, we employ the integral representation from Lemma 1.5.1 in combination with the usual isotropic local law (1.4.17) (see also (1.5.32)) to infer

$$\big|\big(G_1E_-G_2\big)_{\boldsymbol{xy}}\big| < 1 + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2}. \tag{1.5.55}$$

Combining (1.5.54) and (1.5.55) with the decomposition (1.5.53) and the usual averaged local law (1.4.17), we find that (1.5.52) can be bounded by

$$\sum_\sigma \left(\big|\langle (G_1 - M_1)\big(\mathring{A}_1^{1,2}E_\sigma\big)^{\circ_{1,1}}\rangle\big| + \frac{|e_1 - \sigma e_2| + |\eta_1 - \eta_2|}{N\eta_1}\right)\left(1 + \frac{1}{|e_1 - \sigma e_2| + \eta_1 + \eta_2}\right).$$

Using the definition of Ψ_1^{av} in (1.4.15) and the apriori bound $\Psi_1^{\mathrm{av}} \prec \psi_1^{\mathrm{av}}$, this immediately implies the estimate

$$|(1.5.50)| \prec \frac{1}{N\eta} + \frac{1}{\sqrt{N\eta}} \frac{\psi_1^{\mathrm{av}}}{(N\eta)^{1/2}}. \tag{1.5.56}$$

Estimating (1.5.51). For the term (1.5.51), we first note that the two prefactors $c_\sigma(\mathcal{X}_{12}[A_1^{\circ 1,2}]M_2)$ and $c_\sigma(\Phi_\sigma)$ are bounded. However, in each of the two cases $\sigma = \pm$, the bound on *one* of the prefactors needs to be improved: In the first case, $\sigma = +$, we use (1.B.11) and compute

$$c_+(\Phi_+) = \frac{\langle M_1\rangle\big(1 - \langle M_1 M_2\rangle\big)}{\langle M_1 M_2\rangle} = \mathcal{O}\big(|e_1 - e_2| + \eta_1 + \eta_2\big)$$

from (1.5.41) and (1.5.43). Combining this with the bound

$$\Big|\big(G_1 G_2 - M(w_1, E_+, w_2)\big)_{\boldsymbol{xy}}\Big| \prec \left(\frac{1}{\sqrt{N\eta_1}} + \frac{1}{\sqrt{N\eta_2}}\right) \cdot \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2}$$

which is obtained completely analogous to (1.5.54), we conclude that (1.5.51) for $\sigma = +$ can be estimated by $1/\sqrt{N\eta}$ (recall $\eta := \min\{\eta_1, \eta_2\}$). Similarly, in the second case, $\sigma = -$, we perform a computation similar to the one leading to (1.5.19) and use (1.B.11) in order to obtain that $c_-(\mathcal{X}_{12}[\mathring{A}_1^{1,2}]M_2)$ equals

$$\frac{\mathrm{i}}{2}\frac{\langle M_1 \mathring{A}_1^{1,2} M_2^* E_-\rangle}{\langle M_1 E_- M_2^* E_-\rangle} + \frac{1}{2\mathrm{i}}\frac{\langle M_1 \mathring{A}_1^{1,2} M_2 E_-\rangle}{\langle M_1 E_- M_2^* E_-\rangle}\frac{1 + \langle M_1 E_- M_2^* E_-\rangle}{1 + \langle M_1 E_- M_2 E_-\rangle} = \mathcal{O}\big(|e_1 + e_2| + \eta_1 + \eta_2\big)$$

Combining this with the bound

$$\Big|\big(G_1 E_- G_2 - M(w_1, E_-, w_2)\big)_{\boldsymbol{xy}}\Big| \prec \frac{1}{\sqrt{N\eta}} \cdot \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2}$$

which is obtained completely analogous to (1.5.55), we conclude that (1.5.51) can be estimated by $1/\sqrt{N\eta}$ – now in both cases $\sigma = \pm$.

Conclusion. Summarizing our investigations, we have shown that

$$\big(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\big)_{\boldsymbol{xy}} = -\big(\underline{G_1 \mathring{A}_1' W G_2}\big)_{\boldsymbol{xy}} + \mathcal{O}_\prec\big(\mathcal{E}_1^{\mathrm{iso}}\big),$$

where we used the shorthand notation

$$\mathring{A}_1' := \big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^\circ + \sum_\sigma \frac{\mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[\mathring{A}_1]M_2)}{1 - \mathbf{1}_\delta^\sigma c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)}\big(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2\big)^\circ \tag{1.5.57}$$

in the underlined term. Combining (1.5.56) and the bound on (1.5.51) established above with the usual single resolvent local laws (1.4.17) and the bounds on deterministic approximations in Lemma 1.4.3, we collected all the error terms from (1.5.47) in (1.5.26). □

1.5.3 Proof of the third master inequality (1.4.26c)

Let $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ for $j \in [2]$ be spectral parameters and A_1 a regular matrix w.r.t. (w_1, w_2) and A_2 a regular matrix w.r.t. (w_2, w_1) (see Definition 1.4.2). By conjugation with E_-, we again assume w.l.o.g. that $\operatorname{Im} w_1 > 0$ and $\operatorname{Im} w_2 < 0$. Just as in Section 1.5.2, we use the notations $e_j \equiv \operatorname{Re} w_j$, $\eta_j := |\operatorname{Im} w_j|$ for $j \in [2]$ and define $1 \geq \eta := \min_j |\operatorname{Im} w_j|$. We also assume that (1.4.25) holds.

Lemma 1.5.8. (Representation as full underlined)
For any (w_1, w_2)-regular matrix $A_1 = \mathring{A}_1$ and (w_2, w_1)-regular matrix $A_2 = \mathring{A}_2$, we have that

$$\big\langle\big(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\big)\mathring{A}_2\big\rangle = -\big\langle\underline{W G_1 \mathring{A}_1 G_2 \mathring{A}_2'}\big\rangle + \mathcal{O}_\prec\big(\mathcal{E}_2^{\mathrm{av}}\big) \tag{1.5.58}$$

for some (w_2, w_1)*-regular matrix* $A_2' = \mathring{A}_2'$*, which linearly depends on* $A_2 = \mathring{A}_2$ *(analogously to* (1.5.57)*, see* (1.E.18) *for an explicit formula). For the error term in* (1.5.58)*, we used the shorthand notation*

$$\mathcal{E}_2^{\mathrm{av}} := \frac{1}{N\eta}\left(1 + \frac{(\psi_1^{\mathrm{av}})^2}{N\eta} + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right). \tag{1.5.59}$$

Note that similarly to Lemma 1.5.2 but contrary to Lemma 1.5.6, we again expanded the first resolvent G_1. Otherwise, the proof of Lemma 1.5.8, given in Appendix 1.E, is very similar to the one of Lemma 1.5.6. We only mention that the quadratic error $(\psi_1^{\mathrm{av}})^2$ stems from terms of the form

$$\langle \mathcal{S}[G_1 \mathring{A}_1^{1,2} G_2](G_2 - M_2)\mathring{A}_2^{2,1}\rangle,$$

appearing in the analogue of (1.5.47) (see (1.E.9) in Appendix 1.E). Having the approximate representation (1.5.58), we turn to the proof of (1.4.26c) via cumulant expansion of the full underlined term.

Proof of (1.4.26c). Let $p \in \mathbf{N}$. Starting from (1.5.6), we obtain, as in the proofs of (1.4.26a) and (1.4.26b),

$$\begin{aligned} &\mathbf{E}\big|\langle (G_1\mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2))\mathring{A}_2\rangle\big|^{2p} \\ &\lesssim \mathbf{E}\,\widetilde{\Xi}_2^{\mathrm{av}}\big|\langle (G_1\mathring{A}_1 G_2 - M(\ldots))\mathring{A}_2\rangle\big|^{2p-2} \\ &\quad + \sum_{|\boldsymbol{l}|+\sum(J\cup J_*)\geq 2} \mathbf{E}\,\Xi_2^{\mathrm{av}}(\boldsymbol{l}, J, J_*)\big|\langle (G_1\mathring{A}_1 G_2 - M(\ldots))\mathring{A}_2\rangle\big|^{2p-1-|J\cup J_*|} + \mathcal{O}_{\prec}\big((\mathcal{E}_2^{\mathrm{av}})^{2p}\big), \end{aligned} \tag{1.5.60}$$

where

$$\widetilde{\Xi}_2^{\mathrm{av}} := \frac{1}{N^2}\sum_{\sigma}\big|\langle G_1\mathring{A}_1 G_2\mathring{A}_2 G_1 E_\sigma G_1\mathring{A}_1 G_2\mathring{A}_2' E_\sigma\rangle\big| + \cdots$$

with the other terms being analogous, just 1 and 2 in the first half $G_1\mathring{A}_1 G_2\mathring{A}_2 G_1$ of the chain interchanged or the entire half taken as adjoint, and $\Xi_2^{\mathrm{av}}(\boldsymbol{l}, J, J_*)$ is defined as

$$\begin{aligned} \Xi_2^{\mathrm{av}} := N^{-(|\boldsymbol{l}|+\sum(J\cup J_*)+3)/2}\sum_{ab} R_{ab}|\partial^{\boldsymbol{l}}(G_1\mathring{A}_1 G_2\mathring{A}_2')_{ba}| \\ \times \prod_{\boldsymbol{j}\in J}|\partial^{\boldsymbol{j}}\langle G_1\mathring{A}_1 G_2\mathring{A}_2\rangle| \prod_{\boldsymbol{j}\in J_*}|\partial^{\boldsymbol{j}}\langle G_2^*\mathring{A}_2^* G_1^*\mathring{A}_1^*\rangle|. \end{aligned} \tag{1.5.61}$$

As in Sections 1.5.1 and 1.5.2, in the remainder of the proof, we need to analyze the rhs. of (1.5.60). We begin with the second line and study the terms involving Ξ_2^{av} from (1.5.61) afterwards.

Gaussian contribution: second line of (1.5.60)**.** Along the principal strategy outlined in Remark 1.5.3, we need to analyze in total eight terms, each of which carries one of the summands in the definition of $\widetilde{\Xi}_2^{\mathrm{av}}$ as a factor. Since their treatment is very similar, we focus on the exemplary term

$$\langle G_1\mathring{A}_1^{w_1,w_2} G_2\mathring{A}_2^{w_2,w_1} G_1 G_1\mathring{A}_1^{w_1,w_2} G_2(\mathring{A}_2')^{w_2,w_1}\rangle. \tag{1.5.62}$$

Now, we represent $G_1 G_1$ via the integral representation from Lemma 1.5.1 with

$$\tau = +, \quad J = \mathbf{B}_{\ell\kappa_0}, \quad \text{and} \quad \tilde{\eta} = \frac{\ell}{\ell+1}\eta,$$

for which we recall that $w \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$, i.e. in particular $\eta \geq (\ell+1)N^{-1+\epsilon_0}$ and hence $\tilde{\eta} \geq \ell N^{-1+\epsilon_0}$. After splitting the contour integral and bounding the individual contributions as described in (1.5.14), we obtain, with the aid of Lemma 1.4.3,

$$\big|(1.5.62)\big| \prec \frac{1}{\eta^2}\left(1 + \frac{\psi_4^{\mathrm{av}}}{N\eta}\right) + \int_{\mathbf{B}_{\ell\kappa_0}} \frac{\big|\langle G_1\mathring{A}_1^{w_1,w_2} G_2\mathring{A}_2^{w_2,w_1} G(x+\mathrm{i}\tilde{\eta})\mathring{A}_1^{w_1,w_2} G_2(\mathring{A}_2')^{w_2,w_1}\rangle\big|}{(x-e_1)^2+\eta_1^2}\mathrm{d}x.$$

Next, we decompose $\mathring{A}_2^{w_2,w_1}$ and $\mathring{A}_1^{w_1,w_2}$ in the integrand as (recall the notation in Footnote 15)

$$\begin{aligned}\mathring{A}_2^{w_2,x+\mathrm{i}\tilde{\eta}} &= \mathring{A}_2^{w_2,w_1} + \sum_\sigma \mathcal{O}_\sigma(|x-e_1|+|\eta_1-\tilde{\eta}|)E_\sigma \\ \mathring{A}_1^{x+\mathrm{i}\tilde{\eta},w_2} &= \mathring{A}_1^{w_1,w_2} + \sum_\sigma \mathcal{O}_\sigma(|x-e_1|+|\eta_1-\tilde{\eta}|)E_\sigma\,.\end{aligned} \tag{1.5.63}$$

While the properly regularised term contributes an $\eta^{-2}\big(1+\psi_4^{\mathrm{av}}/(N\eta)\big)$-error, a typical cross term shall be estimated as

$$\int_{\mathbf{B}_{\ell\kappa_0}} \frac{\big|\big\langle G_1\mathring{A}_1^{w_1,w_2}G_2\mathring{A}_2^{w_2,x+\mathrm{i}\tilde{\eta}}[G(x+\mathrm{i}\tilde{\eta})-G_2](\mathring{A}_2')^{w_2,w_1}\big\rangle\big|}{\big(|x-e_1|+\eta_1\big)\big(|x-e_2|+\eta_2\big)} \prec \frac{1}{\eta^2}\left(1+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right) \tag{1.5.64}$$

where in the second step we wrote out the averaged trace and estimated each summand in isotropic form with the aid of Lemma 1.4.3, using ψ_2^{iso} instead of ψ_3^{av}.

Finally, for 'error × error'-type terms are bounded by η^{-2}, simply by using a trivial Schwarz inequality in combination with a Ward identity and the usual local law from Theorem 1.2.6 to infer

$$\big|\langle G_1B_1G_2B_2\big| \le \sqrt{\langle G_1B_1B_1^*G_1^*\rangle\langle G_2B_2B_2^*G_2^*\rangle} \le \frac{1}{\eta}\sqrt{\langle \operatorname{Im}G_1B_1B_1^*\rangle\langle \operatorname{Im}G_2B_2B_2^*\rangle} \prec \frac{1}{\eta}\,,$$

which is valid for arbitrary bounded matrices $\|B_1\|,\|B_2\| \lesssim 1$.

This finishes the estimate for the Gaussian contribution from the second line of (1.5.60), for which, collecting the above estimates, we have shown that

$$\Xi_2^{\mathrm{av}} \prec \frac{1}{N^2\eta^2}\left(1+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}+\frac{\psi_4^{\mathrm{av}}}{N\eta}\right). \tag{1.5.65}$$

We are now left with the terms from the last line of (1.5.60) resulting from higher order cumulants.

Higher order cumulants and conclusion. The estimate stemming from higher order cumulants is given in (1.5.74c) in Section 1.5.5. Then, plugging (1.5.65) and (1.5.74c) into (1.5.60), we find, similarly to Section 1.5.1, that

$$\Psi_2^{\mathrm{av}} \prec 1+\frac{(\psi_1^{\mathrm{av}})^2+(\psi_1^{\mathrm{iso}})^2+\psi_2^{\mathrm{av}}}{N\eta}+\frac{\psi_2^{\mathrm{iso}}+(\psi_4^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}}+\frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}+\frac{(\psi_3^{\mathrm{iso}})^{3/8}+(\psi_4^{\mathrm{iso}})^{3/8}}{(N\eta)^{3/16}}\,.$$

The bound given in Proposition 1.4.9 is an immediate consequence after a trivial Young inequality. □

1.5.4 Proof of the fourth master inequality (1.4.26d)

Let $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ for $j\in[3]$ be spectral parameters and A_1 a regular matrix w.r.t. (w_1,w_2) and A_2 a regular matrix w.r.t. (w_2,w_3) (see Definition 1.4.2). By conjugation with E_-, we will assume w.l.o.g. that $\operatorname{Im}w_1>0$, $\operatorname{Im}w_2<0$, and $\operatorname{Im}w_3>0$. As before, we use the notations $e_j\equiv\operatorname{Re}w_j$, $\eta_j:=|\operatorname{Im}w_j|$ for $j\in[3]$ and define $1\ge\eta:=\min_j|\operatorname{Im}w_j|$. We also assume that (1.4.25) holds.

Lemma 1.5.9. (Representation as full underlined)
For $\|\boldsymbol{x}\|,\|\boldsymbol{y}\|\le 1$ and any (w_1,w_2)-regular matrix $A_1=\mathring{A}_1$ and (w_2,w_3)-regular matrix $A_2=\mathring{A}_2$, we have that

$$\big(G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1,\mathring{A}_1,w_2,\mathring{A}_2,w_3)\big)_{\boldsymbol{xy}} = -\big(\underline{G_1\mathring{A}_1'WG_2\mathring{A}_2G_3}\big)_{\boldsymbol{xy}} + \mathcal{O}_\prec\big(\mathcal{E}_2^{\mathrm{iso}}\big) \tag{1.5.66}$$

for some other (w_1,w_2)-regular matrix $A_1'=\mathring{A}_1'$, which linearly depends on $A_1=\mathring{A}_1$ (analogously to (1.5.57)*, see* (1.E.33) *for an explicit formula). For the error term in* (1.5.66)*, we used the shorthand notation*

$$\mathcal{E}_2^{\mathrm{iso}} := \frac{1}{\sqrt{N\eta^3}}\left(1+\psi_1^{\mathrm{iso}}+\frac{\psi_1^{\mathrm{av}}\psi_1^{\mathrm{iso}}}{N\eta}+\frac{\psi_2^{\mathrm{iso}}}{N\eta}\right). \tag{1.5.67}$$

Note that similarly to (1.5.25), we again expanded the second resolvent. The proof of Lemma 1.5.9, given in Appendix 1.E, is very similar to the one of Lemma 1.5.6. We only mention that the errors carrying $\psi_1^{\mathrm{iso}}\psi_1^{\mathrm{av}}$ and ψ_1^{iso} stem from terms of the form

$$\big(G_1\mathcal{S}[(G_1-M_1)A_1^{\circ_{1,2}}]G_2\mathring{A}_2G_3\big)_{\boldsymbol{xy}} \quad \text{and}$$
$$c_\sigma(\mathcal{X}_{12}[\mathring{A}_1]M_2)c_\sigma(\Phi_\sigma)\big(G_1E_\sigma G_2\mathring{A}_2G_3 - M(w_1,E_\sigma,w_2,\mathring{A}_2,w_3)\big)_{\boldsymbol{xy}},$$

respectively, appearing in the analogue of (1.5.47) (see (1.E.24) and (1.E.26) in Appendix 1.E). Having the representation (1.5.66) we turn to the proof of (1.4.26d) via cumulant expansion of the underlined term.

Proof of (1.4.26d). Let $p\in\mathbf{N}$. Then, starting from (1.5.66), we obtain

$$\begin{aligned}&\mathbf{E}\big|\big(G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1,\mathring{A}_1,w_2,\mathring{A}_2,w_3)\big)_{\boldsymbol{xy}}\big|^{2p} \\ &\lesssim \mathbf{E}\,\widetilde{\widetilde{\Xi}}_2^{\mathrm{iso}}\big|\big(G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(\dots)\big)_{\boldsymbol{xy}}\big|^{2p-2} + \mathcal{O}_<\big((\mathcal{E}_1^{\mathrm{iso}})^{2p}\big) \\ &\quad + \sum_{|\boldsymbol{l}|+\sum(J\cup J_*)\geq 2}\mathbf{E}\,\Xi_2^{\mathrm{iso}}(\boldsymbol{l},J,J_*)\big|\big(G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(\dots)\big)_{\boldsymbol{xy}}\big|^{2p-1-|J\cup J_*|},\end{aligned} \tag{1.5.68}$$

where

$$\begin{aligned}\widetilde{\widetilde{\Xi}}_2^{\mathrm{iso}} :=& \frac{\sum_\sigma\sum_{j=1}^3\big|\big(G_1\mathring{A}_1'E_\sigma G_j\mathring{A}_j\dots G_3\big)_{\boldsymbol{xy}}\big(G_1\mathring{A}_1\dots\mathring{A}_{j-1}G_jE_\sigma G_2\mathring{A}_2G_3\big)_{\boldsymbol{xy}}\big|}{N} \\ &+ \frac{\sum_\sigma\sum_{j=1}^3\big|\big(G_1\mathring{A}_1'E_\sigma G_j^*\mathring{A}_{j-1}^*\dots\mathring{A}_1^*G_1^*\big)_{\boldsymbol{xx}}\big(G_3^*\dots\mathring{A}_j^*G_j^*E_\sigma G_2\mathring{A}_2G_3\big)_{\boldsymbol{yy}}\big|}{N}\end{aligned}$$

and $\Xi_2^{\mathrm{iso}}(\boldsymbol{l},J,J_*)$ is defined as

$$\begin{aligned}\Xi_2^{\mathrm{iso}} :=\, & N^{-(|\boldsymbol{l}|+\sum(J\cup J_*)+1)/2}\sum_{ab}R_{ab}\big|\partial^{\boldsymbol{l}}\big[(G_1\mathring{A}_1')_{\boldsymbol{x}a}\big(G_2\mathring{A}_2G_3\big)_{b\boldsymbol{y}}\big]\big| \\ &\times\prod_{\boldsymbol{j}\in J}\big|\partial^{\boldsymbol{j}}\big(G_1\mathring{A}_1G_2\mathring{A}_2G_3\big)_{\boldsymbol{xy}}\big|\prod_{\boldsymbol{j}\in J_*}\big|\partial^{\boldsymbol{j}}\big(G_3^*\mathring{A}_2^*G_2^*\mathring{A}_1^*G_2^*\big)_{\boldsymbol{yx}}\big|.\end{aligned} \tag{1.5.69}$$

We need to analyze the rhs. of the inequality derived in (1.5.68). We begin with the second line.

Gaussian contribution: second line of (1.5.68)**.** Following Remark 1.5.3, we need to analyze in total twelve terms, each of which carries one of the summands in the definition of $\widetilde{\widetilde{\Xi}}_2^{\mathrm{iso}}$ as a factor. Again, using Lemma 1.3.3 for the A's, we pick two exemplary terms

$$\big(G_1\mathring{A}_1^{w_1,w_2}G_2\mathring{A}_2^{w_2,w_3}G_3E_-G_2\mathring{A}_2^{w_2,w_3}G_3\big)_{\boldsymbol{xy}}\big(G_1(\mathring{A}_1')^{w_1,w_2}E_-G_3\big)_{\boldsymbol{xy}} \tag{1.5.70}$$

$$\big(G_1(\mathring{A}_1')^{w_1,w_2}G_2^*(\mathring{A}_1^*)^{\bar w_2,\bar w_1}G_1^*\big)_{\boldsymbol{xx}}\big(G_3^*(\mathring{A}_2^*)^{\bar w_3,\bar w_2}G_2^*G_2\mathring{A}_2^{w_2,w_3}G_3\big)_{\boldsymbol{yy}} \tag{1.5.71}$$

which shall be treated in more detail. The other terms are analogous and hence omitted.

The term (1.5.70). In the first factor, we use (1.2.16), Lemma 1.3.3, Lemma 1.4.3 and Lemma 1.5.1 with parameters

$$\tau=+,\quad J=\mathbf{B}_{(\ell+\frac12)\kappa_0},\quad \text{and}\quad \tilde\eta=\frac{2\ell}{2\ell+1}\eta,$$

(in order to have some flexibility before approaching the boundary of the domain $\mathbf{D}_\ell^{(\epsilon_0,\kappa_0)}$) to bound it as

$$\big|\big(G_1\mathring{A}_1^{w_1,w_2}G_2\mathring{A}_2^{w_2,w_3}G_3E_-G_2\mathring{A}_2^{w_2,w_3}G_3\big)_{\boldsymbol{xy}}\big| < \frac{1}{\eta^{3/2}}\left(1+\frac{\psi_3^{\mathrm{iso}}}{\sqrt{N\eta}}\right)$$

$$+\int_{\mathbf{B}_{(\ell+\frac{1}{2})\kappa_0}} \frac{\big|\big(G_1 \mathring{A}_1^{w_1,w_2} G_2 \mathring{A}_2^{w_2,w_3} G(x+\mathrm{i}\tilde{\eta})(E_- \mathring{A}_2)^{-w_2,w_3} G_3\big)_{\boldsymbol{xy}}\big|}{\big(|x-e_3|+\eta_3\big)\big(|x+e_2|+\eta_2\big)}\mathrm{d}x\,.$$

Next, we decompose $\mathring{A}_2^{w_2,w_3}$ and $(E_- \mathring{A}_2)^{-w_2,w_3}$ according to the integration variable with the aid of Lemma 1.3.3 (iii), analogously to (1.5.63). This leaves us with four terms, which shall be estimated separately. While the fully regularised term gives

$$\frac{1}{\eta^{3/2}}\left(1+\frac{\psi_3^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1+\frac{1}{|e_2+e_3|+\eta_2+\eta_3}\right),$$

the cross terms can be estimated as

$$\frac{1}{\eta^2}\left(1+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right),$$

analogously to (1.5.64). As an exemplary error term, we consider

$$\int_{\mathbf{B}_{(\ell+\frac{1}{2})\kappa_0}} \big|\big(G_1 \mathring{A}_1^{w_1,w_2} G_2 E_+ G(x+\mathrm{i}\tilde{\eta}) E_- G_3\big)_{\boldsymbol{xy}}\big|\mathrm{d}x \tag{1.5.72}$$

and use Lemma 1.5.1 with new parameters

$$\tau = -\,,\quad J = \mathbf{B}_{\ell\kappa_0}\,,\quad \tilde{\eta} = \frac{\ell}{\ell+1}\eta\,,$$

to find, dropping the integration domains for ease of notation,

$$\begin{aligned}\big|(1.5.72)\big| &\prec \frac{1}{\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)+\int \mathrm{d}x\int \mathrm{d}y \frac{\big|\big(G_1 \mathring{A}_1^{w_1,w_2} G(y-\mathrm{i}\tilde{\eta})\big)_{\boldsymbol{x}(E_-\boldsymbol{y})}\big|}{\big(|y-e_2|+\eta_2\big)\big(|y+x|+\eta\big)\big(|y+e_3|+\eta_3\big)}\\ &\prec \frac{1}{\eta^{3/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1+\frac{1}{|e_2+e_3|+\eta_2+\eta_3}\right),\end{aligned}$$

where in the last step we used Lemma 1.3.3 for decomposing $\mathring{A}_1^{w_1,w_2}$ accordingly, and Lemma 1.4.3. This finishes the bound on the first factor in (1.5.70). The second factor can easily be estimated as

$$\big|\big(G_1(\mathring{A}_1')^{w_1,w_2} E_- G_3\big)_{\boldsymbol{xy}}\big| \prec \frac{1}{\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)+\frac{|e_2+e_3|+\eta_2+\eta_3}{\eta}$$

using (1.2.16), Lemma 1.3.3, and Lemma 1.4.3. Notice the cancellation of $|e_2+e_3|$ between the two factors.

The term (1.5.71). For the first factor in (1.5.71), we realise that $(\mathring{A}_1')^{w_1,w_2} = (\mathring{A}_1')^{w_1,\bar{w}_2}$, which without approximation immediately yields that

$$\big|\big(G_1(\mathring{A}_1')^{w_1,w_2} G_2^* (\mathring{A}_1^*)^{\bar{w}_2,\bar{w}_1} G_1^*\big)_{\boldsymbol{xx}}\big| \prec \frac{1}{\eta}\left(1+\frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right)$$

with the aid of Lemma 1.4.3.

In the second factor, we apply a Ward identity to $G_2^* G_2$ and again use that the regularisation is insensitive to complex conjugation in the second spectral parameter. In this way, and decomposing

$$\mathring{A}_2^{w_2,w_3} = \mathring{A}_2^{\bar{w}_2,w_3} + \mathcal{O}\big(|e_2-e_3|+|\eta_2-\eta_3|\big)E_+ + \mathcal{O}\big(|e_2+e_3|+|\eta_2-\eta_3|\big)E_-$$

by means of Lemma 1.3.3 (ii), we find that the second factor is stochastically dominated by

$$\frac{1}{\eta^2}\left(1+\frac{\psi_1^{\mathrm{iso}}+\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right).$$

This finishes the estimate for the Gaussian contribution from the second line of (1.5.68), for which, collecting the above estimates, we have shown that

$$\widetilde{\Xi}_2^{\mathrm{iso}} \prec \frac{1}{N\eta^3}\left[\left(1+\frac{\psi_3^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1+\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta}}\right)+\left(1+\frac{\psi_1^{\mathrm{iso}}+\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right)^2\right]. \tag{1.5.73}$$

We are now left with the terms from the last line of (1.5.68) resulting from higher order cumulants.

Higher order cumulants and conclusion. The estimate stemming from higher order cumulants is given in (1.5.74d) in Section 1.5.5. Then, plugging (1.5.73) and (1.5.74d) into (1.5.68), we find, similarly to Section 1.5.1, that

$$\Psi_2^{\mathrm{iso}} \prec 1+\psi_1^{\mathrm{iso}}+\frac{\psi_1^{\mathrm{av}}\psi_1^{\mathrm{iso}}+(\psi_1^{\mathrm{iso}})^2+\psi_2^{\mathrm{iso}}}{N\eta}+\frac{\psi_2^{\mathrm{iso}}+(\psi_1^{\mathrm{iso}}\psi_3^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/2}}+\frac{(\psi_3^{\mathrm{iso}})^{3/8}+(\psi_4^{\mathrm{iso}})^{3/8}}{(N\eta)^{3/16}}$$

The bound given in Proposition 1.4.9 is an immediate consequence after a trivial Young inequality. □

1.5.5 Contributions from higher order cumulants

The goal of the present section is to estimate the terms originating from higher order cumulants in (1.5.11), (1.5.27), (1.5.60), and (1.5.68). In order to do so, we assume that (1.4.25) holds.

Lemma 1.5.10. *For any $J, J_* \subset \mathbf{Z}^2_{\geq 0}\setminus\{(0,0)\}$, $\boldsymbol{l}\in\mathbf{Z}^2_{\geq 0}$ with $|\boldsymbol{l}|+\sum(J\cup J_*)\geq 2$ it holds that*

$$\left(\Xi_1^{\mathrm{av}}\right)^{1/(1+\sum(J\cup J_*))} \prec \frac{1}{N\eta^{1/2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{(N\eta)^{1/2}}+\frac{(\psi_2^{\mathrm{iso}})^{1/4}}{(N\eta)^{1/8}}\right), \tag{1.5.74a}$$

$$\left(\Xi_1^{\mathrm{iso}}\right)^{1/(1+\sum(J\cup J_*))} \prec \frac{1}{\sqrt{N\eta^2}}\left(1+\frac{\psi_1^{\mathrm{iso}}}{(N\eta)^{1/2}}+\frac{(\psi_2^{\mathrm{iso}})^{1/4}}{(N\eta)^{1/8}}\right), \tag{1.5.74b}$$

$$\left(\Xi_2^{\mathrm{av}}\right)^{1/(1+\sum(J\cup J_*))} \prec \frac{1}{N\eta}\left(1+\frac{(\psi_1^{\mathrm{iso}})^2}{N\eta}+\frac{\psi_2^{\mathrm{iso}}}{(N\eta)^{1/2}}+\frac{(\psi_3^{\mathrm{iso}})^{3/8}+(\psi_4^{\mathrm{iso}})^{3/8}}{(N\eta)^{3/16}}\right), \tag{1.5.74c}$$

$$\left(\Xi_2^{\mathrm{iso}}\right)^{1/(1+\sum(J\cup J_*))} \prec \frac{1}{\sqrt{N\eta^3}}\left(1+\frac{(\psi_1^{\mathrm{iso}})^2}{N\eta}+\frac{\psi_2^{\mathrm{iso}}}{(N\eta)^{1/2}}+\frac{(\psi_3^{\mathrm{iso}})^{3/8}+(\psi_4^{\mathrm{iso}})^{3/8}}{(N\eta)^{3/16}}\right). \tag{1.5.74d}$$

For $k=1,2$, $\boldsymbol{l}\in\mathbf{Z}^2_{\geq 0}$ and a multiset $J\subset\mathbf{Z}^2_{\geq 0}\setminus\{(0,0)\}$ we now define slightly (notationally) simplified versions of $\Xi_k^{\mathrm{av/iso}}$, namely

$$\Xi_k^{\mathrm{av}}(\boldsymbol{l},J) := N^{-(|\boldsymbol{l}|+\sum J+3)/2}\sum_{ab}\left|\partial^{\boldsymbol{l}}((GA)^{k-1}GA')_{ba}\right|\prod_{\boldsymbol{j}\in J}\left|\partial^{\boldsymbol{j}}\langle(GA)^k\rangle\right|, \tag{1.5.75}$$

$$\Xi_k^{\mathrm{iso}}(\boldsymbol{l},J) := N^{-(|\boldsymbol{l}|+\sum J+1)/2}\sum_{ab}\left|\partial^{\boldsymbol{l}}[(GA)_{\boldsymbol{x}a}(G(AG)^{k-1})_{b\boldsymbol{y}}]\right|\prod_{\boldsymbol{j}\in J}\left|\partial^{\boldsymbol{j}}((GA)^kG)_{\boldsymbol{xy}}\right|, \tag{1.5.76}$$

where $\sum J := \sum_{\boldsymbol{j}\in J}|\boldsymbol{j}|$, $|(j_1,j_2)| := j_1+j_2$ and $\partial^{(j_1,j_2)} := \partial_{ab}^{j_1}\partial_{ba}^{j_2}$. Here, for notational simplicity, we do not carry the dependence on the spectral parameters of the resolvents but assume that implicitly each resolvent has its own spectral parameter and that each A is correctly regularised with respect to its neighboring resolvents. In particular compared to (1.5.12), (1.5.28), (1.5.61), and (1.5.69), it is not necessary to distinguish the sets J, J^*.

Proof of Lemma 1.5.10. Throughout the proof, we denote $\phi_k := \psi_k^{\mathrm{iso}}/\sqrt{N\eta}$. The *naive estimate* for the derivatives simply is

$$\begin{aligned}|\partial^{\boldsymbol{l}}((GA)^{k-1}GA')_{ba}| &\prec \eta^{-(k-1)/2}\Big(1+\phi_{k-1}\Big),\\ |\partial^{\boldsymbol{j}}\langle(GA)^k\rangle| &\prec \frac{1}{N\eta^{k/2}}\sum_{k_1+k_2+\cdots=k}\prod_i\Big(1+\phi_{k_i}\Big)\end{aligned} \tag{1.5.77}$$

due to (1.4.10) and recalling (1.4.16). Here a single derivative just splits the chain with k resolvents into two chains with a total of $k+1$ resolvents, leaving the estimates on the main and the error term unchanged. Using (1.5.77) in (1.5.75) we obtain

$$\begin{aligned}
|\Xi_1^{\mathrm{av}}| &\prec (N\eta^{1/2})^{-1-|J|} N^{(2-|\boldsymbol{l}|-\sum J)} \sqrt{N\eta}\Big(1+\phi_1\Big)^{|J|}, \\
|\Xi_2^{\mathrm{av}}| &\prec (N\eta)^{-1-|J|} N^{(2-|\boldsymbol{l}|-\sum J)} \sqrt{N\eta}\Big(1+\phi_1\Big)\Big(1+\phi_2+\phi_1^2\Big)^{|J|}, \\
|\Xi_1^{\mathrm{iso}}| &\prec (\sqrt{N}\eta)^{-1-|J|} \eta^{1+|J|/2} N^{(4-|\boldsymbol{l}|+|J|-\sum J)/2}\Big(1+\phi_1\Big)^{|J|}, \\
|\Xi_2^{\mathrm{iso}}| &\prec (\sqrt{N}\eta^{3/2})^{-1-|J|} \eta^{1+|J|/2} N^{(4-|\boldsymbol{l}|+|J|-\sum J)/2}\Big(1+\phi_1\Big)\Big(1+\phi_2+\phi_1^2\Big)^{|J|},
\end{aligned} \tag{1.5.78}$$

and therefore have proved (1.5.74a) and (1.5.74c) in all cases except $|\boldsymbol{l}| + \sum J = 2$ and (1.5.74b) and (1.5.74d) in all cases except $|\boldsymbol{l}| + \sum J - |J| < 4$. For the remaining cases we need a more refined estimate using the following *Ward lemma*:

Lemma 1.5.11. *Let $\boldsymbol{x}$ be any deterministic vector of bounded norm, let $w_1, \dots, w_k \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ be spectral parameters and $A_1, \dots, A_k$ deterministic matrices of bounded norm. Then for $G_i = G(w_i)$ it holds that*

$$\frac{1}{N}\sum_a \Big|(G_1 \mathring{A}_1^{w_1,w_2} \cdots \mathring{A}_{k-1}^{w_{k-1},w_k} G_k A_k)_{\boldsymbol{x}a}\Big| \prec \frac{1}{\sqrt{N\eta}} \frac{1}{\eta^{(k-1)/2}} \Big(1+\phi_1+\cdots+\phi_{2k}\Big)^{1/2},$$

which improves upon the term-wise bound by a factor of $(N\eta)^{-1/2}$ at the expense of replacing $1+\phi_k$ by $1+\sqrt{\phi_1+\cdots+\phi_{2k}}$.

The proof of the above *Ward lemma* is largely based on yet another more general estimate.

Lemma 1.5.12. *Let $\boldsymbol{x}, \boldsymbol{y}$ be normalised vectors, let $w_1, \dots, w_{k+1} \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$ be spectral parameters and $A_1, \dots, A_k$ be deterministic matrices of bounded norm such that a of them are regular, i.e. $\mathring{A}_i^{w_i,w_{i+1}} = A_i$ for all $i \in \mathcal{I}$ for some $\mathcal{I} \subset [k]$ of cardinality a. Then with $G_i = G(w_i)$ it holds that*

$$|(G_1 A_1 G_2 \cdots A_k G_{k+1})_{\boldsymbol{x}\boldsymbol{y}}| \prec \frac{1}{\eta^{k-a/2}}\Big(1+\phi_1+\cdots+\phi_a\Big). \tag{1.5.79}$$

We defer the proof of Lemma 1.5.12 to the end of this section.

Proof of Lemma 1.5.11. By Cauchy-Schwarz and the norm bound on the middle A_k we have

$$\begin{aligned}
&\Big(\frac{1}{N}\sum_a \Big|(G_1 \mathring{A}_1^{w_1,w_2} \cdots \mathring{A}_{k-1}^{w_{k-1},w_k} G_k A_k)_{\boldsymbol{x}a}\Big|\Big)^2 \\
&\quad\lesssim \frac{1}{N}\Big(G_1 \mathring{A}_1^{w_1,w_2} \cdots \mathring{A}_{k-1}^{w_{k-1},w_k} G_k G_k^* \mathring{A}_{k-1}^{\bar{w}_k,\bar{w}_{k-1}} \cdots \mathring{A}_1^{\bar{w}_2,\bar{w}_1} G_1^*\Big)_{\boldsymbol{x}\boldsymbol{x}} \\
&\quad\prec \frac{1}{N\eta^k}\Big(1+\phi_1+\cdots+\phi_{2k}\Big)
\end{aligned}$$

due to Lemma 1.5.12 for $2k$ resolvents and $a = 2k-2$ regularised A-matrices. □

The rest of the proof is split into several cases.

Treatment of (1.5.74a) *and* (1.5.74c) *for $|\boldsymbol{l}| + \sum J = 2$:* For the case $|\boldsymbol{l}| + \sum J = 2$ we either have $|\boldsymbol{l}| \in \{0, 2\}$ or $\sum J = 1 = |J|$. In the former case an off-diagonal resolvent is guaranteed to be present in the first factor of (1.5.75) (by parity) and in the latter case the second factor consists of a single

off-diagonal resolvent chain. In either case we may use Lemma 1.5.11 to gain a factor of $1/\sqrt{N\eta}$ compared to (1.5.77) and obtain

$$\begin{aligned}|\Xi_1^{\mathrm{av}}| &\prec (N\eta^{1/2})^{-1-|J|}(1+\phi_1)^{(|J|-1)_+}\Big(1+\phi_1+\mathbf{1}(|J|\geq 1)\phi_2^{1/2}\Big),\\ |\Xi_2^{\mathrm{av}}| &\prec (N\eta)^{-1-|J|}(1+\phi_1^2+\phi_2)^{(|J|-1)_+}\Big(1+\phi_1^3+\phi_2^{3/2}+\mathbf{1}(|J|\geq 1)(\phi_3+\phi_4)^{3/4}\Big),\end{aligned} \tag{1.5.80}$$

where we used the fact that for $|J| = 0$ only a single factor of $(1+\phi_1)$ needs to be replaced by a factor of $(1+(\phi_1+\phi_2)^{1/2})$ for Ξ_2^{av} and no factor needs to be replaced for Ξ_1^{av}. Moreover, we used $\phi_1(\phi_3+\phi_4)^{1/2}+\phi_1^2\phi_2^{1/2}\lesssim\phi_1^3+\phi_2^{3/2}+(\phi_3+\phi_4)^{3/4}$ by a simple Young inequality. Now (1.5.80) implies (1.5.74a) and (1.5.74c) by another simple Young inequality.

Treatment of (1.5.74b) *and* (1.5.74d) *for* $|\boldsymbol{l}|+\sum J-|J|\in\{2,3\}$*:* In this case we can simply use Lemma 1.5.11 for the two resolvent chains in the first factor of (1.5.76) involving $\boldsymbol{x},\boldsymbol{y}$ to gain a factor of $(N\eta)^{-1}$ compared to (1.5.77) at the expense of replacing $1+\phi_1$ by $1+\phi_1^{1/2}+\phi_2^{1/2}$ in case of Ξ_2^{iso} which proves (1.5.74b) and (1.5.74d) in this case.

Treatment of (1.5.74b) *and* (1.5.74d) *for* $|\boldsymbol{l}|+\sum J-|J|=0$*:* In this case we necessarily have $|\boldsymbol{l}|=0$ and $|J|\geq 2$ and $|\boldsymbol{j}|=1$ for all $\boldsymbol{j}\in J$. In particular all factors of (1.5.76) consist of two resolvent chains evaluated in $(\boldsymbol{x},a),(\boldsymbol{y},b)$ or $(\boldsymbol{x},b),(\boldsymbol{y},a)$, respectively. This allows to use Lemma 1.5.11 four times (twice for the a- and twice for the b-summation) to gain a factor of $(N\eta)^{-2}$ compared to (1.5.77) at the expense of replacing

$$\text{one factor of}\quad (1+\phi_1)\quad\text{by}\quad (1+(\phi_1+\phi_2)^{1/2})$$

in case of Ξ_1^{iso} and

$$\text{one factor of}\quad (1+\phi_1)(1+\phi_1^2+\phi_2)\quad\text{by}\quad (1+(\phi_1+\phi_2)^{1/2})(1+\phi_1+\phi_2+(\phi_3+\phi_4)^{1/2}) \tag{1.5.81}$$

in case of Ξ_2^{iso}. This concludes the proof in case of Ξ_1^{iso} and together with

$$(1+(\phi_1+\phi_2)^{1/2})(1+\phi_1+\phi_2+(\phi_3+\phi_4)^{1/2})\lesssim 1+(\phi_1+\phi_2)^{3/2}+(\phi_3+\phi_4)^{3/4}$$

also in case of Ξ_2^{iso}.

Treatment of (1.5.74b) *and* (1.5.74d) *for* $|\boldsymbol{l}|+\sum J-|J|=1$*:* In this case we necessarily have $|J|\geq 1$ and either $|\boldsymbol{l}|=0$ or $|\boldsymbol{j}|=1$ for all $\boldsymbol{j}\in J$. In either case we can use Lemma 1.5.11 twice for the first factor and once for some other factor in (1.5.76) to gain a factor of $(N\eta)^{-3/2}$ compared to (1.5.77) at the expense of replacing (1.5.81) in case of Ξ_1^{iso} and

$$\text{one factor of}\quad (1+\phi_1)(1+\phi_1^2+\phi_2)\quad\text{by}\quad (1+(\phi_1+\phi_2)^{1/2})((1+\phi_1)(1+\phi_1+\phi_2)^{1/2}+(\phi_3+\phi_4)^{1/2})$$

in case of Ξ_2^{iso}. Together with

$$(1+(\phi_1+\phi_2)^{1/2})((1+\phi_1)(1+\phi_1+\phi_2)^{1/2}+(\phi_3+\phi_4)^{1/2})\lesssim 1+(\phi_3+\phi_4)^{3/4}+\phi_2^{3/2}+\phi_1^2$$

this concludes the proof also in this case. □

It remains to give the proof of Lemma 1.5.12.

Proof of Lemma 1.5.12. The proof is via induction, i.e. we assume that (1.5.79) has been established for resolvent chains of up to k resolvents. For $k+1$ resolvents and $a=k$, i.e. in case when all deterministic matrices are regular, the claim follow by definition of ψ_k^{iso}. Therefore we may assume that some A_j is not regular which we decompose into its regular component $\mathring{A}_j^{w_j,w_{j+1}}$ and a linear combination of $E_\pm$. By linearity it thus suffices to check (1.5.79) for the cases $A_j=E_\pm$, and

moreover, by chiral symmetry $G_jE_-G_{j+1} = -E_-G(-w_j)E_+G_{j+1}$ and $\mathring{A}^{w_{j-1},w_j}E_- = \mathring{A}^{w_{j-1},-w_j}$ (recall Lemma 1.3.3) the estimate for E_- follows from the estimate for E_+ upon replacing w_j by $-w_j$. Therefore it suffices to check (1.5.79) in case $A_j = E_+$.

If $\mathfrak{s}_j = -\operatorname{sgn}(\operatorname{Im} w_j \operatorname{Im} w_{j+1}) = +$, i.e. the adjacent spectral parameters lie in opposite half-planes, then we use the resolvent identity (1.3.21) to write

$$A_{j-1}G_jE_+G_{j+1}A_{j+1}G_{j+2} = A_{j-1}\frac{G_j - G_{j+1}}{w_j - w_{j+1}}A_{j+1}G_{j+2}\,.$$

We discuss each of the two resulting summands separately. For the summand involving G_{j+1}, if A_{j-1} was not counted as regularised, i.e. $j-1 \notin \mathcal{I}$, then the claim follows by induction and the trivial estimate $|w_j - w_{j+1}| \geq \eta$ since k has been reduced by one, while a has been preserved. On the other hand, if A_{j-1} was correctly regularised, then we use Lemma 1.3.3 to write

$$\mathring{A}_{j-1}^{w_{j-1},w_j} = \mathring{A}_{j-1}^{w_{j-1},\bar{w}_j} = \mathring{A}_{j-1}^{w_{j-1},w_{j+1}} + \mathcal{O}(|\bar{w}_j - w_{j+1}|)E_+ + \mathcal{O}(|\bar{w}_j - w_{j+1}|)E_-\,. \tag{1.5.82}$$

Inserting (1.5.82) into $A_{j-1}G_{j+1}A_{j+1}G_{j+2}/(w_j - w_{j+1})$ the claimed bound follows from induction since for the $\mathring{A}_{j-1}^{w_{j-1},w_{j+1}}$-term a has been preserved and k has been reduced by one compensating for $|w_j - w_{j+1}| \geq \eta$, while for $E_\pm$ both k, a have been reduced by one and $|\bar{w}_j - w_{j+1}|/|w_j - w_{j+1}| \leq 1$. Next, for the summand involving G_j, the argument is completely analogous, apart from the two error terms in

$$\begin{aligned}\mathring{A}_{j+1}^{w_j,w_{j+1}} = \mathring{A}_{j+1}^{w_j,w_{j+2}} &+ \mathcal{O}(|w_j - \bar{w}_{j+1}| + |w_j - \mathfrak{s}_{j+1}w_{j+2}|)E_{\mathfrak{s}_{j+1}} \\ &+ \mathcal{O}(|w_j - \bar{w}_{j+1}| + |w_j + \mathfrak{s}_{j+1}\bar{w}_{j+2}|)E_{-\mathfrak{s}_{j+1}}\,,\end{aligned} \tag{1.5.83}$$

appearing for an $A_{j+1} = \mathring{A}_{j+1}^{w_{j+1},w_{j+2}}$, which has been correctly regularised. Here, we applied Lemma 1.3.3 and denoted, as usual, $\mathfrak{s}_{j+1} = -\operatorname{sgn}(\operatorname{Im} w_{j+1}\operatorname{Im} w_{j+2})$. Now, for the error terms, we assume that the second summand in each $\mathcal{O}(...)$ is non-zero (otherwise we are back to (1.5.82)) and argue by induction: Indeed, using (1.2.16) and applying a resolvent identity (1.3.21), we find

$$\begin{aligned}&\frac{|w_j - \bar{w}_{j+1}| + |w_j - \mathfrak{s}_{j+1}w_{j+2}|}{w_j - w_{j+1}}G_jE_{\mathfrak{s}_{j+1}}G_{j+2} \\ &\qquad = \frac{|w_j - \bar{w}_{j+1}| + |w_j - \mathfrak{s}_{j+1}w_{j+2}|}{(w_j - w_{j+1})\,(w_j - \mathfrak{s}_{j+1}w_{j+2})}\mathfrak{s}_{j+1}\big(G(w_j) - G(\mathfrak{s}_{j+1}w_{j+2})\big)E_{\mathfrak{s}_{j+1}}\,,\end{aligned} \tag{1.5.84}$$

such that, in the resulting chain we have reduced k by *two* and a by one, and the prefactor in (1.5.84) is bounded by $1/\eta$. The argument for the second error in (1.5.83) is completely analogous, after realizing that $\big(|w_j - \bar{w}_{j+1}| + |w_j + \mathfrak{s}_{j+1}\bar{w}_{j+2}|\big)/\big(|w_j - w_{j+1}|\,|w_j + \mathfrak{s}_{j+1}w_{j+2}|\big) \leq 1/\eta$.

On the contrary, if $\mathfrak{s}_j = -\operatorname{sgn}(\operatorname{Im} w_j \operatorname{Im} w_{j+1}) = -$, i.e. the adjacent spectral parameters lie the same half-plane (without loss of generality the upper one), then we use the integral representation from Lemma 1.5.1 to write

$$A_{j-1}G_jE_+G_{j+1}A_{j+1} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{A_{j-1}G(z)A_{j+1}}{(z - w_j)(z - w_{j+1})}\mathrm{d}z\,, \tag{1.5.85}$$

where Γ is an appropriately chosen contour. If $j-1, j+1 \notin \mathcal{I}$, i.e. both A_{j-1}, A_{j+1} were not counted as regularised, then the claim follows by induction and estimating the integral by η^{-1} (up to log factors) since k has been reduced by one, and a has been preserved. On the other hand, if both A_{j-1}, A_{j+1} were counted as regularised, then we use Lemma 1.3.3 to write them as

$$\begin{aligned}\mathring{A}_{j-1}^{w_{j-1},w_j} &= \mathring{A}_{j-1}^{w_{j-1},z} + \mathcal{O}(|w_j - z|)E_+ + \mathcal{O}(|w_j - z|)E_-\,, \\ \mathring{A}_{j+1}^{w_{j+1},w_{j+2}} &= \mathring{A}_{j+1}^{z,w_{j+2}} + \mathcal{O}(|w_{j+1} - z|)E_+ + +\mathcal{O}(|w_{j+1} - z|)E_-\,.\end{aligned} \tag{1.5.86}$$

The resulting term with $\mathring{A}_{j-1}^{w_{j-1},z}, \mathring{A}_j^{z,w_{j+2}}$ can be estimated by induction since k has been reduced by one, a has been preserved and the integral may be estimated by η^{-1}. The other terms with either one or two $E_\pm$ can also be estimated by induction since the integral is at most logarithmically divergent, k has been reduced by one and a by at most two. Finally, if in (1.5.85) one of A_{j-1}, A_{j+1} were counted as regularised, then we use the relevant expansion from (1.5.86), so that for the resulting term with $\mathring{A}$, k has been reduced by one, and a has been preserved, so that the η^{-1} estimate on the integral is affordable. The other term with $E_\pm$ can also be estimated by induction with both a, k reduced by one, and the integral being at most logarithmically divergent. This concludes the proof. □

1.6 Proof of the reduction inequalities, Lemma 1.4.10

During the proof of Lemma 1.4.10, we will heavily rely on the following integral representation for the absolute value $|G|$ of a resolvent (see also [168, Lemma 5.1]).

Lemma 1.6.1. (Integral representation for the absolute value of a resolvent)
Let $w = e + \mathrm{i}\eta \in \mathbf{C} \setminus \mathbf{R}$. Then the absolute value of the resolvent $G(w)$ can be represented as

$$|G(e+\mathrm{i}\eta)| = \frac{2}{\pi}\int_0^\infty \operatorname{Im} G(e+\mathrm{i}\sqrt{\eta^2+s^2})\frac{\mathrm{d}s}{\sqrt{\eta^2+s^2}}\,. \tag{1.6.1}$$

Proof. This immediately follows from the functional calculus for H and the identity

$$\frac{1}{|x-\mathrm{i}\eta|} = \frac{1}{\mathrm{i}\pi}\int_0^\infty\left(\frac{1}{x-\mathrm{i}(\eta^2+s^2)^{1/2}} - \frac{1}{x+\mathrm{i}(\eta^2+s^2)^{1/2}}\right)\frac{\mathrm{d}s}{\sqrt{\eta^2+s^2}}\,. \qquad \square$$

Proof of Lemma 1.4.10. To keep the notation simpler within this proof we may often denote

$$A_i = \mathring{A}_i = \mathring{A}_i^{w_i,w_{i+1}}\,,$$

i.e. sometimes we drop the spectral parameters $w_i = e_i + \mathrm{i}\eta_i$.

We start with the proof of (1.4.27), for which, similarly to [168, Lemma 3.6], we get

$$\Psi_4^{\mathrm{av}} \lesssim N\eta + N^2\eta^2\left(\langle|G_1|A_1|G_2|A_1^*\rangle\langle|G_2|A_2|G_3|A_2^*\rangle\langle|G_3|A_3|G_4|A_3^*\rangle\langle|G_4|A_4|G_1|A_4^*\rangle\right)^{1/2}, \tag{1.6.2}$$

by Lemma 1.4.3, spectral decomposition, and a Schwarz inequality. Next, we use (1.6.1) to write

$$\langle|G_1|A_1|G_2|A_1^*\rangle = \frac{4}{\pi^2}\iint_0^\infty \langle \operatorname{Im} G(w_{1,s})\mathring{A}_1^{w_1,w_2}\operatorname{Im} G(w_{2,t})(\mathring{A}_1^{w_1,w_2})^*\rangle\,\frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}}\,, \tag{1.6.3}$$

where we defined $w_{i,s} := e_i + \mathrm{i}\sqrt{\eta_i^2+s^2}$. The very large s,t–regimes in (1.6.3) can be easily shown to be negligible (e.g. see [168, Proof of Lemma 5.1]), i.e. even if not stated explicitly we assume that the upper integration limit can be replaced by N^{100}. Additionally, we can restrict to the case when $\eta := \min_j|\operatorname{Im} w_j| \le 1$, when this is not the case we use the local law in the regime $\eta > 1$ from Theorems 1.4.4–1.4.5 (see [168, Proof of Lemma 5.1] for a detailed argument). We remark that this argument is not circular since in the proof of the local law for $\eta > 1$ sketched below Remark 1.4.6 one does not use the reduction inequalities in (1.4.27)–(1.4.28).

In order to estimate the rhs. of (1.6.3) we write $\operatorname{Im} G = \frac{1}{2\mathrm{i}}(G - G^*)$ for both $\operatorname{Im} G$ to obtain four terms with two resolvents; to keep the presentation concise we only present the estimate for one of them. From now on we consider only the term $\langle|G_1|A_1|G_2|A_1^*\rangle$, the bound for all the other terms in

the last line of (1.6.2) is completely analogous and so omitted. In the following we will often use the approximations from Lemma 1.3.3 (omitting the trivial $\wedge 1$ in the errors for notational simplicity):

$$\begin{aligned} \mathring{A}^{w_1,w_2} &= \mathring{A}^{w_{1,s},w_{2,t}} + \mathcal{O}(|\sqrt{\eta_1^2+s^2}-\eta_1| + |\sqrt{\eta_2^2+t^2}-\eta_2|)E_+ \\ &\quad + \mathcal{O}(|\sqrt{\eta_1^2+s^2}-\eta_1| + |\sqrt{\eta_2^2+t^2}-\eta_2|)E_- \,, \\ (\mathring{A}^{w_1,w_2})^* &= (\mathring{A}^*)^{w_{2,t},w_{1,s}} + \mathcal{O}(|e_1-e_2| + \sqrt{\eta_1^2+s^2}+\sqrt{\eta_2^2+t^2})E_+ \\ &\quad + \mathcal{O}(|e_1+e_2| + \sqrt{\eta_1^2+s^2}+\sqrt{\eta_2^2+t^2})E_- \,. \end{aligned} \tag{1.6.4}$$

We point out that when taking the adjoint of the first formula to arrive at the second we used that for any w_1, w_2 it holds $(\mathring{A}^{w_1,w_2})^* = (\mathring{A}^*)^{\overline{w_2},\overline{w_1}}$, see Lemma 1.3.3. Recall that within this proof we always assume that $\eta \le 1$. From now on for the error terms we will always use the bounds

$$|\sqrt{\eta_1^2+s^2}-\eta_1| \lesssim s\,, \qquad \sqrt{\eta_1^2+s^2} \le \eta_1 + s\,, \tag{1.6.5}$$

and a similar bound with η_1, s replaced with η_2, t. The first bound is not optimal for small η_1, but good enough for our estimates. Then using (1.6.4) we write

$$\begin{aligned} &\iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_1,w_2} G(w_{2,t})(\mathring{A}_1^{w_1,w_2})^*\rangle \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &= \iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_{1,s},w_{2,t}} G(w_{2,t})(\mathring{A}_1^{*})^{w_{2,t},w_{1,s}}\rangle \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &+ \sum_{\sigma\in\{+,-\}} \iint_0^\infty \langle G(w_{1,s})E_\sigma G(w_{2,t})(\mathring{A}_1^{*})^{w_{2,t},w_{1,s}}\rangle \mathcal{O}(\eta_1+\eta_2+s+t) \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &+ \sum_{\sigma\in\{+,-\}} \iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_{1,s},w_{2,t}} G(w_{2,t})E_\sigma\rangle \mathcal{O}(\eta_1+\eta_2+s+t) \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &+ \sum_{\sigma,\tau\in\{+,-\}} \iint_0^\infty \langle G(w_{1,s})E_\sigma G(w_{2,t})E_\tau\rangle \mathcal{O}(\eta_1^2+\eta_2^2+s^2+t^2) \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &+ \iint_0^\infty \langle G(w_{1,s})\big[\sum_\sigma \mathcal{O}(|e_1-\sigma e_2|)E_\sigma\big] G(w_{2,t})(\mathring{A}_1^{*})^{w_{2,t},w_{1,s}}\rangle \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &+ \iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_{1,s},w_{2,t}} G(w_{2,t})[\mathcal{O}(|e_1-e_2|)E_+ + \mathcal{O}(|e_1+e_2|)E_-]\rangle \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \\ &+ \iint_0^\infty \langle G(w_{1,s})\big[\sum_\sigma \mathcal{O}(|e_1-\sigma e_2|)E_\sigma\big] G(w_{2,t})\big[\sum_\tau \mathcal{O}(|e_1-\tau e_2|)E_\tau\big]\rangle \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}}\,. \end{aligned} \tag{1.6.6}$$

We now estimate the terms in the rhs. of (1.6.6) one by one. In the following estimates we will always omit $\log N$-factors. We start with

$$\left|\iint_0^\infty \langle G(w_{1,s})\mathring{A}_1^{w_{1,s},w_{2,s}} G(w_{2,t})(\mathring{A}_1^{*})^{w_{2,t},w_{1,s}}\rangle \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}}\right| \prec 1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\,,$$

which readily follows by the definition of Ψ_2^{av} in (1.4.15) and from the assumption $\Psi_2^{\mathrm{av}} \prec \psi_2^{\mathrm{av}}$. For the third to the fifth line in (1.6.6) we use the bound

$$\begin{aligned} &\left|\iint_0^\infty \langle G(w_{1,s})E_\sigma G(w_{2,t})B\rangle \mathcal{O}(\eta_1+\eta_2+s+t) \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}}\right| \\ &\qquad \prec \iint_0^\infty \left(\frac{1}{\sqrt{\eta_1^2+s^2}} \wedge \frac{1}{\sqrt{\eta_2^2+t^2}}\right)\big[\eta_1+\eta_2+s+t\big] \frac{\mathrm{d}s\mathrm{d}t}{\sqrt{\eta_1^2+s^2}\sqrt{\eta_2^2+t^2}} \lesssim 1\,, \end{aligned} \tag{1.6.7}$$

for any deterministic norm bounded matrices B and for $\sigma \in \{+,-\}$. For the fifth line of (1.6.6) we used the bound $(s^2+t^2)\wedge 1 \le (s+t)\wedge 1$ (recall that $\wedge 1$ is omitted in the error terms in (1.6.6) for notational simplicity). Note that here we used:

$$|\langle G(w_{1,s})E_\sigma G(w_{2,t})B\rangle| \prec \frac{1}{\sqrt{\eta_1^2+s^2}} \wedge \frac{1}{\sqrt{\eta_2^2+t^2}}\,, \tag{1.6.8}$$

which holds uniformly in matrices with $\|B\| \lesssim 1$. We point out that to obtain the bound (1.6.8) we used spectral decomposition of the resolvents and that $\langle \boldsymbol{w}_i, E_\sigma \boldsymbol{w}_j\rangle = \delta_{i,\sigma j}$ to bound[16]

$$\begin{aligned}|\langle G(w_{1,s})E_\sigma G(w_{2,t})B\rangle| &= \left|\frac{1}{2N}\sum_i \frac{\langle \boldsymbol{w}_i, B\boldsymbol{w}_{\sigma i}\rangle}{(\lambda_i - w_{1,s})(\lambda_i - \sigma w_{2,t})}\right| \\ &\lesssim \frac{1}{N}\sum_i \frac{1}{|\lambda_i - w_{1,s}||\lambda_i - \sigma w_{2,t}|} \\ &\prec \frac{1}{|\operatorname{Im} w_{1,s}| \vee |\operatorname{Im} w_{2,t}|}\,,\end{aligned}$$

where in the last inequality we used the single resolvent local law.

Finally, for the last three lines in (1.6.6) we use that for any norm bounded matrix B, by resolvent identity (1.3.21), we have (recall that $E_+ = I$)

$$|\langle G(w_{1,s})BG(w_{2,t})\rangle| \prec \frac{1}{|w_{1,s}-w_{2,t}|}\,, \qquad |\langle G(w_{1,s})BG(w_{2,t})E_-\rangle| \prec \frac{1}{|w_{1,s}+w_{2,t}|}\,, \tag{1.6.9}$$

which after the integration in (1.6.6) gives a bound of order one, as a consequence of

$$\frac{|e_1 \pm e_2|}{|w_{1,s} \pm w_{2,t}|} \lesssim 1\,.$$

Note that here it is important that the error terms in (1.6.6) involving $|e_1 - e_2|$ are always multiplied with the matrix E_+, while errors of order $|e_1 + e_2|$ are in the direction of E_-.

Combining the computations in (1.6.6)–(1.6.9) we conclude that

$$|\langle |G_1|A_1|G_2|A_1^*\rangle| \prec 1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}, \tag{1.6.10}$$

which, after plugging it in the rhs. of (1.6.2), clearly implies (1.4.27) .

For (1.4.28) for Ψ_3^{iso}, we find

$$\Psi_3^{\mathrm{iso}} \lesssim \sqrt{N\eta} + N\eta^2\Big(\big(G_1A_1|G_2|A_1^*G_1^*\big)_{\boldsymbol{xx}}\big(G_4^*A_3^*|G_3|A_3G_4\big)_{\boldsymbol{yy}}\langle |G_2|A_2|G_3|A_2^*\rangle\Big)^{1/2}, \tag{1.6.11}$$

again by Lemma 1.4.3, spectral decomposition, and a Schwarz inequality. Then, using again the integral representation (1.6.1), we find that

$$\big(G_1A_1|G_2|A_1^*G_1^*\big)_{\boldsymbol{xx}} = \frac{2}{\pi}\int_0^\infty \big(G_1A_1\operatorname{Im}G(w_{2,s})A_1^*G_1^*\big)_{\boldsymbol{xx}}\frac{\mathrm{d}s}{\sqrt{\eta_2^2+s^2}}\,,$$

recalling the notation $w_{2,s} = e_2 + \mathrm{i}\sqrt{\eta_2^2+s^2}$. The estimate for this term is fairly similar to the one in (1.6.3), hence we present only the main differences and skip the details; actually the current case is easier since we now have only one $|G|$.

[16] We point out that here $\boldsymbol{w}_i$ denotes an eigenvector of H^Λ, and it should not be confused with the spectral parameters $w_{1,s}, w_{2,t}$.

After splitting $\operatorname{Im} G = \frac{1}{2\mathrm{i}}(G - G^*)$ and handling both terms separately, we can write, similarly to (1.6.6) and using (1.6.4)–(1.6.5), the following approximation:

$$\begin{aligned}&\int_0^\infty \big(G_1 A_1 G(w_{2,s}) A_1^* G_1^*\big)_{\boldsymbol{xx}} \frac{\mathrm{d}s}{\sqrt{\eta_2^2 + s^2}} \\ &= \int_0^\infty \big(G_1 \mathring{A}_1^{w_1, w_{2,s}} G(w_{2,s}) (\mathring{A}_1^*)^{w_{2,s}, w_1} G_1^*\big)_{\boldsymbol{xx}} \frac{\mathrm{d}s}{\sqrt{\eta_2^2 + s^2}} + \mathcal{E}\,.\end{aligned} \tag{1.6.12}$$

Here $\mathcal{E}$ is an error coming from all the errors in (1.6.4). For the first term in the second line of (1.6.12) we use the bound

$$\left| \int_0^\infty \big(G_1 \mathring{A}_1^{w_1, w_{2,s}} G(w_{2,s}) (\mathring{A}_1^*)^{w_{2,s}, w_1} G_1^*\big)_{\boldsymbol{xx}} \frac{\mathrm{d}s}{\sqrt{\eta_2^2 + s^2}} \right| \prec \frac{1}{\eta}\left(1 + \frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right), \tag{1.6.13}$$

which follows by the definition of Ψ_2^{iso}. For the error term we do not write the details, since once we replace (1.6.8)–(1.6.9) with (here B, B_1, B_2 are deterministic norm bounded matrices)

$$\begin{aligned}&|\big(G_1 B_1 G(w_{2,s}) B_2 G_1^*\big)_{\boldsymbol{xx}}| \le \big(G_1 B_1 B_1^* G_1^*\big)_{\boldsymbol{xx}}^{1/2} \big(G_1 B_2^* G(w_{2,s}) G(w_{2,s})^* B_2 G_1^*\big)_{\boldsymbol{xx}}^{1/2} \prec \frac{1}{\eta\sqrt{\eta_2^2 + s^2}} \\ &|\big(G_1 E_\sigma G(w_{2,s}) B G_1^*\big)_{\boldsymbol{xx}}| \prec \frac{1}{\eta |w_1 - w_{2,s}|}\,,\end{aligned} \tag{1.6.14}$$

respectively, the estimate

$$|\mathcal{E}| \prec \frac{1}{\eta} \tag{1.6.15}$$

follows completely analogously. The estimates (1.6.14) follow by repeated applications of the resolvent identity (1.3.21) (after commuting E_σ with G in case of the second formula), the trivial bound $\|G\| \le 1/\eta$ and the single resolvent local law. Combining, (1.6.13)–(1.6.15) we conclude

$$\left|\big(G_1 A_1 | G_2 | A_1^* G_1^*\big)_{\boldsymbol{xx}}\right| \prec \frac{1}{\eta}\left(1 + \frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right). \tag{1.6.16}$$

The bound in (1.6.16), together with (1.6.10) to estimate the averaged term in (1.6.11), concludes the proof (1.4.28) for Ψ_3^{iso}.

Analogously to (1.6.11), for Ψ_4^{iso} we find that

$$\begin{aligned}\Psi_4^{\mathrm{iso}} &\lesssim \sqrt{N\eta} + N\eta^{5/2}\Big(\big(G_1 A_1 | G_2 | A_1^* G_1^*\big)_{\boldsymbol{xx}} \big(G_5^* A_4^* | G_4 | A_4 G_5\big)_{\boldsymbol{yy}} \langle |G_2| A_2 G_3 A_3 | G_4 | A_3^* G_3^* A_2^* \rangle\Big)^{1/2} \\ &\lesssim \sqrt{N\eta} + N^{3/2}\eta^{5/2}\Big(\big(G_1 A_1 | G_2 | A_1^* G_1^*\big)_{\boldsymbol{xx}} \big(G_5^* A_4^* | G_4 | A_4 G_5\big)_{\boldsymbol{yy}}\Big)^{1/2} \\ &\quad \times \Big(\langle |G_2| A_2 | G_3 | A_2^* \rangle \langle |G_3| A_3 | G_4 | A_3^* \rangle \langle |G_4| A_3^* | G_3 | A_3 \rangle \langle |G_3| A_2^* | G_2 | A_2 \rangle\Big)^{1/4}\end{aligned}$$

where in the last inequality we used spectral decomposition and a bound as in [168, Proof of Lemma 3.6] to bound the trace with four G's and four A's in terms of a product of traces containing only two G's and two A's. Finally, using the bounds (1.6.10), (1.6.16), we conclude the proof of (1.4.28) for Ψ_4^{iso} as well. □

1.A Motivating derivations of the regularisation

In this appendix, we shall motivate and derive the regularisation (1.3.2) introduced in Definition 1.3.1 by considering two basic examples. We also use these examples to present two different approaches

to guess the right regularisation. Before the details, we give an informal summary of these two model calculations.

First, in Section 1.A.1, we compute

$$\mathbf{E}\left|\underline{\langle WG(\mathrm{i}\eta)A\rangle}\right|^2, \tag{1.A.1}$$

which is the leading contribution to $\langle (G-M)B\rangle$ in the single-resolvent local law, with $A = \mathcal{X}[B]M$, see (1.5.18). We will show that, in order to be able to reduce its naive size $1/(N\eta)^2$ to the target $1/(N^2\eta)$, we *need* that $\langle A, V_\pm\rangle = 0$, i.e. we need $A \in \mathbf{C}^{2N\times 2N}$ to be orthogonal to two certain directions $V_\pm$ in $\mathbf{C}^{2N\times 2N}$. For simplicity, we chose the spectral parameter $w = \mathrm{i}\eta$ to be on the imaginary axis, assuming that $0 \in \mathbf{B}_\kappa$ for some $\kappa > 0$. In this case, both cutoff functions (1.4.7) in the actual definition of the regularisation satisfy $\mathbf{1}^\pm_\delta(\mathrm{i}\eta, \mathrm{i}\eta) = 0$ for $\eta > 0$ small enough. Hence, at least *a posteriori*, we really catch both directions $V_\pm$ and not only one. This calculation is rather *foundational* and unambiguously reveals two directions $V_\pm$, for which we need that $\langle A, V_\pm\rangle = 0$, in order to reduce the naive size of (1.A.1).

Second, in Section 1.A.2, we consider the averaged chain with two resolvents

$$\langle G^{\Lambda_1}(w_1)A_1G^{\Lambda_2}(w_2)A_2\rangle\,, \tag{1.A.2}$$

where the resolvents are even allowed to have generally *different*[17] deformations, Λ_1 and Λ_2. Let $M_1 := M^{\Lambda_1}(w_1)$ and $M_2 := M^{\Lambda_2}(w_2)$. For simplicity, we will assume that the stability operators

$$\mathcal{B}_{m^{(*)}n^{(*)}} := 1 - M_m^{(*)}\mathcal{S}[\cdot]M_n^{(*)}\,, \quad m,n \in [2]\,, \tag{1.A.3}$$

for all constellations of adjoints, have at most one *critical* eigenvalue $\beta_{m^{(*)}n^{(*)}}$ which is *not* of order one (with associated right and left eigenvectors $R_{m^{(*)}n^{(*)}}$ and $L_{m^{(*)}n^{(*)}}$, respectively, cf. (1.B.16) later). As we will show in Lemma 1.B.5 (c), this is the case, e.g., if $\Lambda \equiv \Lambda_1 = \Lambda_2$ and $\mathrm{Re}\, w_1, \mathrm{Re}\, w_2 \in \mathbf{B}^\Lambda_\kappa$, and actually remains true for other more general random matrix models with a *flat* [17] self-energy operator $\mathcal{S}[\cdot]$. Recall that $\mathcal{S}[\cdot]$ is flat if

$$c\langle R\rangle \le \mathcal{S}[R] \le C\langle R\rangle \tag{1.A.4}$$

for some constants $c, C > 0$ and any positive semi-definite matrix $R \ge 0$.

Again, the main question is what special property A_1, A_2 must have so that (1.A.2) be smaller than its naive size of order $1/\eta$ obtained from a simple Schwarz inequality. Similarly to (1.A.1), we could directly compute the second moment of the corresponding underline term (see Lemma 1.5.8), but for pedagogical reason we present an alternative argument. Quite *pragmatically*, we start the usual proof via cumulant expansion for a bound on (1.A.2) and find that certain deterministic terms are too big for general A_1, A_2. We shall see that there exist two matrices $\tilde V_\pm \in \mathbf{C}^{2N\times 2N}$ (which turn out to be certain right eigenvectors $R_{m^{(*)}n^{(*)}}$ of (1.A.3), see (1.A.15) and (1.A.18) later), such that, if $\langle A_i, \tilde V_\pm\rangle = 0$, these critical terms are smaller. This suggests a *pragmatic ansatz* of the form (1.3.2) on the regularisation. We will observe that, for the situation $\Lambda_1 = \Lambda_2$ and $w_1 = w_2 = \mathrm{i}\eta$, the expressions for $\tilde V_\pm$ in fact *coincide* with those for $V_\pm$ obtained in Section 1.A.1. Notice that the imaginary part of the single-resolvent setup leading to (1.A.1) is a special case of the two-resolvent setup (1.A.2) since

$$\mathrm{Im}\,\langle GB\rangle = \langle (\mathrm{Im}\, G)B\rangle = \eta\langle GBG^*E_+\rangle$$

for self-adjoint B. Hence the regularity of B tested against $\mathrm{Im}\, G(\mathrm{i}\eta)$ is the same as the regularity of B between $G(\mathrm{i}\eta)$ and $G^*(\mathrm{i}\eta) = G(-\mathrm{i}\eta)$. This shows, at least in this special case, that the *foundational* and the *pragmatic* approaches lead to the same regularisation. Similar conclusion about the equivalence of both approaches holds in general.

[17] All results in the current paper concern the $\Lambda_1 = \Lambda_2$ case; the generalisation $\Lambda_1 \ne \Lambda_2$ is mentioned only to stress that our method is also valid beyond the scope of the current paper.

Finally, in Section 1.A.3, motivated by the previous tandem of *foundational* and *pragmatic* computations in Sections 1.A.1 and 1.A.2, respectively, we list generally valid (i.e. for arbitrary w_1, w_2 also away from the imaginary axis) explicit formulas for the directions $V_\pm$ regularising (1.A.2) in case that $\Lambda_1 = \Lambda_2$. These explicit formulas are identical to those used in the regularisation introduced in Definition 1.3.1.

1.A.1 Variance calculation of (1.A.1)

In the following, we simply write $G = G(\mathrm{i}\eta)$ for ease of notation. Then, using a cumulant expansion and neglecting cumulants of order at least three (or assuming that X is Ginibre), one gets

$$\begin{aligned}\mathbf{E}\big|\langle \underline{WGA}\rangle\big|^2 &= \frac{1}{N}\sum_{ab} R_{ab}\mathbf{E}\langle \Delta^{ab}GA\rangle \partial_{ba}\langle \underline{A^*G^*W}\rangle \\ &= \frac{1}{N}\sum_{ab} R_{ab}\mathbf{E}\langle \Delta^{ab}GA\rangle\langle GA^*G^*\Delta^{ba}\rangle \\ &\quad + \frac{1}{N^2}\sum_{abcd} R_{ab}R_{cd}\mathbf{E}\langle \Delta^{ab}G\Delta^{dc}GA\rangle\langle A^*G^*\Delta^{ba}G^*\Delta^{cd}\rangle \\ &= \frac{1}{N^2}\sum_{\sigma}\sigma\mathbf{E}\langle E_\sigma GAE_\sigma A^*G^*\rangle + \frac{1}{N^2}\sum_{\sigma\tau}\sigma\tau\mathbf{E}\langle E_\sigma G^*E_\tau GA\rangle\langle E_\sigma GE_\tau(GA)^*\rangle\,.\end{aligned} \tag{1.A.5}$$

The rescaled cumulant $R_{ab} := N\kappa(ab,ba)$ has been introduced above (1.5.11) and $\Delta^{ab} \in \mathbf{C}^{2N\times 2N}$ contains only one non-zero entry at position (a,b), i.e. $(\Delta^{ab})_{cd} = \delta_{ac}\delta_{bd}$.

As we will show, the cumulant expansion (1.A.5) yields that (up to a constant)

$$\mathbf{E}\big|\langle \underline{WGA}\rangle\big|^2 \approx \frac{\mathbf{E}\big|\langle \operatorname{Im} GA\rangle\big|^2}{(N\eta)^2} + \frac{\mathbf{E}\big|\langle \operatorname{Im} GAE_-\rangle\big|^2}{(N\eta)^2} + \mathcal{O}\Big(\frac{1}{N^2\eta}\Big). \tag{1.A.6}$$

Indeed, the first summand in the last line of (1.A.5) is estimated by $1/(N^2\eta)$, the target size, with the aid of a trivial Schwarz inequality and a Ward identity using Theorem 1.2.6. By writing out the summation in the last summand, we get in total four terms. Since their treatment is very similar, we focus on two exemplary terms with $\sigma = \tau = +$ (analogous to $\sigma = \tau = -$) and $\sigma = -\tau = -$ (analogous to $\sigma = -\tau = +$).

For the former, we apply a Ward identity and find it to be given by

$$\frac{\mathbf{E}\big|\langle \operatorname{Im} GA\rangle\big|^2}{(N\eta)^2}\,, \tag{1.A.7}$$

which, without any further information on A, using that $\langle GA\rangle \sim 1$ from Theorem 1.2.6, is too big, compared to the targeted $1/(N^2\eta)$-size. However, this drastically improves if $\langle \operatorname{Im} M, A\rangle = 0$ (recall that $\operatorname{Im} M$ is self adjoint): Since $\langle (G-M)A\rangle$ and $\langle \underline{WGA}\rangle$ are roughly of the same size (see (1.5.18)), the contribution (1.A.7) basically becomes a lower-order correction. We have thus identified the first of the two directions $V_\pm$, to which A has to be orthogonal to in order to reduce the naive size of (1.A.1), namely

$$V_+ = \alpha_+ \operatorname{Im} M \quad \text{for some non-zero} \quad \alpha_+ \in \mathbf{C}\,. \tag{1.A.8}$$

The latter case, $\sigma = -\tau = -$, is slightly more involved due to the asymmetry of the two factors in the last summand in the last line of (1.A.5): For the first factor, again a Ward identity is sufficient. In the second factor, we use (1.2.16) together with the integral representation [168, Eq. (3.14)]

$$G^*G^* = \int_{\mathbf{R}} \frac{\operatorname{Im} G(x+\mathrm{i}\eta/2)}{(x+\mathrm{i}\eta/2)^2}\mathrm{d}x\,,$$

similar to Lemma 1.5.1, in the approximate form $G^*G^* \sim \operatorname{Im} G/\eta$. This follows (at least as an effective upper bound) by replacing the Cauchy kernel in the integral

$$|\langle G^*G^*E_-A^*\rangle| \le \int_{\mathbf{R}} \frac{|\langle \operatorname{Im} G(x+\mathrm{i}\eta/2)E_-A^*\rangle|}{x^2+(\eta/2)^2}\mathrm{d}x \sim \frac{\operatorname{Im} G(\mathrm{i}\eta)}{\eta}$$

by a δ-distribution. Overall, this leaves us (roughly) with

$$\frac{\mathbf{E}\left|\langle \operatorname{Im} GAE_-\rangle\right|^2}{(N\eta)^2} \tag{1.A.9}$$

for the second case. Hence, arguing for (1.A.9) completely analogous as done for (1.A.7), we find the second direction V_-, to which A has to be orthogonal to, in order to reduce the naive size of (1.A.1), namely

$$V_- = \alpha_- \operatorname{Im} ME_- \quad \text{for some non-zero} \quad \alpha_- \in \mathbf{C}\,. \tag{1.A.10}$$

We point out that the first term in (1.A.6) would have worked in the exact same way also for spectral parameters $w = e+\mathrm{i}\eta$ with $e \neq 0$. However, the second direction V_- would *not* have been visible in this scenario, since the second term in (1.A.6) would have been replaced by (at least for an upper bound)

$$\frac{\mathbf{E}\left|\langle \operatorname{Im} G(e+\mathrm{i}\eta)AE_-\rangle\right|^2}{N^2\eta\,(|e|+\eta)} + \frac{\mathbf{E}\left|\langle \operatorname{Im} G(e+\mathrm{i}\eta)AE_-\rangle\langle \operatorname{Im} G(-e+\mathrm{i}\eta)E_-A^*\rangle\right|}{N^2\eta\,(|e|+\eta)}\,.$$

1.A.2 General structural regularisation in (1.A.2)

We begin with the general rather *structural* regularizing decomposition of a matrix A (recall (1.3.2)), which shall be conducted as (dropping the tilde, which has been temporarily introduced below (1.A.3))

$$\boxed{A^\circ \equiv \mathring{A} := A - \langle V_+, A\rangle U_+ - \langle V_-, A\rangle U_-} \tag{1.A.11}$$

for some $U_\sigma, V_\sigma \in \mathbf{C}^{2N\times 2N}$ to be determined but subject to the conditions $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$ and $\langle U_\sigma, U_\sigma\rangle = 1$. We point out, that the following calculations are largely insensitive to the form of the self-energy operator $\mathcal{S}[\cdot]$ (but see Footnote 18) and hence the conclusions for U_σ and V_σ derived in this section are valid beyond our concrete model (up to the fact that, due to the chiral symmetry (1.2.16), the regularisation involves a *two*-dimensional projection).

The goal of the present subsection is to *show* that $V_\pm$ must be chosen as certain right eigenvectors $R_{m^{(*)}n^{(*)}}$ of (1.A.3). This follows by expanding (1.A.2) and identifying several terms, whose size is too big for general deterministic matrices. Now, these terms can be neutralised, if $\langle A_i, R_{m^{(*)}n^{(*)}}\rangle = 0$ for certain right eigenvectors. However, as already mentioned in Section 1.3, for the directions $U_\pm$ there are *a priori* no further constraints or conditions (apart from orthogonality and normalisation). Hence, as it turns out to be convenient for our proofs, we will choose the matrices U_σ in such a way, that a resolvent identity, i.e. the transformation of a product into a difference,

$$\boxed{G^{\Lambda_1}(w_1)U_\sigma G^{\Lambda_2}(w_2) \approx \big(G^{\Lambda_1}(w_1) - G^{\Lambda_2}(\sigma w_2)\big)U_\sigma\,,}$$

can be applied (here, the symbol '$\approx$' neglects lower order terms). Finally, the condition $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$ will guarantee that the regularisation is idempotent, i.e. $(\mathring{A})^\circ = \mathring{A}$. Note that our general ansatz (1.A.11) is restricted to the non-degenerate situation, where U_σ and V_σ are non-orthogonal, $\langle V_\sigma, U_\sigma\rangle \sim 1$. This is guaranteed for our concrete model with deformations $\Lambda_1 = \Lambda_2$ (see Section 1.A.3) but requires some non-trivial arguments in more general cases.

Although the regularisation is inherently two-dimensional (at least for our model), we also define

$$\mathring{A}^\sigma = A^{\circ\sigma} := A - \langle V_\sigma, A\rangle U_\sigma\,, \quad \sigma \in \{+,-\}\,,$$

and refer to $A^{\circ\sigma}$ as the *σ-regular component* (or *σ-regularisation*) of A and to $\langle V_\sigma, A\rangle U_\sigma$ as its *σ-singular component*. Note that $(A^{\circ_+})^{\circ_-} = (A^{\circ_-})^{\circ_+} = \mathring{A}$, since $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$.

As usual, we use the common notation $\eta_i := |\operatorname{Im} w_i|$ for $i \in [2]$ and abbreviate (see (1.3.8))

$$\mathfrak{s}_i := -\operatorname{sgn}(\operatorname{Im} w_i \operatorname{Im} w_{i+1}), \quad i \in [2], \tag{1.A.12}$$

where the indices are understood cyclically modulo 2 (cf. Definition 1.4.2). This means that, in particular, $\mathfrak{s}_1 = \mathfrak{s}_2$ due to the short length of the chain (1.A.2). In the following, we will drop the arguments by writing, e.g., $M_1 = M^{\Lambda_1}(w_1)$ and $G_2 = G^{\Lambda_2}(w_2)$. Moreover, we take $A_1 = \mathring{A}_1$ and $A_2 = \mathring{A}_2$ to be regular, i.e. orthogonal to some yet to be specified $V_\pm$.

Now, by means of

$$G_1 = M_1 - M_1\underline{WG_1} + M_1\mathcal{S}[G_1 - M_1]G_1,$$

we immediately find

$$G_1A_1G_2 = M_1A_1G_2 - M_1\underline{WG_1}A_1G_2 + M_1\mathcal{S}[G_1 - M_1]G_1A_1G_2,$$

from which we conclude that

$$\begin{aligned}\mathcal{B}_{12}[G_1A_1G_2] = {} & M_1A_1M_2 + M_1A_1(G_2 - M_2) - M_1\underline{WG_1A_1G_2} \\ & + M_1\mathcal{S}[G_1 - M_1]G_1A_1G_2 + M_1\mathcal{S}[G_1A_1G_2](G_2 - M_2).\end{aligned}$$

This implies

$$\begin{aligned}\langle (G_1A_1G_2 - M_{12}^{A_1})A_2\rangle = {} & \langle M_1A_1(G_2 - M_2)\mathcal{X}_{21}[A_2]\rangle - \langle M_1\underline{WG_1A_1G_2}\mathcal{X}_{21}[A_2]\rangle \\ & + \langle M_1\mathcal{S}[G_1 - M_1]G_1A_1G_2\mathcal{X}_{21}[A_2]\rangle \\ & + \langle M_1\mathcal{S}[G_1A_1G_2](G_2 - M_2)\mathcal{X}_{21}[A_2]\rangle\end{aligned}$$

where we defined

$$M_{12}^{A_1} := \mathcal{B}_{12}^{-1}[M_1A_1M_2] = M_1\mathcal{X}_{12}[A_1]M_2 = M(w_1, A_1, w_2) \tag{1.A.13}$$

(recall (1.4.4) and Definition 1.4.1) and used the shorthand notation

$$\mathcal{X}_{mn}[B] = \big((\mathcal{B}_{nm}^*)^{-1}[B^*]\big)^* = (\mathcal{B}_{m^*n^*}^{-1})^*[B], \quad B \in \mathbf{C}^{2N\times 2N}.$$

The adjoint of $\mathcal{B}_{nm}$ is understood with respect to the standard (normalised) inner product $\langle S, T\rangle := \langle S^*T\rangle$ for $S, T \in \mathbf{C}^{2N\times 2N}$, which is given by

$$\mathcal{B}^* \equiv \mathcal{B}^*(w_1, w_2)[\cdot] := 1 - \mathcal{S}[(M(w_1))^* \cdot (M(w_2))^*]. \tag{1.A.14}$$

So far, the regularisation of A_1 and A_2 has been rather *structural*. To make it more concrete, we must allow V_σ and U_σ to be potentially different depending on which of the A_i is regularised. In order to do so, we also temporarily introduce the additional index i, referring to the considered A_i. That is, we will write $V_{\sigma,i}$ instead of V_σ.

The matrices $V_{\mathfrak{s}_i,i}$ (recall (1.A.12) for the definition of $\mathfrak{s}_i$) shall be determined by requiring that

$$\|M_{12}^{A_1}\| = \|M_1\mathcal{X}_{12}[A_1]M_2\| \lesssim \|A_1\| \quad \text{for} \quad i = 1 \quad \text{and} \quad \|\mathcal{X}_{21}[A_2]\| \lesssim \|A_2\| \quad \text{for} \quad i = 2,$$

meaning that the (adjoint of the) stability operator has a bounded inverse on regular observables (i.e. subtracting the $\mathfrak{s}_i$-singular component amounts to removing the 'bad direction' of the stability operators $\mathcal{X}_{12}$ and $\mathcal{X}_{12}$, respectively). From this condition, we find the characterisation of $V_{\mathfrak{s}_1,1}$ and $V_{\mathfrak{s}_2,2}$, namely

$$\boxed{V_{\mathfrak{s}_1,1} = R_{1^*2^*} = (R_{21})^* \qquad \text{and} \qquad V_{\mathfrak{s}_2,2} = R_{2^*1^*} = (R_{12})^*,} \tag{1.A.15}$$

up to a normalisation constant, which can be specified only after determining U_σ (recall that $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$ and $\langle U_\sigma, U_\sigma\rangle = 1$). Recall from (1.A.3), that we denote by $R_{m^{(*)}n^{(*)}}$ and $L_{m^{(*)}n^{(*)}}$ the (normalised) right and left eigenvectors of $\mathcal{B}_{m^{(*)}n^{(*)}}$ corresponding to the (potentially) *critical eigenvalue* $\beta_{m^{(*)}n^{(*)}}$.

Indeed, in order to verify that (1.A.15) is the right choice for $V_{\mathfrak{s}_i,i}$, we use the decomposition

$$\mathcal{X}_{mn} = (\mathcal{B}^{-1}_{m^*n^*})^* = \frac{1}{\bar\beta_{m^*n^*}} |L_{m^*n^*}\rangle\langle R_{m^*n^*}| + \mathcal{O}(1)\,, \tag{1.A.16}$$

where $\mathcal{O}(1)$ is a shorthand notation for a linear operator $\mathcal{E}: \mathbf{C}^{2N\times 2N} \to \mathbf{C}^{2N\times 2N}$ satisfying $\|\mathcal{E}[B]\| \lesssim \|B\|$. This linear operator is represented by a contour integration of the form

$$\frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}z}{z - \mathcal{B}^*_{m^*n^*}}$$

where the contour encircles all non-critical eigenvalues of $\mathcal{B}^*_{m^*n^*}$ and remains at an order one distance from the entire spectrum. Note that for general non-Hermitian operators the resolvent $(z - \mathcal{B}^*_{m^*n^*})^{-1}$ would not necessarily be bounded (independently of N) just because z is well away from the eigenvalues. However, the explicit form of $\mathcal{S}$ (see (1.2.21)) implies[18] that $\mathcal{B}^*_{m^*n^*} = 1 + T$ where T is a rank-two operator. For such operators elementary linear algebra shows that

$$\left\| \frac{1}{z - \mathcal{B}^*_{m^*n^*}} \right\| \lesssim \left[\mathrm{dist}\big(z, \mathrm{Spec}(\mathcal{B}^*_{m^*n^*})\big)\right]^{-2},$$

i.e. the non-Hermitian instability only affects a two-dimensional subspace.

Using (1.A.16) we find

$$\mathcal{X}_{12}[\mathring{A}_1^{\mathfrak{s}_1}] = \frac{1}{\bar\beta_{1^*2^*}}\big(\langle R_{1^*2^*}, A_1\rangle - \langle V_{\mathfrak{s}_1,1}, A_1\rangle\langle R_{1^*2^*}, U_{\mathfrak{s}_1,1}\rangle\big) L_{1^*2^*} + \mathcal{O}(1)[A_1]$$

for the decomposition of A_1 and

$$\mathcal{X}_{21}[\mathring{A}_2^{\mathfrak{s}_2}] = \frac{1}{\bar\beta_{2^*1^*}}\big(\langle R_{2^*1^*}, A_2\rangle - \langle V_{\mathfrak{s}_2,2}, A_2\rangle\langle R_{2^*1^*}, U_{\mathfrak{s}_2,2}\rangle\big) L_{2^*1^*} + \mathcal{O}(1)[A_2]\,,$$

for the decomposition of A_2. This implies that for $\big(\cdots\big)$ to be vanishing for every $\mathring{A}_i^{\mathfrak{s}_i}$, the matrix $V_{\mathfrak{s}_i,i}$ has to be chosen according to (1.A.15) (recall $\langle V_{\sigma,i}, U_{\tau,i}\rangle = \delta_{\sigma,\tau}$).[19] Overall, subtracting the $\mathfrak{s}_i$-singular component already accounts for removing the 'bad direction' of a involved stability operator and thus – in particular – reduces the naive size of the deterministic approximation (1.A.13).

However, removing the $\mathfrak{s}_i$-singular component is not sufficient: Although $\langle V_{\mathfrak{s}_i,i}, U_{-\mathfrak{s}_i,i}\rangle = 0$ and thus $U_{-\mathfrak{s}_i,i}$ is $\mathfrak{s}_i$-regular, we observe that

$$\langle G_1 U_{-\mathfrak{s}_1,1} G_2 U_{-\mathfrak{s}_2,2}\rangle \tag{1.A.17}$$

still (potentially) has large fluctuations: In our concrete i.i.d. model, take $z \equiv z_1 = z_2$ (to be suppressed from the notation) and $w \equiv w_1 = -w_2$ with $e = \mathrm{Re}\, w_1$ and $\eta = \mathrm{Im}\, w_1 > 0$ w.l.o.g., which implies

[18] This is the only place in Section 1.A.2 where the special form of $\mathcal{S}$ is currently used. For more general $\mathcal{S}$ operator an appropriate generalisation of the symmetrised (saturated) self-energy operator [17, Def. 4.5] to two different spectral parameters is needed, see [392, Eq. (2.30)] in the commutative case.

[19] In case that $\Lambda_1 = \Lambda_2$, by the lower bound (1.B.15), the choices in (1.A.15) not necessarily have to be made *exact*, but tolerate an error of the order given in the rhs. of (1.B.15). Having such a tolerance might be important if one treats the $\Lambda_1 \neq \Lambda_2$ case (contrary to $\Lambda_1 = \Lambda_2$ as done in this paper) and still has to satisfy the constraints $\langle V_\sigma, U_\tau\rangle = \delta_{\sigma,\tau}$ and $\langle U_\sigma, U_\sigma\rangle = 1$.

that $\mathfrak{s}_1 = \mathfrak{s}_2 = +$ and $U_\sigma = E_\sigma$ for $\sigma = \pm$ (see the discussion below (1.3.4)). In this situation, we use (1.2.16) and thus (1.A.17) takes the form

$$\langle G(e+\mathrm{i}\eta)E_- G(-e-\mathrm{i}\eta)E_-\rangle = -\langle G(e+\mathrm{i}\eta)G(e+\mathrm{i}\eta)\rangle\,.$$

By construction of $V_{\mathfrak{s}_i,i}$, the corresponding deterministic approximation (1.A.13) is bounded by one, but this is dominated by the fluctuation of order $1/(N\eta^2)$ in the relevant small regime $\eta \sim N^{-1+\epsilon}$. This example shows again, what we have already established in Section 1.A.1: For our concrete model, at least close to the imaginary axis, the regularisation (1.3.2) is *necessarily a two-dimensional operation*.

For determining the other directions $V_{-\mathfrak{s}_i,i}$, we note that the regularisation should be designed in such a way, that it covers also the cases where one (or both) of the resolvents G_1, G_2 are taken as an adjoint (see, e.g., (1.5.13) and (1.6.10)). Hence, requiring that the same arguments leading to (1.A.15) should also be followed for (i) $\langle G_1 A_1 G_2^* A_2\rangle$ and (ii) $\langle G_1^* A_1 G_2 A_2\rangle$ (considering $\langle G_1^* A_1 G_2^* A_2\rangle$ would again lead to a conclusion for $V_{\mathfrak{s}_i,i}$ as the relative sign of imaginary parts is preserved), we find that $V_{-\mathfrak{s}_1,1} = (R_{2^*1})^*$ and $V_{-\mathfrak{s}_2,2} = (R_{12^*})^*$ in case (i), and $V_{-\mathfrak{s}_1,1} = (R_{21^*})^*$ and $V_{-\mathfrak{s}_2,2} = (R_{1^*2})^*$ in case (ii). In general, the right eigenvectors for these two cases are not the same. However, as pointed out in Footnote 19, there is a certain *tolerance* in choosing the $V_\pm$. Therefore, within this tolerance and in order to have a consistent and conceptually simple choice, we take $V_{-\mathfrak{s}_1,1}$ from case (i) and $V_{-\mathfrak{s}_2,2}$ from case (ii), i.e.

$$\boxed{V_{-\mathfrak{s}_1,1} = R_{1^*2} = (R_{2^*1})^* \qquad \text{and} \qquad V_{-\mathfrak{s}_2,2} = R_{2^*1} = (R_{1^*2})^*\,.} \tag{1.A.18}$$

Here, in both situations the spectral parameter being the *right* neighbour of A_i receives a complex conjugate. In comparison, if we took $V_{-\mathfrak{s}_1,1}$ from case (ii) and $V_{-\mathfrak{s}_2,2}$ from case (i), we would have ended up with the alternative regularisation from Footnote 10, where the *left* neighbor of A_i received a complex conjugate. Again, the relations in (1.A.18) are understood up to a normalizing constant, which can be specified only after determining U_σ.

Now, it is very important to observe that, for our concrete model with $\Lambda_1 = \Lambda_2$ and $w_1 = w_2 = \mathrm{i}\eta$ (in particular, $\mathfrak{s}_1 = \mathfrak{s}_2 = -$), our choices for $V_\pm$ in (1.A.15) and (1.A.18) *agree* with those in (1.A.8) and (1.A.10) obtained from a variance calculation with only a single resolvent. This follows from the explicit formulas for the critical right eigenvector given later in (1.B.16), Lemma 1.B.4 (a), and (1.2.19)

1.A.3 Explicit formulas for our concrete model and $\Lambda_1 = \Lambda_2$

In this subsection, we will give explicit formulas for $V_\pm$ and $U_\pm$ for our concrete model with one fixed deformation Λ. In fact, for $\Lambda_1 = \Lambda_2$, the so far unspecified matrices U_σ can be characterised by requiring that, jointly with the symmetry relation $E_- G^z(-w)E_- = -G^z(w)$, a resolvent identity (see (1.3.21) for the standard resolvent identity) can be applied to $G_2 U_\sigma G_1$. This yields, together with the normalisation $\langle U_\sigma, U_\sigma\rangle = 1$, that[20]

$$U_+ = E_+ \quad \text{and} \quad U_- = E_-\,.$$

The singular (or critical) eigenvectors of the stability operators characterizing $V_{\mathfrak{s}_i,i}$ can also be explicitly calculated. Using (1.A.15) and (1.A.18), we infer, by means of (1.B.16) and the normalisation/orthogonality condition $\langle V_{\sigma,i}, U_{\tau,i}\rangle = \delta_{\sigma,\tau}$, that

$$\begin{aligned} V_{\mathfrak{s}_1,1} &= \frac{M_2 E_{\mathfrak{s}_1} M_1}{\langle M_2 E_{\mathfrak{s}_1} M_1 E_{\mathfrak{s}_1}\rangle}\,, \qquad V_{-\mathfrak{s}_1,1} = \frac{M_2^* E_{-\mathfrak{s}_1} M_1}{\langle M_2^* E_{-\mathfrak{s}_1} M_1 E_{-\mathfrak{s}_1}\rangle}\,, \\ V_{\mathfrak{s}_2,2} &= \frac{M_1 E_{\mathfrak{s}_2} M_2}{\langle M_1 E_{\mathfrak{s}_2} M_2 E_{\mathfrak{s}_2}\rangle}\,, \qquad V_{-\mathfrak{s}_2,2} = \frac{M_1^* E_{-\mathfrak{s}_2} M_2}{\langle M_1^* E_{-\mathfrak{s}_2} M_2 E_{-\mathfrak{s}_2}\rangle}\,, \end{aligned} \tag{1.A.19}$$

[20] Note that the assignment of $\pm$ is *a priori* not determined, but we chose it in that way. This is also reflected in (1.A.15) and (1.A.18).

matching the definition of the regularisation given in (1.4.8) and (1.3.7). The normalisation is obvious and the orthogonality readily follows from (1.2.19) in combination with Lemma 1.B.4.

Finally, we remark that in order to define the regularisation (1.3.7) and work with (1.A.15) and (1.A.18), it is *not* necessary to have the explicit forms for $V_{\sigma,i}$ at hand. Instead, the single instance of relevant *explicit* formulas is the proof of Theorem 1.2.7, more precisely, the bound in Proposition 1.3.4, where one needs that for $|\operatorname{Im} w_1| \sim N^{-1+\epsilon}$, e.g., $(R_{1*1})^*$ is close to $\operatorname{Im} M_1$ (up to a normalisation). But this is true beyond our model, as easily follows after taking the imaginary part of the general *matrix Dyson equation* (see [236])

$$-\frac{1}{M} = w - A + \mathcal{S}[M]\,, \qquad \operatorname{Im} w \cdot \operatorname{Im} M > 0$$

with self-adjoint *matrix of expectations* $A = A^*$ and (flat) *self-energy operator* $\mathcal{S}[\cdot]$. In fact, this yields

$$\big(1 - M\mathcal{S}[\cdot]M^*\big)(\operatorname{Im} M) = (\operatorname{Im} w)\, MM^*\,,$$

i.e. for $|\operatorname{Im} w| \ll 1$ very small, $\operatorname{Im} M$ is an approximate right eigenvector of the stability operator $1 - M\mathcal{S}[\cdot]M^*$ corresponding to the *critical* eigenvalue (recall the discussion below (1.A.3)).

1.B Properties of the MDE and the stability operator: Proof of Lemma 1.3.3

In the first part of this appendix, we derive several elementary properties of the MDE

$$-\frac{1}{M} = w - \hat{\Lambda} + \mathcal{S}[M]\,, \qquad w \in \mathbf{C} \setminus \mathbf{R}\,, \tag{1.B.1}$$

(recall (1.2.20)) and its unique solution M (under the usual constraint $\operatorname{Im} M \cdot \operatorname{Im} w > 0$) where the operator $\mathcal{S}$ was given in (1.2.21) and $\hat{\Lambda} \in \mathbf{C}^{2N\times 2N}$ is from (1.2.2). Afterwards, in the second part, we turn to the associated two-body stability operator

$$\mathcal{B} \equiv \mathcal{B}(w_1, w_2)[\cdot] := 1 - M(w_1)\mathcal{S}[\cdot]M(w_2) \tag{1.B.2}$$

and its adjoint $\mathcal{B}^*$ (see (1.A.14)). Moreover, we also explain the relation between the regularisation from Definition 1.3.1 and the stability operator.

Finally, after proving and combining Lemmas 1.B.1 and 1.B.4 with Lemma 1.B.6 on M and $\mathcal{B}$, respectively, we will complete the proof of Lemma 1.3.3.

1.B.1 The Matrix Dyson Equation (1.B.1) and its solution

Existence and uniqueness of the solution $M = M(w)$ to (1.B.1) with $\operatorname{Im} M \cdot \operatorname{Im} w > 0$ has already been shown in [332]. By [17, Prop. 2.1], this solution can also be represented as the Stieltjes transform of a compactly supported semi-definite matrix-valued probability measure on $\mathbf{R}$, which has the immediate consequence that $\|M(w)\| \le |\operatorname{Im} w|^{-1}$.

Lemma 1.B.1. *Let M be the unique solution to* (1.B.1) *and write its 2×2-block representation as*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{1.B.3}$$

Then we have the following:

(a) The average trace $\langle M\rangle$ coincides with the solution m of (1.2.4)*, $\langle M(w)\rangle = m(w)$, and the blocks in* (1.B.3) *are given by* (1.2.17)–(1.2.18)*. We have $M^*(w) = M(\bar{w})$.*

(b) The solution has a continuous extension to the real line from the upper half plane, denoted by $M(e) := \lim_{\eta\downarrow 0} M(e+\mathrm{i}\eta)$*; the limit from the lower half plane is* $M^*(e)$*. The self-consistent density of states of the MDE, defined as* $\rho(e) = \frac{1}{\pi}\langle \operatorname{Im} M(e)\rangle$*, is identical to the free convolution of* $\mu_{\hat\Lambda} \boxplus \mu_{sc}$ *from* (1.2.3)*. Both* ρ *and its Stieltjes transform* m *are Hölder continuous with a small universal exponent* c*, i.e.*

$$|\rho(e_1) - \rho(e_2)| \le C|e_1 - e_2|^c, \qquad e_1, e_2 \in \mathbf{R},$$

and

$$|m(w_1) - m(w_2)| \le C'|w_1 - w_2|^c, \qquad w_1, w_2 \in \mathbf{C}_+, \tag{1.B.4}$$

where C, C' *depend only on* $\|\Lambda\|$*.*

(c) We have the chiral symmetry

$$E_- M(w) = -M(-w)E_-\,. \tag{1.B.5}$$

In particular, for purely imaginary spectral parameter, $w = \mathrm{i}\operatorname{Im} w$*, it holds that* $m = \mathrm{i}\operatorname{Im} m$ *as well as* $M_{11} = \mathrm{i}\operatorname{Im} M_{11}$ *and* $M_{22} = \mathrm{i}\operatorname{Im} M_{22}$*. Moreover, the off-diagonal blocks of* $\operatorname{Im} M$ *are vanishing on the imaginary axis.*

(d) Fix $\kappa > 0$*. For any spectral parameter in the* κ*-bulk,* $w \in \mathbf{C} \setminus \mathbf{R}$ *with* $\operatorname{Re} w \in \mathbf{B}_\kappa$*, we have*

$$\|M(w)\| \le C(\kappa, \|\Lambda\|) \tag{1.B.6}$$

for some constant depending only on κ *and an upper bound on the norm* $\|\Lambda\|$*. Moreover,* $\rho(e)$ *is real analytic on* $\mathbf{B}_\kappa$ *with derivatives controlled uniformly*

$$\max\{|\partial^k \rho(e)| \,:\, e \in \mathbf{B}_\kappa\} \le C(k, \kappa, \|\Lambda\|) \tag{1.B.7}$$

with a constant $C(k, \kappa, \|\Lambda\|)$ *for any* $k \in \mathbf{N}$*.*

Proof. For part (a), a direct computation shows that M from (1.B.3) with the blocks given in (1.2.17)–(1.2.18) indeed solves (1.B.1) if m is replaced with $\langle M\rangle$ in these formulas. The calculation uses the simple observation that $\langle M_{11}\rangle = \langle M_{22}\rangle$ from (1.2.18), hence $\mathcal{S}[M] = \langle M\rangle$. Furthermore, the MDE also implies that $\langle M\rangle$ solves (1.2.4), but this equation has a unique solution by the theory of free convolutions with a semicircular density, hence $m = \langle M\rangle$. Finally $M^*(w) = M(\bar w)$ follows from $\bar m(w) = m(\bar w)$. This proves (a).

For part (b), since $\mathcal{S}[M] = \langle M\rangle$, we observe that M solves

$$-\frac{1}{M} = w - \hat\Lambda + \langle M\rangle\,,$$

which is exactly the MDE for a deformed Wigner matrix model.[21] The point is that the Hermitised H from (1.2.15) does not satisfy the uniform lower bound in the *flatness* condition on the self-energy operator, i.e. $\mathcal{S}[T] \ge c\langle T\rangle$ does not hold in general. Nevertheless, for the purpose of computing M we can replace H with the deformed Wigner model $W + \hat\Lambda$ with self-energy given $\mathcal{S}[T] = \langle T\rangle$ and which is *flat*. Thus we can use several results from the analysis of the MDE with flatness condition. The Hölder-continuity of the scDos was proven in [17, Prop. 2.2], which easily extends to the Hölder-continuity of its Stieltjes transform m, see e.g. [16, Lemma A.7]. In particular $\langle M(w)\rangle$ extends continuously to the real line and thus the scDos $\rho(e) := \frac{1}{\pi}\langle \operatorname{Im} M(e)\rangle = \frac{1}{\pi}\operatorname{Im} m(e)$ is well defined. Since it has the same Stieltjes transform as the free convolution (1.2.3) by part (a), we proved that the scDos defined via MDE is the same as the free convolution (1.2.3).

[21] That is, a matrix $H = W + \hat\Lambda$, where W is a Hermitian matrix with normalised i.i.d. (up to the symmetry) entries of variance $1/(2N)$.

The continuous extension of M (and not only its trace) requires an additional argument. For any open interval $I \in \mathbf{R}$ define

$$\|M\|_I := \sup\{\|M(e+\mathrm{i}\eta)\| \,:\, e \in I, \eta > 0\}\,.$$

Suppose for some open $I \in \mathbf{R}$ we have $\|M\|_I < \infty$, then we have the Lipschitz continuity

$$\|M(w_1) - M(w_2)\| \le \|M\|_I^2 |w_1 - w_2|\,, \qquad \mathrm{Re}\, w_1, \mathrm{Re}\, w_2 \in I$$

following from the resolvent identity applied to $M(w) = (\hat{\Lambda} - w - m)^{-1}$. Thus $M(w)$ continuously extends to any $e \in I$.

So the key question for the extension (and for many other results on the MDE) is the boundedness $\|M\|_I < \infty$. In the bulk spectrum, i.e. for any $e \in \mathbf{R}$ with $\rho(e) > 0$, we can use the bound

$$\|M(w)\| \le |\mathrm{Im}\, m(w) + \mathrm{Im}\, w|^{-1}$$

that is obtained by taking the imaginary part of (1.B.1), yielding

$$\mathrm{Im}\, M = (\mathrm{Im}\, w + \langle \mathrm{Im}\, M\rangle) MM^*\,,$$

and using $\|MM^*\| = \|M\|^2$ and $\|\mathrm{Im}\, M\| \le \|M\|$. By the Hölder continuity (1.B.4) in small neighborhood I of e (whose size depend on the lower bound on $\rho(e)$) we obtain $\|M\|_I \lesssim \rho(e)^{-2} < \infty$. Thus M continuously extends to I with the same bound and it is locally Lipschitz continuous with a Lipschitz constant of order $\rho(e)^{-2}$. In the entire κ-bulk this extension is controlled by a constant depending only on κ and $\|\Lambda\|$ (via (1.B.4)). This proves (1.B.6).

Near the spectral edges we have only an N-independent upper bound for $\|M\|$. Using the spectral decomposition of $\hat{\Lambda}$ with eigenvalues ν_i and normalised eigenvectors $\boldsymbol{y}_i$, $i \in \pm[N]$, we have

$$M(w) = \sum_i \frac{|\boldsymbol{y}_i\rangle\langle \boldsymbol{y}_i|}{\nu_i - w - m(w)}, \quad \text{thus} \quad \|M(w)\| \le \frac{2N}{\min_i |\nu_i - w - m(w)|}\,. \tag{1.B.8}$$

On the other hand the imaginary part of (1.2.4) implies

$$\mathrm{Im}\, m = \frac{1}{2N} \sum_i \frac{\mathrm{Im}\, m + \mathrm{Im}\, w}{|\nu_i - w - m|^2}$$

thus

$$\frac{1}{2N} \sum_i \frac{1}{|\nu_i - w - m|^2} = \frac{\mathrm{Im}\, m}{\mathrm{Im}\, m + \mathrm{Im}\, w} \le 1$$

so $|\nu_i - w - m| \ge 1/\sqrt{2N}$. From (1.B.8) this gives the uniform bound

$$\|M(w)\| \le (2N)^{3/2}, \qquad w \in \mathbf{C} \setminus \mathbf{R},$$

which guarantees the continuous extension of M to the real line with a uniform Lipschitz constant $(2N)^{3/2}$. As we have seen, in the bulk this regularity can be improved.[22]

For part (c), the symmetry $\rho(e) = \rho(-e)$ immediately implies the symmetry $m(w) = -m(-w)$ for its Stieltjes transform. Then (1.B.5) is an immediate consequences of the formulas (1.2.17)–(1.2.18).

Finally, for part (d), the bound (1.B.6) was already proven above. The real analyticity of ρ and m in the bulk with the bounds on the derivative (1.B.7) follows from taking derivatives in (1.2.4) and using again the lower bound on $\mathrm{Im}\, m$. □

[22] We remark that under some extra condition on Λ further improvements away from the bulk are possible for m but not for M. For example, if the singular values ν_i of Λ are $1/2$-Hölder continuous in the sense that $|\nu_i - \nu_j| \le C_0[|i-j|/N]^{1/2}$, then m is also uniformly bounded and $1/3$-Hölder continuous with a constant depending on C_0, see Section 11.4 of [16].

Finally, we prove some regularity property of the κ-bulk, see (1.2.23).

Lemma 1.B.2. *Let $0 < \kappa' < \kappa$ be two small constants, then*

$$\operatorname{dist}(\partial \mathbf{B}_{\kappa'}, \mathbf{B}_{\kappa}) \geq \mathfrak{c}(\kappa - \kappa') \tag{1.B.9}$$

with some N-independent constant $\mathfrak{c} = \mathfrak{c}(\|\Lambda\|) > 0$. Moreover, $\mathbf{B}_{\kappa}$ is a finite union of disjoint compact intervals; the number of these components *depends only on κ and $\|\Lambda\|$.*

Proof. As in the proof of Lemma 1.B.1, we interpret $\mathbf{B}_{\kappa}$ as the κ-bulk of the deformed Wigner matrix $W + \hat{\Lambda}$, i.e. a model with the flatness condition. The statement on the number of components directly follows from the real analyticity of ρ and (1.B.7).

The same argument would also imply (1.B.9) with a constant $\mathfrak{c}$ that depends on κ and an upper bound on $\|\Lambda\|$. To remove the κ-dependence, we need to use the detailed *shape analysis* for ρ from [22]. In particular, the flatness condition and $\|M\|_I < C(\kappa)$ for any interval $I \subset \mathbf{B}_{\kappa}$ (equivalent to [22, Eq. (4.16)]) implies that Assumption 4.5 in [22] holds. Therefore Theorem 7.2 in [22] applies to our case. This theorem says that in the regime where ρ is small, it is approximately given by explicit $1/3$-Hölder continuous functions, moreover ρ itself is $1/3$-Hölder continuous with Hölder constant depending only on the so-called model parameters of the problem, which in our case is just an upper bound on Λ (note that [22] was written for much more complicated self-energy operators to include the MDE analysis for random matrices with correlated entries). Noticing the $\kappa^{1/3}$ power in the definition of $\mathbf{B}_{\kappa}$ in (1.2.6), this means that the boundaries of $\mathbf{B}_{\kappa}$ are Lipschitz continuous functions of κ when κ is small with a Lipschitz constant depending only on an upper bound on $\|\Lambda\|$. □

Remark 1.B.3. *Note that the proof of the independence of $\mathfrak{c} = \mathfrak{c}(\|\Lambda\|)$ of κ required a much more sophisticated analysis. However, for our main proof, $\mathfrak{c} = \mathfrak{c}(\kappa, \|\Lambda\|) > 0$ in* (1.B.9) *is sufficient, note that* (1.B.9) *is only used in choosing δ in* (1.4.22) *appropriately. More precisely, for fixed $L = L(\epsilon)$ and $\kappa_0 > 0$, given the family $(\ell\kappa_0)_{\ell \in [L]}$ of parameters for the domains $\mathbf{D}_{\ell}^{(\epsilon_0, \kappa_0)}$, we would have that $\operatorname{dist}(\partial \mathbf{B}_{(\ell-1)\kappa_0}, \mathbf{B}_{\ell\kappa_0}) \geq \mathfrak{c}(\ell\kappa_0, \|\Lambda\|)\kappa_0$. Now, the cutoff parameter δ in* (1.4.22) *is chosen much smaller than $\mathfrak{c}(\ell\kappa_0, \|\Lambda\|)\kappa_0$ for every $\ell \leq L(\epsilon)$.*

1.B.2 The stability operator (1.B.2) and its spectral properties

Throughout the entire paper, the two-body stability operator (1.B.2) and its adjoint (1.A.14) play a crucial role. These operators depend on two (a priori) different spectral parameters w_1, w_2 via the solutions $M_1 = M(w_1)$ and $M_2 = M(w_2)$ of the MDE (1.B.1). For these solutions, we have the following basic lemma.

Lemma 1.B.4. *Let $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ be two spectral parameters and $M_1 = M(w_1), M_2 = M(w_2)$ the corresponding solutions to* (1.B.1).

*(a) Then we have the M-*Ward identity,

$$M_1 - M_2 = \big[(w_1 - w_2) + (\langle M_1 \rangle - \langle M_2 \rangle)\big] M_2 M_1 \,. \tag{1.B.10}$$

In particular, M_1 and M_2 commute and it holds that

$$(1 - \langle M M^* \rangle)\, \langle \operatorname{Im} M \rangle = \operatorname{Im} w \, \langle M M^* \rangle \,. \tag{1.B.11}$$

(b) Fix $\kappa > 0$ and let $\operatorname{Re} w_1, \operatorname{Re} w_2 \in \mathbf{B}_{\kappa}$. Then, for $\operatorname{Im} w_1 \operatorname{Im} w_2 > 0$, we have the perturbative estimate

$$\|M(w_1) - M(w_2)\| = \mathcal{O}(|w_1 - w_2| \wedge 1) \,.$$

Proof. Part (a) is an immediate consequence of the MDE (1.B.1) using the fact that

$$M = \big(\hat{\Lambda} - (w+m)\big)^{-1}$$

is a resolvent of $\hat{\Lambda}$. The special case (1.B.11) follows from (1.B.10) with $w_1 = w$ and $w_2 = \bar{w}$, and taking a trace.

For part (b), we focus on the case of small imaginary parts for the spectral parameters (the complementary regime being trivial) and use that M is analytic away from the real axis and differentiate (1.B.1) w.r.t. w, such that we find

$$\partial_w M = \frac{1}{1 - \langle M^2 \rangle} M^2$$

by means of $\mathcal{S}[M^2] = \langle M^2 \rangle$ as follows from the explicit form of M in (1.2.17)–(1.2.18). Next, using (1.B.11), the denominator is lower bounded as

$$\big|1 - \langle M^2 \rangle\big| = \big|\big(1 - \langle MM^* \rangle\big) - 2\mathrm{i}\langle M \operatorname{Im} M \rangle\big| \geq 2\big|\langle (\operatorname{Im} M)^2 \rangle\big| \geq 2\langle \operatorname{Im} M \rangle^2 , \tag{1.B.12}$$

which shows that $\|\partial_w M\| \lesssim 1$ in the bulk. Now the claim follows from the fundamental theorem of calculus together with the boundedness of M, see (1.2.22). □

Armed with this information, we can now turn to the following lemma, collecting several basic spectral properties stability operator $\mathcal{B}$. Its proof will be given at the end of this section.

Lemma 1.B.5. *Let $w_1, w_2 \in \mathbf{C} \smallsetminus \mathbf{R}$ and M_1, M_2 be the respective solutions of* (1.B.1).

(a) The associated two-body stability operator

$$\mathcal{B} = 1 - M_1 \mathcal{S}[\cdot] M_2$$

has two non-trivial eigenvalues $\beta_\pm$ (the other $(2N)^2 - 2$ are equal to one), given by

$$\beta_\pm = 1 \mp \langle M_1 E_\pm M_2 E_\pm \rangle . \tag{1.B.13}$$

The corresponding right- and left-eigenvectors

$$\mathcal{B}[R_\pm] = \beta_\pm R_\pm , \qquad \mathcal{B}^*[L_\pm^*] = \bar{\beta}_\pm L_\pm^* ,$$

take the explicit form

$$R_\pm = M_1 E_\pm M_2 , \qquad L_\pm = E_\pm , \tag{1.B.14}$$

up to a normalisation ensuring that $\langle L_\pm, R_\pm \rangle = 1$.

(b) The eigenvalues (1.B.13) *can be lower bounded as*

$$|\beta_\pm| \gtrsim \big(|\operatorname{Re} w_1 \mp \operatorname{Re} w_2| + |\operatorname{Im} w_1| + |\operatorname{Im} w_2|\big) \wedge 1 . \tag{1.B.15}$$

In particular, the inverse stability operator $\mathcal{B}^{-1}$ exists.

(c) Fix $\kappa > 0$ and denote $\mathfrak{s} := -\operatorname{sgn}(\operatorname{Im} w_1 \operatorname{Im} w_2)$. Then, for $\operatorname{Re} w_1, \operatorname{Re} w_2 \in \mathbf{B}_\kappa$, we have that $|\beta_{-\mathfrak{s}}| \gtrsim 1$.

By the last item, given $\mathfrak{s} := -\operatorname{sgn}(\operatorname{Im} w_1 \operatorname{Im} w_2)$, we will always refer to

$$\big(\beta := 1 - \mathfrak{s}\langle M_1 E_{\mathfrak{s}} M_2 E_{\mathfrak{s}}\rangle\,,\ R := M_1 E_{\mathfrak{s}} M_2\,,\ L := E_{\mathfrak{s}}\big) \tag{1.B.16}$$

as the *critical eigentriple* (and accordingly β as the *critical eigenvalue* etc.), consisting of the eigenvalue and the corresponding right- and left-eigenvector. Moreover, the estimate (1.B.15) shows that, if we have (recall (1.3.6))

$$\mathbf{1}_\delta^{\pm}(w_1, w_2) := \phi_\delta(\operatorname{Re} w_1 \mp \operatorname{Re} w_2)\ \phi_\delta(\operatorname{Im} w_1)\ \phi_\delta(\operatorname{Im} w_2) = 0$$

for some $\delta > 0$, then the inverse stability operator $\mathcal{B}^{-1}$ is bounded and none of the eigenvalues $\beta_\pm$ is really critical. In the complementary regime, $\mathbf{1}_\delta^{\pm}(w_1, w_2) = 1$, and $\operatorname{Re} w_1, \operatorname{Re} w_2 \in \mathbf{B}_\kappa$, we shall now explain the interplay between the critical eigentriple (1.B.16) and the regularisation (1.3.7).

Lemma 1.B.6. *Let $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ with $\operatorname{Re} w_1, \operatorname{Re} w_2 \in \mathbf{B}_\kappa$ for some fixed $\kappa > 0$ and denote the relative sign of imaginary parts by $\mathfrak{s} := -\operatorname{sgn}(\operatorname{Im} w_1 \operatorname{Im} w_2)$. Moreover, let $M_1 = M(w_1), M_2 = M(w_2)$ be the respective solutions of (1.B.1) and $A \in \mathbf{C}^{2N\times 2N}$ a bounded deterministic matrix.*

(a) If $\mathbf{1}_\delta^{\mathfrak{s}}(w_1, w_2) = 1$ for some $\delta > 0$ small enough, the critical left- and right-eigenvectors (1.B.16) are normalised as $\langle L, R\rangle \sim 1$. In particular, if $\mathbf{1}_\delta^{\pm}(w_1, w_2) = 1$, the respective denominator in the regularisation $\mathring{A}^{w_1,w_2}$ (see (1.3.7)) is bounded away from zero.

(b) The operator $\mathcal{X}_{12}$, acting as

$$\mathcal{X}_{12}[B] := \big((\mathcal{B}_{12}^*)^{-1}[B^*]\big)^* = \big(1 - \mathcal{S}[M_1 \cdot M_2]\big)^{-1}[B]\,, \quad B \in \mathbf{C}^{2N\times 2N}\,,$$

where $\mathcal{B}_{12} := 1 - M_1\mathcal{S}[\cdot]M_2$, is well defined and bounded on the $\mathfrak{s}$-regular component $\mathring{A}^{\mathfrak{s}}$ (w.r.t. the pair of spectral parameters (w_1, w_2)) of any bounded A. This means, for

$$\mathring{A}^{\mathfrak{s}} := A - \mathbf{1}_\delta^{\mathfrak{s}}(w_1, w_2)\frac{\langle M_1 A M_2 E_{\mathfrak{s}}\rangle}{\langle M_1 E_{\mathfrak{s}} M_2 E_{\mathfrak{s}}\rangle} E_{\mathfrak{s}} \tag{1.B.17}$$

it holds that $\|\mathcal{X}_{12}[\mathring{A}^{\mathfrak{s}}]\| \lesssim 1$.

In particular, combining Lemma 1.B.4 (b) with Lemma 1.B.6 (a), (1.2.19), and Lemma 1.B.4 (a), we conclude the perturbative statements from Lemma 1.3.3.

Proof of Lemma 1.B.6. For part (a), similarly to the proof of Lemma 1.B.5 (c) given below, we focus on the extreme case $w_2 = \mathfrak{s}\bar{w}_1$, where the critical eigentriple is given by

$$\big(\beta = 1 - \mathfrak{s}\langle M(w_1) E_{\mathfrak{s}} M(\mathfrak{s}\bar{w}_1) E_{\mathfrak{s}}\rangle\,,\ R = M(w_1) E_{\mathfrak{s}} M(\mathfrak{s}\bar{w}_1)\,,\ L = E_{\mathfrak{s}}\big)\,. \tag{1.B.18}$$

Now by means of the chiral symmetry (1.2.19), we readily obtain

$$\langle L, R\rangle = \mathfrak{s}\langle M_1 M_1^*\rangle = \mathfrak{s}\frac{\langle \operatorname{Im} M_1\rangle}{\operatorname{Im} w_1 + \langle \operatorname{Im} M_1\rangle} \sim 1\,,$$

where we used (1.B.11) in the second step. This principal normalisation of order persists after small perturbation of w_2 around the extreme case, but as long as $\mathbf{1}_\delta^{\mathfrak{s}}(w_1, w_2) = 1$. Our claim for the denominators in the regularisation (1.3.7) follows immediately from the representation in (1.B.18).

For part (b), we first note that, by means of Lemma 1.B.5, the statement is trivial for constellations of spectral parameters w_1, w_2 satisfying $\mathbf{1}_\delta^{\mathfrak{s}}(w_1, w_2) = 0$ and we can hence focus on the complementary extreme case $\mathbf{1}_\delta^{\mathfrak{s}}(w_1, w_2) = 1$. Then it follows from the explicit form

$$\mathcal{X}_{12}[B] = B + \sum_\sigma \sigma \frac{\langle M_1 B M_2 E_\sigma\rangle}{1 - \sigma\langle M_1 E_\sigma M_2 E_\sigma\rangle} E_\sigma$$

and Lemma 1.B.5 that

$$\mathcal{X}_{12}[B] = \mathfrak{s}\frac{1}{\beta}\langle M_1 B M_2 E_{\mathfrak{s}}\rangle E_{\mathfrak{s}} + \mathcal{O}(1)[B]\,, \tag{1.B.19}$$

where $\mathcal{O}(1)$ is a shorthand notation for a linear operator $\mathcal{E}: \mathbf{C}^{2N\times 2N} \to \mathbf{C}^{2N\times 2N}$ satisfying $\|\mathcal{E}[B]\| \lesssim \|B\|$. Now, plugging $\mathring{A}^{\mathfrak{s}}$ from (1.B.17) into (1.B.19) yields the desired. □

It remains to give the proof of Lemma 1.B.5.

Proof of Lemma 1.B.5. For (a), we first observe that, due to the simple structure of $\mathcal{S}[\cdot]$, indeed $(2N)^2 - 2$ of the $(2N)^2$ eigenvalues of $\mathcal{B}$ are equal to one. The expressions (1.B.13) and (1.B.14) can be verified by direct computation, invoking Lemma 1.B.4 in combination with the chiral symmetry (1.2.19).

For (b) with $w_1 \neq \pm w_2$, we first find that

$$\frac{1}{\beta_\pm} = \frac{1}{1 \mp \langle M_1 E_\pm M_2 E_\pm\rangle} = 1 + \frac{\langle M_1\rangle \mp \langle M_2\rangle}{w_1 \mp w_2} \tag{1.B.20}$$

as a consequence of Lemma 1.B.4 (a) and the chiral symmetry. Now, using that $\big|\langle M\rangle\big| \leq \langle MM^*\rangle^{1/2} < 1$, which follows from $MM^* = \operatorname{Im} M/(\operatorname{Im} w + \langle \operatorname{Im} M\rangle)$ (see Lemma 1.B.4 (a)), we conclude that

$$|\beta_\pm| \gtrsim |\operatorname{Re} w_1 \mp \operatorname{Re} w_2| \wedge 1 \tag{1.B.21}$$

by application of a triangle inequality in (1.B.20). Next, we estimate

$$\min\big\{|\beta_+|\,, |\beta_-|\big\} \geq \big|1 - \langle M_1 M_1^*\rangle^{1/2}\langle M_2 M_2^*\rangle^{1/2}\big| \gtrsim \big(|\operatorname{Im} w_1| + |\operatorname{Im} w_2|\big) \wedge 1\,, \tag{1.B.22}$$

where in the first step we used $\langle MM^*\rangle < 1$ together with a Schwarz inequality, and (1.B.11) in the second step. Combining (1.B.21) and (1.B.22) yields the claim.

Finally, for (c), we consider the case of small imaginary parts for the spectral parameters (the complementary regime being trivial) and focus on the extreme case $w_1 = -\mathfrak{s}w_2$. Then, using (1.2.19) and (1.B.12), we obtain

$$\big|\beta_{-\mathfrak{s}}\big| = \big|1 - \langle M_1^2\rangle\big| \geq 2\langle \operatorname{Im} M_1\rangle^2 \gtrsim 1\,. \tag{1.B.23}$$

This principal lower bound persists after small perturbations of w_2, and the complementary regime can be dealt with by (1.B.15). □

1.C Proof of Theorem 1.2.6

In this appendix, we give a short proof of the usual single resolvent local law in the bulk given in Theorem 1.2.6. In the literature, bulk local laws are established under the usual *flatness* assumption (see [243, Assumption E]) on the self-energy operator $\mathcal{S}$ (recall (1.A.4)). However, for our model, the stability operator $\mathcal{S}[R] = \sum_\sigma \sigma\langle RE_\sigma\rangle E_\sigma$ *violates* the lower bound in the flatness condition (1.A.4), which is why we need to provide a separate argument. The main idea is that lacking of the lower bound in (1.A.4) is compensated by the orthogonality relation $\langle GE_-\rangle = \langle ME_-\rangle = 0$ as a consequence of (1.5.5).

The following argument heavily relies on [243, Theorem 4.1], where a general high-moment bound on the underlined term in

$$\langle (G-M)B\rangle = -\langle \underline{WG}\mathcal{X}[B]M\rangle + \langle G-M\rangle\langle (G-M)\mathcal{X}[B]M\rangle \tag{1.C.1}$$

and its isotropic counterpart (see (1.C.2) below) has been shown. We stress that this estimate from [243] does *not* require the lower bound in (1.A.4) for the self-energy operator $\mathcal{S}$. As usual,

we suppressed the spectral parameter $w \in \mathbf{C} \setminus \mathbf{R}$ satisfying $\operatorname{Re} w \in \mathbf{B}_\kappa$ for some fixed $\kappa > 0$ from the notation. The expansion (1.C.1) for an arbitrary deterministic matrix $B \in \mathbf{C}^{2N\times 2N}$ has already been established in (1.5.18), where we introduced the linear operator $\mathcal{X}[B] := \big(1 - \mathcal{S}[M \cdot M]\big)^{-1}[B]$ acting on matrices.

For given B, we now decompose it into its $(-)$-regular and $(-)$-singular component (see (1.B.17), the cutoff function being irrelevant here),

$$B = \mathring{B}^- + \frac{\langle MBME_-\rangle}{\langle ME_-ME_-\rangle} E_- \,,$$

respectively. For the second summand, we note that $\langle GE_-\rangle = \langle ME_-\rangle = 0$, and we can hence focus on the regular component, i.e. assume that $B = \mathring{B}^-$ is $(-)$-regular.

In this case, for a bounded deterministic $\|B\| \lesssim 1$ we thus have $\|\mathcal{X}[B]\| \lesssim 1$ from Lemma 1.B.6. With the high-moment bound on the underlined term from [243, Theorem 4.1, part (b)] one can conclude the proof of Theorem 1.2.6 in the averaged case, $|\langle (G-M)B\rangle| \prec (N\eta)^{-1}$, by a standard *bootstrap* argument (see, e.g., [243, Sections 5.3 and 5.4]).

In the isotropic case, we evaluate (1.C.1) for $B = 2N\,|\boldsymbol{y}\rangle\langle \boldsymbol{x}|$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^{2N}$ are deterministic vectors in with $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$. More precisely, we subtract its $(-)$-singular component (which can be dealt with separately as explained above) and insert

$$B = \mathring{B}^- = 2N\,|\boldsymbol{y}\rangle\langle \boldsymbol{x}| - \frac{\langle \boldsymbol{x}, ME_-M\boldsymbol{y}\rangle}{\langle ME_-ME_-\rangle} E_-$$

in the expansion (1.C.1), which leaves us with

$$\begin{aligned}\big(G-M\big)_{\boldsymbol{xy}} = &-\big(\underline{WG}\big)_{\boldsymbol{x}(M\boldsymbol{y})} + \langle G-M\rangle\big(G-M\big)_{\boldsymbol{x}(M\boldsymbol{y})} \\ &+ \left[\frac{\langle \boldsymbol{x}, ME_-M\boldsymbol{y}\rangle}{\langle ME_-ME_-\rangle} + \frac{\langle \boldsymbol{x}, M^2\boldsymbol{y}\rangle}{1-\langle M^2\rangle}\right]\big[\langle \underline{WG}E_-M\rangle - \langle G-M\rangle\langle (G-M)E_-M\rangle\big]\,.\end{aligned} \tag{1.C.2}$$

After realizing that the denominators in (1.C.2) are bounded away from zero (see Lemma 1.B.5 and Lemma 1.B.6), the proof of Theorem 1.2.6 in the isotropic case, $\big|\big(G-M\big)_{\boldsymbol{xy}}\big| \prec (N\eta)^{-1/2}$, can be concluded again by a standard *bootstrap* argument, now using the high-moment bound from [243, Theorem 4.1, part (a)] and the already proven averaged law $|\langle (G-M)B\rangle| \prec (N\eta)^{-1}$ with $\|B\| \lesssim 1$ as an input.

1.D Bounds on the deterministic approximations: Proof of Lemma 1.4.3

The goal of this appendix is to prove the bounds from Lemma 1.4.3 on the deterministic approximation

$$M(w_1, B_1, w_2, ..., B_{k-1}, w_k)$$

to a resolvent chain

$$G(w_1)B_1G(w_2)\cdots B_{k-1}G(w_k)\,.$$

While $M(w_1, ..., w_k)$ has been introduced for an arbitrary number k of spectral parameters $w_1, ..., w_k$ in Definition 1.4.1, the bounds in Lemma 1.4.3 shall be proven for k at most five and the deterministic matrices $B_1, ..., B_{k-1}$ being regular w.r.t. to the surrounding spectral parameters.

As a preparation for the proof of Lemma 1.4.3, we shall now show that $M(w_1, ..., w_k)$ from (1.4.2) satisfies multiple recursive relations, called *recursive Dyson equations*, by using a so-called *meta argument*, that relies on the fact that $M(w_1, ..., w_k)$ actually approximates a chain of products of

resolvents. In fact, we only picked one of the recursive relations (namely (1.D.1) with $j = 1$) for actually *defining* $M(w_1, ..., w_k)$ in Definition 1.4.1. Although the second recursion relation (1.D.2) will not be used in the proof of Lemma 1.4.3, it is obtained completely analogous to (1.D.1) and we hence give it for completeness. A similar meta argument has been done several times, see e.g. [181]. For convenience of the reader we repeat it in our setup.

Lemma 1.D.1. (Recursive Dyson equations for $M(w_1, ..., w_k)$, see [168, Lemma 4.1])
Fix $k \in \mathbf{N}$. Let $w_1, ..., w_k \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters and $B_1, ..., B_{k-1} \in \mathbf{C}^{2N\times 2N}$ deterministic matrices. Then for any $1 \le j \le k$ we have the relations

$$\begin{aligned} M(w_1,...,w_k) &= M(w_1, ..., w_{j-1}, B_{j-1}M(w_j)B_j, w_{j+1}, ..., w_k) \\ &+ \sum_{\sigma=\pm}\sum_{l=1}^{j-1} \sigma M(w_1, ..., B_{l-1}, w_l, E_\sigma, w_j, B_j, ..., w_k)\langle M(w_l, ..., w_{j-1})B_{j-1}M(w_j)E_\sigma\rangle \\ &+ \sum_{\sigma=\pm}\sum_{l=j+1}^{k} \sigma M(w_1, ..., B_{j-1}M(w_j)E_\sigma, w_l, B_l..., w_k)\langle M(w_j, ..., w_l)E_\sigma\rangle \end{aligned} \tag{1.D.1}$$

and

$$\begin{aligned} M(w_1,...,w_k) &= M(w_1, ..., w_{j-1}, B_{j-1}M(w_j)B_j, w_{j+1}, ..., w_k) \\ &+ \sum_{\sigma=\pm}\sum_{l=1}^{j-1} \sigma M(w_1, ..., B_{l-1}, w_l, E_\sigma M(w_j)B_j, ..., w_k)\langle M(w_l, ..., w_j)E_\sigma\rangle \\ &+ \sum_{\sigma=\pm}\sum_{l=j+1}^{k} \sigma M(w_1, ..., B_{j-1}, w_j, E_\sigma, w_l, B_l, ..., w_k)\langle M(w_j)B_jM(w_{j+1}, ..., w_l)E_\sigma\rangle\,. \end{aligned} \tag{1.D.2}$$

If $j = 1$ or $j = k$, we define $B_0 = E_+$ resp. $B_k = E_+$ in (1.D.1) *and* (1.D.2).

The formulas (1.D.1) and (1.D.2) shall be derived by expanding the j^{th} resolvent G_j in the resolvent chain $G_1B_1 \cdots G_jB_j \cdots B_{k-1}G_k$ corresponding to $M(w_1, ..., w_k)$ in an underlined term, once to the right (for (1.D.1), see (1.D.8)) and once to the left (for (1.D.2), see (1.D.10)). Altogether, this yields $2k$ different recursions for $M(w_1, ..., w_k)$, which are listed in the above lemma. Moreover, it would be possible to prove directly that all these different recursions define the same $M(w_1, ..., w_k)$. This strategy has been used in a much simpler setup [170] dealing with Wigner matrices. Here, we find it simpler to use the alternative meta argument.

Proof. The principal idea is to derive the respective relations (1.D.1) and (1.D.2) on the level of resolvent chains $G_1B_1 \cdots B_{k-1}G_k$, which, after taking the expectation and using that $G_i \approx M_i$ from Theorem 1.2.6, yields the same relation on the level of the deterministic approximations. For the purpose of proving identities about $M(w_1, ..., w_k)$, we may use the most convenient distribution for X, namely Gaussian. For the sake of this proof, we thus assume the single entry distribution χ of X to be a standard complex Gaussian $\chi = \mathcal{N}_{\mathbf{C}}(0, 1)$, i.e. X in Assumption 1.2.1 is a complex Ginibre matrix, in which case it holds that (recall the discussion below (1.5.3))

$$\mathbf{E}\,\underline{f(W)Wg(W)} = 0\,. \tag{1.D.3}$$

Let $w_1, ..., w_k \in \mathbf{C} \setminus \mathbf{R}$ be arbitrary (but fixed!) spectral parameters. We now conduct the *meta argument*, consisting of three steps.

Step 1. We consider the resolvent chain

$$G_1B_1 \cdots B_{k-1}G_k\,. \tag{1.D.4}$$

Expanding G_1 via the identity

$$G_1 = M_1 - M_1\underline{WG_1} + M_1\mathcal{S}[G_1 - M_1]G_1$$

and using $\mathcal{S}[G_1 - M_1] = \langle G_1 - M_1 \rangle$ from (1.5.5), we find that

$$\begin{aligned} & G_1 B_1 \cdots B_{k-1} G_k \\ =& M_1 B_1 \cdots B_{k-1} G_k - M_1 \underline{W G_1} B_1 \cdots B_{k-1} G_k + \langle G_1 - M_1 \rangle M_1 G_1 B_1 \cdots B_{k-1} G_k \\ =& M_1 B_1 \cdots B_{k-1} G_k + \sum_{\sigma=\pm} \sum_{l=2}^{k-1} \sigma M_1 \langle G_1 B_1 \cdots B_{l-1} G_l E_\sigma \rangle E_\sigma G_l B_l \cdots B_{k-1} G_k \\ & - M_1 \underline{W G_1 B_1 \cdots B_{k-1} G_k} + \langle G_1 - M_1 \rangle M_1 G_1 B_1 \cdots B_{k-1} G_k + M_1 \mathcal{S}[G_1 B_1 \cdots B_{k-1} G_k] M_k \,, \end{aligned} \tag{1.D.5}$$

where in the last step we distributed the derivatives coming from the definition of the underline in (1.5.3) according to the Leibniz rule. Now, (1.D.9) can be rewritten as

$$\begin{aligned} & G_1 B_1 \cdots B_{k-1} G_k \\ & = (\mathcal{B}_{1k})^{-1} \Bigg[M_1 B_1 \cdots B_{k-1} G_k + \sum_{\sigma=\pm} \sum_{l=2}^{k-1} \sigma M_1 \langle G_1 B_1 \cdots B_{l-1} G_l E_\sigma \rangle E_\sigma G_l B_l \cdots B_{k-1} G_k \\ & \quad - M_1 \underline{W G_1 B_1 \cdots B_{k-1} G_k} + \langle G_1 - M_1 \rangle M_1 G_1 B_1 \cdots B_{k-1} G_k \Bigg]. \end{aligned} \tag{1.D.6}$$

Apart from the last two terms in (1.D.6), this is the exact same relation on the level of resolvents as in Definition 1.4.1 for $M(w_1, ..., w_k)$.

Step 2. Let the original matrix size N be fixed. For any $d \in \mathbf{N}$, we consider the $dN \times dN$ Ginibre random matrix $\boldsymbol{X}^{(d)}$ with entries having variance $1/(dN)$, and the deformation $\boldsymbol{\Lambda}^{(d)} := \Lambda \otimes I_d \in \mathbf{C}^{dN \times dN}$, where $I_d \in \mathbf{C}^{d \times d}$ is the identity matrix. Analogously to (1.2.2) and (1.2.15), we also define the Hermitisations $\hat{\boldsymbol{\Lambda}}^{(d)}$ and $\boldsymbol{W}^{(d)}$, as well as the resolvents $\boldsymbol{G}_i^{(d)} = \boldsymbol{G}^{(d)}(w_i) := (\boldsymbol{W}^{(d)} + \hat{\boldsymbol{\Lambda}}^{(d)} - w_i)^{-1}$. It is crucial to observe that the correspondingly modified MDE

$$-\frac{1}{\boldsymbol{M}^{(d)}} = w - \hat{\boldsymbol{\Lambda}}^{(d)} + \mathcal{S}^{(d)}[\boldsymbol{M}^{(d)}]$$

under the usual $\operatorname{Im} w \operatorname{Im} \boldsymbol{M}^{(d)} > 0$ constraint with

$$\mathcal{S}^{(d)}[R] := \widetilde{\mathbf{E}}\, \widetilde{\boldsymbol{W}}^{(d)} R \widetilde{\boldsymbol{W}}^{(d)} = \sum_\sigma \sigma \langle R\, \boldsymbol{E}_\sigma^{(d)} \rangle \boldsymbol{E}_\sigma^{(d)} \,, \quad \text{where} \quad \boldsymbol{E}_\sigma^{(d)} := E_\sigma \otimes I_d \,,$$

has the *unique solution* $\boldsymbol{M}^{(d)} = M \otimes I_d$, where M is the unique solution of the MDE (1.2.20) on $\mathbf{C}^{2N \times 2N}$. In particular, if we define $\boldsymbol{B}_i^{(d)} := B_i \otimes I_d$ for all $i \in [k]$, then it holds that (1.4.2) defined with $\boldsymbol{M}_i^{(d)}$ and $\boldsymbol{B}_i^{(d)}$ as inputs, also satisfies $\boldsymbol{M}^{(d)}(w_1, \boldsymbol{B}_1^{(d)}, ..., \boldsymbol{B}_{k-1}^{(d)}, w_k) = M(w_1, B_1, ..., B_{k-1}, w_k) \otimes I_d$.

We now multiply the analogue of (1.D.6) in boldface matrices by some $\boldsymbol{B}_k^{(d)} = B_k \otimes I_d$ with $B_k \in \mathbf{C}^{2N \times 2N}$ and take the averaged trace. Next, by means of (1.D.3), taking the expectation of the resulting expression removes the underlined term. Hence, using the one-to-one correspondence between the terms in the second line of (1.D.6) and the terms on the rhs. of (1.4.2), mentioned below (1.D.6), it follows by telescopic replacement and a simple induction on the length k of the chain, that

$$\lim_{d \to \infty} \mathbf{E} \left\langle \boldsymbol{G}_1^{(d)} \boldsymbol{B}_1^{(d)} \cdots \boldsymbol{G}_k^{(d)} \boldsymbol{B}_k^{(d)} \right\rangle = \langle M(w_1, B_1, ..., w_k) B_k \rangle \tag{1.D.7}$$

by means of the usual *global law* [243, Theorem 2.1] for the last term on the rhs. of (1.D.6). In fact, due to the tensorisation, we have that $|\langle \boldsymbol{G}_1^{(d)} - \boldsymbol{M}_1^{(d)} \rangle| < 1/(Nd)$ since $|\operatorname{Im} w_1| \gtrsim 1$, where the implicit constant potentially depends on N but not on d.

We emphasise that the tensorisation by I_d is indeed a necessary step, since the matrices M_i and B_i are N-dependent and hence one cannot take the limit $N \to \infty$ in (1.D.7) for $d = 1$.

Step 3. Having (1.D.7) at hand, the recursive relations in (1.D.1) and (1.D.2) can be proven as follows: For (1.D.1), let $1 \le j \le k$ and expand G_j in (1.D.4) according to

$$G_j = M_j - M_j \underline{W G_j} + M_j \mathcal{S}[G_j - M_j] G_j \,, \tag{1.D.8}$$

which yields, analogously to (1.D.5),

$$\begin{aligned} G_1 \cdots B_{j-1} G_j B_j \cdots G_k &= G_1 \cdots B_{j-1} M_j B_j \cdots G_k \\ &+ \sum_{\sigma=\pm} \sum_{l=1}^{j-1} \sigma G_1 \cdots B_{l-1} G_l \langle G_l \cdots G_{j-1} B_{j-1} M_j E_\sigma \rangle E_\sigma G_j B_j \cdots G_k \\ &+ \sum_{\sigma=\pm} \sum_{l=j+1}^{k} \sigma G_1 \cdots B_{j-1} M_j \langle G_j B_j \cdots B_{l-1} G_l E_\sigma \rangle E_\sigma G_l B_l \cdots G_k \\ &- \underline{G_1 \cdots B_{j-1} M_j W G_j B_j \cdots G_k} + \langle G_j - M_j \rangle G_1 \cdots B_{j-1} M_j G_j B_j \cdots G_k \,. \end{aligned} \tag{1.D.9}$$

Hence, after taking the trace against some arbitrary $B_k \in \mathbf{C}^{2N \times 2N}$, by performing the tensorisation from **Step 2**, taking an expectation, and using (1.D.7), we obtain (1.D.1), but in a trace against B_k. However, since B_k was arbitrary, we conclude the desired.

For the second recursion (1.D.2), the argument is identical except from the fact that we expand G_j in (1.D.4) according to

$$G_j = M_j - \underline{G_j W} M_j + G_j \mathcal{S}[G_j - M_j] M_j \,. \tag{1.D.10}$$

□

The recursive relations from Lemma 1.D.1 can be used to show the bounds from Lemma 1.4.3 on the deterministic counterparts in the definition of $\Psi_k^{\mathrm{av/iso}}$ in (1.4.15) resp. (1.4.16) for $k \le 4$. Recall that all deterministic matrices A_i appearing in the respective averaged or isotropic chain are regular in the sense of Definition 1.4.2.

Proof of Lemma 1.4.3. In the following, we will distinguish the two regimes $\eta \le 1$ and $\eta > 1$ and argue for each of them separately, iteratively using Lemma 1.D.1. Before going into the iteration, recall that $\|M(w_1)\| \lesssim \min(1, \frac{1}{|\operatorname{Im} w_1|})$ from Lemma 1.B.1, which immediately yields (1.4.11) for $k = 1$.

Regime $\eta \le 1$. Using (1.D.1) for $k = j = 2$, we find that

$$M(w_1, A_1, w_2) = M(w_1) \mathcal{X}_{12}[A_1] M(w_2) = \mathcal{B}_{12}^{-1}[M(w_1) A_1 M(w_2)] \,, \tag{1.D.11}$$

where $\mathcal{X}_{12}[B] := \big(1 - \mathcal{S}[M(w_1) \cdot M(w_2)]\big)^{-1}[B]$ for $B \in \mathbf{C}^{2N \times 2N}$. Since A_1 is regular, we conclude (1.4.10) for $k = 1$ (by means of Lemma 1.B.6 (b)), which immediately translates to (1.4.11) for $k = 2$.

Next, for (1.4.10) and $k = 2$, we again use (1.D.1) with $j = 2$, such that we obtain

$$\begin{aligned} M(w_1, A_1, w_2, A_2, w_3) =& M(w_1, \mathcal{X}_{12}[A_1] M(w_2) A_2, w_3) \\ &+ \sum_\sigma \sigma M(w_1, \mathcal{X}_{12}[A_1] M(w_2) E_\sigma, w_3) \langle M(w_2, A_2, w_3) E_\sigma \rangle \,. \end{aligned} \tag{1.D.12}$$

Moreover, using (1.4.11) for $k = 2$ in combination with (1.D.11) and the lower bound (1.B.15) on the eigenvalues of the stability operator $\mathcal{B}$, (1.4.10) for $k = 2$ readily follows.

For (1.4.11) and $k = 3$ we need a different representation of $M(w_1, A_1, w_2, A_2, w_3)$ as

$$\mathcal{B}_{13}^{-1}\big[M(w_1) A_1 M(w_2, A_2, w_3) + \sum_\sigma \sigma M(w_1) E_\sigma M(w_2, A_2, w_3) \langle M(w_1, A_1, w_2) E_\sigma \rangle\big] \,,$$

which follows from (1.D.1) with $j = 1$ (or simply by Definition 1.4.1). This implies

$$\langle \mathcal{B}_{13}^{-1}[\cdots]A_3\rangle = \langle [\cdots]\mathcal{X}_{31}[A_3]\rangle$$

and thus, since $\|[\cdots]\| \lesssim 1$ from (1.4.10) with $k = 1$ and $\|\mathcal{X}_{31}[A_3]\| \lesssim 1$ (recall Lemma 1.B.6 (b)), we have proven (1.4.11) for $k = 3$.

In order to see (1.4.10) for $k = 3$, we first need to show that (1.4.10) for $k = 2$ remains valid, if only *one* of the two involved matrices A_1, A_2 is regular. Henceforth, we will assume that $A_1 = \mathring{A}_1$ and A_2 is arbitrary, the other case being similar and hence omitted. We start with (1.D.12) and use the lower bound (1.B.15) on the eigenvalues of $\mathcal{B}$ in the first term in (1.D.12), such that the remaining terms to be investigated are in the last line of (1.D.12), where we study each factor separately. Thereby, we focus on the case $\operatorname{Im} w_1 > 0$ and $\mathfrak{s}_1 = \mathfrak{s}_2 = +$ (recall (1.3.8)), other constellations being completely analogous. Now, in the second factor in the last line of (1.D.12) we use

$$\big|\langle M(w_2, A_2, w_3)E_-\rangle\big| = \big|\langle M(w_2)A_2M(w_3)\mathcal{X}_{32}[E_-]\rangle\big| \lesssim 1$$

for $\sigma = -$. For $\sigma = +$, we find, using cyclicity of the trace, that $\big|\langle M(w_2, A_2, w_3)E_+\rangle\big|$ equals

$$\big|\langle A_2M(w_3, E_+, w_2)\rangle\big| = \frac{1}{|w_3 - w_2|}\big|\langle A_2\big(M(w_3) - M(w_2)\big)\rangle\big| \lesssim 1 + \frac{1}{|w_3 - w_2|}\,.$$

In the first factor in the last line of (1.D.12), we use the usual bound (1.B.15) for $\sigma = -$ and conclude the desired estimate together with the bound on the second factor for $\sigma = -$. However, for $\sigma = +$, the argument is slightly more involved: Using the usual notations $e_j = \operatorname{Re} w_j$ and $\eta_j = |\operatorname{Im} w_j|$, recall from the proof of Lemma 1.5.6 (see the estimate of (1.5.51)) that

$$\langle M_1\mathcal{X}_{12}[A_1^{\circ 1,2}]M_2M_2^*E_-\rangle = \mathcal{O}\big(|e_1 + e_2| + \eta_1 + \eta_2\big)\,,$$

which readily implies that

$$\langle M_1\mathcal{X}_{12}[A_1^{\circ 1,2}]M_2M_3E_-\rangle = \mathcal{O}\big(|e_2 - e_3| + |e_1 + e_2| + \eta_1 + \eta_2 + \eta_3\big) \tag{1.D.13}$$

by means of Lemma 1.B.4 (b). Employing the associated decomposition in the first factor in the last line of (1.D.12) (and using the analogous $c_\tau(...)$-notation as in (1.5.41)), we find it being equal to

$$M\big(w_1, \big(\mathcal{X}_{12}[A_1]M(w_2)\big)^{\circ 1,3}, w_3\big) + \sum_\tau c_\tau(\mathcal{X}_{12}[A_1^{\circ 1,2}]M_2)M(w_1, E_\tau, w_3)\,.$$

The first summand is easily bounded by one, as follows from (1.4.10) for $k = 1$. Using (1.D.11), the term with $\tau = +$ is also bounded by one. The remaining term with $\tau = -$ can be estimated with the aid of (1.D.13) as

$$\frac{|e_2 - e_3| + |e_1 + e_2| + \eta_1 + \eta_2 + \eta_3}{|w_1 + w_3|}\,.$$

Collecting all the estimates from above, we find that $\|M(w_1, \mathring{A}_1, w_2, A_2, w_3)\|$ is bounded by

$$\frac{1}{\eta} + \left(1 + \frac{|e_1 + e_3| + |e_2 - e_3| + \eta_1 + \eta_2 + \eta_3}{|e_1 + e_3| + \eta_1 + \eta_3}\right)\left(1 + \frac{1}{|e_3 - e_2| + \eta_2 + \eta_3}\right) \lesssim \frac{1}{\eta}\,,$$

which shows that (1.4.10) remains valid if only one of the two involved matrices A_1, A_2 is regular.

Having this at hand, we can now turn to the proof of (1.4.10) for $k = 3$. In fact, by (1.D.1) for $k = 4$, we find

$$\begin{aligned} M(w_1, .., w_4) =& M(w_1, \mathcal{X}_{12}[A_1]M(w_2), A_2, w_3, A_3, w_4) \\ &+ \sum_\sigma \sigma M(w_1, \mathcal{X}_{12}[A_1]M(w_2)E_\sigma, w_3, A_3, w_4)\langle M(w_2, A_2, w_3)E_\sigma\rangle \end{aligned} \tag{1.D.14}$$

$$+\sum_\sigma \sigma M(w_1, \mathcal{X}_{12}[A_1]M(w_2)E_\sigma, w_4)\langle M(w_2, A_2, w_3, A_3, w_4)E_\sigma\rangle\,,$$

where the first and second line of (1.D.14) are bounded by $\frac{1}{\eta}$ and we can thus focus on the last line. Structurally, this term is the analog of the last line in (1.D.12) and also proving it being bounded by $\frac{1}{\eta}$ is completely analogous to the arguments above. This concludes the proof of (1.4.10) for $k = 3$, from which (1.4.11) for $k = 4$ immediately follows.

Finally, we turn to the proof of (1.4.10) for $k = 4$. By (1.D.1) for $j = 1$ (or simply by Definition 1.4.1) we find the different representation

$$\begin{aligned} M(w_1,...,w_5) =& \mathcal{B}_{15}^{-1}\big[M(w_1)A_1M(w_2,...,w_5) \\ &+\sum_\sigma \sigma M(w_1)E_\sigma M(w_2,...,w_5)\langle M(w_1,A_1,w_2)E_\sigma\rangle \\ &+\sum_\sigma \sigma M(w_1)E_\sigma M(w_3,...,w_5)\langle M(w_1,...,w_3)E_\sigma\rangle \\ &+\sum_\sigma \sigma M(w_1)E_\sigma M(w_4,A_4,w_5)\langle M(w_1,...,w_4)E_\sigma\rangle\big]\,. \end{aligned}$$

Combining $\|[\cdots]\| \lesssim \eta^{-1}$, as follows from (1.4.10) for $k \in [3]$ and (1.4.11) for $k \in [4]$, with the usual bound (1.B.15), we conclude the desired. This finishes the proof in the first regime where $\eta \le 1$.

Regime $\eta > 1$. In this second regime, we note that all inverses of stability operators are bounded (see (1.B.15)). Moreover, it easily follows from (1.D.1) that every summand in the definition of $M(w_1,...,w_k)$ carries at least k factors of (different) $M(w_i)$. Now, as mentioned in the beginning of the proof, we have $\|M(w_i)\| \lesssim 1/\eta$, which implies the desired bound. □

1.E Proof of Lemmas 1.5.8 and 1.5.9

In this appendix, we carry out the proofs of the two Lemmas 1.5.8 and 1.5.9.

Proof of Lemma 1.5.8. Similarly to the proof of Lemma 1.5.6, we get from Appendix 1.A and (1.4.4) that

$$\begin{aligned} &\langle (G_1A_1G_2 - M_1\mathcal{X}_{12}[A_1]M_2)A_2\rangle \\ &= \langle M_1A_1(G_2-M_2)\mathcal{X}_{21}[A_2]\rangle - \langle M_1\underline{WG_1A_1G_2}\mathcal{X}_{21}[A_2]\rangle \\ &\quad + \langle M_1\mathcal{S}[G_1-M_1]G_1A_1G_2\mathcal{X}_{21}[A_2]\rangle + \langle M_1\mathcal{S}[G_1A_1G_2](G_2-M_2)\mathcal{X}_{21}[A_2]\rangle\,. \end{aligned} \tag{1.E.1}$$

We note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ and $\|\mathcal{X}_{21}[\mathring{A}_2]\| \lesssim 1$ by means of Lemma 1.B.6.

Then, analogously to (1.5.40), we need to further decompose $\mathcal{X}_{21}[A_2]M_1$ in the last three terms in (1.5.39) as

$$\mathcal{X}_{21}[\mathring{A}_2]M_1 = (\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ + \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{A}_2]M_1)E_\sigma\,,$$

where we again suppressed the spectral parameters (and the relative sign of their imaginary parts, which has been fixed by $\operatorname{Im} w_1 > 0$ and $\operatorname{Im} w_2 < 0$) in the notation for the linear functionals $c_\sigma(\cdot)$ on $\mathbf{C}^{2N\times 2N}$ defined as

$$c_+(B) := \frac{\langle M_2BM_1\rangle}{\langle M_2M_1\rangle} \qquad \text{and} \qquad c_-(B) := \frac{\langle M_2BM_1^*E_-\rangle}{\langle M_2E_-M_1^*E_-\rangle}\,. \tag{1.E.2}$$

Continuing the expansion of (1.E.1), we arrive at

$$\begin{aligned} &\langle M_1\mathring{A}_1(G_2-M_2)\mathcal{X}_{21}[\mathring{A}_2]\rangle - \langle \underline{WG_1\mathring{A}_1G_2}(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle \\ &+ \langle \mathcal{S}[G_1-M_1]G_1\mathring{A}_1G_2(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle + \langle \mathcal{S}[G_1\mathring{A}_1G_2](G_2-M_2)(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle \end{aligned}$$

$$+\sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{A}_2]M_1)\big[-\langle\underline{WG_1\mathring{A}_1G_2}U_\sigma\rangle+\langle\mathcal{S}[G_1-M_1]G_1\mathring{A}_1G_2E_\sigma\rangle \\ +\langle\mathcal{S}[G_1\mathring{A}_1G_2](G_2-M_2)E_\sigma\rangle\big].$$

We emphasise that, in case of $\mathring{A}_2$ and its linear dependents, the regular component is defined w.r.t. the pair of spectral parameters (w_2,w_1).

Next, analogously to the proof of Lemma 1.5.6, we undo the underline in $\big[\cdots\big]$, such that our expansion of (1.E.1) becomes

$$\begin{aligned}&\langle(G_1\mathring{A}_1G_2-M_1\mathcal{X}_{12}[\mathring{A}_1]M_2)\mathring{A}_2\rangle\\ &=\langle M_1\mathring{A}_1(G_2-M_2)\mathcal{X}_{21}[\mathring{A}_2]\rangle-\langle\underline{WG_1\mathring{A}_1G_2}(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle \qquad (1.E.3)\\ &\quad+\langle\mathcal{S}[G_1-M_1]G_1\mathring{A}_1G_2(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle+\langle\mathcal{S}[G_1\mathring{A}_1G_2](G_2-M_2)(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle\\ &\quad+\sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{A}_2]M_1)\big[-\langle\mathring{A}_1G_2E_\sigma\rangle+\langle G_1\mathring{A}_1G_2\mathring{\Phi}_\sigma\rangle+c_\sigma(\Phi_\sigma)\langle G_1\mathring{A}_1G_2E_\sigma\rangle\big],\end{aligned}$$

where

$$\Phi_\sigma\coloneqq E_\sigma\frac{1}{M_1}-\mathcal{S}[M_2E_\sigma] \qquad (1.E.4)$$

was further decomposed with the aid of $c_\sigma(\Phi_\tau)\sim\delta_{\sigma,\tau}$ and we used the notation (1.E.2).

We can now write (1.E.3) for both, $\mathring{A}_2=\mathring{\Phi}_+$ and $\mathring{A}_2=\mathring{\Phi}_-$, and solve the two resulting equation for $\langle G_1\mathring{A}_1G_2\mathring{\Phi}_\sigma\rangle$ and $\langle G_1\mathring{A}_1G_2\mathring{\Phi}_-\rangle$. Observe that by means of

$$c_\tau(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)\sim\delta_{\sigma,\tau}\,,$$

the original *system* of linear equations boils down to two separate ones. Thus, plugging the solutions for $\langle G_1\mathring{A}_1G_2\mathring{\Phi}_\pm\rangle$ back into (1.E.3) we arrive at

$$\begin{aligned}&\langle(G_1\mathring{A}_1G_2-M_1\mathcal{X}_{12}[\mathring{A}_1]M_2)\mathring{A}_2\rangle\\ &=-\langle\underline{WG_1\mathring{A}_1G_2}(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle+\langle G_1-M_1\rangle\langle G_1\mathring{A}_1G_2(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle\\ &\quad+\langle M_1\mathring{A}_1(G_2-M_2)\mathcal{X}_{21}[\mathring{A}_2]\rangle+\langle\mathcal{S}[G_1\mathring{A}_1G_2](G_2-M_2)(\mathcal{X}_{21}[\mathring{A}_2]M_1)^\circ\rangle \qquad (1.E.5)\\ &\quad+\sum_\sigma\frac{\mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{A}_2]M_1)}{1-\mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)}\Big[-\langle\underline{WG_1\mathring{A}_1G_2}(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)^\circ\rangle \qquad (1.E.6)\\ &\quad+\langle G_1-M_1\rangle\langle G_1\mathring{A}_1G_2(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)^\circ\rangle+\langle M_1\mathring{A}_1(G_2-M_2)\mathcal{X}_{21}[\mathring{\Phi}_\sigma]\rangle\\ &\quad+\langle\mathcal{S}[G_1\mathring{A}_1G_2](G_2-M_2)(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)^\circ\rangle \qquad (1.E.7)\\ &\quad-\langle\mathring{A}_1(G_2-M_2)E_\sigma\rangle+c_\sigma(\Phi_\sigma)\langle(G_1\mathring{A}_1G_2-M_{12}^{\mathring{A}_1})E_\sigma\rangle\Big]. \qquad (1.E.8)\end{aligned}$$

We now need to check that the denominators in (1.E.6) are bounded away from zero.

Lemma 1.E.1. *For small enough $\delta>0$, we have that*

$$\big|1-\mathbf{1}_\delta^\sigma(w_2,w_1)\,c_\sigma(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)\big|\gtrsim 1\qquad\textit{for}\quad\sigma=\pm\,.$$

Proof. Completely analogous to Lemma 1.5.7. □

Next, there are two particular terms, namely the ones of the form

$$\langle\mathcal{S}[G_1\mathring{A}_1^{1,2}G_2](G_2-M_2)\mathring{A}_2^{2,1}\rangle\,, \qquad (1.E.9)$$

appearing in (1.E.5) and (1.E.7), and

$$c_\sigma(\mathcal{X}_{21}[\mathring{A}_2^{2,1}]M_1)c_\sigma(\Phi_\sigma)\langle(G_1\mathring{A}_1^{1,2}G_2 - M_1\mathcal{X}_{12}[\mathring{A}_1^{1,2}]M_2)E_\sigma\rangle\,, \tag{1.E.10}$$

appearing in (1.E.8), whose naive size $1/(N\eta^2)$ does not match the target. Hence, they have to be discussed in more detail. In (1.E.9) and (1.E.10), we emphasised the pair of spectral parameters with respect to which the regularisation has been conducted. Moreover, for the following estimates, we recall the a priori bounds (1.4.25).

Estimating (1.E.9). We begin by expanding

$$\langle\mathcal{S}[G_1\mathring{A}_1^{1,2}G_2](G_2 - M_2)\mathring{A}_2^{2,1}\rangle = \sum_\sigma \sigma\,\langle G_1\mathring{A}_1^{1,2}G_2E_\sigma\rangle\langle(G_2 - M_2)\mathring{A}_2^{2,1}E_\sigma\rangle \tag{1.E.11}$$

and note that, analogously to (1.5.53),

$$\mathring{A}_i^{i,j}E_\sigma = \big(\mathring{A}_i^{i,j}E_\sigma\big)^{\circ_{i,i}} + \mathcal{O}\big(|e_i - \sigma e_j| + |\eta_i - \eta_j|\big)E_+ + \mathcal{O}\big(|e_i - \sigma e_j| + |\eta_i - \eta_j|\big)E_- \tag{1.E.12}$$

as well as

$$\mathring{A}_i^{i,j}E_\sigma = \big(\mathring{A}_i^{i,j}E_\sigma\big)^{\circ_{j,j}} + \mathcal{O}\big(|e_i - \sigma e_j| + |\eta_i - \eta_j|\big)E_+ + \mathcal{O}\big(|e_i - \sigma e_j| + |\eta_i - \eta_j|\big)E_- \tag{1.E.13}$$

for $i \neq j \in [2]$ and $\sigma = \pm$.

In the first term in (1.E.11), for $\sigma = +$ and $E_\sigma = E_+$, we use a resolvent identity (1.3.21) and the usual averaged local law (1.4.17) in combination with (1.E.12), (1.E.13) and (1.4.8), in order to bound it as

$$\big|\langle G_1\mathring{A}_1^{1,2}G_2\rangle\big| \prec 1 + \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2}\max_{i\in[2]}|\langle(G_i - M_i)(\mathring{A}_1^{1,2})^{\circ_{i,i}}\rangle|\,. \tag{1.E.14}$$

For $\sigma = -$ and $E_\sigma = E_-$, we first add and subtract the corresponding deterministic approximation $\langle M(w_1, \mathring{A}_1^{1,2}, w_2)E_-\rangle$, which itself is bounded by means of Lemma 1.4.3. In the difference term, we use (1.2.16) and employ the integral representation from Lemma 1.5.1 with

$$\tau = +\,, \quad J = \mathbf{B}_{\ell\kappa_0}\,, \quad \text{and} \quad \tilde\eta = \frac{\ell}{\ell+1}\eta\,,$$

for which we recall that $w_j \in \mathbf{D}_{\ell+1}^{(\epsilon_0,\kappa_0)}$, i.e. in particular $\eta \geq (\ell+1)N^{-1+\epsilon_0}$ and hence $\tilde\eta \geq \ell N^{-1+\epsilon_0}$. Note that Lemma 1.5.1 is also true on the level of the corresponding deterministic approximations, as can be seen, e.g., by a meta argument similarly to the proof of Lemma 1.D.1. Hence, after splitting the contour integral and bounding the individual contributions as described in (1.5.14), we obtain

$$\begin{aligned}\big|\langle G_1A_1^{\circ_{1,2}}G_2E_-\rangle\big| &\prec 1 + \int_{\mathbf{B}_{\ell\kappa_0}}\frac{\big|\big\langle\big(G(x+\mathrm{i}\tilde\eta) - M(x+\mathrm{i}\tilde\eta)\big)A_1^{\circ_{1,2}}E_-\big\rangle\big|}{\big|\big(x - e_1 - \mathrm{i}(\eta_1 - \tilde\eta)\big)\big(x + e_2 - \mathrm{i}(\eta_2 - \tilde\eta)\big)\big|}\mathrm{d}x\\ &\prec 1 + \int_{\mathbf{B}_{\ell\kappa_0}}\frac{\big|\big\langle\big(G(x+\mathrm{i}\tilde\eta) - M(x+\mathrm{i}\tilde\eta)\big)\big(A_1^{\circ_{1,2}}E_-\big)^{\circ_{x+\mathrm{i}\tilde\eta,x+\mathrm{i}\tilde\eta}}\big\rangle\big|}{\big|\big(x - e_1 - \mathrm{i}(\eta_1 - \tilde\eta)\big)\big(x + e_2 - \mathrm{i}(\eta_2 - \tilde\eta)\big)\big|}\mathrm{d}x\,,\end{aligned}$$

where in the second step we used (1.E.12) and (1.E.13), and absorbed logarithmic corrections from the integral into '$\prec$'. This finally yields that

$$\big|\langle G_1A_1^{\circ_{1,2}}G_2E_-\rangle\big| \prec 1 + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2}\cdot\frac{\psi_1^{\mathrm{av}}}{N\eta^{1/2}}\,. \tag{1.E.15}$$

Combining (1.E.14) and (1.E.15) with the estimate

$$\big|\langle(G_2 - M_2)A_2^{\circ_{2,1}}E_\sigma\rangle\big| \prec \frac{|e_1 - \sigma e_2| + |\eta_1 - \eta_2|}{N\eta} + \frac{\psi_1^{\mathrm{av}}}{N\eta^{1/2}} \tag{1.E.16}$$

for the second term in (1.E.11),which readily follows from (1.E.12) and (1.4.17), we find that (1.E.9) can be bounded as

$$\big|\langle \mathcal{S}[G_1 A_1^{\circ 1,2} G_2](G_2 - M_2) A_2^{\circ 2,1}\rangle\big| \prec \frac{1}{N\eta} + \frac{(\psi_1^{\mathrm{av}})^2}{(N\eta)^2}\,, \tag{1.E.17}$$

where we used the trivial estimate $\psi_1^{\mathrm{av}} \prec \eta^{-1/2}$.

Estimating (1.E.10). For the term (1.E.10), we first note that the two prefactors $c_\sigma(\mathcal{X}_{21}[A_2^{\circ 2,1}]M_1)$ and $c_\sigma(\Phi_\sigma)$ are bounded. However, completely analogous to the proof of Lemma 1.5.6, in each of the two cases $\sigma = \pm$, the bound on *one* of the prefactors can be improved: In the first case, $\sigma = +$, we use (1.B.11) and compute

$$c_+(\Phi_+) = \frac{\langle M_1\rangle\big(1 - \langle M_1 M_2\rangle\big)}{\langle M_1 M_2\rangle} = \mathcal{O}\big(|e_1 - e_2| + \eta_1 + \eta_2\big)\,.$$

$$\big|\langle G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\rangle\big| \prec \frac{1}{N\eta} + \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2} \max_{i\in[2]} |\langle (G_i - M_i)(A_1^{\circ 1,2})^{\circ i,i}\rangle|$$

which is obtained completely analogous to (1.E.14), we conclude that (1.E.10) for $\sigma = +$ can be estimated by $1/(N\eta)$. Similarly, in the second case, $\sigma = -$, we perform a computation similar to the one leading to (1.5.19) and use (1.B.11) in order to obtain that $c_-(\mathcal{X}_{12}[A_1^{\circ 1,2}]M_2)$ equals

$$\frac{\mathrm{i}}{2}\frac{\langle M_1 A_1^{\circ 1,2} M_2^* E_-\rangle}{\langle M_1 E_- M_2^* E_-\rangle} + \frac{1}{2\mathrm{i}}\frac{\langle M_1 A_1^{\circ 1,2} M_2 E_-\rangle}{\langle M_1 E_- M_2^* E_-\rangle}\frac{1 + \langle M_1 E_- M_2^* E_-\rangle}{1 + \langle M_1 E_- M_2 E_-\rangle} = \mathcal{O}\big(|e_1 + e_2| + \eta_1 + \eta_2\big)$$

Combining this with the bound

$$\big|\langle \big(G_1 A_1^{\circ 1,2} G_2 - M(w_1, A_1^{\circ 1,2}, w_2)\big) E_-\rangle\big| \prec \frac{1}{N\eta} + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2}\cdot\frac{\psi_1^{\mathrm{av}}}{N\eta^{1/2}}$$

which is obtained completely analogous to (1.E.15), we conclude that (1.E.10) can be estimated by $1/(N\eta)$ – now in both cases $\sigma = \pm$.

Conclusion. Summarizing our investigations, we have shown that

$$\big\langle \big(G_1 \mathring{A}_1 G_2 - M(w_1, \mathring{A}_1, w_2)\big)\mathring{A}_2\big\rangle = -\big\langle \underline{W G_1 \mathring{A}_1 G_2 \mathring{A}_2'}\big\rangle + \mathcal{O}_\prec\big(\mathcal{E}_2^{\mathrm{av}}\big)\,,$$

where we used the shorthand notation

$$\mathring{A}_2' := \big(\mathcal{X}_{21}[\mathring{A}_2]M_1\big)^\circ + \sum_\sigma \frac{\mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{A}_2]M_1)}{1 - \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1)}\big(\mathcal{X}_{21}[\mathring{\Phi}_\sigma]M_1\big)^\circ \tag{1.E.18}$$

in the underlined term. Combining (1.E.17) and the bound on (1.E.10) established above with the usual single resolvent local laws (1.4.17) and the bounds on deterministic approximations in Lemma 1.4.3, we collected all the error terms from the expansion around (1.E.5)–(1.E.8) in (1.5.59). □

Proof of Lemma 1.5.9. We denote $A_i \equiv \mathring{A}_i$, except we wish to emphasise A_i being regular. As usual, we use the customary shorthand notations and start with

$$G_2 = M_2 - M_2\underline{W G_2} + M_2\mathcal{S}[G_2 - M_2]G_2\,,$$

such that we get

$$G_1\tilde{A}_1 G_2 \mathring{A}_2 G_3 = G_1\tilde{A}_1 M_2 \mathring{A}_2 G_3 - G_1\tilde{A}_1 M_2\underline{W G_2}\mathring{A}_2 G_3 + G_1\tilde{A}_1 M_2\mathcal{S}[G_2 - M_2]G_2\mathring{A}_2 G_3$$

for $\tilde{A}_1 = \mathcal{X}_{12}[A_1]$ with $A_1 = \mathring{A}_1$ (note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ by Lemma 1.B.6) and the linear operator $\mathcal{X}_{12}$ has been introduced in (1.5.38). The definition of $\mathcal{X}_{23}$ is completely analogous.

Extending the underline to the whole product, we obtain

$$\begin{aligned} &G_1\big(\tilde{A}_1 - \mathcal{S}[M_1\tilde{A}_1M_2]\big)G_2\mathring{A}_2G_3 \\ &\quad = G_1\tilde{A}_1M_2\mathring{A}_2G_3 - \underline{G_1\tilde{A}_1M_2WG_2\mathring{A}_2G_3} + G_1\tilde{A}_1M_2\mathcal{S}[G_2\mathring{A}_2G_3]G_3 \\ &\qquad + G_1\tilde{A}_1M_2\mathcal{S}[G_2 - M_2]G_2\mathring{A}_2G_3 + G_1\mathcal{S}[(G_1 - M_1)\tilde{A}_1M_2]G_2\mathring{A}_2G_3\,, \end{aligned}$$

which leaves us with

$$\begin{aligned} &G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1, A_1, w_2, A_2, w_3) \qquad (1.E.19)\\ &= \big(G_1\big[\mathcal{X}_{12}[\mathring{A}_1]M_2(\mathring{A}_2 + \mathcal{S}[M_2\mathcal{X}_{23}[\mathring{A}_2]M_3])\big]G_3 - M(w_1, [\cdots], w_3)\big) \\ &\quad - \underline{G_1\mathcal{X}_{12}[\mathring{A}_1]M_2WG_2\mathring{A}_2G_3} + G_1\mathcal{X}_{12}[\mathring{A}_1]M_2\mathcal{S}[G_2 - M_2]G_2\mathring{A}_2G_3 \\ &\quad + G_1\mathcal{S}[(G_1 - M_1)\mathcal{X}_{12}[\mathring{A}_1]M_2]G_2\mathring{A}_2G_3 + G_1\mathcal{X}_{12}[\mathring{A}_1]M_2\mathcal{S}[G_2\mathring{A}_2G_3 - M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]G_3\,, \end{aligned}$$

where we used Lemma 1.D.1 for assembling the purely deterministic terms on the l.h.s. To continue, we first note that $\|\mathcal{X}_{12}[\mathring{A}_1]\| \lesssim 1$ and $\|\mathcal{X}_{23}[\mathring{A}_2]\| \lesssim 1$ (again, the matrices being regular removes the potentially 'bad direction' of the stability operators $\mathcal{X}_{12}$ and $\mathcal{X}_{23}$).

Then, we need to further decompose $\mathcal{X}_{12}[A_1]M_2$ in the last four terms in (1.E.19) as

$$\mathcal{X}_{12}[A_1]M_2 = \big(\mathcal{X}_{12}[A_1]M_2\big)^\circ + \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)E_\sigma\,, \qquad (1.E.20)$$

where, similarly as for $\cdot^\circ$, we suppressed the spectral parameters w_1, w_2 in the notation for the linear functionals $c_\sigma(...)$, which have been defined in see (1.5.41). Now, plugging (1.E.20) into (1.E.19) we find

$$\begin{aligned} &G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1, \mathring{A}_1, w_2, \mathring{A}_2, w_3) \qquad (1.E.21)\\ &= \big(G_1\big[\mathcal{X}_{12}[\mathring{A}_1]M_2(\mathring{A}_2 + \mathcal{S}[M_2\mathcal{X}_{23}[\mathring{A}_2]M_3])\big]G_3 - M(w_1, [\cdots], w_3)\big) \\ &\quad - \underline{G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^\circ WG_2\mathring{A}_2G_3} + G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^\circ\mathcal{S}[G_2 - M_2]G_2\mathring{A}_2G_3 \\ &\quad + G_1\mathcal{S}[(G_1 - M_1)\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^\circ]G_2\mathring{A}_2G_3 + G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^\circ\mathcal{S}[G_2\mathring{A}_2G_3 - M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]G_3 \\ &\quad + \sum_\sigma \mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[\mathring{A}_1]M_2)\Big[- \underline{G_1E_\sigma WG_2\mathring{A}_2G_3} + G_1E_\sigma\mathcal{S}[G_2 - M_2]G_2\mathring{A}_2G_3 \\ &\qquad\qquad + G_1\mathcal{S}[(G_1 - M_1)E_\sigma]G_2\mathring{A}_2G_3 + G_1E_\sigma\mathcal{S}[G_2\mathring{A}_2G_3 - M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]G_3\Big]. \end{aligned}$$

Next, as in the earlier sections (see, e.g., the display above (1.E.4)), in the last line of (1.E.21) we now undo the underline and find the bracket $\big[\cdots\big]$ to equal (the negative of)

$$G_1E_\sigma\big(\mathring{A}_2 + \mathcal{S}[M(w_2, \mathring{A}_2, w_3)]\big)G_3 - G_1\Phi_\sigma G_2\mathring{A}_2G_3\,,$$

where we denoted

$$\Phi_\sigma := E_\sigma\frac{1}{M_2} - \mathcal{S}[M_1E_\sigma]\,.$$

It is apparent from the expansion (1.E.21) (and it can also be checked by hand) that

$$M(w_1, E_\sigma\mathring{A}_2 + E_\sigma\mathcal{S}[M(w_2, \mathring{A}_2, w_3)], w_3) = M(w_1, \Phi_\sigma, w_2, \mathring{A}_2, w_3)\,,$$

which finally yields

$$\begin{aligned} &G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1, \mathring{A}_1, w_2, \mathring{A}_2, w_3) \qquad (1.E.22)\\ &= \big(G_1\big[\mathcal{X}_{12}[\mathring{A}_1]M_2(\mathring{A}_2 + \mathcal{S}[M_2\mathcal{X}_{23}[\mathring{A}_2]M_3])\big]G_3 - M(w_1, [\cdots], w_3)\big) \end{aligned}$$

$$
\begin{aligned}
&- \underline{G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}WG_2\mathring{A}_2G_3} + G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}\mathcal{S}[G_2-M_2]G_2\mathring{A}_2G_3\\
&+ G_1\mathcal{S}[(G_1-M_1)\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}]G_2\mathring{A}_2G_3 + G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}\mathcal{S}[G_2\mathring{A}_2G_3 - M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]G_3\\
&+ \sum_{\sigma}\mathbf{1}_{\delta}^{\sigma}\,c_{\sigma}(\mathcal{X}_{12}[\mathring{A}_1]M_2)\Big[- \big(G_1E_{\sigma}\big(\mathring{A}_2+\mathcal{S}[M(w_2,\mathring{A}_2,w_3)]\big)G_3 - M(w_1,[\cdots]w_3)\big)\\
&+ \big(G_1\mathring{\Phi}_{\sigma}G_2\mathring{A}_2G_3 - M(w_1,\mathring{\Phi}_{\sigma},w_2,\mathring{A}_2,w_3)\big) + \sum_{\sigma}c_{\sigma}(\Phi_{\sigma})\big(G_1E_{\sigma}G_2\mathring{A}_2G_3 - M(w_1,E_{\sigma},w_2,\mathring{A}_2,w_3)\big)\Big],
\end{aligned}
$$

where we further decomposed Φ_σ in the last line of (1.E.22) (while using the first relation in (1.5.45)) just as $\mathcal{X}_{12}[A_1]M_2$ in (1.E.20).

Next, we write (1.E.22) for both, $A_1 = \mathring{A}_1 = \mathring{\Phi}_+$ and $A_1 = \mathring{A}_1 = \mathring{\Phi}_-$, and solve the two resulting linear equations for $G_1\mathring{\Phi}_{\pm}G_2 - M(w_1,\mathring{\Phi}_{\pm},w_2)$. Observe that by means of the second relation in (1.5.45) the original system of linear equations boils down to two separate ones. Thus, plugging the solutions for $G_1\mathring{\Phi}_{\pm}G_2\mathring{A}_2G_3 - M(w_1,\mathring{\Phi}_{\pm},w_2,\mathring{A}_2,w_3)$ back into (1.E.22), we arrive at

$$
\begin{aligned}
&G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1,\mathring{A}_1,w_2,\mathring{A}_2,w_3) \qquad\qquad (1.E.23)\\
&= \big(G_1\big[\mathcal{X}_{12}[\mathring{A}_1]M_2(\mathring{A}_2+\mathcal{S}[M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]\big)\big]G_3 - M(w_1,\big[\cdots\big],w_3)\big)\\
&\quad - \underline{G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}WG_2\mathring{A}_2G_3} + G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}\mathcal{S}[G_2-M_2]G_2\mathring{A}_2G_3\\
&\quad + G_1\mathcal{S}[(G_1-M_1)\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}]G_2\mathring{A}_2G_3 + G_1\big(\mathcal{X}_{12}[\mathring{A}_1]M_2\big)^{\circ}\mathcal{S}[G_2\mathring{A}_2G_3 - M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]G_3\\
&\quad + \sum_{\sigma}\frac{\mathbf{1}_{\delta}^{\sigma}\,c_{\sigma}(\mathcal{X}_{12}[\mathring{A}_1]M_2)}{1-\mathbf{1}_{\delta}^{\sigma}\,c_{\sigma}(\mathcal{X}_{12}[\mathring{\Phi}_{\sigma}]M_2)}\Big[- \big(G_1\big[E_{\sigma}\big(\mathring{A}_2+\mathcal{S}[M(w_2,\mathring{A}_2,w_3)]\big)\big]G_3 - M(w_1,\big[\cdots\big]w_3)\big)\\
&\quad + \big(G_1\big[\mathcal{X}_{12}[\mathring{\Phi}_{\sigma}]M_2(\mathring{A}_2+\mathcal{S}[M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]\big)\big]G_3 - M(w_1,\big[\cdots\big],w_3)\big)\\
&\quad - \underline{G_1\big(\mathcal{X}_{12}[\mathring{\Phi}_{\sigma}]M_2\big)^{\circ}WG_2\mathring{A}_2G_3} + G_1\big(\mathcal{X}_{12}[\mathring{\Phi}_{\sigma}]M_2\big)^{\circ}\mathcal{S}[G_2-M_2]G_2\mathring{A}_2G_3\\
&\quad + G_1\mathcal{S}[(G_1-M_1)\big(\mathcal{X}_{12}[\mathring{\Phi}_{\sigma}]M_2\big)^{\circ}]G_2\mathring{A}_2G_3 + G_1\big(\mathcal{X}_{12}[\mathring{\Phi}_{\sigma}]M_2\big)^{\circ}\mathcal{S}[G_2\mathring{A}_2G_3 - M_2\mathcal{X}_{23}[\mathring{A}_2]M_3]G_3\\
&\quad + c_{\sigma}(\Phi_{\sigma})\big(G_1E_{\sigma}G_2\mathring{A}_2G_3 - M(w_1,E_{\sigma},w_2,\mathring{A}_2,w_3)\big)\Big].
\end{aligned}
$$

It has been shown in Lemma 1.5.7 that the denominators are bounded away from zero.

Next, we take the scalar product of (1.E.23) with two deterministic vectors $\boldsymbol{x},\boldsymbol{y}$ satisfying $\|\boldsymbol{x}\|,\|\boldsymbol{y}\|\le 1$. In the resulting expression, in case that $\mathbf{1}_{\delta}^{\sigma}(w_1,w_2)=1$, there are three particular terms, namely the ones of the form

$$
\big(G_1\mathcal{S}[(G_1-M_1)A_1^{\circ_{1,2}}]G_2\mathring{A}_2G_3\big)_{\boldsymbol{xy}}, \qquad (1.E.24)
$$

as appearing twice, in the fourth and second to last line,

$$
\big(G_1A_1^{\circ_{1,2}}\mathcal{S}[G_2\mathring{A}_2G_3 - M(w_2,\mathring{A}_2,w_3)]G_3\big)_{\boldsymbol{xy}}, \qquad (1.E.25)
$$

as appearing, again twice, in the fourth and second to last line,

$$
c_{\sigma}(\mathcal{X}_{12}[\mathring{A}_1]M_2)c_{\sigma}(\Phi_{\sigma})\big(G_1E_{\sigma}G_2\mathring{A}_2G_3 - M(w_1,E_{\sigma},w_2,\mathring{A}_2,w_3)\big)_{\boldsymbol{xy}}, \qquad (1.E.26)
$$

as appearing in the last line, whose naive sizes $1/(N\eta^3)$, $1/(N\eta^3)$, and $1/\sqrt{N\eta^4}$ do not match the target. Hence, they have to be discussed in more detail.

Estimating (1.E.24). For the terms of the first type, we begin by expanding

$$
\big(G_1\mathcal{S}[(G_1-M_1)A_1^{\circ_{1,2}}]G_2\mathring{A}_2G_3\big)_{\boldsymbol{xy}} = \sum_{\sigma}\sigma\langle(G_1-M_1)A_1^{\circ_{1,2}}E_{\sigma}\rangle\big(G_1E_{\sigma}G_2\mathring{A}_2G_3\big)_{\boldsymbol{xy}}
$$

and recall from (1.E.16) that first factor can be estimated by

$$
|\langle(G_1-M_1)A_1^{\circ_{1,2}}E_{\sigma}\rangle| < \frac{|e_1-\sigma e_2|+|\eta_1-\eta_2|}{N\eta} + \frac{\psi_1^{\mathrm{av}}}{N\eta^{1/2}}. \qquad (1.E.27)
$$

In the second factor, we distinguish the two cases $\sigma = \pm$. For $\sigma = +$, we find

$$G_1 G_2 A_2^{\circ 2,3} G_3 = \frac{G_1 A_2^{\circ 2,3} G_3 - G_2 A_2^{\circ 2,3} G_3}{(e_1 - e_2) + \mathrm{i}(\eta_1 + \eta_2)}$$

by a simple resolvent identity (1.3.21), which together with

$$\begin{aligned} \mathring{A}_2^{w_2,w_3} = \mathring{A}_2^{w_1,w_3} &+ \mathcal{O}\big(|e_1 - e_2| + |\eta_1 - \eta_2| + |e_1 - e_3| + |\eta_1 - \eta_3|\big) E_+ \\ &+ \mathcal{O}\big(|e_1 - e_2| + |\eta_1 - \eta_2| + |e_1 + e_3| + |\eta_1 - \eta_3|\big) E_- \end{aligned}$$

from Lemma 1.3.3 (note the difference between the E_+-error and the E_--error!) and the usual isotropic law (1.4.17) yields the estimate

$$\big|\big(G_1 G_2 A_2^{\circ 2,3} G_3\big)_{\boldsymbol{xy}}\big| \prec \frac{1}{\eta} + \frac{1}{|e_1 - e_2| + \eta_1 + \eta_2} \left(1 + \frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta^2}}\right), \tag{1.E.28}$$

where we again used the a priori bound (1.4.25). For $\sigma = -$ we employ the integral representation from Lemma 1.5.1 and argue similarly as for (1.E.15) such that we finally obtain

$$\big|\big(G_1 E_- G_2 A_2^{\circ 2,3} G_3\big)_{\boldsymbol{xy}}\big| \prec \frac{1}{\eta} + \frac{1}{|e_1 + e_2| + \eta_1 + \eta_2} \left(1 + \frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta^2}}\right). \tag{1.E.29}$$

Now, combining (1.E.27) with (1.E.28) and (1.E.29), we find

$$\big|\big(G_1 \mathcal{S}[(G_1 - M_1) A_1^{\circ 1,2}] G_2 \mathring{A}_2 G_3\big)_{\boldsymbol{xy}}\big| \prec \frac{1}{\sqrt{N\eta^3}} \left(1 + \frac{\psi_1^{\mathrm{av}} \psi_1^{\mathrm{iso}}}{N\eta}\right), \tag{1.E.30}$$

where we used that $\psi_1^{\mathrm{av}} \prec \eta^{-1/2}$ trivially by (1.4.17).

Estimating (1.E.25). For terms of the second type, we again start by expanding

$$\begin{aligned} &\big(G_1 A_1^{\circ 1,2} \mathcal{S}[G_2 \mathring{A}_2 G_3 - M(w_2, \mathring{A}_2, w_3)] G_3\big)_{\boldsymbol{xy}} \\ &\qquad = \sum_\sigma \sigma \big\langle \big(G_2 \mathring{A}_2 G_3 - M(w_2, \mathring{A}_2, w_3)\big) E_\sigma \big\rangle \big(G_1 A_1^{\circ 1,2} E_\sigma G_3\big)_{\boldsymbol{xy}}. \end{aligned}$$

Then, for the first factor, we recall from the estimate of (1.E.9) that

$$\big|\big\langle \big(G_2 A_2^{\circ 2,3} G_3 - M(w_2, A_2^{\circ 2,3}, w_3)\big) E_\sigma \big\rangle\big| \prec \frac{1}{N\eta} + \frac{1}{|e_2 - \sigma e_3| + \eta_2 + \eta_3} \cdot \frac{\psi_1^{\mathrm{av}}}{N\eta^{1/2}}.$$

Treating the second factor analogously to (1.E.28) and (1.E.29) above, we find

$$\big|\big(G_1 A_1^{\circ 1,2} E_\sigma G_3\big)_{\boldsymbol{xy}}\big| \prec \frac{|e_2 - \sigma e_3| + |\eta_2 - \eta_3|}{\eta} + \left(1 + \frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta^2}}\right).$$

Combining the two estimates, we have shown that

$$\big|\big(G_1 A_1^{\circ 1,2} \mathcal{S}[G_2 \mathring{A}_2 G_3 - M(w_2, \mathring{A}_2, w_3)] G_3\big)_{\boldsymbol{xy}}\big| \prec \frac{1}{\sqrt{N\eta^3}} \left(1 + \frac{\psi_1^{\mathrm{iso}}}{N\eta} + \frac{\psi_1^{\mathrm{av}} \psi_1^{\mathrm{iso}}}{N\eta}\right) \tag{1.E.31}$$

where we again used that $\psi_1^{\mathrm{av}} \prec \eta^{-1/2}$ trivially by (1.4.17).

Estimating (1.E.26). For the third term, we recall the (improved) estimates

$$c_+(\Phi_+) = \mathcal{O}\big(|e_1 - e_2| + \eta_1 + \eta_2\big)$$

$$c_-(\mathcal{X}_{12}[\mathring{A}_1]M_2) = \mathcal{O}\big(|e_1+e_2|+\eta_1+\eta_2\big)$$

on the anyway bounded prefactors, which have been shown in the course of estimating (1.5.51). By arguing analogously to (1.E.28) and (1.E.29), we also find

$$\Big|\big(G_1E_\sigma G_2\mathring{A}_2G_3 - M(w_1,E_\sigma,w_2,\mathring{A}_2,w_3)\big)_{\boldsymbol{xy}}\Big| \prec \frac{1}{\sqrt{N\eta^3}} + \frac{1}{|e_1-\sigma e_2|+\eta_2+\eta_3}\,\frac{\psi_1^{\mathrm{iso}}}{\sqrt{N\eta^2}}\,.$$

Now, combining these estimates, we conclude

$$\big|(1.E.26)\big| \prec \frac{1}{\sqrt{N\eta^3}}\big(1+\psi_1^{\mathrm{iso}}\big)\,. \tag{1.E.32}$$

Conclusion. Summarizing our investigations, we have shown that

$$\big(G_1\mathring{A}_1G_2\mathring{A}_2G_3 - M(w_1,\mathring{A}_1,w_2,\mathring{A}_2,w_3)\big)_{\boldsymbol{xy}} = -\big(\underline{G_1\mathring{A}_1'WG_2\mathring{A}_2G_3}\big)_{\boldsymbol{xy}} + \mathcal{O}_\prec\big(\mathcal{E}_2^{\mathrm{iso}}\big)\,,$$

where we used the shorthand notation

$$\mathring{A}_1' = \big(\mathcal{X}_{12}[A_1]M_2\big)^\circ + \sum_\sigma \frac{\mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[A_1]M_2)}{1-\mathbf{1}_\delta^\sigma\, c_\sigma(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2)}\big(\mathcal{X}_{12}[\mathring{\Phi}_\sigma]M_2\big)^\circ \tag{1.E.33}$$

in the underlined term. Combining (1.E.30), (1.E.31), and (1.E.32) with the usual single resolvent local laws (1.4.17) and the bounds on deterministic approximations in Lemma 1.4.3, we collected all the error terms from (1.E.23) in (1.5.67). □

Chapter 2

Gaussian fluctuations in the equipartition principle for Wigner matrices

This chapter contains the paper [152]:

G. Cipolloni, L. Erdős, J. Henheik, and O. Kolupaiev. Gaussian fluctuations in the equipartition principle for Wigner matrices. *Forum Math., Sigma*, 11, 2023

Abstract. The total energy of an eigenstate in a composite quantum system tends to be distributed equally among its constituents. We identify the quantum fluctuation around this equipartition principle in the simplest disordered quantum system consisting of linear combinations of Wigner matrices. As our main ingredient, we prove the Eigenstate Thermalisation Hypothesis and Gaussian fluctuation for general quadratic forms of the bulk eigenvectors of Wigner matrices with an arbitrary deformation.

Content

2.1 Introduction

The general principle of the equipartition of energy for a classical ergodic system asserts that in equilibrium the total energy is equally distributed among all elementary degrees of freedom. A similar principle for the kinetic energy has recently been verified for a general quantum system coupled to a heat bath [602], see also the previous works on the free Brownian particle and a dissipative harmonic oscillator in [70, 534, 71] and extensive literature therein. Motivated by E. Wigner's original vision to model any sufficiently complex quantum system by random matrices, a particularly strong *microcanonical* version of the equipartition principle for Wigner matrices was first formulated and proven in [45]. In its simplest form, consider a fully mean field random Hamilton operator H acting on the high dimensional quantum state space $\mathbf{C}^N$ that consists of the sum of two independent $N \times N$ Wigner matrices,

$$H = W_1 + W_2 \,,$$

as two constituents of the system. Recall that Wigner matrices $W = (w_{ij})$ are real or complex Hermitian random matrices with independent (up to the symmetry constraint $w_{ij} = \bar{w}_{ji}$), identically distributed entries. Let $\boldsymbol{u}$ be a normalised eigenvector of H with eigenvalue (energy) $\lambda = \langle \boldsymbol{u}, H\boldsymbol{u}\rangle$, then equipartition asserts that $\langle \boldsymbol{u}, W_l \boldsymbol{u}\rangle \approx \frac{1}{2}\lambda$ for $l = 1, 2$. In [45, Theorem 3.4] even a precise error bound was proven, i.e. that

$$\Big|\langle \boldsymbol{u}, W_l \boldsymbol{u}\rangle - \frac{1}{2}\lambda\Big| \le \frac{N^\epsilon}{\sqrt{N}} \,, \qquad l = 1, 2 \,, \tag{2.1.1}$$

holds with very high probability for any fixed $\epsilon > 0$; this estimate is optimal up the N^ϵ factor. The main result of the current paper is to identify the fluctuation in (2.1.1), more precisely we will show that

$$\sqrt{N}\Big[\langle \boldsymbol{u}, W_l \boldsymbol{u}\rangle - \frac{1}{2}\lambda\Big]$$

converges to a centred normal distribution as $N \to \infty$. We also compute its variance that turns out to be independent of the energy λ but depends on the symmetry class (real or complex). The result can easily be extended to the case when H is a more general linear combination of several independent Wigner matrices.

The estimate (2.1.1) is reminiscent to the recently proven *Eigenstate Thermalisation Hypothesis (ETH)*, also known as the *Quantum Unique Ergodicity (QUE)*,[1] for Wigner matrices in [165, Theorem 2.2] (see also [169, Theorem 2.6] for an improvement) which asserts that

$$\Big|\langle \boldsymbol{u}, A\boldsymbol{u}\rangle - \langle A\rangle\Big| \le \frac{N^\epsilon}{\sqrt{N}} \,, \qquad \langle A\rangle := \frac{1}{N}\mathrm{Tr}A \,, \tag{2.1.2}$$

holds for any bounded *deterministic* matrix A. In fact, even the Gaussian fluctuation of

$$\sqrt{N}\Big[\langle \boldsymbol{u}, A\boldsymbol{u}\rangle - \langle A\rangle\Big] \tag{2.1.3}$$

was proven in [167, Theorem 2.2] and [169, Theorem 2.8], see also [62] and the recent generalisation [61] to off-diagonal elements as well as joint Gaussianity of several quadratic forms. Earlier

[1] ETH for Wigner matrices was first conjectured by Deutsch [221]. Quantum ergodicity has a long history in the context of the quantisations of chaotic classical dynamical systems starting from the fundamental theorem by Šnirel'man [528]. For more background and related literature, see the Introduction of [165].

results on ETH [247, 369, 88, 63] and its fluctuations [112, 368, 552, 435] for Wigner matrices typically concerned rank one or finite rank observables A.

Despite their apparent similarities, the quadratic form in (2.1.1) is essentially different from that in (2.1.2) since W_l is strongly correlated with $\boldsymbol{u}$ while A in (2.1.2) is deterministic. This explains the difference in the two leading terms; note that $\frac{1}{2}\lambda$ in (2.1.1) is energy dependent and it is far from the value $\langle W_l\rangle \approx 0$, which might be erroneously guessed from (2.1.2). Still, the basic approach leading to ETH (2.1.2) is very useful to study $\langle \boldsymbol{u}, W_l\boldsymbol{u}\rangle$ as well. The basic idea is to condition on one of the Wigner matrices, say W_2, and consider $H = W_1 + W_2$ in the probability space of W_1 as a Wigner matrix with an additive *deterministic* deformation W_2. Assume that we can prove the generalisation of ETH (2.1.2) for such *deformed Wigner* matrices. In the space of W_1 this would result in a concentration of $\langle \boldsymbol{u}, W_2\boldsymbol{u}\rangle$ around some quantity $f(W_2)$ depending only on W_2; however, the answer will nontrivially depend on the deformation, i.e. it will not be simply $\langle W_2\rangle$. Once the correct form of $f(W_2)$ is established, we can find its concentration in the probability space of W_2, yielding the final answer.

To achieve these results we prove more general ETH and fluctuation results for eigenvector overlaps of deformed Wigner matrices of the general form $H = W + D$, where W is an $N \times N$ Wigner matrix and D is arbitrary, bounded deterministic matrix. The goal is to establish the concentration and the fluctuation of the quadratic form $\langle \boldsymbol{u}, A\boldsymbol{u}\rangle$ for a normalised eigenvector $\boldsymbol{u}$ of H with a bounded deterministic matrix A. We remark that for the special case of a rank one matrix $A = |\boldsymbol{q}\rangle\langle\boldsymbol{q}|$, ETH is equivalent to the *complete isotropic delocalisation* of the eigenvector $\boldsymbol{u}$, i.e. that $|\langle \boldsymbol{q}, \boldsymbol{u}\rangle| \le N^\epsilon/\sqrt{N}$ for any deterministic vector $\boldsymbol{q}$ with $\|\boldsymbol{q}\| = 1$. For a diagonal deformation D this has been achieved in [405, 395] and for the general deformation D in [243]. The normal fluctuation of $\langle \boldsymbol{u}, A\boldsymbol{u}\rangle$ for a finite rank A and diagonal D was obtained in [59].

It is well known that for very general mean-field type random matrices H their resolvent $G(z) = (H - z)^{-1}$ concentrates around a deterministic matrix $M = M(z)$; such results are called *local laws*, and we will recall them precisely in (2.4.1). Here M is a solution of the *matrix Dyson equation (MDE)*, which, in case of $H = W + D$, reads as

$$-\frac{1}{M(z)} = z - D + \langle M(z)\rangle\,. \tag{2.1.4}$$

Given M, it turns out that

$$\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle \approx \delta_{i,j}\frac{\langle \operatorname{Im} M(\lambda_i)A\rangle}{\langle \operatorname{Im} M(\lambda_i)\rangle}\,, \tag{2.1.5}$$

where λ_i is the eigenvalue corresponding to the normalised eigenvector $\boldsymbol{u}_i$, i.e. $H\boldsymbol{u}_i = \lambda_i\boldsymbol{u}_i$. Since the eigenvalues are *rigid*, i.e. they fluctuate only very little, the right hand side of (2.1.5) is essentially deterministic and in general it depends on the energy. Similarly to (2.1.3), we will also establish the Gaussian fluctuation around the approximation (2.1.5). For zero deformation, $D = 0$, the matrix M is constant and (2.1.5) recovers (2.1.2)–(2.1.3) as a special case. For simplicity, in the current paper we establish all these results only in the *bulk* of the spectrum, but similar results may be obtained at the *edge* and at the possible *cusp regime* of the spectrum as well; the details are deferred to later works.

We now comment on the new aspects of our methods. The proof of (2.1.5) relies on a basic observation about the local law for $H = W + D$. Its *average* form asserts that

$$\Big|\big\langle (G(z) - M(z))A\big\rangle\Big| \le \frac{N^\epsilon}{N\eta}\,, \qquad \eta := |\operatorname{Im} z| \gg \frac{1}{N}\,, \tag{2.1.6}$$

holds with very high probability and the error is essentially optimal for any bounded deterministic matrix A. However, there is a codimension one subspace of the matrices A for which the estimate improves to $N^\epsilon/(N\sqrt{\eta})$, gaining a $\sqrt{\eta}$ factor in the relevant small $\eta \ll 1$ regime. For Wigner

matrices $H = W$ without deformation, the *traceless* matrices A played this special role. The key idea behind the proof of ETH for Wigner matrices in [165] was to decompose any deterministic matrix as $A =: \langle A \rangle + \mathring{A}$ into its tracial and traceless parts and prove multi-resolvent generalisations of the local law (2.1.6) with an error term distinguishing whether the deterministic matrices are traceless or not. For example, for a typical A we have

$$\langle G(z) A G(z)^* A \rangle \sim \frac{1}{\eta}$$

but for the traceless part of A we have

$$\langle G(z) \mathring{A} G(z)^* \mathring{A} \rangle \sim 1\,, \qquad \eta \gg \frac{1}{N}\,,$$

with appropriately matching error terms. In general, each traceless A improves the typical estimate by a factor $\sqrt{\eta}$. ETH then follows from the spectral theorem,

$$\frac{1}{N} \sum_{i,j=1}^{N} \frac{\eta}{(\lambda_i - e)^2 + \eta^2} \frac{\eta}{(\lambda_j - e)^2 + \eta^2} |\langle \boldsymbol{u}_i, \mathring{A} \boldsymbol{u}_j \rangle|^2 = \langle \operatorname{Im} G(z) \mathring{A} \operatorname{Im} G(z) \mathring{A} \rangle \le N^{\epsilon}\,; \tag{2.1.7}$$

choosing $z = e + \mathrm{i}\eta$ appropriately with $\eta \sim N^{-1+\epsilon}$ we obtain that $|\langle \boldsymbol{u}_i, \mathring{A} \boldsymbol{u}_j \rangle|^2 \le N^{-1+3\epsilon}$ which even includes an off-diagonal version of (2.1.2).

To extend this argument to deformed Wigner matrices requires to identify the appropriate *singular* ("tracial") and *regular* ("traceless") parts of an arbitrary matrix. It turns out that the improved local laws around an energy $e = \operatorname{Re} z$ hold if A is orthogonal[2] to $\operatorname{Im} M(e)$, see (2.2.11) for the new definition of $\mathring{A}$, which denotes the regular part of A. In this theory the matrix $\operatorname{Im} M$ emerges as the critical eigenvector of a *linear stability operator* $\mathcal{B} = I - M\langle \cdot \rangle M^*$ related to the MDE (2.1.4). The major complication compared with the pure Wigner case in [165] is that now the regular part of a matrix becomes energy dependent. In particular, in a multi-resolvent chain $\langle G(z_1) A_1 G(z_2) A_2 \dots \rangle$ it is *a priori* unclear at which spectral parameters the matrices A_i should be regularised; it turns out that the correct regularisation depends on both z_i and z_{i+1}, see (2.4.10) later. A similar procedure was performed for the Hermitisation of non-Hermitian i.i.d. matrices with a general deformation in Chapter 1, see Appendix 1.A for a more conceptual presentation. Having identified the correct regularisation, we derive a system of *master inequalities* for the error terms in multi-resolvent local laws for regular observables; a similar strategy (with minor modifications) have been used in [168, 169] for Wigner matrices and in Chapter 1 for i.i.d. matrices. To keep the presentation short, here we will not aim at the most general local laws with optimal errors unlike in [168, 169]. Although these would be achievable with our methods, here we prove only what is needed for our main results on the equipartition.

The proof of the fluctuation around the ETH uses *Dyson Brownian motion (DBM)* techniques, namely the *Stochastic Eigenstate Equation* for quadratic forms of eigenvectors. This theory has been gradually developed for Wigner matrices in [112, 115, 435], we closely follow the presentation in [167, 169]. The extension of this technique to deformed Wigner matrices is fairly straightforward, so our presentation will be brief. The necessary inputs for this DBM analysis follow from the multi-resolvent local laws that we prove for deformed Wigner matrices.

In a closing remark we mention that the original proof of (2.1.1) in [45] was considerably simpler than that of (2.1.2). This may appear quite surprising due to the complicated correlation between $\boldsymbol{u}$ and $\mathcal{W}$, but a special algebraic cancellation greatly helped in [45]. Namely, with the notation $\mathcal{W} := W_1 - W_2$ and $G(z) := (H - z)^{-1}$, $z = e + \mathrm{i}\eta \in \mathbf{C}_+$, a relatively straightforward cumulant expansion showed that $\langle \operatorname{Im} G(z) \mathcal{W} \operatorname{Im} G(z) \mathcal{W} \rangle$ is essentially bounded[3] even for spectral parameters z very

[2] The space of matrices is equipped with the usual Hilbert-Schmidt scalar product.

[3] This means up to an N^{ϵ} factor with arbitrary small ϵ.

close to the real axis, $\eta \geq N^{-1+\epsilon/2}$. Within this cumulant expansion an algebraic cancellation emerged due to the special form of $\mathcal{W}$. Then, exactly as in (2.1.7) we obtain $|\langle \boldsymbol{u}, \mathcal{W}\boldsymbol{u}\rangle|^2 \lesssim N^{1+\epsilon}\eta^2 = N^{-1+2\epsilon}$. In particular, it shows that $\langle \boldsymbol{u}, \mathcal{W}\boldsymbol{u}\rangle = \langle \boldsymbol{u}, W_1\boldsymbol{u}\rangle - \langle \boldsymbol{u}, W_2\boldsymbol{u}\rangle$ is essentially of order $N^{\epsilon}/\sqrt{N}$ for every eigenvector of H. Since $\langle \boldsymbol{u}, W_1\boldsymbol{u}\rangle + \langle \boldsymbol{u}, W_2\boldsymbol{u}\rangle = \langle \boldsymbol{u}, H\boldsymbol{u}\rangle = \lambda$, we immediately obtain the equipartition (2.1.1). Similar idea proved the more general case, see (2.2.4) later. Note, however, that this trick does not help in establishing the fluctuations of $\langle \boldsymbol{u}, W_l\boldsymbol{u}\rangle$. In fact, the full ETH analysis for deformed Wigner matrices needs to be performed to establish the necessary *a priori* bounds for the Dyson Brownian motion arguments.

Notations and conventions

For positive quantities f, g we write $f \lesssim g$ and $f \sim g$ if $f \leq Cg$ or $cg \leq f \leq Cg$, respectively, for some constants $c, C > 0$ which depend only on the constants appearing in the moment condition, see (2.2.1) later. For any natural number n we set $[n] := \{1, 2, \ldots, n\}$.

We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, for some $N \in \mathbf{N}$. Vector and matrix norms, $\|\boldsymbol{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $N \times N$ matrix A we use the notation $\langle A\rangle := N^{-1}\mathrm{Tr}A$ to denote the normalised trace of A. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and matrices $A \in \mathbf{C}^{N\times N}$ we define the scalar product

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \sum_{i=1}^{N} \overline{x}_i y_i \,.$$

Finally, we will use the concept of "with very high probability" *(w.v.h.p.)* meaning that for any fixed $D > 0$ the probability of an N-dependent event is bigger than $1 - N^{-D}$ for $N \geq N_0(D)$. We introduce the notion of *stochastic domination* (see e.g. [241]): given two families of non-negative random variables

$$X = \left(X^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right)$$

indexed by N (and possibly some parameter u in some parameter space $U^{(N)}$), we say that X is stochastically dominated by Y, if for all $\xi, D > 0$ we have

$$\sup_{u\in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\xi}Y^{(N)}(u)\right] \leq N^{-D} \tag{2.1.8}$$

for large enough $N \geq N_0(\xi, D)$. In this case we use the notation $X \prec Y$ or $X = \mathcal{O}_{\prec}(|Y|)$. We also use the convention that $\xi > 0$ denotes an arbitrary small exponent which is independent of N.

Acknowledgement

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2.2 Main results

We consider $N \times N$ real symmetric or complex Hermitian Wigner matrices $W = W^*$ having single-entry distributions $w_{ab} = N^{-1/2}\chi_{\mathrm{od}}$, for $a > b$, and $w_{aa} = N^{-1/2}\chi_d$, where χ_{od} and χ_{od} are two independent random variables satisfying the following assumptions:

Assumption 2.2.1. *We assume that χ_{d} is a real centred random variable, that χ_{od} is a real or complex random variable such that $\mathbf{E}\,\chi_{\mathrm{od}} = 0$ and $\mathbf{E}\,|\chi_{\mathrm{od}}|^2 = 1$; additionally in the complex case we also assume that $\mathbf{E}\,\chi_{\mathrm{od}}^2 = 0$. Customarily, we use the parameter $\beta = 1, 2$ to indicate the real*

or complex case, respectively. Furthermore, we assume that all the moments of χ_{od} and χ_{d} exist, i.e. for any $p \in \mathbf{N}$ there exists a constant $C_p > 0$ such that

$$\mathbf{E}\,|\chi_{\mathrm{od}}|^p + \mathbf{E}\,|\chi_{\mathrm{d}}|^p \leq C_p. \tag{2.2.1}$$

For definiteness, in the sequel we perform the entire analysis for the complex case; the real case being completely analogous and hence omitted.

The equipartition principle concerns linear combinations of Wigner matrices. Fix $k \in \mathbf{N}$ and consider

$$H := p_1 W_1 + \cdots + p_k W_k, \tag{2.2.2}$$

for some fixed N-independent vector $\boldsymbol{p} = (p_1, ..., p_k) \in \mathbf{R}^k$ of weights and for k independent $N \times N$ Wigner matrices W_l, belonging to the *same* symmetry class (i.e. the off-diagonal random variables χ_{od} are either real or complex for each of the W_l, $l \in [k]$). Then, denoting by $\{\lambda_i\}_{i\in[N]}$ the eigenvalues of H, arranged in increasing order, with associated normalised eigenvectors $\{\boldsymbol{u}_i\}_{i\in[N]}$, the total energy $\langle \boldsymbol{u}_i, H\boldsymbol{u}_i\rangle = \lambda_i$ of the composite system (2.2.2) is *proportionally* distributed among the k constituents, i.e.

$$\langle \boldsymbol{u}_i, p_l W_l\, \boldsymbol{u}_i\rangle \approx \frac{p_l^2}{\|\boldsymbol{p}\|^2}\lambda_i \tag{2.2.3}$$

for every $l \in [k]$, where $\|\boldsymbol{p}\| := \big(\sum_{l=1}^k |p_l|^2\big)^{1/2}$ denotes the usual ℓ^2-norm. This phenomenon, known as *equipartition*, was first proven in [45, Theorem 3.4], with an optimal error estimate:

$$\left|\langle \boldsymbol{u}_i,\, p_l W_l \boldsymbol{u}_j\rangle - \delta_{i,j}\frac{p_l^2}{\|\boldsymbol{p}\|^2}\lambda_i\right| \prec \frac{1}{\sqrt{N}}\,. \tag{2.2.4}$$

Our main result is the corresponding Central Limit Theorem to (2.2.4) for $i = j$, i.e. the proof of Gaussian fluctuations in Equipartition for Wigner matrices – for energies in the *bulk* of the spectrum of H.

Theorem 2.2.2 (Gaussian Fluctuations in Equipartition)**.**
Fix $k \in \mathbf{N}$. Let $W_1, \ldots, W_k$ be independent Wigner matrices satisfying Assumption 2.2.1, all of which being in the same real ($\beta = 1$) or complex ($\beta = 2$) symmetry class, and $\boldsymbol{p} = (p_1, \ldots, p_k) \in \mathbf{R}^k$ be N-independent. Define H as in (2.2.2) *and denote by $\{\lambda_i\}_{i\in[N]}$ the eigenvalues of H, arranged in increasing order, with associated normalised eigenvectors $\{\boldsymbol{u}_i\}_{i\in[N]}$. Then, for fixed $\kappa > 0$, every $l \in [k]$ and for every bulk index $i \in [\kappa N, (1-\kappa)N]$ it holds that*

$$\sqrt{\frac{\beta N}{2}\,\frac{\|\boldsymbol{p}\|^2}{p_l^2\left(\|\boldsymbol{p}\|^2 - p_l^2\right)}}\left[\langle \boldsymbol{u}_i,\, p_l W_l \boldsymbol{u}_i\rangle - \frac{p_l^2}{\|\boldsymbol{p}\|^2}\lambda_i\right] \Longrightarrow \mathcal{N}(0,1) \tag{2.2.5}$$

in the sense of moments,[4] *where $\mathcal{N}(0,1)$ denotes a real standard Gaussian.*

By polarisation we will also obtain the following:

Corollary 2.2.3. *Under the assumptions from Theorem 2.2.2, the random vector $\boldsymbol{X} = (X_1, ..., X_k) \in \mathbf{R}^k$ with*

$$X_l := \sqrt{\frac{\beta N}{2}}\left[\langle \boldsymbol{u}_i,\, p_l W_l \boldsymbol{u}_i\rangle - \frac{p_l^2}{\|\boldsymbol{p}\|^2}\lambda_i\right], \quad l \in [k]\,, \tag{2.2.6}$$

is approximately (in the sense of moments) jointly Gaussian with covariance structure

$$\mathrm{Cov}(X_l, X_m) = \frac{p_l^2\big(\delta_{l,m}\|\boldsymbol{p}\|^2 - p_m^2\big)}{\|\boldsymbol{p}\|^2}\,.$$

[4] Given a sequence of N-dependent random variables, we say that X_N converges to X_∞ in the sense of moments if for any $k \in \mathbf{N}$ it holds $\mathbf{E}\,|X_N|^k = \mathbf{E}\,|X_\infty|^k + \mathcal{O}(N^{-c(k)})$, for some small possibly k–dependent constant $c(k) > 0$.

Remark 2.2.4. *We stated Theorem 2.2.2 only for diagonal overlaps for simplicity. However, one can see that following the proof in [61, Section 3] it is possible to obtain an analogous Central Limit Theorem (CLT) for off–diagonal overlaps as well:*

$$\sqrt{\frac{\|\boldsymbol{p}\|^2\beta N}{p_l^2\left(\|\boldsymbol{p}\|^2-p_l^2\right)}}\,\left|\langle \boldsymbol{u}_i,\, p_l W_l \boldsymbol{u}_j\rangle\right| \Longrightarrow \left|\mathcal{N}(0,1)\right|. \tag{2.2.7}$$

This also gives an analogous version of (2.2.3) *for off-diagonal overlaps. Furthermore, again following [61, Theorem 2.2], it is also possible to derive a multivariate CLT jointly for diagonal and off–diagonal overlaps. See also Remark 2.2.10 below for further explanation.*

Theorem 2.2.2 and Corollary 2.2.3 will follow as a corollary to the Eigenstate Thermalisation Hypothesis (ETH) and its Gaussian fluctuations for *deformed* Wigner matrices, which we present as Theorem 2.2.7 and Theorem 2.2.9 in the following subsection.

Remark 2.2.5. *By a quick inspection of our proof of Theorem 2.2.2, given in Section 2.3, it is possible to generalise the Equipartition principle* (2.2.4) *as well as its Gaussian fluctuations* (2.2.5) *to linear combinations of* deformed Wigner matrices*, i.e. each W_l in* (2.2.2) *being replaced by $W_l + D_l$, where $D_l = D_l^*$ is an essentially arbitrary bounded deterministic matrix (see Assumption 2.2.8 later). However, for brevity of the current paper, we refrain from presenting this extension explicitly.*

2.2.1 ETH and its fluctuations for deformed Wigner matrices

In this section, we consider *deformed Wigner matrices*, $H = W + D$, with increasingly ordered eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ and corresponding orthonormal eigenvectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$. Here, $D = D^* \in \mathbf{C}^{N\times N}$ is a self-adjoint matrix with uniformly bounded norm, i.e. $\|D\| \le C_D$ for some N-independent constant $C_D > 0$. While the *Eigenstate Thermalisation Hypothesis (ETH)* will be shown to hold for general deformations D, we shall require slightly stronger assumptions for proving the Gaussian fluctuations (see Assumption 2.2.8 below).

In order to state our results on the ETH and its fluctuations (Theorems 2.2.7 and 2.2.9, respectively), we need to introduce the concept of *regular observables*, first in a simple form in Definition 2.2.11 (later along the proofs we will need a more general version in Definition 2.4.2). For this purpose we introduce $M(z)$ being the unique solution of the Matrix Dyson Equation (MDE):[5]

$$-\frac{1}{M(z)} = z - D + \langle M(z)\rangle, \qquad \operatorname{Im} M(z)\operatorname{Im} z > 0. \tag{2.2.8}$$

The *self consistent density of states (scDos)* is then defined as

$$\rho(e) := \frac{1}{\pi}\lim_{\eta\downarrow 0}\langle \operatorname{Im} M(e+\mathrm{i}\eta)\rangle\,. \tag{2.2.9}$$

We point out that not only $\langle \operatorname{Im} M(e+\mathrm{i}\eta)\rangle$ has an extension to the real axis, but the whole matrix $M(e) := \lim_{\eta\downarrow 0} M(e+\mathrm{i}\eta)$ is well defined (see Lemma 1.B.1 (b)). The scDos ρ is a compactly supported Hölder-$1/3$ continuous function on $\mathbf{R}$ which is real-analytic on the set $\{\rho > 0\}$[6]. Moreover, for any small $\kappa > 0$ (independent of N) we define the *κ-bulk* of the scDos as

$$\mathbf{B}_\kappa = \left\{x \in \mathbf{R} \,:\, \rho(x) \ge \kappa^{1/3}\right\}, \tag{2.2.10}$$

[5] The MDE for very general mean field random matrices has been introduced in [17] and further analysed in [22]. The properties we use here have been summarised in Appendix 1.B.

[6] The scDos has been thoroughly analysed in increasing generality in [16, 17, 22]. It is supported on finitely many finite intervals. Roughly speaking there are three regimes: the *bulk*, where ρ is well separated away from 0, the *edge* where ρ vanishes as a square root at the edges of each supporting interval that are well separated, and the *cusp* where two supporting intervals (almost) meet and ρ behaves (almost) as a cubic root. Correspondingly, ρ is locally real analytic, Hölder-$1/2$, or Hölder-$1/3$ continuous, respectively. Near the singularities, it has an approximately universal shape. No other singularity type can occur and for typical deformation D there is no cusp regime.

which is a finite union of disjoint compact intervals, see Lemma 1.B.2. For $\mathrm{Re}\, z \in \mathbf{B}_\kappa$ it holds that $\|M(z)\| \lesssim 1$, as easily follows by taking the imaginary part of (2.2.8).

Definition 2.2.6 (Regular observables – One-point regularisation)**.** *Fix $\kappa > 0$ and an energy $e \in \mathbf{B}_\kappa$ in the bulk. Given a matrix $A \in \mathbf{C}^{N\times N}$, we define its* one-point regularisation w.r.t. the energy e, *denoted by $\mathring{A}^e$, as*

$$\mathring{A} = \mathring{A}^e := A - \frac{\langle \mathrm{Im}\, M(e) A\rangle}{\langle \mathrm{Im}\, M(e)\rangle}\,. \tag{2.2.11}$$

Moreover, we call A regular w.r.t. the energy e, *if and only if $A = \mathring{A}^e$.*

Notice that in the analysis of Wigner matrices without deformation, $D = 0$, in [165, 170, 168, 169], M was a constant matrix and the regular observables were simply given by *traceless* matrices, i.e. $\mathring{A} = A - \langle A\rangle$. For deformed Wigner matrices the concept of regular observables depends on the energy.

Next, we define the *quantiles* γ_i of the density ρ implicitly by

$$\int_{-\infty}^{\gamma_i} \rho(x)\,\mathrm{d}x = \frac{i}{N}, \qquad i \in [N]. \tag{2.2.12}$$

We can now formulate the ETH in the bulk for deformed Wigner matrices which generalises the same result for Wigner matrices, $D = 0$, from [165].

Theorem 2.2.7 (Eigenstate Thermalisation Hypothesis)**.** *Let $\kappa > 0$ be an N-independent constant and fix a bounded deterministic $D = D^* \in \mathbf{C}^{N\times N}$. Let $H = W + D$ be a deformed Wigner matrix, where W satisfies Assumption 2.2.1, and denote the orthonormal eigenvectors of H by $\{\boldsymbol{u}_i\}_{i\in[N]}$. Then, for any deterministic $A \in \mathbf{C}^{N\times N}$ with $\|A\| \lesssim 1$, it holds that*

$$\max_{i,j} \left|\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i} \boldsymbol{u}_j\rangle\right| = \max_{i,j} \left|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\frac{\langle A\mathrm{Im}\, M(\gamma_i)\rangle}{\langle \mathrm{Im}\, M(\gamma_i)\rangle}\right| \prec \frac{1}{\sqrt{N}}\,, \tag{2.2.13}$$

where the maximum is taken over all $i, j \in [N]$ such that the quantiles $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$ defined in (2.2.12) *are in the κ-bulk of the scDos ρ.*

This "Law of Large Numbers"-type result (2.2.13) is complemented by the corresponding Central Limit Theorem (2.2.14), which requires slightly strengthened assumptions on the deformation D.

Assumption 2.2.8. *We assume that $D \in \mathbf{C}^{N\times N}$ is a bounded self-adjoint deterministic matrix such that*

(i) the unique solution $M(z)$ to (2.2.8) *is uniformly bounded in norm, i.e. $\sup_{z\in\mathbf{C}} \|M(z)\| \le C_M$ for some N-independent constant $C_M > 0$;*

(ii) the scDos ρ is Hölder-$1/2$ regular, i.e. it does not have any cusps (see Footnote 6).

The requirements on D in Assumption 2.2.8 are natural and they hold for typical applications, see Remark 2.2.12 later for more details. We can now formulate our result on the Gaussian fluctuations in the ETH which generalises the analogous result for Wigner matrices, $D = 0$ from [167].

Theorem 2.2.9 (Fluctuations in ETH)**.** *Fix $\kappa, \sigma > 0$ N-independent constants and let $H = W + D$ be a deformed Wigner matrix, where W satisfies Assumption 2.2.1 and D satisfies Assumption 2.2.8. Denote the orthonormal eigenvectors of H by $\{\boldsymbol{u}_i\}_{i\in[N]}$ and fix an index $i \in [N]$, such that the quantile $\gamma_i \in \mathbf{B}_\kappa$ defined in* (2.2.12) *is in the bulk. Then, for any deterministic Hermitian matrix*

$A \in \mathbf{C}^{N\times N}$ with $\|A\| \lesssim 1$ (which we assume to be real in the case of a real Wigner matrix) satisfying $\langle (A - \langle A\rangle)^2\rangle \geq \sigma$, it holds that

$$\sqrt{\frac{\beta N}{2\operatorname{Var}_{\gamma_i}(A)}}\left[\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle - \frac{\langle A\operatorname{Im} M(\gamma_i)\rangle}{\langle \operatorname{Im} M(\gamma_i)\rangle}\right] \Longrightarrow \mathcal{N}(0,1) \tag{2.2.14}$$

in the sense of moments (see Footnote 4), where[7]

$$\operatorname{Var}_{\gamma_i}(A) := \frac{1}{\langle \operatorname{Im} M(\gamma_i)\rangle^2}\left(\left\langle \left(\mathring{A}^{\gamma_i}\operatorname{Im} M(\gamma_i)\right)^2\right\rangle - \frac{1}{2}\operatorname{Re}\left[\frac{\left\langle \left(M(\gamma_i)\right)^2 \mathring{A}^{\gamma_i}\right\rangle^2}{1-\left\langle \left(M(\gamma_i)\right)^2\right\rangle}\right]\right). \tag{2.2.15}$$

This variance is strictly positive with an effective lower bound

$$\operatorname{Var}_{\gamma_i}(A) \geq c\langle (A - \langle A\rangle)^2\rangle \tag{2.2.16}$$

for some constant $c = c(\kappa, \|D\|) > 0$.

Remark 2.2.10. *We stated Theorem 2.2.9 only for diagonal overlaps to keep the statement simple, but a corresponding CLT for off–diagonal overlaps as well as a multivariate CLT for any finite family of diagonal and off–diagonal overlaps can also be proven.*

We decided not to give a detailed proof of these facts in the current paper in order to present the main new ideas in the analysis of deformed Wigner matrices in the simplest possible setting consisting of only diagonal overlaps. But we remark that following an analysis similar to [61, Section 3], combined with the details presented in Section 2.5, would give an analogous result to [61, Theorem 2.2] also in the deformed Wigner matrices setup. However, this would require introducing several new notations that would obfuscate the main novelties in the analysis of deformed Wigner matrices compared to the Wigner case, which instead are clearer in the simpler setup of Section 2.5.

In the following two remarks we comment on the condition $\langle (A-\langle A\rangle)^2\rangle \geq \sigma$ and on Assumption 2.2.8.

Remark 2.2.11. *The restriction to matrices satisfying $\langle (A - \langle A\rangle)^2\rangle \geq \sigma$, i.e. $A - \langle A\rangle$ being of high rank, is technical. It is due to the fact that our multi-resolvent local laws for resolvent chains $\langle G(z_1)A_1G(z_2)A_2\ldots\rangle$ in Proposition 2.4.4 are non-optimal in terms of the norm for the matrices A_i; they involve the Euclidean norm $\|A_i\|$ and not the smaller Hilbert-Schmidt norm $\sqrt{\langle |A_i|^2\rangle}$ which would be optimal. For the Wigner ensemble, this subtlety is the main difference between the main result in [167] for high rank observable matrices A and its extension to any low rank A in [169]. Following the technique in [169] it would be possible to achieve the estimate with the optimal norm of A also for deformed Wigner matrices. However, we refrain from doing so, since in our main application, Theorem 2.2.2, A itself will be a Wigner matrix which has high rank.*

Remark 2.2.12. *We have several comments on Assumption 2.2.8.*

(i) The boundedness of $\|M(z)\|$ is automatically fulfilled in the bulk $\mathbf{B}_\kappa$ (see remark below (2.2.10)) or when $\operatorname{Re} z$ away from the support of the scDos ρ (see [22, Proposition 3.5]) without any further condition. However, the uniform (in z) estimate formulated in Assumption 2.2.8 does not hold for arbitrary D. A sufficient condition for the boundedness of $\|M\|$ in terms of the spectrum of D is given in [22, Lemma 9.1 (i)]. This especially applies if the eigenvalues $\{d_i\}_{i\in[N]}$ of D (in increasing order) form a piecewise Hölder-$1/2$ regular sequence,[8] *see [22, Lemma 9.3]. In particular, by eigenvalue rigidity [17, 243], it is easy to see that any "Wigner-like" matrix D has Hölder-$1/2$ regular sequence of eigenvalues with very high probability. This is important for the applicability of Theorem 2.2.9 below in the proof of our main result, Theorem 2.2.2, given in Section 2.3.*

[7] See the first paragraph of Section 2.5 for an explanation of why the variance takes this specific form.

[8] In this context, Hölder-$1/2$ regularity means that $|d_i - d_j| \leq C_0(|i-j|/N)^{1/2}$ for some universal constant $C_0 > 0$.

(ii) The assumption that ρ does not have any cusps is a typical condition and of technical nature (needed in the local law (2.4.1) and in Lemma 2.A.2). In case that the sequence of matrices $D = D_N$ has a limiting density of states with single interval support, then also ρ, the scDos of $W + D$, has single interval support [81], in particular, ρ has no cusps [22]. Again, this is important for the applicability of Theorem 2.2.9 in the proof of our main result, in which case D is a Wigner matrix with a semicircle as the limiting density of states.

In the following Section 2.3, we will prove our main result, Theorem 2.2.2, assuming Theorems 2.2.7 and 2.2.9 on deformed Wigner matrices as inputs. These will be proven in Sections 2.4 and 2.5, respectively. Both proofs crucially rely on an averaged local law for two resolvents and two *regular* observables, Proposition 2.4.4, which we prove in Section 2.6. Several additional technical and auxiliary results are deferred to the Appendix.

2.3 Fluctuations in Equipartition: Proof of Theorem 2.2.2

It is sufficient to prove Theorem 2.2.2 only for $k = 2$ with $p_1, p_2 \neq 0$, since we can view the sum (2.2.2) as the sum of $p_1 W_1$ and

$$\sum_{l=2}^{k} p_l W_l \stackrel{\mathrm{d}}{=} \left(\sum_{l=2}^{k} p_l^2 \right)^{1/2} \widetilde{W} \,,$$

where $\widetilde{W}$ is a Wigner matrix independent of W_1 and the equality is understood in distribution.

As a main step, we shall prove the following lemma, where we condition on W_2.

Lemma 2.3.1. *Under the assumptions of Theorem 2.2.2 with $k = 2$ it holds that*

$$\mathbf{E}_{W_1} \langle \boldsymbol{u}_i, p_2 W_2 \boldsymbol{u}_i \rangle = \frac{p_2^2}{\|\boldsymbol{p}\|^2} \gamma_i + \mathcal{O}_{\prec}\left(N^{-1/2-\epsilon} \right) , \tag{2.3.1}$$

$$\frac{\beta N}{2} \mathrm{Var}_{W_1}[\langle \boldsymbol{u}_i, p_2 W_2 \boldsymbol{u}_i \rangle] = \frac{p_1^2 p_2^2}{\|\boldsymbol{p}\|^2} + \mathcal{O}_{\prec}\left(N^{-\epsilon} \right) , \tag{2.3.2}$$

for any $\epsilon > 0$, where γ_i is the i^{th} quantile of the semicircular density with radius $2\|\boldsymbol{p}\|$, i.e.

$$\frac{1}{2\pi \|\boldsymbol{p}\|^2} \int_{-\infty}^{\gamma_i} \sqrt{[4\|\boldsymbol{p}\|^2 - x]_+} \mathrm{d}x = \frac{i}{N} \,.$$

Expectation and variance are taken in the probability space of W_1, conditioned on W_2 being in an event of very high probability, while the stochastic domination in the error terms are understood in the probability space of W_2.

Proof of Theorem 2.2.2. First, we note that all requirements for applying Theorem 2.2.9 to $H = p_1 W_1 + D$, with $D = p_2 W_2$ for some fixed realisation of W_2 in a very high probability event, are satisfied. This follows from Remark 2.2.12 and $\langle (W_2 - \langle W_2 \rangle)^2 \rangle \gtrsim 1$ with very high probability. Next, observe that replacing γ_i in (2.3.1) by the eigenvalue λ_i appearing in (2.2.5) is trivial by the usual eigenvalue rigidity $|\gamma_i - \lambda_i| \prec 1/N$ for Wigner matrices in the bulk [248]. Thus, Theorem 2.2.9 shows that, conditioned on a fixed realisation of W_2,

$$\sqrt{\frac{\beta N}{2}} \left[\langle \boldsymbol{u}_i, p_2 W_2 \boldsymbol{u}_i \rangle - \frac{p_2^2}{\|\boldsymbol{p}\|^2} \lambda_i \right] \tag{2.3.3}$$

is approximately Gaussian with an approximately constant variance (independent of W_2) given in (2.3.2). Since this holds with very high probability w.r.t. W_2, this proves (2.2.5) for $l = 2$; the proof for $l = 1$ is the same. □

Proof of Corollary 2.2.3. We formulated Theorem 2.2.9 as a CLT for overlaps $\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle$ for a single deterministic matrix A, but by standard polarisation it also shows the joint approximate Gaussianity of any p-vector

$$\big(\langle \boldsymbol{u}_i, A_1\boldsymbol{u}_i\rangle, \langle \boldsymbol{u}_i, A_2\boldsymbol{u}_i\rangle, \dots, \langle \boldsymbol{u}_i, A_p\boldsymbol{u}_i\rangle\big) \tag{2.3.4}$$

for any fixed k and deterministic observables $A_1, A_2, \dots A_p$ satisfying $\langle (A_j - \langle A_j\rangle)^2\rangle \geq c$, $j \in [p]$. Namely, using Theorem 2.2.9 to compute the moments of $\langle \boldsymbol{u}_i, A(\boldsymbol{t})\boldsymbol{u}_i\rangle$ for the linear combination $A(\boldsymbol{t}) = \sum_j t_j A_j$ with any real vector $\boldsymbol{t} = (t_1, t_2, \dots, t_p)$, we can identify any joint moments of the coordinates of the vector in (2.3.4) and we find that they satisfy the (approximate) Wick theorem.

Now we can follow the above proof of Theorem 2.2.2, but without the simplification $k = 2$. Conditioning on $W_2, \dots, W_k$ and working in the probability space of W_1, by the polarisation argument above we find that not only each X_l from (2.2.6) is asymptotically Gaussian with a variance independent of $W_2, \dots, W_k$, but they are jointly Gaussian for $l = 2, 3, \dots, k$. This is sufficient for the joint Gaussianity of the entire vector $\boldsymbol{X}$ since $\sum_l X_l = 0$. This completes the proof of Corollary 2.2.3. □

The proof of Lemma 2.3.1 is divided into the computation of the expectation (2.3.1) and the variance (2.3.2).

2.3.1 Computation of the expectation (2.3.1)

As in the proof of Theorem 2.2.2 above, we condition on W_2 and work in the probability space of W_1 i.e. we consider p_2W_2 as a deterministic deformation of p_1W_1. This allows us to use Theorem 2.2.9 as[9]

$$\mathbf{E}_{W_1}\langle \boldsymbol{u}_i, p_2W_2\boldsymbol{u}_i\rangle = \frac{\langle p_2W_2 \operatorname{Im} M_2(\gamma_{i,2}))\rangle}{\langle \operatorname{Im} M_2(\gamma_{i,2})\rangle} + \mathcal{O}_{\prec}\left(N^{-1/2-\epsilon}\right) \tag{2.3.5}$$

for some constant $\epsilon > 0$. Here $M_2(z)$, depending on W_2, is the unique solution of the MDE

$$-\frac{1}{M_2(z)} = z - p_2W_2 + p_1^2\langle M_2(z)\rangle\,, \tag{2.3.6}$$

corresponding to the matrix $p_1W_1 + p_2W_2$, where p_2W_2 is considered a deformation, and $\gamma_{i,2}$ is the i^{th} quantile of the scDos ρ_2 corresponding to (2.3.6). The subscript '2' for M_2, ρ_2 and $\gamma_{i,2}$ in (2.3.5) and (2.3.6) indicates that these objects are dependent on W_2 and hence random.

The Stieltjes transform $m_2(z)$ of ρ_2 is given by the implicit equation

$$m_2(z) := \langle M_2(z)\rangle = \frac{1}{p_2}\cdot\left\langle \frac{1}{W_2 - \frac{1}{p_2}(z + p_1^2 m_2(z))}\right\rangle$$

with the usual side condition $\operatorname{Im} z \cdot \operatorname{Im} m_2(z) > 0$. Applying the standard local law for the resolvent of W_2 on the right hand side shows that

$$\left|m_2(z) - \frac{1}{p_2}m_{\text{sc}}(w_2)\right| \prec \frac{1}{N|\operatorname{Im} w_2|}, \qquad w_2 := \frac{1}{p_2}(z + p_1^2 m_2(z)). \tag{2.3.7}$$

where m_{sc} is the Stieltjes transform of the standard semicircle law, i.e. it satisfies the quadratic equation

$$m_{\text{sc}}(w)^2 + w m_{\text{sc}}(w) + 1 = 0 \tag{2.3.8}$$

with the side condition $\operatorname{Im} w \cdot \operatorname{Im} m_{\text{sc}}(w) > 0$. Note that in (2.3.7) w_2 is random, it depends on W_2, but the local law for $\langle (W_2 - w)^{-1}\rangle$ holds uniformly in the spectral parameter $|\operatorname{Im} w| \geq N^{-1}$, hence a

[9] Note that Theorem 2.2.7 alone would prove (2.3.5) only with $\epsilon = 0$, but the convergence in the sense of moments from Theorem 2.2.9 gains a factor $N^{-\epsilon}$ with a positive ϵ.

standard grid argument and the Lipschitz continuity of the resolvents shows that it holds for any (random) w with $|\mathrm{Im}\, w| \geq N^{-1+\xi}$ with any fixed $\xi > 0$.

Applying (2.3.8) at $w = w_2$ together with (2.3.7) implies that

$$-\frac{1}{\|\boldsymbol{p}\|\, m_2(z)} = \frac{z}{\|\boldsymbol{p}\|} + \|\boldsymbol{p}\|\, m_2(z) + \mathcal{O}_{\prec}\Big(\frac{1}{N|\mathrm{Im}\, w_2|}\Big). \tag{2.3.9}$$

We view this relation as a small additive perturbation of the exact equation

$$-\frac{1}{m_{\mathrm{sc}}\big(\frac{z}{\|\boldsymbol{p}\|}\big)} = \frac{z}{\|\boldsymbol{p}\|} + m_{\mathrm{sc}}\big(\tfrac{z}{\|\boldsymbol{p}\|}\big) \tag{2.3.10}$$

to conclude

$$\Big|\|\boldsymbol{p}\|\, m_2(z) - m_{\mathrm{sc}}\big(\tfrac{z}{\|\boldsymbol{p}\|}\big)\Big| \prec \frac{1}{N|\mathrm{Im}\, z|}\,, \qquad |\mathrm{Im}\, z| \geq N^{-1+\xi}\,, \tag{2.3.11}$$

using that $|\mathrm{Im}\, w_2| \gtrsim |\mathrm{Im}\, z|$ from the definition of w_2 in (2.3.7) and that $\mathrm{Im}\, z \cdot \mathrm{Im}\, m_2(z) > 0$. The conclusion (2.3.11) requires a standard continuity argument, starting from a z with a large imaginary part and continuously reducing the imaginary part by keeping the real part fixed (the same argument is routinely used in the proof of the local law for Wigner matrices, see, e.g., [248]).

The estimate (2.3.11) implies that the quantiles of ρ_2 satisfy the usual rigidity estimate, i.e.

$$|\gamma_{2,i} - \gamma_i| \prec \frac{1}{N} \tag{2.3.12}$$

for bulk indices $i \in [\kappa N, (1-\kappa)N]$ with any N-independent $\kappa > 0$. Moreover, (2.3.11) also implies that for any z in the bulk of the semicircle, i.e. $|\mathrm{Im}\, m_{sc}(z)| \geq c > 0$ for some $c > 0$, independent of N, we have $|\mathrm{Im}\, m_2(z)| \geq c/2$ as long as $|\mathrm{Im}\, z| \geq N^{-1+\xi}$. Using the definition of w_2 in (2.3.7) again, this shows $|\mathrm{Im}\, w_2| \sim |\mathrm{Im}\, w|$ for the deterministic $w := \frac{1}{p_2}(z + p_1^2 m_{\mathrm{sc}}(z))$ for any z with $|\mathrm{Im}\, z| \geq N^{-1+\xi}$. Feeding this information into (2.3.9) and viewing it again as a perturbation of (2.3.10) but with the improved deterministic bound $\mathcal{O}_{\prec}(1/(N|\mathrm{Im}\, w|))$, we obtain

$$\Big|\|\boldsymbol{p}\|\, m_2(z) - m_{\mathrm{sc}}\big(\tfrac{z}{\|\boldsymbol{p}\|}\big)\Big| \prec \frac{1}{N|\mathrm{Im}\, w|}\,, \qquad \text{with} \quad w = \frac{1}{p_2}(z + p_1^2 m_{\mathrm{sc}}(z))\,, \tag{2.3.13}$$

uniformly in $|\mathrm{Im}\, z| \geq N^{-1+\xi}$. In particular, when z is in the bulk of the semicircle, then we have that

$$\Big|\|\boldsymbol{p}\|\, m_2(z) - m_{\mathrm{sc}}\big(\tfrac{z}{\|\boldsymbol{p}\|}\big)\Big| \prec \frac{1}{N}$$

and this relation holds even down to the real axis by the Lipschitz continuity (in fact, real analyticity) of the Stieltjes transform $m_2(z)$ in the bulk.

In the following, we will use the shorthand notation $A \approx B$ for two (families of) random variables A and B if and only if $|A - B| \prec N^{-1}$. Evaluating (2.3.6) at $z = \gamma_{i,2}$, we have

$$M_2(\gamma_{i,2}) = \frac{1}{p_2} \cdot \frac{1}{W_2 - w_{i,2}}\,, \qquad w_{i,2} := \frac{1}{p_2}(\gamma_{i,2} + p_1^2\, m_2(\gamma_{i,2}))\,, \tag{2.3.14}$$

and note that $w_{i,2} \approx w_i := \frac{1}{p_2}\left(\gamma_i + \frac{p_1^2}{\|p\|}\, m_{\mathrm{sc}}\left(\frac{\gamma_i}{\|p\|}\right)\right)$ by (2.3.13) and since $\gamma_{i,2} \approx \gamma_i$ in the bulk by rigidity (2.3.12).

Now we are ready to evaluate the rhs. of (2.3.5). By elementary manipulations using (2.3.14), we can now write the rhs. of (2.3.5) as

$$\frac{\langle p_2 W_2 \mathrm{Im}\, M_2(\gamma_{i,2})\rangle}{\langle \mathrm{Im}\, M_2(\gamma_{i,2})\rangle} = \gamma_{i,2} + \frac{p_1^2}{p_2}\frac{\mathrm{Im}\left[\langle (W_2 - w_{i,2})^{-1})^2\right]}{\mathrm{Im}\,\langle (W_2 - w_{i,2})^{-1}\rangle}\,. \tag{2.3.15}$$

Using (2.3.13), we obtain

$$\langle (W_2 - w_{i,2})^{-1} \rangle \approx \langle (W_2 - w_i)^{-1} \rangle \approx m_{\mathrm{sc}}(w_i) \tag{2.3.16}$$

with very high W_2-probability. Continuing with (2.3.15) and using $\gamma_{i,2} \approx \gamma_i$, we thus find

$$\frac{\langle p_2 W_2 \mathrm{Im}\, M_2(\gamma_{i,2}))\rangle}{\langle \mathrm{Im}\, M_2(\gamma_{i,2})\rangle} \approx \gamma_i + \frac{p_1^2}{p_2} \frac{\mathrm{Im}\left[m_{\mathrm{sc}}(w_i)^2\right]}{\mathrm{Im}\, m_{\mathrm{sc}}(w_i)}\,. \tag{2.3.17}$$

Next, we combine (2.3.8) with $p_2 m_2(\gamma_i) \approx m_{\mathrm{sc}}(w_i)$ from (2.3.7), (2.3.14) and (2.3.16) and find that

$$m_{\mathrm{sc}}(w_i)^2 \approx -\frac{p_2^2}{p_1^2 + p_2^2}\left(1 + \frac{1}{p_2}\gamma_i\, m_{\mathrm{sc}}(w_i)\right). \tag{2.3.18}$$

Hence, plugging (2.3.18) into (2.3.17) we deduce

$$\frac{\langle p_2 W_2 \mathrm{Im}\, M_2(\gamma_{i,2}))\rangle}{\langle \mathrm{Im}\, M_2(\gamma_{i,2})\rangle} \approx \left(1 - \frac{p_1^2}{p_1^2 + p_2^2}\right)\gamma_i = \frac{p_2^2}{\|\boldsymbol{p}\|^2}\gamma_i\,.$$

This completes the proof of (2.3.1).

2.3.2 Computation of the variance (2.3.2)

As in the calculation of the expectation in Section 2.3.1, we first condition on W_2 and work in the probability space of W_1. So, we apply Theorem 2.2.9 to the matrix $p_1 W_1 + p_2 W_2$, where the second term is considered a fixed deterministic deformation. Indeed, using the same notations as in Section 2.3.1, this gives that the lhs. of (2.3.2) equals

$$p_2^2 \,\mathrm{Var}_{\gamma_{i,2}}(W_2) = p_2^2 \frac{1}{\langle \mathrm{Im}\, M_2(\gamma_{i,2})\rangle^2}\left(\left\langle \left(\mathring{W}_2^{\gamma_{i,2}} \mathrm{Im}\, M_2(\gamma_{i,2})\right)^2\right\rangle - \frac{p_1^2}{2}\mathrm{Re}\left[\frac{\left\langle \left(M_2(\gamma_{i,2})\right)^2 \mathring{W}_2^{\gamma_{i,2}}\right\rangle^2}{1 - p_1^2\left\langle \left(M_2(\gamma_{i,2})\right)^2\right\rangle}\right]\right) \tag{2.3.19}$$

up to an additive error of order $\mathcal{O}_{\prec}\big(N^{-\epsilon}\big)$, which will appear on the rhs. of (2.3.2). The factor p_1^2 in the second term of (2.3.19) is a natural rescaling caused by applying Theorem 2.2.9 to a deformation of $p_1 W_1$ instead of a Wigner matrix W_1. Further we express M_2 in terms of a Wigner resolvent $G := (W_2 - w)^{-1}$ and use *local laws* not only for a single resolvent $\langle G\rangle$ but also their multi-resolvent versions for $\langle G^2\rangle$ and $\langle GG^*\rangle$ (see [168]). With a slight abuse of notation we shall henceforth drop the subscript '2' in $\gamma_{i,2}$ and $w_{i,2}$ and replace them by their deterministic values γ_i and w_i, respectively, at a negligible error of order N^{-1} exactly as in Section 2.3.1. Note that $\mathrm{Im}\, w_i \gtrsim 1$ for bulk indices i, so all resolvents below are stable and all denominators are well separated away from zero; this is needed to justify the $\approx$ relations below.

The first term in (2.3.19) can be rewritten as (here $G = G(w_i)$ and $m_{\mathrm{sc}} := m_{\mathrm{sc}}(w_i)$ for brevity)

$$\begin{aligned}
\left\langle \left(\mathring{W}_2^{\gamma_i} \mathrm{Im}\, M_2(\gamma_i)\right)^2\right\rangle &\approx \frac{1}{2}\mathrm{Re}\left(\left|\frac{w_i}{p_2} - \frac{\gamma_i}{p_1^2 + p_2^2}\right|^2 \langle GG^*\rangle - \left(\frac{w_i}{p_2} - \frac{\gamma_i}{p_1^2 + p_2^2}\right)^2 \langle G^2\rangle\right) \\
&\approx \frac{1}{2}\mathrm{Re}\left(\left|\frac{w_i}{p_2} - \frac{\gamma_i}{p_1^2 + p_2^2}\right|^2 \frac{|m_{\mathrm{sc}}|^2}{1 - |m_{\mathrm{sc}}|^2} - \left(\frac{w_i}{p_2} - \frac{\gamma_i}{p_1^2 + p_2^2}\right)^2 \frac{m_{\mathrm{sc}}^2}{1 - m_{sc}^2}\right) \\
&\approx \frac{p_1^4}{2p_2^2(p_1^2 + p_2^2)^2}\mathrm{Re}\left(\frac{1}{1 - |m_{\mathrm{sc}}|^2} - \frac{1}{1 - m_{\mathrm{sc}}^2}\right),
\end{aligned} \tag{2.3.20}$$

where in the last step we used (2.3.18). Similarly for the second term in (2.3.19), we have

$$\frac{\left\langle \left(M_2(\gamma_i)\right)^2 \mathring{W}_2^{\gamma_i}\right\rangle^2}{1 - p_1^2\left\langle \left(M(\gamma_i)\right)^2\right\rangle} \approx \frac{\left[\frac{1}{p_2}\left\langle \frac{1}{p_2}G + \left(\frac{w_i}{p_2} - \frac{\gamma_i}{p_1^2 + p_2^2}\right)G^2\right\rangle\right]^2}{1 - \frac{p_1^2}{p_2^2}\langle G^2\rangle} \approx \frac{m_{\mathrm{sc}}^2(p_2^2 - (p_1^2 + p_2^2)m_{\mathrm{sc}}^2)}{p_2^2(p_1^2 + p_2^2)^2(1 - m_{\mathrm{sc}}^2)}\,. \tag{2.3.21}$$

Plugging (2.3.20) and (2.3.21) into (2.3.19) we obtain

$$p_2^2 \operatorname{Var}_{\gamma_i}(W_2) \approx \frac{2p_2^2 + p_2\gamma_i \operatorname{Re} m_{\mathrm{sc}}}{(\operatorname{Im} m_{\mathrm{sc}})^2} \cdot \frac{p_1^2 p_2^2}{2(p_1^2+p_2^2)^2}\,. \tag{2.3.22}$$

Taking the imaginary part of (2.3.18), we find that $|m_{\mathrm{sc}}|^2 \approx \frac{p_2^2}{p_1^2+p_2^2}$ and hence, using (2.3.8) again, we infer

$$\frac{1}{p_1^2+p_2^2}\frac{2p_2^2 + p_2\gamma_i \operatorname{Re} m_{sc}}{(\operatorname{Im} m_{\mathrm{sc}})^2} \approx \frac{\operatorname{Re}\left[|m_{\mathrm{sc}}|^2 - m_{\mathrm{sc}}^2\right]}{(\operatorname{Im} m_{\mathrm{sc}})^2} = 2\,. \tag{2.3.23}$$

Combining (2.3.22) and (2.3.23) with (2.3.19), this completes the proof of (2.3.2). This proves Lemma 2.3.1.

2.4 Multi–resolvent local laws: Proof of Theorem 2.2.7

To study the eigenvectors of H we analyse its resolvent $G(z) := (H-z)^{-1}$, with $z \in \mathbf{C} \setminus \mathbf{R}$. It is well known [243, 23] that $G(z)$ becomes approximately deterministic in the large N limit. Its deterministic approximation (as a matrix) is given by $M(z)$, the unique solution of (2.2.8), in the following *averaged* and *isotropic* sense:

$$|\langle (G(z)-M(z))B\rangle| \prec \frac{1}{N|\operatorname{Im} z|}, \qquad |\langle \boldsymbol{x}, (G(z)-M(z))\boldsymbol{y}\rangle| \prec \frac{1}{\sqrt{N|\operatorname{Im} z|}}\,, \tag{2.4.1}$$

uniformly in deterministic vectors $\|\boldsymbol{x}\| + \|\boldsymbol{y}\| \lesssim 1$ and deterministic matrices $\|B\| \lesssim 1$. To be precise, while the local laws (2.4.1) hold for $\operatorname{Re} z \in \mathbf{B}_\kappa$ and $\operatorname{dist}(\operatorname{Re} z, \operatorname{supp}(\rho)) \gtrsim 1$ for *arbitrary* bounded self-adjoint deformations $D = D^*$ (see [243, Theorem 2.2]), the complementary regime requires the strengthened Assumption 2.2.8 on D (see [23]). Note that cusps for ρ have been excluded in Assumption 2.2.8, hence the complementary regime only consists of edges, which are covered in [23, Theorem 2.6], under the requirement that $\|M\|$ is bounded – which was also supposed in Assumption 2.2.8.

The isotropic bound $\langle \boldsymbol{x}, \operatorname{Im} G(z)\boldsymbol{x}\rangle \prec 1$ from (2.4.1) immediately gives an (almost) optimal bound on the delocalisation of eigenvectors: $|\langle \boldsymbol{u}_i, \boldsymbol{x}\rangle| \prec N^{-1/2}$ [405, 395, 243, 23, 62]. However, these estimates are not precise enough to conclude optimal bounds for eigenvector overlaps and generic matrices A as in Theorem 2.2.7; in fact by (2.4.1) we can only obtain the trivial bound $|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle| \prec 1$. Instead of the single resolvent local law (2.4.1), we rely on the fact that (see (2.4.17) below)

$$N\left|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle\right|^2 \lesssim \langle \operatorname{Im} G(\gamma_i + \mathrm{i}\eta) A \operatorname{Im} G(\gamma_j + \mathrm{i}\eta) A^*\rangle\,, \tag{2.4.2}$$

for $\eta \sim N^{-1+\epsilon}$, where $\epsilon > 0$ is small but fixed, and $\gamma_i, \gamma_j \in \mathbf{B}_\kappa$ are in the bulk and we estimate the rhs. of (2.4.2). In particular, to prove Theorem 2.2.7 we will use the *multi-resolvent local laws* from Proposition 2.4.4 below.

Multi-resolvent local laws are natural generalisations of (2.4.1) and they assert that longer products

$$G_1 B_1 G_2 \cdots G_{k-1} B_{k-1} G_k \tag{2.4.3}$$

of resolvents $G_i := G(z_i)$ and deterministic matrices[10] $B_1, ..., B_{k-1}$ also become approximately deterministic both in average and isotropic sense in the large N limit as long as $N|\operatorname{Im} z_i| \gg 1$. The deterministic approximation to the chain (2.4.3) is denoted by

$$M(z_1, B_1, z_2, ..., z_{k-1}, B_{k-1}, z_k). \tag{2.4.4}$$

It is not simply $M(z_1)B_1M(z_2)B_2\ldots$, i.e. it cannot be obtained by mechanically replacing each G with M as (2.4.1) might incorrectly suggest. Instead, it is defined recursively in the length k of the chain as follows (see Definition 1.4.1):

[10] We will use the the notational convention, that the letter B denotes arbitrary (generic) matrices, while A is reserved for *regular* matrices, in the sense of Definition 2.4.2 below.

Definition 2.4.1. *Fix $k \in \mathbf{N}$ and let $z_1, ..., z_k \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters. As usual, the corresponding solutions to* (2.2.8) *are denoted by $M(z_j)$, $j \in [k]$. Then, for deterministic matrices $B_1, ..., B_{k-1}$ we recursively define*

$$M(z_1, B_1, ...B_{k-1}, z_k) = \big(\mathcal{B}_{1k}\big)^{-1}\Bigg[M(z_1)B_1M(z_2, ..., z_k) + \sum_{l=2}^{k-1} M(z_1)\langle M(z_1, ..., z_l)\rangle M(z_l, ..., z_k)\Bigg], \tag{2.4.5}$$

where we introduced the shorthand notation

$$\mathcal{B}_{mn} \equiv \mathcal{B}(z_m, z_n) = 1 - M(z_m)\langle\cdot\rangle M(z_n) \tag{2.4.6}$$

for the stability operator acting on the space of $N \times N$ matrices.

It turns out that the size of $M(z_1, B_1, z_2, \dots, z_k)$ in the relevant regime of small $\eta := \min_j |\mathrm{Im}\, z_j|$ is roughly η^{-k+1} in the worst case, with a matching error term in the corresponding local law. This blow-up in the small η regime comes recursively from the large norm of the inverse of the stability operator $\mathcal{B}_{1k}$ in (2.4.5). However, for a special subspace of observable matrices B_i, called *regular* matrices, the size of $M(z_1, B_1, z_2, \dots, z_k)$ is much smaller. For *Wigner matrices*, i.e. for $D = 0$, the regular observables are simply the *traceless* matrices, i.e. observables B such that $\langle B\rangle = 0$. In [165, 170, 168, 169] it was shown that when the matrices B_i are all traceless, then $M(z_1, B_1, z_2, \dots, z_k)$ hence (2.4.3) are smaller by an $\eta^{k/2}$-factor than for general B_i;'s.

The situation for deformed Wigner matrices is more complicated, since the concept of *regular observables* will be dependent on the precise location in the spectrum of H, i.e. dependent on the energy. More precisely, we will require that the trace of A tested against a deterministic energy dependent matrix has to vanish; this reflects the inhomogeneity introduced by D. Analogously to the Wigner case, in Proposition 2.4.4 below we will show that resolvent chains (2.4.3) are much smaller when the deterministic matrices B_i are regular.

Next, we give the definition of regular matrices in the chain (2.4.3). Using the notation A for regular matrices, we will consider chains of resolvents and deterministic matrices of the form

$$\langle G_1A_1 \cdots G_kA_k\rangle \tag{2.4.7}$$

in the averaged case, or

$$\big(G_1A_1 \cdots A_kG_{k+1}\big)_{\boldsymbol{xy}} \tag{2.4.8}$$

in the isotropic case, with $G_i := G(z_i)$ and A_i being *regular* matrices according to the following Definition 2.4.2 (cf. Definition 1.4.2), which generalises the earlier Definition 2.2.6.

Definition 2.4.2 (Regular observables – Two-point regularisation in chains)**.** *Fix a parameter $\kappa > 0$ and let $\delta = \delta(\kappa, \|D\|) > 0$ be small enough (see the discussion below). Consider one of the two expressions* (2.4.7) *or* (2.4.8) *for some fixed length $k \in \mathbf{N}$ and bounded matrices $\|A_i\| \lesssim 1$ and let $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ be spectral parameters with $\mathrm{Re}\, z_j \in \mathbf{B}_\kappa$. For any $j \in [k]$, we denote*

$$\mathbf{1}_\delta(z_j, z_{j+1}) := \phi_\delta(\mathrm{Re}\, z_j - \mathrm{Re}\, z_{j+1})\ \phi_\delta(\mathrm{Im}\, z_j)\ \phi_\delta(\mathrm{Im}\, z_{j+1}) \tag{2.4.9}$$

where $0 \le \phi_\delta \le 1$ is a smooth symmetric bump function on $\mathbf{R}$ satisfying $\phi_\delta(x) = 1$ for $|x| \le \delta/2$ and $\phi_\delta(x) = 0$ for $|x| \ge \delta$. Here and in the following, in case of (2.4.7)*, the indices in* (2.4.9) *are understood cyclically modulo k.*

(a) For $j \in [k]$, denoting $\mathfrak{s}_j := -\mathrm{sgn}(\mathrm{Im}\, z_j \mathrm{Im}\, z_{j+1})$, we define the (two-point) regularisation *of A_j from* (2.4.7) *or* (2.4.8) w.r.t. the spectral parameters (z_j, z_{j+1}) *as*

$$\mathring{A}_j^{z_j, z_{j+1}} := A_j - \mathbf{1}_\delta(z_j, z_{j+1})\frac{\langle M(\mathrm{Re}\, z_j + \mathrm{i}\mathrm{Im}\, z_j)A_jM(\mathrm{Re}\, z_{j+1} + \mathfrak{s}_j\mathrm{i}\mathrm{Im}\, z_{j+1})\rangle}{\langle M(\mathrm{Re}\, z_j + \mathrm{i}\mathrm{Im}\, z_j)M(\mathrm{Re}\, z_{j+1} + \mathfrak{s}_j\mathrm{i}\mathrm{Im}\, z_{j+1})\rangle}\,. \tag{2.4.10}$$

(b) Moreover, we call A_j regular w.r.t. (z_j, z_{j+1}) *if and only if* $\mathring{A}_j^{z_j,z_{j+1}} = A_j$.

As already indicated above, the two-point regularisation generalises Definition 2.2.6 in the sense that

$$\mathring{A}^{e\pm\mathrm{i}\eta,e\pm\mathrm{i}\eta} \longrightarrow \mathring{A}^{e}, \quad \text{and} \quad \mathring{A}^{e\pm\mathrm{i}\eta,e\mp\mathrm{i}\eta} \longrightarrow \mathring{A}^{e}, \quad \text{as} \quad \eta \downarrow 0, \tag{2.4.11}$$

with a linear speed of convergence, for $e \in \mathbf{B}_\kappa$ and any bounded deterministic $A \in \mathbf{C}^{N\times N}$, where we used that, by taking the imaginary part of (2.2.8), $M(z)M(z)^* = \operatorname{Im} M(z)/(\langle \operatorname{Im} M(z)\rangle + \operatorname{Im} z)$.

Moreover, we point out, that the above Definition 2.4.2 of the regularisation is identical to Definitions 1.3.1 and 1.4.2 when dropping the summand with $\mathfrak{s}\tau = -1$ in (1.3.7). In particular, for spectral parameters z_j, z_{j+1} satisfying $\mathbf{1}_\delta(z_j, z_{j+1}) > 0$ (for some $\delta > 0$ small enough), it holds that the denominator in (2.4.10) is bounded away from zero, which shows that the linear map $A \mapsto \mathring{A}$ is bounded. Additionally, we have the following Lipschitz property (see Lemma 1.3.3):

$$\mathring{A}^{z_1,z_2} = \mathring{A}^{w_1,w_2} + \mathcal{O}\big(|z_1 - w_1| + |z_2 - w_2|\big)I, \tag{2.4.12}$$

for any $z_1, z_2, w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ such that $\operatorname{Im} z_i \operatorname{Im} w_i > 0$. It is important that the error in (2.4.12) is a constant times the identity matrix, indicated by $\mathcal{O}(\cdot)I$.

Next, we give bounds on the size of $M(z_1, A_1, \ldots A_{k-1}, z_k)$, the deterministic approximation to the chain $G_1 A_1 \cdots A_{k-1} G_k$ introduced in Definition 2.4.1; the proof of this lemma is presented in Appendix 2.A.

Lemma 2.4.3. *Fix* $\kappa > 0$. *Let* $k \in [4]$ *and* $z_1, \ldots, z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ *be spectral parameters with* $\operatorname{Re} z_j \in \mathbf{B}_\kappa$. *Set* $\eta := \min_j |\operatorname{Im} z_j|$. *Then, for bounded* regular *deterministic matrices* $A_1, \ldots, A_k$ *(according to Definition 2.4.2), we have the bounds*

$$\|M(z_1, A_1, \ldots, A_k, z_{k+1})\| \lesssim \begin{cases} \frac{1}{\eta^{\lfloor k/2 \rfloor}} & \text{if } \eta \le 1 \\ \frac{1}{\eta^{k+1}} & \text{if } \eta > 1 \end{cases}, \tag{2.4.13}$$

$$|\langle M(z_1, A_1, \ldots, A_{k-1}, z_k) A_k\rangle| \lesssim \begin{cases} \frac{1}{\eta^{\lfloor k/2 \rfloor - 1}} \vee 1 & \text{if } \eta \le 1 \\ \frac{1}{\eta^k} & \text{if } \eta > 1 \end{cases}. \tag{2.4.14}$$

For the presentation of Proposition 2.4.4, the main technical result underlying the proof of Theorem 2.2.7, we would only need (2.4.13) and (2.4.14) for $k \in [2]$ from the previous lemma. However, the remaining bounds covered by Lemma 2.4.3 will be instrumental in several parts of our proofs (see Section 2.6 and Appendix 2.A).

Proposition 2.4.4. *Fix* $\epsilon > 0$, $\kappa > 0$, $k \in [2]$, *and consider* $z_1, \ldots, z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ *with* $\operatorname{Re} z_j \in \mathbf{B}_\kappa$. *Consider regular matrices* $A_1, \ldots, A_k$ *with* $\|A_i\| \le 1$, *deterministic vectors* $\boldsymbol{x}, \boldsymbol{y}$ *with* $\|\boldsymbol{x}\| + \|\boldsymbol{y}\| \lesssim 1$, *and set* $G_i := G(z_i)$. *Then, uniformly in* $\eta := \min_j |\operatorname{Im} z_j| \ge N^{-1+\epsilon}$, *we have the* averaged local law

$$\big|\langle \big(G_1 A_1 \ldots G_k - M(z_1, A_1, \ldots, z_k)\big) A_k\rangle\big| \prec \begin{cases} \frac{N^{k/2-1}}{\sqrt{N\eta}} & \text{if } \eta \le 1 \\ \frac{1}{N\eta^{k+1}} & \text{if } \eta > 1 \end{cases} \tag{2.4.15a}$$

and the isotropic local law

$$\big|\langle \boldsymbol{x}, \big(G_1 A_1 \ldots G_{k+1} - M(z_1, A_1, \ldots, z_{k+1})\big)\boldsymbol{y}\rangle\big| \prec \begin{cases} \frac{N^{(k-1)/2}}{\sqrt{N\eta^2}} & \text{if } \eta \le 1 \\ \frac{1}{\sqrt{N}\eta^{k+2}} & \text{if } \eta > 1 \end{cases}. \tag{2.4.15b}$$

In Section 2.6, we will carry out the proof of Proposition 2.4.4 in the much more involved $\eta \le 1$ regime. For $\eta > 1$, the bound simply follows by induction on the number of resolvents in a chain by invoking the trivial estimate $\|M(z)\| \lesssim 1/|\operatorname{Im} z|$. The detailed argument has been carried out in [168, Appendix B] for the case of Wigner matrices. Having Proposition 2.4.4 at hand, we can now prove Theorem 2.2.7.

Proof of Theorem 2.2.7. By (2.4.15a) and (2.4.14) for $k = 2$ it follows that

$$|\langle G_1 A_1 G_2 A_2)| \prec 1\,, \tag{2.4.16}$$

for arbitrary regular matrices $A_1 = \mathring{A}_1^{z_1,z_2}$ and $A_2 = \mathring{A}_2^{z_2,z_1}$. Now, using that (see Lemma 1.3.6 for an analogous statement; see also (2.4.11) and (2.4.12))

$$\mathring{A}^{\gamma_i} = \mathring{A}^{\gamma_i \pm \mathrm{i}\eta, \gamma_j \pm 2\mathrm{i}\eta} + \mathcal{O}\big(|\gamma_i - \gamma_j| + \eta\big) I = \mathring{A}^{\gamma_i \pm \mathrm{i}\eta, \gamma_j \mp 2\mathrm{i}\eta} + \mathcal{O}\big(|\gamma_i - \gamma_j| + \eta\big) I\,,$$

and analogously for $(\mathring{A}^*)^{\gamma_i}$, we obtain (cf. Section 1.3.3)

$$\langle \operatorname{Im} G(\gamma_i + \mathrm{i}\eta) \mathring{A}^{\gamma_i} \operatorname{Im} G(\gamma_j + 2\mathrm{i}\eta) (\mathring{A}^*)^{\gamma_i} \rangle \prec 1\,.$$

Moreover, by spectral decomposition, together with the rigidity of eigenvalues (see e.g. [17, 243]) it follows that (cf. Lemma 1.3.5)

$$N |\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i} \boldsymbol{u}_j \rangle|^2 \prec (N\eta)^2 \langle \operatorname{Im} G(\gamma_i + \mathrm{i}\eta) \mathring{A}^{\gamma_i} \operatorname{Im} G(\gamma_j + 2\mathrm{i}\eta) (\mathring{A}^*)^{\gamma_i} \rangle \prec (N\eta)^2\,. \tag{2.4.17}$$

Choosing $\eta = N^{-1+\xi/2}$ for some arbitrary small $\xi > 0$, we conclude the desired. □

2.5 Dyson Brownian motion: Proof of Theorem 2.2.9

The main observation we used to prove Theorem 2.2.7 in Section 2.4 is the relation (2.4.2), i.e. we related the eigenvector overlaps with a trace of the product of two resolvents and two deterministic matrices. For Theorem 2.2.7 we only needed an upper bound on the size of the eigenvector overlaps, however to prove Theorem 2.2.9 we need to identify their size. For this purpose the main input is the relation

$$\frac{1}{N^{2\epsilon}} \sum_{\substack{|i-i_0| \le N^\epsilon \\ |j-j_0| \le N^\epsilon}} N |\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i} \boldsymbol{u}_j \rangle|^2 \sim \langle \operatorname{Im} G(\gamma_{i_0} + \mathrm{i}\eta) \mathring{A}^{\gamma_{i_0}} \operatorname{Im} G(\gamma_{j_0} + 2\mathrm{i}\eta) (\mathring{A}^*)^{\gamma_{i_0}} \rangle, \tag{2.5.1}$$

with $\eta = N^{-1+\epsilon}$, for some small fixed $\epsilon > 0$, and i_0, j_0 being some fixed bulk indices. The relation (2.5.1) is clearly not enough to identify the fluctuations of the individual eigenvector overlaps, but it gives a hint on the expression of the variance of these overlaps. More precisely, to identify the fluctuations of $N |\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i} \boldsymbol{u}_i \rangle|^2$ we will rely on a Dyson Brownian motion analysis which will reveal that

$$N\, \mathbf{E}[|\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i} \boldsymbol{u}_i \rangle|^2] \approx \frac{1}{\langle \operatorname{Im} M(\gamma_i) \rangle^2}\, \mathbf{E} \langle \operatorname{Im} G(\gamma_i + \mathrm{i}\eta) \mathring{A}^{\gamma_i} \operatorname{Im} G(\gamma_i + 2\mathrm{i}\eta) (\mathring{A}^*)^{\gamma_i} \rangle, \tag{2.5.2}$$

and a similar relation holds for higher moments as well. Finally, the rhs. of (2.5.2) is computed using a multi–resolvent local law (see e.g. (2.4.15a) for $k = 2$), and after some algebraic manipulation (see (2.5.52)–(2.5.55) below) this results in $\mathrm{Var}_{\gamma_i}(A)$ as defined in (2.2.15).

Given the optimal a priori bound (2.2.13), the proof of Theorem 2.2.9 is very similar to the analysis of the *Stochastic Eigenstate Equation (SEE)* in [167, Sections 3-4] and [169, Section 4]. Even if very similar to those papers, to make the presentation clearer, here we write out the main steps of the proof and explain the differences, but we do not write the details; we defer the interested reader to [167]. We also remark that the proof in [167, 169] heavily relies on the analysis of SEE developed in [112] and extend in [115, 435].

Similarly to [167, 169] we only consider the real case, the complex case is completely analogous and so omitted. We prove Theorem 2.2.9 dynamically, i.e. we consider the flow

$$\mathrm{d}W_t = \frac{\mathrm{d}\widetilde{B}_t}{\sqrt{N}}\,, \qquad W_0 = W\,, \tag{2.5.3}$$

with $\widetilde{B}_t$ a real symmetric matrix valued Brownian motion (see e.g. [112, Definition 2.1]). Note that W_t has a Gaussian component of size $\sqrt{t}$, i.e.

$$W_t \stackrel{\mathrm{d}}{=} W_0 + \sqrt{t}U\,,$$

with U being a GOE matrix independent of W_0. Denoting by $\lambda_i(t)$ the eigenvalues of W_t (labeled in increasing order) and by $\boldsymbol{u}_i(t)$ the corresponding orthonormal eigenvectors, we will prove Theorem 2.2.9 for the eigenvectors $\boldsymbol{u}_i(T)$, with $T = N^{-1+\omega}$, for some small fixed $\omega > 0$. Since T is very small, the Gaussian component added in the flow (2.5.3) can easily be removed by a standard Green function comparison (GFT) argument as in [169, Appendix B].

By [112], it is known that the eigenvalues $\lambda_i(t)$ and the eigenvectors $\boldsymbol{u}_i(t)$ are the unique strong solution of the following system of stochastic differential equations (SDEs):

$$\mathrm{d}\lambda_i(t) = \frac{\mathrm{d}B_{ii}(t)}{\sqrt{N}} + \frac{1}{N}\sum_{j\neq i}\frac{1}{\lambda_i(t)-\lambda_j(t)}\mathrm{d}t \tag{2.5.4}$$

$$\mathrm{d}\boldsymbol{u}_i(t) = \frac{1}{\sqrt{N}}\sum_{j\neq i}\frac{\mathrm{d}B_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}\boldsymbol{u}_j(t) - \frac{1}{2N}\sum_{j\neq i}\frac{\boldsymbol{u}_i}{(\lambda_i(t)-\lambda_j(t))^2}\mathrm{d}t\,, \tag{2.5.5}$$

where the matrix $B(t) = (B_{ij}(t))_{i,j=1}^N$ is a standard real symmetric Brownian motion (see e.g. [112, Definition 2.1]).

Even if in Theorem 2.2.9 we want to prove a CLT only for diagonal overlaps $\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle$, by (2.5.5), it follows that there is no closed equation for such quantities. For this reason, following [115, Section 2.3], we study the evolution of the *perfect matching observable* (see (2.5.7) below) along the flow (2.5.5).

2.5.1 Perfect matching observable and proof of Theorem 2.2.9

We introduce the notation

$$p_{ij} = p_{ij}(t) = \langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}C_0\,, \tag{2.5.6}$$

with A being a fixed real symmetric deterministic matrix A and C_0 being a fixed constant independent of i. Note that compared to [167, 169] in (2.5.6) we define the diagonal p_{ii} without subtracting their expectation (see (2.2.13) above), but rather a generic constant C_0 which we will choose later (see (2.5.48) below). The reason behind this choice is that in the current setting, unlike in the Wigner case [167, 169], the expectation of p_{ii} is now i–dependent, hence the flow (2.5.10) below would not be satisfied if we had defined (2.5.7) with the centred p_{ii}'s.

To study moments of the p_{ij}'s we use the *particle representation* introduced in [112] and further developed in [115, 435].. A particle configuration, corresponding to a certain monomials of p_{ij}'s, can be encoded by a function $\boldsymbol{\eta} : [N] \to \mathbf{N}_0$. The image $\eta_j = \boldsymbol{\eta}(j)$ denotes the number of particle at the site j, and $\sum_j \eta_j = n$ denotes the total number of particles. Additionally, given a particle configuration $\boldsymbol{\eta}$, by $\boldsymbol{\eta}^{ij}$, with $i \neq j$, we denote a new particle configuration in which a particle at the site i moved to a new site j, if there is no particle in i then $\boldsymbol{\eta}^{ij} = \boldsymbol{\eta}$. We denote the set of such configuration by Ω^n.

Fix a configuration $\boldsymbol{\eta}$, then we define the *perfect matching observable* (see [115, Section 2.3]):

$$f_{\boldsymbol{\lambda},t,C_0,C_1}(\boldsymbol{\eta}) := \frac{N^{n/2}}{[2C_1]^{n/2}}\frac{1}{(n-1)!!}\frac{1}{\mathcal{M}(\boldsymbol{\eta})}\mathbf{E}\left[\sum_{G\in\mathcal{G}_{\boldsymbol{\eta}}} P(G)\middle|\boldsymbol{\lambda}\right], \quad \mathcal{M}(\boldsymbol{\eta}) := \prod_{i=1}^N (2\eta_i-1)!!\,, \tag{2.5.7}$$

with n being the total number of particles in the configuration $\boldsymbol{\eta}$. The sum in (2.5.7) is taken over $\mathcal{G}_{\boldsymbol{\eta}}$, which denotes the set of perfect matchings on the complete graph with vertex set

$$\mathcal{V}_{\boldsymbol{\eta}} := \{(i,a) : 1 \le i \le n, 1 \le a \le 2\eta_i\}\,.$$

We also introduced the short–hand notation

$$P(G) := \prod_{e\in\mathcal{E}(G)} p(e), \qquad p(e) := p_{i_1 i_2}\,, \tag{2.5.8}$$

where $e = \{(i_1,a_1),(i_2,a_2)\} \in \mathcal{V}_{\boldsymbol{\eta}}^2$, and $\mathcal{E}(G)$ denotes the edges of G. Note that in (2.5.7) we took the conditional expectation with respect to the entire trajectories of the eigenvalues, $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}(t)\}_{t\in[0,T]}$ for some fixed $0 < T \ll 1$. We also remark that the definition (2.5.7) differs slightly from [167, Eq. (3.9)] and [169, Eq. (4.6)], since we now do not normalise by $\langle (A - \langle A\rangle)^2\rangle$ but using a different constant C_1 which we will choose later in the proof (see (2.5.48) below); this is a consequence of the fact that the diagonal overlaps p_{ii} are not correctly centred and normalised. Note that we did not incorporate the factor 2 in (2.5.7) into the constant C_1, since C_1 will be chosen has a normalisation constant to compensate the size of the matrix A, whilst the factor 2 represents the fact that diagonal overlaps, after the proper centering and normalisation depending on A and i, would be centred Gaussian random variable of variance two. Furthermore, we consider eigenvalues paths $\{\boldsymbol{\lambda}(t)\}_{t\in[0,T]}$ which lie in the event

$$\widetilde{\Omega} = \widetilde{\Omega}_\xi, := \Big\{ \sup_{0\le t\le T} \max_{i\in[N]} \eta_{\mathrm{f}}(\gamma_i(t))^{-1} |\lambda_i(t) - \gamma_i(t)| \le N^{\xi} \Big\} \tag{2.5.9}$$

for any $\xi > 0$, where $\eta_{\mathrm{f}}(\gamma_i(t))$ is the local fluctuation scale defined as in [244, Definition 2.4]. In most instances we will use this rigidity estimate in the bulk regime when $\eta_{\mathrm{f}}(\gamma_i(t)) \sim N^{-1}$; at the edges $\eta_{\mathrm{f}}(\gamma_i(t)) \sim N^{-2/3}$. We recall that here $\gamma_i(t)$ denote the quantiles of ρ_t defined as in (2.2.12). The fact that the event $\widetilde{\Omega}$ holds with very high probability follows by [23, Corollary 2.9].

By [115, Theorem 2.6] it follows that $f_{\boldsymbol{\lambda},t}$ is a solution of the parabolic discrete partial differential equation (PDE):

$$\partial_t f_{\boldsymbol{\lambda},t} = \mathcal{B}(t) f_{\boldsymbol{\lambda},t}\,, \tag{2.5.10}$$

$$\mathcal{B}(t) f_{\boldsymbol{\lambda},t} = \sum_{i\neq j} c_{ij}(t) 2\eta_i(1+2\eta_j)\big(f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta}^{ij}) - f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta})\big)\,. \tag{2.5.11}$$

where

$$c_{ij}(t) := \frac{1}{N(\lambda_i(t) - \lambda_j(t))^2}\,. \tag{2.5.12}$$

In the remainder of this section we may often omit $\boldsymbol{\lambda}$ from the notation since the paths of the eigenvalues are fixed within this proof.

The main result of this section is the following Proposition 2.5.1, which will readily prove Theorem 2.2.9. For this purpose we define a version of $f_t(\boldsymbol{\eta})$ with centred and rescaled p_{ii}:

$$q_{\boldsymbol{\lambda},t}(\boldsymbol{\eta}) := \left(\prod_{i=1}^N \frac{1}{\mathrm{Var}_{\gamma_i}(A)^{\eta_i/2}}\right) \frac{N^{n/2}}{2^{n/2}(n-1)!!} \frac{1}{\mathcal{M}(\boldsymbol{\eta})}\, \mathbf{E}\left[\sum_{G\in\mathcal{G}_{\boldsymbol{\eta}}} Q(G)\,\middle|\,\boldsymbol{\lambda}\right] \tag{2.5.13}$$

with $\mathring{A}^{\gamma_i}$ denoting the regular component of A defined as in (2.2.11):

$$\mathring{A}^{\gamma_i} := A - \frac{\langle A \mathrm{Im}\, M(\gamma_i)\rangle}{\langle \mathrm{Im}\, M(\gamma_i)\rangle},$$

and

$$Q(G) := \prod_{e\in\mathcal{E}(G)} q(e), \qquad q(e) := \langle \boldsymbol{u}_{i_1}, \mathring{A}^{\gamma_{i_1}} \boldsymbol{u}_{i_2}\rangle. \tag{2.5.14}$$

Note that the definition in (2.5.14) is not asymmetric for $i_1 \neq i_2$, since in this case $\langle \boldsymbol{u}_{i_1}, \mathring{A}^{\gamma_{i_1}} \boldsymbol{u}_{i_2}\rangle = \langle \boldsymbol{u}_{i_1}, \mathring{A}^{\gamma_{i_2}} \boldsymbol{u}_{i_2}\rangle$.

We now comment on the main difference between q_t and f_t from (2.5.13) and (2.5.7), respectively. First of all we notice the $q(e)$'s in (2.5.14) are slightly different compared with the $p(e)$'s from (2.5.6). In particular, we choose the $q(e)$'s in such a way that the diagonal overlaps have very small expectation (i.e. much smaller than their fluctuations size). The price to pay for this choice is that the centering is i–dependent, hence q_t is not a solution of an equation of the form (2.5.10)–(2.5.11). We also remark that later within the proof, C_0 from (2.5.6) will be chosen as

$$C_0 = \frac{\langle A\mathrm{Im}\, M(\gamma_{i_0})\rangle}{\langle \mathrm{Im}\, M(\gamma_{i_0})\rangle}$$

for some fixed i_0 such that $\gamma_{i_0} \in \mathbf{B}_\kappa$ is in the bulk (recall (2.2.10)). The idea behind this choice is that the analysis of the flow (2.5.10)–(2.5.11) will be completely local, we can thus fix a base point i_0 and ensure that the corresponding overlap is exactly centred, then the nearby overlaps for indices $|i - i_0| \le K$, for some N–dependent $K > 0$, will not be exactly centred, but their expectation will be very small compared to the size of their fluctuations:

$$\frac{\langle A\mathrm{Im}\, M(\gamma_{i_0})\rangle}{\langle \mathrm{Im}\, M(\gamma_{i_0})\rangle} - \frac{\langle A\mathrm{Im}\, M(\gamma_i)\rangle}{\langle \mathrm{Im}\, M(\gamma_i)\rangle} = \mathcal{O}\left(\frac{K}{N}\right).$$

A consequence of this choice is also that the normalisation for q_t and f_t is different: for q_t we chose a normalisation that is i's dependent, whilst for f_t the normalisation C_1 is i–independent and later, consistently with the choice of C_0, it will be chosen as

$$C_1 = \mathrm{Var}_{\gamma_{i_0}}(A)^{n/2},$$

which is exactly the normalisation that makes $f_t(\boldsymbol{\eta}) = 1$ when $\boldsymbol{\eta}$ is such that $\eta_{i_0} = n$ and zero otherwise.

Proposition 2.5.1. *For any $n \in \mathbf{N}$ there exists $c(n) > 0$ such that for any $\epsilon > 0$, and for any $T \ge N^{-1+\epsilon}$ it holds*

$$\sup_{\boldsymbol{\eta}} |q_T(\boldsymbol{\eta}) - \mathbf{1}(n \text{ even})| \lesssim N^{-c(n)}, \tag{2.5.15}$$

with very high probability. The supremum is taken over configurations $\boldsymbol{\eta}$ supported on bulk indices and the implicit constant in (2.5.15) *depends on n and ϵ.*

Proof of Theorem 2.2.9. Fix $n \in \mathbf{N}$, an index i such that $\gamma_i \in \mathbf{B}_\kappa$ is in the bulk, and choose a configuration $\boldsymbol{\eta}$ such that $\eta_i = n$ and $\eta_j = 0$ for any $j \ne i$. Then by Proposition 2.5.1, we conclude that

$$\mathbf{E}\left[\sqrt{\frac{N}{2\mathrm{Var}_{\gamma_i}(A)}}\langle \boldsymbol{u}_i(T), \mathring{A}^{\gamma_i}\boldsymbol{u}_i(T)\rangle\right]^n = \mathbf{1}(n \text{ even})(n-1)!! + \mathcal{O}\left(N^{-c(n)}\right),$$

with $T = N^{-1+\epsilon}$, for some very small fixed $\epsilon > 0$, and $c(n) > 0$. Here $\mathring{A}^{\gamma_i}$ is defined in (2.2.11) and $\mathrm{Var}_{\gamma_i}(A)$ is defined in (2.2.15). Then, by a standard GFT argument (see e.g. [169, Appendix B]), we se that

$$\mathbf{E}\left[\sqrt{\frac{N}{2\mathrm{Var}_{\gamma_i}(A)}}\langle \boldsymbol{u}_i(T), \mathring{A}^{\gamma_i}\boldsymbol{u}_i(T)\rangle\right]^n = \mathbf{E}\left[\sqrt{\frac{N}{2\mathrm{Var}_{\gamma_i}(A)}}\langle \boldsymbol{u}_i(0), \mathring{A}^{\gamma_i}\boldsymbol{u}_i(0)\rangle\right]^n + \mathcal{O}\left(N^{-c(n)}\right).$$

This shows that the Gaussian component added by the dynamics (2.5.3) can be removed at the price of a negligible error implying (2.2.14).

The lower bound on the variance (2.2.16) is an explicit calculation relying on the the definition of M from (2.2.8). In particular, we use that

(i) A and hence $\mathring{A}^{\gamma_i}$ are self-adjoint;

(ii) $\operatorname{Im} M(\gamma_i) \geq g$ for some $g = g(\kappa, \|D\|) > 0$ since we are in the bulk;

(iii) $\langle \mathring{A}^{\gamma_i} \operatorname{Im} M(\gamma_i) \rangle = 0$ by definition of the regularisation;

(iv) $[\operatorname{Re} M(\gamma_i), \operatorname{Im} M(\gamma_i)] = 0$ from (2.2.8).

Then, after writing $\operatorname{Var}_{\gamma_i}(A)$ as a sum of squares and abbreviating $\operatorname{Im} M = \operatorname{Im} M(\gamma_i)$, we find

$$\operatorname{Var}_{\gamma_i}(A) \geq \frac{\left\langle \left(\sqrt{\operatorname{Im} M} \left[A - \frac{\langle A(\operatorname{Im} M)^2 \rangle}{\langle (\operatorname{Im} M)^2 \rangle} \right] \sqrt{\operatorname{Im} M} \right)^2 \right\rangle}{\langle (\operatorname{Im} M)^2 \rangle} \geq g^2 \frac{\left\langle \left[A - \frac{\langle A(\operatorname{Im} M)^2 \rangle}{\langle (\operatorname{Im} M)^2 \rangle} \right]^2 \right\rangle}{\langle (\operatorname{Im} M)^2 \rangle} \geq \frac{g^2}{\langle (\operatorname{Im} M)^2 \rangle} \langle (A - \langle A \rangle)^2 \rangle ,$$

where in the last step we used the trivial variational principle $\langle (A - \langle A \rangle)^2 \rangle = \inf_{t \in \mathbf{R}} \langle (A - t)^2 \rangle$. This completes the proof of Theorem 2.2.9. □

2.5.2 DBM analysis

Similarly to [167, Section 4.1] and [169, Section 4.2] we introduce an equivalent particle representation to encode moments of the p_{ij}'s. In particular, here, and previously in [167, 169], we relied on the particle representation (2.5.16)–(2.5.18) below since our arguments heavily builds on [435], which use this latter representation.

Consider a particle configuration $\boldsymbol{\eta} \in \Omega^n$, for some fixed $n \in \mathbf{N}$, i.e. $\boldsymbol{\eta}$ is such that $\sum_j \eta_j = n$. We now define the new configuration space

$$\Lambda^n := \{ \boldsymbol{x} \in [N]^{2n} : n_i(\boldsymbol{x}) \text{ is even for every } i \in [N] \}, \tag{2.5.16}$$

where

$$n_i(\boldsymbol{x}) := |\{ a \in [2n] : x_a = i \}| \tag{2.5.17}$$

for all $i \in \mathbf{N}$.

By the correspondence

$$\boldsymbol{\eta} \leftrightarrow \boldsymbol{x} \qquad \eta_i = \frac{n_i(\boldsymbol{x})}{2} . \tag{2.5.18}$$

it is easy to see that these two representations are basically equivalent. The only difference is that $\boldsymbol{x}$ uniquely determines $\boldsymbol{\eta}$, but $\boldsymbol{\eta}$ determines only the coordinates of $\boldsymbol{x}$ as a multi-set and not its ordering.

From now on, given a function f defined on Ω^n, we will always consider functions g on $\Lambda^n \subset [N]^{2n}$ defined by

$$f(\boldsymbol{\eta}) = f(\phi(\boldsymbol{x})) = g(\boldsymbol{x}),$$

with $\phi \colon \Lambda^n \to \Omega^n$, $\phi(\boldsymbol{x}) = \boldsymbol{\eta}$ being the projection from the $\boldsymbol{x}$-configuration space to the $\boldsymbol{\eta}$-configuration space using (2.5.18). We thus defined the observable

$$g_t(\boldsymbol{x}) = g_{\boldsymbol{\lambda},t}(\boldsymbol{x}) := f_{\boldsymbol{\lambda},t}(\phi(\boldsymbol{x})) , \tag{2.5.19}$$

with $f_{\boldsymbol{\lambda},t}$ from (2.5.7). Note that $g_t(\boldsymbol{x})$ is equivariant under permutation of the arguments, i.e. it depends on $\boldsymbol{x}$ only as a multi–set. Similarly we define

$$r_t(\boldsymbol{x}) = r_{\boldsymbol{\lambda},t}(\boldsymbol{x}) := q_{\boldsymbol{\lambda},t}(\phi(\boldsymbol{x})) . \tag{2.5.20}$$

We remark that g_t and r_t are the counterpart of f_t and q_t, respectively, in the $\boldsymbol{x}$-configuration space.

We can thus now write the flow (2.5.10)–(2.5.11) in the $\boldsymbol{x}$–configuration space:

$$\partial_t g_t(\boldsymbol{x}) = \mathcal{L}(t) g_t(\boldsymbol{x}) \tag{2.5.21}$$

$$\mathcal{L}(t) := \sum_{j\neq i} \mathcal{L}_{ij}(t), \quad \mathcal{L}_{ij}(t)g(\boldsymbol{x}) := c_{ij}(t)\frac{n_j(\boldsymbol{x})+1}{n_i(\boldsymbol{x})-1}\sum_{a\neq b\in[2n]}\big(g(\boldsymbol{x}^{ij}_{ab})-g(\boldsymbol{x})\big), \tag{2.5.22}$$

where

$$\boldsymbol{x}^{ij}_{ab} := \boldsymbol{x} + \delta_{x_a i}\delta_{x_b i}(j-i)(\boldsymbol{e}_a+\boldsymbol{e}_b), \tag{2.5.23}$$

with $\boldsymbol{e}_a \in \mathbf{R}^{2n}$ denoting the standard unit vector, i.e. $\boldsymbol{e}_a(b) = \delta_{ab}$. We remark that this flow is map on functions defined on $\Lambda^n \subset [N]^{2n}$ which preserves equivariance.

For the following analysis it is convenient to define the scalar product and the natural measure on Λ^n:

$$\langle f,g\rangle_{\Lambda^n} = \langle f,g\rangle_{\Lambda^n,\pi} := \sum_{\boldsymbol{x}\in\Lambda^n}\pi(\boldsymbol{x})\bar{f}(\boldsymbol{x})g(\boldsymbol{x}), \qquad \pi(\boldsymbol{x}) := \prod_{i=1}^{N}((n_i(\boldsymbol{x})-1)!!)^2, \tag{2.5.24}$$

as well as the norm on $L^p(\Lambda^n)$:

$$\|f\|_p = \|f\|_{L^p(\Lambda^n,\pi)} := \left(\sum_{\boldsymbol{x}\in\Lambda^n}\pi(\boldsymbol{x})|f(\boldsymbol{x})|^p\right)^{1/p}. \tag{2.5.25}$$

The operator $\mathcal{L} = \mathcal{L}(t)$ is symmetric with respect to the measure π and it is a negative in $L^2(\Lambda^n)$, with associated Dirichlet form (see [434, Appendix A.2]):

$$D(g) = \langle g, (-\mathcal{L})g\rangle_{\Lambda^n} = \frac{1}{2}\sum_{\boldsymbol{x}\in\Lambda^n}\pi(\boldsymbol{x})\sum_{i\neq j}c_{ij}(t)\frac{n_j(\boldsymbol{x})+1}{n_i(\boldsymbol{x})-1}\sum_{a\neq b\in[2n]}\big|g(\boldsymbol{x}^{ij}_{ab})-g(\boldsymbol{x})\big|^2.$$

Finally, by $\mathcal{U}(s,t)$ we denote the semigroup associated to $\mathcal{L}$, i.e. for any $0\le s\le t$ it holds

$$\partial_t\mathcal{U}(s,t) = \mathcal{L}(t)\mathcal{U}(s,t), \quad \mathcal{U}(s,s) = I. \tag{2.5.26}$$

2.5.3 Short range approximation

As a consequence of the singularity of the coefficients $c_{ij}(t)$ in (2.5.22), the main contribution to the flow (2.5.21) comes from nearby eigenvalues, hence its analysis will be completely local. For this purpose we define the sets

$$\mathcal{J} = \mathcal{J}_\kappa := \{i\in[N] : \gamma_i(0)\in\mathbf{B}_\kappa\}, \tag{2.5.27}$$

which correspond to indices with quantiles $\gamma_i(0)$ (recall (2.2.12)) in the bulk.

Fix a point $\boldsymbol{y}\in\mathcal{J}^{2n}$, and an N-dependent parameter K such that $1\ll K\ll\sqrt{N}$. We remark that $\boldsymbol{y}\in\mathcal{J}^{2n}$ will be fixed for the rest of the analysis. Next, we define the *averaging operator* as a simple multiplication operator by a "smooth" cut-off function:

$$\mathrm{Av}(K,\boldsymbol{y})h(\boldsymbol{x}) := \mathrm{Av}(\boldsymbol{x};K,\boldsymbol{y})h(\boldsymbol{x}), \qquad \mathrm{Av}(\boldsymbol{x};K,\boldsymbol{y}) := \frac{1}{K}\sum_{j=K}^{2K-1}\mathbf{1}(\|\boldsymbol{x}-\boldsymbol{y}\|_1<j), \tag{2.5.28}$$

with $\|\boldsymbol{x}-\boldsymbol{y}\|_1 := \sum_{a=1}^{2n}|x_a-y_a|$. For notational simplicity we may often omit $K,\boldsymbol{y}$ from the notation since they are fixed throughout the proof:

$$\mathrm{Av}(\boldsymbol{x}) = \mathrm{Av}(\boldsymbol{x};K,\boldsymbol{y})h(\boldsymbol{x}), \qquad \mathrm{Av}h(\boldsymbol{x}) = \mathrm{Av}((\boldsymbol{x}))h((\boldsymbol{x})). \tag{2.5.29}$$

Additionally, fix an integer ℓ with $1\ll\ell\ll K$, and define the short range coefficients

$$c^{\mathcal{S}}_{ij}(t) := \begin{cases} c_{ij}(t) & \text{if } i,j\in\mathcal{J} \text{ and } |i-j|\le\ell \\ 0 & \text{otherwise,}\end{cases} \tag{2.5.30}$$

where $c_{ij}(t)$ is defined in (2.5.12). The parameter ℓ is the length of the short range interaction.

We now define a short–range approximation of r_t, with r_t defined in (2.5.20). Note that in the definition of the short–range flow (2.5.31) below there is a slight notational difference compared to [167, Section 4.2] and [169, Section 4.2.1]: we now choose an initial condition h_0 which depends on r_0 rather than g_0. This minor difference is caused by the fact that in [167, 169] the observable g_t was already centred and rescaled, while in the current case the centred and rescaled version of g_t is given by r_t, hence the definition in (2.5.31) is still conceptually the same as the one in [167, 169] (see also the paragraph above (2.5.47) for a more detailed explanation). We point out that we make this choice to ensure that the infinite norm of the short range approximation is always bounded by N^ξ (see below (2.5.32)). The short range approximation $h_t = h_t(\boldsymbol{x})$ is defined as the unique solution of the parabolic equation

$$\begin{aligned} \partial_t h_t(\boldsymbol{x};\ell,K,\boldsymbol{y}) &= \mathcal{S}(t)h_t(\boldsymbol{x};\ell,K,\boldsymbol{y}) \\ h_0(\boldsymbol{x};\ell,K,\boldsymbol{y}) = h_0(\boldsymbol{x};K,\boldsymbol{y}) &:= \mathrm{Av}(\boldsymbol{x};K,\boldsymbol{y})(r_0(\boldsymbol{x}) - \mathbf{1}(n \text{ even})), \end{aligned} \tag{2.5.31}$$

where

$$\mathcal{S}(t) := \sum_{j\neq i}\mathcal{S}_{ij}(t), \quad \mathcal{S}_{ij}(t)h(\boldsymbol{x}) := c_{ij}^{\mathcal{S}}(t)\frac{n_j(\boldsymbol{x})+1}{n_i(\boldsymbol{x})-1}\sum_{a\neq b\in[2n]}\big(h(\boldsymbol{x}_{ab}^{ij}) - h(\boldsymbol{x})\big). \tag{2.5.32}$$

In the remainder of this section we may often omit K, $\boldsymbol{y}$ and ℓ from the notation, since they are fixed for the rest of the proof. We conclude this section defining the transition semigroup $\mathcal{U}_{\mathcal{S}}(s,t) = \mathcal{U}_{\mathcal{S}}(s,t;\ell)$ associated to the short range generator $\mathcal{S}(t)$. Note that $\|h_t\|_\infty \le N^\xi$, for any $t \ge 0$ and any small $\xi > 0$, since $\mathcal{U}_{\mathcal{S}}(s,t)$ is a contraction and $\|h_0\|_\infty \le N^\xi$ by (2.2.13), as a consequence of $h_t(\boldsymbol{x})$ being supported on $\boldsymbol{x} \in \mathcal{J}^{2n}$.

2.5.4 L^2–estimates

To prove the L^∞–bound in Proposition 2.5.1, we first prove an L^2–bound in Proposition 2.5.3 below and then use an ultracontractivity argument for the parabolic PDE (2.5.21) (see [167, Section 4.4]) to get an L^∞–bound. To get an L^2–bound we will analyse h_t, the short–range version of the observable g_t from (2.5.19), and then we will show that h_t and g_t are actually close to each other using the following finite speed of propagation (see [167, Proposition 4.2, Lemmas 4.3–4.4]):

Lemma 2.5.2. *Let $0 \le s_1 \le s_2 \le s_1 + \ell N^{-1}$, and f be a function on Λ^n, then for any $\boldsymbol{x} \in \Lambda^n$ supported on $\mathcal{J}$ it holds*

$$\Big|(\mathcal{U}(s_1,s_2) - \mathcal{U}_{\mathcal{S}}(s_1,s_2;\ell))f(\boldsymbol{x})\Big| \lesssim N^{1+n\xi}\frac{s_2-s_1}{\ell}\|f\|_\infty\,, \tag{2.5.33}$$

for any small $\xi > 0$. The implicit constant in (2.5.15) *depends on n, ϵ, δ.*

To estimate several terms in the analysis of (2.5.31) we will rely on the multi–resolvent local laws from Proposition 2.4.4 (in combination with the extensions in Lemma 2.A.1 in Lemma 2.A.2). For this purpose, for a small $\omega > 2\xi > 0$, we define the very high probability event (see Lemmas 2.A.1–2.A.2)

$$\begin{aligned} \widehat{\Omega} = \widehat{\Omega}_{\omega,\xi} := \\ \bigcap_{\substack{e_i\in\mathbf{B}_\kappa,\\ |\mathrm{Im}\, z_i|\ge N^{-1+\omega}}} \Bigg[\bigcap_{k=2}^{n}\left\{\sup_{0\le t\le T}\big|\langle G_t(z_1)\mathring{A}_1\ldots G_t(z_k)\mathring{A}_k\rangle - \mathbf{1}(k=2)\langle M(z_1,\mathring{A}_1,z_2)\mathring{A}_2\rangle\big| \le \frac{N^{\xi+k/2-1}}{\sqrt{N\eta}}\right\} \\ \cap\left\{\sup_{0\le t\le T}\big|\langle G_t(z_1)\mathring{A}_1\rangle\big| \le \frac{N^\xi}{N\sqrt{|\mathrm{Im}\, z_1|}}\right\}\Bigg] \\ \bigcap_{\substack{z_1,z_2;e_1\in\mathbf{B}_\kappa,\\ |e_1-e_2|\ge c_1,|\mathrm{Im}\, z_i|\ge N^{-1+\omega}}} \left\{\sup_{0\le t\le T}\big|\langle (G_t(z_1)B_1G_t(z_2)B_2\rangle\big| \le N^\xi\right\}, \end{aligned} \tag{2.5.34}$$

where $\mathring{A}_1, \dots, \mathring{A}_k$ are regular matrices defined as in Definition 2.4.2 (here we used the short–hand notation $\mathring{A}_i = \mathring{A}_i^{z_i, z_{i+1}}$),

$$\langle M(z_1, \mathring{A}_1, z_2)\mathring{A}_2\rangle = \langle M(z_1)\mathring{A}_1 M(z_2)\mathring{A}_2\rangle + \frac{\langle M(z_1)\mathring{A}_1 M(z_2)\rangle\langle M(z_2)\mathring{A}_2 M(z_1)\rangle}{1 - \langle M(z_1)M(z_2)\rangle}, \tag{2.5.35}$$

$\eta := \min\{|\mathrm{Im}\, z_i| : i \in [k]\}$, $c_1 > 0$ is a fixed small constant, and B_1, B_2 are norm bounded deterministic matrices. We remark that for $|e_1 - e_2| \geq c_1$ we have the norm bound $\|M(z_1, B_1, z_2)\| \lesssim 1$, with $M(z_1, B_1, z_2)$ being defined in (2.4.4). Then, by standard arguments (see e.g. [169, Eq. (4.30)]), we conclude the bound (recall that $\widetilde{\Omega}_\xi$ from (2.5.9) denotes the rigidity event)

$$\max_{i,j\in\mathcal{J}} |\langle \boldsymbol{u}_i(t), A\boldsymbol{u}_j(t)\rangle| \leq \frac{N^\omega}{\sqrt{N}} \qquad \text{on } \widehat{\Omega}_{\omega,\xi} \cap \widetilde{\Omega}_\xi, \tag{2.5.36}$$

simultaneously for all $i, j \in \mathcal{J}$ and $0 \leq t \leq T$. Additionally, using the notation $\rho_{i,t} := |\mathrm{Im}\,\langle M_t(z_i)\rangle|$, on $\widehat{\Omega}_{\omega,\xi} \cap \widetilde{\Omega}_\xi$ it also holds that

$$|\langle \boldsymbol{u}_i(t), A\boldsymbol{u}_j(t)\rangle| \leq N^\omega \sqrt{\frac{\langle \mathrm{Im}\, G(\gamma_i(t) + \mathrm{i}\eta) A \mathrm{Im}\, G(\gamma_j(t) + \mathrm{i}\eta) A\rangle}{N\rho_{i,t}\rho_{j,t}}} \lesssim \frac{N^\omega}{N^{1/4}}, \tag{2.5.37}$$

when one among i and j is in the bulk and $|i - j| \geq cN$, for some small constant c depending on c_1 from (2.5.34). Here we used that for the index in the bulk, say i, we have $\rho_{i,t} \sim 1$ and for the other index $\rho_{j,t} \gtrsim N^{-1/2}$ as a consequence of $\eta \gg N^{-1}$. We point out that this non optimal bound $N^{-1/4}$, instead of the optimal $N^{-1/2}$, follows from the fact that the bound from Lemma 2.A.2 is not optimal when one of the two spectral parameters in close to an edge; this is exactly the same situation as in [169, Eq. (4.31)] where we get an analogous non optimal bound for overlaps of eigenvectors that are not in the bulk.

We are now ready to prove the main technical proposition of this section. Note the additional term $KN^{-1/2}$ in the error $\mathcal{E}$ in (2.5.39) compared to [167, Proposition 4.2] and [169, Proposition 4.4]; this is a consequence of the fact the p_{ii}'s in (2.5.6) are not correctly centred. We stress that the base point $\boldsymbol{y}$ in Proposition 2.5.3 is fixed throughout the remainder of this section.

Proposition 2.5.3. *For any parameters satisfying $N^{-1} \ll \eta \ll T_1 \ll \ell N^{-1} \ll KN^{-1} \ll N^{-1/2}$, and any small $\epsilon, \xi > 0$ it holds*

$$\|h_{T_1}(\cdot; \ell, K, \boldsymbol{y})\|_2 \lesssim K^{n/2}\mathcal{E}, \tag{2.5.38}$$

with

$$\mathcal{E} := N^{n\xi}\left(\frac{N^\epsilon \ell}{K} + \frac{NT_1}{\ell} + \frac{N\eta}{\ell} + \frac{N^\epsilon}{\sqrt{N\eta}} + \frac{1}{\sqrt{K}} + \frac{K}{\sqrt{N}}\right), \tag{2.5.39}$$

uniformly in particle configurations $\boldsymbol{y}$ such that $y_a = i_0$, for any $a \in [2n]$ and $i_0 \in \mathcal{J}$, and eigenvalue trajectory $\boldsymbol{\lambda}$ in the high probability event $\widetilde{\Omega}_\xi \cap \widehat{\Omega}_{\omega,\xi}$.

Proof. The proof of this proposition is very similar to the one of [167, Proposition 4.2] and [169, Proposition 4.4], we thus only explain the main differences here. In the following by the star over $\sum$ we denote that the summation runs over two n-tuples of fully distinct indices. The key idea in this proof is that in order to rely on the multi–resolvent local laws (2.5.34) we replace the operator $\mathcal{S}(t)$ in (2.5.31) with the new operator

$$\mathcal{A}(t) := \sum_{\boldsymbol{i},\boldsymbol{j}\in[N]^n}^{*} \mathcal{A}_{\boldsymbol{ij}}(t), \quad \mathcal{A}_{\boldsymbol{ij}}(t)h(\boldsymbol{x}) := \frac{1}{\eta}\left(\prod_{r=1}^{n} a^{\mathcal{S}}_{i_r,j_r}(t)\right) \sum_{\boldsymbol{a},\boldsymbol{b}\in[2n]^n}^{*} (h(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) - h(\boldsymbol{x})), \tag{2.5.40}$$

where

$$a_{ij} = a_{ij}(t) := \frac{\eta}{N((\lambda_i(t) - \lambda_j(t))^2 + \eta^2)}, \tag{2.5.41}$$

and $a_{ij}^{\mathcal{S}}$ are their short range version defined as in (2.5.30) with $c_{ij}(t)$ replaced with $a_{ij}(t)$, and

$$\boldsymbol{x}_{\boldsymbol{ab}}^{\boldsymbol{ij}} := \boldsymbol{x} + \left(\prod_{r=1}^{n} \delta_{x_{a_r} i_r} \delta_{x_{b_r} i_r}\right) \sum_{r=1}^{n} (j_r - i_r)(\boldsymbol{e}_{a_r} + \boldsymbol{e}_{b_r}). \tag{2.5.42}$$

The main idea behind this replacement is that infinitesimally $\mathcal{S}(t)$ averages only in one direction at a time, whilst $\mathcal{A}(t)$ averages in all direction simultaneously. This is expressed by the fact that $\boldsymbol{x}_{ab}^{ij}$ from (2.5.23) changes two entries of $\boldsymbol{x}$ per time, instead $\boldsymbol{x}_{\boldsymbol{ab}}^{\boldsymbol{ij}}$ changes all the coordinates of $\boldsymbol{x}$ at the same time, i.e. let $\boldsymbol{i} := (i_1, \dots, i_n), \boldsymbol{j} := (j_1, \dots, j_n) \in [N]^n$, with $\{i_1, \dots, i_n\} \cap \{j_1, \dots, j_n\} = \varnothing$, then $\boldsymbol{x}_{\boldsymbol{ab}}^{\boldsymbol{ij}} \neq \boldsymbol{x}$ if and only if for all $r \in [n]$ it holds that $x_{a_r} = x_{b_r} = i_r$. Technically, the replacement of $\mathcal{S}(t)$ by $\mathcal{A}(t)$ can be performed at the level of Dirichlet forms.

Lemma 2.5.4 (Lemma 4.6 of [167]). *Let $\mathcal{S}(t)$, $\mathcal{A}(t)$ be defined in* (2.5.32) *and* (2.5.40)*, respectively, and let μ denote the uniform measure on Λ^n for which $\mathcal{A}(t)$ is reversible. Then there exists a constant $C(n) > 0$ such that*

$$\langle h, \mathcal{S}(t) h \rangle_{\Lambda^n, \pi} \leq C(n) \langle h, \mathcal{A}(t) h \rangle_{\Lambda^n, \mu} \leq 0, \tag{2.5.43}$$

for any $h \in L^2(\Lambda^n)$, on the very high probability event $\widetilde{\Omega}_\xi \cap \widehat{\Omega}_{\omega,\xi}$.

We start noticing the fact that by (2.5.31) it follows

$$\partial_t \|h_t\|_2^2 = 2 \langle h_t, \mathcal{S}(t) h_t \rangle_{\Lambda^n}. \tag{2.5.44}$$

Then, combining this with (2.5.43), and using that $\boldsymbol{x}_{\boldsymbol{ab}}^{\boldsymbol{ij}} = \boldsymbol{x}$ unless $\boldsymbol{x}_{a_r} = \boldsymbol{x}_{b_r} = i_r$ for all $r \in [n]$, we conclude that

$$\begin{aligned} \partial_t \|h_t\|_2^2 &\leq C(n) \langle h_t, \mathcal{A}(t) h_t \rangle_{\Lambda^n, \mu} \\ &= \frac{C(n)}{2\eta} \sum_{\boldsymbol{x} \in \Lambda^n} \sum_{\boldsymbol{i}, \boldsymbol{j} \in [N]^n}^{*} \left(\prod_{r=1}^{n} a_{i_r j_r}^{\mathcal{S}}(t)\right) \sum_{\boldsymbol{a}, \boldsymbol{b} \in [2n]^n}^{*} \overline{h_t}(\boldsymbol{x}) \big(h_t(\boldsymbol{x}_{\boldsymbol{ab}}^{\boldsymbol{ij}}) - h_t(\boldsymbol{x})\big) \left(\prod_{r=1}^{n} \delta_{x_{a_r} i_r} \delta_{x_{b_r} i_r}\right). \end{aligned} \tag{2.5.45}$$

Then, proceeding as in the proof of [167, Proposition 4.5] (see also [169, Eq. (4.40)]), we conclude that

$$\partial_t \|h_t\|_2^2 \leq -\frac{C_1(n)}{2\eta} \langle h_t \rangle_2^2 + \frac{C_3(n)}{\eta} \mathcal{E}^2 K^n, \tag{2.5.46}$$

which implies $\|h_{T_1}\|_2^2 \leq C(n) \mathcal{E}^2 K^n$, by a simple Gronwall inequality, using that $T_1 \gg \eta$.

We point out that to go from (2.5.45) to (2.5.46) the proof is completely analogous to [167, Proof of Proposition 4.5], with the only exception being the proof of [167, Eqs. (4.41), (4.43)]. We thus now explain how to obtain the analog of [167, Eqs. (4.41), (4.43)] in the current case as well. The fact that we now have the bound (2.5.37) rather than the stronger bound $N^{-1/3}$ as in [169, Eq. (4.31)] does not cause any difference in the final estimate. We thus focus on the main new difficulty in the current analysis, i.e. that in (2.5.31) we choose the initial condition depending on r_0 rather than g_0. We recall that the difference between r_0 and g_0 is that r_0 is defined in such a way all the eigenvector overlaps are precisely centred and normalised in an i–dependent way, whilst for g_0 we can choose the i–independent constant C_0, C_1 so that only the overlap corresponding to a certain base point i_0 is exactly centred and normalised, whilst the nearby overlaps are centred and normalised only modulo a negligible error K/N (see also the paragraph below (2.5.14) for a detailed explanation). This additional difficulty requires that to prove the analog of [167, Eqs. (4.41), (4.43)] we need to estimate the error produced by this mismatch.

Using that the function $f(\boldsymbol{x}) \equiv \mathbf{1}(n \text{ even})$ is in the kernel of $\mathcal{L}(t)$, for any fixed $\boldsymbol{x} \in \Gamma$, and for any fixed $\boldsymbol{i}$, $\boldsymbol{a}$, $\boldsymbol{b}$, we conclude (recall the notation from (2.5.29))

$$\begin{aligned}
&h_t(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) \\
&= \mathcal{U}_{\mathcal{S}}(0,t)\big((\mathrm{Av} r_0)(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) - (\mathrm{Av}\mathbf{1}(n \text{ even}))(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}})\big) \\
&= \mathrm{Av}(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}})\big(\mathcal{U}_{\mathcal{S}}(0,t) r_0(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) - \mathbf{1}(n \text{ even})\big) + \mathcal{O}\left(\frac{N^{\epsilon+n\xi}\ell}{K}\right) \\
&= \left(\mathrm{Av}(\boldsymbol{x}) + \mathcal{O}\left(\frac{\ell}{K}\right)\right)\left(\mathcal{U}(0,t) r_0(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) - \mathbf{1}(n \text{ even}) + \mathcal{O}\left(\frac{N^{1+n\xi}t}{\ell}\right)\right) + \mathcal{O}\left(\frac{N^{\epsilon+n\xi}\ell}{K}\right) \\
&= \mathrm{Av}(\boldsymbol{x})\big(g_t(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) - \mathbf{1}(n \text{ even})\big) + \mathcal{O}\left(\frac{N^{\epsilon+n\xi}\ell}{K} + \frac{N^{1+n\xi}t}{\ell} + \frac{N^{\xi}K}{\sqrt{N}}\right),
\end{aligned} \tag{2.5.47}$$

where in the definition of g_t from (2.5.7), (2.5.19) we chose

$$C_0 := \frac{\langle A \mathrm{Im}\, M(\gamma_{i_0})\rangle}{\langle \mathrm{Im}\, M(\gamma_{i_0})\rangle}, \qquad C_1 := \mathrm{Var}_{\gamma_{i_0}}(A), \tag{2.5.48}$$

and the error terms are uniform in $\boldsymbol{x} \in \Gamma$. Here i_0 is the index defined below (2.5.39). The first three inequalities are completely analogous to [167, Eq. (4.41)]. We now explain how to obtain the last inequality at the price of the additional negligible error $KN^{-1/2}$. Recall the definition of r_t from (2.5.13), (2.5.20), then we now show that for any $\boldsymbol{x}$ supported in the bulk it holds

$$\|r_0(\boldsymbol{x}) - g_0(\boldsymbol{x})\|_\infty \lesssim \frac{N^{\xi}K}{\sqrt{N}}, \tag{2.5.49}$$

for C_0, C_1 chosen as in (2.5.48). Using (2.5.49), together with $\mathcal{U}(0,t)g_0 = g_t$, this proves the last equality in (2.5.47). The main input in the proof of (2.5.49) is the following approximation result

$$\mathring{A}^{\gamma_{i_0}} - \mathring{A}^{\gamma_{i_r}} = \mathcal{O}(|\gamma_{i_0} - \gamma_{i_r}|)I = \mathcal{O}(KN^{-1})I. \tag{2.5.50}$$

We now explain the proof of (2.5.49); for simplicity we present the proof only in the case $n = 2$. To prove (2.5.49) we see that

$$2|\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_{i_0}}\boldsymbol{u}_j\rangle|^2 + \langle \boldsymbol{u}_i, \mathring{A}^{\gamma_{i_0}}\boldsymbol{u}_i\rangle\langle \boldsymbol{u}_j, \mathring{A}^{\gamma_{i_0}}\boldsymbol{u}_j\rangle = 2|\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i}\boldsymbol{u}_j\rangle|^2 + \langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i}\boldsymbol{u}_i\rangle\langle \boldsymbol{u}_j, \mathring{A}^{\gamma_j}\boldsymbol{u}_j\rangle + \mathcal{O}\left(\frac{1}{N}\cdot\frac{N^{\xi}K}{\sqrt{N}}\right),$$

where we used (2.5.50) to replace the "wrong" $\mathring{A}^{\gamma_{i_0}}$ with the "correct" $\mathring{A}^{\gamma_i}$ together with the a priori a bound $\langle \boldsymbol{u}_i, \mathring{A}^{\gamma_i}\boldsymbol{u}_j\rangle \le N^{\xi}N^{-1/2}$. Then multiplying this relation by N we obtain (2.5.49). Additionally, since r_0 and g_0 contain a different rescaling in terms of $\mathrm{Var}_{\gamma_{i_0}}(A)$ and $\mathrm{Var}_{\gamma_i}(A)$, we also used that by similar computations

$$\mathrm{Var}_{\gamma_{i_0}}(A) = \mathrm{Var}_{\gamma_i}(A) + \mathcal{O}\left(\frac{N^{\xi}K}{\sqrt{N}}\right).$$

In particular, we used this approximation to compensate the mismatch that only the diagonal overlaps corresponding to the index i_0 are properly centred and normalised in the definition of g_0, whilst for nearby indices we use this approximation to replace the approximate centering C_0 and normalisation C_1 from (2.5.48) with the correct one, which is the one in the definition of r_0.

Then proceeding as in the proof of [169, Eq. (4.41)] we conclude the analog of [167, Eq. (4.43)]:

$$\begin{aligned}
&\sideset{}{^*}\sum_{\boldsymbol{j}}\left(\prod_{r=1}^{n} a^{\mathcal{S}}_{i_r j_r}(t)\right)\big(g_t(\boldsymbol{x}^{\boldsymbol{ij}}_{\boldsymbol{ab}}) - \mathbf{1}(n \text{ even})\big) \\
&= \sum_{\boldsymbol{j}}\left(\prod_{r=1}^{n} a_{i_r j_r}(t)\right)\left(\frac{N^{n/2}}{\mathrm{Var}_{\gamma_{i_0}}(A)^{n/2}2^{n/2}(n-1)!!}\sum_{G\in\mathcal{G}_{\boldsymbol{\eta j}}} P(G) - \mathbf{1}(n \text{ even})\right) \\
&\quad + \mathcal{O}\left(\frac{N^{n\xi}}{N\eta} + \frac{N^{1+n\xi}\eta}{\ell} + \frac{N^{\xi}K}{\sqrt{N}}\right).
\end{aligned} \tag{2.5.51}$$

Given (2.5.51), the remaining part of the proof is completely analogous to [167, Eqs. (4.44)–(4.51)] except for the slightly different computation

$$\begin{aligned}
&\frac{\langle \operatorname{Im} G(\lambda_{i_{r_1}}+\mathrm{i}\eta)\mathring{A}^{\gamma_{i_0}} \operatorname{Im} G(\lambda_{i_{r_2}}+\mathrm{i}\eta)\mathring{A}^{\gamma_{i_0}}\rangle}{\operatorname{Var}_{\gamma_{i_0}}(A)} \\
&\quad = -\frac{1}{4\operatorname{Var}_{\gamma_{i_0}}(A)} \sum_{\sigma,\tau\in\{+,-\}} \langle G(\lambda_{i_{r_1}}+\sigma\mathrm{i}\eta)\mathring{A}^{\gamma_{i_0}} G(\lambda_{i_{r_2}}+\tau\mathrm{i}\eta)\mathring{A}^{\gamma_{i_0}}\rangle \\
&\quad = -\frac{1}{4\operatorname{Var}_{\gamma_{i_0}}(A)} \sum_{\sigma,\tau\in\{+,-\}} \Big\langle G(\lambda_{i_{r_1}}+\mathrm{i}\sigma\eta)\mathring{A}^{\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta,\gamma_{i_{r_2}}+\mathrm{i}\tau\eta} G(\lambda_{i_{r_2}}+\mathrm{i}\tau\eta)\mathring{A}^{\gamma_{i_{r_2}}+\mathrm{i}\tau\eta,\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta}\Big\rangle \\
&\qquad + \mathcal{O}\left(\frac{K}{N^2\eta^{3/2}}+\frac{K^2}{N^2\eta}+\frac{1}{N\sqrt{\eta}}\right) \\
&\quad = \langle \operatorname{Im} M(\gamma_{i_0})\rangle^2 + \mathcal{O}\left(\frac{K}{N^2\eta^{3/2}}+\frac{K^2}{N^2\eta}+\frac{1}{\sqrt{N\eta}}\right),
\end{aligned} \tag{2.5.52}$$

which replaces [167, Eqs. (4.47)]. Here $\mathring{A}^{\gamma_{i_{r_1}}\pm\mathrm{i}\eta,\gamma_{i_{r_2}}\pm\mathrm{i}\eta}$ is defined as in Definition 2.4.2. We also point out that in the second equality we used the approximation (see (2.4.12))

$$\mathring{A}^{\gamma_{i_0}} = \mathring{A}^{\gamma_{i_{r_1}}\pm\mathrm{i}\eta,\gamma_{i_{r_2}}\pm\mathrm{i}\eta} + \mathcal{O}\big(|\gamma_{i_{r_1}}-\gamma_{i_0}|+|\gamma_{i_{r_2}}-\gamma_{i_0}|+\eta\big)I = \mathring{A}^{\gamma_{i_{r_1}},\gamma_{i_{r_2}}} + \mathcal{O}\left(\frac{K}{N}+\eta\right)I, \tag{2.5.53}$$

together with (here we present the estimate only for one representative term, the other being analogous)

$$\begin{aligned}
\langle G(\lambda_{i_{r_1}}+\mathrm{i}\eta)G(\lambda_{i_{r_2}}+\mathrm{i}\eta)\mathring{A}^{\gamma_{i_0}}\rangle &= \langle G(\lambda_{i_{r_1}}+\mathrm{i}\eta)G(\lambda_{i_{r_2}}+\mathrm{i}\eta)\mathring{A}^{\gamma_{i_{r_2}}+\mathrm{i}\eta,\gamma_{i_{r_2}}+\mathrm{i}\eta}\rangle + \mathcal{O}\left(1+\frac{K}{N\eta}\right) \\
&= \langle M(\gamma_{i_{r_2}}+\mathrm{i}\eta, \mathring{A}^{\gamma_{i_{r_2}}+\mathrm{i}\eta,\gamma_{i_{r_1}}+\mathrm{i}\eta}, \gamma_{i_{r_1}}+\mathrm{i}\eta)\rangle + \mathcal{O}\left(1+\frac{1}{N\eta^{3/2}}+\frac{K}{N\eta}\right) \\
&= \mathcal{O}\left(1+\frac{1}{N\eta^{3/2}}+\frac{K}{N\eta}\right),
\end{aligned} \tag{2.5.54}$$

which follows by (2.5.34) and Lemma 2.4.3 to estimate the deterministic term, together with the integral representation from Lemma 1.5.1 (see also (2.6.17) later), to bound the error terms arising from the replacement (2.5.53). Additionally, in the third equality we used the local for two resolvents from (2.5.34):

$$\begin{aligned}
&-\frac{1}{4}\sum_{\sigma,\tau\in\{+,-\}} \Big\langle G(\lambda_{i_{r_1}}+\mathrm{i}\sigma\eta)\mathring{A}^{\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta,\gamma_{i_{r_2}}+\mathrm{i}\tau\eta} G(\lambda_{i_{r_2}}+\mathrm{i}\tau\eta)\mathring{A}^{\gamma_{i_{r_2}}+\mathrm{i}\tau\eta,\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta}\Big\rangle \\
&\quad = -\frac{1}{4}\sum_{\sigma,\tau\in\{+,-\}} \Big\langle M(\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta, \mathring{A}^{\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta,\gamma_{i_{r_2}}+\mathrm{i}\tau\eta}, \gamma_{i_{r_2}}+\mathrm{i}\tau\eta)\mathring{A}^{\gamma_{i_{r_2}}+\mathrm{i}\tau\eta,\gamma_{i_{r_1}}+\mathrm{i}\sigma\eta}\Big\rangle + \mathcal{O}\left(\frac{1}{\sqrt{N\eta}}\right) \\
&\quad = \langle \operatorname{Im} M(\gamma_{i_0})\rangle^2 \operatorname{Var}_{\gamma_{i_0}}(A) + \mathcal{O}\left(\frac{1}{\sqrt{N\eta}}+\frac{K}{N}\right),
\end{aligned} \tag{2.5.55}$$

with the deterministic term defined in (2.5.35), and in the second equality we used the approximation (2.5.53) again. We remark that here we presented this slightly different (compared to [167, 169]) computation only for chain of length $k = 2$, the computation for longer chains is completely analogous and so omitted. This concludes the proof of (2.5.38).

□

We conclude this section with the proof of Proposition 2.5.1.

Proof of Proposition 2.5.1. Combining the L^2–bound on h_t from Proposition 2.5.3 and the finite speed of propagation estimates in Lemma 2.5.2 we can enhance this L^2–bound to an L^∞–bound completely analogously to the proof of [167, Proposition 3.2] presented in [167, Section 4.4].

□

2.6 Proof of Proposition 2.4.4

Our strategy for proving Proposition 2.4.4 (in the much more involved $\eta \le 1$ regime) is to derive a system of *master inequalities* (Proposition 2.6.3) for the errors in the local laws by cumulant expansion, then use an iterative scheme to gradually improve their estimates. The cumulant expansion naturally introduces longer resolvent chains, potentially leading to an uncontrollable hierarchy, so our master inequalities are complemented by a set of *reduction inequalities* (Lemma 2.6.4) to estimate longer chain in terms of shorter ones. We have used a similar strategy in [168, 169] for Wigner matrices, but now, analogously to Chapter 1, dealing with non-Hermitian i.i.d. matrices, many new error terms due to several adjustments of the z-dependent two-point regularisations need to be handled. By the strong analogy to Chapter 1, our proof of the master inequalities formulated in Proposition 2.6.3 and given in Section 2.6.2 will be rather short and focus on the main differences between Chapter 1 and the current setup.

As the basic control quantities, analogously to [168] and Chapter 1, in the sequel of the proof, we introduce the normalised differences

$$\Psi_k^{\mathrm{av}}(\boldsymbol{z}_k, \boldsymbol{A}_k) := N\eta^{k/2}|\langle G_1 A_1 \cdots G_k A_k - M(z_1, A_1, ..., z_k)A_k)\rangle|\,, \tag{2.6.1}$$

$$\Psi_k^{\mathrm{iso}}(\boldsymbol{z}_{k+1}, \boldsymbol{A}_k, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta^{k+1}}\left|\big(G_1 A_1 \cdots A_k G_{k+1} - M(z_1, A_1, ..., A_k, z_{k+1})\big)_{\boldsymbol{xy}}\right| \tag{2.6.2}$$

for $k \in \mathbf{N}$, where we used the short hand notations

$$G_i := G(z_i)\,, \quad \eta := \min_i |\mathrm{Im}\, z_i|\,, \quad \boldsymbol{z}_k := (z_1, ..., z_k)\,, \quad \boldsymbol{A}_k := (A_1, ..., A_k)\,.$$

The deterministic matrices $\|A_i\| \le 1$, $i \in [k]$, are assumed to be *regular* (i.e., $A_i = \mathring{A}^{z_i, z_{i+1}}$, see Definition 2.4.2) and the deterministic counterparts used in (2.6.1) and (2.6.2) are given recursively in Definition 2.4.1. For convenience, we extend the above definitions to $k = 0$ by

$$\Psi_0^{\mathrm{av}}(z) := N\eta|\langle G(z) - M(z)\rangle|\,, \quad \Psi_0^{\mathrm{iso}}(z, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta}\big|\big(G(z) - M(z)\big)_{\boldsymbol{xy}}\big|$$

and observe that

$$\Psi_0^{\mathrm{av}} + \Psi_0^{\mathrm{iso}} \prec 1 \tag{2.6.3}$$

is the usual single-resolvent local law from (2.4.1), where here and in the following the arguments of $\Psi_k^{\mathrm{av/iso}}$ shall occasionally be omitted. We remark that the index k counts the number of regular matrices in the sense of Definition 2.4.2.

Throughout the entire argument, let $\epsilon > 0$ and $\kappa > 0$ be *arbitrary* but fixed, and let

$$\mathbf{D}^{(\epsilon,\kappa)} := \left\{ z \in \mathbf{C} : \mathrm{Re}\, z \in \mathbf{B}_\kappa\,,\ N^{100} \ge |\mathrm{Im}\, z| \ge N^{-1+\epsilon} \right\} \tag{2.6.4}$$

be the *spectral domain*, where the κ-bulk $\mathbf{B}_\kappa$ has been introduced in (2.2.10). Strictly speaking, we would need to define an entire (finite) *family* of slightly enlarged spectral domains along which the above mentioned iterative scheme for proving Proposition 2.4.4 is conducted. Since this has been carried out in detail in Chapter 1 (see, in particular, Figure 1.4.1), we will neglect this technicality and henceforth assume all bounds on $\Psi_k^{\mathrm{av/iso}}$ to be *uniform* on $\mathbf{D}^{(\epsilon,\kappa)}$ in the following sense.

Definition 2.6.1 (Uniform bounds in the spectral domain)**.** *Let $\epsilon > 0$ and $\kappa > 0$ as above and let $k \in \mathbf{N}$. We say that the bounds*

$$\begin{aligned} \big|\langle G(z_1)B_1 \cdots G(z_k)B_k - M(z_1, B_1, ..., z_k)B_k\rangle\big| &\prec \mathcal{E}^{\mathrm{av}}\,, \\ \Big|\big(G(z_1)B_1 \cdots B_k G(z_{k+1}) - M(z_1, B_1, ..., B_k, z_{k+1})\big)_{\boldsymbol{xy}}\Big| &\prec \mathcal{E}^{\mathrm{iso}} \end{aligned} \tag{2.6.5}$$

hold (ϵ,κ)-uniformly *(or simply* uniformly*) for some deterministic control parameters $\mathcal{E}^{\mathrm{av/iso}} = \mathcal{E}^{\mathrm{av/iso}}(N,\eta)$, depending only on N and $\eta := \min_i |\mathrm{Im}\, z_i|$, if the implicit constant in* (2.6.5) *are uniform in bounded deterministic matrices $\|B_j\| \le 1$, deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \le 1$, and* admissible *spectral parameters $z_j \in \mathbf{D}^{(\epsilon,\kappa)}$ satisfying $1 \ge \eta := \min_j |\mathrm{Im}\, z_j|$.*

Moreover, we may allow for additional restrictions on the deterministic matrices. For example, we may talk about uniformity under the additional assumption that some (or all) of the matrices are regular *(in the sense of Definition 2.4.2).*

Note that (2.6.5) is stated for a fixed choice of spectral parameters z_j in the left hand side, but it is in fact equivalent to an apparently stronger statement, when the same bound holds with a supremum over the spectral parameters (with the same constraints). While one implication is trivial, the other direction follows from (2.6.5) by a standard *grid argument* (see, e.g., the discussion after [168, Definition 3.1]).

We can now formulate Proposition 2.4.4, in the language of our basic control quantities $\Psi_k^{\mathrm{av/iso}}$.

Lemma 2.6.2 (Estimates on $\Psi_1^{\mathrm{av/iso}}$ and $\Psi_2^{\mathrm{av/iso}}$)**.** *For any $\epsilon > 0$ and $\kappa > 0$ we have*

$$\Psi_1^{\mathrm{av}} + \Psi_1^{\mathrm{iso}} \prec 1 \qquad \textit{and} \qquad \Psi_2^{\mathrm{av}} + \Psi_2^{\mathrm{iso}} \prec \sqrt{N\eta} \tag{2.6.6}$$

(ϵ,κ)-uniformly in regular matrices.

Proof of Proposition 2.4.4. The $\eta \ge 1$ case was already explained right after Proposition 2.4.4. The more critical $\eta \le 1$ case immediately follows from Lemma 2.6.2. □

2.6.1 Master inequalities and reduction lemma: Proof of Lemma 2.6.2

We now state the relevant part of a non-linear infinite hierarchy of coupled master inequalities for Ψ_k^{av} and Ψ_k^{iso}. In fact, for our purposes, it is sufficient to have only the inequalities for $k \in [2]$. Slightly simplified versions of this master inequalities will be used in Appendix 2.A for general $k \in \mathbf{N}$. The proof of Proposition 2.6.3 is given in Section 2.6.2

Proposition 2.6.3 (Master inequalities, see Proposition 1.4.9)**.** *Assume that for some deterministic control parameters $\psi_j^{\mathrm{av/iso}}$ we have that*

$$\Psi_j^{\mathrm{av/iso}} \prec \psi_j^{\mathrm{av/iso}}\,, \quad j \in [4]\,, \tag{2.6.7}$$

holds uniformly in regular matrices. Then we have

$$\Psi_1^{\mathrm{av}} \prec 1 + \frac{\psi_1^{\mathrm{av}}}{N\eta} + \frac{\psi_1^{\mathrm{iso}} + (\psi_2^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}\,, \tag{2.6.8a}$$

$$\Psi_1^{\mathrm{iso}} \prec 1 + \frac{\psi_1^{\mathrm{iso}} + \psi_1^{\mathrm{av}}}{(N\eta)^{1/2}} + \frac{(\psi_2^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}\,, \tag{2.6.8b}$$

$$\Psi_2^{\mathrm{av}} \prec 1 + \frac{(\psi_1^{\mathrm{av}})^2 + (\psi_1^{\mathrm{iso}})^2 + \psi_2^{\mathrm{av}}}{N\eta} + \frac{\psi_2^{\mathrm{iso}} + (\psi_4^{\mathrm{av}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\mathrm{iso}})^{1/2} + (\psi_4^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}\,, \tag{2.6.8c}$$

$$\Psi_2^{\mathrm{iso}} \prec 1 + \psi_1^{\mathrm{iso}} + \frac{\psi_1^{\mathrm{av}}\psi_1^{\mathrm{iso}} + (\psi_1^{\mathrm{iso}})^2}{N\eta} + \frac{\psi_2^{\mathrm{iso}} + (\psi_1^{\mathrm{iso}}\psi_3^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/2}} + \frac{(\psi_3^{\mathrm{iso}})^{1/2} + (\psi_4^{\mathrm{iso}})^{1/2}}{(N\eta)^{1/4}}, \tag{2.6.8d}$$

again uniformly in regular matrices.

As shown in the above proposition, resolvent chains of length $k = 1, 2$ are estimated by resolvent chains up to length $2k$. In order to avoid the indicated infinite hierarchy of master inequalities with higher and higher k indices, we will need the following *reduction lemma*.

Lemma 2.6.4 (Reduction inequalities, see Lemma 1.4.10)**.** *As in* (2.6.7)*, assume that* $\Psi_j^{\mathrm{av/iso}} \prec \psi_j^{\mathrm{av/iso}}$ *holds for* $1 \le j \le 4$ *uniformly in regular matrices. Then we have*

$$\Psi_4^{\mathrm{av}} \prec (N\eta)^2 + (\psi_2^{\mathrm{av}})^2, \tag{2.6.9}$$

uniformly in regular matrices, and

$$\begin{aligned} \Psi_3^{\mathrm{iso}} &\prec N\eta\left(1 + \frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right)^{1/2}, \\ \Psi_4^{\mathrm{iso}} &\prec (N\eta)^{3/2}\left(1 + \frac{\psi_2^{\mathrm{iso}}}{\sqrt{N\eta}}\right)\left(1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right) \end{aligned} \tag{2.6.10}$$

again uniformly in regular matrices.

Proof. This is completely analogous to Lemma 1.4.10 and hence omitted. The principal idea is to write out the lhs. of (2.6.9) and (2.6.10) by spectral decomposition and tacitly employ a Schwarz inequality. This leaves us with shortened chains, where certain resolvents G are replaced with absolute values $|G|$, which can be handled by means of a suitable integral representation (see Lemma 1.6.1). □

Now the estimates (2.6.6) follow by combining Proposition 2.6.3 and Lemma 2.6.4 in an iterative scheme, which has been carried out in detail in Section 1.4.3. This completes the proof of Lemma 2.6.2.

2.6.2 Proof of the master inequalities in Proposition 2.6.3

The proof of Proposition 2.6.3 is very similar to the proof of the master inequalities in Proposition 1.4.9. Therefore, we shall only elaborate on (2.6.8a) as a showcase in some detail and briefly discuss (2.6.8b)–(2.6.8d) afterwards.

First, we notice that Lemma 1.5.2 also holds for deformed Wigner matrices (see Lemma 2.6.5 below). In order to formulate it, recall the definition of the *second order renormalisation*, denoted by underline, from [154, Equation (5.3)]. For a function $f(W)$ of the Wigner matrix W, we define

$$\underline{Wf(W)} := Wf(W) - \widetilde{\mathbb{E}}\big[\widetilde{W}(\partial_{\widetilde{W}} f)(W)\big], \tag{2.6.11}$$

where $\partial_{\widetilde{W}}$ denotes the directional derivative in the direction of $\widetilde{W}$, which is a GUE matrix that is independent of W. The expectation is taken w.r.t. the matrix $\widetilde{W}$. Note that, if W itself a GUE matrix, then $\mathbf{E}\underline{Wf(W)} = 0$, while for W with general single entry distributions, this expectation is independent of the first two moments of W. In other words, the underline renormalises the product $Wf(W)$ to second order.

We note that $\widetilde{\mathbf{E}}\widetilde{W}R\widetilde{W} = \langle R\rangle$ and furthermore, that the directional derivative of the resolvent is given by $\partial_{\widetilde{W}} G = -G\widetilde{W}G$. For example, in the special case $f(W) = (W + D - z)^{-1} = G$, we thus have

$$\underline{WG} = WG + \langle G\rangle G$$

by definition of the underline in (2.6.11).

Lemma 2.6.5. *Under the assumption* (2.6.7)*, for any regular matrix* $A = \mathring{A}$ *we have that*

$$\langle (G - M)\mathring{A} \rangle = -\langle \underline{WG\mathring{A}'} \rangle + \mathcal{O}_{\prec}(\mathcal{E}_1^{\mathrm{av}}) \,, \tag{2.6.12}$$

for some other regular matrix $A' = \mathring{A}'$*, which linearly depends on* A *(see* (2.6.21) *for an explicit formula). Here* $G = G(z)$ *and* $\eta := |\mathrm{Im}\, z|$*. For the error term we used the shorthand notation*

$$\mathcal{E}_1^{\mathrm{av}} := \frac{1}{N\eta^{1/2}} \left(1 + \frac{\psi_1^{\mathrm{av}}}{N\eta}\right). \tag{2.6.13}$$

By simple complex conjugation of (2.6.12), we may henceforth assume that $z = e + \mathrm{i}\eta$ with $\eta > 0$. The representation (2.6.12) will be verified later. Now, using (2.6.12) we compute the even moments of $\langle (G - M)\mathring{A} \rangle$ as

$$\mathbf{E}\, |\langle (G - M)A \rangle|^{2p} = \left| - \mathbf{E} \langle \underline{WG} A' \rangle \langle (G - M)A \rangle^{p-1} \langle (G - M)^* A^* \rangle^p \right| + \mathcal{O}_{\prec}\left((\mathcal{E}_1^{\mathrm{av}})^{2p} \right) \tag{2.6.14}$$

and then apply a so-called *cumulant expansion* to the first summand. More precisely, we write out the averaged traces and employ an integration by parts (see, e.g., [165, Eq. (4.14)])

$$\mathbf{E}\, w_{ab} f(W) = \mathbf{E}\, |w_{ab}|^2\, \mathbf{E}\, \partial_{w_{ba}} f(W) + \ldots \quad \text{with} \quad \mathbf{E}\, |w_{ab}|^2 = \frac{1}{N} \,, \tag{2.6.15}$$

indicating higher derivatives and an explicit error term, which can be made arbitrarily small, depending on the number of involved derivatives (see, e.g., [243, Proposition 3.2]). We note that, if W were a GUE matrix, the relation (2.6.15) would be exact without higher derivatives, which shall be discussed below.

Considering the explicitly written *Gaussian term* in (2.6.15) for the main term in (2.6.14), we find that it is bounded from above by (a p-dependent constant times)

$$\mathbf{E}\left[\frac{|\langle GGA'GA \rangle| + |\langle G^* GA' G^* A^* \rangle|}{N^2} |\langle (G - M)A \rangle|^{2p-2} \right]. \tag{2.6.16}$$

The main technical tool to estimate (2.6.16) is the following contour integral representation for the square of resolvent (see Lemma 1.5.1). This is given by

$$G(z)^2 = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta \,, \tag{2.6.17}$$

where the contour $\Gamma = \Gamma(z)$ is the boundary of a finite disjoint union of half bands $J \times [\mathrm{i}\tilde{\eta}, \mathrm{i}\infty)$ which are parametrised counter-clockwise. Here J is a finite disjoint union[11] of closed intervals, which we take as $\mathbf{B}_{\kappa'}$ for a suitable $\kappa' \in (0, \kappa)$ – to be chosen below – and hence contains $e = \mathrm{Re}\, z$; the parameter $\tilde{\eta}$ is chosen to be smaller than, say, $\eta/2$. Applying the integral identity (2.6.17) to the product GG in the term $\langle GGA'GA \rangle$ yields that

$$\left| \langle GGA'GA \rangle \right| \lesssim \left| \int_{\Gamma} \frac{\langle G(\zeta) A' G A \rangle}{(\zeta - z)^2} \mathrm{d}\zeta \right|. \tag{2.6.18}$$

Now, we split the contour Γ in three parts, i.e.

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \,. \tag{2.6.19}$$

[11] Note that, for our concrete setting (2.6.17), one closed interval (i.e. half of Figure 2.6.1) would be sufficient. However, we formulated it more generally here in order to ease the relevant modifications for the (omitted) proofs of (2.6.8b)–(2.6.8d).

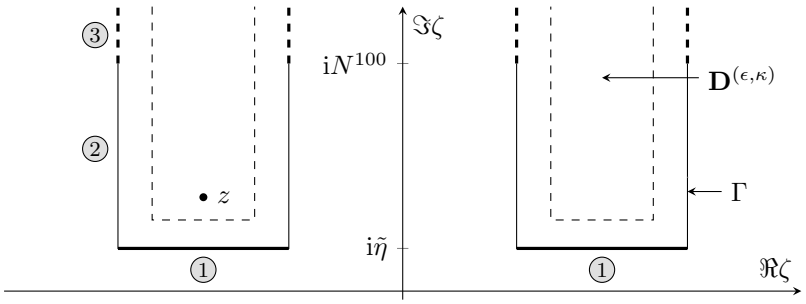

Figure 2.6.1: The contour Γ is split into three parts (see (2.6.19)). Depicted is the situation, where the bulk $\mathbf{B}_\kappa$ consists of two components. The boundary of the associated domain $\mathbf{D}^{(\epsilon,\kappa)}$ is indicated by the two U-shaped dashed lines. Modified version of Figure 1.5.2.

As depicted in Figure 2.6.1, the first part, Γ_1, of the contour consists of the entire horizontal part of Γ. The second part, Γ_2, covers the vertical components up to $|\mathrm{Im}\,\zeta| \le N^{100}$. Finally, Γ_3 consists of the remaining part with $|\mathrm{Im}\,\zeta| > N^{100}$. The contribution coming from Γ_3 can be estimated with a trivial norm bound on G. In order to estimate the integral over Γ_2, we choose the parameter κ' in the definition of $J = \mathbf{B}_{\kappa'}$ in such a way that the distance between z and Γ_2 is greater than $\delta > 0$ from Definition 2.4.2. Hence, for $\zeta \in \Gamma_2$, every matrix is considered regular w.r.t. (z,ζ).

Therefore, after splitting the contour and estimating each contribution as just described, we find, with the aid of Lemma 2.4.3,

$$\big|\langle GGA'GA\rangle\big| \prec \left(1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right) + \int_J \frac{|\langle G(x+\mathrm{i}\tilde\eta)A'G(e+\mathrm{i}\eta)A\rangle|}{(x-e)^2+\eta^2}\mathrm{d}x\,.$$

In this integral over $J = \mathbf{B}_{\kappa'}$, the horizontal part Γ_1, we decompose A and A' in accordance to the spectral parameters of the adjacent resolvents and use the following regularity property:

$$\begin{aligned} A &= \mathring{A}^{e+\mathrm{i}\eta,e+\mathrm{i}\eta} = \mathring{A}^{e+\mathrm{i}\eta,x+\mathrm{i}\tilde\eta} + \mathcal{O}(|x-e|+\eta)I\,,\\ A' &= \big(\mathring{A'}\big)^{e+\mathrm{i}\eta,e+\mathrm{i}\eta} = \big(\mathring{A'}\big)^{x+\mathrm{i}\tilde\eta,e+\mathrm{i}\eta} + \mathcal{O}(|x-e|+\eta)I\,. \end{aligned}$$

Now the integral over J is represented as a sum of four integrals: one of them contains two regular matrices, the rest have at least one identity matrix with a small error factor. For the first one we use the same estimates as for the integral over the vertical part Γ_2. For the other terms, thanks to the identity matrix, we use a resolvent identity, e.g. for $G(x+i\tilde\eta)IG(e+i\eta)$ and note that the$\big(|x-e|+\eta\big)$-error improves the original $1/\eta$-blow up of $\int_J \frac{1}{(x-e)^2+\eta^2}\mathrm{d}x$ to an only $|\log\eta|$-divergent singularity, which is incorporated into '$\prec$'.

For the term $\langle G^*GA'G^*A^*\rangle$ from (2.6.16), we use a similar strategy: After an application of the Ward identity $G^*G = \mathrm{Im}\,G/\eta$, we decompose the deterministic matrices A, A' according to the spectral parameters of their neighbouring resolvents in the product. This argument gives us that each of terms $\langle GGA'GA\rangle$ and $\langle G^*GA'G^*A^*\rangle$ is stochastically dominated by

$$\frac{1}{\eta}\left(1 + \frac{\psi_2^{\mathrm{av}}}{N\eta}\right).$$

Contributions stemming from higher order cumulants in (2.6.16) are estimated exactly in the same way as in Section 1.5.5. The proof of (2.6.8a) is concluded by applying Young inequalities to (2.6.16) (see Section 1.5.1).

Finally we show that (2.6.12) holds:

Proof of Lemma 2.6.5. The proof of this representation is simpler than the proof of its analogue Lemma 1.5.2 because in the current setting all terms in Lemma 1.5.2 containing the particular chiral symmetry matrix E_- are absent. In the same way as in Chapter 1 we arrive at the identity

$$\langle (G-M)A\rangle = -\langle \underline{WG}\mathcal{X}\,[A]\,M\rangle + \langle G-M\rangle\langle (G-M)\mathcal{X}[A]M\rangle\,, \tag{2.6.20}$$

where we introduced the bounded linear operator $\mathcal{X}[B] := \big(1-\langle M\cdot M\rangle\big)^{-1}[B]$. Indeed, boundedness follows from the explicit formula

$$\mathcal{X}[B] = B + \frac{\langle MBM\rangle}{1-\langle M^2\rangle}$$

by means of the lower bound

$$|1-\langle M^2\rangle| = |(1-\langle MM^*\rangle) - 2\mathrm{i}\langle M\mathrm{Im}\,M\rangle| \geq 2\langle \mathrm{Im}\,M\rangle^2 \gtrsim 1\,,$$

obtained by taking the imaginary part of the MDE (2.2.8), in combination with $\|M\|\lesssim 1$.

Next, completely analogously to Lemma 1.5.4, we find the decomposition

$$\mathcal{X}[A]M = \big(\mathcal{X}[A]M\big)^\circ + \mathcal{O}(\eta)I = \mathring{A}' + \mathcal{O}(\eta)I\,, \qquad A' := \mathcal{X}[A]M\,. \tag{2.6.21}$$

Plugging this into (2.6.20), we thus infer

$$\langle (G-M)A\rangle = -\langle \underline{WG}A'\rangle + \langle G-M\rangle\langle (G-M)\mathring{A}'\rangle + \big(-\langle \underline{WG}\rangle + \langle G-M\rangle^2\big)\mathcal{O}(\eta), \tag{2.6.22}$$

The second term in the rhs. of (2.6.22) is obviously bounded by $\psi_1^{\mathrm{av}}/(N^2\eta^{3/2})$ and in the third term we use the usual local law (2.4.1) to estimate it by $\mathcal{O}_\prec\big(N^{-1}\big)$. Combining these information gives (2.6.12). □

Notice that the above arguments leading to (2.6.8a) are completely identical to the ones required in the proof of the analogous master inequality in Proposition 1.4.9 with one minor but key modification: Every term involving the chiral symmetry matrix E_- in Chapter 1 is simply absent and hence, with the notation of Chapter 1, all sums over signs $\sum_{\sigma=\pm}\cdots$ collapse to a single summand with $E_+\equiv I$. With this recipe, the proofs of (2.6.8b)–(2.6.8d) are completely analogous to the ones given in Sections 1.5.2–1.5.4 and hence omitted.

2.A Additional technical Lemmas

In this appendix, we prove several technical lemmas underlying the proofs our main results.

2.A.1 Bounds on averaged multi-resolvent chains

For the proof of Theorem 2.2.9 we need to extend the key estimate $\big|\big\langle G(z_1)\mathring{A}_1G(z_2)\mathring{A}_2\big\rangle\big| < 1$ for $\mathrm{Re}\,z_1,\mathrm{Re}\,z_2\in\mathbf{B}_\kappa$ from (2.4.16), which underlies the proof of the ETH in Theorem 2.2.7, in two directions. First, we need to consider chains with an arbitrary number of resolvents in Lemma 2.A.1, but we will only need a quite weak suboptimal bound which makes its proof quite direct and short. Second, in Lemma 2.A.2, we no longer restrict $\mathrm{Re}\,z_1,\mathrm{Re}\,z_2\in\mathbf{B}_\kappa$ to the bulk, but assume that $|\mathrm{Re}\,z_1-\mathrm{Re}\,z_2|+|\mathrm{Im}\,z_1|+|\mathrm{Im}\,z_2|\geq\nu$ for some N-independent constant $\nu>0$ and additionally allow for arbitrary (non-regular) matrices A_1,A_2. This second extension requires Assumption 2.2.8 on the deformation D, i.e. the boundedness of $M(z)$ also for $\mathrm{Re}\,z\notin\mathbf{B}_\kappa$; this is a slightly stronger requirement than just the boundedness of D assumed in Theorem 2.2.7. Both extensions are relevant for constructing the high probability event $\widehat{\Omega}$ in (2.5.34) for the DBM analysis. The proofs of Lemma 2.A.1 and Lemma 2.A.2 are simple extensions and slight adjustments of the arguments used in Proposition 2.4.4 and they will only be sketched.

Lemma 2.A.1. *Fix $\epsilon > 0$, $\kappa > 0$, $k \in \mathbf{N}$, and consider $z_1, \dots, z_k \in \mathbf{C} \setminus \mathbf{R}$ with $\operatorname{Re} z_j \in \mathbf{B}_\kappa$. Consider regular matrices $A_1, \dots, A_k$ with $\|A_i\| \le 1$, deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$ with $\|\boldsymbol{x}\| + \|\boldsymbol{y}\| \lesssim 1$, and set $G_i := G(z_i)$. Define*

$$\mathcal{G}_k := \widehat{G}_1 A_1 \dots A_{k-1} \widehat{G}_k A_k, \qquad \widehat{G}_j \in \{G_j, |G_j|\}. \tag{2.A.1}$$

Then, uniformly in $\eta := \min_j |\operatorname{Im} z_j| \ge N^{-1+\epsilon}$, we have

$$\big|\langle \mathcal{G}_k \rangle\big| \prec \frac{N^{k/2-1}}{\sqrt{N\eta}}. \tag{2.A.2}$$

Proof. We only consider the case when $\mathcal{G}_k = G_1 A_1 \dots A_{k-1} G_k A_k$, the general case when some G_j is replaced with $|G_j|$ is completely analogous and so omitted. To keep the notation short with a slight abuse of notation we will often denote $(GA)^k = G_1 A_1 \dots A_{k-1} G_k A_k$.

We split the proof into three steps. In Step (i) we first prove the slightly weaker bound $\big|\langle \mathcal{G}_k \rangle\big| \prec N^{k/2-1}$ for any $k \ge 3$ and a similar bound in isotropic sense; then, using Step (i) as an input, we will prove the better estimate (2.A.2) for $k = 3, 4$; finally we prove (2.A.2) for any $k \ge 3$.

Step (i): Similarly to the proof of the reduction inequalities in Lemma 2.6.4 (see Lemma 1.4.10) we readily see that for $k = 2j$ (we omit the indices):

$$\langle (GA)^{2j} \rangle \lesssim N \begin{cases} \langle |G|A(GA)^{j/2-1}|G|A(G^*A)^{j/2-1} \rangle^2 & j \text{ even}, \\ \langle |G|A(GA)^{(j-1)/2}|G|A(G^*A)^{(j-1)/2} \rangle \langle |G|A(GA)^{(j-3)/2}|G|A(G^*A)^{(j-3)/2} \rangle & j \text{ odd}, \end{cases} \tag{2.A.3}$$

and for $k = 2j - 1$:

$$\langle (GA)^{2j-1} \rangle \lesssim \langle |G|A(GA)^{j-2}|G|A(G^*A)^{j-2} \rangle^{1/2} \langle |G|A(GA)^{j-1}|G|A(G^*A)^{j-1} \rangle^{1/2}. \tag{2.A.4}$$

We proceed by induction on the length of the chain. First we use (2.A.3) for $j = 2$, together with $\langle GAGA \rangle \prec 1$ from Proposition 2.4.4, to get the bound $\langle \mathcal{G}_4 \rangle \prec N$, and then use this bound as an input to obtain $\langle \mathcal{G}_3 \rangle \prec N^{1/2}$ using (2.A.4). Then proceeding exactly in the same way we prove that if $\langle \mathcal{G}_l \rangle \prec N^{l/2-1}$ holds for any $l \le k$, then the same bound holds for chains of length $l = k + 1$ and $k + 2$ as well. Similarly, in the isotropic chains we prove $\langle \boldsymbol{x}, \mathcal{G}_k \boldsymbol{y} \rangle \prec N^{(k-1)/2}$; this concludes Step (i).

Step (ii): Given the bounds $\langle \mathcal{G}_k \rangle \prec N^{k/2-1}$, $\langle \boldsymbol{x}, \mathcal{G}_k \boldsymbol{y} \rangle \prec N^{(k-1)/2}$, the estimate in (2.A.2) for $k = 3, 4$ immediately follows by writing the equation for $\mathcal{G}_k$, performing cumulant expansion and using the corresponding bounds on $M(z_1, ..., z_k)$ from Lemma 2.4.3. This was done in [168, Proof of Proposition 3.5], hence we omit the details.

Step (iii): The proof of (2.A.2) for $k \ge 5$ proceeds by induction. We first show that it holds for $k = 5, 6$ and then we prove that if it holds for $k - 2$ and $k - 1$ then it holds for k and $k + 1$ as well.

By Step (ii), it follows that (2.A.2) holds for $k = 3, 4$, and for $k = 2$ we have $\langle GAGA \rangle \prec 1$. Then by (2.A.3) we immediately conclude that the same bound is true for $k = 6$, which together with (2.A.4) also imply the desired bound for $k = 5$. The key point is that (2.A.3) splits a longer k-chain (k even) into a product of shorter chains of length k_1, k_2 with $k_1 + k_2 = k$. As long as $k \ge 5$, at least one of the shorter chain has already length at least three, so we gain the factor $(N\eta)^{-1/2}$. In fact chains of length $k = 2$, from which we do not gain any extra factor, $|\langle GAGA \rangle| \prec 1$, appear only once when we apply (2.A.3) for $k = 6$. But in this case the other factor is a chain of length four with a gain of a $(N\eta)^{-1/2}$ factor. Similarly, (2.A.4) splits the long k chain (k odd) into the square root of two chains of length $k - 1$, and $k + 1$, and for $k \ge 5$ we have the $(N\eta)^{-1/2}$ factor from both. The induction step then readily follows by using again (2.A.3)–(2.A.4) as explained above; this concludes the proof. In fact, in most steps of the induction we gain more than one factor $(N\eta)^{-1/2}$; this would allow us to improve the bound (2.A.2), but for the purpose of the present paper the suboptimal estimate (2.A.2) is sufficient. □

We now turn to the second extension of (2.4.16) allowing for arbitrary spectral parameters, i.e. not necessarily in the bulk, but separated by a safe distance $\nu > 0$.

Lemma 2.A.2. *Fix $\epsilon, \nu > 0$ and let the deformation $D \in \mathbf{C}^{N\times N}$ satisfy Assumption 2.2.8. Let $z_1, z_2 \in \mathbf{C}\setminus\mathbf{R}$ be spectral parameters with $\Delta := |\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2| + |\mathrm{Im}\, z_1| + |\mathrm{Im}\, z_2| \geq \nu > 0$ and $B_1, B_2 \in \mathbf{C}^{N\times N}$ bounded deterministic matrices. Then, uniformly in $\eta := \min\big(|\mathrm{Im}\, z_1|, |\mathrm{Im}\, z_2|\big) \geq N^{-1+\epsilon}$, it holds that*

$$\big|\big\langle G(z_1)B_1G(z_2)B_2\big\rangle\big| \prec 1\,. \tag{2.A.5}$$

Proof. The proof is very similar to that of Proposition 2.4.4, relying on a system of *master inequalities* (Proposition 2.6.3) complemented by the *reduction inequalities* (Lemma 2.6.4), we just comment on the minor differences.

Recall that the naive size, in averaged sense, of a chain

$$G_1B_1G_2B_2\ldots G_{k-1}B_{k-1}G_k \tag{2.A.6}$$

with k resolvents and arbitrary deterministic matrices in between is of order η^{-k+1}; generically this is the size of the corresponding deterministic term in the usual multi-resolvent local law (see [168, Theorem 2.5] with $a = 0$ for the case of Wigner matrices)

$$\begin{aligned} |\langle G_1B_1\cdots G_kB_k - M(z_1, B_1, ..., z_k)B_k\rangle| &\prec \frac{1}{N\eta^k} \\ \Big|\big(G_1B_1\cdots B_kG_{k+1} - M(z_1, B_1, ..., B_k, z_{k+1})\big)_{\boldsymbol{xy}}\Big| &\prec \frac{1}{\sqrt{N\eta}\,\eta^k} \end{aligned} \tag{2.A.7}$$

with the customary short hand notations

$$G_i := G(z_i)\,,\quad \eta := \min_i |\mathrm{Im}\, z_i|\,,\quad \boldsymbol{z}_k := (z_1, ..., z_k)\,,\quad \boldsymbol{B}_k := (B_1, ..., B_k)\,.$$

In the following, we will consider every deterministic matrix B_j together with its neighbouring resolvents, $G_jB_jG_{j+1}$, which we will call the *unit* of B_j. Two units are called *distinct* if they do not share a common resolvent and we will count the number of such distinct units.[12] The main mechanism for the improvement over (2.A.7) in Proposition 2.4.4 for *regular* matrices was that for every distinct unit $G_jB_jG_{j+1}$ in the initial resolvent chain with a *regular* B_j, the naive size of M gets reduced by an η-factor, yielding the bound $\eta^{-\lfloor k/2\rfloor+1}$ in (2.4.14) when all matrices are regular.[13] In the most relevant regime of small $\eta \sim N^{-1+\epsilon}$ this improvement in M is (almost) matched by the corresponding improvement in the error term, see $N^{k/2-1}$ in (2.4.15a) (except that for odd k, the error is bigger by an extra $\eta^{-1/2} \sim N^{1/2}$).

The key point is that if the spectral parameters z_j and z_{j+1} are "far away" in the sense that

$$\Delta_j := |\mathrm{Re}\, z_j - \mathrm{Re}\, z_{j+1}| + |\mathrm{Im}\, z_j| + |\mathrm{Im}\, z_{j+1}| \geq \nu > 0\,, \tag{2.A.8}$$

then *any* matrix B_j in the chain $\ldots G_jB_jG_{j+1}\ldots$ behaves as if it were regular. The reason is that the corresponding stability operator $\mathcal{B}_{j,j+1}$ from (2.4.6) (explicitly given in (2.A.14) and (2.A.15) below) has no singular direction, its inverse is bounded, i.e.

$$\|\mathcal{B}_{j,j+1}^{-1}[R]\| \lesssim \|R\| \quad \text{for all} \quad j \in [k]\,,\quad R \in \mathbf{C}^{N\times N}\,.$$

For example, using the definition (2.4.5), we have

$$\|M(z_1, B, z_2)\| \lesssim 1 \tag{2.A.9}$$

[12] However, in the averaged case, one of the k resolvents can be "reused" in this counting.

[13] This improvement was also termed as the $\sqrt{\eta}$-rule, asserting that every regular matrix improves the M bound and the error in the local law by a factor $\sqrt{\eta}$. This formulation is somewhat imprecise, the M bound always involves integer $1/\eta$-powers; the correct counting is that each *distinct* unit of regular matrices yields a factor η. However, the $\sqrt{\eta}$-rule applies to the error term.

whenever $\Delta_{12} \geq \nu$, hence $\mathcal{B}_{12}^{-1}$ is bounded. Therefore, when mimicking the proof of Proposition 2.4.4, instead of counting regular matrices with distinct units, we need to count the distinct units within the chain (2.A.6) for which the corresponding spectral parameters are far away; their overall effects are the same – modulo a minor difference, that now the errors for odd k do not get increased by $\eta^{-1/2}$ when compared to the M-bound (see later).

To be more precise, in our new setup we introduce the modified[14] normalised differences

$$\widetilde{\Psi}_k^{\mathrm{av}}(\boldsymbol{z}_k, \boldsymbol{B}_k) := N\eta^{\lfloor k/2 \rfloor} |\langle G_1 B_1 \cdots G_k B_k - M(z_1, B_1, ..., z_k) B_k \rangle| , \tag{2.A.10}$$

$$\widetilde{\Psi}_k^{\mathrm{iso}}(\boldsymbol{z}_{k+1}, \boldsymbol{B}_k, \boldsymbol{x}, \boldsymbol{y}) := \sqrt{N\eta}\, \eta^{\lfloor k/2 \rfloor} \left| \big(G_1 B_1 \cdots B_k G_{k+1} - M(z_1, B_1, ..., B_k, z_{k+1}) \big)_{\boldsymbol{xy}} \right| \tag{2.A.11}$$

for $k \in \mathbf{N}$ as a new set of basic control quantities (cf. (2.6.1) and (2.6.2)). The deterministic counterparts M used in (2.A.10) and (2.A.11) are again given recursively in Definition 2.4.1. Contrary to (2.6.1) and (2.6.2), the deterministic matrices $\|B_j\| \leq 1$, $j \in [k]$, are *not* assumed to be regular. This "lack of regularity" is compensated by the requirement that consecutive spectral parameters z_j, z_{j+1} of the unit of B_j satisfy (2.A.8). Just as in Definition 2.4.2, in case of (2.A.10), the indices in (2.A.8) are understood cyclically modulo k. Chains satisfying (2.A.8) for all $j \in [k]$ are called *good*. Hence, in a good chain one can potentially gain a factor η from every unit $G_j B_j G_{j+1}$. Therefore, analogous to the regularity requirement for *all* deterministic matrices in Definition 2.6.1, the normalised differences $\widetilde{\Psi}_k^{\mathrm{av/iso}}$ in (2.A.10) and (2.A.11) will only be used for good chains.

As already indicated above, the analogy between our new setup and the setup of Proposition 2.4.4 is not perfect due to the following reason: For $k = 1$ the error bounds in (2.A.7) improve by $\sqrt{\eta}$ for B_1 being a regular matrix, but for $\Delta_1 \geq \nu > 0$, the improvement is by a full power of η.[15] This discrepancy causes slightly different η-powers for odd k in all estimates (cf. (2.A.10) and (2.A.11)).

We now claim that for good chains, the requirement (2.A.8) for all $j \in [k]$ reduces the naive sizes of the errors in the usual multi-resolvent local laws (2.A.7) at least by a factor $\eta^{\lceil k/2 \rceil}$ for $k = 1, 2$. Previously, in the proof of Proposition 2.4.4, these sizes got reduced by a factor $\sqrt{\eta}$ for every matrix B_j which was regular in the sense of Definition 2.4.2. Now, compared to this *regularity gain*, the main effect for our new, say, *ν-gain* for good chains is that for every $j \in [k]$, the inverse of the stability operator (2.4.6) (explicitly given in (2.A.14) and (2.A.15) below) is bounded, i.e.

$$\|\mathcal{B}_{j,j+1}^{-1}[R]\| \lesssim \|R\| \quad \text{for all} \quad j \in [k], \quad R \in \mathbf{C}^{N \times N} . \tag{2.A.12}$$

Armed with (2.A.12), completely analogously to Proposition 2.4.4, one then starts a proof of the master inequalities (similar to those in Proposition 2.6.3): First, one establishes suitable *underlined lemmas* (cf. Lemma 2.6.5 and Lemmas 1.5.2, 1.5.6, 1.5.8, and 1.5.9), where now no *splitting* of observables into singular and regular parts (see, e.g., (2.6.21) and (1.5.35)) is necessary, since the bounded matrices B_j are arbitrary. Afterwards, the proof proceeds by cumulant expansion (see (2.6.16)), where resolvent chains of length k are estimated by resolvent chains of length up to $2k$. This potentially infinite hierarchy is truncated by suitable reduction inequalities, as in Lemma 2.6.4. Along this procedure, we also create non-good chains, but a direct inspection[16] shows that there are always sufficiently many good chains left that provide the necessary improvements, exactly as in the proof of Proposition 2.4.4.

Just to indicate this mechanism, consider, for example, the Gaussian term appearing in the cumulant expansion of (2.A.10) for $k = 2$, analogous to (2.6.16). In this case we encounter the following term

[14] Notice that for odd k, the η-power in the prefactor is slightly different from those in (2.6.1) and (2.6.2).

[15] For the averaged case, this improvement is really artificial, since the $\Delta_1 \geq \nu$-requirement means that $\eta \gtrsim 1$.

[16] We spare the reader from presenting the case by case checking for the new setup, but we point out that this is doable since Lemma 2.A.2, as well as Proposition 2.4.4, concern chains of length at most $k \leq 2$. Extending these local laws for general k is possible, but it would require a more systematic power-counting of good chains.

with five resolvents that we immediately estimate in terms of chains with four resolvents:

$$\frac{|\langle G_2B_2G_1B_1G_2G_1B_1G_2B_2\rangle|}{N^2} \prec \frac{|\langle G_2B_2G_1B_1G_2B_1G_2B_2\rangle| + |\langle G_2B_2G_1B_1G_1B_1G_2B_2\rangle|}{N^2}. \quad (2.A.13)$$

Here we used that $\Delta = |\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2| + |\mathrm{Im}\, z_1| + |\mathrm{Im}\, z_2| \geq \nu > 0$ to reduce G_2G_1 to a single G term. Strictly speaking, the estimate (2.A.13) directly follows from the resolvent identity $G_2G_1 = (G_2 - G_1)/(z_1 - z_2)$ only when $|z_1 - z_2| \sim |\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2| + |\mathrm{Im}\, z_1 - \mathrm{Im}\, z_2| \gtrsim \nu$; this latter condition follows from $\Delta \geq \nu$ only if $\mathrm{Im}\, z_1 \cdot \mathrm{Im}\, z_2 < 0$. In the remaining case, when $z_1 \approx z_2$ but both with a large imaginary part (since $\Delta \geq \nu$), we can use an appropriate contour integral representation

$$G(z_2)G(z_1) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{G(\zeta)}{(\zeta - z_1)(\zeta - z_2)} \mathrm{d}\zeta$$

similar to (2.6.17) with a contour well separated from z_1, z_2. Hence we obtain a four-resolvent chain $\langle G_2B_2G_1B_1G(\zeta)B_1G_2B_2\rangle$ on the rhs. of (2.A.13), where the spectral parameter ζ of the new $G(\zeta)$ resolvent is "far away" from the other spectral parameters, and it can be treated as $\langle G_2B_2G_1B_1G_jB_1G_2B_2\rangle$, $j = 1, 2$. In fact, in our concrete application of Lemma 2.A.2 we always know that not only $\Delta \geq \nu$, but already $|\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2| \geq \nu$, hence the argument with the resolvent identity is always sufficient.

Note that the two chains on the rhs. of (2.A.13) cannot be directly cast in the form (2.A.10), since not every unit has well separated spectral parameters (e.g. we have $G_2B_1G_2$), hence these chains are not good. However, after application of a reduction inequality (see (2.A.3)), we find that

$$|\langle G_2B_2G_1B_1G_2B_1G_2B_2\rangle| \prec N\big(\langle |G_2|B_2|G_1|B_2^*\rangle\langle |G_1|B_1|G_2|B_1^*\rangle\langle |G_2|B_1|G_2|B_1^*\rangle\langle |G_2|B_2|G_2|B_2^*\rangle\big)^{1/2}$$

and analogously for the second summand in (2.A.13). Estimating the two shorter non-good chains involving only G_2 by $1/\eta$ via a trivial Schwarz inequality, this yields that

$$\frac{|\langle G_2B_2G_1B_1G_2G_1B_1G_2B_2\rangle|}{N^2} \prec \frac{\big(\langle |G_2|B_2|G_1|B_2^*\rangle\langle |G_1|B_1|G_2|B_1^*\rangle\big)^{1/2}}{N\eta},$$

where the remaining shorter chains are good and can be estimated in terms of $\widetilde{\Psi}_2^{\mathrm{av}}$. A similar mechanism works for any other term. This completes the discussion of the discrepancies between the current setup and Proposition 2.4.4.

Notice that this argument always assumes that we have a single resolvent local law and that M's are bounded. At potential cusps in the scDos ρ we do not have a single resolvent local law (see the discussion below (2.4.1)) and the estimates on $\|M(z)\|$ for $\mathrm{Re}\, z$ close to edges (and cusps) of ρ may deteriorate for general deformation D. However, these two phenomena are simply excluded by Assumption 2.2.8 on the deformation D (see also Remark 2.2.12). In particular, this assumption allows us to show exactly same estimates on $M(z_1, ..., z_k)$ as given in Lemma 2.4.3, which serve as an input in the proof sketched above.

To conclude, similarly to Proposition 2.4.4, our method again shows that

$$\big|\big\langle \big(G(z_1)B_1G(z_2) - M(z_1, B_1, z_2)\big)B_2\big\rangle\big| \prec \frac{1}{(N\eta)^{1/2}}$$

which, together with the corresponding bound (2.A.9), immediately yields the desired bound and completes the proof of Lemma 2.A.2. □

2.A.2 Proof of Lemma 2.4.3

The proof is completely analogous to the proof of Lemma 1.4.3, hence we only show how the three main technical aspects of the latter should be adjusted to our setup of deformed Wigner matrices. In general, the setup of Chapter 1 is more complicated due to the chiral symmetry which involves summations over signs $\sigma = \pm$. As a rule of thumb, we can obtain the necessary M-formulas for our current case just by mechanically using the corresponding formulas in Chapter 1 and drop the $\sigma = -1$ terms.

Recursive Relations: The principal idea is to derive several different recursive relations for $M(z_1, ..., z_k)$ (which itself is defined by one of those in Definition 2.4.1) by a so-called *meta argument* [181, 154]. These alternative recursions can then be employed to prove Lemma 2.4.3 iteratively in the number of spectral parameters. These recursive relations are identical to those in Lemma 1.D.1 when dropping the $\sigma = -1$ terms in Eqs. (D.1) and (D.2) therein and writing the $N \times N$ identity instead of E_+.

Stability Operator: The inverse of the stability operator (2.4.6) can be expressed in the following explicit form

$$\mathcal{B}_{12}^{-1}[R] = R + \frac{\langle R \rangle}{1 - \langle M_1 M_2 \rangle} M_1 M_2 = R + \frac{1}{\beta_{12}} \langle R \rangle M_1 M_2 \,, \tag{2.A.14}$$

where $\beta_{12} := 1 - \langle M_1 M_2 \rangle$ is the only non-trivial eigenvalue of $\mathcal{B}_{12}$. Completely analogously to Lemma 1.B.5 (b), it holds that

$$|\beta_{12}| \gtrsim \big(|\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2| + |\mathrm{Im}\, z_1| + |\mathrm{Im}\, z_2| \big) \wedge 1 \,, \tag{2.A.15}$$

which, in combination with (2.A.14), in particular implies (2.4.13) for $k = 1$ and (2.4.14) for $k = 2$.

Longer Chains: In order to prove (2.4.13) for $k = 3$, similarly to Chapter 1 we verify at first (2.4.13) for $k = 2$ in the case when exactly one observable is regular. For this purpose we again use the recursive relation of the form (1.D.12):

$$M(z_1, A_1, z_2, A_2, z_3) = M(z_1, \mathcal{X}_{12}[A_1] M_2 A_2, z_3) + M(z_1, \mathcal{X}_{12}[A_1] M_2, z_3) \langle M(z_2, A_2, z_3) \rangle \,,$$

where we denoted the linear operator $\mathcal{X}_{mn}$ as

$$\mathcal{X}_{mn}[R] := \big(1 - \langle M_m \cdot M_n \rangle \big)^{-1} [R] \qquad \text{for} \quad R \in \mathbf{C}^{N \times N} \,. \tag{2.A.16}$$

Now, similarly to the arguments around (1.D.13), we observe a balancing cancellation in the last term, which comes from the continuity with respect to one of spectral parameters of the regular part of a deterministic matrix when another spectral parameter is fixed (see (2.4.12)). □

Chapter 3

Eigenstate thermalisation at the edge for Wigner matrices

This chapter contains the paper [150]:

G. Cipolloni, L. Erdős, and J. Henheik. Eigenstate thermalisation at the edge for Wigner matrices. *arXiv:2309.05488*, 2023

Abstract. We prove the Eigenstate Thermalisation Hypothesis for Wigner matrices uniformly in the entire spectrum, in particular near the spectral edges, with a bound on the fluctuation that is optimal for any observable. This complements earlier works of Cipolloni et. al. [165, 169] and Benigni et. al. [62, 60] that were restricted either to the bulk of the spectrum or to special observables. As a main ingredient, we prove a new multi-resolvent local law that optimally accounts for the edge scaling.

Content

3.1 Introduction

In the physics literature, the *Eigenstate Thermalisation Hypothesis (ETH)* asserts that each eigenfunction of a sufficiently chaotic quantum system is uniformly distributed in the phase space. This concept was coined by Srednicki [535] after similar ideas appeared in the seminal paper of Deutsch [221]. While the original physics setup concerns genuine many-body systems, especially a small system in a heat bath described by standard statistical mechanics, Deutsch has also formulated a phenomenological version of ETH for the simplest chaotic quantum system, the Wigner ensemble. In this form, ETH asserts that for any deterministic observable (matrix) A and for any normalised eigenvector $\boldsymbol{u}$ of a large $N \times N$ Wigner matrix, the quadratic form $\langle \boldsymbol{u}, A\boldsymbol{u}\rangle$ is very close to its statistical average, which, in the Wigner case, is the normalized trace $\langle A\rangle := \frac{1}{N}\operatorname{Tr} A$:

$$|\langle \boldsymbol{u}, A\boldsymbol{u}\rangle - \langle A\rangle| \lesssim \frac{\|A\|}{\sqrt{N}}. \tag{3.1.1}$$

The $1/\sqrt{N}$ speed of convergence is optimal and it is in agreement with the earlier predictions of Feingold and Peres [258], see also [234]. For more physics background and references, see the introduction of [165].

In the mathematics literature the same phenomenon is known as the *Quantum Unique Ergodicity (QUE)*. In precise mathematical terms it was formulated by Rudnick and Sarnak [510] for standard quantisations of ergodic classical dynamical systems and proved only in some special cases [424, 533, 349, 123], often as a purely limiting statement without optimizing the speed of convergence. The key point is to control the behaviour of *all* eigenfunctions; a similar result for *most* eigenfunctions (called *Quantum Ergodicity*) is much easier and has been earlier discovered by Šnirel'man [528], see also [177, 600].

Motivated by the paper of Deutsch [221] and the novel technical developments in random matrix theory, the ETH for Wigner matrices in the form (3.1.1) has been the object of intense study in recent years. An important question is the precise dependence of the error term in the right hand side on A. The first proof of (3.1.1) given in [165] involved the operator norm $\|\mathring{A}\|$ of the traceless part $\mathring{A} := A - \langle A\rangle$ of A, but this estimate is far from optimal for low rank observables. For example, if $A = |\boldsymbol{q}\rangle\langle\boldsymbol{q}|$ is a rank–one projection onto a normalised vector $\boldsymbol{q} \in \mathbf{C}^N$, then $\langle \boldsymbol{u}, A\boldsymbol{u}\rangle = |\langle \boldsymbol{u}, \boldsymbol{q}\rangle|^2$ which is known to be essentially of order $1/N$ by the *complete delocalisation of eigenvectors*, see [246, 249, 368, 88, 63].

However the result in [165] gives only the suboptimal estimate $|\langle \boldsymbol{u}, \boldsymbol{q}\rangle|^2 \lesssim 1/\sqrt{N}$ for this special observable.

In the Gaussian (GUE/GOE) case, the eigenvectors are uniformly Haar distributed, hence explicit moment calculations for $\langle \boldsymbol{u}, A\boldsymbol{u}\rangle$ are possible by Weingarten calculus. The result indicates the following optimal form of (3.1.1):

$$|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\langle A\rangle| \lesssim \frac{\langle |\mathring{A}|^2\rangle^{1/2}}{\sqrt{N}}. \tag{3.1.2}$$

Note that this estimate involves the (normalised) Hilbert-Schmidt norm $\langle |\mathring{A}|^2\rangle^{1/2}$ instead of the operator norm[1], and it can also be extended to different eigenvectors $\boldsymbol{u}_i, \boldsymbol{u}_j$. In particular, (3.1.2) recovers the optimal delocalisation bound for eigenvectors as a special case.

The optimal form of ETH (3.1.2) for any Wigner matrix was proved for the special case when A is a projection in [62, 60], and for arbitrary A but only in the bulk of the spectrum[2] in [169]. In fact, $\sqrt{N}[\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\langle A\rangle]$ is asymptotically normal with variance proportional to $\langle |\mathring{A}|^2\rangle^{1/2}$ (see [167, 169]) in the bulk, showing that the Hilbert-Schmidt norm $\langle |\mathring{A}|^2\rangle^{1/2}$ is indeed the optimal one. In the main theorem of the current paper (see Theorem 3.2.2 below) we prove (3.1.2) for all observables and all eigenfunctions, giving the optimal ETH for Wigner matrices in all spectral regimes.

We remark that ETH is expected to hold for much more general random matrix ensembles. For example the approach in [165] could be directly generalized to a special class of generalized Wigner matrices in [5]. Furthermore, ETH in the bulk has recently been extended to deformed random matrix ensembles (see Chapters 1 and 2), where both the leading term $\delta_{ij}\langle A\rangle$ and the correct replacement for the traceless part of A in the right hand side of (3.1.2) became considerably more involved, in particular they are energy dependent. The edge regime and the optimal norm of A in the error term are still open questions for these ensembles, but we leave them to further works and for simplicity we focus on the Wigner case here.

The key tool to achieve our ETH is a new *multi–resolvent local law* with traceless observables that is optimal at the spectral edges. Multi–resolvent local laws in general refer to concentration results for alternating products of resolvents of a random matrix and deterministic matrices (observables). Their proofs are typically more difficult at the spectral edges since, besides correctly accounting for the traceless observables, their optimal form also requires to exploit a delicate cancellation mechanism; the smallness of the local density of eigenvalues must accurately compensate for the linear instability of a nonlinear equation that governs the fluctuation. In contrast to the previous proofs of local laws behind ETH results, here we apply a dynamical approach, the *method characteristic flow* complemented with a Green function comparison argument. While this method has already been extensively tested for single resolvent local laws [105, 353, 6, 392, 7, 394, 14], only two papers concern the multi-resolvent situation [161, 111], neither of them focuses on the critical edge behaviour. On top of the edge, we will need to track another key aspect of the local law; in order to obtain the Hilbert-Schmidt norm in (3.1.2), the same norm must appear in the local law as well. Typically, errors in the local laws involve the operator norm of the deterministic matrices between the resolvents, the control in the much harder Hilbert-Schmidt sense was considered only very recently in [169]. However, this work did not track the optimal edge behaviour. Our new local law is simultaneously optimal in both aspects. We will explain the strength of this new local law in the context of previous works in Section 3.2.1.

[1] Note that $\langle |\mathring{A}|^2\rangle^{1/2}$ is substantially smaller than $\|\mathring{A}\|$ for matrices $\mathring{A}$ of low rank; in fact, if $\mathrm{rank}(\mathring{A}) = 1$, then $\|\mathring{A}\| = \sqrt{N}\langle |\mathring{A}|^2\rangle^{1/2}$, losing the entire $\sqrt{N}$ factor in (3.1.1) compared with the optimal (3.1.2).

[2] We point out that the end of the proof of Theorem 2.2 in the published version of [169] contained a small error; a correct and in fact simpler argument was given in the updated arXiv:2203.01861 version of the paper.

Notations

By $\lceil x \rceil := \min\{m \in \mathbf{Z} : m \geq x\}$ and $\lfloor x \rfloor := \max\{m \in \mathbf{Z} : m \leq x\}$ we denote the upper and lower integer part of a real number $x \in \mathbf{R}$. We set $[k] := \{1, ..., k\}$ for $k \in \mathbf{N}$ and $\langle A \rangle := d^{-1}\mathrm{Tr}(A)$, $d \in \mathbf{N}$, for the normalised trace of a $d \times d$-matrix A. For positive quantities A, B we write $A \lesssim B$ resp. $A \gtrsim B$ and mean that $A \leq CB$ resp. $A \geq cB$ for some N-independent constants $c, C > 0$ that depend only on the basic control parameters of the model in Assumption 3.2.1 below. Moreover, for N-dependent positive quantities A, B, we write $A \ll B$ whenever $A/B \to 0$ as $N \to \infty$.

We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, for some $N \in \mathbf{N}$, and define

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_i \bar{x}_i y_i \,, \qquad A_{\boldsymbol{xy}} := \langle \boldsymbol{x}, A\boldsymbol{y} \rangle \,.$$

Matrix entries are indexed by lower case Roman letters $a, b, c, ..., i, j, k, ...$ from the beginning or the middle of the alphabet and unrestricted sums over those are always understood to be over $\{1, ..., N\}$.

Finally, we will use the concept *'with very high probability'*, meaning that for any fixed $D > 0$, the probability of an N-dependent event is bigger than $1 - N^{-D}$ for all $N \geq N_0(D)$. Also, we will use the convention that $\xi > 0$ denotes an arbitrarily small positive exponent, independent of N. Moreover, we introduce the common notion of *stochastic domination* (see, e.g., [241]): For two families

$$X = \left(X^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right)$$

of non-negative random variables indexed by N, and possibly a parameter u, we say that X is stochastically dominated by Y, if for all $\epsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\epsilon} Y^{(N)}(u)\right] \leq N^{-D}$$

for large enough $N \geq N_0(\epsilon, D)$. In this case we write $X \prec Y$. If for some complex family of random variables we have $|X| \prec Y$, we also write $X = O_{\prec}(Y)$.

Acknowledgment.

We thank Volodymyr Riabov for his help with creating Figure 3.3.1 and for pointing out the missing condition $|\sigma| < 1$ in Assumption 3.2.1.

3.2 Main results

We consider $N \times N$ Wigner matrices W, i.e. W is a random real symmetric or complex Hermitian matrix $W = W^*$ with independent entries (up to the Hermitian symmetry) and with single entry distributions $w_{aa} \overset{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{d}}$, and $w_{ab} \overset{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{od}}$, for $a > b$. The random variables $\chi_{\mathrm{d}}, \chi_{\mathrm{od}}$ satisfy the following assumptions.[3]

Assumption 3.2.1. *The off-diagonal distribution χ_{od} is a real or complex centered random variable, $\mathbf{E}\,\chi_{\mathrm{od}} = 0$, with $\mathbf{E}\,|\chi_{\mathrm{od}}|^2 = 1$, and we have that $\sigma := \mathbf{E}\,\chi_{\mathrm{od}}^2$ satisfies $|\sigma| < 1$. The diagonal distribution is a real centered random variable, $\mathbf{E}\,\chi_{\mathrm{d}} = 0$. Furthermore, we assume the existence of high moments, i.e. for any $p \in \mathbf{N}$ there exists $C_p > 0$ such that*

$$\mathbf{E}\left[|\chi_{\mathrm{d}}|^p + |\chi_{\mathrm{od}}|^p\right] \leq C_p \,.$$

[3] By inspecting our proof, it is easy to see that actually we do not need to assume that the off-diagonal entries of W are identically distributed. We only need that they all have the same second moments, but higher moments can be different.

Our main result is the optimal form of the eigenstate thermalization hypothesis (ETH) for Wigner matrices uniformly in the spectrum, in particular, including the spectral edges. Its proof is given in Section 3.2.2 and it is based on a new *multi-resolvent local law*, Theorem 3.2.4 below.

Theorem 3.2.2 (Eigenstate Thermalization Hypothesis)**.** *Let W be a Wigner matrix satisfying Assumption 3.2.1 with orthonormalized eigenvectors $\boldsymbol{u}_1, ..., \boldsymbol{u}_N$ and let $A \in \mathbf{C}^{N\times N}$ be deterministic. Then*

$$\max_{i,j\in[N]} |\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \delta_{ij}\langle A\rangle| \prec \frac{\langle |\mathring{A}|^2\rangle^{1/2}}{\sqrt{N}} \tag{3.2.1}$$

where $\mathring{A} := A - \langle A\rangle$ denotes the traceless part of A.

3.2.1 Multi-resolvent local laws

Consider the resolvent $G(z) := (W - z)^{-1}$, with $z \in \mathbf{C}\setminus\mathbf{R}$. It is well known that in the limit $N \to \infty$ the resolvent becomes deterministic, with its deterministic approximation $m_{\mathrm{sc}}(z)\cdot I$, where m_{sc} is the Stieltjes transform of the semicircular law:

$$m(z) := m_{\mathrm{sc}}(z) = \int_{\mathbf{R}} \frac{1}{x-z}\rho_{\mathrm{sc}}(x)\,\mathrm{d}x, \qquad \rho_{\mathrm{sc}}(x) := \frac{1}{2\pi}\sqrt{[4-x^2]_+}. \tag{3.2.2}$$

This holds even in the local regime as long as $|\mathrm{Im}\, z| \gg N^{-1}$; such concentration results are commonly called *local laws*.

The single resolvent local law, in its simplest form[4], asserts that

$$\big|\langle (G(z) - m(z))A\rangle\big| \prec \frac{\|A\|}{N\eta}, \qquad \eta := |\mathrm{Im}\, z|, \tag{3.2.3}$$

holds for any deterministic matrix (*observable*) A. The $1/N\eta$ error is optimal for $A = I$ in the relevant $\eta \lesssim 1$ regime and $N\eta\langle G(z) - m(z)\rangle$ is approximately Gaussian with variance of order one [329]. However, for traceless observables, i.e. $\langle A\rangle = 0$, hence $A = \mathring{A}$, the bound in (3.2.3) improves to the optimal form,

$$\big|\langle (G(z) - m(z))A\rangle\big| = \big|\langle G(z)A\rangle\big| \prec \frac{\sqrt{\rho(z)}}{N\sqrt{\eta}}\langle |A|^2\rangle^{1/2}, \qquad \rho(z) := \frac{1}{\pi}|\mathrm{Im}\, m(z)|.$$

The improvement in the η-power together and the additional density factor $\rho(z)$ relevant near the spectral edges were first observed in [165], while the optimal dependence on the Hilbert-Schmidt norm of A was proved in [169]. Single resolvent local laws, however, are not sufficient to control the eigenfunction overlaps as in (3.2.1). While the local law, via the spectral decomposition of $\mathrm{Im}\, G = \frac{1}{2\mathrm{i}}(G - G^*)$,

$$\langle \mathrm{Im}\, G(z)A\rangle = \frac{1}{N}\sum_i \frac{\eta}{(\lambda_i - E)^2 + \eta^2}\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle, \qquad z = E + \mathrm{i}\eta, \tag{3.2.4}$$

gives an effectively local average of approximately $N\eta$ diagonal overlaps $\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle$, inferring the size of a single overlap is not possible just from this average since $\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle$ may change sign as i varies.

Two-resolvent local laws are much more powerful. In particular, using

$$\langle \mathrm{Im}\, G(z_1)A\mathrm{Im}\, G(z_2)A^*\rangle = \frac{1}{N}\sum_{i,j} \frac{\eta}{(\lambda_i - E_1)^2 + \eta^2}\frac{\eta}{(\lambda_j - E_2)^2 + \eta^2}|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle|^2, \quad z_l = E_l + \mathrm{i}\eta,\ \ l = 1, 2, \tag{3.2.5}$$

[4] Traditionally [249, 368, 88], local laws did not consider arbitrary test matrix A, but only $A = I$ or special rank one projections leading the *isotropic local laws*. General A was included later, e.g. in [243].

we see that for a traceless observable, $\langle A\rangle = 0$, a bound of the form

$$\langle \operatorname{Im} G(z_1) A \operatorname{Im} G(z_2) A^*\rangle \prec \|A\|^2 \tag{3.2.6}$$

at $\eta \sim N^{-1+\xi}$, $\xi > 0$, would imply that a local average (in both indices) of $|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle|^2$ is bounded by $N^{-1+2\xi}\|A\|^2$. Since $|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle|^2$ is positive (unlike $\langle \boldsymbol{u}_i, A\boldsymbol{u}_i\rangle$ in (3.2.4)), we can deduce the optimal bound $|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle|^2 \prec \frac{1}{N}\|A\|^2$ for each overlap. This argument in this form is valid only in the bulk; near the spectral edges the right hand side of (3.2.6) needs to be improved to $\rho(z_1)\rho(z_2)\|A\|^2$; this was already achieved in [165]. However, to obtain the optimal Hilbert-Schmidt norm of the observable in (3.2.1) a second improvement to the form

$$\langle \operatorname{Im} G(z_1) A \operatorname{Im} G(z_2) A^*\rangle \prec \rho(z_1)\rho(z_2)\langle |A|^2\rangle, \qquad \langle A\rangle = 0, \tag{3.2.7}$$

is necessary. The main achievement of the current paper is to extract both types of improvement *simultaneously.*

While Theorem 3.2.2 requires only the upper bound (3.2.7) for $\operatorname{Im} GA\operatorname{Im} GA$, along its proof other alternating products of resolvents (with or without Im) and deterministic matrices emerge. More precisely, setting $G_i := G(z_i)$ and considering deterministic matrices B_i, the main object of interest is

$$G_1 B_1 G_2 B_2 G_3 \dots B_{k-1} G_k \tag{3.2.8}$$

for some fixed k. We will call expressions of the form (3.2.8) *(resolvent) chains*. We will show a *multi-resolvent local law*, i.e. that any chain (3.2.8) concentrates around a deterministic object and give an upper bound on the fluctuation. The interesting regime is the local one, i.e. when $|\operatorname{Im} z_i| \ll 1$. We will also consider the case when some of the G_i's are replaced by their imaginary part $\operatorname{Im} G_i$, and we will show that in this case the fluctuations are reduced close to the edge of the spectrum by some factor of $|\operatorname{Im} m(z_i)|$ which is essentially the density ρ_{sc} at $\operatorname{Re} z_i$.

It turns out [165] that the sizes of both the deterministic limit of (3.2.8) and its fluctuation are substantially reduced if some of the matrices B_i are traceless. Therefore, in the main part of the paper we study (3.2.8) when all the matrices B_i are traceless, $\langle B_i\rangle = 0$, this will also imply a local law for (3.2.8) for generic B_i's using that any matrix B can be decomposed into a constant and a traceless part as $B = \langle B\rangle \cdot I + \mathring{B}$.

We will prove local laws that are optimal simultaneously in the two different aspects mentioned above in addition to account for the improvement owing to the traceless observables. The first aspect is to trace the improvement near the spectral edges in terms of additional ρ-powers; in general the presence of each $\operatorname{Im} G$ provides an additional ρ factor. Second, instead of the technically much easier Euclidean matrix norm (operator norm) of the B_i's, we need to use the more sophisticated Hilbert-Schmidt norm. One additional advantage of using the Hilbert-Schmidt norm is that it enables us to test the chain in (3.2.8) against rank one matrices and still get optimal bounds. In particular, testing it against the projection $|\boldsymbol{x}\rangle\langle \boldsymbol{y}|$ immediately gives the so-called *isotropic local laws*, i.e. concentration for the individual matrix elements $\langle \boldsymbol{x}, G_1 B_1 \dots B_{k-1} G_k \boldsymbol{y}\rangle$, for any deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$.

Our results also hold for the case when the spectral parameters z_i's are different, but we will not explore the additional potential improvements from this fact since it is not needed for ETH. While in some part of the argument we track the different values of $|\operatorname{Im} z_i|$ precisely (instead of overestimating them by the worst one), we will not exploit the additional gain from possibly different real parts $\operatorname{Re} z_i$; this study is left for future investigations.

Multi-resolvent local laws for chains (3.2.8) with traceless deterministic matrices have been the object of interest in several recent papers, however in each of these works only one aspect of the fluctuations of (3.2.8) was taken into consideration: either the problem was optimal only in the bulk of the spectrum [169], hence missing ρ factors were ignored, or the error term was estimated using

the crude operator norm of the B_i [165, 168], or only chains of length one ($k = 1$) had an optimal error term in both aspects [172]. Our new result (Theorem 3.2.4 below) does not have any of these restriction: we give a bound on the fluctuation of (3.2.8) uniformly in the spectrum with optimal N- and ρ-powers and with the Hilbert-Schmidt norm on the traceless B_i's.

3.2.1.1 Preliminaries on the deterministic approximation

Before stating our main technical result we introduce some additional notation. Given a non-crossing partition π of the set $[k] := \{1, \dots, k\}$ arranged in cyclic order, the partial trace pTr_π of an ordered set of matrices $B_1, \dots, B_{k-1}$ is defined as

$$\mathrm{pTr}_\pi(B_1, \dots, B_{k-1}) := \prod_{S \in \pi \setminus \mathfrak{B}(k)} \left\langle \prod_{j \in S} B_j \right\rangle \prod_{j \in \mathfrak{B}(k) \setminus \{k\}} B_j, \tag{3.2.9}$$

with $\mathfrak{B}(k) \ni \pi$ denoting the unique block containing k. Then, for generic B_i's, the deterministic approximation of (3.2.8) is given by [170, Theorem 3.4]:

$$M_{[1,k]} = M(z_1, B_1, \dots, B_{k-1}, z_k) := \sum_{\pi \in \mathrm{NC}([k])} \mathrm{pTr}_{K(\pi)}(B_1, \dots, B_{k-1}) \prod_{S \in \pi} m_\circ[S], \tag{3.2.10}$$

where $\mathrm{NC}([k])$ denotes the non-crossing partitions of the set $[k]$, and $K(\pi)$ denotes the Kreweras complement of π (see [170, Definition 2.4] and [384]). Furthermore, for any subset $S \subset [k]$ we define $m[S] := m_{\mathrm{sc}}[\boldsymbol{z}_S]$ as the iterated divided difference of m_{sc} evaluated in $\boldsymbol{z}_S := \{z_i : i \in S\}$ which can also be written as

$$m[S] = m_{\mathrm{sc}}[\boldsymbol{z}_S] = m_{\mathrm{sc}}[\{z_i : i \in S\}] = \int_{-2}^{2} \rho_{\mathrm{sc}}(x) \prod_{i \in S} \frac{1}{x - z_i} \mathrm{d}x. \tag{3.2.11}$$

We denote by $m_\circ[\cdot]$ the free-cumulant transform of $m[\cdot]$ which is uniquely defined implicitly by the relation

$$m[S] = \sum_{\pi \in \mathrm{NC}(S)} \prod_{S' \in \pi} m_\circ[S'], \qquad \forall S \subset [k], \tag{3.2.12}$$

e.g. $m_\circ[i, j] = m[\{i, j\}] - m[\{i\}]m[\{j\}]$. For example, for $k = 2$ we have

$$\begin{aligned} M(z_1, B_1, z_2) &= \langle B_1 \rangle (m_{\mathrm{sc}}[z_1, z_2] - m_{\mathrm{sc}}(z_1) m_{\mathrm{sc}}(z_2)) + B_1 m_{\mathrm{sc}}(z_1) m_{\mathrm{sc}}(z_2) \\ &= \frac{\langle B_1 \rangle}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - x^2}}{(x - z_1)(x - z_2)} \mathrm{d}x + (B_1 - \langle B_1 \rangle) m_{\mathrm{sc}}(z_1) m_{\mathrm{sc}}(z_2). \end{aligned} \tag{3.2.13}$$

The main objects of interest within this section are general resolvent chains

$$\mathcal{G}_1 B_1 \mathcal{G}_2 B_2 \dots B_{k-1} \mathcal{G}_k \tag{3.2.14}$$

where $\mathcal{G}_i \in \{G_i, \operatorname{Im} G_i\}$, and we denote by $\mathfrak{I}_k \subset [k]$ the set of the indices for which $\mathcal{G}_i = \operatorname{Im} G_i$. Note that some resolvents may be replaced with their imaginary parts. In order to generalize (3.2.10), for any subset $\mathfrak{I}_k \subset [k]$ we define[5]

$$\mathcal{M}_{[1,k]} = \mathcal{M}(z_1, B_1, \dots, B_{k-1}, z_k; \mathfrak{I}_k) := \sum_{\pi \in \mathrm{NC}([k])} \mathrm{pTr}_{K(\pi)}(B_1, \dots, B_{k-1}) \prod_{S \in \pi} m_\circ^{(\mathfrak{I}_k)}[S], \tag{3.2.15}$$

with $m_\circ^{(\mathfrak{I}_k)}[S]$ implicitly defined as in (3.2.12) with $m[S]$ replaced with $m^{(\mathfrak{I}_k)}[S]$, where

$$m^{(\mathfrak{I}_k)}[S] = m^{(\mathfrak{I}_k)}[\{z_i : i \in S\}] := \int_{-2}^{2} \rho_{\mathrm{sc}}(x) \left(\prod_{i \in \mathfrak{I}_k \cap S} \operatorname{Im} \frac{1}{x - z_i} \right) \left(\prod_{i \in S \setminus \mathfrak{I}_k} \frac{1}{x - z_i} \right) \mathrm{d}x. \tag{3.2.16}$$

We now give some bounds on the deterministic approximations in the case where all matrices in (3.2.15) are traceless, $\langle B_i \rangle = 0$.[6] The proof of the following lemma is presented in Appendix 3.A.

[5] Calligraphic letters like $\mathcal{G}, \mathcal{M}$ indicate that we may consider $\operatorname{Im} G$ instead of some resolvents G in the chain.

[6] From now on we use the convention that traceless matrices are denoted by A, while general deterministic matrices are denoted by B.

Lemma 3.2.3 (M-bounds)**.** *Fix $k \geq 1$. Consider spectral parameters $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ and traceless matrices $A_1, ..., A_k \in \mathbf{C}^{N\times N}$. Moreover, let*

$$\eta_j := |\operatorname{Im} z_j|\,, \qquad m_j := m_{\mathrm{sc}}(z_j)\,, \qquad \rho_j := \frac{1}{\pi}|\operatorname{Im} m_j|\,.$$

(a) Denoting $\ell := \min_{j\in[k]}\big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_k))\big]$ and assuming $N\ell \geq 1$, we have the average bound

$$|\langle \mathcal{M}(z_1, A_1, ..., A_{k-1}, z_k; \mathfrak{I}_k)A_k\rangle| \lesssim \left(\prod_{i\in\mathfrak{I}_k}\rho_i\right) N^{k/2-1} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}\,. \tag{3.2.17}$$

(b) Denoting $\ell := \min_{j\in[k+1]}\big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_{k+1}))\big]$ and assuming $N\ell \geq 1$, we have the isotropic bound[7]

$$|\langle \boldsymbol{x}, \mathcal{M}(z_1, A_1, ..., A_k, z_{k+1}; \mathfrak{I}_{k+1})\boldsymbol{y}\rangle| \lesssim \left(\prod_{i\in\mathfrak{I}_{k+1}}\rho_i\right) N^{k/2} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}\,. \tag{3.2.18}$$

for arbitrary bounded deterministic vectors $\|\boldsymbol{x}\|\,, \|\boldsymbol{y}\| \lesssim 1$.

Note that (3.2.17) already reflects the different aspects of our local law: it correctly accounts for the ρ-powers for each $\operatorname{Im} G$, it involves the Hilbert-Schmidt norm of the observables and it is not hard to see that the N-power is also optimal. Note that the isotropic bound (3.2.18) is stated separately for convenience, although it will be a straightforward consequence of the average bound (3.2.17).

3.2.1.2 Multi-resolvent local law

As our main input for Theorem 3.2.2, we will prove the following multi-resolvent local law, optimally accounting for the decay of the density at the edge.

Theorem 3.2.4 (Multi-resolvent local law with optimal edge dependence)**.** *Let W be a Wigner matrix satisfying Assumption 3.2.1, and fix $k \in \mathbf{N}$. Consider spectral parameters $z_1, \ldots, z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$, the associated resolvents $G_j = G(z_j) := (W - z_j)^{-1}$ with $\mathcal{G}_j \in \{G_j, \operatorname{Im} G_j\}$, and traceless matrices $A_1, \ldots, A_k \in \mathbf{C}^{N\times N}$. Finally, let*

$$\eta_j := |\operatorname{Im} z_j|\,, \qquad m_j := m_{\mathrm{sc}}(z_j)\,, \qquad \rho_j := \frac{1}{\pi}|\operatorname{Im} m_j|\,, \qquad j \in [k+1]. \tag{3.2.19}$$

(a) Denote by $\mathfrak{I}_k$ the set of indices $j \in [k]$ where $\mathcal{G}_j = \operatorname{Im} G_j$. Then, setting

$$\ell := \min_{j\in[k]}\big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_k))\big],$$

we have the average law

$$\big|\langle \mathcal{G}_1 A_1 \mathcal{G}_2 \ldots \mathcal{G}_k A_k\rangle - \langle \mathcal{M}_{[1,k]} A_k\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_k}\rho_i\right) \wedge \max_{i\in[k]}\sqrt{\rho_i}\right] \frac{N^{k/2-1}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}\,, \tag{3.2.20}$$

uniformly in spectral parameters satisfying $\min_j N\eta_j\rho_j \geq N^{\epsilon}$ and $\max_j |z_j| \leq N^{1/\epsilon}$ for some $\epsilon > 0$.

[7] The isotropic bound for $|\langle \boldsymbol{x}, \mathcal{M}\boldsymbol{y}\rangle|$ in (3.2.18) is the same as the norm bound $\|\mathcal{M}\|$.

(b) Denote by $\mathfrak{I}_{k+1}$ the set of indices $j \in [k+1]$ where $\mathcal{G}_j = \operatorname{Im} G_j$. Then, setting

$$\ell := \min_{j\in[k+1]} \big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_{k+1}))\big],$$

we have the isotropic law

$$\big|\langle \boldsymbol{x}, \mathcal{G}_1 A_1 \mathcal{G}_2 \dots A_k \mathcal{G}_{k+1} \boldsymbol{y}\rangle - \langle \boldsymbol{x}, \mathcal{M}_{[1,k+1]} \boldsymbol{y}\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_{k+1}} \rho_i\right) \wedge \max_{i\in[k+1]} \sqrt{\rho_i}\right] \frac{N^{k/2}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}, \tag{3.2.21}$$

uniformly in bounded deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$ and spectral parameters satisfying $\min_j N\eta_j\rho_j \geq N^\epsilon$ and $\max_j |z_j| \leq N^{1/\epsilon}$ for some $\epsilon > 0$.

Observe that, in the regime $N\ell \gg 1$, the error terms in (3.2.20) and (3.2.21) are smaller by an additional small $(N\ell)^{-1/2}$-factor compared to the size of the leading terms in (3.2.17) and (3.2.18), respectively.

Remark 3.2.5 (Optimality)**.** *The bounds* (3.2.20) *and* (3.2.21) *are optimal (up to the N^ξ factor hidden in the $\prec$-relation) in the class of bounds that involve only the parameters N, η_i, ρ_i and the Hilbert-Schmidt norm of A_i's. This fact can be seen by computing the variance of the left hand sides in the case when W is a GUE matrix. The resolvents can be written out by spectral theorem, similarly to* (3.2.5)*, and the variance with respect to the eigenvectors can be explicitly computed by Weingarten calculus, while the variance with respect to the eigenvalues (that are independent of the eigenvectors) can be identified from well-known central limit theorems for linear statistics of eigenvalues. For example, for $k = 2$, $A_1 = A_2 = A$, $z_1 = z_2 = z$ and $\mathfrak{I}_k = \varnothing$, in this way we obtain*

$$\sqrt{\mathbf{E}\big|\langle GAGA\rangle - m^2\langle A^2\rangle\big|^2} \sim \frac{1}{N\eta}\langle A^2\rangle + \frac{\sqrt{\rho}}{N\sqrt{\eta}}\langle A^4\rangle^{1/2}. \tag{3.2.22}$$

After estimating $\langle A^4\rangle \leq N\langle A^2\rangle^2$, which may saturate for certain A, we see the optimality of (3.2.20) *for this case. The general case is a similar, albeit somewhat tedious calculation.*

Remark 3.2.6 (Interpretations)**.** *We have two further comments on Theorem 3.2.4.*

(i) For $\mathfrak{I}_k = \varnothing$ and $\mathfrak{I}_{k+1} = \varnothing$ both bounds, (3.2.20) *and* (3.2.21)*, have already been proven in [169, Theorem 2.2 and Corollary 2.4]. In the complementary cases $\mathfrak{I}_k \neq \varnothing$ and $\mathfrak{I}_{k+1} \neq \varnothing$, we point out that the minimum $\big[\dots \wedge \dots\big]$ in* (3.2.20) *and* (3.2.21) *is realized by the product $\prod_{i\in\mathfrak{I}} \rho_i$ since $\rho_i \lesssim 1$. In particular, as a rule of thumb, every index j for which $\mathcal{G}_j = \operatorname{Im} G_j$, decreases both the size of the deterministic approximation* (3.2.17)*–*(3.2.18) *and the size of the error* (3.2.20)*–*(3.2.21) *by a factor ρ_j, with $\rho_j \leq 1$, compared to the case when $\mathcal{G}_j = G_j$. An exception to this rule is* (3.2.20) *for $k = 1$; here the bounds for $\langle GA\rangle$ and $\langle \operatorname{Im} GA\rangle$ are identical.*

(ii) The estimates in Theorem 3.2.4 remain valid if we replace

$$\begin{aligned} \langle \mathcal{M}_{[1,k]} A_k\rangle &\longrightarrow \left(\prod_{i\in\mathfrak{I}_k} \operatorname{Im} m_i\right)\left(\prod_{i\notin\mathfrak{I}_k} m_i\right)\langle A_1 \dots A_k\rangle \\ \langle \boldsymbol{x}, \mathcal{M}_{[1,k+1]}\boldsymbol{y}\rangle &\longrightarrow \left(\prod_{i\in\mathfrak{I}_{k+1}} \operatorname{Im} m_i\right)\left(\prod_{i\notin\mathfrak{I}_{k+1}} m_i\right)\langle \boldsymbol{x}, A_1 \dots A_k \boldsymbol{y}\rangle \end{aligned} \tag{3.2.23}$$

in (3.2.20) *and* (3.2.21)*, respectively, i.e., if we consider only the trivial partition into singletons π in the definition* (3.2.15) *of $\mathcal{M}_{[1,k]}$. This is simply due to the fact that all other summands in* (3.2.15) *are explicitly smaller than the error terms in* (3.2.20)*–*(3.2.21)*. A proof of this fact is given in Appendix 3.A.*

Remark 3.2.7 (Generalisations)**.** *We mention a few direct generalisations of Theorem 3.2.4 whose proofs are omitted as they are straightforward.*

(i) In Theorem 3.2.4 each $\mathcal{G}$ can be replaced by a product of $\mathcal{G}$'s and an individual $\mathcal{G}$ may also stand for $|G|$, not only for G or $\operatorname{Im} G$ (see [168, Lemma 3.2], [169, Lemma 3.1], and also Lemma 3.4.5 below). We refrain from stating these results explicitly as they are easily obtained using appropriate integral representations of general products of such $\mathcal{G}$'s in terms of a single $\operatorname{Im} G$.

(ii) We stated the multi–resolvent local laws in Theorem 3.2.4 only for $\mathcal{G}_j \in \{G_j, \operatorname{Im} G_j\}$, however, inspecting the proof, one can easily see that it also leads to a local law for $\mathcal{G}_j \in \{G_j, \operatorname{Im} G_j, G_j^{\mathrm{t}}, \operatorname{Im} G_j^{\mathrm{t}}\}$, where G^{t} stands for the transpose of G. In particular, this implies that the ETH in Theorem 3.2.2 can also be extended to

$$\max_{i,j\in[N]} |\langle \overline{\boldsymbol{u}_i}, A\boldsymbol{u}_j\rangle - \langle A\rangle \langle \overline{\boldsymbol{u}_i}, \boldsymbol{u}_j\rangle| \prec \frac{\langle |\mathring{A}|^2\rangle^{1/2}}{\sqrt{N}}. \tag{3.2.24}$$

Furthermore, setting $\sigma := \mathbf{E}\chi_{\mathrm{od}}^2$, for $|\sigma| < 1$ we have (see [165, Theorem 2.3])

$$\left|\langle \overline{\boldsymbol{u}_i}, \boldsymbol{u}_j\rangle\right| \prec \frac{C_\sigma}{\sqrt{N}}.$$

In two extreme cases $\sigma = \pm 1$, we have $|\langle \overline{\boldsymbol{u}_i}, \boldsymbol{u}_j\rangle| = \delta_{i,j}$ if $\sigma = 1$ and $|\langle \overline{\boldsymbol{u}_i}, \boldsymbol{u}_j\rangle| = \delta_{i,N-j+1}$ if $\sigma = -1$ and $\mathbf{E}(W_{aa}^2) = 0$ (see [165, Remark 2.4]). We remark that here $\boldsymbol{u}_i$ denotes the eigenvector corresponding to the eigenvalue λ_i, with the λ_i's labeled in increasing order.

3.2.2 Proof of Theorem 3.2.2

Fix $\epsilon > 0$, pick $E \in [-2, 2]$ and define $\eta(E)$ implicitly by

$$N\eta(E)\rho(E + \mathrm{i}\eta(E)) = N^{\epsilon}.$$

Let A be a traceless matrix $\langle A\rangle = 0$, then by spectral decomposition (3.2.5) and the well-known eigenvalue rigidity[8] (see, e.g., [249]) it is easy to see that (see [165, Lemma 1] for more details)

$$\max_{i,j\in[N]} N\,|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle|^2 \prec N^{2\epsilon} \sup_{E_1,E_2\in[-2,2]} \frac{\left|\langle \operatorname{Im} G(E_1 + \mathrm{i}\eta(E_1))A \operatorname{Im} G(E_2 + \mathrm{i}\eta(E_2))A^*\rangle\right|}{\rho(E_1 + \mathrm{i}\eta(E_1))\rho(E_2 + \mathrm{i}\eta(E_2))} \prec N^{2\epsilon}\langle |A|^2\rangle\,.$$

We point out that in the last inequality we used (3.2.20) for $k = 2$ and $\mathfrak{I}_2 = [2]$:

$$\left|\langle \operatorname{Im} G_1 A \operatorname{Im} G_2 A^*\rangle - \operatorname{Im} m_1 \operatorname{Im} m_2 \langle |A|^2\rangle\right| \prec \frac{\rho_1\rho_2}{\sqrt{N\ell}}\langle |A|^2\rangle\,.$$

The fact that this bound holds simultaneously for all $E_1 = \operatorname{Re} z_1 \in [-2, 2]$ and $E_2 = \operatorname{Re} z_2 \in [-2, 2]$ follows by a simple grid argument together with the Lipschitz regularity of the resolvent (with Lipschitz constant of order N at spectral parameters with imaginary part bigger than $1/N$). This completes the proof of Theorem 3.2.2. □

The rest of the paper is devoted to the proof of the multi-resolvent local law, Theorem 3.2.4.

[8] Rigidity asserts that the increasingly ordered eigenvalues λ_i are very close to the i-th N-quantile γ_i of the semicircle density ρ_{sc} in the sense $|\lambda_i - \gamma_i| \prec N^{-2/3}[i \wedge (N + 1 - i)]^{-1/3}$, i.e. each eigenvalue is strongly concentrated around the corresponding quantile essentially on the scale of the local eigenvalue spacing.

3.3 Multi–resolvent local law: Proof of Theorem 3.2.4

In this section we prove the *multi-resolvent local laws* in Theorem 3.2.4 via the following three steps:

1. **Global law.** Prove a multi-resolvent *global law*, i.e. for spectral parameters "far away" from the spectrum, $\min_j \operatorname{dist}(z_j, [-2,2]) \geq \delta$ for some small $\delta > 0$ (see Proposition 3.3.1).
2. **Characteristic flow.** Propagate the global law to a *local law* by considering the evolution of the Wigner matrix W along the Ornstein-Uhlenbeck flow, thereby introducing an almost order one Gaussian component (see Proposition 3.3.3). The spectral parameters evolve from the global regime to the local regime according to the *characteristic (semicircular) flow*. The simultaneous effect of these two evolutions is a key cancellation of two large terms.
3. **Green function comparison.** Remove the Gaussian component by a Green function comparison (GFT) argument (see Proposition 3.3.4).

As the first step, we have the following global law. Its proof, which is analogous to the proofs presented in [168, Appendix B] and [169, Appendix A], is given in Appendix 3.A.2 for completeness. We point out that all these proofs do not use the system of master inequalities and the bootstrapped error analysis that form the technical backbone of [168, 169], they use simple norm bounds on the resolvents. In particular, Proposition 3.3.1 holds for general deterministic matrices since the traceless condition plays no role in this case.

Proposition 3.3.1 (Step 1: Global law)**.** *Let W be a Wigner matrix satisfying Assumption 3.2.1, and fix any $k \in \mathbf{N}$ and $\delta > 0$. Consider spectral parameters $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$, the associated resolvents $G_j = G(z_j) := (W - z_j)^{-1}$, with $\mathcal{G}_j \in \{G_j, \operatorname{Im} G_j\}$, and deterministic matrices $B_1, ..., B_k \in \mathbf{C}^{N\times N}$. Denote $\eta_i := |\operatorname{Im} z_i|$ and $\rho_i := \pi^{-1}|\operatorname{Im} m_{\mathrm{sc}}(z_i)|$. Then, uniformly in deterministic matrices B_i and in spectral parameters satisfying $\operatorname{dist}(z_j, [-2,2]) \geq \delta$, the following holds.*

(a) Let $\mathfrak{I}_k$ be the set of indices $j \in [k]$ where $\mathcal{G}_j = \operatorname{Im} G_j$, and define $\ell := \min_{j\in[k]} \big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_k))\big]$. Then we have the averaged bound

$$\big|\langle \mathcal{G}_1 B_1 ... \mathcal{G}_k B_k\rangle - \langle \mathcal{M}_{[1,k]} B_k\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_k} \rho_i\right) \wedge \max_{i\in[k]} \sqrt{\rho_i}\right] \frac{N^{k/2-1}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |B_j|^2\rangle^{\frac{1}{2}}. \tag{3.3.1}$$

(b) Let $\mathfrak{I}_{k+1}$ be the set of indices $j \in [k+1]$ where $\mathcal{G}_j = \operatorname{Im} G_j$, and define $\ell := \min_{j\in[k+1]} \big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_{k+1}))\big]$. Then, for deterministic unit vectors $\boldsymbol{x}, \boldsymbol{y}$, we have the isotropic bound

$$\big|(\mathcal{G}_1 B_1 ... B_k \mathcal{G}_{k+1})_{\boldsymbol{xy}} - (\mathcal{M}_{[1,k+1]})_{\boldsymbol{xy}}\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_{k+1}} \rho_i\right) \wedge \max_{i\in[k+1]} \sqrt{\rho_i}\right] \frac{N^{k/2}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |B_j|^2\rangle^{\frac{1}{2}}. \tag{3.3.2}$$

In the next Proposition 3.3.3, using Proposition 3.3.1 as an input, we derive Theorem 3.2.4 for Wigner matrices which have an order one Gaussian component. For this purpose we consider the evolution of the Wigner matrix W along the Ornstein-Uhlenbeck flow

$$\mathrm{d}W_t = -\frac{1}{2} W_t \mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad W_0 = W, \tag{3.3.3}$$

with B_t being real symmetric or complex Hermitian Brownian motion[9] with entries having t times the same first two moments of W, and define its resolvent $G_t(z) := (W_t - z)^{-1}$ with $z \in \mathbf{C} \setminus \mathbf{R}$.

[9] Strictly speaking, we use this Brownian motion only when $\sigma := \mathbf{E}\,\chi_{\mathrm{od}}^2$ is real and $\mathbf{E}\,\chi_{\mathrm{d}}^2 = 1 + \sigma$, otherwise we need a small modification, see later in Section 3.4.

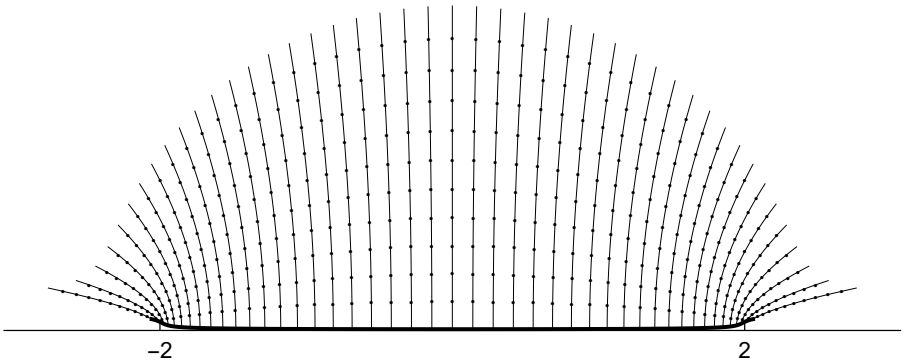

Figure 3.3.1: Several trajectories for solutions of (3.3.5) are depicted. We chose ten reference times, indicated by dots, showing that the rate of change along the flow strongly depends on ρ. The solid black line is the graph of $E \mapsto \eta(E)$ with $\eta(E)$ implicitly defined via $\eta(E)\rho(E + \mathrm{i}\eta(E)) = \mathrm{const.}$ for a small positive constant. A similar picture also appeared in [105, Figure 1].

Even if not stated explicitly we will always consider this flow only for short times, i.e. for $0 \le t \le T$, where the maximal time T is smaller than γ, for some small constant $\gamma > 0$. Note that along the flow (3.3.3) the first two moments of W_t are preserved, and so the self-consistent density of states of W_t is unchanged; it remains the standard semicircle law. We now want to compute the deterministic approximation of product of resolvents and deterministic matrices with trace zero,

$$\mathcal{G}_t(z_1)A_1\mathcal{G}_t(z_2)A_2\mathcal{G}_t(z_3)A_3\ldots, \qquad \langle A_i\rangle = 0, \tag{3.3.4}$$

and have a very precise estimate of the error term.

In fact, we also let the spectral parameters evolve with time with a carefully chosen equation that conveniently cancels some leading error terms in the time evolution of (3.3.4). The corresponding equation is the characteristic equation for the semicircular flow, i.e. given by the first order ODE (see Figure 3.3.1):

$$\partial_t z_{i,t} = -m(z_{i,t}) - \frac{z_{i,t}}{2}. \tag{3.3.5}$$

Define $\eta_{i,t} := |\mathrm{Im}\, z_{i,t}|$ and $\rho_{i,t} := \pi^{-1}|\mathrm{Im}\, m(z_{i,t})|$. Note that along the characteristics we have

$$\partial_t m(z_{i,t}) = -\partial_z m(z_{i,t})\left(m(z_{i,t}) + \frac{z_{i,t}}{2}\right) = -\partial_z m(z_{i,t})\left(-\frac{1}{2m(z_{i,t})} + \frac{m(z_{i,t})}{2}\right) = \frac{m(z_{i,t})}{2}, \tag{3.3.6}$$

where in the last two equalities we used the defining equation $m(z)^2 + zm(z) + 1 = 0$ of the Stieltjes transform of the semicircular law. In particular, taking the imaginary part of (3.3.6) we get $\rho_{i,s} \sim \rho_{i,t}$ for any $0 \le s \le t$, while the behavior of the $\eta_{i,t}$ depends on the regime: in the bulk $\eta_{i,t}$ decreases linearly in time with a speed of order one, close to the edge $\eta_{i,t}$ decreases still linearly, but with a speed depending on ρ, i.e. it is slower near the edges. By standard ODE theory we obtain the following lemma:

Lemma 3.3.2. *Fix an N–independent $\gamma > 0$, fix $0 < T < \gamma$, and pick $z \in \mathbf{C} \setminus \mathbf{R}$. Then there exists an initial condition z_0 such that the solution z_t of* (3.3.5) *with this initial condition z_0 satisfies $z_T = z$. Furthermore, there exists a constant $C > 0$ such that $\mathrm{dist}(z_0, [-2, 2]) \ge CT$.*

The spectral parameters evolving by (3.3.3) will have the property that

$$\mathcal{G}_t(z_{1,t})A_1\ldots A_{k-1}\mathcal{G}_t(z_{k,t}) - \mathcal{M}_{[1,k],t} \approx \mathcal{G}_0(z_{1,0})A_1\ldots A_{k-1}\mathcal{G}_0(z_{k,0}) - \mathcal{M}_{[1,k],0}, \tag{3.3.7}$$

with $\mathcal{M}_{[1,k],t} := \mathcal{M}(z_{1,t}, A_1, \ldots, A_{k-1}, z_{k,t})$, for any $0 \le t \le T$. Note that the deterministic approximation $\mathcal{M}_{[1,k],t}$ depends on time only through the time dependence of the spectral parameters. The deterministic approximation of (3.3.4) with fixed spectral parameters is unchanged along the whole flow (3.3.3) since the Wigner semicircular density is preserved under the OU flow (3.3.3).

Proposition 3.3.3 (Step 2: Characteristic flow)**.** *Fix $\epsilon, \gamma > 0$, $0 \le T \le \gamma$, $K \in \mathbf{N}$. Consider $z_{1,0}, \dots, z_{K+1,0} \in \mathbf{C} \setminus \mathbf{R}$ as initial conditions of the solution $z_{j,t}$ of (3.3.5) for $0 \le t \le T$, define $G_{j,t} := G_t(z_{j,t})$ and let $\mathcal{G}_{j,t} \in \{G_{j,t}, \operatorname{Im} G_{j,t}\}$. Let $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$ be bounded deterministic vectors.*

(a) For any $k \le K$ let $\mathfrak{I}_k$ be the set of indices $j \in [k]$ where $\mathcal{G}_{j,t} = \operatorname{Im} G_{j,t}$, and define $\ell_t := \min_{j\in[k]} \big[\eta_{j,t}(\rho_{j,t} + \mathbf{1}(j \notin \mathfrak{I}_k))\big]$, the time dependent analogue[10] of ℓ. Then, assuming that

$$\big|\langle \mathcal{G}_{1,0} A_1 \dots \mathcal{G}_{k,0} A_k\rangle - \langle \mathcal{M}_{[1,k],0} A_k\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_k} \rho_{i,0}\right) \wedge \max_{i\in[k]} \sqrt{\rho_{i,0}}\right] \frac{N^{k/2-1}}{\sqrt{N\ell_0}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2} \tag{3.3.8}$$

holds uniformly for any $k \le K$, any choice of $A_1, \dots, A_k$ traceless deterministic matrices and any choice of $z_{i,0}$'s such that $N\eta_{i,0}\rho_{i,0} \ge N^\epsilon$ and $|z_{i,0}| \le N^{1/\epsilon}$, then we have

$$\big|\langle \mathcal{G}_{1,T} A_1 \dots \mathcal{G}_{k,T} A_k\rangle - \langle \mathcal{M}_{[1,k],T} A_k\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_k} \rho_{i,T}\right) \wedge \max_{i\in[k]} \sqrt{\rho_{i,T}}\right] \frac{N^{k/2-1}}{\sqrt{N\ell_T}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}, \tag{3.3.9}$$

for any $k \le K$, again uniformly in traceless matrices A_i and in spectral parameters satisfying $N\eta_{i,T}\rho_{i,T} \ge N^\epsilon$, $|z_{i,T}| \le N^{1/\epsilon}$.

(b) Let $\mathfrak{I}_{k+1}$ be the set of indices $j \in [k+1]$ where $\mathcal{G}_{j,t} = \operatorname{Im} G_{j,t}$, and define $\ell_{j,t} := \min_{j\in[k+1]} \big[\eta_{j,t}(\rho_{j,t} + \mathbf{1}(j \notin \mathfrak{I}_{k+1}))\big]$. Then, assuming that

$$\big|\langle \boldsymbol{x}, \mathcal{G}_{1,0} A_1 \dots A_k \mathcal{G}_{k+1,0} \boldsymbol{y}\rangle - \langle \boldsymbol{x}, \mathcal{M}_{[1,k+1],0} \boldsymbol{y}\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_{k+1}} \rho_{i,0}\right) \wedge \max_{i\in[k+1]} \sqrt{\rho_{i,0}}\right] \frac{N^{k/2}}{\sqrt{N\ell_0}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2} \tag{3.3.10}$$

holds for any $k \le K$, uniformly in A's and in the spectral parameters as in part (a), and in deterministic vectors, then we have

$$\big|\langle \boldsymbol{x}, \mathcal{G}_{1,T} A_1 \dots A_k \mathcal{G}_{k+1,T} \boldsymbol{y}\rangle - \langle \boldsymbol{x}, \mathcal{M}_{[1,k+1],T} \boldsymbol{y}\rangle\big| \prec \left[\left(\prod_{i\in\mathfrak{I}_{k+1}} \rho_{i,T}\right) \wedge \max_{i\in[k+1]} \sqrt{\rho_{i,T}}\right] \frac{N^{k/2}}{\sqrt{N\ell_T}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}, \tag{3.3.11}$$

for any $k \le K$, again uniformly in A's and in spectral parameters as in part (a), and in deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$.

Proposition 3.3.3 is proven in Section 3.4. As the third and final step, we show that the additional Gaussian component introduced in Proposition 3.3.3 can be removed using a Green function comparison (GFT) argument. The proof of this proposition is presented in Section 3.5.

Proposition 3.3.4 (Step 3: Green function comparison)**.** *Let $H^{(\boldsymbol{v})}$ and $H^{(\boldsymbol{w})}$ be two $N \times N$ Wigner matrices with matrix elements given by the random variables v_{ab} and w_{ab}, respectively, both satisfying Assumption 3.2.1 and having matching moments up to third order,[11] i.e.*

$$\mathbf{E}\, \bar{v}_{ab}^u v_{ab}^{s-u} = \mathbf{E}\, \bar{w}_{ab}^u w_{ab}^{s-u}, \quad s \in \{0,1,2,3\}, \quad u \in \{0, \dots, s\}. \tag{3.3.12}$$

Fix $K \in \mathbf{N}$ and consider spectral parameters $z_1, \dots, z_{K+1} \in \mathbf{C} \setminus \mathbf{R}$ satisfying $\min_j N\eta_j\rho_j \ge N^\epsilon$ and $\max_j |z_j| \le N^{1/\epsilon}$ for some $\epsilon > 0$ and the associated resolvents $G_j^{(\#)} = G^{(\#)}(z_j) := (H^{(\#)} - z_j)^{-1}$ with $\mathcal{G}_j^{(\#)} \in \{G_j^{(\#)}, \operatorname{Im} G_j^{(\#)}\}$ and $\# = \boldsymbol{v}, \boldsymbol{w}$. Pick traceless matrices $A_1, \dots, A_K \in \mathbf{C}^{N\times N}$.

Assume that, for $H^{(\boldsymbol{v})}$, we have the following bounds (writing $\mathcal{G}_j \equiv \mathcal{G}_j^{(\boldsymbol{v})}$ for brevity).

[10] We point out that the index j realizing the minimum may change along the time evolution. Additionally, by (3.3.6) and the text below it, we note that if $\min_i N\eta_i\rho_i \ge N^\epsilon$ then $\min_i N\eta_{i,t}\rho_{i,t} \ge N^\epsilon$ for any $0 \le t \le T$.

[11] This condition can easily be relaxed to being matching up to an error of size N^{-2} as done, e.g., in [248, Theorem 16.1].

(a) *For any $k \le K$, consider any subset of cardinality k of the $K+1$ spectral parameters and, similarly, consider any subset of cardinality k of the K traceless matrices. Relabel both of them by $[k]$, and denote the set of indices $j \in [k]$ by $\mathfrak{I}_k$ where $\mathcal{G}_j = \operatorname{Im} G_j$. Setting $\ell := \min_{j\in[k]} \left[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_k))\right]$ we have that*

$$\left|\langle \mathcal{G}_1 A_1 ... \mathcal{G}_k A_k\rangle - \langle \mathcal{M}_{[1,k]} A_k\rangle\right| \prec \left[\left(\prod_{i\in\mathfrak{I}_k} \rho_i\right) \wedge \max_{i\in[k]} \sqrt{\rho_i}\right] \frac{N^{k/2-1}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}, \tag{3.3.13}$$

uniformly in all choices of subsets of z's and A's.

(b) *For any $k \le K$, consider any subset of cardinality $k+1$ of the $K+1$ spectral parameters and, similarly, consider any subset of cardinality k of the K traceless matrices. Relabel them by $[k+1]$ and $[k]$, respectively, and denote the set of indices $j \in [k+1]$ by $\mathfrak{I}_{k+1}$ where $\mathcal{G}_j = \operatorname{Im} G_j$. Setting $\ell := \min_{j\in[k+1]} \left[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_{k+1}))\right]$ we have that*

$$\left|\langle \boldsymbol{x}, \mathcal{G}_1 A_1 ... A_k \mathcal{G}_{k+1} \boldsymbol{y}\rangle - \langle \boldsymbol{x}, \mathcal{M}_{[1,k+1]} \boldsymbol{y}\rangle\right| \prec \left[\left(\prod_{i\in\mathfrak{I}_{k+1}} \rho_i\right) \wedge \max_{i\in[k+1]} \sqrt{\rho_i}\right] \frac{N^{k/2}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}, \tag{3.3.14}$$

uniformly in all choices of subsets of z's and A's as in part (a) and in bounded deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$.

Then, (3.3.13)–(3.3.14) *also hold for the ensemble $H^{(w)}$, uniformly all choices of subsets of z's and A's and in bounded deterministic vectors.*

We are now ready to finally conclude the proof of Theorem 3.2.4. Fix $T > 0$, and fix $z_1, \dots, z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ such that $\min_i N\eta_i\rho_i \ge N^\epsilon$, and let $z_{i,0}$ be the initial conditions of the characteristics (3.3.5) chosen so that $z_{i,T} = z_i$ (this is possible thanks to Lemma 3.3.2). Then, the assumption (3.3.8) of Proposition 3.3.3 is satisfied for those $z_{i,0}$ by Proposition 3.3.1 with $\delta = CT$, where $C > 0$ is the constant from Lemma 3.3.2. We can thus use Proposition 3.3.3 to show that (3.3.9) and (3.3.11) hold. Finally, the Gaussian component added in Proposition 3.3.3 is removed using Proposition 3.3.4 with the aid of a complex version of the standard moment-matching lemma [248, Lemma 16.2], see Lemma 3.A.2 in Appendix 3.A.3 for more details. □

3.4 Characteristic flow: Proof of Proposition 3.3.3

In this section we present the proof of Proposition 3.3.3. The argument is structured as follows:

(i) In Section 3.4.1 we begin by proving the average part, Proposition 3.3.3 (a), in the case when $\mathcal{G}_{j,t} = \operatorname{Im} G_{j,t}$ for each $j \in [k]$, i.e., we prove (3.3.9) for chains containing only $\operatorname{Im} G$'s. Along the flow (3.3.3) new resolvents without imaginary part arise, so the pure $\operatorname{Im} G$ structure cannot be directly maintained. However, we can use the integral representation (see, e.g. [168, Eq. (3.14)]),

$$\prod_{j=1}^m G(z_j) = \frac{1}{\pi} \int_{\mathbf{R}} \operatorname{Im} G(x + \mathrm{i}\eta) \prod_{j=1}^m \frac{1}{x - z_j + \operatorname{sgn}(\operatorname{Im} z_j)\mathrm{i}\eta} \mathrm{d}x, \tag{3.4.1}$$

(that is valid for any $0 < \eta < \min_j \operatorname{Im} z_j$ or $\max_j \operatorname{Im} z_j < -\eta < 0$) to express each G in terms of $\operatorname{Im} G$, thus the flow for purely $\operatorname{Im} G$ chains will be self-contained.

(ii) Afterwards, in the very short Section 3.4.2, we prove the isotropic part, Proposition 3.3.3 (b) again first in the case when $\mathcal{G}_{j,t} = \operatorname{Im} G_{j,t}$ for each $j \in [k+1]$. Due to the Hilbert-Schmidt error terms, the isotropic bound (3.3.11) will directly follow from (3.3.9) proven in Section 3.4.1.

(iii) Finally, using the integral representation (3.4.1) in the special case $m = 1$, we derive the general case of mixed chains from the purely $\operatorname{Im} G$'s case in Section 3.4.3.

Without loss of generality, to keep the presentation simpler, throughout this section we will assume that $\sigma := \mathbf{E}\,\chi_{\mathrm{od}}^2$ is real and $\mathbf{E}\,\chi_{\mathrm{d}}^2 = 1 + \sigma$ (recall that $\chi_{\mathrm{d}}, \chi_{\mathrm{od}}$ are the distribution of the diagonal and off-diagonal matrix elements of W, respectively). At the end, in Section 3.4.4, we will explain how to lift these two restrictions.

We recall our choice of the characteristics

$$\partial_t z_{i,t} = -m(z_{i,t}) - \frac{z_{i,t}}{2}. \tag{3.4.2}$$

Additionally, we record the following trivially checkable integration rules for any $\alpha \geq 1$:

$$\int_0^t \frac{1}{\eta_{i,s}^{\alpha}}\,\mathrm{d}s \lesssim \frac{\log N}{\eta_{i,t}^{\alpha-1}\rho_{i,t}} \qquad \text{and} \qquad \int_0^t \frac{1}{\eta_s^{\alpha}}\,\mathrm{d}s \lesssim \frac{\log N}{\eta_t^{\alpha-2}\hat{\ell}_t} \quad \text{with} \quad \eta_t := \min_i \eta_{i,t}\,, \quad \hat{\ell}_t := \min_i \eta_{i,t}\rho_{i,t}\,. \tag{3.4.3}$$

Note that, in general, $\hat{\ell}$ differs from ℓ, introduced in Theorem 3.2.4. However, in case that every resolvent $\mathcal{G}$ in a given chain is an $\operatorname{Im} G$, i.e. $\mathfrak{I}$ is the full set of indices, then it holds that $\hat{\ell} = \ell$. The notation 'hat' will be consistently used to indicate that a chain contains only $\operatorname{Im} G$'s (see (3.4.6)–(3.4.7) below).

Using the short–hand notation $G_{i,t} := (W_t - z_{i,t})^{-1}$ with W_t being the solution of (3.3.3), we now compute the derivative (recall (3.2.15))

$$\mathrm{d}\langle(\operatorname{Im} G_{1,t}A_1 ... \operatorname{Im} G_{k,t} - \mathcal{M}(z_{1,t}, A_1, ..., z_{k,t}; [k]))A_k\rangle = ... \tag{3.4.4}$$

along the characteristics with the aid of Itô's formula. We point out that the following derivation of the flow holds for any deterministic matrices A_i, i.e. in this derivation we do not assume that $\langle A_i\rangle = 0$. We will assume again that A_i are traceless only later starting from the beginning of Section 3.4.1.

The evolution for (3.4.4) (see (3.4.9) below) is obtained by multilinearity from the analogous formula for the time derivative of a resolvent chain without any imaginary parts. So first we compute

$$\begin{aligned}\mathrm{d}\langle(G_{[1,k],t} - M_{[1,k],t})A_k\rangle = {} & \frac{1}{\sqrt{N}}\sum_{a,b=1}^{N}\partial_{ab}\langle G_{[1,k],t}A_k\rangle \mathrm{d}B_{ab,t} + \frac{k}{2}\langle G_{[1,k],t}A_k\rangle \mathrm{d}t + \sum_{\substack{i,j=1\\ i<j}}^{k}\langle G_{[i,j],t}\rangle\langle G_{[j,i],t}\rangle \mathrm{d}t \\ & + \sum_{i=1}^{k}\langle G_{i,t} - m_{i,t}\rangle\langle G_{[1,k],t}^{(i)}A_k\rangle \mathrm{d}t - \partial_t\langle M_{[1,k],t}A_k\rangle \mathrm{d}t + \frac{\sigma}{N}\sum_{\substack{i,j=1\\ i\leq j}}^{k}\langle G_{[i,j],t}G_{[j,i],t}^{\mathrm{t}}\rangle \mathrm{d}t\,,\end{aligned} \tag{3.4.5}$$

where $\partial_{ab} := \partial_{w_{ab}}$ denotes the direction derivative in the direction $w_{ab} = w_{ab}(t) := (W_t)_{ab}$. Here we introduced the notation

$$G_{[i,j],t} := \begin{cases} G_{i,t}A_i \dots A_{j-1}G_{j,t} & \text{if} \quad i < j \\ G_{i,t} & \text{if} \quad i = j \\ G_{i,t}A_{i,t} \dots \operatorname{Im} G_{k,t}A_k G_{1,t}A_1 \dots A_{j-1}G_{j,t} & \text{if} \quad i > j\,, \end{cases}$$

and analogously for the deterministic approximation $M_{[i,j],t}$ (cf. (3.2.10)). Furthermore, we define $G_{[i,j],t}^{(l)}$ exactly as $G_{[i,j],t}$ but with the l–th factor $G_{l,t}$ being substituted by $G_{l,t}^2$. For the last term in (3.4.5) we used the convention that $\langle G_{[i,j],t}^{\mathrm{t}}G_{[j,i],t}\rangle = \langle G_{i,t}^{\mathrm{t}}G_{i,t}A_{i+1}G_{[i+1,i],t}\rangle$ for $j = i$.

In order to write the derivative (3.4.4) in a manageable form, we need to introduce some further short–hand notations. Set

$$\widehat{G}_{[\hat{i},\hat{j}],t} := \begin{cases} \operatorname{Im} G_{i,t} A_i \dots A_{j-1} \operatorname{Im} G_{j,t} & \text{if} \quad i < j \\ \operatorname{Im} G_{i,t} & \text{if} \quad i = j \\ \operatorname{Im} G_{i,t} A_{i,t} \dots \operatorname{Im} G_{k,t} A_k \operatorname{Im} G_{1,t} A_1 \dots A_{j-1} \operatorname{Im} G_{j,t} & \text{if} \quad i > j, \end{cases} \tag{3.4.6}$$

and define $\widehat{G}^{(l)}_{[\hat{i},\hat{j}],t}$ exactly as $\widehat{G}_{[\hat{i},\hat{j}],t}$ except the l–th factor $\operatorname{Im} G_{l,t}$ is substituted with $G_{l,t}\operatorname{Im} G_{l,t}$. Similarly, $\widehat{G}^{(l^*)}_{[\hat{i},\hat{j}],t}$ is defined as $\widehat{G}_{[\hat{i},\hat{j}],t}$ but with the l–th $\operatorname{Im} G_{l,t}$ is substituted by $G^*_{l,t}\operatorname{Im} G_{l,t}$. Furthermore, we also define

$$\widehat{G}_{[\hat{i},j],t} := \begin{cases} \operatorname{Im} G_{i,t} A_i \dots A_{j-1} G_{j,t} & \text{if} \quad i < j \\ G_{i,t} & \text{if} \quad i = j \\ \operatorname{Im} G_{i,t} A_{i,t} \dots \operatorname{Im} G_{k,t} A_k \operatorname{Im} G_{1,t} A_1 \dots A_{j-1} G_{j,t} & \text{if} \quad i > j; \end{cases} \tag{3.4.7}$$

note the absent hat on the j index indicates that the last resolvent $G_{j,t}$ is without imaginary part. We also define $\widehat{G}_{[i^*,\hat{j}],t}$ replacing $\operatorname{Im} G_{i,t}$ with $G^*_{i,t}$ in (3.4.6) and similarly $\widehat{G}_{[i^*,j],t}$ is defined by replacing $\operatorname{Im} G_{i,t}$ with $G^*_{i,t}$ and $\operatorname{Im} G_{j,t}$ with $G_{j,t}$ in (3.4.6). In particular, the 'decorations' of i and j indicate, whether $G_{i,t}$ and $G_{j,t}$ are really taken as plain resolvents (no decoration) or as adjoints (star) or with imaginary part (hat). We point out that throughout this entire section 'hat' on G indicates that the chain contains only $\operatorname{Im} G_i$ unless specified as in (3.4.7). Finally, we use similar notations for the corresponding deterministic approximation $\widehat{M}_{[i^\#,j^\#],t}$ whose 'undecorated' version was defined in (3.2.10). Here $\#$ indicates one of the possible 'decorations', i.e. star, hat or no decoration and the corresponding change entails modifying the factor $(x - z_i)^{-1}$ in (3.2.11) to $(x - \bar{z}_i)^{-1}$ in case of star, and to $\operatorname{Im}(x - \bar{z}_i)^{-1}$ in case of hat (as in (3.2.15)–(3.2.16)).

The time derivative of the deterministic term in (3.4.4) is obtained by the following lemma, the proof of which is given in Appendix 3.A.

Lemma 3.4.1. *For any $k \geq 1$ we have*

$$\begin{aligned} \partial_t \langle \widehat{M}_{[\hat{1},\hat{k}],t} A_k \rangle = \frac{k}{2} \langle \widehat{M}_{[\hat{1},\hat{k}],t} A_k \rangle + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle \widehat{M}_{[\hat{i},j],t} \rangle \langle \widehat{M}_{[\hat{j},i],t} \rangle + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle \widehat{M}_{[i^*,\hat{j}],t} \rangle \langle \widehat{M}_{[j^*,\hat{i}],t} \rangle \\ + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle \widehat{M}_{[\hat{i},\hat{j}],t} \rangle \langle \widehat{M}_{[j^*,i],t} \rangle + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle \widehat{M}_{[i^*,j],t} \rangle \langle \widehat{M}_{[\hat{j},\hat{i}],t} \rangle. \end{aligned} \tag{3.4.8}$$

Hence, by Itô's formula, for any $k \geq 1$, the evolution of $\widehat{G}_{[\hat{1},\hat{k}],t}$ is given by (for brevity we omit the $\mathrm{d}t$ differentials)

$$\begin{aligned} &\mathrm{d}\langle (\widehat{G}_{[\hat{1},\hat{k}],t} - \widehat{M}_{[\hat{1},\hat{k}],t}) A_k \rangle \\ &= \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} \partial_{ab} \langle \widehat{G}_{[\hat{1},\hat{k}],t} A_k \rangle \mathrm{d}B_{ab} + \frac{k}{2} \langle (\widehat{G}_{[\hat{1},\hat{k}],t} - \widehat{M}_{[\hat{1},\hat{k}],t}) A_k \rangle + \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_\sigma \\ &\quad + \sum_{i=1}^{k} \langle G_{i,t} - m_{i,t} \rangle \langle \widehat{G}^{(i)}_{[\hat{1},\hat{k}],t} A_k \rangle + \sum_{i=1}^{k} \langle G^*_{i,t} - \overline{m_{i,t}} \rangle \langle \widehat{G}^{(i^*)}_{[\hat{1},\hat{k}],t} A_k \rangle + \langle \widehat{G}_{[\hat{1},\hat{k}],t} A_k \rangle \sum_{i=1}^{k} \frac{\langle \operatorname{Im} G_{i,t} - \operatorname{Im} m_{i,t} \rangle}{\operatorname{Im} z_{i,t}}, \end{aligned} \tag{3.4.9}$$

where

$$\Omega_1 := \sum_{\substack{i,j=1\\ i<j}}^{k} \Big[\langle \widehat{G}_{[\hat{i},j],t} - \widehat{M}_{[\hat{i},j],t}\rangle\langle \widehat{M}_{[\hat{j},i],t}\rangle + \langle \widehat{M}_{[\hat{i},j],t}\rangle\langle \widehat{G}_{[\hat{j},i],t} - \widehat{M}_{[\hat{j},i],t}\rangle + \langle \widehat{G}_{[\hat{i},j],t} - \widehat{M}_{[\hat{i},j],t}\rangle\langle \widehat{G}_{[\hat{j},i],t} - \widehat{M}_{[\hat{j},i],t}\rangle\Big],$$

$$\Omega_2 := \sum_{\substack{i,j=1\\ i<j}}^{k} \Big[\langle \widehat{G}_{[i^*,\hat{j}],t} - \widehat{M}_{[i^*,\hat{j}],t}\rangle\langle \widehat{M}_{[j^*,\hat{i}],t}\rangle + \langle \widehat{M}_{[i^*,\hat{j}],t}\rangle\langle \widehat{G}_{[j^*,\hat{i}],t} - \widehat{M}_{[j^*,\hat{i}],t}\rangle + \langle \widehat{G}_{[i^*,\hat{j}],t} - \widehat{M}_{[i^*,\hat{j}],t}\rangle\langle \widehat{G}_{[j^*,\hat{i}],t} - \widehat{M}_{[j^*,\hat{i}],t}\rangle\Big],$$

$$\Omega_3 := \sum_{\substack{i,j=1\\ i<j}}^{k} \Big[\langle \widehat{G}_{[\hat{i},\hat{j}],t} - \widehat{M}_{[\hat{i},\hat{j}],t}\rangle\langle \widehat{M}_{[j^*,i],t}\rangle + \langle \widehat{M}_{[\hat{i},\hat{j}],t}\rangle\langle \widehat{G}_{[j^*,i],t} - \widehat{M}_{[j^*,i],t}\rangle + \langle \widehat{G}_{[\hat{i},\hat{j}],t} - \widehat{M}_{[\hat{i},\hat{j}],t}\rangle\langle \widehat{G}_{[j^*,i],t} - \widehat{M}_{[j^*,i],t}\rangle\Big],$$

$$\Omega_4 := \sum_{\substack{i,j=1\\ i<j}}^{k} \Big[\langle \widehat{G}_{[i^*,j],t} - \widehat{M}_{[i^*,j],t}\rangle\langle \widehat{M}_{[\hat{j},\hat{i}],t}\rangle + \langle \widehat{M}_{[i^*,j],t}\rangle\langle \widehat{G}_{[\hat{j},\hat{i}],t} - \widehat{M}_{[\hat{j},\hat{i}],t}\rangle + \langle \widehat{G}_{[i^*,j],t} - \widehat{M}_{[i^*,j],t}\rangle\langle \widehat{G}_{[\hat{j},\hat{i}],t} - \widehat{M}_{[\hat{j},\hat{i}],t}\Big],$$

$$\Omega_\sigma := \frac{\sigma}{N}\sum_{\substack{i,j=1\\ i\le j}}^{k} \Big[\langle G_{[\hat{i},j],t} G^{\mathrm{t}}_{[\hat{j},i],t}\rangle + \langle G_{[i^*,\hat{j}],t} G^{\mathrm{t}}_{[j^*,\hat{i}],t}\rangle + \langle G_{[\hat{i},\hat{j}],t} G^{\mathrm{t}}_{[j^*,i],t}\rangle + \langle G_{[i^*,j],t} G^{\mathrm{t}}_{[\hat{j},\hat{i}],t}\rangle\Big]. \tag{3.4.10}$$

Observe that the flow (3.4.9) for imaginary parts $\operatorname{Im} G$ contains much more terms compared to a flow for plain resolvents G (see (3.4.5)). This is a simple consequence of the fact that each time an $\operatorname{Im} G$ is differentiated it creates two terms, i.e. $\partial_{ab}\operatorname{Im} G = G\Delta^{ab}\operatorname{Im} G + \operatorname{Im} G\Delta^{ab}G^*$, with Δ^{ab} being a matrix consisting of all zeroes except for the (a,b)–entry which is equal to one. Furthermore, the novel last term in (3.4.9) comes from applying a Ward identity, $GG^* = \operatorname{Im} G/\operatorname{Im} z$. We now write out the random part $\mathrm{d}\langle \widehat{G}_{[\hat{1},\hat{k}],t} A_k\rangle$ of the flow (3.4.9) for the simpler cases $k = 1$ and $k = 2$ to show its main structure. Here we used that $\widehat{M}_{\hat{1},t} = \operatorname{Im} m_{1,t}$ with $m_i := m(z_{i,t})$.

Example 3.4.2. *For $k = 1$ we have the evolution*

$$\begin{aligned}\mathrm{d}\langle \operatorname{Im} GA\rangle = &\sum_{a,b=1}^{N} \partial_{ab}\langle \operatorname{Im} GA\rangle \frac{\mathrm{d}B_{ab}}{\sqrt{N}} + \left(\frac{1}{2} + \frac{\langle \operatorname{Im} G - \operatorname{Im} m\rangle}{\operatorname{Im} z_t}\right)\langle \operatorname{Im} GA\rangle + \langle G - m\rangle\langle \operatorname{Im} GAG\rangle \\ &+ \overline{\langle G-m\rangle}\langle \operatorname{Im} GAG^*\rangle + \frac{\sigma}{N}\langle \operatorname{Im} GAGG^{\mathrm{t}}\rangle + \frac{\sigma}{N}\langle (G^*)^{\mathrm{t}} G^* A \operatorname{Im} GA\rangle + \frac{\sigma}{N}\langle \operatorname{Im} G^{\mathrm{t}} G^* AG\rangle,\end{aligned} \tag{3.4.11}$$

and for $k = 2$ we get (for keeping the formula somewhat short, we assume $\sigma = 0$)

$$\begin{aligned}\mathrm{d}\langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 A_2\rangle = &\sum_{a,b=1}^{N} \partial_{ab}\langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 A_2\rangle \frac{\mathrm{d}B_{ab}}{\sqrt{N}} + \langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 A_2\rangle \\ &+ \left(\frac{\langle \operatorname{Im} G_1 - \operatorname{Im} m_1\rangle}{\operatorname{Im} z_{1,t}} + \frac{\langle \operatorname{Im} G_2 - \operatorname{Im} m_2\rangle}{\operatorname{Im} z_{2,t}}\right)\langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 A_2\rangle + \langle G_2^* A_2 G_1\rangle\langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 \\ &+ \langle G_1^* A_1 G_2\rangle\langle \operatorname{Im} G_2 A_2 \operatorname{Im} G_1\rangle + \langle \operatorname{Im} G_1 A_1 G_2\rangle\langle \operatorname{Im} G_2 A_2 G_1\rangle + \langle G_2^* A_2 \operatorname{Im} G_1\rangle\langle G_1^* A_1 \operatorname{Im} G_2\rangle \\ &+ \langle G_1 - m_1\rangle\langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 A_2 G_1\rangle + \langle G_2 - m_2\rangle\langle \operatorname{Im} G_2 A_2 \operatorname{Im} G_1 A_1 G_2\rangle \\ &+ \langle G_1^* - \overline{m_1}\rangle\langle \operatorname{Im} G_1 A_1 \operatorname{Im} G_2 A_2 G_1^*\rangle + \langle G_2^* - \overline{m_2}\rangle\langle \operatorname{Im} G_2 A_2 \operatorname{Im} G_1 A_1 G_2^*\rangle.\end{aligned} \tag{3.4.12}$$

Note that (3.4.11)–(3.4.12) *combined with* (3.4.8) *give* (3.4.9) *for the special cases $k = 1, 2$.*

3.4.1 Proof of Proposition 3.3.3 (a) for pure $\operatorname{Im} G$-chains

The goal of this section is to prove

$$\langle \widehat{G}_{[\hat{1},\hat{k}],T} A_k\rangle - \langle \widehat{M}_{[\hat{1},\hat{k}],T} A_k\rangle = \langle \widehat{G}_{[\hat{1},\hat{k}],0} A_k\rangle - \langle \widehat{M}_{[\hat{1},\hat{k}],0} A_k\rangle + \mathcal{O}_{\prec}\left(\Big(\prod_{i\in[k]}\rho_{i,T}\Big)\frac{N^{k/2-1}}{\sqrt{N\ell_T}}\prod_{j\in[k]}\langle |A_j|^2\rangle^{1/2}\right), \tag{3.4.13}$$

uniformly in the spectrum and in the choice of traceless matrices A_i. We may assume that all the A_i's are Hermitian; the general case follows by multilinearity.

3.4.1.1 Master inequalities

For the purpose of proving (3.4.13), recall the notation $\hat{\ell}_t = \min \eta_{i,t}\rho_{i,t}$ from (3.4.3) and define

$$\Phi_1(t) := \frac{N\sqrt{\hat{\ell}_t}}{\rho_t \langle |A|^2\rangle^{1/2}} \big|\langle G_t A\rangle\big|; \tag{3.4.14a}$$

and for $k \geq 2$

$$\Phi_k(t) := \frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)\prod_{j\in[k]}\langle |A_j|^2\rangle^{1/2}} \big|\langle (\widehat{G}_{[\widehat{1},\widehat{k}],t} - \widehat{M}_{[\widehat{1},\widehat{k}],t})A_k\rangle\big|. \tag{3.4.14b}$$

Note that we defined $\Phi_1(t)$ in a slightly different way than $\Phi_k(t)$ for $k \geq 2$, this is a consequence of the fact that for $k = 1$ we have $|\langle GA\rangle| \sim |\langle \operatorname{Im} GA\rangle|$, i.e. for this special case the imaginary part does not reduce the fluctuation unlike for longer chains (see also Remark 3.2.6 (ii)). The prefactors in (3.4.14) are chosen such that we expect $\Phi_k(t)$ to be an essentially order one quantity, see (3.4.13). The goal is to show exactly this, i.e. that $\Phi_k(t) \prec 1$, uniformly in time $t \leq T$ for any $k \geq 1$. Note that by (3.3.8) it follows

$$\Phi_k(0) \prec 1, \tag{3.4.15}$$

for any $k \geq 1$.

To prove $\Phi_k(t) \prec 1$, we will derive a series of *master inequalities* for these quantities with the following structure. We assume that

$$\Phi_k(t) \prec \phi_k \tag{3.4.16}$$

holds for some deterministic control parameter ϕ_k, *uniformly* in $0 \leq t \leq T$, in spectral parameters satisfying $N\hat{\ell}_t \geq N^\epsilon$ and in traceless deterministic matrices A_j (we stress that ϕ_k's depend neither on time, nor on the spectral parameters $z_{i,t}$, nor on the matrices A_j). Given this input, we will then show that $\Phi_k(t)$'s also satisfy a better upper bound in terms of ϕ's. Iterating this procedure we will arrive at the final bound $\Phi_k(t) \prec 1$. Furthermore, without loss of generality, we assume that $\phi_k \geq 1$.

Proposition 3.4.3 (Master inequalities)**.** *Fix $k \in \mathbf{N}$. Assume that $\Phi_l(t) \prec \phi_l$ for any $1 \leq l \leq k+1$ uniformly in $t \in [0,T]$, in spectral parameters with $N\hat{\ell}_t \geq N^\epsilon$ and in traceless deterministic matrices A_j. Set $\phi_0 := 1$. Then we have the* master inequalities

$$\Phi_k(t) \prec 1 + \frac{1}{(N\hat{\ell}_T)^{1/4}} \sum_{l=1}^{k} \tilde{\phi}_l \tilde{\phi}_{k-l}\,, \tag{3.4.17}$$

uniformly (in the sense explained below (3.4.16)*) in $t \in [0,T]$, where we introduced the shorthand notation*

$$\tilde{\phi}_l := \phi_l + \mathbf{1}(l \text{ is odd})\sqrt{\phi_{l+1}\phi_{l-1}}\,. \tag{3.4.18}$$

Using the master inequalities, we conclude this section with the proof of (3.3.9) for pure $\operatorname{Im} G$ chains.

Proof of Proposition 3.3.3 (a) for pure $\operatorname{Im} G$ chains. We now consider the master inequalities (3.4.17) for $t = T$, with T the time defined in the statement of Proposition 3.3.3.

We use a two-step induction. The base case consists of the cases $k = 1, 2$:

$$\Phi_1(t) \prec 1 + \frac{\phi_1 + \sqrt{\phi_2}}{(N\hat{\ell}_T)^{1/4}}, \qquad \Phi_2(t) \prec 1 + \frac{\phi_1^2 + \phi_2}{(N\hat{\ell}_T)^{1/4}}, \tag{3.4.19}$$

uniformly in $t \in [0,T]$.

The following abstract iteration lemma shows how to use the master inequalities for improving the bound on Φ.

Lemma 3.4.4 (Iteration). *Let $X = X_N(\hat{\ell})$ be an N-dependent random variable depending also on the parameter $\hat{\ell}$. Fix $\epsilon, \delta > 0$. Suppose that for any $l \in \mathbf{N}$ and any $x > 0$ the fact that $X \prec x$ uniformly for $\hat{\ell} \geq N^{-1+l\epsilon}$ implies*

$$X \prec A + \frac{x}{B} + x^{1-\alpha}C^{\alpha}, \tag{3.4.20}$$

uniformly for $\hat{\ell} \geq N^{-1+(l+l')\epsilon}$, for some constants $l' \in \mathbf{N}$, $B \geq N^{\delta} > 0$, $A, C > 0$, and $\alpha \in (0,1)$, and suppose we also know that $X \prec N^{D}$ uniformly[12] in $\hat{\ell} \geq N^{-1+\epsilon}$. Then

$$X \prec A + C,$$

uniformly for $\hat{\ell} \geq N^{-1+(1+\kappa l')\epsilon}$, for some $\kappa = \kappa(\alpha, D, \delta)$.

Proof. The proof is a simple iteration of (3.4.20) κ times; it is immediate to see that κ depends only on α, D, δ. □

Notice that using Lemma 3.4.4 reduces the domain of parameters η_i, ρ_i for which the master inequalities (3.4.17) hold, e.g. from $\hat{\ell}_t \geq N^{-1+l\epsilon}$ to $\hat{\ell}_t \geq N^{-1+(l+l')\epsilon}$, and so on. However, this can happen only finitely many times, and so it does not affect the estimates in the sense of stochastic domination that always allows for a small N-power tolerance that can be achieved by adjusting ϵ small enough. For simplicity, we ignore this subtlety here, see Sections 1.4.1–1.4.3 for a more detailed explanation.

Using iteration from Lemma 3.4.4 we obtain

$$\Phi_1(t) \prec 1 + \frac{\sqrt{\phi_2}}{(N\hat{\ell}_T)^{1/4}} \quad \text{and} \quad \Phi_2(t) \prec 1 + \frac{\phi_1^2}{(N\hat{\ell}_T)^{1/4}}$$

uniformly in $t \in [0,T]$. We can thus summarize what we proved so far as follows (all the following statements hold uniformly in $t \in [0,T]$):

$$\begin{cases} \Phi_1(t) \prec \phi_1 \\ \Phi_2(t) \prec \phi_2 \end{cases} \quad \Longrightarrow \quad \begin{cases} \Phi_1(t) \prec 1 + \frac{\sqrt{\phi_2}}{(N\hat{\ell}_T)^{1/4}} \\ \Phi_2(t) \prec 1 + \frac{\phi_1^2}{(N\hat{\ell}_T)^{1/4}} \end{cases} \tag{3.4.21}$$

for any deterministic control parameters ϕ_1, ϕ_2.

Next, we use (3.4.21) again replacing the control parameters ϕ_1, ϕ_2 in the input with the new

$$\phi_1' := 1 + \frac{(\phi_2)^{1/2}}{(N\hat{\ell}_T)^{1/4}}, \qquad \phi_2' := 1 + \frac{\phi_1^2}{(N\hat{\ell}_T)^{1/4}}. \tag{3.4.22}$$

We thus obtain

$$\begin{cases} \Phi_1(t) \prec 1 + \frac{(\phi_2')^{1/2}}{(N\hat{\ell}_T)^{1/4}} \lesssim 1 + \frac{\phi_1}{(N\hat{\ell}_T)^{3/8}} \\ \Phi_2(t) \prec 1 + \frac{(\phi_1')^2}{(N\hat{\ell}_T)^{1/4}} \lesssim 1 + \frac{\phi_2}{(N\hat{\ell}_T)^{3/4}} \end{cases} \tag{3.4.23}$$

where in the second inequalities we used the definitions (3.4.22).

Finally, using that $\Phi_i(t) \prec \phi_i$ by assumption and applying Lemma 3.4.4 once again, we obtain

$$\Phi_1(t) \prec 1 \quad \text{and} \quad \Phi_2(t) \prec 1\,.$$

uniformly in $t \in [0,T]$. To prove the same relation for $\Phi_l(t)$ with $l \geq 3$, we use a step-two induction. Fix an even $k \geq 4$ and assume as our induction hypothesis that $\Phi_l(t) \prec 1$ for any $1 \leq l \leq k-2$,

[12] We remark that D, δ, α are N–independent constants, all the other quantities may depend on N.

uniformly in $t \in [0,T]$. We now prove that $\Phi_l(t) \prec 1$ also holds for $l = k-1, k$, uniformly in $t \in [0,T]$. From (3.4.17), using the induction hypothesis $\Phi_l(t) \prec \phi_l := 1$ for $1 \le l \le k-2$, we have

$$\Phi_{k-1}(t) \prec 1 + \frac{\phi_{k-1} + \sqrt{\phi_k}}{(N\hat{\ell}_T)^{1/4}}, \qquad \Phi_k(t) \prec 1 + \frac{\phi_k + \phi_{k-1} + \sqrt{\phi_k}}{(N\hat{\ell}_T)^{1/4}}$$

uniformly in $t \in [0,T]$. Then using iteration from Lemma 3.4.4, we obtain

$$\Phi_{k-1}(t) \prec 1 + \frac{\sqrt{\phi_k}}{(N\hat{\ell}_T)^{1/4}} \quad \text{and} \quad \Phi_k(t) \prec 1 + \frac{\phi_{k-1}}{(N\hat{\ell}_T)^{1/4}}$$

uniformly in $t \in [0,T]$. Proceeding similarly to (3.4.21)–(3.4.23), we thus obtain

$$\Phi_{k-1}(t) \prec 1 \qquad \text{and} \qquad \Phi_k(t) \prec 1\,.$$

This concludes the induction step and hence, by setting $t = T$, the proof of Proposition 3.3.3 (a) modulo the proof of Proposition 3.4.3, which will be done next. □

3.4.1.2 Proof of Proposition 3.4.3

As a preparation for the proof of the master inequalities (Proposition 3.4.3), we recall that $t \mapsto \eta_{i,t}$ is decreasing and $\rho_{i,s} \sim \rho_{i,t}$ for any $0 \le s \le t \lesssim 1$ (see (3.3.6), (3.4.2), and the paragraphs around).

Proof of Proposition 3.4.3. We begin with the case $k = 1$. Hence, for $A_1 = A$, we start by rewriting the flow (3.4.11) with $\operatorname{Im} G$ replaced by $G = G_t(z_t)$ (recall (3.4.14)):

$$\mathrm{d}\langle GA\rangle = \sum_{a,b=1}^{N} \partial_{ab}\langle GA\rangle \frac{\mathrm{d}B_{ab}}{\sqrt{N}} + \frac{1}{2}\langle GA\rangle \mathrm{d}t + \langle G - m\rangle\langle G^2 A\rangle \mathrm{d}t + \frac{\sigma}{N}\langle GAGG^{\mathrm{t}}\rangle \mathrm{d}t\,. \tag{3.4.24}$$

We point out that the additional term $\frac{1}{2}\langle GA\rangle$ in the rhs. of (3.4.24) can be incorporated into the lhs. by differentiating $e^{-t/2}\langle GA\rangle$; the extra exponential factor is irrelevant since $e^{t/2} \sim 1$ for our times $t \lesssim 1$. Note that the same argument applies to the term

$$\frac{k}{2}\langle (\widehat{G}_{[\hat{1},\hat{k}],t} - \widehat{M}_{[\hat{1},\hat{k}],t})A_k\rangle$$

appearing in (3.4.9) for general k. We are now ready to obtain the master inequality for $\Phi_1(t)$.

Assume $\Phi_k(t) \prec \phi_k$ for $k = 1, 2$, in the sense of uniformity explained after (3.4.16) (recall that $\Phi_1(0) \prec 1$ by (3.4.15)), and we will prove improved bounds on $\Phi_1(t)$. We first consider the third summand in (3.4.24). Here, we use the integral representation (see also Lemma 1.5.1)

$$G^2(z) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{G(w)}{(w-z)^2}\mathrm{d}w\,, \tag{3.4.25}$$

which simply follows from residue calculus. Here, Γ is a tiny circle of radius $|\operatorname{Im} z|/2$ around $z \in \mathbf{C} \setminus \mathbf{R}$, which ensures that $|\operatorname{Im} w||\operatorname{Im} m(w)| \sim |\operatorname{Im} z||\operatorname{Im} m(z)|$ as follows by elementary continuity properties of $m(w)$. In this way, applying (3.4.25) for every fixed time $s \le t$ and using the fact that the deterministic approximation of $\langle G^2 A\rangle$ vanishes as $\langle A\rangle = 0$, we obtain (with the $G_s := G_s(z_s)$, $m_s := m(z_s)$ notation)

$$\left|\langle G_s^2 A\rangle\right| \prec \frac{1}{\eta_s}\frac{\rho_s\langle |A|^2\rangle^{1/2}}{N\sqrt{\hat{\ell}_s}}\phi_1\,.$$

Hence, in combination with the single resolvent local law $|\langle G_s - m_s\rangle| \prec 1/(N\eta_s)$, we find

$$\frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle |A|^2\rangle^{1/2}}\int_0^t \langle G_s - m_s\rangle\langle G_s^2 A\rangle\,\mathrm{d}s \prec \frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle |A|^2\rangle^{1/2}}\int_0^t \phi_1\,\frac{\rho_s\langle |A|^2\rangle^{1/2}}{N^2\eta_s^2\hat{\ell}_s^{1/2}}\,\mathrm{d}s \lesssim \frac{\phi_1}{N\hat{\ell}_t}\log N. \tag{3.4.26}$$

In the last step we used the integration estimate (3.4.3) and the fact that along the characteristics $\hat{\ell}_s \gtrsim \hat{\ell}_t$ for $0 \le s \le t$. The prefactor $N\sqrt{\hat{\ell}_t}/(\rho_t\langle|A|^2\rangle^{1/2})$ is included in anticipation of the same prefactor in the definition of Φ_1 in (3.4.14).

Then we proceed with the estimate of the quadratic variation of the martingale term in (3.4.24):

$$\frac{1}{N}\sum_{a,b=1}^{N}\left[\left|\partial_{ab}\langle G_sA\rangle\right|^2+\sigma\partial_{ab}\langle G_sA\rangle\overline{\partial_{ba}\langle G_sA\rangle}\right]\mathrm{d}t\lesssim\frac{1}{N^3}\sum_{a,b=1}^{N}\left|(G_sAG_s)_{ab}\right|^2\mathrm{d}t$$
$$=\frac{1}{N^2}\langle G_sAG_sG_s^*AG_s^*\rangle\mathrm{d}t=\frac{1}{N^2\eta_t^2}\langle\mathrm{Im}\,G_sA\mathrm{Im}\,G_sA\rangle\mathrm{d}t,$$

where we used that $\mathrm{d}[B_{ab},\overline{B_{cd}}]=\delta_{ac}\delta_{bd}+\sigma\delta_{ad}\delta_{bc}$ and the Ward identity $GG^*=\frac{\mathrm{Im}\,G}{\mathrm{Im}\,z}$. Then, we write

$$\langle\mathrm{Im}\,G_sA\mathrm{Im}\,G_sA\rangle=\langle\widehat{M}_{[\widehat{1},\widehat{2}],s}A\rangle+\left(\langle\mathrm{Im}\,G_sA\mathrm{Im}\,G_sA\rangle-\langle\widehat{M}_{[\widehat{1},\widehat{2}],s}A\rangle\right)\prec\rho_s^2\langle|A|^2\rangle+\frac{\rho_s^2\langle|A|^2\rangle}{\sqrt{N\hat{\ell}_s}}\phi_2\,.$$

Here we used that the deterministic approximation $\langle\widehat{M}_{[\widehat{1},\widehat{2}],s}A\rangle$ is bounded by $\rho_s^2\langle|A|^2\rangle$ and we used (3.4.14) together with $\Phi_2(s)\prec\phi_2$. For the time integration of the quadratic variation term, with the appropriate prefactor, we obtain

$$\begin{aligned}&\frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle|A|^2\rangle^{1/2}}\left(\int_0^t\frac{\langle\mathrm{Im}\,G_sA\mathrm{Im}\,G_sA\rangle}{N^2\eta_s^2}\,\mathrm{d}s\right)^{1/2}\\&\qquad\prec\frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle|A|^2\rangle^{1/2}}\left(\int_0^t\frac{\rho_s^2\langle|A|^2\rangle}{N^2\eta_s^2}\left(1+\frac{\phi_2}{(N\hat{\ell}_s)^{1/2}}\right)\mathrm{d}s\right)^{1/2}\lesssim 1+\frac{\sqrt{\phi_2}}{(N\hat{\ell}_t)^{1/4}}\,.\end{aligned}\tag{3.4.27}$$

Here in the last inequality we used that along the characteristics $\hat{\ell}_s \gtrsim \hat{\ell}_t$ for $0 \le s \le t$ and the integration rule (3.4.3). Using the Burkholder-Davis-Gundy (BDG) inequality we conclude exactly the same estimate (3.4.27) for the stochastic term in (3.4.24) in high probability as in quadratic variation.

Next, we estimate the last term in the rhs. of (3.4.24):

$$\begin{aligned}\frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle|A|^2\rangle^{1/2}}\int_0^t\frac{|\sigma|}{N}\left|\langle G_sAG_sG_s^{\mathrm{t}}\rangle\right|\mathrm{d}s&\le\frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle|A|^2\rangle^{1/2}}\int_0^t\frac{1}{N\eta_s^{3/2}}\langle\mathrm{Im}\,G_sA\mathrm{Im}\,G_sA\rangle^{1/2}\langle\mathrm{Im}\,G_s\rangle^{1/2}\,\mathrm{d}s\\&\prec\frac{N\sqrt{\hat{\ell}_t}}{\rho_t\langle|A|^2\rangle^{1/2}}\int_0^t\frac{\rho_s^{1/2}}{N\eta_s^{3/2}}\left(\langle|A|^2\rangle\rho_s^2+\frac{\langle|A|^2\rangle\rho_s^2\phi_2}{\sqrt{N\hat{\ell}_s}}\right)^{1/2}\mathrm{d}s\\&\lesssim 1+\frac{\sqrt{\phi_2}}{(N\hat{\ell}_t)^{1/4}},\end{aligned}\tag{3.4.28}$$

where in the first inequality we used Schwarz inequality together with several Ward identities, and in the second inequality the single resolvent local law $|\langle G_s-m_s\rangle|\prec 1/(N\eta_s)$ to show that $\langle\mathrm{Im}\,G_s\rangle\prec\rho_s$ (recall that we consider the regime $N\eta_s\rho_s\ge N^\epsilon$, so $1/(N\eta_s)\le\rho_s$).

Putting all these estimates together, and using that $\Phi_1(0)\prec 1$ by (3.4.15) to bound the initial condition after integration, we obtain the first master inequality

$$\Phi_1(t)\prec 1+\frac{\phi_1}{N\hat{\ell}_t}+\frac{\sqrt{\phi_2}}{(N\hat{\ell}_t)^{1/4}},\tag{3.4.29}$$

again in the sense of uniformity explained after (3.4.16). This gives (3.4.17) using $N\widehat{\ell}_t\ge 1$ and $\hat{\ell}_T\lesssim\hat{\ell}_t$[13].

[13]This follows from the fact that by (3.3.5) we have $\rho_{i,s}\sim\rho_{i,t}$ and $\eta_{i,t}\lesssim\eta_{i,s}$, for $s\le t$.

For the proof of the master inequalities (3.4.17) with $k \geq 2$, a fundamental input for the estimates of the various terms in (3.4.9) is the following G^2-*Lemma*. Recall that even if we are interested only in pure $\operatorname{Im} G$ chains, their evolution equation (3.4.9) necessarily contains mixed chains as well. The G^2-Lemma turns them back to pure $\operatorname{Im} G$ chains. It expresses how to estimate *not* strictly alternating purely $\operatorname{Im} G$ chains in terms of strictly alternating purely $\operatorname{Im} G$ chains based upon the integral representation (3.4.1). Note that this formula involves the non-analytic function $\operatorname{Im} G$ hence simple and flexible contour deformations are prohibited, contrary to the $k = 1$ case, where we did not care about preserving $\operatorname{Im} G$'s and the contour integral (3.4.25) with the analytic $G(z)$ was applicable.

For brevity we will state the G^2-Lemma for spectral parameters $z_1, ..., z_k$ without time dependence, but eventually we will use them for $z_{1,t}, ..., z_{k,t}$ at any fixed time along the flow. The proof is given in Section 3.4.1.3 below.

Lemma 3.4.5 (G^2-Lemma)**.** *Fix $k \geq 2$. Let $i, j \in [k]$ with $j - i \geq 1$ and assume that $\Phi_l \prec \phi_l$ holds uniformly (in the sense explained after* (3.4.16)*) for some control parameters $\phi_l \geq 1$ for $l = 1, 2, \dots, k$. Then, for all versions of $\widehat{G}_{[i^\#, j^\#]}$ and $\widehat{M}_{[i^\#, j^\#]}$, i.e. for any choice of $\#$ indicating star (adjoint), hat (imaginary part) or simply no 'decoration', we have the following:*[14]

$$\left|\langle \widehat{M}_{[i^\#, j^\#]} \rangle\right| \prec \left(\frac{\rho_i \rho_j}{\eta_i \eta_j}\right)^{1/2} \Big(\prod_{n=i+1}^{j-1} \rho_n \Big) N^{\frac{j-i}{2}-1} \Big(\prod_{m=i}^{j-1} \langle |A_m|^2 \rangle^{1/2} \Big) \tag{3.4.30}$$

and (the decorations at the indices i and j on $\widehat{G}$ and on $\widehat{M}$ have to be matching)

$$\left|\langle \widehat{G}_{[i^\#, j^\#]} - \widehat{M}_{[i^\#, j^\#]} \rangle\right| \prec \left(\frac{\rho_i \rho_j}{\eta_i \eta_j}\right)^{1/2} \Big(\prod_{n=i+1}^{j-1} \rho_n \Big) \frac{N^{\frac{j-i}{2}-1}}{\sqrt{N\hat{\ell}}} \Big(\prod_{m=i}^{j-1} \langle |A_m|^2 \rangle^{1/2} \Big) \tilde{\phi}_{j-i}\,, \tag{3.4.31}$$

where we used the notation $\tilde{\phi}_{j-i} = \phi_{j-i} + \mathbf{1}(j - i \text{ odd}) \sqrt{\phi_{j-i-1} \phi_{j-i+1}}$ (as in (3.4.18)*).*

Moreover, it holds that (now $\#$ indicates star (adjoint) or no 'decoration')

$$\left|\langle \widehat{G}^{(i^\#)}_{[\hat{1}, \hat{k}]} A_k \rangle\right| \prec \left(\frac{\rho_i}{\eta_i}\right)^{1/2} \left|\langle \operatorname{Im} G_i \big(A_i \operatorname{Im} G_{i+1} \dots A_{i-1} \big) \operatorname{Im} G_i \big(A_i \operatorname{Im} G_{i+1} \dots A_{i-1} \big)^* \rangle\right|^{1/2}. \tag{3.4.32}$$

Since all resolvent chains and their M-approximations are multi-linear in the A's, by a simple scaling we may assume, without loss of generality, that $\langle |A_j|^2 \rangle = 1$ for all $j \in [k]$. This shortens some formulas.

We start our estimates on $\Phi_k(t)$ with bounding the quadratic variation of the martingale term in (3.4.9):

$$\begin{aligned} &\frac{1}{N} \sum_{a,b=1}^{N} \Big[\big| \partial_{ab} \langle \widehat{G}_{[\hat{1}, \hat{k}]} A_k \rangle \big|^2 + \sigma \partial_{ab} \langle \widehat{G}_{[\hat{1}, \hat{k}]} A_k \rangle \overline{\partial_{ba} \langle \widehat{G}_{[\hat{1}, \hat{k}]} A_k \rangle} \Big] \\ &\qquad \lesssim \frac{1}{N^2} \sum_{i=1}^{k} \Big\langle \big(G_i A_i \operatorname{Im} G_{i+1} \dots A_{i-1} G_i \big) \big(G_i A_i \operatorname{Im} G_{i+1} \dots A_{i-1} G_i \big)^* \Big\rangle \qquad (3.4.33) \\ &\qquad = \sum_{i=1}^{k} \frac{\big\langle \operatorname{Im} G_i \big(A_i \operatorname{Im} G_{i+1} \dots A_{i-1} \big) \operatorname{Im} G_i \big(A_i \operatorname{Im} G_{i+1} \dots A_{i-1} \big)^* \big\rangle}{N^2 \eta_i^2}\,, \end{aligned}$$

where we omitted the time dependence. Notice that the quadratic variation in (3.4.33) naturally contains chains of length $2k$. In order to get a closed system of inequalities containing only chains up to length k we rely on the following *reduction inequality*; its proof is given in Appendix 3.A.

[14] Note that we use the $\prec$-notation to purely deterministic quantities. The reason is that it conveniently absorbs irrelevant $|\log \eta| \lesssim (\log N)$-factors coming from slightly singular integrals, see Footnote 15.

Lemma 3.4.6 (Reduction inequality). *Fix $k \geq 2$, and assume that $\Phi_l(t) \prec \phi_l$ holds uniformly in $t \in [0,T]$ for $0 \leq l \leq 2k$. Then, uniformly in $t \in [0,T]$ we have*

$$\Phi_{2k}(t) \prec \begin{cases} (N\hat{\ell}_t)^{1/2} + \frac{1}{(N\hat{\ell}_t)^{1/2}}\phi_k^2 & k \text{ even} \\ (N\hat{\ell}_t)^{1/2} + \phi_{k-1} + \phi_{k+1} + \frac{1}{(N\hat{\ell}_t)^{1/2}}\phi_{k+1}\phi_{k-1} & k \text{ odd.} \end{cases} \tag{3.4.34}$$

In the remainder of the proof we will always use (3.4.34) in the following simplified form

$$\Phi_{2k}(t) \prec (N\widehat{\ell}_t)^{1/2} + \tilde{\phi}_k^2, \tag{3.4.35}$$

with $\tilde{\phi}_k$ being defined in (3.4.18). Note that (3.4.35) follows from (3.4.34) using that $\phi_l \geq 1$. Furthermore, we remark that (3.4.35) holds for k being even and odd.

Adding the prefactor from the definition of $\Phi_{2k}(s)$, we find that

$$\begin{aligned} &\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\left(\int_0^t \frac{\big\langle \operatorname{Im} G_{i,s}(...)\operatorname{Im} G_{i,s}(...)^*\big\rangle}{N^2\eta_{i,s}^2}\,\mathrm{d}s\right)^{1/2} \\ &\prec \frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\left(\int_0^t \frac{N^{k-2}\Big(\prod_{i\in[k]}\rho_{i,s}\Big)^2}{N\eta_s^2}\left(1+\frac{\Phi_{2k}(s)}{(N\hat{\ell}_s)^{1/2}}\right)\mathrm{d}s\right)^{1/2} \lesssim 1 + \frac{\tilde{\phi}_k}{(N\hat{\ell}_t)^{1/4}}\,, \end{aligned} \tag{3.4.36}$$

analogously to (3.4.27), where we again used that along the characteristics $\hat{\ell}_s \gtrsim \hat{\ell}_t$ for $0 \leq s \leq t$ and the integration rule (3.4.3). Additionally, in the last inequality we used (3.4.35) for $\Phi_{2k}(s)$. Then, using the BDG inequality we conclude the same estimate in high probability for the martingale term in (3.4.9).

Next, we bound the first two terms in the last line of (3.4.9). We have

$$\begin{aligned} &\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t \big|\langle G_{i,s}-m_{i,s}\rangle\langle \widehat{G}^{(i)}_{[\hat{1},\hat{k}],s}A_k\rangle\big|\,\mathrm{d}s \\ &\prec \frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t \frac{\rho_{i,s}^{1/2}}{N\eta_{i,s}^{3/2}}N^{(k-1)/2}\Big(\prod_{i\in[k]}\rho_{i,s}\Big)\left(1+\frac{\Phi_{2k}(s)}{(N\hat{\ell}_s)^{1/2}}\right)^{1/2}\mathrm{d}s \lesssim 1+\frac{\tilde{\phi}_k}{(N\hat{\ell}_t)^{1/4}}\,, \end{aligned} \tag{3.4.37}$$

where we used the bound in (3.4.32) together with a usual single resolvent local law $|\langle G_{i,s}-m_{i,s}\rangle| \prec (N\eta_{i,s})^{-1}$ and applied a similar reasoning as for (3.4.36), and in the last inequality we used (3.4.35).

Then, we estimate the terms in Ω_σ of (3.4.9). For $j \neq i$ we have

$$\begin{aligned} &\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t \frac{1}{N}\big|\langle G_{[\widehat{i},j],s}G^{\mathrm{t}}_{[\widehat{j},i],s}\rangle\big|\,\mathrm{d}s \\ &\leq \frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t \frac{1}{N}\langle G_{[\widehat{i},j],s}G^*_{[\widehat{i},j],s}\rangle^{1/2}\langle G^*_{[\widehat{j},i],s}G_{[\widehat{j},i],s}\rangle^{1/2}\,\mathrm{d}s \\ &\prec \sqrt{N\hat{\ell}_t}\int_0^t \frac{1}{N\eta_{i,s}\eta_{j,s}}\left(1+\frac{\Phi_{2(j-i)}(s)}{\sqrt{N\hat{\ell}_t}}\right)^{1/2}\left(1+\frac{\Phi_{2(k-j+i)}(s)}{\sqrt{N\hat{\ell}_t}}\right)^{1/2}\mathrm{d}s \\ &\lesssim \frac{1}{\sqrt{N\hat{\ell}_t}}+\frac{\tilde{\phi}_{j-i}}{(N\hat{\ell}_t)^{3/4}}+\frac{\tilde{\phi}_{k-j+i}}{(N\hat{\ell}_t)^{3/4}}+\frac{\tilde{\phi}_{j-i}\tilde{\phi}_{k-j+i}}{N\hat{\ell}_t}\,, \\ &\lesssim \frac{1}{\sqrt{N\hat{\ell}_t}}+\frac{\tilde{\phi}_{j-i}\tilde{\phi}_{k-j+i}}{(N\hat{\ell}_t)^{3/4}}, \end{aligned} \tag{3.4.38}$$

where in the first inequality we used Schwarz,in the second inequality the Ward identity (see (3.4.28) for similar computations in a simpler case), in the third inequality we used the reduction inequality (3.4.35) for $j-i\geq 2$ and that $\tilde{\phi}_1=\phi_1+\sqrt{\phi_2}$ for $j-i=1$, and in the last inequality we used that $\tilde{\phi}_l\geq 1$ for any $l\geq 0$. Similarly, for $j=i$ we get a bound $1+\tilde{\phi}_k/(N\hat{\ell}_t)^{1/4}$. To combine these two cases in a simpler bound we just estimate

$$\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t\frac{1}{N}\big|\langle G_{[\widehat{i},j],s}G^{\mathrm{t}}_{[\widehat{j},i],s}\rangle\big|\,\mathrm{d}s\lesssim 1+\frac{\tilde{\phi}_{j-i}\tilde{\phi}_{k-j+i}}{(N\hat{\ell}_t)^{1/4}}\,. \tag{3.4.39}$$

We are now left with the terms $\Omega_1,\Omega_2,\Omega_3,\Omega_4$ of (3.4.9). We write out the estimates for Ω_1 as all the other Ω_a, $a=2,3,4$, are completely analogous. Using (3.4.30)–(3.4.31) for $i<j$ we estimate

$$\begin{aligned}
&\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t\big|\langle\widehat{G}_{[\hat{i},j],s}-\widehat{M}_{[\hat{i},j],s}\rangle\langle\widehat{M}_{[\hat{j},i],s}\rangle\big|\,\mathrm{d}s\\
&\prec\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t\frac{N^{(j-i)/2-1}}{\sqrt{N\hat{\ell}_s}}\Big(\prod_{n\in[i+1,j-1]}\rho_{n,s}\Big)\frac{\rho_{i,s}\rho_{j,s}}{\eta_{i,s}\eta_{j,s}}\Big(\prod_{n\in[i,j]^c}\rho_{n,s}\Big)N^{(k-j+i)/2-1}\tilde{\phi}_{j-i}\,\mathrm{d}s\\
&\lesssim\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t\frac{\tilde{\phi}_{j-i}}{N\eta_s^2}\frac{N^{k/2-1}}{\sqrt{N\hat{\ell}_s}}\Big(\prod_{i\in[k]}\rho_{i,s}\Big)\mathrm{d}s\lesssim\frac{\tilde{\phi}_{j-i}}{N\hat{\ell}_t},
\end{aligned} \tag{3.4.40}$$

where $[i,j]^c:=[1,i-1]\cup[j+1,k]$. Similarly, we bound

$$\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t\big|\langle\widehat{G}_{[\hat{i},j],s}-\widehat{M}_{[\hat{i},j],s}\rangle\langle\widehat{G}_{[\hat{j},i],s}-\widehat{M}_{[\hat{j},i],s}\rangle\big|\,\mathrm{d}s\prec\frac{\tilde{\phi}_{j-i}\tilde{\phi}_{k-j+i}}{(N\hat{\ell}_t)^{3/2}}\,. \tag{3.4.41}$$

Finally, we estimate the last term in the last line of the rhs. of (3.4.9) as

$$\frac{\sqrt{N\hat{\ell}_t}}{N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,t}\Big)}\int_0^t\left|\langle\widehat{G}_{[\hat{1},\hat{k}],s}A_k\rangle\frac{\langle\operatorname{Im}G_{i,s}-\operatorname{Im}m_{i,s}\rangle}{\eta_{i,s}}\right|\mathrm{d}s\prec\frac{1}{\sqrt{N\hat{\ell}_t}}+\frac{\phi_k}{N\hat{\ell}_t}\,, \tag{3.4.42}$$

where we again used the usual single resolvent local law, the integration rule (3.4.3) and

$$\big|\langle\widehat{G}_{[\hat{1},\hat{k}],s}A_k\rangle\big|\prec N^{k/2-1}\Big(\prod_{i\in[k]}\rho_{i,s}\Big)\left(1+\frac{\phi_k}{\sqrt{N\hat{\ell}_s}}\right).$$

Putting all these estimates (3.4.36)–(3.4.42) together, we thus obtain

$$\Phi_k(t)\prec 1+\frac{1}{N\hat{\ell}_t}\sum_{l=1}^k\tilde{\phi}_l+\frac{1}{(N\hat{\ell}_t)^{3/2}}\sum_{l=1}^{k-1}\tilde{\phi}_l\tilde{\phi}_{k-l}+\frac{|\sigma|}{(N\hat{\ell}_t)^{1/4}}\sum_{l=1}^k\tilde{\phi}_l\tilde{\phi}_{k-l}. \tag{3.4.43}$$

Finally, using that $\tilde{\phi}_l\geq 1$, $|\sigma|\leq 1$, and that $N\widehat{\ell}_t>1$, $\hat{\ell}_T\lesssim\hat{\ell}_t$, we thus conclude (3.4.17). This finishes the proof of Proposition 3.4.3, modulo the proof of Lemma 3.4.5 that will be done next. □

3.4.1.3 Proof of Lemma 3.4.5

As a preparation for our proof, we observe that the estimate (3.2.17) (modulo logarithmic corrections in ℓ) even holds true if the condition $N\ell \geq 1$ with

$$\ell = \min_i [\eta_i(\rho_i + \mathbf{1}(i \notin \mathfrak{I}_k))] = \eta_{i_{\min}}(\rho_{i_{\min}} + \mathbf{1}(i_{\min} \notin \mathfrak{I}_k))]$$

is violated, but the *second smallest*

$$\ell_2 := \min_{i \neq i_{\min}} [\eta_i(\rho_i + \mathbf{1}(i \notin \mathfrak{I}_k))]$$

satisfies $N\ell_2 \geq 1$. More precisely, under this weaker assumption, we still have that

$$|\langle \mathcal{M}(z_1, A_1, ..., A_{k-1}, z_k; \mathfrak{I}_k) A_k \rangle| \lesssim \big(1 + \mathbf{1}(i_{\min} \notin \mathfrak{I}_k) |\log \ell|\big) \bigg(\prod_{i \in \mathfrak{I}_k} \rho_i \bigg) N^{k/2-1} \prod_{j \in [k]} \langle |A_j|^2 \rangle^{1/2} . \tag{3.4.44}$$

This simply follows by realizing that the key estimate within the proof of (3.2.17), namely (3.A.4) in Appendix 3.A, can alternatively be estimated as

$$\Big| m^{(\mathfrak{I}_k)}[S] \Big| \lesssim \big(1 + \mathbf{1}(i_{\min} \notin \mathfrak{I}_k, i_{\min} \in S) |\log \ell|\big) \frac{\prod_{i \in S \cap \mathfrak{I}_k} \rho_i}{\ell_2^{|S|-1}},$$

and following the steps leading to the proof of Lemma 3.2.3 (a).[15] We now turn to the actual proof of Lemma 3.4.5 and again assume that, by simple scaling, $\langle |A_m|^2 \rangle = 1$ for all $m \in [k]$.

We start with the proof of (3.4.30) for both $\#$'s indicating no decoration and assuming, for definiteness, that $\eta_i = \operatorname{Im} z_i > 0$ and $\eta_j = \operatorname{Im} z_j > 0$; all other cases can be treated similarly and are hence omitted. In this case, we use the integral representation [168, Eq. (3.15)] (which simply follows from (3.2.15)–(3.2.16) using multilinearity)[16]

$$\big\langle \widehat{M}_{[i,j]} \big\rangle = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\big\langle \widehat{M}(x + \mathrm{i}\zeta, A_i, z_{i+1}, ..., z_{j-1}) A_{j-1} \big\rangle}{(x - z_i + \mathrm{i}\zeta)(x - z_j + \mathrm{i}\zeta)} \mathrm{d}x \tag{3.4.45}$$

with $\zeta := (\eta_i \wedge \eta_j)/2$. To estimate the x-integration in (3.4.45), we will apply the following basic lemma, which shall frequently be used in the sequel. Its proof is omitted as it is a simple Hölder's inequality and elementary calculus using basic properties of $\rho(z)$.

Lemma 3.4.7. *Under the setting and assumptions described above, for any $\alpha \in [0,1]$, we have that*

$$\frac{1}{\zeta^\alpha} \int_{\mathbf{R}} \frac{\big(\rho(x + \mathrm{i}\zeta)\big)^{1-\alpha}}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \mathrm{d}x \prec \frac{1}{(\eta_i \eta_j)^{1/2}} \bigg(\frac{\rho_i \rho_j}{\big((\eta_i \rho_i)(\eta_j \rho_j)\big)^\alpha} \bigg)^{1/2} . \tag{3.4.46}$$

Therefore, plugging in (3.4.44) with $\mathbf{1}(...) = 0$ for the numerator in (3.4.45) and then using (3.4.46), we obtain

$$\big| \big\langle \widehat{M}_{[i,j]} \big\rangle \big| \lesssim \bigg(\prod_{n=i+1}^{j-1} \rho_n \bigg) N^{(j-i)/2-1} \int_{\mathbf{R}} \frac{\rho(x + \mathrm{i}\zeta)}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \mathrm{d}x \prec \bigg(\frac{\rho_i \rho_j}{\eta_i \eta_j} \bigg)^{1/2} \bigg(\prod_{n=i+1}^{j-1} \rho_n \bigg) N^{(j-i)/2-1} , \tag{3.4.47}$$

completing the proof of (3.4.30).

[15] The logarithmic corrections are stemming from the estimate $\int_{\mathbf{R}} \frac{\rho(x)}{|x-z|} \mathrm{d}x \lesssim 1 + \big|\log |\operatorname{Im} z|\big|$ (cf. (3.2.16)).

[16] Alternatively, this can also be obtained using (3.4.1) for $m = 2$: The resolvent chain, which is approximated by $\langle \widehat{M}_{[i,j]} \rangle$ contains a $G_j G_i$-factor after cyclicity of the trace. Applying (3.4.1) for $m = 2$ to this part of the chain and using a *meta argument* like in Appendix 3.A.4, we can also conclude (3.4.45).

We now turn to the proof of (3.4.31), again focusing on the case where both #'s indicate no decoration and assuming that $\eta_i = \operatorname{Im} z_i > 0$ and $\eta_j = \operatorname{Im} z_j > 0$. As the first step, we apply the integral representations (3.4.1) and (3.4.45) (see [168, Eqs. (3.14) and (3.15)]) to find

$$\left|\left\langle \widehat{G}_{[i,j]} - \widehat{M}_{[i,j]} \right\rangle\right| \lesssim \int_{\mathbf{R}} \frac{\left|\left\langle \left(\operatorname{Im} G(x+\mathrm{i}\zeta) A_i \widehat{G}_{[\widehat{i+1},\widehat{j-1}]} - \widehat{M}(x+\mathrm{i}\zeta, A_i, \ldots)\right) A_{j-1}\right\rangle\right|}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \mathrm{d}x \tag{3.4.48}$$

with $\zeta = (\eta_i \wedge \eta_j)/2$ and split the integral into an *above the scale* and a *below the scale* part. This concept refers to spectral regimes $x \in \mathbf{R}$ where the typical eigenvalue spacing $\rho(x+\mathrm{i}\zeta)/N$ is larger or smaller than the given ζ. More precisely, we fix an arbitrarily small $\xi > 0$ and decompose $\mathbf{R}$ into[17]

$$\left\{x : N\rho(x+\mathrm{i}\zeta)\zeta \geq N^{\xi}\right\} \cup \left\{x : N\rho(x+\mathrm{i}\zeta)\zeta < N^{\xi}\right\} =: I_{\mathrm{above}} \cup I_{\mathrm{below}}\,. \tag{3.4.49}$$

For the *above the scale* part, we use that $\Phi_{j-i} \prec \phi_{j-i}$ and estimate this part of the integral (3.4.48) by

$$\int_{I_{\mathrm{above}}} \frac{\rho(x+\mathrm{i}\zeta)}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \left(\prod_{n=i+1}^{j-1} \rho_n\right) \frac{N^{(k-i)/2-1}}{\sqrt{N\hat{\ell}(x)}} \phi_{j-i} \mathrm{d}x\,, \tag{3.4.50}$$

where we emphasized that now $\hat{\ell}(x) = \zeta\rho(x+\mathrm{i}\zeta) \wedge \min_{n\in[i+1,j-1]} \eta_n\rho_n$ depends on the integration variable x since the integrated chain in (3.4.48) contains a resolvent at spectral parameter $x+\mathrm{i}\zeta$. Next, we further split I_{above} into two parts $I_{\mathrm{above}} = I_{\mathrm{above},=} \cup I_{\mathrm{above},<}$ with

$$I_{\mathrm{above},=} := \left\{x : \hat{\ell}(x) = \rho(x+\mathrm{i}\zeta)\zeta\right\} \quad \text{and} \quad I_{\mathrm{above},<} := \left\{x : \hat{\ell}(x) < \rho(x+\mathrm{i}\zeta)\zeta\right\}, \tag{3.4.51}$$

depending on whether the minimum is attained at the special spectral argument $x+\mathrm{i}\zeta$ or not, and estimate each of them separately. In this way, we obtain the contribution from $I_{\mathrm{above},=}$ to (3.4.50) to equal

$$\frac{1}{\sqrt{N}} \left[\frac{1}{\zeta^{1/2}} \int_{I_{\mathrm{above},=}} \frac{\left(\rho(x+\mathrm{i}\zeta)\right)^{1/2}}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \mathrm{d}x\right] \rho_{i+1} \ldots \rho_{j-1} N^{(j-i)/2-1} \phi_{j-i}\,. \tag{3.4.52}$$

By means of Lemma 3.4.7 with $\alpha = 1/2$ applied to the integral in $\left[\cdots\right]$, this can be bounded as

$$\frac{1}{(\eta_i\eta_j)^{1/2}} \frac{1}{\sqrt{N}} \sum_{s=1}^{j-i} \frac{\sqrt{\rho_i}\,\rho_{i+1} \ldots \rho_{j-1}\sqrt{\rho_j}}{\sqrt{(\eta_i\rho_i)^{1/2}}\sqrt{(\eta_j\rho_j)^{1/2}}} N^{(j-i)/2-1} \phi_{j-i} \leq \left(\frac{\rho_i\rho_j}{\eta_i\eta_j}\right)^{1/2} \left(\prod_{n=i+1}^{j-1} \rho_n\right) \frac{N^{(j-i)/2-1}}{\sqrt{N\hat{\ell}}} \phi_{j-i}\,. \tag{3.4.53}$$

For $I_{\mathrm{above},<}$ the argument is completely analogous, yielding exactly the same bound as in (3.4.53). This completes the bound for the *above the scale* part.

For the *below the scale* part, we estimate the two terms in the numerator in (3.4.48) separately; in this regime the local law is anyway not effective in the sense that $G - M$ is not smaller than G. For the $\widehat{M}$-term, we recall the bound (3.4.44), and estimate

$$\begin{aligned}\int_{I_{\mathrm{below}}} \frac{\left|\left\langle \widehat{M}(x+\mathrm{i}\zeta, A_i, \ldots) A_{j-1}\right\rangle\right|}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \mathrm{d}x &\lesssim N^{(j-i)/2-1} \rho_{i+1} \ldots \rho_{j-1} \left[\int_{I_{\mathrm{below}}} \frac{\rho(x+\mathrm{i}\zeta)}{|x - z_i + \mathrm{i}\zeta||x - z_j + \mathrm{i}\zeta|} \mathrm{d}x\right] \\ &\prec \frac{1}{(\eta_i\eta_j)^{1/2}} \frac{1}{N} \frac{\sqrt{\rho_i}\,\rho_{i+1} \ldots \rho_{j-1}\sqrt{\rho_j}}{(\eta_i\rho_i)^{1/2}(\eta_j\rho_j)^{1/2}} N^{(j-i)/2-1} \lesssim \left(\frac{\rho_i\rho_j}{\eta_i\eta_j}\right)^{1/2} \left(\prod_{n=i+1}^{j-1} \rho_n\right) \frac{N^{(j-i)/2-1}}{N\hat{\ell}}\,.\end{aligned} \tag{3.4.54}$$

[17] To be precise, in the integral (3.4.48) we first need to cut-off the regime where $|x| \geq N^{100}$, say, and estimate this contribution by a simple norm bound using that the spectrum of the Wigner matrix is contained in $[-2-\epsilon, 2+\epsilon]$ with very high probability [249]. Such technicality about the irrelevant, very far out x-regime will henceforth be ignored.

To go from the second to the third line, we used that $\rho(x+\mathrm{i}\zeta)\zeta \prec N^{-1}$ for $x \in I_{\mathrm{below}}$ (recall that $\xi > 0$ in the definition (3.4.49) may be chosen arbitrarily small) and employed Lemma 3.4.7 with $\alpha = 1$. In the ultimate step, we utilized $\eta_i\rho_i \wedge \eta_j\rho_j \geq \hat{\ell}$ together with $N\hat{\ell} \gtrsim 1$. This concludes the discussion of the $\widehat{M}$-term.

Next, we turn to the $\widehat{G}$-term in (3.4.48) in the regime $x \in I_{\mathrm{below}}$ and first focus on the case where $j-i$ is even. Here, we employ a Schwarz inequality in order to be able to exploit

$$\begin{aligned}\big|\big\langle \mathrm{Im}\, G(x+\mathrm{i}\zeta) A_i \widehat{G}_{[\widehat{i+1},\widehat{j-1}]} A_{j-1}\big\rangle\big| \\ \leq \frac{\zeta_x}{\zeta}\big|\big\langle \mathrm{Im}\, G(x+\mathrm{i}\zeta_x)(A_i \ldots \mathrm{Im}\, G_{r-1}A_{r-1})\mathrm{Im}\, G_r (A_i \ldots \mathrm{Im}\, G_{r-1}A_{r-1})^*\big\rangle\big|^{1/2} \\ \times \big|\big\langle \mathrm{Im}\, G_r(A_r \ldots \mathrm{Im}\, G_{j-1}A_{j-1})\mathrm{Im}\, G(x+\mathrm{i}\zeta_x)(A_r \ldots \mathrm{Im}\, G_{j-1}A_{j-1})^*\big\rangle\big|^{1/2}\end{aligned} \tag{3.4.55}$$

where $\zeta_x > \zeta$ is implicitly defined via $N\rho(x+\mathrm{i}\zeta_x)\zeta_x = N^{\xi}$ and we denoted $r := (i+j)/2$. After application of a Schwarz inequality, we find this part of (3.4.48) to be bounded by

$$\begin{aligned}&\left(\int_{I_{\mathrm{below}}} \frac{\zeta_x}{\zeta} \frac{\big|\big\langle \mathrm{Im}\, G(x+\mathrm{i}\zeta_x)(A_i \ldots \mathrm{Im}\, G_{r-1}A_{r-1})\mathrm{Im}\, G_{\frac{j+i}{2}}(A_i \ldots \mathrm{Im}\, G_{r-1}A_{r-1})^*\big\rangle\big|}{|x-z_i+\mathrm{i}\zeta||x-z_j+\mathrm{i}\zeta|}\mathrm{d}x\right)^{1/2} \\ &\times\left(\int_{I_{\mathrm{below}}} \frac{\zeta_x}{\zeta} \frac{\big|\big\langle \mathrm{Im}\, G_r(A_r \ldots \mathrm{Im}\, G_{j-1}A_{j-1})\mathrm{Im}\, G(x+\mathrm{i}\zeta_x)(A_r \ldots \mathrm{Im}\, G_{j-1}A_{j-1})^*\big\rangle\big|}{|x-z_i+\mathrm{i}\zeta||x-z_j+\mathrm{i}\zeta|}\mathrm{d}x\right)^{1/2}\end{aligned} \tag{3.4.56}$$

Adding and subtracting the respective $\widehat{M}$-terms for both resolvent chains in (3.4.56), we are left with two terms for each integral. For concreteness, we estimate the one in the first line in (3.4.56), the second line the same. The first line in (3.4.56) is bounded by (the square root of)

$$\begin{aligned}\frac{1}{\zeta}\int_{I_{\mathrm{below}}} \mathrm{d}x \frac{\zeta_x \rho(x+\mathrm{i}\zeta_x)}{|x-z_i+\mathrm{i}\zeta||x-z_j+\mathrm{i}\zeta|}\left(\prod_{n=i+1}^{r-1}\rho_n\right)^2 \rho_r\, N^{\frac{j-i}{2}-1}(1+\phi_{j-i}) \\ \prec (1+\phi_{j-i})\left(\frac{\rho_i\rho_j}{\eta_i\eta_j}\right)^{1/2}\left(\prod_{n=i+1}^{r-1}\rho_n\right)^2 \rho_r \frac{N^{\frac{j-i}{2}-1}}{N\hat{\ell}}\,.\end{aligned}$$

Here, we used that $N\rho(x+\mathrm{i}\zeta_x)\zeta_x = N^{\xi}$ for arbitrarily small $\xi > 0$ and employed Lemma 3.4.7 (with $\alpha = 1$) in estimates analogous to (3.4.53) and (3.4.54). Combining this with the identical estimate for the second line of (3.4.56) and using $N\hat{\ell} \geq 1$ and $\phi_{j-i} \geq 1$, we finally deduce that

$$(3.4.56) \prec \phi_{j-i}\left(\frac{\rho_i\rho_j}{\eta_i\eta_j}\right)^{1/2}\left(\prod_{n=i+1}^{j-1}\rho_n\right)\frac{N^{(j-i)/2-1}}{\sqrt{N\hat{\ell}}}\,. \tag{3.4.57}$$

For $j-i$ being odd, only the monotonicity argument (3.4.55) is different:

$$\begin{aligned}\big|\big\langle \mathrm{Im}\, G(x+\mathrm{i}\zeta) A_i \widehat{G}_{[\widehat{i+1},\widehat{j-1}]} A_{j-1}\big\rangle\big| \\ \leq \frac{\zeta_x}{\zeta}\big|\big\langle \mathrm{Im}\, G(x+\mathrm{i}\zeta_x)(A_i \ldots \mathrm{Im}\, G_{r-1}A_{r-1})\mathrm{Im}\, G_r (A_i \ldots \mathrm{Im}\, G_{r-1}A_{r-1})^*\big\rangle\big|^{1/2} \\ \times \big|\big\langle \mathrm{Im}\, G_r(A_{r+1} \ldots \mathrm{Im}\, G_{j-1}A_{j-1})\mathrm{Im}\, G(x+\mathrm{i}\zeta_x)(A_{r+1} \ldots \mathrm{Im}\, G_{j-1}A_{j-1})^*\big\rangle\big|^{1/2},\end{aligned}$$

where we now denoted $r := (i+j+1)/2$. This asymmetry in the lengths of the resolvent chains now leads to the term $\sqrt{\phi_{j-i+1}\phi_{j-i-1}}$ in (3.4.31), the rest of the argument is identical.

Finally, we turn to the proof of (3.4.32). Again, we focus on the case where # indicates no decoration. By application of a Schwarz inequality, we find

$$\begin{aligned}\left|\langle \widehat{G}^{(i)}_{[\hat{1},\hat{k}]} A_k\rangle\right| &= \left|\langle \mathrm{Im}\, G_1 A_1 ... \mathrm{Im}\, G_{i-1} A_{i-1} \mathrm{Im}\, G_i G_i A_i ... \mathrm{Im}\, G_k A_k\rangle\right| \\ &\le |\langle G_i G_i^*\rangle|^{1/2} \left|\langle \mathrm{Im}\, G_i (A_i \mathrm{Im}\, G_{i+1} \dots A_{i-1}) \mathrm{Im}\, G_i (A_i \mathrm{Im}\, G_{i+1} \dots A_{i-1})^*\rangle\right|^{1/2} \\ &\prec \left(\frac{\rho_i}{\eta_i}\right)^{1/2} \left|\langle \mathrm{Im}\, G_i (A_i \mathrm{Im}\, G_{i+1} \dots A_{i-1}) \mathrm{Im}\, G_i (A_i \mathrm{Im}\, G_{i+1} \dots A_{i-1})^*\rangle\right|^{1/2},\end{aligned} \tag{3.4.58}$$

where in the last step we used the Ward identity $GG^* = \mathrm{Im}\, G/\eta$ together with the usual single resolvent local law applied to $\mathrm{Im}\, G_i$. This concludes the proof of Lemma 3.4.5 which was the last missing piece for the proof of Proposition 3.3.3 (a) for pure $\mathrm{Im}\, G$ chains. □

3.4.2 Proof of Proposition 3.3.3 (b) for pure $\mathrm{Im}\, G$-chains

In this section, we briefly explain how to derive Proposition 3.3.3 (b) from Proposition 3.3.3 (a). For fixed spectral parameters and bounded deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$, we have

$$\left|\langle \boldsymbol{x}, (\widehat{G}_{[\hat{1},\widehat{k+1}]} - \widehat{M}_{[\hat{1},\widehat{k+1}]})\boldsymbol{y}\rangle\right| \lesssim \left|\langle (\widehat{G}_{[\hat{1},\widehat{k+1}]} - \widehat{M}_{[\hat{1},\widehat{k+1}]}) A_{k+1}\rangle\right| + \left|\langle \widehat{G}_{[\hat{1},\widehat{k+1}]} - \widehat{M}_{[\hat{1},\widehat{k+1}]}\rangle\right| \tag{3.4.59}$$

with the special choice $A_{k+1} := N\boldsymbol{y}\boldsymbol{x}^* - \langle \boldsymbol{x}, \boldsymbol{y}\rangle$. Next, using that $\langle |A_{k+1}|^2\rangle^{1/2} \lesssim N^{1/2}$ we find from Proposition 3.3.3 (a) for pure $\mathrm{Im}\, G$ chains the first term in (3.4.59) to be bounded as

$$\left|\langle (\widehat{G}_{[\hat{1},\widehat{k+1}]} - \widehat{M}_{[\hat{1},\widehat{k+1}]}) A_{k+1}\rangle\right| \prec \Big(\prod_{i\in[k+1]} \rho_i\Big) \frac{N^{k/2}}{\sqrt{N\hat{\ell}}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}.$$

For the second term, we apply (3.4.31) from Lemma 3.4.5 (note that by Proposition 3.3.3 (a) for pure $\mathrm{Im}\, G$ chains, we have $\Phi_k \prec \phi_k := 1$ and hence also $\tilde{\phi}_k = 1$) and obtain

$$\left|\langle \widehat{G}_{[\hat{1},\widehat{k+1}]} - \widehat{M}_{[\hat{1},\widehat{k+1}]}\rangle\right| \prec \left(\frac{\rho_1\rho_{k+1}}{\eta_1\eta_{k+1}}\right)^{1/2} \Big(\prod_{i=2}^{k} \rho_i\Big) \frac{N^{k/2-1}}{\sqrt{N\hat{\ell}}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2} \le \Big(\prod_{i\in[k+1]} \rho_i\Big) \frac{N^{k/2}}{\sqrt{N\hat{\ell}}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2},$$

where in the last step we used $\eta_1\rho_1 \wedge \eta_{k+1}\rho_{k+1} \ge \hat{\ell}$ and $N\hat{\ell} \ge 1$. This concludes the proof of Proposition 3.3.3 (b) for pure $\mathrm{Im}\, G$ chains.

3.4.3 Proof of Proposition 3.3.3 (b) for mixed chains

We consider mixed resolvent chains

$$\mathcal{G}_1 A_1 ... \mathcal{G}_k A_k$$

with $\mathcal{G}_j \in \{G_j, \mathrm{Im}\, G_j\}$ and traceless matrices $A_1, ..., A_k \in \mathbf{C}^{N\times N}$, and explain how the respective bounds in (3.2.20)–(3.2.21) are obtained from the multi-resolvent local law for pure $\mathrm{Im}\, G$-chains derived in Sections 3.4.1–3.4.2. We will henceforth focus on the average case, the isotropic bounds can immediately be obtained from those by following Section 3.4.2.

Recalling

$$\ell = \min_{j\in[k]} \big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_k))\big]$$

where $\mathfrak{I}_k$ denotes the set of indices $j \in [k]$ where $\mathcal{G}_j = \mathrm{Im}\, G_j$, the goal of this section is to prove that

$$\left|\langle \mathcal{G}_1 A_1 ... \mathcal{G}_k A_k\rangle - \langle \mathcal{M}_{[1,k]} A_k\rangle\right| \prec \left[\left(\prod_{i\in\mathfrak{I}_k} \rho_i\right) \wedge \max_{i\in[k]} \sqrt{\rho_i}\right] \frac{N^{k/2-1}}{\sqrt{N\ell}} \prod_{j\in[k]} \langle |A_j|^2\rangle^{1/2}. \tag{3.4.60}$$

In order to do so, we iteratively apply the integral representation (3.4.1) with $m = 1$ for every $\mathcal{G}_j$ such that $j \notin \mathfrak{I}_k$. In Section 3.4.3.1, this procedure will immediately yield the claimed bound (3.4.60) for $\mathfrak{I}_k \neq \varnothing$ (recall from Remark 3.2.6 (ii), that in this case the minimum in (3.4.60) is always realized by the product). In the complementary case, $\mathfrak{I}_k = \varnothing$, which has already been studied in [169], the outcome of iteratively applying (3.4.1) is the natural continuation of the pattern obtained for $\mathfrak{I}_k \neq \varnothing$. However, in this way we only find the weaker bound, where in (3.4.60) the minimum $\big[... \wedge ...\big]$ replaced by one. The improvement to include the small factor $\max_{i\in[k]} \sqrt{\rho_i}$ requires a short separate argument, which we provide in Section 3.4.3.2.

3.4.3.1 The case $\mathfrak{I}_k \neq \varnothing$

For concreteness, we consider the case where $\mathfrak{I}_k = [k-1]$, i.e. $\mathcal{G}_k = G_k$ with $\operatorname{Im} z_k > 0$ w.l.o.g. and all other $\mathcal{G}$'s are $\operatorname{Im} G$'s. Then, using the integral representation (3.4.1) with $m = 1$ and $\eta = \zeta = \operatorname{Im} z_k/2$, and its analog for the deterministic approximation (see [168, Eqs. (3.14) and (3.15)] and (3.4.48) above), we find that

$$\begin{aligned}&|\langle \operatorname{Im} G_1 A_1 ... G_k A_k\rangle - \langle \mathcal{M}(z_1, A_1, ..., z_k; [k-1])A_k\rangle| \\ &\quad \lesssim \int_{\mathbf{R}} \frac{|\langle \operatorname{Im} G_1 A_1 ... \operatorname{Im} G(x+\mathrm{i}\zeta)A_k\rangle - \langle \mathcal{M}(z_1, A_1, ..., x+\mathrm{i}\zeta; [k])A_k\rangle|}{|x - z_k + \mathrm{i}\zeta|}\,\mathrm{d}x\end{aligned}$$

We then follow the steps in the proof of Lemma 3.4.5 starting from (3.4.48) in order to estimate the integral. In particular, we split the integration region into I_{above} and I_{below}, just as in (3.4.49). In the treatment of these regimes, the two main differences compared to the proof of Lemma 3.4.5 are the following:

(i) We use the M-bound in (3.4.44) with logarithmic corrections, which can be absorbed into $\prec$.

(ii) Lemma 3.4.7 gets replaced by the bound

$$\int_{\mathbf{R}} \frac{\big(\rho(x+\mathrm{i}\zeta)\big)^{\alpha}}{|x - z_k + \mathrm{i}\zeta|}\,\mathrm{d}x \prec 1 \qquad \text{for all} \qquad \alpha > 0\,,$$

which can easily be seen using that $\operatorname{Im} z_k \geq N^{-1}$ and $\rho(w)$ decays polynomially as $|w| \to \infty$.

For example, instead of (3.4.52) we estimate (recall that I_{above} is further split into $I_{\mathrm{above},=}$ and $I_{\mathrm{above},<}$ in (3.4.51))

$$\frac{1}{\sqrt{N}}\left[\frac{1}{\zeta^{1/2}}\int_{I_{\mathrm{above},=}} \frac{\big(\rho(x+\mathrm{i}\zeta)\big)^{1/2}}{|x - z_k + \mathrm{i}\zeta|}\mathrm{d}x\right]\rho_1 \ldots \rho_{k-1} N^{k/2-1} \prec \left(\prod_{i\in[k-1]} \rho_i\right)\frac{N^{k/2-1}}{\sqrt{N\ell}}\,,$$

neglecting the product of Hilbert-Schmidt norms. We point out that, compared to the estimates in the pure $\operatorname{Im} G$-case, now $\ell := \min_{j\in[k]}\big[\eta_j(\rho_j + \mathbf{1}(j \neq k))\big]$ and ρ_k disappeared from the rhs. Therefore, as a result, we find the claimed bound (3.4.60) for $\mathfrak{I}_k = [k-1]$. All other cases with $\mathfrak{I}_k \neq \varnothing$ follow by iteratively applying this strategy. This completes the proof of Proposition 3.3.3 (b) if $\mathfrak{I}_k \neq \varnothing$.

3.4.3.2 The case $\mathfrak{I}_k = \varnothing$

As mentioned above, in order to obtain the improvement by $\max_{i\in[k]} \sqrt{\rho_i}$, we now give a separate argument. We thereby closely follow the steps in Section 3.4.1 and point out only the main differences. In particular, we now use the flow (3.4.5), together with the following lemma proven in Appendix 3.A.4, instead of (3.4.9). Here, similarly to (3.4.5), the absence of hats indicates that *none* of the resolvents $\mathcal{G}$ in the chain approximated by M is an $\operatorname{Im} G$.

Lemma 3.4.8. *We have*

$$\partial_t \langle M_{[1,k],t} A_k \rangle = \frac{k}{2} \langle M_{[1,k],t} A_k \rangle + \sum_{\substack{i,j=1,\\ i<j}}^{k} \langle M_{[i,j],t} \rangle \langle M_{[j,i],t} \rangle .$$

Moreover, using the shorthand notations

$$\eta_t := \min_{i\in[k]} \eta_{i,t} \quad \text{and} \quad \rho_t := \max_{i\in[k]} \rho_{i,t} ,$$

we introduce the new normalized differences

$$\Psi_k(t) := \frac{\sqrt{N\eta_t}}{N^{k/2-1}\sqrt{\rho_t}\prod_{j\in[k]}\langle |A_j|^2\rangle^{1/2}} \big| \langle (G_{[1,k],t} - M_{[1,k],t}) A_k \rangle \big| \tag{3.4.61}$$

for every $k \in \mathbf{N}$. The Ψ_k's introduced here are the no-$\operatorname{Im} G$-analogs of the Φ_k's defined in (3.4.14), i.e. all hats are removed and we replaced $\hat{\ell}_t \to \eta_t$ as well as $\prod_i \rho_{i,t} \to \sqrt{\rho_t}$.

In the following, we will derive master inequalities for the Ψ_k's, analogously to Proposition 3.4.3. However, compared to the proof in Section 3.4.1, we now have two major simplifications:

(i) Since the bound (3.4.60) for $\mathfrak{I}_k \neq \varnothing$ is already proven, the contribution of the quadratic variation term in (3.4.5), which automatically carries two $\operatorname{Im} G$'s, is easily estimated as (again assuming $\langle |A_j|^2 \rangle = 1$ for all $j \in [k]$ henceforth)

$$\begin{aligned} \frac{\sqrt{N\eta_t}}{N^{k/2-1}\sqrt{\rho_t}} \left(\int_0^t \frac{\big\langle \operatorname{Im} G_{i,s} \big(A_i G_{i+1,s} \ldots A_{i-1} \big) \operatorname{Im} G_{i,s} \big(A_i G_{i+1,s} \ldots A_{i-1} \big)^* \big\rangle}{N^2 \eta_{i,s}^2} \, \mathrm{d}s \right)^{1/2} \\ \prec \frac{\sqrt{N\eta_t}}{N^{k/2-1}\sqrt{\rho_t}} \left(\int_0^t \frac{N^{k-2}\rho_{i,s}^2}{N\eta_{i,s}^2} \, \mathrm{d}s \right)^{1/2} \lesssim \sqrt{\frac{\rho_{i,t}\eta_t}{\rho_t \eta_{i,t}}} \le 1 , \end{aligned}$$

analogously to (3.4.36). Note that in the first step, we did not use the overestimate $1/\eta_{i,s} \le 1/\eta_s$ inside the integral as done in (3.4.36). The same reasoning applies to the analog of the first two terms in the last line of (3.4.9) and the terms contained in Ω_σ (cf. the estimates in (3.4.37)–(3.4.39)). We point out that, in this section, the already proven bounds for resolvent chains containing at least one $\operatorname{Im} G$ make the usage of reduction inequalities as in Lemma 3.4.6 obsolete.

(ii) For treating the analogues of $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ in (3.4.9), it is not necessary to "restore" $\operatorname{Im} G$'s via the integral representation (3.4.1) as in the proof of the G^2-Lemma 3.4.5. Instead, in the course of proving an analog of Lemma 3.4.5 (again suppressing the time dependence of the z's as well as η and ρ) it is sufficient to apply resolvent identities for $|z_i - z_j| \ge \eta$ and the integral representation

$$G(z_i)G(z_j) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{G(w)}{(w-z_i)(w-z_j)} \mathrm{d}w ,$$

for $|z_i - z_j| \le \eta$. In this case z_i and z_j are necessarily on the same halfplane ($\operatorname{Im} z_i \operatorname{Im} z_j > 0$) and, just as in (3.4.25), Γ is a tiny contour encircling $z_i, z_j \in \mathbf{C} \setminus \mathbf{R}$ in such a way that $\operatorname{dist}(\Gamma, \{z_i, z_j\}) \sim \eta$, which ensures that $|\operatorname{Im} m(w)| \lesssim \max_{i\in[k]} \rho_i$ on Γ as follows by elementary continuity properties of $m(w)$.

As a consequence, for fixed $k \in \mathbf{N}$, we find, assuming $\Psi_l \prec \psi_l$ for some control parameters $\psi_l \ge 1$ for $l = 1, 2, \ldots, k$ in the usual sense of uniformity explained below (3.4.16), that

$$\big| \big\langle M_{[i,j]} \big\rangle \big| \prec \frac{1}{\eta} N^{\frac{j-i}{2}-1} \Big(\prod_{m=i}^{j-1} \langle |A_m|^2 \rangle^{1/2} \Big) ,$$

as an analog of (3.4.30) and

$$\big|\big\langle G_{[i,j]} - M_{[i,j]}\big\rangle\big| \prec \frac{1}{\eta}\, N^{\frac{j-i}{2}-1}\sqrt{\frac{\rho}{N\eta}}\Big(\prod_{m=i}^{j-1}\langle|A_m|^2\rangle^{1/2}\Big)\psi_{j-i}\,,$$

as an analog of (3.4.31), for all $i, j \in [k]$ with $j - i \geq 1$.

Overall, using the above two simplifications and following the arguments in (3.4.36)–(3.4.42), we arrive at the following new set of master inequalities.

Proposition 3.4.9 (Master inequalities II)**.** *Fix $k \in \mathbf{N}$. Assume that $\Psi_l(t) \prec \psi_l$ for any $1 \leq l \leq k$ uniformly in $t \in [0,T]$, in spectral parameters with $N\hat{\ell}_t \geq N^{\epsilon}$ (recall $\hat{\ell}_t = \min_{i\in[k]} \eta_{i,t}\rho_{i,t}$ from* (3.4.3)*; not to be confused with the ℓ used around* (3.4.60)*!) and in traceless deterministic matrices A_j for some $\psi_l \geq 1$ and set $\psi_0 := 1$. Then we have the* master inequalities

$$\Psi_k(t) \prec 1 + \frac{1}{(N\hat{\ell}_T)^{1/4}}\sum_{l=1}^{k}\psi_l\psi_{k-l} \tag{3.4.62}$$

uniformly (in the sense explained below (3.4.16)*) in $t \in [0,T]$.*

Using that $N\hat{\ell}_T \geq N^{\epsilon}$ and iteration (Lemma 3.4.4), analogously to Section 3.4.1.1, we can immediately deduce that $\Psi_k(T) \prec 1$ where T is the time defined in the statement of Proposition 3.3.3. This concludes the proof of Proposition 3.3.3 (b) for the remaining case $\mathfrak{I}_k = \varnothing$.

3.4.4 Modifications for general $\sigma = \mathbf{E}\chi_{\mathrm{od}}^2$.

The proof of Proposition 3.3.3 presented so far assumed for simplicity that $\sigma = \mathbf{E}\chi_{\mathrm{od}}^2$ is real and $\mathbf{E}\chi_{\mathrm{d}}^2 = 1 + \sigma$. We now explain how to prove the general case, when these two restrictions are lifted. The only changes concern the choice of the initial condition and of the evolution B_t in the flow (3.3.3).

If σ is not real, we modify the evolution in (3.3.3) in such a way the entries of B_t are $\sqrt{t}$ times a standard complex Gaussian, and we modify the initial condition in (3.3.3) from $W_0 = W$ to $W_0 = \widetilde{W}_T$, with another Wigner matrix $\widetilde{W}_T$ prepared such that

$$e^{-T/2}\widetilde{W}_T + \sqrt{1-e^{-T}}U \overset{\mathrm{d}}{=} W. \tag{3.4.63}$$

Here U is a GUE matrix, which is independent of $\widetilde{W}_T$ (here we used that $|\sigma| < 1$). We point out that the limiting eigenvalue density of $\widetilde{W}_T$ does not change along the flow (3.3.3) as a consequence of the fact that $\mathbf{E}|(W_t)_{ab}|^2$, for $a > b$, is preserved, and only

$$\mathbf{E}(W_t)_{ab}^2 = e^{-t}\,\mathbf{E}(\widetilde{W}_T)_{ab}^2, \qquad \mathbf{E}(W_t)_{aa}^2 = e^{-t/2}\,\mathbf{E}(\widetilde{W}_T)_{aa}^2 + \frac{1}{N}\sqrt{1-e^{-t}}\,, \qquad t \in [0,T]\,,$$

change. The fact that $\mathbf{E}(W_t)_{ab}^2$ and $\mathbf{E}(W_t)_{aa}^2$ do change along the flow contributes to a change of order $1/N$ in the averaged Stieltjes transform of W_t; such change is easily seen to be negligible for the precision of the local laws we are considering here. If $\sigma \in \mathbf{R}$ but $\mathbf{E}\chi_{\mathrm{d}}^2 \neq 1+\sigma$, similarly to (3.4.63), we choose B_t so that its entries have variance t times the variance of W for the off–diagonal entries and $\mathbf{E}(B_t)_{aa}^2 = (1+\sigma)t$, and we can prepare yet another Wigner matrix $\widehat{W}_T$ such that

$$e^{-T/2}\widehat{W}_T + \sqrt{1-e^{-T}}\widehat{U} \overset{\mathrm{d}}{=} W, \tag{3.4.64}$$

with $\widehat{U}$ being independent of $\widehat{W}_T$ and having the same entries distribution as W except for the diagonal entries having variance $\mathbf{E}\widehat{U}_{aa}^2 = \frac{1}{N}(1+\sigma)$. The second moments of $(\widehat{W}_t)_{ab}$ are preserved and only the diagonal changes

$$\mathbf{E}(\widehat{W}_t)_{aa}^2 = e^{-t/2}\,\mathbf{E}(\widehat{W}_T)_{aa}^2 + \frac{1}{N}\sqrt{1-e^{-t}}(1+\sigma);$$

hence the limiting eigenvalue distribution is still given by the semicircular law.

3.5 Green function comparison: Proof of Proposition 3.3.4

In this section, we remove the Gaussian component introduced in Propositions 3.3.3 by a Green function comparison (GFT) argument, i.e. we prove Proposition 3.3.4. For simplicity, we will write the detailed proof only in the case of no imaginary parts, i.e. $\mathfrak{I}_k = \varnothing$ and $\mathfrak{I}_{k+1} = \varnothing$ in the average and isotropic case, respectively. The minor modifications needed for handling the other cases will be briefly discussed in Section 3.5.4 below.

Before entering the proof, we point out that typical GFT arguments (starting from [551]) are used to compare the distribution of a genuinely fluctuating observable under two different matrix ensembles whose single entry distributions have matching first few moments. Technically, a family of interpolating ensembles is constructed which may be finite (e.g. Lindeberg replacement strategy) or continuous (e.g. along an Ornstein-Uhlenbeck flow) and the change of the distribution in question is closely monitored along the interpolation. In this standard setup for GFT, however, local laws serve as *a priori* bounds obtained by independent methods and they assumed to hold for all interpolating ensembles in between. In other words, concentration–type information about resolvents $G(z)$ with $\operatorname{Im} z$ well above the eigenvalue spacing are turned into information on the distribution of $G(z)$ with $\operatorname{Im} z$ at, or even slightly below the eigenvalue spacing. Our application of GFT is different in spirit, since we aim to prove local laws for one ensemble knowing them for the other one. Thus GFT needs to be done *self-consistently* with monitoring a carefully designed quantity that satisfies a Gronwall-type inequality along the interpolation.

We remark that more than ten years ago Knowles and Yin in [370] used GFT in a similar spirit to prove single resolvent local law for ensembles where the deterministic approximation M to G is not a multiple of identity matrix (for example deformed Wigner matrices). Later a much more direct and generally applicable alternative method based upon the matrix Dyson equation [17, 23] has been developed to prove such local laws without GFT. Our current dynamical approach revives the idea of a self-consistent GFT, since it naturally serves as a counterpart of the characteristic flow to remove the Gaussian component added along that flow. In fact, the approach of [370] also used a tandem of gradual reduction of $\eta = \operatorname{Im} z$ (called *bootstrapping* steps) and a self-consistent GFT (called *interpolation* steps), see Fig. 1.1 in [370]. However, the bootstrapping step in [370] was much less effective than the characteristic flow which does the η-reduction in one step even for a much more complex multi-resolvent chain. In the GFT step, we use the simple entry-by-entry Lindeberg replacement strategy that is better adjustable to our complicated resolvent chains instead of a special continuous interpolation as in [370], but the core of both techniques is a self-consistent Gronwall argument. The main technical challenge in our proof is that the error in one step of the Lindeberg replacement is not always sufficiently small, but by carefully monitoring the errors in each step, we gain from summing them up explicitly. We will explain this mechanism in Example 3.5.11.

Now we turn to the actual proof. Recalling the notations

$$\eta := \min_i |\operatorname{Im} z_i| \quad \text{and} \quad \rho := \pi^{-1} \max_i |\operatorname{Im} m_i|, \tag{3.5.1}$$

we begin by distinguishing the *averaged* and *isotropic* control quantities

$$\Psi_k^{\mathrm{av}} := \frac{\sqrt{N\eta}}{N^{k/2-1}\sqrt{\rho}} \big|\big\langle \big(G_1 A_1 \ldots G_k - M_{[1,k]}\big) A_k \big\rangle\big| \tag{3.5.2}$$

$$\Psi_k^{\mathrm{iso}}(\boldsymbol{x}, \boldsymbol{y}) := \frac{\sqrt{N\eta}}{N^{k/2}\sqrt{\rho}} \big|\big(G_1 A_1 \ldots A_k G_{k+1} - M_{[1,k+1]}\big)_{\boldsymbol{xy}}\big|, \tag{3.5.3}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ are unit deterministic vectors and the traceless matrices $A_i \in \mathbf{C}^{N\times N}$ are assumed to have normalized Hilbert-Schmidt norms, $\langle |A_i|^2 \rangle^{1/2} = 1$. Recall that, in (3.5.2)–(3.5.3), we only consider chains without $\operatorname{Im} G$'s, the more general cases will be discussed later in Section 3.5.4. Finally, we point out that our notation in (3.5.2)–(3.5.3) already suppressed the dependence on the

spectral parameters and deterministic matrices, since the sets of these are considered fixed along the argument. In the following, we will often say that an estimate on Ψ holds *uniformly*, by which we will always mean uniformity in all unit deterministic vectors and all choices of subsets of spectral parameters and deterministic matrices as explained in Proposition 3.3.4 (a).

Now, the goal of this section is to prove Proposition 3.3.4. More precisely, we will show that, if the optimal multi-resolvent local laws

$$\Psi_k^{\mathrm{av}} + \Psi_k^{\mathrm{iso}} \prec 1, \quad \text{for all fixed} \quad k \in \mathbf{N}, \tag{3.5.4}$$

hold *uniformly* for a Wigner matrix with some given single entry distributions, then they also hold for every other Wigner matrix with different single entry distributions, again *uniformly*, provided that the *first three moments* of the entries of these two ensembles *match*. A fundamental input for our proof is that the corresponding single resolvent local laws hold for *every* Wigner matrix ensemble [249, 369, 88], i.e. the following Green function comparison argument is not needed for them.

Theorem 3.5.1. *For fixed $\epsilon > 0$, we have*

$$|\langle G - m \rangle| \prec \frac{1}{N\eta}, \qquad \big|\big(G - m\big)_{\boldsymbol{xy}}\big| \prec \sqrt{\frac{\rho}{N\eta}} + \frac{1}{N\eta} \tag{3.5.5}$$

uniformly in unit deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$ and at spectral parameter $z \in \mathbf{C} \setminus \mathbf{R}$ with $\eta = |\operatorname{Im} z| \geq N^{-1+\epsilon}$ and $\operatorname{Re} z \in \mathbf{R}$, where $\rho = \pi^{-1} |\operatorname{Im} m(z)|$.

For convenience, these single resolvent laws will be expressed in the compact form

$$\Psi_0^{\mathrm{av}} + \Psi_0^{\mathrm{iso}} \prec 1,$$

which extends (3.5.2)–(3.5.3) when no traceless matrices A are present (see, e.g., [168, 169]).

Before starting the proof, we recall some notation which has already been used in the statement of Proposition 3.3.4. We will distinguish between the two ensembles compared in the GFT argument by using different letters, v_{ab} and w_{ab}, for their matrix elements, and we shall occasionally use the notation $H^{(\boldsymbol{v})}$ and $H^{(\boldsymbol{w})}$ to indicate the difference. Alternatively, one could denote the matrix elements by a universal letter h_{ab} and distinguish the two ensembles in the underlying measure, especially in the expectations $\mathbf{E}_{\boldsymbol{v}}$ and $\mathbf{E}_{\boldsymbol{w}}$. However, since the proof of Proposition 3.3.4 works by replacing the matrix elements one-by-one in $N(N+1)/2$ steps, we use the first notation, analogously to [248, Section 16].

3.5.1 Preliminaries

The principal idea of the proof is as follows: First, we fix a bijective ordering

$$\phi : \{(i,j) \in [N]^2 : i \leq j\} \to [\gamma(N)], \qquad \gamma(N) := \frac{N(N+1)}{2} \tag{3.5.6}$$

on the index set of independent entries of a Wigner matrix. Then, according to the induced ordering, the matrix elements are swapped one-by-one from the distribution v_{ab} to w_{ab} in $\gamma(N) \sim N^2$ steps. In particular, at step $\gamma \in \{0\} \cup [\gamma(N)]$ in this replacement procedure, the resulting matrix $H^{(\gamma)}$ has entries which are distributed according to w_{ij} whenever $\phi\big((i,j)\big) \leq \gamma$ and according to v_{ij} whenever $\phi\big((i,j)\big) > \gamma$, i.e. $H^{(0)} = H^{(\boldsymbol{v})}$ and $H^{(\gamma(N))} = H^{(\boldsymbol{w})}$. This one-by-one replacement of the matrix elements naturally requires understanding the *isotropic* law (3.3.14), as already indicated in (3.5.3).

In order to derive (3.5.4) also for $H^{(\boldsymbol{w})}$, we compute high moments of $\Psi_k^{\mathrm{av/iso}}$ for $H^{(\gamma)}$ and $H^{(\gamma-1)}$ for general $\gamma \in [\gamma(N)]$ and compare the results. Given sufficiently good one-step bounds, a telescopic

argument will yield the estimate (3.5.4) also for $H^{(w)}$. These "sufficiently good" one-step bounds are essentially required to accommodate the large number $O(N^2)$ of necessary replacements in order to arrive at $H^{(\gamma(N))}$. A key feature of our proof, in contrast to previous applications of the replacement strategy, is that the error will not always be $o(N^{-2})$ in each step, but their cumulative size after summation is still $o(1)$.

The proof of Proposition 3.3.4 is divided in two main parts: At first, in **Part (a)**, Section 3.5.2, we show the isotropic part of (3.5.4), that is $\Psi_k^{\mathrm{iso}} \prec 1$, via a double induction on the number $k \in \mathbf{N}$ of traceless matrices and the moment $p \in \mathbf{N}$ taken of Ψ_k^{iso}, i.e. $\mathbf{E}|\Psi_k^{\mathrm{iso}}|^p$. Thereby, we crucially use that the $\prec$-bound is (essentially) equivalent to controlling arbitrarily high moments up to an N^{ξ}-error with arbitrarily small $\xi > 0$. Afterwards, in **Part (b)**, Section 3.5.3, using Part (a) as an input, we will demonstrate $\Psi_k^{\mathrm{av}} \prec 1$ (and thus conclude the proof of Proposition 3.3.4 for $\mathfrak{I}_{k+1} = \varnothing$ resp. $\mathfrak{I}_k = \varnothing$) for every fixed k via a single induction on the moment p. The main reason for this order of the argument is that the one-by-one replacement in step γ is conducted via resolvent expansion focusing on the differing matrix entries at positions $(i,j) = \phi^{-1}(\gamma)$ and (j,i), and thereby it naturally produces isotropic quantities (see Lemma 3.5.3 below). Hence, the argument for Ψ_k^{av} cannot be self-contained and must rely on Ψ_k^{iso}, which in fact will not involve the averaged local laws at all.

We fix some further notation. We have an initial Wigner matrix $H^{(0)} := H^{(v)}$ and iteratively define

$$H^{(\gamma)} := H^{(\gamma-1)} - \frac{1}{\sqrt{N}}\Delta_V^{(\gamma)} + \frac{1}{\sqrt{N}}\Delta_W^{(\gamma)}, \tag{3.5.7}$$

a sequence of Wigner matrices for $\gamma \in [\gamma(N)]$, where we denoted[18]

$$\Delta_V^{(\gamma)} := \sqrt{N}\frac{E^{(ij)}(H^{(v)})_{ij} + E^{(ji)}(H^{(v)})_{ji}}{1+\delta_{ij}} \quad \text{and} \quad \Delta_W^{(\gamma)} := \sqrt{N}\frac{E^{(ij)}(H^{(w)})_{ij} + E^{(ji)}(H^{(w)})_{ji}}{1+\delta_{ij}}. \tag{3.5.8}$$

Here, $\phi\big((i,j)\big) = \gamma$ and $E^{(ij)}$ denotes the matrix whose matrix elements are zero everywhere except at position (i,j), i.e. $(E^{(ij)})_{k\ell} = \delta_{ik}\delta_{j\ell}$. The denominator $1+\delta_{ij}$ is introduced to account for the factor of two in the numerator occurring for diagonal indices. Note that $H^{(\gamma)}$ and $H^{(\gamma-1)}$ differ only in the (i,j) and (j,i) matrix elements, and they can be written as

$$H^{(\gamma-1)} = \check{H}^{(\gamma)} + \frac{1}{\sqrt{N}}\Delta_V^{(\gamma)} \quad \text{and} \quad H^{(\gamma)} = \check{H}^{(\gamma)} + \frac{1}{\sqrt{N}}\Delta_W^{(\gamma)} \tag{3.5.9}$$

with a matrix $\check{H}^{(\gamma)}$ whose matrix element is zero at the (i,j) and (j,i) positions. Similarly, we denote the corresponding resolvents at spectral parameter $z_j \in \mathbf{C}\setminus\mathbf{R}$ by

$$G_j^{(\gamma)} := (H^{(\gamma)} - z_j)^{-1}, \quad G_j^{(\gamma-1)} := (H^{(\gamma-1)} - z_j)^{-1}, \quad \text{and} \quad \check{G}_j^{(\gamma)} := (\check{H}^{(\gamma)} - z_j)^{-1}. \tag{3.5.10}$$

Observe that, at each step γ in the replacement procedure, the deterministic approximation to a resolvent chain involving $G^{(\gamma)}$ is the same. This is because only the first two moments of the matrix elements of $H^{(\gamma)}$ determine this approximation, symbolically denoted by M, via the *Matrix Dyson Equation (MDE)*, see, e.g., [236]. For a chain in the checked resolvents $\check{G}$, the approximating M is *in principle* differing from the non-checked ones, simply because the self-energy operator $\check{\mathcal{S}}^{(\gamma)}[R] = \mathbf{E}[\check{H}^{(\gamma)}R\check{H}^{(\gamma)}]$ associated with $\check{H}^{(\gamma)}$ is no longer exactly the averaged trace $\langle\cdot\rangle$. However, since this discrepancy introduces an error of size $1/N$ in the MDE, which is a stable equation, this will not be visible in the local laws (3.5.4). Therefore, we shall henceforth ignore this minor point and shall just define the normalized differences

$$\Psi_k^{\mathrm{av},(\gamma)}, \quad \check{\Psi}_k^{\mathrm{av},(\gamma)}, \quad \Psi_k^{\mathrm{iso},(\gamma)}(\boldsymbol{x},\boldsymbol{y}), \quad \text{and} \quad \check{\Psi}_k^{\mathrm{iso},(\gamma)}(\boldsymbol{x},\boldsymbol{y}),$$

[18] Observe that in this normalization, the non-zero entries of $\Delta_V^{(\gamma)}$ and $\Delta_W^{(\gamma)}$ are of order one random variables.

exactly as in (3.5.2)–(3.5.3), but with G_j replaced by $G_j^{(\gamma)}$ and $\check{G}_j^{(\gamma)}$, respectively. We emphasize again that the deterministic counterparts in all of the normalized differences are the *same*.

We can now turn to the actual proof.

3.5.2 Part (a): Proof of the isotropic law

In this first part, we exclusively work with isotropic quantities and we shall hence drop the superscript $^{\mathrm{iso}}$ in the entire Section 3.5.2. As already mentioned above, we shall prove the claim by a *double induction* on k and the moment p taken of Ψ_k, i.e. $\mathbf{E}|\Psi_k|^p$.

Thereby, the primary induction parameter is k and our goal is to show that, if for some $k \in \mathbf{N}$ we have

$$\max_{\gamma\le\gamma(N)} \Psi_{k'}^{(\gamma)} + \max_{\gamma\le\gamma(N)} \check{\Psi}_{k'}^{(\gamma)} \prec 1\,, \qquad \forall\, k' \in \{0,...,k-1\}\,, \tag{3.5.11}$$

then also

$$\max_{\gamma\le\gamma(N)} \Psi_{k}^{(\gamma)} + \max_{\gamma\le\gamma(N)} \check{\Psi}_{k}^{(\gamma)} \prec 1\,. \tag{3.5.12}$$

Within the proof of (3.5.12), for a fixed k, we will then crucially use that the $\prec$-bound is equivalent to controlling arbitrarily high moments $\mathbf{E}|\Psi_k|^p$ up to an N^{ξ}-error for an arbitrarily small $\xi > 0$. Therefore, we use another secondary induction on the moment p. More precisely, in order to establish (3.5.12) from (3.5.11), our goal is to show that, for any fixed $k \in \mathbf{N}$, if for some $p \in \mathbf{N}$ we have that

$$\max_{\gamma\le\gamma(N)} \big\|\Psi_{k}^{(\gamma)}\big\|_{p-1} + \max_{\gamma\le\gamma(N)} \big\|\check{\Psi}_{k}^{(\gamma)}\big\|_{p-1} \lesssim N^{\xi}$$

for any $\xi > 0$, then also

$$\max_{\gamma\le\gamma(N)} \big\|\Psi_{k}^{(\gamma)}\big\|_{p} + \max_{\gamma\le\gamma(N)} \big\|\check{\Psi}_{k}^{(\gamma)}\big\|_{p} \lesssim N^{\xi} \tag{3.5.13}$$

holds for any $\xi > 0$, where implicit constants depend on k, p and ξ. Here for a random variable X we used the definition $\|X\|_p := [\mathbf{E}|X|^p]^{1/p}$.

To summarize, as the *induction hypothesis*, given some arbitrary fixed $p, k \in \mathbf{N}$, we will assume that

$$\max_{\gamma\le\gamma(N)} \Psi_{k'}^{(\gamma)} + \max_{\gamma\le\gamma(N)} \check{\Psi}_{k'}^{(\gamma)} \prec 1 \quad \text{and} \quad \max_{\gamma\le\gamma(N)} \big\|\Psi_{k}^{(\gamma)}\big\|_{p-1} + \max_{\gamma\le\gamma(N)} \big\|\check{\Psi}_{k}^{(\gamma)}\big\|_{p-1} \le C_{k,p,\xi} N^{\xi} \tag{3.5.14}$$

hold uniformly for all $k' \in \{0,...,k-1\}$ and $\xi > 0$ with an appropriate N-independent constant. Then we will conclude (3.5.13).

The overall *base case* ($k = 1$, $p = 1$) is easy to verify: it solely consists of the usual isotropic law (the first estimate in (3.5.14) for $k' = 0$) and the trivial bound $\mathbf{E}|\Psi_1|^0 = 1$ (the second estimate in (3.5.14) for $k = 1$ and $p = 1$).

We start with two arbitrary but fixed bounded deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$ and introduce the set

$$I_{\boldsymbol{xy}} := \{\boldsymbol{x}, \boldsymbol{y}\} \cup \{\boldsymbol{e}_a : a \in [N]\} \subset \mathbf{C}^N \tag{3.5.15}$$

of vectors, which will naturally arise along the argument (see (3.5.32) below), where $\boldsymbol{e}_a$ denotes the standard basis vector in the coordinate direction a. Note that the cardinality of $I_{\boldsymbol{xy}}$ is $N + 2$. After defining[19]

$$\Omega_k^p(\gamma) := \max_{\boldsymbol{u},\boldsymbol{v}\in I_{\boldsymbol{xy}}} \|\Psi_k^{(\gamma)}(\boldsymbol{u},\boldsymbol{v})\|_p^p \tag{3.5.16}$$

(we omitted the dependence on $\boldsymbol{x}, \boldsymbol{y}$ in the notation, as they are considered fixed along the whole argument), the principal goal of the induction step is to prove the following proposition.

[19] Here, p is a superscript, not a power.

Proposition 3.5.2 (Gronwall estimate). *Fix $p, k \in \mathbf{N}$ and assume* (3.5.14) *holds. Then, for any $\xi > 0$, there exist some constants $C_1, C_2 > 0$ (depending on p, k, and ξ, but independent of N, $\boldsymbol{x}$, and $\boldsymbol{y}$) such that*

$$\Omega_k^p(\gamma_0) \le C_1 \frac{1}{N^2} \sum_{\gamma<\gamma_0} \Omega_k^p(\gamma) + C_2 N^\xi \tag{3.5.17}$$

for every $\gamma_0 \in [\gamma(N)]$.

Note that (3.5.17) is a discrete Gronwall inequality for $\Omega_k^p(\gamma)$. Hence, having Proposition 3.5.2 at hand (note that, in particular, $\Omega_k^p(0) \le C_2 N^\xi$), we obtain

$$\max_{\gamma\le\gamma(N)} \Omega_k^p(\gamma) \le C_2 \mathrm{e}^{C_1} N^\xi \le C_3(k,p,\xi) N^\xi \,, \tag{3.5.18}$$

uniformly in $\boldsymbol{x}$ and $\boldsymbol{y}$ and all choices of spectral parameters and traceless deterministic matrices, which then implies the Ψ-part of (3.5.13). In the next subsections we present auxiliary results necessary for the proof of Proposition 3.5.2 which will then be concluded in Section 3.5.2.5. The $\breve{\Psi}$-part of (3.5.13) and thus the induction step will finally be completed in Section 3.5.2.6.

In order to simplify notation, we shall henceforth drop the subscripts for all resolvents and deterministic matrices, i.e. write $G_j = G$ and $A_j = A$ instead.

3.5.2.1 Preliminaries

The fundamental building block of our proof is the following elementary lemma on resolvent expansion. Note that we need to express $G^{(\gamma-1)}, G^{(\gamma)}$ in terms of the "unperturbed" resolvent $\breve{G}^{(\gamma)}$ of $\breve{H}^{(\gamma)}$ that has zero elements in the γ-th position, and conversely, we need to express $\breve{G}^{(\gamma)}$ in terms of both "perturbed" resolvents using $\Delta_V^{(\gamma)}$ and $\Delta_W^{(\gamma)}$ from (3.5.8) as perturbations, see (3.5.10). We work with finite resolvent expansions up to some order m, independent of N, to be determined later. The last term therefore always contains the original resolvent as well and it will have to be estimated deterministically by its norm but if m is large enough this will be affordable.

Lemma 3.5.3 (Resolvent expansions). *For every fixed $m \in \mathbf{N}$, it holds that*

$$\breve{G}^{(\gamma)} = \sum_{\ell=0}^{m} N^{-\ell/2} \big(G^{(\gamma)} \Delta_W^{(\gamma)}\big)^\ell G^{(\gamma)} + N^{-(m+1)/2} \big(G^{(\gamma)} \Delta_W^{(\gamma)}\big)^{m+1} \breve{G}^{(\gamma)} \tag{3.5.19a}$$

and

$$G^{(\gamma)} = \sum_{\ell=0}^{m} (-1)^\ell N^{-\ell/2} \big(\breve{G}^{(\gamma)} \Delta_W^{(\gamma)}\big)^\ell \breve{G}^{(\gamma)} + (-1)^{(m+1)} N^{-(m+1)/2} \big(\breve{G}^{(\gamma)} \Delta_W^{(\gamma)}\big)^{m+1} G^{(\gamma)} \,. \tag{3.5.19b}$$

These relations also hold verbatim when replacing $G^{(\gamma)} \to G^{(\gamma-1)}$ and $\Delta_W^{(\gamma)} \to \Delta_V^{(\gamma)}$. □

We now expand each $G^{(\gamma)}$ in

$$\big|\Psi_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\big|^p = \left(\frac{N\eta}{\rho}\right)^{p/2} N^{-pk/2} \big|\big((G^{(\gamma)}A)^k G^{(\gamma)} - M_{[1,k+1]}\big)_{\boldsymbol{xy}}\big|^p \tag{3.5.20}$$

and each $G^{(\gamma-1)}$ in

$$\big|\Psi_k^{(\gamma-1)}(\boldsymbol{x},\boldsymbol{y})\big|^p = \left(\frac{N\eta}{\rho}\right)^{p/2} N^{-pk/2} \big|\big((G^{(\gamma-1)}A)^k G^{(\gamma-1)} - M_{[1,k+1]}\big)_{\boldsymbol{xy}}\big|^p \tag{3.5.21}$$

according to (3.5.19b) (for some $m \ge 4$ to be determined below, depending on p and k; see (3.5.49)) and sort the resulting terms by their power $r = 0, 1, 2, \ldots$ of $N^{-1/2}$. Then we take the expectation

with respect to w_{ij} and v_{ij}, respectively (recall that $\phi\big((i,j)\big) = \gamma$), and use the moment matching condition (3.3.12). As a result, we find that the terms with a prefactor $N^{-r/2}$ for $r = 0, 1, 2, 3$ are algebraically *exactly the same* for both (3.5.20) and (3.5.21). The conclusion of this argument is formalized in the following lemma.

Lemma 3.5.4. *For any fixed $(i,j) \in [N]^2$ with $i \le j$ and $\gamma = \phi(i,j)$ we have that*

$$\mathbf{E}_{w_{ij}} \big|\Psi_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\big|^p = \sum_{r=0}^{3} N^{-r/2} \alpha_{k,r}^{(\gamma)}(\boldsymbol{x},\boldsymbol{y}) \big|\breve{\Psi}_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\big|^{p-r} + \text{higher order terms} \tag{3.5.22}$$

$$\mathbf{E}_{v_{ij}} \big|\Psi_k^{(\gamma-1)}(\boldsymbol{x},\boldsymbol{y})\big|^p = \sum_{r=0}^{3} N^{-r/2} \alpha_{k,r}^{(\gamma)}(\boldsymbol{x},\boldsymbol{y}) \big|\breve{\Psi}_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\big|^{p-r} + \text{higher order terms} \tag{3.5.23}$$

for some identical *coefficients $\alpha_{k,r}^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})$ independent of v_{ij} and w_{ij} whose precise values are (mostly) irrelevant. Here "higher order terms" denote terms with prefactor $N^{-r/2}$ with $r \ge 4$.*

In the following Sections 3.5.2.2–3.5.2.4, preparing the conclusion of the proof of Proposition 3.5.2 in Section 3.5.2.5, we will discuss the higher order terms in (3.5.22) and (3.5.23). These have to be estimated individually by size when we will consider the difference of (3.5.22) and (3.5.23). Recall that, we will eventually compare $\Psi_k^{(0)}(\boldsymbol{x},\boldsymbol{y})$ and $\Psi_k^{(\gamma(N))}(\boldsymbol{x},\boldsymbol{y})$ in $\gamma(N) = O(N^2)$ many steps, which is why the higher order terms must all be bounded by $1/N^2$, roughly said. More precisely, we will use the following telescopic summation: For every $\gamma_0 \in [\gamma(N)]$, it holds that

$$\Big| \|\Psi_k^{(\gamma_0)}(\boldsymbol{x},\boldsymbol{y})\|_p^p - \|\Psi_k^{(0)}(\boldsymbol{x},\boldsymbol{y})\|_p^p \Big| \le \sum_{1 \le \gamma \le \gamma_0} \Big| \|\Psi_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\|_p^p - \|\Psi_k^{(\gamma-1)}(\boldsymbol{x},\boldsymbol{y})\|_p^p \Big| . \tag{3.5.24}$$

In the next Section 3.5.2.2, we will explain the term with $r = 4$ in Lemma 3.5.4, i.e. with N^{-2}-prefactor, in detail. All other higher order terms with $r \ge 5$ but still involving only the resolvent $\breve{G}^{(\gamma)}$ are completely analogous, in fact easier (see Section 3.5.2.3 later for some detail). Afterwards, in Section 3.5.2.4, we will discuss, how the maximal order m of the resolvent expansion (3.5.19b) has to be chosen in order to accommodate the remainder term involving a non-checked resolvent $G^{(\gamma)}$ (resp. $G^{(\gamma-1)}$).

Throughout the following argument we shall focus on the higher order terms in (3.5.22), the treatment of (3.5.23) is exactly the same. Whenever it does not lead to confusion, we shall henceforth drop the superscript γ.

3.5.2.2 Fourth order terms in Lemma 3.5.4

The goal of the current Section 3.5.2.2 is to show that the terms of order $r = 4$ arising in the telescopic summation (3.5.24) can be bounded by the rhs. of (3.5.17).

In the following, we denote (cf. (3.5.8))

$$\Delta = \Delta^{(\gamma)} = \frac{E^{(ij)} + E^{(ji)}}{1 + \delta_{ij}} \tag{3.5.25}$$

and find, similarly to (3.5.8), after taking the full expectation, the $r = 4$ (i.e. $1/N^2$) prefactor of the higher order terms in (3.5.22) to be bounded by (a constant times)

$$\mathbf{E} \sum_{d=1}^{4\wedge p} \big|\breve{\Psi}_k(\boldsymbol{x},\boldsymbol{y})\big|^{p-d} \left(\frac{N\eta}{\rho}\right)^{d/2} N^{-dk/2} \sum_{4\Delta \rightsquigarrow d} \Big| \underbrace{\big(...\Delta...\Delta...\big)_{\boldsymbol{xy}} \dots \big(...\Delta...\big)_{\boldsymbol{xy}}}_{\text{four } \Delta \text{ in a total } d \text{ chains}} \Big| . \tag{3.5.26}$$

Here d counts the number of formerly "intact" resolvent chains $\big((\check{G}A)^k\check{G}\big)_{\boldsymbol{xy}}$, which have been 'destroyed' by at least one replacement $\check{G} \to \check{G}\Delta\check{G}$ due to the expansion (3.5.19b). The symbol

$$\sum_{4\Delta \rightsquigarrow d} \tag{3.5.27}$$

indicates that we sum over all possibilities to destroy exactly d chains by four Δ's. Note that a chain may be "destroyed" by more than one Δ, therefore d may be less than four. After using the explicit form of Δ, altogether we arrive at a finite sum of $4+d$ chains.

Example 3.5.5. *For example, for $d = 1$ we have that*

$$\begin{aligned}
&\sum_{4\Delta \rightsquigarrow 1} \big|\big(...\Delta...\Delta...\Delta...\Delta...\big)_{\boldsymbol{xy}}\big| \\
&= \sum_{\substack{k_1,\dots,k_5\geq 0:\\ \sum_l k_l = k}} \big|\big((\check{G}A)^{k_1}\check{G}\Delta(\check{G}A)^{k_2}\check{G}\Delta(\check{G}A)^{k_3}\check{G}\Delta(\check{G}A)^{k_4}\check{G}\Delta(\check{G}A)^{k_5}\check{G}\big)_{\boldsymbol{xy}}\big| \\
&= \sum_{\substack{k_1,\dots,k_5\geq 0:\\ \sum_l k_l = k}} \Big[\big|\big((\check{G}A)^{k_1}\check{G}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big((\check{G}A)^{k_2}\check{G}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big((\check{G}A)^{k_3}\check{G}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big((\check{G}A)^{k_4}\check{G}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big((\check{G}A)^{k_5}\check{G}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big| + ...\Big]
\end{aligned} \tag{3.5.28}$$

with the neglected summands being analogous, only having different distributions of $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ occurring, which can be produced by the structure of Δ.

For general d, in each of the $4+d$ resolvent chains in the rhs. of (3.5.26), we now add and subtract the corresponding deterministic M-term, $(\check{G}A)^k\check{G} = ((\check{G}A)^k\check{G} - M_{k+1}) + M_{k+1}$ (see also (3.5.33) below), schematically written as $G = (G-M)+M$. In the sequel, we will distinguish the following two complementary cases:

Case (i): At least d of the $d+4$ resolvent chains are replaced by their fluctuating part, $G-M$.

Case (ii): At least five of the $d+4$ resolvent chains are replaced by their deterministic counterpart, M.

Case (i): In case (i), we first separate those possibilities from (3.5.27), where the destruction of the d chains $\big((\check{G}A)^k\check{G}\big)_{\boldsymbol{xy}}$ in fact *preserves* d resolvent chains each with k traceless matrices A, but with deterministic vectors, which are not $\boldsymbol{x}$ and $\boldsymbol{y}$. This happens when all four Δ's are placed at the ends of the chains. For example, if $d=1$, we separate these possibilities as

$$\begin{aligned}
&\check{G}_{\boldsymbol{x}\boldsymbol{e}_i}\check{G}_{\boldsymbol{e}_j\boldsymbol{e}_j}\check{G}_{\boldsymbol{e}_i\boldsymbol{e}_i}\check{G}_{\boldsymbol{e}_j\boldsymbol{e}_j}\big((\check{G}A)^k\check{G}\big)_{\boldsymbol{e}_i\boldsymbol{y}} + ... \quad \text{or} \\
&\check{G}_{\boldsymbol{x}\boldsymbol{e}_i}\check{G}_{\boldsymbol{e}_j\boldsymbol{e}_j}\big((\check{G}A)^k\check{G}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\check{G}_{\boldsymbol{e}_j\boldsymbol{e}_j}\check{G}_{\boldsymbol{e}_i\boldsymbol{y}} + ...
\end{aligned} \tag{3.5.29}$$

In the following, we shall focus on the first exemplary term in (3.5.29). Its fluctuating part

$$\big((\check{G}A)^k\check{G} - M_{[1,k+1]}\big)_{\boldsymbol{e}_i\boldsymbol{y}} \tag{3.5.30}$$

can then be paired with the leftover $\big(N\eta/\rho\big)^{1/2}N^{-k/2}$ in (3.5.26) and thereby produces a further full $\big|\check{\Psi}_k^{(\gamma)}(\boldsymbol{e}_j,\boldsymbol{y})\big|$; the remaining terms coming from a single resolvent in (3.5.29) are simply estimated by one,

$$|\check{G}_{\boldsymbol{u}\boldsymbol{v}}| \prec 1\,, \qquad \boldsymbol{u},\boldsymbol{v} \in I_{\boldsymbol{xy}} \quad \text{cf. (3.5.15)}\,, \tag{3.5.31}$$

by the usual isotropic law (3.5.5). All these terms stemming from (3.5.26) and constituting a full $\big|\breve{\Psi}_k^{(\gamma)}\big|$ (or $\big|\breve{\Psi}_k^{(\gamma)}\big|^d$ for general $d \in [4 \wedge p]$) can then be estimated by

$$\breve{\Omega}_k^p(\gamma) := \max_{\boldsymbol{u},\boldsymbol{v}\in I_{\boldsymbol{xy}}} \|\breve{\Psi}_k^{(\gamma)}(\boldsymbol{u},\boldsymbol{v})\|_p^p . \tag{3.5.32}$$

Now, after having separated the possibilities from (3.5.27), where the destruction preserves d resolvent chains with k deterministic matrices in between, we are left with those which solely create *strictly shorter* chains by the procedure $4\Delta \rightsquigarrow d$. These terms can entirely be treated by our *induction hypothesis* (3.5.14): The power of $\breve{\Psi}_k^{(\gamma)}$ has been reduced by (at least) one (cf. the second estimate in (3.5.14)) and $\breve{\Psi}_{k'}^{(\gamma)} + \Psi_{k'}^{(\gamma)} < 1$ uniformly in γ for *strictly* shorter chains, $k' < k$, has already been shown (first estimate in (3.5.14)).

Example 3.5.6. *Writing*

$$M_{j-i+1} \equiv M_{[i,j]} \quad \textit{for} \quad 1 \le i < j \le k+1 , \tag{3.5.33}$$

with a slight abuse of notation, we estimate the $d = 1$ term in (3.5.26) *(after having split off the cases when one of the k_l's equals k and all others are zero in* (3.5.29)*) as*

$$\begin{aligned}
&\mathbf{E}\big|\breve{\Psi}_k(\boldsymbol{x},\boldsymbol{y})\big|^{p-1} \left(\frac{N\eta}{\rho}\right)^{1/2} N^{-k/2} \times \\
&\quad \times \sum_{\substack{0\le k_l\le k-1:\\ \sum_l k_l = k}} \Big[\big| \big((\breve{G}A)^{k_1}\breve{G} - M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i} \big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j} \big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i} \big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j} \big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}} \big| \\
&\qquad + \big| \big((\breve{G}A)^{k_1}\breve{G} - M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i} \big((\breve{G}A)^{k_2}\breve{G} - M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j} \big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i} \big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j} \big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}} \big| + \ldots \Big] \\
&\lesssim N^{\xi} \left(\frac{N\eta}{\rho}\right)^{1/2} N^{-k/2} \sum_{\substack{0\le k_l\le k-1:\\ \sum_l k_l = k}} \left[\left(\frac{\rho}{N\eta}\right)^{1/2} N^{\sum_l k_l/2} + \left(\frac{\rho}{N\eta}\right) N^{\sum_l k_l/2} + \ldots \right] \lesssim N^{\xi} ,
\end{aligned} \tag{3.5.34}$$

where analogous summands (i.e. having further $G - M$ factors instead of M, or other arrangements of standard basis vectors $\boldsymbol{e}_i, \boldsymbol{e}_j$ stemming from (3.5.25)*) are again indicated by dots. In the first estimate, we used that $\big|(M_{j+1})_{\boldsymbol{uv}}\big| \lesssim N^{j/2}$ for all $\boldsymbol{u},\boldsymbol{v} \in I_{x,y}$ from Lemma 3.2.3 (b) together with the induction hypothesis* (3.5.14)*.*

In the general case, $d \ge 1$, the argument works analogously to the above example: The minimal number of d fluctuating terms carrying an $(\rho/N\eta)^{1/2}$-factor cancel the leftover $(N\eta/\rho)^{d/2}$-factor in (3.5.26). The remaining $N^{k_l/2}$-factors can then be handled by a simple power counting.

Overall, we find that, all the terms in (3.5.26) summarized in Case (i), can be bounded by

$$C_1\breve{\Omega}_k^p(\gamma) + C_2 N^{\xi} \tag{3.5.35}$$

for some positive constants $C_1, C_2 > 0$, which shall henceforth be used generically, i.e. their value might change from line to line (but remain uniformly bounded in γ).

Case (ii): For the second case, we recall that all the purely deterministic terms are *independent* of γ, i.e., as emphasized above, at each replacement step the deterministic approximation to a resolvent chain is the same. However, it is *not* sufficient to just estimate every M-term blindly via $\big|(M_{j+1})_{\boldsymbol{uv}}\big| \lesssim N^{j/2}$, as done in (3.5.34). Instead, we need to *gain from the summation* in (3.5.24) over all replacement positions. This is the main new element of our proof compared with previous GFT arguments.

Example 3.5.7. *We again look at our $d = 1$ example. Using the notation* (3.5.33)*, we find the* trivial estimate

$$\begin{aligned}
&\mathbf{E}\left|\breve{\Psi}_k(\boldsymbol{x},\boldsymbol{y})\right|^{p-1}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\sum_{\substack{0\le k_l\le k:\\ \sum_l k_l=k}}\Big[\big|\big(M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big|+\ldots\Big]\\
&\lesssim N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\sum_{\substack{0\le k_l\le k:\\ \sum_l k_l=k}}\Big[N^{\sum_l k_l/2}+\ldots\Big]\lesssim N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2},
\end{aligned}\tag{3.5.36}$$

where we again used the induction hypothesis (3.5.14) *and $\big|(M_{j+1})_{\boldsymbol{u}\boldsymbol{v}}\big|\lesssim N^{j/2}$. This bound is off by a factor $(N\eta/\rho)^{1/2}$, which we will now improve on.*

Indeed, the point in gaining from the summation *is that, although at each individual step γ, the deterministic terms in* (3.5.36) *might be large,* on average *over γ their contribution is bounded. More precisely, fixing one constellation of k_l's in* (3.5.36) *and using $\mathbf{E}\left|\breve{\Psi}_k\right|^{p-1}\lesssim N^{\xi}$, we find the average of the first line in* (3.5.36) *over all $i,j\in[N]$ to be bounded by (a constant times)*

$$\begin{aligned}
&N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\frac{1}{N^2}\sum_{i,j}\Big[\big|\big(M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big|+\ldots\Big]\\
&\lesssim N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}\frac{1}{N^2}\sum_{i,j}\left[\frac{\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|}{N^{k_1/2}}+\ldots\right]\\
&\lesssim N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}\frac{1}{N}\sqrt{N}\left[\frac{\sqrt{\big(|M_{k_1+1}|^2\big)_{\boldsymbol{x}\boldsymbol{x}}}}{N^{k_1/2}}+\ldots\right]\lesssim N^{\xi}\left(\frac{\eta}{\rho}\right)^{1/2}\lesssim N^{\xi}.
\end{aligned}\tag{3.5.37}$$

To go from the first to the second line, we used $\big|(M_{j+1})_{\boldsymbol{u}\boldsymbol{v}}\big|\lesssim N^{j/2}$ for all but the first M factor. Next, we used a Schwarz inequality for the i-summation, which involves the off-diagonal term $(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}$:

$$\sum_i\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|\le\sqrt{N}\left(\sum_i\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|^2\right)^{1/2}\le\sqrt{N}\sqrt{\big(|M_{k_1+1}|^2\big)_{\boldsymbol{x}\boldsymbol{x}}}\,.\tag{3.5.38}$$

In the penultimate estimate, we used that

$$\sqrt{\big(|M_{j+1}|^2\big)_{\boldsymbol{u}\boldsymbol{u}}}\lesssim N^{j/2},\tag{3.5.39}$$

as follows from the fact that $N^{j/2}$ is in fact the operator norm bound for M_{j+1}, and the final estimate in (3.5.37) *simply used the general fact $\eta/\rho\lesssim 1$.*

We point out that we even could have gained another $1/\sqrt{N}$-factor from the i-summation by not estimating $\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}$ trivially by $N^{k_5/2}$ but using

$$\begin{aligned}
\sum_i\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big|&\le\left(\sum_i\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|^2\right)^{1/2}\left(\sum_i\big|(M_{k_5+1})_{\boldsymbol{e}_i\boldsymbol{y}}\big|^2\right)^{1/2}\\
&\le\sqrt{\big(|M_{k_1+1}|^2\big)_{\boldsymbol{x}\boldsymbol{x}}}\sqrt{\big(|M_{k_5+1}|^2\big)_{\boldsymbol{y}\boldsymbol{y}}}\,.
\end{aligned}\tag{3.5.40}$$

instead of (3.5.38)*. However, we do not need this additional factor $1/\sqrt{N}$ here. Finally, note that the j-summation in* (3.5.37) *would have been useless, since the j-terms are diagonal. The summation gain is effective only for off-diagonal terms as in* (3.5.38)*.*

The above example indicates the following general mechanism: After estimating all the $G-M$-type terms with the aid of the induction hypothesis (3.5.14), and estimating the M-factors just trivially by their size, we are left with an excess $(N\eta/\rho)^{u/2}$-factor, for some $u \in [4]$. In order to remove this leftover factor, we need at least u *(collectively) summable bounded M-terms* like

$$\frac{\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|}{N^{k_1/2}} \tag{3.5.41}$$

in (3.5.37) (see also (3.5.39)). In fact, each of these collectively summable factors will gain one $1/\sqrt{N}$ compared to the trivial estimate, like the one in (3.5.36). Here, the notion "collective" refers to particular index structures, which allow an effective summation. Denoting terms like (3.5.41) symbolically by $M_{\boldsymbol{x}\boldsymbol{e}_i}$ for brevity, by *(collectively) summable bounded M-terms* we mean the following possible index structures

$$\begin{aligned}
u=1:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}| \quad \text{or} \quad \sum_{i,j}|M_{\boldsymbol{e}_j\boldsymbol{y}}| \quad \text{or} \quad \dots\\
u=2:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{e}_j\boldsymbol{y}}| \quad \text{or} \quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{e}_i\boldsymbol{y}}| \quad \text{or} \quad \dots\\
u=3:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{e}_i\boldsymbol{y}}||M_{\boldsymbol{e}_j\boldsymbol{y}}| \quad \text{or} \quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{e}_j\boldsymbol{y}}|^2 \quad \text{or} \quad \dots\\
u=4:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{x}\boldsymbol{e}_j}||M_{\boldsymbol{e}_i\boldsymbol{y}}||M_{\boldsymbol{e}_j\boldsymbol{y}}| \quad \text{or} \quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}|^2|M_{\boldsymbol{e}_j\boldsymbol{y}}|^2 \quad \text{or} \quad \dots
\end{aligned} \tag{3.5.42}$$

where dots are always indicating other similar terms, obtained from trivial exchanges $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$ or $i \leftrightarrow j$.

In principle, every summation over i and j potentially gains a full $1/N$-factor each – provided that there are enough M's with suitable indices as in (3.5.42). The existence of u *collectively summable bounded M-terms* then ensures that of this potential $1/N^2$-improvement at least a $1/N^{u/2}$-gain is effective. More precisely, as an example, for the first column of terms in (3.5.42) we have that

$$\begin{aligned}
u=1:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}| \le N^{3/2}\bigg(\sum_i|M_{\boldsymbol{x}\boldsymbol{e}_i}|^2\bigg)^{1/2} \lesssim N^{2-1/2}\\
u=2:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{e}_j\boldsymbol{y}}| \le N\bigg(\sum_i|M_{\boldsymbol{x}\boldsymbol{e}_i}|^2\bigg)^{1/2}\bigg(\sum_j|M_{\boldsymbol{e}_j\boldsymbol{y}}|^2\bigg)^{1/2} \lesssim N^{2-2/2}\\
u=3:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{e}_i\boldsymbol{y}}||M_{\boldsymbol{e}_j\boldsymbol{y}}|\\
&\qquad \le N^{1/2}\bigg(\sum_i|M_{\boldsymbol{x}\boldsymbol{e}_i}|^2\bigg)^{1/2}\bigg(\sum_i|M_{\boldsymbol{e}_i\boldsymbol{y}}|^2\bigg)^{1/2}\bigg(\sum_j|M_{\boldsymbol{e}_j\boldsymbol{y}}|^2\bigg)^{1/2} \lesssim N^{2-3/2}\\
u=4:&\quad \sum_{i,j}|M_{\boldsymbol{x}\boldsymbol{e}_i}||M_{\boldsymbol{x}\boldsymbol{e}_j}||M_{\boldsymbol{e}_i\boldsymbol{y}}||M_{\boldsymbol{e}_j\boldsymbol{y}}|\\
&\qquad \le \bigg(\sum_i|M_{\boldsymbol{x}\boldsymbol{e}_i}|^2\bigg)^{1/2}\bigg(\sum_i|M_{\boldsymbol{e}_i\boldsymbol{y}}|^2\bigg)^{1/2}\bigg(\sum_j|M_{\boldsymbol{e}_j\boldsymbol{y}}|^2\bigg)^{1/2}\bigg(\sum_j|M_{\boldsymbol{x}\boldsymbol{e}_j}|^2\bigg)^{1/2} \lesssim N^{2-4/2}
\end{aligned} \tag{3.5.43}$$

by application of Schwarz inequalities like in (3.5.38)–(3.5.40) and using that $\|M\| \lesssim 1$. We point out that the $\eta/\rho \le 1$ factor within each excess $(N\eta/\rho)^{1/2}$ would not be able to compensate for excess N-factors; but the *gains from the summation* are obtained solely on the level of N's.

It follows from a simple counting argument (or simply by considering all cases directly), that for any $u \in [4]$, we find an appropriately summable index structure within the at least five purely deterministic terms, as in (3.5.42)–(3.5.43). Hence, we deduce that

$$(3.5.26) \le C_1\breve{\Omega}_k^p(\gamma) + C_2N^{\xi}\bigg(1+\sum_{u=1}^4\bigg(\frac{N\eta}{\rho}\bigg)^{u/2}\Big|\big[u\,\text{sum. bdd. } M\text{-terms}\big]_{\boldsymbol{x},\boldsymbol{y}}^{(\gamma)}\Big|\bigg), \tag{3.5.44}$$

where

$$\big[u\,\text{sum. bdd. } M\text{-terms}\big]^{(\gamma)}_{\boldsymbol{x},\boldsymbol{y}} \tag{3.5.45}$$

stands symbolically for a product of u *collectively summable bounded deterministic terms*, like (3.5.41), for which we have just shown the following.

Lemma 3.5.8. *It holds that*

$$\sum_{\gamma\in[\gamma(N)]} \Big|\big[u\,\text{sum. bdd. } M\text{-terms}\big]^{(\gamma)}_{\boldsymbol{x},\boldsymbol{y}}\Big| \lesssim N^{2-u/2}\,. \tag{3.5.46}$$

Combining (3.5.44) with (3.5.46), this concludes the argument for the fourth order terms in (3.5.22).

3.5.2.3 Further higher order terms in Lemma 3.5.4

Just as in the previous Section 3.5.2.2, the goal of the current Section 3.5.2.3 is to show that the terms of order $r \geq 5$ arising in the telescopic summation (3.5.24) can be bounded by the rhs. of (3.5.17).

For these other higher order terms in (3.5.22) with $r \geq 5$ and involving *only* $\breve{G}$ (and not G), the two cases distinguished above for $r = 4$ generalize to the following.

Case (i'): At least d of the $d+r$ resolvent chains are replaced by their fluctuating part, $G - M$.

Case (ii'): At least $r+1$ of the $d+r$ resolvent chains are replaced by their deterministic counterpart, M.

For Case (i'), we separate a $1/N^2$-prefactor and find that the remaining part can be estimated by

$$C_1 N^{-(r-4)/2}\breve{\Omega}^p_k(\gamma) + C_2 N^{\xi} N^{-(r-4)/2}\,, \tag{3.5.47}$$

completely analogously to (3.5.35). In fact, we gain an additional $N^{-(r-4)/2} \ll 1$ factor in both terms. This reflects the idea that more $G - M$ terms are better because their presumed bounds carry a factor $(\rho/N\eta)^{1/2}$ (encoded in the prefactor $(N\eta/\rho)^{1/2}$ in the definition of Ψ^{iso}_k in (3.5.3)).

For Case (ii'), we include the additional $N^{-(r-4)/2}$ (after having separated a $1/N^2$-prefactor) into our counting of the leftover $(N\eta/\rho)^{u/2}$-factor (recall the discussion below (3.5.41)). In this way, we find that the maximal number of such leftover factors is $r - (r-4) = 4$. Hence, for every $u \in [4]$, we find an appropriately summable index structure, completely analogously to (3.5.42), and deduce that (leaving out the separated $1/N^2$-prefactor)

$$\begin{aligned} &r^{\text{th}}\,\text{order term in (3.5.22)} \\ &\quad\leq C_1 N^{-(r-4)/2}\breve{\Omega}^p_k(\gamma) + C_2 N^{\xi}\bigg(N^{-(r-4)/2} + \sum_{u=1}^{4}\Big(\frac{N\eta}{\rho}\Big)^{u/2}\Big|\big[u\,\text{sum. bdd. } M\text{-terms}\big]^{(\gamma)}_{\boldsymbol{x},\boldsymbol{y}}\Big|\bigg), \end{aligned} \tag{3.5.48}$$

which can be directly incorporated into (3.5.44) after adjusting the constants. Note that while the contributions form Case (i') improve by larger r, the terms from Case (ii') that carry many M-factors, do not.

Combining (3.5.48) with (3.5.46), this concludes the argument for the higher order terms in (3.5.22).

3.5.2.4 Truncation of the resolvent expansion

It remains to discuss the *truncation terms*, which involve *not* only $\breve{G}$, but also G, i.e. the order $m \in \mathbf{N}$ for the truncation of the resolvent expansion (3.5.19b). Also here, our goal is to show that the contribution of these terms arising in the telescopic summation (3.5.24) can be bounded by the rhs. of (3.5.17). After expanding each resolvent in (3.5.20) via (3.5.19b), for every fixed $q \geq 1$, we collect those terms which contain the final summand in (3.5.19b) (the *truncation term*), and hence G exactly q times. For these terms with $q \geq 1$ fixed, we then proceed as follows: Estimate those chains within the truncation term in which G appears trivially by norm, $\|G\| \leq 1/\eta$ (note that there are at most $k+1$ resolvents in such chains and we can afford estimating all of them by $1/\eta$ not just the last one G) and use $\|A\| \leq \sqrt{N}\langle |A|^2\rangle^{1/2}$ (recall that we assumed $\langle |A|^2\rangle^{1/2} = 1$ around (3.5.2)–(3.5.3)), and treat the other factors by our induction hypothesis (3.5.14) (resulting in an N^{ξ} factor).

In this way, we conclude the estimate

$$\big[q\,\text{truncation terms}\big] \lesssim N^{\xi}\frac{(N\eta/\rho)^{p/2}}{\big(N^{\frac{m+1}{2}}\big)^q}\left(\frac{N^{k/2}}{\eta^{k+1}}\right)^q = \frac{N^{\xi}}{N^{2q}}\frac{1}{N^{p(q-1)/2}}\left(\frac{\eta}{\rho}\right)^{p/2}\frac{1}{(N\eta)^{(k+1)q}} \lesssim \frac{N^{\xi}}{N^2} \tag{3.5.49}$$

when choosing $m = p + 3k + 5$, where in the last step we used that $\eta/\rho \lesssim 1$ and $N\eta \gg 1$. We remark that $(N\eta/\rho)^{p/2}$ in (3.5.49) comes from the prefactor of Ψ_k, $\big(N^{\frac{m+1}{2}}\big)^{-q}$ from the cumulant order of the truncation terms and $\big(N^{k/2}/\eta^{k+1}\big)^q$ from the trivial bounds.

3.5.2.5 Proof of Proposition 3.5.2

As mentioned above (3.5.25), the treatment of the higher order terms in (3.5.23) is identical to our discussion above. Therefore, summarizing Sections 3.5.2.2–3.5.2.4, we have proven the following.

Lemma 3.5.9. *Fix $p, k \in \mathbf{N}$ and assume that the induction hypothesis* (3.5.14) *holds. Then, for every $\gamma \in [\gamma(N)]$, we have that*

$$\Big|\|\Psi_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\|_p^p - \|\Psi_k^{(\gamma-1)}(\boldsymbol{x},\boldsymbol{y})\|_p^p\Big| \leq \frac{C_1}{N^2}\breve{\Omega}_k^p(\gamma) + C_2\frac{N^{\xi}}{N^2}\left(1 + \sum_{u=1}^{4}\left(\frac{N\eta}{\rho}\right)^{u/2}\Big|\big[u\,\text{sum. bdd. } M\text{-terms}\big]_{\boldsymbol{x},\boldsymbol{y}}^{(\gamma)}\Big|\right),$$

where $\big[u\,\text{sum. bdd. } M\text{-terms}\big]_{\boldsymbol{x},\boldsymbol{y}}^{(\gamma)}$ is understood as explained below (3.5.45)*.*

Next, employing the telescopic summation from (3.5.24) we find that

$$\|\Psi_k^{(\gamma_0)}(\boldsymbol{x},\boldsymbol{y})\|_p^p \leq C_1\frac{1}{N^2}\sum_{\gamma<\gamma_0}\breve{\Omega}_k^p(\gamma) + C_2N^{\xi} + \frac{N^{\xi}}{N^2}\sum_{\gamma<\gamma_0}\left(\sum_{u=1}^{4}\left(\frac{N\eta}{\rho}\right)^{u/2}\Big|\big[u\,\text{sum. bdd. } M\text{-terms}\big]_{\boldsymbol{x},\boldsymbol{y}}^{(\gamma)}\Big|\right) \tag{3.5.50}$$

after having absorbed $\|\Psi_k^{(0)}(\boldsymbol{x},\boldsymbol{y})\|_p^p$ into C_2N^{ξ} by our initial assumption that we have multi-resolvent local laws (3.5.4) for the Wigner matrix $H^{(\mathbf{v})} = H^{(0)}$. We are left with discussing the first and last term on the rhs. of (3.5.50).

For the first term, we rely on the following lemma, which says that, in particular, we can replace each $\breve{\Omega}_k^p(\gamma)$ in (3.5.50) by $\Omega_k^p(\gamma)$, absorbing the additional error into C_2.

Lemma 3.5.10. *Fix $p, k \in \mathbf{N}$. Then, for every fixed $\gamma \in [\gamma(N)]$, the expressions (recall* (3.5.32)*)*

$$\Omega_k^p(\gamma)\,, \quad \Omega_k^p(\gamma-1)\,, \quad \textit{and} \quad \breve{\Omega}_k^p(\gamma)$$

are comparable up to an additive error of order N^{ξ} for arbitrarily small $\xi > 0$.

Proof. We give a sketch of the simple argument based on Lemma 3.5.9 in combination with Lemma 3.5.4: Similarly to the proof of Lemma 3.5.9, we first expand $G^{(\gamma)}$ (resp. $G^{(\gamma-1)}$) in $\|\Psi_k^{(\gamma)}(\boldsymbol{x},\boldsymbol{y})\|_p^p$ (resp. $\|\Psi_k^{(\gamma-1)}(\boldsymbol{x},\boldsymbol{y})\|_p^p$) by means of (3.5.19b) and realize that $\alpha_{k,0}^{(\gamma)}(\boldsymbol{x},\boldsymbol{y}) = 1$ in (3.5.22)–(3.5.23). The various terms arising in the expansion (now for all $r \geq 1$ and not only for $r \geq 4$) are dealt with as explained in Sections 3.5.2.2–3.5.2.4.

However, there is a major simplification, since we do not need to gain from the summation as in Case (ii) in Section 3.5.2.2: The maximal excess power u of the leftover $(N\eta/\rho)^{1/2}$-factor is bounded by the order r of the expansions in (3.5.22)–(3.5.23) (simply because at order r, there are at most $d = r$ destroyed resolvent chains), such that the characteristic $1/N^{r/2}$-factor at order r balances this excess. Finally, we take a maximum over all $\boldsymbol{u},\boldsymbol{v} \in I_{\boldsymbol{x},\boldsymbol{y}}$ for all $\|\check{\Psi}_k^{(\gamma)}(\boldsymbol{u},\boldsymbol{v})\|_p^p$ arising through the expansion (see (3.5.32)).

This finishes the sketch of the proof of Lemma 3.5.10. □

For the last term in (3.5.50), we extend the summation $\sum_{\gamma<\gamma_0}$ to all indices $i,j \in [N]$; it is an upper bound as we only sum positive terms. Then, for every fixed $u \in [4]$, we need to gain from this summation of $\big[u\,\mathrm{sum.\ bdd.}\,M\text{-terms}\big]_{\boldsymbol{x},\boldsymbol{y}}^{(\gamma)}$ over all $\gamma \in [\gamma(N)]$ precisely $N^{-u/2}$ compared to the naive N^2-size of the summation. This was achieved in Lemma 3.5.8 by the index structure (3.5.42) of the factors and application of several Schwarz inequalities (3.5.43).

Hence, combining (3.5.50) with Lemma 3.5.10 and Lemma 3.5.8, we find that

$$\|\Psi_k^{(\gamma_0)}(\boldsymbol{x},\boldsymbol{y})\|_p^p \leq C_1 \frac{1}{N^2} \sum_{\gamma<\gamma_0} \Omega_k^p(\gamma) + C_2 N^{\xi}\,.$$

Since the rhs. is independent of the elements in $I_{\boldsymbol{xy}}$ (recall (3.5.32)), we can as well maximize over those on the lhs. and arrive at Proposition 3.5.2. □

3.5.2.6 Conclusion of the induction step

Having Proposition 3.5.2 and hence (3.5.18) at hand, we can immediately deduce

$$\max_{\gamma\leq\gamma(N)} \check{\Omega}_k^p(\gamma) \lesssim N^{\xi}$$

from Lemma 3.5.10 above. This proves the $\check{\Psi}$-part of (3.5.13) and thus finishes the induction step.

Therefore, using uniformity of this bound, we conclude the proof of the isotropic multi-resolvent local laws (3.3.14).

3.5.3 Part (b): Proof of the averaged law.

The general idea of the proof of the averaged law is exactly the same as in the previous section: We replace all matrix elements one-by-one in $\gamma(N) \sim N^2$ steps and sum up the changes over all positions $\gamma \in [\gamma(N)]$ (cf. (3.5.24)). However, there are a several (minor) differences in the averaged case compared to Section 3.5.2, which we will explain in the following.

Since both, averaged and isotropic normalized differences, (3.5.2) and (3.5.3), appear, we shall henceforth reintroduce the superscripts $^{\mathrm{av}}$ and $^{\mathrm{iso}}$. Moreover, contrary to the isotropic proof, in this part it is sufficient to consider an arbitrary fixed $k \in \mathbf{N}$ and perform a *single induction* on the moment p taken of Ψ_k^{av}, i.e. $\mathbf{E}|\Psi_k^{\mathrm{av}}|^p = \|\Psi_k^{\mathrm{av}}\|_p^p$. We point out that the induction on k used in the previous section is not needed, because the proof of the isotropic laws has already been concluded (see (3.5.53) later). Hence, as the *induction hypothesis*, we will assume that

$$\max_{\gamma\leq\gamma(N)} \|\Psi_k^{\mathrm{av},(\gamma)}\|_{p-1} + \max_{\gamma\leq\gamma(N)} \|\check{\Psi}_k^{\mathrm{av},(\gamma)}\|_{p-1} \lesssim N^{\xi} \tag{3.5.51}$$

holds uniformly in traceless matrices for all $\xi > 0$, and our goal is to prove the same relation with p replacing $p-1$. The base case is thus simply the trivial bound ($p = 1$) given by $\mathbf{E}|\Psi_k^{\mathrm{av}}|^0 = 1$. To ease notation, just as in Section 3.5.2, we will drop the subscripts for all resolvents and deterministic matrices, i.e. write $G_j = G$ and $A_j = A$ instead. Moreover, whenever it does not lead to confusion, we will drop all further sub- and superscripts.

Completely analogously to Section 3.5.2, we use resolvent expansions from Lemma 3.5.3 to prove the exact agreement of the orders $r \in \{0,1,2,3\}$ as in Lemma 3.5.4. For the higher order terms (again focusing on the most critical fourth order ones, see Section 3.5.2.2), we argue completely analogously to (3.5.26), but now we have an additional effect: Whenever an intact averaged chain gets destroyed by a replacement $G \to G\Delta G$ from a derivative, we obtain (a sum of) isotropic chains with a $1/N$ prefactor from the normalization of the trace, i.e.

$$\langle (GA)^k \rangle \longrightarrow \langle G\Delta(GA)^k \rangle = \frac{1}{N}\big((GA)^k G\big)_{\boldsymbol{e}_i\boldsymbol{e}_j} + \frac{1}{N}\big((GA)^k G\big)_{\boldsymbol{e}_j\boldsymbol{e}_i}\,. \tag{3.5.52}$$

In this way, the analogue of (3.5.26) reads

$$\mathbf{E}\sum_{d=1}^{4\wedge p}\big|\breve{\Psi}_k^{\mathrm{av}}(\boldsymbol{x},\boldsymbol{y})\big|^{p-d}\left(\frac{N\eta}{\rho}\right)^{d/2} N^{-d(k/2-1)}\frac{1}{N^d}\sum_{(4-d)\,\Delta\rightsquigarrow d}\Big|\underbrace{\big(...\Delta...\Delta...\big)_{\boldsymbol{e}_i\boldsymbol{e}_j}\cdot...\cdot\big(...\Delta...\big)_{\boldsymbol{e}_j\boldsymbol{e}_i}}_{(4-d)\ \Delta\text{ in a total }d\text{ iso chains}}\Big|\,, \tag{3.5.53}$$

where the isotropic chains referred to in (3.5.53), are precisely those obtained in (3.5.52). In particular, one Δ has already been "used" for each destroyed averaged chain, hence only $(4-d)$ Δ's are placed in the isotropic chains (recall (3.5.27)). Observe that, after writing $N^{-d(k/2-1)}/N^d = N^{-dk/2}$, beside from the unit vectors in the isotropic chains, the structure of (3.5.53) is exactly the same for (3.5.26).

Next, in each of the resulting four resolvent chains in the rhs. of (3.5.53), as before we add and subtract the corresponding M-term, again schematically written as $G = (G-M)+M$. Exactly as in the previous section, we have to distinguish two cases.

Case (i): At least d of the 4 resolvent chains are replaced by their fluctuating part, $G-M$.

Case (ii): At least $5-d$ of the 4 resolvent chains are replaced by their deterministic counterpart, M.

Case (i): First, we note that, since there are only strictly lower moments of Ψ_k^{av} appearing in (3.5.53) after the resolvent expansion, we can directly employ the *induction hypothesis* (3.5.51), i.e. there is no possibility of preserving the destroyed chains unlike in (3.5.29). Therefore, by additionally applying the already established isotropic laws from the previous section in combination with $\big|(M_{j+1})_{\boldsymbol{u}\boldsymbol{v}}\big| \lesssim N^{j/2}$ (recall also (3.5.33)), we find that

$$\text{Case (i) terms of (3.5.53)} \lesssim N^{\xi}\sum_{d=1}^{4\wedge p}\left(\frac{N\eta}{\rho}\right)^{d/2} N^{-dk/2}\left[N^{dk/2}\left(\frac{\rho}{N\eta}\right)^{d/2}+...\right] \lesssim N^{\xi}\,, \tag{3.5.54}$$

indicating terms with more than d factors of $G-M$ by dots. This concludes the discussion of Case (i).

Case (ii): For the second case, we again recall that all purely deterministic terms are independent of the replacement step γ. Moreover, completely analogously to Case (ii) in Section 3.5.2.2, it is not sufficient to just estimate every isotropic M-term blindly – instead we again need to *gain from the summation* over all replacement positions. We again illustrate this by an example.

Example 3.5.11. *We first consider $d = 1$ and use the notation* (3.5.33)*. Then, by means of the induction hypothesis* (3.5.51)*, we have the trivial estimate*

$$\begin{aligned}\mathbf{E}\big|\breve{\Psi}_k^{\mathrm{av}}(\boldsymbol{x},\boldsymbol{y})\big|^{p-1}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\sum_{\substack{0\le k_l\le k:\\ \sum_l k_l=k}}\Big[\big|\big(M_{k_1+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big|+\ldots\Big]\\ \lesssim N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\sum_{\substack{0\le k_l\le k:\\ \sum_l k_l=k}}\Big[N^{\sum_l k_l/2}+\ldots\Big]\lesssim N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2},\end{aligned} \tag{3.5.55}$$

analogously to (3.5.36)*. Again, this bound is off by a factor $(N\eta/\rho)^{1/2}$, which can be improved on by averaging over all replacement position.*

Compared to the isotropic case, we can no longer gain from summing over off-diagonal terms of the form $M_{\boldsymbol{x}\boldsymbol{e}_i}$. Instead, now we sum over squares of terms of the form $M_{\boldsymbol{e}_i\boldsymbol{e}_i}$ and estimate it by

$$\sum_i\big|M_{\boldsymbol{e}_i\boldsymbol{e}_i}\big|^2\le\sum_{i,j}\big|M_{\boldsymbol{e}_i\boldsymbol{e}_j}\big|^2\le\sum_i\big(|M|^2\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}=N\langle|M|^2\rangle\,, \tag{3.5.56}$$

similarly to (3.5.38)–(3.5.40)*. Note that* (3.5.56) *is better than the trivial bound, which would give $N\|M\|^2$. The key for exploiting this improvement is the following lemma, the proof of which is given in Appendix 3.A.*

Lemma 3.5.12. *Using the assumptions and notations from Lemma 3.2.3 and the normalization $\langle|A_i|^2\rangle = 1$, we have that*

$$\Big\langle\big|\mathcal{M}(z_1,A_1,\ldots,A_k,z_{k+1};\mathbf{I}_{k+1})\big|^2\Big\rangle\lesssim N^k\left(\prod_{i\in\mathbf{I}_{k+1}}\rho_i\right)^2\left[\left(\frac{\max_{i\in[k+1]}\big(\rho_i+\mathbf{1}(i\notin\mathfrak{I}_{k+1})\big)}{N\ell}\right)^2\vee\frac{1}{N}\right]. \tag{3.5.57}$$

Applying (3.5.57) *for $k = k_l$ and $\mathfrak{I}_{k_l+1} = \varnothing$ (recall* (3.2.10)*,* (3.2.15)*, and* (3.5.33)*), we see the bound*

$$\langle|M_{k_l+1}|^2\rangle\lesssim N^{k_l}\left[\left(\frac{\rho}{N\eta}\right)^2\vee\frac{1}{N}\right]. \tag{3.5.58}$$

We remark that this estimate is better by the factor $\big[\big(N\eta/\rho\big)^{-2}\vee N^{-1}\big]\ll 1$ compared to the naive norm bound $|(M_{k_l+1})_{\boldsymbol{u}\boldsymbol{v}}|^2\le\|M_{k_l+1}\|^2\lesssim N^{k_l}$ from Lemma 3.2.3 (b) employed in (3.5.55)*. Hence, fixing one constellation of k_l's in* (3.5.55)*, we find the average of the first line in* (3.5.55) *over all $i,j\in[N]$ to be bounded by*

$$\begin{aligned}&N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\frac{1}{N^2}\sum_{i,j}\Big[\big|\big(M_{k_1+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big|+\ldots\Big]\\ \lesssim&N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\frac{1}{N^2}\left[\prod_{l\in[4]}\left(\sum_i\big|\big(M_{k_l+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big|^2\right)^{1/2}+\ldots\right]\\ \lesssim&N^{\xi}\left(\frac{N\eta}{\rho}\right)^{1/2}N^{-k/2}\frac{1}{N^2}\left[\left(\prod_{l\in[4]}N^{k_l+1}\left[\left(\frac{\rho}{N\eta}\right)^2\vee\frac{1}{N}\right]\right)^{1/2}+\ldots\right]\\ \lesssim&N^{\xi}\left[\left(\frac{\rho}{N\eta}\right)^{7/2}\vee\left(\frac{\eta}{\rho}\right)^{1/2}\frac{1}{N^{3/2}}\right]\lesssim N^{\xi}.\end{aligned} \tag{3.5.59}$$

To go from the first to the second line, we employed a trivial Schwarz inequality. To go to the penultimate line, we used (3.5.56) *with $M = M_{k_l+1}$. For the final estimate, we employed $(\prod_{l\in[4]}N^{k_l+1})^{1/2} = N^{k/2+2}$.*

Next, we consider one example for $d = 4$, where all four resolvent chains are replaced by their deterministic counterpart. In this case, the analog of (3.5.59) *reads*

$$\begin{aligned}
&N^{\xi}\left(\frac{N\eta}{\rho}\right)^2 N^{-2k}\frac{1}{N^2}\sum_{i,j}\Big[\big|\big(M_{k+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_j}\big(M_{k+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_i}\big(M_{k+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_j}\big(M_{k+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_i}\big| + ...\Big]\\
\lesssim &N^{\xi}\left(\frac{N\eta}{\rho}\right)^2 N^{-k}\frac{1}{N^2}\left[\sum_{i,j}\big|\big(M_{k+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_j}\big|^2 + ...\right]\\
\lesssim &N^{\xi}\left(\frac{N\eta}{\rho}\right)^2 N^{-k}\frac{1}{N^2}\left[N^{k+1}\left[\left(\frac{\rho}{N\eta}\right)^2\vee\frac{1}{N}\right] + ...\right]\lesssim N^{\xi}\left[\frac{1}{N}\vee\left(\frac{\eta}{\rho}\right)^2\right]\lesssim N^{\xi}\,.
\end{aligned}$$

To go from the first to the second line, we estimated two factors of M_{k+1} by their norm, $\big|(M_{k+1})_{\boldsymbol{uv}}\big|\lesssim N^{k/2}$. Next, to go to the third line, we employed (3.5.56) *and Lemma 3.5.12. The final estimate used $\eta/\rho\lesssim 1$.*

The above examples showcase the general mechanism for the terms in Case (ii): After estimating all the $(G-M)$-type terms with the aid of the induction hypothesis (3.5.51), we are left with an excess $(N\eta/\rho)^{u/2}$-factor, for some $u\in[4]$. Analogously to (3.5.41)–(3.5.42), this leftover factor is then controlled by *gaining from the summation* like in (3.5.57). We skip the simple counting argument ensuring this gain.

The treatment of the further higher order terms and the truncation of the resolvent expansion is completely analogous to Sections 3.5.2.3 and 3.5.2.4, respectively. Therefore, by telescopic summation like in (3.5.24), we find that

$$\max_{\gamma\le\gamma(N)}\|\Psi_k^{\mathrm{av},(\gamma)}\|_p^p+\max_{\gamma\le\gamma(N)}\|\breve{\Psi}_k^{\mathrm{av},(\gamma)}\|_p^p\lesssim\|\Psi_k^{\mathrm{av},(0)}\|_p^p+N^{\xi}\lesssim N^{\xi}$$

where in the last step we absorbed $\|\Psi_k^{\mathrm{av},(0)}\|_p^p$ into N^{ξ} by our initial assumption that we have multi-resolvent local laws (3.5.4) for the matrix $H^{(\boldsymbol{v})}=H^{(0)}$. The checked version is obtained completely analogously to Lemma 3.5.10.

This completes the proof of the induction step. We have thus finished the argument for the averaged case and hence the proof of Proposition 3.3.4. □

3.5.4 The case $\mathfrak{I}_k\neq\varnothing$ (resp. $\mathfrak{I}_{k+1}\neq\varnothing$)

In this section, we explain how to adjust the above argument for proving Proposition 3.3.4 in the case that at least one of the resolvents in the chains of interests

$$\langle\mathcal{G}_1A_1...\mathcal{G}_kA_k\rangle\qquad\text{and}\qquad\big(\mathcal{G}_1A_1...\mathcal{G}_kA_k\mathcal{G}_{k+1}\big)_{\boldsymbol{xy}}$$

is an imaginary part, i.e. $\mathcal{G}_i=\operatorname{Im}G_i$ for at least one index $i\in[k]$ (resp. $i\in[k+1]$). Recall the local laws for the average and isotropic chain from (3.3.13) and (3.3.14), respectively. Compared to the case of no imaginary parts, handled in the previous Sections 3.5.2–3.5.3, there are now two changes: First, the bound contains the product $\prod_{i\in\mathfrak{I}}\rho_i$ (instead of one). Second, the smallness factor $(N\eta/\rho)^{-1/2}$ from before is now replaced by $(N\ell)^{-1/2}$

For adjusting the first change, the simple but key insight is, that when applying the resolvent expansion from Lemma 3.5.3 to both G and G^* in $\operatorname{Im}G=\frac{1}{2\mathrm{i}}(G-G^*)$, we can always "restore" exactly one $\operatorname{Im}G$ on the rhs. More precisely, taking (3.5.19b) for concreteness and using $\Delta=\Delta^*$,

we have that

$$\begin{aligned}\operatorname{Im} G = \frac{1}{2\mathrm{i}}\big[G - G^*\big] = \frac{1}{2\mathrm{i}}\Big[&\Big(\check{G} - N^{-1/2}\check{G}\Delta\check{G} + N^{-1}\check{G}\Delta\check{G}\Delta\check{G} + \ldots\Big) \\ &- \Big(\check{G}^* - N^{-1/2}\check{G}^*\Delta\check{G}^* + N^{-1}\check{G}^*\Delta\check{G}^*\Delta\check{G}^* + \ldots\Big)\Big] \\ = \operatorname{Im}\check{G} &- N^{-1/2}\big(\operatorname{Im}\check{G}\Delta\check{G} + \check{G}^*\Delta\operatorname{Im}\check{G}\big) \\ &+ N^{-1}\big(\operatorname{Im}\check{G}\Delta\check{G}\Delta\check{G} + \check{G}^*\Delta\operatorname{Im}\check{G}\Delta\check{G} + \check{G}^*\Delta\check{G}^*\Delta\operatorname{Im}\check{G}\big) + \ldots\end{aligned}$$

In this way, the imaginary parts in the original chain are "preserved" by the resolvent expansion. Recall that $|\operatorname{Im}\check{G}_{\boldsymbol{uv}}(z)| \prec \rho(z)$ (as a consequence of (3.5.5) for $N|\operatorname{Im} z|\rho(z) \gg 1$; recall $N\hat{\ell} \gg 1$), which improves (3.5.31). In particular, using Lemma 3.2.3, we find that the factor $\big(\prod_{i\in\mathfrak{J}}\rho_i\big)^{-d}$ stemming from the correct normalisation of the analog of $\Psi_k^{\mathrm{av/iso}}$ in (3.5.2)–(3.5.3) and thus appearing in the expression analogous to (3.5.26) is naturally compensated by a product of ρ's stemming from the destroyed chains.

For adjusting to the second change, it suffices to replace every η/ρ appearing in Sections 3.5.2–3.5.3 by ℓ and realize that the complement of the interesting regime, i.e. the regime $\ell \geq 1$ is already proven in Proposition 3.3.1.

3.A Additional technical results

In this section we prove several additional technical results which are used in the main sections.

3.A.1 Bounds on the deterministic approximations

Proofs of Lemma 3.2.3 and the claim in Remark 3.2.6 (ii). We will first proof the following stronger bound in Lemma 3.A.1, from which we immediately deduce Lemma 3.2.3 and the claim in Remark 3.2.6 (ii). The proof of the following lemma is given at the end of the current section.

Lemma 3.A.1. *Fix $k \geq 1$. Consider spectral parameters $z_1, \ldots, z_k \in \mathbf{C} \setminus \mathbf{R}$ and traceless matrices $A_1, \ldots, A_k \in \mathbf{C}^{N\times N}$, and define for every $j \in [k]$*

$$\eta_j := |\operatorname{Im} z_j|, \qquad \rho_j := \frac{1}{\pi}|\operatorname{Im} m_{\mathrm{sc}}(z_j)|, \qquad \ell := \min_j\big[\eta_j(\rho_j + \mathbf{1}(j \notin \mathfrak{I}_k))\big].$$

Then, for every $1 \leq s \leq \lfloor k/2\rfloor$ and $\pi \in \mathrm{NC}([k])$ with $|\pi| = k+1-s$, it holds that

$$\Big|\langle \mathrm{pTr}_{K(\pi)}(A_1, \ldots, A_{k-1})A_k\rangle \prod_{S\in\pi} m_\circ^{(\mathfrak{I}_k)}[S]\Big| \lesssim \Big(\prod_{j\in\mathfrak{I}_k}\rho_j\Big)\frac{1}{\ell^{s-1}} \prod_{\substack{S\in K(\pi)\\ |S|\geq 2}}\prod_{j\in S}\Big\langle |A_j|^{|S|}\Big\rangle^{\frac{1}{|S|}}. \tag{3.A.1}$$

with $m_\circ^{(\mathfrak{J})}[S]$ being defined above (3.2.16)*. For $s > \lfloor k/2\rfloor$ the lhs. of* (3.A.1) *equals zero.*

For the proof of Lemma 3.2.3 (a) and the claim in Remark 3.2.6 (ii) concerning (3.2.20) we use that $\langle |A|^p\rangle^{1/p} \leq N^{\frac{p-2}{2p}}\langle |A|^2\rangle^{1/2}$ for any $p \geq 2$, and hence

$$\text{rhs. of (3.A.1)} \lesssim N^{k/2-1}\Big(\prod_{j\in\mathfrak{I}_k}\rho_j\Big)\Big(\prod_{j=1}^k\langle |A_j|^2\rangle^{1/2}\Big)\frac{1}{(N\ell)^{s-1}}.$$

This shows that, in particular, all terms with $s > 1$ in (3.A.1) are explicitly smaller than the error term in (3.2.20), where we used that $N\ell \gg 1$. The $s = 1$ term exactly constitutes the deterministic

approximation in (3.2.23), i.e. the sum in (3.A.1) contains exactly one term

$$\sum_{\substack{\pi\in\mathrm{NC}([k]):\\ |\pi|=k}} \langle \mathrm{pTr}_{K(\pi)}(A_1,\dots,A_{k-1})A_k\rangle \prod_{S\in\pi} m_\circ^{(\mathfrak{I}_k)}[S] = \Bigg(\prod_{j\in\mathfrak{I}_k} \mathrm{Im}\, m_j\Bigg)\Bigg(\prod_{j\notin\mathfrak{I}_k} m_j\Bigg)\langle A_1 ... A_k\rangle\,.$$

Here we used that $|\pi| = k$ implies that the Kreweras complement consists of the full set, $K(\pi) = [k]$.

Finally, for the proof of Lemma 3.2.3 (b) and the claim in Remark 3.2.6 (ii) concerning (3.2.21) (i.e. the corresponding isotropic bounds) we argue completely analogously to Section 3.4.2. □

It remains to prove Lemma 3.A.1.

Proof of Lemma 3.A.1. Fix an arbitrary non-crossing partition $\pi \in \mathrm{NC}([k])$ consisting of $|\pi| = k+1-s$ blocks.

First, note that, in order to get a non-vanishing partial trace

$$\langle \mathrm{pTr}_{K(\pi)}(A_1,\dots,A_{k-1})A_k\rangle = \prod_{S\in K(\pi)} \Big\langle \prod_{j\in S} A_j \Big\rangle$$

the minimal size of a block $S \in K(\pi)$ is two (using that the A_i's are traceless). Therefore, by application of Hölder's inequality,

$$\big|\langle \mathrm{pTr}_{K(\pi)}(A_1,\dots,A_{k-1})A_k\rangle\big| \le \prod_{\substack{S\in K(\pi)\\ |S|\ge 2}} \prod_{j\in S} \big\langle |A_j|^{|S|}\big\rangle^{\frac{1}{|S|}}\,. \tag{3.A.2}$$

In order to estimate $\prod_{S\in\pi} m_\circ^{(\mathfrak{I}_k)}[S]$, we recall the Möbius inversion formula [170, Lemma 2.16]

$$m_\circ^{(\mathfrak{I}_k)}[S] = m^{(\mathfrak{I}_k)}[S] + \sum_{\substack{\pi\in\mathrm{NC}(S)\\ |\pi|\ge 2}} (-1)^{|\pi|-1}\Bigg(\prod_{T\in K(\pi)} C_{|T|-1}\Bigg)\prod_{U\in\pi} m^{(\mathfrak{I}_k)}[U] \tag{3.A.3}$$

where C_n is the n^{th} Catalan number. Hence, it suffices to bound the iterated divided differences $m^{(\mathfrak{I}_k)}[S]$ for a subset $S \subset [k]$ as

$$\big|m^{(\mathfrak{I}_k)}[S]\big| \lesssim \frac{\prod_{i\in\mathfrak{I}_k\cap S}\rho_i}{\ell^{|S|-1}} \tag{3.A.4}$$

which is a direct consequence of the integral representation (3.2.16). Indeed, combining (3.A.3) with (3.A.4) and using that the sum in (3.A.3) is restricted to partitions of S with at least two blocks, we obtain

$$\Bigg|\prod_{S\in\pi} m_\circ^{(\mathfrak{I}_k)}[S]\Bigg| \lesssim \Bigg(\prod_{i\in\mathfrak{I}_k}\rho_i\Bigg)\frac{1}{\ell^{s-1}} \tag{3.A.5}$$

where we additionally used that the original non-crossing partition $\pi \in \mathrm{NC}([k])$ consists of exactly $k+1-s$ blocks. Combining (3.A.5) with (3.A.2), we conclude the proof of (3.A.1).

For $s > \lfloor k/2\rfloor$, we note that the Kreweras complement $K(\pi)$ necessary contains singletons, and hence the lhs. of (3.A.1) vanishes since $\langle A_i\rangle = 0$. □

We conclude this section by giving the proof of Lemma 3.5.12.

Proof of Lemma 3.5.12. The principal idea of the proof is very similar to the previous ones given in this section, hence we provide only a brief argument.

Recalling (3.2.15)–(3.2.16), we have that

$$\left\langle \left| \mathcal{M}(z_1, A_1, \dots, A_k, z_{k+1}; \mathfrak{I}_{k+1}) \right|^2 \right\rangle \lesssim \sum_{\pi \in \mathrm{NC}([k+1])} \left\langle \left| \mathrm{pTr}_{K(\pi)}(A_1, \dots, A_k) \right|^2 \right\rangle \left| \prod_{S \in \pi} m_\circ^{(\mathfrak{I}_{k+1})}[S] \right|^2 . \tag{3.A.6}$$

Next, analogously to Lemma 3.A.1 above, we decompose the summation over all partitions π into groups, where $|\pi| = k + 2 - s$ with $1 \le s \le \lceil (k+1)/2 \rceil$ is fixed (note that $\lfloor \cdot \rfloor$ got replaced by $\lceil \cdot \rceil$ due to the presence of a non-traceless identity matrix). Moreover, for fixed s we distinguish two cases in (3.A.6) (recall (3.2.9)): For Case (i), we assume that the unique block $\mathfrak{B}(k+1) \in K(\pi)$ containing $k+1$ contains no other elements, i.e. $\mathfrak{B}(k+1) \setminus \{k+1\} = \varnothing$. For Case (ii), we assume that $\mathfrak{B}(k+1) \setminus \{k+1\} \neq \varnothing$.

Case (i). First, we note that necessarily $s \ge 2$ in this case. Then, we have that

$$\left\langle \left| \mathrm{pTr}_{K(\pi)}(A_1, \dots, A_k) \right|^2 \right\rangle \le \left(\prod_{\substack{S \in K(\pi) \setminus \mathfrak{B}(k+1) \\ |S| \ge 2}} \prod_{j \in S} \left\langle |A_j|^{|S|} \right\rangle^{\frac{1}{|S|}} \right)^2 \le \left(\frac{N^{k/2}}{N^{s-1}} \right)^2 ,$$

analogously to (3.A.2). Since in Case (i), z_1 and z_{k+1} are always together in one block $S \in \pi$ with $|\pi| = k + 2 - s$, we obtain

$$\left| \prod_{S \in \pi} m_\circ^{(\mathfrak{I}_{k+1})}[S] \right|^2 \lesssim \left[\frac{\left(\prod_{i \in \mathfrak{I}_{k+1}} \rho_i \right) \wedge \max_{i \in [k+1]} \rho_i}{\ell^{s-1}} \right]^2$$

analogously to (3.A.5) by means of (3.A.3) and the integral representation (3.2.16). The additional $\wedge \max_{i \in [k+1]} \rho_i$, which is effective only for $\mathfrak{I}_{k+1} = \varnothing$, comes from the estimate

$$\int_{\mathbf{R}} \frac{\rho(x)}{|x - z_1|\,|x - z_{k+1}|} \mathrm{d}x \lesssim \frac{\rho_1 \vee \rho_{k+1}}{\ell} ,$$

easily obtained by a Schwarz inequality.

Case (ii). In this case, the above estimates of the two factors in (3.A.6) modify to

$$\left\langle \left| \mathrm{pTr}_{K(\pi)}(A_1, \dots, A_k) \right|^2 \right\rangle \le \left(\prod_{\substack{S \in K(\pi) \\ |S| \ge 2}} \left(\prod_{j \in S_1} \left\langle |A_j|^{2(|S_1|-1)} \right\rangle^{\frac{1}{2(|S_1|-1)}} \right) \left(\prod_{i=2}^{s} \prod_{j \in S_i} \left\langle |A_j|^{|S_i|} \right\rangle^{\frac{1}{|S_i|}} \right) \right)^2 ,$$

assuming that $S_1 = \mathfrak{B}(k+1)$, and

$$\left| \prod_{S \in \pi} m_\circ^{(\mathfrak{I}_{k+1})}[S] \right|^2 \lesssim \left[\frac{\prod_{i \in \mathfrak{I}_{k+1}} \rho_i}{\ell^{s-1}} \right]^2 .$$

Putting the two cases together and using $\langle |A|^p \rangle^{1/p} \le N^{\frac{p-2}{2p}} \langle |A|^2 \rangle^{1/2}$ for any $p \ge 2$ together with $N\ell > 1$ and the normalization $\langle |A_j|^2 \rangle = 1$, we find that

$$\left\langle \left| \mathcal{M}(z_1, A_1, \dots, A_k, z_{k+1}; \mathbf{I}_{k+1}) \right|^2 \right\rangle \lesssim N^k \left(\prod_{i \in \mathbf{I}_{k+1}} \rho_i \right)^2 \left[\left(\frac{\max_{i \in [k+1]} \left(\rho_i + \mathbf{1}(i \notin \mathfrak{I}_{k+1}) \right)}{N\ell} \right)^2 + \frac{1}{N} \right] .$$

□

3.A.2 Proof of the global law in Proposition 3.3.1

We only discuss the proof of the average case (3.3.1), the isotropic case (3.3.2) is analogous and hence omitted. Set $d := \min_i \operatorname{dist}(z_i, [-2,2])$ and recall that $d \geq \delta \gtrsim 1$.

The case of no $\operatorname{Im} G$'s, i.e. $\mathfrak{I}_k = \varnothing$, has already been dealt with in [169, Appendix A] and yielded the bound (3.3.1) with a factor $d^{-(k+1)}$ instead of $\sqrt{\max_i \rho_i/\ell}$. In the $d \gtrsim 1$ regime, this bound is in fact stronger, $d^{-(k+1)} \lesssim d^{-1} \lesssim \sqrt{\max_i \rho_i/\ell}$, since $|\rho(z)| \sim |\operatorname{Im} z|/\operatorname{dist}(z,[-2,2])^2$ and $\ell \sim \min_i |\operatorname{Im} z_i|$.

In case of $\mathfrak{I}_k \neq \varnothing$ we need to gain from the fact that the original chain contained $\operatorname{Im} G$'s. The principal idea is analogous to [168, Appendix B] and [169, Appendix A], as we employ a cumulant expansion and argue by induction on the length k of the initial chain. However, in order to gain from the imaginary parts, the key observation is that within the cumulant expansion, the total number of Im's is preserved, as becomes apparent from the formula

$$\partial_{ab} \operatorname{Im} G = G\Delta^{ab} \operatorname{Im} G + \operatorname{Im} G \Delta^{ab} G^*$$

for the derivative of an $\operatorname{Im} G$ factor. Here, ∂_{ab} denotes the partial derivative w.r.t. the matrix entry w_{ab} of the Wigner matrix W and Δ^{ab} is a matrix consisting of all zeroes except for the (a,b)–entry which is equal to one. Using the norm bounds $\|\operatorname{Im} G_j\| \leq |\operatorname{Im} z_j|/\operatorname{dist}(z_j,[-2,2])^2 \sim \rho_j$ and $\|G_j\| \leq 1/d$ by spectral decomposition, we obtain (3.3.1) but with a factor $d^{k+1-|\mathfrak{I}_k|}$ instead of $\sqrt{\ell}$, analogously to [169, Eq. (A.2)]. Finally, since $\sqrt{\ell} \lesssim d \lesssim d^{k+1-|\mathfrak{I}_k|}$, this concludes the proof. □

3.A.3 Complex moment matching

In order to conduct the third step of our proof, the Green function comparison (GFT) of Proposition 3.3.4, we need to guarantee the moment matching condition (3.3.12) of the single entry distributions. For real random variables (or complex ones with independent real and imaginary parts), the argument ensuring this (and even an approximately matching fourth moment) is standard (see, e.g., [248, Lemma 16.2]) and based on an explicit construction of a distribution supported on three points in $\mathbf{R}$. However, for general complex random variables, this construction is not sufficient; we now present its complex variant.

Let Z be a complex random variable and denote its moments by

$$m_{i,j} = m_{i,j}(Z) := \mathbf{E}\left[\overline{Z}^i Z^j\right] \qquad \text{for} \qquad i,j \in \mathbf{N}_0\,, \tag{3.A.7}$$

and call $i+j$ the *order* of $m_{i,j}$. Clearly $m_{0,0} = 1$ and $m_{i,j} = \overline{m}_{j,i}$, so we can focus on $m_{i,j}$ with $i \leq j$.

Lemma 3.A.2. *Let $m_{0,2}, m_{0,3}, m_{1,2} \in \mathbf{C}$ with $|m_{0,2}| \leq 1$. Then there exists a complex random variable Z supported on at most eleven points $z_1, \ldots, z_{11} \in \mathbf{C}$, such that its moments* (3.A.7) *are given by*

$$m_{0,1}(Z) = 0\,, \quad m_{1,1}(Z) = 1\,, \quad m_{0,2}(Z) = m_{0,2}\,, \quad m_{0,3}(Z) = m_{0,3}\,, \quad \text{and} \quad m_{1,2}(Z) = m_{1,2}\,. \tag{3.A.8}$$

Remark 3.A.3. *A generalized version of this problem (constructing an atomic measure with arbitrary number of prescribed moments), known as the* truncated complex K-moment problem*, has been solved by Curto and Fialkow in [187]. To keep our result self-contained, we give a simple independent proof for the special case of three moments that we need here.*

Having Lemma 3.A.2 at hand, one can easily see that there exists a random variable that has the prescribed first three moments and it has an independent Gaussian component of given variance $\gamma > 0$. More precisely, given $m_{0,1} = 0$, $m_{1,1} = 1$, $m_{0,2}$, $m_{0,3}$, and $m_{1,2}$ with $|m_{0,2}| \leq 1$ as the set of moments of χ_{od}, we look for a representation of Z in the form

$$Z := (1-\gamma)^{1/2} Z' + \gamma^{1/2} \xi_G \quad \text{with} \quad \gamma \in (0,1) \quad \text{fixed}$$

with some random variable Z' to be constructed, where ξ_G is a centered complex Gaussian random variable having second moments $m_{0,2}(\xi_G) = m_{0,2}$ and $m_{1,1}(\xi_G) = 1$. The moments of Z' thus satisfy the relations

$$m_{i,j} = (1-\gamma)^{(i+j)/2} m_{i,j}(Z') + \gamma^{(i+j)/2} m_{i,j}(\xi_G) \quad \text{with} \quad 1 \le i+j \le 3. \tag{3.A.9}$$

In particular, $|m_{0,2}(Z')| = |m_{0,2}| \le 1$, so the moment sequence $m_{i,j}(Z')$ from (3.A.9) satisfy the only nontrivial condition of Lemma 3.A.2. Therefore, by Lemma 3.A.2, we can construct the random variable Z'. Finally, we remark that all random variables involved have arbitrarily high moments (cf. Assumption 3.2.1). This moment matching argument shows how to choose the distribution of the initial condition W_0 of the Ornstein-Uhlenbeck flow (3.3.3) so that after time $T = \gamma$ it will match with the distribution of the original matrix W up to three moments.

Proof of Lemma 3.A.2. We only outline the construction of the points $z_1, ..., z_{11} \in \mathbf{C}$, the precise computations are a simple exercise in calculus and linear algebra and hence omitted.

We set $z_{11} = 0$ to be the origin. The remaining ten points are then placed on five lines through the origin, carrying two points each, i.e. we put

$$z_j = r_j e^{i\varphi_j} \quad \text{and} \quad z_{11-j} = \hat{z}_j := -\hat{r}_j e^{i\varphi_j} \quad \text{with} \quad r_j, \hat{r}_j \ge 0\,, \varphi_j \in [0, 2\pi) \quad \text{for} \quad j \in [5]\,.$$

For simplicity, we can even prescribe four of the five angular variables in such a way that the corresponding points lie on the real and imaginary axis and the two diagonals, i.e. set $\varphi_j := j\pi/4$ for $j \in [4]$.

We then take the law of Z to be of the form

$$\sum_{j\in[5]} \left(p_j \delta_{z_j} + \hat{p}_j \delta_{\hat{z}_j}\right) + \left(1 - \sum_{j\in[5]} (p_j + \hat{p}_j)\right)\delta_0$$

for weights $p_j, \hat{p}_j \ge 0$ satisfying $\sum_{j\in[5]}(p_j + \hat{p}_j) \le 1$. As mentioned above, it is a simple exercise to show that the remaining parameters $r_j, \hat{r}_j, p_j, \hat{p}_j \ge 0$ for $j \in [5]$ and $\varphi_5 \in [0, 2\pi)$ can be chosen in such a way to accommodate (3.A.8). More precisely, taking $A_j := p_j r_j = \hat{p}_j \hat{r}_j \ge 0$ for $j \in [5]$ (this ensures $m_{0,1}(Z) = 0$), $r_5 = \hat{r}_5$, and using our choices of $\varphi_j = j\pi/4$ for $j \in [4]$, the two complex conditions $m_{0,3}(Z) = m_{0,3}$ and $m_{1,2}(Z) = m_{1,2}$ turn into four real linear equations for the variables $C_j := B_j(r_j - \hat{r}_j) \in \mathbf{R}$ for $j \in [4]$ with $B_j := A_j(r_j + \hat{r}_j) \ge 0$. The determinant of this linear systems can easily seen to be non-vanishing and it thus determines the difference variables $r_j - \hat{r}_j \in \mathbf{R}$ for $j \in [4]$. Finally, the independent variables $\varphi_5 \in [0, 2\pi)$ and $B_j := A_j(r_j + \hat{r}_j) \ge 0$ for $j \in [5]$ can easily be chosen to satisfy $m_{1,1}(Z) = 1$ and $m_{0,2}(Z) = m_{0,2}$. □

3.A.4 Additional proofs for Section 3.4

Proofs of Lemmas 3.4.1 and 3.4.8. The claim of Lemma 3.4.1 follows by multi-linearity from Lemma 3.4.8.

For the proof of Lemma 3.4.8, we will use a *tensorization argument* (or *meta argument*) similar to [181] and the proof of Lemma 1.D.1. Throughout this proof the size N of W is fixed. For $d \in \mathbf{N}$ consider the $(Nd) \times (Nd)$ Wigner matrix $\boldsymbol{W}^{(d)}$, i.e. the entries of $\boldsymbol{W}^{(d)}$ have variance $1/(Nd)$. Let $\boldsymbol{W}_t^{(d)}$ be the Ornstein-Uhlenbeck flow as in (3.3.3) with initial condition $\boldsymbol{W}_0^{(d)} = \boldsymbol{W}^{(d)}$, and define its resolvent $\boldsymbol{G}_{i,t}^{(d)} := (\boldsymbol{W}_t^{(d)} - z_{i,t})^{-1}$, then the deterministic approximation of the resolvent is still given by m_1, the Stieltjes transform of the semicircular law.

We now explain that also the deterministic approximation of products of resolvents and deterministic matrices is unchanged. For $1 \le i \le k$, define $\boldsymbol{A}_i^{(d)} := A_i \otimes I_d$, with I_d denoting the d–dimensional identity, then for $\boldsymbol{M}_{[1,k],t}^{(d)}$ defined as in (3.2.10) with $\boldsymbol{M}_{i,t}^{(d)}$ and $\boldsymbol{A}_i^{(d)}$ we have

$$\boldsymbol{M}_{[1,k],t}^{(d)} := \boldsymbol{M}^{(d)}(z_{1,t}, \boldsymbol{A}_1^{(d)}, \ldots, \boldsymbol{A}_{k-1}^{(d)}, z_{k,t}) = M(z_{1,t}, A_1, \ldots, A_{k-1}, z_{k,t}) \otimes I_d. \tag{3.A.10}$$

Fix $0 < s < t$, then integrating (3.4.5) for the bold faced resolvent and deterministic matrices, in time from s to t and taking the expectation we obtain

$$\begin{aligned}
&\langle \boldsymbol{M}^{(d)}_{[1,k],t}\boldsymbol{A}_k\rangle - \langle \boldsymbol{M}^{(d)}_{[1,k],s}\boldsymbol{A}_k\rangle \\
&= -\mathbf{E}\langle(\boldsymbol{G}_{[1,k],t} - \boldsymbol{M}^{(d)}_{[1,k],t})\boldsymbol{A}_k\rangle + \mathbf{E}\langle(\boldsymbol{G}_{[1,k],s} - \boldsymbol{M}^{(d)}_{[1,k],s})\boldsymbol{A}_k\rangle + \frac{k}{2}\int_s^t \mathbf{E}\langle \boldsymbol{G}_{[1,k],r}\boldsymbol{A}_k\rangle\,\mathrm{d}r \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{k}\int_s^t \mathfrak{E}\langle \boldsymbol{G}_{[i,j],r}\rangle\langle \boldsymbol{G}_{[j,i],r}\rangle\,\mathrm{d}r + \sum_{i=1}^{k}\int_s^t \mathfrak{E}\langle \boldsymbol{G}_{i,r} - m_{i,r}\rangle\langle \boldsymbol{G}^{(i)}_{[1,k],r}\boldsymbol{A}_k\rangle\,\mathrm{d}r + \frac{\sigma}{Nd}\sum_{\substack{i,j=1\\ i\le j}}^{k}\int_s^t \mathfrak{E}\langle \boldsymbol{G}_{[i,j]r}\boldsymbol{G}^{\mathrm{t}}_{[j,i],r}\rangle\,\mathrm{d}r\,.
\end{aligned} \tag{3.A.11}$$

Using the global law in Proposition 3.3.1 and (3.A.10), and taking the limit $d \to \infty$, this implies that for $|\mathrm{Im}\, z_i| \gtrsim 1$ we have

$$\langle M_{[1,k],t}A_k\rangle - \langle M_{[1,k],s}A_k\rangle = \frac{k}{2}\int_s^t \langle M_{[1,k],r}A_k\rangle\,\mathrm{d}r + \sum_{\substack{i,j=1,\\ i<j}}^{k-1}\int_s^t \langle M_{[i,j],r}\rangle\langle M_{[j,i],r}\rangle\,\mathrm{d}r. \tag{3.A.12}$$

Finally, dividing (3.A.12) by $t-s$ and taking the limit $s \to t$, we conclude the proof of Lemma 3.4.8. □

Proof of Lemma 3.4.6. The proof of this lemma is very similar to [169, Lemma 3.3]. Hence we give the argument only for the case where k is even, if k is odd the proof is completely analogous. Moreover, for notational simplicity we henceforth drop the time dependence and the precise indices of G_s's and A's, i.e. write $\mathrm{Im}\, G_s \equiv \mathrm{Im}\, G_{i,s}$, $A \equiv A_j$, $\rho \equiv \rho_i$ and so on. Then, by application of the general bound

$$|\langle B_1B_2B_3B_4\rangle| \le N\prod_{i=1}^{4}\langle |B_i|^2\rangle^{1/2} \quad \text{for all} \quad B_i \in \mathbf{C}^{N\times N}$$

applied to $B_i = \sqrt{|\mathrm{Im}\, G_s|}A(\mathrm{Im}\, G_sA)^{k/2-1}\sqrt{|\mathrm{Im}\, G_s|}$ and with the aid of (3.2.17), we find that

$$\begin{aligned}
\Phi_{2k}(s) &= \frac{\sqrt{N\hat{\ell}_s}}{N^{k-1}\,\rho_s^{2k}\,\langle |A|^2\rangle^k}\left|\left\langle (\mathrm{Im}\, G_sA)^{2k} - \widehat{M}_{[\hat{1},\widehat{2k}],s}A\right\rangle\right| \\
&\lesssim \sqrt{N\hat{\ell}_s} + \frac{\sqrt{N\hat{\ell}_s}}{N^{k-1}\,\rho_s^{2k}\,\langle |A|^2\rangle^k}N\left|\left\langle (\mathrm{Im}\, G_sA)^k\right\rangle\right|^2 \\
&\prec \sqrt{N\hat{\ell}_s} + \frac{\sqrt{N\hat{\ell}_s}}{N^{k-1}\,\rho_s^{2k}\,\langle |A|^2\rangle^k}N\left[N^{k/2-1}\rho_s^k\langle |A|^2\rangle^{k/2}\left(1 + \frac{\phi_k}{\sqrt{N\hat{\ell}_s}}\right)\right]^2 \\
&\lesssim \sqrt{N\hat{\ell}_s} + \frac{\phi_k^2}{\sqrt{N\hat{\ell}_s}}\,.
\end{aligned}$$

We remark that, in order to bound $\left\langle (\mathrm{Im}\, G_sA)^k\right\rangle$ in terms of ϕ_k, we added and subtracted the corresponding M-term and used the assumption that $\Phi_k(s) \prec \phi_k$. □

Chapter 4

Out-of-time-ordered correlators for Wigner matrices

This chapter contains the paper [151]:

G. Cipolloni, L. Erdős, and J. Henheik. Out-of-time-ordered correlators for Wigner matrices. *Adv. Theor. Math. Phys.*, 2024. Accepted, arXiv: 2402.17609

Abstract. We consider the time evolution of the *out-of-time-ordered correlator* (OTOC) of two general observables A and B in a mean field chaotic quantum system described by a random Wigner matrix as its Hamiltonian. We rigorously identify three time regimes separated by the physically relevant *scrambling* and *relaxation* times. The main feature of our analysis is that we express the error terms in the optimal Schatten (tracial) norms of the observables, allowing us to track the exact dependence of the errors on their rank. In particular, for significantly overlapping observables with low rank the OTOC is shown to exhibit a significant local maximum at the scrambling time, a feature that may not have been noticed in the physics literature before. Our main tool is a novel multi-resolvent local law with Schatten norms that unifies and improves previous local laws involving either the much cruder operator norm (cf. [168]) or the Hilbert-Schmidt norm (cf. [169]).

Content

4.1 Introduction

A basic feature of a strongly interacting quantum system is that local initial states become non-local along the unitary time evolution, in particular they become increasingly harder to distinguish by local observables. The simplest way to detect this chaotic behavior is to monitor the overlap $\langle A(t)B\rangle$ of the Heisenberg time evolution $A(t) := \mathrm{e}^{\mathrm{i}tH}Ae^{-\mathrm{i}tH}$ of an observable A with another static observable B, where H is the Hamiltonian and A, B are Hermitian operators. Here $\langle M\rangle := \frac{1}{N}\operatorname{Tr} M$ denotes the normalized trace of an $N \times N$ matrix, N is the dimension of the quantum state space. As time goes on, the overlap between two local observables converges to its stationary value in a process called *quantum thermalisation*. Since this stationary value is practically[1] factorized, $\langle A\rangle\langle B\rangle$, the original observable A becomes hardly detectable from its time evolution by local observables B.

A more refined measure of the dynamically evolving quantum chaos is the *out-of-time-ordered correlator (OTOC)* of two observables, defined as[2]

$$\mathcal{C}_{A,B}(t) := \frac{1}{2}\Big\langle \big|[A(t), B]\big|^2 \Big\rangle \tag{4.1.1}$$

measuring the evolution of the commutator of $A(t)$ and B. Starting with commuting observables, $[A, B] = 0$, this quantity initially grows, expressing how the time evolution $A(t)$ of a local observable spreads (or *scrambles*) to non-local degrees of freedom expressed by B. The moment when this growth stops is called[3] the *scrambling time* t_*. Scrambling is closely related to thermalisation, but it typically involves non-local observables B. Thus in a quantum system with *local* interactions, the thermalisation time is smaller than t_* and it is independent of the system size, while the scrambling takes place on a longer time scale until local information is shared with all degrees of freedom in the system. Beyond the scrambling time, the OTOC settles to a constant value at a larger time scale called the *relaxation time*, and then it remains essentially unchanged.

The fine distinction between thermalisation and scrambling became very popular in physics about 15 years ago motivated by the fundamental papers by Hayden and Preskill [328] and Sekino and Susskind [516] related to the black hole information paradox. The concept of the OTOC in quantum chaos

[1] In a closed quantum system with finitely many degrees of freedom the initial state is never fully lost as the stationary value still slightly depends on the original overlap $\langle AB\rangle$, but it is suppressed by N; e.g. it follows from (4.1.2) that

$$\lim_{t\to\infty} \mathbf{E}_{GUE}\langle A(t)B\rangle = \Big(1 - \frac{1}{N+1}\Big)\langle A\rangle\langle B\rangle + \frac{1}{N+1}\langle AB\rangle.$$

[2] In the physics literature, the OTOC is usually defined without the factor $1/2$. We chose it, however, for convenience.

[3] We remark that some papers use slightly different definition, here we follow the terminology of [279], [183, Section 3.3].

research was introduced in Kitaev's lectures [366] on the connection between the *Sachdev-Ye-Kitaev (SYK) model* and black holes. Owing to these fascinating connections, the physics literature on OTOC in various interacting quantum systems has become enormous; we refer the reader to the reviews [588, 279] and extensive references therein. In contrast, OTOC has basically not been considered in the mathematical literature apart from [413] that studies a very different model than our current random matrix setup.

Besides the OTOC, quantum chaos has several other signatures: the conventional one is the spectral statistics of the Hamiltonian. Following E. Wigner's groundbreaking observation, in a sufficiently chaotic quantum system the local eigenvalue statistics are given by the universal Wigner-Dyson distribution that depends only on the basic symmetries of the system. In the physics literature the spectral statistics are often described by the *spectral form factor (SFF)*, or *two point spectral correlator*, defined as $r_2(t) := \mathbf{E}|\langle \mathrm{e}^{\mathrm{i}tH}\rangle|^2$, where $\mathbf{E}$ indicates a statistical averaging over an ensemble of Hamiltonians. It is well known that the SFF tends to become universal for large times[4], while it still reflects properties of the actual quantum system (especially its density of states) for shorter times. A good physics summary is found in [183], while a recent mathematical analysis of the SFF for general Wigner matrices was given in [158]; more precise formulas are available for exactly solvable ensembles [265, 266, 267].

The OTOC is a more refined description of quantum chaos than the SFF, as it also involves observables. In particular, the SFF misses important features like the sensitivity of chaos to the locality of the observables or *early time chaos*, i.e. the exponential growth of the OTOC for certain strongly interacting systems like SYK (called *fast scramblers* [516]) versus the polynomial growth for *slow scramblers* like certain weakly chaotic systems (see, e.g. [268, Section II] and references therein). Note that the SFF can be recovered from the OTOC by averaging, either over the observables or over the unitary group in case of unitarily invariant Hamiltonians, see [183]. For example, if H is a GUE random matrix, then [183, Eq. (57)-(58)])

$$\mathbf{E}_{GUE}\langle A(t)B\rangle = \langle A\rangle\langle B\rangle + \frac{N^2 r_2(t)-1}{N^2-1}\big[\langle AB\rangle - \langle A\rangle\langle B\rangle\big], \tag{4.1.2}$$

with

$$r_2(t) = \Big(\frac{J_1(2t)}{t}\Big)^2 + \frac{1}{N} - \frac{1}{N}\Big(1-\frac{t}{2N}\Big)\mathbf{1}(t\le 2N),$$

where J_1 is the Bessel function of the first kind of order one. Thus $r_2(t)$ can be expressed from $\langle A(t)B\rangle$. A similar relation holds between the OTOC and the four point spectral correlator.

The main goal of the current paper is to give a comprehensive mathematical analysis of the OTOC with general observables A, B, when the Hamiltonian H is a Wigner matrix. Wigner matrices represent the Hamiltonian of the most chaotic quantum systems with matrix elements being independent, identically distributed (i.i.d.) random variables. In the physics literature, random matrix theory (RMT) is often used as a test case to see to what extent this relatively simple model mimics the physics of more complicated systems such as interacting many-body models (like SYK) or even models with nontrivial spatial structure (like spin chains). Spectral statistics are remarkably robust, especially the universality of the large time (so-called *plateau*) regime of the SFF has proved to be ubiquitous in many different chaotic quantum systems, in accordance with the celebrated Bohigas-Giannoni-Schmit conjecture [89]. The OTOC is a more delicate quantity and, admittedly, its several interesting features that appear in more realistic strongly coupled systems are not captured by RMT. For example, the early time exponential growth of the OTOC is not present in RMT and there is no qualitative difference between the thermalisation and scrambling times since H is mean field (see [183, Section 3.3] for a detailed analysis). The difference between chaotic and integrable systems or the effect of their possible coexistence on the OTOC (studied e.g. in [268]) are also not visible in RMT since Wigner matrices are fully chaotic. Nevertheless, RMT becomes a good description

[4] See the celebrated *slope-dip-ramp-plateau* picture, e.g. in [414].

beyond the scrambling time as claimed in [183, Section 3.3] and demonstrated in [183, Section 6]. The calculations are performed under the unitary invariance assumption and without controlling the error terms.

In the main Theorem 4.2.2 of the current work, using very different methods, we rigorously describe the behaviour of $\mathcal{C}_{A,B}(t)$ up to very long times for general Wigner matrices (no unitary invariance assumed). We mimic the locality of the observables by considering matrices A, B that are far from being full rank and track this effect throughout all error terms by using tracial norms that are sensitive to the rank. We distinguish three time regimes (see Figure 4.1.1 and Section 4.2.1); for short times (before the scrambling time t_*) we find a quadratic growth in t; for intermediate times we find that $\mathcal{C}_{A,B}(t)$ heavily depends on the ranks of A, B and their overlap AB. In particular, we detect a remarkable high peak of $\mathcal{C}_{A,B}(t)$ when the ranks of A and B are small but their overlap $\langle AB \rangle$ is still relatively large. To our knowledge, this observation may be new even in the physics literature. We then identify the *relaxation time*, t_{**}, when the OTOC *saturates*, i.e. it becomes essentially constant (equal its thermal limiting value) with small oscillations. As expected, in the last regime, after t_{**}, our model behaves universally; a qualitatively similar behaviour has been demonstrated for several more complicated systems in the physics literature both theoretically and numerically, see e.g. [183, 268, 588, 436, 279]. While for technical reasons we cannot consider infinite times, our analysis is valid for sufficiently long times to see all physically relevant features. For brevity we carry out the proofs at infinite temperature, but our methods can easily be extended to any finite temperature and we will give the corresponding formulas (see Section 4.2.2 below).

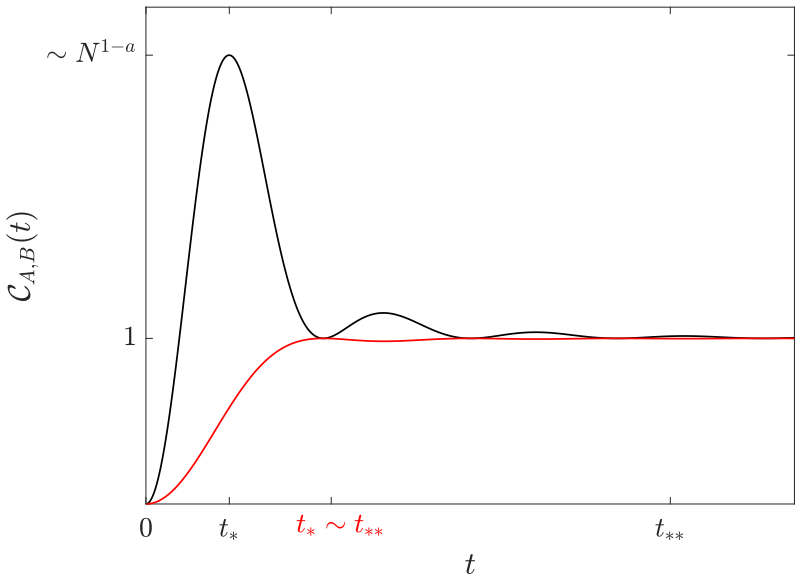

Figure 4.1.1: The two curves show the behaviour of $\mathcal{C}_{A,B}(t)$ in two different scenarios for two commuting traceless observables A, B normalized to $\langle A^2 \rangle = \langle B^2 \rangle = 1$. The black curve represents the case $A = B$ with $\mathrm{rank}(A) = N^a$, $a \in [0, 1]$ where the OTOC exhibits a large peak of size N^{1-a} around the scrambling time $t_* \sim 1$. Afterwards, it decays to its thermal limiting value (normalized to one) around the relaxation time $t_{**} \sim N^{\frac{1-a}{3}}$. The red curve represents the case where $AB = 0$. Here, both t_* and t_{**} are of order one, independent of the ranks of A and B. For more details see Section 4.2.1.

From the mathematical point of view, our work is the closest to [170, 168], where the deterministic leading term of traces of products of observables at different times, $\langle A_1(t_1)A_2(t_2)A_3(t_3)....\rangle$, were computed (see also [499, 498], where even the Gaussian fluctuations of such chains were proven). Clearly the OTOC is a special case of (the difference of two) such chains. The main novelty is that

now we use only Schatten (tracial) norms[5] of the observables in the estimates, while [170, 499, 498] used the much cruder operator norm. In particular, we can extend the time scale for the validity of our description. More importantly, note that the interesting features of the OTOC are manifested for small rank observables for which the operator norm is a major overestimate and conceptually is an overkill.

The main tool is a concentration result, called *multi-resolvent local law*, for alternating products of resolvents of random matrices and deterministic matrices. More precisely, setting $G_i := G(z_i)$ and considering deterministic matrices A_i, the main object of interest is

$$G_1 A_1 G_2 A_2 G_3 \dots A_{k-1} G_k \tag{4.1.3}$$

for some fixed k. We will show that (4.1.3) concentrates around a deterministic object and gives an upper bound on the fluctuation. The interesting regime is the local one, i.e. when $|\operatorname{Im} z_i| \ll 1$. Resolvents can then be converted to unitary time evolution $\mathrm{e}^{\mathrm{i}tH}$ by standard contour integration.

Local laws in general assert that resolvents $G(z)$ tend to become deterministic (with high probability) in the large N limit even if the spectral parameter z is very close to the real axis (typically for any $|\operatorname{Im} z| \gg N^{-1}$ in the bulk spectrum). For example, typical *single resolvent local laws* for Wigner matrices assert that, for any fixed $\xi > 0$,[6]

$$\big|\langle (G(z) - m(z))A\rangle\big| \le \frac{N^\xi \|A\|}{N\eta}, \qquad \big|\langle \boldsymbol{x}, (G(z) - m(z))\boldsymbol{y}\rangle\big| \le \frac{N^\xi \|\boldsymbol{x}\|\|\boldsymbol{y}\|}{\sqrt{N\eta}}, \tag{4.1.4}$$

with $\eta := |\operatorname{Im} z|$ for a deterministic matrix $A \in \mathbf{C}^{N\times N}$ and deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ with very high probability as N becomes large. Here, $m(z)$ is the Stieltjes-transform of the Wigner semicircle law:

$$m(z) = m_{\mathrm{sc}}(z) := \int_{\mathbf{R}} \frac{1}{(x-z)} \rho_{\mathrm{sc}}(x)\,\mathrm{d}x, \qquad \rho_{\mathrm{sc}}(x) := \frac{1}{2\pi}\sqrt{[4-x^2]_+}. \tag{4.1.5}$$

However, the deterministic limit of a multi-resolvent chain (4.1.3) is not simply $m(z_1)m(z_2)A_1A_2\dots$, i.e, one cannot mechanically replace each G by a scalar m, the actual formula is much more complicated, see (4.3.2) below. As to the accuracy of this deterministic approximation, besides N and the imaginary part of the spectral parameter, $\eta = \operatorname{Im} z$, the error term crucially depends on the appropriate norms of A_i as well as on the distinction whether A_i is traceless or not. The fact that traceless observables substantially reduce both the size of the deterministic limit of (4.1.3) and of its fluctuation has first been observed and exploited in [165], see also [168] for a comprehensive analysis of arbitrary long chains. The results in [168] were optimal both in N and η, but they all used the simplest operator norm of A_i's in the error term which is far from being optimal for low rank observables.

Concerning the more accurate norms, only very recent papers [169] and the one in Chapter 3 started deviating from the operator norm in the error terms. The main purpose of these papers was to prove the Eigenstate Thermalisation Hypothesis (ETH) for random matrices (originally posed by Deutsch in [221]) in its most optimal form, including low rank observables. Moreover, the key point in Chapter 3 was to obtain ETH also uniformly in the spectrum, including the critical edge regime. This required to focus on a local law for $\langle \operatorname{Im} GA \operatorname{Im} GA\rangle$ and to extract the smallness of order ρ^2 at the spectral edge owing to the *imaginary part* of the resolvents (here $\rho = \pi^{-1}|\operatorname{Im} m_{\mathrm{sc}}|$ is the local density of states). However, [169] and Chapter 3 *exclusively* used the Hilbert-Schmidt (HS) norm, $\langle |A|^2\rangle^{1/2}$, which caused suboptimal $(N\eta)$-dependence. For the purpose of [169] and Chapter 3, this suboptimality in $(N\eta)$ was irrelevant since the proof of ETH relies on local laws in the η–regime when $N\eta$ is practically order one.

[5] The (normalized) p-th Schatten norm of a matrix $A \in \mathbf{C}^{N\times N}$ is defined as $\langle |A|^p\rangle^{1/p}$ for $p \in [1,\infty)$, where $|A| := (AA^*)^{1/2}$.

[6] Traditionally [249, 369, 88], local laws did not consider arbitrary test matrix A, but only $A = I$ or special rank one projections $A = \boldsymbol{y}\boldsymbol{x}^*$ leading to the *isotropic local law* in (4.1.4). General A were included later, e.g. in [243].

In contrast to [169] and Chapter 3, for studying the OTOC at shorter times, we need local laws in the regime where $N\eta$ is large (since $\eta \sim 1/t$ dictated by the contour integration and $t \ll N$). In the current paper we prove local laws that are optimal both in N and η *and* use the optimal Schatten norms of the observables. Especially, this allows for a more accurate description of the OTOC in the physically relevant regime of small rank observables.[7] We, however, do not need to track the ρ-dependence or pay attention to the imaginary parts. Therefore, the current work and Chapter 3 are complementary; they effectively handle very different aspects of the local law. While the fundamental idea of these two works is similar, both use the *Zigzag strategy* described in Section 4.4, the actual proofs are quite different. The main focus in Chapter 3 was to design and handle contour integral representations that allowed us to reduce every estimate to resolvent chains involving only $\operatorname{Im} G$'s. In the current paper $\operatorname{Im} G$ plays no role, but we need to track the precise Schatten norms very carefully.

To illustrate the strength of our new result in comparison with the previous bounds, we present the following three estimates for the simplest case $k = 2$, with $A_1 = A_2 = A$, $z_1 = z_2$ in the bulk, and ignoring N^{ξ}-factors for some arbitrarily small $\xi > 0$:

$$\left|\langle GAGA\rangle - m^2\langle A^2\rangle\right| \lesssim \begin{cases} \dfrac{\|A\|^2}{N\eta} & \text{from [168, Theorem 2.5];} \\ \sqrt{N\eta}\dfrac{\langle|A|^2\rangle}{N\eta} & \text{from [169, Theorem 2.2];} \\ \dfrac{\langle|A|^2\rangle}{N\eta} + \dfrac{\langle|A|^4\rangle^{1/2}}{N\sqrt{\eta}} & \text{from Theorem 4.3.3.} \end{cases} \tag{4.1.6}$$

Note that our current result in the last line of (4.1.6) implies both previous results since $\langle|A|^4\rangle^{1/2} \le \sqrt{N}\langle|A|^2\rangle$ and $\langle|A|^p\rangle^{1/p} \le \|A\|$. While the former bound saturates for low rank observables, the latter saturates for high rank ones. Therefore, our new result with Schatten norms optimally interpolates between these two, at least in the bulk regime.

Notations

By $\lceil x\rceil := \min\{m \in \mathbf{Z}\colon m \ge x\}$ and $\lfloor x\rfloor := \max\{m \in \mathbf{Z}\colon m \le x\}$ we denote the upper and lower integer part of a real number $x \in \mathbf{R}$. For $k \in \mathbf{N}$ we set $[k] := \{1, ..., k\}$, and $\langle A\rangle := d^{-1}\operatorname{Tr}(A)$, $d \in \mathbf{N}$, for the normalised trace of a $d \times d$-matrix A, while $\operatorname{rk} A \equiv \operatorname{rank} A$ denotes its rank. For positive quantities f, g we write $f \lesssim g$ resp. $f \gtrsim g$ and mean that $f \le Cg$ resp. $f \ge cg$ for some N-independent constants $c, C > 0$ that may depend only on the basic control parameters C_p, see (4.2.1) in Assumption 4.2.1 below. Moreover, we will also write $f \sim g$ in case that $f \lesssim g$ and $g \lesssim f$.

We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, for some $N \in \mathbf{N}$, and define

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \sum_i \bar{x}_i y_i\,, \qquad A_{\boldsymbol{xy}} := \langle \boldsymbol{x}, A\boldsymbol{y}\rangle\,.$$

Matrix entries are indexed by lower case Roman letters $a, b, c, ..., i, j, k, ...$ from the beginning or the middle of the alphabet and unrestricted sums over those are always understood to be over $\{1, ..., N\}$.

We will use the concept *'with very high probability'*, meaning that for any fixed $D > 0$, the probability of an N-dependent event is bigger than $1 - N^{-D}$ for all $N \ge N_0(D)$. Also, we will use the convention that $\xi > 0$ denotes an arbitrarily small positive exponent, independent of N. Moreover, we introduce the common notion of *stochastic domination* (see, e.g., [241]): For two families

$$X = \left(X^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right) \text{ and } Y = \left(Y^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right)$$

[7] We point out that, in principle, also the multi-resolvent local laws in the earlier papers [168, 169] and the one in Chapter 3 allow studying the OTOC. However, since these papers exclusively use crude operator norms or HS norms in the error terms, the accessible times and/or observables are strongly restricted compared to the current paper. See Section 4.2.1 for a more detailed comparison in the context of two examples.

of non-negative random variables indexed by N, and possibly an additional parameter u from a parameter space $U^{(N)}$, we say that X is stochastically dominated by Y, if for all $\epsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\epsilon} Y^{(N)}(u)\right] \leq N^{-D}$$

for large enough $N \geq N_0(\epsilon, D)$. In this case we write $X \prec Y$. If for some complex family of random variables we have $|X| \prec Y$, we write $X = O_{\prec}(Y)$.

4.2 Main results

We consider $N \times N$ Wigner matrices W, i.e. W is a random real symmetric or complex Hermitian matrix $W = W^*$ with independent entries (up to the Hermitian symmetry) and with single entry distributions $w_{aa} \stackrel{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{d}}$, and $w_{ab} \stackrel{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{od}}$, for $a > b$. The random variables $\chi_{\mathrm{d}}, \chi_{\mathrm{od}}$ satisfy the following assumptions.[8]

Assumption 4.2.1. *The off-diagonal distribution χ_{od} is a real or complex centered random variable, $\mathbf{E}\,\chi_{\mathrm{od}} = 0$, satisfying $\mathbf{E}\,|\chi_{\mathrm{od}}|^2 = 1$. The diagonal distribution is a real centered random variable, $\mathbf{E}\,\chi_{\mathrm{d}} = 0$. Furthermore, we assume the existence of high moments, i.e. for any $p \in \mathbf{N}$ there exists $C_p > 0$ such that*

$$\mathbf{E}\left[|\chi_{\mathrm{d}}|^p + |\chi_{\mathrm{od}}|^p\right] \leq C_p\,. \tag{4.2.1}$$

Our main result, Theorem 4.2.2 below, concerns the Heisenberg time evolution $A(t) := \mathrm{e}^{\mathrm{i}Wt} A \mathrm{e}^{-\mathrm{i}Wt}$ of a fixed deterministic self-adjoint observable $A = A^* \in \mathbf{C}^{N\times N}$ governed by the Wigner matrix W. More precisely, we consider the *out-of-time-ordered correlator (OTOC)*

$$\mathcal{C}_{A,B}(t) := \frac{1}{2}\left\langle \left|[A(t), B]\right|^2 \right\rangle = \left\langle A(t)^2 B^2 \right\rangle - \left\langle A(t) B A(t) B \right\rangle \tag{4.2.2}$$

with another self-adjoint observable $B = B^* \in \mathbf{C}^{N\times N}$, consisting of a two-point and a four-point[9] part,

$$\mathcal{D}_{A,B}(t) := \left\langle A(t)^2 B^2 \right\rangle, \quad \mathcal{F}_{A,B}(t) := \left\langle A(t) B A(t) B \right\rangle, \quad \text{respectively.} \tag{4.2.3}$$

In the formulation of Theorem 4.2.2, a key role is played by the Fourier transform of the semicircular density (4.1.5),

$$\varphi(t) := \widehat{\rho_{\mathrm{sc}}}(t) = \int_{-2}^{2} \mathrm{d}x \rho_{\mathrm{sc}}(x) \mathrm{e}^{\mathrm{i}xt} = \frac{J_1(2t)}{t}\,, \quad t \in \mathbb{R}, \tag{4.2.4}$$

where we recall that J_1 is the Bessel function of the first kind of order one. We note that by standard asymptotics of the Bessel functions on the real line, it holds that

$$J_1(s) = \begin{cases} -\operatorname{sgn}(s)\cos\left(|s| + \dfrac{\pi}{4}\right)\sqrt{\dfrac{2}{\pi|s|}} + \mathcal{O}\left(\dfrac{1}{|s|^{3/2}}\right) & s \to \pm\infty\,, \\ \dfrac{s}{2} - \dfrac{1}{2}\left(\dfrac{s}{2}\right)^3 + \mathcal{O}(|s|^5) & s \to 0\,. \end{cases} \tag{4.2.5}$$

For simplicity, we formulate our main result only for traceless matrices, $\langle A \rangle = \langle B \rangle = 0$ and at infinite temperature. For general observables, see Remark 4.2.3 and for finite temperature, see Section 4.2.2.

[8] By inspecting our proof, it is easy to see that actually we do not need to assume that the off-diagonal entries of W are identically distributed. We only need that they all have the same second moments, but higher moments can be different.

[9] We remark that some papers (see, e.g., [485]) refer to the four-point part $\mathcal{F}_{A,B}(t)$ alone as the OTOC.

Theorem 4.2.2 (OTOC for Wigner matrices)**.** *Let W be a Wigner matrix satisfying Assumption 4.2.1 and let $A, B \in \mathbf{C}^{N\times N}$ be self-adjoint deterministic matrices which are traceless, $\langle A\rangle = \langle B\rangle = 0$. Fix any $\epsilon > 0$. Then, the OTOC* (4.2.2) *satisfies*

$$\begin{aligned}\mathcal{C}_{A,B}(t) =& \langle A^2\rangle\langle B^2\rangle\big[1-\varphi(t)^2\big] + 2\langle AB\rangle^2\varphi(t)^2\big[\varphi(2t)-\varphi(t)^2\big] \\ &+ \langle A^2B^2\rangle\,\varphi(t)^2 - \langle ABAB\rangle\varphi(t)^4 + \mathcal{O}_{\prec}\big(\mathcal{E}_{A,B}(t,N)\big)\end{aligned} \tag{4.2.6}$$

with an error term $\mathcal{E} \equiv \mathcal{E}_{A,B}(t,N)$ given by

$$\mathcal{E} := \mathrm{e}^{t/N^{1/2-\epsilon}}\left(\frac{|t|^4\langle A^2\rangle^2}{N} + \frac{|t|\langle A^8\rangle^{1/2}}{N}\right)^{1/2}\left(\frac{|t|^4\langle B^2\rangle^2}{N} + \frac{|t|\langle B^8\rangle^{1/2}}{N}\right)^{1/2}. \tag{4.2.7}$$

The proof of Theorem 4.2.2 is given in Section 4.3.3 below. It is based on a novel *multi-resolvent local law* with error terms involving optimal Schatten norms, see Theorem 4.3.3.

Remark 4.2.3. *We have several comments on Theorem 4.2.2:*

(i) [Non-traceless observables] *For general observables A, B, we can decompose them into a tracial and traceless part, $A =: \langle A\rangle + \mathring{A}$, and similarly for B. The tracial parts, $\langle A\rangle$ and $\langle B\rangle$, then commute with the unitary time evolution $\mathrm{e}^{\mathrm{i}tW}$ and one straightforwardly obtains a result similar to Theorem 4.2.2 (see also Remark 4.3.5).*

(ii) [Variance of fluctuations] *The size of the fluctuations around the deterministic leading term in* (4.2.6)*, i.e. the variance of $\mathcal{C}_{A,B}(t)$, is explicitly computable, following the arguments leading to [499, Lemma 2.5]. The result is expressible purely in terms of Schatten norms of A and B (cf. [499, Lemma 2.5 and Definition 3.4]), however in [499, 498] the error terms are still in terms of crude operator norms.*

(iii) [Gaussianity] *It is also possible to prove Gaussianity of the fluctuations of $\mathcal{C}_{A,B}(t)$ (cf. [499, Theorem 2.7 and Corollary 2.12]) by showing an approximate Wick theorem for resolvent chains, similarly to [499, Theorem 3.6]. However, we refrain from doing so for brevity of this paper.*

4.2.1 Physical interpretation of Theorem 4.2.2 by two examples

We will now discuss the behavior of $\mathcal{C}_{A,B}(t)$ in two exemplary and extreme situations of observables A, B. In the first example we will set the two observables identical, in the second we will assume that their product vanishes. More concretely, we define

$$\begin{aligned} A &= N^{\frac{1-a}{2}}\operatorname{diag}(1,-1,...,1,-1,0,...,0)\,, \qquad a \in [0,1)\,, \\ B &= N^{\frac{1-b}{2}}\operatorname{diag}(0,...,0,1,-1,...,1,-1)\,, \qquad b \in [0,1)\,, \end{aligned} \tag{4.2.8}$$

in such a way that $AB = BA = 0$, and $\langle A\rangle = \langle B\rangle = 0$ as well as $\langle A^2\rangle = \langle B^2\rangle = 1$, i.e. A (resp. B) contains N^a-many (resp. N^b-many) non-zero entries on the diagonal.

Example 1. For the first example, we have, using $\langle A^2\rangle = 1$,

$$\begin{aligned}\mathcal{C}_{A,A}(t) =& 1-\varphi(t)^2\big\{1-2\varphi(2t)+2\varphi(t)^2\big\} \\ &+ \langle A^4\rangle\big[\varphi(t)^2-\varphi(t)^4\big] + \mathcal{O}_{\prec}\big(\mathcal{E}_{A,A}(t,N)\big)\,.\end{aligned} \tag{4.2.9}$$

Example 2. For the second example, we have, using $\langle A^2\rangle = \langle B^2\rangle = 1$,

$$\mathcal{C}_{A,B}(t) = 1-\varphi(t)^2 + \mathcal{O}_{\prec}\big(\mathcal{E}_{A,B}(t,N)\big)\,. \tag{4.2.10}$$

The key features of these two examples (4.2.9)–(4.2.10) are briefly summarized in Table 4.1. Ignoring the respective error terms, $\mathcal{C}_{A,A}(t)$ and $\mathcal{C}_{A,B}(t)$ are schematically depicted in Figure 4.1.1. Now we comment on each time regime.

(i) [**Short-time regime**] By the asymptotics in (4.2.5), we have the short-time asymptotic

$$\mathcal{C}_{A,A}(t) = t^2\big(\langle A^4\rangle - 1\big) + \mathcal{O}\big(|t|^4\langle A^4\rangle\big) + \mathcal{O}_{\prec}\big(\mathcal{E}\big) \qquad \text{(Example 1)}\,,$$
$$\mathcal{C}_{A,B}(t) = t^2 + \mathcal{O}\big(|t|^4\big) + \mathcal{O}_{\prec}\big(\mathcal{E}\big) \qquad \text{(Example 2)}\,.$$

Note that $\varphi(t)$ in (4.2.9)–(4.2.10) depends polynomially on t (cf. (4.2.4)–(4.2.5)) up to the scrambling time t_*. This shows that the OTOC for Wigner matrices does *not* exhibit the exponential increase $\sim \mathrm{e}^{2\lambda t}$ between the perturbative regime and t_*, observed for quantum systems with a *classically* chaotic analogue, where λ is the Lyapunov exponent of the classical system. In fact, the indicated polynomial dependence of the OTOC (4.2.2) up to t_* has already been observed for certain quantum chaotic systems *without* a classical analogue (e.g. for some spin-$\frac{1}{2}$ chains [268]).

(ii) [**Scrambling time**] The monotonous growth of both $\mathcal{C}_{A,A}(t)$ and $\mathcal{C}_{A,B}(t)$ stops at a time of order one (using elementary properties of φ from (4.2.4)), hence the scrambling time is $t_* \sim 1$. However, the maximally attained value strongly differs for the two examples: While $\mathcal{C}_{A,A}(t_*) \sim N^{1-a}$ heavily depends on a (i.e. the rank of A), the peak of $\mathcal{C}_{A,B}(t_*) \sim 1$ is independent of the ranks of A and B.

(iii) [**Intermediate regime up to the relaxation time**] The following interval of intermediate times is characterized by a decay of the OTOC (4.2.6) towards its thermal limiting value $\langle A^2\rangle\langle B^2\rangle = 1$ up to the relaxation time t_{**}. This regime is also quite different for the two examples (4.2.9)–(4.2.10): While for Example 1 the interval of intermediate times is given by $t \in [t_*, t_{**}]$ where $t_{**} \sim N^{(1-a)/3}$, in Example 2 the relaxation time $t_{**} \sim 1$ is comparable with the scrambling time. However, for technical reasons, the *entire* interval of intermediate times is only accessible if $a > 5/11$ for Example 1, and $a + b > 2/3$ for Example 2, otherwise the leading terms in (4.2.9)–(4.2.10) become smaller than their respective error term. In comparison, computing the OTOC with the operator norm in the error terms [168, Corollary 2.7], would lead to the (more restrictive) conditions $a > 5/8$ for Example 1, and $a + b > 1$ for Example 2.

(iv) [**Long-time regime**] In the consecutive long-time regime, i.e. $t \gg N^{(1-a)/3}$ for Example 1 and $t \gg 1$ for Example 2, we find the OTOC (4.2.6) to concentrate around its thermal limiting value $\langle A^2\rangle\langle B^2\rangle = 1$ with small oscillations. This confirms the expectation, that the OTOC in strongly chaotic systems exhibits only small fluctuations for long times. These are accessible up to

$$t \le N^{\min\{\frac{1}{4}, \frac{3a-1}{2}\}-\epsilon} \qquad \text{and} \qquad t \le N^{\min\{\frac{1}{4}, \frac{3a+1}{10}, \frac{3b+1}{10}, \frac{3(a+b)-2}{4}\}-\epsilon}$$

for Example 1 and 2, respectively. Again, in comparison with Theorem 4.2.2, the operator norm error terms from [168, Cor. 2.7] would lead to the constraints $t \le N^{\frac{2a-1}{2}-\epsilon}$ for Example 1, and $t \le N^{\frac{a+b-1}{2}-\epsilon}$ for Example 2.

4.2.2 Finite temperature case

Theorem 4.2.2 can easily be extended to the case of finite temperature, $\beta = 1/T > 0$. The OTOC (4.2.2) now is given by

$$\mathcal{C}^{(\beta)}_{A,B}(t) := \frac{1}{2}\frac{\mathrm{Tr}\big[|[A(t), B]|^2\mathrm{e}^{-\beta W}\big]}{Z} \tag{4.2.11}$$

Ex. 1: $A = B$, $\operatorname{rk} A = N^a$	Ex. 2: $AB = 0$, $\operatorname{rk} A = N^a$, $\operatorname{rk} B = N^b$
$\mathcal{C}_{A,A}(t) \sim t^2 \left(\langle A^4 \rangle - \langle A^2 \rangle^2 \right)$	$\mathcal{C}_{A,B}(t) \sim t^2 \langle A^2 \rangle \langle B^2 \rangle$
$t_* \sim 1$ and $\mathcal{C}_{A,A}(t_*) \sim N^{1-a}$	$t_* \sim 1$ and $\mathcal{C}_{A,B}(t_*) \sim 1$
full access if $a > 5/11$	full access if $a + b > 2/3$
$t_{**} \sim N^{\frac{1-a}{3}}$ and $\mathcal{C}_{A,A}(t_{**}) \sim \langle A^2 \rangle^2$	$t_{**} \sim 1$ and $\mathcal{C}_{A,B}(t_{**}) \sim \langle A^2 \rangle \langle B^2 \rangle$
up to $t \leq N^{\min\{\frac{1}{4}, \frac{3a-1}{2}\}-\epsilon}$	up to $t \leq N^{\min\left\{\frac{1}{4}, \frac{3a+1}{10}, \frac{3b+1}{10}, \frac{3(a+b)-2}{4}\right\}-\epsilon}$

Table 4.1: Overview of the two examples (4.2.9) and (4.2.10).

with partition function $Z = \operatorname{Tr}\left[\mathrm{e}^{-\beta W} \right]$. In this case, the analog of (4.2.6) in the regime[10] $\beta \ll \log N$ reads

$$\begin{aligned} \mathcal{C}_{A,B}^{(\beta)}(t) &= \langle A^2 \rangle \langle B^2 \rangle \left[1 - \varphi(t)^2 \right] + \langle A^2 B^2 \rangle \varphi(t)^2 - \langle ABAB \rangle \frac{\varphi(t)^3 \operatorname{Re}\left[\varphi(t + \mathrm{i}\beta) \right]}{\varphi(\mathrm{i}\beta)} \\ &+ \langle AB \rangle^2 \frac{\varphi(t)}{\varphi(\mathrm{i}\beta)} \operatorname{Re}\Big[\varphi(2t)\varphi(t + \mathrm{i}\beta) + \varphi(t)\varphi(2t + \mathrm{i}\beta) - 2\varphi(t)^2 \varphi(t + \mathrm{i}\beta) \Big] + \mathcal{O}_{\prec}(\mathcal{E}) \,, \end{aligned} \tag{4.2.12}$$

where $\mathcal{E}$ is from (4.2.7). Here $\varphi(z)$ is the complex extension of $\varphi(t)$ for $z \in \mathbf{C}$; note that $\varphi(z)$ is generically complex but $\varphi(\mathrm{i}\beta)$ is real.

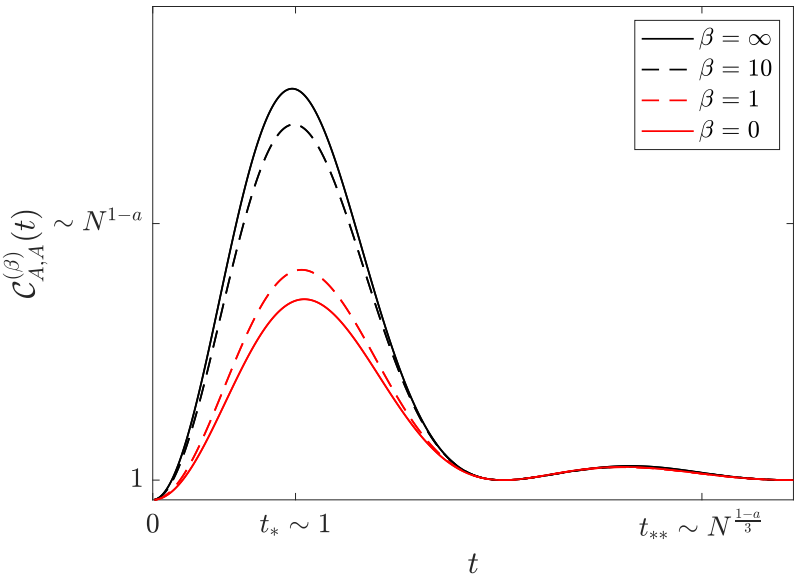

Figure 4.2.1: Depicted are four curves illustrating the influence of $\beta = 1/T$ on the OTOC $\mathcal{C}_{A,A}^{(\beta)}(t)$ up to intermediate times for Example 1 from Section 4.2.1 (i.e. normalized to $\langle A^2 \rangle = 1$ with $\operatorname{rank} A = N^{\frac{1-a}{2}}$ and $a \in [0, 1]$). As β increases, the characteristic rank-dependent peak of size $\sim N^{1-a}$ around the scrambling time $t_* \sim 1$ becomes more pronounced and very slightly shifted to the left.

Moreover, using the asymptotics of the Bessel function in the complex plane, we have, as $|z| \to \infty$ with $|\arg z| < \pi$,

$$\varphi(z) = -\sqrt{\frac{1}{\pi z^3}} \left(\cos\left(2z + \frac{\pi}{4} \right) + \mathrm{e}^{2|\operatorname{Im} z|} \mathcal{O}\left(|z|^{-1} \right) \right) .$$

[10] We restrict to this regime for simplicity, as all the error terms can be absorbed into $\prec$.

In particular, the thermal limiting value of $\mathcal{C}^{(\beta)}_{A,B}(t)$ is *independent* of β at least in our regime $\beta \ll \log N$. Note that for much larger $\beta \gtrsim \sqrt{N}$ physics calculations predict a temperature dependence of the thermal limiting value of the OTOC , see [436, Eqs. (3.8)–(3.9)].

However, before the long-time regime and neglecting the error term in (4.2.12), we find a strong dependence of the OTOC on temperature for Example 1 from Section 4.2.1 (cf. (4.2.8)–(4.2.9)) as illustrated in Figure 4.2.1. In contrast to that, for Example 2 from Section 4.2.1 (cf. (4.2.8) and (4.2.10)), the whole OTOC curve (as depicted in Figure 4.1.1) is independent of temperature at any time. This is because all the β dependent terms in (4.2.12) drop out since $AB = BA = 0$.

4.3 Proof of Theorem 4.2.2: Multi-resolvent local law with Schatten norms

Theorem 4.2.2 relies on a new multi-resolvent local law for alternating chains (4.1.3) with deterministic matrices via simple contour integration (see Section 4.3.3). Its main novelty, using Schatten norms for the observables and still keeping optimality in N and η, has already been explained in the Introduction. After collecting some preliminary information, in Section 4.3.2 we present our new local law, Theorem 4.3.3, then in Section 4.3.3 we quickly complete the proof of Theorem 4.2.2. Starting from Section 4.4 we will focus on the proof of Theorem 4.3.3.

4.3.1 Preliminaries on the deterministic approximation

Before stating our main technical result, we introduce some additional notation. Given a non-crossing partition π of the set $[k] := \{1, \dots, k\}$ arranged in cyclic order, the partial trace pTr_π is defined as

$$\mathrm{pTr}_\pi(B_1, \dots, B_{k-1}) := \prod_{S \in \pi \setminus \mathfrak{B}(k)} \left\langle \prod_{j \in S} B_j \right\rangle \prod_{j \in \mathfrak{B}(k) \setminus \{k\}} B_j, \tag{4.3.1}$$

with $\mathfrak{B}(k) \in \pi$ denoting the unique block containing k. Then, for generic B_i's, the deterministic approximation of (4.1.3) is given by (see [170, Theorem 3.4])

$$M_{[1,k]} = M(z_1, B_1, \dots, B_{k-1}, z_k) := \sum_{\pi \in \mathrm{NC}([k])} \mathrm{pTr}_{K(\pi)}(B_1, \dots, B_{k-1}) \prod_{S \in \pi} m_\circ[S], \tag{4.3.2}$$

where $\mathrm{NC}([k])$ denotes the non-crossing partitions of the set $[k]$, and $K(\pi)$ denotes the Kreweras complement of π (see [170, Definition 2.4] and [384]). Furthermore, for any subset $S \subset [k]$ we define $m[S] := m_{\mathrm{sc}}[\boldsymbol{z}_S]$ as the iterated divided difference of m_{sc} evaluated in $\boldsymbol{z}_S := \{z_i : i \in S\}$, and by $m_\circ[\cdot]$ denote the free-cumulant transform of $m[\cdot]$ which is uniquely defined implicitly by the relation

$$m[S] = \sum_{\pi \in \mathrm{NC}(S)} \prod_{S' \in \pi} m_\circ[S'], \qquad \forall S \subset [k], \tag{4.3.3}$$

e.g. $m_\circ[i,j] = m[\{i,j\}] - m[\{i\}]m[\{j\}]$. Throughout the paper, we will often use the fact that $m_{\mathrm{sc}}[\boldsymbol{z}_S]$ can be written as follows

$$m_{\mathrm{sc}}[\{z_i : i \in S\}] = \int_{-2}^{2} \rho_{\mathrm{sc}}(x) \prod_{i \in S} \frac{1}{(x - z_i)} \mathrm{d}x. \tag{4.3.4}$$

In order to formulate bounds on the deterministic approximation M as well as the local law bounds in a concise way, we introduce the following *weighted Schatten norms.*

Definition 4.3.1 (ℓ-weighted Schatten norms)**.** *For $B \in \mathbf{C}^{N \times N}$, $\ell > 0$ and $p \in [2, \infty]$, we define the* ℓ-weighted $(2,p)$-Schatten norm *as*

$$\|\!|B|\!\|_{p,\ell} := \frac{\langle |B|^2 \rangle^{1/2}}{\ell^{1/2}} + \frac{\langle |B|^p \rangle^{1/p}}{\ell^{1/p}}, \quad p < \infty, \qquad \|\!|B|\!\|_{\infty,\ell} := \frac{\langle |B|^2 \rangle^{1/2}}{\ell^{1/2}} + \|B\|. \tag{4.3.5}$$

By elementary inequalities, we have, for $2 \le p \le q \le \infty$,

$$\|B\|_{p,\ell} \lesssim \|B\|_{q,\ell}, \qquad \|B\|_{q,\ell} \lesssim \left[1 + (N\ell)^{\frac{q-p}{pq}}\right] \|B\|_{p,\ell}. \tag{4.3.6}$$

The next lemma gives a bound on the deterministic approximation $M_{[1,k]}$ with traceless observables; its proof is given in Appendix 4.A. We will henceforth follow the convention that the letter B is used for generic matrices, while A is reserved for traceless ones.

Lemma 4.3.2 (M-bound)**.** *Fix* $k \ge 1$. *Consider spectral parameters* $z_1, ..., z_k \in \mathbf{C} \setminus \mathbf{R}$ *and traceless matrices* $A_1, ..., A_k \in \mathbf{C}^{N\times N}$. *Denoting* $\ell := \min_{j\in[k]} \eta_j\rho_j$ *with* $\eta_j := |\mathrm{Im}\, z_j|$ *and* $\rho_j := \pi^{-1}|\mathrm{Im}\, m(z_j)|$, *it holds that*[11]

$$\left|\langle M_{[1,k]}(z_1, A_1, ..., z_k)A_k\rangle\right| \lesssim \mathbf{1}(k \ge 2)\ \ell \prod_{i\in[k]} \|A_i\|_{k,\ell}, \tag{4.3.7}$$

where $\|A_i\|_{k,\ell}$ *was defined in* (4.3.5).

It is easy to see that the bound (4.3.7) is optimal for k even (cf. Footnote 11) and in case that $A_i = A$ for all $i \in [k]$ and $z_i \in \{z, \bar{z}\}$ for some $z \in \mathbf{C} \setminus \mathbf{R}$ in the *bulk*, i.e. $\mathrm{Re}\, z \in [-2+\delta, 2-\delta]$ for some $\delta > 0$.

4.3.2 Multi-resolvent local law

We now formulate our main technical result, an *averaged multi-resolvent local law with Schatten norms*, in Theorem 4.3.3; its proof is given in Section 4.4. The corresponding *isotropic multi-resolvent local law* will be formulated in Theorem 4.6.2 later.

Theorem 4.3.3 (Averaged multi-resolvent local law with Schatten norms)**.** *Let* W *be a Wigner matrix satisfying Assumption 4.2.1, and fix* $k \in \mathbf{N}$. *Consider spectral parameters* $z_1, ..., z_k \in \mathbf{C} \setminus \mathbf{R}$, *the associated resolvents* $G_j = G(z_j) := (W - z_j)^{-1}$, *and traceless deterministic matrices* $A_1, ..., A_k \in \mathbf{C}^{N\times N}$. *Denote*

$$\eta_j := |\mathrm{Im}\, z_j|, \qquad \rho_j := \pi^{-1}|\mathrm{Im}\, m(z_j)|, \qquad \ell := \min_{j\in[k]} \eta_j\rho_j,$$

and let $M_{[1,k]}$ *be given in* (4.3.2). *Then, for any fixed* $\epsilon > 0$ *and recalling* (4.3.5), *we have*

$$\left|\langle G_1A_1...G_kA_k\rangle - \langle M_{[1,k]}A_k\rangle\right| \prec \frac{1}{N} \prod_{i\in[k]} \|A_i\|_{2k,\ell}, \tag{4.3.8}$$

uniformly in spectral parameters satisfying $N\ell \ge N^\epsilon$ *and* $\max_j |z_j| \le N^{1/\epsilon}$.

Notice that, in the bulk regime, ℓ essentially agrees with $\eta := \min_{i\in[k]} |\mathrm{Im}\, z_i|$, since $\rho_i \sim 1$. However, for (4.3.8) to be valid in any regime, the standard condition $N\eta \gg 1$ for the local law in the bulk needs to be replaced by $N\ell \gg 1$. This ensures that we are at the *mesoscopic scale*, i.e. there are many eigenvalues in a local window of size η_i around each $\mathrm{Re}\, z_i$.

As already mentioned in the Introduction around (4.1.6), Theorem 4.3.3 unifies and improves the previous local laws (in the bulk spectrum) with an error term involving either the operator norm (see [168, Eq. (2.11a) in Theorem 2.5]) or Hilbert-Schmidt norm (see [169, Theorem 2.2]). This follows by estimating (in the relevant $\ell \lesssim 1$ regime)

$$\|A_i\|_{2k,\ell} \lesssim \begin{cases} \dfrac{\|A_i\|}{\ell^{1/2}} & \text{for [168, Eq. (2.11a), Thm. 2.5]} \\ (N\ell)^{\frac{k-1}{2k}} \dfrac{\langle |A_i|^2\rangle^{1/2}}{\ell^{1/2}} & \text{for [169, Theorem 2.2]} \end{cases} \tag{4.3.9}$$

in (4.3.8) for $N\ell > 1$ and every $i \in [k]$ by means of elementary inequalities.

[11] We point out that the bound can be improved to $\ell \prod_{i\in[k]} \left(\ell^{(1-k)/(2k)}\langle |A_i|^3\rangle^{1/k}\langle |A_i|^2\rangle^{(k-3)/(2k)} + \ell^{-1/k}\langle |A_i|^k\rangle^{1/k}\right)$ in case of odd $k \ge 3$, but we do not follow this improvement for brevity and ease of notation.

Remark 4.3.4 (Optimality). *The bound* (4.3.8) *is optimal in case that* $A_i = A$ *for all* $i \in [k]$ *and* $z_i \in \{z, \bar{z}\}$ *for some* $z \in \mathbf{C} \setminus \mathbf{R}$ *with* $|\operatorname{Im} z| \geq N^{-1+\epsilon}$ *in the bulk. For* W *being GUE, this can easily be checked by spectral decomposition of the resolvents and using Weingarten calculus [179] for the Haar-distributed eigenvectors. The so-called "ladder diagram" gives the (first) Hilbert-Schmidt term in the estimate*

$$\left|\langle (GA)^k \rangle - \langle M_{[1,k]} A \rangle\right| \prec \frac{\|A\|_{2k,\ell}^k}{N} \sim \frac{\langle |A|^2 \rangle^{k/2}}{N\ell^{k/2}} + \frac{\langle |A|^{2k} \rangle^{1/2}}{N\ell^{1/2}}\,. \tag{4.3.10}$$

Interestingly, in some regimes the second term in (4.3.10) *(non ladder diagram) gives the main contribution, defying the general belief that always the ladder diagrams are the leading terms.*

Remark 4.3.5 (Extensions). *In Theorem 4.3.3, each* G *may also be replaced by a product of* G*'s or even* $|G|$*'s (absolute value). We refrain from stating these results explicitly, as they can be easily obtained from appropriate integral representations (see* (4.5.23) *and* (4.5.27) *below). We formulate only the following example for* $k = 2$ *and identical observables* $B_1 = B_2$ *for illustration. Let* $B \in \mathbf{C}^{N\times N}$ *be an arbitrary (i.e. not necessarily traceless) deterministic matrix. Then, decomposing* $B = \mathring{B} + \langle B \rangle$*, we have*

$$\begin{aligned} \langle G_1 B G_2 B \rangle &= \frac{m_1 m_2 \langle B \rangle^2}{1 - m_1 m_2} + m_1 m_2 \langle \mathring{B}^2 \rangle \\ &\quad + \mathcal{O}_\prec\left(\frac{|\langle B \rangle|^2}{N\ell^2} + \frac{|\langle B \rangle| \langle |\mathring{B}|^2 \rangle^{1/2}}{N\ell^{3/2}} + \frac{\langle |\mathring{B}|^2 \rangle}{N\ell} + \frac{\langle |\mathring{B}|^4 \rangle^{1/2}}{N\ell^{1/2}} \right). \end{aligned} \tag{4.3.11}$$

The statement for different observables B_1, B_2 *is analogous.*

4.3.3 Out-of-time-ordered correlators: Proof of Theorem 4.2.2

In order to prove Theorem 4.2.2, we distinguish the following three time regimes,

$$\text{(i) } |t| < 1\,, \qquad \text{(ii) } 1 \leq |t| \leq N^{(1-\epsilon)/2}\,, \quad \text{and} \quad \text{(iii) } |t| > N^{(1-\epsilon)/2} \tag{4.3.12}$$

for some small fixed $\epsilon > 0$ from Theorem 4.2.2. In the following, we focus on the most complicated case (ii) and discuss the other two cases briefly at the end of this section. Since the arguments in this section are fairly standard, we will leave some irrelevant technical details to the reader.

For $1 \leq |t| \leq N^{(1-\epsilon)/2}$, we employ the integral representation

$$\mathrm{e}^{\pm \mathrm{i} t W} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \mathrm{e}^{\pm \mathrm{i} t z} G(z) \mathrm{d}z \tag{4.3.13}$$

with the contour

$$\Gamma \equiv \Gamma_{t,R} := \partial\big([-R, R] \times \mathrm{i}[-|t|^{-1}, |t|^{-1}]\big) \tag{4.3.14}$$

parametrized counterclockwise and the parameter R chosen as $R = N^\kappa$ for some arbitrarily small (but fixed) $\kappa > 0$. In this way, we can write the four-point part $\mathcal{F}_{A,B}(t)$ in (4.2.3) of the OTOC (4.2.2) as

$$\mathcal{F}_{A,B}(t) = \left(\prod_{i\in[4]} \oint_\Gamma \frac{\mathrm{d}z_i}{2\pi \mathrm{i}} \right) \mathrm{e}^{\mathrm{i}t(z_1+z_3-z_2-z_4)} \langle G(z_1) A G(z_2) B G(z_3) A G(z_4) B \rangle\,. \tag{4.3.15}$$

For the part, where *all* z_i's, $i \in [4]$, run on the horizontal parts of the contour Γ, we can replace $\langle G(z_1)A...G(z_4)B\rangle$ by $\langle M(z_1, A, ..., z_4)B\rangle$ at the price of an error $\mathcal{O}_\prec(\mathcal{E})$ with $\mathcal{E} = \mathcal{E}_{A,B}(t)$ from (4.2.7) by means of Theorem 4.3.3 for $k = 4$. Here, we used $\ell \gtrsim N^{-2\kappa}|t|^{-2}$, which follows by $\rho(z_i) \gtrsim |\operatorname{Im} z_i|(1 + \operatorname{dist}(z_i, [-2, 2])^2)^{-1}$, and that $\kappa > 0$ is arbitrarily small and hence N^κ can be absorbed into $\prec$.

If *all* the z_i's run on the vertical parts of Γ, we employ the global law version of our main result, Proposition 4.4.1 below, to replace $\langle G(z_1)A...G(z_4)B\rangle$ by $\langle M(z_1,A,...,z_4)B\rangle$, now at the price of an error $\mathcal{O}_\prec\big(\langle A^8\rangle^{1/4}\langle B^8\rangle^{1/4}/N\big)$, which can easily be included in $\mathcal{O}_\prec(\mathcal{E})$. We point out that Theorem 4.3.3 cannot be used in this regime, since $\ell = 0$ when one of the z_i's crosses the real axis.

In the remaining cases, where some of the z_i's run on the horizontal parts and others run on the vertical parts, we can no longer cleanly apply either the local or global law (Theorem 4.3.3 and Proposition 4.4.1, respectively). Instead, we treat this situation by expanding around the case where all z_i are on the horizontal parts. More precisely, in case that, say, z_1 runs on the right vertical part of the contour Γ and $z_2,....,z_4$ are fixed on the horizontal parts, we employ analyticity of $G(z_1)$ away from $[-3,3]$ (since $\|W\| \le 2+\epsilon$ with very high probability). This enables us to write $g(z_1) := \langle G(z_1)A...G(z_4)B\rangle$ at $z_1 = R+\mathrm{i}q$ with $q \in [-|t|^{-1},|t|^{-1}]$, by Taylor expansion around $\zeta := R+\mathrm{i}|t|^{-1}$, as

$$g(R+\mathrm{i}q) = \sum_{j=0}^{K}\frac{(q-|t|^{-1})^j}{2\pi\mathrm{i}}\oint_{C_{1/(2|t|)}}\frac{g(w)}{(w-\zeta)^{j+1}}\mathrm{d}w + \mathcal{O}\big(\widetilde{\mathcal{E}}\big) \tag{4.3.16}$$

with $\widetilde{\mathcal{E}} := \langle A^4\rangle^{1/2}\langle B^4\rangle^{1/2}/(R^{K+1}|t|^{K-2})$ for any $K \in \mathbf{N}$ to be chosen below, where $C_{1/(2|t|)}$ is the circle of radius $1/(2|t|)$ centered around ζ. In (4.3.16), we used that the $(K+1)^{\text{th}}$ derivative of $g(z_1)$ on the vertical part of the contour is (deterministically) bounded as

$$\big|g^{(K+1)}(z_1)\big| \lesssim \Big|\big\langle\big(G(z_1)\big)^{K+1}AG(z_2)BG(z_3)AG(z_4)B\big\rangle\Big| \lesssim \frac{\langle A^4\rangle^{1/2}\langle B^4\rangle^{1/2}|t|^3}{R^{K+1}},$$

since $\|G(z_i)\| \le |t|$ for $i \in \{2,3,4\}$. Using the representation (4.3.16), we can now replace $g(w)$ by $h(w) := \langle M(w,A,...,z_4)B\rangle$, again at the expense of an error $\mathcal{O}_\prec(\mathcal{E})$ by means of Theorem 4.3.3. As one can easily see that h is also analytic around $R+\mathrm{i}q$ and satisfies the same relation as g in (4.3.16), we find that $|g(R+\mathrm{i}q)-h(R+\mathrm{i}q)| \prec \mathcal{E}+\widetilde{\mathcal{E}}$. Hence, in order to absorb $\widetilde{\mathcal{E}}$ into $\mathcal{E}$, it remains to choose K in (4.3.16) as $K := \lceil\kappa^{-1}\rceil$.

Therefore, since integrating the $\mathcal{O}_\prec(\mathcal{E})$-bound in (4.3.15) only adds some N^κ-factors from the length of Γ (note that $|\mathrm{e}^{\mathrm{i}t(z_1+z_3-z_2-z_4)}| \lesssim 1$), which can easily be absorbed into $\prec$, we find that

$$\mathcal{F}_{A,B}(t) = \left(\prod_{i\in[4]}\oint_\Gamma\frac{\mathrm{d}z_i}{2\pi\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}t(z_1+z_3-z_2-z_4)}\langle M(z_1,A,z_2,B,z_3,A,z_4)B\rangle + \mathcal{O}_\prec\big(\mathcal{E}\big)\,. \tag{4.3.17}$$

Similarly, decomposing $A^2 = (A^2-\langle A^2\rangle)+\langle A^2\rangle$ and analogously for B, the two-point part $\mathcal{D}_{A,B}(t)$ from (4.2.3) is given by, again for times $1 \le |t| \le N^{(1-\epsilon)/2}$,

$$\mathcal{D}_{A,B}(t) = \langle A^2\rangle\langle B^2\rangle + \left(\prod_{i\in[2]}\oint_\Gamma\frac{\mathrm{d}z_i}{2\pi\mathrm{i}}\right)\mathrm{e}^{\mathrm{i}t(z_1-z_2)}\langle M(z_1,\mathring{(A^2)},z_2)\mathring{(B^2)}\rangle + \mathcal{O}_\prec(\mathcal{E})\,, \tag{4.3.18}$$

where $\mathring{(A^2)} = A^2-\langle A^2\rangle$ and analogously for B.

In the other two regimes, $|t|<1$ and $|t|>N^{(1-\epsilon)/2}$, we follow the above arguments for $1 \le |t| \le N^{1/2-\epsilon}$, but replace the contour Γ from (4.3.14) by[12]

$$\{z\in\mathbf{C}:\mathrm{dist}(z,[-2,2]) = |t|^{-1}\} \quad\text{and}\quad \partial\big([-R,R]\times\mathrm{i}[-N^{(1-\epsilon)/2},N^{(1-\epsilon)/2}]\big)\,,$$

respectively. This results in error terms in identities analogous to (4.3.17)–(4.3.18), that can easily be seen to be bounded by $\mathcal{O}_\prec(\mathcal{E})$, just as before, by application of Proposition 4.4.1 and Theorem 4.3.3, respectively.

It remains to explicitly compute the deterministic terms in (4.3.17)–(4.3.18), which, by using the formulas (4.3.1)–(4.3.4), straightforwardly results in the expression given in (4.2.6). This concludes the proof of Theorem 4.2.2. □

[12] We choose the first contour only for $t \neq 0$. If $t = 0$, the lhs. of (4.2.6) carries no randomness, and we find $\mathcal{C}_{A,B}(0) = \langle A^2B^2\rangle - \langle ABAB\rangle$.

4.4 Zigzag strategy: Proof of Theorem 4.3.3

In this section we prove our main technical result from Section 4.3, the multi-resolvent local law in Theorem 4.3.3. Its proof is conducted via the *characteristic flow method* [7, 105, 353, 6, 394, 14] followed by a *Green function comparison* (GFT) argument. A combination of these tools, which we coin the *Zigzag strategy*, has first been used in [161, 162] and Chapter 3. It consists of the following three steps:

1. **Global law.** Proof of a multi-resolvent *global law*, i.e. for spectral parameters "far away" from the spectrum, $\min_j \operatorname{dist}(z_j, [-2,2]) \geq \delta$ for some small $\delta > 0$ (see Proposition 4.4.1 below).

2. **Characteristic flow.** Propagate the global law to a *local law* by considering the evolution of the Wigner matrix W along the Ornstein-Uhlenbeck flow

$$\mathrm{d}W_t = -\frac{1}{2} W_t \mathrm{d}t + \frac{\mathrm{d}B_t}{\sqrt{N}}, \qquad W_0 = W\,, \tag{4.4.1}$$

   with B_t a standard real symmetric or complex Hermitian Brownian motion, thereby introducing an order one Gaussian component (see Proposition 4.4.3). The spectral parameters of the resolvents evolve from the global regime to the local regime according to the *characteristic (semicircular) flow*

$$\partial_t z_{i,t} = -m(z_{i,t}) - \frac{z_{i,t}}{2}\,. \tag{4.4.2}$$

   The simultaneous effect of these two evolutions is a key cancellation of two large terms.

3. **Green function comparison.** Remove the Gaussian component by a Green function comparison (GFT) argument (see Proposition 4.4.4).

As the first step, we have the following global law, the proof of which is completely analogous to the proofs presented in [168, Appendix B], [169, Appendix A], and Appendix 3.A.2, and so omitted. Proposition 4.4.1 is stated for a general deterministic matrix B since the traceless condition plays no role in this case.

Proposition 4.4.1 (Step 1: Global law)**.** *Let W be a Wigner matrix satisfying Assumption 4.2.1, and fix $k \in \mathbf{N}$ and $\delta > 0$. Consider spectral parameters $z_1, \ldots, z_k \in \mathbf{C} \setminus \mathbf{R}$, the associated resolvents $G_j = G(z_j) := (W - z_j)^{-1}$, and deterministic matrices $B_1, \ldots, B_k \in \mathbf{C}^{N\times N}$. Then, uniformly in spectral parameters satisfying $d := \min_{j\in[k]} \operatorname{dist}(z_j, [-2,2]) \geq \delta$ and deterministic matrices $B_1, \ldots, B_k$, it holds that*

$$\left|\langle G_1 B_1 \ldots G_k B_k\rangle - \langle M_{[1,k]} B_k\rangle\right| \prec \frac{1}{N} \prod_{i\in[k]} \|B_i\|_{2k,d}\,. \tag{4.4.3}$$

Next, using Proposition 4.4.1 as an input, we derive Theorem 4.3.3 for Wigner matrices which have an order one Gaussian component, as formulated in Proposition 4.4.3. For this purpose we consider the evolution of the Wigner matrix W along the Ornstein-Uhlenbeck flow (4.4.1) and define its resolvent $G_t(z) := (W_t - z)^{-1}$ with $z_i \in \mathbf{C} \setminus \mathbf{R}$. Even if not stated explicitly we will always consider this flow only for short times, i.e. for $0 \leq t \leq T$, where the maximal time T is smaller than some $\gamma > 0$. Note that the first two moments of W_t are preserved along the flow (4.4.1), and hence the self-consistent density of states of W_t is unchanged; it remains the standard semicircle law. We now want to compute the deterministic approximation to an alternating product of resolvents and deterministic matrices $A_1, A_2, \ldots$ with trace zero

$$G_t(z_1) A_1 G_t(z_2) A_2 G_t(z_3) A_3 \ldots, \tag{4.4.4}$$

and have a very precise estimate of the error term.

In fact, we will let the spectral parameters evolve with time with a carefully chosen equation that will conveniently cancel some error terms in the time evolution of (4.4.4). The corresponding equation will be the characteristic equation for the semicircular flow, i.e. given by the first order ODE (4.4.2) (see Figure 3.3.1 for an illustration of this flow). We remark that, along the characteristics we have

$$\begin{aligned}\partial_t m(z_{i,t}) &= -\partial_z m(z_{i,t})\left(m(z_{i,t}) + \frac{z_{i,t}}{2}\right)\\ &= -\partial_z m(z_{i,t})\left(-\frac{1}{2m(z_{i,t})} + \frac{m(z_{i,t})}{2}\right) = \frac{m(z_{i,t})}{2},\end{aligned} \tag{4.4.5}$$

where in the last two equalities we used the defining equation $m(z)^2 + zm(z) + 1 = 0$ of the Stieltjes transform of the semicircular law. In particular, this implies that $\rho_{i,s} \sim \rho_{i,t}$ for any $0 \le s \le t$, where we denoted $\rho_{i,t} := \pi^{-1}|\mathrm{Im}\, m(z_{i,t})|$. In contrast to that, the behavior of the $\eta_{i,t} := |\mathrm{Im}\, z_{i,t}|$ depends on the regime: in the bulk $\eta_{i,t}$ decreases linearly in time with a speed of order one, at the edge the decay is still linear but with a speed depending on the size of the local density of states. By standard ODE theory we obtain the following lemma:

Lemma 4.4.2 (see Lemma 3.3.2)**.** *Fix an N–independent $\gamma > 0$, fix $0 < T < \gamma$, and pick $z \in \mathbf{C} \setminus \mathbf{R}$. Then there exists an initial condition z_0 such that the solution z_t of* (4.4.2) *with this initial condition z_0 satisfies $z_T = z$. Furthermore, there exists a constant $C > 0$ such that* $\mathrm{dist}(z_0, [-2, 2]) \ge CT$.

The spectral parameters evolving by (4.4.1) will have the property that

$$G_t(z_{1,t})A_1 \dots G_t(z_{k,t}) - M_{[1,k],t} \approx G_0(z_{1,0})A_1 \dots G_0(z_{k,0}) - M_{[1,k],0}, \tag{4.4.6}$$

with $M_{[1,k],t} := M(z_{1,t}, A_1, \dots, A_{k-1}, z_{k,t})$, for any $0 \le t \le T$. Note that the deterministic approximation $M_{[1,k],t}$ depends on time only through the time dependence of the spectral parameters, the deterministic approximation of (4.4.4) with fixed spectral parameters does not depend on time, i.e. it is unchanged along the whole flow (4.4.1).

Proposition 4.4.3 (Step 2: Characteristic flow)**.** *Fix any $\epsilon, \gamma > 0$, a time $0 \le T \le \gamma$, and $K \in \mathbf{N}$. Consider $z_{1,0}, \dots, z_{K,0} \in \mathbf{C} \setminus \mathbf{R}$ as initial conditions to the solution $z_{j,t}$ of* (4.4.2) *for $0 \le t \le T$ and define $G_{j,t} := G_t(z_{j,t})$ as well as $\eta_{j,t} := |\mathrm{Im}\, z_{j,t}|$ and $\rho_{j,t} := \pi^{-1}|\mathrm{Im}\, m(z_{j,t})|$.*

Let $k \le K$, define $\ell_t := \min_{j\in[k]} \eta_{j,t}\rho_{j,t}$ and recall (4.3.5)*. Then, assuming that*

$$\left|\langle G_{1,0}A_1 \dots G_{k,0}A_k\rangle - \langle M_{[1,k],0}A_k\rangle\right| \prec \frac{1}{N}\prod_{i\in[k]} |\!|\!|A_i|\!|\!|_{2k,\ell_0}\,, \tag{4.4.7}$$

holds uniformly for any $k \le K$, any choice of deterministic traceless $A_1, \dots, A_k$ and any choice of $z_{i,0}$'s such that $N\ell_0 \ge N^\epsilon$ and $\max_{j\in[k]} |z_{j,0}| \le N^{1/\epsilon}$, then we have

$$\left|\langle G_{1,T}A_1 \dots G_{k,T}A_k\rangle - \langle M_{[1,k],T}A_k\rangle\right| \prec \frac{1}{N}\prod_{i\in[k]} |\!|\!|A_i|\!|\!|_{2k,\ell_T} \tag{4.4.8}$$

for any $k \le K$, again uniformly in traceless matrices A_i and in spectral parameters satisfying $N\ell_T \ge N^\epsilon$ and $\max_{j\in[k]} |z_{j,T}| \le N^{1/\epsilon}$.

The proof of Proposition 4.4.3 is given in Section 4.5. As the third and final step, we show that the additional Gaussian component introduced in Proposition 4.4.3 can be removed using a Green function comparison (GFT) argument at the price of a negligible error. The proof of this proposition is presented in Section 4.6.

Proposition 4.4.4 (Step 3: Green function comparison)**.** *Let $H^{(v)}$ and $H^{(w)}$ be two $N \times N$ Wigner matrices with matrix elements given by the random variables v_{ab} and w_{ab}, respectively, both satisfying Assumption 4.2.1 and having matching moments up to third order,*[13] *i.e.*

$$\mathbf{E}\,\bar{v}_{ab}^{u} v_{ab}^{s-u} = \mathbf{E}\,\bar{w}_{ab}^{u} w_{ab}^{s-u}\,, \quad s \in \{0,1,2,3\}\,, \quad u \in \{0,...,s\}\,. \tag{4.4.9}$$

Fix $K \in \mathbf{N}$ and consider spectral parameters $z_1,...,z_K \in \mathbf{C} \setminus \mathbf{R}$ that satisfy $\min_j N\eta_j\rho_j \geq N^{\epsilon}$ and $\max_j |z_j| \leq N^{1/\epsilon}$ for some $\epsilon > 0$ and the associated resolvents $G_j^{(\#)} = G^{(\#)}(z_j) := (H^{(\#)} - z_j)^{-1}$ with $\# = \boldsymbol{v}, \boldsymbol{w}$. Pick traceless matrices $A_1,...,A_K \in \mathbf{C}^{N\times N}$.

Assume that, for $H^{(v)}$, we have the following bounds (writing $G_j \equiv G_j^{(\#)}$ for brevity): For any $k \leq K$, consider any subset of cardinality k of the K spectral parameters, and similarly, consider any subset of cardinality k of the deterministic matrices. Relabeling both of them by $[k]$, setting $\ell := \min_{j\in[k]} \eta_j\rho_j$ and recalling (4.3.5)*, we have that*

$$\left|\langle G_1 A_1 ... G_k A_k\rangle - \langle M_{[1,k]} A_k\rangle\right| \prec \frac{1}{N}\prod_{i\in[k]} \|A_i\|_{2k,\ell} \tag{4.4.10}$$

uniformly in all choices of subsets of z's and A's.

Then, (4.4.10) *also holds for the ensemble $H^{(w)}$, uniformly all choices of subsets of z's and A's.*

We are now ready to finally conclude the proof of Theorem 4.3.3. Fix $T > 0$, and fix $z_1, \ldots, z_k \in \mathbf{C} \setminus \mathbf{R}$ such that $\min_i N\eta_i\rho_i \geq N^{\epsilon}$, and let $z_{i,0}$ be the initial conditions of the characteristics (4.4.2) chosen so that $z_{i,T} = z_i$ (this is possible thanks to Lemma 4.4.2). Then, the assumption (4.4.7) of Proposition 4.4.3 is satisfied for those $z_{i,0}$ by Proposition 4.4.1 with $\delta = CT$, since $d \gtrsim \ell_0$ and where $C > 0$ is the constant from Lemma 4.4.2. We can thus use Proposition 4.4.3 to show that (4.4.8) holds. Finally, the Gaussian component added in Proposition 4.4.3 is removed using Proposition 4.4.4 with the aid of a complex version (see Lemma 3.A.2) of the standard moment-matching lemma [248, Lemma 16.2]. □

4.5 Characteristic flow: Proof of Proposition 4.4.3

In this section, we give the proof of Proposition 4.4.3. The argument is divided into three parts.

(i) We begin by introducing several deterministic and stochastic control quantities, which play a fundamental role throughout the rest of this paper. The stochastic control quantities (some normalized differences between a resolvent chain and the corresponding M-term, see (4.5.11)–(4.5.12) below) satisfy a system of *master inequalities* (see Propositions 4.5.2–4.5.3).

(ii) Taking these master inequalities together with Proposition 4.4.1 as inputs, we will prove Proposition 4.4.3 in Section 4.5.1, thus concluding Step 2 of the argument in Section 4.4.

(iii) Afterwards, based on Proposition 4.4.1 and several relations among the *deterministic* control quantities (see Lemma 4.5.1), we will prove the master inequalities in Section 4.5.2.

To keep the presentation simpler, within this section we assume that $\mathbf{E}\,\chi_{\mathrm{od}}^2 = 0$ and that $\mathbf{E}\,\chi_{\mathrm{d}}^2 = 1$, i.e. we consider only complex Wigner matrices. The modifications for the general case are analogous to Section 3.4, and so omitted.

We recall our choice of the characteristics $\partial_t z_{i,t} = -m(z_{i,t}) - \frac{z_{i,t}}{2}$ from (4.4.2) and note that $t \mapsto \eta_{i,t}$ is decreasing and that $\rho_{i,t} \sim \rho_{i,s}$ for any $0 \leq s \leq t$ (see (4.4.5) and the paragraph below it for more

[13] This condition can easily be relaxed to being matching up to an error of size N^{-2} as done, e.g., in [248, Theorem 16.1].

details). Additionally, we record the following, trivially checkable, simple integration rule which will be used many times in this section:

$$\int_0^t \frac{1}{\eta_s^\alpha}\mathrm{d}s \lesssim \frac{\log N}{\eta_t^{\alpha-2}\ell_t} \quad \text{with} \quad \eta_s := \min_i \eta_{i,s}\,, \quad \ell_s := \min_i \eta_{i,s}\rho_{i,s}\,. \tag{4.5.1}$$

Along the characteristics, using the short–hand notation $G_{i,t} = (W_t - z_{i,t})^{-1}$, with W_t being the solution of (4.4.1), by Itô's formula, we have[14]

$$\begin{aligned}\mathrm{d}\langle (G_{[1,k],t} - M_{[1,k],t})A\rangle &= \frac{1}{\sqrt{N}}\sum_{a,b=1}^N \partial_{ab}\langle G_{[1,k],t}A\rangle \mathrm{d}B_{ab,t}\\ &+ \frac{k}{2}\langle G_{[1,k],t}A\rangle \mathrm{d}t + \sum_{\substack{i,j=1\\ i<j}}^k \langle G_{[i,j],t}\rangle\langle G_{[j,i],t}\rangle \mathrm{d}t\\ &+ \sum_{i=1}^k \langle G_{i,t} - m_{i,t}\rangle\langle G^{(i)}_{[1,k],t}A\rangle \mathrm{d}t - \partial_t\langle M_{[1,k],t}A\rangle \mathrm{d}t,\end{aligned} \tag{4.5.2}$$

where ∂_{ab} denotes the directional derivative $\partial_{w_{ab}}$. We also set

$$G_{[i,j],t} := \begin{cases} G_{i,t}A_i \dots A_{j-1}G_{j,t} & \text{if} \quad i<j\\ G_{i,t} & \text{if} \quad i=j\\ G_{i,t}A_{i,t}\dots G_{k,t}A_kG_{1,t}A_1\dots A_{j-1}G_{j,t} & \text{if} \quad i>j,\end{cases} \tag{4.5.3}$$

and defined $G^{(l)}_{[i,j],t}$ as $G_{[i,j],t}$ but with the l–th resolvent substituted by $G^2_{l,t}$. The last definition for $i>j$ reflects the cyclicity of the trace, since $G_{[i,j],t}$ will be needed in a tracial situation. For any $i,j\in[k]$, we denote the deterministic approximation of $G_{[i,j],t}$ by $M_{[i,j],t}$, with $M_{[i,j],t}$ being defined as in (4.3.2) with $[1,k]$ replaced by $[i,j]$ if $i<j$ and by $[1,j]\cup[i,k]$ if $i>j$.

Deterministic control quantities: mean and standard deviation size. We now introduce two deterministic control quantities that measure the size of the mean of long chains $\langle G_1A_1...G_kA_k\rangle$ and their standard deviation, respectively, in terms of the ℓ-weighted Schatten norms of $A_1,...,A_k$ (recall Definition 4.3.1) and the spectral parameters of the resolvents $G_i = G(z_i)$. These are given by

$$\mathbf{m}_k(\ell;\boldsymbol{A}_{J_k}) := \mathbf{1}(k\geq 2)\prod_{\alpha\in J_k}\|A_\alpha\|_{k,\ell}\,, \quad \mathfrak{s}_k(\ell;\boldsymbol{A}_{J_k}) := \prod_{\alpha\in J_k}\|A_\alpha\|_{2k,\ell}\,, \tag{4.5.4}$$

and will be called the *mean size* and the *standard deviation size*, respectively. They are functions of a positive number ℓ (usually given by $\ell = \min_i \eta_i\rho_i$) and a multiset $\boldsymbol{A}_{J_k} = (A_\alpha)_{\alpha\in J_k}$ of deterministic matrices of cardinality $|J_k| = k$.

In the following, we shall frequently drop the symbol $\boldsymbol{A}$ from the definitions in (4.5.4), i.e. write $\mathfrak{m}_k(\ell;J_k)$ and $\mathfrak{s}_k(\ell;J_k)$ instead of $\mathfrak{m}_k(\ell;\boldsymbol{A}_{J_k})$ and $\mathfrak{s}_k(\ell;\boldsymbol{A}_{J_k})$, respectively. Moreover, for time dependent spectral parameters, we will also use the notation $\mathfrak{m}_k(t) = \mathfrak{m}_k(t;J_k) = \mathfrak{m}_k(\ell_t;\boldsymbol{A}_{J_k})$, with $\ell_t = \min_i \eta_{i,t}\rho_{i,t}$, and a similar notation for the standard deviation size. In particular, we may often omit the dependence on the deterministic matrix A. (More generally, we will omit every argument of (4.5.4), whenever it is clear what they are and it does not lead to confusion.) For example, for $i<j$, we have, with $\eta_t := \min_i \eta_{i,t}$,

$$|\langle M_{[i,j],t}A_j\rangle| \lesssim \ell_t\,\mathfrak{m}_{j-i+1}(t), \qquad |\langle M_{[i,j],t}\rangle| \lesssim (\ell_t/\eta_t)\mathfrak{m}_{j-i}(t). \tag{4.5.5}$$

The first bound follows from (4.3.7); the second bound in (4.5.5) can easily be obtained by arguments entirely analogous to the ones leading to Lemma 4.A.1 in Appendix 4.A. The bounds (4.5.5) justify the

[14] We point out that (4.5.2) holds for any matrix $A_i\in\mathbf{C}^{N\times N}$, i.e. we did not use that the A_i's are traceless.

terminology that $\mathfrak{m}_k$ is the mean size of a resolvent chain $\langle G_1 A_1 ... G_k A_k \rangle$, which is well approximated by $\langle M_{[1,k]} A_k \rangle$. The additional ℓ_t factor was introduced to simplify formulas later.

In the remainder of this section we will often use the following lemma about some relations among the sizes $\mathfrak{m}_k(t)$, $\mathfrak{s}(t)$ from (4.5.4). The proofs are immediate consequences of (4.3.6) and $\ell_s \gtrsim \ell_t$ for $s \le t$ (recall the discussion below (4.4.5)) and hence omitted.

Lemma 4.5.1 ($\mathfrak{m}/\mathfrak{s}$-relations)**.** *Let $k \ge 1$ and consider time-dependent spectral parameters $z_{1,t}, ..., z_{k,t}$ and a multiset of traceless matrices $\boldsymbol{A}_{J_k}$ as above. Then, the mean size $\mathfrak{m}_k$ and the standard variation size $\mathfrak{s}_k$ satisfy the following inequalities.*

(i) Super-multiplicativity, i.e. for any $1 \le j \le k-1$ we have

$$\mathfrak{m}_j(t; J_j)\, \mathfrak{m}_{k-j}(t; J_{k-j}) \lesssim \mathfrak{m}_k(t; J_k)\,, \quad \mathfrak{s}_j(t; J_j)\, \mathfrak{s}_{k-j}(t; J_{k-j}) \lesssim \mathfrak{s}_k(t; J_k) \tag{4.5.6}$$

for all disjoint decompositions $J_k = J_j \dot\cup J_{k-j}$.

(ii) The mean size can be upper bounded by the standard deviation size as

$$\mathfrak{m}_k(t; J_k) \lesssim \mathfrak{s}_k(t; J_k) \qquad \text{and} \qquad \sqrt{\mathfrak{m}_{2k}(t; J_k \cup J_k)} \lesssim \mathfrak{s}_k(t; J_k)\,, \tag{4.5.7}$$

where $J_k \cup J_k$ denotes the union of the multiset J_k with itself.

(iii) The standard deviation size satisfies the doubling inequality (recall $\ell_t = \min_{j\in[k]} \eta_{j,t}\rho_{j,t}$)

$$\mathfrak{s}_{2k}(t; J_k \cup J_k) \lesssim [1 + \sqrt{N\ell_t}]\big(\mathfrak{s}_k(t; J_k)\big)^2\,. \tag{4.5.8}$$

(iv) Monotonicity in time: for any $0 \le s \le t$ we have

$$\mathfrak{m}_k(s; J_k)\, \ell_s^\alpha \lesssim \mathfrak{m}_k(t; J_k)\, \ell_t^\alpha \qquad \mathfrak{s}_k(s; J_k)\, \ell_s^\beta \lesssim \mathfrak{s}_k(t; J_k)\, \ell_t^\beta \tag{4.5.9}$$

for all $\alpha \in [0,1]$ and $\beta \in [0, 1/2]$.

Stochastic control quantities: Master and reduction inequalities. Using the notation from (4.5.4), the goal is thus to prove that

$$\langle G_{[1,k],T} A_k \rangle - \langle M_{[1,k],T} A_k \rangle = \langle G_{[1,k],0} A_k \rangle - \langle M_{[1,k],0} A_k \rangle + \mathcal{O}_\prec\Big(\frac{\mathfrak{s}_k(T)}{N}\Big)\,, \tag{4.5.10}$$

uniformly in the spectrum and uniformly in traceless deterministic matrices A_i, for some fixed $T \lesssim 1$. We may henceforth assume that all the A_i's are Hermitian; the general case follows by multilinearity.

For the purpose of proving (4.5.10), recall the notation $\ell_t := \min_i \eta_{i,t}\rho_{i,t}$ from (4.5.1) and define

$$\Phi_k(t) = \Phi_k(t; \boldsymbol{z}_{[1,k]}; \boldsymbol{A}_{[1,k]}) := \frac{N}{\mathfrak{s}_k(\ell_t; \boldsymbol{A}_{[1,k]})}\big|\langle (G_{[1,k],t} - M_{[1,k],t}) A_k \rangle\big| \tag{4.5.11}$$

for all $k \ge 1$. Here, $\boldsymbol{A}_{[1,k]} = (A_1, \dots, A_k)$ and $\boldsymbol{z}_{[1,k]} = (z_1, ..., z_k)$ with $z_i = z_{i,0}$ (initial condition) denote multisets of deterministic matrices and spectral parameters, respectively. We now briefly comment on the definition (4.5.11). We chose the pre–factor in the definition of $\Phi_k(t)$ so that eventually $\Phi_k(t)$ will be of order one with high probability (cf. (4.5.10)). However, we will not be able to prove this directly, we first prove that $\Phi_k(t) \prec \sqrt{N\ell_t}$ and then, using this bound as an input, we prove the desired bound $\Phi_k(t) \prec 1$. To implement this technically we introduce another quantity

$$\Psi_k(t) = \Psi_k(t; \boldsymbol{z}_{[1,k]}; \boldsymbol{A}_{[1,k]}) := \frac{N}{\mathfrak{s}_k(\ell_t; \boldsymbol{A}_{[1,k]})\sqrt{N\ell_t}}\big|\langle (G_{[1,k],t} - M_{[1,k],t}) A_k \rangle\big| \tag{4.5.12}$$

for all $k \geq 1$, which we will show to be bounded by one (note $\Psi_k(t) \prec 1$ implies $\Phi_k(t) \prec \sqrt{N\ell_t}$), and then show that $\Psi_k(t) \prec 1$ in fact implies $\Phi_k(t) \prec 1$.

We start considering $\Psi_k(t)$; note that by (4.4.7) it follows

$$\Psi_k(0) \lesssim \Phi_k(0) \prec 1, \tag{4.5.13}$$

for any $k \geq 1$. For this purpose we will derive a series of *master inequalities* for these quantities with the following structure. We assume that

$$\Psi_k(t) \prec \psi_k \tag{4.5.14}$$

holds for some deterministic control parameter $\psi_k \geq 1$, uniformly in deterministic matrices A_i, times $0 \leq t \leq T$ and in spectral parameters with $\ell_t \geq N^{-1+\epsilon}$ for some small fixed $\epsilon > 0$ (we stress that the ψ_k's depend neither on time, nor on the spectral parameters $z_{i,t}$, nor on the deterministic matrix A). Starting from (4.5.14) we derive an improved upper bound for $\Psi_k(t)$ and show that, by *iterating* these inequalities, we indeed obtain the desired $\Psi_k(t) \prec 1$. The main inputs to prove this fact are the *master inequalities* in (4.5.15), which informally states that if $\Psi_l(t) \prec \psi_l$ for any $l = 1, \dots, k + \mathbf{1}(k \,\mathrm{odd})$, then this actually implies the improved bound (4.5.15).

Proposition 4.5.2 (Master inequalities)**.** *Fix $k \in \mathbf{N}$. Assume that $\Psi_l(t) \prec \psi_l$ for some deterministic control parameters ψ_l, for any $1 \leq l \leq k + \mathbf{1}(k \,\mathrm{odd})$, uniformly in $t \in [0,T]$. Then*

$$\Psi_k(t) \prec 1 + \sum_{j=1}^{k-2} \psi_j + \frac{1}{(N\ell_T)^{1/4}} \sum_{j=1}^{k} \widetilde{\psi}_j \widetilde{\psi}_{k-j}, \tag{4.5.15}$$

uniformly in $t \in [0,T]$. Here we set $\widetilde{\psi}_l := \psi_l + \mathbf{1}(l \,\mathrm{odd}) \sqrt{\psi_{l-1}\psi_{l+1}}$, $\psi_0 := 1$.

Using (4.5.15), in the next section we show that $\Psi_k(t) \prec 1$ by an iterative procedure. Then, to conclude $\Phi_k(t) \prec 1$, we rely on the following proposition, which will eventually prove Proposition 4.4.3.

Proposition 4.5.3. *Fix $k \in \mathbf{N}$, assume that $\Phi_l(t) \prec 1$, for $l \leq k-2$ uniformly in $t \in [0,T]$, and that $\Phi_l(t) \prec \sqrt{N\ell_t}$ for $1 \leq l \leq 2k$ and $t \in [0,T]$. Then, uniformly in $t \in [0,T]$,*

$$\Phi_k(t) \prec 1\,. \tag{4.5.16}$$

4.5.1 Closing the hierarchy: Proof of Proposition 4.4.3

To show that Proposition 4.5.2 in fact implies $\Psi_k(t) \prec 1$ we rely on the following procedure, which we refer to as *iteration* (see, e.g., Lemma 1.4.11).

Lemma 4.5.4 (Iteration)**.** *Fix $k \in \mathbf{N}$, $T > 0$, and N-independent constants $\epsilon, \delta > 0$, $\alpha \in (0,1)$ and $D > 0$. Let X be a random variable depending on k time dependent spectral parameters $z_{1,t}, ..., z_{k,t}$, $t \in [0,T]$, and recall that $\ell_t = \min_{j \in [k]} \eta_{j,t}\rho_{j,t}$. Assume that the a-priori bound $X \prec N^D$ holds uniformly in $t \in [0,T]$, $\ell_t \geq N^{-1+\epsilon}$. Suppose that there is a deterministic quantity x (may depend on ℓ_T and N) such that for any fixed $l \in \mathbf{N}$ the fact that $X \prec x$ uniformly for $t \in [0,T]$ and $\ell_t \geq N^{-1+l\epsilon}$ implies*[15]

$$X \prec A + \frac{x}{B} + x^{1-\alpha} C^{\alpha}, \tag{4.5.17}$$

uniformly for $t \in [0,T]$ and $\ell_t \geq N^{-1+(l+l')\epsilon}$, for some constants[16] *$l' \in \mathbf{N}$, $B \geq \delta > 0$, $A, C > 0$. Then, iterating* (4.5.17) *finitely many times, we obtain*

$$X \prec A + C,$$

uniformly for $t \in [0,T]$ and $\ell_t \geq N^{-1+(1+Kl')\epsilon}$, for some $K = K(\alpha, D, \delta)$.

[15] Here the scalar $A > 0$ should not be confused with the matrices A_i which appear throughout the proof.

[16] The constants A, B, C may depend on N, while l and l' are independent of N.

We are now ready to present the proof of Proposition 4.4.3.

Proof of Proposition 4.4.3. The proof of this proposition is divided into two steps: we first prove that $\Psi_k(t) \prec 1$ for any k and then use this information as an input to conclude that $\Phi_k(t) \prec 1$ for any k. These two bounds are quite different in spirit. The first one is an a-priori bound so the iterative procedure behind its proof must be self-improving. This is reflected in the triangular structure of (4.5.15): the right hand side contains quantities with index at most k (or $k+1$ when k is odd) and terms with the highest index come with a small prefactor (recall that $N\ell_T$ is large). This makes the system (4.5.15) closable. The iteration is done essentially for each fixed k and then we use an induction on k. Owing to the parity issue in the definition of $\widetilde{\psi}$, we use a step-two induction, but this is just a small technicality. The second bound $\Phi_k(t) \prec 1$ is quite different, since its proof relies on the a-priori bound obtained in the first step. The key point is that in order to prove $\Phi_k(t) \prec 1$ for some fixed k, we need to know the a-priori bound $\Phi_l(t) \prec \sqrt{N\ell_t}$ for any $l \le 2k$, i.e. without the a-priori bound the system of inequalities behind the proof of $\Phi_k(t) \prec 1$ would not be closable. This explains why we need to proceed in two stages.

We now start we the proof of $\Psi_k(t) \prec 1$. We first prove this for $k = 1, 2$ and then, using an inductive argument, we show that the same bound holds for any $k \ge 3$. By (4.5.15) for $k = 1, 2$ we have

$$\Psi_1(t) \prec 1 + \frac{\psi_1 + \sqrt{\psi_2}}{(N\ell_T)^{1/4}}, \qquad \Psi_2(t) \prec 1 + \frac{\psi_1^2 + \psi_2}{(N\ell_T)^{1/4}} \tag{4.5.18}$$

Using iteration we then obtain (all estimates are uniform in $t \in [0, T]$)

$$\Psi_1(t) \prec 1 + \frac{\sqrt{\psi_2}}{(N\ell_T)^{1/4}}, \qquad \Psi_2(t) \prec 1 + \frac{\psi_1^2}{(N\ell_T)^{1/4}}. \tag{4.5.19}$$

Plugging the first inequality into the second one and using iteration, this immediately gives that $\Psi_1(t) + \Psi_2(t) \prec 1$.

Next, we proceed with the induction step. Fix an even $k \in \mathbf{N}$, and assume that $\Psi_l(t) \prec 1$, for $l \le k-2$, then by (4.5.15) we have

$$\Psi_{k-1}(t) \prec 1 + \frac{\psi_{k-1} + \sqrt{\psi_k}}{(N\ell_T)^{1/4}}, \qquad \Psi_2(t) \prec 1 + \psi_{k-1} + \frac{\psi_k + \sqrt{\psi_k}}{(N\ell_T)^{1/4}} \tag{4.5.20}$$

By iteration, we have

$$\Psi_{k-1}(t) \prec 1 + \frac{\sqrt{\psi_k}}{(N\ell_T)^{1/4}}, \qquad \Psi_2(t) \prec 1 + \psi_{k-1}, \tag{4.5.21}$$

which concludes $\Psi_l(t) \prec 1$, for $l \le k$, by plugging the first inequality into the second one and using iteration once again.

Finally, to conclude $\Phi_k(t) \prec 1$, we proceed by a step-two induction on k. For $k = 1, 2$, the assumption $\Phi_l(t) \prec \sqrt{N\ell_t}$, for $l \le 4$, of Proposition 4.5.3 is satisfied, and so we have $\Phi_1(t) + \Phi_2(t) \prec 1$. Then we proceed with the induction step, i.e. for a fixed even $k \in \mathbf{N}$ we assume that $\Phi_l(t) \prec \sqrt{N\ell_t}$, for $l \le 2k$, and $\Phi_l(t) \prec 1$, for $l \le k-2$. Then, by Proposition 4.5.3, we have $\Phi_l(t) \prec 1$ for $l \le k$; this concludes the induction step, hence the proof. □

4.5.2 Master and reduction inequalities: Proofs of Propositions 4.5.2–4.5.3

We first present the proof of Proposition 4.5.2 in detail and then at the end of this section we explain the minor changes to obtain Proposition 4.5.3.

Proof of Proposition 4.5.2. To keep the notation simple, from now on we often omit the t–dependence from the resolvents $G_i = G_{i,t}$ and from their deterministic approximations $M_{[i,j]} = M_{[i,j],t}$, but we will still keep the t–dependence in the spectral parameters, in $\Psi_k(t)$ and in the quantities (4.5.4); additionally, we stress that ψ_k does not depend on time. All estimates are uniform in the time parameters $t, s \in [0, T]$.

Note that in (4.5.12) we defined $\Psi_k(t)$ only for alternating chains of single resolvents and deterministic A's, i.e. no G_i^k appears. However, in (4.5.2) we naturally get chains involving G_i^2. For these terms we use the estimate (recall the estimate for $\langle M_{[i,j]} \rangle$ from (4.5.5)):

$$\left| \langle G_{[i,j]} - M_{[i,j]} \rangle \right| \prec \frac{\psi_{j-i} \mathfrak{s}_{j-i}(t)}{N\eta_t}. \tag{4.5.22}$$

For $\operatorname{Im} z_{i,t} \operatorname{Im} z_{j,t} > 0$ this trivially follows by integral representation

$$G_j G_i = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{G(z)}{(z - z_{i,t})(z - z_{j,t})} \, \mathrm{d}z, \tag{4.5.23}$$

for "linearizing" the product of the first and last G's in $\langle G_{[i,j]} \rangle = \langle G_i A_i \dots A_{j-i} G_j \rangle$ after using cyclicity of the trace. Here Γ is a contour which lies in the region $\{z \in \mathbf{C} : |\operatorname{Im} z| \rho(z) \geq \ell_t/2\}$. For $\operatorname{Im} z_{i,t} \operatorname{Im} z_{j,t} < 0$ we "linearize" by the resolvent identity. Using (4.5.23) will change the value of the imaginary part of the spectral parameters, so the domain on which the inequalities below hold (characterized by $N\ell \geq N^\epsilon$) may change from time to time, i.e., say $\epsilon \to \epsilon/2$. However, this can happen only finitely many times as it does not affect the $\prec$–bound (see Figure 1.4.1 for a detailed discussion of this minor technicality).

To describe the evolution of $M_{[1,k],t}$ in (4.5.2) we rely on the following lemma.

Lemma 4.5.5 (Lemma 3.4.8)**.** *For any deterministic matrices $A_i \in \mathbf{C}^{N \times N}$ (i.e. not necessarily traceless), we have*

$$\partial_t \langle M_{[1,k],t} A_k \rangle = \frac{k}{2} \langle M_{[1,k],t} A_k \rangle + \sum_{\substack{i,j=1, \\ i<j}}^{k} \langle M_{[i,j],t} \rangle \langle M_{[j,i],t} \rangle. \tag{4.5.24}$$

Adding and subtracting the deterministic approximation of each term in (4.5.2) and using Lemma 4.5.5 we thus obtain

$$\begin{aligned}
\mathrm{d}\langle (G_{[1,k],t} - M_{[1,k],t}) A_k \rangle = {} & \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} \partial_{ab} \langle G_{[1,k],t} A_k \rangle \mathrm{d}B_{ab,t} \\
& + \frac{k}{2} \langle (G_{[1,k],t} - M_{[1,k],t}) A_k \rangle \mathrm{d}t \\
& + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle G_{[i,j],t} - M_{[i,j],t} \rangle \langle M_{[j,i],t} \rangle \mathrm{d}t \\
& + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle M_{[i,j],t} \rangle \langle G_{[j,i],t} - M_{[j,i],t} \rangle \mathrm{d}t \\
& + \sum_{\substack{i,j=1 \\ i<j}}^{k} \langle G_{[i,j],t} - M_{[i,j],t} \rangle \langle G_{[j,i],t} - M_{[j,i],t} \rangle \mathrm{d}t \\
& + \sum_{i=1}^{k} \langle G_{i,t} - m_{i,t} \rangle \langle G^{(i)}_{[1,k],t} A_k \rangle \mathrm{d}t \, .
\end{aligned} \tag{4.5.25}$$

We point out that, using a simple change of variables, the term $\langle (G_{[1,k]} - M_{[1,k]})A_k \rangle$ amounts to a simple rescaling $e^{kt/2}$, so we will ignore it. The quadratic variation of the martingale term in (4.5.25) is given by

$$\begin{aligned} &\frac{1}{N}\sum_{a,b=1}^{N} \left|\partial_{ab}\langle G_{[1,k]}A_k\rangle\right|^2 \\ &\quad \lesssim \sum_{i=1}^{k} \frac{\langle \operatorname{Im} G_i (A_i \operatorname{Im} G_{i+1} \dots A_{i-1}) \operatorname{Im} G_i (A_i \operatorname{Im} G_{i+1} \dots A_{i-1})^* \rangle}{N^2 \eta_{i,t}^2}, \end{aligned} \tag{4.5.26}$$

where we also used the *Ward-identity* $G_i G_i^* = \eta_i^{-1} \operatorname{Im} G_i$. Notice that the rhs. of (4.5.26) naturally contains chains of $2k$ resolvents. However, to have a closed system of master inequalities for products of resolvents of length k, we split the chain of length $2k$ into the product of two chains of length k. For this purpose we use the following *reduction inequality* which will be proven at the end of this section. For any fixed matrices $R, Q \in \mathbf{C}^{N\times N}$, and spectral parameters $z, w \in \mathbf{C} \setminus \mathbf{R}$, we have[17]

$$\begin{aligned} &\langle \operatorname{Im} G(z) Q G(w) R \operatorname{Im} G(z) R^* G(w)^* Q^* \rangle \\ &\qquad \lesssim N \langle \operatorname{Im} G(z) Q | G(w) | Q^* \rangle \langle \operatorname{Im} G(z) R^* | G(w) | R \rangle. \end{aligned} \tag{4.5.28}$$

We now focus on the case k being even for notational simplicity. Combining (4.5.26) with (4.5.28) used for $z = z_1, w = z_{k/2}$ and

$$Q = A_1 G_2 \dots G_{k/2-1} A_{k/2-1}\,, \qquad\qquad R = A_{k/2} G_{k/2+1} \dots G_k A_k,$$

together with $\Psi_k(t) \prec \psi_k$, we obtain the following bound for the quadratic variation

$$\frac{1}{N}\sum_{a,b=1}^{N} \left|\partial_{ab}\langle G_{[1,k],s} A_k\rangle\right|^2 \prec \frac{\mathfrak{m}_k(s)^2}{N\eta_s^2} + \frac{N\ell_t \mathfrak{s}(s)^2 \psi_k^2}{N^3 \eta_s^2}. \tag{4.5.29}$$

Then, using the Burkholder–Davis–Gundy (BDG) inequality, we conclude that the martingale term in (4.5.25) is bounded by

$$\begin{aligned} &\frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}} \left[\int_0^t \left(\frac{\ell_s \mathfrak{m}_k(s)^2}{N\eta_s^2} + \frac{N\ell_s \mathfrak{s}_k(s)^2 \psi_k^2}{N^3 \eta_s^2} \right) \mathrm{d}s \right]^{1/2} \\ &\qquad \lesssim \frac{\mathfrak{m}_k(t)}{\mathfrak{s}_k(t)} + \frac{\psi_k}{\sqrt{N}} \left(\int_0^t \frac{1}{\eta_s^2}\,\mathrm{d}s \right)^{1/2} \lesssim 1 + \frac{\psi_k}{\sqrt{N\ell_t}}, \end{aligned} \tag{4.5.30}$$

with very high probability. Here in the first inequality we used (4.5.9), and in the second inequality we used (4.5.7) for the first term and (4.5.1) for the second term.

Similarly, for odd k, using (4.5.28) for $z = z_1, w = z_{(k-1)/2}$ and

$$Q = A_1 G_2 \dots G_{(k-3)/2} A_{(k-3)/2}\,, \qquad\qquad R = A_{(k-1)/2} G_{(k+1)/2} \dots G_k A_k$$

we get a bound $1 + \sqrt{\psi_{k-1}\psi_{k+1}}/(N\ell_t)^{1/4}$ for the martingale term.

[17] We point out that the bound $\Psi_k(t) \le \psi_k$ holds also for chains when some resolvents G's are replaced by their absolute value $|G|$. This can be easily seen using the integral representation

$$|G(E+\mathrm{i}\eta)| = \frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im} G(E + \mathrm{i}\sqrt{\eta^2+v^2})}{\sqrt{\eta^2+v^2}}\,\mathrm{d}v\,, \tag{4.5.27}$$

together with the bound (4.5.12) for chains containing only resolvents (see e.g. [168, Lemma 5.1]).

Next, using (4.5.22), we estimate the contribution of the last line in (4.5.25)

$$\begin{aligned}&\frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}}\int_0^t \langle G_{i,s}-m_{i,s}\rangle\langle G^{(i)}_{[1,k],s}A_k\rangle\,\mathrm{d}s\\&\quad\prec \frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}}\int_0^t \frac{1}{N\eta_s^2}\left(\ell_s\mathfrak{m}_k(s)+\frac{\psi_k\mathfrak{s}_k(s)\sqrt{N\ell_s}}{N}\right)\mathrm{d}s\lesssim \frac{1}{\sqrt{N\ell_t}}+\frac{\psi_k}{N\ell_t},\end{aligned}\tag{4.5.31}$$

where in the first inequality we used $|\langle G_{i,s}-m_{i,s}\rangle|\prec (N\eta_s)^{-1}$ from (4.1.4), and in the second inequality we used (4.5.7), (4.5.9) and (4.5.1).

Using again (4.5.22), we estimate the third and fourth line of (4.5.25) as follows:

$$\begin{aligned}&\frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}}\int_0^t \sum_{\substack{i,j=1\\ i<j}}^{k-1}\langle M_{[i,j],s}\rangle\langle G_{[j,i],s}-M_{[j,i],s}\rangle\,\mathrm{d}s\\&\prec \frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}}\int_0^t \sum_{\substack{i,j=1\\ i+1<j}}^{k-1}\frac{\ell_s\mathfrak{m}_{j-i}(s)}{\eta_s}\frac{\psi_{k-(j-i)}\mathfrak{s}_{k-j+i}(s)\sqrt{N\ell_s}}{N\eta_s}\,\mathrm{d}s\lesssim\sum_{j=1}^{k-2}\psi_j,\end{aligned}\tag{4.5.32}$$

where in the first inequality we used that $\langle M_{[i,i+1],s}\rangle=0$ and (4.5.5) to estimate $\langle M_{[i,j],s}\rangle$, and in the second inequality we used (4.5.6)–(4.5.7), (4.5.9) and (4.5.1).

Finally, for the fifth line of (4.5.25) we estimate:

$$\begin{aligned}&\frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}}\int_0^t \sum_{\substack{i,j=1\\ i<j}}^{k-1}\langle G_{[i,j],s}-M_{[i,j],s}\rangle\langle G_{[j,i],s}-M_{[j,i],s}\rangle\,\mathrm{d}s\\&\prec \frac{N}{\mathfrak{s}_k(t)\sqrt{N\ell_t}}\sum_{\substack{i,j=1\\ i<j}}^{k-1}\int_0^t \frac{\psi_{j-i}\mathfrak{s}_{j-i}(s)\sqrt{N\ell_s}}{N\eta_s}\frac{\psi_{k-(j-i)}\mathfrak{s}_{k-j+i}(s)\sqrt{N\ell_s}}{N\eta_s}\,\mathrm{d}s\\&\lesssim\frac{1}{\sqrt{N\ell_t}}\sum_{j=1}^{k-1}\psi_j\psi_{k-j},\end{aligned}\tag{4.5.33}$$

where in the second inequality we used (4.5.6), (4.5.9) and (4.5.1). Collecting all these bounds, using that $\ell_t\gtrsim\ell_T$ and $N\ell_T\geq 1$, we obtain

$$\Psi_k(t)\prec 1+\frac{\psi_k+\mathbf{1}(k\,\mathrm{odd})\sqrt{\psi_{k-1}\psi_{k+1}}}{(N\ell_T)^{1/4}}+\sum_{j=1}^{k-2}\psi_j+\frac{1}{(N\ell_T)^{1/2}}\sum_{j=1}^{k-1}\psi_j\psi_{k-j}.$$

Finally, using that $\psi_l\geq 1$ and $N\ell_T\geq 1$, we conclude (4.5.15). □

Proof of Proposition 4.5.3. The proof of (4.5.16) is very similar to the one of (4.5.15), for this reason we only explain the minor differences. All the terms in (4.5.25) are estimated exactly in the same way as in the proof of Proposition 4.5.2 with the exception of the martingale term. In fact, this is the only step where the estimate for Ψ_k (first step) differs from the estimate on Φ_k (second step). Estimating a longer chain by two smaller ones loses a certain $N\ell$ factor; this loss is unavoidable in the first step, but it can be avoided in the second one, once the a-priori bound is available.

More precisely, the estimates in (4.5.31), (4.5.33), after multiplying them by a factor $\sqrt{N\ell_t}$, are the same with the only minor difference that we can now use $\psi_l=1$. The estimate (4.5.32) becomes (recall that $\langle M_{[i,i+1],s}\rangle=0$)

$$\begin{aligned}&\frac{N}{\mathfrak{s}_k(t)}\int_0^t \sum_{\substack{i,j=1\\ i+1<j}}^{k-1}\langle M_{[i,j],s}\rangle\langle G_{[j,i],s}-M_{[j,i],s}\rangle\,\mathrm{d}s\\&\qquad\prec \frac{N}{\mathfrak{s}_k(t)}\int_0^t \sum_{\substack{i,j=1\\ i+1<j}}^{k-1}\frac{\ell_s\mathfrak{m}_{j-i}(s)}{\eta_s}\frac{\Phi_{k-(j-i)}\mathfrak{s}_{k-j+i}(s)}{N\eta_s}\,\mathrm{d}s\lesssim 1,\end{aligned}$$

where the only difference with (4.5.32) is that now in the second inequality we used that $\Phi_l(t) \prec 1$, for $l \le k-2$, by the induction assumption. Instead of the bound for the quadratic variation (4.5.26), using that $\Phi_{2k}(t) \prec \sqrt{N\ell_t}$, the estimate in (4.5.30) is replaced by

$$\begin{aligned}&\frac{N}{\mathfrak{s}_k(t)}\left[\int_0^t \frac{\langle \operatorname{Im} G_{1,s}(A_1\dots A_k)\operatorname{Im} G_{1,s}(A_k\dots A_1)^*\rangle}{N^2\eta_s^2}\,\mathrm{d}s\right]^{1/2}\\ \prec&\frac{N}{\mathfrak{s}_k(t)}\left[\int_0^t\left(\frac{\mathfrak{m}_{2k}(s)\ell_s}{N^2\eta_s^2}+\frac{\Phi_{2k}(s)\mathfrak{s}_{2k}(s)}{N^3\eta_s^2}\right)\mathrm{d}s\right]^{1/2}\lesssim\left[\int_0^t\left(\frac{\ell_t}{\eta_s^2}+\frac{\sqrt{N\ell_t}}{N\eta_s^2}\right)\mathrm{d}s\right]^{1/2}\lesssim 1\,,\end{aligned}$$

where in the second inequality we used (4.5.7), (4.5.9) and in the last inequality we used (4.5.1). This concludes the proof of Proposition 4.5.3. □

We conclude this section with the proof of (4.5.28).

Proof of (4.5.28). By spectral decomposition of W ($\lambda_i, \boldsymbol{u}_i$ being its eigenvalues and eigenvectors) we have

$$\begin{aligned}&\langle \operatorname{Im} G(z)QG(w)R\operatorname{Im} G(z)R^*G(w)^*Q^*\rangle\\ =&\frac{|\operatorname{Im} z|^2}{N}\sum_{i,j,k,l=1}^{N}\frac{\langle \boldsymbol{u}_i, Q\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j, R\boldsymbol{u}_k\rangle\langle \boldsymbol{u}_k, R^*\boldsymbol{u}_l\rangle\langle \boldsymbol{u}_l, Q^*\boldsymbol{u}_i\rangle}{|\lambda_i-z|^2(\lambda_j-w)|\lambda_k-z|^2(\lambda_l-\overline{w})}\\ \lesssim& N\langle \operatorname{Im} G(z)Q|G(w)|Q^*\rangle\langle \operatorname{Im} G(z)R^*|G(w)|R\rangle,\end{aligned}$$

where in the last step we used the Schwarz inequality

$$\begin{aligned}&\left|\langle \boldsymbol{u}_i, Q\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j, R\boldsymbol{u}_k\rangle\langle \boldsymbol{u}_k, R^*\boldsymbol{u}_l\rangle\langle \boldsymbol{u}_l, Q^*\boldsymbol{u}_i\rangle\right|\\ &\qquad\lesssim\left|\langle \boldsymbol{u}_i, Q\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_k, R^*\boldsymbol{u}_l\rangle\right|^2+\left|\langle \boldsymbol{u}_j, R\boldsymbol{u}_k\rangle\langle \boldsymbol{u}_l, Q^*\boldsymbol{u}_i\rangle\right|^2.\end{aligned}$$

□

4.6 Green function comparison: Proof of Proposition 4.4.4

In this section, we remove the Gaussian component introduced in Proposition 4.4.3 using a Green function comparison (GFT) argument, i.e. we prove Proposition 4.4.4. The basic idea is the same as in Section 3.5: We perform a *self-consistent* GFT (i.e., given a local law for one ensemble, we aim to prove it for a different one) using an entry-by-entry Lindeberg replacement strategy in $O(N^2)$ many steps. Note that, unlike here, in typical applications of GFT to answer universality questions, the local law is given as an a-priori input. Prior to Chapter 3 the GFT has been used in a similar spirit by Knowles and Yin [369] in order to prove a single resolvent local law for ensembles, where the deterministic approximation M to G is no longer a multiple of the identity (e.g. deformed Wigner matrices). In contrast to our approach, they used a continuous interpolation between ensembles, but we stick with the entrywise Lindeberg replacement, which is easier to adjust to multiple resolvents, similarly as in Chapter 3.

A characteristic property of the Lindeberg strategy, is that along the replacement procedure *isotropic resolvent chains* naturally arise. In particular, we have to consider the isotropic analog of Theorem 4.3.3, the average local law, as well, and need to show that also

$$\left(G_1A_1\dots G_kA_kG_{k+1}\right)_{\boldsymbol{xy}} \tag{4.6.1}$$

concentrates around a deterministic value $\left(M(z_1, A_1, \dots, z_k, A_k, z_{k+1})\right)_{\boldsymbol{xy}}$ with M given by (4.3.2). This will also be done via the Zigzag strategy (recall the outline in the beginning of Section 4.4) in Section 4.6.1.

First, analogously to Lemma 4.3.2, we have the following bound on the deterministic approximation of (4.6.1), the proof of which is deferred to Appendix 4.A.

Lemma 4.6.1 (Isotropic M-bounds)**.** *Assume the setting of Lemma 4.3.2 but with $k+1$ (instead of k) spectral parameters $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$. Then, for deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ with $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$, it holds that*[18]

$$\left|\langle \boldsymbol{x}, M_{[1,k+1]}(z_1, A_1, ..., A_k, z_{k+1})\boldsymbol{y}\rangle\right| \lesssim \prod_{i\in[k]} \|\!|A_i|\!\|_{\infty,\ell}\,, \tag{4.6.2}$$

where $\|\!|A_i|\!\|_{\infty,\ell}$ has been introduced in Definition 4.3.1.

The analog of Theorem 4.3.3 is the following *isotropic multi-resolvent local law*. The proof is given in Section 4.6.1.

Theorem 4.6.2 (Isotropic multi-resolvent local laws)**.** *Assume the setting and notations of Theorem 4.3.3 but with $k+1$ (instead of k) spectral parameters $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ be deterministic vectors with $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$. Then it holds that*

$$\left|\langle \boldsymbol{x}, G_1 A_1 ... A_k G_{k+1}\boldsymbol{y}\rangle - \langle \boldsymbol{x}, M_{[1,k+1]}\boldsymbol{y}\rangle\right| \prec \frac{1}{\sqrt{N\ell}} \prod_{i\in[k]} \|\!|A_i|\!\|_{\infty,\ell}\,, \tag{4.6.3}$$

where $\|\!|A_i|\!\|_{\infty,\ell}$ has been introduced in Definition 4.3.1.

Analogously to Theorem 4.3.3 and (4.3.9), this unifies and improves the previous local laws with operator norm (see [168, Eq. (2.11b) in Theorem 2.5]) and Hilbert-Schmidt norm (see [169, Corollary 2.4]) in the bulk of the spectrum. This follows by estimating (in the relevant $\ell \lesssim 1$ regime)

$$\|\!|A_i|\!\|_{\infty,\ell} \lesssim \begin{cases} \dfrac{\|A_i\|}{\ell^{1/2}} & \text{for [168, Eq. (2.11b), Thm. 2.5]} \\ (N\ell)^{1/2}\dfrac{\langle |A_i|^2\rangle^{1/2}}{\ell^{1/2}} & \text{for [169, Corollary 2.4]} \end{cases} \tag{4.6.4}$$

in (4.6.3) for $N\ell > 1$ and every $i \in [k]$ by means of elementary inequalities.

Note that in the isotropic law (4.6.3) we use the $\|\!|A_i|\!\|_{\infty,\ell}$ norm instead of the $\|\!|A_i|\!\|_{2k,\ell}$ norm in the corresponding averaged law (4.3.8). By taking $A_{k+1} \coloneqq N\boldsymbol{y}\boldsymbol{x}^*$ (assume for simplicity that $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$) in (4.3.8) for $k \to k+1$, it would in fact be possible to obtain an isotropic law immediately from Theorem 4.3.3. However, the bound provided in (4.6.3) is stronger than that, as can be seen by means of (4.3.6) and $\|\!|A_{k+1}|\!\|_{2(k+1),\ell} \sim N/(N\ell)^{\frac{1}{2(k+1)}}$, which yield

$$\frac{1}{\sqrt{N\ell}} \prod_{i\in[k]} \|\!|A_i|\!\|_{\infty,\ell} \lesssim \frac{1}{N} \prod_{i\in[k+1]} \|\!|A_i|\!\|_{2(k+1),\ell}\,. \tag{4.6.5}$$

In case that all A_i for $i \in [k]$ have large rank, the lhs. of (4.6.5) is in fact much smaller (by some inverse $(N\ell)$-power) than the rhs. of (4.6.5).

As for Theorem 4.3.3, we now give a concrete example how to use Theorem 4.6.2 for general (i.e. not necessarily traceless) matrices.

Example 4.6.3. *For $k = 1$, by* (4.6.3)*, we have*

$$\left|\langle \boldsymbol{x}, G_1 B G_2 \boldsymbol{y}\rangle - m_1 m_2 B_{\boldsymbol{x}\boldsymbol{y}} - \frac{m_1 m_2 \langle B\rangle \langle \boldsymbol{x}, \boldsymbol{y}\rangle}{1 - m_1 m_2}\right| \prec \frac{|\langle B\rangle|}{\sqrt{N\ell^3}} + \frac{\langle |\mathring{B}|^2\rangle^{1/2}}{\sqrt{N\ell^2}} + \frac{\|\mathring{B}\|}{\sqrt{N\ell}}\,,$$

for a general matrix $B = \mathring{B} + \langle B\rangle$, completely analogously to (4.3.11)*.*

[18] Analogously to Footnote 11, we point out that the case of k odd admits an improved bound by $\prod_{i\in[k]} \|A_i\|^{\frac{1}{k}} \|\!|A|\!\|_{\infty,\ell}^{1-\frac{1}{k}}$, but we do not follow this improvement for simplicity.

4.6.1 Isotropic law: Proof of Theorem 4.6.2

Analogously to the proof of the averaged law, Theorem 4.3.3, the proof of the isotropic law, Theorem 4.6.2, is also conducted via the Zigzag strategy with natural modifications, so we will be very brief. The initial step, the global law, has already been proven in [168, Theorem 2.5] (see Eq. (2.11b) in the $d \geq 1$ regime, which is even stronger than (4.6.6) below).

Proposition 4.6.4 (Step 1: Isotropic global law)**.** *Let W be a Wigner matrix satisfying Assumption 4.2.1, and fix $k \in \mathbf{N}$ and $\delta > 0$. Consider spectral parameters $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$, the associated resolvents $G_j = G(z_j) := (W - z_j)^{-1}$, and deterministic matrices $B_1, ..., B_k \in \mathbf{C}^{N\times N}$. Then, uniformly in spectral parameters satisfying $d := \min_{j\in[k+1]} \operatorname{dist}(z_j, [-2, 2]) \geq \delta$, deterministic matrices $B_1, ..., B_k$ and deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$ with $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$, it holds that*

$$\left|\left(G_1 B_1 ... B_k G_{k+1}\right)_{\boldsymbol{xy}} - \left(M_{[1,k+1]}\right)_{\boldsymbol{xy}}\right| \prec \frac{\prod_{i\in[k]} \|B_i\|_{\infty,d}}{\sqrt{Nd}}\,. \tag{4.6.6}$$

In the second step, the global law is propagated to a local law through the characteristic flow (4.4.1)–(4.4.2), thereby introducing an order one Gaussian component. The proof of Proposition 4.6.5 is postponed to Appendix 4.A.2.1.

Proposition 4.6.5 (Step 2: Isotropic characteristic flow)**.** *Fix any $\epsilon, \gamma > 0$, a time $0 \leq T \leq \gamma$, and $K \in \mathbf{N}$. Consider $z_{1,0}, \ldots, z_{K+1,0} \in \mathbf{C} \setminus \mathbf{R}$ as initial conditions to the solution $z_{j,t}$ of (4.4.2) for $0 \leq t \leq T$ and define $G_{j,t} := G_t(z_{j,t})$ as well as $\eta_{j,t} := |\operatorname{Im} z_{j,t}|$ and $\rho_{j,t} := \pi^{-1}|\operatorname{Im} m(z_{j,t})|$.*

Let $k \leq K$, define $\ell_t := \min_{j\in[k+1]} \eta_{j,t}\rho_{j,t}$ and recall (4.3.5). Assuming that

$$\left|\left(G_{1,0} A_1 ... A_k G_{k+1,0}\right)_{\boldsymbol{xy}} - \left(M_{[1,k+1],0}\right)_{\boldsymbol{xy}}\right| \prec \frac{1}{\sqrt{N\ell_0}} \prod_{i\in[k]} \|A_i\|_{\infty,\ell_0} \tag{4.6.7}$$

holds uniformly for any $k \leq K$, any choice of deterministic traceless $A_1, ..., A_k$, any choice of $z_{i,0}$'s such that $N\ell_0 \geq N^\epsilon$ and $\max_{j\in[k]} |z_{j,0}| \leq N^{1/\epsilon}$, and all deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$, then we have

$$\left|\left(G_{1,T} A_1 ... A_k G_{k+1,T}\right)_{\boldsymbol{xy}} - \left(M_{[1,k+1],T}\right)_{\boldsymbol{xy}}\right| \prec \frac{1}{\sqrt{N\ell_T}} \prod_{i\in[k]} \|A_i\|_{\infty,\ell_T} \tag{4.6.8}$$

for any $k \leq K$, again uniformly in traceless matrices A_i, in deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$ with $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$ and in spectral parameters satisfying $N\ell_T \geq N^\epsilon$ and $\max_{j\in[k]} |z_{j,T}| \leq N^{1/\epsilon}$.

In the third and final step, we remove the Gaussian component introduced in Proposition 4.6.5 by a GFT argument. The proof of Proposition 4.6.6 is given in Section 4.6.2.1 below.

Proposition 4.6.6 (Step 3: Isotropic Green function comparison)**.** *Let $H^{(v)}$ and $H^{(w)}$ be two $N \times N$ Wigner matrices with matrix elements given by the random variables v_{ab} and w_{ab}, respectively, both satisfying Assumption 4.2.1 and having matching moments up to third order, i.e.*

$$\mathbf{E}\, \bar{v}_{ab}^u v_{ab}^{s-u} = \mathbf{E}\, \bar{w}_{ab}^u w_{ab}^{s-u}\,, \quad s \in \{0, 1, 2, 3\}\,, \quad u \in \{0, ..., s\}\,. \tag{4.6.9}$$

Fix $K \in \mathbf{N}$ and consider spectral parameters $z_1, ..., z_{K+1} \in \mathbf{C} \setminus \mathbf{R}$ satisfying $\min_j N\eta_j\rho_j \geq N^\epsilon$ and $\max_j |z_j| \leq N^{1/\epsilon}$ for some $\epsilon > 0$ and the associated resolvents $G_j^{(\#)} = G^{(\#)}(z_j) := (H^{(\#)} - z_j)^{-1}$ with $\# = \boldsymbol{v}, \boldsymbol{w}$. Pick traceless matrices $A_1, ..., A_K \in \mathbf{C}^{N\times N}$.

For any $k \leq K$, consider any subset of cardinality $k+1$ of the $K+1$ spectral parameters, and similarly, consider any subset of cardinality k of the deterministic matrices. Relabeling them by $[k+1]$ and $[k]$, respectively, setting $\ell := \min_{j\in[k+1]} \eta_j\rho_j$ and recalling (4.3.5), we assume that for $G_j = G_j^{(v)}$ we have

$$\left|\left(G_1 A_1 ... A_k G_{k+1}\right)_{\boldsymbol{xy}} - \left(M_{[1,k+1]}\right)_{\boldsymbol{xy}}\right| \prec \frac{1}{\sqrt{N\ell}} \prod_{i\in[k]} \|A_i\|_{\infty,\ell}\,, \tag{4.6.10}$$

uniformly in all choices of subsets of z's and A's and in bounded deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$.

Then, (4.6.10) *also holds for the ensemble $H^{(w)}$, i.e. for $G_j = G_j^{(w)}$, uniformly in all choices of subsets of z's and A's and in bounded deterministic vectors $\|\boldsymbol{x}\|, \|\boldsymbol{y}\| \lesssim 1$.*

Based on Propositions 4.6.4–4.6.6, the proof of Theorem 4.6.2 is concluded in the same way as the proof of Theorem 4.3.3 in the end of Section 4.4. □

4.6.2 GFT argument: Proof of Propositions 4.4.4 and 4.6.6

The principal idea of the GFT argument is the same as in Section 3.5 and even the detailed argument almost directly translates to our case. The main difference is that we now use the conceptual *mean and standard deviation sizes* $\mathfrak{m}/\mathfrak{s}$ (recall (4.5.4)) and $\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$ (see (4.6.12) and (4.A.5) below) as basic deterministic control quantities; while in Section 3.5 they were not introduced explicitly as they essentially boiled down to simple N-powers. More precisely, in view of the bounds (3.2.17)–(3.2.18) and (3.2.20)–(3.2.21), we simply replace (ignoring all the ρ-factors and using the normalization $\langle |A_i|^2 \rangle = 1$ for the just mentioned terms in Chapter 3)

$$\frac{N^{k/2}}{N\ell} \longrightarrow \mathfrak{m}_k\,, \quad \frac{N^{k/2}}{(N\ell)^{1/2}} \longrightarrow \mathfrak{s}_k\,, \quad N^{k/2} \longrightarrow \mathfrak{m}_k^{\mathrm{iso}}\,, \quad \frac{N^{k/2}}{\ell^{1/2}} \longrightarrow \mathfrak{s}_k^{\mathrm{iso}}\,. \tag{4.6.11}$$

Given this similarity (4.6.11) we will henceforth be very brief and only point out a few minor adaptions of the proof in Section 3.5 to our new conceptual notations in Sections 4.6.2.1–4.6.2.2, discussing the isotropic and averaged case, respectively. For further simplicity of notation, we shall drop all irrelevant sub-scripts of resolvents and deterministic matrices, i.e. write $G = G_j$ and $A = A_j$. Additionally, we will also drop the $\breve{G}$- and $G^{(\gamma)}$-notations (see (3.5.10)), indicating the precise step in the replacement procedure, as it will be irrelevant for the modifications discussed below.

4.6.2.1 Part (a): Proof of the isotropic law, Proposition 4.6.6 (cf. Section 3.5.2)

We start with some preliminaries. In order to express the bounds in (4.6.2) and (4.6.3) concisely, we employ the $\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$-notation (the *isotropic mean and standard deviation sizes*, analogously to (4.5.4) in the average case) to write (4.6.2) and (4.6.3) as

$$\begin{aligned} \big|(M_{[1,k+1]})_{\boldsymbol{xy}}\big| &\lesssim (\|\!|A|\!\|_{\infty,\ell})^k =: \mathfrak{m}_k^{\mathrm{iso}}\,, \\ \Big|\big((GA)^k G\big)_{\boldsymbol{xy}} - \big(M_{[1,k+1]}\big)_{\boldsymbol{xy}}\Big| &< \frac{(\|\!|A|\!\|_{\infty,\ell})^k}{\sqrt{N\ell}} =: \frac{\mathfrak{s}_k^{\mathrm{iso}}}{N^{1/2}}\,, \end{aligned} \tag{4.6.12}$$

respectively (the general definition of $\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$ is given in (4.A.5)). Moreover, we also write $M_{j-i+1} \equiv M_{[i,j]}$ for all $1 \le i < j \le k+1$ with a slight abuse of notation (see (3.5.33)). Lastly, we will heavily use the following relations, proven in parts (i)–(ii) of Lemma 4.A.2,

$$\mathfrak{m}_j^{\mathrm{iso}}\,\mathfrak{m}_{k-j}^{\mathrm{iso}} \lesssim \mathfrak{m}_k^{\mathrm{iso}}\,, \qquad \mathfrak{s}_j^{\mathrm{iso}}\,\mathfrak{s}_{k-j}^{\mathrm{iso}} \lesssim \ell^{-1/2}\mathfrak{s}_k^{\mathrm{iso}}\,, \quad \text{and} \quad \mathfrak{m}_k^{\mathrm{iso}} \lesssim \ell^{1/2}\,\mathfrak{s}_k^{\mathrm{iso}} \tag{4.6.13}$$

for all $k \in \mathbf{N}$ and $0 \le j \le k$ using the conventions $\mathfrak{m}_0^{\mathrm{iso}} := 1$ and $\mathfrak{s}_0^{\mathrm{iso}} := \ell^{-1/2}$ (recall $\ell = \min_i \eta_i\rho_i$). With (4.6.12)–(4.6.13) at hand, we can now discuss the two bits of the argument in Section 3.5.2, which are not completely straightforward to adapt to our current setting (cf. Case (i) and Case (ii) in Section 3.5.2.2). We will refer to explicit equation numbers within a longer proof in Chapter 3 that we do not repeat here, hence the reader needs to be familiar with Section 3.5.2 to follow the details. However, to facilitate a high level understanding without going into details, we recall the novel idea in Chapter 3. Traditional GFT proofs via the Lindeberg strategy estimated the change of, say, $\mathbf{E}\big|\big((GA)^kG\big)_{\boldsymbol{xy}}\big|^p$ after *each* replacement $\boldsymbol{v}_{ab} \to \boldsymbol{w}_{ab}$ and showed that it is bounded by $o(N^{-2})$-times

its natural target size, so N^2 replacements were affordable. In other words the telescopic summation over $(a,b) \in [N]^2$ was done trivially. This does not hold for the self-consistent GFT argument in Chapter 3: the replacement for some (a,b) pairs may be too large, but their sum is still affordable. We will refer to it as the *gain from the summation* idea. This is related to our efforts to control the observables in tracial norms; the simplest toy example is the estimate

$$\frac{1}{N^2}\sum_{ab}|A_{ab}| \le N^{-1/2}\langle |A|^2\rangle^{1/2} \tag{4.6.14}$$

which is optimal. However, if we use the best available bound $|A_{ab}| \le \|A\|$ for each summand individually, then the lhs. of (4.6.14) is overestimated by $\|A\|$ which is much bigger than the rhs. With keeping this idea in mind, we now return to the detailed modifications in Section 3.5.2.2.

The terms considered in Case (i) are the ones for which there are enough $(G-M)$-type terms to balance the N-prefactor (cf. (4.6.15) and (4.6.16)) by solely using (4.6.13). The terms in Case (ii) require to *gain from the summation* over all steps in the replacement procedure.

Case (i): First, the analog of (3.5.34) becomes (neglecting the irrelevant $|\Psi_k|^{p-1}$- and N^ξ-factors)

$$\begin{aligned}
&\frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}}\sum_{\substack{0\le k_l\le k-1:\\ \sum_l k_l=k}}\mathbf{E}\Big[\big|\big((GA)^{k_1}G-M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\ldots\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big| \\
&\qquad +\big|\big((GA)^{k_1}G-M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big((GA)^{k_2}G-M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\ldots\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big|+\ldots\Big] \\
&\lesssim \frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}}\sum_{\substack{0\le k_l\le k-1:\\ \sum_l k_l=k}}\left[\frac{\mathfrak{s}_{k_1}^{\mathrm{iso}}}{N^{1/2}}\left(\prod_{l=2}^5\mathfrak{m}_{k_l}^{\mathrm{iso}}\right)+\frac{\mathfrak{s}_{k_1}^{\mathrm{iso}}\mathfrak{s}_{k_2}^{\mathrm{iso}}}{N}\left(\prod_{l=3}^5\mathfrak{m}_{k_l}^{\mathrm{iso}}\right)+\ldots\right]\lesssim\left[1+\frac{1}{(N\ell)^{1/2}}+\ldots\right]\lesssim 1\,.
\end{aligned} \tag{4.6.15}$$

In the penultimate step we used the relations in (4.6.13) multiple times to estimate $\mathfrak{s}_{k_1}^{\mathrm{iso}}\prod_{l=2}^5\mathfrak{m}_{k_l}^{\mathrm{iso}}\lesssim \mathfrak{s}_k^{\mathrm{iso}}$ and $\mathfrak{s}_{k_1}^{\mathrm{iso}}\mathfrak{s}_{k_l}^{\mathrm{iso}}\prod_{l=3}^5\mathfrak{m}_{k_l}^{\mathrm{iso}}\lesssim \ell^{-1/2}\mathfrak{s}_k^{\mathrm{iso}}$, and similarly for the other analogous terms indicated by dots. The last step in (4.6.15) is due to $N\ell>1$.

Case (ii): Second, the key trick in Section 3.5.2.2, the *gain from summations* described in Example 3.5.7, turns into the following. The *trivial estimate*, cf. (3.5.36), reads (again neglecting the irrelevant $|\Psi_k|^{p-1}$- and N^ξ-factors)

$$\begin{aligned}
&\frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}}\sum_{\substack{0\le k_l\le k:\\ \sum_l k_l=k}}\left[\big|\big(M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\ldots\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big|+\ldots\right] \\
&\lesssim \frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}}\sum_{\substack{0\le k_l\le k:\\ \sum_l k_l=k}}\left[\left(\prod_{l=1}^5\mathfrak{m}_{k_l}^{\mathrm{iso}}\right)+\ldots\right]\lesssim (N\ell)^{1/2}\,,
\end{aligned} \tag{4.6.16}$$

where we again used (4.6.13) multiple times. Analogously to (3.5.37), this can be improved on by averaging over all replacement steps: Fixing one constellation of k_l's in (4.6.16), we find

$$\begin{aligned}
&\frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}}\frac{1}{N^2}\sum_{i,j}\left[\big|\big(M_{k_1+1}\big)_{\boldsymbol{x}\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\ldots\big(M_{k_5+1}\big)_{\boldsymbol{e}_i\boldsymbol{y}}\big|+\ldots\right] \\
&\qquad\lesssim (N\ell)^{1/2}\frac{1}{N^2}\sum_{i,j}\left[\frac{\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|}{\mathfrak{m}_{k_1}^{\mathrm{iso}}}+\ldots\right]\lesssim (N\ell)^{1/2}\frac{1}{N^{1/2}}\lesssim 1
\end{aligned} \tag{4.6.17}$$

in the relevant $\ell\le 1$ regime (the opposite regime being covered by Proposition 4.6.4). To go to the second line, we employed (4.6.13). In the penultimate step, we used

$$\sum_i\big|(M_{k_1+1})_{\boldsymbol{x}\boldsymbol{e}_i}\big|\le\sqrt{N}\sqrt{\big(|M_{k_1+1}|^2\big)_{\boldsymbol{x}\boldsymbol{x}}}\lesssim\sqrt{N}\,\mathfrak{m}_{k_1}^{\mathrm{iso}}\,, \tag{4.6.18}$$

which follows by a Schwarz inequality, completely analogously to (3.5.38)–(3.5.39).

With these two slight adjustments, which rest on the estimates in (4.6.13), the proof in Section 3.5.2 can entirely be translated to our current setting in a straightforward way, so we omit further details. This concludes the proof of Proposition 4.6.6. □

4.6.2.2 Part (b): Proof of the averaged law, Proposition 4.4.4 (cf. Section 3.5.3)

For the averaged law, in addition to the bounds in (4.6.12)–(4.6.13), we will also use the relations

$$\mathfrak{m}_k \lesssim \mathfrak{m}_k^{\mathrm{iso}}\,, \qquad \mathfrak{s}_k \lesssim \ell^{1/2}\,\mathfrak{s}_k^{\mathrm{iso}}\,, \quad \text{and} \quad \mathfrak{s}_k^{\mathrm{iso}} \lesssim N\big[1+(N\ell)^{-1/2}\big]\mathfrak{s}_k \tag{4.6.19}$$

from (4.A.11)–(4.A.12) in Lemma 4.A.3 for all $k \in \mathbf{N}$. Just as in the case of Section 4.6.2.1, there are two bits of the argument in Section 3.5.3, that are not entirely straightforward to adjust to the setting of the current paper (cf. Case (i) and Case (ii) in Section 3.5.3).

Analogously to the isotropic case in Section 4.6.2.1, while the terms considered in Case (i) solely use (4.6.13) and (4.6.19), the terms in Case (ii) require to *gain from the summation* over all steps in the replacement procedure.

Case (i): First, consider (3.5.54) with $d = 2$ (for concreteness). Then, the estimate turns into (neglecting the irrelevant N^{ξ}-factor)

$$\Big(\frac{N}{\mathfrak{s}_k}\Big)^2 \frac{1}{N^2} \sum_{\substack{k_1+k_2=k\\ k_1'+k_2'=k}} \bigg[\mathfrak{m}_{k_1}^{\mathrm{iso}} \frac{\mathfrak{s}_{k_2}^{\mathrm{iso}}}{N^{1/2}} \mathfrak{m}_{k_1'}^{\mathrm{iso}} \frac{\mathfrak{s}_{k_2'}^{\mathrm{iso}}}{N^{1/2}} + \mathfrak{m}_{k_1}^{\mathrm{iso}} \mathfrak{m}_{k_2}^{\mathrm{iso}} \frac{\mathfrak{s}_{k_1'}^{\mathrm{iso}}}{N^{1/2}} \frac{\mathfrak{s}_{k_2'}^{\mathrm{iso}}}{N^{1/2}} + \ldots\bigg] \lesssim \bigg[1+\frac{1}{N\ell}\bigg] \lesssim 1\,,$$

where we used (4.6.13) and (4.6.19) to bound $\mathfrak{m}_{k_1}^{\mathrm{iso}}\mathfrak{s}_{k_2}^{\mathrm{iso}} \lesssim \ell^{-1/2}[1+(N\ell)^{1/2}]\mathfrak{s}_k$ as well as $\mathfrak{m}_{k_1}^{\mathrm{iso}}\mathfrak{m}_{k_2}^{\mathrm{iso}} \lesssim [1+(N\ell)^{1/2}]\mathfrak{s}_k$ and $\mathfrak{s}_{k_1'}^{\mathrm{iso}}\mathfrak{s}_{k_2'}^{\mathrm{iso}} \lesssim \ell^{-1}[1+(N\ell)^{1/2}]\mathfrak{s}_k$ in the penultimate step, and the last step used $N\ell > 1$.

Case (ii): Second, we again discuss how to *gain from the summation*, as originally explained in Example 3.5.11. The *trivial estimate* from [150, Eq. (5.55)], again neglecting the irrelevant $|\Psi_k|^{p-1}$ factor and fixing one constellation of k_l's summing up to k, becomes (neglecting the irrelevant N^{ξ}-factor)

$$\begin{aligned}(\mathfrak{s}_k)^{-1}\Big[\big|\big(M_{k_1+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big| + \ldots\Big]\\ \lesssim (\mathfrak{s}_k)^{-1}\bigg[\prod_l \mathfrak{m}_{k_l}^{\mathrm{iso}} + \ldots\bigg] \lesssim (N\ell)^{1/2}\,,\end{aligned} \tag{4.6.20}$$

where we used (4.6.13) and (4.6.19). Analogously to (3.5.59), we can again improve upon this by averaging over all replacement positions. The key for this *gain from the summation* is the bound

$$\big\langle |M_{k+1}|^2 \big\rangle \lesssim (\mathfrak{s}_k)^2 \qquad \forall k \in \mathbf{N}\,, \tag{4.6.21}$$

which can be obtained completely analogously to Lemma 3.5.12. Armed with (4.6.21), the analog of the improved bound (3.5.59) now reads (again neglecting the irrelevant N^{ξ}-factor)

$$\begin{aligned}&(\mathfrak{s}_k)^{-1}\frac{1}{N^2}\sum_{i,j}\Big[\big|\big(M_{k_1+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_2+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big(M_{k_3+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big(M_{k_4+1}\big)_{\boldsymbol{e}_j\boldsymbol{e}_j}\big| + \ldots\Big]\\ \lesssim&(\mathfrak{s}_k)^{-1}\bigg[\prod_{l\in[4]}\bigg(\frac{1}{N}\sum_i \big|\big(M_{k_l+1}\big)_{\boldsymbol{e}_i\boldsymbol{e}_i}\big|^2\bigg)^{1/2} + \ldots\bigg] \lesssim (\mathfrak{s}_k)^{-1}\bigg[\prod_{l\in[4]}\mathfrak{s}_{k_l} + \ldots\bigg] \lesssim 1\,,\end{aligned} \tag{4.6.22}$$

where in the first step we employed a trivial Schwarz inequality.

With the above two slight adjustments at hand, which basically rest on the estimates in (4.6.13), (4.6.19) and (4.6.21), we can straightforwardly translate the entire proof of the averaged part in Section 3.5.3 to our current setting. This finishes our discussion of the adjustments of the arguments from Section 3.5 and thus the proof of Proposition 4.4.4. □

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4.A Additional technical results

In this section we prove several additional technical results which were used in the main sections.

4.A.1 Bound on the deterministic approximation: Proofs of Lemmas 4.3.2 and 4.6.1

We will deduce Lemmas 4.3.2 and 4.6.1 from the following stronger bound. Part (a) of Lemma 4.A.1 is already proven in Lemma 3.A.1 applied to the special case $\mathfrak{I}_k = \varnothing$. The proof of part (b) is analogous to that and hence omitted.

Lemma 4.A.1 (cf. Lemma 3.A.1)**.** *Fix $k \geq 1$ and let $A_1, ..., A_k \in \mathbf{C}^{N\times N}$ be traceless deterministic matrices.*

(a) Consider spectral parameters $z_1, ..., z_k \in \mathbf{C} \setminus \mathbf{R}$ and define $\eta := \min_{j\in[k]} |\mathrm{Im}\, z_j|$. Then, for every $1 \leq s \leq \lfloor k/2 \rfloor$ and $\pi \in \mathrm{NC}([k])$ with $|\pi| = k + 1 - s$, it holds that

$$\left|\langle \mathrm{pTr}_{K(\pi)}(A_1, \ldots, A_{k-1})A_k\rangle \prod_{S\in\pi} m_\circ[S]\right| \lesssim \frac{1}{\eta^{s-1}} \left(\prod_{\substack{S\in K(\pi)\\ |S|\geq 2}} \prod_{j\in S} \langle |A_j|^{|S|}\rangle^{\frac{1}{|S|}} \right). \tag{4.A.1}$$

with $m_\circ[S]$ being defined in (4.3.3)*. For $s > \lfloor k/2 \rfloor$ the lhs. of* (4.A.1) *equals zero.*

(b) Consider spectral parameters $z_1, ..., z_{k+1} \in \mathbf{C} \setminus \mathbf{R}$ and define $\eta := \min_{j\in[k+1]} |\mathrm{Im}\, z_j|$. Then, for every $1 \leq s \leq \lceil (k+1)/2 \rceil$ and $\pi \in \mathrm{NC}([k+1])$ with $|\pi| = k + 1 - s$, it holds that

$$\left\| \mathrm{pTr}_{K(\pi)}(A_1, \ldots, A_k) \prod_{S\in\pi} m_\circ[S] \right\| \lesssim \frac{1}{\eta^{s-1}} \left(\prod_{\substack{S\in K(\pi)\setminus\mathfrak{B}:\\ |S|\geq 2}} \prod_{j\in S} \langle |A_j|^{|S|}\rangle^{\frac{1}{|S|}} \right) \left(\prod_{j\in\mathfrak{B}\setminus\{k+1\}} \|A_j\| \right). \tag{4.A.2}$$

where $\mathfrak{B} \equiv \mathfrak{B}(k+1) \in K(\pi)$ being the unique block containing $k+1$ (recall (4.3.1)*). For $s > \lceil (k+1)/2 \rceil$ the lhs. of* (4.A.2) *equals zero.*

Given Lemma 4.A.1, analogously to Appendix 3.A.1, the proofs of Lemmas 4.3.2 and 4.6.1 immediately follow after realizing that $\eta \gtrsim \ell$ and applying Hölder's and Young's inequality. To show this mechanism, consider (4.A.1) for an example where $k = 6$ and $s = 2$, and focus on the non-crossing partition $\pi = \{15|2|3|4|6\}$ for which $K(\pi) = \{1234|56\}$. In this case, the rhs. of (4.A.1) can be estimated as

$$\begin{aligned}
&\eta^{-1} \left(\prod_{i=1}^{4} \langle |A_i|^4\rangle^{1/4} \right) \left(\prod_{i=5}^{6} \langle |A_i|^2\rangle^{1/2} \right) \\
&\lesssim \ell \left(\prod_{i=1}^{4} \left[\langle |A_i|^6\rangle^{1/8} \ell^{-1/8} \right] \left[\langle |A_i|^2\rangle^{1/8} \ell^{-1/8} \right] \right) \left(\prod_{i=5}^{6} \langle |A_i|^2\rangle^{1/2} \ell^{-1/2} \right) \\
&\lesssim \ell \prod_{i\in[6]} \left(\frac{\langle |A_i|^6\rangle^{1/6}}{\ell^{1/6}} + \frac{\langle |A_i|^2\rangle^{1/2}}{\ell^{1/2}} \right).
\end{aligned}$$

In the first step, we employed $\eta \gtrsim \ell$ and Hölder's inequality in the form $\langle |A_i|^4\rangle^{1/4} \leq \langle |A_i|^6\rangle^{1/8} \langle |A_i|^2\rangle^{1/8}$ for $i \in [4]$. In the second step, we then used Young's inequality for $i \in [4]$ and added $\langle |A_i|^6\rangle^{1/6} \ell^{-1/6}$ for $i = 5, 6$ to complete the $\|\!|A_i|\!\|_{6,\ell}$ norm for all $i \in [6]$ (recall (4.3.5) and (4.3.7)). □

4.A.2 GFT and isotropic local law: Additional proofs for Section 4.6

4.A.2.1 Isotropic law: Proof of Theorem 4.6.2

In this section we want to study chains of the form

$$(G_1 A_1 \ldots A_k G_{k+1})_{\boldsymbol{xy}} := \langle \boldsymbol{x}, G_1 A_1 \ldots A_k G_{k+1} \boldsymbol{y}\rangle, \tag{4.A.3}$$

for unit deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$.

Following the notation (4.5.3), by $G_{[1,k+1],t}$ we denote the evolution of the quantity in (4.A.3) along the Ornstein-Uhlenbeck flow (4.4.1) with the characteristic equation (4.4.2). Then, by (4.5.2) and Lemma 4.5.5 (used for $k \to k+1$), choosing $A_{k+1} = N\boldsymbol{y}\boldsymbol{x}^*$, we obtain the flow

$$\begin{aligned}
\mathrm{d}(G_{[1,k+1],t} - M_{[1,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} = {} & \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} \partial_{ab}(G_{[1,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}B_{ab,t} + \frac{k}{2}(G_{[1,k+1],t} - M_{[1,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}t \\
& + \sum_{\substack{i,j=1 \\ i<j}}^{k+1} \langle G_{[i,j],t} - M_{[i,j],t} \rangle (M_{[1,i]\cup[j,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}t \\
& + \sum_{\substack{i,j=1 \\ i<j}}^{k+1} \langle M_{[i,j],t} \rangle (G_{[1,i]\cup[j,k+1],t} - M_{[1,i]\cup[j,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}t \\
& + \sum_{\substack{i,j=1 \\ i<j}}^{k+1} \langle G_{[i,j],t} - M_{[i,j],t} \rangle (G_{[1,i]\cup[j,k+1],t} - M_{[1,i]\cup[j,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}t \\
& + \sum_{i=1}^{k+1} \langle G_{i,t} - m_{i,t} \rangle (G^{(i)}_{[1,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}\,,
\end{aligned} \tag{4.A.4}$$

where we used the notation

$$(G_{[1,i]\cup[j,k+1],t})_{\boldsymbol{x}\boldsymbol{y}} \coloneqq (G_{1,t}A_1 \dots A_{i-1}G_{i,t}G_{j,t}A_j \dots A_{k-1}G_{k+1,t})_{\boldsymbol{x}\boldsymbol{y}}$$

and similarly for $M_{[1,i]\cup[j,k+1],t}$.

Deterministic control quantities. Next, we introduce the new *isotropic* control quantities, the analogues of the *averaged* quantities defined in (4.5.4) (recall (4.3.5) for the definition of $\|A_\alpha\|_{\infty,\ell}$):

$$\mathfrak{m}_k^{\mathrm{iso}}(\ell; \boldsymbol{A}_{J_k}) \coloneqq \prod_{\alpha\in J_k} \|A_\alpha\|_{\infty,\ell}\,, \quad \mathfrak{s}_k^{\mathrm{iso}}(\ell; \boldsymbol{A}_{J_k}) \coloneqq \ell^{-1/2} \prod_{\alpha\in J_k} \|A_\alpha\|_{\infty,\ell}\,, \tag{4.A.5}$$

which will be called the *isotropic mean size* and *isotropic standard deviation size*, respectively. They are functions of a positive number ℓ (usually given by $\ell = \min_i \eta_i\rho_i$ for some spectral parameters z_i) and a multiset $\boldsymbol{A}_{J_k} = (A_\alpha)_{\alpha\in J_k}$ of deterministic matrices of cardinality $|J_k| = k$.

Similarly to the paragraph below (4.5.4), we may often omit the arguments of $\mathfrak{m}_k^{\mathrm{iso}}$ and $\mathfrak{s}_k^{\mathrm{iso}}$, write J_k instead of $\boldsymbol{A}_{J_k}$, and for time dependent spectral parameters we use the short–hand notations $\mathfrak{m}_k^{\mathrm{iso}}(t) = \mathfrak{m}_k^{\mathrm{iso}}(t; J_k) = \mathfrak{m}_k^{\mathrm{iso}}(\ell_t; \boldsymbol{A}_{J_k})$ and $\mathfrak{s}_k^{\mathrm{iso}}(t) = \mathfrak{s}_k^{\mathrm{iso}}(t; J_k) = \mathfrak{s}_k^{\mathrm{iso}}(\ell_t; \boldsymbol{A}_{J_k})$. Note that with this definitions, by (4.6.2), we have (analogously to (4.5.5)), with $\eta_t \coloneqq \min_i \eta_{i,t}$,

$$\big|(M_{[i,j],t})_{\boldsymbol{x}\boldsymbol{y}}\big| \lesssim \mathfrak{m}_{j-i}^{\mathrm{iso}}(t) \qquad \text{and} \qquad \big|(M_{[1,i]\cup[j,k+1],t})_{\boldsymbol{x}\boldsymbol{y}}\big| \lesssim \eta_t^{-1}\mathfrak{m}_{k-(j-i)}^{\mathrm{iso}}(t)\,. \tag{4.A.6}$$

We now record the following relations about $\mathfrak{m}_k^{\mathrm{iso}}/\mathfrak{s}_k^{\mathrm{iso}}$, analogously to Lemma 4.5.1. The proof is again an immediate consequence of (4.3.6) and $\ell_s \gtrsim \ell_t$ for $s \le t$ (recall the discussion below (4.4.5)) and hence omitted.

Lemma 4.A.2 ($\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$-relations)**.** *Let $k \ge 1$ and consider time-dependent spectral parameters $z_{1,t}, ..., z_{k+1,t}$ and a multiset of traceless matrices $\boldsymbol{A}_{J_k}$ as above. Set $\ell_t \coloneqq \min_{j\in[k+1]} \eta_{j,t}\rho_{j,t}$. Then, the mean size $\mathfrak{m}_k^{\mathrm{iso}}$ and the standard deviation size $\mathfrak{s}_k^{\mathrm{iso}}$ satisfy the following inequalities (using the conventions $\mathfrak{m}_0^{\mathrm{iso}}(t) \coloneqq 1$ and $\mathfrak{s}_0^{\mathrm{iso}}(t) \coloneqq \ell_t^{-1/2}$).*

(i) Super-multiplicativity; i.e. for any $0 \le j \le k$ it holds that

$$\mathfrak{m}_j^{\mathrm{iso}}(t; J_j)\,\mathfrak{m}_{k-j}^{\mathrm{iso}}(t; J_{k-j}) \lesssim \mathfrak{m}_k^{\mathrm{iso}}(t; J_k)\,, \quad \mathfrak{s}_j^{\mathrm{iso}}(t; J_j)\,\mathfrak{s}_{k-j}^{\mathrm{iso}}(t; J_{k-j}) \lesssim \ell_t^{-1/2}\mathfrak{s}_k^{\mathrm{iso}}(t; J_k) \tag{4.A.7}$$

for all disjoint decompositions $J_k = J_j \cup J_{k-j}$.

(ii) The mean size can be upper bounded by the standard deviation size as

$$\mathfrak{m}_k^{\mathrm{iso}}(t;J_k) \lesssim \ell_t^{1/2}\,\mathfrak{s}_k^{\mathrm{iso}}(t;J_k)\,. \tag{4.A.8}$$

(iii) The standard deviation size satisfies the doubling inequality

$$\mathfrak{s}_{2k}^{\mathrm{iso}}(t;J_k\cup J_k) \lesssim \ell_t^{1/2}\,\big(\mathfrak{s}_k^{\mathrm{iso}}(t;J_k)\big)^2\,, \tag{4.A.9}$$

where $J_k \cup J_k$ denotes the union of the multiset J_k with itself.

(iv) Monotonicity in time: for any $0 \le s \le t$ and $\alpha \in [0,1/2]$, we have

$$\mathfrak{m}_k^{\mathrm{iso}}(s;J_k) \lesssim \mathfrak{m}_k^{\mathrm{iso}}(t;J_k) \qquad \text{and} \qquad \mathfrak{s}_k^{\mathrm{iso}}(s;J_k)\,\ell_s^{\alpha} \lesssim \mathfrak{s}_k^{\mathrm{iso}}(t;J_k)\,\ell_t^{\alpha}\,. \tag{4.A.10}$$

Moreover, we have the following relations among $\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$ and $\mathfrak{m}/\mathfrak{s}$ from (4.5.4), whose proofs are again immediate from (4.3.6) and hence omitted.

Lemma 4.A.3 ($\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$-$\mathfrak{m}/\mathfrak{s}$-relations)**.** *Let $k \ge 1$, consider a multiset $\boldsymbol{A}_{J_k}$ of traceless matrices as above and fix $\ell > 0$. Then, the mean sizes $\mathfrak{m}_k^{\mathrm{iso}}$ and $\mathfrak{m}_k$, and the standard deviation sizes $\mathfrak{s}_k^{\mathrm{iso}}$ and $\mathfrak{s}_k$, satisfy the following inequalities.*

(i) The average sizes $\mathfrak{m}/\mathfrak{s}$ *are bounded by the isotropic sizes $\mathfrak{m}^{\mathrm{iso}}/\mathfrak{s}^{\mathrm{iso}}$:*

$$\mathfrak{m}_k(\ell;J_k) \lesssim \mathfrak{m}_k^{\mathrm{iso}}(\ell;J_k)\,, \quad \mathfrak{s}_k(\ell;J_k) \lesssim \ell^{1/2}\,\mathfrak{s}_k^{\mathrm{iso}}(\ell;J_k)\,. \tag{4.A.11}$$

(ii) The isotropic standard deviation size can be upper bounded by the average standard deviation size

$$\mathfrak{s}_k^{\mathrm{iso}}(\ell;J_k) \lesssim N\big[1+(N\ell)^{-1/2}\big]\,\mathfrak{s}_k(\ell;J_k)\,. \tag{4.A.12}$$

Stochastic control quantities. Consider deterministic vectors $\boldsymbol{x},\boldsymbol{y} \in \mathbf{C}^N$ with $\|\boldsymbol{x}\|,\|\boldsymbol{y}\| \lesssim 1$, traceless matrices $A_1,\dots,A_k$, and for $k \ge 1$ define, analogously to (4.5.11)–(4.5.12), $\Psi_k^{\mathrm{iso}}(t) \equiv \Psi_k^{\mathrm{iso}}(t;\boldsymbol{x},\boldsymbol{y};\boldsymbol{z}_{[1,k+1]};\boldsymbol{A}_{[1,k]})$ and $\Phi_k^{\mathrm{iso}}(t) \equiv \Phi_k^{\mathrm{iso}}(t;\boldsymbol{x},\boldsymbol{y};\boldsymbol{z}_{[1,k+1]};\boldsymbol{A}_{[1,k]})$ as

$$\begin{aligned}
\Psi_k^{\mathrm{iso}}(t) &:= \frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}(\ell_t;\boldsymbol{A}_{[1,k]})(N\ell_t)^{1/4}}\big|(G_{[1,k+1],t}-M_{[1,k+1],t})_{\boldsymbol{xy}}\big|\,,\\
\Phi_k^{\mathrm{iso}}(t) &:= \frac{N^{1/2}}{\mathfrak{s}_k^{\mathrm{iso}}(\ell_t;\boldsymbol{A}_{[1,k]})}\big|(G_{[1,k+1],t}-M_{[1,k+1],t})_{\boldsymbol{xy}}\big|\,,
\end{aligned} \tag{4.A.13}$$

where we denoted the multisets of deterministic matrices by $\boldsymbol{A}_{[1,k]}$ and $\boldsymbol{z}_{[1,k+1]} = (z_1,\dots,z_{k+1})$ with $z_i = z_{i,0}$ (initial condition), respectively. Note that by (4.1.4), using the convention $\mathfrak{s}_0^{\mathrm{iso}}(t) := \ell_t^{-1/2}$, we have $\Phi_0(t) \prec 1$, and that by (4.6.6) it follows

$$\Phi_k^{\mathrm{iso}}(0) \prec 1, \tag{4.A.14}$$

for any $k \ge 1$. Similarly to the averaged case (see the two paragraphs around (4.5.11)–(4.5.12)), also in the isotropic case we introduced the two quantities in (4.A.13) as we will first prove $\Psi_k^{\mathrm{iso}}(t) = \Phi_k^{\mathrm{iso}}(t)/(N\ell_t)^{1/4} \prec 1$ using the master inequalities in Proposition 4.A.4, and then use this as an input to prove $\Phi_k^{\mathrm{iso}}(t) \prec 1$ in Proposition 4.A.5.

Proposition 4.A.4 (Isotropic master inequalities)**.** *Fix $k \in \mathbf{N}$, assume that $\Psi_l^{\mathrm{iso}}(t) \prec \psi_l^{\mathrm{iso}}$, with $\psi_0^{\mathrm{iso}} := 1$ and $\Phi_l(t) \prec 1$ (with $\Phi_l(t)$ from (4.5.12)), for any $0 \le l \le k+\mathbf{1}(k\,\mathrm{odd})$ uniformly in $t \in [0,T]$. Then*

$$\Psi_k^{\mathrm{iso}}(t) \prec 1 + \frac{\psi_k^{\mathrm{iso}} + \psi_{k+\mathbf{1}(k\,\mathrm{odd})}^{\mathrm{iso}}}{(N\ell_T)^{1/8}} + \sum_{l=1}^{k-1} \psi_l^{\mathrm{iso}} \tag{4.A.15}$$

uniformly in $t \in [0,T]$.

Analogously to Section 4.5, using Proposition 4.A.4 we show that $\Psi_k^{\mathrm{iso}}(t) \prec 1$ and then use this information as input to conclude $\Phi_k^{\mathrm{iso}}(t) \prec 1$:

Proposition 4.A.5. *Fix $k \in \mathbf{N}$, assume that $\Phi_l^{\mathrm{iso}}(t) \prec 1$, for $0 \le l \le k-2$, $\Phi_l(t) \prec 1$ for $0 \le l \le k$ (with $\Phi_l(t)$ from (4.5.11)), and $\Phi_k(t) \prec (N\ell_t)^{1/4}$ for $l \le 2k$, uniformly in $t \in [0,T]$. Then*

$$\Phi_k^{\mathrm{iso}}(t) \prec 1 \tag{4.A.16}$$

uniformly in $t \in [0,T]$.

Closing the hierarchy. We start showing that (4.A.15), using *iteration* as in Lemma 4.5.4, in fact implies $\Psi_k(t) \prec 1$. We start with $k = 1, 2$. By (4.A.15), we have

$$\Psi_1^{\mathrm{iso}}(t) \prec 1 + \frac{\psi_1^{\mathrm{iso}} + \psi_2^{\mathrm{iso}}}{(N\ell_T)^{1/8}}, \qquad \Psi_2^{\mathrm{iso}}(t) \prec 1 + \frac{\psi_2^{\mathrm{iso}}}{(N\ell_T)^{1/8}} + \psi_1^{\mathrm{iso}}. \tag{4.A.17}$$

Then, by iteration, we obtain

$$\Psi_1^{\mathrm{iso}}(t) \prec 1 + \frac{\psi_2^{\mathrm{iso}}}{(N\ell_T)^{1/8}}, \qquad \Psi_2^{\mathrm{iso}}(t) \prec 1 + \psi_1^{\mathrm{iso}}. \tag{4.A.18}$$

Finally, plugging the first inequality into the second one and using iteration once again we obtain $\Psi_1^{\mathrm{iso}}(t) + \Psi_2^{\mathrm{iso}}(t) \prec 1$.

Next, we proceed by induction. Fix an even $k \in \mathbf{N}$, and assume that $\Psi_l^{\mathrm{iso}}(t) \prec 1$ for $l \le k-2$, then by (4.A.15) we have

$$\Psi_{k-1}^{\mathrm{iso}}(t) \prec 1 + \frac{\psi_{k-1}^{\mathrm{iso}} + \psi_k^{\mathrm{iso}}}{(N\ell_T)^{1/8}}, \qquad \Psi_k^{\mathrm{iso}}(t) \prec 1 + \frac{\psi_k^{\mathrm{iso}}}{(N\ell_T)^{1/8}} + \psi_{k-1}^{\mathrm{iso}}. \tag{4.A.19}$$

Proceeding exactly as in (4.A.17)–(4.A.18), by (4.A.19), we conclude that $\Psi_l^{\mathrm{iso}}(t) \prec 1$ for any $l \le k$. Finally, using that as a consequence of $(N\ell_t)^{1/4}\Phi_k^{\mathrm{iso}}(t) = \Psi_l^{\mathrm{iso}}(t) \prec 1$ the hypothesis of Proposition 4.A.5 are satisfied, we conclude that $\Phi_k^{\mathrm{iso}}(t) \prec 1$ and so the proof of Theorem 4.6.2. □

4.A.2.2 Isotropic master inequalities: Proofs of Props. 4.A.4–4.A.5

Proof of Proposition 4.A.4. Note that the second term in the first line of (4.A.4) can be incorporated into the lhs. by differentiating $e^{-kt/2}(G_{[1,k+1,t]} - M_{[1,k+1,t]})_{\boldsymbol{xy}}$. The exponential factor $e^{kt/2} \sim 1$ is irrelevant, we thus neglect this term from the analysis. In the following we will often use the simple bound (4.5.1) even if not stated explicitly. Additionally, every time that two resolvents get next to each other in a chain we use the integral representation (4.5.23) to reduce their number by one at the price of an additional $1/\eta$–factor (see e.g. (4.5.22)).

We now start with the computation of the quadratic variation of the martingale term in (4.A.4):

$$\frac{1}{N}\sum_{i=1}^{k+1} \big[G_{[1,i],t}(G_{[1,i],t})^*\big]_{\boldsymbol{xx}} \big[(G_{[i,k+1],t})^* G_{[i,k+1],t}\big]_{\boldsymbol{yy}}\,\mathrm{d}t. \tag{4.A.20}$$

Similarly to the averaged case (see (4.5.26)–(4.5.30)), to estimate the quadratic variation of the martingale term in (4.A.4) we rely on the following *reduction inequality* for $k \in \mathbf{N}$ even and $j \le k$ (here for simplicity we drop the indices of G's and A's):

$$\big|(G_{[1,k+j+1]})_{\boldsymbol{xy}}\big| \lesssim \sqrt{N}\langle |G|A(GA)^{j-1}|G|(AG^*)^{j-1}A\rangle^{1/2} \prod_{\boldsymbol{v}\in\{\boldsymbol{x},\boldsymbol{y}\}} [(GA)^{k/2}|G|(AG^*)^{k/2}]_{\boldsymbol{vv}}^{1/2}. \tag{4.A.21}$$

The proof of (4.A.21) is postponed to the end of this section. Recall the conventions $\psi_0^{\mathrm{iso}} = 1$ and $\mathfrak{s}_0^{\mathrm{iso}}(t) = \ell_t^{-1/2}$, then using (4.A.21) for $j = k-2i+2$, $i \le k/2$ for even k (and $j = k-2i+1$, $i \le (k+1)/2$ for odd k) and then (4.A.8), (4.5.7), to bound (recall (4.5.27) in Footnote 17)

$$\begin{aligned}
&\big[(G_{[i,k+1],s})^* G_{[i,k+1],s}\big]_{\boldsymbol{yy}} \\
&\prec \sqrt{N}\left(\mathfrak{m}_k^{\mathrm{iso}}(s) + \frac{\mathfrak{s}_k^{\mathrm{iso}}(s)(N\ell_s)^{1/4}}{\sqrt{N}}\right)\left(\ell_s \mathfrak{m}_{2(k-2i+2)}(s) + \frac{\mathfrak{s}_{2(k-2i+2)}(s)}{N}\right)^{1/2} \\
&\lesssim \mathfrak{s}_k^{\mathrm{iso}}(s)\sqrt{\mathfrak{s}_{2(k-2i+2)}(s)}\ell_s^{1/2}\big(\sqrt{N\ell_t} + (N\ell_t)^{1/4}\psi_k^{\mathrm{iso}}\big),
\end{aligned} \tag{4.A.22}$$

we estimate each term of (4.A.20) by

$$\begin{aligned}
&\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\left(\frac{1}{N}\sum_{i=1}^{k+1}\int_0^t \big[G_{[1,i],s}(G_{[1,i],s})^*\big]_{\boldsymbol{xx}}\big[(G_{[i,k+1],s})^* G_{[i,k+1],s}\big]_{\boldsymbol{yy}}\,\mathrm{d}s\right)^{1/2} \\
&\prec \frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\Bigg(\sum_{i=1}^{\lceil k/2\rceil}\int_0^t \frac{\mathfrak{s}_k^{\mathrm{iso}}(s)}{N\eta_s^2}\left(\mathfrak{m}_{2(i-1)}^{\mathrm{iso}}(s) + \frac{\psi_{2(i-1)}^{\mathrm{iso}}\mathfrak{s}_{2(i-1)}^{\mathrm{iso}}(s)(N\ell_s)^{1/4}}{\sqrt{N}}\right) \\
&\qquad \times \big(\sqrt{N\ell_s} + \psi_{k+\mathbf{1}(k\,\mathrm{odd})}^{\mathrm{iso}}(N\ell_s)^{1/4}\big)\sqrt{\ell_s \mathfrak{s}_{2(k-i+1)}(s)}\,\mathrm{d}s\Bigg)^{1/2} \\
&\lesssim \frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\Bigg(\sum_{i=1}^{\lceil k/2\rceil}\int_0^t \frac{\mathfrak{s}_k^{\mathrm{iso}}(s)\mathfrak{s}_{k-i+1}^{\mathrm{iso}}(s)\mathfrak{s}_{i-1}^{\mathrm{iso}}(s)}{N\eta_s^2}\ell_s^{3/2} \\
&\qquad \times \left(1 + \frac{\psi_{2(i-1)}^{\mathrm{iso}}}{(N\ell_s)^{1/4}}\right)\big(\sqrt{N\ell_s} + \psi_{k+\mathbf{1}(k\,\mathrm{odd})}^{\mathrm{iso}}(N\ell_s)^{1/4}\big)\,\mathrm{d}s\Bigg)^{1/2} \\
&\lesssim 1 + \sum_{i=1}^{\lceil k/2\rceil}\frac{\sqrt{\psi_{2(i-1)}^{\mathrm{iso}}}}{(N\ell_s)^{1/8}} + \frac{\sqrt{\psi_{k+\mathbf{1}(k\,\mathrm{odd})}^{\mathrm{iso}}}}{(N\ell_t)^{1/8}} + \sum_{i=1}^{\lceil k/2\rceil}\frac{\sqrt{\psi_{2(i-1)}^{\mathrm{iso}}\psi_{k+\mathbf{1}(k\,\mathrm{odd})}^{\mathrm{iso}}}}{(N\ell_t)^{1/4}}.
\end{aligned} \tag{4.A.23}$$

Here in the second inequality (4.A.8)–(4.A.9), (4.A.11), and in the last inequality we used (4.A.7), (4.A.10).

In the following computations, when two G's, with spectral parameters having imaginary parts of the same sign appear next to each other (i.e. without a A in between), we use the integral representation (4.5.23) to reduce the number of G's by one at the price of an additional $1/\eta$. If the imaginary parts of the spectral parameters have different signs, we use resolvent identity (see e.g. around (4.5.22)–(4.5.23) in the averaged case). For the terms in the second line of (4.A.4) we estimate

$$\begin{aligned}
&\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t \langle G_{[i,j],s} - M_{[i,j],s}\rangle (M_{[1,i]\cup[j,k+1],s})_{\boldsymbol{xy}}\,\mathrm{d}s \\
&\qquad \prec \frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t \frac{\mathfrak{s}_{j-i}(s)}{N\eta_s}\frac{\mathfrak{m}_{k-j+i}^{\mathrm{iso}}(s)}{\eta_s}\,\mathrm{d}s \lesssim \frac{1}{(N\ell_t)^{3/4}},
\end{aligned} \tag{4.A.24}$$

where we used (4.A.6)–(4.A.8), the second inequality in (4.A.11), and (4.A.10).

For the terms in the third line of (4.A.4) we estimate

$$\begin{aligned}
&\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t \langle M_{[i,j],s}\rangle (G_{[1,i]\cup[j,k+1],s} - M_{[1,i]\cup[j,k+1],s})_{\boldsymbol{xy}}\,\mathrm{d}s \\
&\prec \frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t \frac{\mathfrak{m}_{j-i}(s)}{\eta_s}\frac{\mathfrak{s}_{k-j+i}^{\mathrm{iso}}(s)\psi_{k-j+i}^{\mathrm{iso}}(N\ell_s)^{1/4}}{\sqrt{N}\eta_s}\,\mathrm{d}s \lesssim \psi_{k-j+i}^{\mathrm{iso}},
\end{aligned} \tag{4.A.25}$$

where in the last inequality we used the first estimate of (4.A.11), (4.A.7)–(4.A.8).

For the terms in the fourth line of (4.A.4) we estimate

$$\begin{aligned}&\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t\langle G_{[i,j],s}-M_{[i,j],s}\rangle(G_{[1,i]\cup[j,k+1],s}-M_{[1,i]\cup[j,k+1],s})_{\boldsymbol{xy}}\,\mathrm{d}s\\&\prec\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t\frac{\mathfrak{s}_{j-i}(s)}{N\eta_s}\frac{\mathfrak{s}_{k-j+i}^{\mathrm{iso}}(s)\psi_{k-j+i}^{\mathrm{iso}}(N\ell_s)^{1/4}}{\sqrt{N}\eta_s}\,\mathrm{d}s\lesssim\frac{1}{N\ell_t}\psi_{k-j+i}^{\mathrm{iso}},\end{aligned}$$

where in the last step we used the second inequality of (4.A.11) and the second inequality of (4.A.7).

Finally, for the term in the last line of (4.A.4) we estimate

$$\begin{aligned}&\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t\langle G_{i,s}-m_{i,s}\rangle(G^{(i)}_{[1,k+1],s})_{\boldsymbol{xy}}\,\mathrm{d}s\\&\prec\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)(N\ell_t)^{1/4}}\int_0^t\frac{1}{N\eta_s}\left(\frac{\mathfrak{m}_k^{\mathrm{iso}}(s)}{\eta_s}+\frac{\mathfrak{s}_k^{\mathrm{iso}}(s)\psi_k^{\mathrm{iso}}(N\ell_s)^{1/4}}{\sqrt{N}\eta_s}\right)\mathrm{d}s\\&\lesssim\frac{1}{(N\ell_t)^{3/4}}+\frac{\psi_k^{\mathrm{iso}}}{N\ell_t},\end{aligned}\tag{4.A.26}$$

where we used (4.A.6) and (4.A.8) together with (4.A.8). Combining (4.A.23)–(4.A.26), recalling that $\langle M_{[i,i+1],s}\rangle = 0$ in (4.A.25), using (4.A.14) to bound $\Psi_k^{\mathrm{iso}}(0) \prec 1$ and $N\ell_T \geq 1$, $\psi_l^{\mathrm{iso}} \geq 1$, $\ell_t \gtrsim \ell_T$, we conclude (4.A.15). □

Proof of Proposition 4.A.5. The proof of this proposition is analogous to the proof of Proposition 4.A.5. All the terms except for the martingale one are estimated as in the proof of Proposition 4.A.5 after multiplying each line by $(N\ell_t)^{1/4}$ and setting $\psi_l^{\mathrm{iso}} = 1$. We conclude the proof pointing out that the only difference is in the estimate of the quadratic variation of the martingale term, i.e. (4.A.23) has to be replaced by

$$\begin{aligned}&\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)}\left(\frac{1}{N}\sum_{i=1}^{k+1}\int_0^t\left[G_{[1,i],s}(G_{[1,i],s})^*\right]_{\boldsymbol{xx}}\left[(G_{[i,k+1],s})^*G_{[i,k+1],s}\right]_{\boldsymbol{yy}}\mathrm{d}s\right)^{1/2}\\&\prec\frac{\sqrt{N}}{\mathfrak{s}_k^{\mathrm{iso}}(t)}\left(\int_0^t\frac{\ell_s\mathfrak{s}_{2(i-1)}^{\mathrm{iso}}(s)\mathfrak{s}_{2k-2(i-1)}^{\mathrm{iso}}(s)}{N\eta_s^2}\,\mathrm{d}s\right)^{1/2}\lesssim 1,\end{aligned}$$

where in the first inequality we used the definition (4.A.13) together with (4.A.6), (4.A.8), and in the second inequality we used (4.A.7), (4.A.9)–(4.A.10). □

Proof of (4.A.21). By spectral decomposition we estimate

$$\begin{aligned}\left|(Q_1G(z)Q_2G(w)Q_3)_{\boldsymbol{xy}}\right| &= \left|\sum_{ij}\frac{\langle\boldsymbol{x},Q_1\boldsymbol{u}_i\rangle\langle\boldsymbol{u}_i,Q_2\boldsymbol{u}_j\rangle\langle\boldsymbol{u}_j,Q_3\boldsymbol{u}_j\rangle}{(\lambda_i-z)(\lambda_j-w)}\right|\\&\lesssim\sqrt{N}(Q_1|G(z)|Q_1^*)_{\boldsymbol{xx}}(Q_3^*|G(w)|Q_3)_{\boldsymbol{yy}}\langle|G(z)|Q_2|G(w)|Q_2^*\rangle^{1/2},\end{aligned}\tag{4.A.27}$$

where in the last inequality we used Schwarz inequality to separate Q_2 from Q_1, Q_3. Choosing $z = z_{k/2+1}$, $w = z_{k/2+j+1}$, and

$$Q_1 = G_1A_1\dots A_{k/2},\quad Q_2 = A_{k/2+1}G_{k/2+2}\dots A_{k/2+j},\quad Q_3 = A_{k/2+j+1}G_{k/2+j+2}\dots G_{k+1},$$

this concludes the proof. □

Chapter 5

Eigenvector decorrelation for random matrices

This chapter contains the paper [153]:

G. Cipolloni, L. Erdős, J. Henheik, and O. Kolupaiev. Eigenvector decorelation for random matrices. *arXiv:2410.10718*, 2024

Abstract. We study the sensitivity of the eigenvectors of random matrices, showing that even small perturbations make the eigenvectors almost orthogonal. More precisely, we consider two deformed Wigner matrices $W + D_1$, $W + D_2$ and show that their bulk eigenvectors become asymptotically orthogonal as soon as $\mathrm{Tr}(D_1 - D_2)^2 \gg 1$, or their respective energies are separated on a scale much bigger than the local eigenvalue spacing. Furthermore, we show that quadratic forms of eigenvectors of $W + D_1$, $W + D_2$ with any deterministic matrix $A \in \mathbf{C}^{N\times N}$ in a specific subspace of codimension one are of size $N^{-1/2}$. This proves a generalization of the Eigenstate Thermalization Hypothesis to eigenvectors belonging to two different spectral families.

Content

5.1 Introduction

5.1.1 Main result

The behavior of the eigenvectors of a Hermitian matrix under perturbations is known to be quite subtle: even a small change in the matrix may lead to a significant rotation of the eigenvectors due to resonances. This phenomenon is ubiquitous in a broad range of numerical and statistical applications, see, e.g., [146, 200, 202, 253, 471, 598]. For example, in the classical paper [200] Davis and Kahan give a deterministic upper bound for the deviation of the eigenvectors from the unperturbed ones in terms of the spectral gap, while in [471, Theorem 2] the authors show that there always exists a perturbation causing a big change in the eigenvectors. When the magnitude of the perturbation exceeds the local eigenvalue spacing of the initial matrix, standard perturbation theory does not control the eigenbasis of the perturbed matrix any more and the behavior of the eigenvectors is highly sensitive to the properties of the original matrix. While in some rare cases even such larger perturbations still cause only a small change, typically the perturbed eigenbasis is completely decoupled from the initial one. In this paper, we show that indeed this typical scenario occurs for random matrices with very high probability.

More precisely, we consider two *deformed Wigner matrices* of the form $H_1 = W + D_1$, $H_2 = W + D_2$, where W is a *Wigner matrix*[1] and D_1, D_2 are Hermitian deterministic *deformations*, which we assume to be traceless without loss of generality. Denote the eigenvalues (*energies*) of H_l in increasing

[1] A Wigner matrix is a Hermitian $N \times N$ matrix $W = W^*$ with independent, identically distributed centered entries (up to the Hermitian symmetry) with $\mathbf{E}|W_{ab}|^2 = 1/N$; see also Assumption 5.2.1.

order[2] by $\lambda_1^l \le \lambda_2^l \le \ldots \le \lambda_N^l$, $l = 1, 2$, and let $\boldsymbol{u}_1^l, \boldsymbol{u}_2^l, \ldots, \boldsymbol{u}_N^l$ be the corresponding orthonormal eigenvectors. We measure the distance between the families $\{\boldsymbol{u}_i^1\}_{i=1}^N$ and $\{\boldsymbol{u}_j^2\}_{j=1}^N$ by looking at the *eigenvector overlaps* $\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle$ for a deterministic observable matrix A.

Our first main result (Theorem 5.2.4) is the decomposition

$$\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle = \langle VA\rangle \langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle + \mathcal{O}\left(\frac{\|A\|}{\sqrt{N}}\right), \tag{5.1.1}$$

for bulk indices[3] i, j, where $\langle X\rangle := \frac{1}{N}\mathrm{Tr}\, X$ denotes the averaged trace of $X \in \mathbf{C}^{N\times N}$. Here V is an appropriately chosen deterministic matrix depending on the deformations D_1, D_2 and the (typical locations of the) energies λ_i^1, λ_j^2 with $\|V\| \lesssim 1$ (see (5.2.11) for its definition). The $N^{-1/2}$ error term in (5.1.1) is optimal.

As our second main result (Theorem 5.2.6), we give an upper bound on the overlap $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ in (5.1.1). In the special case $D_1 = D_2$ we trivially have $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle = \delta_{ij}$, hence (5.1.1) is just the Eigenstate Thermalization Hypothesis (ETH) for deformed Wigner matrices proven in Chapter 2. However, in general, when we consider two different deformations, the overlap $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ is non-trivial. In fact, it subtly depends on two effects; the difference in deformations, $D_1 - D_2$, and the difference in energy $\lambda_i^1 - \lambda_j^2$. In order to study the decorrelation properties of $\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle$ we thus need to give an estimate on this eigenvector overlap in terms of these two differences. More precisely, in Theorem 5.2.6, we prove the optimal bound

$$\left|\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle\right|^2 \lesssim \frac{1}{N} \cdot \frac{1}{\langle (D_1 - D_2)^2\rangle + \mathrm{LT} + |\lambda_i^1 - \lambda_j^2|^2}, \tag{5.1.2}$$

where the so-called *linear term* LT is (the absolute value of) a specific linear combination of $D_1 - D_2$ and $\lambda_i^1 - \lambda_j^2$ and its precise definition will be given in (5.2.17). The estimate (5.1.2) manifests the interplay of the two decay effects in three different terms, which can make the eigenvectors $\boldsymbol{u}_i^1, \boldsymbol{u}_j^2$ almost orthogonal. The identification of the decay in $D_1 - D_2$ is the main new result in this paper. It captures the effect that the spectral resolutions of $W + D_1, W + D_2$ become more and more independent as $\langle (D_1 - D_2)^2\rangle$ grows. We describe the relation between the three terms in (5.1.2) in more details below Theorem 5.2.6. Here we only comment on the optimality of our proven decay in terms of $\langle (D_1 - D_2)^2\rangle$. Standard second order perturbation theory (outlined in Remark 5.2.7) indicates that $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^2\rangle \approx 1$ in the regime $\langle (D_1 - D_2)^2\rangle \ll 1/N$. Our bound (5.1.2) shows that $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^2\rangle \approx 0$ in the opposite regime $\langle (D_1 - D_2)^2\rangle \gg 1/N$.

Putting together our two main results, (5.1.1) and (5.1.2), we see that the overlap $\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle$ can be small on two different grounds: Either the observable matrix A is (nearly) orthogonal to V, i.e. $\langle VA\rangle \approx 0$, or the overlap $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ is small as estimated in (5.1.2). We coin the first the *regularity effect* and the second the *overlap decay effect*. All results hold with very large probability.

5.1.2 Previous related results

To put our results (5.1.1)–(5.1.2) into context we now describe several related results, which partially explored only one of the two smallness effects at a time. We stress that our results (5.1.1)–(5.1.2) manage to catch both these effects in a unified and optimal manner. In fact, prior to this work, the regularity effect (5.1.1) was only studied in the context of the same matrix H, i.e. $D_1 = D_2$ (and possibly both equal to zero), to prove the ETH in the setting of random matrices. The ETH, posed by Deutsch in [221] as a signature of chaos in quantum systems, states that quadratic forms of eigenfunctions of chaotic Hamiltonians can be described purely by macroscopic quantitites and

[2] The upper index $l = 1, 2$ of the eigenvalues λ and other related quantities should not be confused with a power.

[3] We say that an index i is in the bulk of the spectrum if the density of states around λ_i^l is strictly positive; see (5.2.7) for the precise definition.

that the (pseudo-random) fluctuations are entropically suppressed. In the context of a *single* random matrix ensemble the ETH reads as

$$\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle = \langle VA\rangle\delta_{ij} + \mathcal{O}\left(\frac{\|A\|}{\sqrt{N}}\right), \tag{5.1.3}$$

where $\boldsymbol{u}_i$ are the orthonormal eigenvectors of an $N\times N$ random matrix H. The ETH in the form (5.1.3) was first proven for Wigner matrices (i.e. $D_1 = D_2 = 0$, in which case $V = I$) in [165] (see also [112, 115] for previous partial results). We point out that even the Gaussianity of the fluctuations in the $N^{-1/2}$-term is known for Wigner matrices for special observables [112, 115], for general observables [167, 61, 60], and for deformed Wigner matrices (see Chapter 2). The result (5.1.3) was extended in several directions: to more general random matrix ensembles [5, 538, 245] (see also Chapter 2), where V becomes energy dependent, to d–regular graphs [52, 51] and to improvement of the error term in (5.1.3) from $\|A\|$ to $\langle A^2\rangle^{1/2}$ [61, 169] (see also Chapter 3). Related to (5.1.3), we also mention that in the past few years there has been great interest in studying eigenvector overlaps of different nature in several other contexts, including tensor principal component analysis (PCA) [482], shrinkage estimators [224, 423], noise detection [21, 129], minors [31], the equipartition principle [45], and many body physics [192].

On the other hand, estimates of the form (5.1.2), focusing on the $D_1 - D_2$ behavior, were previously studied only for Hermitian matrices in the very special case when $D_i = x_i D$, for a scalar x_i, in [159], and in the context of decorrelation estimates for the Hermitization of non–Hermitian matrices in [157, 156, 161, 162], where the deformation has a very special 2×2 block structure with zero diagonal blocks and off–diagonal blocks being constant multiples of the identity. As a related problem, sensitivity of the top eigenvector for a Wigner matrix to resampling of a small portion of the matrix elements was studied in [100] and extended to sparse matrices in [99].

5.1.3 Multi–resolvent local laws

Local laws in general are concentration estimates for a single resolvent G of a random matrix, or alternating chains of resolvents and deterministic matrices A, i.e. $GAGAGA\ldots$. The main technical tool that we use to prove the decorrelation estimates for eigenvectors in (5.1.1)–(5.1.2) is a *two-resolvent local law*, which is stated in Theorem 5.3.2 below.

We now first describe our new multi-resolvent local law and then relate it to previous results. Let us denote the resolvent of $W + D_j$ at $z_j \in \mathbf{C}\setminus\mathbf{R}$ by $G_j := (W + D_j - z_j)^{-1}$ and let A be a deterministic $N\times N$ matrix. Then our new multi–resolvent local law asserts that, as N tends to infinity, the matrix product G_1AG_2 concentrates around its deterministic approximation, denoted by M_{12}^A, which is explicitly given by[4]

$$M_{12}^A = M_1AM_2 + \frac{\langle M_1AM_2\rangle}{1-\langle M_1M_2\rangle}M_1M_2.$$

Here M_i denotes the deterministic approximation of the single resolvent G_i obtained as the unique solution $M_i = M^{D_i}(z_i)$ to the *Matrix Dyson Equation* (MDE)

$$-M_i^{-1} = z_i - D_i + \langle M_i\rangle \tag{5.1.4}$$

under the constraint $\operatorname{Im} M_i \operatorname{Im} z_i > 0$. We optimally control the fluctuation of G_1AG_2 around M_{12}^A in terms of $D_1 - D_2$ and $z_1 - z_2$, showing that typically the size of the fluctuation around M_{12}^A is smaller than the size of M_{12}^A itself. For this reason, in Proposition 5.3.1 below, we give the following bound on M_{12}^A:

$$\|M_{12}^A\| \lesssim \frac{1}{\gamma}, \qquad \gamma := \langle (D_1-D_2)^2\rangle + |\operatorname{Re} z_1 - \operatorname{Re} z_2|^2 + \mathrm{LT} + |\operatorname{Im} z_1| + |\operatorname{Im} z_2|, \tag{5.1.5}$$

[4]Note that G_1AG_2 is *not* close to M_1AM_2, indicating that multi-resolvent local laws are not simple consequences of the single resolvent local law.

where the $\mathrm{LT} \geq 0$ behaves as (the absolute value of) a linear combination of $D_1 - D_2$ and $z_1 - z_2$ (a precise definition will be given in (5.2.17) later). The interesting regime is when $\gamma \ll 1$. However, when $A \in \mathbf{C}^{N\times N}$ lies in a specific subspace of codimension one, the bound in (5.1.5) improves to $\|M_{12}^A\| \lesssim 1$. We call such matrices *regular* and establish an improved local law for $G_1 A G_2$ in this case. When one deals with Wigner matrices, i.e. $D_1 = D_2 = 0$, then A is regular if and only if $\operatorname{Tr} A = 0$. However, when the deformations D_1, D_2 are non–zero, the notion of regularity depends on D_1, D_2, as well as on the spectral parameters z_1, z_2, in a nontrivial way; see Definition 5.2.2 for the precise definition.

We now informally discuss the structure of the bounds in the multi–resolvent local laws in Theorem 5.3.2 and Proposition 5.4.16 with a concrete example. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ be deterministic unit vectors. When $D_1 = D_2$, it was shown in Proposition 2.4.4 that for $\|A\| \lesssim 1$ we have[5]

$$|\langle \boldsymbol{x}, (G_1 A G_2 - M_{12}^A)\boldsymbol{y}\rangle| \lesssim \begin{cases} \frac{1}{\sqrt{N\eta}} \cdot \frac{1}{\eta} = (N\eta^3)^{-1/2}, & A \text{ is general}, \\ \frac{1}{\sqrt{N\eta}} \cdot \frac{1}{\eta} \cdot \sqrt{\eta} = (N\eta^2)^{-1/2}, & A \text{ is regular} \end{cases} \tag{5.1.6}$$

for $N\eta \gg 1$, where $\eta := |\operatorname{Im} z_1| \wedge |\operatorname{Im} z_2|$ is small in the interesting *local* regime. Note that the bound in the case of regular A is $\sqrt{\eta}$ times better than in the general case. This improvement is known as a $\sqrt{\eta}$*–rule* and was initially observed in [168] in the context of Wigner matrices. This rule correctly predicts the size of an arbitrarily long resolvent chain $G_1 A_1 G_2 \cdots A_{k-1} G_k$: each regular A_i accounts for an additional $\sqrt{\eta}$ improvement compared with the bound uniform in $\operatorname{Re} z_1, \operatorname{Re} z_2$ and all bounded observables.

In this paper we make a step further and show how (5.1.6) improves once we start taking into account the distance between spectral parameters and between deformations. We also show how this decay effect can be combined with the effect that the matrix A is regular. Namely, we prove that (see Proposition 5.4.16 below)

$$|\langle \boldsymbol{x}, (G_1 A G_2 - M_{12}^A)\boldsymbol{y}\rangle| \lesssim \begin{cases} \frac{1}{\sqrt{N\eta}} \cdot \frac{1}{\eta} \cdot \sqrt{\frac{\eta}{\gamma}} = (N\eta^2\gamma)^{-1/2}, & A \text{ is general}, \\ \frac{1}{\sqrt{N\eta}} \cdot \frac{1}{\eta} \cdot \sqrt{\frac{\eta}{\gamma}} \cdot \sqrt{\gamma} = (N\eta^2)^{-1/2}, & A \text{ is regular}. \end{cases} \tag{5.1.7}$$

Note that for the control parameter γ from (5.1.5) we have $\sqrt{\eta/\gamma} \lesssim 1$, showing that in fact this additional factor in (5.1.7), compared to (5.1.6), gives additional smallness.

From (5.1.7), we can thus draw the following two rules of thumb, refining the previous $\sqrt{\eta}$-rule.

$\sqrt{\eta/\gamma}$–rule (*Decay effect*): For each pair of neighboring resolvents with different indices, G_1, G_2, we gain an additional (small) factor $\sqrt{\eta/\gamma}$.

$\sqrt{\gamma}$–rule (*Regularity effect*): For each regular matrix we gain an additional (small) factor $\sqrt{\gamma}$.

Note that when both effects are present, we gain back the $\sqrt{\eta/\gamma}\sqrt{\gamma} = \sqrt{\eta}$-rule. Thus with the proper definition of regularity no additional gain can be obtained from the decay effect; this is natural since the $\sqrt{\eta/\gamma}$-rule comes from the unique unstable direction of the two-body stability operator (5.2.12), while the concept of regularity exactly removes this worst direction.

In (5.1.6)–(5.1.7) we presented the example of the two-resolvent isotropic law for clarity of presentation, but in Theorem 5.3.2 and Proposition 5.4.16 we prove analogous results also in the averaged case and for isotropic chains containing three resolvents, respectively. Longer chains can also be handled by our method and our two new rules correctly predict their size, but we refrain from doing so, since they are not needed for the eigenvector overlap. In fact, on a heuristic level one could deduce the results in Theorem 5.3.2, Proposition 5.4.16 by using the $\sqrt{\gamma}$– and $\sqrt{\eta/\gamma}$–rules for each

[5] By $\langle \cdot, \cdot \rangle$ we denote the inner product in $\mathbf{C}^N$.

unit G_1AG_2 and multiplying the gains from them. In particular, in the averaged case $\langle G_1AG_2A\rangle$ one can extract the gain from both units G_1AG_2 and G_2AG_1 because of the cyclicity of the trace.

Our paper is the first instance when both the decay and the regularity effects are considered together, previously only at most one of them was identified at a time. In fact, the study of multi-resolvent local laws started in the context of Wigner matrices where none of these two effects were exploited [170]; see also [330, 329] for concrete cases when some decay in $|\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2|$ was identified in the context of central limit theorems for linear eigenvalue statistics. After [170], there has been great progress in proving multi-resolvent local laws either for regular observables [165, 168, 169, 5, 538, 245, 480] (see also the previous Chapters 1–4) or for different deformations of a specific form for Hermitian matrices [159] and for the Hermitization of non–Hermitian matrices [157, 161, 162].

We conclude this section by pointing out that the multi-resolvent local laws mentioned above have also been used in several other important problems in random matrix theory; we now name some of them. They played a key role in the recent solution of the bulk universality conjecture for non–Hermitian random matrices [431, 481, 226], as well as in proving universality of the distribution of diagonal overlaps of left/right non–Hermitian eigenvectors [480] and of their entries [227, 479]. Two–resolvents local laws have also been used to prove decorrelation estimates for the resolvent of the Hermitization of non–Hermitian matrices in the context of space–time correlation of linear statistics of non–Hermitian eigenvalues [106], and to compute the leading order asymptotic of the log-determinant of non–Hermitian matrices [173]. Lastly we point out that similar decorrelation estimates, proven in [159], have been used in [466] to study random hives associated to the eigenvalues of GUE matrices.

5.1.4 The method of characteristics

We prove multi-resolvent local laws in Theorem 5.3.2 using the so–called *zigzag* strategy [150, 161], which involves three key steps. First, we prove a concentration bound on the global scale (*global law*), i.e. when the spectral parameters are at a distance of order one from the spectrum. Then we propagate this bound down to the real line by evolving the matrix W along the Ornstein–Uhlenbeck flow, while the spectral parameters z_1, z_2 and the deformations D_1, D_2 evolve according to a certain deterministic evolution, called *characteristic equations* (see (5.4.8) below for the definition). Along this flow the imaginary part of the spectral parameters is reduced (*zig step*). This second step establishes local laws for spectral parameters with small imaginary parts, though only for matrices with a Gaussian component, added by the Ornstein-Uhlenbeck flow. Finally, the last step of the zigzag strategy eliminates this Gaussian component, again dynamically, via a *Green function comparison* argument (*zag step*). We point out that zig and zag steps are used many times in tandem to decrease the distance of the spectral parameters to the spectrum step by step.

While the zigzag strategy is a well–established method which has been worked out in many instances, there are several important novelties in our current approach. The first novelty is that we perform the proof for an abstract control parameter satisfying certain general conditions which we precisely describe in Definition 5.4.4. We do this since the structure of the upper bounds in Theorem 5.2.6 is fairly complicated and we thus need to keep track of different effects at the same time. The second novelty is the self–improving estimates in the zag step stated in Lemmas 5.4.11 and 5.4.12. In fact, we need to perform several zigzag steps to prove the optimal $1/\sqrt{\gamma}$ decay, instead of the $1/\sqrt{\eta}$ in (5.1.7). We do this gradually: We first prove (5.1.7) with $1/\sqrt{\gamma}$ replaced by $1/\sqrt{\eta^{1-b}\gamma^b}$ for some $b \in (0,1)$ and then, using this bound as an input, we improve it to $1/\sqrt{\eta^{1-b'}\gamma^{b'}}$ for some $b' > b$. Iterating this procedure finitely many times we finally obtain the desired $1/\sqrt{\gamma}$ in (5.1.7). As an additional third novelty, we extend the delicate analysis of the two-body stability from [237] to include the new linear term LT.

We conclude this section with a brief historical discussion of the use of the *method of characteristics* (zig step) in random matrix theory[6]. The idea to study the evolution of the resolvent along the

[6]We point out that, even if we do not mention it, some of the following references also use a comparison step

characteristic flow was first introduced in [488, 353, 582, 6, 105] to prove local laws for single resolvents in the bulk and the edge of the spectrum, though only for matrices which have a Gaussian component. In the edge regime a similar version of the characteristics was used before to prove Tracy–Widom universality for the largest eigenvalue of deformed Wigner matrices [407]. In the context of single resolvent local laws, this method was later extended to cover also the cusp regime [7, 136, 239]. All the results mentioned above concern single resolvent local laws. Only more recently the method of characteristics was used to prove local laws for products of two or more resolvents. The first instances of multi–resolvent local laws proven with this method are for the unitary Brownian motion [111] and for the product of resolvents of the Hermitization of non–Hermitian matrices at different spectral parameters [161]. Since then this method has been very successful in proving a multitude of multi–resolvent local laws for regular matrices or for matrices with specific different deformations [538, 162, 245, 480, 163] (see also Chapters 3 and 4). In the current work we show that this method is also effective to optimally catch both the decay and the regularity effect at the same time. Finally, we mention that the method of characteristics was also useful to prove central limit theorems for linear eigenvalues statistics [353, 6, 393, 392, 394], to study their time correlations [106], as well as to study certain extremal statistics [173].

Notations and conventions

We set $[k] := \{1, ..., k\}$ for $k \in \mathbf{N}$ and $\langle A \rangle := N^{-1}\mathrm{Tr}(A)$, $N \in \mathbf{N}$, for the normalized trace of an $N \times N$-matrix A. For positive quantities f, g we write $f \lesssim g$, $f \gtrsim g$, to denote that $f \leq Cg$ and $f \geq cg$, respectively, for some N-independent constants $c, C > 0$ that depend only on the basic control parameters of the model in Assumption 5.2.1 below. We denote the complex upper–half plane by $\mathbf{H} := \{z \in \mathbf{C} : \mathrm{Im}\, z > 0\}$

We denote vectors by bold-faced lower case Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, for some $N \in \mathbf{N}$. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and a matrix $A \in \mathbf{C}^{N\times N}$ we define

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_i \bar{x}_i y_i \,, \qquad A_{\boldsymbol{xy}} := \langle \boldsymbol{x}, A\boldsymbol{y} \rangle \,.$$

Matrix entries are indexed by lower case Roman letters $a, b, c, ..., i, j, k, ...$ from the beginning or the middle of the alphabet and unrestricted sums over those are always understood to be over $\{1, ..., N\}$.

Finally, we will use the concept *with very high probability*, meaning that for any fixed $D > 0$, the probability of an N-dependent event is bigger than $1 - N^{-D}$ for all $N \geq N_0(D)$. We will use the convention that $\xi > 0$ denotes an arbitrarily small positive exponent, independent of N. Moreover, we introduce the common notion of *stochastic domination* (see, e.g., [241]): For two families

$$X = \left(X^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) \mid N \in \mathbf{N}, u \in U^{(N)}\right)$$

of non-negative random variables indexed by N, and possibly a parameter u, we say that X is stochastically dominated by Y, if for all $\epsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\epsilon} Y^{(N)}(u)\right] \leq N^{-D}$$

for large enough $N \geq N_0(\epsilon, D)$. In this case we write $X \prec Y$. If for some complex family of random variables we have $|X| \prec Y$, we also write $X = O_{\prec}(Y)$.

5.2 Main results

We consider an $N \times N$ deformed Wigner matrix of the form $H = D + W$, where $D = D^*$ is a deterministic deformation and W is a Wigner matrix, i.e. real symmetric or complex Hermitian

similar to the zag step to remove the additional Gaussian component added via the zig step.

matrix $W = W^*$ with independent entries (up to the symmetry constraint) having distribution

$$W_{aa} \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{N}}\chi_{\mathrm{d}}, \qquad W_{ab} \stackrel{\mathrm{d}}{=} \frac{1}{\sqrt{N}}\chi_{\mathrm{od}}, \quad a > b. \tag{5.2.1}$$

On the (N-independent) random variables $\chi_{\mathrm{d}} \in \mathbf{R}$, $\chi_{\mathrm{od}} \in \mathbf{C}$ we formulate the following assumptions:

Assumption 5.2.1. *Both χ_{d}, χ_{od} are centered $\mathbf{E}\chi_{\mathrm{d}} = \mathbf{E}\chi_{\mathrm{od}} = 0$ and have unit variance $\mathbf{E}\chi_{\mathrm{d}}^2 = \mathbf{E}|\chi_{\mathrm{od}}|^2 = 1$. In the complex case we also assume[7] that $\mathbf{E}\chi_{\mathrm{od}}^2 = 0$. Furthermore, we assume the existence of high moments, i.e. for any $p \in \mathbf{N}$ there exists a constant $C_p > 0$ such that*

$$\mathbf{E}\left[|\chi_{\mathrm{d}}|^p + |\chi_{\mathrm{od}}|^p\right] \le C_p. \tag{5.2.2}$$

Our main goal is to study the decorrelation of the eigenvectors of $W + D_1$, $W + D_2$ for two different Hermitian deformations $D_1, D_2 \in \mathbf{C}^{N\times N}$. For simplicity, we will always assume that the deformations D_1, D_2 are traceless, i.e. that $\langle D_1\rangle = \langle D_2\rangle = 0$. This is not restrictive, since the spectrum of $W + D_l$, for $l = 1, 2$, differs from the spectrum of $W + (D_l - \langle D_l\rangle)$ only by a shift of size $\langle D_l\rangle$ to the right. In particular, all the results presented below also hold without the restriction to traceless deformations, one just needs to shift the spectral parameters properly.

Before stating our main result we introduce some useful notations and definitions. Let $D = D^* \in \mathbf{C}^{N\times N}$ with $\|D\| \lesssim 1$, denote its empirical eigenvalue density by

$$\mu(D) := \frac{1}{N}\sum_{i=1}^{N}\delta_{d_i}, \tag{5.2.3}$$

with $d_1, \dots, d_N$ denoting the eigenvalues of D. Let μ_{sc} be the semicircular distribution with density $\rho_{\mathrm{sc}}(x) := (2\pi)^{-1}\sqrt{(4-x^2)_+}$; we recall that ρ_{sc} is the limiting density of the eigenvalues of a Wigner matrix W. Then the limiting eigenvalue density of $W + D$ is given by the free convolution (see [81] for a detailed discussion)

$$\mu_D = \mu_{\mathrm{sc}} \boxplus \mu(D), \tag{5.2.4}$$

which is a probability distribution on $\mathbf{R}$. Let m_D be the Stieltjes transform of μ_D, i.e. for $z \in \mathbf{C}\setminus\mathbf{R}$ we have

$$m_D(z) := \int_{\mathbf{R}} \frac{\mu_D(\mathrm{d}x)}{x - z}, \tag{5.2.5}$$

and define the corresponding density by

$$\rho_D(x) := \lim_{\eta\to 0^+}\rho_D(x + \mathrm{i}\eta), \qquad \rho_D(z) := \frac{1}{\pi}\big|\operatorname{Im} m_D(z)\big|. \tag{5.2.6}$$

Next, fix a small $\kappa > 0$, and define the *κ-bulk* of the density ρ_D by

$$\mathbf{B}_\kappa(D) := \{x \in \mathbf{R} : \rho_D(x) \ge \kappa\}. \tag{5.2.7}$$

Furthermore, we define the *quantiles* γ_i^D of ρ_D implicitly via

$$\int_{-\infty}^{\gamma_i^D}\rho_D(x)\,\mathrm{d}x = \frac{i}{N}, \qquad i \in [N]. \tag{5.2.8}$$

From the *eigenvalue rigidity* it is known [17, 243] that γ_i^D very well approximates the ith eigenvalue λ_i of $W + D$.

We are now ready to state our two main results.

[7] We make this further assumption just to keep the presentation cleaner and shorter. In fact, inspecting the proof of Sections 5.5 and 5.6 it is clear that this assumption can easily be removed. This was explained in detail in Section 3.4.4.

5.2.1 First main result: Regular observables and eigenstate thermalization (Theorem 5.2.4)

In order to prove the decomposition in (5.1.1) with such a precise estimate of the error term, we need to find the appropriate one-codimensional set of observables, $A = A(D_1, D_2, \gamma_i^{D_1}, \gamma_j^{D_2})$, depending both on D_1, D_2 as well as on the approximate eigenvalues so that $\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle$ can be bounded by $N^{-1/2}$. In Definition 5.2.2 we characterize the family of such matrices. This result can be thought as a generalization of the ETH for eigenvectors belonging to two different spectral families.

We start by introducing the notion of *regular observables*, a concept, which in this generality was first introduced in Definition 1.3.1 and later in Definition 2.4.2.

Definition 5.2.2 (Regular observables)**.** *Let $A \in \mathbf{C}^{N\times N}$ be a deterministic matrix, let $z_1, z_2 \in \mathbf{C}\setminus\mathbf{R}$ be spectral parameters, and let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be deterministic deformations. Fix a small constant*[8] *$\delta > 0$ depending on κ from* (5.2.7) *and $\|D_1\|, \|D_2\|$. Introduce the short–hand notation $\nu_l := (z_l, D_l)$, $l = 1, 2$, we will call ν_l a* spectral pair*. Set*

$$\phi(\nu_1,\nu_2) = \phi_\delta(\nu_1,\nu_2) := \chi_\delta(\operatorname{Re} z_1 - \operatorname{Re} z_2)\chi_\delta(\langle (D_1 - D_2)^2\rangle)\chi_\delta(\operatorname{Im} z_1)\chi_\delta(\operatorname{Im} z_2), \tag{5.2.9}$$

where $0 \le \chi_\delta(x) \le 1$ is a symmetric bump function such that it is equal to one for $|x| \le \delta/2$ and equal to zero for $|x| \ge \delta$.

We define the (ν_1, ν_2)–regular component *of A by*

$$\mathring{A}^{\nu_1,\nu_2} := A - \phi(\nu_1,\nu_2)\langle VA\rangle I, \tag{5.2.10}$$

where we used the short–hand notation

$$V = V(\nu_1,\nu_2) := \frac{M^{D_2}(\operatorname{Re} z_2 + \mathrm{i}\mathfrak{s}\operatorname{Im} z_2)M^{D_1}(\operatorname{Re} z_1 + \mathrm{i}\operatorname{Im} z_1)}{\langle M^{D_1}(\operatorname{Re} z_1 + \mathrm{i}\operatorname{Im} z_1)M^{D_2}(\operatorname{Re} z_2 + \mathrm{i}\mathfrak{s}\operatorname{Im} z_2)\rangle}. \tag{5.2.11}$$

In (5.2.11) *the relative sign of the imaginary parts is defined as*

$$\mathfrak{s} = \mathfrak{s}(z_1, z_2) := -\operatorname{sgn}(\operatorname{Im} z_1 \operatorname{Im} z_2).$$

We say that A is a regular observable with respect to (ν_1, ν_2) *if $A = \mathring{A}^{\nu_1,\nu_2}$.*

Note that our definition of regularity is *asymmetric* in the two spectral pairs. In particular, while $\mathring{A}^{\nu_1,\nu_2} = \mathring{A}^{\nu_1,\bar{\nu}_2}$, it does *not* necessarily hold that $\mathring{A}^{\nu_1,\nu_2}$ equals $\mathring{A}^{\bar{\nu}_1,\nu_2}$. The way of regularization presented in Definition 5.2.2 is not the only possible one. Alternatively, one could exchange the indices 1 and 2, or put $\mathfrak{s}$ on the other argument. It is also possible to define a regularization which is symmetric in ν_1, ν_2, hence may look more canonical, however we do not proceed in this direction since the definition (5.2.10) which we use is technically more manageable.

Remark 5.2.3 (On the choice of V)**.** *The convenience of our choice of V and thus the definition of regular observables in* (5.2.10) *lies in the fact that V is the right eigenvector $R_{12} = R(\nu_1, \nu_2)$ corresponding to the smallest (in absolute value) eigenvalue of the operator $\mathcal{X}_{12}$, which is defined by*

$$\mathcal{X}_{12}[\cdot] := \left[([\mathcal{B}_{12}]^{-1})^*[\cdot^*]\right]^*, \qquad \mathcal{B}_{12}[\cdot] := 1 - M^{D_1}(z_1)\langle\cdot\rangle M^{D_2}(z_2) = 1 - M_1\langle\cdot\rangle M_2, \tag{5.2.12}$$

with M_l from (5.1.4)*. Here $\mathcal{B}_{12}$ denotes the two–body stability operator that naturally appears when solving the analog of the Dyson equation for the deterministic approximation M_{12}^A of the two-resolvent chain G_1AG_2. With the above choice $\mathring{A}^{\nu_1,\nu_2}$ is defined so that $\langle \mathring{A}^{\nu_1,\nu_2} R_{12}\rangle = 0$, i.e. $V = R_{12}$.*

[8] The precise dependence of δ on κ and $\|D_1\|, \|D_2\|$ is discussed in the last paragraph of the proof of Theorem 5.2.6.

The operator $\mathcal{X}_{12}$ has a single very large eigenvalue if and only if $D_1 \approx D_2$, $z_1 \approx \bar{z}_2$ and $|\operatorname{Im} z_1|, |\operatorname{Im} z_2|$ are small. Regular observables are defined precisely such that the action of $\mathcal{X}_{12}$ (and also $\mathcal{X}_{1\bar{2}}$) remain bounded on them. This also explains the role of the cutoff function ϕ in (5.2.9)*: regularity is a nontrivial concept only when $\phi \neq 0$; in the complementary regime $\phi = 0$ every matrix A is regular.*

We are now ready to state our first main result.

Theorem 5.2.4 (Generalized Eigenstate Thermalization)**.** *Fix any $\kappa > 0$ and fix $D_1, D_2 \in \mathbf{C}^{N\times N}$ with $\|D_l\| \lesssim 1$. Let W be a Wigner matrix satisfying Assumption 5.2.1, and, for $l = 1, 2$, denote by $\boldsymbol{u}_1^l, \dots, \boldsymbol{u}_N^l$ the orthonormal eigenvectors of $W + D_l$. Fix indices i, j such that the quantiles $\gamma_i^{D_1} \in \mathbf{B}_\kappa(D_1)$ and $\gamma_j^{D_2} \in \mathbf{B}_\kappa(D_2)$ are in the κ–bulk of the corresponding densities. Let $A \in \mathbf{C}^{N\times N}$ be a deterministic matrix which is regular with respect to $(\nu_1, \nu_2) := ((\gamma_i^{D_1} + \mathrm{i}0^+, D_1), (\gamma_j^{D_2} + \mathrm{i}0^+, D_2))$. Then,*

$$\left|\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle\right| \prec \frac{\|A\|}{\sqrt{N}}. \tag{5.2.13}$$

More generally, for arbitrary observables $A \in \mathbf{C}^{N\times N}$, we have

$$\left|\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle - \langle VA\rangle \phi_{ij} \langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle\right| \prec \frac{\|A\|}{\sqrt{N}}, \tag{5.2.14}$$

where $V = V(\nu_1, \nu_2)$ is defined in (5.2.11) *and satisfies $\|V\| \lesssim 1$. Here, for a fixed small $\delta = \delta(\kappa) > 0$, we defined*

$$\phi_{ij} = \phi_{ij}(\delta) := \mathbf{1}(|\gamma_i^{D_1} - \gamma_j^{D_2}| \le \delta)\mathbf{1}(\langle (D_1 - D_2)^2\rangle \le \delta). \tag{5.2.15}$$

The bounds (5.2.13) *and* (5.2.14) *are uniform in the indices i, j such that $\gamma_i^{D_1} \in \mathbf{B}_\kappa(D_1)$ and $\gamma_j^{D_2} \in \mathbf{B}_\kappa(D_2)$.*

Example 5.2.5 (Eigenstate Thermalization)**.** *Some special cases of* (5.2.14) *recover previously known results:*

(i) For $D_1 = D_2 = 0$ (5.2.14) *is the ETH bound for Wigner matrices [165, Theorem 2.2], as in this case $V = I$, yielding*

$$\left|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \langle A\rangle \delta_{ij}\right| \prec \frac{\|A\|}{\sqrt{N}}. \tag{5.2.16}$$

Here $\{\boldsymbol{u}_i\}_{i=1}^N$ denote the orthonormal eigenvectors of W. Though (5.2.14) *implies* (5.2.16) *only for bulk indices, in [165],* (5.2.16) *was proven for all $i, j \in [N]$.*

(ii) More generally, when $D_1 = D_2 = D \in \mathbf{C}^{N\times N}$, we have

$$V = \frac{M(\gamma_i)M^*(\gamma_j)}{\langle M(\gamma_i)M^*(\gamma_j)\rangle},$$

with $\gamma_i := \gamma_i^D$. In this case, (5.2.14) *is the ETH bound for deformed Wigner matrices as given in Theorem 2.2.7:*

$$\left|\langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle - \frac{\langle \operatorname{Im} M(\gamma_i)A\rangle}{\langle \operatorname{Im} M(\gamma_i)\rangle}\delta_{ij}\right| \prec \frac{\|A\|}{\sqrt{N}}$$

for bulk indices, where we used that $V = \operatorname{Im} M(\gamma_i)/\langle \operatorname{Im} M(\gamma_i)\rangle$. Here $\{\boldsymbol{u}_i\}_{i=1}^N$ denote the orthonormal eigenvectors of $W + D$.

In the next section, we will estimate the overlap $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ appearing in (5.2.14).

5.2.2 Second main result: Optimal eigenvector decorrelation (Theorem 5.2.6)

In (5.2.14) we showed that for general observables (matrices) A the overlap $\langle \boldsymbol{u}_i^1, A\boldsymbol{u}_j^2\rangle$ can be decomposed as $\langle VA\rangle\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ plus a very small error. However, while $\|V\| \lesssim 1$ is deterministic, the overlap $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ is in general still random. This naturally raises the question if we can give a non-trivial bound on the overlap $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$. We positively answer this question in Theorem 5.2.6 below. In particular, we show that the size of the overlaps $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ is typically smaller when D_1, D_2 are more separated. Another effect is that the overlap becomes smaller when we consider eigenvectors corresponding to well separated eigenvalues. To quantify these types of decay we introduce the *linear term*, defined as

$$\mathrm{LT}(z_1, z_2) := \begin{cases} \left| z_1 - z_2 - \frac{\langle M_1(D_1 - D_2)M_2\rangle}{\langle M_1 M_2\rangle}\right| \wedge 1, & \text{if } \operatorname{Im} z_1 \operatorname{Im} z_2 < 0, \\ \left| z_1 - \bar{z}_2 - \frac{\langle M_1(D_1 - D_2)M_2^*\rangle}{\langle M_1 M_2^*\rangle}\right| \wedge 1, & \text{if } \operatorname{Im} z_1 \operatorname{Im} z_2 > 0. \end{cases} \tag{5.2.17}$$

Here, $M_l = M^{D_l}(z_l)$, for $l = 1, 2$, is the unique solution [332, Theorem 2.1] of the MDE (5.1.4) under the constraint $\operatorname{Im} M_l \operatorname{Im} z_l > 0$. We also mention that from (5.1.4) one can recover (5.2.5) by $m_{D_l}(z_l) = \langle M_l(z_l)\rangle$. From the definition (5.2.17) and the fact that M_l and D_l commute it follows that $\mathrm{LT}(z_1, z_2) = \mathrm{LT}(z_1, \bar{z}_2)$ and $\mathrm{LT}(z_1, z_2) = \mathrm{LT}(\bar{z}_1, z_2)$ for any $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$. Therefore, (5.2.17) extends continuously to the real line, i.e. $\mathrm{LT}(z_1, z_2)$ is well-defined for $z_1, z_2 \in \mathbf{R}$.

We are now ready to state our second main result.

Theorem 5.2.6 (Optimal eigenvector decorrelation)**.** *Fix any $\kappa > 0$ and fix $D_1, D_2 \in \mathbf{C}^{N\times N}$ Hermitian with $\|D_l\| \lesssim 1$. Let W be a Wigner matrix satisfying Assumption 5.2.1, and, for $l = 1, 2$, denote by $\boldsymbol{u}_1^l, \ldots, \boldsymbol{u}_N^l$ the orthonormal eigenvectors of $W + D_l$. Then,*

$$\left|\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle\right|^2 \prec \frac{1}{N} \cdot \frac{1}{\langle (D_1 - D_2)^2\rangle + \mathrm{LT}(\gamma_i^{D_1}, \gamma_j^{D_2}) + |\gamma_i^{D_1} - \gamma_j^{D_2}|^2} \wedge 1, \tag{5.2.18}$$

uniformly over indices i, j such the quantiles $\gamma_i^{D_l} \in \mathbf{B}_\kappa(D_l)$, for $l = 1, 2$, are in the κ–bulk of the density ρ_{D_l}.

We now briefly comment on (5.2.18). There are several effects that make the eigenvectors almost orthogonal; these are manifested by the various terms in the denominator on the rhs. of (5.2.18). The main novel effect is expressed by the term $\langle (D_1 - D_2)^2\rangle$ that measures the decay due to the fact that the spectra of $W + D_1$, $W + D_2$ become more and more independent as $\langle (D_1 - D_2)^2\rangle$ increases. Focusing on this effect only, (5.2.18) simplifies to

$$\left|\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle\right|^2 \prec \frac{1}{N\langle (D_1 - D_2)^2\rangle}, \tag{5.2.19}$$

uniformly for bulk indices. The second effect appears when the corresponding eigenvalues (*energies*), which are well approximated by the quantiles γ^D, are far away. This effect is trivially present even for a single deformation, $D_1 = D_2 = D$, in which case $\langle \boldsymbol{u}_i^D, \boldsymbol{u}_j^D\rangle = \delta_{ij}$. Finally, the combination of these two effects is more delicate. The last term in (5.2.18) shows that the *square* of the energy difference, $|\gamma_i^{D_1} - \gamma_j^{D_2}|$, is always present in the estimate. This is improved to *linear* decay, contained in the term LT, but for the difference of the *renormalized energies* that are the energies $\gamma_i^{D_l}$ shifted with $\langle M_1 D_l M_2^{(*)}\rangle / \langle M_1 M_2^{(*)}\rangle$.

Remark 5.2.7 (Eigenvector correlation in perturbative regime)**.** *As discussed above, we showed that the overlaps $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle$ are much smaller than $\|\boldsymbol{u}_i^1\| \cdot \|\boldsymbol{u}_j^2\| = 1$ when $\langle (D_1 - D_2)^2\rangle \gg 1/N$. Here, for simplicity, we only consider diagonal overlaps, i.e. $i = j$. We point out that the smallness of* (5.2.18) *may be due also to the other two terms in the denominator of the right–hand side of* (5.2.18),

however we do not consider these effects in this remark to keep the presentation simpler. We now show that this condition is necessary, in fact we claim that for $\langle (D_1 - D_2)^2 \rangle \ll 1/N$ *we have*

$$\langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^2 \rangle = 1 + o(1). \tag{5.2.20}$$

We now describe how to obtain (5.2.20)*. By second order perturbation theory we have*

$$\langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^2 \rangle = \langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^1 \rangle + \sum_{j \neq i} \frac{|\langle \boldsymbol{u}_i^1, (D_1 - D_2)\boldsymbol{u}_j^1 \rangle|^2}{(\lambda_i^1 - \lambda_j^1)^2} + \dots. \tag{5.2.21}$$

Since $\langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^1 \rangle = 1$*, we only need to estimate the second term in the right–hand side of* (5.2.21)*. Higher order terms in the perturbation series* (5.2.21) *can be estimated similarly but we omit them for simplicity. In order to deduce* (5.2.20)*, we need to give a lower bound on the denominator and an upper bound on the numerator in the rhs. of* (5.2.21)*.*

For the lower bound we have

$$(\lambda_i^1 - \lambda_j^1)^2 \gtrsim \frac{|i-j|^2}{N^2} \tag{5.2.22}$$

with high probability. To see this, in case of $|i-j| \geq N^\xi$ *with an arbitrary small* $\xi > 0$*, we employ the rigidity estimate [17, 243]. For nearby indices, say* $i < j \leq i + N^\xi$*, we use*

$$\mathbf{P}\big(|\lambda_i^1 - \lambda_j^1| \leq N^{-1-\omega}\big) \leq \mathbf{P}\big(|\lambda_i^1 - \lambda_{i+1}^1| \leq N^{-1-\omega}\big) \leq N^{-c\omega},$$

for some small fixed $c, \omega > 0$*. In the last step we used the universality of the eigenvalue gaps for deformed Wigner matrices*[9] *and the explicit level repulsion bound for GOE/GUE matrix:*

$$\mathbf{P}^{\mathrm{GOE/GUE}}\big(|\lambda_i^1 - \lambda_{i+1}^1| \leq N^{-1-\omega}\big) \leq N^{-c\omega}.$$

For the upper bound, we employ ETH for deformed Wigner matrix $W + D_1$ *in the Hilbert-Schmidt norm form:*

$$\left| \langle \boldsymbol{u}_i^1, (D_1 - D_2)\boldsymbol{u}_j^1 \rangle - \frac{\langle (D_1 - D_2)\mathrm{Im}\, M^{D_1}(\gamma_i^1) \rangle}{\langle \mathrm{Im}\, M^{D_1}(\gamma_i^1) \rangle} \delta_{ij} \right| \lesssim \frac{\langle (D_1 - D_2)^2 \rangle^{1/2}}{\sqrt{N}}, \tag{5.2.23}$$

with M^{D_1} *from* (5.1.4) *and* $\gamma_i^1 := \gamma_i^{D_1}$ *being the quantiles from* (5.2.8)*. In Chapter 2 we proved ETH for deformed Wigner matrices in the form*

$$\left| \langle \boldsymbol{u}_i^1, (D_1 - D_2)\boldsymbol{u}_j^1 \rangle - \frac{\langle (D_1 - D_2)\mathrm{Im}\, M^{D_1}(\gamma_i^1) \rangle}{\langle \mathrm{Im}\, M^{D_1}(\gamma_i^1) \rangle} \delta_{ij} \right| \lesssim \frac{\|D_1 - D_2\|}{\sqrt{N}}, \tag{5.2.24}$$

i.e with the operator norm $\|D_1 - D_2\|$ *instead of the Hilbert-Schmidt norm of* $D_1 - D_2$*. Strictly speaking, the improved bound* (5.2.23) *is nowhere proven for the eigenvectors of deformed Wigner matrix* $W + D_1$*,* $D_1 \neq 0$*, however this can be easily obtained using a similar (in fact much simpler) zigzag approach as the one presented in Sections 5.5–5.6 of this paper. We also point out that a bound similar to* (5.2.23) *has already been obtained, using similar arguments, for Wigner matrices* ($D_1 = 0$) *in Chapter 3 and for Wigner–type matrices with a diagonal deformation in [245].*

Finally, combining (5.2.22) *with* (5.2.23)*, from* (5.2.21) *we obtain*

$$\langle \boldsymbol{u}_i^1, \boldsymbol{u}_i^2 \rangle = 1 + \mathcal{O}\left(N \langle (D_1 - D_2)^2 \rangle\right)$$

which directly implies the desired claim (5.2.20)*. Every step of this argument can easily be made rigorous but we omit details for brevity.*

[9] The first bulk universality result in terms of correlation functions for deformed Wigner matrices with diagonal deformations was given in [409]. The gap universality in full generality was given, e.g., in Corollary 2.6 of [243].

Remark 5.2.8 (Independence of eigenvalue gaps)**.** *We point out that using the eigenvector overlap bound* (5.2.18) *we can prove that the eigenvalue gaps in the bulk of the spectrum of* $W + D_1$*,* $W + D_2$ *are independent as long as* $\langle (D_1 - D_2)^2 \rangle \gg 1/N$*. In fact, following verbatim [157, Section 7] and its adaptation to the Hermitian case in [159], we can prove the desired independence via the study of weakly correlated Dyson Brownian motions. The only input required for this proof is the overlap bound* $|\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2 \rangle| \ll 1$*.*

We point out that the bounds (5.2.14) and (5.2.18) are optimal except for the N^ϵ-factor (for any $\epsilon > 0$) coming from the $\prec$ bound. This can be seen by the fact that a local N^δ-average of eigenvectors

$$\frac{1}{N^{2\delta}} \sum_{\substack{|i-i_0| \le N^\delta \\ |j-j_0| \le N^\delta}} N \left| \langle \boldsymbol{u}_i^1, A \boldsymbol{u}_j^2 \rangle \right|^2$$

for some small $\delta > 0$ is proportional to products of resolvents, as shown in the rhs. of (5.3.15) below, for which we precisely compute the deterministic approximation in Theorem 5.3.2.

We stated our main results Theorems 5.2.4 and 5.2.6 only for indices in the bulk of the spectra of $W + D_1$, $W + D_2$ and estimated the error in Theorem 5.2.4 in terms of the operator norm $\|A\|$. In Section 5.3.3 below, we comment on possible extensions and improvements.

5.3 Proofs of the Main Results: Multi-resolvent local laws

In this section we present several technical tools and preliminary results that will be often used in this paper. More precisely, in Section 5.3.1 we study lower bounds on the stability operator, which are one of the fundamental input to obtain the decay in the rhs. of (5.2.18). Then, in Section 5.3.2, we state our main technical result (Theorem 5.3.2 below), which is a *multi–resolvent local law* for the product of the resolvents of $W + D_1$ and $W + D_2$, with $D_1, D_2 \in \mathbf{C}^{N \times N}$. Lastly, in Section 5.3.3 we comment on the optimality and discuss some possible extension of Theorem 5.3.2.

5.3.1 Preliminaries on the stability operator

Recall the definition of the stability operator from (5.2.12). One can easily see that its smallest (in absolute value) eigenvalue is $1 - \langle M_1 M_2 \rangle$ with associated eigenvector $M_1 M_2$; the only other eigenvalue, trivially equal to one, is highly degenerate. Here, $M_1 = M^{D_1}(z_1)$, $M_2 = M^{D_2}(z_2)$ are the solutions of the MDE (5.1.4). In this section we give a lower bound on its absolute value

$$\beta(z_1, z_2) := \left| 1 - \langle M_1 M_2 \rangle \right|. \tag{5.3.1}$$

The main control parameters in the following statements are $\langle (D_1 - D_2)^2 \rangle$ and the linear term $\mathrm{LT}(z_1, z_2)$ which is defined as in (5.2.17), for $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$. The proof of the following proposition and comments about its optimality are postponed to Section 5.A.1.

Proposition 5.3.1 (Stability bound)**.** *Fix a (large) constant* $L > 0$*. Let* $D_1, D_2 \in \mathbf{C}^{N \times N}$ *be Hermitian matrices with* $\langle D_l \rangle = 0$ *and* $\|D_l\| \le L$ *for* $l = 1, 2$*. For* $z_l = E_l + \mathrm{i}\eta_l \in \mathbf{H}$*,* $l = 1, 2$*, recall the notation* $\rho_l := \pi^{-1} \langle \operatorname{Im} M^{D_l}(z_l) \rangle$ *and denote*

$$\beta_* := \beta_*(z_1, z_2) = \beta(z_1, z_2) \wedge \beta(z_1, \bar{z}_2), \tag{5.3.2}$$

$$\widehat{\gamma} := \widehat{\gamma}(z_1, z_2) = \langle (D_1 - D_2)^2 \rangle + \mathrm{LT} + |E_1 - E_2|^2 \wedge 1 + \frac{\eta_1}{\rho_1} \wedge 1 + \frac{\eta_2}{\rho_2} \wedge 1. \tag{5.3.3}$$

Then uniformly in $z_1, z_2 \in \mathbf{H}$ *it holds that*

$$(\rho_1 + \rho_2)^2 \lesssim \beta(z_1, z_2). \tag{5.3.4}$$

Moreover, fix a (large) constant $C_0 > 0$ and assume that for some intervals $\mathbf{I}_1, \mathbf{I}_2 \subset \mathbf{R}$ we have

$$\sup_{\mathrm{Re}\, z_l \in \mathbf{I}_l} \|M^{D_l}(z_l)\| \le C_0, \quad l = 1, 2. \tag{5.3.5}$$

Then uniformly in $z_l = E_l + \mathrm{i}\eta_l \in \mathbf{H}$ with $E_l \in \mathbf{I}_l$, $l = 1, 2$, it holds that

$$\widehat{\gamma} \lesssim \beta_* \lesssim \widehat{\gamma}^{1/4}, \tag{5.3.6}$$

where the implicit constants depend only on L and C_0.

Note that (5.3.5) is automatically satisfied for $\mathbf{I}_l = \mathbf{B}_\kappa(D_l)$ with the constant C_0 depending only on κ. This follows from the bound

$$\|M_l^D(z_l)\| \le \big(|\mathrm{Im}\, z_l| + |\langle \mathrm{Im}\, M^{D_l}(z_l)\rangle|\big)^{-1} \le C\kappa^{-1}.$$

We point out that, even if not highlighted in the notation, the quantities β, β_* and $\widehat{\gamma}$ also depend on the deformations D_1, D_2. We will often omit this dependence in notations when it is clear what the arguments are.

The most relevant part of Proposition 5.3.1 is the lower bound $\beta_* \gtrsim \widehat{\gamma}$. This bound in a weaker form (more precisely, without LT included in $\widehat{\gamma}$) has already appeared in [237, Proposition 4.2]. It should be viewed as an upper bound on the two–body stability operator in terms of simpler control parameters collected in $\widehat{\gamma}$. In the inequality $\beta_* \lesssim \widehat{\gamma}^{1/4}$ we do not pursue getting the optimal power for $\widehat{\gamma}$. In fact, any positive exponent would work for our purpose.

5.3.2 Multi–resolvent local law: Proofs of Theorems 5.2.6 and 5.2.4

The main idea to give a bound on single eigenvector overlaps as in Theorems 5.2.6–5.2.4 is to upper bound the overlaps by traces of products of two resolvents, and then prove a bound for these quantities (see e.g. (5.3.15) below). For this reason in this section we first recall the traditional single resolvent local law, and then state our new *multi–resolvent local laws*, which are our main technical result.

Let $D \in \mathbf{C}^{N\times N}$, with $\|D\| \lesssim 1$, and let W be a Wigner matrix satisfying Assumption 5.2.1. Then, for $z \in \mathbf{C} \setminus \mathbf{R}$ we define the resolvent of $W + D$ by $G(z) = G^D(z) := (W + D - z)^{-1}$. It is well known that in the limit $N \to \infty$ the resolvent becomes approximately deterministic $G(z) \approx M(z)$, with $M(z) = M^D(z)$ being the solution of (5.1.4). This is expressed by the following single resolvent local law [243, Theorem 2.1]

$$\big|\langle (G(z) - M(z))A\rangle\big| \prec \frac{1}{N|\mathrm{Im}\, z|}, \qquad \big|\langle \boldsymbol{x}, (G(z) - M(z))\boldsymbol{y}\rangle\big| \prec \frac{1}{\sqrt{N|\mathrm{Im}\, z|}}, \tag{5.3.7}$$

uniformly in deterministic matrices $A \in \mathbf{C}^{N\times N}$ with $\|A\| \le 1$, unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, and spectral parameters z in the bulk regime, i.e. $\mathrm{Re}\, z \in \mathbf{B}_\kappa(D)$ for some fixed $\kappa > 0$.

The main topic of this section, however, is to compute the deterministic approximation of the products of two resolvents $G_1 A G_2$, with $G_l := G^{D_l}(z_l)$ for $l = 1, 2$ and a deterministic observable in between. While $G^{D_l}(z_l) \approx M^{D_l}(z_l)$, the deterministic approximation of $G_1 A G_2$ is not given by the product of the deterministic approximations $M_1 A M_2$, but, as we will see from our result, rather by

$$M^A_{\nu_1,\nu_2} := \mathcal{B}_{12}^{-1}\big[M_1 A M_2\big], \tag{5.3.8}$$

with $\nu_l = (z_l, D_l)$, $M_l = M^{D_l}(z_l)$ and with $\mathcal{B}_{12}$ being the stability operator defined in (5.2.12). We will stick to the following notational convention. In most cases we will simplify the notation $M^A_{\nu_1,\nu_2}$ to M^A_{12} when it is clear from the context what the arguments are. Moreover, if ν_1, ν_2 depend on an

additional parameter t, i.e. $\nu_1 = \nu_1(t)$, $\nu_2 = \nu_2(t)$, we will denote the dependence of (5.3.8) on t in two equivalent ways:

$$M^A_{\nu_1(t),\nu_2(t)} = M^A_{12,t}. \tag{5.3.9}$$

On the deterministic approximation defined in (5.3.8) we have the bound (see Proposition 5.4.6 below)

$$\|M^A_{\nu_1,\nu_2}\| \lesssim \frac{\|A\|}{\beta_*}, \tag{5.3.10}$$

with β_* from (5.3.2). In the case when A is (ν_1,ν_2)-regular, i.e. $A = \mathring{A}^{\nu_1,\nu_2}$, (5.3.10) improves to $\|M^A_{\nu_1,\nu_2}\| \lesssim \|A\|$. For precise statement see Proposition 5.4.6. We are now ready to state our main technical result.

Theorem 5.3.2 (Average two-resolvent local laws in the bulk)**.** *Fix $L, \epsilon, \kappa > 0$. Let W be a Wigner matrix satisfying Assumption 5.2.1, and let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be Hermitian matrices such that $\langle D_l\rangle = 0$ and $\|D_l\| \le L$ for $l = 1,2$. For spectral parameters $z_1, z_2 \in \mathbf{C}\setminus\mathbf{R}$, denote $\eta_l := |\operatorname{Im} z_l|$ and $\eta_* := \eta_1 \wedge \eta_2 \wedge 1$. Finally, let $\widehat{\gamma} = \widehat{\gamma}(z_1, z_2)$ be defined as in* (5.3.3)*. Then, the following holds:*

Part 1. [General case] *For deterministic $B_1, B_2 \in \mathbf{C}^{N\times N}$ we have*

$$\left|\left\langle \left(G^{D_1}(z_1)B_1G^{D_2}(z_2) - M^{B_1}_{\nu_1,\nu_2}\right)B_2\right\rangle\right| \prec \left(\frac{1}{N\eta_1\eta_2} \wedge \frac{1}{\sqrt{N\eta_*\widehat{\gamma}}}\right)\|B_1\|\|B_2\| \tag{5.3.11}$$

uniformly in B_1, B_2, spectral parameters satisfying $\operatorname{Re} z_l \in \mathbf{B}_\kappa(D_l)$, $|z_l| \le N^{100}$, for $l = 1,2$, and $\eta_* \ge N^{-1+\epsilon}$.

Part 2. [Regular case] *Consider deterministic $A_1, A_2, B \in \mathbf{C}^{N\times N}$. Moreover, recalling* (5.2.10)*, let A_1 be (ν_1,ν_2)-regular and A_2 be (ν_2,ν_1)-regular. Then,*

$$\left|\left\langle \left(G^{D_1}(z_1)A_1G^{D_2}(z_2) - M^{A_1}_{\nu_1,\nu_2}\right)B\right\rangle\right| \prec \left(\frac{1}{N\eta_1\eta_2} \wedge \frac{1}{\sqrt{N\eta_*\widehat{\gamma}}}\right)\|A_1\|\|B\|, \tag{5.3.12}$$

$$\left|\left\langle \left(G^{D_1}(z_1)A_1G^{D_2}(z_2) - M^{A_1}_{\nu_1,\nu_2}\right)A_2\right\rangle\right| \prec \left(\frac{1}{N\eta_1\eta_2} \wedge \frac{1}{\sqrt{N\eta_*}}\right)\|A_1\|\|A_2\|, \tag{5.3.13}$$

uniformly in A_1, A_2, B, spectral parameters satisfying $\operatorname{Re} z_l \in \mathbf{B}_\kappa(D_l)$, $|z_l| \le N^{100}$, for $l = 1,2$, and $\eta_* \ge N^{-1+\epsilon}$.

One important technical tool needed for the proof of Part 2 Theorem 5.2.4 is the content of the following lemma, which compares regularizations of a deterministic matrix with respect to different pairs of spectral pairs. We point out that this is not a type of continuity statement about the dependence of $\mathring{A}^{\nu_1,\nu_2}$ on (ν_1,ν_2) like in Lemma 1.3.3. We postpone the proof of Lemma 5.3.3 to Appendix 5.A.4.

Lemma 5.3.3 (Comparison of different regularizations)**.** *Fix (large) $L > 0$ and (small) $\kappa > 0$. Let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be Hermitian deformations. Moreover, assume that $\langle D_1\rangle = \langle D_2\rangle = 0$ and $\|D_1\| \le L$, $\|D_2\| \le L$. Take spectral parameters $z_1, z_2 \in \mathbf{H}$ such that $\min\{\rho_1(z_1), \rho_2(z_2)\} \ge \kappa$, where $\rho_l(z_l) := \langle \operatorname{Im} M^{D_l}(z_l)\rangle\pi^{-1}$, $l = 1,2$. For $y_1, y_2 \ge 0$ denote $z_l' := z_l + \mathrm{i}y_l \in \mathbf{H}$, $l = 1,2$. Additionally we use notations $\nu_l := (z_l, D_l)$, $\nu_l' := (z_l', D_l)$ and $\bar{\nu}_l' := (\bar{z}_l', D_l)$ for $l = 1,2$. Then for any observable $A \in \mathbf{C}^{N\times N}$ we have*

$$\begin{aligned}
&\|\mathring{A}^{\nu_1',\nu_2'} - \mathring{A}^{\nu_1,\nu_2}\| \lesssim \|A\|\sqrt{\widehat{\gamma}(z_1', z_2')}, \qquad &&\|\mathring{A}^{\bar{\nu}_1',\nu_2'} - \mathring{A}^{\nu_1,\nu_2}\| \lesssim \|A\|\sqrt{\widehat{\gamma}(z_1', z_2')},\\
&\|\mathring{A}^{\nu_2',\nu_1'} - \mathring{A}^{\nu_1,\nu_2}\| \lesssim \|A\|\sqrt{\widehat{\gamma}(z_1', z_2')}, \qquad &&\|\mathring{A}^{\bar{\nu}_2',\nu_1'} - \mathring{A}^{\nu_1,\nu_2}\| \lesssim \|A\|\sqrt{\widehat{\gamma}(z_1', z_2')}.
\end{aligned} \tag{5.3.14}$$

The implicit constants in (5.3.14) *depend only on L and κ. Recall that when the complex conjugation falls on the second spectral pair, we have $\mathring{A}^{\nu_1',\bar{\nu}_2'} = \mathring{A}^{\nu_1',\nu_2'}$ by definition.*

In the rhs. of (5.3.14) we do not aim to get the optimal power of $\widehat{\gamma}$, but rather formulate Lemma 5.3.3 minimalistically and collect only those bounds which will be used later.

Given Theorem 5.3.2 and Lemma 5.3.3 we immediately conclude the proof of Theorems 5.2.6 and 5.2.4.

Proof of Theorems 5.2.4 and 5.2.6. We first prove Theorem 5.2.6. Consider $i, j \in [N]$ such that $\gamma_i^{D_1} \in \mathbf{B}_\kappa(D_1)$ and $\gamma_j^{D_2} \in \mathbf{B}_\kappa(D_2)$, and let $\eta = N^{-1+\epsilon}$ for a small fixed $\epsilon > 0$. Then, by spectral decomposition we readily obtain

$$N\big|\langle \boldsymbol{u}_i^1, \boldsymbol{u}_j^2\rangle\big|^2 \prec (N\eta)^2\big|\big\langle \operatorname{Im} G^{D_1}(\gamma_i^{D_1} + \mathrm{i}\eta)\operatorname{Im} G^{D_2}(\gamma_j^{D_2} + \mathrm{i}\eta)\big\rangle\big|. \tag{5.3.15}$$

We point out that to prove (5.3.15) we also use the standard rigidity bound from [17, 243]:

$$\big|\lambda_i^D - \gamma_i^D\big| \prec \frac{1}{N}, \qquad \gamma_i^D \in \mathbf{B}_\kappa(D). \tag{5.3.16}$$

Finally, combining (5.3.15) with (5.3.11), using (5.3.10) and

$$\|M^{D_l}(z_l)\| = \left\|\frac{1}{D_l - z_l - \langle M^{D_l}(z_l)\rangle}\right\| \le |\langle \operatorname{Im} M^{D_l}(z_l)\rangle|^{-1} \le C_\kappa \tag{5.3.17}$$

in the κ–bulk of the spectrum, and the definition of $\widehat{\gamma}$ from (5.3.3), we immediately conclude (5.2.18).

Now we discuss how to adjust the argument above to prove Theorem 5.2.4. The fact that $\|V\| \lesssim 1$ in the regime when $\phi_{ij} \neq 0$ follows by simple perturbation theory if δ is chosen sufficiently small in terms of κ from (5.2.7) and in terms of $\|D_1\|, \|D_2\|$. In the complementary regime this bound is trivial. One more input which is needed for the proof of Theorem 5.2.4 is Lemma 5.3.3 specialized to the case $y_1 = y_2 = 0$. In fact, Lemma 5.3.3 implies that if A is (ν_1, ν_2)–regular, then it is close in operator norm to $\mathring{A}^{\nu_2,\nu_1}$. The rest of the details are omitted for the sake of brevity (see the proof of Theorem 1.2.2 for the details of a very similar proof). □

We conclude this section commenting on the optimality of Theorem 5.3.2.

5.3.3 Optimality and possible extensions of Theorem 5.3.2

In this section explain in what sense Theorem 5.3.2 is optimal and that it can be extended to energies where the limiting eigenvalue density is small. We also comment on the possibility of replacing the operator norm in the rhs. of the estimates in Theorem 5.3.2 with the typically smaller Hilbert–Schmidt norm. All these improvements and extensions can be achieved following our *zigzag* strategy of first proving the desired result for matrices with a fairly large Gaussian component, as in Section 5.5 (zig step), and then prove it for general matrices using a dynamical comparison argument, as in Section 5.6 (zag step). We omit the details of these proofs to keep the presentation simple and short, in fact, the main focus of this paper is to develop techniques to handle different deformations D_1, D_2 and we do it in the simpler cases of the bulk of the spectrum proving estimates in terms of the operator norm. In the following we give more precise references to papers where similar analyses were already performed in detail.

We first consider the bound (5.3.11). In the bulk of the spectrum, this bound is optimal except for the fact that $1/(\sqrt{N\eta_*}\widehat{\gamma})$ should be replaced by $1/(N\eta_*\widehat{\gamma})$. Notice, that once the bound $1/(N\eta_*\widehat{\gamma})$ is achieved, the term $1/(N\eta_1\eta_2)$ in (5.3.11) is obsolete, as it is always bigger. This improvement can be achieved by proving (weaker) local laws also for products of longer resolvents; see, e.g., [168] and Chapters 3 and 4 for similar arguments. In fact, this overestimate is due to the fact that the four resolvents chains, appearing e.g. in the quadratic variation of the stochastic term in (5.5.2) below are currently estimated in terms of products of traces of two resolvents using certain crude reduction

inequalities (see e.g. (5.5.27)). Following the evolution of these longer chains more carefully would give the improvement $1/(N\eta_*\widehat{\gamma})$.

We also believe that assuming that $M^{D_l}(z_l)$ are bounded throughout the spectrum (see e.g. condition (5.3.5) with $\mathbf{I} = \mathbf{R}$ and Remark 5.A.2 below) one can extend the local laws (5.3.11)–(5.3.13) to hold uniformly in the spectrum with the similar zigzag strategy. In fact, in this case we expect an additional gain $\sqrt{\rho_1 + \rho_2}$ in their rhs.; see Theorem 3.2.4 for a similar argument. We postpone the details to future work.

Finally, the operator norm in (5.3.11)–(5.3.13) can be replaced by the typically smaller Hilbert–Schmidt norm. Again, this can be achieved following our proof in Sections 5.5–5.6, but we omit a detailed proof of brevity. A similar proof was carried out in full detail in Chapter 3 in the simpler setting of Wigner matrices using the Lindeberg swapping technique, but it can be readily adapted to the current case. Additionally, we expect that the Lindeberg technique can be replaced by a dynamical argument similar to Section 5.6.

5.4 Zigzag strategy: Proof of Theorem 5.3.2

To prove the multi–resolvent local law in Theorem 5.3.2 we follow the *zigzag strategy*, similarly to [161] and Chapters 3–4. That is, we prove Theorem 5.3.2 by running in tandem the *characteristic flow* associated to a matrix valued Ornstein–Uhlenbeck process, and a *Green's function comparison theorem (GFT)*.

More precisely, the zigzag strategy consists of the following three steps:

1. **Global law:** Prove a *global law* for spectral parameters z_j that are "far away" from the self-consistent spectrum, $\min_j \operatorname{dist}(z_j, \operatorname{supp}(\rho^{D_j})) \geq \delta$ (see Section 5.4.1).
2. **Characteristic flow:** Propagate the bound from large distances to a smaller one by considering the evolution of the Wigner matrix W along an Ornstein-Uhlenbeck flow, thereby introducing a Gaussian component (see Section 5.4.2). The spectral parameters evolve according to the *characteristic flow* defined in (5.4.8). The simultaneous effect of these two evolutions is a key cancellation of two large terms.
3. **Green function comparison:** Remove the Gaussian component by a Green function comparison (GFT) argument (see Section 5.4.3).

In order to reduce the distance of the spectral parameters down to the optimal scale for the local law, Steps 2 and 3 will be applied many times in tandem. This inductive argument is carried out in Proposition 5.4.17 in Section 5.4.4.

While Theorem 5.3.2 states local laws only for average quantities, within the GFT, isotropic resolvent chains of the form

$$(GBG)_{\boldsymbol{xy}} \quad \text{or} \quad (GBGBG)_{\boldsymbol{xy}} \tag{5.4.1}$$

naturally arise, which requires to analyze them as well. That is, we necessarily need to perform the zigzag strategy for such quantities in an analogous way.

Throughout the entire argument, all process will run for times t in a fixed interval $[0, T]$ for some terminal time $T > 0$ of order one, which we will choose below in (5.4.48).

5.4.1 Input global laws

Here we state the necessary global laws that will be used as an input to prove Theorem 5.3.2. Note that in the global regime no restriction to the bulk is necessary.

Proposition 5.4.1 (Global law)**.** *Fix $L, \epsilon, \delta > 0$. Let W be a Wigner matrix satisfying Assumption 5.2.1, and let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be bounded Hermitian matrices, i.e. $\|D_l\| \le L$ for $l = 1, 2$. For spectral parameters $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ with $\min_j \operatorname{dist}(z_j, \operatorname{supp}(\rho^{D_j})) \ge \delta$, deterministic unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and matrices $B_1, B_2 \in \mathbf{C}^{N\times N}$, we have (recall $G_j := (W + D_j - z_j)^{-1}$)*

$$\left|\left\langle \left(G_1 B_1 G_2 - M^{B_1}_{\nu_1,\nu_2}\right) B_2\right\rangle\right| \prec \frac{\|B_1\|\|B_2\|}{N}, \tag{5.4.2}$$

$$\left|\left\langle \boldsymbol{x}, \left(G_1 B_1 G_2 - M^{B_1}_{\nu_1,\nu_2}\right) \boldsymbol{y}\right\rangle\right| \prec \frac{\|B_1\|}{\sqrt{N}}, \tag{5.4.3}$$

$$\left|\left\langle \boldsymbol{x}, G_1 B_1 G_2 B_2 G^{(*)}_{1,s} \boldsymbol{y}\right\rangle\right| \prec \|B_1\|\|B_2\|. \tag{5.4.4}$$

Proof. The proof of these global laws is very similar to the one presented in [168, Appendix B], we thus omit several details and just present the main steps. To keep the presentation short and simple we only present the proof in the averaged case.

In the following we will often use the fact that

$$\|G_j\| \lesssim \frac{1}{\min_j \operatorname{dist}(z_j, \operatorname{supp}(\rho^{D_j}))} \le \frac{1}{\delta} \lesssim 1. \tag{5.4.5}$$

By explicit computations it is easy to see that

$$\begin{aligned}(1 - M_1\langle\cdot\rangle M_2)\left(G_1 B_1 G_2 - M^{B_1}_{12}\right) = M_1 B_1 (G_2 - M_2) - M_1 \underline{W G_1 B_1 G_2} \\ + M_1\langle G_1 B_1 G_2\rangle (G_2 - M_2) + M_1\langle G_1 - M_1\rangle G_1 B_1 G_2,\end{aligned} \tag{5.4.6}$$

where

$$\underline{W G_1 B_1 G_2} := W G_1 G_2 + \langle G_1\rangle G_1 B_1 G_2 + \langle G_1 B_1 G_2\rangle G_2.$$

Taking the trace in (5.4.6) against B_2, by the single resolvent local law (5.3.7), the norm bound (5.4.5), and the fact that $(1 - M_1\langle\cdot\rangle M_2)^{-1}$ is bounded in the regime $\min_j \operatorname{dist}(z_j, \operatorname{supp}(\rho^{D_j})) \ge \delta$, we have

$$\langle (G_1 B_1 G_2 - M^{B_1}_{12}) B_2\rangle = -\langle M_1 \underline{W G_1 B_1 G_2 B_2}\rangle + \mathcal{O}_\prec\left(\frac{\|B_1\|\|B_2\|}{N}\right). \tag{5.4.7}$$

Finally, using a minimalistic cumulant expansion as in [168, (B.4)–(B.8)], we conclude $|\langle M_1 \underline{W G_1 B_1 G_2 B_2}\rangle| \prec N^{-1}\|B_1\|\|B_2\|$ and so (5.4.2). □

5.4.2 Zig step: Propagating bounds via the characteristic flow

For Hermitian $D_j \in \mathbf{C}^{N\times N}$ with $\langle D_j\rangle = 0$, spectral parameters $z_j \in \mathbf{C} \setminus \mathbf{R}$, $j = 1, 2$, and fixed $T > 0$ the characteristic flow is defined by the following ODEs (see also [161, (5.3)]):

$$\partial_t D_{j,t} := -\frac{1}{2} D_{j,t}, \qquad \partial_t z_{j,t} = -\langle M^{D_{j,t}}(z_{j,t})\rangle - \frac{z_{j,t}}{2}, \qquad j = 1, 2, \tag{5.4.8}$$

with terminal conditions $D_{j,T} = D_j$ and $z_{j,T} = z_j$.

We will often use the following short–hand notations:

$$M_{j,t} := M^{D_{j,t}}(z_{j,t}), \quad \rho_{j,t} = \rho_{j,t}(z_{j,t}) := \frac{1}{\pi}\left|\langle \operatorname{Im} M^{D_{j,t}}(z_{j,t})\rangle\right|, \quad \eta_{j,t} := |\operatorname{Im} z_{j,t}|, \qquad j = 1, 2.$$

Even if our main results in Sections 5.2 and 5.3.2 are presented only in the bulk of the spectrum, in the case of general observables we study the zig step uniformly in the spectrum, since this does not present significant additional difficulties (see Part 1 of Proposition 5.4.8 below). For this reason, several definition in the remainder of this section will be presented uniformly in the spectrum. In the zig step for regular observables and in the zag step for both types of observables we still restrict ourselves to the bulk.

5.4.2.1 Preliminaries on the characteristic flow and admissible control parameters

Before formulating the fundamental building blocks for Step 2 of the zigzag strategy in Section 5.4.2.2 below, we collect a few preliminaries concerning the characteristic flow and introduce *admissible control parameters* γ, generalizing the concretely chosen $\widehat{\gamma}$ in (5.3.3).

First, we define a time–dependent version of the spectral domains on which we prove the local law from Theorem 5.3.2 along the flow.

Definition 5.4.2 (Spectral domains)**.** *We define the time dependent spectral domains as follows:*

(i) [Unrestricted domains] *Fix a (small)* $\epsilon > 0$*. For* $j \in [2]$ *define*

$$\Omega_T^j := \left\{ z \in \mathbf{C} \setminus \mathbf{R} : |\operatorname{Im} z \cdot \rho_{j,T}(z)| \geq N^{-1+\epsilon}, |\operatorname{Im} z| \leq N^{100}, |\operatorname{Re} z| \leq N^{200} \right\}. \tag{5.4.9}$$

For $s, t \in [0,T]$ *denote by* $\mathfrak{F}_{t,s}^j$ *the evolution operator along the flow* (5.4.8)*, i.e.* $\mathfrak{F}_{t,s}^j(z_{j,s}) = z_{j,t}$*. Then we construct the family of* unrestricted spectral domains $\{\Omega_t^j\}_{t\in[0,T]}$*,* $j \in [2]$*, by* $\Omega_t^j := \mathfrak{F}_{t,T}^j(\Omega_T^j)$*.*

(ii) [Bulk-restricted domains] *Fix additionally a (small)* $\kappa > 0$ *and recall* (5.2.7) *for the definition of the* κ*-bulk* $\mathbf{B}_\kappa(D) = \cup_{r=1}^m I_r$*. Here* $I_r = [a_r, b_r] \subset \mathbf{R}$ *are closed non-intersecting intervals and* $b_r < a_{r+1}$ *for* $r \in [1, m-1]$*. We also denote* $b_0 := -\infty$ *and* $a_{m+1} := +\infty$*. Then we define the family of* bulk-restricted spectral domains *as*

$$\Omega_{\kappa,T}^j := \Omega_T^j \setminus \left(\bigcup_{r=0}^m \{ z \in \mathbf{C} \setminus \mathbf{R} : \operatorname{Re} z \in [b_r, a_{r+1}], |\operatorname{Im} z| \leq |\operatorname{Re} z - a_{r+1}| \wedge |\operatorname{Re} z - b_r| \} \right) \tag{5.4.10}$$

and $\Omega_{\kappa,t}^j := \mathfrak{F}_{t,T}^j(\Omega_{\kappa,T}^j)$ *for* $t \in [0,T]$*.*

The bulk-restricted spectral domains $\Omega_{\kappa,t}^j$ are depicted in Figure 5.4.1.

Next, we state some trivially checkable properties of the characteristics flow (5.4.8). Since Lemma 5.4.3 (i) holds for $j = 1$ and $j = 2$, we drop the index j in z_j, D_j, Ω_t^j and related quantities. In particular, we use the notation z_t for $z_{j,t}$.

Lemma 5.4.3 (Elementary properties of the characteristic flow)**.** *We have the following.*

(i) Let $z_0 \in \Omega_0$ *be given. Then we have*

(1) $M_t(z_t) = \mathrm{e}^{t/2} M_0(z_0)$*.*

(2) The map $t \mapsto \eta_t$ *is monotone decreasing.*

(3) The solution to the second equation in (5.4.8) *is explicitly given by*

$$z_t = \mathrm{e}^{-t/2} z_0 - 2\langle M_0(z_0) \rangle \sinh \frac{t}{2}. \tag{5.4.11}$$

(4) For any $m > 1$ *and* $t \in [0,T]$ *we have*

$$\int_0^t \frac{\rho_s}{\eta_s^m} \mathrm{d}s \leq \frac{1}{(m-1)\eta_t^{m-1}}, \qquad \int_0^t \frac{\rho_s}{\eta_s} \mathrm{d}s \leq \log\left(\frac{\eta_0}{\eta_t}\right). \tag{5.4.12}$$

(5) We have

$$\frac{\eta_t}{\rho_t} = e^{s-t} \frac{\eta_s}{\rho_s} - \pi(1 - e^{s-t}). \tag{5.4.13}$$

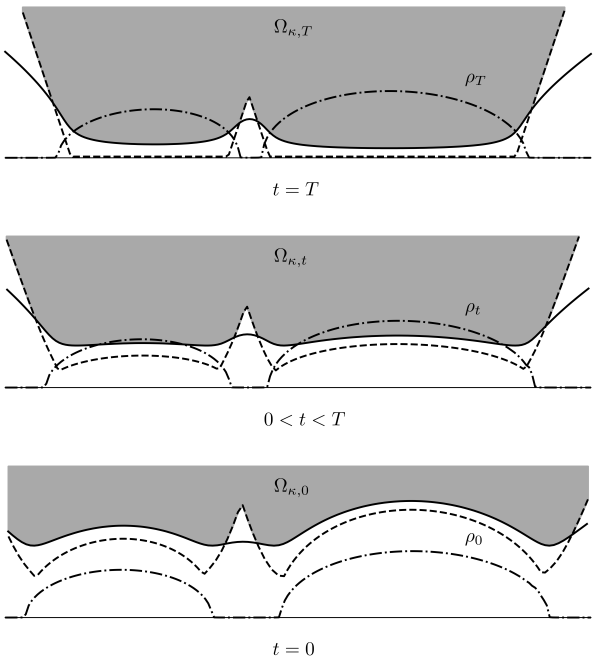

Figure 5.4.1: In gray, we illustrated the $\operatorname{Im} z > 0$ part of the bulk-restricted spectral domains $\Omega_{\kappa,t}$ for three times, $t = 0$, $t \in (0,T)$, and $t = T$ (the $\operatorname{Im} z < 0$ part is obtained by reflection). On each of the panels, the graph of the density ρ_t is superimposed in a dash-dotted style. The solid curve in the $t = T$ panel represents the implicitly defined curve $|\operatorname{Im} z|\rho(z) = N^{-1+\epsilon}$, above which one has the unrestricted domain $\Omega_T \supset \Omega_{\kappa,T}$. On the same $t = T$ panel, the region below the dashed curve is removed in the rhs. of (5.4.10). For $t < T$ the solid and the dashed curves are the images of the corresponding curves at $t = T$ under the flow $\mathfrak{F}_{t,T}$.

(ii) Let $z_j \in \Omega_0^j$ and $D_j = D_j^* \in \mathbf{C}^{N\times N}$ be given for $j \in [2]$. Denote $\nu_{j,t} := (z_{j,t}, D_{j,t})$ for $t \in [0,T]$ and assume that $\phi(\nu_{1,T}, \nu_{2,T}) = 1$ (recall (5.2.9) for its definition). Let $A \in \mathbf{C}^{N\times N}$ be a regular observable with respect to $(\nu_{1,T}, \nu_{2,T})$. Then A is regular with respect to $(\nu_{1,t}, \nu_{2,t})$ for any $t \in [0,T]$.

Notice that the error terms in Theorem 5.3.2 are expressed in terms of the control parameter $\widehat{\gamma}$. In Theorem 5.3.2, $\widehat{\gamma}$ is explicitly given, however, in order to make the argument more transparent, we collect in Definition 5.4.4 all properties of $\widehat{\gamma}$ which are needed for the proof of Theorem 5.3.2, arriving to the definition of an *admissible control parameter*. In Proposition 5.4.5 we show that $\widehat{\gamma}$ on its own is an admissible control parameter. Further in Section 5.5 we work in this more general framework using a general admissible parameter γ instead of $\widehat{\gamma}$.

Definition 5.4.4 (Admissible control parameter)**.** *Let $\gamma : \big(\mathbf{C}\setminus\mathbf{R}\big)^2 \times \big(\mathbf{C}^{N\times N}\big)^2 \to (0,+\infty)$ be uniformly bounded in N and assume that $\gamma(z_1,z_2,D_1,D_2) = \gamma(\bar{z}_1,z_2,D_1,D_2)$ and the same for $z_2 \to \bar{z}_2$. Moreover, for $t \in [0,T]$, let $\gamma_t : \Omega_0^1 \times \Omega_0^2 \times \big(\mathbf{C}^{N\times N}\big)^2 \to (0,\infty)$ with*

$$\gamma_t(z_1,z_2,D_1,D_2) := \gamma(z_{1,t},z_{2,t},D_{1,t},D_{2,t}) \tag{5.4.14}$$

be the time-dependent version of γ, and $\beta_{*,t} : \Omega_0^1 \times \Omega_0^2 \times \big(\mathbf{C}^{N\times N}\big)^2 \to (0,\infty)$ with

$$\beta_{*,t}(z_1,z_2,D_1,D_2) := \beta_*(z_{1,t},z_{2,t},D_{1,t},D_{2,t}). \tag{5.4.15}$$

the time-dependent version of β_* (recall (5.3.2) and (5.3.1)). In (5.4.14) and (5.4.15), $z_{j,t}$ and $D_{j,t}$ are the solutions to (5.4.8) with $z_{j,0} = z_j$ and $D_{j,0} = D_j$ for $j \in [2]$.

Let $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathbf{C}^{N\times N}$ be N-dependent families of $N \times N$ Hermitian matrices. We say that γ is a $(\mathfrak{D}_1, \mathfrak{D}_2)$-admissible control parameter *if the following conditions hold uniformly in $D_j \in \mathfrak{D}_j$, $z_j \in \Omega_0^j$, $j \in [2]$, $t \in [0,T]$ and N:*

(1) [γ is a lower bound on the stability operator] *It holds that*

$$(|\operatorname{Im} z_{1,t}|/\rho_{1,t}(z_{1,t}) + |\operatorname{Im} z_{2,t}|/\rho_{2,t}(z_{2,t})) \wedge 1 \lesssim \gamma_t \lesssim \beta_{*,t}\,, \tag{5.4.16}$$

where both γ_t and $\beta_{*,t}$ are evaluated at (z_1, z_2, D_1, D_2).

(2) [Monotonicity in time] *Uniformly in $0 \le s \le t \le T$, we have*

$$\gamma_s(z_1, z_2, D_1, D_2) \sim \gamma_t(z_1, z_2, D_1, D_2) + t - s. \tag{5.4.17}$$

(3) [Vague monotonicity in imaginary part] *Uniformly in $z_1, z_2 \in \mathbf{H}$ and $x \in [0,\infty)$ it holds that*

$$\gamma(z_1, z_2, D_{1,t}, D_{2,t}) \lesssim \gamma(z_1, z_2 + \mathrm{i}x, D_{1,t}, D_{2,t}) \wedge \gamma(z_1 + \mathrm{i}x, z_2, D_{1,t}, D_{2,t}). \tag{5.4.18}$$

We now verify that $\widehat{\gamma}$ is an admissible control parameter in the sense of Definition 5.4.4.

Proposition 5.4.5 (Admissibility of $\widehat{\gamma}$)**.** *Fix $L, C_0 > 0$. Let $\mathfrak{D}$ be a set of all traceless $N \times N$ Hermitian matrices such that any $D \in \mathfrak{D}$ satisfies* (5.3.5) *for $\mathbf{I} = \mathbf{R}$ with constant C_0 and $\|D\| \le L$. Then $\widehat{\gamma}$ defined in* (5.3.3) *is a $(\mathfrak{D}, \mathfrak{D})$-admissible control parameter.*

The proof of Proposition 5.4.5 and a sufficient condition for D to satisfy (5.3.5) for $\mathbf{I} = \mathbf{R}$ are given in Appendix 5.A.2.

As discussed around (5.4.1), during the proof of Theorem 5.3.2 we need to handle resolvent products of the form $GBGBG$. More precisely, let $D_l \in \mathbf{C}^{N\times N}$ be Hermitian deformations and $z_l \in \mathbf{C} \setminus \mathbf{R}$ for $l \in [3]$. Denote $G_l := (W + D_l - z_l)^{-1}$, $\nu_l := (z_l, D_l)$ and $M_l := M^{D_l}(z_l)$. We define the deterministic approximation of $G_1B_1G_2B_2G_3$ by (see Definition 2.4.1)

$$M^{B_1,B_2}_{\nu_1,\nu_2,\nu_3} := \mathcal{B}_{13}^{-1}\left[M_1B_1M^{B_2}_{\nu_2,\nu_3} + \langle M^{B_1}_{\nu_1,\nu_2}\rangle M_1 M^{B_2}_{\nu_2,\nu_3}\right], \tag{5.4.19}$$

where $\mathcal{B}_{13}$ is defined in (5.2.12), i.e. $\mathcal{B}_{13}[\cdot] = 1 - M_1\langle\cdot\rangle M_3$. In the case when ν_l depend on additional parameter t, i.e. $\nu_l = \nu_l(t)$, $l \in [3]$, we adhere the analogue of the convention (5.3.9) for $M^{B_1,B_2}_{\nu_1,\nu_2,\nu_3}$. Namely, we use the shorthand notation

$$M^{B_1,B_2}_{123,t} := M^{B_1,B_2}_{\nu_1(t),\nu_2(t),\nu_3(t)}. \tag{5.4.20}$$

We are now ready to state bounds on the deterministic approximation of products of two and three resolvents:

Proposition 5.4.6 (Bounds on M)**.** *Fix $L > 0$. Let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be Hermitian deformations with $\langle D_j\rangle = 0$ and $\|D_j\| \le L$ for $j = 1, 2$. For spectral parameters $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ denote the corresponding spectral pairs by $\nu_j = (z_j, D_j)$. Additionally we denote $\bar{\nu}_j := (\bar{z}_j, D_j)$ and*

$$\ell(z_1, z_2) := \eta_1\rho_1(z_1) \wedge \eta_1\rho_2(z_2), \quad \textit{where} \quad \eta_j = |\operatorname{Im} z_j|,\ \rho_j(z_j) = \pi^{-1}|\langle \operatorname{Im} M^{D_j}(z_j)\rangle|,\ j = 1, 2.$$

Part 1: [General case] *Fix additionally $C_0 > 0$ and assume that D_1, D_2 satisfy* (5.3.5) *for $\mathbf{I} = \mathbf{R}$ with constant C_0. Then uniformly in $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ and deterministic matrices $B_1, B_2 \in \mathbf{C}^{N\times N}$ it holds that*

$$\left\|M^{B_1}_{\nu_1,\nu_2}\right\| \lesssim \frac{\|B_1\|}{\beta_*(z_1, z_2)}, \tag{5.4.21a}$$

$$\|M^{B_1,B_2}_{\nu_1,\nu_2,\nu_1}\| + \|M^{B_1,B_2}_{\nu_1,\nu_2,\tilde{\nu}_1}\| \lesssim \frac{\|B_1\|\|B_2\|}{\ell(z_1,z_2)\beta_*(z_1,z_2)}. \tag{5.4.21b}$$

Here the implicit constants depend only on C_0 and L.

Part 2: [Regular case] *Fix $\kappa > 0$. Uniformly in $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ with $\rho_j(z_j) \geq \kappa$ for $j = 1, 2$, in (ν_1, ν_2)-regular $A_1 \in \mathbf{C}^{N\times N}$ and general $B_2 \in \mathbf{C}^{N\times N}$ we have*

$$\left\|M^{A_1}_{\nu_1,\nu_2}\right\| \lesssim \|A_1\|, \tag{5.4.22a}$$

$$\|M^{A_1,B_2}_{\nu_1,\nu_2,\nu_1}\| + \|M^{A_1,B_2}_{\nu_1,\nu_2,\tilde{\nu}_1}\| \lesssim \frac{\|A_1\|\|B_2\|}{\ell(z_1,z_2)\sqrt{\beta_*(z_1,z_2)}}, \tag{5.4.22b}$$

$$\|M^{A_1,A_2}_{\nu_1,\nu_2,\nu_1}\| + \|M^{A_1,A_2}_{\nu_1,\nu_2,\tilde{\nu}_1}\| \lesssim \frac{\|A_1\|\|A_2\|}{\ell(z_1,z_2)}. \tag{5.4.22c}$$

We point out that $\ell \sim \eta_*$, since z_1, z_2 satisfy $\rho_i(z_i) \geq \kappa$. The implicit constants in (5.4.22a)-(5.4.22c) *depend only on L and κ,* (5.4.22b) *also holds when the second observable is (ν_2, ν_1)-regular and the first one is general.*

The proof of Proposition 5.4.6 is given in Appendix 5.A.3.

5.4.2.2 Propagating local law bounds

The general setting for propagating local law bounds is the following:

Setting 5.4.7 (Zig step)**.** *Fix large constant $L > 0$ and let $\mathfrak{D}_1, \mathfrak{D}_2$ be sets of $N \times N$ traceless Hermitian matrices such that $\|D\| \leq L$ for any $D \in \mathfrak{D}_j$, $j \in [2]$. Let γ be a $(\mathfrak{D}_1, \mathfrak{D}_2)$-admissible control parameter as in Definition 5.4.4.*

Fix a terminal time $T > 0$, let $z_{j,0} \in \Omega^j_0$, $D_{j,0} \in \mathfrak{D}_j$ for $j \in [2]$, and denote their time evolutions (5.4.8) *by $z_{j,t} \in \Omega^j_t$ and $D_{j,t} \in \mathfrak{D}_j$, respectively. Moreover, let $s \in [0, T]$ be an initial time for the Ornstein-Uhlenbeck process. That is, for $t \in [s, T]$, let W_t be the solution to* (5.4.23)*,*

$$\mathrm{d}W_t = -\frac{1}{2}W_t\mathrm{d}t + \frac{\mathrm{d}B_{t-s}}{\sqrt{N}} \quad \text{with initial condition} \quad W_s = W\,. \tag{5.4.23}$$

Here, B_{t-s} is a real symmetric ($\beta = 1$) or complex Hermitian ($\beta = 2$) matrix-valued Brownian motion with entries having variance equal to $(t - s)$ times those of a GOE/GUE matrix. Finally, we denote the resolvent of $W_t + D_{j,t}$ at $z_{j,t}$ by

$$G_{j,t} := (W_t + D_{j,t} - z_{j,t})^{-1} \tag{5.4.24}$$

and introduce the abbreviations γ_t from Definition 5.4.4, and

$$\eta_{*,t} := \eta_{1,t} \wedge \eta_{2,t} \wedge 1 \quad \text{and} \quad \ell_t := (\eta_{1,t}\rho_{1,t}) \wedge (\eta_{2,t}\rho_{2,t})\,. \tag{5.4.25}$$

The following proposition formalizes the propagation of local laws along the evolution from Setting 5.4.7.

Proposition 5.4.8 (Average and isotropic zig step for two and three resolvents)**.** *In the Setting 5.4.7, we have the following.*

Part 1: [General case] *Consider Setting 5.4.7 with $\mathfrak{D}_1, \mathfrak{D}_2$ such that* (5.3.5) *is satisfied for $\mathbf{I} = \mathbf{R}$ with some constant C_0 for any matrix $D \in \mathfrak{D}_1 \cup \mathfrak{D}_2$. Assume that, for fixed initial time $s \in [0,T]$, we have*[10]

$$\left|\left\langle \left(G_{1,s}B_1G_{2,s} - M_{12,s}^{B_1}\right)B_2\right\rangle\right| \prec \left(\frac{1}{N\eta_{1,s}\eta_{2,s}} \wedge \frac{1}{\sqrt{N\ell_s\gamma_s}}\right)\|B_1\|\|B_2\|, \tag{5.4.26a}$$

$$\left|\left\langle \boldsymbol{x}, \left(G_{1,s}B_1G_{2,s} - M_{12,s}^{B_1}\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\ell_s}} \cdot \frac{1}{\sqrt{\eta_{*,s}\gamma_s}}\|B_1\|, \tag{5.4.26b}$$

$$\left|\left\langle \boldsymbol{x}, G_{1,s}B_1G_{2,s}B_2G_{1,s}^{(*)}\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell_s} \cdot \frac{1}{\gamma_s}\|B_1\|\|B_2\|, \tag{5.4.26c}$$

uniformly in $z_{j,s} \in \Omega_s^j$, $j \in [2]$, deterministic matrices B_1, B_2 and unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$. Then it holds that

$$\left|\left\langle \left(G_{1,t}B_1G_{2,t} - M_{12,t}^{B_1}\right)B_2\right\rangle\right| \prec \left(\frac{1}{N\eta_{1,t}\eta_{2,t}} \wedge \frac{1}{\sqrt{N\ell_t\gamma_t}}\right)\|B_1\|\|B_2\|, \tag{5.4.27a}$$

$$\left|\left\langle \boldsymbol{x}, \left(G_{1,t}B_1G_{2,t} - M_{12,t}^{B_1}\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\ell_t}} \cdot \frac{1}{\sqrt{\eta_{*,t}\gamma_t}}\|B_1\|, \tag{5.4.27b}$$

$$\left|\left\langle \boldsymbol{x}, G_{1,t}B_1G_{2,t}B_2G_{1,t}^{(*)}\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell_t} \cdot \frac{1}{\gamma_t}\|B_1\|\|B_2\|, \tag{5.4.27c}$$

uniformly in $t \in [s,T]$, $z_{j,t} \in \Omega_t^j$, $j \in [2]$, matrices B_1, B_2 and unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$.

Part 2: [Regular case] *Assume that the result of Part 1 holds in $[0,T]$ and consider the slightly modified Setting 5.4.7 with $z_{j,0} \in \Omega_{\kappa,0}^j$ for some bulk parameter $\kappa > 0$. Assume that for fixed initial time $s \in [0,T]$, we have*

$$\left|\left\langle \left(G_{1,s}A_1G_{2,s} - M_{12,s}^{A}\right)B\right\rangle\right| \prec \left(\frac{1}{N\eta_{1,s}\eta_{2,s}} \wedge \frac{1}{\sqrt{N\ell_s\gamma_s}}\right)\|A_1\|\|B\|, \tag{5.4.28a}$$

$$\left|\left\langle \left(G_{1,s}A_1G_{2,s} - M_{12,s}^{A_1}\right)A_2\right\rangle\right| \prec \left(\frac{1}{N\eta_{1,s}\eta_{2,s}} \wedge \frac{1}{\sqrt{N\ell_s}}\right)\|A_1\|\|A_2\|, \tag{5.4.28b}$$

$$\left|\left\langle \boldsymbol{x}, \left(G_{1,s}A_1G_{2,s} - M_{12,s}^{A_1}\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\ell_s}} \cdot \frac{1}{\sqrt{\eta_{*,s}}}\|A_1\|, \tag{5.4.28c}$$

$$\left|\left\langle \boldsymbol{x}, G_{1,s}A_1G_{2,s}BG_{1,s}^{(*)}\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell_s} \cdot \frac{1}{\sqrt{\gamma_s}}\|A_1\|\|B\|, \tag{5.4.28d}$$

$$\left|\left\langle \boldsymbol{x}, G_{1,s}A_1G_{2,s}A_2G_{1,s}^{(*)}\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell_s}\|A_1\|\|A_2\|, \tag{5.4.28e}$$

uniformly in $z_{j,s} \in \Omega_{\kappa,s}^j$, $j \in [2]$, deterministic $B \in \mathbf{C}^{N\times N}$, (ν_1,ν_2)-regular A_1 and (ν_2,ν_1)-regular A_2 and unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$. Then it holds that

$$\left|\left\langle \left(G_{1,t}A_1G_{2,t} - M_{12,t}^{A}\right)B\right\rangle\right| \prec \left(\frac{1}{N\eta_{1,t}\eta_{2,t}} \wedge \frac{1}{\sqrt{N\ell_t\gamma_t}}\right)\|A_1\|\|B\|, \tag{5.4.29a}$$

$$\left|\left\langle \left(G_{1,t}A_1G_{2,t} - M_{12,t}^{A_1}\right)A_2\right\rangle\right| \prec \left(\frac{1}{N\eta_{1,t}\eta_{2,t}} \wedge \frac{1}{\sqrt{N\ell_t}}\right)\|A_1\|\|A_2\|, \tag{5.4.29b}$$

$$\left|\left\langle \boldsymbol{x}, \left(G_{1,t}A_1G_{2,t} - M_{12,t}^{A_1}\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\ell_t}} \cdot \frac{1}{\sqrt{\eta_{*,t}}}\|A_1\|, \tag{5.4.29c}$$

$$\left|\left\langle \boldsymbol{x}, G_{1,t}A_1G_{2,t}BG_{1,t}^{(*)}\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell_t} \cdot \frac{1}{\sqrt{\gamma_t}}\|A_1\|\|B\|, \tag{5.4.29d}$$

$$\left|\left\langle \boldsymbol{x}, G_{1,t}A_1G_{2,t}A_2G_{1,t}^{(*)}\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell_t}\|A_1\|\|A_2\|, \tag{5.4.29e}$$

[10] The notation $G^{(*)}$ indicates both choices of adjoint G^* and no adjoint G.

uniformly in $t \in [s,T]$, $z_{j,t} \in \Omega^j_{\kappa,t}$, $j \in [2]$, *deterministic* $B \in \mathbf{C}^{N\times N}$, (ν_1,ν_2)*-regular* A_1 *and* (ν_2,ν_1)*-regular* A_2 *and unit vectors* $\boldsymbol{x},\boldsymbol{y} \in \mathbf{C}^N$.

Note that while the case of general observables is self–contained (i.e. it does not require any information about regular observables), the cases of one or two regular observables have to be done in tandem. In fact, when computing the quadratic variation for the stochastic term in (5.5.6) for traces with only one regular observable one gets a trace with two regular observables. On the other hand, the case of two regular observables is not self–contained either, because of the Lin_t term in (5.5.6).

5.4.3 Zag step: Removing Gaussian components via a GFT

As already explained below (5.4.8), from now on we constrain the argument to the *bulk*, i.e. we assume the spectral parameters to be in bulk-restricted domains $\Omega^j_{\kappa,t}$, where it holds that $\ell \sim \eta_*$. For ease of notation, we shall also write $\eta \equiv \eta_*$.

The general setting for removing a Gaussian component in Lemmas 5.4.10–5.4.14 is the following.

Setting 5.4.9 (Zag step)**.** *Fix a large constant* $L > 0$ *and let* $\mathfrak{D}_1, \mathfrak{D}_2$ *be sets of* $N \times N$ *traceless Hermitian matrices such that any* $D \in \mathfrak{D}_j$, $j \in [2]$, *satisfies* $\|D\| \le L$. *Let* γ *be a* $(\mathfrak{D}_1,\mathfrak{D}_2)$*-admissible control parameter as in Definition 5.4.4.*

Fix some $\kappa > 0$ *(bulk parameter) and a terminal time* $T > 0$, *let* $z_{j,0} \in \Omega^j_{\kappa,0}$, $D_{j,0} \in \mathfrak{D}_j$ *for* $j \in [2]$, *and denote their time evolutions* (5.4.8) *by* $z_{j,t} \in \Omega^j_{\kappa,t}$ *and* $D_{j,t} \in \mathfrak{D}_j$, *respectively. Now, take two fixed times* $s,t \in [0,T]$ *with* $s \le t$ *and consider the Ornstein-Uhlenbeck process*

$$\mathrm{d}W_r = -\frac{1}{2}W_r \mathrm{d}r + \frac{\mathrm{d}B_{r-s}}{\sqrt{N}} \quad \textit{with initial condition} \quad W_s = W \tag{5.4.30}$$

for times $r \in [s,t]$. *Finally, we denote the resolvent of* $W_r + D_{j,t}$ *at* $z_{j,t}$ *(note that the* t *index is fixed!) by*

$$G_{j,r} := (W_r + D_{j,t} - z_{j,t})^{-1}\,. \tag{5.4.31}$$

The times s,t *and hence, in particular, the spectral parameters* $z_{j,t} \in \Omega^j_{\kappa,t}$ *remain fixed through the Lemmas 5.4.10–5.4.14 below. Thus, dropping the time arguments, we denote* $\eta_j = |\mathrm{Im}\, z_{j,t}|$, $\eta := \min_j \eta_j$, *and* $\gamma = \gamma_t(z_1,z_2,D_1,D_2)$.

Contrary to the the *zig* step in Section 5.4.2, where all three local law bounds (two resolvent average, two and three resolvent isotropic) are propagated together (cf. Proposition 5.4.8), the Gronwall estimates needed to remove the Gaussian component will be done separately in a carefully chosen order. More precisely, we begin with an *unconditional* Gronwall estimate for isotropic two resolvents, see Lemma 5.4.10. We call it unconditional, because the differential inequality obtained in (5.4.33) does not require any input. The differential inequalities (5.4.35), (5.4.39), and (5.4.43) in Lemmas 5.4.11–5.4.13, however, require certain inputs, which are obtained from integrating the differential inequalities in time. Since Lemmas 5.4.11–5.4.13 require inputs, we call them *conditional* Gronwall estimates. Moreover, we point out that the proof of the two resolvent and three resolvent isotropic bounds contain an internal recursion. In fact, Lemmas 5.4.11–5.4.12 are used several times to gradually improve the bound (the exponent b is improved to b'). Finally, in Lemma 5.4.14 we explain how the conditional Gronwall estimates change in case of *regular observables*.

All estimates in Lemmas 5.4.10–5.4.14 hold *uniformly* in all spectral parameters $z_{j,t} \in \Omega^j_{\kappa,t}$ for the fixed time t.

5.4.3.1 Unconditional Gronwall estimate for isotropic two-resolvent chains

We begin with an unconditional Gronwall estimate for isotropic two resolvents.

Lemma 5.4.10 (Unconditional Gronwall estimate for isotropic two resolvents)**.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ be bounded and set*

$$R_r := \left| \left(G_{1,r} B_1 G_{2,r} - M_{12}^{B_1} \right)_{\boldsymbol{xy}} \right| \quad \text{and} \quad \mathcal{E}_0 := \frac{1}{\sqrt{N\eta}} \frac{1}{\eta}, \tag{5.4.32}$$

Then, for $p \in \mathbf{N}$ and any $\xi > 0$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r} \mathbf{E} |R_r|^{2p} \lesssim \left(1 + \frac{1}{\sqrt{N\eta}\,\eta} \right) \left(\mathbf{E} |R_r|^{2p} + N^{\xi} \mathcal{E}_0^{2p} \right). \tag{5.4.33}$$

The proof of Lemma 5.4.10 is given in Section 5.6.

5.4.3.2 Conditional Gronwall estimates: general case

In this section, we collect our conditional Gronwall estimates for general observables. The initial input, i.e. (5.4.34) for $b = 0$, will be obtained from integrating (5.4.33) in time. A similar approach was introduced in parallel in [163].

Lemma 5.4.11 (Conditional Gronwall estimate for isotropic two resolvents)**.** *Assume that for some fixed $b \in [0,1]$ it holds that*

$$R_r := \left| \left(G_{1,r} B_1 G_{2,r} - M_{12}^{B_1} \right)_{\boldsymbol{xy}} \right| \prec \mathcal{E}_0 \quad \text{with} \quad \mathcal{E}_0 := \frac{1}{\sqrt{N\eta}} \frac{1}{\eta^{1-b/2} \gamma^{b/2}}, \tag{5.4.34}$$

uniformly in bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and $r \in [s,t]$. Then, for $p \in \mathbf{N}$ and any $\xi > 0$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r} \mathbf{E} |R_r|^{2p} \lesssim \left(1 + \frac{1}{\sqrt{N\eta}\,\eta} \right) \left(\mathbf{E} |R_t|^{2p} + N^{\xi} \mathcal{E}_1^{2p} \right), \tag{5.4.35}$$

where we denoted

$$\mathcal{E}_1 := \frac{1}{\sqrt{N\eta}} \frac{1}{\eta^{1-b'/2} \gamma^{b'/2}} \quad \text{with} \quad b' := (b + 1/3) \wedge 1. \tag{5.4.36}$$

The proof of Lemma 5.4.11 is given in Section 5.6. The conditional Gronwall estimate concerning isotropic three resolvents is given in the following lemma.

Lemma 5.4.12 (Conditional Gronwall estimate for isotropic three resolvents)**.** *Assume that for some fixed $b \in [0,1]$ it holds that*

$$R_r := \left| \left(G_{1,r} B_1 G_{2,r} B_2 G_{1,r} \right)_{\boldsymbol{xy}} \right| \prec \mathcal{E}_0 \quad \text{with} \quad \mathcal{E}_0 := \frac{1}{\eta} \frac{1}{\eta^{1-b} \gamma^b}, \tag{5.4.37}$$

uniformly bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and $r \in [s,t]$. Moreover, suppose that

$$\left| \left(G_{1,r} B_1 G_{2,r} - M_{12}^{B_1} \right)_{\boldsymbol{xy}} \right| \prec \frac{1}{\sqrt{N\eta}\,\eta^{1-b'/2} \gamma^{b'/2}} \quad \text{with} \quad b' := (b + 1/3) \wedge 1, \tag{5.4.38}$$

uniformly in $r \in [s,t]$, bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, and $B_1 \in \mathbf{C}^{N \times N}$ (and the same for indices 1 and 2 interchanged). Then, for $p \in \mathbf{N}$ and any $\xi > 0$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r} \mathbf{E} |R_t|^{2p} \lesssim \left(1 + \frac{1}{\sqrt{N\eta}\,\eta} \right) \left(\mathbf{E} |R_r|^{2p} + N^{\xi} \mathcal{E}_1^{2p} \right), \tag{5.4.39}$$

where we denoted

$$\mathcal{E}_1 := \frac{1}{\eta} \frac{1}{\eta^{1-b'} \gamma^{b'}}. \tag{5.4.40}$$

The proof of Lemma 5.4.12 is completely analogous to that of Lemma 5.4.11 and so omitted.

We point out that the input bound (5.4.37) with $b = 0$ is trivially satisfied since (neglecting the time dependence)

$$\left|\left(G_1 B_1 G_2 B_2 G_1\right)_{\boldsymbol{xy}}\right| \le \frac{\|B_2\|}{\eta}\sqrt{\left(G_1 B_2 \mathrm{Im}\, G_2 B_1^* G_1^*\right)_{\boldsymbol{xx}}\left(\mathrm{Im}\, G_1\right)_{\boldsymbol{yy}}} \prec \frac{1}{\eta^2} \tag{5.4.41}$$

by a simple Schwarz inequality together with Ward identities, the trivial bound $\|G\| \le \eta^{-1}$, and a single resolvent local law giving $|G_{\boldsymbol{uv}}| \prec 1$ for $\boldsymbol{u}, \boldsymbol{v}$ of bounded norm. The other input (5.4.37) will be obtained by integrating the differential inequality (5.4.35) from Lemma 5.4.11.

The time integrated versions of the differential inequalities from Lemmas 5.4.11–5.4.12 both serve as inputs for the following lemma concerning average two resolvents.

Lemma 5.4.13 (Conditional Gronwall estimate for average two resolvents)**.** *Assume that*

$$\left|\left(G_{1,r} B_1 G_{2,r} - M_{12}^{B_1}\right)_{\boldsymbol{xy}}\right| \prec \frac{1}{\sqrt{N\eta}\,\eta^{1/2}\gamma^{1/2}} \quad \text{and} \quad \left|\left(G_{1,r} B_1 G_{2,r} B_2 G_{1,r}\right)_{\boldsymbol{xy}}\right| \prec \frac{1}{\eta\,\gamma} \tag{5.4.42}$$

uniformly in $r \in [s,t]$, bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, and $B_1, B_2 \in \mathbf{C}^{N\times N}$. Then, defining

$$R_t := \left|\left\langle\left(G_{1,t} B_1 G_{2,t} - M_{12}^{B_1}\right) B_2\right\rangle\right|,$$

for $p \in \mathbf{N}$ and any $\xi > 0$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathbf{E}\,|R_r|^{2p} \lesssim \left(1 + \frac{1}{\sqrt{N}\,\eta}\right)\left(\mathbf{E}\,|R_r|^{2p} + N^{\xi}\mathcal{E}_1^{2p}\right), \quad \text{where} \quad \mathcal{E}_1 := \frac{1}{N\eta_1\eta_2} \wedge \frac{1}{\sqrt{N\eta}\,\gamma}. \tag{5.4.43}$$

The proof of Lemma 5.4.13 is given in Section 5.6.

5.4.3.3 Conditional Gronwall estimates: regular case

For *regular observables*, the desired local law enjoys a further improvement in accordance with the $\sqrt{\gamma}$-rule (see the discussion in Section 5.1) for such observables. In order to remove the Gaussian component introduced in the characteristic flow step, we again employ conditional Gronwall estimates.

Lemma 5.4.14 (Conditional Gronwall estimates for regular observables)**.** *Let $A_1, A_2 \in \mathbf{C}^{N\times N}$ be bounded matrices and assume that A_1 is (ν_1, ν_2)-regular and A_2 is (ν_2, ν_1)-regular. Then we have the following:*

(i) Upon replacing $B_1 \to A_1$ and $\gamma \to 1$, Lemma 5.4.11 holds verbatim.

(ii) Upon replacing $B_i \to A_i$, for $i \in [2]$, and $\gamma \to 1$, Lemma 5.4.12 holds verbatim.

Moreover, in case that only one *of the general observables B_i is replaced by a regular one A_i, and the assumption* (5.4.38) *is suitably adjusted (namely replacing $\gamma \to 1$ only for the case with a regular observable), Lemma 5.4.12 holds with $\gamma \to \sqrt{\gamma}$ in the definition of $\mathcal{E}_0$ and $\mathcal{E}_1$ in* (5.4.37) *and* (5.4.40)*, respectively.*

(iii) Upon replacing $B_i \to A_i$, for $i \in [2]$, and $\gamma \to 1$, Lemma 5.4.13 holds verbatim.

Moreover, in case that only one *of the general observables B_i is replaced by a regular one A_i, and the assumption* (5.4.42) *is suitably adjusted (as described in item (iii) above), the conclusion* (5.4.43) *holds with $\gamma \to \sqrt{\gamma}$.*

Proof. The proof of Lemma 5.4.14 works in the exact same way as the proofs of Lemmas 5.4.11–5.4.13, with the only difference that the bound (5.6.6) gets complemented by the improved estimates

$$\left\|M_{12}^{A_1}\right\| \lesssim \|A_1\| \quad \text{and} \quad \left\|M_{21}^{A_2}\right\| \lesssim \|A_2\| \tag{5.4.44}$$

from Proposition 5.4.6 (note that there is no γ^{-1} on the rhs. of (5.4.44)). The rest of the argument is identical. □

5.4.4 Conclusion of the zigzag strategy: Proof of Theorem 5.3.2

We start with the following trivially checkable lemma, which follows by standard ODE theory and (5.4.13).

Lemma 5.4.15 (Initial conditions). *Fix $0 \le T < 1$, and pick a spectral parameter $|z| \lesssim 1$ and a matrix $\|D\| \lesssim 1$. Then there exist initial conditions z_0, D_0 such that the solutions z_t, D_t of* (5.4.8)*, with initial conditions z_0, D_0, satisfies $z_T = z$ and $D_T = D$. Additionally, we have $\operatorname{dist}(z_0, \operatorname{supp}(\rho_{D_0})) \ge cT$, for some universal constant $c > 0$.*

Along the proof of Theorem 5.3.2, we will also prove the following proposition.

Proposition 5.4.16 (Isotropic two- and three-resolvent local laws). *Fix $L, C_0, \epsilon > 0$. Let W be a Wigner matrix satisfying Assumption 5.2.1, and let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be bounded Hermitian matrices. For spectral parameters $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$, denote $\eta_l := |\operatorname{Im} z_l|$, $\rho_l := \pi^{-1}|\langle \operatorname{Im} M_l \rangle|$, and $\ell := \min_{l\in[2]} \eta_l\rho_l$. Finally, let $\widehat{\gamma} = \widehat{\gamma}(z_1, z_2)$ be defined as in* (5.3.3)*. Then, the following holds:*

Part 1: [General case] *For bounded $B_1, B_2 \in \mathbf{C}^{N\times N}$ and unit $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, we have*

$$\left|\left\langle \boldsymbol{x}, \left(G_1 B_1 G_2 - M^{B_1}_{z_1,z_2}\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\ell}} \cdot \frac{1}{\sqrt{\eta_*\gamma}}, \tag{5.4.45a}$$

$$\left|\left\langle \boldsymbol{x}, G_1 B_1 G_2 B_2 G_1^{(*)} \boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell\gamma}, \tag{5.4.45b}$$

uniformly in spectral parameters satisfying $|z_1|, |z_2| \le N^{100}$ and $N\ell \ge N^\epsilon$.

Part 2: [Regular case] *Recall* (5.2.10)*, let $A_1 \in \mathbf{C}^{N\times N}$ be (ν_1, ν_2)–regular and let $A_2 \in \mathbf{C}^{N\times N}$ be (ν_2, ν_1)–regular. Then, for bounded $A_1, A_2, B \in \mathbf{C}^{N\times N}$ and unit $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, we have*

$$\left|\left\langle \boldsymbol{x}, \left(G_1 A_1 G_2 - M^{A_1}_{z_1,z_2}\right)\boldsymbol{y}\right\rangle\right| \prec \frac{1}{\sqrt{N\ell}} \cdot \frac{1}{\sqrt{\eta_*}}, \tag{5.4.46a}$$

$$\left|\left\langle \boldsymbol{x}, G_1 A_1 G_2 B G_1^{(*)} \boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell\sqrt{\gamma}}, \tag{5.4.46b}$$

$$\left|\left\langle \boldsymbol{x}, G_1 A_1 G_2 A_2 G_1^{(*)} \boldsymbol{y}\right\rangle\right| \prec \frac{1}{\ell}, \tag{5.4.46c}$$

uniformly in spectral parameters satisfying $|z_1|, |z_2| \le N^{100}$ and $N\ell \ge N^\epsilon$.

5.4.4.1 General case: Proof of Part 1 of Theorem 5.3.2 and Proposition 5.4.16

Fix a bulk parameter $\kappa > 0$ and $\epsilon > 0$. For $j \in [2]$, we now define sequences of domains in the following way: Consider the monotonically increasing sequence $(a_k)_{k\in\mathbf{N}_0} \subset [0,1]$ defined recursively as

$$a_{k+1} := \frac{2}{3}a_k + \frac{1}{3} \quad \text{with} \quad a_0 = 0. \tag{5.4.47}$$

Moreover, set

$$\eta_k := N^{-a_k}$$

and let $K \in \mathbf{N}$ be the smallest integer satisfying $\eta_K < N^{-1+\epsilon}$ (note that $K = O(|\log\epsilon|)$ is independent of N). By Lemma 5.4.15, choose the terminal time $T > 0$ in such a way that

$$\Omega^j_{\kappa,0} \subset \{z \in \mathbf{C} : |\operatorname{Im} z| \ge c\} \quad \text{for} \quad j \in [2]. \tag{5.4.48}$$

Here, $c > 0$ depends only on L and κ via Lemma 5.5.4 (ii). Next, let $(t_k)_{k=0}^K \subset [0,T]$ be monotonically increasing sequence of times with $t_0 = 0$, $t_K = T$ and, for $k \in [K-1]$, we define t_k as the largest time in $[0,T]$ satisfying

$$\Omega^j_k := \Omega_{\kappa,t_k} \subset \{z \in \mathbf{C} : |\operatorname{Im} z| \ge \eta_k\} \quad \text{for} \quad j \in [2]. \tag{5.4.49}$$

After having set up these sequences of domains, the key for proving the target local laws is the following *induction argument*, which we prove below.

Proposition 5.4.17 (Induction on scales)**.** *Assume that the local laws* (5.3.11) *and* (5.4.45a)–(5.4.45b) *hold uniformly on* Ω_k^j *for the deformed Wigner matrices* $W + D_{j,t_k}$*. Then they also hold uniformly on* Ω_{k+1}^j *for the deformed Wigner matrices* $W + D_{j,t_{k+1}}$*.*

The input for $k = 0$ is ensured by the global law in Proposition 5.4.1. Then, applying Proposition 5.4.17 in total K times, we arrive at Part 1 of Theorem 5.3.2 and Proposition 5.4.16.

Proof of Proposition 5.4.17. Given the assumption in Proposition 5.4.17, we find from Proposition 5.4.8 with $s = t_k$ and $t = t_{k+1}$, the local laws to hold on Ω_{k+1}^j at the cost of having introduced a Gaussian component of order $t_{k+1} - t_k$. We now remove this Gaussian component in several steps. Here, we will frequently employ Gronwall's Lemma to integrate the differential inequalities (5.4.33), (5.4.35), (5.4.39), and (5.4.43), and thereby use that (by construction)

$$t_{k+1} - t_k \lesssim 1 \quad \text{and} \quad \frac{t_{k+1} - t_k}{\sqrt{N}|\mathrm{Im}\, z|^{3/2}} \lesssim 1 \quad \text{uniformly for} \quad z \in \Omega_{k+1}^j\,,\ j \in [2]\,. \tag{5.4.50}$$

The steps are as follows:

1. With the aid of Lemma 5.4.10, integrating (5.4.33) ending at $t = t_{k+1}$, we infer (5.4.34) with $b = 0$ and $s = t_k$.
2. By Lemma 5.4.11, integrating (5.4.35) ending at $t = t_{k+1}$, we infer (5.4.38) for $b = 0$ (i.e. $b' = 1/3$) and $s = t_k$.
3. By Lemma 5.4.12 (and using (5.4.41)), integrating (5.4.39) ending at $t = t_{k+1}$, we obtain (5.4.37) with $b = 1/3$.
4. In order to improve the exponent b, repeat steps 2 and 3 for two more times, giving us (5.4.37)–(5.4.38) for $b = 1$ and $s = t_k$, $t = t_{k+1}$. That, is we proved (5.4.45a)–(5.4.45b) to hold on Ω_{k+1}^j.
5. Finally, by application of Lemma 5.4.13 (note that (5.4.42) is obtained in Step 4), we integrate (5.4.43) ending at $t = t_{k+1}$ to infer (5.3.11) to hold on Ω_{k+1}^j.

This concludes the proof of Proposition 5.4.17. □

5.4.4.2 Regular case: Proof of Part 2 of Theorem 5.3.2 and Proposition 5.4.16

The proof of Theorem 5.3.2 for regular observables (Part 2) follows very similar steps to those in the proof of Part 1, with the only exception that the local laws for chains with one and two regular observables have to be propagated together. We thus omit this proof for the sake of brevity. □

5.5 Zig step: Proof of Proposition 5.4.8

In the current section we present the proof of Proposition 5.4.8. Firstly we do the zig step for average two-resolvent chains in Section 5.5.1. This is done self-consistently, i.e. without involving isotropic quantities or longer chains. However the single resolvent local law is used, which states that for any fixed $\zeta > 0$ and for any $z \in \mathbf{C} \setminus \mathbf{R}$ such that $N|\mathrm{Im}\, z|\rho(z) \geq N^\zeta$, it holds that

$$\big|\langle (G(z) - M(z))A\rangle\big| \prec \frac{1}{N|\mathrm{Im}\, z|}, \qquad \big|\langle \boldsymbol{x}, (G(z) - M(z))\boldsymbol{y}\rangle\big| \prec \sqrt{\frac{\rho}{N|\mathrm{Im}\, z|}}. \tag{5.5.1}$$

Note that (5.5.1) coincides with (5.3.7) when $\mathrm{Re}\, z$ is in the bulk, however (5.5.1) is more general since it is uniform in the spectrum. The local law (5.5.1) was proven near the edge in [243] and was later extended to the cusp regime in [239]. In fact, for the proof of Proposition 5.4.8 we do not need (5.5.1) itself, but just a weaker statement that (5.5.1) propagates along the zig flow, which can be directly proven by the methods described below in Section 5.5.1. Thus our proof can be easily made independent of [243, 239], but for simplicity in the current presentation we will rely on them as they are already available.

Later in Section 5.5.2 we work with isotropic two- and three-resolvent chains and prove (5.4.27b), (5.4.27c) relying on the result of Section 5.5.1. Finally, in Section 5.5.3 we explain how the proofs of (5.4.27a)–(5.4.27c) should be modified in the setting when one or several of observables are regular in the sense of Definition 5.2.2.

Throughout the entire section we will assume without loss of generality that all matrices A_j, B_j, $j = 1, 2$ are bounded in operator norm by 1, i.e. $\|A_j\| \le 1$, $\|B_j\| \le 1$. Also by $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ we will mean unit vectors. Moreover, for simplicity we present the proof for $s = 0$ and $t = T$. To keep the presentation short we often omit the time dependence in $G_{j,s}$ and simply write G_j. That is, we use the shorthand notation

$$G_j = (W_s + D_{j,s} - z_{j,s})^{-1}, \quad j \in [1, 2],$$

whenever the time s is clear from the context. For all other time dependent variables, such as $z_{j,s}$, $D_{j,s}$, and ℓ_s, we keep the time dependence explicitly.

5.5.1 Average two-resolvent chains: Proof of (5.4.27a) in Proposition 5.4.8

By Itô calculus, for any deterministic observables $R_1, R_2 \in \mathbf{C}^{N\times N}$, recalling (5.4.24), (5.4.8) and (5.4.23), we have take the real case. Probably the Setting(5.4.7) should also include the real case and introduce $\beta = 1, 2$ and then very soon in this proof we focus only on the complex case.]

$$\begin{aligned}
\mathrm{d}\langle G_{1,t}R_1G_{2,t}R_2\rangle &= \mathrm{d}\mathcal{E}_t + \langle G_{1,t}R_1G_{2,t}R_2\rangle \mathrm{d}t + \langle G_{1,t}R_1G_{2,t}\rangle\langle G_{2,t}R_2G_{1,t}\rangle \mathrm{d}t \\
&\quad + \langle G_{1,t} - M_{1,t}\rangle\langle G_{1,t}^2R_1G_{2,t}R_2\rangle \mathrm{d}t + \langle G_{2,t} - M_{2,t}\rangle\langle G_{1,t}R_1G_{2,t}^2R_2\rangle \mathrm{d}t, \\
&\quad + \frac{\mathbf{1}(\beta = 1)}{N}\Big[\langle G_{1,t}^{\mathrm{t}}G_{1,t}R_1G_{2,t}R_2G_{1,t}\rangle \mathrm{d}t + \langle G_{2,t}^{\mathrm{t}}G_{2,t}R_2G_{1,t}R_1G_{2,t}\rangle \mathrm{d}t \\
&\quad + \langle (G_{1,t}R_1G_{2,t})^{\mathrm{t}}G_{2,t}R_2G_{1,t}\rangle \mathrm{d}t\Big],
\end{aligned} \tag{5.5.2}$$

where the *martingale term* in the first line of (5.5.2) is given by

$$\mathrm{d}\mathcal{E}_t = \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} \partial_{ab}\langle G_{1,t}R_1G_{2,t}R_2\rangle \mathrm{d}B_{ab}. \tag{5.5.3}$$

Here $\partial_{ab} = \partial_{w_{ab}(t)}$ stands for the directional derivative in the direction of $w_{ab}(t)$ (here $w_{ab}(t)$ denote the entries of W_t), $\beta = 1$, $\beta = 2$ denote the real and complex case, respectively, and t denotes the transposition. From now on to keep the presentation short and simple we only consider the complex case $\beta = 2$, since the real case $\beta = 1$ is very similar, it only requires to estimate a few more terms in (5.5.2), whose estimate does not require any new idea. We refer to Chapter 3 for a similar case when the additional terms present in the real case were estimated carefully.

The differential in (5.5.2) is complemented by the time derivative of the corresponding deterministic approximation (recall the shorthand notation $M_{12,t}^R$ from (5.3.9)) given in the next lemma. Its proof is completely analogous to [161, Lemma 5.5] and hence omitted.

Lemma 5.5.1 (Time derivative of M_{12})**.** *For any $t \in [0, T]$ it holds that*

$$\partial_t\langle M_{12,t}^{R_1}R_2\rangle = \langle M_{12,t}^{R_1}R_2\rangle + \langle M_{12,t}^{R_1}\rangle\langle M_{21,t}^{R_2}\rangle. \tag{5.5.4}$$

Then, using the shorthand notation

$$g_t^{R_1,R_2} := \left\langle \left(G_{1,t} R_1 G_{2,t} - M_{12,t}^{R_1} \right) R_2 \right\rangle, \tag{5.5.5}$$

we find, subtracting (5.5.4) from (5.5.2), that

$$\mathrm{d}g_t^{R_1,R_2} = \left(1 + (2 - k(R_1,R_2))\langle M_{12,t}^I \rangle\right) g_t^{R_1,R_2} \mathrm{d}t + \mathrm{d}\mathcal{E}_t + \mathcal{F}_t \mathrm{d}t\,. \tag{5.5.6}$$

Here, we introduced the notation $\mathcal{F}_t = \mathrm{Lin}_t + \mathrm{Err}_t$ for the *forcing term*, where the *linear term* and *error term* are given by

$$\begin{aligned} \mathrm{Lin}_t &= k(R_1)\langle M_{12,t}^{R_1} \rangle g_t^{I,R_2} + k(R_2)\langle M_{21,t}^{R_2} \rangle g_t^{R_1,I}, \\ \mathrm{Err}_t &= g_t^{I,R_2} g_t^{R_1,I} + \langle G_{1,t} - M_{1,t} \rangle \langle G_{1,t}^2 R_1 G_{2,t} R_2 \rangle + \langle G_{2,t} - M_{2,t} \rangle \langle G_{1,t} R_1 G_{2,t}^2 R_2 \rangle\,, \end{aligned} \tag{5.5.7}$$

respectively. Moreover, we denoted

$$k(R_1, \ldots, R_m) := \#\{j \in [1,m] : R_j \neq I\} \tag{5.5.8}$$

for deterministic $R_1, \ldots, R_m \in \mathbb{C}^{N \times N}$.

Recall the exponent $\epsilon > 0$ which is fixed in Theorem 5.3.2. The current Setting 5.4.7 depends on ϵ through the definition of spectral domains (5.4.9). Take any $\xi_0, \xi_1, \xi_2 \in (0, \epsilon/10)$ such that $\xi_0 < \xi_1/2 < \xi_2/4$ and define the stopping time

$$\begin{aligned} \tau^{R_1,R_2} &:= \sup\{t \in [0,T] : \max_{s \in [0,t]} \max_{z_{j,0} \in \Omega_0^j} \alpha_s^{-1} \left| g_s^{R_1,R_2} \right| \leq N^{2\xi_{k(R_1,R_2)}}\}, \\ \tau &:= \min\{\tau^{R_1,R_2} : R_1, R_2 \in \mathfrak{S}\}\,, \quad \text{with} \quad \mathfrak{S} := \{I, B_1, B_1^*, B_2, B_2^*\}\,, \end{aligned} \tag{5.5.9}$$

where we introduced the shorthand notation

$$\alpha_t := \frac{1}{N\eta_{1,t}\eta_{2,t}} \wedge \frac{1}{\sqrt{N\ell_t}\gamma_t}\,.$$

We point out that both g_s and α_s in (5.5.9) depend on the $z_{j,s}$'s and thus on the $z_{j,0}$'s via the flow as its initial condition.

In the analysis of (5.5.6) the following two quantities play significant role

$$f_r := 2\mathrm{Re}\,\langle M_{12,r}^I \rangle \wedge 0, \tag{5.5.10}$$

$$\beta_r := |1 - \langle M_{1,r} M_{2,r} \rangle|. \tag{5.5.11}$$

These functions depend on time $r \in [0,T]$ and initial conditions $z_{j,0} \in \Omega_0^j$, $D_{j,0} \in \mathbb{C}^{N \times N}$, $j \in [1,2]$, but we will omit the dependence on initial conditions in notations when this does not cause an ambiguity. Also note that (5.5.11) is the time-dependent version of (5.3.1) where $\beta(z_1, z_2)$ is defined. Clearly f_t is essentially the coefficient of $g_t^{R_1,R_2}$ in the linear ODE (5.5.6) with forcing terms, so its exponential plays the role of the propagator. We stress that the notation $k(R_1, \ldots, R_m)$ introduced in (5.5.8) serves only the purpose of covering all possible cases $R_1, R_2 \in \mathfrak{S}$ in one formula (5.5.6). We do not exploit the fact that for $k(R_1, R_2) > 0$ the propagator with $1 + (1 - k(R_1,R_2)/2) f_t$ in the rhs. of (5.5.6) becomes smaller than $1 + f_t$, but rather estimate the propagator from above by the exponential of $1 + f_t$ in all cases.

We now state two important technical lemmas whose proofs are postponed to Section 5.A.5 and after concluding the proof of (5.4.27a), respectively. Lemma 5.5.2 controls the propagator of (5.5.6).

Lemma 5.5.2 (Bound on the propagator)**.** *We have the following:*

1. *For any spectral pairs ν_1, ν_2 it holds that*

$$2|\langle M_{\nu_1,\nu_2}^I \rangle| \leq \pi\rho_1/\eta_1 + \pi\rho_2/\eta_2, \quad \textit{with} \quad \rho_j(z) = \pi^{-1}|\langle \mathrm{Im}\, M_j(z) \rangle|,\ j \in [1,2]. \tag{5.5.12}$$

2. *For any* $z_{j,0} \in \Omega_0^j$, $j \in [1,2]$, *there exists* $s_0 = s_0(z_{1,0}, z_{2,0}) \in [0,T]$ *such that* $f_r > 0$ *for all* $r < s_0$ *and* $f_r = 0$ *for all* $r > s_0$. *Note that* s_0 *may be an endpoint of* $[0,T]$.

3. *For any* $s,t \in [0,T]$, $s < t$, *we have*

$$\int_s^t f_r \mathrm{d}r \le \log \frac{\eta_{1,s}\eta_{2,s}}{\eta_{1,t}\eta_{2,t}}, \tag{5.5.13a}$$

$$\int_s^t f_r \mathrm{d}r = 2\log \frac{\beta_{s\wedge s_0}}{\beta_{t\wedge s_0}}. \tag{5.5.13b}$$

4. *For any* $s,t \in [0,T]$, $s \le t$, *it holds* $\beta_s \sim \beta_t + (t-s)$.

The following lemma controls the forcing terms of (5.5.6), i.e. the martingale term, the linear term and error term.

Lemma 5.5.3 (Bound on the forcing terms)**.** *Consider* $R_1, R_2 \in \mathfrak{S}$. *Denote the quadratic variation of the martingale term* $\mathrm{d}\mathcal{E}_t$ (5.5.3) *by*

$$\mathrm{QV}\big[g_t^{R_1,R_2}\big] := \frac{1}{N}\sum_{a,b=1}^{N} \left|\partial_{ab}\langle G_{1,t}R_1G_{2,t}R_2\rangle\right|^2. \tag{5.5.14}$$

Then for any $\zeta > 0$ *it holds, with very high probability, that*

$$\begin{aligned}\left(\int_0^{t\wedge\tau} \mathrm{QV}\big[g_s^{R_1,R_2}\big]\mathrm{d}s\right)^{1/2} + \int_0^{t\wedge\tau} |\mathcal{F}_s|\mathrm{d}s \\ \lesssim \alpha_{t\wedge\tau}\left(k(R_1)N^{2\xi_{k(R_2)}} + k(R_2)N^{2\xi_{k(R_1)}} + N^{\zeta}\right)\log N\end{aligned} \tag{5.5.15}$$

uniformly in $t \in [0,T]$, $z_{j,0} \in \Omega_0^j$ *and* $\|B_j\| \le 1$, $j \in [1,2]$.

In the following, we will consider (5.5.6) as a system of equations for $g_t^{R_1,R_2}$, $R_1, R_2 \in \mathfrak{S}$. For each choice of $R_1, R_2 \in \mathfrak{S}$, we use the *stochastic Gronwall argument* from [162, Lemma 5.6] with (5.5.15) as an input to show that $\tau^{R_1,R_2} > \tau$ unless $\tau^{R_1,R_2} = T$. This would readily imply that $\tau = T$ with very high probability, i.e. (5.4.27a) holds.

Take any $R_1, R_2 \in \mathfrak{S}$ and denote $g_s := g_s^{R_1,R_2}$, $\xi := \xi_{k(R_1,R_2)}$. Consider (5.5.15) for some $\zeta < \xi$. Due to the choice of ξ_j, $j \in [0,2]$ the rhs. of (5.5.15) is upper bounded by $\alpha_{t\wedge\tau}N^{\xi}$, where we ignored the irrelevant $\log N$ factor. Then [162, Lemma 5.6] with $d = 1$ applied for the scalar equation (5.5.6) asserts that for any arbitrary small $\zeta > 0$ and for any $t \ge 0$ we have

$$\sup_{0\le s\le t\wedge\tau}|g_s|^2 \lesssim |g_0|^2 + N^{2\xi+3\zeta}\alpha_{t\wedge\tau}^2 + \int_0^{t\wedge\tau}\left(|g_0|^2 + N^{2\xi+3\zeta}\alpha_s^2\right) f_s \exp\left(2\left(1+N^{-\zeta}\right)\int_s^{t\wedge\tau} f_r\mathrm{d}r\right)\mathrm{d}s. \tag{5.5.16}$$

It follows from (5.4.26a) that $|g_0|^2 \lesssim N^{3\zeta}\alpha_0^2 \le N^{3\zeta}\alpha_s^2$ with very high probability. Also (5.5.13a) implies that

$$\exp\left(2N^{-\zeta}\int_s^{t\wedge\tau} f_r\mathrm{d}r\right) \le \exp\left(CN^{-\zeta}\log N\right) \lesssim 1.$$

Therefore, (5.5.16) simplifies to

$$\sup_{0\le s\le t\wedge\tau}|g_s|^2 \lesssim N^{2\xi+3\zeta}\alpha_{t\wedge\tau}^2 + N^{2\xi+3\zeta}\int_0^{t\wedge\tau}\alpha_s^2 f_s \exp\left(2\int_s^{t\wedge\tau} f_r\mathrm{d}r\right)\mathrm{d}s. \tag{5.5.17}$$

Take $\zeta < \xi/3$. Then for the purpose of showing that $\tau = T$ with very high probability it suffices to verify the inequality

$$\int_0^{t\wedge\tau}\alpha_s^2 f_s \exp\left(2\int_s^{t\wedge\tau} f_r\mathrm{d}r\right)\mathrm{d}s \lesssim \alpha_{t\wedge\tau}^2\log N. \tag{5.5.18}$$

We first check that the lhs. of (5.5.18) has an upper bound of order $\log N/(N\eta_{1,t\wedge\tau}\eta_{2,t\wedge\tau})^2$. In order to see this, we employ (5.5.12) and (5.5.13a) along with $\alpha_s \le 1/(N\eta_{1,s}\eta_{2,s})$ and find that

$$\int_0^{t\wedge\tau} \alpha_s^2 f_s \exp\Big(2\int_s^{t\wedge\tau} f_r \mathrm{d}r\Big)\mathrm{d}s \le \left(\frac{1}{N\eta_{1,t\wedge\tau}\eta_{2,t\wedge\tau}}\right)^2 \int_0^{t\wedge\tau}\left(\frac{\rho_{1,s}}{\eta_{1,s}}+\frac{\rho_{2,s}}{\eta_{2,s}}\right)\mathrm{d}s \lesssim \frac{\log N}{(N\eta_{1,t\wedge\tau}\eta_{2,t\wedge\tau})^2}. \tag{5.5.19}$$

To establish the upper bound of order $\log N/(N\ell_{t\wedge\tau}\gamma_{t\wedge\tau}^2)$ for the lhs. of (5.5.18) we split the region of integration into two parts $[0,s_*]$ and $[s_*,t\wedge\tau]$, where s_* is defined as

$$s_* := \inf\{s\in[0,t\wedge\tau] : \min\{\eta_{1,s}/\rho_{1,s},\eta_{2,s}/\rho_{2,s}\}\le\gamma_{t\wedge\tau}\}. \tag{5.5.20}$$

Since $\eta_{j,s}/\rho_{j,s}$, $j\in[2]$, are monotonically decreasing functions in s, it holds that $\eta_{j,s}/\rho_{j,s}\le\gamma_{t\wedge\tau}$, $j\in[2]$, for $s\in[s_*,t\wedge\tau]$. Another property of s_* which will be used is that

$$t\wedge\tau - s_* \lesssim \gamma_{t\wedge\tau}. \tag{5.5.21}$$

We postpone the proof of (5.5.21) until the end of the proof of Part 1 of Proposition 5.4.8. In combination with (5.4.16) and the fourth statement of Lemma 5.5.2, (5.5.21) gives that

$$\beta_s \sim \beta_{t\wedge\tau}, \quad \forall s\in[s_*,t\wedge\tau]. \tag{5.5.22}$$

Armed with (5.5.22), we are now ready to complete the proof of (5.5.18). We may assume w.l.o.g. that $t\le s_0$, since $f_s=0$ for $s>s_0$ (recall Lemma 5.5.2 (2)). First, in the regime $s\in[0,s_*]$ we use that $\exp\big(\int_{s_*}^{t\wedge\tau} f_r\mathrm{d}r\big)\sim 1$ by means of (5.5.13b) and (5.5.22), and thus an estimate similar to (5.5.19) yields

$$\int_0^{s_*}\alpha_s^2 f_s\exp\Big(2\int_s^{t\wedge\tau} f_r\mathrm{d}r\Big)\mathrm{d}s \lesssim \frac{\log N}{(N\eta_{1,s_*}\eta_{2,s_*})^2}\lesssim\frac{\log N}{N\ell_{s_*}\gamma_{s_*}^2}\lesssim\frac{\log N}{N\ell_{t\wedge\tau}\gamma_{t\wedge\tau}^2}. \tag{5.5.23}$$

Second, in the regime $s\in[s_*,t\wedge\tau]$ use (5.5.13b), $\alpha_s\le 1/(\sqrt{N l_s}\gamma_s)$ and the bound $f_s\lesssim\beta_s^{-1}$ to get

$$\int_{s_*}^{t\wedge\tau}\alpha_s^2 f_s e^{2\int_s^{t\wedge\tau} f_r\mathrm{d}r}\mathrm{d}s \lesssim \frac{1}{N\ell_{t\wedge\tau}\gamma_{t\wedge\tau}^2}\cdot\frac{1}{\beta_{t\wedge\tau}^4}\int_{s_*}^{t\wedge\tau}\beta_s^3\mathrm{d}s \sim \frac{\beta_{t\wedge\tau}^3(t\wedge\tau-s_*)}{N\ell_{t\wedge\tau}\gamma_{t\wedge\tau}^2\beta_{t\wedge\tau}^4}\lesssim\frac{1}{N\ell_{t\wedge\tau}\gamma_{t\wedge\tau}^2}. \tag{5.5.24}$$

Here we used (5.5.22) in the last but one inequality and (5.5.21) in the last one. This finishes the proof of (5.5.18).

Now we verify (5.5.21). For any $r,s\in[0,T]$ from the definition of the characteristic flow we have that

$$e^r\eta_{j,r}/\rho_{j,r} - \mathrm{e}^s\eta_{j,s}/\rho_{j,s} = -(\mathrm{e}^r-\mathrm{e}^s)\pi/2. \tag{5.5.25}$$

For $r=t\wedge\tau$, $s=s_*$ and j such that $\gamma_{t\wedge\tau}\ge\eta_{j,s_*}/\rho_{j,s_*}$ we find that

$$|t\wedge\tau - s_*\wedge\tau| \sim |e^{t\wedge\tau}-e^{s_*}| \lesssim |\eta_{j,t\wedge\tau}/\rho_{j,t\wedge\tau}| + |\eta_{j,s_*}/\rho_{j,s_*}| \lesssim \gamma_{t\wedge\tau}.$$

This concludes the proof of the average part (5.4.27a) of Part 1 of Proposition 5.4.8. □

To prepare for the proof of Lemma 5.5.3 in the next proposition we show that the spectral domains Ω_t^j and $\Omega_{\kappa,t}^j$ (see (5.4.9) and (5.4.10)) for $t\in[0,T]$ and $j\in[2]$ satisfy the *ray property*. Informally this means that for every z in these domains with $\operatorname{Im} z>0$ (resp. $\operatorname{Im} z<0$) the vertical ray going off toward $\operatorname{Re} z+\mathrm{i}\infty$ (resp. $\operatorname{Re} z-\mathrm{i}\infty$) is essentially contained in the domain. Since the result holds both for $j=1,2$, we will neglect j in notations. The proof of Lemma 5.5.4 is given in Appendix 5.A.5.

Lemma 5.5.4 (Ray property for time dependent spectral domains)**.** *Fix a (large) $L>0$ and let $D\in\mathbf{C}^{N\times N}$ be a self-adjoint deformation with $\|D\|\le L$. Then we have the following.*

(i) [Unrestricted spectral domains] *For any $t \in [0,T]$, $z \in \Omega_t$ and $x \geq 0$ such that $|\mathrm{Im}\, z| + x \leq N^{100}$ it holds that $z + \mathrm{sgn}(\mathrm{Im}\, z)\mathrm{i}x \in \Omega_t$. That is, for $\mathrm{Im}\, z > 0$ ($\mathrm{Im}\, z < 0$) the vertical ray which starts at z, goes up (down) and leaves Ω_t only after reaching points with imaginary part larger than N^{100} (smaller than $-N^{100}$).*

(ii) [Bulk-restricted spectral domains] *Fix a (small) $\kappa > 0$. Then there exists $t_* \in [0,T]$ such that the previous part of the statement holds for $\Omega_{\kappa,t}$ for any $t \in [t_*, T]$. Namely, For any $t \in [t_*, T]$, $z \in \Omega_{\kappa,t}$ and $x \geq 0$ such that $|\mathrm{Im}\, z| + x \leq N^{100}$ it holds that $z + \mathrm{sgn}(\mathrm{Im}\, z)\mathrm{i}x \in \Omega_{\kappa,t}$. Moreover, $T - t_* \sim 1$ with implicit constants which depend only on κ and L.*

Proof of Lemma 5.5.3. Recall that the target bound in (5.5.15) consists of three parts: The quadratic variation QV of the martingale term and the two contributions Lin and Err to $\mathcal{F}$. We will discuss each part separately.

Before going into the proof, we point out that all bounds below hold with very high probability and for times $s \in [0, t \wedge \tau]$. We often omit in notations the dependence of resolvent chains and their deterministic approximations on time when this does not lead to an ambiguity.

Bound on QV: By computing the derivatives ∂_{ab} in (5.5.14), using Schwarz inequality and Ward identity, we get

$$\mathrm{QV}\big[g_s^{R_1,R_2}\big] \lesssim \frac{1}{N^2}\left(\frac{1}{\eta_{1,s}^2}\langle \mathrm{Im}\, G_1 R_1 G_2 R_2 \mathrm{Im}\, G_1 R_2^* G_2^* R_1^*\rangle + \frac{1}{\eta_{2,s}^2}\langle \mathrm{Im}\, G_2 R_2 G_1 R_1 \mathrm{Im}\, G_2 R_1^* G_1^* R_2^*\rangle\right). \tag{5.5.26}$$

In the following, we will focus on the first of the two terms in (5.5.26), since the estimates for the second one are identical. Firstly we give an upper bound which does not depend on γ:

$$\langle \mathrm{Im}\, G_1 R_1 G_2 R_2 \mathrm{Im}\, G_1 R_2^* G_2^* R_1^*\rangle \leq \langle \mathrm{Im}\, G_1\rangle \| R_1 G_2 R_2 \mathrm{Im}\, G_1 R_2^* G_2^* R_1^*\| \lesssim \rho_{1,s}/(\eta_{1,s}\eta_{2,s}^2),$$

where we used the averaged version of the single–resolvent local law from (5.5.1). We thus conclude the bound

$$\frac{1}{N^2}\int_0^{t\wedge\tau} \frac{1}{\eta_{1,s}^2}\langle \mathrm{Im}\, G_1 R_1 G_2 R_2 \mathrm{Im}\, G_1 R_2^* G_2^* R_1^*\rangle \mathrm{d}s \lesssim \frac{1}{N^2}\int_0^{t\wedge\tau} \frac{\rho_{1,s}}{\eta_{1,s}^3\eta_{2,s}^2}\mathrm{d}s \lesssim \left(\frac{1}{N\eta_{1,t\wedge\tau}\eta_{2,t\wedge\tau}}\right)^2.$$

Next, we aim to reduce the average four resolvent chain from (5.5.26) to a product of average two resolvent chains. In order to do so, we introduce the shorthand notations $S := R_2 \mathrm{Im}\, G_1 R_2^*$, $T := R_1^* \mathrm{Im}\, G_1 R_1$ and note that $S, T \geq 0$. Let $\{\lambda_i^{(2)}\}_{i\in[N]}$ be the eigenvalues of $W + D_2$ and $\boldsymbol{u}_i^2$ the corresponding normalized eigenvectors. By spectral decomposition of G_2 we can write

$$\begin{aligned} |\langle G_2 S G_2^* T\rangle| &= \frac{1}{N}\left|\sum_{i,j\in[N]} \frac{\langle \boldsymbol{u}_i^2, S\boldsymbol{u}_j^2\rangle\langle \boldsymbol{u}_j^2, T\boldsymbol{u}_i^2\rangle}{\big(\lambda_i^{(2)} - z_2\big)\big(\lambda_j^{(2)} - \bar{z}_2\big)}\right| \lesssim \frac{1}{N}\sum_{i,j\in[N]} \frac{\langle \boldsymbol{u}_i^2, S\boldsymbol{u}_i^2\rangle\langle \boldsymbol{u}_j^2, T\boldsymbol{u}_j^2\rangle}{\big|\lambda_i^{(2)} - z_2\big|\cdot\big|\lambda_j^{(2)} - z_2\big|} \\ &= N\langle |G_2|S\rangle\langle |G_2|T\rangle = N\langle \mathrm{Im}\, G_1 R_2^* | G_2 | R_2\rangle\langle \mathrm{Im}\, G_1 R_1 | G_2 | R_1^*\rangle. \end{aligned} \tag{5.5.27}$$

In the end of the first line we used the positive definiteness of S, T and the elementary estimate

$$\begin{aligned} \langle \boldsymbol{u}_i^2, S\boldsymbol{u}_j^2\rangle\langle \boldsymbol{u}_j^2, T\boldsymbol{u}_i^2\rangle &\leq \big(\langle \boldsymbol{u}_i^2, S\boldsymbol{u}_i^2\rangle\langle \boldsymbol{u}_j^2, S\boldsymbol{u}_j^2\rangle\langle \boldsymbol{u}_i^2, T\boldsymbol{u}_i^2\rangle\langle \boldsymbol{u}_j^2, T\boldsymbol{u}_j^2\rangle\big)^{1/2} \\ &\lesssim \langle \boldsymbol{u}_i^2, S\boldsymbol{u}_i^2\rangle\langle \boldsymbol{u}_j^2, T\boldsymbol{u}_j^2\rangle + \langle \boldsymbol{u}_j^2, S\boldsymbol{u}_j^2\rangle\langle \boldsymbol{u}_i^2, T\boldsymbol{u}_i^2\rangle. \end{aligned}$$

In order to deal with absolute values of resolvents we employ the integral representation [168, Eq. (5.4)]:

$$|G(E + \mathrm{i}\eta)| = \frac{2}{\pi}\int_0^\infty \frac{\mathrm{Im}\, G(E + \mathrm{i}\sqrt{\eta^2 + x^2})}{\sqrt{\eta^2 + x^2}}\mathrm{d}x. \tag{5.5.28}$$

of $|G|$ in terms of $\operatorname{Im} G$ along the ray $z + \mathrm{i}\operatorname{sgn}(z)x$ for $x \geq 0$ (cf. Lemma 5.5.4). Hence, using (5.5.28) for the first factor on the rhs. of (5.5.27) we get

$$\langle \operatorname{Im} G_1 R_2^* |G_2| R_2 \rangle = \frac{2}{\pi} \int_0^\infty \langle \operatorname{Im} G_1 R_2^* \operatorname{Im} G_2(E_{2,s} + \mathrm{i}\zeta_{2,s,x}) R_2 \rangle \zeta_{2,s,x}^{-1} \mathrm{d}x, \tag{5.5.29}$$

where we abbreviated $\zeta_{2,s,x} := (\eta_{2,s}^2 + x^2)^{1/2}$. We now split the region of integration $[0,\infty)$ into two parts: S_1 corresponds to the regime $\zeta_{2,s,x} \leq N^{100}$ and S_2 is the complementary regime, i.e. $S_2 := [0,\infty) \setminus S_1$. Now, for any $x \in S_1$, by Lemma 5.5.4 it holds that $E_{2,s} + \mathrm{i}\zeta_{2,s,x} \in \Omega_s^2$. Thus we conclude

$$\langle \operatorname{Im} G_1 R_2^* \operatorname{Im} G_2(E_{2,s} + \mathrm{i}\zeta_{2,s,x}) R_2 \rangle \lesssim \frac{1}{\gamma(z_{1,s}, E_{2,s} + \mathrm{i}\zeta_{2,s,x})} \left(1 + \frac{N^{2\xi_2}}{\sqrt{N\ell(z_{1,s}, E_{2,s} + \mathrm{i}\zeta_{2,s,x})}} \right) \tag{5.5.30}$$

where we abbreviated $\ell(z,z') := |\operatorname{Im} z| \rho_{1,s}(z) \wedge |\operatorname{Im} z'| \rho_{2,s}(z')$ for $z, z' \in \mathbf{C} \setminus \mathbf{R}$. Along with the vague monotonicity of γ in imaginary part (5.4.18) this inequality implies that

$$\int_{S_1} \langle \operatorname{Im} G_1 R_2^* \operatorname{Im} G_2(E_{2,s} + \mathrm{i}\zeta_{2,s,x}) R_2 \rangle \zeta_{2,s,x}^{-1} \mathrm{d}x \lesssim \frac{\log N}{\gamma(z_{1,s}, z_{2,s})}. \tag{5.5.31}$$

In the complementary regime we simply bound the integrand of (5.5.29) by the product of operator norms of resolvents. This gives an upper bound of order $\eta_{1,s}^{-1} N^{-100}$ for the integral over S_2. In particular, this is smaller than γ_s^{-1} since γ is a bounded function of $(z_1, z_2, D, D_2) \in \left(\mathbf{C} \setminus \mathbf{R}\right)^2 \times \left(\mathbf{C}^{N \times N}\right)^2$ (see Setting 5.4.7 and Definition 5.4.4).

Arguing similarly for the second factor in (5.5.27) we get

$$\frac{1}{N^2} \int_0^{t \wedge \tau} \frac{1}{\eta_{1,s}^2} \langle \operatorname{Im} G_1 R_1 G_2 R_2 \operatorname{Im} G_1 R_2^* G_2^* R_1^* \rangle \mathrm{d}s \lesssim \frac{1}{N} \int_0^{t \wedge \tau} \frac{1}{\eta_{1,s}^2 \gamma_s^2} \mathrm{d}s \lesssim \frac{1}{N \ell_{t \wedge \tau} \gamma_{t \wedge \tau}}.$$

This concludes the desired bound on the quadratic variation.

Bound on Lin_t. Recalling the definition from (5.5.7), in order to verify

$$\int_0^{t \wedge \tau} \mathrm{Lin}_s \mathrm{d}s \lesssim \alpha_{t \wedge \tau} \left(k(R_1) N^{2\xi_{k(R_2)}} + k(R_2) N^{2\xi_{k(R_1)}} \right) \log N$$

it is sufficient to notice that α_s decreases along the flow and that by (5.3.6) and (5.4.12) we have

$$\int_0^{t \wedge \tau} \left| \langle M_{12,s}^{R_j} \rangle \right| \mathrm{d}s \lesssim \int_0^{t \wedge \tau} \left(\frac{\rho_{1,s}}{\eta_{1,s}} \wedge 1 \right) \mathrm{d}s \lesssim \log N, \quad j \in [1,2].$$

Bound on Err_t. For the first term in Err_t in (5.5.7) by means of (5.4.12) we easily find

$$\int_0^{t \wedge \tau} \left| g_s^{I,R_2} g_s^{R_1,I} \right| \mathrm{d}s \lesssim \frac{N^{2\xi_{k(R_1)} + 2\xi_{k(R_2)}}}{N} \int_0^{t \wedge \tau} \frac{\alpha_s}{\eta_{1,s} \eta_{2,s}} \mathrm{d}s \lesssim \frac{N^{2\xi_{k(R_1)} + 2\xi_{k(R_2)}}}{N \ell_{t \wedge \tau}} \alpha_{t \wedge \tau} \lesssim \alpha_{t \wedge \tau}.$$

The remaining two terms in Err_t can be treated completely analogously, hence we focus on the first of the two for concreteness.

As the first step, we separate the first G_1 from the rest of the factors in $\langle G_1^2 R_1 G_2 R_2 \rangle$ via a Cauchy-Schwarz inequality followed by a Ward identity:

$$|\langle G_1^2 R_1 G_2 R_2 \rangle| \leq \frac{\langle \operatorname{Im} G_1 \rangle^{1/2} \langle \operatorname{Im} G_1 R_1 G_2 R_2 R_2^* G_2^* R_1^* \rangle^{1/2}}{\eta_{1,s}} \leq \frac{\langle \operatorname{Im} G_1 \rangle \| R_1 G_2 R_2 R_2^* G_2^* R_1^* \|^{1/2}}{\eta_{1,s}} \leq \frac{\rho_{1,s}}{\eta_{1,s} \eta_{2,s}}. \tag{5.5.32}$$

The last estimate follows from the usual averaged single-resolvent local law for $\operatorname{Im} G_1$ (5.5.1) and holds with very high probability. In order to get an upper bound for $\langle G_1^2 R_1 G_2 R_2\rangle$ in terms of γ we use the *reduction bound*

$$|\langle G_1^2 R_1 G_2 R_2\rangle| \le \langle |G_1|R_1|G_2|R_1^*\rangle^{1/2}\langle |G_1|G_1^* R_2^*|G_2|R_2 G_1\rangle^{1/2} \le \langle |G_1|R_2^*|G_2|R_2\rangle \eta_{1,s}^{-1}, \tag{5.5.33}$$

obtained analogously to (5.5.27), where in the final estimate we additionally used the commutativity of $|G_1|^{1/2}$, G_1 and G_1^* together with $\|G_1 G_1^*\| \le \eta_{1,s}^{-2}$. By means of (5.5.28), using similar arguments as around (5.5.31) we hence find

$$\langle |G_1|R_1|G_2|R_1^*\rangle \lesssim \gamma_s^{-1}\log N, \quad \langle |G_1|R_2^*|G_2|R_2\rangle \lesssim \gamma_s^{-1}\log N\,, \tag{5.5.34}$$

and thus

$$|\langle G_1^2 R_1 G_2 R_2\rangle| \lesssim (\eta_{1,s}\gamma_s)^{-1}\log N. \tag{5.5.35}$$

Finally, combining (5.5.32) and (5.5.35) with the single-resolvent local law $|\langle G_{1,s} - M_{1,s}\rangle| \lesssim N^\zeta/(N\eta_{1,s})$ we find, with very high probability that

$$\int_0^{t\wedge\tau} |\langle G_{1,s} - M_{1,s}\rangle\langle G_{1,s}^2 R_1 G_{2,s} R_2\rangle|\mathrm{d}s \lesssim \int_0^{t\wedge\tau} \frac{N^\zeta}{N\eta_{1,s}}\left(\frac{\rho_{1,s}}{\eta_{1,s}\eta_{2,s}} \wedge \frac{1}{\eta_{1,s}\gamma_s}\right)\mathrm{d}s \lesssim N^\zeta \alpha_{t\wedge\tau}. \tag{5.5.36}$$

This finishes the proof of Lemma 5.5.3. □

5.5.2 Isotropic two- and three-resolvent chains: Proof of (5.4.27b)–(5.4.27c) in Proposition 5.4.8

Consider deterministic matrices $B_1, B_2 \in \mathbf{C}^{N\times N}$ and unit vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$. The argument below proves (5.4.27b), (5.4.27c) uniformly in B_1, B_2, $\boldsymbol{x}, \boldsymbol{y}$. For notational simplicity we omit the dependence of z_j and G_j on t. Furthermore, to keep the presentation simple, we give the proof only in the complex case ($\beta = 2$) just as in Section 5.5.1 and again refer to Chapter 3 for a detailed treatment of the case $\beta = 1$.

To start with, the analog of (5.5.6) for isotropic two resolvents is (recall (5.5.8) for the definition of $k(R_1)$)

$$\mathrm{d}\big(G_{1,t}R_1G_{2,t} - M_{12,t}^{R_1}\big)_{\boldsymbol{vw}} = \big(1 + (1-k(R_1))\langle M_{12,t}^I\rangle\big)\big(G_{1,t}R_1G_{2,t} - M_{12,t}^{R_1}\big)_{\boldsymbol{vw}}\mathrm{d}t + \mathrm{d}\mathcal{E}_t^{(2)} + \mathcal{F}_t^{(2)}\mathrm{d}t, \tag{5.5.37}$$

for any deterministic vectors $\boldsymbol{v}, \boldsymbol{w}$, where $\mathrm{d}\mathcal{E}_t^{(2)}$ is the martingale term

$$\mathrm{d}\mathcal{E}_t^{(2)} = \frac{1}{\sqrt{N}}\sum_{a,b=1}^N \partial_{ab}\left(G_{1,t}R_1G_{2,t}\right)_{\boldsymbol{vw}}\mathrm{d}B_{ab},$$

and the forcing term $\mathcal{F}_t^{(2)} = \mathrm{Lin}_t^{(2)} + \mathrm{Err}_t^{(2)}$ is the sum of the linear term $\mathrm{Lin}_t^{(2)}$ and the error term $\mathrm{Err}_t^{(2)}$,

$$\begin{aligned}
\mathrm{Lin}_t^{(2)} &:= k(R_1)\langle M_{12,t}^{R_1}\rangle\big(G_{1,t}G_{2,t} - M_{12,t}^I\big)_{\boldsymbol{vw}},\\
\mathrm{Err}_t^{(2)} &:= \big\langle G_{1,t}R_1G_{2,t} - M_{12,t}^{R_1}\big\rangle\left(G_{1,t}G_{2,t}\right)_{\boldsymbol{vw}} + \langle G_{1,t} - M_{1,t}\rangle\left(G_{1,t}^2R_1G_{2,t}\right)_{\boldsymbol{vw}}\\
&\quad + \langle G_{2,t} - M_{2,t}\rangle\left(G_{1,t}R_1G_{2,t}^2\right)_{\boldsymbol{vw}},
\end{aligned} \tag{5.5.38}$$

respectively. Recalling the short notation $M_{121,t}^{R_1,R_2}$ from (5.4.20) we similarly get that

$$\begin{aligned}
&\mathrm{d}\big(G_{1,t}R_1G_{2,t}R_2G_{1,t} - M_{121,t}^{R_1,R_2}\big)_{\boldsymbol{vw}}\\
&= \big(\tfrac{3}{2} + (2-k(R_1,R_2))\langle M_{12,t}^I\rangle\big)\big(G_{1,t}R_1G_{2,t}R_2G_{1,t} - M_{121,t}^{R_1,R_2}\big)_{\boldsymbol{vw}}\mathrm{d}t + \mathrm{d}\mathcal{E}_t^{(3)} + \mathcal{F}_t^{(3)}\mathrm{d}t\,,
\end{aligned} \tag{5.5.39}$$

where now the martingale term is given by

$$\mathrm{d}\mathcal{E}_t^{(3)} = \frac{1}{\sqrt{N}} \sum_{a,b=1}^{N} \partial_{ab} \left(G_{1,t} R_1 G_{2,t} R_2 G_{1,t}\right)_{\boldsymbol{vw}} \mathrm{d}B_{ab}$$

and the summands of $\mathcal{F}_t^{(3)} = \mathrm{Lin}_t^{(3)} + \sum_{i=1}^{3} \mathrm{Err}_{i,t}^{(3)}$ read

$$\begin{aligned}
\mathrm{Lin}_t^{(3)} &= k(R_1)\langle M_{12,t}^{R_1}\rangle (G_{1,t}G_{2,t}R_2G_{1,t} - M_{121,t}^{I,R_2})_{\boldsymbol{vw}} + k(R_2)\langle M_{21,t}^{R_2}\rangle \left(G_{1,t}R_1G_{2,t}G_{1,t} - M_{121,t}^{R_1,I}\right)_{\boldsymbol{vw}}, \\
\mathrm{Err}_{1,t}^{(3)} &= \left\langle G_{1,t}R_1G_{2,t}R_2G_{1,t} - M_{121,t}^{R_1,R_2}\right\rangle \left(G_{1,t}^2\right)_{\boldsymbol{vw}} + \left\langle M_{121,t}^{R_1,R_2}\right\rangle \left(G_{1,t}^2 - M_{11,t}^{I}\right)_{\boldsymbol{vw}}, \\
\mathrm{Err}_{2,t}^{(3)} &= \left\langle G_{1,t}R_1G_{2,t} - M_{12,t}^{R_1}\right\rangle (G_{1,t}G_{2,t}R_2G_{1,t})_{\boldsymbol{vw}} + \left\langle G_{2,t}R_2G_{1,t} - M_{21,t}^{R_2}\right\rangle (G_{1,t}R_1G_{2,t}G_{1,t})_{\boldsymbol{vw}}, \\
\mathrm{Err}_{3,t}^{(3)} &= \langle G_{1,t} - M_{1,t}\rangle (G_{1,t}^2R_1G_{2,t}R_2G_{1,t})_{\boldsymbol{vw}} + \langle G_{2,t} - M_{2,t}\rangle (G_{1,t}R_1G_{2,t}^2R_2G_{1,t})_{\boldsymbol{vw}} \\
&\quad + \langle G_{1,t} - M_{1,t}\rangle (G_{1,t}R_1G_{2,t}R_2G_{1,t}^2)_{\boldsymbol{vw}}\,.
\end{aligned}$$

In the following analysis, we will need several tolerance exponents $\theta_0, \theta_1, \xi_0, \xi_1, \xi_2 \in (0, \epsilon/10)$, which we required to satisfy the relations

$$\xi_0 < \xi_1 < \xi_2 < 2\xi_1 < \theta_0 < \theta_1\,. \tag{5.5.40}$$

We then define the stopping times

$$\begin{aligned}
\tau^{R_1} &:= \sup\left\{t \in [0,T] : \max_{s\in[0,t]} \max_{\boldsymbol{v},\boldsymbol{w}\in\{\boldsymbol{x},\boldsymbol{y}\}} \max_{z_{j,0}\in\Omega_0^j} \sqrt{N\ell_s\eta_{*,s}\gamma_s}\left|\left(G_{1,s}R_1G_{2,s} - M_{12,s}^{R_1}\right)_{\boldsymbol{vw}}\right| \le N^{2\theta_{k(R_1)}}\right\}, \\
\tau^{R_1,R_2} &:= \sup\left\{t \in [0,T] : \max_{s\in[0,t]} \max_{\boldsymbol{v},\boldsymbol{w}\in\{\boldsymbol{x},\boldsymbol{y}\}} \max_{z_{j,0}\in\Omega_0^j} \ell_s\gamma_s\left|\left(G_{1,s}R_1G_{2,s}R_2G_{1,s}^{(*)}\right)_{\boldsymbol{vw}}\right| \le N^{2\xi_{k(R_1,R_2)}}\right\}, \\
\tau &:= \min\left\{\tau^{R_1}, \tau^{R_1,R_2} : R_1, R_2 \in \mathfrak{S}\right\}, \quad \text{recalling} \quad \mathfrak{S} = \{I, B_1, B_1^*, B_2, B_2^*\}
\end{aligned}$$

from (5.5.9). As in Section 5.5.1, the goal is to show that $\tau = T$. First note that $\tau > 0$ by initial conditions (5.4.26b), (5.4.26c).

To prove our goal, we control the terms on the rhs. of (5.5.37) and (5.5.39). In particular we claim that uniformly in $t \in [0,T]$ we have

$$\left(\int_0^{t\wedge\tau} \mathrm{QV}_s^{(2)}\mathrm{d}s\right)^{1/2} + \int_0^{t\wedge\tau} |\mathcal{F}_s^{(2)}|\mathrm{d}s \lesssim \frac{N^{\xi_{2k(R_1)}} + k(R_1)N^{2\theta_0}}{\sqrt{N\ell_{t\wedge\tau}\eta_{*,t\wedge\tau}\gamma_{t\wedge\tau}}} \log N, \tag{5.5.41a}$$

$$\left(\int_0^{t\wedge\tau} \mathrm{QV}_s^{(3)}\mathrm{d}s\right)^{1/2} + \int_0^{t\wedge\tau} |\mathcal{F}_s^{(3)}|\mathrm{d}s \lesssim \frac{\sum_{j=1,2} k(R_j)N^{2\xi_{k(R_{3-j})}} + N^{\xi_0} + k(R_1,R_2)N^{\xi_2}}{\ell_{t\wedge\tau}\gamma_{t\wedge\tau}} \log N \tag{5.5.41b}$$

with very high probability, where $\mathrm{QV}_s^{(2)}$ and $\mathrm{QV}_s^{(3)}$ are quadratic variations of $\mathrm{d}\mathcal{E}_s^{(2)}$ and $\mathrm{d}\mathcal{E}_s^{(3)}$ respectively.

For brevity, we omit the proof of (5.5.41a). From the proof of (5.5.41b), we discuss only the quadratic variation term (first term in the lhs. of (5.5.41b)) and $\mathrm{Err}_3^{(3)}$ (part of the second term in the lhs. of (5.5.41b)). Along the way, the relations in (5.5.40) are used several times in order to accommodate error terms originating from the quadratic variation and the error terms $\mathrm{Lin}_t^{(2)}$ and $\mathrm{Lin}_t^{(3)}$. We leave the rest of the technicalities to the reader and refer to [163] where they are carefully carried out. However we point out that there are no new methods needed for analysis of the terms which we do not discuss here.

Firstly, for $\mathrm{QV}_s^{(3)}$ we have

$$\begin{aligned} N\cdot \mathrm{QV}_s^{(3)} &= \sum_{a,b=1}^{N} \left|\partial_{ab}\left(G_1R_1G_2R_2G_1\right)_{\boldsymbol{vw}}\right|^2 \\ &\lesssim \eta_{1,s}^{-2}\left(\mathrm{Im}\, G_1\right)_{\boldsymbol{vv}}\left(G_1^*R_2^*G_2^*R_1^*\mathrm{Im}\, G_1R_1G_2R_2G_1\right)_{\boldsymbol{ww}} \\ &\quad + \eta_{2,s}^{-2}\left(G_1R_1\mathrm{Im}\, G_2R_1^*G_1^*\right)_{\boldsymbol{vv}}\left(G_1^*R_2^*\mathrm{Im}\, G_2R_2G_1\right)_{\boldsymbol{ww}} \\ &\quad + \eta_{1,s}^{-2}\left(G_1R_1G_2R_2\mathrm{Im}\, G_1R_2^*G_2^*R_1^*G_1^*\right)_{\boldsymbol{vv}}\left(\mathrm{Im}\, G_1\right)_{\boldsymbol{ww}}. \end{aligned} \tag{5.5.42}$$

The long resolvent chains in the first and third term on the rhs. of (5.5.42) have to be reduced to shorter ones via a suitable reduction inequality. In the first term on the rhs. of (5.5.42), we employ the polar decomposition of G_2, i.e. represent G_2 as $|G_2|U$, where U is a unitary matrix. Denoting

$$\boldsymbol{x} := U|G_2|^{1/2}R_2G_1\boldsymbol{w} \quad \text{and} \quad S := |G_1|^{1/2}R_1^*\mathrm{Im}\, G_1R_1|G_1|^{1/2}$$

and using that $S \geq 0$ we get

$$\begin{aligned} \left(G_1^*R_2^*G_2^*R_1^*\mathrm{Im}\, G_1R_1G_2R_2G_1\right)_{\boldsymbol{ww}} &= (S)_{\boldsymbol{xx}} \\ &\leq N\langle S\rangle\|x\|^2 = N\langle \mathrm{Im}\, G_1R_1|G_2|R_1^*\rangle\left(G_1^*R_2^*|G_2|R_2G_1\right)_{\boldsymbol{ww}}. \end{aligned} \tag{5.5.43}$$

By using the integral representation (5.5.28) for $|G_2|$ in both factors in the rhs. of (5.5.43) we then find

$$\langle \mathrm{Im}\, G_1R_1|G_2|R_1^*\rangle \lesssim \frac{\log N}{\gamma_s} \quad \text{and} \quad \left(G_1^*R_2^*|G_2|R_2G_1\right)_{\boldsymbol{ww}} \lesssim \frac{N^{2\xi_2}\log N}{\ell_s\gamma_s} \tag{5.5.44}$$

with very high probability. Here, to obtain the upper bound for $\langle \mathrm{Im}\, G_1R_1|G_2|R_1^*\rangle$ we employed (5.4.27a), which was already proven in Section 5.5.1. Note that in the proof of the second part of (5.5.44) we encounter the resolvent chains of the form

$$\left(G_1^*R_2^*|\widetilde{G}_2|R_2G_1\right)_{\boldsymbol{ww}}, \quad \text{where} \quad \widetilde{G}_2 = \widetilde{G}_{2,s} := \left(W_s - D_{2,s} - z_{2,s} - ix\right)^{-1},\ x\cdot\mathrm{sgn}(\mathrm{Im}\, z) \geq 0.$$

These chains are bounded by $(\ell\gamma)^{-1}$ with very high probability, where ℓ and γ are evaluated at $(z_{1,s}, z_{2,s} + \mathrm{i}x)$. So in order to argue similarly to (5.5.29)-(5.5.30) we need to know that $\ell(z_{1,s}, z_{2,s}) \lesssim \ell(z_{1,s}, z_{2,s} + ix)$. This indeed holds since $\eta \mapsto \eta\rho(E + \mathrm{i}\eta)$ is an increasing function, which can be easily seen from the Stieltjes representation of $\rho(E + \mathrm{i}\eta)$.

For the third term in the rhs. of (5.5.42) the argument is the same, while for the second term we use Proposition 5.4.6 in combination with the bound on the fluctuation of 3G isotropic chain which is available for $s \leq \tau$. Thus using (5.4.12) we have

$$\begin{aligned} \int_0^{t\wedge\tau} \mathrm{QV}_s^{(3)}\mathrm{d}s &\lesssim \int_0^{t\wedge\tau}\left(\frac{\max\{N^{2\xi_{2k(R_1)}}, N^{2\xi_{2k(R_2)}}\}}{\eta_{1,s}^2\ell_s\gamma_s^2}(\log N)^2 + \frac{N^{-1+2\xi_{2k(R_1)}+2\xi_{2k(R_2)}}}{\eta_{2,s}^2\ell_s^2\gamma_s^2}\right)\mathrm{d}s \\ &\lesssim \frac{N^{2\xi_0} + k(R_1,R_2)N^{2\xi_2}}{\left(\ell_{t\wedge\tau}\gamma_{t\wedge\tau}\right)^2}\left((\log N)^2 + \frac{N^{2\xi_2}}{N\eta_{2,t\wedge\tau}\rho_{2,t\wedge\tau}}\right) \\ &\lesssim \left(\frac{\left(N^{\xi_0} + k(R_1,R_2)N^{\xi_2}\right)\log N}{\ell_{t\wedge\tau}\gamma_{t\wedge\tau}}\right)^2. \end{aligned}$$

Next we give the upper bound for $\mathrm{Err}_{3,t}^{(3)}$ providing the argument for the last term, for the other terms in $\mathrm{Err}_{3,t}^{(3)}$ the proof is similar. Use the usual averaged single resolvent local law from (5.5.1) for the first factor $\langle G_1 - M_1\rangle$. In the second factor we employ the reduction estimate

$$\left|\left(G_1R_1G_2R_2G_1^2\right)_{\boldsymbol{vw}}\right| \leq \frac{N}{\eta_{1,s}}\left(\langle|G_1|R_1|G_2|R_1^*\rangle\langle \mathrm{Im}\, G_1R_2^*|G_2|R_2\rangle|G_1|_{\boldsymbol{vv}}\left(\mathrm{Im}\, G_1\right)_{\boldsymbol{ww}}\right)^{1/2}.$$

Applying (5.5.28) to the absolute values of resolvents and arguing similarly to (5.5.29) we get that

$$\int_0^{t\wedge\tau} \left|\langle G_{1,s} - M_{1,s}\rangle (G_{1,s}^2 R_1 G_{2,s} R_2 G_{1,s})_{\boldsymbol{vw}}\right| \mathrm{d}s \lesssim \int_0^{t\wedge\tau} \frac{N^\zeta}{N\eta_{1,s}} \cdot \frac{N(\log N)^2}{\eta_{1,s}\gamma_s} \mathrm{d}s \lesssim \frac{N^{2\zeta}}{\ell_{t\wedge\tau}\gamma_{t\wedge\tau}}$$

for any $\zeta > 0$.

Once we have (5.5.41a),(5.5.41b) in hand, we argue similarly to (5.5.16) – (5.5.17).Thus in order to complete the proof of (5.4.27b)-(5.4.27c) it suffices to verify the inequalities (recall (5.5.10) for the definition of f_r)

$$\int_0^{t\wedge\tau} \frac{1}{N\ell_s\eta_{*,s}\gamma_s} f_s \exp\left(\int_s^{t\wedge\tau} f_r \mathrm{d}r\right) \mathrm{d}s \lesssim \frac{\log N}{N\ell_{t\wedge\tau}\eta_{*,t\wedge\tau}\gamma_{t\wedge\tau}}, \tag{5.5.45a}$$

$$\int_0^{t\wedge\tau} \frac{1}{(\ell_s\gamma_s)^2} f_s \exp\left(2\int_s^{t\wedge\tau} f_r \mathrm{d}r\right) \mathrm{d}s \lesssim \frac{\log N}{(\ell_{t\wedge\tau}\gamma_{t\wedge\tau})^2}, \tag{5.5.45b}$$

where (5.5.45a) corresponds to the propagation of the upper bound on the lhs. of (5.4.27b) and (5.5.45b) is the analog of (5.4.27c). The proof of (5.5.45a), (5.5.45b) is analogous to the proof of (5.5.18) and is based on splitting of the interval of integration into $[0, s_*]$ and $[s_*, t\wedge\tau]$, where s_* is defined in (5.5.20). The only difference is that in the regime $s \in [0, s_*]$ one needs to use the bound $\gamma_s \geq \eta_{j,s}/\rho_{j,s}$, $j \in [2]$.

This concludes the proof of the isotropic parts (5.4.27b)–(5.4.27c) of Part 1 of Proposition 5.4.8. □

5.5.3 Modifications for the regular case: Proof of Part 2 of Proposition 5.4.8

Several steps in this proof are very similar to the ones presented in Sections 5.5.1–5.5.2 we thus omit several details and present the proof only in the averaged case to illustrate the main differences in the simplest possible setting. Moreover, we work in the bulk-restricted spectral domains (5.4.10) unlike in Sections 5.5.1–5.5.2 where the proof is presented uniformly in the spectrum. In particular, it holds that $\ell_t \sim \eta_{1,t} \wedge \eta_{2,t} \wedge 1$.

Fix matrices $A_1, A_2 \in \mathbf{C}^{N\times N}$ and take any $R_1, R_2 \in \{I, A_1, A_1^*, A_2, A_2^*\}$. For initial conditions $z_{j,0} \in \Omega_{\kappa,0}^j$, $D_{j,0} \in \mathfrak{D}_j$, $j = 1, 2$, we will use the shorthand notation $\mathring{R}_j^{12} = \mathring{R}_j^{\nu_{1,t},\nu_{2,t}}$ and $\mathring{R}_j^{21} = \mathring{R}_j^{\nu_{2,t},\nu_{1,t}}$ whenever the time t can be unambiguously determined from the context. Here we denoted $\nu_{j,t} := (z_{j,t}, D_{j,t})$ where $z_{j,t}, D_{j,t}$ is the solution to the characteristic flow equation (5.4.8) with initial conditions $z_{j,0}, D_{j,0}$.

We first consider the case when one observable is regularized and compute the differential $\mathrm{d}g_t^{\mathring{R}_1^{12},R_2}$. Similarly to (5.5.6) we have

$$\begin{aligned}
\mathrm{d}g_t^{\mathring{R}_1^{12},R_2} &= g_t^{\mathring{R}_1^{12},R_2}\mathrm{d}t + \langle M_{12,t}^{\mathring{R}_1^{12}}\rangle g_t^{I,R_2}\mathrm{d}t + \langle M_{21,t}^{R_2}\rangle g_t^{\mathring{R}_1^{12},I}\mathrm{d}t + \mathrm{d}\mathcal{E}_t + \mathrm{Err}_t\mathrm{d}t + \mathrm{Reg}_t^{(1)}\mathrm{d}t,\\
\mathrm{d}\mathcal{E}_t &= \frac{1}{\sqrt{N}}\sum_{a,b=1}^N \partial_{ab}\langle G_{1,t}\mathring{R}_1^{12}G_{2,t}R_2\rangle \mathrm{d}B_{ab},\\
\mathrm{Err}_t &= g_t^{I,R_2} g_t^{\mathring{R}_1^{12},I} + \langle G_{1,t} - M_{1,t}\rangle\langle G_{1,t}^2\mathring{R}_1^{12}G_{2,t}R_2\rangle + \langle G_{2,t} - M_{2,t}\rangle\langle G_{1,t}\mathring{R}_1^{12}G_{2,t}^2R_2\rangle,\\
\mathrm{Reg}_t^{(1)} &= -\partial_t\left[\phi(\nu_{1,t},\nu_{2,t})\right]\frac{\langle M_{1,t}R_1M_{2,t}^{(*)}\rangle}{\langle M_{1,t}M_{2,t}^{(*)}\rangle}\langle G_{1,t}G_{2,t}R_2\rangle,
\end{aligned} \tag{5.5.46}$$

for the definition of ϕ see (5.2.9). In the third line in (5.5.46) the star above $M_{2,t}$ is present if and only if $\mathrm{Im}\, z_{1,t}\mathrm{Im}\, z_{2,t} > 0$. The only difference of (5.5.46) from (5.5.6) is the additional error term $\mathrm{Reg}_t^{(1)}$ which comes from the differentiation of $\mathring{R}_1^{\nu_{1,t},\nu_{2,t}}$ in t. We point out that only the artificial cutoff gives a contribution to $\mathrm{Reg}_t^{(1)}$. If the regular component (5.2.10) was defined without ϕ,

then $\mathring{R}_1^{\nu_{1,t},\nu_{2,t}}$ would be independent of t (see also Lemma 5.4.3(ii)). Note that $\mathrm{Reg}_t^{(1)} = 0$ when $\phi(\nu_{1,t},\nu_{2,t}) \in \{0,1\}$ and in the complementary regime $\widehat{\gamma}(z_{1,t},z_{2,t}) \sim 1$. Employing (5.4.26a) which is already proven in Sec. 5.5.1 we get

$$\int_0^t \left|\mathrm{Reg}_s^{(1)}\right| \mathrm{d}s \lesssim \frac{1}{\sqrt{N\ell_t}}, \quad \forall t \in [0,T]. \tag{5.5.47}$$

Beside (5.5.46) we also consider the case when both observables are regularized according to Definition 5.2.2. These two cases have to be considered together since their equations are coupled. The differential $\mathrm{d}g_t^{\mathring{R}_1^{12},\mathring{R}_2^{21}}$ is completely analogous to (5.5.46) except for the term $\mathrm{Reg}_t^{(1)}$ which should be replaced by

$$\mathrm{Reg}_t^{(2)} := -\partial_t\left[\phi(\nu_{1,t},\nu_{2,t})\right]\left(\frac{\langle M_{1,t}R_1M_{2,t}^{(*)}\rangle}{\langle M_{1,t}M_{2,t}^{(*)}\rangle}\langle G_{1,t}G_{2,t}\mathring{R}_2^{21}\rangle + \frac{\langle M_{2,t}R_2M_{1,t}^{(*)}\rangle}{\langle M_{2,t}M_{1,t}^{(*)}\rangle}\left\langle G_{1,t}\mathring{R}_1^{12}G_{2,t}\right\rangle\right)$$

with the same notational convention about $(*)$ as in (5.5.46). It is easy to see that $\mathrm{Reg}^{(2)}$ satisfies the bound (5.5.47).

From now on, to further simplify the presentation, in the case when only one among R_1, R_2 is regularized we assume that the matrix which is not regularized equals to identity. If this is not the case, then one can proceed as in Section 5.5.1 where $k(R_1,R_2)$ was introduced in (5.5.8) in order to distinguish between identity and non-identity observables.

Introduce the stopping time

$$\begin{aligned}
\tau^{R_1} &:= \sup\left\{t\in[0,T] : \max_{s\in[0,t]}\max_{z_{j,0}\in\Omega_{\kappa,0}^j} \alpha_{1,s}^{-1}\left(\left|g_s^{\mathring{R}_1^{12},I}\right| + \left|g_s^{I,\mathring{R}_1^{21}}\right|\right) \le N^{2\xi_1}\right\},\\
\tau^{R_1,R_2} &:= \sup\left\{t\in[0,T] : \max_{s\in[0,t]}\max_{z_{j,0}\in\Omega_{\kappa,0}^j} \alpha_{2,s}^{-1}\left|g_s^{\mathring{R}_1^{12},\mathring{R}_2^{21}}\right| \le N^{2\xi_2}\right\},\\
\tau &:= \min\left\{\tau^{R_1},\tau^{R_1,R_2} : R_1,R_2\in\{A_1,A_1^*,A_2,A_2^*\}\right\}
\end{aligned} \tag{5.5.48}$$

for some small $0 < \xi_1 < \xi_2 < \epsilon/10$, where

$$\alpha_{1,s} := \frac{1}{N\eta_{1,s}\eta_{2,s}} \wedge \frac{1}{\sqrt{N\ell_s\gamma_s}}, \qquad \alpha_{2,s} := \frac{1}{N\eta_{1,s}\eta_{2,s}} \wedge \frac{1}{\sqrt{N\ell_s}}. \tag{5.5.49}$$

The estimates for $g_t^{I,\mathring{R}_1^{21}}$ and $g_t^{\mathring{R}_1^{12},I}$ are completely analogous, in the following we thus consider only $g_t^{\mathring{R}_1^{12},I}$. Note that in the definition of the stopping time τ in (5.5.48) we only consider quantities with at least one regular observable, since the case of no regular observables already follows by the results of Part 1 of this proof.

The argument around (5.5.47) shows that whenever $\phi \neq 1$, it contributes with controllable and irrelevant error terms $\mathrm{Reg}_t^{(1)}$ and $\mathrm{Reg}_t^{(2)}$ to (5.5.46) and to the analogue of (5.5.46) in the case of two regularized observables, respectively. Hence, for simplicity we may assume that for initial conditions $\nu_{j,0} = (z_{j,0}, D_{j,0})$, $j = 1,2$, it holds that

$$\phi(\nu_{1,t},\nu_{2,t}) = 1, \quad \forall t \in [0,T]. \tag{5.5.50}$$

The main advantage of this simplification is that in this way the concept of regularization becomes time independent. More precisely, recalling the definition of the regular component (5.2.10) and using Lemma 5.4.3(i) along with (5.5.50) we see that the regularization with respect to $(\nu_{1,t},\nu_{2,t})$ does not depend on t, i.e.

$$\mathring{R}_1^{\nu_{1,t},\nu_{2,t}} = R_1^{\nu_{1,T},\nu_{2,T}} \quad \text{and} \quad \mathring{R}_2^{\nu_{2,t},\nu_{1,t}} = R_2^{\nu_{2,T},\nu_{1,T}}$$

for all $t \in [0,T]$. We will further assume that in (5.5.48) R_1 is $(\nu_{1,T},\nu_{2,T})$-regular and R_2 is $(\nu_{2,T},\nu_{1,T})$-regular.

The fact that we can achieve the bound $1/(N\eta_{1,t}\eta_{2,t})$ for $g_t^{A_1,I}$ follows directly by the arguments in Section 5.5.1 for any general observable A_1. In the remainder of the proof we thus focus on proving the bounds $1/\sqrt{N\ell_t\gamma_t}$ and $1/\sqrt{N\ell_t}$ in the case of one or two regular observables, respectively. Throughout this section we use the properties of the characteristic flow from Lemma 5.4.3 even if we do not mention it explicitly.

First, we notice that the first term in the rhs. of the differential equation in (5.5.46) can be neglected as it only amounts to a negligible rescaling $e^{-t}g_t^{\mathring{R}_1^{12},R_2}$. Then, we consider the stochastic term in (5.5.46). To estimate this term we first bound its quadratic variation, denoted by $\mathrm{QV}[\cdot]$ as follows (we only write one representative term):

$$\begin{aligned}\mathrm{QV}\big[g_t^{A_1,I}\big] &\lesssim \frac{1}{N^2\eta_{1,t}}\langle \mathrm{Im}\, G_1A_1G_2\mathrm{Im}\, G_1A_1G_2^*\rangle \lesssim \frac{1}{N\eta_{1,t}^2}\langle \mathrm{Im}\, G_1A_1|G_2|A_1\rangle\langle \mathrm{Im}\, G_1|G_2|\rangle,\\ \mathrm{QV}\big[g_t^{A_1,A_2}\big] &\lesssim \frac{1}{N^2\eta_{1,t}}\langle \mathrm{Im}\, G_1A_1G_2A_2\mathrm{Im}\, G_1A_2G_2^*A_1\rangle \lesssim \frac{1}{N\eta_{1,t}^2}\langle \mathrm{Im}\, G_1A_1|G_2|A_1\rangle\langle \mathrm{Im}\, G_1A_2|G_2|A_2\rangle,\end{aligned} \tag{5.5.51}$$

where we used the reduction inequalities from (5.5.27). Here we restricted the argument to the case $R_1 = A_1$ when one observable is regularized and $R_1 = A_1$, $R_2 = A_2$ when both are regularized. In general, one needs to consider $R_1, R_2 \in \{A_1, A_1^*, A_2, A_2^*\}$, but all these cases are analogous to the one considered in (5.5.51) and thus omitted. Notice that in the rhs. of (5.5.51) also $|G|$ appeared. Products of traces with some G's replaced by $|G|$ were already handled in (5.5.29) using the integral representation (5.5.28), however, the situation here is more delicate as we need to ensure that it is still possible to gain the additional smallness coming from A_1, A_2 being regular along the whole vertical line (5.5.29). This analysis was already performed in full detail in (1.6.3)–(1.6.10), we thus not repeat it here. We point out that in Chapter 1 this was done for fixed spectral parameters, however, given Lemma 5.3.3, the fact that z_t now changes in time does not cause any complication as assuming (5.5.50) the notion of regularity does not depend on time. Proceeding as in (1.6.3)–(1.6.10), using (5.3.10), (5.4.22a), and (5.4.27a), we thus conclude

$$\mathrm{QV}\big[g_t^{A_1,I}\big] \lesssim \frac{1}{N\eta_{1,t}^2\gamma_t}, \qquad \mathrm{QV}\big[g_t^{A_1,A_2}\big] \lesssim \frac{1}{N\eta_{1,t}^2}. \tag{5.5.52}$$

By the path-wise Burkholder-Davis-Gundy inequality (see [523, Appendix B.6, Eq. (18)] with $c = 0$ for continuous martingale) we thus obtain

$$\sup_{0\le t\le T}\left|\int_0^{t\wedge\tau}\mathrm{d}\mathcal{E}_s\right| \lesssim N^{\xi_j}\alpha_{j,t\wedge\tau}, \tag{5.5.53}$$

with $j = 1$ in the case of one regularized observable and $j = 2$ when both R_1, R_2 are regularized. This convention will be used throughout this proof even if not mentioned explicitly.

Next, proceeding as in (5.5.32)–(5.5.36), using the bound (5.4.22a) for the deterministic terms, it is easy to see that

$$\int_0^{t\wedge\tau}\mathrm{Err}_s\,\mathrm{d}s \prec \frac{N^{\xi_j}}{N\ell_{t\wedge\tau}}\alpha_{j,t\wedge\tau} + \frac{N^{4\xi_j}}{N\gamma_{t\wedge\tau}}. \tag{5.5.54}$$

We point out that also in the proof of (5.5.54) we need to use the integral representation (5.5.29) as discussed above (see, e.g., (5.5.31)); we omit the details for brevity.

Combining (5.5.49) and (5.5.53), for any $0 \le s \le t$, by integrating the differential equation (5.5.46) from s to $t \wedge \tau$, we thus obtain

$$\begin{aligned} g^{A_1,I}_{t\wedge\tau} &= g^{A_1,I}_s + \int_s^{t\wedge\tau} \langle M^{A_1}_{12,r}\rangle g^{I,I}_r \mathrm{d}r + \int_s^{t\wedge\tau} \langle M^{I}_{12,r}\rangle g^{A_1,I}_r \mathrm{d}r + \mathcal{O}\left(N^{\xi_1}\alpha_{1,t\wedge\tau}\right), \\ g^{A_1,A_2}_{t\wedge\tau} &= g^{A_1,A_2}_s + \int_s^{t\wedge\tau} \langle M^{A_1}_{12,r}\rangle g^{I,A_2}_r \mathrm{d}r + \int_s^{t\wedge\tau} \langle M^{A_2}_{12,r}\rangle g^{A_1,I}_r \mathrm{d}r + \mathcal{O}\left(N^{\xi_2}\alpha_{2,t\wedge\tau}\right). \end{aligned} \tag{5.5.55}$$

The terms in (5.5.55) evaluated at time s are estimated using (5.4.28a) and (5.4.28b), respectively. Using (5.4.22a) for the first integral in the first line of (5.5.55) and for both integrals in the second line of (5.5.55), we thus obtain

$$g^{A_1,I}_{t\wedge\tau} = \int_s^{t\wedge\tau} \langle M^{I}_{12,r}\rangle g^{A_1,I}_r \mathrm{d}r + \mathcal{O}\left(N^{\xi_1}\alpha_{1,t\wedge\tau}\right), \qquad g^{A_1,A_2}_{t\wedge\tau} = \mathcal{O}\left(N^{\xi_2}\alpha_{2,t\wedge\tau}\right). \tag{5.5.56}$$

To conclude the estimate of $g^{A_1,I}_{t\wedge\tau}$ we apply Gronwall inequality and obtain

$$g^{A_1,I}_{t\wedge\tau} \lesssim N^{\xi_1}\alpha_{1,t\wedge\tau} + N^{\xi_1}\int_s^{t\wedge\tau} \alpha_{1,r}\frac{1}{\gamma_r}\frac{\beta_r}{\beta_{t\wedge\tau}}\,\mathrm{d}r \lesssim N^{\xi_1}\alpha_{1,t\wedge\tau}, \tag{5.5.57}$$

where in the first inequality we used (5.5.10)–(5.5.13b) and the second inequality follows by computations similar to (5.5.23)–(5.5.24). This shows that $\tau = T$ and thus it concludes the proof. □

5.6 Zag step: Proof of (un)conditional Gronwall estimates in Lemmas 5.4.10–5.4.13

In this section, we prove the Gronwall estimates from Section 5.4.3. Throughout their proofs, we will extensively use that, for a smooth function f and W_t solving (5.4.30), by Itô's formula, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\, f(W_t) = -\frac{1}{2}\sum_{a,b}\mathbf{E}\, w_{ab}(r)(\partial_{ab}f)(W_t) + \frac{1}{2}\sum_{a,b}\sum_{\alpha\in\{ab,ba\}}\kappa(ab,\alpha)\,\mathbf{E}(\partial_{ab}\partial_\alpha f)(W_t), \tag{5.6.1}$$

and hence by a cumulant expansion (see, e.g., [243, Prop. 3.2])

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\, f(W_t) = \sum_{k=2}^{K-1}\sum_{a,b}\sum_{\boldsymbol{\alpha}\in\{ab,ba\}^k}\frac{\kappa(ab,\boldsymbol{\alpha})}{k!}\,\mathbf{E}(\partial_{ab}\partial_{\boldsymbol{\alpha}}f)(W_t) + \Omega_K \tag{5.6.2}$$

with some explicit error term Ω_K. Here, for a k-tuple of double indices $\boldsymbol{\alpha} = (\alpha_1,...,\alpha_k)$ we used the shorthand notation $\kappa(ab,(\alpha_1,...,\alpha_k)) = \kappa(w_{ab},w_{\alpha_1},...,w_{\alpha_k})$ for the joint cumulant of $w_{ab},w_{\alpha_1},...,w_{\alpha_k}$ and set $\partial_{\boldsymbol{\alpha}} = \partial_{w_{\alpha_1}}...\partial_{w_{\alpha_k}}$ and $\partial_{ab} = \partial_{w_{ab}}$.

In order to simplify the following presentation, we will henceforth assume that there is no difference for off-diagonal ($a \neq b$) and diagonal ($a = b$) cumulants $\kappa(ab,\boldsymbol{\alpha})$ in (1.A.5). The general case can be handled with straightforward minor modifications and is thus left to the reader.

5.6.1 Conditional Gronwall estimates: Proof of Lemmas 5.4.11 and 5.4.13

We begin by proving the conditional Gronwall estimates in Lemmas 5.4.11 and 5.4.13.

Proof of Lemma 5.4.11. By Ito's formula and a cumulant expansion, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\,|R_t|^{2p} \lesssim \sum_{k=3}^{K}\frac{1}{N^{k/2}}\sum_{l=0}^{k}\left|\sum_{a,b}\mathbf{E}(\partial_{ab}^l\partial_{ba}^{k-l}|R_t|^{2p})\right| + \mathcal{O}(N^{-100p}) \tag{5.6.3}$$

where we truncated the expansion at $K = \mathcal{O}(p)$ and used the trivial bound $\|G\| \le \eta^{-1}$ to estimate the error term in (5.6.3).

Throughtout the proof, we will frequently use that

$$\frac{1}{\sqrt{N\eta}\eta^{1/2}\gamma^{1/2}} \lesssim \mathcal{E}_1 \lesssim \mathcal{E}_0 \lesssim \frac{1}{\sqrt{N\eta}\eta}, \quad \mathcal{E}_0/\mathcal{E}_1 \lesssim \eta^{-1/6}, \quad \text{and} \quad \eta \lesssim \gamma$$

as well as $\eta \lesssim 1$ and $N\eta \geq 1$, without further mentioning.

Third order terms: We begin by estimating the term of order $k = 3$ in (5.6.3), as these are the most delicate ones. Distributing the derivatives according to the Leibniz rule, we see that there are three types of terms, namely (i) $(\partial^3 R)|R|^{2p-1}$, (ii) $(\partial^2 R)(\partial R)|R|^{2p-2}$, and (iii) $(\partial R)^3|R|^{2p-3}$. For ease of notation, we shall henceforth drop the subscript t of R_t as well as the index of G_i, whenever it does not lead to confusion, or is irrelevant. Moreover, we will not distinguish between R and $\overline{R}$ (and hence G and G^*) as their treatment is exactly the same.

For terms of type (i), we focus on two exemplary constellations of indices; other terms are estimated analogously and are hence omitted. First, we consider

$$N^{-3/2}\left|\sum_{a,b} G_{\boldsymbol{x}a}G_{bb}G_{aa}(G_1B_1G_2)_{b\boldsymbol{y}}\right||R|^{2p-1}. \tag{5.6.4}$$

For each of the four factors within the sum in (5.6.4), we now employ either the the isotropic single resolvent law $G_{\boldsymbol{uv}} = M_{\boldsymbol{uv}} + \mathcal{O}_{\prec}\big((N\eta)^{-1/2}\big)$ or (5.4.34). The resulting eight terms are then estimated by application of Schwarz inequalities (for the off-diagonal terms $M_{\boldsymbol{x}a}$ and $(M_{12}^{B_1})_{b\boldsymbol{y}}$) and *isotropic resummation*, e.g. as

$$N^{-3/2}\left|\sum_{a,b} M_{\boldsymbol{x}a}M_{bb}M_{aa}(M_{12}^{B_1})_{b\boldsymbol{y}}\right| \lesssim N^{-1/2}\sqrt{\sum_a |M_{\boldsymbol{x}a}|^2}\sqrt{\sum_b \big|(M_{12}^{B_1})_{b\boldsymbol{y}}\big|^2} \lesssim \frac{1}{\sqrt{N}\,\gamma} \tag{5.6.5}$$

or, now using isotropic resummation for $(G-M)_{\boldsymbol{x}a}$,

$$N^{-3/2}\left|\sum_{a,b}(G-M)_{\boldsymbol{x}a}M_{bb}M_{aa}(M_{12}^{B_1})_{b\boldsymbol{y}}\right| \prec N^{-1}\big|(G-M)_{\boldsymbol{x}\boldsymbol{m}}\big|\sqrt{\sum_b \big|(M_{12}^{B_1})_{b\boldsymbol{y}}\big|^2} \lesssim \frac{1}{N\sqrt{\eta}\,\gamma},$$

where we denoted $\boldsymbol{m} = (M_{aa})_{a\in[N]}$ and used that $\|\boldsymbol{m}\| \lesssim \sqrt{N}$, or

$$N^{-3/2}\left|\sum_{a,b} M_{\boldsymbol{x}a}(G-M)_{bb}M_{aa}(G_1B_1G_2 - M_{12}^{B_1})_{b\boldsymbol{y}}\right| \prec \sqrt{\sum_a |M_{\boldsymbol{x}a}|^2}\frac{\mathcal{E}_0}{\sqrt{N\eta}} \lesssim \frac{\mathcal{E}_0}{\sqrt{N\eta}}.$$

In the above estimates we frequently used the bound

$$\big\|M_{12}^{B_1}\big\| \lesssim \|B_1\|\,\gamma^{-1} \tag{5.6.6}$$

from Proposition 5.4.6.

Collecting all the terms, we thus find by application of Young's inequality and using $\eta \lesssim 1$, that

$$\mathbf{E}\,[(5.6.4)] \lesssim N^{\xi/2p}\left(\frac{1}{\sqrt{N\eta}\,\gamma} + \frac{\mathcal{E}_0}{\sqrt{N\eta}}\right)\mathbf{E}\,|R|^{2p-1} \lesssim \left(1 + \frac{1}{\sqrt{N}\eta^{3/2}}\right)\big(\mathbf{E}\,|R|^{2p} + N^{\xi}\mathcal{E}_1^{2p}\big)$$

for any $\xi > 0$. Secondly, we consider

$$N^{-3/2}\left|\sum_{a,b} G_{\boldsymbol{x}a}G_{ab}(G_1B_1G_2)_{bb}G_{a\boldsymbol{y}}\right||R|^{2p-1}. \tag{5.6.7}$$

Following the strategy explained below (5.6.4), we estimate, e.g.,

$$N^{-3/2}\left|\sum_{a,b} M_{\boldsymbol{x}a}(G-M)_{ab}(M_{12}^{B_1})_{bb}(G-M)_{a\boldsymbol{y}}\right| \prec \frac{1}{N\eta\gamma}\sqrt{\sum_a |M_{\boldsymbol{x}a}|^2} \lesssim \mathcal{E}_1$$

or

$$N^{-3/2}\left|\sum_{a,b}(G-M)_{\boldsymbol{x}a}(G-M)_{ab}(G_1B_1G_2-M_{12}^{B_1})_{bb}(G-M)_{a\boldsymbol{y}}\right| \prec N^{1/2}\frac{\mathcal{E}_0}{(N\eta)^{3/2}} \lesssim \frac{\mathcal{E}_1}{\sqrt{N}\eta^{7/6}}\,,$$

such that we conclude for any $\xi>0$, just as above,

$$\mathbf{E}\left[(5.6.7)\right] \lesssim \left(1+\frac{1}{\sqrt{N}\eta^{3/2}}\right)\left(\mathbf{E}\,|R|^{2p}+N^{\xi}\mathcal{E}_1^{2p}\right)\,.$$

For terms of type (ii), we again focus on two exemplary constellations of indices and omit the other ones, as they can be treated analogously. First, we consider

$$N^{-3/2}\left|\sum_{a,b}G_{\boldsymbol{x}a}(G_1B_1G_2)_{b\boldsymbol{y}}G_{\boldsymbol{x}a}G_{bb}(G_1B_1G_2)_{a\boldsymbol{y}}\right||R|^{2p-2}\,, \tag{5.6.8}$$

which we estimate as described below (5.6.4). An exemplary term (ignoring $|R|^{2p-2}$) is bounded as

$$\begin{aligned} &N^{-3/2}\left|\sum_{a,b}M_{\boldsymbol{x}a}(M_{12}^{B_1})_{b\boldsymbol{y}}(G-M)_{\boldsymbol{x}a}M_{bb}(G_1B_1G_2-M_{12}^{B_1})_{a\boldsymbol{y}}\right| \\ &\prec N^{-1}\sqrt{\sum_a|M_{\boldsymbol{x}a}|^2}\left|(M_{12}^{B_1})_{\boldsymbol{m}\boldsymbol{y}}\right|\frac{\mathcal{E}_0}{\sqrt{N\eta}} \lesssim N^{-1/2}\frac{\mathcal{E}_0}{\sqrt{N\eta\gamma}} \lesssim \mathcal{E}_1^2\,, \end{aligned} \tag{5.6.9}$$

where we used $\|\boldsymbol{m}\|\lesssim\sqrt{N}$ and (5.6.6). Secondly, we consider

$$N^{-3/2}\left|\sum_{a,b}G_{\boldsymbol{x}a}(G_1B_1G_2)_{b\boldsymbol{y}}G_{\boldsymbol{x}b}(G_1B_1G_2)_{aa}G_{b\boldsymbol{y}}\right||R|^{2p-2}\,. \tag{5.6.10}$$

Again, an exemplary term (following the strategy below (5.6.4)) can be estimated as

$$\begin{aligned} &N^{-3/2}\left|\sum_{a,b}(G-M)_{\boldsymbol{x}a}(G_1B_1G_2-M_{12}^{B_1})_{b\boldsymbol{y}}M_{\boldsymbol{x}b}(G_1B_1G_2-M_{12}^{B_1})_{aa}(G-M)_{a\boldsymbol{y}}\right| \\ &\prec\sqrt{\sum_b|M_{\boldsymbol{x}b}|^2}\,\frac{\mathcal{E}_0^2}{N\eta}\lesssim\frac{\mathcal{E}_1^2}{N\eta^{4/3}}\,. \end{aligned}$$

In total, for terms of type (ii) we find, by means of Young's inequality, that, for any $\xi>0$,

$$\mathbf{E}\left[(5.6.8)+(5.6.10)\right] \lesssim \left(1+\frac{1}{\sqrt{N}\eta^{3/2}}\right)\left(\mathbf{E}\,|R|^{2p}+N^{\xi}\mathcal{E}_1^{2p}\right)\,.$$

Lastly, for third order terms in (5.6.3), we turn to terms of type (iii). One exemplary and representative index constellation is given by

$$N^{-3/2}\left|\sum_{a,b}G_{\boldsymbol{x}a}(G_1B_1G_2)_{b\boldsymbol{y}}G_{\boldsymbol{x}a}(G_1B_1G_2)_{b\boldsymbol{y}}G_{\boldsymbol{x}b}(G_1B_1G_2)_{a\boldsymbol{y}}\right||R|^{2p-3}\,, \tag{5.6.11}$$

which we again estimate as described below (5.6.4), e.g., as (neglecting $|R|^{2p-3}$)

$$\begin{aligned} &N^{-3/2}\left|\sum_{a,b}(G-M)_{\boldsymbol{x}a}(G_1B_1G_2-M_{12}^{B_1})_{b\boldsymbol{y}}M_{\boldsymbol{x}a}(M_{12}^{B_1})_{b\boldsymbol{y}}(G-M)_{\boldsymbol{x}b}(M_{12}^{B_1})_{a\boldsymbol{y}}\right| \\ &\prec N^{-1}\frac{\mathcal{E}_0}{N\eta\gamma^2}\lesssim\mathcal{E}_1^3\,. \end{aligned}$$

In total, for terms of type (iii) we find, by means of Young's inequality, that, for any $\xi > 0$,

$$\mathbf{E}[(5.6.11)] \lesssim \left(1 + \frac{1}{\sqrt{N}\eta^{3/2}}\right)\left(\mathbf{E}\,|R|^{2p} + N^{\xi}\mathcal{E}_1^{2p}\right).$$

Therefore, collecting all the estimates for terms of type (i), (ii), and (iii), we have

$$N^{-3/2}\sum_{l=0}^{3}\left|\sum_{a,b}\mathbf{E}(\partial_{ab}^{l}\partial_{ba}^{k-l}|R|^{2p})\right| \lesssim \left(1 + \frac{1}{\sqrt{N}\eta^{3/2}}\right)\left(|R|^{2p} + N^{\xi}\mathcal{E}_1^{2p}\right).$$

Higher order terms: We now discuss the higher order terms in (5.6.3) with $k \geq 4$ and distinguish two cases: First, we consider the case where the k derivatives hit $m \leq k-2$ different factors of R's. Afterwards, we discuss the remaining case $m \in \{k-1, k\}$ (note that necessarily $m \leq k$).

Indeed, for $m \leq k-2$ different R factors that are hit by a derivative, we employ the estimates ($\boldsymbol{u}, \boldsymbol{v}$ are arbitrary vectors of bounded norm)

$$|G_{\boldsymbol{uv}}| \prec 1 \quad \text{and} \quad |(G_1B_1G_2)_{\boldsymbol{uv}}| \prec \gamma^{-1} + \mathcal{E}_0 \tag{5.6.12}$$

for all but two off-diagonal terms. In this way, modulo changing one or more of the a, b or $\boldsymbol{x}, \boldsymbol{y}$ indices to b, a or $\boldsymbol{y}, \boldsymbol{x}$, respectively (which are all treated completely analogously), and ignoring the "untouched" $|R|^{2p-m}$ factor, we arrive at

$$N^{-k/2}\sum_{ab}|G_{\boldsymbol{x}a}|\big|G_{b\boldsymbol{y}}\big|\big(\gamma^{-1} + \mathcal{E}_0\big)^m \tag{5.6.13}$$

for $m \geq 2$, or, for $m = 1$,

$$N^{-k/2}\sum_{ab}|G_{\boldsymbol{x}a}|\big|(G_1B_1G_2)_{b\boldsymbol{y}}\big|. \tag{5.6.14}$$

Following the strategy explained below (5.6.4), we then find

$$\begin{aligned}(5.6.13) + (5.6.14) &\prec \left(\frac{1}{N^{(k-2)/2}\gamma} + \frac{\mathcal{E}_0}{N^{(k-3)/2}\eta^{1/2}}\right)\mathbf{1}(m=1) + \frac{1}{N^{(k-2)/2}\eta}\big(\gamma^{-1} + \mathcal{E}_0\big)^m\mathbf{1}(m \geq 2)\\ &\lesssim \frac{1}{N^{(k-2)/2}\eta\gamma^m} + \frac{\mathcal{E}_0\,\mathbf{1}(m=1)}{N^{(k-3)/2}\eta^{1/2}} + \frac{\mathcal{E}_0^m}{N^{(k-3)/2}\eta^{1/2}} \lesssim \left(1 + \frac{1}{\sqrt{N}\eta}\right)\mathcal{E}_1^m.\end{aligned}$$

Next, for $m \in \{k-1, k\}$, we note that (by simple combinatorics) there are at least two R's, which are hit by a derivative exactly once. Therefore, using (5.6.12) for all the terms originating from the other $m-2$ differentiated R's, and ignoring the "untouched" $|R|^{2p-m}$ factor, we arrive at

$$N^{-k/2}\sum_{ab}|G_{\boldsymbol{x}a}|\big|(G_1B_1G_2)_{b\boldsymbol{y}}\big|\big|G_{\boldsymbol{x}a}\big|\big|(G_1B_1G_2)_{b\boldsymbol{y}}\big|\big(\gamma^{-1} + \mathcal{E}_0\big)^{m-2} \tag{5.6.15}$$

or with a, b in the last two terms interchanged. Similarly to above, we now estimate

$$(5.6.15) \prec \left(\frac{1}{N^{k/2}\eta\gamma^2} + \frac{\mathcal{E}_0}{N^{(k-1)/2}\eta\gamma} + \frac{\mathcal{E}_0^2}{N^{(k-2)/2}\eta}\right)\big(\gamma^{-1} + \mathcal{E}_0\big)^{m-2} \lesssim \left(1 + \frac{1}{\sqrt{N}\eta^{7/6}}\right)\mathcal{E}_1^m.$$

Therefore, collecting all the terms of order $k \geq 4$, we have, by means of Young's inequality and using $\eta \lesssim 1$,

$$N^{-k/2}\sum_{a,b}\sum_{l=0}^{k}\big|\mathbf{E}(\partial_{ab}^{l}\partial_{ba}^{k-l}|R|^{2p})\big| \lesssim \left(1 + \frac{1}{\sqrt{N}\eta^{3/2}}\right)\left(\mathbf{E}\,|R|^{2p} + N^{\xi}\mathcal{E}_1^{2p}\right),$$

for any $\xi > 0$. This concludes the proof of (5.4.35). □

Proof of Lemma 5.4.13. Just as in (5.6.3), we compute by Ito's formula and a cumulant expansion (truncated at order $K = \mathcal{O}(p)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{E}\,|R_t|^{2p} \lesssim \sum_{k=3}^{K} \frac{1}{N^{k/2}} \sum_{l=0}^{k} \left| \sum_{a,b} \mathbf{E}(\partial_{ab}^{l} \partial_{ba}^{k-l} |R_t|^{2p}) \right| + \mathcal{O}(N^{-100p})\,. \tag{5.6.16}$$

Just as in the proof of Lemma 5.4.11, for ease of notation, we shall henceforth drop the subscript t of R_t as well as the index of G_i, whenever it does not lead to confusion, or is irrelevant. Moreover, we will not distinguish between R and $\overline{R}$ (and hence G and G^*) as their treatment is exactly the same. We will first focus on the case where $\mathcal{E}_1 = 1/(\sqrt{N\eta}\gamma)$ (recall (5.4.43)).

By direct computation, using (5.4.42) and $\eta \lesssim \gamma$, we find that (the N^{-1} comes from the normalized trace in the definition of R)

$$\left|\partial_{ab}^{l} \partial_{ba}^{k-l} R\right| \prec \frac{1}{N\eta\gamma} \tag{5.6.17}$$

for all $k \in \mathbf{N}$, $l \in [k] \cup \{0\}$.

Let $m \le k$ be the number of R-factors in (5.6.16), that are hit by a derivative. For $k = 3$ and $m \ge 2$, as well as $k \ge 4$ and $m \in [k]$ in (5.6.16), the estimate (5.6.17) allows to bound these terms as (recall $\mathcal{E}_1$ from (5.4.43))

$$N^{-(k-4)/2} \left(\frac{1}{N\eta\gamma}\right)^{m} |R|^{2p-m} \lesssim \frac{1}{\sqrt{N}\eta} \left(|R|^{2p} + \mathcal{E}_1^{2p}\right) \tag{5.6.18}$$

where we bounded the a, b summations in (5.6.16) trivially, employed Young's inequality and used $N\eta \ge 1$.

The remaining case with $k = 3$ and $m = 1$ is now discussed separately. Note that, by explicit computation, in this case there is at least one off-diagonal term, i.e. of the form G_{ab}, $(GBG)_{ab}$, or $(GBGBG)_{ab}$, resulting from three derivatives hitting a single R factor. In the first case, using (5.4.42) together with a Schwarz inequality, a Ward identity, and Young's inequality, we can bound these terms as

$$N^{-5/2} \frac{1}{\eta\gamma} \sum_{ab} |G_{ab}| |R|^{2p-1} \prec \frac{1}{N\eta^{3/2}\gamma} |R|^{2p-1} \lesssim \frac{1}{\sqrt{N}\eta} \left(|R|^{2p} + \mathcal{E}_1^{2p}\right)\,.$$

In the second case, the bound works completely analogously, using

$$N^{-5/2}\gamma^{-1} \sum_{ab} |(G_1 B G_2)_{ab}| \lesssim \frac{1}{N\eta^{1/2}\gamma} \sqrt{\left(G_1 B \operatorname{Im} G_2 B^* G_1^*\right)_{aa}} \prec \frac{1}{N\eta\gamma^{3/2}} \lesssim \frac{1}{\sqrt{N}\eta}\,\mathcal{E}_1$$

instead. In third case, however, we need to use *isotropic resummation*: Since $(GBGBG)_{ab}$ is the only off-diagonal term (otherwise one could apply one of the first two cases), we necessarily deal with a term having the following index structure (ignoring the untouched $|R|^{2p-1}$)

$$N^{-5/2} \sum_{ab} (G_1 B_1 G_2 B_2 G_1)_{ab} G_{aa} G_{bb}\,. \tag{5.6.19}$$

We now write $G_{aa} = M_{aa} + \mathcal{O}_{\prec}((N\eta)^{-1/2})$, and similarly for G_{bb}, and estimate the resulting four terms separately. For the $M_{aa}M_{bb}$-term, we can isotropically sum up both indices a, b as

$$N^{-5/2} \left| \sum_{ab} (G_1 B_1 G_2 B_2 G_1)_{ab} M_{aa} M_{bb} \right| \lesssim N^{-5/2} \left|(G_1 B_1 G_2 B_2 G_1)_{\boldsymbol{mm}}\right| \prec \frac{1}{N^{3/2}\eta\gamma} \lesssim \mathcal{E}_1$$

where we denoted $\boldsymbol{m} = (M_{aa})_{a\in[N]}$ and used that $\|\boldsymbol{m}\| \lesssim \sqrt{N}$. For the other three terms, we use (5.4.42) and estimate the a, b summations trivially such that we find them to be bounded by

$$(N\eta^{3/2}\gamma)^{-1} \lesssim \mathcal{E}_1/(\sqrt{N}\eta)\,.$$

Thus, collecting all the terms and employing Young's inequality, we conclude (5.4.43) for the case $\mathcal{E}_1 = 1/(\sqrt{N\eta}\gamma)$.

In the other case, when $\mathcal{E}_1 = 1/(N\eta_1\eta_2)$, we only need to estimate the terms with $k = 3$ slightly more carefully. In fact, for $k \geq 4$ the bound (5.6.18) is sufficient, since, by definition of γ in Definition 5.4.4, it holds that $\gamma \gtrsim \eta_1 \vee \eta_2$. Now, the main difference compared to the discussion above is that since $\mathcal{E}_1 = 1/(N\eta_1\eta_2)$ has N (instead of $\sqrt{N}$ as in the first case) in the denominator, the summations over a and b have to be carried out more effectively, i.e. by exploiting as many off-diagonal terms as possible and by *isotropic resummation*. In order to do this, we schematically decompose a diagonal resolvent chain as $G_{aa} = M_{aa} + \text{fluctuation}$, similarly to (5.6.19). This is sufficient to treat all the terms arising for $k = 3$ and $m = 1$.

For $m = 2, 3$, however, there is an additional twist if the only off-diagonal terms are of the form $(G_1B_1G_2B_2G_1)_{ab}$, since we have no effective decomposition for longer isotropic chains. In this case, for $m = 3$, we estimate

$$\begin{aligned} & N^{-9/2}\left|\sum_{a,b}\left((G_1B_1G_2B_2G_1)_{ab}\right)^3\right| \\ & \prec N^{-7/2}\frac{1}{\eta\gamma}\max_a\left(G_1B_1G_2B_2G_1G_1^*B_2^*G_2^*B_1^*G_1^*\right)_{aa} \\ & \lesssim \frac{1}{N^{7/2}\eta_1^3\eta_2^2}\max_a\left(G_1B_1\mathrm{Im}\,G_2B_1^*G_1^*\right)_{aa} \lesssim \frac{1}{\sqrt{N}\eta}\,\frac{1}{(N\eta_1\eta_2)^3} = \frac{\mathcal{E}_1^3}{\sqrt{N}\eta}\,. \end{aligned} \tag{5.6.20}$$

To go to the second line, we estimate one of the three factors $(G_1B_1G_2B_2G_1)_{ab}$ by (5.4.42). Next, we used the operator norm bound $\|B_2G_1G_1^*B_2^*\| \lesssim \eta_1^{-2}$ and a Ward identity. In the penultimate step, we used (5.4.42) and the fact that $\gamma \gtrsim \eta_1 \vee \eta_2$. Similar terms arising for $m = 2$ are treated analogously to (5.6.20) and are hence left to the reader.

This concludes the proof of Lemma 5.4.13. □

5.6.2 Unconditional Gronwall estimate: Proof of Lemma 5.4.10

The proof of Lemma 5.4.10 is very similar to that of Lemma 5.4.11 and we freely use the simplified notations introduced there. The only difference compared to Lemma 5.4.11 is the following: In that proof we used the input estimate

$$\left(G_1B_1G_2\right)_{\boldsymbol{uv}} = \left(M_{12}^{B_1}\right)_{\boldsymbol{uv}} + \mathcal{O}_\prec(\mathcal{E}_0) \tag{5.6.21}$$

from (5.4.34) and effectively summed up the M-term (see, e.g., (5.6.5)). In the current proof, we not use the splitting in (5.6.21) but instead employ the trivial estimate $\left|(G_1B_1G_2)_{\boldsymbol{uv}}\right| \prec \eta^{-1}$ (as follows by a Schwarz inequality together with a Ward identity and a single resolvent local law) or sum it up, e.g., as

$$\sum_a |(G_1B_1G_2)_{\boldsymbol{x}a}| \leq N^{1/2}\sqrt{\sum_a |(G_1B_1G_2)_{\boldsymbol{x}a}|^2} \prec \frac{N^{1/2}}{\eta^{3/2}}\,, \tag{5.6.22}$$

where the final estimate follows from a Ward identity and (5.4.41).

To illustrate the changes in a more concrete example, we consider (5.6.8), and estimate it as

$$\begin{aligned} & N^{-3/2}\left|\sum_{a,b} G_{\boldsymbol{x}a}(G_1B_1G_2)_{b\boldsymbol{y}}M_{\boldsymbol{x}a}G_{bb}(G_1B_1G_2)_{a\boldsymbol{y}}\right| \\ & \qquad \prec N^{-3/2}\frac{1}{\eta}\sum_a |G_{\boldsymbol{x}a}|^2\sum_b |(G_1B_1G_2)_{b\boldsymbol{y}}| \prec \frac{1}{N\eta^3} = \mathcal{E}_0^2 \end{aligned}$$

by using a Schwarz inequality together with a Ward identity, the bound $|G_{\boldsymbol{uv}}| \prec 1$, and (5.6.22).

All the other terms can be treated with completely analogous simple modifications, hence we omit their detailed discussion. □

5.A Proofs of additional technical results

In this appendix, we collect several results of technical results, that were used in the main text.

5.A.1 Proof of Proposition 5.3.1 and about its optimality

In this section we first demonstrate the optimality of the lower bound on β_* from (5.3.6) given in Proposition 5.3.1 and then present the proof of Proposition 5.3.1 itself. Throughout this section, we will use the shorthand notation

$$\Delta^2 := \langle (D_1 - D_2)^2 \rangle .$$

Proposition 5.A.1 (Optimality of the stability bound in the bulk)**.** *Fix a (small) $\kappa > 0$ and a (large) $L > 0$. Let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be traceless Hermitian matrices with $\|D_l\| \le L$, $l = 1, 2$. Then uniformly in $E_1, E_2 \in \mathbf{R}$ with $\max\{\rho_1(E_1), \rho_2(E_2)\} \ge \kappa$ it holds that*

$$\beta_*(E_1 + \mathrm{i}0, E_2 + \mathrm{i}0) \sim \widehat{\gamma}(E_1 + \mathrm{i}0, E_2 + \mathrm{i}0). \tag{5.A.1}$$

In (5.A.1) *implicit constants depend only on κ and L.*

Proof. In the regime $\Delta^2 + |E_1 - E_2| \le c$, the estimate (5.A.1) follows from a straightforward perturbative calculation for $\beta(E_1 + \mathrm{i}0, E_2 - \mathrm{i}0)$. Here, the implicit constant $c > 0$ depends only on κ and L. In the complementary regime, we have $\widehat{\gamma} \sim 1$ and also $\beta_* \sim 1$ by Proposition 5.3.1. Therefore, it holds that $\beta_* \sim \widehat{\gamma}$. The rest of the proof of Proposition 5.A.1 is elementary and thus omitted. □

Proof of Proposition 5.3.1: Assume for simplicity that $\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{R}$. Since $\|D_j\| \le L$, we have that $\mathrm{supp}\rho_j \subset [-L - 2, L + 2]$ for $j = 1, 2$. In the following, we will distinguish the two cases (i) $\max\{|z_1|, |z_2|\} \ge L + 3$ and (ii) $\max\{|z_1|, |z_2|\} \le L + 3$.

Case (i): We will show that $\beta_*(z_1, z_2) \sim 1$ and $\widehat{\gamma}(z_1, z_2) \sim 1$, which imply, in particular (5.3.6) and (5.3.4). Assume w.l.o.g. that $|z_1| \ge L + 3$. Denote

$$d_1 := \mathrm{dist}(z_1, \mathrm{supp}\rho_1) = \min\{|z_1 - x| : x \in \mathrm{supp}\rho_1\}.$$

Using the integral representation

$$\langle \mathrm{Im}\, M_1 \rangle = \int_{\mathbf{R}} \frac{\eta_1}{|x - z_1|^2} \rho_1(x) \mathrm{d}x, \tag{5.A.2}$$

we find that $\langle \mathrm{Im}\, M_1(z_1) \rangle \le \eta_1 / d_1^2$. Therefore,

$$\langle M_1 M_1^* \rangle = \frac{\langle \mathrm{Im}\, M_1 \rangle}{\eta_1 + \langle \mathrm{Im}\, M_1 \rangle} \le \frac{1}{1 + d^2}.$$

This allows us to show that $\beta_*(z_1, z_2) \sim 1$, as follows from

$$1 \gtrsim \beta_*(z_1, z_2) \ge 1 - \max\{|\langle M_1 M_2 \rangle|, |\langle M_1 M_2^* \rangle|\} \ge 1 - \langle M_1 M_1^* \rangle^{1/2} \langle M_2 M_2^* \rangle^{1/2} \ge \frac{d^2}{1 + d^2} \gtrsim 1.$$

Here we used that $\langle M_2 M_2^* \rangle \le 1$ and $d \ge 1$. Moreover, $\eta_1 / \rho_1 \gtrsim d^2$, which implies $\widehat{\gamma}(z_1, z_2) \sim 1$. Thus $\beta_*(z_1, z_2) \sim \widehat{\gamma}(z_1, z_2)$.

Case (ii): For $|z_j| \le L + 3$, $j = 1, 2$, we split the proof in two parts: the lower bound on β_*, and the upper bound on β_*.

Lower bound on β_*. Taking into account [237, Proposition 4.2] it is sufficient to show that $\mathrm{LT} \lesssim \beta_*$. Subtracting (5.1.4) for M_1 from (5.1.4) for M_2^* we get that

$$z_1 - \bar{z}_2 - \frac{\langle M_1 (D_1 - D_2) M_2^* \rangle}{\langle M_1 M_2^* \rangle} = \frac{(1 - \langle M_1 M_2^* \rangle)(\langle M_1 \rangle - \langle M_2^* \rangle)}{\langle M_1 M_2^* \rangle}.$$

Therefore, we can rewrite LT as

$$\mathrm{LT} = \left| \frac{(1 - \langle M_1 M_2^* \rangle)(\langle M_1 \rangle - \langle M_2^* \rangle)}{\langle M_1 M_2^* \rangle} \right| \wedge 1. \tag{5.A.3}$$

If $|\langle M_1 M_2^* \rangle| \geq 1/2$, (5.A.3) implies the bound $\mathrm{LT} \lesssim |1 - \langle M_1 M_2^* \rangle|$, where we used that $|\langle M_1 \rangle - \langle M_2^* \rangle| \lesssim 1$. In the complementary regime, i.e. when $|\langle M_1 M_2^* \rangle| < 1/2$ we have $\beta(z_1, \bar{z}_2) > 1/2 \gtrsim \mathrm{LT}$.

Now we prove that $\mathrm{LT} \lesssim \beta(z_1, z_2)$. First, consider the case $|\langle M_1 M_2^* \rangle| \geq 1/2$. Again it is convenient to work with LT represented in the form (5.A.3). For the first factor in the numerator of (5.A.3) it holds that

$$|1 - \langle M_1 M_2^* \rangle| \leq |1 - \langle M_1 M_2 \rangle| + 2\|M_1\| \cdot |\langle \operatorname{Im} M_2 \rangle| \lesssim |1 - \langle M_1 M_2 \rangle|^{1/2}. \tag{5.A.4}$$

In the last step we used (5.3.4) which is proven in [237, Proposition 4.2]. For the second factor we use the bound

$$\begin{aligned} |\langle M_1 \rangle - \langle M_2^* \rangle|^2 &\leq \langle (M_1 - M_2^*)(M_1^* - M_2) \rangle \\ &= \langle M_1 M_1^* \rangle + \langle M_2 M_2^* \rangle - 2\operatorname{Re} \langle M_1 M_2 \rangle \lesssim |1 - \langle M_1 M_2 \rangle|. \end{aligned} \tag{5.A.5}$$

Therefore, (5.A.3) along with (5.A.4) and (5.A.5) implies $\mathrm{LT} \lesssim \beta(z_1, z_2)$.

Second, we consider the case $|\langle M_1 M_2^* \rangle| < 1/2$. Then

$$|1 - \langle M_1 M_2 \rangle| \geq |1 - \langle M_1 M_2^* \rangle| - 2|\langle M_1 \operatorname{Im} M_2 \rangle| \geq 1/2 - 2C_0 |\langle \operatorname{Im} M_1 \rangle| \tag{5.A.6}$$

for some constant C_0. In case that $|\langle \operatorname{Im} M_1 \rangle| < 1/(8C_0)$, (5.A.6) shows that $\beta(z_1, z_2) \geq 1/4 \gtrsim \mathrm{LT}$. If $|\langle \operatorname{Im} M_1 \rangle| \geq 1/(8C_0)$, we use (5.3.4) to get $\beta(z_1, z_2) \geq |\langle \operatorname{Im} M_1 \rangle|^2 \gtrsim 1 \gtrsim \mathrm{LT}$.

Upper bound on β_*. Firstly we have

$$\beta_* \leq |1 - \langle M_1 M_2^* \rangle| \leq |1 - \langle M_1 M_1^* \rangle| + |\langle M_1^* (M_1 - M_2) \rangle|. \tag{5.A.7}$$

The first term on the rhs. of (5.A.7) has an upper bound of order $\widehat{\gamma}$, as follows from

$$|1 - \langle M_1 M_1^* \rangle| = \frac{\eta_1}{\eta_1 + \langle \operatorname{Im} M_1 \rangle} \lesssim \frac{\eta_1}{\rho_1} \wedge 1 \leq \widehat{\gamma}. \tag{5.A.8}$$

The second term on the rhs. of (5.A.7) can be rewritten as

$$\left| \frac{(z_1 - z_2 - \langle M_1 (D_1 - D_2) M_2 \rangle) \langle M_1^* M_1 M_2 \rangle}{1 - \langle M_1 M_2 \rangle} \right| \lesssim \frac{|E_1 - E_2| + \eta_1 + \eta_2 + \Delta}{\beta_*} \lesssim \frac{\widehat{\gamma}^{1/2}}{\beta_*}. \tag{5.A.9}$$

Now, combining (5.A.7) with (5.A.8) and (5.A.9) we get that $\beta_* \lesssim \widehat{\gamma}^{1/4}$.

This concludes the proof of Proposition 5.3.1. □

5.A.2 Proof of Proposition 5.4.5

Before we turn to the proof of Proposition 5.4.5, we explain some sufficient condition for M being bounded on the whole complex plane.

Remark 5.A.2 (Sufficient condition for (5.3.5) with $\mathbf{I} = \mathbf{R}$)**.** *As pointed out below Proposition 5.3.1, the bound* (5.3.5) *holds trivially in the bulk of the spectrum. We now give some sufficient conditions to ensure that* (5.3.5) *holds uniformly in the spectrum. Denote the eigenvalues of any self-adjoint deformation D by $\{d_j\}_{j=1}^N$ labeled in increasing order, $d_j \leq d_k$ for $j < k$. Fix a large positive constant $L > 0$. The set $\mathcal{M}_L$ of admissible self-adjoint deformations D is defined as follows: we say that $D \in \mathcal{M}_L$ if $\|D\| \leq L$ and there exists an N-independent partition $\{I_s\}_{s=1}^m$ of $[0,1]$ in at most L segments such that for any $s \in [1, m]$ and any $j, k \in [1, N]$ with $j/N, k/N \in I_s$ we have $|d_j - d_k| \leq L|j/N - k/N|^{1/2}$. Since the operator $\mathcal{S} = \langle \cdot \rangle$ is flat, condition $D \in \mathcal{M}_L$ implies that D satisfies* (5.3.5) *for $\mathbf{I} = \mathbf{R}$ with some $C_0 < \infty$ by means of [22, Lemma 9.3].*

Proof of Proposition 5.4.5. In order to prove Proposition 5.4.5, we need to verify the properties of an *admissible control parameter* from Definition 5.4.4. Note that in Proposition 5.3.1 we have already shown that $\widehat{\gamma}$ satisfies (5.4.16), i.e. $\widehat{\gamma}$ is a lower bound on the stability operator. It thus remains to check items (2) and (3) of Definition 5.4.4, i.e. monotonicity in time and vague monotonicity in imaginary part. In the rest of the proof, let $z_1, z_2 \in \mathbf{H}$ and $w_2 := z_2 + \mathrm{i}x$ with $x \geq 0$.

Monotonicity in time: In order to prove monotonicity in time, we claim that

$$\langle (D_{1,s} - D_{2,s})^2 \rangle \sim \langle (D_{1,t} - D_{2,t})^2 \rangle, \tag{5.A.10a}$$

$$\mathrm{LT}_s \lesssim \mathrm{LT}_t + t - s, \quad \mathrm{LT}_t \lesssim \mathrm{LT}_s + t - s, \tag{5.A.10b}$$

$$|E_{1,s} - E_{2,s}|^2 \lesssim |E_{1,t} - E_{2,t}|^2 + (t-s)^2, \quad |E_{1,t} - E_{2,t}|^2 \lesssim |E_{1,s} - E_{2,s}|^2 + (t-s)^2, \tag{5.A.10c}$$

$$\frac{\eta_{j,s}}{\rho_{j,s}} \wedge 1 \sim \frac{\eta_{j,t}}{\rho_{j,t}} \wedge 1 + t - s, \quad j \in [2], \tag{5.A.10d}$$

uniformly in $s, t \in [0, T]$, $s \leq t$.

The first assertion (5.A.10a) is a direct consequence of (5.4.8), (5.A.10c) follows from (5.4.11) and (5.A.10d) follows from (5.5.25). To verify (5.A.10b), we again use (5.4.11) for $z_{1,s}, z_{2,s} \in \mathbf{H}$ to get

$$\begin{aligned} z_{1,t} - \bar{z}_{2,t} - \frac{\langle M_{1,t}(D_{1,t} - D_{2,t})M_{2,t}^* \rangle}{\langle M_{1,t} M_{2,t}^* \rangle} = \mathrm{e}^{-\frac{t-s}{2}} \left(z_{1,s} - \bar{z}_{2,s} - \frac{\langle M_{1,s}(D_{1,s} - D_{2,s})M_{2,s}^* \rangle}{\langle M_{1,s} M_{2,s}^* \rangle} \right) \\ - 2\big(\langle M_{1,s} \rangle - \langle M_{2,s}^* \rangle\big) \sinh \frac{t-s}{2}. \end{aligned}$$

Armed with (5.A.10a)-(5.A.10d) we obtain $\widehat{\gamma}_s + t - s \sim \widehat{\gamma}_t + t - s$. Moreover, by (5.A.10d) it holds that $\widehat{\gamma}_s \gtrsim t - s$, and thus $\widehat{\gamma}_s \sim \widehat{\gamma}_t + t - s$.

Vague monotonicity in space: Note that $\widehat{\gamma}$ has the symmetry $\widehat{\gamma}(z_1, z_2, D_1, D_2) = \widehat{\gamma}(z_2, z_1, D_2, D_1)$. Thus it is sufficient to prove the first part of (5.4.18). In the following, we will distinguish between the two cases (i) $|\langle M_1(z_1) M_2^*(z_2) \rangle| \geq 1/2$ and (ii) $|\langle M_1(z_1) M_2^*(z_2) \rangle| < 1/2$. The exact choice of the threshold separating this two cases is not important, $1/2$ may be replaced by any $c \in (0, 1)$. The proof in case (ii) is much simpler, since it corresponds to the situation when $\beta_*(z_1, z_2) \gtrsim 1$ and one only needs to show that $\beta_*(z_1, w_2) \gtrsim 1$. The proof in case (i), however, is much more involved.

Case (i): For $|\langle M_1(z_1) M_2^*(z_2) \rangle| \geq 1/2$, we first note that the integral representation (5.A.2) implies $\operatorname{Im} z_2/\rho_2(z_2) \leq \operatorname{Im} w_2/\rho_2(w_2)$, i.e. we have monotonicity of this summand in the definition of (5.3.3).

It is thus left to show that

$$\mathrm{LT}(z_1, z_2) \lesssim \widehat{\gamma}(z_1, w_2). \tag{5.A.11}$$

First, suppose that $|\langle M_1(z_1) M_2^*(w_2) \rangle| \geq 1/2$. If $\mathrm{LT}(z_1, z_2) \leq \Delta^2$, then (5.A.11) obviously holds. Thus we may assume that $\mathrm{LT}(z_1, z_2) > \Delta^2$. Using the shorthand notations $M_j := M_j(z_j)$, $j \in [2]$, $\widetilde{M}_2 := M_2(w_2)$ and $\Sigma := D_1 - D_2$, it is easy to see that

$$\begin{aligned} &|\mathrm{LT}(z_1, z_2) - \mathrm{LT}(z_1, w_2)| \\ &\quad \leq |z_2 - w_2| + \left| \frac{\langle M_1 \Sigma (M_2^* - \widetilde{M}_2^*) \rangle}{\langle M_1 M_2^* \rangle} \right| + \left| \frac{\langle M_1 \Sigma M_2^* \rangle \langle M_1 (M_2^* - \widetilde{M}_2^*) \rangle}{\langle M_1 M_2^* \rangle \langle M_1 \widetilde{M}_2^* \rangle} \right| \\ &\quad = |z_2 - w_2| + \left| \frac{\langle M_1 \Sigma M_2^* \widetilde{M}_2^* \rangle (z_2 - w_2)}{\langle M_1 M_2^* \rangle (1 - \langle M_2^* \widetilde{M}_2^* \rangle)} \right| + \left| \frac{\langle M_1 \Sigma M_2^* \rangle \langle M_1 M_2^* \widetilde{M}_2^* \rangle (z_2 - w_2)}{\langle M_1 M_2^* \rangle \langle M_1 \widetilde{M}_2^* \rangle (1 - \langle M_2^* \widetilde{M}_2^* \rangle)} \right| \\ &\quad \leq |z_2 - w_2| + (2L^3 + 4L^5) \Delta \left| \frac{z_2 - w_2}{1 - \langle M_2 \widetilde{M}_2 \rangle} \right|. \end{aligned} \tag{5.A.12}$$

If $|\langle M_2\widetilde{M}_2\rangle| > 1/2$, then $\left|(z_2-w_2)(1-\langle M_2\widetilde{M}_2\rangle)^{-1}\right| \sim |z_2-w_2|$. In the complementary case, $|\langle M_2\widetilde{M}_2\rangle| \le 1/2$, note that

$$\left|\frac{z_2-w_2}{1-\langle M_2\widetilde{M}_2\rangle}\right| = \left|\frac{\langle M_2\rangle - \langle\widetilde{M}_2\rangle}{\langle M_2\widetilde{M}_2\rangle}\right|.$$

Since D_2 satisfies (5.3.5) with $\mathbf{I}=\mathbf{R}$, there exists $C_0'>0$ which depends only on L such that

$$|\langle M_2(\xi)\rangle - \langle M_2(\zeta)\rangle| \le C_0'|\xi-\zeta|^{1/3} \tag{5.A.13}$$

for any $\xi,\zeta\in\mathbf{H}$ with $|\xi|,|\zeta|<L$. Therefore, in both cases, $|\langle M_2\widetilde{M}_2\rangle|>1/2$ and $|\langle M_2\widetilde{M}_2\rangle|\le 1/2$, we have

$$|\mathrm{LT}(z_1,z_2)-\mathrm{LT}(z_1,w_2)| \le |z_2-w_2| + C_1\Delta|z_2-w_2|^{1/3} \tag{5.A.14}$$

for some constant C_1 which only depends on L. Next we distinguish between several regimes based on the relation of $|z_2-w_2|$, Δ and $\rho_2(w_2)$.

(1) First, assume that $|z_2-w_2|\ge\Delta^{3/2}$. Then, as a consequence of (5.A.14), we have

$$|\mathrm{LT}(z_1,z_2)-\mathrm{LT}(z_1,w_2)| \le (C_1+1)|z_2-w_2|.$$

This immediately implies (5.A.11) in the case $|z_2-w_2|<\mathrm{LT}(z_1,z_2)/(2(C_1+1))$. In the complementary regime we have

$$\operatorname{Im} w_2/\rho_2(w_2) \ge \operatorname{Im} w_2 \ge |w_2-z_2| \ge (2(C_1+1))^{-1}\mathrm{LT}(z_1,z_2),$$

which allows to conclude (5.A.11) as well.

(2) Next, assume that $|z_2-w_2|<\Delta^{3/2}$ and $\rho_2(w_2)<C_2(\operatorname{Im} w)^{1/3}$, where $C_2>2C_0$ is a large positive constant depending only on L. From (5.A.14) we have

$$|\mathrm{LT}(z_1,z_2)-\mathrm{LT}(z_1,w_2)| \le (C_1+1)\Delta|z_2-w_2|^{1/3}$$

which gives (5.A.11) for $|z_2-w_2|\le(\mathrm{LT}(z_1,z_2)/(2(C_1+1)\Delta))^3$. If w_2 does not satisfy this inequality, then it holds that

$$\mathrm{LT}(z_1,z_2)/2 < (C_1+1)\Delta|z_2-w_2|^{1/3} \le (C_1+1)\mathrm{LT}^{1/2}(z_1,z_2)|z_2-w_2|^{1/3}. \tag{5.A.15}$$

Therefore,

$$(\operatorname{Im} w_2)^{2/3} \ge |w_2-z_2|^{2/3} \ge (2(C_1+1))^{-2}\mathrm{LT}(z_1,z_2).$$

In combination with the bound $\rho_2(w_2)<C_2(\operatorname{Im} w_2)^{1/3}$ this implies (5.A.11).

(3) Finally, assume that $|z_2-w_2|<\Delta^{3/2}$ and $\rho_2(w_2)\ge C_2(\operatorname{Im} w)^{1/3}$. It follows from (5.A.13) that for any ζ from the segment $\mathcal{I}$ connecting z_2 and w_2 we have $\rho_2(\zeta)\ge\rho_2(w_2)/2$. Hence

$$|\langle M_2(z_2)\rangle - \langle M_2(w_2)\rangle| = \left|\int_{\mathcal{I}}\frac{\langle M_2^2(\zeta)\rangle}{1-\langle M_2^2(\zeta)\rangle}\mathrm{d}\zeta\right| \le \frac{|z_2-w_2|}{\min_{\zeta\in\mathcal{I}}|1-\langle M_2^2(\zeta)\rangle|} \le \frac{C_3|z_2-w_2|}{\rho_2(w)^2},$$

where C_3 depends only on L. Combine this bound with (5.A.12). The case when the lhs. of (5.A.12) has an upper bound of order $|z_2-w_2|$ was already considered above. Thus we may assume that

$$|\mathrm{LT}(z_1,z_2)-\mathrm{LT}(z_1,w_2)| \le C_4\Delta\frac{|z_2-w_2|}{\rho_2(w)^2}, \tag{5.A.16}$$

where $C_4>0$ depends only on L. If the rhs. of (5.A.16) is bounded from above by $\mathrm{LT}(z_1,z_2)/2$, we conclude the desired (5.A.11). Otherwise, similarly to (5.A.15) we get

$$\mathrm{LT}(z_1,z_2) < \left(2C_4\frac{|z_2-w_2|}{\rho_2^2(w_2)}\right)^2 \le (2C_4)^2\frac{\operatorname{Im} w_2}{\rho_2^2(w_2)}\cdot\frac{\operatorname{Im} w_2}{\rho(w_2)} \lesssim \frac{\operatorname{Im} w_2}{\rho(w_2)}$$

since $\rho_2(w_2) \geq C_2(\operatorname{Im} w)^{1/3}$.

After having treated the case $|\langle M_1(z_1)M_2^*(w_2)\rangle| \geq 1/2$, we may assume that $|\langle M_1(z_1)M_2^*(w_2)\rangle| < 1/2$. In this case, we have $\beta(z_1, \bar{w}_2) \geq 1/2$. Notice that

$$|\beta(z_1, \bar{w}_2) - \beta(z_1, w_2)| \leq 2L\rho_2(w_2). \tag{5.A.17}$$

If $\rho_2(w_2) < 1/(8L)$, then (5.A.17) gives $\beta(z_1, w_2) \geq 1/4$. Otherwise by [237, Proposition 4.2] it holds that $\beta(z_1, w_2) \gtrsim \rho_2(w_2)^2 \gtrsim 1$. This means that $\beta_*(z_1, w_2) \sim 1$. Therefore, by Proposition 5.3.1 $\gamma_0(z_1, w_2) \sim 1$, which immediately implies (5.A.11).

Case (ii): In order to verify (5.4.18) in the case $|\langle M_1(z_1)M_2^*(z_2)\rangle| < 1/2$, it is sufficient to show that $\beta_*(z_1, w_2) \sim 1$. Indeed, once we have this, the bound $\beta_*(z_1, w_2) \lesssim \widehat{\gamma}^{1/4}(z_1, w_2)$ from Proposition 5.3.1 gives that $\widehat{\gamma}(z_1, w_2) \sim 1$, i.e. (5.4.18) holds. Using the Hölder 1/3-regularity (5.A.13) of $\langle M_2\rangle$ in a similar way as in the argument above (5.A.13) we get that $\beta(z_1, \bar{w}_2) \sim 1$ for $|z_2 - w_2| \leq c$ for some small positive constant $c \sim 1$ which depends only on L. For $|z_2 - w_2| > c$ by (5.3.6) we have

$$\beta(z_1, \bar{w}_2) \gtrsim \operatorname{Im} w_2/\rho_2(w_2) \geq \operatorname{Im} w_2 \gtrsim 1.$$

Thus we have shown the existence of a (small) constant $c_0 > 0$ which depends only on L such that $\beta(z_1, \bar{w}_2) \geq c_0$. Similarly to the proof around (5.A.17) we argue that $\beta(z_1, w_2) \sim 1$.

This concludes the proof of Proposition 5.4.5. □

5.A.3 Proof of Proposition 5.4.6:

The proof is split in two parts.

Part 1: The bound (5.4.21a) is the direct consequence of (5.3.8). In order to verify (5.4.21b) note that

$$\|M^{B_1,B_2}_{\nu_1,\nu_2,\nu_1}\| \lesssim \frac{\|B_1\| \cdot \|B_2\|}{|1 - \langle M_1M_2\rangle|^2|1 - \langle M_1^2\rangle|}.$$

Then use the lower bounds $|1 - \langle M_1M_2\rangle| \gtrsim \eta_1/\rho_1$ from Proposition 5.3.1 and $|1 - \langle M_1^2\rangle| \gtrsim \rho_1^2$ from (5.3.4) to get the desired result. For the upper bound on $\|M^{B_1,B_2}_{\nu_1,\nu_2,\bar{\nu}_1}\|$ the argument is similar, but one needs to use instead $|1 - \langle M_1M_2\rangle| \vee |1 - \langle M_1M_2^*\rangle| \gtrsim \rho_1^2$ and $|1 - \langle M_1M_1^*\rangle| \gtrsim \eta_1/\rho_1$ from Proposition 5.3.1.

Part 2: At first we prove (5.4.22a). Inverting $\mathcal{B}_{12}$ defined in (5.2.12) and using (5.3.8) we get that

$$M^{A_1}_{\nu_1,\nu_2} = M_1A_1M_2 + \frac{\langle M_1A_1M_2\rangle}{1 - \langle M_1M_2\rangle}M_1M_2.$$

If $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$, then by (5.3.4) $\beta(z_1, z_2) \gtrsim \kappa^2$, so (5.4.22a) holds. Assume further that $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$. Since A_1 is (ν_1, ν_2)-regular, either $\phi(\nu_1, \nu_2)$ defined in (5.2.9) vanishes or $\langle M_1A_1M_2\rangle = 0$. In the first case $\widehat{\gamma} \sim 1$, so by Proposition 5.3.1 $\beta(z_1, z_2) \sim 1$. In the second case $M^{A_1}_{\nu_1,\nu_2} = M_1A_1M_2$. In both cases $\|M^{A_1}_{\nu_1,\nu_2}\| \lesssim \|A_1\|$, i.e. (5.4.22a) holds.

The proofs of (5.4.22c) and of the part of (5.4.22b) which addresses $\|M^{A_1,B_2}_{\nu_1,\nu_2,\nu_1}\|$ go along the same lines. The only non-trivial bound is an upper bound (5.4.22b) on $\|M^{A_1,B_2}_{\nu_1,\nu_2,\bar{\nu}_1}\|$ in the case when $\operatorname{Im} z_1 \operatorname{Im} z_2 > 0$ and $\langle M_1A_1M_2^*\rangle = 0$. Using explicit formulas for $\mathcal{B}_{13}^{-1}$ and for two-resolvent deterministic approximations we see that it is sufficient to verify the following cancellation between two terms:

$$\left|\frac{\langle M_1M_1^*A_1M_2\rangle}{1 - \langle M_1^*M_2\rangle} + \frac{\langle M_1A_1M_2\rangle\langle M_1M_1^*M_2\rangle}{(1 - \langle M_1^*M_2\rangle)(1 - \langle M_1M_2\rangle)}\right| \lesssim \frac{1}{\sqrt{|1 - \langle M_1^*M_2\rangle|}}. \tag{5.A.18}$$

By (5.3.4) $|1 - \langle M_1 M_2\rangle| \sim 1$. We further rewrite (5.A.18) as

$$|\langle M_1 A_1 M_2\rangle(1 - \langle M_1^* M_2\rangle) - \langle M_1^* A_1 M_2\rangle(1 - \langle M_1 M_2\rangle)| \lesssim \sqrt{|1 - \langle M_1^* M_2\rangle|},$$

which immediately follows from Lemma 5.3.3 applied to $y_1 = y_2 = 0$. This finishes the verification of (5.4.22b). □

5.A.4 Proof of Lemma 5.3.3:

Let $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$ be any spectral parameters and denote $\nu_j^\# := (w_j, D_j)$, $j = 1, 2$, and $\mathcal{A} := \mathring{A}^{\nu_1,\nu_2}$. We have

$$\mathring{\mathcal{A}}^{\nu_1^\#,\nu_2^\#} = \mathring{A}^{\nu_1^\#,\nu_2^\#} + (\mathcal{A} - A)(1 - \phi(\nu_1^\#, \nu_2^\#)).$$

Using the fact that $\widehat{\gamma}(w_1, w_2) \sim 1$ when $\phi(\nu_1^\#, \nu_2^\#) \neq 1$ we get

$$\left\| \mathring{\mathcal{A}}^{\nu_1^\#,\nu_2^\#} - \mathring{A}^{\nu_1^\#,\nu_2^\#} \right\| \lesssim \|A\| \widehat{\gamma}(w_1, w_2).$$

Thus we may assume that $A = \mathcal{A}$, i.e. that A is (ν_1, ν_2)-regular.

As usual we will denote $M_l(z) := M^{D_l}(z)$ for $l = 1, 2$. Since A is (ν_1, ν_2)-regular, either (i) $\phi(\nu_1, \nu_2) = 0$ or (ii) $\langle M_1(z_1) A M_2^*(z_2)\rangle = 0$. In case (i), it is a direct consequence of the definition (5.2.9) of ϕ that $\phi(\nu_1', \nu_2') = 0$. Therefore, since the lhs. of (5.3.14) vanishes, (5.3.14) trivially holds. Thus, we will henceforth assume that $\langle M_1(z_1) A M_2^*(z_2)\rangle = 0$.

In the following, we will focus on showing that

$$\|\mathring{A}^{\nu_2',\nu_1'} - A\| \lesssim \|A\| \sqrt{\widehat{\gamma}(z_1', z_2')} \tag{5.A.19}$$

since the argument for the other bounds claimed in Lemma 5.3.3 are similar and thus are omitted. Firstly note that (5.A.19) is trivial in the case $\phi(\nu_1', \nu_2') = 0$. In the complementary regime, where $\phi(\nu_1', \nu_2') \neq 0$, we have $|\langle M_2(z_2') M_1^*(z_1')\rangle| \sim 1$ and it is sufficient to prove that

$$|\langle M_2(z_2') A M_1^*(z_1')\rangle| \lesssim \|A\| \sqrt{\widehat{\gamma}(z_1', z_2')}. \tag{5.A.20}$$

Using the (ν_1, ν_2)-regularity of A we rewrite the lhs. of (5.A.20) as

$$\langle M_2(z_2') A M_1^*(z_1')\rangle = \langle M_2(z_2') A (M_1(z_1') - M_2(z_2))^*\rangle - \langle (M_1(z_1) - M_2(z_2')) A M_2^*(z_2)\rangle. \tag{5.A.21}$$

Subtracting (5.1.4) for $M_2(z_2)$ from (5.1.4) for $M_1(z_1')$ we get

$$M_1(z_1') - M_2(z_2) = \frac{(z_1' - z_2) - \langle M_1(z_1')(D_1 - D_2) M_2(z_2)\rangle}{1 - \langle M_1(z_1') M_2(z_2)\rangle} M_1(z_1') M_2(z_2) - M_1(z_1')(D_1 - D_2) M_2(z_2). \tag{5.A.22}$$

Since $\rho_2(z_2) \geq \kappa$, the denominator in (5.A.22) has a lower bound of order one by the means of (5.3.4). Plugging (5.A.22) into the first term on the rhs. of (5.A.21) we arrive at

$$|\langle M_2(z_2') A (M_1(z_1') - M_2(z_2))^*\rangle| \lesssim \|A\| \left(|z_1' - z_2| + \langle (D_1 - D_2)^2\rangle^{1/2} \right) \lesssim \|A\| \widehat{\gamma}(z_1', z_2) \lesssim \|A\| \widehat{\gamma}(z_1', z_2').$$

In the last step we used that $\widehat{\gamma}$ is an admissible control parameter (cf. Proposition 5.4.5) and hence satisfies the monotonicity property (5.4.18). By a similar argument for the second term on the rhs. of (5.A.21) we conclude (5.A.20) and thus the proof of Lemma 5.3.3. □

5.A.5 Proofs of technical results in the proof of Proposition 5.4.8

In this section we present the proofs of Lemma 5.5.2 and Lemma 5.5.4.

Proof of Lemma 5.5.2. We will verify each item in Lemma 5.5.2 separately.
Item (1): In order to prove (5.5.12) it is sufficient to show that

$$\frac{\langle M_1 M_1^*\rangle^{1/2}\langle M_2 M_2^*\rangle^{1/2}}{1-\langle M_1 M_1^*\rangle^{1/2}\langle M_2 M_2^*\rangle^{1/2}} \le \frac{\pi}{2}\left(\frac{\rho_1}{\eta_1}+\frac{\rho_2}{\eta_2}\right) \tag{5.A.23}$$

since $|\langle M_1 M_2\rangle| \le \langle M_1 M_1^*\rangle^{1/2}\langle M_2 M_2^*\rangle^{1/2}$. Using the shorthand notations $x := \pi\rho_1/\eta_1 > 0$ and $y := \pi\rho_2/\eta_2 > 0$, we have

$$\langle M_1 M_1^*\rangle\langle M_2 M_2^*\rangle = xy(x+1)^{-1}(y+1)^{-1}.$$

Then (5.A.23) is equivalent to

$$\left((1+1/x)^{1/2}(1+1/y)^{1/2}-1\right)^{-1} \le (x+y)/2\,,$$

which can be rewritten as

$$(x+y)^2+(x+y)^2(1/x+1/y)+(x+y)^2/(xy) \ge (x+y)^2+4(x+y)+4.$$

This inequality holds true since $(x+y)(1/x+1/y) \ge 4$ and $(x+y)^2 \ge 4xy$. Thus, (5.5.12) holds.
Item (2): Under the characteristic flow, $M_{j,t}$ evolves as $M_{j,t} = \mathrm{e}^{t/2}M_{j,0}$, cf. Lemma 5.4.3 (i). Thus

$$\operatorname{Re}\langle M_{12,r}^I\rangle = \mathrm{e}^r\frac{\operatorname{Re}\left[\langle M_{1,0}M_{2,0}\rangle\right]-\mathrm{e}^r|\langle M_{1,0}M_{2,0}\rangle|^2}{|1-\langle M_{1,r}M_{2,r}\rangle|^2}.$$

Since $\operatorname{Re}\left[\langle M_{1,0}M_{2,0}\rangle\right]-\mathrm{e}^r|\langle M_{1,0}M_{2,0}\rangle|^2$ is monotonically decreasing in r and the denominator is positive, the second statement of Lemma 5.5.2 holds.
Item (3): In order to conclude (5.5.13a), we integrate (5.5.12) to get

$$\int_s^t f_r\mathrm{d}r \le 2\int_s^t|\langle M_{12,r}^I\rangle|\mathrm{d}r \le \int_s^r\left(\frac{\pi\rho_{1,r}}{\eta_{1,r}}+\frac{\pi\rho_{2,r}}{\eta_{2,r}}\right)\mathrm{d}r \le \log\frac{\eta_{1,s}\eta_{2,s}}{\eta_{1,t}\eta_{2,t}}.$$

Here we used that $\pi\rho_{j,r} \le -\partial_r\eta_{j,r}$, $j=1,2$. To derive (5.5.13b), assume for notational simplicity that $s_0 \ge t$. Then

$$\frac{1}{2}\int_s^t f_r\mathrm{d}r = \operatorname{Re}\int_s^t\frac{\mathrm{e}^r\langle M_{1,0}M_{2,0}\rangle}{1-\mathrm{e}^r\langle M_{1,0}M_{2,0}\rangle}\mathrm{d}r = \operatorname{Re}\log\frac{1-\langle M_{1,s}M_{2,s}\rangle}{1-\langle M_{1,t}M_{2,t}\rangle} = \log\frac{\beta_s}{\beta_t}.$$

Item (4): For any $0 \le s \le t \le T$ it holds that

$$\beta_s = |1-\langle M_{1,s}M_{2,s}\rangle| = |1-\mathrm{e}^{s-t}\langle M_{1,t}M_{2,t}\rangle| = |\mathrm{e}^{s-t}(1-\langle M_{1,t}M_{2,t}\rangle)+(1-\mathrm{e}^{s-t})| \sim \beta_t+t-s,$$

where in the last implication we used that $1-\mathrm{e}^{s-t} \ge 0$ and $\operatorname{Re}(1-\langle M_{1,t}M_{2,t}\rangle) \ge 0$. □

Proof of Lemma 5.5.4. We prove the two parts of Lemma 5.5.4 separately.
Part (i): At first we show that the constraint $|\operatorname{Re} z| \le N^{200}$ may be removed from the definition (5.4.9) of Ω_T. More precisely, we prove that if

$$|\operatorname{Im} z|\rho_T(z) \ge N^{-1+\epsilon} \quad \text{and} \quad |\operatorname{Im} z| \le N^{100}, \tag{5.A.24}$$

then $|\mathrm{Re}\, z| \le N^{200}$. Assume the opposite, i.e. that there exists $z = E + \mathrm{i}\eta \in \mathbf{C} \setminus \mathbf{R}$ as in (5.A.24) such that $|\mathrm{Re}\, z| > N^{200}$. We have

$$N^{-1+\epsilon} \lesssim |\mathrm{Im}\, z| \rho_T(z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\eta^2}{(x-E)^2+\eta^2} \rho(x) \mathrm{d}x \sim \frac{\eta^2}{\eta^2+E^2}.$$

In the last step we used that $(x-E)^2 + \eta^2 \sim E^2 + \eta^2$ for any $x \in \mathrm{supp}\, \rho_T$ once the distance from E to the support of ρ has a lower bound of order 1. Therefore it holds that

$$|E| \lesssim |\eta| N^{(1-\epsilon)/2} \le N^{200},$$

which contradicts to the assumption $|E| > N^{200}$.

Now we are ready to prove the first part of Lemma 5.5.4. The ray property of Ω_T follows from the monotonicity of the function

$$[0, \infty) \ni \eta \mapsto \eta \rho_T(E + \mathrm{i}\eta) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\eta^2}{(x-E)^2+\eta^2} \rho_T(x) \mathrm{d}x$$

for any fixed E. Moreover, since this function increases from 0 at $\eta = 0$ to 1 at $\eta \to +\infty$, for any $E \in \mathbf{R}$ there exists a unique $\eta = \eta(E) > 0$ such that

$$\eta(E) \rho_T(E + \mathrm{i}\eta(E)) = N^{-1+\epsilon}. \tag{5.A.25}$$

In particular, the part of the boundary of $\Omega_T \cap \mathbf{H}$ which is not introduced by the constraint $|\mathrm{Im}\, z| \le N^{100}$ is a graph of a function $E \mapsto \eta(E)$. Differentiating the defining equation (5.A.25) for $\eta(E)$ in E, we get that

$$\eta'(E) = \int_{\mathbf{R}} \frac{\eta(E-x)}{((x-E)^2+\eta^2)^2} \rho_T(x) \mathrm{d}x \left(\int_{\mathbf{R}} \frac{(x-E)^2}{((x-E)^2+\eta^2)^2} \rho_T(x) \mathrm{d}x \right)^{-1}. \tag{5.A.26}$$

Armed with these preliminaries, we will obtain Lemma 5.5.4 (i) by contradiction, so assume that for some $t \in [0, T)$ the ray property is violated. Then there exist two points $z_{1,t}, z_{2,t}$ with $\mathrm{Im}\, z_{j,t} < N^{100}$, $j = 1, 2$, on the boundary of $\Omega_t \cap \mathbf{H}$ such that the vertical ray which enters Ω_t through one of this points leaves it through the other one. Denote $z_{j,T} := \mathfrak{F}_{T,t} z_{j,t}$ and $E_j := \mathrm{Re}\, z_{j,T}$, $j = 1, 2$. Without loss of generality assume that $E_1 < E_2$. Then we have

$$\mathrm{Re}\, [\mathfrak{F}_{t,T} z_{1,T}] = \mathrm{Re}\, [\mathfrak{F}_{t,T} z_{2,T}]. \tag{5.A.27}$$

Since $z_{j,t} \in \partial\Omega_t$, $\mathrm{Im}\, z_{j,t} < N^{100}$ and $\mathrm{Im}\, z_{j,T} < \mathrm{Im}\, z_{j,t}$ by Lemma 5.4.3, it holds that $\mathrm{Im}\, z_{j,T} = \eta(E_j)$, where $\eta(E)$ is defined in (5.A.25). Combining (5.A.27) with (5.4.11) we see that

$$\mathrm{e}^{(T-t)/2} E_1 + 2\mathrm{Re}\, \langle M_T(z_{1,T}) \rangle \sinh \frac{T-t}{2} = \mathrm{e}^{(T-t)/2} E_2 + 2\mathrm{Re}\, \langle M_T(z_{2,T}) \rangle \sinh \frac{T-t}{2}. \tag{5.A.28}$$

This is equivalent to

$$\frac{\mathrm{Re}\, \langle M_T(z_{1,T}) \rangle - \mathrm{Re}\, \langle M_T(z_{2,T}) \rangle}{E_1 - E_2} = -\frac{1}{(1 - \mathrm{e}^{-(T-t)})}. \tag{5.A.29}$$

While the rhs. of (5.A.29) is strictly smaller than -1, the lhs. equals

$$(E_2 - E_1)^{-1} \int_{E_1}^{E_2} \partial_E \mathrm{Re}\, \langle M(E + \mathrm{i}\eta(E)) \rangle \mathrm{d}E. \tag{5.A.30}$$

Further, denoting $z := E + \mathrm{i}\eta(E)$ we have

$$\begin{aligned} \partial_E \mathrm{Re}\, \langle M(E + \mathrm{i}\eta(E)) \rangle &= \partial_E \mathrm{Re}\, \langle M(z) \rangle + \partial_\eta \mathrm{Re}\, \langle M(z) \rangle \eta'(E) \\ &= \partial_E \mathrm{Re}\, \langle M(z) \rangle + 2 \int_{\mathbb{R}} \frac{(E-x)\eta}{((x-E)^2+\eta^2)^2} \rho_T(x) \mathrm{d}x \eta'(E) \ge \partial_E \mathrm{Re}\, \langle M(z) \rangle. \end{aligned}$$

In the last inequality we used (5.A.26) to show that the second term is positive. Since

$$\partial_E \mathrm{Re}\,\langle M(z)\rangle = \mathrm{Re}\,\frac{\langle M^2\rangle}{1-\langle M^2\rangle} = -1 + \frac{1-\mathrm{Re}\,\langle M^2\rangle}{|1-\langle M^2\rangle|^2} \geq -1,$$

the lhs. of (5.A.29) is lower bounded by -1. Hence, we arrive at a contraction and hence Lemma 5.5.4 (i) holds.

Part (ii): Now we prove the second part of Lemma 5.5.4 concerning the bulk-restricted domains $\Omega_{\kappa,t}$. We aim to prove that there exists $t_* \in [0,T)$ with $T - t_* \sim 1$ such that $\Omega_{\kappa,t}$ has the ray property for all $t \in [t_*, T]$. By construction (5.4.10) $\Omega_{\kappa,T}$ satisfies the ray property. As in the argument above assume that $\Omega_{\kappa,t}$ does not satisfy the ray property for some $t \in [0,T)$. However, unlike in the previous part of the proof we do not argue by contradiction, but rather prove that $T - t \gtrsim 1$.

Similarly to (5.A.28), we find $z_{1,T}$, $z_{2,T}$ on the boundary of $\Omega_{\kappa,T} \cap \mathbf{H}$ such that (5.A.29) holds, where we denoted $E_j := \mathrm{Re}\, z_{j,T}$, $j \in [2]$. Moreover, by choosing the time t, for which the ray property of $\Omega_{\kappa,t}$ is violated, sufficiently close to T, one can find such $z_{1,T}$, $z_{2,T}$ meeting the following additional condition: Either $E_1, E_2 \in [b_r, (b_r + a_{r+1})/2]$ or $E_1, E_2 \in [(b_r + a_{r+1})/2, a_{r+1}]$ for some $r \in [m-1]$, where we freely used the notations a_r, b_r from Definition 5.4.2. Without loss of generality we may assume that the first of these two options holds and that $E_1 < E_2$. Then (5.A.29) reads

$$\frac{1}{E_2 - E_1} \int_0^{E_2 - E_1} \partial_x \mathrm{Re}\,\langle M_T(z_{1,T} + x + \mathrm{i}x)\rangle \mathrm{d}x = -\frac{1}{1 - \mathrm{e}^{T-t}}.$$

Therefore, we have

$$\sup_{x \in [E_1, E_2]} \left| \frac{\langle M_T^2\rangle}{1 - \langle M_T^2\rangle} \right| \gtrsim (T-t)^{-1}, \tag{5.A.31}$$

where M_T is evaluated at $z_{1,T} + x + \mathrm{i}x$. We view (5.A.31) as a lower bound on $T - t$ and are hence left to show that the lhs. of (5.A.31) has an upper bound of order one.

For the numerator it holds that $|\langle M_T^2\rangle| \leq 1$, while Proposition 5.3.1 applied to the denominator gives that

$$|1 - \langle M_T^2(z)\rangle| \gtrsim \mathrm{Im}\, z + \rho_T^2(z), \quad z = z_{1,T} + x + \mathrm{i}x,\ x \in [0, E_2 - E_1]. \tag{5.A.32}$$

Recall that $b_r \in \mathbf{B}_\kappa$, i.e. $\rho_T(b_r) \geq \kappa$. Then there exists $c_0 > 0$ which depends only on κ and L such that for any $y \in [0, c_0]$ we have $\rho_T(b_r + y + \mathrm{i}y) \geq \kappa/2$. This is a simple consequence of the differential inequality

$$\partial_y \rho_T(b_r + y + \mathrm{i}y) \lesssim \frac{1}{|1 - \langle M_T^2(b_r + y + \mathrm{i}y)\rangle|} \lesssim \frac{1}{\rho_T^2(b_r + y + \mathrm{i}y)},$$

where in the last step we again used Proposition 5.3.1. Take $x \in [0, E_2 - E_1]$ and choose $y := E_1 - b_r + x$, which guarantees that $z := b_r + y + \mathrm{i}y = z_{1,T} + x + \mathrm{i}x$. If $y \in [0, c_0]$, then $\rho_T(z) \geq \kappa/2$ and (5.A.32) shows that $|1 - \langle M_T^2(z)\rangle| \gtrsim 1$. In the case $y > c_0$ we have $\mathrm{Im}\, z \gtrsim 1$ and derive the same conclusion $|1 - \langle M_T^2(z)\rangle| \gtrsim 1$ from (5.A.32).

This finishes the proof of Lemma 5.5.4. □

Chapter 6

Cusp universality for correlated random matrices

This chapter contains the paper [239]:

L. Erdős, J. Henheik, and V. Riabov. Cusp universality for correlated random matrices. *arXiv:2410.06813*, 2024

Abstract. For correlated real symmetric or complex Hermitian random matrices, we prove that the local eigenvalue statistics at any cusp singularity are universal. Since the density of states typically exhibits only square root edge or cubic root cusp singularities, our result completes the proof of the Wigner-Dyson-Mehta universality conjecture in all spectral regimes for a very general class of random matrices. Previously only the bulk and the edge universality were established in this generality [23], while cusp universality was proven only for Wigner-type matrices with independent entries [244, 155]. As our main technical input, we prove an optimal local law at the cusp using the *Zigzag strategy*, a recursive tandem of the characteristic flow method and a Green function comparison argument. Moreover, our proof of the optimal local law holds uniformly in the spectrum, thus we also provide a significantly simplified alternative proof of the local eigenvalue universality in the previously studied bulk [243] and edge [23] regimes.

Content

6.1 Introduction

The celebrated Wigner-Dyson-Mehta (WDM) conjecture asserts that the local eigenvalue statistics of large random matrices become *universal*: they depend only on the symmetry class of the matrix and not on the precise details of its distribution. This remarkable effect is extremely robust and manifests in all spectral regimes. The correlation functions of the eigenvalues are governed by one of three universal determinantal processes, whose kernel functions depend on the local shape of the eigenvalue density. As proven by Dyson, Gaudin and Mehta [446] for the Gaussian GOE/GUE ensembles, the local statistics of the eigenvalues in the *bulk* of the spectrum are driven by the *sine kernel*. At the spectral edges, where the density of states vanishes like a square root, Tracy and Widom [562, 563] computed that the correlation functions for GOE/GUE are given by the *Airy kernel*. As was first observed by Wigner [586], and formalized as a conjecture for standard Wigner matrices by Dyson and Mehta in the 1960's, these statistics hold well beyond the Gaussian ensembles. After the first proofs for standard Wigner matrices [109, 251, 252, 531, 551, 550], these universality results in the bulk and at the edge saw rapid development and were gradually extended[1] to ensembles of ever greater generality: for Wigner matrices with diagonal [406, 409] and non-diagonal deformations [370], Wigner-type ensembles with not necessarily identically distributed but still independent entries [15], and even to random matrices allowing for substantial correlations among the entries [17, 243, 23].

The third and final class of universal local statistics emerges at the *cusp-like* singularities of the density with cubic-root behavior. There, the eigenvalues form a *Pearcey process*, which was first identified by Brézin and Hikami for a Gaussian unitary (GUE) matrix with a special deterministic deformation [120, 121]. Compared to the bulk and edge, the cusp regime is less understood and universality in this most delicate spectral regime was established only recently in [244, 155], however only for a special class of random matrices. More precisely, these proofs were restricted to Wigner-type ensembles with independent entries and diagonal deformations, and did not cover the broadest class of correlated ensembles, for which bulk and edge universality had already been proven.

[1] In another direction of generalization, sparse matrices [9, 250, 354, 408], adjacency matrices of regular graphs [50], band matrices [110, 115, 529], and dynamically defined matrices [8] have also been considered. In parallel to that, universal statistics in the bulk and at the edge have been established for invariant β-ensembles (see, e.g., [25, 54, 109, 108, 209, 208, 385, 486, 487, 520, 521, 575]) and their discrete analogs [41, 104, 304, 360], although often using very different methods.

Our main result completes the picture by proving the universality of the local eigenvalues statistics at the cusp for random matrices with correlated entries and an arbitrary deformation, as stated in our main result, Theorem 6.2.13. The proof follows the *three-step strategy*, a general method for proving universality of local spectral statistics, summarized in [248]. The first step in this strategy is the *local law*, which asserts that the resolvent $G(z) = (H - z)^{-1}$ at $z = E + \mathrm{i}\eta \in \mathbb{H}$ of the random matrix H concentrates around a deterministic matrix $M(z)$ as the dimension of the matrix tends to infinity. This concentration estimate holds for η just above the local eigenvalue spacing at E, resolving the empirical distribution of eigenvalues at this scale. The second step is to establish universality for ensembles with a tiny Gaussian component, and the third step is a perturbative argument that removes the Gaussian component. Crucially, the optimal local law is used as a key input for both the second and third steps. These latter two steps have proven to be extremely robust and essentially model-independent tools [243, 23, 244, 155]. Nevertheless, the critical first step, the proof of the *local law*, remains highly model-dependent.

As our main technical result, Theorem 6.2.8, we prove the *optimal average and isotropic local laws* for correlated random matrices. These local laws assert that for any fixed $\xi > 0$, any deterministic matrix B and test vectors $\boldsymbol{x}, \boldsymbol{y}$, the bounds

$$\big|\big\langle \big(G(z) - M(z)\big)B\big\rangle\big| \lesssim N^{\xi}\frac{\|B\|_{\mathrm{hs}}}{N\eta} \quad \text{and} \quad \big|\big(G(z) - M(z)\big)_{\boldsymbol{xy}}\big| \lesssim N^{\xi}\sqrt{\frac{\rho(z)}{N\eta}}\|\boldsymbol{x}\|\,\|\boldsymbol{y}\| \tag{6.1.1}$$

hold with very high probability. Here N is the dimension of the random matrix H, $\langle\cdot\rangle \coloneqq N^{-1}\operatorname{Tr}[\cdot]$ denotes the normalized trace, and $\rho(z) \coloneqq \pi^{-1}\langle \operatorname{Im} M(z)\rangle > 0$ is the *self consistent density of states.* Moreover, Theorem 6.2.8 provides further optimal improvements to the right-hand sides of (6.1.1) for spectral parameters $z = E + \mathrm{i}\eta$ with energy E outside of the self-consistent spectrum. We point out that the local laws in (6.1.1) are optimal in terms of their dependence on $\rho(z)$ and the (normalized) Hilbert–Schmidt norm of the observable matrix B. In many cases, such as for low-rank observables, the Hilbert–Schmidt norm $\|B\|_{\mathrm{hs}} \coloneqq \langle BB^*\rangle^{1/2}$ is much smaller than the operator norm $\|B\|$, which has traditionally been used in previous single-resolvent local laws [243, 23, 244]. Thus, our local law (6.1.1) unifies and improves upon the previous local laws, even in the Wigner-type case.

Traditional proofs of the local laws relied on solving an approximate self-consistent equation for the difference $G - M$. They consisted of two parts: a stability analysis of the underlying deterministic *Dyson equation* and a probabilistic estimate on the fluctuations. Both steps become quite cumbersome beyond the simple Wigner matrices. In particular, for general Wigner-type [15, 244] and correlated random matrices [243, 23], the stability analysis became intricate [16, 22], and the probabilistic part relied on sophisticated Feynman graph expansions. Recently, a completely new approach, the *Zigzag strategy* [161, 150, 162, 151, 245, 136] (see also Chapters 3–5), has been developed. This approach consists of an iterated application of two steps in tandem (cf. Figure 6.3.3 below): the *characteristic flow method* [105, 6, 353, 392, 7, 394, 14], coined the *zig-step*, and a Green function comparison (GFT) argument driven by an Ornstein-Uhlenbeck flow, called the *zag-step*. Remarkably, the Zigzag strategy circumvents many of the difficulties that arise along the more traditional local law proofs. It even removes the key obstacles that previously hindered the proof of the optimal local law at the cusp for the most general correlated matrices. We now explain this crucial aspect in more detail.

For traditional proofs of the local laws, the bulk regime is the easiest since the underlying Dyson equation is stable when $\rho(z)$ is separated away from zero. In the regime where the density $\rho(z)$ vanishes, this stability deteriorates – specifically, the corresponding stability factor behaves like $\rho(z)^{-1}$ at a square-root edge and as $\rho(z)^{-2}$ at a cubic-root cusp. This blow-up had to be compensated by a fine control on the error term in the approximate Dyson equation. On the probabilistic side, obtaining the optimal very-high-probability estimate on the fluctuation error required a high moment calculation that exploited various *fluctuation averaging* mechanisms, even in the simplest bulk regime. In the edge regime, an additional factor $\rho(z)$ needed to be extracted, which essentially relied on the

emergence of the imaginary part of the resolvent via the *Ward identity*, $GG^* = \operatorname{Im} G/\eta$. However, for cusp singularities, an additional *second order* cancellation effect was necessary. This delicate effect, coined the *cusp fluctuation averaging* [244], arises from a finite set of critical Feynman subdiagrams, called the *σ-cells*. Roughly speaking, a σ-cell consists of four resolvents interconnected through the deterministic approximation M and the correlation four-tensor of the matrix elements. In the case of Wigner-type matrices with diagonal deformations, M becomes a *diagonal* matrix, leading to a simplification of the original *matrix* Dyson equation into a *vector* equation. Moreover, since the entries of a Wigner-type matrix are independent, the correlation tensor is reduced to a matrix acting on the diagonal. These substantial simplifications facilitated the intricate extraction of σ-cells, effectively capturing the second order cancellation effect. Identifying the analog of the σ-cells for correlated matrices, when M is no longer diagonal and the correlation is a full-fledged four-tensor remains out of reach.

In this paper, we leverage the *Zigzag strategy* to conveniently avoid the complicated graphical expansions and, more importantly, circumvent the extraction of σ-cells. The only stability input required is a trivial bound of the form $\rho(z)/\eta$, that is precisely tracked by the Ward identity. The characteristic flow at the heart of the Zigzag strategy has previously proved itself to be effective in dealing with a *first order* blow-up of the stability factor, such as at the edge of Wigner matrices (see Chapter 3), and in capturing the $z_1 - z_2$ decorrelation effect for the Hermitizations of non-Hermitian i.i.d. matrices [162, 163]. The current work demonstrates that the Zigzag strategy is even capable of circumnavigating general *second order* instabilities arising at the cusp. Evidence of this feature of the characteristic flow has already been observed for unitary Brownian motion [8] and in a special non-Hermitian setting [136], where an additional symmetry was available.

Besides unraveling this remarkable power of the Zigzag approach in full generality, our paper is the first to implement the method in a correlated setting, which requires adjustments to the Zigzag dynamics. The GFT argument at the core of the zag step requires an a-priori bound on the resolvent as an input, which typically stems from a single resolvent local law. This, however, would render our argument circular. Hence, to remedy the situation, we augment the zag step with an internal induction[2] (*bootstrap*) in η. Furthermore, our result has two additional features: (i) for the averaged law in (6.1.1), we obtain the optimal estimate on the observable B in terms of its Hilbert–Schmidt norm, and (ii) we extend the Zigzag approach beyond the typical *above the scale* regime of $N\eta\rho(z) \geq N^{\varepsilon}$ (see Section 6.6). We emphasize that, in addition to covering the missing cusp regime, our proof also provides a unified approach to optimal local laws for the most general class of random matrices with correlated entries, completely eliminating any dependence of the proof on the specific spectral regime. The price we pay for our simple and self-contained Zigzag proof of the local law is assuming *fullness* of the correlated random matrix (cf. Assumption 6.2.4), rather than the slightly weaker *flatness* condition (cf. [243, Assumption (E)]). However, this stronger assumption is justified because fullness is necessary for deducing universality using the three-step strategy, regardless of how the local law is proven.

Notations and conventions

We use the notation $[N]$ to represent the index set $\{1, \dots, N\}$. The letters a, b, j, and k are used to denote integer indices, while α (with various subscripts) denotes elements of $[N]^2$. All unrestricted summations of the form $\sum_a$ and $\sum_\alpha$ are understood to run over $a \in [N]$ and $\alpha \in [N]^2$, respectively.

We denote vectors in $\mathbb{C}^{N\times N}$ using boldface letters, e.g., $\boldsymbol{x}$. The scalar product on $\mathbb{C}^N$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \sum_{j=1}^N \overline{x_j} y_j$, and the corresponding Euclidean norm is denoted by $\|\boldsymbol{x}\| := \langle \boldsymbol{x}, \boldsymbol{x}\rangle^{1/2}$.

Matrices are denoted by capital letters. Unless explicitly stated otherwise, all matrices we consider are $N \times N$. For a matrix $A \in \mathbf{C}^{N\times N}$, the angle brackets $\langle A\rangle := N^{-1}\operatorname{Tr}[A]$ denote its normalized

[2] This argument is reminiscent of [370] and we also refer to [553] for an alternative approach.

trace. We use the following notations for the matrix norms:

$$\|A\|_{\max} := \max_{a,b} |A_{ab}|, \quad \|A\| := \sup_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\|, \quad \|A\|_{\mathrm{hs}} := \big\langle |A|^2 \big\rangle^{1/2},$$

where $|A|^2 := AA^*$. Furthermore, for any $a \in [N]$ and vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we use the following notation:

$$A_{\boldsymbol{xy}} := \langle \boldsymbol{x}, A\boldsymbol{y} \rangle, \quad A_{\boldsymbol{x}a} := \langle \boldsymbol{x}, A\boldsymbol{e}_a \rangle, \quad A_{a\boldsymbol{y}} := \langle \boldsymbol{e}_a, A\boldsymbol{y} \rangle,$$

where $\boldsymbol{e}_a$ is the standard a-th basis vector of $\mathbb{C}^N$. We denote the complex upper half-plane by $\mathbb{H}$, that is, $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and its closure by $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R}$. For a complex number $z \in \mathbb{C}$, we use the notation $\langle z \rangle := 1 + |z|$.

We use c and C to denote unspecified, positive constants—small and large, respectively—that are independent of N and may change from line to line. Various tolerance exponents are denoted by Greek letters such as $\varepsilon, \xi, \delta, \zeta, \mu, \nu$. The notation $\xi \ll \varepsilon$ means that there exists a small absolute constant $c > 0$ such that $\xi \le c\varepsilon$. We use $\nu > 0$ to denote arbitrary small tolerance exponents.

For two positive quantities $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \lesssim \mathcal{Y}$ if there exists a constant $C > 0$ that depends only on the *model parameters* in Assumptions 6.2.1–6.2.5 (unless explicitly stated otherwise), such that $\mathcal{X} \le C\mathcal{Y}$. We use the notation $\mathcal{X} \sim \mathcal{Y}$ if both $\mathcal{X} \lesssim \mathcal{Y}$ and $\mathcal{Y} \lesssim \mathcal{X}$ hold. For an arbitrary quantity $\mathcal{X}$ and a positive quantity $\mathcal{Y}$, we use the notation $\mathcal{X} = \mathcal{O}(\mathcal{Y})$ to indicate that $|\mathcal{X}| \lesssim \mathcal{Y}$.

Let $\Omega := \{\Omega^{(N)}(u) \,|\, N \in \mathbb{N}, u \in \mathcal{U}^{(N)}\}$ be a family of events depending on N and possibly on a parameter u that varies over some parameter set $\mathcal{U}^{(N)}$. We say that Ω holds *with very high probability* (w.v.h.p.) uniformly in $u \in \mathcal{U}^{(N)}$ if, for any $D > 0$,

$$\sup_{u \in \mathcal{U}^{(N)}} \mathbb{P}\big[\Omega^{(N)}(u)\big] \ge 1 - N^{-D},$$

for any $N \ge N_0(D)$. We often discard the explicit dependence of $\Omega^{(N)}$ and $\mathcal{U}^{(N)}$ on N, and simply refer to Ω as a very-high-probability event. A bound is said to hold w.v.h.p. if it holds on a very-high-probability event.

6.2 Main results

We consider real symmetric or complex Hermitian random matrices H of the form

$$H = A + W, \qquad \mathbf{E}\, W = 0, \tag{6.2.1}$$

where $A \in \mathbf{C}^{N \times N}$ is a bounded deterministic matrix (cf. Assumption 6.2.1 below) and W has sufficiently fast decaying correlations between its matrix elements (cf. Assumption 6.2.3 below).

For any random matrix H, we define the *self-energy operator* $\mathcal{S}_H$ corresponding to H by its action on any deterministic matrix $X \in \mathbb{C}^{N \times N}$,

$$\mathcal{S}_H[X] := \mathbf{E}\big[(H - \mathbf{E}\, H)X(H - \mathbf{E}\, H)\big]. \tag{6.2.2}$$

The Matrix Dyson Equation (MDE) with a *data pair* $(A, \mathcal{S})$ is given by

$$-M(z)^{-1} = z - A + \mathcal{S}\big[M(z)\big] \tag{6.2.3}$$

for the unknown matrix valued function $M(z)$, $z \in \mathbf{C} \setminus \mathbf{R}$. It is well known (Theorem 2.1 [16]) that the MDE has a unique solution under the constraint that $(\operatorname{Im} z)\operatorname{Im} M(z) > 0$, where $\operatorname{Im} M = \frac{1}{2\mathrm{i}}(M - M^*)$. The corresponding *self-consistent density of states* (scDOS) ρ is a probability density function on the real line defined via the Stieltjes inversion formula,

$$\rho(x) := \lim_{\eta \to +0} \frac{1}{\pi} \big\langle \operatorname{Im} M(x + \mathrm{i}\eta) \big\rangle. \tag{6.2.4}$$

We define $\rho(z) := \pi^{-1}\langle \operatorname{Im} M(z)\rangle$ to be the harmonic extension of the scDOS to the complex upper-half plane. With a slight abuse of notation, we also refer to $\rho(z)$ as scDOS. As shown in [22], under suitable assumptions (which are formulated precisely in Section 6.2.1 below) on the data pair $(A, \mathcal{S})$ and the solution M of the MDE (6.2.3), the scDOS ρ is $1/3$-Hölder continuous. Furthermore, the set where the scDOS is positive, $\{x \in \mathbf{R} : \rho(x) > 0\}$, splits into finitely many connected components, that are called *bands*. Inside the bands, the density is real-analytic with a square root growth behavior at the *edges*. If two bands touch, however, a cubic root *cusp* emerges. These are the only two possible types of singularities. Precise universal asymptotic formulas in the *almost cusp regime* are given, e.g., in [244, Eqs. (2.4a)–(2.4e)].

As the main result of this paper, Theorem 6.2.13, we show the universality of the local eigenvalue statistics of correlated real symmetric and complex Hermitian random matrices at cusp-like singularities. As mentioned in the introduction, the proof of cusp universality follows the *three-step strategy* [248], the first step of which is a *local law* (see Theorem 6.2.8) identifying the empirical eigenvalue distribution on a scale slightly above the typical eigenvalue spacing, with very high probability. After precisely formulating the assumptions that we impose on the random matrix (6.2.1) in Section 6.2.1, we present our novel local law in Section 6.2.2. Afterwards, in Section 6.2.3, we formulate our main result on cusp universality and other consequences of the local law, such as eigenvector delocalization and eigenvalue rigidity.

6.2.1 Assumptions

In this section, we precisely formulate the assumptions, under which our main result, Theorem 6.2.8, holds, and comment on them.

Assumption 6.2.1 (Bounded expectation)**.** *There exists a constant $C_A > 0$ such that $\|A\| \le C_A$, uniformly in N.*

Assumption 6.2.2 (Finite moments)**.** *For every $p \in \mathbf{N}$, there exists a constant μ_p such that $\mathbf{E}|\sqrt{N}w_\alpha|^p \le \mu_p$ for all $\alpha \in [N]^2$.*

Before formulating our assumption on the correlation structure of the random matrix W, we introduce some custom notation to keep the definition of the norms of the (normalized) *cumulants*[3],

$$\kappa(\alpha_1, ..., \alpha_k) \equiv \kappa(\sqrt{N}w_{\alpha_1}, ..., \sqrt{N}w_{\alpha_k}), \tag{6.2.5}$$

relatively compact. If, instead of an index $a \in [N]$, we write a dot $(\cdot)$ in a scalar quantity, then we consider it as an N-vector indexed by the coordinate in place of the dot. As an example, $\kappa(a_1\cdot, a_2b_2)$ is an N-vector, whose i-entry is $\kappa(a_1 i, a_2 b_2)$ and $\|\kappa(a_1\cdot, a_2b_2)\|$ is its Euclidean (vector) norm. Similarly, $\|X(*,*)\|$ refers to the operator norm of the $N^2 \times N^2$ matrix with entries $X(\alpha_1, \alpha_2)$. We also introduce a combination of these conventions. In particular, $\big\|\|\kappa(\boldsymbol{x}*, \cdot*)\|\big\|$ denotes the operator norm $\|Y\|$ of the matrix Y with entries $Y(i,j) = \|\kappa(\boldsymbol{x}i, \cdot j)\| = \|\sum_a x_a \kappa(ai, \cdot j)\|$. Since the operator norm is invariant under transposition of the matrix, this does not lead to ambiguity regarding the order of i and j. Note that we use dot $(\cdot)$ as a placeholder for the variable related to the inner norm, and star $(*)$ for the outer norm.

[3] Let $\boldsymbol{w} = (w_1, ..., w_k)$ be a random vector. Recall that its joint cumulants, $\kappa_{\boldsymbol{m}}$ with $\boldsymbol{m} \in \mathbf{N}_0^k$, are traditionally given as the coefficients of the $\log$-characteristic function

$$\log \mathbf{E}\, \mathrm{e}^{\mathrm{i}\boldsymbol{w}\cdot\boldsymbol{t}} = \sum_{\boldsymbol{m}} \kappa_{\boldsymbol{m}} \frac{(\mathrm{i}\boldsymbol{t})^{\boldsymbol{m}}}{\boldsymbol{m}!}\,.$$

For $\boldsymbol{w} = (\sqrt{N}w_{\alpha_1}, ..., \sqrt{N}w_{\alpha_k})$ we use the notation $\kappa(\alpha_1, ..., \alpha_k) \equiv \kappa(\sqrt{N}w_{\alpha_1}, ..., \sqrt{N}w_{\alpha_k}) := \kappa_{(1,...,1)}$ and note that, by construction, $\kappa(\alpha_1, ..., \alpha_k)$ is invariant under permutations of its arguments. For example, for $k = 2$, $\kappa(\alpha_1, \alpha_2) = N\,\mathbf{E}[w_{\alpha_1} w_{\alpha_2}]$.

The following assumption on the correlation structure of W is formulated in the real symmetric case. For complex Hermitian matrices, we require the cumulant norms introduced below to be bounded for all choices of real and imaginary in each of the arguments of a cumulant, i.e. for $\kappa(\alpha_1^{\mathfrak{X}_1}, ..., \alpha_k^{\mathfrak{X}_k}) = \kappa(\sqrt{N}\mathfrak{X}_1 w_{\alpha_1}, ..., \sqrt{N}\mathfrak{X}_k w_{\alpha_k})$ and all choices of $\mathfrak{X}_i \in \{\mathrm{Re}, \mathrm{Im}\}$ (see [243, Appendix C] for a more detailed discussion).

Assumption 6.2.3 (Correlation structure)**.** *The correlations among the matrix entries $(w_\alpha)_\alpha$ of W satisfy the following.*

(i) The cumulants $\kappa(\alpha_1, ..., \alpha_k)$ have bounded matrix norms (viewed as an $N^2 \times N^2$ matrix), i.e. for all $k \geq 2$ there exists a constant $C_k > 0$ such that[4]

$$\|\kappa\|_k := \left\| \sum_{\alpha_1, ..., \alpha_{k-2}} |\kappa(\alpha_1, ..., \alpha_{k-2}, *, *)| \right\| \leq C_k \,. \tag{6.2.6}$$

Moreover, we suppose that

$$\|\kappa\|_2^{\mathrm{iso}} := \inf_{\kappa = \kappa_{\mathrm{c}} + \kappa_{\mathrm{d}}} \left(\|\kappa_{\mathrm{c}}\|_c + \|\kappa_{\mathrm{d}}\|_d \right) \leq C_2 \,, \tag{6.2.7}$$

where the infimum is taken over all decompositions of κ in two functions $\kappa_{\mathrm{c}}, \kappa_{\mathrm{d}}$, where the subscripts stand for "direct" and "cross" (see [243, Remark 2.8] for an explanation of this terminology) and the corresponding norms are defined as

$$\|\kappa\|_d := \sup_{\|\boldsymbol{x}\| \leq 1} \Big\| \|\kappa(\boldsymbol{x}*, \cdot *)\| \Big\|, \quad \textit{and} \quad \|\kappa\|_c := \sup_{\|\boldsymbol{x}\| \leq 1} \Big\| \|\kappa(\boldsymbol{x}*, *\cdot)\| \Big\| \,.$$

Finally, we assume that

$$\|\kappa\|_3^{\mathrm{av}} := N^{-3/2} \sup_{\substack{X, Y, Z \in \mathbf{C}^{N \times N}: \\ \|X\|, \|Y\| \leq 1, \ \|Z\|_{\mathrm{hs}} \leq 1}} \sum_{ab} \sum_{a_1 b_1} \sum_{a_2 b_2} \big| \kappa(ab, a_1 b_1, a_2 b_2) \big| |X_{b_1 a_2}| |Y_{b_2 a_3}| |Z_{b_3 a_1}| \leq C_3 \,. \tag{6.2.8}$$

(ii) There exists a positive $\mu > 0$, such that for every α there exists an index set $\mathcal{N}(\alpha)$ of cardinality $|\mathcal{N}(\alpha)| \leq N^{1/2-\mu}$ with the property that $w_\alpha \perp w_\beta$ for all $\beta \notin \mathcal{N}(\alpha)$. That is, every element is correlated with at most $N^{1/2-\mu}$ other matrix elements and is independent of the rest.

The first part of Assumption 6.2.3 is needed to control every finite order term in a cumulant expansion in Proposition 6.5.2, analogously to Assumption (C) in [243]. The condition in (6.2.8) is needed only since we are dealing with Hilbert–Schmidt norm error terms and thus did not appear in [243], where the observables were bounded in terms of their operator norm. In Example 6.2.6 below, we present a prototypical class of models with a polynomially decaying metric correlation structure satisfying Assumption 6.2.3 (i). Complementary to Assumption 6.2.3 (i), the only purpose of the second part of Assumption 6.2.3 is to ensure that the cumulant expansion can be truncated. In [243], this was guaranteed by a more complicated and slightly more general condition on the correlation decay (cf. [243, Assumption (D)]). We believe, however, that our proof of Theorem 6.2.8 works under this condition as well, but we refrain from doing so for brevity.

Assumption 6.2.4 (Fullness)**.** *We say that a random matrix H satisfies the fullness condition with a constant $c > 0$ if*

$$N \, \mathbf{E}\big[|\operatorname{Tr}[(H - \mathbf{E}\, H)X]|^2 \big] \geq c \operatorname{Tr}[X^2], \tag{6.2.9}$$

for any deterministic matrix X of the same symmetry class as H (real symmetric or complex Hermitian).

We assume that there exists a constant $c_{\mathrm{full}} > 0$ such that the random matrix H satisfies the fullness condition as in (6.2.9) *with the constant $c := c_{\mathrm{full}}$.*

[4] We remark that the constants C_k in the bounds (6.2.6)–(6.2.8) could also be replaced by $C_{k,\nu} N^\nu$ for any $\nu > 0$, where $C_{k,\nu}$ is a positive constant. All our proofs hold under this more general condition, but we omit it for simplicity.

Assumption 6.2.5 (Bounded self-consistent Green function)**.** *Fix $C_M, c_M > 0$ and define the set of* admissible energies *as*

$$\mathcal{I} \equiv \mathcal{I}_{C_M,c_M} := \{e \in \mathbf{R} : \|M(z)\| \le C_M \langle z\rangle^{-1} \quad \text{for all} \quad z \in \mathbf{C} \quad \text{with} \quad \operatorname{Re} z \in [e - c_M, e + c_M]\}\,. \tag{6.2.10}$$

We assume that $\mathcal{I} \neq \varnothing$.

Recall that we refer to the constants in Assumptions 6.2.1–6.2.5 as *model parameters*.

Example 6.2.6 (Polynomially Decaying Metric Correlation Structure)**.** *A prime example of correlated random matrix satisfying the Assumption 6.2.3 (i) is the polynomially decaying model. For second order cumulants, we assume that*

$$\big|\kappa(a_1 b_1, a_2 b_2)\big| \le \frac{C_2}{1 + d(a_1 b_1, a_2 b_2)^s}, \tag{6.2.11a}$$

for some $s > 2$, where we define the distance d on the set of labels $[N]^2$ as

$$d(a_1 b_1, a_2 b_2) := \min\big\{|a_1 - a_2| + |b_1 - b_2|, |a_1 - b_2| + |b_1 - a_2|\big\}. \tag{6.2.11b}$$

For cumulants of order $k \ge 3$, we assume the following decay condition

$$\big|\kappa(\alpha_1, \ldots, \alpha_k)\big| \le C_k \prod_{e \in \mathfrak{T}_{\min}} \frac{1}{1 + d(e)^s}, \tag{6.2.11c}$$

where $\mathfrak{T}_{\min}$ is a minimal spanning tree, i.e., a spanning tree for which the sum of the edge weights is minimal, in a complete graph with vertices $\alpha_1, \alpha_2, \ldots, \alpha_k$ and edge weights induced by the distance d, defined in (6.2.11b)*. The validity of* (6.2.6)–(6.2.7) *was asserted in Example 2.10 of [243], and we verify the new condition* (6.2.8) *in Appendix 6.B.*

6.2.2 Local law

In this section, we formulate our main technical result, the optimal local laws in Theorem 6.2.8. These show that $G(z) = (H - z)^{-1}$ is very well approximated by $M(z)$ in the $N \to \infty$ limit, with optimal convergence rate even at all singular points of the scDOS down to the typical eigenvalue spacing. We now define the scale on which the eigenvalues are predicted to fluctuate around a given energy e_0.

Definition 6.2.7 (Local fluctuation scale)**.** *Let $e_0 \in \mathcal{I}$ be an admissible energy. We define the self-consistent* fluctuation scale $\eta_{\mathfrak{f}} = \eta_{\mathfrak{f}}(e_0) > 0$ *(indicated by subscript $\mathfrak{f}$) at energy e_0 via*

$$\int_{-\eta_{\mathfrak{f}}}^{\eta_{\mathfrak{f}}} \rho(e_0 + x)\mathrm{d}x = \frac{1}{N}\,, \tag{6.2.12}$$

if $e_0 \in \operatorname{supp}\rho$. In case that $e_0 \notin \operatorname{supp}\rho$, we define $\eta_{\mathfrak{f}}$ as the fluctuation scale at a nearby edge. More precisely, let I be the largest interval with $e_0 \in I \subset \mathbf{R} \setminus \operatorname{supp}\rho$ and set $\Delta := \min\{|I|, 1\}$. Then, $\eta_{\mathfrak{f}}$ satisfies the scaling relation

$$\eta_{\mathfrak{f}} \sim \begin{cases} N^{-2/3}\Delta^{1/9} & \text{if} \quad \Delta > N^{-3/4} \\ N^{-3/4} & \text{if} \quad \Delta \le N^{-3/4}\,. \end{cases} \tag{6.2.13}$$

While for e_0 in the *bulk*, where the scDOS satisfies $\rho \sim 1$, we have $\eta_{\mathfrak{f}} \sim N^{-1}$, it holds that $\eta_{\mathfrak{f}} \sim N^{-2/3}$ at a regular *edge* and $\eta_{\mathfrak{f}} \sim N^{-3/4}$ at an exact *cusp*.

Theorem 6.2.8 (Optimal Local Laws). *Fix small N-independent constants $\varepsilon_0, \xi_0 > 0$. Let $H \in \mathbf{C}^{N\times N}$ be a real symmetric or complex Hermitian correlated random matrix. Suppose that Assumptions 6.2.1–6.2.5 are satisfied, and let $\mathcal{I}$ be the set of admissible energies from* (6.2.10)*. Then, uniformly for all $z \in \mathbb{H}$ with $\operatorname{Re} z \in \mathcal{I}$ and $\operatorname{dist}(z, \operatorname{supp}\rho) \in [N^{\varepsilon_0}\eta_{\mathfrak{f}}(\operatorname{Re} z), N^D]$, the resolvent $G(z) := (H-z)^{-1}$ satisfies the* optimal isotropic local law,

$$\big|\big(G(z) - M(z)\big)_{\boldsymbol{xy}}\big| \le N^{\xi_0}\sqrt{\frac{\rho(z)}{\langle z\rangle^2 N\eta}}\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|\,, \tag{6.2.14a}$$

for any deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, and the optimal average local law,

$$\big|\big\langle\big(G(z) - M(z)\big)B\big\rangle\big| \le \frac{N^{\xi_0}}{\langle z\rangle N\operatorname{dist}\big(z, \operatorname{supp}\rho\big)}\|B\|_{\mathrm{hs}}\,, \tag{6.2.14b}$$

for any deterministic matrix $B \in \mathbf{C}^{N\times N}$, both with very high probability.

6.2.3 Delocalization, rigidity, and universality

The local law in Theorem 6.2.8 is the main input for eigenvector delocalization, eigenvalue rigidity, and universality, as stated below. While Corollaries 6.2.10–6.2.11 and Theorem 6.2.13 are proven as corollaries to Theorem 6.2.8 in Section 6.3.3, the exclusion of eigenvalues outside the support of the scDOS in Theorem 6.2.9 is obtained alongside the proof of Theorem 6.2.8 and presented in Section 6.6.

Theorem 6.2.9 (No eigenvalues outside the support of the scDOS). *Under the assumptions of Theorem 6.2.8 we have the following: Let $e_0 \in \mathcal{I} \setminus \operatorname{supp}\rho$. There exists a constant $c > 0$ such that for any fixed small N-independent constant $\theta_0 > 0$*

$$\operatorname{dist}\big(\operatorname{spec} H \cap [e_0 - c, e_0 + c], \operatorname{supp}\rho\big) \le N^{\theta_0}\eta_{\mathfrak{f}}(e_0), \tag{6.2.15}$$

with very high probability. Here we use the convention that $\operatorname{dist}(\varnothing, \ldots) = 0$.

Corollary 6.2.10 (Eigenvector delocalization). *Let $\boldsymbol{u}_i \in \mathbf{C}^N$ with $\|\boldsymbol{u}_i\| = 1$ be a normalized eigenvector of H corresponding to the eigenvalue λ_i. Then, under the assumptions of Theorem 6.2.8, for any small N-independent constant $\omega_0 > 0$, the estimate*

$$\max_{\substack{i\in[N]:\\ \lambda_i\in\mathcal{I}}}\big|\langle\boldsymbol{x}, \boldsymbol{u}_i\rangle\big| \le \frac{N^{\omega_0}}{\sqrt{N}} \tag{6.2.16}$$

holds with very high probability, uniformly in deterministic vectors $\boldsymbol{x} \in \mathbf{C}^N$ with $\|\boldsymbol{x}\| = 1$.

Corollary 6.2.11 (Band rigidity and eigenvalue rigidity). *Assume the conditions of Theorem 6.2.8 with $\mathcal{I} = \mathbf{R}$ in Assumption 6.2.5. Then, the following holds.*

(a) For any $\theta > 0$, whenever $e_0 \in \mathbf{R} \setminus \operatorname{supp}\rho$ with $\operatorname{dist}(e_0, \operatorname{supp}\rho) \ge N^{\theta}\eta_{\mathfrak{f}}(e_0)$, the number of eigenvalues less than e_0 is deterministic with high probability. More precisely,

$$\big|\operatorname{spec} H \cap (-\infty, e_0)\big| = N\int_{-\infty}^{e_0}\rho(x)\mathrm{d}x, \qquad \textit{w.v.h.p.} \tag{6.2.17}$$

(b) Let $\lambda_1 \le \ldots \le \lambda_N$ denote the ordered eigenvalues of H and assume that $e_0 \in \operatorname{int}(\operatorname{supp}\rho)$. Then, for any small N-independent constant $\chi_0 > 0$, it holds that

$$\big|\lambda_{k(e_0)} - e_0\big| \le N^{\chi_0}\eta_{\mathfrak{f}}(e_0)\,, \tag{6.2.18}$$

with very high probability, where we defined the (self-consistent) eigenvalue index *as $k(e_0) := \lceil N\int_{-\infty}^{e_0}\rho(x)\mathrm{d}x\rceil$.*

Remark 6.2.12 (Integer mass)**.** *We point out that* (6.2.17) *entails the nontrivial fact that, whenever* $e_0 \notin \operatorname{supp}\rho$ *satisfies* $\operatorname{dist}(e_0, \operatorname{supp}\rho) \geq N^{\theta}\eta_{\mathfrak{f}}(e_0)$ *for some* $\theta > 0$*, the integral* $N\int_{-\infty}^{e_0}\rho(x)\mathrm{d}x$ *is always an integer. An immediate consequence is that, for each connected component* $[a,b]$ *of* $\operatorname{supp}\rho$*, it holds that* $N\int_a^b \rho(x)\mathrm{d}x$ *is an integer. That is, each* spectral band *contains that number of eigenvalues with very high probability. For spectral bands which are separated by a distance of order one, this was previously shown in [23, Corollary 2.9]. Our Corollary 6.2.11 improves this to the optimal minimal distance* $N^{\epsilon}\eta_{\mathfrak{f}}(e_0)$*.*

As our last consequence to the optimal local laws in Theorem 6.2.8, we prove cusp universality in Theorem 6.2.13 below. Since universality is already known in the bulk [243] as well as the edge regime [23], we will henceforth focus on the (approximate) cubic-root cusp. However, the optimal local laws of Theorem 6.2.8 can be used as an input for the three-step strategy to yield bulk and edge universality as well. From the in-depth analysis of the MDE (6.2.3) and its solution in [22], we know that the scDOS ρ is described by explicit universal shape functions in the vicinity of local minima with a small value of ρ and near small gaps in the support of ρ; see, e.g., [244, Eqs. (2.4a)–(2.4e)] for precise formulas.

Whenever the local length scale of such an almost cusp shape around a point $\mathfrak{b}$ matches (or is smaller than) the local eigenvalue spacing, i.e. if $\mathfrak{b}$ is a small local minimum, satisfying $\rho(\mathfrak{b}) \lesssim N^{-1/4}$, or a midpoint of a gap with width $\Delta \lesssim N^{-3/4}$, then we call the local shape around $\mathfrak{b}$ a *physical cusp* – reflecting the fact that it becomes indistinguishable from an exact cusp when resolved with a precision (slightly) above the local eigenvalue spacing $\sim N^{-3/4}$. In this case, $\mathfrak{b}$ is called a *physical cusp point*. Besides the local length scale of a physical cusp point $\mathfrak{b}$, the specific shape of the scDOS around $\mathfrak{b}$ is characterized by a single additional parameter $\gamma > 0$, called the *slope parameter*.

In order to formulate our result on cusp universality in Theorem 6.2.13, it is natural to consider the rescaled k-point function $p_k^{(N)}$, which is implicitly defined as

$$\mathbf{E}\binom{N}{k}^{-1}\sum_{\{j_1,\ldots,j_k\}\subset[N]} f(\lambda_{j_1},\ldots,\lambda_{j_k}) =: \int_{\mathbf{R}^k} f(\boldsymbol{x})p_k^{(N)}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \tag{6.2.19}$$

for any test function f. Here, the summation is over all distinct subsets of k integers from $[N]$.

Theorem 6.2.13 (Cusp universality for correlated random matrices)**.** *Let* $H \in \mathbf{C}^{N\times N}$ *be a real symmetric or complex Hermitian correlated random matrix as in* (6.2.1)*. Suppose that Assumptions 6.2.1–6.2.5 are satisfied, assume that a* physical cusp point $\mathfrak{b} \in \mathcal{I}$ *lies in the set of admissible energies* (6.2.10)*, and let* $\gamma > 0$ *be the appropriate slope parameter at* $\mathfrak{b}$*. Then, the local* k*-point correlation function at* $\mathfrak{b}$ *is universal. That is, for every* $k \in \mathbf{N}$ *there exists a* k*-point correlation function* $p_{k,\alpha}^{\mathrm{GOE/GUE}}$ *such that for any test function* $F \in C_c^1(\overline{\Omega})$ *on a bounded open set* $\Omega \subset \mathbf{R}^k$*, it holds that*[5]

$$\int_{\mathfrak{R}^k} F(\boldsymbol{x})\left[\frac{N^{k/4}}{\gamma^k}p_k^{(N)}\left(\mathfrak{b}+\frac{\boldsymbol{x}}{\gamma N^{3/4}}\right) - p_{k,\alpha}^{\mathrm{GOE/GUE}}(\boldsymbol{x})\right]\mathrm{d}\boldsymbol{x} = \mathcal{O}_{k,\Omega}(N^{-c(k)}\|F\|_{C^1})\,, \tag{6.2.20}$$

where the parameter α *depends on* γ*, the local length scale and the specific shape of the scDOS around* $\mathfrak{b}$*, i.e., whether it is an exact cusp, a small gap, or a small minimum (see [244, Eq. (2.6)] or [155, Eq. (2.5)]). The constant* $c(k) > 0$ *in* (6.2.20) *depends only on* k*, and the implicit constant in the error term depends on* k *and the diameter of the set* Ω*.*

Remark 6.2.14 (On $p_{k,\alpha}^{\mathrm{GUE/GOE}}$)**.** *For the universal* k*-point correlation function* $p_{k,\alpha}^{\mathrm{GOE/GUE}}$*, we have the following.*

[5] Here, $\mathfrak{b}$ is identified with the vector $(\mathfrak{b},\ldots,\mathfrak{b}) \in \mathbf{R}^k$.

(i) In the complex Hermitian *symmetry class, the k-point function takes the determinantal form*

$$p_{k,\alpha}^{\mathrm{GUE}}(\boldsymbol{x}) = \det\big(K_\alpha(x_i,x_j)\big)_{i,j=1}^k, \tag{6.2.21}$$

where the extended Pearcey kernel *with parameter $\alpha \in \mathbf{R}$ is given by*

$$K_\alpha(x,y) = \frac{1}{(2\pi\mathrm{i})^2}\int_\Xi \mathrm{d}z \int_\Phi \mathrm{d}w\, \frac{\exp\big(-w^4/4+\alpha w^2/2 - yw + z^4/4 - \alpha z^2/2 + xz\big)}{w-z}. \tag{6.2.22}$$

Here, Ξ is a contour consisting of rays from $\pm\mathrm{e}^{\mathrm{i}\pi/4}$ to 0 and rays from 0 to $\pm\mathrm{e}^{-\mathrm{i}\pi/4}$, and Φ is the ray from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$. See [10, 121, 564] and the references in [244] for more details.

(ii) In the real symmetric *case, the k-point correlation function $p_{k,\alpha}^{\mathrm{GOE}}$ (possibly only a distribution) is not known explicitly, not even if it is Pfaffian. However, $p_{k,\alpha}^{\mathrm{GOE}}$ exists in the dual of C^1 as the limit of correlation functions of a suitable one-parameter family of Gaussian comparison models (see Sec. 3 and in particular Eq. (3.5) of [155]).*

6.3 Zigzag strategy: Proof of the main results

To streamline the presentation, we assume that the set of admissible energies $\mathcal{I}$, defined in (6.2.10) of Assumption 6.2.5, is the entire real line, that is, $\mathcal{I} = \mathbb{R}$. We discuss the straightforward modifications for general $\mathcal{I}$ in Remark 6.3.8.

Definition 6.3.1 (Local Laws)**.** *Let H_u be a random matrix depending on a parameter*[6] *$u \in \mathcal{U}$, and let M_u be the solution to the MDE* (6.2.3) *with the data pair $(\mathbf{E}\,H_u, \mathcal{S}_{H_u})$, where $\mathcal{S}_{H_u}$ is defined in* (6.2.2)*. For all $u \in \mathcal{U}$, let $\mathcal{D}_u \subset \mathbb{H}$ and let $\xi > 0$. We say that the resolvent $G_u(z) := (H_u - z)^{-1}$ satisfies the averaged local law and the isotropic local law, respectively, with data $(\mathcal{D}_u, \xi)$ uniformly in $u \in \mathcal{U}$, if and only if the bounds*

$$\Big|\big\langle\big(G_u(z) - M_u(z)\big)B\big\rangle\Big| \le \frac{N^{3\xi}}{N\eta}, \quad \text{and} \quad \Big|\big(G_u(z) - M_u(z)\big)_{\boldsymbol{xy}}\Big| \le N^{\xi}\bigg(\sqrt{\frac{\rho_u(z)}{N\eta}} + \frac{1}{N\eta}\bigg), \tag{6.3.1}$$

hold uniformly in $z := E + \mathrm{i}\eta \in \mathcal{D}_u$ and in $u \in \mathcal{U}$, with very high probability, for any deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$ with $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$, and any deterministic matrices B with $\|B\|_{\mathrm{hs}} = 1$. Here $\rho_u(z) := \frac{1}{\pi}\langle \operatorname{Im} M_u(z)\rangle$.

The goal of the present section is to prove the local laws in the *above the scale* regime, where $\rho(z)N|\operatorname{Im} z|$ is large. Fix a (small) N-independent constant $\varepsilon > 0$, a large constant $C_L > 0$, and define the spectral domain $\mathcal{D}^{\mathrm{abv}}$ as

$$\mathcal{D}^{\mathrm{abv}} \equiv \mathcal{D}^{\mathrm{abv}}(\varepsilon, C_L) := \big\{z := E + \mathrm{i}\eta \in \mathbb{H} : \rho(z)N\eta \ge N^{\varepsilon},\, |E| \le C_L,\, \eta \le C_L\big\}. \tag{6.3.2}$$

The regime $\rho(z)N\eta \ge N^{\varepsilon}$ is natural for studying the local laws, since $\rho(E+\mathrm{i}\eta)N\eta$ is the typical number of eigenvalues in the interval of size η around the energy E.

Theorem 6.3.2 (Local Laws above the Scale)**.** *Fix a (small) N-independent constant $\varepsilon > 0$, a large constant $C_L > 0$. Let H be a random matrix satisfying the Assumptions 6.2.1–6.2.5, then the resolvent $G(z) := (H - z)^{-1}$ satisfies the local laws* (6.3.1) *with data $(\mathcal{D}^{\mathrm{abv}}, 2\xi)$, for any fixed tolerance exponent $0 < \xi \le \frac{1}{100}\varepsilon$, where $\mathcal{D}^{\mathrm{abv}} = \mathcal{D}^{\mathrm{abv}}(\varepsilon, C_L)$.*

[6] In applications, the parameter u will typically be time and the set $\mathcal{U}$ will be a bounded subinterval of $\mathbf{R}$.

To prove Theorem 6.2.8 in the *below the scale* regime, that is, to handle the case when $\rho(z)N|\operatorname{Im} z|$ is small, we proceed in two steps. In the key first step we use the local laws above the scale of Theorem 6.3.2 to prove Theorem 6.2.9 that asserts the absence of spectrum outside of the support of the scDOS ρ. Then the second step is a routine derivation of (6.2.14b) and (6.2.14a) from (6.2.15) and (6.3.1). Both steps are presented in Section 6.6. In the main part of the proof, we only consider spectral parameters z satisfying $\operatorname{dist}(z, \operatorname{supp}\rho) \lesssim 1$. The easy extension to the regime $\operatorname{dist}(z, \operatorname{supp}\rho) \gtrsim 1$ and the resulting $\langle z\rangle^{-2}$-decay are briefly addressed in Remark 6.6.8.

In the sequel, we treat the constants ε, C_L in (6.3.2) as additional model parameters and omit them from the arguments of $\mathcal{D}^{\mathrm{abv}}$.

Throughout the paper, we consistently use the notation $\varepsilon, \xi, \zeta, \delta$ to represent positive N-independent tolerance exponents, each playing a particular role in the proof. Specifically, ε denotes the tolerance exponent from the definition of the domain $\mathcal{D}^{\mathrm{abv}}$ (see (6.3.2) and (6.3.21) below); ξ and its multiples represent the target tolerance exponents for the local laws above the scale in (6.3.1). The exponent ζ appears in the below-the-scale part of the proof (Section 6.6). Multiples of $-\zeta$ are used in the exclusion estimate (6.6.9) and in the lower bound on $\rho N\eta$ in (6.6.8). The exponent δ refers to the step size used in various inductive arguments. In the sequel, we adhere to the following conventions:

$$\delta \ll \xi \ll \varepsilon, \quad \zeta \ll \xi, \quad \delta < \mu, \tag{6.3.3}$$

where $\mu > 0$ is the constant from Assumption 6.2.3 (ii). We also assume that the arbitrary exponent $\nu > 0$ is much smaller than the other tolerance exponents, that is, $\nu \ll \delta$ and $\nu \ll \zeta$.

6.3.1 Input: Global Laws

Let $\rho(z)$ be the harmonic extension to $\mathbb{H}$ of the scDOS corresponding to a solution of (6.2.3). Given small positive constants $\varepsilon, \xi > 0$, and a large constant $D > 0$, we define the global domain as

$$\mathcal{D}^{\mathrm{glob}} \equiv \mathcal{D}^{\mathrm{glob}}(D, \varepsilon, \xi, \rho) := \big\{ z := E + \mathrm{i}\eta \in \mathbb{H} : |E| \le N^D,\ N^{-1+\varepsilon} \le \eta \le N^D,\ \rho(z)^{-1}\eta \ge N^{-\xi/4} \big\}. \tag{6.3.4}$$

Effectively, the function $\rho(z)^{-1}\eta$ in (6.3.4) controls the proximity of the spectral parameter z to the support ρ.

Proposition 6.3.3. *Let H be a random matrix satisfying the Assumptions 6.2.1–6.2.5, and let $\rho(z)$ be the scDOS arising from the solution to the MDE* (6.2.3) *corresponding to H. Let $\mathcal{D}^{\mathrm{bdd}} := \mathcal{I} + [-c_M, c_M] + \mathrm{i}\mathbb{R} \subset \mathbb{C}$, where $\mathcal{I}$ is defined in* (6.2.10)*. Fix a large constant $D > 0$ and a tolerance exponent $0 < \xi < \frac{1}{10}\varepsilon$. Then the resolvent $G(z) := (H - z)^{-1}$ satisfies*

$$\Big| \big(G(z) - M(z)\big)_{\boldsymbol{xy}} \Big| \le N^{\xi}\Psi(z)\|\boldsymbol{x}\|\|\boldsymbol{y}\|, \tag{6.3.5a}$$

$$\Big| \big\langle \big(G(z) - M(z)\big)B \big\rangle \Big| \le N^{3\xi}\Psi(z)\sqrt{\frac{\langle z\rangle}{N\eta}}\|B\|_{\mathrm{hs}}, \tag{6.3.5b}$$

with very high probability, uniformly in $z := E + \mathrm{i}\eta \in \mathcal{D}^{\mathrm{glob}}(D, \varepsilon, \xi, \rho) \cap \mathcal{D}^{\mathrm{bdd}}$, for any deterministic vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and matrices B. Here the control parameter $\Psi(z)$ is defined as

$$\Psi(z) := \sqrt{\frac{\rho(z)}{\langle z\rangle^2 N\eta}} + \frac{1}{\langle z\rangle^2 N\eta}, \quad \eta := \operatorname{Im} z. \tag{6.3.6}$$

We prove Proposition 6.3.3 in Section 6.7.

6.3.2 Local law via Zigzag strategy: Proof of Theorem 6.3.2

6.3.2.1 Preliminaries: Two Random Matrix Flows

For any random matrix H, we define the covariance tensor Σ_H corresponding to H by its action on any deterministic matrix $X \in \mathbb{C}^{N\times N}$,

$$\Sigma_H[X] := \mathbf{E}\big[\mathrm{Tr}\big[(H-\mathbf{E}\,H)X\big](H-\mathbf{E}\,H)\big]. \tag{6.3.7}$$

Note that Σ_H is different from the self-energy operator (6.2.2), but they both carry equivalent information. Moreover, it is positive definite on the space of matrices equipped with the usual scalar product $(X,Y) = \langle X^*Y\rangle$ and we will denote by $\Sigma^{1/2}$ its square root.

Along the proof, we use two distinct flows in the space of $N\times N$ random matrices: the *zig-flow* (standard Ornstein–Uhlenbeck process), defined as

$$\mathrm{d}H_t = -\frac{1}{2}H_t\mathrm{d}t + \frac{\mathrm{d}\mathfrak{B}_t}{\sqrt{N}}, \qquad t\ge 0; \tag{6.3.8}$$

and the *zag-flow* (modified Ornstein–Uhlenbeck process), distinguished by the superscript t,

$$\mathrm{d}H^t = -\frac{1}{2}\big(H^t - \mathfrak{E}\,H^t\big) + \Sigma_{H^0}^{1/2}\big[\mathrm{d}\mathfrak{B}_t\big], \qquad t\ge 0, \tag{6.3.9}$$

where Σ_{H^0} is the covariance tensor of H^0, defined according to (6.3.7). In both (6.3.8) and (6.3.9), $\mathfrak{B}_t$ denotes the real symmetric or complex Hermitian Brownian motion, depending on the symmetry class of H.

Note that along the zig-flow (6.3.8), the covariance tensor $\Sigma_t := \Sigma_{H_t}$, corresponding to H_t via (6.3.7), satisfies the ordinary differential equation

$$\mathrm{d}\Sigma_t = \big(-\Sigma_t + \Sigma_{\mathrm{G}}\big)\mathrm{d}t, \tag{6.3.10}$$

where Σ_{G} is the covariance tensor of a GOE/GUE matrix in the same symmetry class as H. That is $\Sigma_{\mathrm{G}}[X] = N^{-1}X$ in the complex Hermitian case, and $\Sigma_{\mathrm{G}}[X] = N^{-1}(X + X^{\mathrm{t}})$ in the real-symmetric case, where X^{t} denotes the transpose of X. On the other hand, along the zag-flow (6.3.9), the expectation and the covariance tensor of H_t (and hence the self-energy $\mathcal{S}_{H_t}$) are preserved. Therefore, the deterministic approximation M remains unchanged along the zag-flow.

For any $t\ge 0$, we define the flow maps $\mathfrak{F}^t_{\mathrm{zig}}$ and $\mathfrak{F}^t_{\mathrm{zag}}$ on the space of probability distribution $\mathcal{P}(\mathbb{C}^{N\times N})$ by

$$\mathfrak{F}^t_{\mathrm{zig}}\big[H\big] := H_t, \quad \text{where } H_t \text{ solves (6.3.8) with the initial condition } H_0 = H. \tag{6.3.11}$$

$$\mathfrak{F}^t_{\mathrm{zag}}\big[H\big] := H^t, \quad \text{where } H^t \text{ solves (6.3.9) with the initial condition } H^0 = H. \tag{6.3.12}$$

The key relation between the flow maps $\mathfrak{F}^t_{\mathrm{zig}}$ and $\mathfrak{F}^t_{\mathrm{zag}}$ is captured by the following lemma.

Lemma 6.3.4 (Flow Distribution Surjectivity)**.** *Let H be a random matrix satisfying the fullness condition* (6.2.9) *with a constant $0<c<1$, then there exists a random matrix $\mathfrak{H}_{c,t}(H)$ such that*

$$\mathfrak{F}^t_{\mathrm{zig}}\big[\mathfrak{H}_{c,t}(H)\big] \overset{d}{=} \mathfrak{F}^{s(t)}_{\mathrm{zag}}\big[H\big], \quad 0\le t\le -\log(1-c), \tag{6.3.13}$$

where the function $s(t)\equiv s_c(t)$ is defined as

$$s(t)\equiv s_c(t) := \log c - \log\big(c-1+\mathrm{e}^{-t}\big), \tag{6.3.14}$$

and satisfies

$$s(t)\le 2c^{-1}t, \quad 0\le t\le c/2. \tag{6.3.15}$$

We defer the proof of Lemma 6.3.4 to the Appendix 6.A.

6.3.2.2 Zigzag approach: Iterative application of the characteristic flow and GFT

We consider the time-dependent matrix Dyson equation (MDE),

$$-M_t(z)^{-1} = z - A_t + \mathcal{S}_t\big[M_t(z)\big], \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad (\operatorname{Im} z)\operatorname{Im} M_t(z) > 0, \tag{6.3.16}$$

where the data pair $(A_t, \mathcal{S}_t)$ is given as the unique solutions to the differential equations

$$\mathrm{d}A_t = -\frac{1}{2}A_t\mathrm{d}t\,, \quad \mathrm{d}\mathcal{S}_t = (-\mathcal{S}_t + \langle\cdot\rangle)\mathrm{d}t\,. \tag{6.3.17}$$

with the *terminal conditions* $A_T = A = \mathbf{E}\,H$ and $\mathcal{S}_T = \mathcal{S} = \mathbf{E}[(H-A)(\,\cdot\,)(H-A)]$, respectively.

Given $M_t(z)$, we consider the *characteristic ODE* for the time dependent spectral parameter $z_t \in \mathbf{C}$ (see Figure 6.3.1),

$$\mathrm{d}z_t = -\frac{1}{2}z_t\mathrm{d}t - \big\langle M_t(z_t)\big\rangle\mathrm{d}t. \tag{6.3.18}$$

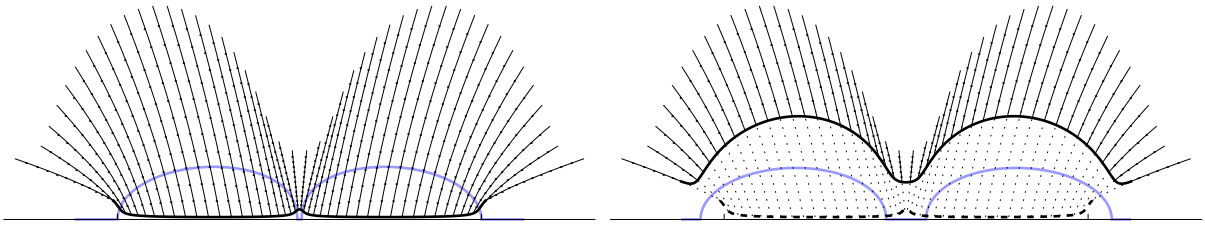

Figure 6.3.1: The left panel depicts several trajectories of the flow (6.3.18) that terminate at the *scale curve* $\rho_T(z)N\operatorname{Im} z = c$ (solid black line), while the the graph of scDOS ρ_T is superimposed in light blue. The right panel depicts trajectories up to an intermediate time $t \in (0,T)$ with their continuations beyond t shown as thin dotted lines. The pre-image of the scale curve at the time t is depicted as a solid black line, and the scale curve itself is depicted as a dashed black line. The graph of scDOS ρ_t is superimposed in light blue. In both panels, the black markers along the trajectories of (6.3.18) are evenly spaced in time.

By trivial ODE arguments, for all $0 \le s \le t$, the corresponding (inverse) flow map $\varphi_{s,t} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is defined uniquely by

$$\varphi_{s,t}(z_t) := z_s, \quad \text{where } z_s \text{ solves (6.3.18).} \tag{6.3.19}$$

It can be directly checked that along the trajectories of (6.3.18), the solution to the time-dependent MDE (6.3.16) satisfies

$$\mathrm{d}M_t(z_t) = \frac{1}{2}M_t(z_t)\mathrm{d}t. \tag{6.3.20}$$

Lemma 6.3.5 (Time-Dependent Domains)**.** *There exist a constant $C' \sim 1$ such that for any constant $0 < c' \le \pi$ and any terminal time $0 < T \lesssim 1$, the time-dependent domains $\mathcal{D}_t^{\mathrm{abv}}$, $t \in [0,T]$, (see Figure 6.3.2), defined as*

$$\begin{aligned}\mathcal{D}_t^{\mathrm{abv}} \equiv \mathcal{D}_t^{\mathrm{abv}}(\varepsilon, C_L, c', T) := \big\{z := E + \mathrm{i}\eta \in \mathbb{H} :\ & \rho_t(z)N\eta \ge N^{\varepsilon},\ |E| \vee \eta \le C_L + C'\cdot(T-t),\\ & \rho_t(z)^{-1}\eta \ge c'\cdot\big(N^{-1+\varepsilon} + T - t\big)\big\},\end{aligned} \tag{6.3.21}$$

satisfy $\varphi_{s,t}(\mathcal{D}_t^{\mathrm{abv}}) \subset \mathcal{D}_s^{\mathrm{abv}}$ for all $0 \le s \le t \le T$, where $\varphi_{s,t}$ is the flow map defined in (6.3.19)*.*

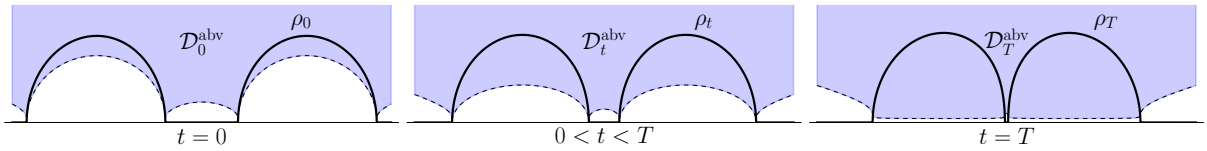

Figure 6.3.2: The time-dependent domain $\mathcal{D}_t^{\mathrm{abv}}$, defined in (6.3.21), is illustrated in blue at three distinct times: the initial time $t = 0$ (left), an intermediate time $0 < t < T$ (center), and the terminal time $t = T$ (right). The graph of the scDOS ρ_t is superimposed in black on each panel (not to scale).

We defer the proof of Lemma 6.3.5 to Appendix 6.A.

As in (6.3.4), the function $\rho_t(z)^{-1}\eta$ in the definition (6.3.21) effectively controls the distance between z and the support of ρ_t. Therefore the time-dependent family of domains $\mathcal{D}_t^{\mathrm{abv}}$ effectively interpolates between the global regime $\mathcal{D}^{\mathrm{glob}}$ and the final target domain $\mathcal{D}^{\mathrm{abv}}$. Indeed, since $\rho(z) \lesssim 1$, by choosing the constant $c' \sim 1$ in (6.3.21) small enough, we can guarantee that $\mathcal{D}^{\mathrm{abv}} \subset \mathcal{D}_T^{\mathrm{abv}}$, where we recall that $\mathcal{D}^{\mathrm{abv}}$ is defined in (6.3.2). On the other hand, it follows from (6.3.4) that by choosing

$$T := CN^{-\xi/4}, \tag{6.3.22}$$

with a sufficiently large constant $C \gtrsim 1$, we can guarantee that $\mathcal{D}_0^{\mathrm{abv}} \subset \mathcal{D}^{\mathrm{glob}}$, where $\mathcal{D}^{\mathrm{glob}}$ is defined in (6.3.4).

We conduct the proof inductively. Fix a tolerance exponent $0 < \xi \ll \varepsilon$, a step size $0 < \delta \ll \xi$ (recall (6.3.3)). For the terminal time T chosen as in (6.3.22), let K be the smallest integer such that $N^{-K\delta}T \le N^{-1+\varepsilon}$, and define a sequence of times $\{t_k\}_{k=0}^K$ as

$$t_0 := 0, \quad t_k := T - N^{-k\delta}T, \quad k \in \{1, \dots, K-1\}, \quad t_K := T. \tag{6.3.23}$$

Let $\{\Delta t_k\}_{k=1}^K$ denote the difference sequence of $\{t_k\}_{k=0}^K$, that is

$$\Delta t_k := t_k - t_{k-1}, \quad k \in \{1, \dots, K\}. \tag{6.3.24}$$

Let Σ_t solve the equation (6.3.10) with the terminal condition $\Sigma_T = \Sigma$, where Σ, defined via (6.3.7), is the covariance tensor of the target matrix H, for which we eventually prove the local laws in Theorem 6.2.8. Observe that for all $0 \le t \le T$, the solution Σ_t satisfies

$$\Sigma_t \ge \widetilde{c}\,\Sigma_{\mathrm{G}}, \quad \widetilde{c} := \frac{c_{\mathrm{flat}}}{2} \wedge 1, \tag{6.3.25}$$

where c_{flat} is the constant in Assumption 6.2.4. Given the target random matrix ensemble H, we construct two sequences of random matrices, $\{H_k\}_{k=0}^K$ and $\{H^k\}_{k=1}^K$ recursively by

$$H_K := H, \quad H^k := \mathfrak{F}_{\mathrm{zag}}^{s(\Delta t_k)}[H_k], \quad H_{k-1} := \mathfrak{H}_{\widetilde{c},\Delta t_k}(H^k), \quad k \in \{1, \dots, K\}, \tag{6.3.26}$$

where $s(t) := s_{\widetilde{\alpha}}(t)$ and $\mathfrak{H}_{\widetilde{c},\Delta t_k}$ are given by Lemma 6.3.4, and $\widetilde{c}$ is the constant in (6.3.25). It follows by a simple backward inductive argument starting at $k = K$ that the covariance tensor of both H_k and H^k is given by Σ_{t_k}, hence by (6.3.25), H_{k-1} is well-defined.

Proposition 6.3.6 (Zig Step). *Fix $k \in \{1, \dots, K\}$, and denote*

$$G_t(z) := \left(\mathfrak{F}_{\mathrm{zig}}^{t-t_{k-1}}[H_{k-1}] - z\right)^{-1}, \quad t_{k-1} \le t \le t_k. \tag{6.3.27}$$

Assume that for some $\xi, \nu > 0$ with $\xi + K\nu \ll \varepsilon$, and $\ell \le 2k$, the resolvent G_t satisfies the local laws (6.3.1) *with data $(\mathcal{D}_t^{\mathrm{abv}}, \xi + \ell\nu)$ at time $t = t_{k-1}$, then the resolvent G_t satisfies the local laws* (6.3.1) *with data $(\mathcal{D}_t^{\mathrm{abv}}, \xi + (\ell+1)\nu)$ uniformly in $t \in [t_{k-1}, t_k]$.*

[7] Note that the zag-flow (6.3.9) acts in the direction opposite to the dashed arrows, i.e., the size of the Gaussian components increases along both zig- and zag- matrix flows, (6.3.8) and (6.3.9), respectively.

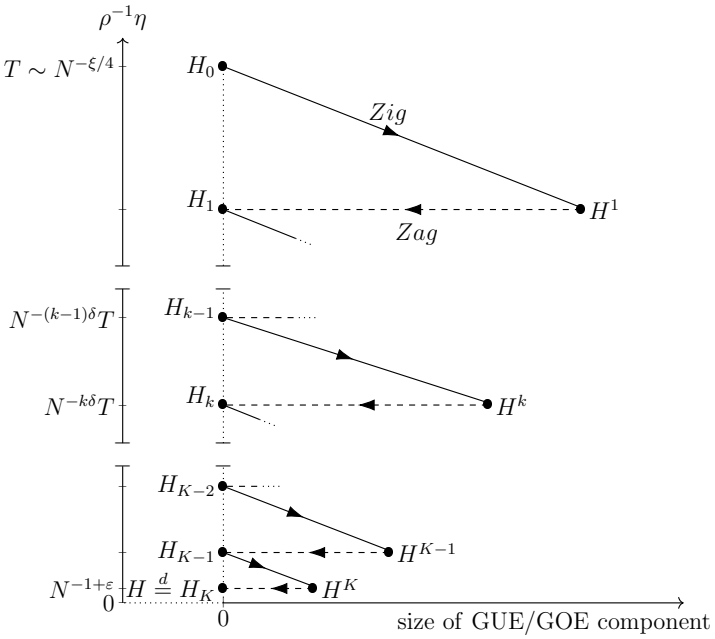

Figure 6.3.3: Schematic representation of the Zigzag induction. The random matrices H_k, H^k, as defined in (6.3.26), are situated within an abstract coordinate system. The horizontal axis represents the size of the Gaussian component, while the vertical axis indicates the lower bound on $\rho(z)^{-1}\eta$ in the domains, c.f. (6.3.21), where we prove the local laws (6.3.1). Solid arrows denote applications of Proposition 6.3.6 (referred to as *Zig* steps), and dashed arrows[7] indicate applications of Proposition 6.3.7 (*Zag* steps).

Proposition 6.3.7 (Zag Step)**.** *Fix $k \in \{1, \dots, K\}$. Let $s_k := s(\Delta t_k)$ be the time defined in* (6.3.14)*, let H_k be the random matrix defined in* (6.3.26)*, and denote*

$$G^s(z) := \big(\mathfrak{F}^s_{\mathrm{zag}}[H_k] - z\big)^{-1}, \quad 0 \le s \le s_k. \tag{6.3.28}$$

Assume that for some $\xi, \nu > 0$ with $\xi + K\nu \ll \varepsilon$, and $\ell \le 2k$, the resolvent G^s satisfies the local laws (6.3.1) *with data $(\mathcal{D}^{\mathrm{abv}}_{t_k}, \xi + \ell\nu)$ at time $s = s_k$, then G^s satisfies the local laws* (6.3.1) *with data $(\mathcal{D}^{\mathrm{abv}}_{t_k}, \xi + (\ell+1)\nu)$ uniformly in $s \in [0, s_k]$.*

Having formulated the cardinal steps of the Zigzag strategy, we now put them together to prove our key theorem on the local laws above the scale. Note that in the above the scale regime $\rho(z)N\eta \ge N^{\varepsilon}$, the term $1/(N\eta)$ in the isotropic bound is (6.3.1) is dominated by $\sqrt{\rho/(N\eta)}$, and hence will be ignored in Sections 6.4 and 6.5.

Proof of Theorem 6.3.2. Recall our choice of the constant $c' \sim 1$ in (6.3.21) and the terminal time $T \sim N^{-\xi/4}$ in (6.3.22) that guarantees the inclusions $\mathcal{D}^{\mathrm{abv}}_0 \subset \mathcal{D}^{\mathrm{glob}}$ and $\mathcal{D}^{\mathrm{abv}} \subset \mathcal{D}^{\mathrm{abv}}_T$. Therefore, Proposition 6.3.3 implies that the resolvent $G_0(z) := (H_0 - z)^{-1}$ of a random matrix H_0, defined in (6.3.26), satisfies the local laws (6.3.1) with data $(\mathcal{D}^{\mathrm{abv}}_0, \xi)$. Using Propositions 6.3.6 and 6.3.7 in tandem K times, we prove by forward induction on k that for any $\nu > 0$, the resolvent $G_k(z) := (H_k - z)^{-1}$ satisfies the local laws (6.3.1) with data $(\mathcal{D}^{\mathrm{abv}}_{t_k}, \xi + 2k\nu)$, for all $k \in \{1, \dots, K\}$. Since $H_K = H$ and $\mathcal{D}^{\mathrm{abv}}_{t_K} = \mathcal{D}^{\mathrm{abv}}_T \supset \mathcal{D}^{\mathrm{abv}}$, this concludes the proof of Theorem 6.3.2. □

Remark 6.3.8 (On Locality of Assumption 6.2.5)**.** *In the case of a general set of admissible energies $\mathcal{I}$, defined in* (6.2.10)*, our proof holds verbatim, except the spectral domains $\mathcal{D}^{\mathrm{glob}}, \mathcal{D}^{\mathrm{abv}}, \mathcal{D}^{\mathrm{abv}}_t$ used along the proof have to be restricted. More precisely, we need the following modifications:*

(i) we restrict the domain $\mathcal{D}^{\mathrm{abv}}_t$, defined in (6.3.21)*, by intersecting it with the region*

$$\mathcal{D}^{\mathrm{bdd}}_t := \big\{z \in \mathbb{C} : \operatorname{dist}(\operatorname{Re} z, \mathcal{I}) \le c_M/2 + C' \cdot (T - t)\big\}, \quad 0 \le t \le T; \tag{6.3.29}$$

(ii) we restrict the domain $\mathcal{D}^{\mathrm{abv}}$, defined in (6.3.2)*, by intersecting it with $\mathcal{D}^{\mathrm{bdd}}_T$;*

(iii) we restrict the global domain $\mathcal{D}^{\mathrm{glob}}$, defined in (6.3.4) *by intersecting it with $\{z \in \mathbb{C} : \operatorname{dist}(\operatorname{Re} z, \mathcal{I}) \le \frac{3}{4}c_M\}$.*

6.3.3 Proofs of Corollaries 6.2.10–6.2.11 and Theorem 6.2.13

In this section, we deduce eigenvector delocalization, band rigidity and eigenvalue rigidity, as well as cusp universality from the local law in Theorem 6.2.8. These arguments are essentially independent of the correlation structure of the random matrix, so we only refer to analogous proofs, which can easily be adjusted to our case with straightforward modifications.

Proof of Corollary 6.2.10 on eigenvector delocalization. As usual, eigenvector delocalization is an immediate consequence of the optimal isotropic local law from Theorem 6.2.8 for $\operatorname{Im} G$; see [243, Proof of Corollary 2.4] or [15, Proof of Corollary 1.14] for this argument. □

Proof of Corollary 6.2.11 on band rigidity and eigenvalue rigidity. The proof of band rigidity was first done for correlated matrices in [23, Proof of Corollary 2.5 in Section 5] but with $\operatorname{dist}(e_0, \operatorname{supp}\rho) \gtrsim 1$. The adjustments for $\operatorname{dist}(e_0, \operatorname{supp}\rho) \geq N^{\theta}\eta_{\mathrm{f}}(e_0)$ are carried out in [244, Proof of Corollary 2.6] for the case of Wigner-type matrices (i.e. without correlations). This argument immediately translates to our setting, hence we omit the details for brevity.

Armed with band rigidity as in (6.2.17), the proof of Corollary 6.2.11 (b) is conducted in the same way as in [15, Proofs of Corollaries 1.10 and 1.11] or [244, Proof of Corollary 2.6]. □

Proof of Theorem 6.2.13 on cusp universality. Given the optimal local law in Theorem 6.2.8, universality at the cusp follows by the *three-step strategy*. This has already been worked out in the general correlated case in both the complex Hermitian [244] and real symmetric [155] symmetry class. More precisely, in [155, Section 3] it is spelled out that only the local law in this paper (as the first step of the three-step strategy) required restricting to Wigner-type matrices with independent entries. Our Theorem 6.2.8 provides the necessary local law for correlated matrices. □

6.4 Characteristic flow: Proof of Proposition 6.3.6

First, we collect the necessary properties of the solution M_t to the time-dependent MDE (6.3.16).

Lemma 6.4.1 (Preliminary bounds on M_t)**.** *Let $(A, \mathcal{S})$ be a data-pair satisfying the Assumptions 6.2.1, 6.2.4, and 6.2.5. Then there exists a threshold $T_* \sim 1$ such that for any terminal time $0 < T < T_*$, the solution M_t to the time-dependent MDE* (6.3.16)*, with the terminal condition on the data pair $(A_T, \mathcal{S}_T) = (A, \mathcal{S})$, satisfies*

$$\|M_t(z)\| \lesssim 1, \quad c\rho_t(z) \leq \operatorname{Im} M_t(z) \leq C\rho_t(z), \tag{6.4.1}$$

uniformly in z with $\operatorname{Re} z \in \mathcal{I}$, where $\mathcal{I}$ is the set of admissible energies from (6.2.10)*. Here the second inequality holds in the sense of quadratic forms, with $1 \lesssim c \leq C \lesssim 1$.*

Essentially, at the terminal time $t = T$, the bounds (6.4.1) follow from the assumptions of the lemma, while at all other times $0 \leq t < T$, the equations (6.3.17) guarantee that the data pair $(A_t, \mathcal{S}_t)$ constitutes only a small perturbation around $(A_T, \mathcal{S}_T)$. We give a more detailed proof of Lemma 6.4.1 in Appendix 6.A.

Equipped with Lemma 6.4.1, we are ready to prove Proposition 6.3.6. We conduct the proof in the complex Hermitian case, the obvious modifications in the real symmetric case[8] are left to the reader. Throughout the proof we consider the step index k to be fixed, and hence omit it from the subscripts.

[8] For a detailed treatment of the real symmetric case in the setting of standard Wigner matrices, we refer the reader to Section 3.4.4. The modifications for more general ensembles are analogous.

It suffices to prove that the resolvent G_t satisfies the local laws (6.3.1) with data $(\mathcal{D}_t^{\mathrm{abv}}, \xi + (\ell+1)\nu)$ for any fixed $t := t_{\mathrm{final}} \in [t_{k-1}, t_k]$ and $z \in \mathcal{D}_{t_{\mathrm{final}}}^{\mathrm{abv}}$, since uniformity in t and z can be obtained by a simple grid argument[9]. Let $t_{\mathrm{init}} := t_{k-1}$, and for all $t \in [t_{\mathrm{init}}, t_{\mathrm{final}}]$, let $z_t := \varphi_{t,t_{\mathrm{final}}}(z)$, where the map φ is defined in (6.3.19). It follows from Lemma 6.3.5 that $z_t \in \mathcal{D}_t^{\mathrm{abv}}$ for all $t \in [t_{\mathrm{init}}, t_{\mathrm{final}}]$. We denote $G_t := (H_t - z_t)^{-1}$, and $M_t := M_t(z_t)$, where M_t is the solution to (6.3.16).

Using Itô's formula, we deduce that for any deterministic $N \times N$ matrix B,

$$\mathrm{d}\big\langle (G_t - M_t)B \big\rangle = \bigg(\frac{1}{2}\big\langle (G_t - M_t)B \big\rangle + \big\langle G_t - M_t \big\rangle \big\langle G_t^2 B \big\rangle \bigg) \mathrm{d}t + \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab} \big\langle G_t B \big\rangle \mathrm{d}\big(\mathfrak{B}_t\big)_{ab}, \tag{6.4.2}$$

where $\partial_{ab} := \partial_{H_{ab,t}}$ denotes the partial derivative with respect to the matrix entry $H_{ab,t}$. In particular, for a fixed pair of deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$ with $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$, setting $B := N\boldsymbol{y}\boldsymbol{x}^*$ we obtain

$$\mathrm{d}\big(G_t - M_t\big)_{\boldsymbol{x}\boldsymbol{y}} = \bigg(\frac{1}{2}\big(G_t - M_t\big)_{\boldsymbol{x}\boldsymbol{y}} + \big\langle G_t - M_t \big\rangle \big(G_t^2\big)_{\boldsymbol{x}\boldsymbol{y}} \bigg) \mathrm{d}t + \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab} \big(G_t\big)_{\boldsymbol{x}\boldsymbol{y}} \mathrm{d}\big(\mathfrak{B}_t\big)_{ab}. \tag{6.4.3}$$

First, we prove that the resolvent $G_{t_{\mathrm{final}}}$ satisfies the isotropic local law and averaged local law in (6.3.1) for $B := I$ with data $(\mathcal{D}_{t_{\mathrm{final}}}^{\mathrm{abv}}, \xi + (\ell + \frac{1}{2})\nu)$. Define a set of deterministic vectors $\mathcal{V} := \{\boldsymbol{x}, \boldsymbol{y}\}$. Define the stopping time τ

$$\begin{aligned} \tau := \sup\bigg\{ t_{\mathrm{init}} \le t \le t_{\mathrm{final}} : \sup_{t_{\mathrm{init}} \le s \le t} \max_{\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}} \Big| \sqrt{\rho_s(z_s)^{-1} N \eta_s} \big(G_s - M_s\big)_{\boldsymbol{u}\boldsymbol{v}} \Big| \le N^{\xi + (\ell + \frac{1}{2})\nu} \bigg\} \\ \wedge \sup\bigg\{ t_{\mathrm{init}} \le t \le t_{\mathrm{final}} : \sup_{t_{\mathrm{init}} \le s \le t} \Big| N\eta_s \big\langle G_s - M_s \big\rangle \Big| \le N^{3\xi + 3(\ell + \frac{1}{2})\nu} \bigg\}, \end{aligned} \tag{6.4.4}$$

where we denote $\eta_t := \operatorname{Im} z_t > 0$.

Computing the quadratic variation of the martingale term in (6.4.2), we obtain

$$\begin{aligned} \bigg[\int_{t_{\mathrm{init}}}^{\cdot} \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab} \big\langle G_s \big\rangle \mathrm{d}\big(\mathfrak{B}_s\big)_{ab} \bigg]_{t \wedge \tau} &\le \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{\big\langle (\operatorname{Im} G_s)^2 \big\rangle}{N^2 \eta_s^2} \mathrm{d}s \le \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{\big\langle \operatorname{Im} G_s \big\rangle}{N^2 \eta_s^3} \mathrm{d}s \\ &\le \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{\big\langle \operatorname{Im} M_s \big\rangle + \frac{1}{2}\eta_s}{N^2 \eta_s^3} \mathrm{d}s + \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{\big\langle \operatorname{Im} G_s - \operatorname{Im} M_s \big\rangle}{N^2 \eta_s^3} \mathrm{d}s, \end{aligned} \tag{6.4.5}$$

where in the penultimate step we used the norm bound $\|\operatorname{Im} G_s\| \le \eta_s^{-1}$, and in the ultimate step we used the fact that $\eta_s > 0$ in $\mathcal{D}_s^{\mathrm{abv}}$. We now estimate the two integrals in the last line of (6.4.5) separately. For the first integral, we use the imaginary part of (6.3.18) to obtain

$$\int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{\big\langle \operatorname{Im} M_s \big\rangle + \frac{1}{2}\eta_s}{N^2 \eta_s^3} \mathrm{d}s = \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{-\mathrm{d}\eta_s}{N^2 \eta_s^3} \le \frac{1}{N^2 \eta_{t \wedge \tau}^2}. \tag{6.4.6}$$

For the second integral, we use the definition (6.4.4) of the stopping time τ, and the imaginary part of (6.3.20) to deduce that

$$\bigg| \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{\big\langle \operatorname{Im} G_s - \operatorname{Im} M_s \big\rangle}{N^2 \eta_s^3} \mathrm{d}s \bigg| \lesssim \bigg| \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{N^{3\xi + 3(\ell + \frac{1}{2}\nu)}}{N^3 \eta_s^4} \mathrm{d}s \bigg| \lesssim \bigg| \int_{t_{\mathrm{init}}}^{t \wedge \tau} \frac{N^{3\xi + 3(\ell + \frac{1}{2}\nu)}}{N^3 \eta_s^4 \langle \operatorname{Im} M_s \rangle} \mathrm{d}\eta_s \bigg| \lesssim \frac{N^{-\varepsilon + 3\xi + 3(\ell + \frac{1}{2}\nu)}}{N^2 \eta_{t \wedge \tau}^2}, \tag{6.4.7}$$

[9] The grid argument relies on two straightforward observations: First, the resolvent $G_t(z)$ with $|\operatorname{Im} z| \ge N^{-1}$ – and, therefore, all quantities we consider – are Lipschitz continuous with a Lipschitz constant $\lesssim N^C$ for some $C > 0$ both in z and in t. Second, for any $C > 0$, the intersection of N^C-many very-high-probability events also occurs with very high probability. Therefore, a uniform very-high probability bounds are first established over a sufficiently fine N^{-C} grid in the domain of z or t, and then extended to the entire domain by Lipschitz continuity.

where in the last inequality we used that $\langle \operatorname{Im} M_s \rangle N\eta_s \sim \rho_s(z_s) N\eta_s \gtrsim N^{\varepsilon}$ for all $t_{\mathrm{init}} \le s \le t_{\mathrm{final}}$ by (6.3.21).

Therefore, using the path-wise Burkholder-Davis-Gundy inequality (see Lemma 5.6 in [162] and Appendix B.6, Eq. (18) in [523]) and the fact that $\xi + K\nu \ll \varepsilon$, we deduce that, with very high probability

$$\max_{t_{\mathrm{init}} \le s \le t} \left| \int_{t_{\mathrm{init}}}^{s\wedge\tau} \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab} \langle G_s \rangle \mathrm{d}\big(\mathfrak{B}_s\big)_{ab} \right| \le \frac{N^{\nu}}{N\eta_{t\wedge\tau}}. \tag{6.4.8}$$

Next, using the Ward identity and the definition (6.4.4) of the stopping time τ, we obtain

$$\left| \frac{1}{2} + \langle G_s^2 \rangle \right| \le \frac{1}{2} + \frac{\langle \operatorname{Im} G_s \rangle}{\eta_s} \le -\frac{1}{\eta_s} \frac{\mathrm{d}\eta_s}{\mathrm{d}s} + \frac{N^{3\xi+3(\ell+\frac{1}{2})\nu}}{N\eta_s^2} \le -\frac{1}{\eta_s} \frac{\mathrm{d}\eta_s}{\mathrm{d}s} \left(1 + CN^{-\varepsilon+3\xi+3(\ell+\frac{1}{2})\nu} \right), \tag{6.4.9}$$

where in the last inequality we used the imaginary part of (6.3.18) and the bound $\langle \operatorname{Im} M_s \rangle N\eta_s \sim \rho_s(z_s) N\eta_s \gtrsim N^{\varepsilon}$ from (6.3.21). By integrating (6.4.2), it follows from the assumption of Proposition 6.3.6 at $t = t_{k-1} = t_{\mathrm{init}}$ and (6.4.8) that the bound

$$\left| \langle G_{t\wedge\tau} - M_{t\wedge\tau} \rangle \right| \le -\left(1 + CN^{-\varepsilon+\xi+(\ell+\frac{1}{2})\nu} \right) \left(\int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{\left| \langle G_s - M_s \rangle \right|}{\eta_s} \frac{\mathrm{d}\eta_s}{\mathrm{d}s} \mathrm{d}s + \frac{N^{3\xi+3\ell\nu}}{N\eta_{t\wedge\tau}} \right), \tag{6.4.10}$$

holds with very high probability. Here we used that $\xi \ll \varepsilon$ from (6.3.3), and the assumption that $\ell\nu \le 2K\nu \ll \varepsilon$. Applying the Gronwall inequality yields the very-high-probability bound,

$$\left| \langle G_{t\wedge\tau} - M_{t\wedge\tau} \rangle \right| \le \frac{N^{3\xi+3(\ell+\frac{1}{4})\nu}}{N\eta_{t\wedge\tau}}, \tag{6.4.11}$$

uniformly in $t_{\mathrm{init}} \le t \le t_{\mathrm{final}}$.

Similarly, computing the quadratic variation of the martingale term in (6.4.3), we obtain

$$\begin{aligned} \left[\int_{t_{\mathrm{init}}}^{\cdot} \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab} \big(G_s\big)_{\boldsymbol{uv}} \mathrm{d}\big(\mathfrak{B}_s\big)_{ab} \right]_{t\wedge\tau} &\le \int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{\big(\operatorname{Im} G_s\big)_{\boldsymbol{uu}} \big(\operatorname{Im} G_s\big)_{\boldsymbol{vv}}}{N\eta_s^2} \mathrm{d}s \\ &\lesssim \int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{\rho_s(z_s)^2}{N\eta_s^2} \left(1 + \frac{N^{\xi+(\ell+1)\nu}}{\sqrt{\rho_s(z_s) N\eta_s}} \right)^2 \mathrm{d}s \lesssim \frac{\rho_{t\wedge\tau}(z_{t\wedge\tau})}{N\eta_{t\wedge\tau}}, \end{aligned} \tag{6.4.12}$$

where we used the imaginary part of (6.3.20) to obtain $\rho_s(z_s) \sim \rho_{t\wedge\tau}(z_{t\wedge\tau})$. Therefore, using the path-wise Burkholder-Davis-Gundy inequality, we deduce the very-high-probability bound

$$\max_{t_{\mathrm{init}} \le s \le t} \left| \int_{t_{\mathrm{init}}}^{s\wedge\tau} \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab} \big(G_s\big)_{\boldsymbol{uv}} \mathrm{d}\big(\mathfrak{B}_s\big)_{ab} \right| \le N^{\nu} \sqrt{\frac{\rho_{t\wedge\tau}(z_{t\wedge\tau})}{N\eta_{t\wedge\tau}}}. \tag{6.4.13}$$

Moreover, using the Ward identity within the Schwarz estimate $\left| \big(G_s^2\big)_{\boldsymbol{uv}} \right| \le \sqrt{(|G_s|^2)_{\boldsymbol{uu}} (|G_s|^2)_{\boldsymbol{vv}}}$, together with (6.4.1), (6.4.4), and the relations $\xi + K\nu \ll \varepsilon$, we deduce that

$$\left| \big(G_s^2\big)_{\boldsymbol{uv}} \right| \lesssim \rho_s(z_s) \left(1 + \frac{N^{\xi+(\ell+\frac{1}{2})\nu}}{\sqrt{\rho_s(z_s) N\eta_s}} \right) \lesssim \rho_s(z_s). \tag{6.4.14}$$

Therefore, from (6.3.21) and the bound (6.4.11), we conclude that

$$\left| \int_{t_{\mathrm{init}}}^{t\wedge\tau} \langle G_s - M_s \rangle \big(G_s^2\big)_{\boldsymbol{uv}} \mathrm{d}s \right| \lesssim \int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{N^{3\xi+3(\ell+\frac{1}{2})\nu}}{N\eta_s} \frac{\rho_s(z_s)}{\eta_s} \mathrm{d}s \le \frac{N^{3\xi+3(\ell+1)\nu}}{N^{\varepsilon/2}} \sqrt{\frac{\rho_{t\wedge\tau}(z_{t\wedge\tau})}{N\eta_{t\wedge\tau}}}. \tag{6.4.15}$$

Integrating (6.4.3), and combining the assumption of Proposition 6.3.6 at time $t = t_{k-1} = t_{\mathrm{init}}$, (6.4.13) and (6.4.15) yields

$$\left|\left(G_{t\wedge\tau} - M_{t\wedge\tau}\right)_{\boldsymbol{uv}}\right| \le N^{\xi+(\ell+\frac{1}{2})\nu}\sqrt{\frac{\rho_{t\wedge\tau}(z)}{N\eta_{t\wedge\tau}}}, \tag{6.4.16}$$

uniformly in $t_{\mathrm{init}} \le t \le t_{\mathrm{final}}$, with very high probability, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. Note that the term $\frac{1}{2}(G_t - M_t)_{\boldsymbol{uv}}$ on the right-hand side of (6.4.3) and can be removed by differentiating $\mathrm{e}^{-t/2}(G_t - M_t)_{\boldsymbol{uv}}$ with the harmless prefactor $\mathrm{e}^{-t/2} = 1 + \mathcal{O}(T)$.

Hence, using (6.4.11) and (6.4.16), we conclude that $\tau = t_{\mathrm{final}}$ with very high probability, therefore establishing the isotropic local law and averaged local law in (6.3.1) for $B := I$ with data $(\mathcal{D}^{\mathrm{abv}}_{t_{\mathrm{final}}}, \xi + (\ell + \frac{1}{2})\nu)$.

For a general observable $B \in \mathbb{C}^{N\times N}$, we use the bound (6.4.11) as input to obtain the very-high-probability estimate

$$\left|\left\langle G_s - M_s\right\rangle\left\langle G_s^2 B\right\rangle\right| \le \frac{N^{3\xi+3(\ell+\frac{1}{4})\nu}}{N\eta_s}\frac{\left\langle \operatorname{Im} G_s\right\rangle^{1/2}\left\langle \operatorname{Im} G_s BB^*\right\rangle^{1/2}}{\eta_s} \lesssim \frac{N^{3\xi+3(\ell+\frac{1}{4})\nu}}{N\eta_s}\frac{\rho_s(z_s)}{\eta_s}\left\|B\right\|_{\mathrm{hs}}. \tag{6.4.17}$$

uniformly in $t_{\mathrm{init}} \le s \le t_{\mathrm{final}}$. Here, in the last step we used the isotropic bound (6.4.16) for the eigenvectors $\boldsymbol{v}_j$ of BB^*, corresponding to the eigenvalues $|\sigma_j|^2$, to conclude that, with very high probability,

$$\left\langle \operatorname{Im} GBB^*\right\rangle = \frac{1}{N}\sum_j |\sigma_j|^2 (\operatorname{Im} G)_{\boldsymbol{v}_j\boldsymbol{v}_j} \lesssim \rho_s(z_s)\|B\|_{\mathrm{hs}}^2. \tag{6.4.18}$$

Similarly, using (6.4.18), we estimate the quadratic variation of the corresponding martingale term in (6.4.2) for general B,

$$\left[\int_{t_{\mathrm{init}}}^{\cdot} \frac{1}{\sqrt{N}}\sum_{ab}\partial_{ab}\left\langle G_s B\right\rangle \mathrm{d}\left(\mathfrak{B}_s\right)_{ab}\right]_{t\wedge\tau} \lesssim \frac{N^{\nu}}{N^2\eta_{t\wedge\tau}^2}\left\|B\right\|_{\mathrm{hs}}. \tag{6.4.19}$$

Combining (6.4.2), (6.4.17), and (6.4.19), we conclude that the resolvent $G_{t_{\mathrm{final}}}$ satisfies the averaged local law in (6.3.1) with data $(\mathcal{D}^{\mathrm{abv}}_{t_{\mathrm{final}}}, \xi + (\ell+1)\nu)$ for any $B \in \mathbb{C}^{N\times N}$. This concludes the proof of Proposition 6.3.6. □

6.5 Green function comparison: Proof of Proposition 6.3.7

The goal of this section is to prove Proposition 6.3.7 and thereby conclude the argument for the *zag* step of our proof. For simplicity, we will carry out the proof only in the real symmetric case; the complex Hermitian case can be dealt with minor modifications and is thus omitted. Moreover, since throughout the argument the time t_k defined in (6.3.23) remains fixed, for the remainder of this section, we drop the superscript t_k from $\mathcal{D}^{\mathrm{abv}}_{t_k}, \rho_{t_k}$, and M_{t_k}. To further condense the notation, we abbreviate $\mathcal{D} := \mathcal{D}^{\mathrm{abv}}_{t_k}$ and $s_{\mathrm{final}} := s(\Delta t_k)$.

The proof will be conducted iteratively along vertical truncations of the domain $\mathcal{D}$, defined as

$$\mathcal{D}_\gamma \equiv \mathcal{D}^{\mathrm{abv}}_{t_k,\gamma} := \left\{z := E + \mathrm{i}\eta \in \mathcal{D} \equiv \mathcal{D}^{\mathrm{abv}}_{t_k} : \eta \ge N^{-1+\gamma}\right\}, \quad 0 < \gamma \le 1. \tag{6.5.1}$$

This is formalized in the following proposition, which we prove in Section 6.5.2.

Proposition 6.5.1 (Zag Bootstrap)**.** *Fix a constant $0 < \gamma_0 \le 1$ and assume that the very-high-probability bounds on the matrix elements of the resolvent* (6.3.28)

$$\left|\left(G^s(z)\right)_{\boldsymbol{uv}}\right| \lesssim 1, \quad \left|\left(\operatorname{Im} G^s(z)\right)_{\boldsymbol{uu}}\right| \lesssim \rho(z), \tag{6.5.2}$$

hold uniformly in $z \in \mathcal{D}_{\gamma_0}$ and $s \in [0, s_{\mathrm{final}}]$, for any deterministic $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^N$ with $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$.

Fix $\gamma_1 \geq \gamma_0 - \delta$ with $\delta < \mu$ satisfying $\delta \ll \xi$, and assume that for some $\nu > 0$ and $\ell \in \mathbb{N}$, the resolvent G^s satisfies the local laws (6.3.1) *with data $(\mathcal{D}_{\gamma_1}, \xi + \ell\nu)$ at time $s = s_{\mathrm{final}}$. Then the resolvent G^s satisfies the local laws* (6.3.1) *with data $(\mathcal{D}_{\gamma_1}, \xi + (\ell+1)\nu)$ uniformly in $s \in [0, s_{\mathrm{final}}]$.*

Armed with Proposition 6.5.1, we can easily conclude Proposition 6.3.7.

Proof of Proposition 6.3.7. The proof goes via induction in $\gamma(k) := 1 - k\delta$ by iteratively applying Proposition 6.5.1. As the base case, clearly, the estimates (6.5.2) hold for $\gamma_0 = \gamma(0) = 1$ as a direct consequence of the bounds $\|G^s(E + \mathrm{i}\eta)\| \leq \eta^{-1}$ and $\rho(E + \mathrm{i}\eta) \sim 1$ for $\eta \sim 1$. Moreover, the resolvent G^s satisfies the local laws (6.3.1) with data $(\mathcal{D}_{\gamma_1}, \xi + \ell\nu)$ with $\gamma_1 = \gamma(1)$ at time $s = s_{\mathrm{final}}$ by assumption. Hence, the resolvent G^s satisfies the local laws (6.3.1) with data $(\mathcal{D}_{\gamma(1)}, \xi + (\ell+1)\nu)$ uniformly in $s \in [0, s_{\mathrm{final}}]$ by Proposition 6.5.1. As a consequence, since $\xi + (\ell+1)\nu \ll \varepsilon$, we have that the resolvent G^s satisfies the bounds (6.5.2) uniformly in $z \in \mathcal{D}_{\gamma(1)}$ and $s \in [0, s_{\mathrm{final}}]$

As the induction step, assume now that for an integer $k \geq 1$ the resolvent G^s satisfies the bounds (6.5.2) uniformly in $z \in \mathcal{D}_{\gamma(k)}$ and $s \in [0, s_{\mathrm{final}}]$. (Recall that, as above, G^s satisfies the local laws (6.3.1) with data $(\mathcal{D}_{\gamma(k)}, \xi + \ell\nu)$ at time $s = s_{\mathrm{final}}$ by assumption.) Therefore, the resolvent G^s satisfies the local laws (6.3.1) with data $(\mathcal{D}_{\gamma(k+1)}, \xi + (\ell+1)\nu)$ uniformly in $s \in [0, s_{\mathrm{final}}]$ by Proposition 6.5.1. Note that after $K' := \lceil (1+\varepsilon)/\delta \rceil \sim 1$ steps, $\mathcal{D}_{\gamma(K')} = \mathcal{D}$ and we have hence proven Proposition 6.3.7. □

It thus remains to prove Proposition 6.5.1. We begin by collecting several preliminaries in Section 6.5.1. Afterwards, in Section 6.5.2 we give the proof of Proposition 6.5.1 based on average and isotropic *Gronwall estimates*. These bounds are proven in Sections 6.5.3.1 and 6.5.3.2, respectively.

6.5.1 Preliminaries

In order to perform the GFT, i.e., compare initial and final W's, given by $W^t = H^t - A$ with H^t being the solution to (6.3.9), we employ Itô's formula: For a C^2-function $f(W^t)$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbf{E}\, f(W^t) = -\frac{1}{2}\,\mathbf{E}\sum_{\alpha} w_\alpha(t)(\partial_\alpha f)(W^t) + \frac{1}{2N}\sum_{\alpha,\beta}\kappa_t(\alpha,\beta)\,\mathbf{E}(\partial_\alpha\partial_\beta f)(W^t)\,, \tag{6.5.3}$$

where $\kappa_t(\alpha,\beta)$ denotes the (normalized, recall (6.2.5)) second order cumulant of $w_\alpha(t)$ and $w_\beta(t)$, the matrix entries of W^t. The first summand on the rhs. of (6.5.3) can now be further treated by cumulant expansion, which is first key ingredient for our proof.

Proposition 6.5.2 (Multivariate cumulant expansion; cf. Proposition 3.2 in [243] and Lemma 3.1 in [329])**.** *Let $f : \mathbf{R}^{N\times N} \to \mathbf{C}$ be a L times differentiable function with bounded derivatives. Let W be a random matrix, whose normalized cumulants satisfy Assumption 6.2.3. Then, for any index $\alpha_0 \in [N]^2$ it holds that*

$$\mathbf{E}\, w_{\alpha_0} f(W) = \sum_{k=0}^{L-1}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(\alpha_0)^k}\frac{\kappa(\alpha_0,\boldsymbol{\alpha})}{N^{(k+1)/2}k!}\,\mathbf{E}(\partial_{\boldsymbol{\alpha}} f)(W) + \Omega_L(f,\alpha_0), \tag{6.5.4}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k)$ and $\partial_{\boldsymbol{\alpha}} = \partial_{w_{\alpha_1}} ... \partial_{w_{\alpha_k}}$ for $k \geq 1$, and for $k = 0$ is considered as the function f itself. Moreover, the error term in (6.5.4) *satisfies*

$$\big|\Omega_L(f,\alpha_0)\big| \lesssim \frac{C_L}{N^{(L+1)/2}}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(\alpha_0)^L}\sup_{\lambda\in[0,1]}\Big(\mathbf{E}\big|(\partial_{\boldsymbol{\alpha}} f)(\lambda W|_{\mathcal{N}(\alpha_0)} + W|_{[N]^2\setminus\mathcal{N}(\alpha_0)})\big|^2\Big)^{1/2}, \tag{6.5.5}$$

for some constant $C_L > 0$ depending only on L. The notation $W|_{\mathcal{N}}$ for $\mathcal{N} \subset [N]^2$ in (6.5.5) *refers to the matrix which equals W at all entries $\alpha \in \mathcal{N}$ and is zero otherwise.*

Note that the $k = 1$ term in the expansion of the first summand on the rhs. of (6.5.3) exactly cancels the second summand on the rhs. of (6.5.3). For Proposition 6.5.2 being practically applicable we need to control (i) every order of the expansion, and (ii) the truncation term Ω. These will be guaranteed by Assumption 6.2.3 above.

The second key input required for the GFT argument is the following monotonicitiy estimate on resolvents.

Lemma 6.5.3 (Monotonicity estimate)**.** *Fix a constant $0 < \gamma_0 \le 1$ and assume that the very-high-probability bounds* (6.5.2) *hold uniformly in $z \in \mathcal{D}_{\gamma_0}$ and $s \in [0, s_{\mathrm{final}}]$, for any deterministic $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^N$ with $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$.*

Fix $\gamma_1 \ge \gamma_0 - \delta$. Then, we have

$$|G^s(E + \mathrm{i}\eta_1)_{\boldsymbol{uv}}| \lesssim \frac{\eta_0}{\eta_1}, \qquad |\mathrm{Im}\, G^s(E + \mathrm{i}\eta_1)_{\boldsymbol{uu}}| \lesssim \rho(E + \mathrm{i}\eta_0)\frac{\eta_0}{\eta_1}, \tag{6.5.6}$$

with very high probability, uniformly in $z := E + \mathrm{i}\eta_1 \in \mathcal{D}_{\gamma_1}$ for any $\eta_0 \ge N^{-1+\gamma_0} \vee \eta_1$, time $s \in [0, s_{\mathrm{final}}]$, and for any deterministic vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^N$ with $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$.

We defer the proof of Lemma 6.5.3 to Appendix 6.A.

6.5.2 Gronwall estimates: Proof of Proposition 6.5.1

In this section, we provide the proof of Proposition 6.5.1 based on two Gronwall estimates, formulated in Propositions 6.5.4–6.5.5 below that will be proven in the next subsection. The isotropic part of Proposition 6.5.1 will be concluded in a self contained way, based entirely on the *isotropic Gronwall estimate* in Proposition 6.5.4. Its conclusion in (6.5.11) then serves as an input for the *average Gronwall estimate* in Proposition 6.5.5.

Proposition 6.5.4 (Isotropic Gronwall estimate)**.** *Assume the conditions of Proposition 6.5.1. Fix $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ of bounded norm, $z := E + \mathrm{i}\eta_1 \in \mathcal{D}_{\gamma_1}$ and $\eta_0 \ge N^{-1+\gamma_0} \vee \eta_1$ such that $\eta_0/\eta_1 \le N^{\delta}$. For $s \in [0, s_{\mathrm{final}}]$, define*

$$S_s := \big(G^s(E + \mathrm{i}\eta_1) - M(E + \mathrm{i}\eta_1)\big)_{\boldsymbol{xy}}. \tag{6.5.7}$$

Then, for any (large) even $p \in \mathbf{N}$, it holds that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\,|S_s|^p\right| \lesssim \left(1 + N^{10\delta}\sqrt{\frac{\rho(E + \mathrm{i}\eta_0)}{\eta_0}}\right)\big[\,\mathbf{E}\,|S_s|^p + (\Psi(\eta_1))^p\,\big], \tag{6.5.8}$$

uniformly in $s \in [0, s_{\mathrm{final}}]$, bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, and $z \in \mathcal{D}_{\gamma_1}$. Here, for $\eta \in [\eta_0, \eta_1]$, we denoted

$$\Psi(\eta) := \sqrt{\frac{\rho(E + \mathrm{i}\eta)}{N\eta}}. \tag{6.5.9}$$

By Gronwall's lemma, uniformly in $s \in [0, s_{\mathrm{final}}]$, from (6.5.8) we find that

$$\begin{aligned}\mathbf{E}\,|S_s|^p &\lesssim \exp\left(\left(1 + N^{10\delta}\sqrt{\frac{\rho(E + \mathrm{i}\eta_0)}{\eta_0}}\right)(s_{\mathrm{final}} - s)\right)\big[\,\mathbf{E}\,|S_{s_{\mathrm{final}}}|^p + (\Psi(\eta_1))^p\,\big] \\ &\lesssim \exp(N^{-\xi/10})\big[\,\mathbf{E}\,|S_{s_{\mathrm{final}}}|^p + (\Psi(\eta_1))^p\,\big] \lesssim \mathbf{E}\,|S_{s_{\mathrm{final}}}|^p + (\Psi(\eta_1))^p.\end{aligned} \tag{6.5.10}$$

Here we used that $\rho(E + \mathrm{i}\eta_0)/\eta_0 \lesssim N^{k\delta}/T$ by (6.3.21), $s_{\mathrm{final}} \lesssim N^{-(k-1)\delta}T$ by (6.3.15), $T \sim N^{-\xi/4}$ from (6.3.22), and $\delta \ll \xi$ by (6.3.3).

To estimate $\mathbf{E}\,|S_{s_{\mathrm{final}}}|^p$, recall that the resolvent G^s satisfies the isotropic local law in (6.3.1) with data $(\mathcal{D}_{\gamma_1}, \xi + \ell\nu)$ at $s = s_{\mathrm{final}}$. Therefore, since p in (6.5.10) was arbitrary, we find that

$$\left|\big(G^s(z) - M(z)\big)_{\boldsymbol{xy}}\right| \le N^{\xi+(\ell+1)\nu}\sqrt{\frac{\rho(z)}{N\eta_1}}, \tag{6.5.11}$$

uniformly in $z := E + \mathrm{i}\eta_1 \in \mathcal{D}_{\gamma_1}$, $s \in [0, s_{\mathrm{final}}]$, and bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, with very high probability.

This proves the isotropic part of Proposition 6.5.1 and we are left with the average part.

Proposition 6.5.5 (Average Gronwall estimate)**.** *Fix $B \in \mathbf{C}^{N\times N}$ of bounded Hilbert–Schmidt norm, $\|B\|_{\mathrm{hs}} \le 1$, $z := E + \mathrm{i}\eta \in \mathcal{D}_{\gamma_1}$, and $\eta_0 \ge N^{-1+\gamma_0} \vee \eta_1$ such that $\eta_0/\eta_1 \le N^{\delta}$. For $s \in [0, s_{\mathrm{final}}]$, define*

$$R_s := \langle (G^s(E + \mathrm{i}\eta_1) - M(E + \mathrm{i}\eta_1))B\rangle\,. \tag{6.5.12}$$

Moreover, suppose that (6.5.11) *holds uniformly in $z := E + \mathrm{i}\eta_1 \in \mathcal{D}_{\gamma_1}$, $s \in [0, s_{\mathrm{final}}]$, and bounded $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$. Then, for any (large) even $p \in \mathbf{N}$ it holds that*

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\,|R_s|^p\right| \lesssim \left(1 + N^{-2\delta}\,\frac{\rho(E + \mathrm{i}\eta_0)}{\eta_0}\right)\left[\mathbf{E}\,|R_s|^p + \left(\frac{N^{3\xi}}{N\eta_1}\right)^p\right], \tag{6.5.13}$$

uniformly in $s \in [0, s_{\mathrm{final}}]$, bounded $B \in \mathbf{C}^{N\times N}$, and $z \in \mathcal{D}_{\gamma_1}$.

Analogously to (6.5.10), by Gronwall's lemma, uniformly in $s \in [0, s_{\mathrm{final}}]$, we find that

$$\begin{aligned}\mathbf{E}\,|R_s|^p &\lesssim \exp\left(\left(1 + N^{-2\delta}\,\frac{\rho(E + \mathrm{i}\eta_0)}{\eta_0}\right)(s_{\mathrm{final}} - s)\right)\left[\,\mathbf{E}\,|R_{s_{\mathrm{final}}}|^p + \left(\frac{N^{3\xi}}{N\eta_1}\right)^p\,\right] \\ &\lesssim \exp(N^{-\delta})\left[\,\mathbf{E}\,|R_{s_{\mathrm{final}}}|^p + (\Psi(\eta_1))^p\,\right] \lesssim \mathbf{E}\,|R_{s_{\mathrm{final}}}|^p + \left(\frac{N^{3\xi}}{N\eta_1}\right)^p.\end{aligned} \tag{6.5.14}$$

Here we used that $\rho(E + \mathrm{i}\eta_0)/\eta_0 \lesssim N^{k\delta}/T$ by (6.3.21), $s_{\mathrm{final}} \lesssim N^{-(k-1)\delta}T$ by (6.3.15), $T \sim N^{-\xi/4}$ by (6.3.22), and $\delta \ll \xi$ by (6.3.3). Note that the small prefactor $N^{-2\delta}$ in (6.5.13) is absolutely essential, unlike in the isotropic case (6.5.10), where a large prefactor $N^{10\delta}$ is affordable thanks to the square root. The linear appearance of ρ/η in (6.5.13) is only due to fact that we estimate B in terms of its Hilbert–Schmidt norm $\|B\|_{\mathrm{hs}}$; cf. the estimate in (6.5.21). For observables with $\|B\| \sim \|B\|_{\mathrm{hs}}$, such as the identity matrix $B = \mathbf{1}$, the linear dependence on ρ/η can be improved to a $\sqrt{\rho/\eta}$. We exploit this fact in (6.6.22) below.

Recall that the resolvent G^s satisfies the average local law in (6.3.1) with data $(\mathcal{D}_{\gamma_1}, \xi + \ell\nu)$ at $s = s_{\mathrm{final}}$. Therefore, since p in (6.5.14) was arbitrary, we find that

$$\left|\left\langle\big(G^s(z) - M(z)\big)B\right\rangle\right| \le \frac{N^{3(\xi+(\ell+1)\nu)}}{N\eta_1},$$

uniformly in $z := E + \mathrm{i}\eta_1 \in \mathcal{D}_{\gamma_1}$, $s \in [0, s_{\mathrm{final}}]$, and $B \in \mathbf{C}^{N\times N}$ with $\|B\|_{\mathrm{hs}} \le 1$, with very high probability.

This concludes the proof of Proposition 6.5.1. □

6.5.3 Cumulant expansion: Proofs of Propositions 6.5.5 and 6.5.4

The proofs of Propositions 6.5.4–6.5.5 are based on the multivariate cumulant expansion from Proposition 6.5.2 and the monotonicity estimate from Lemma 6.5.3. We begin by proving the average Gronwall estimate in Proposition 6.5.5. Moreover, we will henceforth omit the superscript s from the resolvent G^s.

6.5.3.1 Average case: Proof of Proposition 6.5.5

Throughout the proof, we will assume that $\|B\|_{\mathrm{hs}} \lesssim 1$. By (6.5.3) for R_s we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\,|R_s|^p = -\frac{1}{2}\,\mathbf{E}\sum_{\alpha_1} w_{\alpha_1}(s)(\partial_{\alpha_1}|R_s|^p) + \frac{1}{2}\sum_{\alpha_1,\alpha_2}\kappa_s(\alpha_1,\alpha_2)\,\mathbf{E}\big[\partial_{\alpha_1}\partial_{\alpha_2}|R_s|^p\big], \tag{6.5.15}$$

where $w_{\alpha_i}(s)$ is the α_i-th entry of W_s, $\kappa_s(\alpha_1,\alpha_2,\ldots)$ is a joint normalized cumulant of $w_{\alpha_1}(s), w_{\alpha_2}(s),\ldots$ and $\partial_{\alpha_i} = \partial_{w_{\alpha_i}(s)}$ denotes the partial derivative in the direction of $w_{\alpha_i}(s)$.

The first term on the rhs. of (6.5.15) can now be expanded by means of Proposition 6.5.2:

$$\mathbf{E}\big[w_{\alpha_1}(s)(\partial_{\alpha_1}|R_s|^p)\big] = \sum_{k=0}^{L-1}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(\alpha_1)^k}\frac{\kappa_s(\alpha_1,\boldsymbol{\alpha})}{N^{(k+1)/2}\,k!}\,\mathbf{E}\big[\partial_{\alpha_1}\partial_{\boldsymbol{\alpha}}|R_s|^p\big] + \Omega_L\,. \tag{6.5.16}$$

Since L derivatives of $|R_s|^p$ create L additional resolvent matrix elements (where each of them is bounded with the aid of Lemma 6.5.3) and using that $|\mathcal{N}(\alpha_1)| \lesssim N^{1/2-\mu}$ by Assumption 6.2.3 (ii), the error term Ω_L can be estimated as[10]

$$|\Omega_L| \lesssim N^{-\frac{L+1}{2}} N^{L(1/2-\mu)} N^{(p+L)\delta} \lesssim N^{2p\delta + L(\delta-\mu)}. \tag{6.5.17}$$

Using the relation $\mu > \delta$ from (6.3.3) and $L := \lceil((1+\delta)p+2)/(\mu-\delta)\rceil$, we see that $|\Omega_L| \le N^{-2}(N\eta_1)^{-p}$ (the factor N^{-2} is needed to bound the summation over α_1 in (6.5.15)). With this choice of L, the error term Ω_L will henceforth be ignored.

Plugging (6.5.16) into (6.5.15) and using that the $k=0$ term is zero by $\kappa_s(\alpha_1) = \mathbf{E}\,w_{\alpha_1}(s) = 0$, and that the $k=1$ term in (6.5.16) cancels the second term on the rhs. of (6.5.15), we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\,|R_s|^p\right| \lesssim \left|\sum_{k=2}^{L-1}\sum_{\alpha_1}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(\alpha_1)^k}\frac{\kappa_s(\alpha_1,\boldsymbol{\alpha})}{N^{(k+1)/2}\,k!}\,\mathbf{E}(\partial_{\alpha_1}\partial_{\boldsymbol{\alpha}}|R_s|^p)\right| + \left(\frac{1}{N\eta_1}\right)^p. \tag{6.5.18}$$

We will now first estimate the third order cumulant terms (i.e. those with $k=2$ in (6.5.18)), as these are the most delicate, and afterwards turn to the higher order ones that can be handled by simple power counting with a little twist due to the Hilbert–Schmidt norm of the observable B. Moreover, we drop the time dependence of R_s and κ_s whenever it does not lead to confusion. We point out that Assumption 6.2.3 also holds for W^s from (6.3.9), uniformly in $s\in[0,\infty)$. Indeed, adding an independent Gaussian random matrix to W_0 has no effect on cumulants of order $k\ge 3$ (by Gaussianity) and leaves the first two joint moments as well as the independence property of Assumption 6.2.3 (ii) invariant (the covariance tensor Σ is trivial beyond the range $\mathcal{N}(\alpha_1)$) by construction (6.3.9). In particular, we can freely extend the summation over $\boldsymbol{\alpha}\in\mathcal{N}(\alpha_1)^k$ in (6.5.18) to $\boldsymbol{\alpha}\in([N]^2)^k$ and combine the latter two summations in (6.5.18) into $\sum_{\alpha_1,\boldsymbol{\alpha}}$.

Now, for the third order cumulant terms, we aim to control

$$\left|N^{-3/2}\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)\,\mathbf{E}(\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}|R|^p)\right|,$$

[10] To be precise, note some of the $p+L$ resolvents in the error term Ω_L are actually resolvents of the random matrix $W^{(\lambda)} := \lambda W|_{\mathcal{N}(\alpha_0)} + W|_{[N]^2\setminus\mathcal{N}(\alpha_0)}$ (recall (6.5.5)) and we need to guarantee their boundedness as well, uniformly in $\lambda\in[0,1]$. We perform a resolvent expansion of $G^{(\lambda)} := (A + W^{(\lambda)} - z)^{-1}$ up to some order $\tilde{m}\in\mathbf{N}$ around $G^{(1)}$ whose boundedness is known. For each $G^{(\lambda)}$, the m^{th} order term in this expansion can be bounded by $N^{\delta}N^{m(\delta-\mu+\nu)}$ with the aid of Lemma 6.5.3 (to bound $G^{(1)}$ isotropically) and using the norm estimate $\|W|_{\mathcal{N}(\alpha_0)}\| \le N^{-\mu+\nu}$, w.v.h.p. for any $\nu>0$, which is a consequence of Assumption 6.2.3 (ii). By a simple norm bound $\|G^{(\lambda)}\|\le\eta^{-1}$, the last truncation term in the resolvent expansion admits the bound $N^{\delta}N^{\tilde{m}(\delta-\mu+\nu)}\eta^{-1}$. Therefore, since η depends at most polynomially on N and $\mu > \delta+\nu$ for some $\nu>0$ small enough, the resolvent expansion can be truncated at finite order, leaving us with the bound N^{δ} for every matrix element of $G^{(\lambda)}$ employed in (6.5.17), uniformly in $\lambda\in[0,1]$.

which, after employing the Leibniz rule, can be broken up into terms of the form $(\partial_\alpha^3 R)|R|^{p-1}$, $(\partial_\alpha R)(\partial_\alpha^2 R)|R|^{p-2}$, and $(\partial_\alpha R)^3|R|^{p-3}$. To further ease the notation, here and in the following, we neglect the difference between R and $\overline{R}$, as these will be estimated in a completely analogous way.

We begin with the terms of the form $(\partial_\alpha^3 R)|R|^{p-1}$, which requires the bound (6.2.8) in Assumption 6.2.3 (i). Writing $\langle GB\rangle = N^{-1}\sum_j (GB)_{jj}$ and identifying $\alpha_i \equiv (a_i, b_i) \in [N]^2$, we aim to estimate (ignoring the $|R|^{p-1}$-factor)

$$N^{-5/2}\left|\sum_{j,\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{ja_1}G_{b_1a_2}G_{b_2a_3}(GB)_{b_3j}\right| = N^{-5/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{b_1a_2}G_{b_2a_3}(GBG)_{b_3a_1}\right|.$$

For both $G_{b_1a_2}$ and $G_{b_2a_3}$ we write $G_{ba} = M_{ba} + (G-M)_{ba}$ and use $\|M\| \lesssim 1$ for the M-term and the bound (6.5.11) for the $(G-M)$-term. In particular (recalling the notation (6.5.9)),

$$\begin{aligned}
&N^{-5/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)M_{b_1a_2}(G-M)_{b_2a_3}(GBG)_{b_3a_1}\right| \\
&\quad\lesssim N^{-5/2}N^{\xi+(\ell+1)\nu}\Psi(\eta_1)\sum_{\alpha_1,\alpha_2,\alpha_3}|\kappa(\alpha_1,\alpha_2,\alpha_3)||(GBG)_{b_3a_1}| \\
&\quad\lesssim N^{-5/2}N^{\xi+(\ell+1)\nu}\Psi(\eta_1)\sum_{\alpha_1,\alpha_2,\alpha_3}|\kappa(\alpha_1,\alpha_2,\alpha_3)||(GBB^*G^*)_{b_3b_3}|^{1/2}|(GG^*)_{a_1a_1}|^{1/2} \\
&\quad\lesssim N^{\xi+(\ell+1)\nu}\Psi(\eta_1)^2\left\|\sum_{\alpha_2}|\kappa(*,\alpha_2,*)|\right\|\langle GG^*BB^*\rangle^{1/2} \lesssim N^{-\delta}\sqrt{\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}}\,\frac{N^{3\xi}}{N\eta_1},
\end{aligned} \tag{6.5.19}$$

with very high probability. In the second step, we used the Schwarz inequality. In the ultimate step, similarly to (6.4.18), we used (6.2.6), the Ward identity, the spectral decomposition of BB^*, and (6.5.11) together with $(\operatorname{Im} M)_{\boldsymbol{vv}} \lesssim \rho(E+\mathrm{i}\eta_1)$ by (6.4.1), to obtain

$$\langle \operatorname{Im} GBB^*\rangle = \frac{1}{N}\sum_j |\sigma_j|^2 (\operatorname{Im} G)_{\boldsymbol{v}_j\boldsymbol{v}_j} \lesssim \rho(E+\mathrm{i}\eta_1)\|B\|_{\mathrm{hs}}^2, \tag{6.5.20}$$

and used $\delta \ll \xi$ by (6.3.3), and the fact that $\nu > 0$ is arbitrarily small. Note that the small factor $N^{-\delta}$ in the last line of (6.5.19) is balanced by an additional N^{ξ}. The terms with $(G-M)_{b_1a_2}G_{b_2a_3}$ and $(G-M)_{b_1a_2}(G-M)_{b_2a_3}$ are treated analogously and we are thus left with the $M_{b_1a_2}M_{b_2a_3}$-term. Here, using the $\|\!|\kappa|\!\|_3^{\mathrm{av}}$ norm from (6.2.8), we estimate

$$\begin{aligned}
&N^{-5/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)M_{b_1a_2}M_{b_2a_3}(GBG)_{b_3a_1}\right| \\
&\quad\le N^{-1}\|\!|\kappa|\!\|_3^{\mathrm{av}}\|M\|^2\|GBG\|_{\mathrm{hs}} \lesssim \eta_1^{-1/2}\frac{1}{N\eta_1}\langle \operatorname{Im} GBB^*\rangle^{1/2} \lesssim N^{-\delta}\sqrt{\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}}\,\frac{N^{3\xi}}{N\eta_1},
\end{aligned} \tag{6.5.21}$$

with very high probability. In the penultimate step we used the definition of $\|\cdot\|_{\mathrm{hs}}$ together with a Ward identity and the trivial bound $\|G\| \le \eta_1^{-1}$; in the last step we employed (6.5.20) and $\eta_0/\eta_1 \le N^{\delta}$ together with monotonicity of $\eta \mapsto \eta\rho(E+\mathrm{i}\eta)$ and $\delta \ll \xi$. Hence, by two Young inequalities , we thus find

$$\left|N^{-3/2}\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)\,\mathbf{E}\big[(\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}R)|R|^{p-1}\big]\right| \lesssim \left(1+N^{-2\delta}\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}\right)\left[\mathbf{E}\,|R|^p + \left(\frac{N^{3\xi}}{N\eta_1}\right)^p\right], \tag{6.5.22}$$

where we overestimated $N^{-\delta}\sqrt{\rho/\eta_0} \lesssim 1 + N^{-2\delta}\rho/\eta_0$.

Next, we turn to terms of the form $(\partial_\alpha R)(\partial_\alpha^2 R)|R|^{p-2}$. Similarly to (6.5.19), using (6.2.6) for $k = 3$, we find

$$\begin{aligned}
&N^{-7/2}\Bigg|\sum_{j,k,\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{ja_1}G_{b_1a_2}(GB)_{b_2j}G_{ka_3}(GB)_{b_3k}\Bigg| \\
&\quad\lesssim N^{-7/2}\sum_{\alpha_1,\alpha_2,\alpha_3}|\kappa(\alpha_1,\alpha_2,\alpha_3)|\,|(GBG)_{b_3a_3}|\,|(GBG)_{b_2a_1}| \\
&\quad\lesssim N^{-7/2}\sqrt{\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}}\sum_{\alpha_1,\alpha_2,\alpha_3}|\kappa(\alpha_1,\alpha_2,\alpha_3)|\,|(GBG)_{b_3a_3}|\sqrt{(GBB^*G^*)_{b_2b_2}} \\
&\quad\lesssim N^{-7/2}\sqrt{\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}}\,\|\kappa\|_3\sqrt{\sum_{b_3,a_3}|(GBG)_{b_3a_3}|^2}\sqrt{\sum_{b_2a_2}(GBB^*G^*)_{b_2b_2}} \\
&\quad\lesssim \frac{\rho(E+\mathrm{i}\eta_1)^{1/2}}{N^2\eta_1^2}\langle\operatorname{Im}GB\operatorname{Im}GB^*\rangle^{1/2}\langle\operatorname{Im}GBB^*\rangle^{1/2}\lesssim N^{-\delta}\sqrt{\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}}\left(\frac{N^{3\xi}}{N\eta_1}\right)^2,
\end{aligned} \tag{6.5.23}$$

with very high probability. To go to the third line, we used a Schwarz inequality and the estimate $(GG^*)_{a_1a_1}\lesssim\rho/\eta_1$ w.v.h.p. (as follows by a Ward identity and (6.5.11)). In the penultimate step, we again used several Ward identities. In the last step we used $\langle\operatorname{Im}GB\operatorname{Im}GB^*\rangle\le\langle\operatorname{Im}GBB^*\rangle/\eta_1$ and (6.5.20) together with $\eta_0/\eta_1\le N^\delta$, monotonicity of $\eta\mapsto\eta\rho(E+\mathrm{i}\eta)$, and $\delta\ll\xi$ by (6.3.3). Hence, again by Young's inequality and overestimating $N^{-\delta}\sqrt{\rho/\eta_0}\lesssim 1+N^{-2\delta}\rho/\eta_0$, we find

$$\left|N^{-3/2}\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)\,\mathbf{E}\big[(\partial_{\alpha_1}\partial_{\alpha_2}R)(\partial_{\alpha_3}R)|R|^{p-2}\big]\right|\lesssim\left(1+N^{-2\delta}\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}\right)\left[\mathbf{E}|R|^p+N^\xi\left(\frac{N^\delta}{N\eta_1}\right)^p\right]. \tag{6.5.24}$$

Finally, we estimate terms of the form $(\partial_\alpha R)^3|R|^{p-3}$, which are the most critical ones, since they necessarily contribute the $N^{-2\delta}\rho/\eta$ factor as we estimate B by its Hilbert–Schmidt norm $\|B\|_{\mathrm{hs}}$. For terms of the form $(\partial_\alpha R)^3|R|^{p-3}$, similarly to (6.5.19) and (6.5.23), we find

$$\begin{aligned}
&N^{-9/2}\Bigg|\sum_{j,k,\ell,\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{ja_1}(GB)_{b_1j}G_{ka_2}(GB)_{b_2k}G_{\ell a_3}(GB)_{b_3\ell}\Bigg| \\
&\quad\lesssim N^{-9/2}\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\|B\|\sum_{\alpha_1,\alpha_2,\alpha_3}|\kappa(\alpha_1,\alpha_2,\alpha_3)|\,\big|(GBG)_{b_2a_2}\big|\,\big|(GBG)_{b_3a_3}\big| \\
&\quad\lesssim N^{-9/2}\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\|B\|\|\kappa\|_3\sum_{a,b}\big|(GBG)_{ab}\big|^2\lesssim N^{-7/2}\frac{\rho(E+\mathrm{i}\eta_1)^2}{\eta_1^4}\|B\|\|B\|_{\mathrm{hs}}^2 \\
&\quad\lesssim N^{-2\delta}\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}\left(\frac{N^{3\xi}}{N\eta_1}\right)^3.
\end{aligned} \tag{6.5.25}$$

To go to the second line, we used that

$$|(GBG)_{ab}|\le\|B\|\sqrt{(GG^*)_{aa}(GG^*)_{bb}}\lesssim\|B\|\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}, \tag{6.5.26}$$

by a Schwarz inequality, a Ward identity and (6.5.11). In the third line we estimated

$$\sum_{a,b}\big|(GBG)_{ab}\big|^2=\frac{N}{\eta_1^2}\langle\operatorname{Im}GB\operatorname{Im}GB^*\rangle\lesssim\frac{N\rho(E+\mathrm{i}\eta_1)}{\eta_1^3}\|B\|_{\mathrm{hs}}^2, \tag{6.5.27}$$

with very high probability, by means of Ward identities and (6.5.20). To go to the fourth line, we used $\|B\|\le\sqrt{N}\|B\|_{\mathrm{hs}}$ and the fact that $\delta\ll\xi$ by (6.3.3), together with $\eta_0/\eta_1\le N^\delta$ and monotonicity of $\eta\mapsto\eta\rho(E+\mathrm{i}\eta)$. Hence, (6.5.25) together with Young's inequality implies that

$$N^{-3/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\mathbf{E}\big[(\partial_{\alpha_1}R)(\partial_{\alpha_2}R)(\partial_{\alpha_3}R)|R|^{p-3}\big]\right|\lesssim\left(1+N^{-2\delta}\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}\right)\left[\mathbf{E}|R|^p+\left(\frac{N^{3\xi}}{N\eta_1}\right)^p\right]. \tag{6.5.28}$$

For the higher order terms in (6.5.18) with $n = k+1 \geq 4$ we aim to estimate

$$\left| N^{-n/2} \sum_{\alpha_1,\dots,\alpha_n} \kappa(\alpha_1,\dots,\alpha_n)\, \mathbf{E}\big[\partial_{\alpha_1}\dots\partial_{\alpha_n}|R|^p\big] \right| .$$

In case that the n derivatives are distributed on $k \in [n]$ factors of R, we find that, for $n_\ell \in \mathbf{N}$ with $\sum_{\ell=1}^k n_\ell = n$ and identifying $(\alpha_i)_{i\in[n]} \equiv \big((a_{\ell_i}, b_{\ell_i})\big)_{i\in[n_\ell],\ell\in[k]}$,

$$\begin{aligned}
&\left| N^{-n/2}N^{-k} \sum_{j_1,\dots,j_k} \sum_{\alpha_1,\dots,\alpha_n} \kappa(\alpha_1,\dots,\alpha_n) \prod_{\ell=1}^k \big(G_{j_\ell a_{\ell_1}} G_{b_{\ell_1} a_{\ell_2}} \dots G_{b_{\ell_{n_\ell-1}} a_{\ell_{n_\ell}}} (GB)_{b_{\ell_{n_\ell}} j_\ell} \big) \right| \\
&\lesssim N^{-n/2}N^{-k} \sum_{\alpha_1,\dots,\alpha_n} |\kappa(\alpha_1,\dots,\alpha_n)| \prod_{\ell=1}^k \big|(GBG)_{b_{\ell_{n_\ell}} a_{\ell_1}}\big| \qquad (6.5.29) \\
&\lesssim N^{-n/2}N^{-k} \left(\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\right)^{k-2} \|B\|^{k-2} \sum_{\alpha_1,\dots,\alpha_n} |\kappa(\alpha_1,\dots,\alpha_n)| \big|(GBG)_{\tilde b_1 \tilde a_1}\big| \big|(GBG)_{\tilde b_2 \tilde a_2}\big|
\end{aligned}$$

To go to the second line, we performed all the j summations and estimated all the other resolvents without a j index by (6.5.11); to go to the third line, we used (6.5.26) for $k-2$ of the k factors and used a simplified notation for the indices $\tilde a, \tilde b$, which agree with some a_{ℓ_i}, b_{ℓ_j}. The two factors of GBG are kept separately, since we aim for an estimate in terms of Hilbert–Schmidt norm $\|B\|_{\mathrm{hs}}$ of the observable B; otherwise the whole argument for the higher order terms would be a simple power counting. However, now we distinguish two cases: (i) $k \leq n-2$, and (ii) $k \in \{n-1, n\}$.

In the less critical case (i), we use a Schwarz inequality to estimate $\big|(GBG)_{\tilde b \tilde a}\big| \lesssim \sqrt{(GBB^*G^*)_{\tilde b\tilde b}}\sqrt{\rho/\eta_1}$, similarly to (6.5.26). Then, we continue to estimate (6.5.29) as

$$\begin{aligned}
&N^{-n/2}N^{-k} \left(\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\right)^{k-1} \|B\|^{k-2} \sum_{\alpha_1,\dots,\alpha_n} |\kappa(\alpha_1,\dots,\alpha_n)| \sqrt{(GBB^*G^*)_{\tilde b_1\tilde b_1}} \sqrt{(GBB^*G^*)_{\tilde b_2\tilde b_2}} \\
&\leq N^{-n/2}N^{-k} \left(\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\right)^{k-1} \|B\|^{k-2} \|\kappa\|_n \sum_{ab} (GBB^*G^*)_{aa} \\
&\lesssim N^{2-n/2} \left(\frac{\rho(E+\mathrm{i}\eta_1)}{N\eta_1}\right)^{k} \|B\|^{k-2} \|B\|_{\mathrm{hs}}^2 \lesssim \left(\frac{\rho(E+\mathrm{i}\eta_1)}{N\eta_1}\right)^{k} .
\end{aligned} \qquad (6.5.30)$$

While in the second step, we used (6.5.20), the final step follows from $\|B\| \leq \sqrt{N}\|B\|_{\mathrm{hs}} \lesssim \sqrt{N}$ and $k \leq n-2$.

For case (ii), we first note that necessarily $(\tilde a_1, \tilde b_1) = (a_1, b_1) = \alpha_1$, and similarly for index 2, up to permutation of the arguments of κ in (6.5.29). This simply follows, since $n \geq 4$ derivatives hitting each of $k \in \{n-1, n\}$ factors at least once, means that at least two of them are hit exactly once. Therefore, we can continue estimating (6.5.29) as

$$\begin{aligned}
&N^{-n/2}N^{-k} \left(\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\right)^{k-2} \|B\|^{k-2} \sum_{\alpha_1,\dots,\alpha_n} |\kappa(\alpha_1,\dots,\alpha_n)| \big|(GBG)_{b_1 a_1}\big| \big|(GBG)_{b_2 a_2}\big| \\
&\leq N^{-n/2}N^{-k} \left(\frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1}\right)^{k-2} \|B\|^{k-2} \|\kappa\|_n \sum_{ab} \big|(GBG)_{ba}\big|^2 \qquad (6.5.31) \\
&\lesssim N^{(k-n)/2} \left(\frac{1}{N\eta_1}\right)^{k} \frac{\rho(E+\mathrm{i}\eta_1)}{\eta_1} \lesssim N^{-2\delta} \frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0} \left(\frac{N^{3\xi}}{N\eta_1}\right)^{k} .
\end{aligned}$$

Note that for $k = n$ this estimate truly contributes the critical $N^{-2\delta}\rho(E+\mathrm{i}\eta_0)/\eta_0$ factor. Here, in the second step, we used (6.5.27) together with $\|B\| \leq \sqrt{N}\|B\|_{\mathrm{hs}} \lesssim \sqrt{N}$; the final step follows from $\eta_0/\eta_1 \leq N^\delta$ together with monotonicity of $\eta \mapsto \eta\rho(E+\mathrm{i}\eta)$ and $\delta \ll \xi$ by (6.3.3).

Hence, by Young's inequality we deduce

$$\left|N^{-n/2}\sum_{\alpha_1,\dots,\alpha_n}\kappa(\alpha_1,\dots,\alpha_n)\,\mathbf{E}(\partial_{\alpha_1}\dots\partial_{\alpha_n}|R|^p)\right| \lesssim \left(1+N^{-2\delta}\frac{\rho(E+\mathrm{i}\eta_0)}{\eta_0}\right)\left[\mathbf{E}\,|R|^p+\left(\frac{N^{3\xi}}{N\eta_1}\right)^p\right]. \tag{6.5.32}$$

Therefore, combining (6.5.18) with (6.5.22), (6.5.24), (6.5.28), and (6.5.32), we obtain (6.5.13). This finishes the proof of Proposition 6.5.5. □

6.5.3.2 Isotropic case: Proof of Proposition 6.5.4

Similarly to the proof of Proposition 6.5.5, after applying Itô's Lemma and a cumulant expansion, we find

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{E}\,|S_s|^p\right| \lesssim \left|\sum_{k=2}^{L-1}\sum_{\alpha_1}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(\alpha_1)^k}\frac{\kappa_s(\alpha_1,\boldsymbol{\alpha})}{N^{(k+1)/2}\,k!}\,\mathbf{E}\big[\partial_{\alpha_1}\partial_{\boldsymbol{\alpha}}|S_s|^p\big]\right| + \Psi(\eta_1)^p\,. \tag{6.5.33}$$

for some large enough L.

Employing the same notational simplifications as explained below (6.5.18), we again first estimate the third order cumulant terms, given by

$$\left|N^{-3/2}\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)\,\mathbf{E}\big[\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}|S|^p\big]\right|.$$

Distributing the derivatives according to the Leibniz rule, we need to estimate various terms of the forms $(\partial_\alpha^3 S)|S|^{p-1}$, $(\partial_\alpha S)(\partial_\alpha^2 S)|S|^{p-2}$, and $(\partial_\alpha S)^3|S|^{p-3}$. In contrast to the average case treated in the proof of Proposition 6.5.5, there is no term in the cumulant expansion producing the most critical $N^{-2\delta}\rho/\eta$ factor; instead we get $N^{8\delta}\sqrt{\rho/\eta} = N^{1/2+8\delta}\Psi$.

We start with estimating the first type of terms. In this case, identifying $\alpha_i \equiv (a_i, b_i) \in [N]^2$ and using Lemma 6.5.3 together with a Ward identity and Assumption 6.2.3 (i), we find

$$\begin{aligned}
&N^{-3/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{\boldsymbol{x}a_1}G_{b_1a_2}G_{b_2a_3}G_{b_3\boldsymbol{y}}\right| \\
&\quad\lesssim N^{-3/2}N^{2\delta}\sum_{\alpha_1,\alpha_2,\alpha_3}|\kappa(\alpha_1,\alpha_2,\alpha_3)|\,|G_{\boldsymbol{x}a_1}|\,|G_{b_3\boldsymbol{y}}| \\
&\quad\lesssim N^{-3/2}N^{2\delta}\,\|\kappa\|_3\left(\sum_{a_1,b_1}|G_{\boldsymbol{x}a_1}|^2\right)^{1/2}\left(\sum_{a_3,b_3}|G_{b_3\boldsymbol{y}}|^2\right)^{1/2} \lesssim N^{1/2+3\delta}\frac{\rho(E+\mathrm{i}\eta_0)}{N\eta_1}
\end{aligned} \tag{6.5.34}$$

with very high probability. Completely analogously we obtain

$$N^{-3/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{\boldsymbol{x}a_1}G_{b_1\boldsymbol{y}}G_{\boldsymbol{x}a_2}G_{b_2a_3}G_{b_3\boldsymbol{y}}\right| \lesssim N^{1/2+4\delta}\left(\frac{\rho(E+\mathrm{i}\eta_0)}{N\eta_1}\right)^{3/2} \tag{6.5.35}$$

and

$$N^{-3/2}\left|\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)G_{\boldsymbol{x}a_1}G_{b_1\boldsymbol{y}}G_{\boldsymbol{x}a_2}G_{b_2\boldsymbol{y}}G_{\boldsymbol{x}a_3}G_{b_3\boldsymbol{y}}\right| \lesssim N^{1/2+4\delta}\left(\frac{\rho(E+\mathrm{i}\eta_0)}{N\eta_1}\right)^{2}, \tag{6.5.36}$$

again with very high probability. Hence, combining (6.5.34), (6.5.35), and (6.5.36) with Young's inequality and additionally using that $\eta \mapsto \rho(E+\mathrm{i}\eta)/\eta$ is monotonically decreasing, we infer

$$\left|N^{-3/2}\sum_{\alpha_1,\alpha_2,\alpha_3}\kappa(\alpha_1,\alpha_2,\alpha_3)\,\mathbf{E}\big[\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}|S|^p\big]\right| \lesssim N^{1/2+8\delta}\Psi(\eta_0)\big[\,\mathbf{E}\,|S|^p+\Psi(\eta_1)^p\big] \qquad \text{w.v.h.p.}$$

Next, we turn to the higher order terms, where we aim to estimate

$$\left| N^{-n/2} \sum_{\alpha_1,\dots,\alpha_n} \kappa(\alpha_1,\dots,\alpha_n)\, \mathbf{E}\big[\partial_{\alpha_1}\dots\partial_{\alpha_n}|S|^p\big] \right|. \tag{6.5.37}$$

Distributing the n derivatives on $k \in [n]$ factors of S, we find that, for $n_\ell \in \mathbf{N}$ with $\sum_{\ell=1}^k n_\ell = n$ and (w.l.o.g.) $n_1 \le n_2 \le \dots \le n_k$, and identifying $(\alpha_i)_{i\in[n]} \equiv \big((a_{\ell_i}, b_{\ell_i})\big)_{i\in[n_\ell],\ell\in[k]}$, (6.5.37) can be rewritten as (ignoring the factor $|S|^{p-k}$)

$$\left| N^{-n/2} \sum_{\alpha_1,\dots,\alpha_n} \kappa(\alpha_1,\dots,\alpha_n) \prod_{\ell=1}^k \big(G_{\boldsymbol{x}a_{\ell_1}} G_{b_{\ell_1}a_{\ell_2}} \dots G_{b_{\ell_{n_\ell-1}}a_{\ell_{n_\ell}}} G_{b_{\ell_{n_\ell}}\boldsymbol{y}} \big) \right|. \tag{6.5.38}$$

If $n_2 = 1$, since there are now at least two factors of S hit by a single derivative, we find that (similarly to (6.5.31) in the proof of Proposition 6.5.5, cf. also [136, Eqs. (8.82)–(8.85)])

$$\begin{aligned} (6.5.38) &\lesssim N^{-n/2} N^{(n+k-4)\delta} \|\kappa\|_n \sum_{a,b} |G_{\boldsymbol{x}a} G_{b\boldsymbol{y}}|^2 \\ &\lesssim N^{2-n/2} N^{(n+k-2)\delta} \Psi(\eta_1)^4 \le \big[N^{1/2+8\delta}\Psi(\eta_0)\big]\Psi(\eta_1)^k, \end{aligned}$$

with very high probability. If $n_2 > 1$, we find, analogously to (6.5.30) in the proof of Proposition 6.5.5 (cf. also [136, Eqs. (8.86)–(8.87)])

$$\begin{aligned} (6.5.38) &\lesssim N^{-n/2} N^{(n+k-2)\delta} \|\kappa\|_n \sum_{a,b} |G_{\boldsymbol{x}a}|^2 \lesssim N^{2-n/2} N^{(n+k)\delta} \Psi(\eta_1)^2 \\ &\lesssim N^{1/2} N^{(n+k)\delta-(n-4)\xi/2} \Psi(\eta_0)\Psi(\eta_1)^{n-2} \lesssim \big[N^{1/2+8\delta}\Psi(\eta_0)\big]\Psi(\eta_1)^k, \end{aligned}$$

with very high probability. Here, to go to the second line, we used that $N^{-1/2+\xi/2} \le \Psi(\eta_0) \le \Psi(\eta_1)$. In the ultimate step, we used $\Psi(\eta_1) \le 1$ and that, since $n_2 > 1$ and $n_1 \le n_2 \le \dots \le n_k$, we have $n \ge k+2$. Therefore, using Young's inequality, we infer

$$\left| N^{-n/2} \sum_{\alpha_1,\dots,\alpha_n} \kappa(\alpha_1,\dots,\alpha_n)\, \mathbf{E}\big[\partial_{\alpha_1}\dots\partial_{\alpha_n}|S|^p\big] \right| \lesssim N^{1/2+8\delta}\Psi(\eta_0)\big[\,\mathbf{E}\,|S|^p + \Psi(\eta_1)^p\big], \tag{6.5.39}$$

with very high probability, and thus, combining (6.5.34), (6.5.35), and (6.5.36) with (6.5.39), and including the $\Psi(\eta_1)^p$ term from (6.5.33), we obtain (6.5.8). This finishes the proof of Proposition 6.5.4. □

6.6 Local law outside the support of the scDOS

In this section, we prove Theorem 6.2.9, that is, the absence of spectrum inside the gaps in the support of ρ_T of size $\Delta_T \ge N^{-3/4+5\varepsilon}$, where $\varepsilon > 0$ is the exponent from (6.3.2). Recall our choice of the terminal time $T \sim N^{-\xi/4}$ from (6.3.22).

The characteristic flow was used to exclude outliers near a regular square-root edge for Dyson Brownian motion with general β and potential in [6, Section 4]. In [136, Section 8.1], the approach was used at the edge of non-Hermitian i.i.d. matrices, which corresponds to a cusp-like singularity of the hermitization. We present a modified version of the proof that allows us to avoid moment-matching arguments, used in [136] to remove the order one Gaussian component.

6.6.1 Time-Evolution of the Gaps

First, we analyze the dynamics of the gaps in the support the scDOS corresponding to the time-dependent MDE (6.3.16). For all $t \in [0,T]$, define the density $\rho_t : \mathbb{R} \to \mathbb{R}_+$ via the Stieltjes inversion formula, $\rho_t(x) := \pi^{-1} \lim_{\eta\to+0} \langle \operatorname{Im} M_t(x+\mathrm{i}\eta)\rangle$.

Definition 6.6.1 (Endpoints of a Gap)**.** *For a continuous probability density function ρ on $\mathbb{R}$, we say that $\mathfrak{e}^-, \mathfrak{e}^+$ are left and right end-points of a gap in the support of ρ if and only if $\mathfrak{e}^-, \mathfrak{e}^+ \in \partial\{x \in \mathbb{R} : \rho(x) > 0\}$ and $\rho(x) = 0$ for all $x \in [\mathfrak{e}^-, \mathfrak{e}^+]$.*

Once Theorem 6.3.2 is established, the proof of Theorems 6.2.8 and 6.2.9 reduces to considering gaps in the support of ρ_T with at least one end point satisfying $\operatorname{dist}(\mathfrak{e}_T, \mathcal{I}) \le c_M/4$, where $\mathfrak{e}_T \in \{\mathfrak{e}_T^-, \mathfrak{e}_T^+\}$, $\mathcal{I}$ is the set of admissible energies defined in (6.2.10), and $c_M > 0$ is the constant from Assumption 6.2.5. We then distinguish between two relevant cases:

(i) The final gap size $\Delta_T := \mathfrak{e}_T^+ - \mathfrak{e}_T^- \le c_M/4$,

(ii) $\Delta_T > c_M/4$.

We focus on the more challenging case (i), which, in particular, includes all cusp-like singularities in the set of admissible energies. In this case, by Lemma 6.4.1, the solution $M_t(z)$ remains bounded in and around the gap for all times $0 \le t \le T$.

In the simpler case (ii), it is straightforward to verify that the singularity at the endpoint $\mathfrak{e}_t := \varphi_{t,T}(\mathfrak{e}_T)$ is a regular edge-point for all $0 \le t \le T$, where $\varphi_{t,T}$ is the flow map defined in (6.3.19). Consequently, there is no need to track the precise behavior of the opposite endpoint of the gap, and the analysis in Section 6.6 holds with Δ_t replaced by 1. The definition of the sub-scale domain $\mathcal{D}_t^{\mathrm{sub}}$ (see (6.6.8) below) must be adjusted by the condition $\tilde\kappa_t(z) := \operatorname{dist}(\mathfrak{e}_t, z) \le c_M/8 + C'(T-t)$, where $C' \sim 1$ is an appropriate constant (e.g., from Lemma 6.3.5). The rest of the proof then follows verbatim. Therefore, for the remainder of this section, we assume that $\Delta_T \le c_M/4$.

For any $t \in [0,T]$ and any $z := E + \mathrm{i}\eta$ with E lying inside the gap $[\mathfrak{e}_t^-, \mathfrak{e}_t^+]$ in the support of ρ_t, the scDOS $\rho_t(z)$ satisfies (see Remark 7.3 in [22])

$$\rho_t(z) \sim \frac{\eta}{(\tilde\kappa_t(z)+\eta)^{1/2}(\Delta_t + \tilde\kappa_t(z) + \eta)^{1/6}}, \quad \tilde\kappa_t(z) := \operatorname{dist}(E, \mathfrak{e}_t^{\pm}). \tag{6.6.1}$$

In the following lemma, we collect the necessary properties of the quantities $\mathfrak{e}_t^{\pm}$, Δ_t, $\tilde\kappa_t(z_t)$ along the flow (6.3.18), that we later use in the proof of Proposition 6.6.6. Recall that the terminal time is small, $T \sim N^{-\xi/4} \ll 1$ by (6.3.22), and the final gap is also sufficiently small $\Delta_T \le c_M/4$.

Lemma 6.6.2 (Characteristic Flow near Small Gaps)**.** *For any time $0 \le t \le T$, let $\mathfrak{e}_t^-, \mathfrak{e}_t^+$ be the left and right end-points of a gap in the support of ρ_t with size $0 < \Delta_t \lesssim 1$, then for any $0 \le s \le t$, there exist a gap in the support of ρ_s with endpoints $\mathfrak{e}_s^-, \mathfrak{e}_s^+$ and width $\Delta_s := \mathfrak{e}_s^+ - \mathfrak{e}_s^-$, that satisfy*

$$\Delta_s \sim \Delta_t + (t-s)^{3/2}, \tag{6.6.2}$$

$$\mathrm{d}\mathfrak{e}_s^{\pm} = -\frac{1}{2}\mathfrak{e}_s^{\pm}\mathrm{d}s - \langle M_s(\mathfrak{e}_s^{\pm})\rangle \mathrm{d}s. \tag{6.6.3}$$

Pick an $E_t \in (\mathfrak{e}_t^-, \mathfrak{e}_t^+)$ and $\eta_t \lesssim N^{-\nu}\Delta_t$ for some $\nu > 0$. Let $z_s = E_s + \mathrm{i}\eta_s := \varphi_{s,t}(E_t + \mathrm{i}\eta_t)$, as defined in (6.3.19)*, then*

$$\eta_s \lesssim N^{-\nu/2}\Delta_s, \quad E_s \in (\mathfrak{e}_s^-, \mathfrak{e}_s^+), \quad 0 \le s \le t. \tag{6.6.4}$$

Moreover, for any $0 \le s \le t$, recall $\tilde\kappa_s(z) := \operatorname{dist}(\operatorname{Re} z, \mathfrak{e}_s^{\pm})$, and assume that $\tilde\kappa_t(z_t) \gtrsim N^{\nu}\eta_t$, then

$$\eta_s^{-1}\tilde\kappa_s(z_s) \gtrsim \eta_t^{-1}\tilde\kappa_t(z_t), \quad 0 \le s \le t. \tag{6.6.5}$$

Finally, there exists a constant $\mathfrak{c} > 0$, such that for any $0 \le t \le T$, if $E_t \in (\mathfrak{e}_t^-, \mathfrak{e}_t^+)$ and $\eta_t \lesssim N^{-\nu}\tilde\kappa_t$, then $z_s := \varphi_{s,t}(E_t + \mathrm{i}\eta_t)$ satisfies

$$\sqrt{\tilde\kappa_s(z_s)} \ge \sqrt{\tilde\kappa_t(z_t)} + \mathfrak{c}(t-s)\Delta_s^{-1/6}. \tag{6.6.6}$$

We defer the proof of Lemma 6.6.2 to Appendix 6.A.

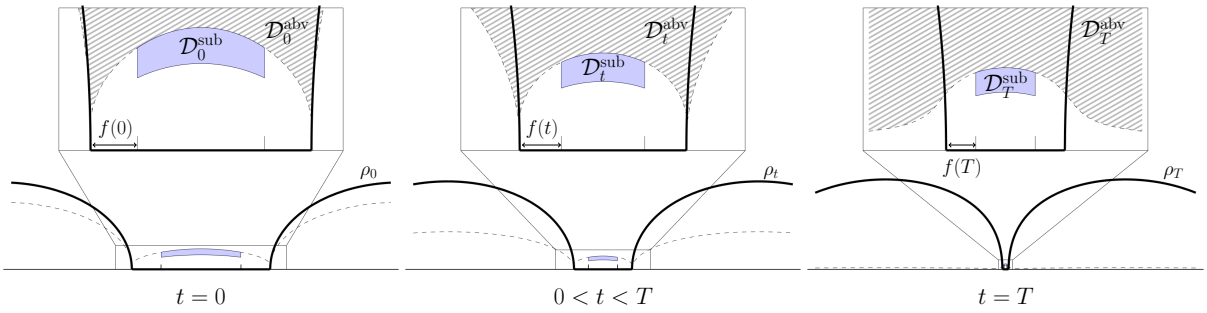

Figure 6.6.1: Shaded in blue is the illustration of the time-dependent domain $\mathcal{D}_t^{\mathrm{sub}}$, defined in (6.6.8), at three distinct times: the initial time $t = 0$ (left), an intermediate time $0 < t < T$ (center), and the terminal time $t = T$ (right). The domain $\mathcal{D}_t^{\mathrm{abv}}$ at the corresponding time t is indicated with crosshatching in the zoomed-in insert, with its boundary indicated by a dashed line in the main plot. The zoomed-in insert also depicts the distance $f(t)$, defined in (6.6.7), between the edge of the support of ρ_t and the corresponding horizontal cut-off of the domain $\mathcal{D}_t^{\mathrm{sub}}$. The graph of the scDOS ρ_t is superimposed in black on each panel (not to scale).

6.6.2 Absence of Spectrum inside Small Gaps. Proof of Theorem 6.2.9

In the sequel, we always assume that the final gap satisfies $\Delta_T \geq N^{-3/4+5\varepsilon}$. Recall the constant ε from (6.3.2), and define the function $f \equiv f_\varepsilon$ by

$$f(t) \equiv f_\varepsilon(t) := \left[\frac{N^{-1+\varepsilon} + \mathfrak{r}(T-t)}{2\Delta_t^{1/6}} \vee N^{\varepsilon}\sqrt{\eta_{\mathfrak{f},t}}\right]^2, \quad \eta_{\mathfrak{f},t} := N^{-2/3}\Delta_t^{1/9}, \quad t \in [0,T], \tag{6.6.7}$$

where we chose the constant $\mathfrak{r}$ satisfying $1 \lesssim \mathfrak{r} \leq \mathfrak{c}$ (where $\mathfrak{c}$ is the constant from (6.6.6)) to be sufficiently small such that $f(t) \leq \frac{1}{4}\Delta_t$. This is indeed possible, since it follows from (6.6.2) that $\Delta_t^{2/3} \gtrsim \Delta_T^{2/3} + (T-t)$, and $\Delta_T^{2/3} \gg N^{-1/2}$ by assumption on the final gap size.

Fix a tolerance exponent $0 < \zeta < \frac{1}{100}\xi$, where ξ is the exponent from (6.3.22), and define the time-dependent sub-scale domain $\mathcal{D}_t^{\mathrm{sub}}$ by

$$\mathcal{D}_t^{\mathrm{sub}} \equiv \mathcal{D}_t^{\mathrm{sub}}(\varepsilon,\zeta) := \left\{z := E + \mathrm{i}\eta \in \mathbb{H} : \tilde{\kappa}_t(z) \geq f(t),\, N^{-\zeta/2} \leq \rho_t(z)N\eta \leq N^{\varepsilon}\right\}, \tag{6.6.8}$$

where we recall $\tilde{\kappa}_t(z) = \mathrm{dist}(\mathrm{Re}\, z, \mathfrak{e}_t^{\pm})$. In the sequel, we omit the arguments ε, ζ of the domain $\mathcal{D}_t^{\mathrm{sub}}$ from the notation.

Definition 6.6.3 (Exclusion Estimate)**.** *Let H_u be a random matrix depending on some parameter*[11] *$u \in \mathcal{U}$, and let M_u be the solution to the MDE* (6.2.3) *with the data pair $(\mathbf{E}\, H_u, \mathcal{S}_u)$, where $\mathcal{S}_u$ is the self-energy operator corresponding to H_u via* (6.2.2)*. For all $u \in \mathcal{U}$, let $\mathcal{D}_u$ be a subset of $\mathbb{C}$, and let $\zeta > 0$. We say that the resolvent $G_u(z) := (H_u - z)^{-1}$ satisfies the* exclusion estimate*, with data $(\mathcal{D}_u, \zeta, \Omega)$ uniformly in $u \in \mathcal{U}$, if and only if the bound*

$$\left|\left\langle G_u(z) - M_u(z)\right\rangle\right| \leq \frac{N^{-\zeta}}{N|\mathrm{Im}\, z|}, \tag{6.6.9}$$

holds uniformly in $z \in \mathcal{D}_u$ and in $u \in \mathcal{U}$, on the event Ω.

The goal of the present subsection is to deduce the following claim.

Claim 6.6.4. *If a random matrix H satisfies the assumptions of Theorem 6.2.8, then for any $0 < \zeta < \frac{1}{100}\xi$, the resolvent $G(z) := (H - z)^{-1}$ satisfies the exclusion estimate* (6.6.9) *with data $(\mathcal{D}_T^{\mathrm{sub}}, 2\zeta, \Omega)$ for some very-high-probability event Ω.*

[11] As in Definition 6.3.1, the parameter u will typically be time and the set $\mathcal{U}$ will be a bounded subinterval of $\mathbf{R}$.

Then, using Claim 6.6.4 as an input, we conclude (6.2.15) using the following lemma.

Lemma 6.6.5 (Eigenvalue Exclusion)**.** *Fix a time $t \in [0,T]$, with the terminal time T as in* (6.3.22)*, and let H be a random matrix satisfying $\mathbf{E}\,H = A_t$ and $\mathcal{S}_H = \mathcal{S}_t$, where $\mathcal{S}_H$ is the self-energy corresponding to H via* (6.2.2)*. Assume that for some tolerance exponent $\nu > 0$ and $\ell \in \mathbb{N}$ with $\ell\nu \ll \zeta$, the resolvent $G(z) := (H-z)^{-1}$ satisfies the exclusion estimate* (6.6.9) *with data $(\mathcal{D}_t^{\mathrm{sub}}, \zeta - \ell\nu, \Omega)$, then*

$$\operatorname{spec}(H) \cap [\mathfrak{e}_t^- + f(t), \mathfrak{e}_t^+ - f(t)] = \varnothing \quad \text{on } \Omega. \tag{6.6.10}$$

We defer the proof of Lemma 6.6.5 to Appendix 6.A.

Proof of Theorem 6.2.9. Choose $\varepsilon := \frac{1}{5}\theta_0$, $\xi := \frac{1}{10}\varepsilon$ and $\zeta < \frac{1}{100}\xi$. It follows from Claim 6.6.4 that $G(z) := (H-z)^{-1}$ satisfies the exclusion estimate (6.6.9) with data $(\mathcal{D}_T^{\mathrm{sub}}, 2\zeta, \Omega)$ for some very-high-probability event Ω, where $\mathcal{D}_T^{\mathrm{sub}} := \mathcal{D}_T^{\mathrm{sub}}(\varepsilon,\xi)$ is defined in (6.6.8). Hence, (6.2.15) follows immediately from (6.6.10) of Lemma 6.6.5, since $f(T) := f_\varepsilon(T) \geq N^{2\varepsilon}\eta_{\mathfrak{f}}(e_0)$ by definition (6.6.7). This concludes the proof of Theorem 6.2.9. □

To prove Claim 6.6.4, we augment the Zigzag induction of Section 6.3 with the following propositions. Recall that the relations (6.3.3) between the fixed tolerance exponents ζ, ξ, ε from (6.3.2), (6.3.22) and (6.6.8), respectively.

Proposition 6.6.6 (Zig Step below the Scale)**.** *Fix $k \in \{1,\dots,K\}$, and recall the definition of t_k from* (6.3.23)*. Let $G_t(z)$ be the time-dependent resolvent defined in* (6.3.27)*. Assume that for some $\nu > 0$ and $\ell \in \mathbb{N}$ with $\ell\nu \ll \zeta$, the resolvent G_t satisfies the exclusion estimate* (6.6.9) *with data $(\mathcal{D}_t^{\mathrm{sub}}, \zeta - \ell\nu, \Omega)$ at time $t = t_{k-1}$, for some very-high-probability event Ω. Then the resolvent G_t satisfies the exclusion estimate* (6.6.9) *with data $(\mathcal{D}_t^{\mathrm{sub}}, \zeta - (\ell+1)\nu, \Omega')$ uniformly in $t \in [t_{k-1}, t_k]$, for some very-high-probability event $\Omega' \subset \Omega$.*

Proposition 6.6.7 (Zag Step below the Scale)**.** *Fix $k \in \{1,\dots,K\}$, and let $G^s(z)$ be the time-dependent resolvent defined in* (6.3.28)*, and let $s_k := s(\Delta t_k)$ be as defined in* (6.3.14)*. Assume that for some $\nu > 0$ and $\ell \in \mathbb{N}$ with $\ell\nu \ll \zeta$, the resolvent $G^s(z)$ satisfies the exclusion estimate* (6.6.9) *with data $(\mathcal{D}_{t_k}^{\mathrm{sub}}, \zeta - \ell\nu, \Omega)$ at time $s = s_k$, for some very-high-probability event Ω, and the isotropic local law in* (6.3.1) *with data $(\mathcal{D}_{t_k}^{\mathrm{abv}}, \xi + \ell\nu)$ uniformly in time $s \in [0, s_k]$. Then the bound $G^s(z)$ satisfies the exclusion estimate* (6.6.9) *with data $(\mathcal{D}_{t_k}^{\mathrm{sub}}, \zeta - (\ell+1)\nu, \Omega')$ uniformly in time $s \in [0, s_k]$, for some very-high-probability event $\Omega' \subset \Omega$.*

Proof of Claim 6.6.4. Claim 6.6.4 follows by induction in k as in Section 6.3 using the tandem of Propositions 6.6.6 and 6.6.7, and using the global law of Proposition 6.3.3 for H_0 as the initial estimate at step $k = 0$. This is indeed sufficient, since for all $z := E + \mathrm{i}\eta \in \mathcal{D}_0^{\mathrm{sub}}$, $N\eta \gtrsim N^{1/4+\varepsilon-\zeta/4}$, hence the right-hand side of (6.3.5b) satisfies

$$N^{3\xi}\Psi(z)\sqrt{\frac{\langle z\rangle}{N\eta}} \lesssim \frac{N^{3\xi}}{N\eta}\sqrt{\frac{1+\rho_0(z)N\eta}{N\eta}} \lesssim \frac{N^{-1/8+3\xi+\zeta/8}}{N\eta} \leq \frac{N^{-\zeta}}{N\eta}. \tag{6.6.11}$$

This concludes the proof of Claim 6.6.4. □

Proof of Proposition 6.6.6. The proof is essentially analogous to that in Sections 6.4, hence we only outline the key differences.

It follows from (6.3.18) that $z_s := \varphi_{s,t}(z_t) \in \mathcal{D}_s^{\mathrm{sub}}$ for all $z_t \in \mathcal{D}_t^{\mathrm{sub}}$ and all $0 \leq s \leq t$. Moreover, using (6.6.1), we conclude that

$$\frac{\operatorname{Im} z}{\tilde{\kappa}_s(z)} \lesssim \left(\frac{\eta_{\mathfrak{f},s}}{\tilde{\kappa}_s(z)}\right)^{3/4}\sqrt{\rho_s(z)N\operatorname{Im} z} \lesssim N^{-\varepsilon}, \quad z \in \mathcal{D}_s^{\mathrm{sub}}. \tag{6.6.12}$$

Therefore, it follows from (6.6.6) that for all $z_t \in \mathcal{D}_t^{\mathrm{sub}}$, the trajectory $z_s := \varphi_{s,t}(z_t)$ satisfies

$$\tilde{\kappa}_s(z_s) - f(s) = \left(\sqrt{\tilde{\kappa}_s(z_s)} + \sqrt{f(s)}\right)\left(\sqrt{\tilde{\kappa}_s(z_s)} - \sqrt{f(s)}\right) \gtrsim \sqrt{\tilde{\kappa}_s(z_s)}\,\frac{t-s}{\Delta_s^{1/6}}, \quad 0 \le s \le t, \tag{6.6.13}$$

where, in the second step, we used (6.6.6) and (6.6.7) to estimate $\sqrt{\tilde{\kappa}_s(z_s)} - \sqrt{f(s)}$.

Let $t_{\mathrm{init}} := t_{k-1}$ and $t_{\mathrm{final}} := t_k$. Define the stopping time τ by

$$\tau := \inf\left\{t_{\mathrm{init}} < t \le t_{\mathrm{final}} : \sup_{t_{\mathrm{init}} \le s \le t} \sup_{z \in \mathcal{D}_t^{\mathrm{sub}}} \left|N\eta_s \langle G_s(z) - M_s(z)\rangle\right| \le N^{-\zeta+(\ell+1)\nu}\right\}. \tag{6.6.14}$$

Statement (6.6.10) of Lemma 6.6.5 then implies that on the event $\Omega := \{t \le \tau\}$, the resolvent G_t satisfies the norm bound

$$\|G_t(z)\| \le \frac{\operatorname{Im} z}{\left(\tilde{\kappa}_t(z) - f(t)\right)^2 + (\operatorname{Im} z)^2}, \quad z \in \mathcal{D}_t^{\mathrm{sub}}. \tag{6.6.15}$$

Therefore, computing the quadratic variation of the martingale term in (6.4.2) with $B = \mathbf{1}$ similarly to (6.4.5) yields

$$\begin{aligned}\left[\int_{t_{\mathrm{init}}}^{\cdot} \frac{1}{\sqrt{N}} \sum_{ab} \partial_{ab}\langle G_s\rangle \mathrm{d}\left(\mathfrak{B}_s\right)_{ab}\right]_{t\wedge\tau} &\le \int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{\left\langle (\operatorname{Im} G_s)^2\right\rangle}{N^2\eta_s^2}\mathrm{d}s \le \int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{\left\langle \operatorname{Im} G_s\right\rangle}{N^2\eta_s^2}\frac{\eta_s}{\left(\tilde{\kappa}_s - f(s)\right)^2 + \eta_s^2}\mathrm{d}s \\ &\lesssim \int_{t_{\mathrm{init}}}^{t\wedge\tau} \frac{1}{N^2\tilde{\kappa}_s^{3/2}\Delta_s^{-1/6}\left((t\wedge\tau - s)^2 + \tilde{\kappa}_s^{-1}\Delta_s^{1/3}\eta_s^2\right)}\mathrm{d}s \\ &\lesssim \frac{1}{N^2\eta_{t\wedge\tau}^2}\frac{\eta_{t\wedge\tau}}{\tilde{\kappa}_{t\wedge\tau}} \lesssim \frac{N^{-\varepsilon}}{N^2\eta_{t\wedge\tau}^2},\end{aligned} \tag{6.6.16}$$

abbreviating $G_s := G_s(z_s)$, $\eta_s := \operatorname{Im} z_s$, and $\tilde{\kappa}_s := \tilde{\kappa}_s(z_s)$. In (6.6.16), in the second step we used (6.6.1), (6.6.13) and (6.6.14), while in the last line we used the fact that $\tilde{\kappa}_s \gtrsim \tilde{\kappa}_t$, $\Delta_s \gtrsim \Delta_t$ and $\tilde{\kappa}_s^{1/2}\Delta_s^{-1/6} \gtrsim \tilde{\kappa}_t^{1/2}\Delta_t^{-1/6}$ for all $s \le t$, that follows from (6.6.1), (6.6.2), (6.6.5) and (6.6.6).

The remainder of the proof follows analogously to Section 6.4. □

Proof of Proposition 6.6.7. Note that by choosing the constant $c' \sim 1$ in (6.3.21) small enough, we can guarantee that for any $t \in [0,T]$ and any $z := E + \mathrm{i}\eta \in \mathcal{D}_t^{\mathrm{out}}$, the point $E + \mathrm{i}\eta(E)$ lies in $\mathcal{D}_t^{\mathrm{abv}}$, where $\eta(E)$ is defined implicitly via $\eta(E)\rho_t(E + \mathrm{i}\eta(E)) = N^{-1+\varepsilon}$. Indeed, we only need to check that $\rho_t(E+\mathrm{i}\eta(E))^{-1}\eta(E) \ge c'(N^{-1+\varepsilon} + T - t)$. However, it follows from (6.6.1) and the definition of $f(t)$ in (6.6.7) that $\rho_t(E+\mathrm{i}\eta(E))^{-1}\eta(E) \gtrsim N^{\varepsilon}\eta_{\mathrm{f},t}^{1/2}\Delta_t^{1/6} + T - t$. Together with $\Delta_t \gtrsim \Delta_T \gtrsim N^{-3/4+5\varepsilon}$, this immediately implies that the inclusion $E + \mathrm{i}\eta(E) \in \mathcal{D}^{\mathrm{abv}}$ for sufficiently small $c' \sim 1$.

Since throughout the proof the time t_k remains fixed, for the remainder of this section, we drop the superscript t_k from $\mathcal{D}_{t_k}^{\mathrm{abv}}, \mathcal{D}_{t_k}^{\mathrm{sub}}, \rho_{t_k}, \tilde{\kappa}_{t_k}, \Delta_{t_k}$, and M_{t_k}.

First, using a monotonicity estimate analogous to Lemma 6.5.3 (see (6.A.10) and (6.A.11) in Remark 6.A.1), we conclude from the isotropic local law in (6.3.1) for $G^s(z)$ that, uniformly in $z \in \mathcal{D}^{\mathrm{sub}}$, in $a, b \in [N]$ and in $s \in [0, s_k]$,

$$\left|(\operatorname{Im} G^s)_{aa}\right| \lesssim \frac{N^{\varepsilon}}{N\eta}, \quad \left|(G^s - M)_{ab}\right| \lesssim \frac{N^{\varepsilon}}{N\eta}, \quad \left|(G^s)_{ab}\right| \lesssim 1, \quad \text{w.v.h.p.} \tag{6.6.17}$$

Moreover, note that for all $z := E + \mathrm{i}\eta \in \mathcal{D}^{\mathrm{sub}}$, we have the estimates

$$\tilde{\kappa}(z)\Delta^{1/3} \gtrsim N^{-1+4\varepsilon}, \quad N\eta \sim N^{1/2}\tilde{\kappa}(z)^{1/4}\Delta^{1/12}\sqrt{\rho(z)N\eta} \gtrsim N^{1/4+\varepsilon-\zeta/4}. \tag{6.6.18}$$

As in Section 6.5, we conduct the proof along the vertical truncations of the domain $\mathcal{D}^{\mathrm{sub}}$, defined as

$$\mathcal{D}^{\mathrm{sub}}_{\gamma} \equiv \mathcal{D}^{\mathrm{sub}}_{t_k,\gamma} := \big\{ z \in \mathcal{D}^{\mathrm{sub}} \mathbin{\mathtt{+}} \mathcal{D}^{\mathrm{sub}}_{t_k} : \operatorname{Im} z \geq N^{-1+\gamma} \big\}, \quad 0 < \gamma \leq 1. \tag{6.6.19}$$

In particular, we assert that if for some constant $\gamma_0 > 0$, the resolvent G^s satisfies the estimate

$$\big\langle \operatorname{Im} G^s(z) \big\rangle \lesssim \rho(z), \tag{6.6.20}$$

with very high probability uniformly in $z \in \mathcal{D}^{\mathrm{sub}}_{\gamma_0} \cup \mathcal{D}^{\mathrm{abv}}$ and in time $s \in [0, s_k]$, then the estimate (6.6.9) holds uniformly in $z \in \mathcal{D}^{\mathrm{sub}}_{\gamma_1}$ for any fixed $\gamma_1 \leq \gamma_0 - (\zeta \wedge \frac{1}{2}\mu)$, and uniformly in time $s \in [0, s_k]$ with very high probability.

To this end, we show that the quantity $R_s(z) := \langle G^s(z) - M(z) \rangle$ satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} \mathbf{E} \, |R_s(z)|^p \right| \lesssim \left(1 + \frac{N^{3\zeta}}{\sqrt{\Delta t_k}} \right) \left[\mathbf{E} \, |R_s(z)|^p + \left(\frac{N^{-\zeta}}{N |\operatorname{Im} z|} \right)^p \right], \quad z \in \mathcal{D}^{\mathrm{sub}}_{\gamma_1}, \tag{6.6.21}$$

where $\Delta t_k := t_k - t_{k-1}$ and t_k are defined in (6.3.23). Note that $N^{3\zeta}\sqrt{\Delta t_k} \leq N^{3\zeta} T^{1/2} \lesssim N^{-\zeta}$, using that $T \sim N^{-\xi/4}$ from (6.3.22). The proof of (6.6.21) is analogous to that of Proposition 6.5.5. The main difference is that for the most critical term (6.5.21), we use the bound

$$\begin{aligned} N^{-5/2} \left| \sum_{\alpha_1,\alpha_2,\alpha_3} \kappa_s(\alpha_1,\alpha_2,\alpha_3) M_{b_1 a_2} M_{b_2 a_3} (G^s G^s)_{b_3 a_1} \right| &\leq N^{-1} \|\kappa\|_3^{\mathrm{av}} \|M\|^2 \|G^s G^s\|_{\mathrm{hs}} \\ &\lesssim \frac{\langle \operatorname{Im} G^s \rangle^{1/2}}{N\eta^{3/2}} \lesssim \frac{N^{-\zeta}}{N\eta} \frac{N^{2\zeta}}{\tilde{\kappa}^{1/4} \Delta^{1/12}} \lesssim \frac{N^{-\zeta}}{N\eta} \frac{N^{2\zeta}}{\sqrt{T - t_k}} \lesssim \frac{N^{-\zeta}}{N\eta} \frac{N^{\frac{5}{2}\zeta}}{\sqrt{\Delta t_k}}, \end{aligned} \tag{6.6.22}$$

where we used (6.6.20) together with the monotonicity of the map $\eta \mapsto \eta \langle \operatorname{Im} G^s(E + \mathrm{i}\eta) \rangle$ for any fixed $E \in \mathbb{R}$ to assert that $\langle \operatorname{Im} G^s(z) \rangle \lesssim N^{2\zeta} \rho(z)$ with very high probability, uniformly in $z \in \mathcal{D}^{\mathrm{sub}}_{\gamma_1}$.

The remainder of the proof follows analogously to Section 6.5 using the estimates (6.6.17) instead of the respective bounds in (6.5.2) and (6.5.11). □

6.6.3 Improved Local Laws away from the Spectrum. Proof of Theorem 6.2.8

Let $\varepsilon := \min\{\frac{1}{5}\varepsilon_0, \frac{1}{2}\xi_0\}$ and $\xi := \frac{1}{10}\varepsilon$. Let $z \in \mathbb{C}$ be a spectral parameter satisfying $N^{\varepsilon_0}\eta_{\mathrm{f}}(E) \leq \operatorname{dist}(z, \operatorname{supp}\rho) \leq N^D$. Without loss of generality, we assume that $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = \|B\|_{\mathrm{hs}} = 1$, and that $z := E + \mathrm{i}\eta$ with $\eta \geq 0$.

First, consider the case $\operatorname{dist}(z, \operatorname{supp}\rho) \leq 2\eta$, then it is straightforward to check using the universal shape of the density ρ (see, e.g., Remark 7.3 in [22]) that $\rho(z) N\eta \gtrsim N^{\varepsilon}$. Therefore, in this regime, Theorem 6.2.8 follows from Theorem 6.3.2 and Proposition 6.3.3.

It remains to consider the regime $\operatorname{dist}(z, \operatorname{supp}\rho) \geq 2\eta$. Clearly, E lies outside of the support of ρ. Let $\mathfrak{e}^-$ and $\mathfrak{e}^+$ be the left and right end-points of the gap that contains E. The assumption $\operatorname{dist}(z, \operatorname{supp}\rho) \geq 2\eta$ implies that $\tilde{\kappa} := \operatorname{dist}(E, \mathfrak{e}^{\pm}) \gtrsim \eta$, hence $\Delta := \mathfrak{e}^+ - \mathfrak{e}^- \geq \tilde{\kappa} \gtrsim N^{\varepsilon_0}\eta_{\mathrm{f}}(E) = N^{-2/3+\varepsilon_0}\Delta^{1/9}$, and thus $\Delta \geq N^{-3/4+9\varepsilon_0/8}$.

Define a local domain $\mathcal{D}^{\mathrm{out}} \equiv \mathcal{D}^{\mathrm{out}}(E)$ as

$$\mathcal{D}^{\mathrm{out}} \equiv \mathcal{D}^{\mathrm{out}}(E) := \{ z' \in \mathbb{C} : |\operatorname{Re} z' - E| \leq \frac{1}{2}\tilde{\kappa}, |\operatorname{Im} z'| \leq \tilde{\kappa} \}, \quad \tilde{\kappa} := \operatorname{dist}(E, \mathfrak{e}^{\pm}), \tag{6.6.23}$$

and observe that $z \in \mathcal{D}^{\mathrm{out}}$. Moreover, by Theorem 6.2.9 with $\theta_0 := \frac{1}{2}\varepsilon_0$, there exists a very-high-probability event Ω, such that $\operatorname{spec}(H) \cap \mathcal{D}^{\mathrm{out}} = \varnothing$ on Ω.

Therefore, on the very-high-probability event Ω, the matrix-valued map $z' \mapsto G(z') - M(z')$ is analytic in the interior of $\mathcal{D}^{\mathrm{out}}$. Using the Cauchy formula, we obtain the contour integral representation

$$G(z) - M(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{G(z') - M(z')}{z - z'} \mathrm{d}z', \tag{6.6.24}$$

where $\Gamma \subset \mathcal{D}^{\mathrm{out}}$ is the contour tracing the boundary of a rectangle centered at z with width $\frac{1}{4}\tilde{\kappa}$ and height $\frac{3}{4}\tilde{\kappa}$. Note that $|z' - z| \gtrsim \tilde{\kappa}$ for all $z' \in \Gamma$. Using a monotonicity estimate analogous to Lemma 6.5.3 (see (6.A.11), (6.A.13) in Remark 6.A.1), we conclude from Proposition 6.3.3 and Theorem 6.3.2 that on a very-high-probability event $\Omega' \subset \Omega$, the resolvent $G(z')$ satisfies

$$\left| \left\langle \big(G(z') - M(z')\big) B \right\rangle \right| \lesssim \frac{N^{\varepsilon}}{N|\mathrm{Im}\, z'|} \wedge \frac{1}{\tilde{\kappa}}, \quad \left| \big(G(z') - M(z')\big)_{\boldsymbol{xy}} \right| \lesssim N^{\varepsilon} \sqrt{\frac{\rho(z')}{N|\mathrm{Im}\, z'|}} + \frac{N^{\varepsilon}}{N|\mathrm{Im}\, z'|} \wedge \frac{1}{\tilde{\kappa}}, \tag{6.6.25}$$

uniformly in $z' \in \Gamma$, where the alternative $\tilde{\kappa}^{-1}$ bound follows from the norm-bound on $\|G(z')\|$ and (6.2.15).

Plugging the bounds (6.6.25) into the representation (6.6.24) and using the comparison relation (6.6.1), we obtain (6.2.14b) and (6.2.14a) at the point z. Here we used (6.6.1) and $\tilde{\kappa} \geq N^{\varepsilon_0} \eta_{\mathfrak{f}}(E)$ to assert that

$$\sqrt{\frac{\rho(z)}{N\eta}} \sim \sqrt{\frac{1}{N\tilde{\kappa}^{1/2}\Delta^{1/6}}} \gtrsim \frac{1}{N\tilde{\kappa}}. \tag{6.6.26}$$

This concludes the proof of Theorem 6.2.8. □

Remark 6.6.8 (Faraway Regime)**.** *Similarly to the away from-the-spectrum part of the proof of Theorem 6.2.8 in Section 6.6.3 above, the global law* (6.3.5b)*, together with the contour integration, can be used to obtain the faraway laws*

$$\left| \left\langle \big(G(z) - M(z)\big) B \right\rangle \right| \lesssim \frac{N^{\xi_0}}{N\langle z \rangle^2} \|B\|_{\mathrm{hs}}, \quad \left| \big(G(z) - M(z)\big)_{\boldsymbol{xy}} \right| \lesssim \frac{N^{\xi_0}}{\sqrt{N}\langle z \rangle^2} \|\boldsymbol{x}\| \|\boldsymbol{y}\|, \tag{6.6.27}$$

in the regime $\mathrm{dist}(z, \mathrm{supp}\, \rho) \in [C, N^D]$ *for some sufficiently large positive* $C \sim 1$*. The proof requires only the global laws of Proposition 6.3.3 as an input, and is conducted without the use of the Zigzag dynamics.*

6.7 Global Laws: Proof of Proposition 6.3.3

We prove Proposition 6.3.3 in two steps. First, in Section 6.7.2, we prove the isotropic local law (6.3.5a). Then, in Section 6.7.3, we conclude the proof of Proposition 6.3.3 by proving the averaged law (6.3.5b), using the isotropic law (6.3.5a) as an input. Before proceeding with the proof, we collect some preliminary bounds on the stability operator and define the appropriate norm for proving the isotropic local law.

6.7.1 Preliminaries for the Global Law

First, for any $z \in \mathbb{C}$, the *stability operator* $\mathcal{B}(z) : \mathbb{C}^{N \times N} \to \mathbb{C}^{N \times N}$ is defined by its action on $X \in \mathbb{C}^{N \times N}$,

$$\mathcal{B}(z)[X] := X - M(z)\mathcal{S}[X]M(z). \tag{6.7.1}$$

We control the inverse of the stability operator $\mathcal{B}$ using the following lemma.

Lemma 6.7.1. *(Proposition 4.4 in [23]) Let* $M(z)$ *be the solution to the MDE* (6.2.3)*, and let* $\mathcal{I}$ *be the set of admissible energies defined in* (6.2.10)*. Then the stability operator* $\mathcal{B}(z)$*, defined in* (6.7.1) *satisfies, for all* $z \in \mathbb{C}$ *with* $\mathrm{dist}(\mathrm{Re}\, z, \mathcal{I}) \leq \frac{3}{4} c_M$*,*

$$\left\|\mathcal{B}^{-1}(z)\right\|_{\mathrm{hs}\to\mathrm{hs}} + \left\|\mathcal{B}^{-1}(z)\right\|_{\|\cdot\|\to\|\cdot\|} \lesssim 1 + \beta(z)^{-1}, \quad \beta(z) := \rho(z)^2 + \rho(z)|\sigma(z)| + \rho(z)^{-1}|\mathrm{Im}\, z|, \tag{6.7.2}$$

where the function[12] $\sigma(z)$ *is defined as*

$$\sigma(z) := \left\langle \operatorname{sign}\big(\operatorname{Re} U(z)\big)\big(\rho(z)^{-1}\operatorname{Im} U(z)\big)^3 \right\rangle, \qquad U := \frac{(\operatorname{Im} M)^{-1/2}(\operatorname{Re} M)(\operatorname{Im} M)^{-1/2} + \mathrm{i}}{\left|(\operatorname{Im} M)^{-1/2}(\operatorname{Re} M)(\operatorname{Im} M)^{-1/2} + \mathrm{i}\right|}, \quad z \in \mathbb{H}. \tag{6.7.3}$$

Note that by definition of $\mathcal{D}^{\mathrm{glob}}$ in (6.3.4), the stability *factor satisfies* $\beta(z) \geq N^{-\xi/4}$ for all $z \in \mathcal{D}^{\mathrm{glob}}$.

Remark 6.7.2 (Local Laws in the Stable Domain)**.** *In Section 6.7 we only use the bound* $\beta(z) \geq \rho(z)^{-1}|\operatorname{Im} z|$*. However, by Remark 10.4 in [22], there exists a function* $\widetilde{\beta}(z)$ *satisfying* $\beta(z) \lesssim \widetilde{\beta}(z) \leq \beta(z)$*, such that the map* $\eta \mapsto \widetilde{\beta}(E + \mathrm{i}\eta)$ *is non-decreasing in* $\eta > 0$ *for any fixed* E*. Therefore, the global domain, defined in* (6.3.4)*, can be replaced by the* stable domain*, defined as*

$$\mathcal{D}^{\mathrm{stab}} := \big\{z := E + \mathrm{i}\eta \in \mathbb{H} : |E| \leq N^D,\, N^{-1+\varepsilon} \leq \eta \leq N^D,\, \widetilde{\beta}(z) \geq N^{-\xi/4}\big\}, \tag{6.7.4}$$

with our proof of Proposition 6.3.3 naturally extending to the larger stable domain*. In particular, the stable domain extends down to the level* $\eta \geq N^{-1+\varepsilon}$ *in the bulk of spectrum, where* $\rho(E) \gtrsim 1$*. Therefore, we provide an independent proof of the local laws in Theorems 2.1 and 2.2 of [243] under the Assumptions 6.2.1–6.2.5 without the complicated graphical expansion machinery.*

Next, for a fixed spectral parameter $z \in \mathcal{D}^{\mathrm{glob}}(\xi, D)$, and a fixed pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$, define a family of sets of vectors,

$$\begin{aligned} &\mathcal{V}_0 \equiv \mathcal{V}_0(z) := \big\{\boldsymbol{e}_a\big\}_{a=1}^N \cup \{\boldsymbol{x}, \boldsymbol{y}\}, \\ &\mathcal{V}_j \equiv \mathcal{V}_j(z) := \mathcal{V}_{j-1} \cup \big\{M\boldsymbol{u}, \kappa_{\mathrm{c}}\big((M\boldsymbol{u})a, \cdot b\big), \kappa_{\mathrm{d}}\big((M\boldsymbol{u})a, b\cdot\big) : \boldsymbol{u} \in \mathcal{V}_{j-1},\, a, b \in [N]\big\}, \quad j \in \{1, \ldots, J\}, \end{aligned} \tag{6.7.5}$$

where $M := M(z)$, and J is an integer satisfying $J \geq 2/\xi$. We use the corresponding isotropic norm (Section 5.1 in [243])

$$\|X\|_* \equiv \|X\|_*^{\boldsymbol{x},\boldsymbol{y},J,z} := \sum_{j=0}^{J} N^{-\frac{j}{2J}} \|X\|_{(j)} + N^{-1/2} \max_{\boldsymbol{v} \in \mathcal{V}_J} \frac{\|X_{\cdot \boldsymbol{v}}\|}{\|\boldsymbol{v}\|}, \quad \|X\|_{(j)} := \max_{\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}_j} \frac{\big|X_{\boldsymbol{u}\boldsymbol{v}}\big|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}. \tag{6.7.6}$$

Note that the cardinality of the sets $\mathcal{V}_j$ is bounded by N^{CJ}, hence we can take the maximum of very-high-probability bounds over these sets.

Finally, recall that for all z with $\operatorname{Re} z$ in the set of admissible energies $\mathcal{I}$ from Assumption 6.2.5, $M(z)$ satisfies the bound

$$\|M(z)\| \lesssim \langle z \rangle^{-1}. \tag{6.7.7}$$

6.7.2 Proof of the Isotropic Bound in Proposition 6.3.3

Proof of the isotropic law in (6.3.5a). Recall the definition of the domain $\mathcal{D}^{\mathrm{glob}}$ from (6.3.4). We conduct the proof iteratively along vertical truncations $\mathcal{D}_\gamma^{\mathrm{glob}}$ of the domain $\mathcal{D}^{\mathrm{glob}}$, defined as

$$\mathcal{D}_\gamma^{\mathrm{glob}} := \big\{z := E + \mathrm{i}\eta \in \mathcal{D}^{\mathrm{glob}} : \eta \geq N^{-1+\gamma}\big\}, \quad \gamma > 0. \tag{6.7.8}$$

Once the local law (6.3.5a) is established in the domain $\mathcal{D}_{\gamma_0}^{\mathrm{glob}}$ for some $\gamma_0 \geq 0$, a simple monotonicity argument analogous to Lemma 6.5.3 (see the proof of Lemma 6.5.3 in Appendix 6.A) implies that the following bounds on the resolvent $G(z)$,

$$\big|G(z)_{\boldsymbol{u}\boldsymbol{v}}\big| \lesssim N^{\delta}\langle z \rangle^{-1}, \quad \big|\big(\operatorname{Im} G(z)\big)_{\boldsymbol{u}\boldsymbol{u}}\big| \lesssim N^{\xi+\delta}\left(\rho(z) + \frac{1}{N\eta}\right), \quad \text{w.v.h.p.}, \tag{6.7.9}$$

[12] Roughly speaking, the quantity $|\sigma(z)|$ measures how close z is to a possible almost cusp, in particular, if x is an exact cusp of the density $\rho(x)$, then $\sigma(x) = 0$.

hold uniformly in $z \in \mathcal{D}^{\mathrm{glob}}_{\gamma_1}$ for any $\gamma_1 \geq \gamma_0 - \delta$ with $\delta \leq \frac{1}{20}\xi$, and for any deterministic $\boldsymbol{u}, \boldsymbol{v}$ with $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$. Therefore, the key step in the iteration is going from estimates on the resolvent $G(z)$ to a bound on $(G(z) - M(z))_{\boldsymbol{xy}}$, that is, using the bounds (6.7.9) as an input to prove the isotropic local law (6.3.5a) in the domain $\mathcal{D}^{\mathrm{glob}}_{\gamma_1}$. This crucial step is based on the following gap in the possible values of $\|G - M\|_*$.

Lemma 6.7.3 (Gap in the Values of $G - M$)**.** *Fix a spectral parameter $z \in \mathcal{D}^{\mathrm{glob}}_{\gamma_1}$, with some $\gamma_1 > 0$ such that* (6.7.9) *holds on $\mathcal{D}^{\mathrm{glob}}_{\gamma_1}$, then*

$$\|G(z) - M(z)\|_* \lesssim N^{-\xi} \text{ w.v.h.p.} \quad \Longrightarrow \quad \|G(z) - M(z)\|_* \lesssim N^{\xi}\Psi(z) \text{ w.v.h.p.} \tag{6.7.10}$$

We initialize the iteration in the domain $\mathcal{D}^{\mathrm{glob}}_{2+\delta}$. Indeed, owing to the very high probability bound $|H_{\boldsymbol{uv}}| \lesssim N^{1/2+\nu}$ for any $\nu > 0$, we have, for any deterministic $\boldsymbol{u}, \boldsymbol{v}$ with $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1$,

$$\|G(z)\| \lesssim \langle z\rangle^{-1}, \quad \big|\big(\operatorname{Im} G(z)\big)_{\boldsymbol{uu}}\big| \lesssim \frac{\eta}{\langle z\rangle^2} \sim \rho(z), \quad z \in \mathcal{D}^{\mathrm{glob}}_{2+\delta}, \quad \text{w.v.h.p.} \tag{6.7.11}$$

Note that the bound $\|G(z) - M(z)\|_* \leq N^{-\xi}$ holds trivially for all z with $\operatorname{Im} z \geq N^{\xi}$. After Lemma 6.7.3 is established, the proof of (6.3.5a) follows the standard continuity argument on a fine grid (see Section 5.4 in [243]). This concludes the proof of the isotropic law in (6.3.5a). □

The remainder of this subsection is devoted to the proof of Lemma 6.7.3. A local law for random matrices with slow correlation decay away from the cusps was already proved in [243] and [23]. We present an independent proof under the Assumptions 6.2.1–6.2.5. We utilize the *minimalistic cumulant expansion*, that was used previously in [408] and [168]. This allows us to avoid the complicated graphical expansions.

Proof of Lemma 6.7.3. Since $z := E + \mathrm{i}\eta$ is fixed, we omit the argument of G, M, Ψ, ρ, β, and $\mathcal{B}$. Assume the very-high-probability bound

$$\|G - M\|_* \lesssim N^{-\xi}. \tag{6.7.12}$$

It suffices to show that $\|G - M\|_* \leq N^{\xi}\Psi$ with very high probability. Assume that for a deterministic control parameter ψ, the quantity $\Psi^{-1}\|G - M\|_*$ satisfies

$$\Psi^{-1}\|G - M\|_* \lesssim \psi, \quad \text{w.v.h.p.} \tag{6.7.13}$$

By definition of the resolvent $G := (H - z)^{-1}$ and the MDE (6.2.3), we difference $G - M$ satisfies

$$G - M = -M\underline{WG} + M\mathcal{S}[G - M]G, \tag{6.7.14}$$

where the matrix[13] $\underline{WG}$ is defined as

$$\underline{WG} := WG + \mathcal{S}[G]G. \tag{6.7.15}$$

Therefore, subtracting $M\mathcal{S}[G - M]M$ from both sides and the inverse of the stability operator $\mathcal{B}$, defined in (6.7.1), yields the equation

$$G - M = -\mathcal{B}^{-1}\big[M\underline{WG}\big] + \mathcal{B}^{-1}\big[M\mathcal{S}[G - M](G - M)\big], \tag{6.7.16}$$

[13] The underline $\underline{WG}$ is a renormalization of WG; for renormalization of general products $f(W)Wg(W)$, see Section 4 in [165].

Observe, that for any $X \in \mathbb{C}^{N\times N}$, (Eq. (5.4c) in [243])

$$\begin{aligned}\big\|\mathcal{B}^{-1}[X]\big\|_{(j)} &\le \|X\|_{(j)} + \left(\|M\|^2 \|\!|\mathcal{S}|\!\| + \|M\|^4 \|\!|\mathcal{S}|\!\|^2 \big\|\mathcal{B}^{-1}\big\|_{\mathrm{hs}\to\mathrm{hs}}\right)\|X\|_{\max} \\ &\lesssim \|X\|_{(j)} + \big(1+\beta^{-1}\big)\|X\|_{(0)}\,,\end{aligned} \tag{6.7.17}$$

where in the last step we used (6.7.2). Here we denote

$$|\!|\!|\mathcal{S}|\!|\!| := \|\mathcal{S}\|_{\max\to\|\cdot\|} \vee \|\mathcal{S}\|_{\mathrm{hs}\to\|\cdot\|}\,. \tag{6.7.18}$$

To control the norm $\|G-M\|_*$, we first bound the $\|\cdot\|_{(j)}$ individually, and then estimate the contribution coming from the last summand in (6.7.6) later. Fix an index $j \in \{0,\dots,J\}$ and fix a pair of vectors $\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}_j$. We compute the p-th (for even p) moment of

$$S_j \equiv S_j^{\boldsymbol{uv}} := N^{\frac{-j}{2J}}(G-M)_{\boldsymbol{uv}}, \tag{6.7.19}$$

using the equation (6.7.16) for a single factor,

$$\mathbf{E}\big[|S_j|^p\big] \le \mathbf{E}\big[N^{\frac{-j}{2J}}\big(\mathcal{B}^{-1}[M\underline{WG}]\big)_{\boldsymbol{uv}}\overline{S_j}|S_j|^{p-2}\big] + \mathbf{E}\big[N^{\frac{-j}{2J}}\big(\mathcal{B}^{-1}[M\mathcal{S}[G-M](G-M)]\big)_{\boldsymbol{uv}}\overline{S_j}|S_j|^{p-2}\big]. \tag{6.7.20}$$

First, we estimate the size of the second term on the right-hand side of (6.7.20). We observe that (Eq. (5.5a), (5.5b) in [243])

$$\|M\mathcal{S}[X]X\|_{(j)} \lesssim |\!|\!|\kappa|\!|\!|_2^{\mathrm{iso}}\,\|M\|\min\left\{\|X\|_{(j+1)}\,,\sqrt{N}\,\|X\|_{(0)}\right\}\|X\|_*\,. \tag{6.7.21}$$

We only use the second mode of the $\min$ bound when $j=J$. Combining (6.7.12), (6.7.17) and (6.7.21), we deduce that $Q_j := N^{\frac{-j}{2J}}\big(\mathcal{B}^{-1}[M\mathcal{S}[G-M](G-M)]\big)_{\boldsymbol{uv}}$ satisfies

$$\begin{aligned}\|Q_j\|_{(j)} &\lesssim \frac{\|G-M\|_*}{\langle z\rangle N^{\frac{j}{2J}}}\left(\|G-M\|_{(j+1)}\,\mathbf{1}_{j<J} + \sqrt{N}\,\|G-M\|_{(0)}\,\mathbf{1}_{j=J} + \big(1+\beta^{-1}\big)\|G-M\|_{(1)}\right) \\ &\lesssim N^{\frac{1}{2J}-\xi}\langle z\rangle^{-1}\big(1+\beta^{-1}\big)\psi\Psi, \quad \text{w.v.h.p.},\end{aligned} \tag{6.7.22}$$

where in the last step we used the estimate (6.7.2), the definition of $\mathcal{D}^{\mathrm{glob}}$ in (6.3.4), assumptions (6.7.12–6.7.13), and the bound $\|X\|_{(j)} \le N^{\frac{j}{2J}}\|X\|_*$ that follows from the definition of $\|\cdot\|_*$ in (6.7.6).

Next, we estimate the first term in (6.7.20). For any $j \in \{0,\dots,J\}$ and any $\boldsymbol{u},\boldsymbol{v} \in \mathcal{V}_j$, using the multivariate cumulant expansion formula from Proposition 6.5.2, we obtain

$$\begin{aligned}\left|\mathbf{E}\big[(M\underline{WG})_{\boldsymbol{uv}}\overline{S_j}|S_j|^{p-2}\big]\right| \le{}& \left|\mathbf{E}\left[\frac{1}{N}\sum_{ab}\sum_{\alpha_1} M_{\boldsymbol{u}a}G_{b\boldsymbol{v}}\kappa(ab,\alpha_1)\partial_{\alpha_1}\big\{\overline{S_j}|S_j|^{p-2}\big\}\right]\right| \\ &+ \sum_{k=2}^{L-1}\left|\mathbf{E}\left[\sum_{ab}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(ab)^k} M_{\boldsymbol{u}a}\frac{\kappa(ab,\boldsymbol{\alpha})}{N^{(k+1)/2}k!}\partial_{\boldsymbol{\alpha}}\big\{G_{b\boldsymbol{v}}\overline{S_j}|S_j|^{p-2}\big\}\right]\right| \\ &+ N^{\frac{j}{2J}}\big|\Omega_{j,L}^{\boldsymbol{uv}}\big|.\end{aligned} \tag{6.7.23}$$

Similarly to (6.5.17), we can choose L large enough such that $|\Omega_{j,L}| \lesssim \big(\Psi\,\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\big)^p$. We note that the $N^{\frac{-j}{2J}}$ factors in (6.7.19) are only relevant for the quadratic term Q_j estimated above, therefore, we do not follow it in the sequel. Moreover, we drop the norms $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ for brevity.

First, we estimate the term involving second-order cumulants on the right-hand side of (6.7.23). Here we estimate the contribution coming from the cross part of the second cumulants κ_{c}, the estimate for the direct part κ_{d} is completely analogous. Ignoring the difference between S_j and $\overline{S_j}$, and dropping the overall $|S_j|^{p-2}$ factor, we obtain the bound

$$\begin{aligned}\left|\frac{1}{N}\sum_{ab}\sum_{\alpha_1}\kappa_{\mathrm{c}}(ab,\alpha_1)M_{\boldsymbol{u}a}G_{b\boldsymbol{v}}\partial_{\alpha_1}S_j\right| &\lesssim \frac{1}{N}\sum_{bb_1}\left|\sum_{a_1}\kappa_{\mathrm{c}}\big((M\boldsymbol{u})\,b,a_1b_1\big)G_{\boldsymbol{u}a_1}\right|\left|G_{b\boldsymbol{v}}G_{b_1\boldsymbol{v}}\right| \\ &\lesssim N^{-1+\delta}\langle z\rangle^{-1}\big\|\|\kappa_{\mathrm{c}}((M\boldsymbol{u})*,\cdot*)\|\big\|\,\|G_{\cdot\boldsymbol{v}}\|^2 \\ &\lesssim \|\kappa\|_2^{\mathrm{iso}}N^{\xi+2\delta}\Psi^2,\quad \text{w.v.h.p.}\end{aligned} \tag{6.7.24}$$

In the ultimate step, we used (6.7.7) and (6.7.9) to assert that, with very high probability,

$$\frac{1}{\langle z\rangle\sqrt{N}}\,\|G_{\cdot\boldsymbol{v}}\| = \sqrt{\frac{(\operatorname{Im}G)_{\boldsymbol{vv}}}{\langle z\rangle^2 N\eta}} \lesssim N^{\frac{\xi+\delta}{2}}\Psi. \tag{6.7.25}$$

Next, we bound term involving third and higher order cumulants in (6.7.23). Consider, for example,

$$\begin{aligned}&\left|\sum_{ab}\sum_{\alpha_1,\alpha_2}\frac{\kappa(ab,\alpha_1,\alpha_2)}{N^{3/2}}M_{\boldsymbol{u}a}G_{b\boldsymbol{v}}(\partial_{\alpha_1}S_j)(\partial_{\alpha_2}S_j)\right| \\ &\quad\lesssim N^{-3/2}\left|\sum_{ab}\sum_{a_1b_1a_2b_2}\kappa(ab,a_1b_1,a_2b_2)M_{\boldsymbol{u}a}G_{b\boldsymbol{v}}G_{\boldsymbol{u}a_1}G_{b_1\boldsymbol{v}}G_{\boldsymbol{u}a_2}G_{b_2\boldsymbol{v}}\right| \\ &\quad\lesssim N^{3\xi/2+7\delta/2}\Psi^3\|\kappa\|_3,\quad \text{w.v.h.p.}\end{aligned} \tag{6.7.26}$$

Note that the structure of the term (6.7.26) is identical to that of (6.5.36). Indeed, the only difference is that the resolvent $G_{\boldsymbol{x}a}$ is replaced by the deterministic approximation $M_{\boldsymbol{u}a}$ ($\boldsymbol{u}$ and $\boldsymbol{v}$ in (6.7.26) play the role of $\boldsymbol{x}$ and $\boldsymbol{y}$ in (6.5.36)). Consequently, the summation over a is bounded using

$$\left(\sum_a|M_{\boldsymbol{u}a}|^2\right)^{1/2} \le \|M\| \lesssim \frac{1}{\langle z\rangle} \quad\text{instead of}\quad \left(\sum_a|G_{\boldsymbol{u}a}|^2\right)^{1/2} \lesssim N^{\frac{\xi+\delta}{2}}\sqrt{\frac{\rho+\frac{1}{N\eta}}{\eta}}, \tag{6.7.27}$$

yielding a saving of a $\sqrt{\rho/\eta}$ factor in terms of the (ρ/η)-power on the right-hand side of (6.7.26) compared to the bound in (6.5.36). All other terms in (6.7.23) with cumulant of order three and higher are bounded analogously to their counterparts in the proof of Proposition 6.5.4, with the additional saving of $\sqrt{\rho/\eta}$ coming from (6.7.27).

Therefore, using a weighted Young inequality to handle the separated $|S_j|^{p-k}$ terms, we deduce that for all $j\in\{0,\dots,J\}$,

$$\mathbf{E}\big[N^{\frac{-j}{2J}}\big(\mathcal{B}^{-1}[M\underline{WG}]\big)_{\boldsymbol{uv}}\overline{S_j}|S_j|^{p-2}\big] \le \big(N^{\xi/2+4\delta}(1+\beta^{-1})\Psi\big)^p + N^{-p\delta}\,\mathbf{E}\big[|S_j|^p\big]. \tag{6.7.28}$$

It follows from (6.7.20), (6.7.22), (6.7.23), and (6.7.28) that

$$\mathbf{E}\big[|S_j|^p\big] \lesssim \big(\Psi\big)^p\left(N^{\xi/2+4\delta}(1+\beta^{-1}) + N^{-\delta}\psi + N^{\frac{1}{2J}-\xi+\delta}\langle z\rangle^{-1}\big(1+\beta^{-1}\big)\psi\right)^p. \tag{6.7.29}$$

Since $J\ge 2/\xi$, and $\delta\le\xi/20$, we have $(1+\beta^{-1})N^{\frac{1}{2J}-\xi+\delta}\le N^{-\delta}$, and we conclude that

$$|S_j| \lesssim N^{\nu}\Psi\big(N^{3\xi/4+4\delta}(1+\beta^{-1}) + N^{-\delta}\psi\big),\quad \text{w.v.h.p.} \tag{6.7.30}$$

Next, we estimate the contribution of the last summand in (6.7.6) to $\|G-M\|_*$. We fix a vector $\boldsymbol{v}\in\mathcal{V}_J$ and compute a the p-th (for even p) moment of

$$S \equiv S^{\boldsymbol{v}} := N^{-1}\,\|(G-M)_{\cdot\boldsymbol{v}}\|^2 = N^{-1}\big((G-M)^*(G-M)\big)_{\boldsymbol{vv}}. \tag{6.7.31}$$

Using the equation (6.7.14) for a single S factor, we obtain

$$\begin{aligned}\mathbf{E}\big[|S|^p\big] \le N^{-1}\big|\mathbf{E}\big[\big((G-M)^*M\underline{WG}\big)_{\boldsymbol{vv}}\overline{S}|S|^{p-2}\big]\big| \\ + N^{-1}\big|\mathbf{E}\big[\big((G-M)^*M\mathcal{S}[G-M]G\big)_{\boldsymbol{vv}}\overline{S}|S|^{p-2}\big]\big|.\end{aligned} \tag{6.7.32}$$

To estimate the term in the second line of (6.7.32), we note the following bound,

$$\big|\big(X^*M\mathcal{S}[X]Y\big)_{\boldsymbol{vv}}\big| \le \|X_{\cdot\boldsymbol{v}}\|\,\|M\|\,\|\mathcal{S}\|_{\max\to\|\cdot\|}\,\|X\|_{(0)}\,\|Y_{\cdot\boldsymbol{v}}\|\,. \tag{6.7.33}$$

Therefore, using (6.7.6), (6.7.12), (6.7.13), (6.7.25), and (6.7.33), we obtain the very-high-probability bound

$$\frac{1}{N}\big|\big((G-M)^*M\mathcal{S}[G-M]G\big)_{\boldsymbol{vv}}\big| \lesssim \frac{1}{\langle z\rangle\sqrt{N}}\,\|G-M\|_*^2\,\|G_{\cdot\boldsymbol{v}}\| \lesssim N^{-\xi+\delta}\Psi^2\psi. \tag{6.7.34}$$

Next, we turn to estimating the first term on the right-hand side of (6.7.32) using the multivariate cumulant expansion formula,

$$\begin{aligned}\frac{1}{N}\big|\mathbf{E}\big[\big((G-M)^*M\underline{WG}\big)_{\boldsymbol{vv}}\overline{S}|S|^{p-2}\big]\big| \lesssim \frac{1}{N^2}\bigg|\sum_{abc}\sum_{\alpha_1}\kappa(ab,\alpha_1)G_{b\boldsymbol{v}}M_{ca}\partial_{\alpha_1}\big\{(G-M)^*_{\boldsymbol{v}c}\overline{S}|S|^{p-2}\big\}\bigg| \\ + \frac{1}{N}\sum_{k=2}^{L}\bigg|\sum_{abc}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(ab)^k}\frac{\kappa(ab,\boldsymbol{\alpha})}{N^{(k+1)/2}k!}M_{ca}\partial_{\boldsymbol{\alpha}}\big\{G_{b\boldsymbol{v}}(G-M)^*_{\boldsymbol{v}c}\overline{S}|S|^{p-2}\big\}\bigg| \\ + \Omega_L^{\boldsymbol{v}},\end{aligned} \tag{6.7.35}$$

where for sufficiently large integer L, the error term $\Omega_L^{\boldsymbol{v}}$ admits the bound $\Omega_L^{\boldsymbol{v}} \lesssim \Psi^{2p}$, and is therefore negligible.

We bound the term involving the second cumulants in (6.7.35). First, for the term containing $\partial_{\alpha_1}(G-M)^*_{\boldsymbol{v}c}$, completely analogously to (6.7.24), we obtain

$$\frac{1}{N^2}\sum_c\bigg|\sum_{ba_1b_1}\kappa\big((M\boldsymbol{e}_c)b,a_1b_1\big)G_{b\boldsymbol{v}}G^*_{\boldsymbol{v}a_1}G^*_{b_1c}\bigg| \lesssim \|\kappa\|_2^{\mathrm{iso}}N^{\xi+2\delta}\Psi^2, \quad \text{w.v.h.p.}, \tag{6.7.36}$$

where the additional summation over the index c is compensated by the N^{-1} prefactor. Next, we estimate the terms arising from $\partial_{\alpha_1}S$. We focus on the term containing $((G-M)^*\partial_{\alpha_1}G)_{\boldsymbol{vv}}$, other terms are estimated similarly. For the cross part κ_{c}, we obtain (ignoring the factor $|S|^{p-2}$ temporarily)

$$\begin{aligned}&\frac{1}{N^3}\bigg|\sum_{cb}\sum_{\alpha_1}\kappa_{\mathrm{c}}\big((M\boldsymbol{e}_c)b,\alpha_1\big)G_{b\boldsymbol{v}}(G-M)^*_{\boldsymbol{v}c}\big((G-M)^*\partial_{\alpha_1}G\big)_{\boldsymbol{vv}}\bigg| \\ &\quad\lesssim \frac{1}{N^2}\sum_{cd}\big|(G-M)^*_{\boldsymbol{v}c}(G-M)^*_{\boldsymbol{v}d}\big|\frac{1}{N}\sum_{bb_1}\bigg|\sum_{a_1}\kappa_{\mathrm{c}}\big((M\boldsymbol{e}_c)b,a_1b_1\big)G_{da_1}\bigg|\big|G_{b\boldsymbol{v}}G_{b_1\boldsymbol{v}}\big| \\ &\quad\lesssim \|\kappa\|_2^{\mathrm{iso}}N^{\xi+2\delta}\Psi^2\frac{1}{N^2}\sum_{cd}\big|(G-M)^*_{\boldsymbol{v}c}(G-M)^*_{\boldsymbol{v}d}\big| \\ &\quad\lesssim \|\kappa\|_2^{\mathrm{iso}}N^{\xi+2\delta}\Psi^2\,\|G-M\|_*^2 \lesssim \|\kappa\|_2^{\mathrm{iso}}N^{\xi+2\delta}\Psi^4\psi^2, \quad \text{w.v.h.p.},\end{aligned} \tag{6.7.37}$$

where in the second step we used the bound analogous to (6.7.24) for each c,d, and in the last step we used (6.7.13).

Similar estimates hold for terms involving higher order cumulants in (6.7.35). For example, identifying $\alpha_i := (a_i, b_i)$,

$$\begin{aligned} N^{-7/2}&\Big|\sum_{abc}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)M_{ca}(G-M)^*_{\boldsymbol{v}c}\big(\partial_{\alpha_1}G\big)_{b\boldsymbol{v}}\big((G-M)^*\partial_{\alpha_2}G\big)_{\boldsymbol{v}\boldsymbol{v}}\Big| \\ &\lesssim N^{-7/2}\sum_{cd}\big|(G-M)^*_{\boldsymbol{v}d}(G-M)^*_{\boldsymbol{v}c}\big|\Big|\sum_{ab}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)M_{ca}G_{ba_1}G_{b_1\boldsymbol{v}}G_{da_2}G_{b_2\boldsymbol{v}}\Big| \\ &\lesssim \|\kappa\|_3\,\|G-M\|_*^2\,N^{\xi+3\delta}\Psi^2 \lesssim \|\kappa\|_3 N^{\xi+3\delta}\langle z\rangle^{-1}\Psi^4\psi^2, \quad \text{w.v.h.p.} \end{aligned} \tag{6.7.38}$$

Therefore, we obtain, using the very-high-probability bound $S \lesssim \psi\Psi$ by (6.7.13),

$$\frac{1}{N}\big|\mathbf{E}\big[\big((G-M)^*M\underline{WG}\big)_{\boldsymbol{v}\boldsymbol{v}}\overline{S}|S|^{p-2}\big]\big| \lesssim \big(\Psi\big)^{2p}\bigg(N^{\xi}N^{8\delta}+N^{-\delta}\psi^2\bigg)^p, \tag{6.7.39}$$

hence, using (6.7.32) and (6.7.34), we deduce that with very high probability,

$$\sqrt{S} \lesssim N^{\nu}\Psi\big(N^{\xi/2+4\delta}+N^{-\delta/2}\psi\big). \tag{6.7.40}$$

It follows from (6.7.6), (6.3.6), (6.7.30) and (6.7.40), that

$$\Psi^{-1}\,\|G-M\|_* \lesssim \psi \text{ w.v.h.p.} \Longrightarrow \Psi \lesssim N^{\xi/2+4\delta+\nu}(1+\beta^{-1}) + N^{-\delta/2+\nu}\psi \text{ w.v.h.p.} \tag{6.7.41}$$

By iteration, this implies that $\Psi^{-1}\,\|G-M\|_* \lesssim N^{\xi/2+4\delta+\nu}(1+\beta^{-1}) \lesssim N^{3\xi/4+4\delta+\nu}$ with very high probability, since $\beta \geq N^{-\xi/4}$ in $\mathcal{D}^{\mathrm{glob}}$. This concludes the proof of Lemma 6.7.3. □

6.7.3 Proof of the Averaged Bound in Proposition 6.3.3

We conclude this section by proving the averaged law in Proposition 6.3.3 using the isotropic law (6.3.5a), proved in Section 6.7.2 above, as an input.

Proof of the averaged law in (6.3.5b). Fix a deterministic matrix B and a spectral parameter $z \in \mathcal{D}^{\mathrm{glob}}$, and let $R := \langle (G-M)B\rangle$. Using the equation (6.7.16), we compute the p-th (for even p) moment of R,

$$\mathbf{E}\big[|R|^p\big] \leq \big|\mathbf{E}\big[\big\langle M\mathcal{S}[G-M](G-M)\widetilde{B}\big\rangle\overline{R}|R|^{p-2}\big]\big| + \big|\mathbf{E}\big[\big\langle M\underline{WG}\widetilde{B}\big\rangle\overline{R}|R|^{p-2}\big]\big|, \tag{6.7.42}$$

where we denote $\widetilde{B} := \big((\mathcal{B}^{-1})^*[B^*]\big)^*$. By (6.7.2) and Lemma 6.7.1, the observable $\widetilde{B}$ satisfies

$$\big\|\widetilde{B}\big\|_{\mathrm{hs}} \lesssim \big(1+\beta^{-1}\big)\,\|B\|_{\mathrm{hs}}\,. \tag{6.7.43}$$

To bound the first term on the right-hand side of (6.7.42), we employ the polar decomposition $\widetilde{B} = \sum_j \sigma_j \boldsymbol{v}_j\boldsymbol{u}_j^*$, where $\sigma_j := \sigma_j(\widetilde{B})$ and $\boldsymbol{u}_j := \boldsymbol{u}_j(\widetilde{B}), \boldsymbol{v}_j := \boldsymbol{v}_j(\widetilde{B})$ are the singular values and corresponding left and right, respectively, singular vectors of $\widetilde{B}$. It follows from (6.3.5a), (6.7.21), and (6.7.43), that with very high probability,

$$\big|\big\langle M\mathcal{S}[G-M](G-M)\widetilde{B}\big\rangle\big| \leq \frac{1}{N}\sum_j|\sigma_j|\big|\big\langle\big(M\mathcal{S}[G-M](G-M)\big)_{\boldsymbol{u}_j\boldsymbol{v}_j}\big\rangle\big| \lesssim N^{2\xi}\big(1+\beta^{-1}\big)\Psi^2\,\|B\|_{\mathrm{hs}}\,, \tag{6.7.44}$$

where $\Psi := \Psi(z)$ is defined in (6.3.6).

Next, we bound the second term on the right-hand side of (6.7.42) using the multivariate cumulant expansion formula from Proposition 6.5.2,

$$\begin{aligned}\Big|\mathbf{E}\big[\big\langle M\underline{WG}\widetilde{B}\big\rangle\overline{R}|R|^{p-2}\big]\Big| &\le \bigg|\mathbf{E}\bigg[\frac{1}{N^2}\sum_{ab}\sum_{\alpha_1}\kappa(ab,\alpha_1)\big(G\widetilde{B}M\big)_{ba}\partial_{\alpha_1}\big\{\overline{R}|R|^{p-2}\big\}\bigg]\bigg| \\ &\quad+\sum_{k=2}^{L}\bigg|\mathbf{E}\bigg[\frac{1}{N}\sum_{ab}\sum_{\boldsymbol{\alpha}\in\mathcal{N}(ab)^k}\frac{\kappa(ab,\boldsymbol{\alpha})}{N^{(k+1)/2}k!}\partial_{\boldsymbol{\alpha}}\big\{\big(G\widetilde{B}M\big)_{ba}\overline{R}|R|^{p-2}\big\}\bigg]\bigg| \\ &\quad+\big|\Omega_L^B\big|.\end{aligned} \tag{6.7.45}$$

Here, once again Ω_L^B is an error term satisfying $|\Omega_L^B| \lesssim (\sqrt{\langle z\rangle/(N\eta)}\Psi\,\|B\|_{\mathrm{hs}})^p$ for large enough L, controlled similarly to (6.5.17). The terms involving second order cumulants admit the bound (ignoring the common $|R|^{p-2}$ factor)

$$\begin{aligned}\bigg|\frac{1}{N^2}\sum_{ab}\sum_{\alpha_1}\kappa(ab,\alpha_1)\big(G\widetilde{B}M\big)_{ba}\partial_{\alpha_1}R\bigg| &\le \bigg|\frac{1}{N^3}\sum_{ab}\sum_{a_1b_1}\kappa(ab,a_1b_1)\big(G\widetilde{B}M\big)_{ba}\big(GBG\big)_{b_1a_1}\bigg| \\ &\le \frac{1}{\langle z\rangle N^2\eta^2}\big\||\kappa(*,*)|\big\|\big\langle\widetilde{B}\widetilde{B}^*\mathrm{Im}\,G\big\rangle^{1/2}\big\langle BB^*\mathrm{Im}\,G\big\rangle^{1/2} \\ &\lesssim N^{\xi}\big(1+\beta^{-1}\big)\|\kappa\|_2\frac{\langle z\rangle}{N\eta}\Psi^2\,\|B\|_{\mathrm{hs}}^2, \quad \text{w.v.h.p.},\end{aligned} \tag{6.7.46}$$

where in the second step we used the norm bound (6.7.7). Here, in the last step, we used the established isotropic law (6.3.5a), the spectral decomposition of $\widetilde{B}\widetilde{B}^*$ and (6.7.43) to assert that, with very high probability,

$$\frac{\big\langle\widetilde{B}\widetilde{B}^*\mathrm{Im}\,G\big\rangle}{N\eta} = \frac{1}{N^2\eta}\sum_j|\sigma_j|^2\big(\mathrm{Im}\,G\big)_{\boldsymbol{u}_j\boldsymbol{u}_j} \lesssim \frac{N^{\xi}}{N^2\eta}\sum_j|\sigma_j|^2\bigg(\rho+\sqrt{\frac{\rho}{N\eta}}+\frac{1}{N\eta}\bigg) \lesssim N^{\xi}\big(1+\beta^{-1}\big)^2\langle z\rangle^2\Psi^2\|B\|_{\mathrm{hs}}^2, \tag{6.7.47}$$

where σ_j and $\boldsymbol{u}_j$ are the singular values and left singular vectors of $\widetilde{B}$. Similar bound without the factor $(1+\beta^{-1})^2$ holds for B instead of $\widetilde{B}$. Note that, unlike for the isotropic law (6.3.5a), for the current proof of the average law there is no need to split the second order cumulant into direct and cross terms, the simpler bound (6.2.6) suffices.

Next, we estimate the terms in (6.7.45) involving third order cumulants. Consider the term containing a single (∂R). Dropping $|R|^{p-2}$, we obtain

$$\begin{aligned}&\bigg|N^{-5/2}\sum_{ab}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)\big(\partial_{\alpha_1}G\widetilde{B}M\big)_{ba}(\partial_{\alpha_2}R)\bigg| \\ &\qquad\lesssim N^{-7/2}\,\|M\|\max_{cd}|G_{cd}|\sum_{ab}\sum_{\alpha_1\alpha_2}\big|\kappa(ab,\alpha_1,\alpha_2)\big|\big|\big(G\widetilde{B}\widetilde{B}^*G^*\big)_{b_1b_1}\big|^{1/2}\big(GBG\big)_{\alpha_2} \\ &\qquad\lesssim \langle z\rangle^{-2}\bigg\|\sum_{ab}|\kappa(ab,*,*)|\bigg\|\sqrt{\frac{\langle\widetilde{B}\widetilde{B}^*\mathrm{Im}\,G\rangle}{N\eta}}\sqrt{\frac{\langle BB^*\mathrm{Im}\,G\rangle}{N^3\eta^3}} \lesssim N^{\xi}\big(1+\beta^{-1}\big)\frac{\|\kappa\|_3}{N\eta}\Psi^2\,\|B\|_{\mathrm{hs}}^2,\end{aligned} \tag{6.7.48}$$

with very high probability, where we used (6.3.5a), (6.7.47), and the bounds

$$\big|(G\widetilde{B}M)_{ab}\big| \le \|M\|\,\big|\big(G\widetilde{B}\widetilde{B}^*G^*\big)_{aa}\big|^{1/2}, \quad \frac{1}{N}\sum_{ab}|(GBG)_{ab}|^2 \le \frac{1}{\eta^3}\langle BB^*\mathrm{Im}\,G\rangle. \tag{6.7.49}$$

The term containing $(\partial^2 R)$ admits a completely analogous estimate.

For the term containing $(\partial R)^2$, we obtain, dropping $|R|^{p-3}$,

$$\begin{aligned}\Big|N^{-5/2}\sum_{ab}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)\big(G\widetilde{B}M\big)_{ba}(\partial_{\alpha_1}R)(\partial_{\alpha_2}R)\Big| \\ &\lesssim N^{-9/2}\max_{\alpha}\big|(GBG)_\alpha\big|\sum_{ab,\alpha_2}\sum_{\alpha_1}\big|\kappa(ab,\alpha_1,\alpha_2)\big|\big(G\widetilde{B}M\big)_{ba}\big(GBG\big)_{\alpha_2}\big| \\ &\lesssim N^{\xi}\langle z\rangle^2\Psi^2\,\|B\|_{\mathrm{hs}}\Big\|\sum_{\alpha_1}|\kappa(*,\alpha_1,*)|\Big\|\sqrt{\frac{\langle\widetilde{B}MM^*\widetilde{B}^*\operatorname{Im}G\rangle}{N\eta}}\sqrt{\frac{\langle BB^*\operatorname{Im}G\rangle}{N^3\eta^3}} \\ &\lesssim N^{2\xi}\big(1+\beta^{-1}\big)\frac{\|\kappa\|_3}{N\eta}\langle z\rangle^3\Psi^4\|B\|_{\mathrm{hs}}^3,\quad \text{w.v.h.p.},\end{aligned} \tag{6.7.50}$$

where we used the local law (6.3.5a) to assert that, with very high probability,

$$\frac{1}{N^{3/2}}\big|(GBG)_{ab}\big|\lesssim\frac{\|B\|}{\sqrt{N}}\frac{\sqrt{\big(\operatorname{Im}G\big)_{aa}\big(\operatorname{Im}G\big)_{bb}}}{N\eta}\lesssim N^{\xi}\langle z\rangle^2\Psi^2\,\|B\|_{\mathrm{hs}}\,. \tag{6.7.51}$$

Note that in estimating $\max_\alpha|(GBG)_\alpha|$, we need to use the operator norm $\|B\|$ since no summation on indices is available. We convert it into $\|B\|_{\mathrm{hs}}$ at a costs of an extra $\sqrt{N}$ factor, as $\|B\|\le\sqrt{N}\,\|B\|_{\mathrm{hs}}$, but this is affordable since we collected sufficiently many powers of $N^{-1/2}$ in the third cumulant term.

Finally, we estimate the term with no (∂R), namely, dropping $|R|^{p-1}$

$$\Big|N^{-5/2}\sum_{ab}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)G_{ba_1}G_{b_1a_2}\big(G\widetilde{B}M\big)_{b_2a}\Big|. \tag{6.7.52}$$

For both G_{ba_1} and $G_{b_1a_2}$, we write $G_{ab}=M_{ab}+(G-M)_{ab}$ and use the bound $|M_{ab}|\lesssim\langle z\rangle^{-1}$, $|(G-M)_{ab}|\lesssim N^{\xi}\Psi$, w.v.h.p., that follow from (6.7.7) and (6.3.5a), respectively, to estimate the contributions coming from the deterministic and the fluctuating part separately. In particular, we obtain the very-high-probability bound,

$$\begin{aligned}\Big|N^{-5/2}\sum_{ab}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)(G-M)_{ba_1}M_{b_1a_2}\big(G\widetilde{B}M\big)_{b_2a}\Big| \\ &\lesssim N^{-1/2+\xi}\Psi\langle z\rangle^{-2}\Big\|\sum_{\alpha_1}|\kappa(*,\alpha_1,*)|\Big\|\big\langle G\widetilde{B}\widetilde{B}^*G^*\big\rangle^{1/2} \\ &\lesssim\langle z\rangle^{-1}N^{2\xi}\big(1+\beta^{-1}\big)\|\kappa\|_3\Psi^2\,\|B\|_{\mathrm{hs}}\,.\end{aligned} \tag{6.7.53}$$

The contributions coming from $M_{ba_1}(G-M)_{b_1a_2}$ and $(G-M)_{ba_1}(G-M)_{b_1a_2}$ admit analogous estimates. Therefore, it remains to bound the contribution coming from $M_{ba_1}M_{b_1a_2}$. Using (6.2.8), we estimate

$$\begin{aligned}\Big|N^{-5/2}\sum_{ab}\sum_{\alpha_1,\alpha_2}\kappa(ab,\alpha_1,\alpha_2)M_{ba_1}M_{b_1a_2}\big(G\widetilde{B}M\big)_{b_2a}\Big|&\le N^{-1}\|\kappa\|_3^{\mathrm{av}}\,\|M\|^2\,\big\|G\widetilde{B}M\big\|_{\mathrm{hs}} \\ &\lesssim N^{\xi/2}\big(1+\beta^{-1}\big)\langle z\rangle^{-2}N^{-1/2}\Psi\,\|B\|_{\mathrm{hs}}\,,\quad\text{w.v.h.p.}\end{aligned} \tag{6.7.54}$$

Putting back the dropped $|R|$ factors into the estimates (6.7.46), (6.7.48), (6.7.50), (6.7.53) and (6.7.54) and using the Young's inequality to separate these factors into an additive $|R|^p$ term with a small multiplicative constant, we see that the second and third order cumulant terms in (6.7.45) can be estimated by $\big(N^{3\xi/2}\langle z\rangle^{1/2}(N\eta)^{-1/2}\Psi\,\|B\|_{\mathrm{hs}}\big)^p+N^{-p\xi/4}|R|^p$. Here we used $\beta\ge N^{-\xi/4}$ from (6.3.4).

Estimating the terms involving forth and higher order cumulants using simple power counting, similarly to (6.5.30)–(6.5.31), we deduce that

$$\mathbf{E}\big[|R|^p\big] \lesssim \left(N^{3\xi/2}\langle z\rangle^{1/2}(N\eta)^{-1/2}\Psi\,\|B\|_{\mathrm{hs}}\right)^p + N^{-p\xi/4}\,\mathbf{E}\big[|R|^p\big]. \tag{6.7.55}$$

This concludes the proof of (6.3.5b). □

6.A Technical Lemmas

In this appendix, we collect the proofs of several technical lemmas used throughout this paper.

Proof of Lemma 6.3.4. Observe that for any $s \geq 0$ and any initial condition H, the distribution of the random matrix $\mathfrak{F}^s_{\mathrm{zag}}[H]$ satisfies

$$\mathbf{F}^s_{\mathrm{zag}}\big[H\big] \overset{d}{=} \mathbf{E}[H] + \mathrm{e}^{-s/2}\big(H - \mathbf{E}\,H\big) + \sqrt{1-\mathrm{e}^{-s}}\,\Sigma_H^{1/2}\big[W_{\mathrm{G}}\big], \tag{6.A.1}$$

where W_{G} is a standard GUE/GOE random matrix (in the same symmetry class as H) independent of H. Moreover, if $\Sigma_H \geq c\Sigma_{\mathrm{G}}$ for some constant $0 < c < 1$, then there exists a random matrix $\widehat{W}$ with $\mathbf{E}\,\widehat{W} = 0$, such that

$$\Sigma_H^{1/2}\big[W_{\mathrm{G}}\big] \overset{d}{=} \widehat{W} + \sqrt{c}\,\widetilde{W}_{\mathrm{G}}, \tag{6.A.2}$$

where $\widetilde{W}_{\mathrm{G}}$ is a GUE/GOE matrix independent of $\widehat{W}$. Therefore,

$$\mathbf{F}^s_{\mathrm{zag}}\big[H\big] \overset{d}{=} \widehat{H}^s + \sqrt{c}\sqrt{1-\mathrm{e}^{-s}}\,\widetilde{W}_{\mathrm{G}}, \quad \widehat{H}^s \overset{d}{=} \mathbf{E}[H] + \mathrm{e}^{-s/2}\big(H - \mathbf{E}\,H\big) + \sqrt{1-\mathrm{e}^{-s}}\,\widehat{W}, \tag{6.A.3}$$

where $\widetilde{W}_{\mathrm{G}}$ is independent of $\widehat{H}^s$. Hence, (6.3.13) follows immediately from (6.3.8) and (6.A.3) for $\mathfrak{H}_{c,t}(H)$ defined as

$$\mathbf{H}_{c,t}(H) := \mathrm{e}^{t/2}\left(\mathbf{E}\,H + \mathrm{e}^{-s(t)/2}\big(H - \mathbf{E}\,H\big) + \sqrt{1-\mathrm{e}^{-s(t)}}\widehat{W}\right), \tag{6.A.4}$$

where the random matrix $\widehat{W}$ independent of H satisfies (6.A.2), and $s(t) \equiv s_c(t)$ is defined in (6.3.14). The estimate (6.3.15) is a direct consequence of (6.3.14). This concludes the proof of Lemma 6.3.4. □

Proof of Lemma 6.3.5. Fix a time $0 \leq t \leq T$ and let z_s denote the solution of (6.3.18) that satisfies $z_t \in \mathcal{D}_t^{\mathrm{abv}}$. It follows from (6.3.18) and (6.3.20) that for all $s \in [0,t]$,

$$\mathrm{d}\big(\eta_s\rho_s(z_s)\big) = -\pi\rho_s(z_s)^2\mathrm{d}s \leq 0, \tag{6.A.5}$$

where we denote $\eta_s := \operatorname{Im} z_s$. A similar computation reveals that

$$\mathrm{d}\big(\rho_s(z_s)^{-1}\eta_s\big) = -\big(\rho_s(z_s)^{-1}\eta_s + \pi\big)\mathrm{d}s \leq -\pi\mathrm{d}s, \tag{6.A.6}$$

since $\rho_s^{-1}(z)\operatorname{Im} z \geq 0$ for all $z \in \mathbb{C}$. Moreover, it follows from (6.2.5) that $|\mathrm{d}z_s/\mathrm{d}s| \lesssim C'$ for all $0 \leq s \leq T$ and all $z_t \in \mathcal{D}_t^{\mathrm{abv}}$, hence, using the estimates (6.A.5) and (6.A.6), we deduce that $z_s \in \mathcal{D}_s^{\mathrm{abv}}$ for all $s \in [0,t]$. This concludes the proof of Lemma 6.3.5. □

Proof of Lemma 6.4.1. Clearly, for terminal times $0 \leq T \lesssim 1$, the solutions to (6.3.17) satisfy $\|A_t - A_T\| \lesssim T - t$ and $\|\mathcal{S}_t - \mathcal{S}_T\|_{\|\cdot\|\to\|\cdot\|} \lesssim T - t$, for all $0 \leq t \leq T$. Therefore, for some sufficiently small threshold $T_* \sim 1$, the first bound in (6.4.1) follows immediately from Assumption 6.2.5 and the stability of the MDE against small perturbations of the data pair, see Section 10 in [22]. Moreover, it follows from the fullness Assumption 6.2.4, that, by possibly shrinking the threshold T_*, we can guarantee that $\mathcal{S}_t[X] \sim \langle X\rangle$ for any Hermitian matrix $X \geq 0$. Hence, the second bound in (6.4.1) follows from Proposition 3.5 in [22] and the first bound in (6.4.1). This concludes the proof of Lemma 6.4.1. □

Proof of Lemma 6.5.3. Throughout the proof, we consider the time $s \in [0, s_{\mathrm{final}}]$ to be fixed, and drop it from the superscript of G^s. The uniformity of all estimates in s follows trivially from the assumptions of Lemma 6.5.3.

First, we prove the second estimate in (6.5.6). The map $\eta \mapsto \eta^2/(x^2+\eta^2)$ is increasing in $\eta > 0$ for any $x \in \mathbb{R}$, hence it follows by spectral decomposition of $\operatorname{Im} G$ that

$$\eta_1 \operatorname{Im} G(E+\mathrm{i}\eta_1) \leq \eta_0 \operatorname{Im} G(E+\mathrm{i}\eta_0), \tag{6.A.7}$$

in the sense of quadratic forms. Therefore, the second estimate in (6.5.6) follows immediately from (6.5.2).

Next, we prove the first estimate in (6.5.6). Using the Schwarz inequality and the Ward identity, we deduce that for all $0 < \eta < \eta_0$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\eta} \big(G(E+\mathrm{i}\eta)\big)_{\boldsymbol{uv}} \right| \lesssim \frac{\left| \big(\operatorname{Im} G(E+\mathrm{i}\eta)\big)_{\boldsymbol{uu}} \big(\operatorname{Im} G(E+\mathrm{i}\eta)\big)_{\boldsymbol{vv}} \right|^{1/2}}{\eta} \lesssim \frac{\eta_0}{\eta^2} \rho(E+\mathrm{i}\eta_0), \tag{6.A.8}$$

where in the second step we used the monotonicity of the maps $\eta \mapsto \eta \operatorname{Im} G(E+\mathrm{i}\eta)$ and $\eta \mapsto \eta\rho(E+\mathrm{i}\eta)$, and the second bound in (6.5.6) established above. Integrating the bound (6.A.8) from η_1 to η_0, we obtain

$$\left| \big(G(E+\mathrm{i}\eta_1)\big)_{\boldsymbol{uv}} \right| \lesssim \left| \big(G(E+\mathrm{i}\eta_0)\big)_{\boldsymbol{uv}} \right| + \frac{\eta_0}{\eta_1} \rho(E+\mathrm{i}\eta_0). \tag{6.A.9}$$

Since $\rho(E+\mathrm{i}\eta_0) \lesssim 1$, the first estimate in (6.5.6) follows immediately from (6.5.2) and (6.A.9). This concludes the proof of Lemma 6.5.3. □

Remark 6.A.1 (Local Laws below the Scale)**.** *Assume that $\rho(E+\mathrm{i}\eta_1)N\eta_1 \leq N^{\varepsilon}$ and $\rho(E+\mathrm{i}\eta_0)N\eta_0 = N^{\varepsilon}$, in particular $\eta_1 \leq \eta_0$. Using* (6.A.7) *with* (6.3.1) *at $z := E+\mathrm{i}\eta_0$ as an input, we obtain the very-high-probability bound*

$$\big(\operatorname{Im} G(E+\mathrm{i}\eta_1)\big)_{\boldsymbol{uu}} \lesssim \frac{\eta_0}{\eta_1} \rho(E+\mathrm{i}\eta_0) \lesssim \frac{\rho(E+\mathrm{i}\eta_0)N\eta_0}{N\eta_1} \lesssim \frac{N^{\varepsilon}}{N\eta_1}. \tag{6.A.10}$$

Using Lemma 6.7.1 and the identity $\mathrm{d}M(z)/\mathrm{d}z = \mathcal{B}^{-1}(z)[M(z)^2]$, that follows by taking the z-derivative of (6.2.3)*, we conclude that $\|\mathrm{d}M(z)/\mathrm{d}z\| \lesssim |\rho(z)/\operatorname{Im} z|$. Hence, differentiating $(G(E+\mathrm{i}\eta) - M(E+\mathrm{i}\eta))_{\boldsymbol{uv}}$ with respect to η, similarly to* (6.A.8)*, we can deduce that*

$$\left| \big(G(E+\mathrm{i}\eta_1) - M(E+\mathrm{i}\eta_1)\big)_{\boldsymbol{uv}} \right| \lesssim \frac{N^{\varepsilon}}{N\eta_1}, \quad \textit{w.v.h.p.} \tag{6.A.11}$$

Analogous reasoning also applies to averaged bounds. Indeed,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\eta} \big\langle \big(G(E+\mathrm{i}\eta) - M(E+\mathrm{i}\eta)\big) B \big\rangle \right| \lesssim \frac{\left| \big\langle \operatorname{Im} G(E+\mathrm{i}\eta) \big\rangle \big\langle \operatorname{Im} G(E+\mathrm{i}\eta) BB^* \big\rangle \right|^{1/2} + \rho(E+\mathrm{i}\eta)\, \|B\|_{\mathrm{hs}}}{\eta}. \tag{6.A.12}$$

Therefore, by integrating (6.A.12) *in η and using* (6.3.1)*, we can deduce that*

$$\left| \big\langle \big(G(E+\mathrm{i}\eta_1) - M(E+\mathrm{i}\eta_1)\big) B \big\rangle \right| \lesssim \frac{N^{\varepsilon}}{N\eta_1} \|B\|_{\mathrm{hs}}, \quad \textit{w.v.h.p.} \tag{6.A.13}$$

These results show that the local laws (6.3.1) *hold at $z = E+\mathrm{i}\eta_1$, for any $0 < \eta_1 \leq \eta_0$, once they hold at $E+\mathrm{i}\eta_0$ with η_0 satisfying $\rho(E+\mathrm{i}\eta_0)N\eta_0 = N^{\varepsilon}$.*

Proof of Lemma 6.6.2. First, we prove (6.6.2). Let σ_t be function defined in (6.7.3), corresponding to the solution M_t of the time-dependent MDE (6.3.16). It follows from Lemma 5.5 in [22] that σ_t admits a uniformly $1/3$-Hölder regular extension $\overline{\mathbb{H}}$. Moreover, it follows from Lemma 7.16 in

[22] that $|\sigma_t(\mathfrak{e}_t^-)| \sim |\sigma_t(\mathfrak{e}_t^+)| \sim \Delta_t^{1/3}$ and it follows from Theorem 7.7 (ii.b) in [22] that $\sigma_t(\mathfrak{e}_t^-) < 0$ and $\sigma_t(\mathfrak{e}_t^+) > 0$. Therefore, there exists a point $x_t \in (\sigma_t(\mathfrak{e}_t^-), \sigma_t(\mathfrak{e}_t^+))$ satisfying $\sigma_t(x_t) = 0$. For any $0 \le s \le t$, let $x_s := \varphi_{s,t}(x_t)$ as defined in (6.3.19). It follows from (6.3.20) that $\sigma_s(x_s) = 0$ for any $s \in [0,t]$.

Furthermore, $1/3$-Hölder regularity of ρ_t in t implies that there exists $c \sim 1$ such that for times s satisfying $0 \le t - s \le c\Delta_t^{1/3}$, the density ρ_s has a gap in the support around x_s of size $\Delta_s > 0$, let $\mathfrak{e}_s^-$ and $\mathfrak{e}_s^+$ denote its endpoints. From $1/3$-Hölder regularity of σ_s, we infer that $\operatorname{dist}(x_s, \mathfrak{e}_s^\pm) \sim \Delta_s$.

On the other hand, the map $h_s : x \mapsto \lim_{\eta \to +0} \rho_s(x + \mathrm{i}\eta)^{-1}\eta$ is also $1/3$-Hölder regular, uniformly in s, hence $h_s(x_s) \sim \operatorname{dist}(x_s, \mathfrak{e}_s^\pm)^{1/2}\Delta_s^{1/6} \sim \Delta_s^{2/3}$ by (6.6.1). Along the trajectories of (6.3.18), $h_s(x_s) \sim h_t(x_t) + (t-s)$ for all s satisfying $0 \le t - s \le c\Delta_t^{1/3}$, therefore (6.6.2) holds for all $0 \le t - s \le c\Delta_t^{1/3}$. In particular, $\Delta_{t - c\Delta^{1/3}} \gtrsim \Delta_t + \Delta_t^{1/2}$, which implies that (6.6.2) holds for all $0 \le s \le t$. This concludes the proof of (6.6.2).

Next, we prove (6.6.3). A similar relation for the evolution of the gaps under the free semicircular flow was studied in Section 5.1 of [244]. To keep the present paper reasonably self-contained, we present a complete proof for the evolution under the characteristic flow (6.3.18). Observe that it follows immediately from (6.3.16) that the density $\rho_s(x)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\rho_s(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2}\rho_s(x) + \langle \operatorname{Re} M_s(x)\rangle \rho_s(x)\right), \quad x \in \mathbb{R}, \quad 0 \le s \le t. \tag{6.A.14}$$

Consider the mass of ρ_s that lies to the left of the point x_s. Equation (6.A.14) implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{-\infty}^{x_s} \rho_s(x)\mathrm{d}x = 0, \quad 0 \le s \le t, \tag{6.A.15}$$

where we used that $\rho_s(x_s) = 0$. Therefore, the mass of the band of ρ_s to the left of $\mathfrak{e}_s^-$ is constant $0 \le s \le t$. For any $r > 0$, define $\gamma_s(r)$ implicitly by

$$\int_{-\infty}^{\gamma_s(r)} \rho_s(x)\mathrm{d}x = \int_{-\infty}^{x_s} \rho_s(x)\mathrm{d}x - r. \tag{6.A.16}$$

Note that by the definition of the edge point $\mathfrak{e}_s^-$ and the structure theorem [22, Theorem 7.2 (ii)] for ρ_s, there exists a constant $\widetilde{c} > 0$ such that $\rho_s(\gamma_s(r)) > 0$ for all $0 \le r \le \widetilde{c}$ and all $0 \le s \le t$. Moreover, $\gamma_s(r) < \mathfrak{e}_s^- \le x_s$. Therefore, it follows from (6.A.14) that (a similar equation for the free semicircular flow was obtained in Section 4.1 of [155])

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma_s(r) = -\frac{1}{2}\gamma_s(r) - \big\langle \operatorname{Re} M_s\big(\gamma_s(r)\big)\big\rangle, \quad 0 \le r \le \widetilde{c}, \quad 0 \le s \le t. \tag{6.A.17}$$

The evolution equation (6.6.3) for $\mathfrak{e}_s^-$ follows by taking the limit $r \to 0$ in (6.A.17). An analogous argument that considers the mass of ρ_s to the right of x_s implies (6.6.3) for $\mathfrak{e}_s^+$.

Next, we prove the first estimate in (6.6.4). By taking the imaginary parts of (6.3.18) and (6.3.20), we obtain

$$\eta_s \sim \eta_t + \rho_t(t-s), \quad \rho_s \sim \rho_t, \quad 0 \le s \le t. \tag{6.A.18}$$

Moreover, it follows form the comparison relation for ρ_t from (6.6.1), that

$$\rho_t \sim \eta_t(\tilde{\kappa}_t + \eta_t)^{-1/2}(\Delta_t + \tilde{\kappa}_t + \eta_t)^{-1/6} \lesssim \eta_t^{1/2}\Delta_t^{-1/6}, \tag{6.A.19}$$

where we used $0 \le \tilde{\kappa}_t \le \Delta_t$ and the assumption that $\eta_t \lesssim N^{-\nu}\Delta_t$. Therefore, using (6.6.2), (6.A.18) and (6.A.19), we obtain

$$\eta_s \lesssim \eta_t^{1/2}\Delta_t^{-1/6}\big(\Delta_t + (t-s)^{3/2}\big)^{2/3} \lesssim N^{-\nu/2}\Delta_t^{1/3}\Delta_s^{2/3} \lesssim N^{-\nu/2}\Delta_s, \quad 0 \le s \le t, \tag{6.A.20}$$

hence the first bound in (6.6.4) is established. To prove the second relation in (6.6.4), observe that it suffices to show that for all $0 \le s \le t$ and all $\eta \lesssim N^{-\nu/2}\Delta_s$, we have the bound

$$\pm\mathrm{Re}\big[F_s^{\pm}(\eta) - F_s^{\pm}(+0)\big] > 0, \quad F_s^{\pm}(\eta) := -\frac{1}{2}(\mathfrak{e}_s^{\pm} + \mathrm{i}\eta) - \langle M_s(\mathfrak{e}_s^{\pm} + \mathrm{i}\eta)\rangle. \tag{6.A.21}$$

Indeed, (6.A.21) implies that along (6.3.18), for all points at level $\eta \lesssim N^{-\nu/2}\Delta_s$ above the ends $\mathfrak{e}_s^{\pm}$ of the gap, their projection onto the real line moves away from the gap, for all times $0 \le s \le t$. Hence no trajectory $z_s = E_s + \mathrm{i}\eta_s$ satisfying $E_t \in (\mathfrak{e}_t^-, \mathfrak{e}_t^+)$ and $\eta_t \lesssim N^{-\nu}\Delta_t$ can violate (6.6.4). To see that (6.A.21) holds, note that the Stieltjes representation for $\langle M_s\rangle$ and the universal shape (see, e.g., [244, Eqs. (2.4a)–(2.4e)] for precise formulas) of the density ρ_s in the vicinity of its singularities $\mathfrak{e}_s^{\pm}$ yields

$$-\mathrm{Re}\big[F_s^{-}(\eta) - F_s^{-}(+0)\big] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{-\eta^2}{x(x^2+\eta^2)}\rho_s(\mathfrak{e}_s^- + x)\mathrm{d}x \ge C\Delta_s^{-1/6}\eta^{1/2} + \mathcal{O}\big(\Delta_s^{-5/3}\eta^2\big) + \mathcal{O}(\eta^2) > 0, \tag{6.A.22}$$

where in the last line we used the assumption $\eta \lesssim N^{-\nu/2}\Delta_s \lesssim N^{-\nu/2}$. The computation for F_s^+ is completely analogous. This concludes the proof of (6.6.4).

Next, we prove (6.6.5). Using the comparison relations (6.A.18) for ρ_s and $\rho_s^{-1}\eta_s$, together with the bound $\eta_s \lesssim N^{-\nu/2}\Delta_s$, and the assumption $\tilde{\kappa}_t(z_t) \gtrsim N^{\nu}\eta_t$, we deduce that

$$1 + \frac{\tilde{\kappa}_s}{\eta_s} \sim \frac{\rho_t^{-2}\eta_t + (t-s)}{\Delta_t^{1/3} + (t-s)^{1/2}} \gtrsim \min\left\{1 + \frac{\tilde{\kappa}_t}{\eta_t}, \frac{\tilde{\kappa}_t^{1/2}\Delta_t^{1/2}}{\eta_t}\right\} \gtrsim \frac{\tilde{\kappa}_t}{\eta_t}, \tag{6.A.23}$$

which implies (6.6.5) immediately.

Finally, we prove (6.6.6). Without loss of generality, we can assume $z_s = \mathfrak{e}_s^- + y_s + \mathrm{i}\eta_s$ with $0 \le y_s \le (1 - C_1)\Delta_s$ for some $1 \lesssim C_1 < 3/4$. Considering the difference of the real part of (6.3.18) and (6.6.3), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}y_s + \frac{1}{2}y_s &= \mathrm{Re}\left\langle M_s(\mathfrak{e}_s^-) - M_s(\mathfrak{e}_s^- + y_s) + M_s(\mathfrak{e}_s^- + y_s) - M_s(z_s)\right\rangle \\ &= -y_s\frac{1}{\pi}\int_{\mathbb{R}}\frac{\rho_s(\mathfrak{e}_s^- + x)\mathrm{d}x}{x(x - y_s)} - \eta_s^2\frac{1}{\pi}\int_{\mathbb{R}}\frac{\rho_s(\mathfrak{e}_s^- + x)\mathrm{d}x}{(y_s - x)\big((y_s - x)^2 + \eta_s^2\big)},\end{aligned} \tag{6.A.24}$$

where in the second line we used the Stieltjes representation for $\langle M_s\rangle$. Using the universal shape of the density ρ_s near the singularities $\mathfrak{e}_s^{\pm}$, we conclude that uniformly in $0 \le s \le t \le T$,

$$y_s\int_{\mathbb{R}}\frac{\rho_s(\mathfrak{e}_s^- + x)\mathrm{d}x}{x(x - y_s)} \gtrsim \frac{y_s^{1/2}}{\Delta_s^{1/6}}, \quad \eta_s^2\left|\int_{\mathbb{R}}\frac{\rho_s(\mathfrak{e}_s^- + x)\mathrm{d}x}{(y_s - x)\big((x - y_s)^2 + \eta_s^2\big)}\right| \lesssim \frac{\eta_s^2}{\Delta_s^{1/6}(y_s + \eta_s)^{3/2}}. \tag{6.A.25}$$

Since $\eta_s \lesssim N^{-\nu/2}\tilde{\kappa}_s$ and $\tilde{\kappa}_s \le y_s$, we conclude from (6.A.24) and (6.A.25) that

$$\frac{\mathrm{d}}{\mathrm{d}s}y_s \lesssim -\frac{y_s^{1/2}}{\Delta_s^{1/6}}, \tag{6.A.26}$$

which, together with (6.6.2), implies (6.6.6) for some constant $\mathfrak{c} > 0$. This concludes the proof of Lemma 6.6.2. □

Proof of Lemma 6.6.5. We restrict our considerations to the event Ω on which the assumed bound (6.6.9) holds.

First, we prove (6.6.10). Assume, to the contrary, that there is an eigenvalue λ of H such that $\lambda \in [\mathfrak{e}_t^- + f(t), \mathfrak{e}_t^+ - f(t)]$, then

$$\big\langle \operatorname{Im} G(\lambda + \mathrm{i}\eta) \big\rangle \geq \frac{1}{N\eta}, \quad \eta > 0. \tag{6.A.27}$$

On the other hand, choosing η implicitly such that $\rho_t(\lambda + \mathrm{i}\eta)N\eta = N^{-\zeta/2}$, implies that $\lambda + \mathrm{i}\eta \in \mathcal{D}_t^{\mathrm{sub}}$ and $\eta \sim \tilde{\kappa}_t(\lambda)^{1/4}\eta_{\mathrm{f},t}^{3/4}N^{-\zeta/4}$. Therefore, using the assumed bound (6.6.9) with data $(\mathcal{D}_t^{\mathrm{sub}}, \zeta + \ell\nu, \Omega)$ yields

$$\big\langle \operatorname{Im} G(\lambda + \mathrm{i}\eta) \big\rangle \lesssim \frac{\rho_t(\lambda + \mathrm{i}\eta)N\eta + N^{-\zeta}(\log N)^{\gamma}}{N\eta} \lesssim \frac{N^{-\zeta/2}}{N\eta}. \tag{6.A.28}$$

Therefore, we conclude by contradiction that (6.6.10) holds on Ω. This concludes the proof of Lemma 6.6.5. □

6.B Polynomially Decaying Metric Correlation Structure

In this section, we verify the last condition in Assumption 6.2.3 (i) for the ensemble in Example 6.2.6. More precisely, we show that (6.2.8) holds under the assumption that (recall (6.2.11c) from Example 6.2.6)

$$\big|\kappa(\alpha_1, \alpha_2, \alpha_3)\big| \leq C_3 \prod_{e \in \mathfrak{T}_{\min}} \frac{1}{1 + d(e)^s}, \tag{6.B.1}$$

for some[14] $s > 2$, where $\mathfrak{T}_{\min}$ is a minimal spanning tree in a complete graph with vertices $\alpha_1, \alpha_2, \alpha_3$ and edge weights induced by distance d, defined in (6.2.11b). That is, out goal is to show that, for all $X, Y, Z \in \mathbb{C}^{N\times N}$, the estimate

$$N^{-3/2} \sum_{\alpha_1,\alpha_2,\alpha_3} \big|\kappa(\alpha_1, \alpha_2, \alpha_3)\big| |X_{b_1a_2}| |Y_{b_2a_3}| |Z_{b_3a_1}| \lesssim C_3 \|X\| \|Y\| \|Z\|_{\mathrm{hs}}, \qquad \alpha_j := (a_j, b_j), \tag{6.B.2}$$

holds for some absolute implicit constant, where C_3 is the constant from (6.B.1).

We estimate the contribution of the case when $(\alpha_1, \alpha_3) \in \mathfrak{T}_{\min}$ and $d(\alpha_1, \alpha_3) = |a_1 - b_3| + |b_1 - a_3|$ in full detail. It is straightforward to check that in all other cases, using the trivial bounds $|X_{b_1a_2}| \leq \|X\|$, $|Y_{b_2a_3}| \leq \|Y\|$ is sufficient. Indeed, since $s > 2$, the indices b_1, a_2, b_2, a_3 can be summed up after using the norm bounds on X and Y; then for the remaining (a_1, b_3) sum, we use $N^{-3/2}\sum_{a_1,b_3} |Z_{a_1b_3}| \lesssim \|Z\|_{\mathrm{hs}}$.

Therefore, it suffices to bound

$$\mathfrak{X} \equiv \mathfrak{X}(X, Y, Z) := C_3 N^{-3/2} \sum_{\alpha_1,\alpha_3} \frac{|Z_{b_3a_1}|}{1 + \big(|a_1 - b_3| + |b_1 - a_3|\big)^s} \sum_{\alpha_2} \frac{|X_{b_1a_2}| |Y_{b_2a_3}|}{1 + \big(|a_1 - a_2| + |b_1 - b_2|\big)^s}, \tag{6.B.3}$$

where we assumed for concreteness that $\mathfrak{T}_{\min} = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3)\}$ and $d(\alpha_1, \alpha_3) = |a_1 - a_2| + |b_1 - b_2|$ (other cases are identical). First, we use the Schwarz inequality in the b_2 summation, to obtain

$$\sum_{b_2} \frac{|Y_{b_2a_3}|}{1 + \big(|a_1 - a_2| + |b_1 - b_2|\big)^s} \lesssim \frac{1}{1 + |a_1 - a_2|^{s-1/2}} \sqrt{\sum_{b_2} |Y_{b_2a_3}|^2} \lesssim \frac{\|Y\|}{1 + |a_1 - a_2|^{s-1/2}}. \tag{6.B.4}$$

Plugging (6.B.4) into the expression for $\mathfrak{X}$ in (6.B.3) and performing the summation in a_3, we obtain the estimates

$$\begin{aligned} \mathfrak{X} &\lesssim C_3 \|Y\| N^{-3/2} \sum_{a_1,b_3} \frac{|Z_{b_3a_1}|}{1 + |b_3 - a_1|^{s-1}} \sum_{a_2} \frac{1}{1 + |a_1 - a_2|^{s-1/2}} \sum_{b_1} |X_{b_1a_2}| \\ &\lesssim C_3 \|X\| \|Y\| N^{-1} \sum_{a_1,b_3} \frac{|Z_{b_3a_1}|}{1 + |b_3 - a_1|^{s-1}} \sum_{a_2} \frac{1}{1 + |a_1 - a_2|^{s-1/2}} \lesssim C_3 \|X\| \|Y\| \|Z\|_{\mathrm{hs}}, \end{aligned} \tag{6.B.5}$$

[14] The estimate (6.B.2) below can be proved under the relaxed summability condition $s > 3/2$. However, $s > 2$ in (6.2.11c) is still necessary for (6.2.6)–(6.2.7).

where in the second step we used Schwarz inequality in b_1, and in the ultimate step we use the fact that $s > 2$ to first sum the convergent series in a_2, and then apply Schwarz in (a_1, b_3). This yields the desired (6.B.2).

Chapter 7

Prethermalization for deformed Wigner matrices

This chapter contains the paper [238]:

L. Erdős, J. Henheik, J. Reker, and V. Riabov. Prethermalization for Deformed Wigner Matrices. *Ann. Henri Poincaré*, pages 1–43, 2024

Abstract. We prove that a class of weakly perturbed Hamiltonians of the form $H_\lambda = H_0 + \lambda W$, with W being a Wigner matrix, exhibits *prethermalization*. That is, the time evolution generated by H_λ relaxes to its ultimate thermal state via an intermediate prethermal state with a lifetime of order λ^{-2}. Moreover, we obtain a general relaxation formula, expressing the perturbed dynamics via the unperturbed dynamics and the ultimate thermal state. The proof relies on a two-resolvent law for the deformed Wigner matrix H_λ.

Content

7.1 Introduction

It is well-known (see, e.g., [286]) that certain macroscopic observables in an isolated quantum system with many interacting degrees of freedom tend to equilibrate, i.e., their expectation values become essentially constant at large times. However, if the system is coupled to the environment (reservoir), then the process of relaxation to equilibrium may take different forms depending on the properties of the initial system and the structure of the perturbation.

In this work, we consider a weakly coupled system of the form

$$H_\lambda := H_0 + \lambda W\,, \tag{7.1.1}$$

where H_0 is a single-body or a many-body *Hamiltonian*, W is an energy preserving (Hermitian) *perturbation*, and λ is a small coupling constant. For our phenomenological study, we consider a mean-field random perturbation that couples all modes. Following the extensive physics literature [586, 221, 468, 189, 192, 193], we choose the perturbation W to be a *Wigner random matrix*, i.e., a random matrix with centered, independent identically distributed (i.i.d.) entries (modulo the Hermitian symmetry).

The central object of our study is the perturbed time evolution of the quantum expectation value

$$\langle A\rangle_{P_\lambda(t)} := \mathrm{Tr}[P_\lambda(t)A] \tag{7.1.2}$$

of an *observable* A, compared to the unperturbed evolution $\langle A\rangle_{P_0(t)} = \mathrm{Tr}[P_0(t)A]$, which is considered known. Here

$$P_\lambda(t) := \mathrm{e}^{-\mathrm{i}tH_\lambda} P \mathrm{e}^{\mathrm{i}tH_\lambda} \qquad \text{resp.} \qquad P_0(t) := \mathrm{e}^{-\mathrm{i}tH_0} P \mathrm{e}^{\mathrm{i}tH_0} \tag{7.1.3}$$

denote the Heisenberg time evolution of an initial *state* P governed by the (un)perturbed Hamiltonian. We point out that the *unperturbed* evolution strongly depends on all its constituents and hence, it might exhibit qualitatively different and generally complex behavior.

In contrast, the *perturbed* system relaxes to equilibrium via a robust mechanism, and it can be described by a fairly simple general *relaxation formula*

$$\langle A\rangle_{P_\lambda(t)} \approx \langle A\rangle_{\widetilde{P}_\lambda} + |g_\lambda(t)|^2\big[\langle A\rangle_{P_0(t)} - \langle A\rangle_{\widetilde{P}_\lambda}\big], \qquad g_\lambda(t) := \mathrm{e}^{-\alpha\lambda^2 t}, \quad \alpha > 0, \tag{7.1.4}$$

where $\widetilde{P}_\lambda$ is the thermal state of the composite system (7.1.1). In this form, Eq. (7.1.4) is first mentioned in [468, Eq. (40)], where it describes the time dependence of the expectation of an observable in a nonintegrable system after perturbation by a random matrix.

The relaxation formula (7.1.4) shows convergence to the thermal state at an exponential rate on time scales of order λ^{-2} (in agreement with *Fermi's golden rule*), but it also carries more refined information about the role of the unperturbed dynamics in the process. A particularly interesting case occurs if both the perturbed and unperturbed systems equilibrate but do not approach the same

limiting value. This often happens if H_0 has an additional symmetry (conserved quantity) that is broken by the perturbation. If the time scale λ^{-2} of the perturbed equilibration is smaller than that of the unperturbed one, then the former robust process eclipses the latter. In particular, the precise form of $\langle A \rangle_{P_0(t)}$ in (7.1.4) is irrelevant whenever the prefactor $|g_\lambda(t)|^2$ is already exponentially small. In the opposite case, however, the equilibration of the perturbed dynamics happens in two stages. This phenomenon, known as *prethermalization* in the physics literature, was first described in a paper by Moeckel and Kehrein [450]. We remark, however, that this terminology was already used to describe a different phenomenon a few years earlier [64].

Nowadays, prethermalization has been extensively studied both experimentally (see, e.g., the review [396]) and theoretically (e.g., in [101, 223, 276, 372, 496, 574, 430], see also the review [455]). Reimann and Dabelow [189] studied the first relaxation stage of a prethermalization process, which is governed by H_0. More precisely, assuming that $P_0(t)$ relaxes to a non-thermal steady state, they find that the perturbed time evolution $\langle A \rangle_{P_\lambda(t)}$ with a sufficiently weak perturbation ($\lambda \ll 1$) closely follows the unperturbed time evolution $\langle A \rangle_{P_0(t)}$ for times $t \ll \lambda^{-2}$. In particular, the perturbed time evolution $\langle A \rangle_{P_\lambda(t)}$ is close to the non-thermal steady state of H_0 for times $1 \ll t \ll \lambda^{-2}$.[1] The authors of [189] further extended their principal approach to a general study of relaxation theory for perturbed quantum dynamics in [192, 193]. These works now include all times and also the *strong* coupling regime (in case of banded matrices), which yields a characteristic power-law time decay (given more precisely by a Bessel function) instead of the exponential decay in (7.1.4). The theoretical model is then applied to several examples and compare the prediction to numerical and experimental works (see also Dabelow's PhD thesis [188] for further details). Finally, we also mention that prethermalization in the form of the existence of an effectively conserved quantity for very long times has been rigorously established in [1] for periodically driven quantum systems if the frequency is large compared with the size of the driving potential.

In this paper, we approach prethermalization from the viewpoint of random matrix theory, interpreting the unperturbed Hamiltonian $H_0 \equiv H_0(N) \in \mathbf{C}^{N\times N}$ as a fixed sequence of bounded self-adjoint deterministic matrices and the perturbation $W \equiv W(N)$ as an $N \times N$ Wigner random matrix. Our Hamiltonian H_λ in the setting of (7.1.1) is also called *deformed Wigner matrix* in random matrix theory, or it can be viewed as a Wigner random matrix with nonzero expectation. Wigner matrices are encountered in many related physics models, e.g., the recent rigorous study of thermalization problems [165, 170, 168]. Here, the key technical result is a strong concentration property of the resolvent $G(z) = (H_\lambda - z)^{-1}$ or products of several resolvents around their deterministic approximation. Such results are commonly called *multi-resolvent global or local laws*, depending on the distance of the spectral parameter from the spectrum. For example, a typical *two-resolvent* law computes

$$\langle\!\langle G(z_1)A_1G(z_2)A_2 \rangle\!\rangle = \langle\!\langle (H_\lambda - z_1)^{-1}A_1(H_\lambda - z_2)^{-1}A_2 \rangle\!\rangle \tag{7.1.5}$$

to leading order in N, where $\langle\!\langle \cdot \rangle\!\rangle$ denotes the normalized trace, $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$, and $A_1, A_2 \in \mathbf{C}^{N\times N}$ are deterministic matrices. Using functional calculus, the resolvents can be replaced by more general and even N-dependent functions, thus linking (7.1.5) to the Heisenberg time evolution. Recent work (see Chapter 2) establishes a multi-resolvent local law for deformed Wigner matrices in the bulk regime of the spectrum, which motivated our study of perturbed quantum systems.

7.1.1 Description of the main results

The principal goal of this work is a rigorous proof of the relaxation formula (Corollary 7.2.7) and prethermalization (Corollary 7.2.10) for perturbed quantum Hamiltonians of the form (7.1.1). We

[1] Unperturbed systems H_0, for which the time evolution $\langle A \rangle_{P_0(t)}$ does not approach the microcanonical prediction of equilibrium statistical mechanics, but nevertheless have a large-t limit (a non-thermal steady state), are studied in [43]. In a sense, these systems exhibit prethermalization, although they do not approach thermal equilibrium after the first steady state is reached.

thereby assume that the unperturbed Hamiltonian H_0 has a (locally) regular limiting density of states ρ_0 around a reference energy E_0 and only energies in a microscopically large but macroscopically small interval $I_\Delta := [E_0 - \Delta, E_0 + \Delta]$ are populated by the initial state P (similar assumptions are made in [189, 192, 193]). We then show the following corollaries of our main Theorem 7.2.4:

Cor. 7.2.7: The relaxation formula (7.1.4) holds generally for short and long kinetic times, i.e. $t \ll \lambda^{-2}$ and $t \gg \lambda^{-2}$, corresponding to $|g_\lambda(t)|^2 \approx 1$ and $|g_\lambda(t)|^2 \approx 0$, respectively. At intermediate times, $t \sim \lambda^{-2}$ it is generally *not* valid, unless the quadratic forms $\langle \boldsymbol{u}_j, A\boldsymbol{u}_j \rangle$ of overlaps with the eigenvectors $\boldsymbol{u}_j$ of H_0 behave regularly in j (cf. Definition 7.2.6). This happens, e.g., if H_0 satisfies the Eigenstate Thermalization Hypothesis (ETH).

Cor. 7.2.10: Assuming that the unperturbed time evolution has a long time limit $\langle A \rangle_{P_0(t)} \overset{t\to\infty}{\longrightarrow} \langle A \rangle_{P_{\mathrm{pre}}}$, such that the *prethermal state* P_{pre} is distinguishable from the thermal state $\widetilde{P}_\lambda$ of the perturbed system, i.e. $\langle A \rangle_{P_{\mathrm{pre}}} \neq \langle A \rangle_{\widetilde{P}_\lambda}$ (cf. Definition 7.2.9), we show the characteristic two-step relaxation of a prethermalization process; see Figure 7.1.1.

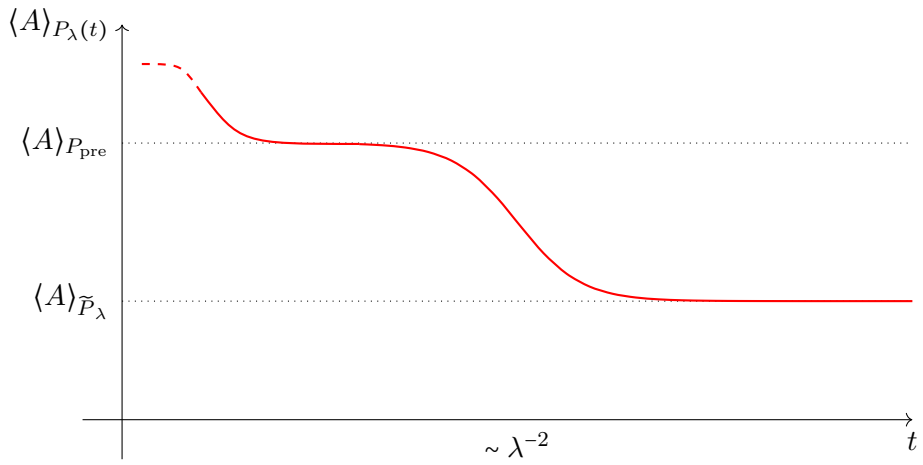

Figure 7.1.1: Depicted is a schematic graph of the prethermalization process: For times $1 \ll t \ll \lambda^{-2}$, the perturbed time evolution of the quantum expectation $\langle A \rangle_{P_\lambda(t)}$ (see (7.1.2)) is close to the quantum expectation of A in the prethermal state P_{pre}, i.e. $\langle A \rangle_{P_\lambda(t)} \approx \langle A \rangle_{P_{\mathrm{pre}}}$. For times $t \gg \lambda^{-2}$, we have that $\langle A \rangle_{P_\lambda(t)}$ is close to the limiting thermal quantum expectation, i.e. $\langle A \rangle_{P_\lambda(t)} \approx \langle A \rangle_{\widetilde{P}_\lambda}$. This value is typically different from the prethermal quantum expectation $\langle A \rangle_{P_{\mathrm{pre}}}$. This means, the ultimate relaxation of $\langle A \rangle_{P_\lambda(t)}$ towards $\langle A \rangle_{\widetilde{P}_\lambda}$ happens via an intermediate *prethermal* value $\langle A \rangle_{P_{\mathrm{pre}}}$ in two steps, whose time scales are separated by λ^{-2}.

7.1.2 Outline of the paper

The general task in this paper is to approximately evaluate the random time evolution $\langle A \rangle_{P_\lambda(t)}$ from (7.1.2). This is carried out in several steps summarized schematically in Figure 7.1.2. First, in Theorem 7.2.4 (a) in Section 7.2.2, the true time evolution $P_\lambda(t)$ is expressed as a linear combination of the unperturbed time evolution $P_0(t)$ and another deterministic time-dependent object $\widetilde{P}_{\lambda,t}$ that is conceptually simpler than $P_\lambda(t)$. Then, in Theorem 7.2.4 (b), we identify a time-independent state $\widetilde{P}_\lambda$ as the large time limit of $\widetilde{P}_{\lambda,t}$. Combining both parts of Theorem 7.2.4, we arrive at Corollary 7.2.7, which establishes the relaxation formula (7.1.4) at small and large kinetic times. As mentioned above, at intermediate times, it holds only for observables having the *local overlap regularity (LOR) property* (see Definition 7.2.6). In the subsequent Section 7.2.3, dropping the LOR property, but assuming additionally that the unperturbed Hamiltonian H_0 and the initial state P have the *prethermalization property* (see Definition 7.2.9), we obtain the characteristic two-scale

relaxation of $P_\lambda(t)$ towards $\widetilde{P}_\lambda$ via an intermediate *prethermal state* P_{pre} (see Corollary 7.2.10 and Figure 7.1.1). As an additional result, in Theorem 7.2.11 in Section 7.2.4 we relate $\widetilde{P}_\lambda$ to the microcanonical ensemble of H_λ, called $P_\lambda^{(\mathrm{mc})}$, which is independent of the initial state P. Finally, our results are illustrated by two simple examples in Section 7.2.5.

While most proofs are given in Section 7.3, some auxiliary results and additional proofs are deferred to Appendix 7.A.

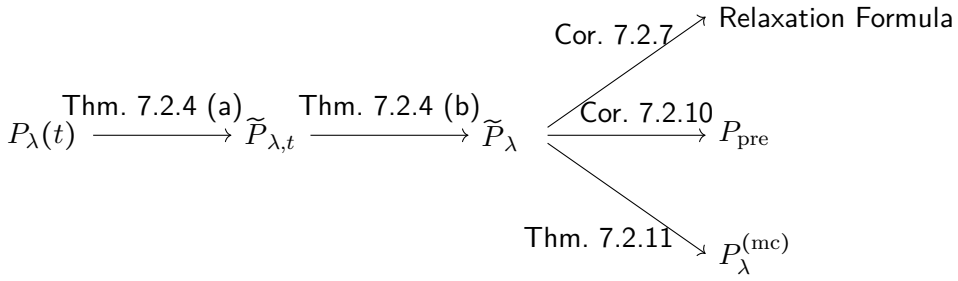

Figure 7.1.2: The structure of our main results.

7.1.3 Notation

For positive quantities f, g we write $f \lesssim g$ (or $f = \mathcal{O}(g)$) and $f \sim g$ if $f \le Cg$ or $cg \le f \le Cg$, respectively, for some constants $c, C > 0$ which only depend on the constants appearing in the moment condition (see (7.2.1)) and the definition of the set of admissible energies (see (7.2.4)). In informal explanations, we frequently use the notation $f \ll g$, which indicates that f is "much smaller" than g. Moreover, we shall also write $w \approx z$ to indicate the closeness of $w, z \in \mathbf{C}$ with a not precisely specified error.

For any natural number n we set $[n] := \{1, 2, \dots, n\}$. Matrix entries are indexed by lowercase Roman letters $a, b, c, \dots$ and $i, j, k, \dots$ from the beginning or the middle of the alphabet and unrestricted sums over $a, b, c, \dots$ and $i, j, k, \dots$ are always understood to be over $[N] = \{1, \dots, N\}$.

We denote vectors by bold-faced lowercase Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, for some $N \in \mathbf{N}$. Vector and matrix norms, $\|\boldsymbol{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $N \times N$ matrices A, B we use the notations $\langle\!\langle A \rangle\!\rangle := N^{-1}\mathrm{Tr}A$ to denote the normalized trace of A and $\langle A \rangle_B := \mathrm{Tr}[AB]$ is the trace of the product AB. We denote the spectrum of a matrix or operator A by $\sigma(A)$. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ and matrices $A \in \mathbf{C}^{N \times N}$ we define

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_i \overline{x}_i y_i \,, \qquad A_{\boldsymbol{xy}} := \langle \boldsymbol{x}, A\boldsymbol{y} \rangle \,.$$

For a unit vector $\boldsymbol{v} \in \mathbf{C}^N$ we shall also use the notation $|\boldsymbol{v}\rangle\langle\boldsymbol{v}|$ for the projection onto the one-dimensional subspace spanned by $\boldsymbol{v}$.

Finally, we use the concept of "with very high probability" *(w.v.h.p.)* meaning that for any fixed $C > 0$, the probability of an N-dependent event is bigger than $1 - N^{-C}$ for $N \ge N_0(C)$. We introduce the notion of *stochastic domination* (see e.g. [241]): given two families of non-negative random variables

$$X = \left(X^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right)$$

indexed by N (and possibly some parameter u in some parameter space $U^{(N)}$), we say that X is stochastically dominated by Y, if for all $\xi, C > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^\xi Y^{(N)}(u)\right] \le N^{-C} \tag{7.1.6}$$

for large enough $N \ge N_0(\xi, C)$. In this case we use the notation $X \prec Y$ or $X = \mathcal{O}_\prec(Y)$.

Acknowledgments

We thank Peter Reimann and Lennart Dabelow for helpful comments.

7.2 Main Results

In Section 7.2.1, we give the precise definition of Wigner random matrices and the assumptions on the Hamiltonian, observables and states under consideration in (7.1.1)–(7.1.3). Afterward, we formulate our main results in Sections 7.2.2–7.2.4. Finally, in Section 7.2.5 we discuss our findings in the context of two simple examples.

7.2.1 Assumptions

We begin with formulating the assumption on the Wigner matrix W.

Assumption 7.2.1 (Wigner matrix)**.** *Let $W \equiv W(N) = (w_{ij})_{i,j\in[N]}$ from (7.1.1) be a real symmetric or complex Hermitian random matrix $W = W^*$ with independent entries distributed according to the laws $w_{ij} \stackrel{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{od}}$ for $i < j$ and $w_{jj} \stackrel{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{d}}$. The random variables χ_{od} and χ_{d} satisfy the following assumptions:[2] We assume that χ_{d} is a centered real random variable, and χ_{od} is a real or complex random variable with $\mathbf{E}\,\chi_{\mathrm{od}} = 0$ and $\mathbf{E}\,|\chi_{\mathrm{od}}|^2 = 1$.*

Furthermore, we assume the existence of higher moments, namely

$$\mathbf{E}\,|\chi_{\mathrm{d}}|^p + \mathbf{E}\,|\chi_{\mathrm{od}}|^p \le C_p, \tag{7.2.1}$$

for all $p \in \mathbf{N}$, where C_p are positive constants.

For concreteness, we focus on the complex case with the additional assumptions $\mathbf{E}\,\chi_{\mathrm{od}}^2 = 0$ and $\mathbf{E}\,|\chi_{\mathrm{d}}|^2 = 1$; all other cases can also be handled as in [165]. The precise conditions on the Wigner matrix only play a role in the underlying *two resolvent global law* (Proposition 7.3.1).

For the Hamiltonian $H_0 \equiv H_0(N)$ in (7.1.1) we assume the following.

Assumption 7.2.2 (H_0 and its density of states)**.** *The Hamiltonian $H_0 \equiv H_0(N)$ is deterministic, self-adjoint $H_0 = H_0^*$, and uniformly bounded $\|H_0\| \lesssim 1$. We denote the resolvent of H_0 at any spectral parameter $z \in \mathbf{C} \setminus \mathbf{R}$ by*

$$M_0(z) := \frac{1}{H_0 - z}\,.$$

Moreover, we assume the following:

(i) There exists a compactly supported measurable function $\rho_0 : \mathbf{R} \to [0, +\infty)$ with

$$\int_{\mathbf{R}} \rho_0(x)\mathrm{d}x = 1$$

and two positive sequences $\epsilon_0(N)$ and $\eta_0(N)$, both converging to zero as $N \to \infty$, such that, uniformly in $z \in \mathbf{C}\backslash\mathbf{R}$ with $\eta := |\mathrm{Im}\, z| \ge \eta_0 \equiv \eta_0(N)$, we have

$$\langle\!\langle M_0(z) \rangle\!\rangle = m_0(z) + \mathcal{O}(\epsilon_0) \quad \textit{with} \quad \epsilon_0 \equiv \epsilon_0(N)\,. \tag{7.2.2}$$

Here,

$$m_0(z) := \int_{\mathbf{R}} \frac{\rho_0(x)}{x - z}\mathrm{d}x \tag{7.2.3}$$

is the Stieltjes transform of ρ_0. We refer to ρ_0 as the limiting density of states, *and to $\mathrm{supp}(\rho_0)$ as the* limiting spectrum *of H_0.*

[2] A careful examination of our proof reveals that the entries of W need not be distributed identically. Indeed, only the matching of the second moments is necessary, but higher moments can differ.

(ii) For small positive constants $\kappa, c > 0$, we define the set of admissible energies $\sigma_{\mathrm{adm}}^{(\kappa,c)}$ *in the limiting spectrum of H_0 by*[3]

$$\sigma_{\mathrm{adm}}^{(\kappa,c)} := \left\{ x \in \mathrm{supp}(\rho_0) : \inf_{|y-x|\le\kappa} \rho_0(y) > c,\ \|\rho_0\|_{C^{1,1}([x-\kappa,x+\kappa])} \le 1/c \right\}. \tag{7.2.4}$$

We assume that for some positive $\kappa, c > 0$, $\sigma_{\mathrm{adm}}^{(\kappa,c)}$ is not empty.

Assuming that the set of admissible energies in (7.2.4) is non-empty ensures that there is a part of the limiting spectrum $\mathrm{supp}(\rho_0)$, where the limiting density of states ρ_0 behaves regularly, i.e. it is strictly positive and sufficiently smooth. Finally, we formulate our assumptions on the states considered in (7.1.2) and (7.1.3).

Assumption 7.2.3 (States)**.** *Given Assumption 7.2.2, we first pick a* reference energy

$$E_0 \in \sigma_{\mathrm{adm}}^{(\kappa_0,c_0)} \quad \text{for some} \quad \kappa_0, c_0 > 0, \tag{7.2.5}$$

and further introduce $I_\delta := [E_0 - \delta, E_0 + \delta]$ for any $0 < \delta < \kappa_0$. Moreover, take an energy width *$\Delta \in (0, \frac{1}{6}\kappa_0)$ and let $\Pi_\Delta := \mathbb{1}_{I_\Delta}(H_0)$ be the spectral projection of H_0 onto the interval I_Δ.*

Then, we assume that the (deterministic) initial state $P \equiv P(N) \in \mathbf{C}^{N\times N}$ in (7.1.3) *is a state in the usual sense ($P = P^*$, $0 \le P \le 1$, and $\mathrm{Tr}[P] = 1$), and is localized in I_Δ, i.e.*

$$P = \Pi_\Delta P \Pi_\Delta \,. \tag{7.2.6}$$

Note that we assume only the state P to be localized in I_Δ and not the (deterministic) observable $A \equiv A(N) \in \mathbf{C}^{N\times N}$. However, by inspecting the proof, we see that A and P play essentially symmetric roles, and thus, our results hold verbatim if we assume localization of A instead of P. Moreover, for ease of notation, we drop the N-dependence of all the involved matrices.

7.2.2 Relaxation of perturbed quantum dynamics

In this section, we present our main result on the time evolution of the random quantum expectation $\langle A\rangle_{P_\lambda(t)}$ from (7.1.2). Its relaxation is described in two steps, hence Parts (a) and (b) in the following theorem. In the first step, we eliminate the randomness and identify the leading deterministic part of $\langle A\rangle_{P_\lambda(t)}$ in terms of M_0, the unperturbed resolvent. In the second step, we consider the short and long-time limits of the leading term. Further explanatory comments come after the theorem and in Remark 7.2.5 below.

Theorem 7.2.4 (Relaxation of perturbed dynamics)**.** *Let $H_\lambda = H_0 + \lambda W$ be a perturbed Hamiltonian like in* (7.1.1) *with $\lambda > 0$, whose constituents satisfy Assumptions 7.2.1 and 7.2.2, respectively. Pick a reference energy E_0 like in* (7.2.5)*. Let P be a state satisfying Assumption 7.2.3 for some energy width $\Delta > 0$ and A a bounded deterministic observable, $\|A\| \lesssim 1$.*

Then, using the notations (7.1.2) *and* (7.1.3)*, we have the following two approximation statements:*

(a) [Relaxation in the kinetic limit] *The perturbed dynamics $\langle A\rangle_{P_\lambda(t)}$ satisfies*

$$\langle A\rangle_{P_\lambda(t)} = |g_\lambda(t)|^2 \langle A\rangle_{P_0(t)} + \langle A\rangle_{\widetilde{P}_{\lambda,t}} + \mathcal{E}\,, \tag{7.2.7}$$

where we denoted

$$g_\lambda(t) := \mathrm{e}^{-\alpha\lambda^2 t} \quad \text{with} \quad \alpha := \pi\rho_0(E_0)\,. \tag{7.2.8}$$

[3] Here, $C^{1,1}(J)$ denotes the set of continuously differentiable functions with a Lipschitz-continuous derivative on an interval J, equipped with the norm $\|f\|_{C^{1,1}(J)} := \|f\|_{C^1(J)} + \sup_{x,y\in J: x\neq y} \frac{|f'(x)-f'(y)|}{|x-y|}$.

Moreover, we introduced[4]

$$\widetilde{P}_{\lambda,t} := \frac{\int_{\mathbf{R}} \int_{\mathbf{R}} \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\, K_{\lambda,t}(x-y)\, \langle \operatorname{Im} M_0(y+\mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x\mathrm{d}y}{\int_{\mathbf{R}} \operatorname{Tr}[\operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)]\, \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x}\,, \tag{7.2.9}$$

with an explicit kernel given by

$$K_{\lambda,t}(u) := \frac{1}{\pi}\frac{2\alpha\lambda^2}{u^2+(2\alpha\lambda^2)^2}\left(2\alpha\lambda^2 t\frac{\sin(tu)}{tu} - \cos(tu) + \mathrm{e}^{-2\alpha\lambda^2 t}\right). \tag{7.2.10}$$

Finally, we have $\mathcal{E} = \mathcal{O}(\mathcal{E}_0) + \mathcal{O}_{\prec}(C(\lambda,t)/\sqrt{N})$ *for some constant* $C(\lambda,t) > 0$ *and for every fixed* $T \in (0,\infty)$*, the deterministic error* $\mathcal{E}_0 = \mathcal{E}_0(\lambda,t,\Delta,N)$ *satisfies*

$$\lim_{\Delta\to 0} \lim_{\substack{t\to\infty \\ \lambda\to 0 \\ \lambda^2 t = T}} \lim_{N\to\infty} \mathcal{E}_0 = 0\,. \tag{7.2.11}$$

(b) [Long and short kinetic time limit] *Defining*

$$\widetilde{P}_{\lambda} := \frac{\int_{\mathbf{R}} \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\, \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x}{\int_{\mathbf{R}} \operatorname{Tr}[\operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)]\, \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x}\,, \tag{7.2.12}$$

it holds that

$$\langle A\rangle_{\widetilde{P}_{\lambda,t}} = \left(1-|g_\lambda(t)|^2\right)\langle A\rangle_{\widetilde{P}_\lambda} + \mathcal{R}\,, \tag{7.2.13}$$

where, for every fixed $T \in (0,\infty)$*, the error term* $\mathcal{R} = \mathcal{R}(\lambda,t,\Delta,N)$ *satisfies*

$$\limsup_{\Delta\to 0} \limsup_{\substack{t\to\infty \\ \lambda\to 0 \\ \lambda^2 t = T}} \limsup_{N\to\infty} |\mathcal{R}| \lesssim T\mathrm{e}^{-2\alpha T}. \tag{7.2.14}$$

We point out that the error $\mathcal{E}$ in (7.2.7) naturally consists of two parts, a deterministic and a stochastic one. The stochastic part of order $\mathcal{O}_{\prec}(C(\lambda,t)/\sqrt{N})$ is obtained from a *global law* for two resolvents of the random matrix H_λ (see (7.3.6) below); the deterministic part $\mathcal{O}(\mathcal{E}_0)$ is obtained from estimating the deterministic leading term in (7.3.7).

Note that the error $\mathcal{R}$ is small compared to the first term in the rhs. of (7.2.13) only in the regime where T is large, in particular $\langle A\rangle_{\widetilde{P}_{\lambda,t}}$ converges to $\langle A\rangle_{\widetilde{P}_\lambda}$ exponentially fast. In the small T regime, both terms on the right-hand side of (7.2.13) vanish linearly in T. We chose the above formulation (7.2.13) because, in this way, it relates directly to the relaxation formula (7.1.4) (see Corollary 7.2.7 below).

Remark 7.2.5. *We have two further comments on Theorem 7.2.4.*

(i) The triple limits in (7.2.11) *and* (7.2.14) *consist of a thermodynamic limit* ($N\to\infty$)*, a kinetic limit or* van Hove limit ($t\to\infty$ *and* $\lambda\to 0$ *while* $\lambda^2 t$ *is fixed), and an infinitesimal spectral localization* ($\Delta\to 0$)*. Note that the* kinetic time parameter $T = \lambda^2 t$ *is natural, as the time scale prescribed for relaxation in the physics literature, e.g., by explicit analysis of the Pauli master equation in [455, Sect. 5.2.6], is* $\mathcal{O}(\lambda^{-2})$*. The* $\Delta\to 0$ *limit is needed only to ensure that the mean level spacing is approximately constant near* E_0 *on the scale* Δ*. If the density of states is flat, the* $\Delta\to 0$ *limit can be omitted. We emphasize that the error terms in Theorem 7.2.4 are explicit in the sense that their dependence on the scaling parameters* N*,* t*,* λ*, and* Δ *is tracked throughout the proof. The limit in* (7.2.11) *is then the natural order of limits in which these errors vanish. Finally, we remark that the explicitly tracked errors allow for certain combined limits, although for simplicity, we do not pursue these extensions.*

[4] Recall that the imaginary part of a matrix $B \in \mathbf{C}^{N\times N}$ is given by $\operatorname{Im} B = \frac{1}{2i}(B - B^*)$.

(ii) The idea behind (7.2.12)–(7.2.14) *is that the kernel* (7.2.10) *is an approximate delta function with $T = \lambda^2 t$-dependent magnitude. More precisely, its Fourier transform*[5]

$$\widehat{K}_{\lambda,t}(p) = \frac{1}{\sqrt{2\pi}} \big(1 - \mathrm{e}^{-2\alpha\lambda^2(t-|p|)}\big)\, \mathbb{1}(|p| \le t)$$

converges uniformly in compact intervals to a constant. That is, for every fixed $T \in (0,\infty)$ and for every compact set $\Omega \subset \mathbf{R}$, we have

$$\lim_{\substack{t\to\infty \\ \lambda\to 0 \\ \lambda^2 t = T}} \sup_{p\in\Omega} \left| \widehat{K}_{\lambda,t}(p) - \frac{1}{\sqrt{2\pi}} \big(1 - \mathrm{e}^{-2\alpha T}\big) \right| = 0\,.$$

However, since $x \mapsto \operatorname{Im} M_0(x + \mathrm{i}\alpha\lambda^2)$ is only regular on scale λ^2, the approximation

$$K_{\lambda,t}(x-y) \approx \big(1 - \mathrm{e}^{-2\alpha\lambda^2 t}\big)\delta(x-y),$$

used in heuristically obtaining (7.2.13) *from* (7.2.9) *and* (7.2.12)*, is* not *generally valid unless T is very small or very large, as* (7.2.14) *indicates.*

In the remaining part of the current Section 7.2.2, we connect Theorem 7.2.4 to the relaxation formula (7.1.4). As a preparation, we formulate the following *local overlap regularity property*, required in Corollary 7.2.7 (c) below. For this purpose, let μ_j and $\boldsymbol{u}_j$ denote the eigenvalues and corresponding normalized eigenvectors of

$$H_0 = \sum_j \mu_j \, |\boldsymbol{u}_j\rangle \, \langle \boldsymbol{u}_j| \,. \tag{7.2.15}$$

Definition 7.2.6 (Local overlap regularity (LOR))**.** *Let the Hamiltonian H_0 be as in Assumption 7.2.2. We say that a bounded deterministic observable A, $\|A\| \lesssim 1$, has the* local overlap regularity (LOR) property *if and only if the eigenvector overlaps $\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle$ are approximately constant in the following sense: There exists a constant $\mathfrak{A} \in \mathbf{R}$ such that*[6]

$$\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle = \mathbf{A} + \mathcal{O}\big(\mathcal{E}_{\mathrm{LOR}}\big) \quad \text{for all} \quad j \in \mathbf{N} \quad \text{with} \quad \mu_j \in I_{2\Delta}\,, \tag{7.2.16}$$

where the error $\mathcal{E}_{\mathrm{LOR}} = \mathcal{E}_{\mathrm{LOR}}(\Delta, N)$ satisfies

$$\lim_{\Delta\to 0} \lim_{N\to\infty} \mathcal{E}_{\mathrm{LOR}} = 0\,.$$

The LOR property (7.2.16) is satisfied, e.g., if H_0 satisfies the Eigenstate Thermalization Hypothesis [221, 535] (see also the discussion in [192]). For general systems, the ETH remains an unproven hypothesis. We remark, however, that it has been rigorously proven for a large class of mean-field random matrices H_0 (see [165] and Chapter 2), including Wigner matrices and their deformations.

The following corollary collects our rigorous results on the relaxation formula (7.1.4). The first two parts (items (a) and (b) below) immediately follow from Theorem 7.2.4 (a) and (b). The third part, item (c), involves the LOR property of the observable A and requires a separate argument, provided in Section 7.3.4.

[5] We use the convention that the Fourier transform of $f \in L^1(\mathbf{R})$ is defined as $\widehat{f}(p) := (2\pi)^{-1/2} \int_{\mathbf{R}} f(x)\mathrm{e}^{-\mathrm{i}px}\mathrm{d}x$.

[6] In fact, it is sufficient to assume that the overlaps are regular in j in the sense that there exists a uniformly equicontinuous sequence $(\mathfrak{A}_N)_{N\in\mathbf{N}}$ of functions $\mathfrak{A}_N : I_{2\Delta} \to \mathbf{R}$ such that

$$\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle = \mathbf{A}_N(\mu_j) + \mathcal{O}\big(\mathcal{E}_{\mathrm{LOR}}\big) \quad \text{for all} \quad j \in \mathbf{N} \quad \text{with} \quad \mu_j \in I_{2\Delta}\,.$$

Alternatively, we could also assume that, for every fixed $\Delta > 0$ small enough, the overlaps $\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle$ are well approximated by $\mathfrak{A}_N(\mu_j)$ in ℓ^p-sense for some $p \ge 1$, i.e.,

$$\frac{1}{N} \sum_{j:\mu_j\in I_{2\Delta}} |\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle - \mathfrak{A}_N(\mu_j)|^p \overset{N\to\infty}{\longrightarrow} 0\,.$$

The case $p = 2$ is reminiscent of the so called *weak ETH* studied in [86, 191].

Corollary 7.2.7 (Relaxation formula)**.** *Under the assumptions and using the notations of Theorem 7.2.4, it holds that*

$$\langle A\rangle_{P_\lambda(t)} = \langle A\rangle_{\widetilde{P}_\lambda} + |g_\lambda(t)|^2\Big[\langle A\rangle_{P_0(t)} - \langle A\rangle_{\widetilde{P}_\lambda}\Big] + \mathcal{R} + \mathcal{E}\,. \tag{7.2.17}$$

In particular, we have the following:

(a) [Short kinetic time behavior] *Let* $0 < T \lesssim 1$*. Then, it holds that*

$$\limsup_{\Delta\to 0}\ \limsup_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t = T}}\ \limsup_{N\to\infty}\big|\langle A\rangle_{P_\lambda(t)} - \langle A\rangle_{P_0(t)}\big| \lesssim T \qquad \text{almost surely (a.s.)} \tag{7.2.18}$$

(b) [Long kinetic time behavior] *Let* $T \gtrsim 1$*. Then it holds that (recall* $\alpha = \pi\rho_0(E_0)$*)*

$$\limsup_{\Delta\to 0}\ \limsup_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t = T}}\ \limsup_{N\to\infty}\Big|\langle A\rangle_{P_\lambda(t)} - \langle A\rangle_{\widetilde{P}_\lambda}\Big| \lesssim T\mathrm{e}^{-2\alpha T} \qquad \text{a.s.} \tag{7.2.19}$$

Moreover, additionally assuming that A *has the LOR property from Definition 7.2.6, we have:*

(c) [Intermediate kinetic times under LOR] *For every fixed* $T \in (0,\infty)$ *it holds that*

$$\lim_{\Delta\to 0}\ \lim_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t = T}}\ \lim_{N\to\infty}\Big[|\mathcal{R}| + |\mathcal{E}|\Big] = 0 \qquad \text{a.s.}\,, \tag{7.2.20}$$

i.e. the relaxation formula (7.1.4) *is valid at* all *kinetic times* $T \in (0,\infty)$*.*

Summarizing Corollary 7.2.7, we have that the relaxation formula (7.1.4) generally holds in the two limiting regimes (a) $|g_\lambda(t)|^2 \approx 0$ and (b) $|g_\lambda(t)|^2 \approx 1$, i.e. $T \ll 1$ or $T \gg 1$, respectively. In between, (7.1.4) is valid under the additional assumption that A has the LOR property from Definition 7.2.6, as this allows for the improved bound (7.2.20) on $\mathcal{R}$ compared to (7.2.14). However, *without* this regularity assumption, only the bound (7.2.14) (i.e. (7.2.18) and (7.2.19)) can hold, which indicates that the relaxation formula (7.1.4) is *not* generally valid for intermediate kinetic times $T \sim 1$. Indeed, it is easy to construct a counterexample. Finally, we remark that Corollary 7.2.7 (c) holds verbatim if the state P satisfies the LOR condition instead of the observable A. This simply follows by inspecting the proof in Section 7.3.4.

Remark 7.2.8. *We have two further comments on Corollary 7.2.7.*

(i) The relaxation formula (7.2.17) *is in perfect agreement with the main result of Dabelow and Reimann, see [192, Eq. (16)]. In fact, the state* $\widetilde{P}_\lambda$ *defined in* (7.2.12) *agrees with* $\widetilde{\rho}$ *from [192, Eq. (16)], named the "'washed out' descendant of the so-called diagonal ensemble" [192].*

(ii) In fact, recalling (7.2.15)*, the proof of Theorem 7.2.4 (b) in Section 7.3.3 reveals that (see* (7.3.71)*) the error* $\mathcal{R}$ *in* (7.2.13) *and* (7.2.17) *is given by*

$$\mathcal{R} = \frac{1}{r}\sum_{j,k}\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_k, P\boldsymbol{u}_k\rangle\mathfrak{R}_{\lambda,t}(\mu_j - \mu_k)\,, \tag{7.2.21}$$

where we denoted

$$r := \int_{\mathbf{R}} \mathrm{Tr}[\operatorname{Im} M_0(x + \mathrm{i}\alpha\lambda^2)]\,\langle \operatorname{Im} M_0(x + \mathrm{i}\alpha\lambda^2)\rangle_P\,\mathrm{d}x \tag{7.2.22}$$

and

$$\mathfrak{R}_{\lambda,t}(u) := \pi \mathrm{e}^{-2\alpha\lambda^2 t} \frac{2\alpha\lambda^2}{u^2 + (2\alpha\lambda^2)^2} \bigg(1 - \cos(tu) - 2\alpha\lambda^2 t \frac{\sin(tu)}{tu} \bigg). \tag{7.2.23}$$

The explicit error term (7.2.21)–(7.2.23) *is in precise agreement with the error term in [192], see Eqs. (17) and (18). In particular, assuming the LOR property* (7.2.16) *for* A *(or* P*), the smallness of* $\mathcal{R}$ *in* (7.2.20) *for all kinetic times* $T \in (0, \infty)$ *is a consequence of the fact that* $\int_{\mathbf{R}} \mathfrak{R}_{\lambda,t}(u) \mathrm{d}u = 0$.

7.2.3 Prethermalization

In this section, we specialize the general relaxation theory of perturbed quantum dynamics from Theorem 7.2.4 to a class of unperturbed Hamiltonians H_0 and states P which *has the prethermalization property* in the following sense.

Definition 7.2.9 (Prethermalization property)**.** *Let the Hamiltonian* H_0 *and the state* P *be defined as in Assumptions 7.2.2 and 7.2.3. We say that* (H_0, P) *has the* prethermalization property *if and only if there exists a state* P_{pre} *(called the* prethermal state*) such that we have the following:*

(a) The unperturbed time evolution $P_0(t)$ *converges to* P_{pre}*, i.e., for all*[7] *deterministic observables* $A \in \mathbf{C}^{N \times N}$ *with* $\|A\| \lesssim 1$*, it holds that*

$$\lim_{t\to\infty} \lim_{N\to\infty} \Big[\langle A \rangle_{P_0(t)} - \langle A \rangle_{P_{\mathrm{pre}}} \Big] = 0 \,. \tag{7.2.24a}$$

(b) There exists a bounded deterministic observable $A_0 \in \mathbf{C}^{N \times N}$ *which distinguishes* P_{pre} *from* $\widetilde{P}_\lambda$ *(cf.* (7.2.12)*), i.e. there exists a constant* $\mathfrak{c}_{\mathrm{pre}} > 0$ *such that*

$$\liminf_{\lambda\to 0} \liminf_{N\to\infty} \Big| \langle A_0 \rangle_{P_{\mathrm{pre}}} - \langle A_0 \rangle_{\widetilde{P}_\lambda} \Big| \geq \mathfrak{c}_{\mathrm{pre}} \,. \tag{7.2.24b}$$

We emphasize that (H_0, P) having the *prethermalization property* is a purely deterministic condition, i.e., in particular, it does not depend on the Wigner matrix W. In the physics literature (see, e.g., [430] but also [455, 396, 372, 43]), the prethermalization property is generally expected to be satisfied if H_0 is an *integrable* Hamiltonian having at least one additional conserved quantity Q for which $[H_0, Q] = 0$.[8] This symmetry is then broken by a generic perturbation W, i.e. $[W, Q] \neq 0$. In the presence of M conserved quantities $(Q_k)_{k=1}^M$, a good candidate for the prethermal state P_{pre} is given by the so called *generalized Gibbs ensemble (GGE)*

$$P_{\mathrm{GGE}} = \frac{\mathrm{e}^{-\sum_{k=1}^M \lambda_k Q_k}}{\operatorname{Tr} \mathrm{e}^{-\sum_{k=1}^M \lambda_k Q_k}} \,,$$

where the parameters λ_k are chosen in such a way that $\operatorname{Tr} Q_k P_{\mathrm{GGE}} = \operatorname{Tr} Q_k P$ for all $k \in [M]$ (see, e.g., [372] and [455, Section 5.1]).

Exemplary pairs (H_0, P) and observables A_0 satisfying the conditions in Definition 7.2.9 are given in Section 7.2.5.

Assuming that (H_0, P) has the prethermalization property, Theorem 7.2.4 reads as follows.

Corollary 7.2.10 (Prethermalization)**.** *Under the assumptions of Theorem 7.2.4, let further* (H_0, P) *have the prethermalization property from Definition 7.2.9. Then, recalling the notations from Theorem 7.2.4, it holds that*

$$\langle A \rangle_{P_\lambda(t)} = \langle A \rangle_{\widetilde{P}_\lambda} + |g_\lambda(t)|^2 \Big[\langle A \rangle_{P_{\mathrm{pre}}} - \langle A \rangle_{\widetilde{P}_\lambda} \Big] + \mathcal{R} + \mathcal{E}' \,. \tag{7.2.25}$$

[7] A common variant of this requirement in the physics literature is to assume the validity of (7.2.24a) only for *local* observables, i.e. supported in a finite region of an underlying space (see, e.g., [455, Section 5.2]).

[8] We use the usual notation for the commutator, i.e. $[A, B] := AB - BA$ for all matrices A, B.

We have $\mathcal{E}' = \mathcal{O}(\mathcal{E}'_0) + \mathcal{O}_\prec(C(\lambda,t)/\sqrt{N})$ *for some constant* $C(\lambda,t) > 0$ *and for every fixed* $T \in (0,\infty)$*, the deterministic errors* $\mathcal{E}'_0 = \mathcal{E}'_0(\lambda,t,\Delta,N)$ *and* $\mathcal{R} = \mathcal{R}(\lambda,t,\Delta,N)$ *satisfy*

$$\lim_{\Delta\to 0}\lim_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t=T}}\lim_{N\to\infty}\mathcal{E}'_0 = 0 \qquad \textit{and} \qquad \limsup_{\Delta\to 0}\limsup_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t=T}}\limsup_{N\to\infty}|\mathcal{R}| \lesssim T\mathrm{e}^{-2\alpha T}\,.$$

We remark that the error term $\mathcal{E}'_0$ contributing in (7.2.25) consists of two parts, $\mathcal{E}'_0 = \mathcal{E}_0 + \mathcal{E}_{\mathrm{pre}}$, with $\mathcal{E}_0 = \mathcal{E}_0(\lambda,t,\Delta,N)$ from Theorem 7.2.4 and $\mathcal{E}_{\mathrm{pre}} = \mathcal{E}_{\mathrm{pre}}(t,N)$ being the (absolute value of the) error in (7.2.24a). Note that (7.2.25) in particular implies the following small and large T behaviors:

$$\begin{aligned}&\limsup_{\Delta\to 0}\limsup_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t=T}}\limsup_{N\to\infty}\left|\langle A\rangle_{P_\lambda(t)} - \langle A\rangle_{P_{\mathrm{pre}}}\right| \lesssim T \quad \text{for} \quad T\lesssim 1 \quad \text{a.s.}\,,\\ \text{and}\quad &\limsup_{\Delta\to 0}\limsup_{\substack{t\to\infty\\ \lambda\to 0\\ \lambda^2 t=T}}\limsup_{N\to\infty}\left|\langle A\rangle_{P_\lambda(t)} - \langle A\rangle_{\widetilde{P}_\lambda}\right| \lesssim T\mathrm{e}^{-2\alpha T} \quad \text{for} \quad T\gtrsim 1 \quad \text{a.s.}\end{aligned} \tag{7.2.26}$$

Moreover, (7.2.24b) ensures that $\langle A\rangle_{P_{\mathrm{pre}}} \neq \langle A\rangle_{\widetilde{P}_\lambda}$ for at least one observable $A = A_0$, which, together with (7.2.26) establishes Figure 7.1.1 as a schematic graph of a prethermalization process.

7.2.4 Connection to the microcanonical ensemble

Under an additional regularity assumption on $x \mapsto \langle\!\langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)A\rangle\!\rangle$ we can relate the state $\widetilde{P}_\lambda$ from (7.2.12) to the microcanonical ensemble.

Theorem 7.2.11 (Microcanonical average)**.** *Under the assumptions of Theorem 7.2.4, let us further assume that*

$$h \equiv h(\lambda,N) : x \mapsto \langle\!\langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)A\rangle\!\rangle$$

is a Lipschitz continuous map on I_Δ *with Lipschitz constant* $\operatorname{Lip}_{I_\Delta}(h)$ *bounded in the sense that*

$$\limsup_{\Delta\to 0}\limsup_{\lambda\to 0}\limsup_{N\to\infty}\operatorname{Lip}_{I_\Delta}(h) \lesssim 1\,. \tag{7.2.27}$$

Then

$$\langle A\rangle_{\widetilde{P}_\lambda} = \langle A\rangle_{P_\lambda^{(\mathrm{mc})}} + \mathcal{E}_{\mathrm{mc}} \qquad \textit{with} \qquad P_\lambda^{(\mathrm{mc})} := \frac{\operatorname{Im} M_0(E_0+\mathrm{i}\alpha\lambda^2)}{\operatorname{Tr}[\operatorname{Im} M_0(E_0+\mathrm{i}\alpha\lambda^2)]}\,, \tag{7.2.28}$$

where the error $\mathcal{E}_{\mathrm{mc}} = \mathcal{E}_{\mathrm{mc}}(\lambda,\Delta,N)$ *satisfies*

$$\lim_{\Delta\to 0}\lim_{\lambda\to 0}\lim_{N\to\infty}\mathcal{E}_{\mathrm{mc}} = 0.$$

We emphasize that $P_\lambda^{(\mathrm{mc})}$ is completely independent of the initial state P. Moreover, as mentioned above and already indicated by the notation, we can interpret $\langle A\rangle_{P_\lambda^{(\mathrm{mc})}}$ from (7.2.28) as the microcanonical average of H_λ at energy E_0. The reason underlying this interpretation is that for any normalized eigenvector $\boldsymbol{v}_\lambda$ of H_λ with eigenvalue E_λ very close to E_0, it holds that[9]

$$\langle \boldsymbol{v}_\lambda, A\boldsymbol{v}_\lambda\rangle \approx \frac{\operatorname{Tr}[\operatorname{Im} M_0(E_0+\mathrm{i}\alpha\lambda^2)A]}{\operatorname{Tr}[\operatorname{Im} M_0(E_0+\mathrm{i}\alpha\lambda^2)]} = \langle A\rangle_{P_\lambda^{(\mathrm{mc})}}\,.$$

This means, $P_\lambda^{(\mathrm{mc})}$ is a close effective approximation to the actual projection $|\boldsymbol{v}_\lambda\rangle\langle\boldsymbol{v}_\lambda|$ onto the eigenspace spanned by $\boldsymbol{v}_\lambda$.

[9] Indeed, taking, say, $E_\lambda \in I_{\Delta/2}$ it can be rigorously shown that the difference $\langle \boldsymbol{v}_\lambda, A\boldsymbol{v}_\lambda\rangle - \langle A\rangle_{P_\lambda^{(\mathrm{mc})}}$ vanishes in the triple limit (7.2.11). More precisely, this follows from the *Eigenstate Thermalization Hypothesis (ETH)* for the random matrix $H_\lambda = H_0 + \lambda W$ (see Theorem 2.2.7) and using assumption (7.2.27) together with (7.3.1) and (7.A.7).

7.2.5 Examples

In this section, we give two examples of physical settings where prethermalization occurs and connect them to our assumptions. Note that both examples are one-dimensional. However, the extension to higher dimensions is straightforward. Moreover, although we do not express the Hamiltonians below as matrices, both act on finite-dimensional Hilbert spaces and can hence naturally be represented as such.

7.2.5.1 Next-nearest neighbor hopping

For $N \in \mathbf{N}$ even, we consider the Laplacian-like Hamiltonian H_0 acting on functions $\psi \in \ell^2(\mathbf{Z}/(N\mathbf{Z}))$ as

$$(H_0\psi)(x) := 2\psi(x) - \psi(x-2) - \psi(x+2) \tag{7.2.29}$$

where $x-2$ and $x+2$ are interpreted mod N. Note that H_0 is similar to the discrete Laplacian with periodic boundary condition but induces next-nearest neighbor hopping instead of nearest neighbor hopping. In particular, H_0 conserves parity in the sense that functions that are only supported on the even or odd points of $\mathbf{Z}/(N\mathbf{Z})$, respectively, remain invariant, and thus its spectrum has an additional two-fold degeneracy. This corresponds to the conserved quantity Q being the projection onto the even sites; clearly $[H_0, Q] = 0$. Similar to the routine computations done for the discrete Laplacian, one can readily check the following:

- The Hamiltonian H_0 is bounded, $\|H_0\| \lesssim 1$.
- Its spectrum is given by $\sigma(H_0) = \{2(1-\cos(2p_j)) : p_j = 2\pi j/N\}_{j\in[N]} \subset [0,4]$.
- The limiting density of states as $N \to \infty$ evaluates to

$$\rho_0(x) = \frac{1}{\pi\sqrt{x(4-x)}}\mathbb{1}_{[0,4]}(x) \tag{7.2.30}$$

  which is compactly supported and satisfies the regularity assumptions in Assumption 7.2.2 for x bounded away from 0 and 4.

In this setting, we fix k such that the eigenvalue $2(1-\cos(2p_k))$ satisfies $p_k \in (0, \pi/2)$. Now take $P := |\boldsymbol{u}_k\rangle\langle\boldsymbol{u}_k|$ with $\boldsymbol{u}_k$ being the normalized eigenvector of H_0 supported on the *even* sub-lattice corresponding to the eigenvalue $2(1-\cos(2p_k))$. By construction, for every bounded observable A we have

$$\langle A\rangle_{P_0(t)} = \langle A\rangle_P = \langle A\rangle_{P_{\mathrm{pre}}}, \quad \text{for all} \quad t \geq 0,$$

since $[P, H_0] = 0$. Hence, the symmetry implies that $P_{\mathrm{pre}} = P$. In particular, for $A := \mathbf{1}_{\mathrm{odd}}$ being the identity operator on the *odd* sub-lattice, its prethermal value is given by $\langle A\rangle_{P_{\mathrm{pre}}} = 0$. Moreover, by spectral decomposition of $H_0 = \sum_j \mu_j |\boldsymbol{u}_j\rangle\langle\boldsymbol{u}_j|$, we obtain

$$\langle\!\langle \operatorname{Im} M_0(x + \mathrm{i}\lambda^2\alpha)A\rangle\!\rangle = \frac{1}{N}\sum_j \langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle \frac{\alpha\lambda^2}{|x-\mu_j|^2 + (\alpha\lambda^2)^2} > 0, \tag{7.2.31}$$

which implies that $\langle A\rangle_{\widetilde{P}_\lambda} \neq \langle A\rangle_{P_{\mathrm{pre}}}$ (recall the definition of $\widetilde{P}_\lambda$ in (7.2.12)) for $A = \mathbf{1}_{\mathrm{odd}}$. Hence, we deduce that (H_0, P) has the prethermalization property from Definition 7.2.9.

7.2.5.2 Free spinless fermions on a lattice

As our second example, we consider a model of spinless fermions in a periodic one-dimensional lattice of even length N (cf. [192, App. B]), which can be seen as a many-body analog of the first example (although with nearest neighbor hopping instead of next-nearest neighbor hopping). Let

$$H_0 = \frac{1}{\sqrt{N}} \sum_j c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j, \tag{7.2.32}$$

where $c_j^\dagger$ and c_j denote the fermionic creation and annihilation operators at site j, and the summation indices are considered modulo N. Note that the Hamiltonian in (7.2.32) conserves the particle number. It is readily checked that H_0 admits a limiting density of states which is not compactly supported but has fast decaying (Gaussian) tails. As the regularity assumptions in Assumption 7.2.2 (ii) are satisfied, this example is still sufficiently close to our theory to be described by it reasonably well.

In this setting, pick ψ_j as the orthonormal eigenfunctions of the discrete Laplacian describing nearest neighbor hopping with periodic boundary conditions (i.e. the analog of (7.2.29) with ± 1 instead of ± 2) corresponding to the eigenvalues $2(1 - \cos(p_j))$ with

$$p_j := \frac{2\pi j}{N}, \quad \frac{j}{N} \in \left[\frac{1}{8}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{7}{8}\right].$$

We then construct $P := |\psi\rangle\langle\psi|$ as a rank-1 projection onto an eigenstate of H_0 by taking

$$\psi := \bigwedge_j \psi_j$$

as a Slater determinant of the $N/2$ one-particle wave functions ψ_j. This ensures that P satisfies Assumption 7.2.3, as the density of states (which is the same as (7.2.30)) is regular in such intervals. Noting that $[P, H_0] = 0$, we obtain

$$\langle A \rangle_{P_0(t)} = \langle A \rangle_P = \langle A \rangle_{P_{\mathrm{pre}}}, \quad \text{for all} \quad t \geq 0,$$

for every bounded observable A. Hence, $P_{\mathrm{pre}} = P$, similar to the first example. In particular, for $A = \mathbf{1}_{\mathcal{H}_{N/2}^\perp}$ being the identity on the orthogonal complement of the $N/2$-particle sector of the Fock space, the prethermal value is given by $\langle A \rangle_{P_{\mathrm{pre}}} = 0$. Moreover, by spectral decomposition of H_0, similarly to (7.2.31), we find that $\langle A \rangle_{\widetilde{P}_\lambda} \neq \langle A \rangle_{P_{\mathrm{pre}}}$. Hence, we deduce that (H_0, P) has the prethermalization property from Definition 7.2.9.

7.3 Proofs

In this section, we provide the proofs of our main results formulated in Section 7.2. We begin by giving the proof of Theorem 7.2.4, which we organize in three steps:

(i) In Section 7.3.1, as the first step, we approximate the random time evolution $\langle A \rangle_{P_\lambda(t)}$ by a deterministic object, up to an error vanishing as $N \to \infty$ with very high probability. This is done using a suitable *global law* for two resolvents of the random matrix H_λ (see Proposition 7.3.1 below).

(ii) The deterministic object resulting from Step (i) consists of two terms, a regular and a singular one. In Section 7.3.2, we evaluate these terms up to errors captured by $\mathcal{E}$ (see Proposition 7.3.2). This proves Theorem 7.2.4 (a).

(iii) As the third and final step in Section 7.3.3, we examine the behavior of the singular term in the limits $T \to 0$ and $T \to \infty$ for $T := \lambda^2 t$ (see Proposition 7.3.6). This proves Theorem 7.2.4 (b).

Afterwards, we give the proofs of Corollary 7.2.7 and Theorem 7.2.11 in Sections 7.3.4 and 7.3.5, respectively. The proof of Corollary 7.2.10 is immediate from Definition 7.2.9 and Theorem 7.2.4 and hence omitted.

7.3.1 Step (i): Global law

Let $\lambda > 0$ and let $H_\lambda := D + \lambda W$ such that $D \in \mathbf{C}^{N\times N}$ is a self-adjoint deterministic matrix with $\|D\| \lesssim 1$ and W is a Wigner matrix satisfying Assumption 7.2.1. We refer to H_λ as a *deformed Wigner matrix*. It is well known [17, 243, 22, 23], that the random resolvent[10] $G_\lambda(z) := (H_\lambda - z)^{-1}$ of H_λ at spectral parameter $z \in \mathbf{C} \setminus \mathbf{R}$ is very well approximated by a deterministic matrix M_λ, which is the unique solution to the *Matrix Dyson Equation (MDE)*[11]

$$-\frac{1}{M_\lambda(z)} = z - D + \lambda^2 \langle\!\langle M_\lambda(z) \rangle\!\rangle, \quad \text{with} \quad \operatorname{Im} M_\lambda(z) \operatorname{Im} z > 0\,. \tag{7.3.1}$$

In particular (see [243, Theorem 2.1]), for $\operatorname{dist}(z, \sigma(D) + [-2\lambda, 2\lambda]) \geq c$ for some N-independent $c > 0$ and $\sigma(D) \subset \mathbf{R}$ denoting the spectrum of D, and arbitrary deterministic matrix $B \in \mathbf{C}^{N\times N}$ with $\|B\| \lesssim 1$, it holds that

$$\left|\langle\!\langle (G_\lambda(z) - M_\lambda(z))B \rangle\!\rangle\right| \prec \frac{1}{N}\,. \tag{7.3.2}$$

For our purposes, it is not sufficient to approximate only a single resolvent in the sense of (7.3.2). Instead, we need to establish the deterministic approximation to $\langle \boldsymbol{x}, G_\lambda(z_1) B G_\lambda(z_2) \boldsymbol{y}\rangle$ with two deterministic vectors $\boldsymbol{x}, \boldsymbol{y}$. This is the content of the following proposition, the proof of which is given in Appendix 7.A.2.

Proposition 7.3.1 (Isotropic two-resolvent global law for deformed Wigner matrices)**.** *Let $\lambda > 0$ and let $H_\lambda := D + \lambda W$ be an $N \times N$ deformed Wigner matrix (as in Assumption 7.2.1) with a bounded self-adjoint deformation $D \in \mathbf{C}^{N\times N}$. Pick $B \in \mathbf{C}^{N\times N}$, a deterministic matrix with $\|B\| \lesssim 1$, deterministic vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ with $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$, and two spectral parameters $z_1, z_2 \in \mathbf{C}$ satisfying $\min_{i\in[2]} \operatorname{dist}(z_i, \sigma(D) + [-2\lambda, 2\lambda]) \geq c$ for some N-independent parameter $c > 0$. Denote further $G_{\lambda,i} := G_\lambda(z_i) = (H_\lambda - z_i)^{-1}$. Then,*

$$\left|\langle \boldsymbol{x}, G_{\lambda,1} B G_{\lambda,2} \boldsymbol{y}\rangle - \left\langle \boldsymbol{x}, \left(M_{\lambda,1} B M_{\lambda,2} + \lambda^2 \frac{M_{\lambda,1} M_{\lambda,2} \langle\!\langle M_{\lambda,1} B M_{\lambda,2} \rangle\!\rangle}{1 - \lambda^2 \langle\!\langle M_{\lambda,1} M_{\lambda,2} \rangle\!\rangle} \right) \boldsymbol{y} \right\rangle \right| \prec \frac{C(\lambda, c)}{\sqrt{N}}\,, \tag{7.3.3}$$

where we denoted $M_{\lambda,i} := M_\lambda(z_i)$ with $M_\lambda(z) \in \mathbf{C}^{N\times N}$ being the solution of (7.3.1). The positive constant $C(\lambda, c)$ in (7.3.3) depends[12] only on its arguments λ and c.

We now apply Proposition 7.3.1 to resolvents $G_\lambda(z) := (H_\lambda - z)^{-1}$ of our concrete deformed Wigner random matrix $H_\lambda = H_0 + \lambda W$. For the proof of Theorem 7.2.4, we use (7.3.3) as follows: Applying residue calculus allows to rewrite $\langle A \rangle_{P_\lambda(t)}$ as the contour integral

$$\langle A \rangle_{P_\lambda(t)} = \langle \mathrm{e}^{-\mathrm{i}tH_\lambda} P \mathrm{e}^{\mathrm{i}tH_\lambda} A \rangle = \frac{1}{(2\pi\mathrm{i})^2} \oint_{\gamma_1} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1 - z_2)} \operatorname{Tr}\big[G_\lambda(z_1) A G_\lambda(z_2) P\big] \mathrm{d}z_1 \mathrm{d}z_2 \tag{7.3.4}$$

where γ_1 and γ_2 are two semicircles (each being the complex conjugate of the other) with some large radius $R \gtrsim 1$ (see Figure 7.3.1 below). We further define the contours such that the distance between the flat part of the semicircles and the real line is t^{-1}. Note that we have

$$\sigma(H_\lambda) \subseteq \sigma(H_0) + [-(2+\epsilon)\lambda, (2+\epsilon)\lambda] \quad \text{w.v.h.p.} \tag{7.3.5}$$

for any fixed $\epsilon > 0$ by standard perturbation theory, using that $\|W\| \leq 2 + \epsilon$ with very high probability (see, e.g., [241, Theorem 7.6]). In particular, the contours encircle the spectrum of H_λ completely if R is chosen large enough.

[10] In the sequel, we consistently use the letter G to denote resolvents of random matrices, for example $G_\lambda(z) := (H_\lambda - z)^{-1}$, and M to denote deterministic resolvents – more specifically, resolvents of H_0.

[11] The MDE in the context of mean-field random matrices was introduced in [17] and extensively analyzed in [22].

[12] In fact, by examining the proof of Proposition 7.3.1, it can easily be seen that the dependence on both small parameters λ and c is at most (inverse) polynomial.

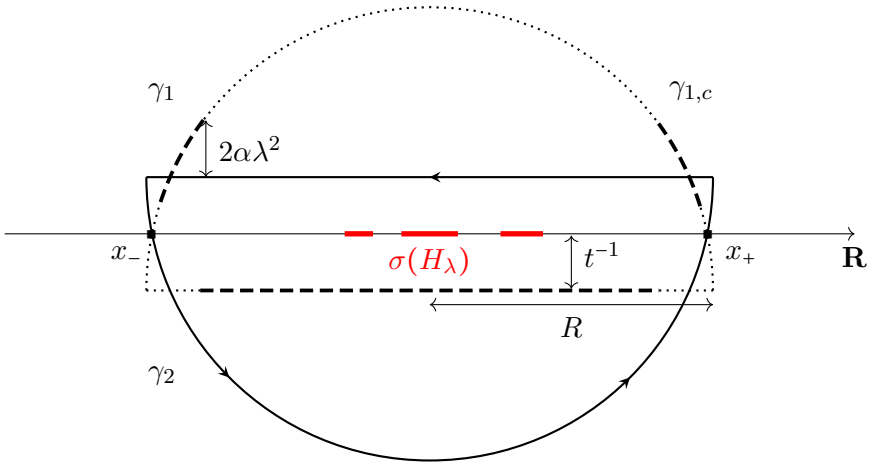

Figure 7.3.1: Sketch of the contours γ_1 and γ_2 from (7.3.4). The contour γ_2 is depicted as a solid black curve, while the contour γ_1 is indicated with dashed and dotted lines. The dotted parts of γ_1 constitute the set $\gamma_{1,c}$, defined in (7.3.32). The intersections of γ_1 (and γ_2) with the real line are denoted by $x_\pm$.

Writing $P = \sum_j p_j |\boldsymbol{p}_j\rangle \langle \boldsymbol{p}_j|$ in spectral decomposition and using that the $p_j \in [0,1]$ sum to one by Assumption 7.2.3, the global law (7.3.3) applied to $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{p}_j$ implies[13]

$$\begin{aligned}\Big| &\mathrm{Tr}[G_\lambda(z_1)AG_\lambda(z_2)P] \\ &- \mathrm{Tr}\left[M_\lambda(z_1)AM_\lambda(z_2)P + \lambda^2 \frac{M_\lambda(z_1)M_\lambda(z_2)P\langle\!\langle M_\lambda(z_1)AM_\lambda(z_2)\rangle\!\rangle}{1-\lambda^2\langle\!\langle M_\lambda(z_1)M_\lambda(z_2)\rangle\!\rangle}\right]\Big| < \frac{C(\lambda,t)}{\sqrt{N}}\end{aligned} \tag{7.3.6}$$

uniformly for z_1, z_2 along the contours γ_1, γ_2 for any fixed $\lambda > 0$. Just as in (7.3.3), $C(\lambda,t)$ denotes a positive constant depending only on λ and t. Therefore, combining (7.3.4) with (7.3.6), we find that

$$\begin{aligned}\langle A\rangle_{P_\lambda(t)} = &\frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_1}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1-z_2)}\,\mathrm{Tr}\big[M_\lambda(z_1)AM_\lambda(z_2)P\big]\mathrm{d}z_1\mathrm{d}z_2 \\ &+ \frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_1}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1-z_2)}\lambda^2 \frac{\langle\!\langle[\,]M_\lambda(z_1)AM_\lambda(z_2)\rangle\!\rangle\,\mathrm{Tr}\big[M_\lambda(z_1)PM_\lambda(z_2)\big]}{1-\lambda^2\langle\!\langle M_\lambda(z_1)M_\lambda(z_2)\rangle\!\rangle}\mathrm{d}z_1\mathrm{d}z_2 \\ &+ \mathcal{O}_\prec\left(\frac{C(\lambda,t)}{\sqrt{N}}\right).\end{aligned} \tag{7.3.7}$$

To establish (7.2.7), our main task thus lies in evaluating the right-hand side of (7.3.7). For simplicity, we refer to the integrals in the first and second line of (7.3.7) as the *regular* and the *singular* term, respectively.

7.3.2 Step (ii): Evaluation of the regular and singular term and proof of Theorem 7.2.4 (a)

We organize the result of our computation of (7.3.7) in the following proposition.

[13] Note that (7.3.6) is an *averaged* two-resolvent local law, but we prove it via an *isotropic* law. Therefore, the error term in (7.3.6) is not optimal: The true bound is of order N^{-1} instead of $N^{-1/2}$. While we could have obtained an optimal error term with the same proof idea, the weaker bound is sufficient for our analysis and we use it for simplicity.

Proposition 7.3.2 (Evaluation of the regular and singular term)**.** *Under the assumptions of Theorem 7.2.4 and letting γ_1, γ_2 be the contours in Fig. 7.3.1, we have (recalling $\alpha = \pi\rho_0(E_0)$)*

$$\begin{aligned}\frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_1}\oint_{\gamma_2}\mathrm{e}^{\mathrm{i}t(z_1-z_2)}\operatorname{Tr}\big[M_\lambda(z_1)AM_\lambda(z_2)P\big]\mathrm{d}z_1\mathrm{d}z_2\\ = \mathrm{e}^{-2\alpha\lambda^2 t}\langle A\rangle_{P_0(t)} + \mathcal{O}(\mathcal{E}_{\mathrm{reg}})\,,\end{aligned} \tag{7.3.8a}$$

for the regular term and

$$\begin{aligned}\widetilde{\mathcal{F}}_{\mathrm{sing}} \coloneqq \frac{\lambda^2}{(2\pi\mathrm{i})^2}\oint_{\gamma_1}\oint_{\gamma_2}\mathrm{e}^{\mathrm{i}t(z_1-z_2)}\frac{\langle\!\langle M_\lambda(z_1)AM_\lambda(z_2)\rangle\!\rangle\operatorname{Tr}\big[M_\lambda(z_1)PM_\lambda(z_2)\big]}{1-\lambda^2\langle\!\langle M_\lambda(z_1)M_\lambda(z_2)\rangle\!\rangle}\mathrm{d}z_1\mathrm{d}z_2\\ = \langle A\rangle_{\widetilde{P}_{\lambda,t}} + \mathcal{O}(\mathcal{E}_{\mathrm{sing}})\end{aligned} \tag{7.3.8b}$$

for the singular term, with some error terms $\mathcal{E}_{\mathrm{reg/sing}} = \mathcal{E}_{\mathrm{reg/sing}}(\lambda, t, \Delta, N)$ in (7.3.8a) *and* (7.3.8b) *satisfying* (7.2.11)*. The explicit form of $\mathcal{E}_{\mathrm{reg}}$ is given in* (7.3.16) *and* (7.3.18)*, while the explicit form of $\mathcal{E}_{\mathrm{sing}}$ is given in* (7.3.26)*.*

Plugging (7.3.8a) and (7.3.8b) into (7.3.7), we immediately conclude Theorem 7.2.4 (a) after setting $\mathcal{E}_0 \coloneqq \mathcal{E}_{\mathrm{reg}} + \mathcal{E}_{\mathrm{sing}}$ and including the error term from (7.3.7) into $\mathcal{E}$. □

It thus remains to give the proof of Proposition 7.3.2, i.e. its two parts (7.3.8a) and (7.3.8b). This is done in Sections 7.3.2.1 and 7.3.2.2, respectively.

7.3.2.1 Proof of (7.3.8a)

The main contribution to the integral in (7.3.8a) comes from the regime[14] where z_1 and z_2 are close to E_0. Hence, as a first approximation we use the replacements $\langle\!\langle M_\lambda(z_1)\rangle\!\rangle \approx \overline{m_0(E_0)}$ and $\langle\!\langle M_\lambda(z_2)\rangle\!\rangle \approx m_0(E_0)$ in (7.3.1), which leads to

$$M_\lambda(z_1) \approx \frac{1}{H_0 - z_1 - \lambda^2\overline{m_0(E_0)}} \qquad \text{and} \qquad M_\lambda(z_2) \approx \frac{1}{H_0 - z_2 - \lambda^2 m_0(E_0)}\,. \tag{7.3.9}$$

Applying the replacements in (7.3.9) for the term in (7.3.8a) yields

$$\begin{aligned}&\frac{1}{(2\pi\mathrm{i})^2}\oint_{\gamma_1}\oint_{\gamma_2}\mathrm{e}^{\mathrm{i}t(z_1-z_2)}\operatorname{Tr}\left[\frac{1}{H_0 - z_1 - \lambda^2\overline{m_0(E_0)}}A\frac{1}{H_0 - z_2 - \lambda^2 m_0(E_0)}P\right]\mathrm{d}z_1\mathrm{d}z_2\\ &= \operatorname{Tr}\left[\mathrm{e}^{\mathrm{i}t(H_0-\lambda^2\overline{m_0(E_0)})}A\mathrm{e}^{-\mathrm{i}t(H_0-\lambda^2 m_0(E_0))}P\right]\\ &= \mathrm{e}^{-2\operatorname{Im} m_0(E_0)\lambda^2 t}\operatorname{Tr}\left[\mathrm{e}^{\mathrm{i}tH_0}A\mathrm{e}^{-\mathrm{i}tH_0}P\right] = \mathrm{e}^{-2\alpha\lambda^2 t}\,\langle A\rangle_{P_0(t)},\end{aligned} \tag{7.3.10}$$

since $\operatorname{Im} m_0(E_0) = \pi\rho_0(E_0)$, from simple residue calculus for $\lambda > 0$ small enough, using that $|m_0(E_0)| \lesssim 1$ and γ_1, γ_2 encircle the spectrum of H_0. We have thus extracted the main term in (7.3.8a), and it remains to estimate the errors resulting from the replacements in (7.3.9).

Recall (see (7.2.15)) that μ_j and $\boldsymbol{u}_j$ are the eigenvalues and the respective orthonormalized eigenvectors of H_0, i.e.

$$H_0 = \sum_j \mu_j\,|\boldsymbol{u}_j\rangle\,\langle\boldsymbol{u}_j|\,. \tag{7.3.11}$$

[14] Recall that the flat pieces of the contours γ_1 and γ_2 from Figure 7.3.1 lie on the lower and upper half plane, respectively.

Then, by means of (7.3.1), spectral decomposition (7.3.11) of H_0 and using Assumption 7.2.3 together with $[H_0, \Pi_\Delta] = 0$ and $\Pi_\Delta^2 = \Pi_\Delta$, we have that

$$\text{lhs. of (7.3.8a)} = \operatorname{Tr}\big[\widetilde{\Theta}_1 A \widetilde{\Theta}_2 P\big] = \sum_{\mu_i, \mu_j \in I_\Delta} \langle \boldsymbol{u}_i, A\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j, P\boldsymbol{u}_i\rangle \widetilde{\vartheta}_1(i)\widetilde{\vartheta}_2(j)\,, \tag{7.3.12}$$

where we denoted

$$\widetilde{\Theta}_1 \coloneqq \sum_{\mu_j \in I_\Delta} |\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j| \widetilde{\vartheta}_1(j) \quad \text{and} \quad \widetilde{\Theta}_2 \coloneqq \sum_{\mu_j \in I_\Delta} |\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j| \widetilde{\vartheta}_2(j) \tag{7.3.13}$$

with

$$\widetilde{\vartheta}_1(j) \coloneqq \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}tz_1}}{\mu_j - z_1 - \lambda^2 \langle\!\langle M_\lambda(z_1)\rangle\!\rangle} \mathrm{d}z_1\,, \quad \widetilde{\vartheta}_2(j) \coloneqq \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_2} \frac{\mathrm{e}^{-\mathrm{i}tz_2}}{\mu_j - z_2 - \lambda^2 \langle\!\langle M_\lambda(z_2)\rangle\!\rangle} \mathrm{d}z_2\,. \tag{7.3.14}$$

Note that, by symmetry of the contours γ_1 and γ_2, we have that $\overline{\widetilde{\vartheta}_1(j)} = \widetilde{\vartheta}_2(j)$ and $\widetilde{\Theta}_1^* = \widetilde{\Theta}_2$.

The key to approximating (7.3.12) is the following lemma, whose proof is given at the end of the current Section 7.3.2.1.

Lemma 7.3.3 (First replacement lemma)**.** *Using the above notations and assumption, denote*

$$\Theta_1 \coloneqq \sum_{\mu_j \in I_\Delta} |\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j| \vartheta_1(j) \quad \textit{with} \quad \vartheta_1(j) \coloneqq \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_1} \frac{\mathrm{e}^{\mathrm{i}tz_1}}{\mu_j - z_1 - \lambda^2 \overline{m_0(E_0)}} \mathrm{d}z_1$$

and $\Theta_2 \coloneqq \Theta_1^*$ *via* $\vartheta_2(j) \coloneqq \overline{\vartheta_1(j)}$*, analogously to* (7.3.13) *and* (7.3.14)*. Then it holds that*

$$\sup_{\mu_i \in I_\Delta} \big|\widetilde{\vartheta}_1(i) - \vartheta_1(i)\big| + \sup_{\mu_j \in I_\Delta} \big|\widetilde{\vartheta}_2(j) - \vartheta_2(j)\big| \lesssim \widetilde{\mathcal{E}}_{\mathrm{reg}} \tag{7.3.15}$$

for sufficiently small $\lambda > 0$ *and* N *large enough (dependent on* λ*, cf. Lemma 7.A.1). Here, recalling* (7.2.2) *for the definition of* ϵ_0*, we denoted*

$$\widetilde{\mathcal{E}}_{\mathrm{reg}} \coloneqq \lambda^2 t\,\Delta + \lambda\,(1 + \lambda^2 t) + \frac{\lambda}{\Delta}\left(1 + \frac{\lambda}{\Delta}\right) + \lambda^2 t\,\epsilon_0\,. \tag{7.3.16}$$

Therefore, by writing $\widetilde{\Theta} = \Theta + (\widetilde{\Theta} - \Theta)$ in (7.3.12), we find the lhs. of (7.3.8a) to be given by

$$\operatorname{Tr}\big[\Theta_1 A \Theta_2 P\big] + \operatorname{Tr}\big[(\widetilde{\Theta}_1 - \Theta_1) A \Theta_2 P\big] + \operatorname{Tr}\big[\Theta_1 A (\widetilde{\Theta}_2 - \Theta_2) P\big] + \operatorname{Tr}\big[(\widetilde{\Theta}_1 - \Theta_1) A (\widetilde{\Theta}_2 - \Theta_2) P\big]\,. \tag{7.3.17}$$

The first term in (7.3.17) precisely yields the result of (7.3.10) using Assumption 7.2.3. Using $\|A\| \lesssim 1$ and $\operatorname{Tr}[P] = 1$, the second and third term in (7.3.17) can be estimated by (a constant times)

$$\|\Theta_1\|\,\|\widetilde{\Theta}_2 - \Theta_2\| + \|\widetilde{\Theta}_1 - \Theta_1\|\,\|\Theta_2\| \lesssim \widetilde{\mathcal{E}}_{\mathrm{reg}}\,.$$

Here we used (7.3.15) and that $\|\Theta_1\| \le 1$ and $\|\Theta_2\| \le 1$ as follows by the explicit expressions

$$\Theta_1 = \mathrm{e}^{\mathrm{i}t(\Pi_\Delta H_0 \Pi_\Delta - \lambda^2 \overline{m_0(E_0)})} \quad \text{and} \quad \Theta_2 = \mathrm{e}^{-\mathrm{i}t(\Pi_\Delta H_0 \Pi_\Delta - \lambda^2 m_0(E_0))}$$

and $\operatorname{Im} m_0(E_0) \ge 0$. Similarly, applying (7.3.15) again, the fourth term in (7.3.17) is bounded by $\mathcal{O}(\widetilde{\mathcal{E}}_{\mathrm{reg}}^2)$. Collecting all four terms of (7.3.16), this concludes the proof of (7.3.8a) with

$$\mathcal{E}_{\mathrm{reg}} \coloneqq \widetilde{\mathcal{E}}_{\mathrm{reg}} + \widetilde{\mathcal{E}}_{\mathrm{reg}}^2\,. \tag{7.3.18}$$

It remains to give the proof of Lemma 7.3.3.

Proof of Lemma 7.3.3. Since $\overline{\widetilde{\vartheta}_1(j)} = \widetilde{\vartheta}_2(j)$ and $\vartheta_2(j) \coloneqq \overline{\vartheta_1(j)}$, we only estimate $\widetilde{\vartheta}_2(j) - \vartheta_2(j)$ for arbitrary but fixed index j such that $\mu_j \in I_\Delta$. Moreover, for ease of notation, we completely drop the subscript 2.

As a first step, we split the contour into three parts:

$$\gamma = \Gamma_1 \dot{+} \Gamma_2 \dot{+} \Gamma_3\,, \tag{7.3.19}$$

where Γ_1 is the horizontal part of γ with $\operatorname{Re} z \in I_{2\Delta}$, Γ_2 is the horizontal part of γ with $\operatorname{Re} z \notin I_{2\Delta}$ and Γ_3 consists of the great arc of radius R (cf. Figure 7.3.1). We now estimate these three parts separately.

For the first part, Γ_1, we have that (using the notation $m_\lambda(z) = \langle\!\langle M_\lambda(z) \rangle\!\rangle$ from Lemma 7.A.1)[15]

$$\begin{aligned} &\left| \int_{\Gamma_1} \mathrm{e}^{-\mathrm{i}tz} \left[\frac{1}{\mu_j - z - \lambda^2 m_\lambda(z)} - \frac{1}{\mu_j - z - \lambda^2 m_0(E_0)} \right] \mathrm{d}z \right| \\ &\qquad \lesssim \int_{\Gamma_1} \frac{\lambda^2 \left(1/t + \lambda + \Delta + \epsilon_0\right)}{|\mu_j - z - \lambda^2 m_\lambda(z)|\,|\mu_j - z - \lambda^2 m_0(E_0)|} |\mathrm{d}z| \lesssim \lambda^2 t \left(1/t + \lambda + \Delta + \epsilon_0\right), \end{aligned} \tag{7.3.20}$$

uniformly in $\mu_j \in I_\Delta$. To go to the second line, we used that $|m_\lambda(z) - m_0(E_0)| \lesssim 1/t + \lambda + \Delta + \epsilon_0$. This follows by adding and subtracting $m_\lambda(E_0)$ and using $|m_\lambda(z) - m_\lambda(E_0)| \lesssim \Delta + 1/t$ (using $|m_\lambda'(z)| \lesssim 1$ for $\operatorname{Re} z \in I_{2\Delta}$; cf. the last estimate in (7.A.2) from Lemma 7.A.1) and $|m_\lambda(E_0) - m_0(E_0)| \lesssim \lambda + \epsilon_0$ (using that (7.A.7) holds down to the real line by combining it with (7.A.8)). For the final bound, we employed a Schwarz inequality for the integral and estimated the resulting integrals

$$\int_{\Gamma_1} \frac{|\mathrm{d}z|}{|\mu_j - z - \lambda^2 m_\lambda(z)|^2} \lesssim (1 + \lambda^2)\, t \lesssim t \quad \text{and} \quad \int_{\Gamma_1} \frac{|\mathrm{d}z|}{|\mu_j - z - \lambda^2 m_0(E_0)|^2} \lesssim t\,,$$

by a change of variables $z \to z + \lambda^2 m_\lambda(z)$ using that $|m_\lambda'(z)| \lesssim 1$ for $z \in \Gamma_1$ by means of (7.A.2) together with $|\operatorname{Im}[z + \lambda^2 m_\lambda(z)]| \geq t^{-1}$, and $|m_0(E_0)| \lesssim 1$ together with $|\operatorname{Im}[z + \lambda^2 m_0(E_0)]| \geq t^{-1}$, respectively.

We now turn to the second part, i.e. the integral similar to the left-hand side of (7.3.20) but on the contour Γ_2. By means of $|m_0(E_0)| \lesssim 1$ and $|m_\lambda(z)| \leq \lambda^{-1}$ (see the first estimate in (7.A.1)) we bound $|m_\lambda(z) - m_0(E_0)| \lesssim \lambda^{-1}$. Using $|m_0(E_0)| \lesssim 1$ and $|m_\lambda(z)| \leq \lambda^{-1}$ again, together with $\operatorname{dist}(\mu_j, \Gamma_2) \gtrsim \Delta$, we find this second part to be bounded by (a constant times)

$$\int_{\Gamma_2} \frac{\lambda}{|\mu_j - z|^2} |\mathrm{d}z| \left(1 + \frac{\lambda}{\Delta}\right) \lesssim \frac{\lambda}{\Delta} \left(1 + \frac{\lambda}{\Delta}\right), \tag{7.3.21}$$

again uniformly in $\mu_j \in I_\Delta$.

Finally, we estimate the third part. By the exact same reasoning as for Γ_2, we arrive at the bound (7.3.21) with Δ replaced by R and Γ_2 replaced by Γ_3. Hence, using that the radius R of the semicircle is larger than one (see Figure 7.3.1), we find the third part to be bounded by $\mathcal{O}(\lambda/R)$, uniformly in $\mu_j \in I_\Delta$.

Combining this with the error terms in (7.3.20) and (7.3.21), this concludes the proof. □

7.3.2.2 Proof of (7.3.8b)

Recall that $\widetilde{\mathcal{F}}_{\mathrm{sing}}$ denotes the singular term defined in (7.3.8b). To carry out the analog of the approximation (7.3.9), we observe that the resolvent identity for H_0 implies

$$\frac{z_{1,\lambda} - z_{2,\lambda}}{z_1 - z_2} M_0(z_{1,\lambda}) M_0(z_{2,\lambda}) = \frac{M_0(z_{1,\lambda}) - M_0(z_{2,\lambda})}{z_1 - z_2} = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\operatorname{Im} M_0(x_\lambda)}{(x - z_1)(x - z_2)} \mathrm{d}x, \tag{7.3.22}$$

[15] Here and in the following, $|\mathrm{d}z|$ denotes the total variation of the complex measure $\mathrm{d}z$.

where we introduce the notation $z_{1,\lambda} := z_1 + \lambda^2\overline{m_0(E_0)}$, $z_{2,\lambda} := z_2 + \lambda^2 m_0(E_0)$, and $x_\lambda := x + \lambda^2 m_0(E_0)$. Note that x_λ is a complex number with $\operatorname{Im} x_\lambda = \lambda^2 \operatorname{Im} m_0(E_0) > 0$. Here, the second equality follows from the contour representation of the resolvent M_0 of H_0, namely

$$M_0(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\operatorname{Im} M_0(x+\mathrm{i}\eta)}{x+\mathrm{i}\eta - z}\mathrm{d}x, \quad \operatorname{Im} z > \eta > 0. \tag{7.3.23}$$

On the other hand, subtracting two instances of the MDE (7.3.1) yields

$$\frac{M_\lambda(z_1)M_\lambda(z_2)}{1-\lambda^2\langle\!\langle M_\lambda(z_1)M_\lambda(z_2)\rangle\!\rangle} = \frac{M_\lambda(z_1)-M_\lambda(z_2)}{z_1-z_2} = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\operatorname{Im} M_\lambda(x)}{(x-z_1)(x-z_2)}\mathrm{d}x, \tag{7.3.24}$$

where we used the matrix-valued analog of the Stieltjes representation for $M_\lambda(z)$ (cf. [22, Prop. 2.1]),

$$M_\lambda(z) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\operatorname{Im} M_\lambda(x)}{x-z}\mathrm{d}x, \quad z \in \mathbf{C}\setminus\mathbf{R}\,. \tag{7.3.25}$$

In particular, identities (7.3.22) and (7.3.24) suggest that the appropriate approximation for the factor $1-\lambda^2\langle\!\langle M_\lambda(z_1)M_\lambda(z_2)\rangle\!\rangle$ in the denominator of (7.3.8b) is $(z_1-z_2)(z_{1,\lambda}-z_{2,\lambda})^{-1}$. Indeed, we prove that the following estimate holds.

Lemma 7.3.4. *Under the Assumption 7.2.2 and 7.2.3, the singular term $\widetilde{\mathcal{F}}_{\mathrm{sing}}$ defined in* (7.3.8b) *satisfies*

$$\left|\widetilde{\mathcal{F}}_{\mathrm{sing}} - \mathcal{F}_{\mathrm{sing}}\right| \lesssim \mathcal{E}_{\mathrm{sing}} := (\Delta+\epsilon_0)(1+\lambda^2 t) + \lambda\big(1+\lambda^2 t + \Delta^{-1}\log t\big)^2, \tag{7.3.26}$$

where the quantity $\mathcal{F}_{\mathrm{sing}}$ is given by

$$\begin{aligned}\mathcal{F}_{\mathrm{sing}} := \frac{\lambda^2}{(2\pi\mathrm{i})^2}\oint_{\gamma_1}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1-z_2)}\frac{z_{1,\lambda}-z_{2,\lambda}}{z_1-z_2} \\ \times\big\langle\!\big\langle M_0(z_{1,\lambda})AM_0(_{2,\lambda})\big\rangle\!\big\rangle \operatorname{Tr}\big[M_0(z_{1,\lambda})PM_0(z_{2,\lambda})\big]\mathrm{d}z_1\mathrm{d}z_2.\end{aligned} \tag{7.3.27}$$

We defer the proof of Lemma 7.3.4 to the end of the current Section 7.3.2.2, and proceed to analyze the right-hand side of (7.3.27).

Applying the identity (7.3.22) to both traces in the integrand of (7.3.27), we obtain the expression

$$\mathcal{F}_{\mathrm{sing}} = \int_{\mathbf{R}}\int_{\mathbf{R}} \big\langle\!\big\langle \operatorname{Im} M_0(x_\lambda)A\big\rangle\!\big\rangle \operatorname{Tr}[\operatorname{Im} M_0(y_\lambda)P]F_{\lambda,t}(x,y)\mathrm{d}x\mathrm{d}y, \tag{7.3.28}$$

where the function $F_{\lambda,t}(x,y)$ is defined as

$$F_{\lambda,t}(x,y) := \frac{\lambda^2}{(2\pi\mathrm{i})^2\pi^2}\oint_{\gamma_1}\oint_{\gamma_2} \frac{\mathrm{e}^{\mathrm{i}t(z_1-z_2)}(z_1-z_2)}{(x-z_1)(x-z_2)(y-z_1)(y-z_2)(z_{1,\lambda}-z_{2,\lambda})}\mathrm{d}z_2\mathrm{d}z_1. \tag{7.3.29}$$

Recall the contours γ_1 and γ_2 from Figure 7.3.1, and that $z_{1,\lambda}-z_{2,\lambda} = z_1-z_2-2\mathrm{i}\alpha\lambda^2$. Evaluating the contour integration over γ_2 in (7.3.29) yields

$$\begin{aligned}F_{\lambda,t}(x,y) = \frac{\lambda^2}{\pi^2}\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_1}\Bigg(\frac{-\mathrm{e}^{\mathrm{i}t(z_1-x)}\chi(x)}{(x-y)(y-z_1)(x_\lambda-z_{1,\lambda})} + \frac{\mathrm{e}^{\mathrm{i}t(z_1-y)}\chi(y)}{(x-y)(x-z_1)(y_\lambda-z_{1,\lambda})} \\ + \frac{2\mathrm{i}\alpha\lambda^2\mathrm{e}^{-2\alpha\lambda^2 t}}{(x-z_1)(y-z_1)(x_\lambda-z_{1,\lambda})(y_\lambda-z_{1,\lambda})}\Bigg)\mathrm{d}z_1 - F^{\mathrm{out}}_{\lambda,t}(x,y),\end{aligned} \tag{7.3.30}$$

where we define $\chi(z) := \mathbb{1}_{\Omega_2}(z)$, and Ω_2 is the compact connected component of $\mathbf{C}\backslash\gamma_2$, and the function $F^{\mathrm{out}}_{\lambda,t}(x,y)$ is defined as

$$F^{\mathrm{out}}_{\lambda,t}(x,y) := \frac{\lambda^2}{\pi^2}\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_1}\frac{2\mathrm{i}\alpha\lambda^2\mathrm{e}^{-2\alpha\lambda^2 t}\big(1-\chi(z_1-2\mathrm{i}\alpha\lambda^2)\big)}{(x-z_1)(y-z_1)(x_\lambda-z_{1,\lambda})(y_\lambda-z_{1,\lambda})}\mathrm{d}z_1. \tag{7.3.31}$$

We proceed to show that $F^{\mathrm{out}}_{\lambda,t}(x,y)$ contributes at most an $\mathcal{O}(\lambda^4(1+|\log\lambda|^2+(\lambda^2 t)^2))$ error to the right-hand side of (7.3.28). Let $\gamma_{1,c}$ denote the set (see Figure 7.3.1)

$$\gamma_{1,c} := \{z_1 \in \gamma_1 : z_1 - 2\mathrm{i}\alpha\lambda^2 \notin \Omega_2\}. \tag{7.3.32}$$

Then the contribution of $F^{\mathrm{out}}_{\lambda,t}(x,y)$ to the integral in (7.3.28) is given by

$$\mathcal{E}_{\gamma_{1,c}} := \frac{\alpha\lambda^4 \mathrm{e}^{-2\alpha\lambda^2 t}}{\pi^3} \int_{\gamma_{1,c}} \int_{\mathbf{R}} \frac{\langle\!\langle \operatorname{Im} M_0(x_\lambda) A \rangle\!\rangle \mathrm{d}x}{(x-z_1)(x_\lambda - z_{1,\lambda})} \int_{\mathbf{R}} \frac{\operatorname{Tr}[\operatorname{Im} M_0(y_\lambda) P]\mathrm{d}y}{(y-z_1)(y_\lambda - z_{1,\lambda})} \mathrm{d}z_1. \tag{7.3.33}$$

Note that by choosing the radii of the contours $R \gtrsim 1$ large enough, we can assume that

$$\operatorname{dist}(\sigma(H_0), \gamma_{1,c}) \gtrsim R. \tag{7.3.34}$$

Using the spectral decomposition of H_0 from (7.3.11), the fact that $\int_{\mathbf{R}} \operatorname{Im}[(\mu_j - x_\lambda)]\mathrm{d}x = \pi$, and Assumption 7.2.3, we conclude that

$$\left| \int_{\mathbf{R}} \langle\!\langle \operatorname{Im} M_0(x_\lambda) A \rangle\!\rangle \mathrm{d}x \right| + \left| \int_{\mathbf{R}} \operatorname{Tr}\big[\operatorname{Im} M_0(y_\lambda) P\big] \mathrm{d}y \right| \lesssim 1 + \|A\| \lesssim 1. \tag{7.3.35}$$

Furthermore, the spectral decomposition for H_0 implies that for all x with $\operatorname{dist}(x, \sigma(H_0)) \gtrsim 1$,

$$\big|\operatorname{Tr}\big[\operatorname{Im} M_0(x_\lambda) P\big]\big| + \big|\langle\!\langle \operatorname{Im} M_0(x_\lambda) A \rangle\!\rangle\big| \lesssim \lambda^2 |x - E_0|^{-2}. \tag{7.3.36}$$

Let $x_\pm$ denote the intersections of the contour γ_1 with $\mathbf{R}$ (see Figure 7.3.1), and define $\mathbb{D}$ to be a union of two small disks of radius ε around $x_\pm$, $\mathbb{D} := \{z \in \mathbf{C} : \min\{|z - x_-|, |z - x_+|\} \le \varepsilon\}$, for a sufficiently small constant $\varepsilon \sim 1$. Applying the Cauchy-Schwarz inequality to the z_1 integration in (7.3.33), and using the estimates (7.3.34)-(7.3.36) to bound the contribution coming from the outside of $\mathbb{D}$, and applying the estimates (7.3.34) and (7.3.36) for all $x, y \in \mathbf{R} \cap \mathbb{D}$, we obtain

$$\big|\mathcal{E}_{\gamma_{1,c}}\big| \lesssim \lambda^4 \int_{\gamma_{1,c} \cap \mathbf{D}} \left| \frac{\lambda^2}{R^2} \int_{\mathbf{R}\cap\mathbb{D}} \frac{\mathrm{d}x}{|x - z_1||x_\lambda - z_{1,\lambda}|} \right|^2 |\mathrm{d}z_1| + \mathcal{O}\big(R^{-3}\lambda^4\big). \tag{7.3.37}$$

For z_1 on the horizontal linear segment of $\gamma_{1,c} \cap \mathbb{D}$, we use that $\operatorname{Im} z_1 = -1/t$ to obtain

$$\frac{\lambda^2}{R^2} \int_{\mathbf{R}\cap\mathbb{D}} \frac{\mathrm{d}x}{|x - z_1||x_\lambda - z_{1,\lambda}|} \lesssim \frac{1 + \lambda^2 t}{R^2}, \tag{7.3.38}$$

On the other hand, for z_1 lying on the circular arc parts of $\gamma_{1,c} \cap \mathbb{D}$, we compute

$$\frac{\lambda^2}{R^2} \int_{\mathbf{R}\cap\mathbb{D}} \frac{\mathrm{d}x}{|x - z_1||x_\lambda - z_{1,\lambda}|} \lesssim \frac{\lambda^2}{R^2} \frac{|\log\lambda^2| + \big|\log|\eta_1|\big| + \big|\log|\eta_1 - 2\alpha\lambda^2|\big|}{|\eta_1| + \lambda^2}, \tag{7.3.39}$$

where $\eta_1 := \operatorname{Im} z_1$. Squaring the estimates (7.3.38) and (7.3.39) and integrating them over the respective parts of $\gamma_{1,c} \cap \mathbb{D}$, we conclude from (7.3.37) that

$$\big|\mathcal{E}_{\gamma_{1,c}}\big| \lesssim R^{-3}\lambda^4\big(1 + (\lambda^2 t)^2 + |\log\lambda|^2\big). \tag{7.3.40}$$

Next, using residue calculus, we compute the first term on the right-hand side of (7.3.30), i.e., the contour integral over γ_1, to obtain

$$\begin{aligned} F_{\lambda,t}(x,y) = {} & \frac{K_{\lambda,t}(x-y)}{\alpha\pi} - F^{\mathrm{out}}_{\lambda,t}(x,y) + \frac{K_{\lambda,t}(x-y)}{\alpha\pi}\big(\chi(x)\chi(y) - 1\big) \\ & + \frac{1}{2\pi^2} \frac{2\lambda^2 \mathrm{e}^{-2\alpha\lambda^2 t}\big(\chi(x) - \chi(y)\big)^2}{|x-y|^2 + (2\alpha\lambda^2)^2} + \frac{1}{\pi^2} \frac{2\mathrm{i}\alpha\lambda^4 \mathrm{e}^{-2\alpha\lambda^2 t}}{|x-y|^2 + (2\alpha\lambda^2)^2} \frac{\chi(x) - \chi(y)}{x - y}, \end{aligned} \tag{7.3.41}$$

where $K_{\lambda,t}$ is the kernel defined in (7.2.10). As we have proved above, the contribution of $F^{\mathrm{out}}_{\lambda,t}(x,y)$ to the integral in (7.3.28) admits the bound (7.3.40). Similarly, using the estimates (7.3.35) and (7.3.36), it is straightforward to check that the third and fourth terms on the right-hand side of (7.3.41) contribute at most $\mathcal{O}(\lambda^4)$ to the right-hand side of (7.3.28), while the last term contributes at most $\mathcal{O}(\lambda^2)$. Therefore,

$$\mathcal{F}_{\mathrm{sing}} = \langle A\rangle_{\widetilde{P}_{\lambda,t}}\big(1+\mathcal{O}(\epsilon_0+\Delta+\lambda^2/\Delta)\big)+\mathcal{O}(\lambda^2), \tag{7.3.42}$$

where we used the estimate $\pi\alpha = \pi^2\rho_0(E_0) = N^{-1}r\big(1+\mathcal{O}(\epsilon_0+\Delta+\lambda^2/\Delta)\big)$ that follows from Lemma 7.A.2 for r defined in (7.2.22), and performed a change of variables $x \to x-\lambda^2\mathrm{Re}\, m_0(E_0)$ and $y \to y-\lambda^2\mathrm{Re}\, m_0(E_0)$. We note that the N^{-1} prefactor results from the different normalization of the trace in (7.2.22) and (7.A.14). Furthermore, Proposition 7.3.6 below implies that under the Assumptions 7.2.2 and 7.2.3, $|\langle A\rangle_{\widetilde{P}_{\lambda,t}}| \lesssim 1$. Since the proof of Proposition 7.3.6 is independent of the statement of (7.3.8b), this concludes the proof of (7.3.8b).

We proceed to prove Lemma 7.3.4.

Proof of Lemma 7.3.4. Define the sequences of overlaps $\mathfrak{a}_j := \langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle$, and $\mathfrak{p}_k := \langle \boldsymbol{u}_k, P\boldsymbol{u}_k\rangle$ where we recall from (7.3.11) that $\boldsymbol{u}_j$'s are the eigenvectors of H_0. Observe that the Assumption 7.2.3 implies that

$$\|\mathfrak{a}\|_\infty \lesssim 1, \quad \mathfrak{p}_k \geq 0 \quad \text{and} \quad \|\mathfrak{p}\|_1 = 1. \tag{7.3.43}$$

Then, using the spectral decomposition (7.3.11) of H_0 and the identity (7.3.24), we rewrite $\widetilde{\mathcal{F}}_{\mathrm{sing}}$ in the following form,

$$\widetilde{\mathcal{F}}_{\mathrm{sing}} = \sum_{j,k}\frac{\mathfrak{a}_j\mathfrak{p}_k}{N}\frac{1}{\pi}\int_{\mathbf{R}}\mathrm{Im}\,\widetilde{\nu}_j(x)\lambda^2\big|\widetilde{\omega}_k(x)\big|^2\mathrm{d}x, \tag{7.3.44}$$

where $\widetilde{\nu}_j(z) := (\mu_j - z - \lambda^2 m_\lambda(z))^{-1}$ for $z \in \mathbf{C}$, and the functions $\widetilde{\omega}_k(x)$ are defined by the improper integrals[16]

$$\widetilde{\omega}_k(x) := \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{\mathrm{e}^{\mathrm{i}tz}\widetilde{\nu}_k(z)}{x-z}\mathrm{d}z, \quad x \in \mathbf{R}. \tag{7.3.45}$$

Here we adhere to the convention $m_\lambda(x) := \lim_{\eta\to+0} m_\lambda(x+\mathrm{i}\eta)$.

The key estimates for proving (7.3.26) are collected in the following Lemma that we prove at the end of the subsection.

Lemma 7.3.5 (Second replacement lemma)**.** *Define the functions*

$$\omega_k(x) := \frac{1}{2\pi\mathrm{i}}\oint_{\gamma_2}\frac{\mathrm{e}^{-\mathrm{i}tz}}{x-z}\frac{1}{\mu_k - z_\lambda}\mathrm{d}z, \quad k \in [N], \tag{7.3.46}$$

where we denote $z_\lambda := z + \lambda^2 m_0(E_0)$. Then under the Assumptions 7.2.2 and 7.2.3, the estimates

$$\big|\omega_k(x) - \widetilde{\omega}_k(x)\big| \lesssim \frac{(\Delta+\epsilon_0)(1+\lambda^2 t)}{|\mu_k - x_\lambda|} + 1 + \lambda^2 t + \Delta^{-1}\log t, \quad x \in I_{3\Delta}, \tag{7.3.47}$$

$$\big|\omega_k(x)\big| + \big|\widetilde{\omega}_k(x)\big| \lesssim \frac{1}{|\mu_k - x_\lambda|} + 1 + \lambda^2 t + \Delta^{-1}\log t, \quad x \in I_{3\Delta}, \tag{7.3.48}$$

$$\big|\omega_k(x)\big| + \big|\widetilde{\omega}_k(x)\big| \lesssim \Delta^{-1}\log t + R^{-1}, \quad x \in [-\tfrac{1}{2}R, \tfrac{1}{2}R]\backslash I_{3\Delta}, \tag{7.3.49}$$

hold for all k with $\mu_k \in I_\Delta$.

[16] The integral in (7.3.45) diverges logarithmically as x approaches the intersection of the contour γ_2 with the real line. However, the contribution of such singularities to the $\mathrm{d}x$ integral on the right-hand side of (7.3.44) is negligible.

Observe that applying the identities (7.3.22), (7.3.24), and the spectral decomposition of H_0 to (7.3.28) yields

$$\widetilde{\mathcal{F}}_{\mathrm{sing}} - \mathcal{F}_{\mathrm{sing}} = \mathcal{E}_{\mathrm{sing},1} + \mathcal{E}_{\mathrm{sing},2}, \tag{7.3.50}$$

where, recalling $x_\lambda := x - \lambda^2 m_0(E_0)$, the quantities $\mathcal{E}_{\mathrm{sing},1}$ and $\mathcal{E}_{\mathrm{sing},2}$ are defined as

$$\mathcal{E}_{\mathrm{sing},1} := \sum_{j,k} \frac{\mathfrak{a}_j \mathfrak{p}_k}{N} \frac{1}{\pi} \int_{\mathbf{R}} \mathrm{Im}\big[\widetilde{\nu}_j(x) - (\mu_j - x_\lambda)^{-1}\big] \lambda^2 \big|\widetilde{\omega}_k(x)\big|^2 \mathrm{d}x. \tag{7.3.51}$$

$$\mathcal{E}_{\mathrm{sing},2} := \sum_{j,k} \frac{\mathfrak{a}_j \mathfrak{p}_k}{N} \frac{1}{\pi} \int_{\mathbf{R}} \mathrm{Im}\big[(\mu_j - x_\lambda)^{-1}\big] \lambda^2 \Big(\big|\widetilde{\omega}_k(x)\big|^2 - \big|\omega_k(x)\big|^2 \Big) \mathrm{d}x. \tag{7.3.52}$$

First, we estimate the quantity $\mathcal{E}_{\mathrm{sing},1}$ defined in (7.3.51). In the regime $x \in I_{3\Delta}$, the bounds in (7.A.2) imply that

$$\big|\widetilde{\nu}_j(x) - (\mu_j - x_\lambda)^{-1}\big| \lesssim (\Delta + \epsilon_0) \frac{\lambda^2}{|\mu_j - x_\lambda|^2}, \quad x \in I_{3\Delta}. \tag{7.3.53}$$

On the other hand, it is straightforward to check that the regime $|x| \geq \frac{1}{2}R$ contributes at most $\mathcal{O}(\lambda^2 R^{-2})$ to the integral on the right-hand side of (7.3.51). We note that the logarithmic singularity resulting from the contour γ_2 intersecting the real line is removed by the x integration.

Therefore, estimates (7.3.43), (7.3.48), (7.3.49), and (7.3.53) imply that

$$\begin{aligned} \big|\mathcal{E}_{\mathrm{sing},1}\big| &\lesssim \sum_{j,k} \frac{|\mathfrak{a}_j| \mathfrak{p}_k}{N} \int_{I_{3\Delta}} \frac{\lambda^2(\Delta + \epsilon_0)}{|\mu_j - x_\lambda|^2} \frac{\lambda^2}{|\mu_k - x_\lambda|^2} \mathrm{d}x \\ &\quad + \lambda^2 (1 + \lambda^2 t + \Delta^{-1} \log t)^2 \sum_j \frac{1}{N} \int_{\mathbf{R}} \mathrm{Im}\big[\widetilde{\nu}_j(x) + (\mu_j - x_\lambda)^{-1}\big] \mathrm{d}x \\ &\lesssim \sum_{j,k} \frac{|\mathfrak{a}_j| \mathfrak{p}_k}{N} \int_{\mathbf{R}} \frac{\lambda^2(\Delta + \epsilon_0)}{|\mu_j - x_\lambda|^2} \frac{\lambda^2}{|\mu_k - x_\lambda|^2} \mathrm{d}x + \lambda^2 \big(1 + \lambda^2 t + \Delta^{-1} \log t\big)^2, \end{aligned} \tag{7.3.54}$$

where in the fist step we used the bound $|\mathrm{Im}\,[\widetilde{\nu}_j(x) - (\mu_j - x_\lambda)^{-1}]| \leq \mathrm{Im}\,[\widetilde{\nu}_j(x) + (\mu_j - x_\lambda)^{-1}]$ to estimate the contribution coming from the second term on the right-hand side of (7.3.48) and the regime $x \in [-\frac{1}{2}R, \frac{1}{2}R] \backslash I_{3\Delta}$; and in the second step we used that $\int_{\mathbf{R}} \mathrm{Im}\,[(\mu_j - x_\lambda)^{-1}] \mathrm{d}x = \pi$ and $\int_{\mathbf{R}} \frac{1}{N} \sum_j \mathrm{Im}\, \widetilde{\nu}_j(x) \mathrm{d}x = \int_R \mathrm{Im}\, m_\lambda(x) \mathrm{d}x = \pi$ (see, e.g., Proposition 2.1 and Eq. (2.9) in [22]). Computing the integral in the second term on the right-hand side of (7.3.54) explicitly, and using the spectral decomposition of H_0, we deduce from the admissibility of E_0 that

$$\sum_{j,k} \frac{|\mathfrak{a}_j| \mathfrak{p}_k}{N} \int_{\mathbf{R}} \frac{\lambda^2(\Delta + \epsilon_0)}{|\mu_j - x_\lambda|^2} \frac{\lambda^2 \mathrm{d}x}{|\mu_k - x_\lambda|^2} \lesssim (\Delta + \epsilon_0) \sup_{\mu_k \in I_\Delta} \big\langle\!\big\langle \mathrm{Im}\, M_0(\mu_k + 2\mathrm{i}\lambda^2 \alpha) \big\rangle\!\big\rangle \lesssim \Delta + \epsilon_0. \tag{7.3.55}$$

Here we employed (7.3.43) and the estimate $\langle \mathfrak{a}, X\mathfrak{p} \rangle \lesssim \|\mathfrak{a}\|_\infty \|\mathfrak{p}\|_1 \sup_{k \in \mathrm{supp}(\mathfrak{p})} \sum_j |X_{jk}|$. Hence, we conclude that

$$\big|\mathcal{E}_{\mathrm{sing},1}\big| \lesssim \Delta + \epsilon_0 + \lambda^2 \big(1 + \lambda^2 t + \Delta^{-1} \log t\big)^2. \tag{7.3.56}$$

We proceed to estimate the quantity $\mathcal{E}_{\mathrm{sing},2}$ defined in (7.3.52). We note again that the contribution of the regime $|x| \geq \frac{1}{2}R$ to the integral on the right-hand side of (7.3.52) is bounded by $\mathcal{O}(\lambda^2 R^{-2})$. Therefore, combining the estimates (7.3.47), (7.3.48), and (7.3.49) yields the bound

$$\big|\mathcal{E}_{\mathrm{sing},2}\big| \lesssim (\Delta + \epsilon_0)(1 + \lambda^2 t) + \lambda \big(1 + \lambda^2 t + \Delta^{-1} \log t\big)^2, \tag{7.3.57}$$

obtained similarly to (7.3.54) and (7.3.56). Together with (7.3.50), the bounds (7.3.56) and (7.3.57) conclude the proof of (7.3.26). □

It remains to prove Lemma 7.3.5.

Proof of Lemma 7.3.5. Throughout the proof we assume that $k \in [N]$ satisfies $\mu_k \in I_\Delta$, and $x \in \mathbf{R}$ satisfies $|x| \le \frac{1}{2}R$. We introduce the auxiliary quantities

$$\check{\nu}_k(z) := \frac{1}{\mu_k - z - \lambda^2 m_\lambda(\mu_k)}, \quad \check{\omega}_k(x) := \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_2} \frac{\mathrm{e}^{-\mathrm{i}tz}}{x-z} \check{\nu}_k(z) \mathrm{d}z. \tag{7.3.58}$$

An explicit computation using the residue calculus reveals that

$$\check{\omega}_k(x) = \frac{\mathrm{e}^{-\mathrm{i}tx} - \mathrm{e}^{-\mathrm{i}t(\mu_k - \lambda^2 m_\lambda(\mu_k))}}{\mu_k - x - \lambda^2 m_\lambda(\mu_k)}, \quad \omega_k(x) = \frac{\mathrm{e}^{-\mathrm{i}tx} - \mathrm{e}^{-\mathrm{i}t(\mu_k - \lambda^2 m_0(E_0))}}{\mu_k - x_\lambda}. \tag{7.3.59}$$

Furthermore, using the bound in (7.A.2), we obtain

$$\big|\omega_k(x) - \check{\omega}_k(x)\big| \lesssim \bigg(\frac{1}{|\mu_k - x_\lambda|^2} + \frac{t}{|\mu_k - x_\lambda|}\bigg)\lambda^2 |\mu_k - E_0| \lesssim \frac{\Delta(1+\lambda^2 t)}{|\mu_k - x_\lambda|}, \tag{7.3.60}$$

where we additionally applied the estimate

$$\big|y + \mathcal{O}(\eta)\big| + \eta \sim |y| + \eta, \quad y \in \mathbf{R}, \eta > 0. \tag{7.3.61}$$

We decompose the contour $\gamma_2 = \Gamma_1 \dot{+} \Gamma_2 \dot{+} \Gamma_3$ according to (7.3.19). It is straightforward to check that for $\nu^{\#}(z)$ denoting one of $\widetilde{\nu}_k(z)$, $\check{\nu}_k(z)$ or $(\mu_k - z_\lambda)^{-1}$,

$$\bigg|\int_{\Gamma_2 \dot{+} \Gamma_3} \frac{\mathrm{e}^{-\mathrm{i}tz}}{x-z} \nu_k^{\#}(z) \mathrm{d}z\bigg| \lesssim \frac{\log t}{\Delta} + \frac{1}{R}, \tag{7.3.62}$$

where we used that $\operatorname{Im} z = t^{-1}$ for all $z \in \Gamma_1$. Therefore, rewriting the left-hand sides of (7.3.47)-(7.3.49) using the integral definitions (7.3.45) and (7.3.46), it suffices to estimate the contributions coming from the segment $\Gamma_1 \subset \gamma_2$.

Using (7.3.61) and (7.A.2), we deduce that for all $z \in \Gamma_1$, defined in (7.3.19),

$$\big|\check{\nu}_k(z) - \widetilde{\nu}_k(z)\big| \lesssim \frac{\lambda^2}{|\mu_k - z_\lambda|}. \tag{7.3.63}$$

Integrating the bound (7.3.63) then yields

$$\bigg|\int_{\Gamma_1} \frac{\mathrm{e}^{-\mathrm{i}tz}}{x-z}\big[\widetilde{\nu}_k(z) - \check{\nu}_k(z)\big]\mathrm{d}z\bigg| \lesssim \big(1 + \lambda^2 t\big)\mathbb{1}_{x \in I_{3\Delta}} + \big(\Delta^{-1}\log t\big)\mathbb{1}_{x \notin I_{3\Delta}}, \tag{7.3.64}$$

which, together with (7.3.60), (7.3.62) immediately implies (7.3.47) after writing $\widetilde{\omega}_k - \omega_k = (\widetilde{\omega}_k - \check{\omega}_k) + (\check{\omega}_k - \omega_k)$.

On the other hand, noting that $|\omega_k(x)| + |\check{\omega}_k(x)| \lesssim |\mu_k - x_\lambda|^{-1}$ by (7.3.59), and combining the estimates (7.3.60) and (7.3.62) yields (7.3.48) and (7.3.49). This concludes the proof of Lemma 7.3.5. □

7.3.3 Step (iii): Limiting behavior of the singular term and proof of Theorem 7.2.4 (b)

We organize the result of approximating $\langle A \rangle_{\widetilde{P}_{\lambda,t}}$ in the following proposition.

Proposition 7.3.6. *Under the assumptions of Theorem 7.2.4, and with $\widetilde{P}_\lambda$ defined as in* (7.2.12)*, we have that, for any fixed $T \in (0,\infty)$ and recalling $\alpha = \pi\rho_0(E_0)$*

$$\limsup_{\Delta \to 0} \limsup_{\substack{t\to\infty \\ \lambda \to 0 \\ \lambda^2 t = T}} \limsup_{N \to \infty} \big|\langle A \rangle_{\widetilde{P}_{\lambda,t}} - (1 - \mathrm{e}^{-2\alpha\lambda^2 t})\langle A \rangle_{\widetilde{P}_\lambda}\big| \lesssim T\mathrm{e}^{-2\alpha T}.$$

Given Proposition 7.3.6, Theorem 7.2.4 (b) immediately follows. □

Proof of Proposition 7.3.6. First, we observe that representing $\operatorname{Im} M_\lambda$ in spectral decomposition of H_0, the quantity $\langle A\rangle_{\widetilde{P}_{\lambda,t}}$ with $P_{\lambda,t}$ defined in (7.2.9), can be rewritten in the from

$$\langle A\rangle_{\widetilde{P}_{\lambda,t}} = \frac{1}{r}\sum_{j,k}\mathfrak{a}_j\mathfrak{p}_k\int_{\mathbf{R}}\phi_{\alpha\lambda^2}(x-\mu_j)\big(K_{\lambda,t}*\phi_{\alpha\lambda^2}\big)(x-\mu_k)\mathrm{d}x, \tag{7.3.65}$$

where $r = \int_{\mathbf{R}}\operatorname{Tr}[\operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)]\langle\operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\rangle_P\mathrm{d}x > 0$ has already been introduced in Remark 7.2.8 (ii), and we denoted $\phi_\eta := \operatorname{Im}[(x-\mathrm{i}\eta)^{-1}]$. Recall that $\mu_j, \boldsymbol{u}_j$ are the eigenvalues and the respective eigenvectors of H_0, and $\mathfrak{a}_j := \langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle$, $\mathfrak{p}_j := \langle \boldsymbol{u}_j, P\boldsymbol{u}_j\rangle$. Applying the Parseval-Plancherel identity to the right-hand side of (7.3.65) yields

$$\langle A\rangle_{\widetilde{P}_{\lambda,t}} = \frac{1}{r}\sum_{j,k}\mathfrak{a}_j\mathfrak{p}_k\Phi_{\lambda,t}(\mu_j-\mu_k),\quad \Phi_{\lambda,t}(u) := \frac{\pi^2}{(2\pi)^{1/2}}\int_{\mathbf{R}}\mathrm{e}^{-2\alpha\lambda^2|q|-\mathrm{i}uq}\,\widehat{K_{\lambda,t}}(q)\mathrm{d}q, \tag{7.3.66}$$

where we used the fact that $\widehat{\phi_\eta}(q) = (\frac{\pi}{2})^{1/2}\mathrm{e}^{-\eta|q|}$, $\eta>0$ (recall Footnote 5).

A direct computation starting with (7.2.10) reveals that

$$\widehat{K_{\lambda,t}}(q) = \begin{cases}(2\pi)^{-1/2}\big(1-\mathrm{e}^{-2\alpha\lambda^2(t-|q|)}\big) & \text{for}\quad |p|\le t\,,\\ 0 & \text{for}\quad |p|>t\,,\end{cases} \tag{7.3.67}$$

and implies that $\Phi_{\lambda,t}(u)$ admits the explicit expression

$$\Phi_{\lambda,t}(u) = \big(1-\mathrm{e}^{-2\alpha\lambda^2 t}\big)\pi\phi_{2\alpha\lambda^2}(u) + \mathfrak{R}_{\lambda,t}(u), \tag{7.3.68}$$

where the function $\mathfrak{R}_{\lambda,t}(u)$ is defined by

$$\mathfrak{R}_{\lambda,t}(u) := \pi\mathrm{e}^{-2\alpha\lambda^2 t}\phi_{2\alpha\lambda^2}(u)\bigg(1-\cos(tu)-2\alpha\lambda^2 t\frac{\sin(tu)}{tu}\bigg). \tag{7.3.69}$$

Observe that the contribution of the first term on the right-hand side of (7.3.68) to $\langle A\rangle_{\widetilde{P}_{\lambda,t}}$ is given by $(1-\mathrm{e}^{-2\lambda^2\alpha t})\langle A\rangle_{\widetilde{P}_\lambda}$, since

$$\langle A\rangle_{\widetilde{P}_\lambda} = \frac{1}{r}\sum_{j,k}\mathfrak{a}_j\mathfrak{p}_k\int_{\mathbf{R}}\phi_{\alpha\lambda^2}(x-\mu_j)\phi_{\alpha\lambda^2}(x-\mu_k)\mathrm{d}x. \tag{7.3.70}$$

Here we used the definition of the state $\widetilde{P}_\lambda$ in (7.2.12), and the Parseval-Plancherel identity.

The key observation is that the contribution of the remaining $\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)$ term

$$\mathcal{R} := \sum_{j,k}\frac{\mathfrak{a}_j\mathfrak{p}_k}{r}\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k) \tag{7.3.71}$$

in (7.3.68) to $\langle A\rangle_{\widetilde{P}_{\lambda,t}}$ admits the bound[17]

$$\big|\mathcal{R}\big| \le \sup_{k\in\operatorname{supp}(\mathfrak{p})}\frac{1}{r}\sum_j\big|\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\big|\cdot\|\mathfrak{a}\|_\infty\|\mathfrak{p}\|_1\,. \tag{7.3.72}$$

Observe that there exists a constant $C>0$ such that for any $\xi>0$ and $t>0$, we have that $\phi_\xi(u)(1-\cos(tu)) \le C\xi t\phi_{1/t}(u)$ for all $u\in\mathbf{R}$. This follows immediately from the fact that the

[17] Inequality (7.3.72) can be interpreted as a discrete analog of Young's convolution inequality, which can not be evoked directly since the eigenvalues μ_j do not form a group under addition.

function $s \mapsto (s^2+1)(1-\cos s)/s^2$ is uniformly bounded on $\mathbf{R}$. Therefore, the function $\mathfrak{R}$ admits the bound

$$|\mathbf{R}_{\lambda,t}(u)| \le 2\pi\alpha\lambda^2 t\,\mathrm{e}^{-2\alpha\lambda^2 t}\Big(C\phi_{1/t}(u)+\phi_{2\alpha\lambda^2}(u)\Big), \quad u\in\mathbf{R}. \tag{7.3.73}$$

Summing the bound (7.3.73) over $u=\mu_j$ yields

$$\frac{1}{r}\sum_j\big|\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\big| \lesssim \lambda^2 t\,\mathrm{e}^{-2\alpha\lambda^2 t}\frac{N}{r}\mathrm{Im}\,\langle\!\langle M_0(\mu_k+2\mathrm{i}\alpha\lambda^2)+M_0(\mu_k+\mathrm{i}/t)\rangle\!\rangle. \tag{7.3.74}$$

Using the localization of the state P as in (7.2.6), the admissibility of E_0 in (7.2.5), and the first line of (7.A.14) to deduce that $r\sim N(1+\mathcal{O}(\epsilon_0+\Delta+\lambda^2/\Delta))$, we obtain

$$\sup_{k\in\mathrm{supp}(\mathfrak{p})}\frac{1}{r}\sum_j\big|\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\big| \lesssim \lambda^2 t\,\mathrm{e}^{-2\alpha\lambda^2 t}\big(1+\epsilon_0\big)\big(1+\epsilon_0+\Delta+\lambda^2/\Delta\big). \tag{7.3.75}$$

This concludes the proof of Proposition 7.3.6. □

7.3.4 Relaxation formula: Proof of Corollary 7.2.7

Estimates (7.2.18) and (7.2.19) in items (a) and (b), respectively, follow immediately from Theorem 7.2.4.

To prove (7.2.20) in item (c), observe that plugging the estimate (7.2.16) from the Definition 7.2.6 of local overlap regularity into (7.3.71) yields

$$\begin{aligned}\big|\mathcal{R}\big| \lesssim |\mathfrak{A}|\sup_{k\in\mathrm{supp}(\mathfrak{p})}\bigg|\frac{1}{N}\sum_{\mu_j\in I_{2\Delta}}\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\bigg| + \sup_{k\in\mathrm{supp}(\mathfrak{p})}\bigg|\sum_{\mu_j\in I_{2\Delta}}\frac{\mathfrak{a}_j-\mathfrak{A}}{N}\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\bigg| \\ + \sup_{k\in\mathrm{supp}(\mathfrak{p})}\bigg|\frac{1}{N}\sum_{\mu_j\notin I_{2\Delta}}\mathfrak{a}_j\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\bigg|,\end{aligned} \tag{7.3.76}$$

where we used that $|r|\sim N\,(1+\mathcal{O}(\epsilon_0+\Delta+\lambda^2/\Delta))$ by the first line of (7.A.14) from Lemma 7.A.2.

Applying the estimates analogous to (7.3.72) and (7.3.75) to the second sum on the right-hand side of (7.3.76), we deduce the bound

$$\sup_{k\in\mathrm{supp}(\mathfrak{p})}\bigg|\sum_{\mu_j\in I_{2\Delta}}\frac{\mathfrak{a}_j-\mathfrak{A}}{N}\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\bigg| \lesssim \big|\mathcal{E}_{\mathrm{LOR}}\big|\cdot\big(1+\lambda^2/\Delta\big) \tag{7.3.77}$$

Note that by (7.3.43) and the uniform bound

$$|\mathfrak{R}_{\lambda,t}(u)| \lesssim \frac{\lambda^2}{\Delta^2}, \quad \text{for} \quad |u|\gtrsim\Delta$$

following from (7.3.73), the tail sum, i.e., the second line of (7.3.76), admits the estimate

$$\sup_{k\in\mathrm{supp}(\mathfrak{p})}\bigg|\frac{1}{N}\sum_{\mu_j\notin I_{2\Delta}}\mathfrak{a}_j\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\bigg| \lesssim \frac{\lambda^2}{\Delta^2}. \tag{7.3.78}$$

Therefore, it remains to estimate the first term on the right-hand side of (7.3.76). Since the function $\mathfrak{R}_{\lambda,t}(u)$ is holomorphic in u for $|\mathrm{Im}\,u|\le\alpha\lambda^2$, we obtain the following series of estimates,

$$\begin{aligned}\frac{1}{N}\sum_{\mu_j\in I_{2\Delta}}\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k) &= \frac{1}{2\pi\mathrm{i}}\oint_\gamma\mathfrak{R}_{\lambda,t}(z-\mu_k)\langle\!\langle M_0(z)\rangle\!\rangle\mathrm{d}z+\mathcal{O}\bigg(\frac{\lambda^2}{\Delta^2}\bigg) \\ &= \frac{1}{2\pi\mathrm{i}}\oint_\gamma\mathfrak{R}_{\lambda,t}(z-\mu_k)m_0(z)\mathrm{d}z+\mathcal{O}\bigg(\eta_0+\frac{\epsilon_0}{\lambda^2}+\frac{\lambda^2}{\Delta^2}\bigg) \\ &= \int_{I_{2\Delta}}\mathfrak{R}_{\lambda,t}(u-\mu_k)\rho_0(u)\mathrm{d}u+\mathcal{O}\bigg(\eta_0+\frac{\epsilon_0}{\lambda^2}+\frac{\lambda^2}{\Delta^2}\bigg),\end{aligned} \tag{7.3.79}$$

where the contour γ is defined to be a rectangle of height $2\eta_0$ and width $4C$ centered at E_0, and the constant $C \sim 1$ is chosen in such a way that $\sigma(H_0) \subset [E_0 - C, E_0 + C]$. Here, in the first step, we used residue calculus and an estimate analogous to (7.3.78) to extend the sum to all μ_j's. The second step follows by integrating the estimate (7.2.2) on the horizontal segments of γ and bounding the contribution of the vertical segments of the contour γ by $\mathcal{O}(\eta_0)$. Finally, the third step is a consequence of the Stieltjes representation (7.2.3) and $|\mathfrak{R}_{\lambda,t}(u)| \lesssim \Delta^{-2}\lambda^2$ for $|u| \gtrsim \Delta$. Using the estimate $\rho_0(u) = \rho_0(E_0) + \mathcal{O}(\Delta)$ for all $u \in I_{2\Delta}$ by admissibility of E_0 as in (7.2.4), we conclude that

$$\int_{I_{2\Delta}} \mathfrak{R}_{\lambda,t}(u-\mu_k)\rho_0(u)\mathrm{d}u = \rho_0(E_0)\int_{I_{2\Delta}} \mathfrak{R}_{\lambda,t}(u-\mu_k)\mathrm{d}u + \mathcal{O}(\Delta)\,, \tag{7.3.80}$$

where we used $\int_{\mathbf{R}} |\mathfrak{R}_{\lambda,t}(u)|\mathrm{d}u \lesssim \lambda^2 t\,\mathrm{e}^{-2\alpha\lambda^2 t} \lesssim 1$ as a consequence of (7.3.73). Moreover, a direct computation starting with (7.3.69) reveals that

$$\int_{\mathfrak{R}} \mathfrak{R}_{\lambda,t}(u-\mu_k)\mathrm{d}u = 0 \qquad \text{and} \qquad \int_{\mathfrak{R}\setminus I_{2\Delta}} \big|\mathfrak{R}_{\lambda,t}(u-\mu_k)\big|\mathrm{d}u \lesssim \frac{\lambda^2}{\Delta}\,. \tag{7.3.81}$$

Hence, combining estimates (7.3.76)–(7.3.81) yields

$$\big|\mathcal{R}\big| \lesssim \Delta + \eta_0 + \lambda^{-2}\epsilon_0 + \Delta^{-2}\lambda^2\,, \tag{7.3.82}$$

which implies the $\mathcal{R}$-part of (7.2.20); the $\mathcal{E}$-part is an immediate consequence of Theorem 7.2.4 (a).

To complete the proof under the weaker assumption on $\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle$, stated in Footnote 6, we first *uniformly* approximate $\mathfrak{A}_N$ by a real analytic function $\mathfrak{A}_{N,\ell} : I_{2\Delta} \to \mathbf{R}$ with $\ell = \ell(\lambda,t) > 0$ (to be chosen below), which can be analytically extended to $\{z \in \mathbf{C} : \mathrm{dist}(z, I_{2\Delta}) < \ell\}$ and satisfy $\sup_{N\in\mathbf{N}} \|\mathfrak{A}_{N,\ell} - \mathfrak{A}_N\|_\infty \to 0$ as $\ell \to 0$. Such $\mathfrak{A}_{N,\ell}$ can be explicitly constructed, e.g., by convolution of $\mathfrak{A}_N$ with a Gaussian having variance of order ℓ. For ease of notation, we shall now drop the subscript N. Then, the error term $\mathfrak{A} - \mathfrak{A}_\ell$ is easily seen to give a vanishing contribution (as $\ell \to 0$) by means of (7.3.75). Indeed, using (7.3.75) a bound analogous to (7.3.72), and $\|\mathfrak{p}\|_1 \le 1$, we find that

$$\left|\sum_{j,k} \frac{\mathfrak{p}_k}{r}\big(\mathfrak{A}(\mu_j) - \mathfrak{A}_\ell(\mu_j)\big)\mathfrak{R}_{\lambda,t}(\mu_j-\mu_k)\right| \lesssim \lambda^2 t\,\mathrm{e}^{-2\alpha\lambda^2 t}\big(1 + \Delta + \lambda^2/\Delta\big)\cdot\|\mathfrak{A} - \mathfrak{A}_\ell\|_\infty. \tag{7.3.83}$$

Next, observe that using analyticity of $\mathfrak{A}_\ell$ and reasoning as in the proof for the case of $\mathfrak{A}$ being constant above, we obtain

$$\begin{aligned}\big|\mathcal{R}\big| \lesssim \sup_{\mu_k\in I_\Delta}\left|\int_{I_{2\Delta}} \big(\mathfrak{A}_\ell(u) - \mathfrak{A}_\ell(\mu_k)\big)\mathfrak{R}_{\lambda,t}(u-\mu_k)\mathrm{d}u\right| \\ + \Delta + \eta_0 + \lambda^{-2}\epsilon_0 + \Delta^{-2}\lambda^2 + \big(1 + \Delta + \lambda^2/\Delta\big)\cdot\|\mathfrak{A} - \mathfrak{A}_\ell\|_\infty.\end{aligned} \tag{7.3.84}$$

Since $\mathfrak{A}_\ell(z)$ is analytic in the strip of width ℓ, (7.3.73) implies that the integral on the right-hand side of (7.3.84) is bounded by

$$\frac{1}{\ell}\int_{I_{2\Delta}}\left(\frac{\lambda^2|u-\mu_k|}{|u-\mu_k|^2 + (2\alpha\lambda^2)^2} + \frac{t^{-1}|u-\mu_k|}{|u-\mu_k|^2 + t^{-2}}\right)\mathrm{d}u \lesssim \frac{\lambda^2|\log\lambda| + t^{-1}\log t}{\ell}, \tag{7.3.85}$$

uniformly in k such that $\mu_k \in I_\Delta$. Hence, choosing, say, $\ell := \lambda + t^{-1/2}$, this concludes the proof of Corollary 7.2.7. □

7.3.5 Microcanonical average: Proof of Theorem 7.2.11

Using (7.2.12), we start by writing out

$$\langle A\rangle_{\widetilde{P}_\lambda} = \frac{\int_{\mathbf{R}} \langle\!\langle \operatorname{Im} M_0(x + \mathrm{i}\alpha\lambda^2)A\rangle\!\rangle \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\rangle_P\,\mathrm{d}x}{\int_{\mathbf{R}} \langle\!\langle \operatorname{Im} M_0(x + \mathrm{i}\alpha\lambda^2)\rangle\!\rangle \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2)\rangle_P\,\mathrm{d}x}\,. \tag{7.3.86}$$

For the denominator, we have

$$\int_{\mathbf{R}} \langle\!\langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2) \rangle\!\rangle \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2) \rangle_P \, \mathrm{d}x = \pi \langle\!\langle \operatorname{Im} M_0(E_0+\mathrm{i}\alpha\lambda^2) \rangle\!\rangle + \mathcal{O}\big(\epsilon_0 + \Delta + \lambda^2/\Delta\big) \tag{7.3.87}$$

from Lemma 7.A.2. For the numerator, we use the assumption that $h(x) = \langle\!\langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2) A \rangle\!\rangle$ has uniformly bounded Lipschitz constant for $x \in I_\Delta$ (recall (7.2.27)). Hence we find

$$\int_{\mathbf{R}} h(x) \langle \operatorname{Im} M_0(x+\mathrm{i}\alpha\lambda^2) \rangle_P \, \mathrm{d}x = \pi h(E_0) + \mathcal{O}\big(\Delta + \lambda^2/\Delta\big) \tag{7.3.88}$$

completely analogously to (7.A.15) and (7.A.16), using Assumption 7.2.3.

Therefore, plugging (7.3.87) and (7.3.88) into (7.3.86), and using Assumption 7.2.2 (ii) together with Lemma 7.A.2, we obtain

$$\langle A \rangle_{\widetilde{P}_\lambda} = \langle A \rangle_{P_\lambda^{(\mathrm{mc})}} + \mathcal{O}(\mathcal{E}_{\mathrm{mc}}) \quad \text{with} \quad \mathcal{E}_{\mathrm{mc}} := \epsilon_0 + \Delta + \lambda^2/\Delta \,.$$

This concludes the proof of Theorem 7.2.11. □

7.A Auxiliary results and additional proofs

7.A.1 Auxiliary results

In this subsection, we derive two technical lemmas, which are frequently used throughout the main text.

The first one (Lemma 7.A.1) is concerned with properties of the self-consistent resolvent $M_\lambda(z)$ from the λ-dependent Quadratic Matrix Equation (7.3.1) using Assumption 7.2.2 on the unperturbed matrix H_0. Recall that $\frac{1}{6}\kappa_0$ denotes the upper bound for the energy width Δ (cf. Assumption 7.2.3).

Lemma 7.A.1. *Let $z := E + \mathrm{i}\eta$ be a spectral parameter in $\mathbf{C}$ with $|z| \le C$, then the solution $M_\lambda(z)$ to* (7.3.1) *satisfies the bounds*

$$\langle\!\langle |M_\lambda(z)|^2 \rangle\!\rangle \le \lambda^{-2}, \quad \langle\!\langle |M_\lambda(z)| \rangle\!\rangle \le \lambda^{-1}, \quad \operatorname{Im} z \ge 0. \tag{7.A.1}$$

Moreover, assuming that $\operatorname{Im} z \ge 0$, $|\operatorname{Re} z - E_0| \le \frac{1}{2}\kappa_0$, and E_0 lies in the admissible spectrum $\sigma_{\mathrm{adm}}^{(\kappa_0, c_0)}$ of H_0 the following estimates

$$\operatorname{Im} m_\lambda(z) \gtrsim 1, \quad |m_\lambda(z)| \lesssim 1, \quad |m_\lambda'(z)| \lesssim 1, \tag{7.A.2}$$

with $m_\lambda(z) := \langle\!\langle M_\lambda(z) \rangle\!\rangle$ hold for any fixed $0 < \lambda \le \lambda_*$, and all $N \ge N_\lambda \in \mathbf{N}$, uniformly in z.

Proof of Lemma 7.A.1. First, we prove the (7.A.1) for $\operatorname{Im} z > 0$. Taking the imaginary part of the MDE (7.3.1), multiplying by $|M_\lambda|^2$ and taking the averaged trace yields

$$\operatorname{Im} m_\lambda(z) = \big(\operatorname{Im} z + \lambda^2 \operatorname{Im} m_\lambda(z)\big) \langle\!\langle |M_\lambda(z)|^2 \rangle\!\rangle, \tag{7.A.3}$$

which immediately implies the first bound in (7.A.1). The second estimate in (7.A.1) follows from the first one by the Cauchy–Schwarz inequality. To extend (7.A.1) down to $\operatorname{Im} z = 0$, we first address the regularity[18] of $m_\lambda(z) = \langle\!\langle M_\lambda(z) \rangle\!\rangle$.

Differentiating the MDE (7.3.1), taking the trace, and invoking the first bound in (7.A.1), we obtain

$$\big|m_\lambda'(z)\big| = \frac{\big|\langle\!\langle M_\lambda(z)^2 \rangle\!\rangle\big|}{\big|1 - \lambda^2 \langle\!\langle M_\lambda(z)^2 \rangle\!\rangle\big|} \le \frac{1}{2\lambda^4 \big\langle\!\big\langle (\operatorname{Im} M_\lambda(z))^2 \big\rangle\!\big\rangle} \lesssim \frac{1}{\lambda^6 (\operatorname{Im} m_\lambda(z))^2}, \tag{7.A.4}$$

[18] The comprehensive analysis of the MDE in [17, 22] shows that for any fixed λ, under the additional boundedness assumption $\|M_\lambda\| \lesssim 1$, the operator $M_\lambda(z)$ is $1/3$-Hölder continuous with a λ-dependent constant. However, for the purposes of proving the regularity of m_λ, the operator norm bound is not necessary.

where we used the imaginary part of (7.3.1) and the positive-definiteness of $\operatorname{Im} M_\lambda(z)$ to obtain the last inequality in (7.A.4). In the penultimate step, we used the first estimate in (7.A.1) to deduce the following chain of inequalities, dropping z for brevity,

$$\big|\lambda^{-2} - \langle\!\langle M_\lambda^2\rangle\!\rangle\big| \geq \lambda^{-2} - \operatorname{Re}\langle\!\langle M_\lambda^2\rangle\!\rangle = \lambda^{-2} - \big\langle\!\!\big\langle |M_\lambda|^2\big\rangle\!\!\big\rangle + 2\big\langle\!\!\big\langle (\operatorname{Im} M_\lambda)^2\big\rangle\!\!\big\rangle[\,] \geq 2\big\langle\!\!\big\langle (\operatorname{Im} M_\lambda)^2\big\rangle\!\!\big\rangle. \tag{7.A.5}$$

Together with the first estimate in (7.A.1), (7.A.4) implies that $\lambda^2 \operatorname{Im} m_\lambda(z)$ is uniformly $1/3$-Hölder continuous in $\{z \in \mathbf{C} : |\operatorname{Im} z| \geq 0, |z| \leq C\}$. Taking the limit $\operatorname{Im} z \to +0$ in (7.A.3), using (7.A.1) with $\operatorname{Im} z > 0$ and the regularity of $\lambda^2 \operatorname{Im} m(z)$, we obtain the first bound in (7.A.1) with $\operatorname{Im} z = 0$. This concludes the proof of (7.A.1) for all $|z| \leq C$ with $\operatorname{Im} z \geq 0$.

Next, we turn to proving the first estimate in (7.A.2). It follows from the MDE (7.3.1) that

$$m_\lambda(z) = \Big\langle\!\!\Big\langle \big(H_0 - z - \lambda^2 m_\lambda(z)\big)^{-1}\Big\rangle\!\!\Big\rangle, \tag{7.A.6}$$

and $\operatorname{Im} z + \lambda^2 \operatorname{Im} m_\lambda(z) \geq \operatorname{Im} z$.

First, assume that $\operatorname{Im} z \geq \eta_0$ (recall (7.2.2)). It follows from (7.A.1) that $\lambda^2 |m_\lambda(z)| \leq \lambda$, hence by suitably shrinking the threshold λ_*, we can assume that $|\operatorname{Re}[z + \lambda^2 m_\lambda(z)] - E_0| \leq \frac{3}{4}\kappa_0$ for all z satisfying $\operatorname{Im} z \geq \eta_0$, $|z| \leq C$ and $|\operatorname{Re} z - E_0| \leq \frac{1}{2}\kappa_0$. Therefore

$$m_\lambda(z) = m_0\big(z + \lambda^2 m_\lambda(z)\big) + \mathcal{O}(\epsilon_0) = m_0(z) + \mathcal{O}(\lambda + \epsilon_0), \tag{7.A.7}$$

where the first step follows by (7.2.2) and (7.A.1), and in the second estimate we used that $E_0 \in \sigma_{\mathrm{adm}}^{(\kappa_0, c_0)}$, defined in (7.2.4). In particular, taking the imaginary part of (7.A.7), and using the positivity of ρ_0 in the admissible spectrum yields $\operatorname{Im} m_\lambda(z) \gtrsim 1 + \mathcal{O}(\lambda + \epsilon_0)$. Hence, from the $1/3$-Hölder continuity of $\lambda^2 \operatorname{Im} m_\lambda(z)$, and (7.A.7) we deduce that

$$\lambda^2 \operatorname{Im} m_\lambda(z) \gtrsim \lambda^2 + \mathcal{O}(\lambda^3 + \lambda^2 \epsilon_0 + \eta_0^{1/3}). \tag{7.A.8}$$

for all z with $|\operatorname{Re} z - E_0| \leq \frac{1}{2}\kappa_0$, $\operatorname{Im} z \geq 0$, and $|z| \leq C$. Therefore, for a suitably small threshold λ_* and any fixed $0 < \lambda \leq \lambda_*$, the first estimate in (7.A.2) is established for all N satisfying $\eta_0(N)^{1/3} \lesssim \lambda^2$ with the implicit constant depending only on the constant in (7.A.4). Since $\eta_0(N)$ converges to zero, there exists $N_\lambda \in \mathbf{N}$ such that all $N \geq N_\lambda$ satisfy $\eta_0(N)^{1/3} \lesssim \lambda^2$. Furthermore, we obtain $\operatorname{Im} z + \lambda^2 \operatorname{Im} m_\lambda(z) \gtrsim \lambda^2 \geq \eta_0$, hence the second estimate in (7.A.2) follows from (7.2.2) and (7.A.6).

Finally, we prove the third estimate in (7.A.2). Differentiating (7.A.6) with respect to z yields

$$m_\lambda'(z) = \Big\langle\!\!\Big\langle \big(H_0 - z - \lambda^2 m_\lambda(z)\big)^{-2}\Big\rangle\!\!\Big\rangle\big(1 + \lambda^2 m_\lambda'(z)\big), \tag{7.A.9}$$

In particular, the first factor on the right-hand side of (7.A.9) is a normalized trace of $(H_0 - \zeta)^{-2}$ with $\zeta := z + \lambda^2 m_\lambda(z)$, satisfying $|\operatorname{Re}\zeta - E_0| \leq \frac{3}{4}\kappa_0$ and $\operatorname{Im}\zeta \geq 2c\lambda^2 \geq \eta_0$ for some positive constant $c \sim 1$ by the first and second estimates in (7.A.2). Hence, integrating (7.2.2), we deduce that

$$\langle\!\langle (H_0 - \zeta)^{-2}\rangle\!\rangle = \int_{\mathbf{R}+\mathrm{i}c\lambda^2} \frac{\operatorname{Im}\langle\!\langle (H_0 - w)^{-1}\rangle\!\rangle}{\pi\,(w-\zeta)^2}\mathrm{d}w = \int_{\mathbf{R}+\mathrm{i}c\lambda^2} \frac{\operatorname{Im} m_0(w)}{\pi\,(w-\zeta)^2}\mathrm{d}w + \mathcal{O}\big(\lambda^{-2}\epsilon_0\big). \tag{7.A.10}$$

Moreover, using the fact that $m_0(w)$ is the Stieltjes transform of the limiting density ρ_0, we conclude that

$$\langle\!\langle (H_0 - \zeta)^{-2}\rangle\!\rangle = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\rho_0(x)}{(x-\zeta)^2}\mathrm{d}x + \mathcal{O}\big(\lambda^{-2}\epsilon_0\big), \tag{7.A.11}$$

In particular, since $\operatorname{Re}\zeta$ lies in the admissible spectrum of H_0, the integral on the right-hand side of (7.A.11) admits the estimate

$$\left|\frac{1}{\pi}\int_{\mathbf{R}} \frac{\rho_0(x)}{(x-\zeta)^2}\mathrm{d}x\right| = \left|\int_J \frac{\rho_0'(x) - \rho_0'(\operatorname{Re}\zeta)}{x-\zeta}\mathrm{d}x\right| + \left|\int_J \frac{\rho_0'(\operatorname{Re}\zeta)\operatorname{Im}\zeta}{|x-\zeta|^2}\mathrm{d}x\right| + \mathcal{O}(\kappa_0^{-2}), \tag{7.A.12}$$

where $J := [\mathrm{Re}\,\zeta - c\kappa_0, \mathrm{Re}\,\zeta + c\kappa_0] \subset I_{\kappa_0}$. Here we used that the kernel $(x - \mathrm{Re}\,\zeta)/|x-\zeta|^2$ is odd around $\mathrm{Re}\,\zeta$ and the integrability of ρ_0 to estimate the integral over $\mathbf{R}\backslash J$, while the boundary term resulting from integration by parts over J is bounded by $\kappa_0^{-1}\mathcal{O}(\|\rho_0\|_{C^1(J)}) \lesssim \kappa_0^{-1}$. Observe that the second term on the right-hand side of (7.A.12) is bounded by $\mathcal{O}(|\rho_0'(\mathrm{Re}\,\zeta)|)$, while the first term is bounded by $\mathcal{O}(L)$, where L is the Lipschitz constant of ρ_0' on the interval $[E_0 - \kappa_0, E_0 + \kappa_0]$. Therefore, $|\langle\!\langle (H_0 - \zeta)^{-2}\rangle\!\rangle| = \mathcal{O}(1)$. Finally, rearranging the identity (7.A.9) now yields

$$m_\lambda'(z) = \frac{\big\langle\!\big\langle \big(H_0 - z - \lambda^2 m_\lambda(z)\big)^{-2}\big\rangle\!\big\rangle}{1 - \lambda^2\big\langle\!\big\langle \big(H_0 - z - \lambda^2 m_\lambda(z)\big)^{-2}\big\rangle\!\big\rangle} = \mathcal{O}(1). \tag{7.A.13}$$

This concludes the proof of Lemma 7.A.1. □

We conclude this section by evaluating the denominator in (7.2.9).

Lemma 7.A.2. *Under Assumptions 7.2.2 and 7.2.3 (recalling the notation $\alpha = \pi\rho_0(E_0)$) it holds that*

$$\begin{aligned}\Big|\int_{\mathbf{R}} \langle\!\langle \mathrm{Im}\, M_0(x + \mathrm{i}\alpha\lambda^2)\rangle\!\rangle \langle \mathrm{Im}\, M_0(x + \mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x - \pi^2\rho_0(E_0)\Big| &= \mathcal{O}\big(\epsilon_0 + \Delta + \lambda^2/\Delta\big), \\ \big|\pi^2\rho_0(E_0) - \pi\langle\!\langle \mathrm{Im}\, M_0(E_0 + \mathrm{i}\alpha\lambda^2)\rangle\!\rangle\big| &= \mathcal{O}\big(\epsilon_0 + \Delta + \lambda^2/\Delta\big).\end{aligned} \tag{7.A.14}$$

Proof. We only prove the first relation in (7.A.14). The argument for the second estimate is analogous and hence omitted.

We have that

$$\begin{aligned}&\int_{\mathbf{R}} \langle\!\langle \mathrm{Im}\, M_0(x + \mathrm{i}\alpha\lambda^2)\rangle\!\rangle \langle \mathrm{Im}\, M_0(x + \mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x \\ =& \int_{\mathbf{R}} \mathrm{Im}\, m_0(x + \mathrm{i}\alpha\lambda^2) \langle \mathrm{Im}\, M_0(x + \mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x + \mathcal{O}(\epsilon_0) \\ =& \sum_{\mu_j \in I_\Delta} \langle \boldsymbol{u}_j, P\boldsymbol{u}_j\rangle \int_{\mathbf{R}} \mathrm{d}y \rho_0(y) \int_{\mathbf{R}} \mathrm{d}x \frac{\alpha\lambda^2}{(x-y)^2 + (\alpha\lambda^2)^2} \frac{\alpha\lambda^2}{(x-\mu_j)^2 + (\alpha\lambda^2)^2} + \mathcal{O}(\epsilon_0) \\ =& \pi^2 \sum_{\mu_j \in I_\Delta} \langle \boldsymbol{u}_j, P\boldsymbol{u}_j\rangle \int_{\mathbf{R}} \mathrm{d}y \rho_0(y) \frac{1}{\pi} \frac{2\alpha\lambda^2}{(y-\mu_j)^2 + (2\alpha\lambda^2)^2} + \mathcal{O}(\epsilon_0),\end{aligned} \tag{7.A.15}$$

where we used $\int_{\mathbf{R}} \langle \mathrm{Im}\, M_0(x + \mathrm{i}\alpha\lambda^2)\rangle_P \,\mathrm{d}x = \pi$ and (7.2.2) to go to the second line. To go to the third line, we employed spectral decomposition (7.3.11) of H_0 and used Assumption 7.2.3. Next, using that $\mu_j \in I_\Delta$ and regularity of ρ_0 within $I_{2\Delta}$, we can evaluate the integral in the last line of (7.A.15) as

$$\int_{\mathbf{R}} \mathrm{d}y \rho_0(y) \frac{1}{\pi} \frac{2\alpha\lambda^2}{(y-\mu_j)^2 + (2\alpha\lambda^2)^2} = \rho_0(E_0) + \mathcal{O}\big(\Delta + \lambda^2/\Delta\big), \tag{7.A.16}$$

by adding and subtracting $\rho_0(E_0)$ and estimating $I_{2\Delta}$ and $I_{2\Delta}^c$ separately.

Combining (7.A.15) with (7.A.16) and using that $\mathrm{Tr}[P] = 1$ concludes the argument. □

7.A.2 Proof of Proposition 7.3.1

For simplicity of the presentation, we carry out the argument only for $\lambda = 1$; the claim in Proposition 7.3.1 follows from a natural rescaling. Our proof closely follows [168, Appendix B], Section 1.5.2, and Section 2.6.2,[19] hence, we only give the main steps. Note that we are only interested in a

[19] In fact, for z_1, z_2 in the *bulk* of the self consistent density of states $\rho(x) := \pi^{-1}\lim_{\eta\to 0^+}\langle\!\langle \mathrm{Im}\, M(x + \mathrm{i}\eta)\rangle\!\rangle$, Proposition 7.3.1 has already been proven for so called *regular* matrices A in Proposition 2.4.4. Here, we provide the proof uniformly in the spectrum and for arbitrary matrices.

global law, i.e. the spectral parameters z_1, z_2 have a distance to the spectrum of $H = D + W$ that is uniformly bounded from below by an N-independent positive constant (see also (7.3.5)). In particular, we can simply afford the norm bounds $\|G_i\| \lesssim 1$ and $\|M_i\| \lesssim 1$.

As a preparation for our argument, we recall the definition of the *second order renormalization*, denoted by *underline*, from [165]. For functions $f(W), g(W)$ of the random matrix W, we define

$$\underline{f(W)Wg(W)} := f(W)Wg(W) - \widetilde{\mathbf{E}}\big[(\partial_{\widetilde{W}} f)(W)\widetilde{W}g(W) + f(W)\widetilde{W}(\partial_{\widetilde{W}} g)(W)\big], \tag{7.A.17}$$

where $\partial_{\widetilde{W}}$ denotes the directional derivative in the direction of the GUE matrix $\widetilde{W}$ that is independent of W. The expectation is taken w.r.t. the matrix $\widetilde{W}$. Note that if W itself is a GUE matrix, then $\mathbf{E}\,\underline{f(W)Wg(W)} = 0$, while for W with a general distribution, this expectation is independent of the first two moments of W. In other words, the underline renormalizes the product $f(W)Wg(W)$ to the second order. We remark that underline (7.A.17) is a well-defined notation if the 'middle' W to which the renormalization refers is unambiguous. This is the case in our proof, since the functions f, g are resolvents, i.e. not involving explicitly monomials in W.

Moreover, we note that $\widetilde{\mathbf{E}}\widetilde{W}R\widetilde{W} = \langle\!\langle R \rangle\!\rangle$ and furthermore, that the directional derivative of the resolvent is given by $\partial_{\widetilde{W}} G = -G\widetilde{W}G$. For example, in the special case $f(W) = 1$ and $g(W) = (W + D - z)^{-1} = G$, we thus have

$$\underline{WG} = WG + \langle\!\langle G \rangle\!\rangle G$$

by definition of the underline in (7.A.17). Using this underline notation in combination with the identity $G(W + D - z) = I$ and the defining equation (7.3.1) for M, we have

$$G = M - M\underline{WG} + M\langle\!\langle G - M \rangle\!\rangle G = M - \underline{GW}M + G\langle\!\langle G - M \rangle\!\rangle M\,. \tag{7.A.18}$$

Moreover, we have the following lemma, the proof of which is given at the end of this section.

Lemma 7.A.3 (Representation as full underlined, cf. Lemma 1.5.6)**.** *Under the notations and assumptions of Proposition 7.3.1, we have that*

$$\left(G_1BG_2 - \left(M_1BM_2 + \frac{M_1M_2\langle\!\langle M_1BM_2 \rangle\!\rangle}{1 - \langle\!\langle M_1M_2 \rangle\!\rangle}\right)\right)_{\boldsymbol{xy}} = -\big(\underline{G_1B'M_2WG_2}\big)_{\boldsymbol{xy}} + \mathcal{O}_{\prec}(\mathcal{E}_{\mathrm{iso}}) \tag{7.A.19}$$

with $\mathcal{E}_{\mathrm{iso}} := 1/\sqrt{N}$ *for some bounded deterministic matrix* $B' \in \mathbf{C}^{N\times N}$.

Having this approximate representation of the lhs. of (7.3.3) as a full underlined term at hand, we turn to the proof of (7.3.3) via a (minimalistic) cumulant expansion; see [168, Eq. (4.32)] and (1.5.27).

Let $p \in \mathbf{N}$ be arbitrary. Then, abbreviating the lhs. of (7.A.19) by $\mathcal{Q}_{\boldsymbol{xy}}$, we obtain

$$\mathbf{E}\big|\mathcal{Q}_{\boldsymbol{xy}}\big|^{2p} \lesssim \mathbf{E}\,\widetilde{\Xi}\big|\mathcal{Q}_{\boldsymbol{xy}}\big|^{2p-2} + \sum_{|\boldsymbol{l}|+\sum(J\cup J_*)\geq 2} \mathbf{E}\,\Xi(\boldsymbol{l}, J, J_*)\big|\mathcal{Q}_{\boldsymbol{xy}}\big|^{2p-1-|J\cup J_*|} + \mathcal{O}_{\prec}\big(\mathcal{E}_{\mathrm{iso}}^{-2p}\big)\,, \tag{7.A.20}$$

where the summation in (7.A.20) is taken over tuples $\boldsymbol{l} \in \mathbf{Z}^2_{\geq 0}$ and multisets of tuples $J, J_* \subset \mathbf{Z}^2_{\geq 0} \setminus \{(0,0)\}$, for which we set $\partial^{(l_1,l_2)} := \partial_{ab}^{l_1}\partial_{ba}^{l_2}$, $|(l_1,l_2)| = l_1 + l_2$, $\sum J = \sum_{\boldsymbol{j}\in J} |\boldsymbol{j}|$. Moreover, we denoted

$$\begin{aligned}\widetilde{\Xi} :=& \frac{\Big|\big(G_1B'G_1BG_2\big)_{\boldsymbol{xy}}\big(G_1G_2\big)_{\boldsymbol{xy}}\Big| + \Big|\big(G_1B'G_2\big)_{\boldsymbol{xy}}\big(G_1BG_2G_2\big)_{\boldsymbol{xy}}\Big|}{N} \\ &+ \frac{\Big|\big(G_1B'G_2^*B^*G_1^*\big)_{\boldsymbol{xx}}\big(G_2^*G_2\big)_{\boldsymbol{yy}}\Big| + \Big|\big(G_1B'G_1^*\big)_{\boldsymbol{xx}}\big(G_2^*A^*G_1^*G_2\big)_{\boldsymbol{yy}}\Big|}{N}\,,\end{aligned} \tag{7.A.21}$$

and defined $\Xi(\boldsymbol{l}, J, J_*)$ via

$$\Xi := N^{-(|\boldsymbol{l}|+\sum(J\cup J_*)+1)/2} \sum_{ab} \big|\partial^{\boldsymbol{l}}\big[(G_1B')_{\boldsymbol{x}a}\big(G_2\big)_{b\boldsymbol{y}}\big]\big| \times \prod_{\boldsymbol{j}\in J} \big|\partial^{\boldsymbol{j}}\big(G_1BG_2\big)_{\boldsymbol{xy}}\big| \prod_{\boldsymbol{j}\in J_*} \big|\partial^{\boldsymbol{j}}\big(G_2^*B^*G_2^*\big)_{\boldsymbol{yx}}\big|. \tag{7.A.22}$$

Now, by a simple norm bound, $\|G\| \lesssim 1$, we find that

$$\widetilde{\Xi} \prec \mathcal{E}_{\mathrm{iso}}^2 . \tag{7.A.23}$$

For Ξ, our goal is to show that

$$\Xi(\boldsymbol{l}, J, J_*) \prec \mathcal{E}_{\mathrm{iso}}^{1+|J\cup J_*|} . \tag{7.A.24}$$

First, we have the naive bounds

$$\big|\partial^{\boldsymbol{l}}\big[(G_1B')_{\boldsymbol{x}a}\big(G_2\big)_{b\boldsymbol{y}}\big]\big| + \big|\partial^{\boldsymbol{j}}\big(G_1BG_2\big)_{\boldsymbol{xy}}\big| + \big|\partial^{\boldsymbol{j}}\big(G_2^*B^*G_2^*\big)_{\boldsymbol{yx}}\big| \prec 1 \tag{7.A.25}$$

and hence

$$\Xi \prec N^{-(|\boldsymbol{l}|+\sum(J\cup J_*)+1)/2} N^2 = N^{(4-|\boldsymbol{l}|)/2} N^{-(1+\sum(J\cup J_*))/2} = N^{(4-|\boldsymbol{l}|)/2} \mathcal{E}_{\mathrm{iso}}^{1+\sum(J\cup J_*)} .$$

Thus, for $|\boldsymbol{l}| \geq 4$, we find the naive bounds (7.A.25) to be sufficient for (7.A.24), since trivially $|J \cup J_*| \leq \sum(J \cup J_*)$. For $|\boldsymbol{l}| \leq 3$, we perform the summation $\sum_{ab}$ a bit more carefully, e.g., recalling the norm bound $\|G\| \lesssim 1$, as

$$\sum_a |G_{\boldsymbol{x}a}| \leq N^{1/2} \sqrt{\sum_a |G_{\boldsymbol{x}a}|^2} = N^{1/2} (|G|^2)_{\boldsymbol{xx}} \lesssim N^{1/2}$$

instead of the naive $\sum_a |G_{\boldsymbol{x}a}| \lesssim N$. Indeed, using the condition $|\boldsymbol{l}| + \sum(J \cup J_*) \geq 2$, we can check all the cases $|\boldsymbol{l}| \leq 3$ explicitly and find

$$\sum_{ab} \big|\partial^{\boldsymbol{l}}\big[(G_1B')_{\boldsymbol{x}a}\big(G_2\big)_{b\boldsymbol{y}}\big]\big| \prod_{\boldsymbol{j}\in J} \big|\partial^{\boldsymbol{j}}\big(G_1BG_2\big)_{\boldsymbol{xy}}\big| \prod_{\boldsymbol{j}\in J_*} \big|\partial^{\boldsymbol{j}}\big(G_2^*B^*G_2^*\big)_{\boldsymbol{yx}}\big| \prec N^2\, N^{-(4-|\boldsymbol{l}|)/2} ,$$

from which we conclude (7.A.24).

Plugging (7.A.24) together with (7.A.23) into (7.A.20), using a Young inequality and recalling that p was arbitrary, we deduce that

$$|\mathcal{Q}_{\boldsymbol{xy}}| \prec \mathcal{E}_{\mathrm{iso}} = \frac{1}{\sqrt{N}} ,$$

i.e. we have proven Proposition 7.3.1. □

It remains to give the proof of Lemma 7.A.3.

Proof of Lemma 7.A.3. Applying (7.A.18) to G_2, we thus find that

$$G_1\widetilde{B}G_2 = G_1\widetilde{B}M_2 - G_1\widetilde{B}M_2\underline{WG_2} + G_1\widetilde{B}M_2\langle\!\langle G_2 - M_2\rangle\!\rangle G_2$$

for $\widetilde{B} = \mathcal{X}_{12}[B]$, where we introduced the linear operator

$$\mathcal{X}_{12}[C] := \big(1 - \langle\!\langle M_1 \cdot M_2\rangle\!\rangle\big)^{-1}[C] \quad \text{for} \quad C \in \mathbf{C}^{2N\times 2N} . \tag{7.A.26}$$

Extending the underline to the whole product, we obtain

$$\begin{aligned} G_1\widetilde{B}G_2 =& M_1\widetilde{B}M_2 + (G_1 - M_1)\widetilde{B}M_2 - \underline{G_1\widetilde{B}M_2WG_2} \\ &+ G_1\widetilde{B}M_2\langle\!\langle G_2 - M_2\rangle\!\rangle G_2 + G_1\langle\!\langle G_1\widetilde{B}M_2\rangle\!\rangle G_2 , \end{aligned}$$

from which we conclude

$$\begin{aligned} G_1BG_2 =& M_1\mathcal{X}_{12}[B]M_2 + (G_1 - M_1)\mathcal{X}_{12}[B]M_2 - \underline{G_1\mathcal{X}_{12}[B]M_2WG_2} \\ &+ G_1\mathcal{X}_{12}[B]M_2\langle\!\langle G_2 - M_2\rangle\!\rangle G_2 + G_1\langle\!\langle (G_1 - M_1)\mathcal{X}_{12}[B]M_2\rangle\!\rangle G_2\,. \end{aligned} \tag{7.A.27}$$

To proceed, we note that

$$B' := \mathcal{X}_{12}[B] = B + \frac{\langle\!\langle M_1BM_2\rangle\!\rangle}{1 - \langle\!\langle M_1M_2\rangle\!\rangle} \tag{7.A.28}$$

has bounded norm, $\|B'\| \lesssim 1$, since for z_1, z_2 away from the spectrum of H, $\mathcal{X}_{12}$ is a bounded operator (see Lemma 1.B.5 and Appendix 2.A.2). This can be seen as follows: First, the denominator in (7.A.28) is estimated as $|1 - \langle\!\langle M_1M_2\rangle\!\rangle| \geq 1 - \langle\!\langle |M_1|^2\rangle\!\rangle^{1/2}\langle\!\langle |M_2|^2\rangle\!\rangle^{1/2}$. Next, we employ the identity $\langle\!\langle |M_i|^2\rangle\!\rangle = \langle\!\langle \operatorname{Im} M_i\rangle\!\rangle/(\operatorname{Im} z_i + \langle\!\langle \operatorname{Im} M_i\rangle\!\rangle)$, as follows from the MDE (7.3.1) and, together with a Schwarz inequality, implies boundedness of the numerator in (7.A.28). Finally, $|1 - \langle\!\langle M_1M_2\rangle\!\rangle| \geq c > 0$ follows from the fact that $|\operatorname{Im} z_i| \sim |\langle\!\langle \operatorname{Im} M_i\rangle\!\rangle|$ in the regime of z_i's with distance to the spectrum of H bounded below by an N-independent positive constant.

Then, using the norm bounds $\|G_i\| \lesssim 1$ and $\|M_i\| \lesssim 1$ together with the *single resolvent global law* (7.3.2) (see also [243, Theorem 2.1]) for the second, fourth and fifth term in (7.A.27), we conclude the desired. □

Chapter 8

Loschmidt echo for deformed Wigner matrices

This chapter contains the paper [237]:

L. Erdős, J. Henheik, and O. Kolupaiev. Loschmidt echo for deformed Wigner matrices. *Lett. Math. Phys.*, 115(1):1–42, 2025

Abstract. We consider two Hamiltonians that are close to each other, $H_1 \approx H_2$, and analyze the time-decay of the corresponding *Loschmidt echo* $\mathfrak{M}(t) := |\langle \psi_0, \mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1}\psi_0\rangle|^2$ that expresses the effect of an imperfect time reversal on the initial state ψ_0. Our model Hamiltonians are deformed Wigner matrices that do not share a common eigenbasis. The main tools are new two-resolvent laws for such H_1 and H_2.

Content

8.1 Introduction

Recent quantum technological advances put quantum mechanical time reversal procedures in the focus of both experimental [308, 415, 425, 546, 306, 473] and theoretical [307, 356, 305, 514, 517, 515, 190, 191] research (see also the review [297] for a concise overview). The basic physical setup consists of an initial (normalized) quantum state ψ_0 and two self-adjoint Hamiltonians close to each other, $H_1 \approx H_2$, each governing the evolution of the system during a time span t. First, the initial state ψ_0 evolves under the Hamiltonian H_1 from time zero to t, resulting in the state $\psi_t = \exp(-\mathrm{i}H_1 t)\psi_0$. Then, during a second evolution between t and $2t$, one applies the Hamiltonian H_2 backward in time, equivalently the Hamiltonian $-H_2$ in forward time, aiming to recover the initial state ψ_0. A schematic summary of this process is given by

$$\psi_0 \xrightarrow[H_1]{t} \psi_t \xrightarrow[-H_2]{t} \psi_0' \,. \tag{8.1.1}$$

Note that, if $H_2 = H_1$, the restoration of ψ_0 would be perfect, $\psi_0' = \psi_0$ for any time t. However, in realistic setup the second Hamiltonian is never a perfect copy of the first one: the non-zero difference between H_1 and H_2 regularly leads to an imperfect recovery ψ_0' of ψ_0 and the discrepancy also depends on time.

This imperfection in the time reversal is captured in the scalar *overlap function* [578, 293, 517] (sometimes also called *fidelity amplitude* [292, 293, 579])

$$\mathfrak{S}(t) = \mathfrak{S}^{(E_0)}_{H_1,H_2}(t) := \left\langle \psi_0, \mathrm{e}^{\mathrm{i}H_2 t}\mathrm{e}^{-\mathrm{i}H_1 t}\psi_0 \right\rangle \tag{8.1.2}$$

where it is assumed that the initial state is supported[1] around its energy $\langle\psi_0, H_1\psi_0\rangle \approx \langle\psi_0, H_2\psi_0\rangle \approx E_0$. The central object of our paper is the absolute value square of the overlap function

$$\mathfrak{M}(t) = \mathfrak{M}^{(E_0)}_{H_1,H_2}(t) := \left|\mathfrak{S}^{(E_0)}_{H_1,H_2}(t)\right|^2 . \tag{8.1.3}$$

This was coined the *fidelity*, e.g., by Gorin *et. al.* in [293], or the *Loschmidt echo* by Peres [491] and Jalabert-Pastawski in [358] owing to its connection to the classical Loschmidt's paradox of time reversibility [427, 98].

In addition to (8.1.2)–(8.1.3), we will also consider an *averaged overlap function* and an *averaged Loschmidt echo*, defined as

$$\overline{\mathfrak{S}}(t) = \overline{\mathfrak{S}}^{(E_0,\eta_0)}_{H_1,H_2}(t) := \mathrm{Av}\big[\mathfrak{S}^{(E)}_{H_1,H_2}(t)\big] \quad \text{and} \quad \overline{\mathfrak{M}}(t) = \overline{\mathfrak{M}}^{(E_0,\eta_0)}_{H_1,H_2}(t) := \left|\overline{\mathfrak{S}}^{(E_0,\eta_0)}_{H_1,H_2}(t)\right|^2 , \tag{8.1.4}$$

respectively. In (8.1.4), by $\mathrm{Av}[...]$, we denoted an averaging over initial states with energies E in a small energy window of size η_0 around E_0 (see (8.2.5) below for a precise implementation of this concept).

[1] This means that, when writing $\psi_0 = \sum_n c_n^{(i)}\phi_n^{(i)}$ in the eigenbasis $\{\phi_n^{(i)}\}_n$ of H_i, only coefficients $c_n^{(i)}$ corresponding to an eigenvalue close to E_0 are non-vanishing.

The Loschmidt echo is a basic object in the study of complex quantum system and has attracted considerable attention in different areas of research, e.g. quantum chaos [491, 358, 305, 307, 356, 517], quantum information theory [280, 278], and statistical mechanics [514, 515, 190, 191]. The Loschmidt echo, as a measurable physical quantity, is observed and predicted to follow a quite universal behavior as a function of time (cf. the discussion of our main results around (8.1.6)–(8.1.7) below). On a high level (see [297]), the reason for the robust universal features is that the subsequent forward and backward evolutions act as a "filter" for irrelevant details. The typical behavior of the Loschmidt echo can be structured in three consecutive phases (see Figure 8.1.1, cf. also [297, Figure 4]): After an initial short-time parabolic decay, $\mathfrak{M}(t) \approx 1 - \gamma t^2$, the Loschmidt echo exhibits an intermediate-time asymptotic exponential decay[2], $\mathfrak{M}(t) \approx \mathrm{e}^{-\Gamma t}$. Finally, at times t beyond the so-called *saturation time* $t_s \sim (\log N)/\Gamma$, where N is the (effective) Hilbert space dimension, it saturates at a value inversely proportional to N, i.e. $\mathfrak{M}(t) \sim 1/N$.

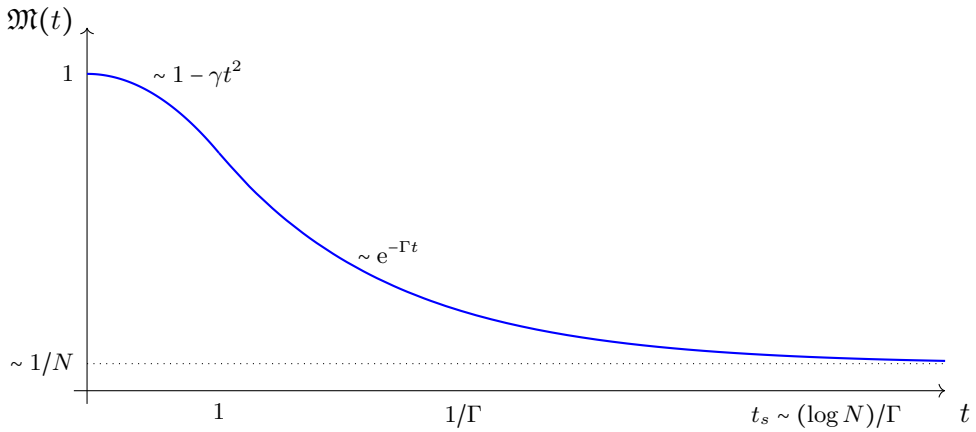

Figure 8.1.1: Illustrated is the typical behavior of the Loschmidt echo in its three consecutive phases: Short-time parabolic decay, intermediate-time asymptotic decay, and long-time saturation. In both of our main results (8.1.6)–(8.1.7), the decay parameters γ and Γ generally satisfy $\gamma \sim \Gamma \sim \langle (H_1 - H_2)^2 \rangle$; cf. (8.1.8).

There are several ways to determine the behavior of the Loschmidt echo in a given system (see the review [297]): One standard option is to employ semi-classical approximations [358, 578, 599], another one is numerical evaluation [548, 203, 576]. Here, following E. Wigner's original vision of describing chaotic quantum systems by large random matrices [586] (see also further extensive physics literature [185, 292, 371, 210, 147, 148, 191]), we model (part of) the Hamiltonian(s) H_1, H_2 by Wigner random matrices with independent entries. In this setup we can give a mathematically rigorous and quite precise analysis of certain features of the Loschmidt echo; some of them have been predicted in the physics literature.

Before defining the precise model, we first discuss where the name *echo* for $\mathfrak{M}(t)$ comes from. Fix any time $t > 0$ and consider the two-step process (8.1.1). For $s \in [0, 2t]$ denote the state at the intermediate time s by ψ_s, namely, $\psi_s = \mathrm{e}^{-\mathrm{i}sH_1}\psi_0$ for $s \in [0, t]$ and $\psi_s = \mathrm{e}^{\mathrm{i}(t-s)H_2}\mathrm{e}^{-\mathrm{i}tH_1}\psi_0$ for $s \in [t, 2t]$. Comparing this notation to (8.1.1) we see that $\psi_{2t} = \psi_0'$. Denote further the (squared) overlap of ψ_0 and ψ_s by

$$\mathfrak{P}_t(s) := |\langle \psi_0, \psi_s \rangle|^2. \tag{8.1.5}$$

This quantity depends also on ψ_0 and H_1, H_2, but we suppress this dependence in notations for simplicity. We call $\mathfrak{P}_t(s)$, $s \in [0, 2t]$, the *Loschmidt echo process*. Clearly, $\mathfrak{P}_t(0) = 1$ and $\mathfrak{P}_t(2t) = \mathfrak{M}(t)$. Later in Corollary 8.2.5 we show that $\overline{\mathfrak{P}}_t(t) \ll \overline{\mathfrak{P}}_t(2t)$ under suitable assumptions,

[2] In the very extreme case, when the difference $H_1 - H_2$ is small compared to the local eigenvalue spacing one observes Gaussian instead of exponential decay, $\mathfrak{M}(t) \approx \mathrm{e}^{-\gamma t^2}$ (see, e.g., [297, Section 2.3.1])

where $\overline{\mathfrak{P}}_t$ is an averaged version of $\mathfrak{P}_t$ defined in (8.2.9a)-(8.2.9b). This result means that typically the original complete overlap $\mathfrak{P}_t(0) = 1$ is partially recovered at the final moment of time $2t$, though at the intermediate time t it is much smaller than $\mathfrak{P}_t(2t)$ (see Figure 8.1.2).

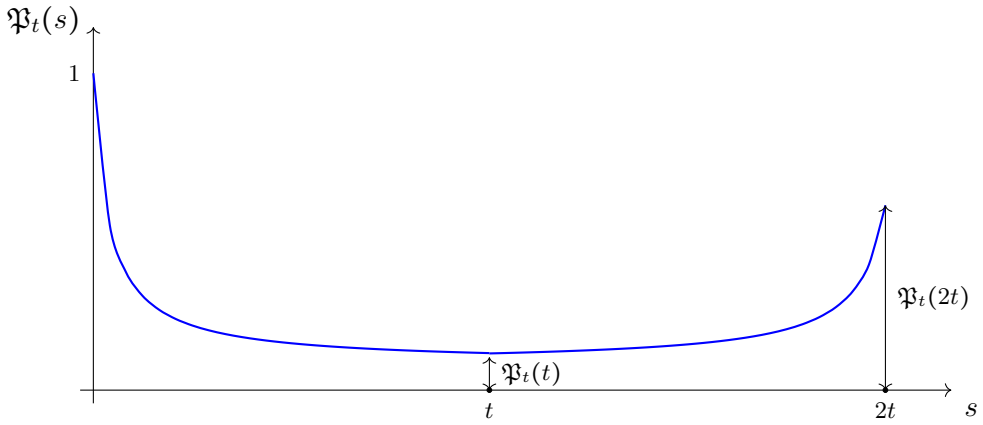

Figure 8.1.2: Schematic behavior of the overlap $\mathfrak{P}_t(s)$ from (8.1.5) for $s \in [0, 2t]$. At the midpoint, $s = t$, typically $\mathfrak{P}_t(t) \ll \mathfrak{P}_t(2t)$, which indicates a partial recovery between time t and $2t$ of the original complete overlap at time $s = 0$.

As our main result, we rigorously prove the decay of the Loschmidt echo for two different physical settings (called *Scenario I* and *Scenario II*), which we now describe somewhat informally (see Section 8.2 for more precise statements containing all the technical details).

For our first result (Scenario I, Theorem 8.2.4), we consider two *deformed* $N \times N$ *Wigner matrices* $H_j = D_j + W$, $j = 1, 2$, with bounded deterministic D_j, satisfying $D_1 \approx D_2$, and W a (common) random Wigner matrix. This setup corresponds to an arbitrary deterministic system modeled by the Hamiltonian D_1 and the time reversed Hamiltonian D_2 nearby, which are both subject to an overall mean-field noise described by the same Wigner matrix W throughout the whole echo process. In this setting, for an energy E_0 in the bulk of the density of states of both H_1 and H_2, we consider the averaged Loschmidt echo (8.1.4). Our result in Theorem 8.2.4 then shows (i) *short-time parabolic decay* and (ii) *intermediate-time asymptotic decay* of the form

$$\overline{\mathfrak{M}}(t) \approx \begin{cases} 1 - \gamma t^2 & \text{for} \quad t \ll 1 \\ \mathrm{e}^{-\Gamma t} & \text{for} \quad 1 \ll t \lesssim \Delta^{-2}. \end{cases} \tag{8.1.6}$$

Both *decay parameters* satisfy $\gamma \sim \Delta^2$ and $\Gamma \sim \Delta^2$, where $\Delta := \langle (D_1 - D_2)^2 \rangle^{1/2}$, and depend on E_0 and the density of states at E_0. Here we introduced the notation $\langle A \rangle := \frac{1}{N} \mathrm{Tr}\, A$ for any $N \times N$ matrix A. We point out that the quadratic relation $\Gamma \sim \Delta^2$ is in perfect agreement with *Fermi's golden rule*.

For our second main result (Scenario II, Theorem 8.2.10), we consider a physically different situation: Now the two Hamiltonians[3] are $H_1 = D$ and $H_2 = D + \lambda W$ with the same deterministic D, a standard Wigner matrix W and a small parameter $|\lambda| \ll 1$. The normalization is chosen such that $\mathbf{E}\langle W^2 \rangle = 1$. Hence, the imperfection along the backward evolution is modeled by a small Wigner matrix λW indicating an additive noise (see, e.g., [190, Eq. (31)] or [191, Eq. (1)]). For a normalized initial state $\psi_0 \in \mathbf{C}^N$ supported in the bulk of the density of states of both H_1 and H_2 with energy $\langle \psi_0, H_1 \psi_0 \rangle \approx \langle \psi_0, H_2 \psi_0 \rangle \approx E_0$, we now consider the usual Loschmidt echo (8.1.3) without averaging. Similarly to (8.1.6), our result in Theorem 8.2.10 then shows (i) *short-time parabolic decay* and (ii)

[3] Within Theorem 8.2.10, the Hamiltonians $H_1 = D$ and $H_2 = D + \lambda W$ will be denoted by H_0 and H_λ, respectively.

intermediate-time asymptotic decay of the form

$$\mathfrak{M}(t) \approx \begin{cases} 1-\gamma t^2 & \text{for} \quad t \ll 1 \\ \mathrm{e}^{-\Gamma t} & \text{for} \quad 1 \ll t \lesssim \lambda^{-2}. \end{cases} \tag{8.1.7}$$

Here the *decay parameters* satisfy $\gamma = \lambda^2$ and $\Gamma = 2\pi\rho_0(E_0)\lambda^2$, where ρ_0 is the (limiting, as $N \to \infty$) density of states of D. Finally, we note that, since $\mathbf{E}\langle W^2\rangle = 1$, in both of our scenarios (8.1.6)–(8.1.7) the decay parameters γ and Γ satisfy the general relation

$$\gamma \sim \Gamma \sim \mathbf{E}\langle (H_1 - H_2)^2\rangle\,. \tag{8.1.8}$$

As corollaries to our main results (8.1.6)–(8.1.7) in Theorems 8.2.4 and 8.2.10, we also consider the *scrambled Loschmidt echo* [190, 387, 515] $\mathfrak{M}_\delta^{\mathrm{sc}}(t)$ and its averaged analog $\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t)$. They are defined from

$$\mathfrak{S}_\delta^{\mathrm{sc}}(t) := \left\langle \psi_0, \mathrm{e}^{\mathrm{i}H_2 t}\mathrm{e}^{-\mathrm{i}\delta V}\mathrm{e}^{-\mathrm{i}H_1 t}\psi_0\right\rangle \tag{8.1.9}$$

and its averaged analog $\overline{\mathfrak{S}}_\delta^{\mathrm{sc}}(t)$ as

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) := \left|\mathfrak{S}_\delta^{\mathrm{sc}}(t)\right|^2 \quad \text{and} \quad \overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) := \left|\overline{\mathfrak{S}}_\delta^{\mathrm{sc}}(t)\right|^2,$$

exactly as in (8.1.3)–(8.1.4), respectively. In (8.1.9), H_1 and H_2 are the two Hamiltonians either from Scenario I or Scenario II. The idea behind the quantity in (8.1.9) is that, between the forward and backward evolution, there is a (short) *scrambling time* δ, in which the system is uncontrolled and governed by another self-adjoint *scrambling Hamiltonian* V [190]. Similarly to (8.1.1), a schematic summary of this process is given by

$$\psi_0 \xrightarrow[H_1]{t} \psi_t \xrightarrow[V]{\delta} \psi_t' \xrightarrow[-H_2]{t} \psi_0'\,. \tag{8.1.10}$$

In Corollaries 8.2.6 and 8.2.11 (of Theorems 8.2.4 and 8.2.10, respectively), we model the scrambling Hamiltonian by another Wigner matrix, $V := \widetilde{W}$, that is *independent* of W; see [190]. As a result, we find that

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) \approx (\varphi(\delta))^2\,\overline{\mathfrak{M}}(t) \quad \text{and} \quad \mathfrak{M}_\delta^{\mathrm{sc}}(t) \approx (\varphi(\delta))^2\,\mathfrak{M}(t) \tag{8.1.11}$$

in the setting of Scenario I and Scenario II, respectively, where we denoted $\varphi(\delta) := J_1(2\delta)/\delta$ and J_1 is the first order Bessel function of the first kind. Note that in (8.1.11) we see the effects of the scrambling Hamiltonian V and the imperfect time reversal of H_1 and H_2 to completely decouple (cf. [190, Eq. (35)]).

We point out that Scenario II, discussed around (8.1.7), and the corollaries described in (8.1.11) are primarily given to provide a more comprehensive view of Loschmidt echoes modeled with Wigner matrices. Technically, these are obtained by simple modifications of earlier results and techniques (see [170], Chapter 7 and the proof in Section 8.7 for details). The mathematically novel principal part of this work therefore consists of Theorem 8.2.4 analyzing Scenario I.

The proof of Theorem 8.2.4 relies on a new *two-resolvent global law*, i.e. a concentration estimate for products of resolvents $G_i(z_i) := (H_i - z_i)^{-1}$ for $z_i \in \mathbf{C}\setminus\mathbf{R}$ as the dimension N of the matrix becomes large. By functional calculus, this can then be used for computing more complicated functions of H_i, like the exponential, and thus connecting to the time evolutions above. A typical global law computes, e.g.,

$$\langle \psi_0, G_2(z_2)G_1(z_1)\psi_0\rangle \tag{8.1.12}$$

to leading order in N with error terms vanishing like $N^{-1/2+\epsilon}$ with very high probability. Note that such an error term prohibits accessing times beyond the saturation time (when the Loschmidt echo

is of order $1/N$), which is why our main results (8.1.6)–(8.1.7) are only valid up to order Δ^{-2} and λ^{-2}, respectively, i.e. well before the saturation time.

The main novelty of this paper is a precise estimate on the deterministic leading term to (8.1.12). While it is well known that $G_i(z_i) \approx M_i(z_i)$, where the deterministic matrix $M_i(z_i)$ is the solution of the *Matrix Dyson Equation* (8.2.1), it does *not* hold that $G_2(z_2)G_1(z_1) \approx M_2(z_2)M_1(z_1)$ owing to correlations between G_1 and G_2. The correct approximation is

$$G_2(z_2)G_1(z_1) \approx \frac{M_2(z_2)M_1(z_1)}{1 - \langle M_1(z_1)M_2(z_2)\rangle}. \tag{8.1.13}$$

To control (8.1.13), we hence need to estimate the denominator of (8.1.13), which is well known in case of $H_1 = H_2$, i.e. $D_1 = D_2$ [165] (see also Chapters 2 and 7). Here, however, the analysis of (8.1.13) is much more intricate, since for general D_1, D_2 the deterministic approximations $M_1(z_1), M_2(z_2)$ do *not* commute. In our main Proposition 8.4.2, we optimally track the dependence of (8.1.13) on the difference $D_1 - D_2$ of the two deformations and on $z_1 - z_2$.

Notations

For positive quantities f, g we write $f \lesssim g$ (or $f = \mathcal{O}(g)$) and $f \sim g$ if $f \le Cg$ or $cg \le f \le Cg$, respectively, for some constants $c, C > 0$ which only depend on the constants appearing in the moment condition (see Assumption 8.2.1), the bound on M in Assumption 8.2.2, the constants from Assumption 8.2.8, or the bulk parameter κ from (8.2.3). In informal explanations, we frequently use the notation $f \ll g$, which indicates that f is "much smaller" than g. Moreover, we shall also write $w \approx z$ to indicate the closeness of $w, z \in \mathbf{C}$ with a not precisely specified error.

For any natural number n we set $[n] := \{1, 2, \dots, n\}$. Matrix entries are indexed by lowercase Roman letters $a, b, c, \dots$ from the beginning of the alphabet. We denote vectors by bold-faced lowercase Roman letters $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$, or lower case Greek letters $\psi, \phi \in \mathbf{C}^N$, for some $N \in \mathbf{N}$. Vector and matrix norms, $\|\boldsymbol{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $N \times N$ matrix A we use the notation $\langle A\rangle := N^{-1}\mathrm{Tr}A$ for its normalized trace and denote the spectrum of A by $\sigma(A)$. Moreover, for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}^N$ we denote their scalar product by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \sum_i \overline{x}_i y_i$. The support of a function f is denoted by $\mathrm{supp}(f)$.

Finally, we use the concept of "with very high probability" *(w.v.h.p.)* meaning that for any fixed $C > 0$, the probability of an N-dependent event is bigger than $1 - N^{-C}$ for $N \ge N_0(C)$. We also introduce the notion of *stochastic domination* (see e.g. [241]): given two families of non-negative random variables

$$X = \left(X^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) : N \in \mathbf{N}, u \in U^{(N)}\right)$$

indexed by N (and possibly some parameter u in some parameter space $U^{(N)}$), we say that X is stochastically dominated by Y, if for all $\xi, C > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^{\xi} Y^{(N)}(u)\right] \le N^{-C} \tag{8.1.14}$$

for large enough $N \ge N_0(\xi, C)$. In this case we use the notation $X \prec Y$ or $X = \mathcal{O}_{\prec}(Y)$.

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8.2 Main results

The key players of our paper are *deformed Wigner matrices*, i.e. matrices of the form $H = D + W$, where $D = D^* \in \mathbf{C}^{N\times N}$ is a bounded deterministic matrix (called *deformation*), $\|D\| \le L$ for some N-independent $L > 0$ and $W = W^* \in \mathbf{C}^{N\times N}$ is a real symmetric or complex Hermitian Wigner matrices. This means, its entries are independently distributed random variables according to the laws[4] $w_{ij} \overset{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{od}}$ for $i < j$ and $w_{jj} \overset{\mathrm{d}}{=} N^{-1/2}\chi_{\mathrm{d}}$. For the single entry distributions χ_{od} and χ_{d} we assume the following.

Assumption 8.2.1 (Wigner matrix)**.** *We assume that χ_{d} is a centered real random variable, and χ_{od} is a real or complex random variable with $\mathbf{E}\,\chi_{\mathrm{od}} = 0$ and $\mathbf{E}\,|\chi_{\mathrm{od}}|^2 = 1$. Furthermore, we assume the existence of higher moments, namely $\mathbf{E}\,|\chi_{\mathrm{d}}|^p + \mathbf{E}\,|\chi_{\mathrm{od}}|^p \le C_p$ for all $p \in \mathbf{N}$, where C_p are positive constants.*

It is well known [243, 17] that the resolvent of H, denoted by $G(z) := (H - z)^{-1}$ for $z \in \mathbf{C} \setminus \mathbf{R}$, becomes approximately deterministic in the large N limit. Its deterministic approximation (as a matrix) is given by $M(z)$, the unique solution of the Matrix Dyson equation (MDE)

$$-\frac{1}{M(z)} = z - D + \langle M(z)\rangle \quad \text{for} \quad z \in \mathbf{C} \setminus \mathbf{R} \quad \text{under the constraint} \quad \operatorname{Im} z \operatorname{Im} M(z) > 0\,, \tag{8.2.1}$$

where $\operatorname{Im} M(z) := [M(z) - M(z)^*]/2\mathrm{i}$ and positivity is understood as a matrix. The corresponding (N-dependent) *self consistent density of states (scDos)* is defined as

$$\rho(e) := \frac{1}{\pi}\lim_{\eta\downarrow 0}\langle \operatorname{Im} M(e + \mathrm{i}\eta)\rangle\,. \tag{8.2.2}$$

This is a compactly supported Hölder-$1/3$ continuous function on $\mathbf{R}$ which is in fact real-analytic on the set $\{\rho > 0\}$[5]. The positive harmonic extension of ρ is denoted by $\rho(z) := \pi^{-1}|\langle \operatorname{Im} M(z)\rangle|$ for $z \in \mathbf{C} \setminus \mathbf{R}$. We point out that not only the tracial quantity $\langle \operatorname{Im} M(e + \mathrm{i}\eta)\rangle$ has an extension to the real axis, but the whole matrix $M(e) := \lim_{\eta\downarrow 0} M(e + \mathrm{i}\eta)$ is well defined (see Lemma 1.B.1 (b)). Moreover, for any small $\kappa > 0$ (independent of N) we define the *κ-bulk* of the scDos (8.2.2) as

$$\mathbf{B}_\kappa(\rho) = \{x \in \mathbf{R} \,:\, \rho(x) \ge \kappa\}\,. \tag{8.2.3}$$

It is a finite union of disjoint compact intervals, cf. Lemma 1.B.2. Note that, for $\operatorname{Re} z \in \mathbf{B}_\kappa$ it holds that $\|M(z)\| \lesssim 1$, as easily follows by taking the imaginary part of (8.2.1).

Now, the resolvent G is close to M from (8.2.1) in the following *averaged* and *isotropic* sense:

$$|\langle (G(z) - M(z))B\rangle| \prec \frac{1}{N|\operatorname{Im} z|}, \qquad |\langle \boldsymbol{x}\,, (G(z) - M(z))\boldsymbol{y}\rangle| \prec \frac{1}{\sqrt{N|\operatorname{Im} z|}}\,, \tag{8.2.4}$$

uniformly in deterministic vectors $\|\boldsymbol{x}\| + \|\boldsymbol{y}\| \lesssim 1$ and deterministic matrices $\|B\| \lesssim 1$. These estimates are called *local laws* when $|\operatorname{Im} z| \ll 1$ and *global laws* when $|\operatorname{Im} z| \gtrsim 1$. To be precise about their validity, we recall that while (8.2.4) holds for $\operatorname{Re} z \in \mathbf{B}_\kappa$ and $\operatorname{dist}(\operatorname{Re} z, \operatorname{supp}(\rho)) \gtrsim 1$ for *arbitrary* bounded self-adjoint deformations $D = D^*$ (see [243, Theorem 2.1]), the complementary regime requires the additional Assumption 8.2.2 on D stated below (see [23, Theorem 2.6] and [239, Theorem 2.8]). A sufficient condition for Assumption 8.2.2 is discussed in Remark 8.2.3; see also [22].

In the remainder of this section, we formulate our main results on the two different Loschmidt echo scenarios described in Section 8.1.

[4] A careful examination of our proof reveals that the entries of W need not be distributed identically. Indeed, only the matching of the second moments is necessary, but higher moments can differ.

[5] In [16, 17, 22], the scDos has been thoroughly analysed in increasing generality of the ensemble. It is supported on finitely many finite intervals and, roughly speaking, there are three different regimes for the behavior or ρ: In the *bulk*, ρ is strictly positive; at the *edge*, ρ vanishes like a square root at the edges of every supporting interval which are well separated; at the *cusp*, where two intervals of support (almost) meet, ρ behaves (almost) as a cubic root. Correspondingly, ρ is locally real analytic, Hölder-$1/2$, or Hölder-$1/3$ continuous, respectively. Near the singularities, it has an approximately universal shape (see (8.A.3a)–(8.A.3d) in the proof of Lemma 8.A.1).

8.2.1 Scenario I: Two deformations of a Wigner matrix

For the first echo scenario, we consider two deformed Wigner matrices, $H_j = D_j + W$, $j \in [2]$, and denote their resolvents and corresponding deterministic approximation (8.2.1) by G_j and M_j, respectively. A natural definition of the averaged Loschmidt echo is

$$\overline{\mathfrak{M}}(t) = \overline{\mathfrak{M}}^{(E_0,\eta_0)}_{H_1,H_2}(t) := \left| \frac{\langle \mathrm{e}^{\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_2} \rangle}{\langle \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0) \rangle} \right|^2 , \tag{8.2.5}$$

since $\operatorname{Im} G / \langle \operatorname{Im} M \rangle$ in (8.2.5) effectively localizes around E_0 and *averages* in a window of size $\eta_0 > 0$ assumed to be independent of N. In Remark 8.2.7 below we comment on the averaging implemented by (8.2.5). Note that in order to match (8.1.2)-(8.1.3) from the introduction we need to replace t by $-t$ in (8.2.5). However, this replacement does not change the quantity (8.2.5) since

$$\left| \langle \mathrm{e}^{\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_2} \rangle \right| = \left| \langle \mathrm{e}^{\mathrm{i}tH_2} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_1} \rangle \right| = \left| \langle \mathrm{e}^{-\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{\mathrm{i}tH_2} \rangle \right| ,$$

where in the last step we used that $\mathrm{e}^{\mathrm{i}tH_1}$ and $\operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0)$ commute. Using this observation we will work with (8.2.5) in the rest of the paper. The same comment applies also to the other versions of the averaged Loschmidt echo defined in Section 8.2.1, namely to (8.2.9a), (8.2.9b) and (8.2.12).

We will henceforth assume that the deformations D_1, D_2 are such that the corresponding solutions M_1, M_2 to (8.2.1) are bounded.

Assumption 8.2.2 (Boundedness of M)**.** *Let D be an $N \times N$ Hermitian matrix and M the solution to* (8.2.1)*. We assume that there exists an N-independent positive constant L such that $\sup_{z \in \mathbf{C} \setminus \mathbf{R}} \|M(z)\| < L$.*

Assumption 8.2.2 is the basis for the *shape theory* of the scDos, which we briefly described in Footnote 5. We now give a sufficient condition on D for Assumption 8.2.2 to hold. It basically requires that its ordered eigenvalue sequence has to be piecewise Hölder-$1/2$ continuous as a function of the label.

Remark 8.2.3 (Sufficient condition for Assumption 8.2.2)**.** *Denote the eigenvalues of any self-adjoint deformation D by $\{d_j\}_{j=1}^N$ labeled in increasing order, $d_j \le d_k$ for $j < k$. Fix a (large) positive constant $L > 0$. The set $\mathcal{M}_L$ of admissible self-adjoint deformations D is defined as follows: we say that $D \in \mathcal{M}_L$ if $\|D\| \le L$ and there exists an N-independent partition $\{I_s\}_{s=1}^m$ of $[0,1]$ in at most L segments such that for any $s \in [1,m]$ and any $j,k \in [1,N]$ with $j/N, k/N \in I_s$ we have $|d_j - d_k| \le L|j/N - k/N|^{1/2}$. Since the operator $\mathcal{S} = \langle \cdot \rangle$ is flat, condition $D \in \mathcal{M}_L$ implies that D satisfies Assumption 8.2.2 for some $L' < \infty$ by means of [22, Lemma 9.3].*

We can now formulate our first main result.

Theorem 8.2.4 (Averaged Loschmidt echo with two deformations)**.** *Let W be a Wigner matrix satisfying Assumption 8.2.1, and $D_1, D_2 \in \mathbf{C}^{N \times N}$ be bounded, traceless*[6] *Hermitian matrices, i.e. $\|D_j\| \le L$ for some $L > 0$ and $\langle D_1 \rangle = \langle D_2 \rangle = 0$, additionally satisfying Assumption 8.2.2. Fix $\eta_0 \le 1$ and let E_0 be an energy in the bulk of the scDos of H_1 and H_2, i.e. assume that there exist $\delta, \kappa > 0$ such that $[E_0 - \delta, E_0 + \delta] \subset \mathbf{B}_\kappa(\rho_1) \cap \mathbf{B}_\kappa(\rho_2)$. We also assume that parameters η_0, κ and δ are N-independent.*

Consider the deformed Wigner matrices $H_j := D_j + W$ for $j \in [2]$ and the corresponding averaged (at energy E_0 in a window of size $\eta_0 > 0$) Loschmidt echo $\overline{\mathfrak{M}}(t)$ for times $t \ge 0$ defined in (8.2.5)*. Then we have the following:*

[6] If D_1 or D_2 had a non-zero trace, it could be absorbed by a simple (scalar) energy shift.

(i) [Short-time parabolic decay] *As $t \to 0$, it holds that*

$$\overline{\mathfrak{M}}(t) = 1 - \gamma t^2 + \mathcal{O}\big(\langle D^2\rangle t^3\big) + \mathcal{O}_{\prec}((N\eta_0)^{-1}) \tag{8.2.6}$$

where the decay parameter *is given by $\gamma := \langle (D - \langle PD\rangle)^2 P\rangle$, where we abbreviated $D := D_2 - D_1$ and $P := \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0)/\langle \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0)\rangle$. It satisfies $\gamma \sim \Delta^2 := \langle D^2\rangle$ and the implicit constant in $\sim$ depends only on κ and L.*

The implicit constants in the error terms in (8.2.6) *depend only on L, δ, κ and the C_p's from Assumption 8.2.1.*

(ii) [Intermediate-time asymptotic decay] *Take a (large) positive K and consider times $1 \le t \le K/\Delta^2$. Then there exists a positive constant c such that whenever $\Delta < c$ and $\eta_0 < \Delta/|\log\Delta|$ it holds that*

$$\overline{\mathfrak{M}}(t) = \mathrm{e}^{-\Gamma t} + \mathcal{O}\left(\mathcal{E}\right) + \mathcal{O}_{\prec}\big(C(t,\eta_0)/N\big), \tag{8.2.7}$$

where the rate Γ *(explicitly given in* (8.4.27)*) satisfies $\Gamma \sim \Delta^2$ with the implicit constant depending only on κ and L. Moreover, we denoted*

$$\mathcal{E} = \mathcal{E}(t,\Delta,\eta_0) := \frac{1+\log t}{t} + \Delta|\log\Delta| + \frac{\eta_0|\log\Delta|}{\Delta} \tag{8.2.8}$$

and $C(t) > 0$ is a positive constant depending only on t.

The implicit constants in the error terms in (8.2.7) *depend only on L, δ, κ, K and the C_p's from Assumption 8.2.1.*

Since $t \le K/\Delta^2$, we find that the leading term $\mathrm{e}^{-\Gamma t}$ in (8.2.7) remains of order one throughout the whole time regime. The error term $\mathcal{E}$ is small compared to this leading term if $t \gg 1$, $\Delta \ll 1$, and $\eta_0 \ll \Delta/|\log\Delta|$, hence these relations define the regime of the parameters where our theorem is meaningful.

The following corollary to Theorem 8.2.4 reveals the key property of the Loschmidt echo, the partial recovery of the initial overlap, as discussed in the introduction; see Figure 8.1.2.

Corollary 8.2.5 (Averaged Loschmidt echo process)**.** *Assume the set-up and the conditions of Theorem 8.2.4. For time $t > 0$ define the* averaged Loschmidt echo process *$\overline{\mathfrak{P}}_t(s)$, $s \in [0, 2t]$, as follows:*

$$\overline{\mathfrak{P}}_t(s) := \left| \frac{\left\langle \mathrm{e}^{\mathrm{i}sH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0)\right\rangle}{\langle \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0)\rangle} \right|^2, \qquad s \in [0,t], \tag{8.2.9a}$$

$$\overline{\mathfrak{P}}_t(s) := \left| \frac{\left\langle \mathrm{e}^{\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}(s-t)H_2}\right\rangle}{\langle \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0)\rangle} \right|^2, \quad s \in (t,2t]. \tag{8.2.9b}$$

Let $\lim^*$ be the simultaneous limit in Δ, η_0, t such that $\Delta, \eta_0 \to 0$ and $t \to \infty$ under constraints $\Delta^2 \ll \eta_0 \ll \Delta/|\log\Delta|$ and $1/\eta_0 \ll t \lesssim 1/\Delta^2$. Here $a \ll b$ means that $a/b \to 0$ in this limit. Then almost surely we have

$$\lim^* \limsup_{N\to\infty} \frac{\overline{\mathfrak{P}}_t(t)}{\overline{\mathfrak{P}}_t(2t)} = \lim^* \limsup_{N\to\infty} \frac{\overline{\mathfrak{P}}_t(t)}{e^{-\Gamma t}} = 0, \tag{8.2.10}$$

where Γ is the same as in Theorem 8.2.4.

Proof of Corollary 8.2.5. Firstly take the limit $N \to \infty$ in the denominator $\overline{\mathfrak{P}}_t(2t) = \overline{\mathfrak{M}}(t)$ of (8.2.10). Recall the definition of $\mathcal{E}$ from (8.2.8). By means of Theorem 8.2.4 we have

$$\liminf_{N\to\infty} \overline{\mathfrak{P}}_t(2t) = \liminf_{N\to\infty} \left(\mathrm{e}^{-\Gamma t} + \mathcal{O}\left(\mathcal{E}(t,\Delta,\eta_0)\right)\right) = \liminf_{N\to\infty} \left(\mathrm{e}^{-\Gamma t}(1 + o(1))\right) \sim 1$$

in the limit $\lim^*$. Here we used that $\Gamma \sim \Delta^2$ and $t \lesssim \Delta^{-2}$, so $\mathrm{e}^{-\Gamma t} \sim 1$. Thus in order to verify (8.2.10) it is sufficient to show that

$$\lim^* \limsup_{N\to\infty} \overline{\mathfrak{P}}_t(t) = 0.$$

From the average single resolvent global law for H_1, see (8.2.4) or [243, Theorem 2.1], we get that

$$\lim_{N\to\infty} \left| \left\langle \mathrm{e}^{\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \right\rangle - \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}tx} \frac{\eta_0}{(x-E_0)^2 + \eta_0^2} \rho_1(x)\mathrm{d}x \right| = 0.$$

Recall that $E_0 \in \boldsymbol{B}_\kappa(\rho_1)$. Thus $\langle \operatorname{Im} M(E_0 + \mathrm{i}\eta_0)\rangle \sim 1$ for $\eta_0 \to 0$ and

$$\limsup_{N\to\infty} \overline{\mathfrak{P}}_t(t) \lesssim \limsup_{N\to\infty} \left| \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}tx} \frac{\eta_0}{(x-E_0)^2 + \eta_0^2} \rho_1(x)\mathrm{d}x \right|^2 \lesssim \left(\frac{1}{\eta_0 t}\right)^2. \tag{8.2.11}$$

In the last inequality we employed integration by parts. Additionally we used that $\rho_1(x)$ is a bounded function of x which is guaranteed by Assumption 8.2.2 and that $\rho_1(x)$ has bounded derivative for $|x - E_0| \le \delta$ (see also Footnote 5), where δ was fixed in Theorem 8.2.4. Both of these bounds (on $\rho_1(x)$ and $\mathrm{d}\rho_1(x)/\mathrm{d}x$) are uniform in N. In the limit $\lim^*$ we have $\eta_0 t \to \infty$, so (8.2.11) finishes the proof of Corollary 8.2.5. □

As mentioned in the introduction, we also have the following corollary to Theorem 8.2.4.

Corollary 8.2.6 (Scrambled averaged Loschmidt echo with two deformations)**.** *Assume the conditions of Theorem 8.2.4 and consider (as a variant of* (8.2.5)*) the* scrambled *averaged Loschmidt echo*

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) := \left| \frac{\left\langle \mathrm{e}^{-\mathrm{i}\delta\widetilde{W}} \mathrm{e}^{\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_2} \right\rangle}{\langle \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0)\rangle} \right|^2, \tag{8.2.12}$$

where $\widetilde{W}$ is a Wigner matrix satisfying Assumption 8.2.1, independent *of W and $0 \le \delta \le N^{2/3-\varepsilon}$ for some fixed $\varepsilon > 0$. Moreover, let φ be the Fourier transform of the semi-circular density of states $\rho_{\mathrm{sc}}(x) := (2\pi)^{-1}\sqrt{[4-x^2]_+}$, which is explicitly given as*

$$\varphi(\delta) := \widehat{\rho_{\mathrm{sc}}}(\delta) = \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}\delta x} \rho_{\mathrm{sc}}(x)\mathrm{d}x = \frac{J_1(2\delta)}{\delta} \tag{8.2.13}$$

where J_1 is the first order Bessel function of the first kind.

Then, instead of (8.2.6)–(8.2.7)*, we have that*

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2 \left[1 - \gamma t^2 + \mathcal{O}\big(\langle D^2\rangle t^3\big) + \mathcal{O}_\prec\big((N\eta_0)^{-1}\big)\right] + \mathcal{O}_\prec\big(\delta/(N\eta_0)\big) \quad \textit{as} \quad t \to 0$$

and

$$\overline{\mathfrak{M}}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2 \left[\mathrm{e}^{-\Gamma t} + \mathcal{O}(\mathcal{E}) + \mathcal{O}_\prec\big(C(t)/N\big)\right] + \mathcal{O}_\prec\big(\delta/(N\eta_0)\big) \quad \textit{for} \quad 1 \le t \le K/\Delta^2$$

in the short and intermediate time regimes, respectively.

Proof of Corollary 8.2.6. Denote $A := \mathrm{e}^{\mathrm{i}tH_1} \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0) \mathrm{e}^{-\mathrm{i}tH_2}$ and observe that $\|A\| \le 1/\eta_0$. Then, by residue calculus with the contour $C_\delta := \{z \in \mathbf{C} : \operatorname{dist}(z, [-2,2]) = \delta^{-1}\}$ and a single resolvent law[7] as in (8.2.4), using only the randomness of $\widetilde{W}$, we find

$$\begin{aligned} \langle \mathrm{e}^{-\mathrm{i}\delta\widetilde{W}} A\rangle &= \frac{1}{2\pi\mathrm{i}} \oint_{C_\delta} \mathrm{e}^{-\mathrm{i}\delta z} \langle A(W-z)^{-1}\rangle \mathrm{d}z = \frac{\langle A\rangle}{2\pi\mathrm{i}} \oint_{C_\delta} \mathrm{e}^{-\mathrm{i}\delta z} m_{\mathrm{sc}}(z)\mathrm{d}z + \mathcal{O}_\prec(\delta/(N\eta_0)) \\ &= \langle A\rangle \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}\delta x} \rho_{\mathrm{sc}}(x)\mathrm{d}x + \mathcal{O}_\prec(\delta/\sqrt{N}) = \langle A\rangle \varphi(\delta) + \mathcal{O}_\prec(\delta/(N\eta_0)). \end{aligned}$$

The rest of the proof follows from Theorem 8.2.4. □

[7] To be precise, when $\delta < 1$, we use the slightly improved average *global* law $|\langle A(W-z)^{-1}\rangle - m(z)\langle A\rangle| \prec \delta^2\|A\|/N$ (see, e.g., [243, Theorem 2.1]).

We close this section by commenting on the effect of the small averaging of the Loschmidt echo over several energy states implemented in (8.2.5). This is a necessary technical step for our proof in Scenario I that relies on a two-resolvent local law. Note that averaging will not be necessary for Scenario II since it uses only single resolvent local law.

Remark 8.2.7 (Averaging of the Loschmidt echo)**.** *We provide two independent non-rigorous arguments for the averaged Loschmidt echo $\overline{\mathfrak{M}}$ and the original Loschmidt echo $\mathfrak{M}$ being close to each other.*

(1) First, by means of the Eigenstate Thermalization Hypothesis (ETH) for a deformed Wigner matrix $H = D + W$, see Theorem 2.2.7, and a single resolvent local law (8.2.4)*, it holds that*

$$\langle \boldsymbol{u}_j, A\boldsymbol{u}_j\rangle \approx \frac{\langle \operatorname{Im} M(E_0+\mathrm{i}\eta_0)A\rangle}{\langle \operatorname{Im} M(E_0+\mathrm{i}\eta_0)\rangle} \approx \frac{\langle \operatorname{Im} G(E_0+\mathrm{i}\eta_0)A\rangle}{\langle \operatorname{Im} M(E_0+\mathrm{i}\eta_0)\rangle}\,. \tag{8.2.14}$$

Here, A is an arbitrary deterministic matrix, $\boldsymbol{u}_j$ is a (normalized) eigenvector of H with eigenvalue $\approx E_0$, and η_0 a small regularization. In this sense, the pure state $|\boldsymbol{u}_j\rangle\langle\boldsymbol{u}_j|$ is weakly *close to $\operatorname{Im} G/\langle \operatorname{Im} M\rangle$ (i.e. if tested against a deterministic A), which heuristically supports the implementation of the averaged Loschmidt echo in* (8.2.5)*. However, the rigorous ETH statements do not allow to choose A depending on the underlying randomness like $A = e^{-\mathrm{i}tH_2}e^{\mathrm{i}tH_1}$.*

(2) Another supporting argument uses the fact that the averaged overlap function $\overline{\mathfrak{S}}^{(E,\eta_0)}(t)$ *(in particular its phase) is approximately constant as long as E varies in a range $|E-E_0| \lesssim \eta_0$. Hence it is irrelevant if one (a) first averages and then takes absolute value square, or (b) does it the other way around. The fact that $\overline{\mathfrak{S}}^{(E,\eta_0)}(t)$ is slowly varying in E follows by a simple computation using that (i) $\overline{\mathfrak{S}}^{(E_0,\eta_0)}(t) \approx I_{E_0,\eta_0}(t)/\langle \operatorname{Im} M_1(E_0+\mathrm{i}\eta_0)\rangle$ (see* (8.4.1) *and* (8.4.7)*), (ii) I_{E_0,η_0} is given by $\mathrm{e}^{\mathrm{i}t\mathfrak{s}_0}\langle \operatorname{Im} M_1(E_0+\mathrm{i}\eta_0)\rangle$ (see* (8.4.26)*), (iii) the exponent $\mathfrak{s}_0$ is Lipschitz continuous on scale Δ (see the last relation of* (8.4.16)*), and (iv) we have $t \lesssim \Delta^{-2}$ and $\eta_0 \ll \Delta$ by assumption.*

Both, the ETH argument (8.2.14) *and the fact that $\overline{\mathfrak{S}}^{(E,\eta_0)}(t)$ is approximately constant as long as $|E-E_0| \lesssim \eta_0$, independently indicate that the averaged Loschmidt echo $\overline{\mathfrak{M}}$ and the non-averaged Loschmidt echo $\mathfrak{M}$ should practically agree with each other. However, neither of them constitutes a rigorous proof, since (1) the observable A in* (8.2.14) *cannot be chosen to depend on the randomness, and (2) we cannot exclude that for some initial fixed energy state ψ_0, $\mathfrak{S}$ in* (8.1.2) *behaves very differently from its typical value computed by local averaging.*

8.2.2 Scenario II: Perturbation by a Wigner matrix

For the second echo scenario, we consider a single deformed Wigner matrix $H_\lambda = H_0 + \lambda W$ and the Loschmidt echo

$$\mathfrak{M}(t) = \mathfrak{M}^{(E_0,\Delta)}_{H_\lambda,H_0}(t) := \left|\langle \psi_0, \mathrm{e}^{\mathrm{i}tH_\lambda}\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle\right|^2 \tag{8.2.15}$$

for some normalized initial state $\psi_0 \in \mathbf{C}^N$ with energy $E_0 = \langle \psi_0, H_0\psi_0\rangle$ and localized in an interval of size Δ around E_0 (see Assumption 8.2.9 below for a precise statement). The localization parameter Δ plays the same role as η_0 in Section 8.2.1, but here we work with a sharp cutoff in the energy.

The unperturbed Hamiltonian H_0 is assumed to satisfy the following.

Assumption 8.2.8 (H_0 and its limiting density of states)**.** *The Hamiltonian H_0 is deterministic, self-adjoint $H_0 = H_0^*$, and uniformly bounded, $\|H_0\| \le C_{H_0}$ for some $C_{H_0} > 0$. We denote the resolvent of H_0 at any spectral parameter $z \in \mathbf{C}\setminus\mathbf{R}$ by $M_0(z) := (H_0 - z)^{-1}$. Moreover, we assume the following:*

(i) There exists a compactly supported measurable function $\rho_0 : \mathbf{R} \to [0, +\infty)$ *with* $\int_{\mathbf{R}} \rho_0(x)\mathrm{d}x = 1$ *and two positive sequences* $\epsilon_0(N)$ *and* $\eta_0(N)$*, both converging to zero as* $N \to \infty$*, such that, uniformly in* $z \in \mathbf{C}\backslash\mathbf{R}$ *with* $\eta := |\mathrm{Im}\, z| \geq \eta_0 \equiv \eta_0(N)$*, we have*

$$\langle M_0(z)\rangle = m_0(z) + \mathcal{O}(\epsilon_0) \quad \textit{with} \quad \epsilon_0 \equiv \epsilon_0(N)\,. \tag{8.2.16}$$

Here,

$$m_0(z) := \int_{\mathbf{R}} \frac{\rho_0(x)}{x - z}\mathrm{d}x \tag{8.2.17}$$

is the Stieltjes transform of ρ_0*. We refer to* ρ_0 *as the* limiting density of states, *and to* $\mathrm{supp}(\rho_0)$ *as the* limiting spectrum *of* H_0*.*

(ii) For small positive constants $\kappa, c > 0$*, we define the set of* admissible energies $\sigma_{\mathrm{adm}}^{(\kappa,c)}$ *in the limiting spectrum of* H_0 *by*[8]

$$\sigma_{\mathrm{adm}}^{(\kappa,c)} := \left\{ x \in \mathrm{supp}(\rho_0) : \inf_{|y-x|\leq\kappa} \rho_0(y) > c,\ \|\rho_0\|_{C^{1,1}([x-\kappa,x+\kappa])} \leq 1/c \right\}. \tag{8.2.18}$$

We assume that for some positive N*-independent* $\kappa, c > 0$*,* $\sigma_{\mathrm{adm}}^{(\kappa,c)}$ *is not empty.*

Assuming that the set of admissible energies in (8.2.18) is non-empty guarantees the limiting spectrum $\mathrm{supp}(\rho_0)$ has a part, where the limiting density of states behaves regularly, i.e. it is sufficiently smooth and strictly positive (in the *bulk*).

Assumption 8.2.9 (Locality of the initial state)**.** *Given Assumption 8.2.8, we first pick a* reference energy

$$E_0 \in \sigma_{\mathrm{adm}}^{(\kappa_0,c_0)} \quad \textit{for some} \quad \kappa_0, c_0 > 0, \tag{8.2.19}$$

and further introduce $I_\delta := [E_0 - \delta, E_0 + \delta]$ *for any* $0 < \delta < \kappa_0$*. Moreover, take an* N*-independent* energy width $\Delta \in (0, \kappa_0/2)$ *and let* $\Pi_\Delta := \mathbf{1}_{I_\Delta}(H_0)$ *be the spectral projection of* H_0 *onto the interval* I_Δ*. Then, we assume that the initial state* $\psi_0 \in \mathbf{C}^N$ *is normalized,* $\|\psi_0\| = 1$*, has energy* $E_0 = \langle\psi_0, H_0\psi_0\rangle$*, and satisfies* $\Pi_\Delta\psi_0 = \psi_0$*, i.e.* ψ_0 *is localized in* I_Δ*.*

Theorem 8.2.10 (Loschmidt echo with a single deformation)**.** *Consider the Loschmidt echo* (8.2.15) *for times* $t \geq 0$ *and assume that its constituents satisfy Assumptions 8.2.1 and 8.2.8–8.2.9. Then we have the following:*

(i) [Short-time parabolic decay] *As* $t \to 0$ *it holds that*

$$\mathfrak{M}(t) = 1 - \lambda^2 t^2 + \mathcal{O}(\lambda^2 t^3) + \mathcal{O}_\prec\big(1/\sqrt{N}\big)\,. \tag{8.2.20}$$

The implicit constants in the error terms in (8.2.20) *only depend on* C_{H_0} *and the* C_p*'s from Assumption 8.2.1.*

(ii) [Intermediate-time asymptotic decay] *For all times* $t \geq 0$ *it holds that*

$$\mathfrak{M}(t) = \mathrm{e}^{-2\pi\rho_0(E_0)\lambda^2 t} + \mathcal{O}(\mathcal{E}) + \mathcal{O}_\prec\big(C(t,\lambda)/\sqrt{N}\big)\,, \tag{8.2.21}$$

where for any fixed $T > 0$ *the error term* $\mathcal{E}$*, explicitly given in* (8.7.8)*, satisfies*

$$\lim_{\Delta\to 0} \lim_{\substack{t\to\infty,\lambda\to 0 \\ \lambda^2 t\leq T}} \lim_{N\to\infty} \mathcal{E} = 0$$

and the constant $C(t,\lambda) > 0$ *depends only on its arguments. The implicit constants in the error terms in* (8.2.21) *depend only on* C_{H_0} *from Assumption 8.2.8,* κ_0, c_0 *from Assumption 8.2.9, and the* C_p*'s from Assumption 8.2.1.*

[8] Here, $C^{1,1}(J)$ denotes the set of continuously differentiable functions with a Lipschitz-continuous derivative on an interval J, equipped with the norm $\|f\|_{C^{1,1}(J)} := \|f\|_{C^1(J)} + \sup_{x,y\in J: x\neq y} \frac{|f'(x)-f'(y)|}{|x-y|}$.

In the small time regime, $t \to 0$, (8.2.20) is surely more precise than (8.2.21), but the latter is more relevant to describe the exponential decay for times of order $t \sim \lambda^{-2}$.

The proof of the following Corollary 8.2.11 is completely analogous to the proof of Corollary 8.2.6 (only using an isotropic law instead of an averaged law) and so omitted.

Corollary 8.2.11 (Scrambled Loschmidt echo with a single deformation)**.** *Assume the conditions of Theorem 8.2.10 and consider (as a variant of* (8.2.15)*) the* scrambled *Loschmidt echo*

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) := \left|\left\langle \psi_0, \mathrm{e}^{\mathrm{i}tH_\lambda} \mathrm{e}^{-\mathrm{i}\delta\widetilde{W}} \mathrm{e}^{-\mathrm{i}tH_0} \psi_0 \right\rangle\right|^2 , \tag{8.2.22}$$

where $\widetilde{W}$ is a Wigner matrix satisfying Assumption 8.2.1, independent *of W and $0 \le \delta \le N^{2/3-\varepsilon}$ for some fixed $\varepsilon > 0$. Moreover, let φ be given by* (8.2.13)*.*

Then, instead of (8.2.20)–(8.2.21)*, we have that*

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2 \left[1 - \lambda^2 t^2 + \mathcal{O}(\lambda^2 t^3) + \mathcal{O}_\prec\big(1/\sqrt{N}\big)\right] + \mathcal{O}_\prec\big(\delta/\sqrt{N}\big) \quad \textit{as} \quad t \to 0$$

and

$$\mathfrak{M}_\delta^{\mathrm{sc}}(t) = (\varphi(\delta))^2 \left[\mathrm{e}^{-2\pi\rho_0(E_0)\lambda^2 t} + \mathcal{O}(\mathcal{E}) + \mathcal{O}_\prec\big(C(t,\lambda)/\sqrt{N}\big)\right] + \mathcal{O}_\prec\big(\delta/\sqrt{N}\big) \quad \textit{for} \quad \lambda^2 t \le T,$$

respectively.

The rest of the paper is devoted to proving Theorems 8.2.4 and 8.2.10. The proof of Theorem 8.2.4 is conducted in Sections 8.3–8.4. In Section 8.7 we prove Theorem 8.2.10. The proof of several technical results from Section 8.4 is deferred to Sections 8.5 and 8.6, and Appendix 8.A.

8.3 Short-time parabolic decay in Scenario I: Proof of Theorem 8.2.4 (i)

In the following, we abbreviate $\widetilde{P} = \operatorname{Im} G_1(E_0 + \mathrm{i}\eta_0)/\langle \operatorname{Im} M_1(E_0 + \mathrm{i}\eta_0)\rangle$, such that $\mathfrak{M}(t)$ can be written as

$$\overline{\mathfrak{M}}(t) = \left|\left\langle \mathrm{e}^{\mathrm{i}tH_1} \widetilde{P} \mathrm{e}^{-\mathrm{i}tH_2} \right\rangle\right|^2 = \left|\left\langle \widetilde{P} \mathrm{e}^{\mathrm{i}tH_1} \mathrm{e}^{-\mathrm{i}tH_2} \right\rangle\right|^2 . \tag{8.3.1}$$

Next, we trivially Taylor expand $\mathrm{e}^{\mathrm{i}tH_1}$ and $\mathrm{e}^{-\mathrm{i}tH_2}$ to second order, leaving us with

$$\mathrm{e}^{\mathrm{i}tH_1} \mathrm{e}^{-\mathrm{i}tH_2} = 1 + \mathrm{i}t(H_1 - H_2) - \frac{t^2}{2}\big((H_1 - H_2)^2 - [H_1, H_2]\big) + \mathcal{O}(t^3) . \tag{8.3.2}$$

Plugging this in (8.3.1), we find

$$\begin{aligned} \overline{\mathfrak{M}}(t) =& \left\langle \widetilde{P} \left(1 - \tfrac{t^2}{2}((H_1 - H_2)^2 - [H_1, H_2])\right)\right\rangle^2 \\ &+ t^2 \langle \widetilde{P}(H_1 - H_2)\rangle^2 + \mathcal{O}(t^3) + \mathcal{O}_\prec((N\eta_0)^{-1}) \\ =& 1 - \left\langle \left(D - \langle PD\rangle\right)^2 P \right\rangle t^2 + \mathcal{O}(t^3) + \mathcal{O}_\prec((N\eta_0)^{-1}) . \end{aligned} \tag{8.3.3}$$

Here, we additionally used that $\langle \widetilde{P}[H_1, H_2]\rangle = 0$ since $\widetilde{P}$ is a function of H_1, $D = H_2 - H_1$, and a single resolvent law in the form $\langle \widetilde{P} A\rangle = \langle PA\rangle + \mathcal{O}_\prec((N\eta_0)^{-1})$ for any A with $\|A\| \lesssim 1$. The fact that the decay parameter $\gamma = \langle (D - \langle PD\rangle)^2 P\rangle$ satisfies $\gamma \sim \Delta^2$ is a simple consequence of the *flatness* of the stability operator for a deformed Wigner matrix (see, e.g., [22, Proposition 3.5]).

In order to conclude (8.2.6), it remains to show that the error term $\mathcal{O}(t^3)$ in (8.3.3) is actually improvable to $\mathcal{O}(\langle D^2\rangle t^3)$. To see this, we (formally)[9] employ the Baker-Campbell-Hausdorff (BCH) formula, to write the exponentials as

$$\begin{aligned} \mathrm{e}^{\mathrm{i}tH_1}\mathrm{e}^{-\mathrm{i}tH_2} &= \mathrm{e}^{K} \quad \text{with} \\ K &= \mathrm{i}t(H_1-H_2)+\frac{t^2}{2}[H_1,H_2]+\frac{\mathrm{i}t^3}{12}\big([H_1,[H_1,H_2]]-[H_2,[H_1,H_2]]\big) \\ &\quad -\frac{t^4}{24}[H_2,[H_1,[H_1,H_2]]]+\dots \end{aligned} \tag{8.3.4}$$

and note that every summand in the expression for K in (8.3.4) can be written as a linear combination of nested commutators of D with $H \equiv H_1$ with one D always being in the innermost commutator. Hence, to conclude the desired, we need to show that, (i) all the terms in e^K containing only a single D vanish, when evaluated in $\langle \widetilde{P}\dots\rangle$, and (ii) all the terms in e^K containing at least two D's lead to an additional $\langle D^2\rangle$-factor in the error term.

For (i), note that the only way to have just a single D in a nested commutator is precisely $\mathrm{ad}_H^n(D)$ with $\mathrm{ad}_H(D) \coloneqq [H,D]$. Evaluated in $\langle \widetilde{P}\dots\rangle$, this vanishes, $\langle \widetilde{P}\mathrm{ad}_H^n(D)\rangle = 0$, since $[\widetilde{P},H]=0$ and hence

$$\langle \widetilde{P}\mathrm{ad}_H^n(D)\rangle = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\langle \widetilde{P}H^{n-k}DH^k\rangle = \langle \widetilde{P}H^nD\rangle\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0\,. \tag{8.3.5}$$

For (ii), we take a product, say, T, of H's and at least two D's, resulting from resolving (a product of) nested commutators, and estimate

$$\big|\langle \widetilde{P}T\rangle\big| \lesssim \langle \widetilde{P}D^2\rangle = \langle PD^2\rangle + \mathcal{O}_{\prec}\big((N\eta_0)^{-1}\big) \lesssim \langle D^2\rangle + \mathcal{O}_{\prec}\big((N\eta_0)^{-1}\big)\,. \tag{8.3.6}$$

In the first step, we estimated all H's and all but two D's in T by their operator norm, additionally using that $\widetilde{P}\geq 0$ and $[H,\widetilde{P}]=0$. In the second step, we employed the single resolvent law (8.2.4), while in the last step we used $\|P\|\lesssim 1$.

We have hence shown, that all the terms of e^K in (8.3.4) carrying at least a third power of t, can in fact be bounded with an additional $\langle D^2\rangle$-factor compared to (8.3.3). This concludes the proof. □

8.4 Asymptotic decay in Scenario I: Proof of Theorem 8.2.4 (ii)

The principal goal of this section is to prove (8.2.7) in Theorem 8.2.4 (ii), i.e. study the behavior of $\overline{\mathfrak{M}}(t)$ defined in (8.2.5) for times $1\leq t\lesssim \Delta^{-2}$. In order to do so, we compute the random quantity $\big\langle \mathrm{e}^{\mathrm{i}tH_1}\mathrm{Im}\,G_1(E_0+\mathrm{i}\eta_0)\mathrm{e}^{-\mathrm{i}tH_2}\big\rangle$ by residue calculus as

$$\begin{aligned} \big\langle \mathrm{e}^{\mathrm{i}tH_1}\mathrm{Im}\,G_1(E_0+\mathrm{i}\eta_0)\mathrm{e}^{-\mathrm{i}tH_2}\big\rangle &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2\oint_{\gamma_1}\oint_{\gamma_2}\mathrm{e}^{\mathrm{i}t(z_1-z_2)}\frac{\eta_0}{(z_1-E_0)^2+\eta_0^2}\,\langle G_1(z_1)G_2(z_2)\rangle\,\mathrm{d}z_1\mathrm{d}z_2 \\ &\quad +\frac{1}{4\pi}\oint_{\gamma_2}\mathrm{e}^{\mathrm{i}t(E_0+i\eta_0-z_2)}\,\langle G_1(E_0+\mathrm{i}\eta_0)G_2(z_2)\rangle\,\mathrm{d}z_2. \end{aligned} \tag{8.4.1}$$

Here, the contours γ_1,γ_2 are chosen to be two semicircles as indicated in Figure 8.4.1. More precisely, we take a (large) constant $R>0$ such that $\mathrm{supp}\rho_1$ and $\mathrm{supp}\rho_2$ are contained in $[-(R-1),R-1]$. The distance of the flat pieces from the real axis are denoted by $\eta_1 \coloneqq \min\{1/t,\eta_0/2\}$ and $0<\eta_2\lesssim 1/t$. The latter will explicitly be chosen later in Section 8.4.3, where we conclude the proof of Theorem 8.2.4 (ii).

[9] In order to guarantee convergence of the BCH expansion (8.3.4), we need the time t to be small enough such that $|t|(\|H_1\|+\|H_2\|)<\log 2$ [544], which can be achieved in an open interval around zero, since $\|D_i\|\lesssim 1$ and $\|W\|\leq 2+\epsilon$ with very high probability.

We decompose both contours into their flat in semicircular parts, $\gamma_j = \gamma_j^{(1)} + \gamma_j^{(2)}$, $j \in [2]$, and parametrize them as follows:

$$\gamma_1^{(1)} : \; z_1 = E_1 - \mathrm{i}\eta_1 \quad \text{with} \quad E_1 \in [-2R, 2R]\,, \qquad \gamma_1^{(2)} : \; z_1 = 2R\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{i}\eta_1 \quad \text{with} \quad \varphi \in [0, \pi] \tag{8.4.2}$$

$$\gamma_2^{(1)} : \; z_2 = E_2 + \mathrm{i}\eta_2 \quad \text{with} \quad E_2 \in [-R, R]\,, \qquad \gamma_2^{(2)} : \; z_2 = R\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{i}\eta_2 \quad \text{with} \quad \varphi \in [0, \pi] \tag{8.4.3}$$

Finally, we point out that, in order to (8.4.1) being valid, γ_1 is chosen in such a way that it encircles $E_0 + \mathrm{i}\eta_0$, but *not* $E_0 - \mathrm{i}\eta_0$.

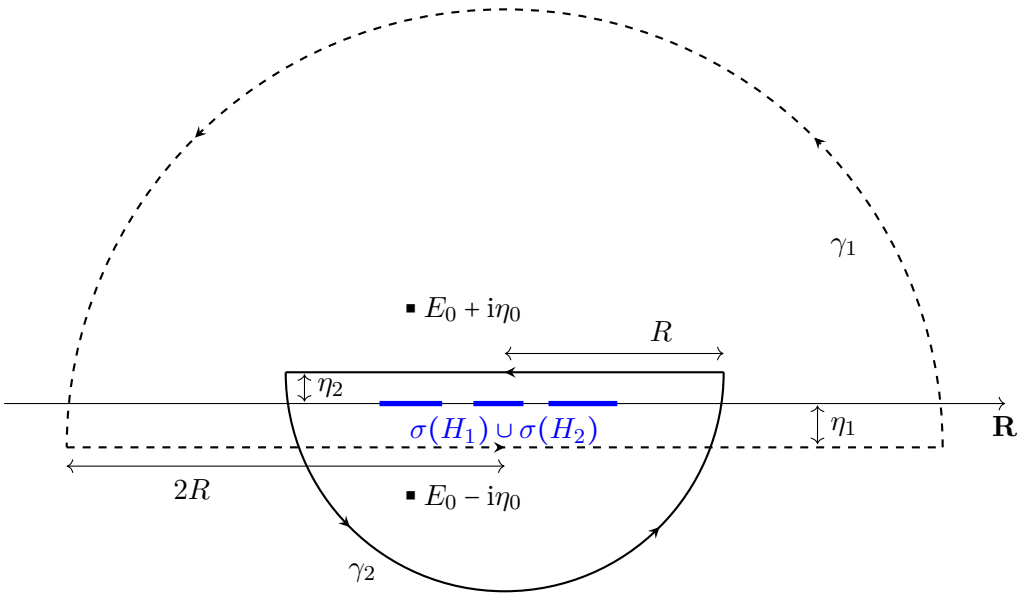

Figure 8.4.1: Sketch of the contours γ_1 (dashed) and γ_2 (full) from (8.4.2)–(8.4.3). The union of the spectra of H_1 and H_2 is indicated in blue.

The following argument leading towards the proof of Theorem 8.2.4 (ii) is split in three parts. First, in Section 8.4.1, we approximate the random contour integrals (8.4.1) by their deterministic counterparts by using an appropriate *two resolvent global law* for two different deformations (Proposition 8.4.1). Afterwards, in Section 8.4.2, we collect some preliminary stability bounds (Proposition 8.4.2) and information on the *shift*, which is the key parameter in our analysis of the Loschmidt echo; see Lemmas 8.4.4–8.4.6. Finally, in Section 8.4.3, we summarize the evaluation of the deterministic contour integrals from Section 8.4.1 in five Lemmas 8.4.7–8.4.11. Combining these with estimates on the shift from Section 8.4.2, we conclude the proof of Theorem 8.2.4 (ii) at the end of Section 8.4.3.

8.4.1 Step (i): Global law with two deformations

The following two resolvent global law will be used to approximate (8.4.1) by its deterministic counterpart.

Proposition 8.4.1 (Average two resolvent global law)**.** *Let $D_1, D_2 \in \mathbf{C}^{N\times N}$ be a bounded Hermitian matrices, i.e. $\|D_j\| \le L$ for some $L > 0$, and W a Wigner matrix satisfying Assumption 8.2.1. Moreover, let $z_1, z_2 \in \mathbf{C}$ be spectral parameters satisfying $\kappa := \min_{i\in[2]} \mathrm{dist}(z_i, [-(L+2), L+2]) \ge \delta > 0$ and denote $G_j(z_j) := (D_j + W - z_j)^{-1}$ for $j \in [2]$. Then it holds that*

$$|\langle G_1(z_1) G_2(z_2)\rangle - \langle M(z_1, z_2)\rangle| \prec \frac{C(\delta)}{N}\,, \tag{8.4.4}$$

where $C(\delta) > 0$ is a constant depending[10] only on its argument (apart from L and the constants from Assumption 8.2.1). In (8.4.4)*, we abbreviated*

$$M(z_1, z_2) = M_{12}(z_1, z_2) := \frac{M_1(z_1)M_2(z_2)}{1 - \langle M_1(z_1)M_2(z_2)\rangle} \tag{8.4.5}$$

and $M_j = M_j(z_j)$, for $j \in [2]$, is the unique solution to the Matrix Dyson equation (MDE)

$$-\frac{1}{M_j} = z_j - D_j + \langle M_j\rangle \quad \textit{with} \quad \operatorname{Im} M_j(z_j)\operatorname{Im} z_j > 0 \quad \textit{for} \quad z_j \in \mathbf{C} \setminus \mathbf{R}\,. \tag{8.4.6}$$

Proof. Using that $\|D_j + W\| \le L + 2 + \epsilon$, $j \in [2]$ with very high probability and the stability bound $|1 - \langle M_1(z_1)M_2(z_2)\rangle|^{-1} \lesssim 1$ for $\kappa := \min_{i\in[2]} \operatorname{dist}(z_i, [-(L+2), L+2]) \gtrsim 1$ from Proposition 8.4.2 below,[11] the proof works in the same way as [238, Proposition 3.1], [168, Appendix B], Section 1.5.2, or Section 2.6.2. We omit the details for brevity. □

Hence, by means of Proposition 8.4.1, we find that the random contour integral (8.4.1) can be approximated by the deterministic quantity

$$\begin{aligned} I_{E_0,\eta_0}(t) := \Big(\frac{1}{2\pi\mathrm{i}}\Big)^2 \oint_{\gamma_1}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1-z_2)} \frac{\eta_0}{(z_1-E_0)^2+\eta_0^2} \langle M(z_1,z_2)\rangle \,\mathrm{d}z_1\mathrm{d}z_2 \\ + \frac{1}{4\pi}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(E_0+i\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle \,\mathrm{d}z_2. \end{aligned} \tag{8.4.7}$$

up to an error of size $\mathcal{O}_\prec\big(C(\eta_1)C(\eta_2)/N\big)$, where we additionally used that the lengths of the contours are bounded, $\ell(\gamma_j) \lesssim 1$ for $j \in [2]$.

8.4.2 Step (ii): Preliminary bounds on the stability operator and the shift

As usual in random matrix theory, local/global laws are governed by a *stability operator*, which, in our case is given by

$$\mathcal{B}_{12}(z_1, z_2)[\cdot] := \mathbf{1} - M_1\langle\cdot\rangle M_2 \quad \text{with} \quad M_j \equiv M_j(z_j)\,. \tag{8.4.8}$$

One can easily see that $\mathcal{B}_{12}(z_1, z_2)$ has a highly degenerate eigenvalue one, and its only non-trivial eigenvalue is given by $1 - \langle M_1 M_2\rangle$ with corresponding eigen"vector" $M_1 M_2$.

The following proposition, whose proof is given in Section 8.5, states an upper bound on the inverse of this non-trivial eigenvalue. A simplified form this stability bound already appeared in [160, Lemma 5.2] for the very special case that $D_1 = \alpha D_2$ for some $\alpha \in \mathbf{R}$.

Proposition 8.4.2 (Stability bound)**.** *Fix a (large) $L > 0$. Uniformly in $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ and traceless Hermitian D_1, D_2 with $|z_j| \le L$, $\|D_j\| \le L$, $j = 1, 2$, it holds that*

$$\left|\frac{1}{1 - \langle M_1 M_2\rangle}\right| \lesssim \frac{1}{\Delta^2 + (\operatorname{Re} z_1 - \operatorname{Re} z_2)^2 + (\operatorname{Im}\langle M_1\rangle + \operatorname{Im}\langle M_2\rangle)^2 + \left|\frac{\operatorname{Im} z_1}{\langle \operatorname{Im} M_1\rangle}\right| + \left|\frac{\operatorname{Im} z_2}{\langle \operatorname{Im} M_2\rangle}\right|} \vee 1\,, \tag{8.4.9}$$

where we denoted $\Delta^2 := \langle (D_1 - D_2)^2\rangle$.

[10] By carefully tracking δ throughout the proof, one can see that the dependence is inverse polynomially, $C(\delta) \lesssim \delta^{-n}$ for some $n \in \mathbf{N}$. This will, however, be completely irrelevant for our purposes.

[11] In view of (8.4.10), note that the supports $\operatorname{supp}(\rho_1), \operatorname{supp}(\rho_2)$ of the scDos of H_1 and H_2 are contained in $[-(L+2), (L+2)]$.

In the current Section 8.4, more precisely, the proof of Proposition 8.4.1 above, only the special case

$$|1 - \langle M_1 M_2 \rangle|^{-1} \lesssim 1 \quad \text{for} \quad \max_{j \in [2]} \operatorname{dist}(z_j, \operatorname{supp}(\rho_j)) \gtrsim 1 \tag{8.4.10}$$

of Proposition 8.4.2 is relevant. However, for later reference, we also point out that, in particular, $|1 - \langle M_1 M_2 \rangle|^{-1} \lesssim |z_1 - z_2|^{-2}$ and that the lhs. of (8.4.9) is bounded by one, whenever z_1, z_2 are in the same half-plane and $\rho_1(z_1) + \rho_2(z_2) \gtrsim 1$ (e.g. if one of them is in the bulk, $\operatorname{Re} z_j \in \mathbf{B}_\kappa(\rho_j)$).

In addition to these bounds, Proposition 8.4.2 also plays an important role in the analysis of the *shift* $\mathfrak{s}(z_1, z_2)$ of the spectral parameters z_1, z_2 in the *(generalized) M-resolvent identity*

$$\langle M_{12} \rangle = \frac{\langle M_1 M_2 \rangle}{1 - \langle M_1 M_2 \rangle} = \frac{\langle M_1 \rangle - \langle M_2 \rangle}{z_1 - z_2 - \mathfrak{s}(z_1, z_2)}, \tag{8.4.11}$$

which can easily be obtained by subtracting MDEs (8.4.6) for M_2 and M_1 from each other. In (8.4.11), the shift is defined as follows.

Definition 8.4.3 (The *shift*)**.** *Let D_1, D_2 be Hermitian traceless matrices and let $M_j(z_j)$ for $j \in [2]$ be the solution of the MDE* (8.4.6)*. Then, we define the* shift *(depending on D_1, D_2 and $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$) as*

$$\mathfrak{s}(z_1, z_2) := \frac{\langle M_1(z_1)(D_1 - D_2)M_2(z_2) \rangle}{\langle M_1(z_1) M_2(z_2) \rangle}, \tag{8.4.12}$$

whenever the denominator does not vanish.

As already mentioned above, the shift $\mathfrak{s}$ is the key parameter in our analysis of the Loschmidt echo. We now collect several estimates on $\mathfrak{s}$ in the following Lemmas 8.4.4–8.4.6. The proofs, which are based on the stability bound in Proposition 8.4.2, are given in Section 8.5.

Lemma 8.4.4 (Properties of $\mathfrak{s}(z_1, z_2)$)**.** *Fix a (small) $\kappa > 0$ and a (large) $L > 0$. Consider spectral parameters $z_1, z_2 \in \mathbf{C} \setminus \mathbf{R}$ such that $\operatorname{Im} z_1 \operatorname{Im} z_2 < 0$ and $|z_j| \le L$, $\|D_j\| \le L$, for $j \in [2]$. Assume that at least one of these parameters is such that the (positive) harmonic extension of the scDos is positive, i.e. $\rho_1(z_1) + \rho_2(z_2) \ge \kappa$. Then there exists a positive constant $\mathfrak{c}$ which depends only on κ, L such that for any Hermitian traceless D_1, D_2 with $\Delta := \langle (D_1 - D_2)^2 \rangle^{1/2} \le \mathfrak{c}$ we have the following:*

1. *The denominator of the shift* (8.4.12) *is of order one, $|\langle M_1(z_1) M_2(z_2) \rangle| \sim 1$. In particular,*

$$|\mathfrak{s}(z_1, z_2)| \lesssim \Delta. \tag{8.4.13}$$

2. *If $\rho_j(z_j) \ge \kappa/2$, then*

$$|\partial_{z_j} \mathfrak{s}(z_1, z_2)| \lesssim \Delta. \tag{8.4.14}$$

Here all implicit constants depend only on κ and L.

We now introduce an auxiliary function f, which exactly detects the influence of the shift on the real part of a spectral parameter.

Lemma 8.4.5 (Definition of f and $\mathfrak{s}_0$)**.** *Fix a (small) $\kappa > 0$ and a (large) $L > 0$. Consider $0 < \eta_1, \eta_2 < L$ and a spectral parameter $z_2 = E_2 + \mathrm{i}\eta_2$ such that $\rho_2(z_2) \ge \kappa$, and satisfying $|z_2| \le L$. Let D_1, D_2 be Hermitian traceless matrices with $\|D_j\| \le L$, $j \in [2]$. Assume that $\Delta := \langle (D_1 - D_2)^2 \rangle^{1/2} \le \mathfrak{c}$, where $\mathfrak{c}$ is the constant from Lemma 8.4.4.*

Then there exists a unique energy renormalization *$f^{\eta_1, \eta_2}(E_2) = f(E_2) \in \mathbb{R}$ with $|f(E_2)| \le L$ such that*

$$\operatorname{Re}\left(f(E_2) - E_2 - \mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2) \right) = 0.$$

Moreover, denoting the renormalized (one point) shift *by*

$$\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) := \mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2), \tag{8.4.15}$$

the functions $f^{\eta_1,\eta_2}(E_2)$ *and* $\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)$ *are differentiable in* η_1, η_2 *and for* $E_2 \in \mathbf{B}_\kappa(\rho_2)$ *in the bulk, and the derivatives satisfy*

$$|\partial_{E_2} f^{\eta_1,\eta_2}(E_2) - 1| \lesssim \Delta, \quad \left|\partial_{\eta_j} f^{\eta_1,\eta_2}(E_2)\right| \lesssim \Delta,\ j \in [2], \quad \textit{and} \quad |\partial_{E_2}\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)| \lesssim \Delta. \tag{8.4.16}$$

Whenever it does not lead to confusion our ambiguities, we will omit the superscripts η_1, η_2 of f^{η_1,η_2} and $\mathfrak{s}_0^{\eta_1,\eta_2}$. Next, we show that the *imaginary part* of the renormalized shift is in fact much smaller than indicated by the upper bounds of order Δ in (8.4.13)–(8.4.14) and (8.4.16).

Lemma 8.4.6 (Behavior of $\operatorname{Im}\mathfrak{s}_0$)**.** *Fix a (small)* $\kappa > 0$ *and a (large)* $L > 0$*. Let* $E \in \mathbf{B}_\kappa(\rho_2)$ *be in the bulk of* ρ_2*. Then there exist positive constants* $c_1, c_2 > 0$ *such that for any Hermitian traceless* D_1, D_2 *with* $\|D_j\| \le L$*,* $j = 1, 2$*,* $\Delta < c_1$ *and for any* $0 < \eta_j \le c_2\Delta$*, for* $j \in [2]$*, it holds that*

$$\operatorname{Im}\mathfrak{s}_0^{\eta_1,\eta_2}(E) \sim \Delta^2. \tag{8.4.17}$$

Here, c_1, c_2 *and the implicit constants in* (8.4.17) *depend only on* κ *and* L*.*

In the following section, armed with the preliminary bounds from Proposition 8.4.2 and Lemmas 8.4.4–8.4.6, we carry out the evaluation of the contour integrals in (8.4.7).

8.4.3 Step (iii): Contour integration of the deterministic approximation

Throughout this section, let $[a, b]$ be an interval with length of order one satisfying $\operatorname{dist}(E_0, [a, b]^c) \gtrsim 1$ and $\operatorname{dist}([a, b], (\operatorname{supp}(\rho_1) \cap \operatorname{supp}(\rho_2))^c) \gtrsim 1$. That is, the energy E_0 from Theorem 8.2.4 is order one away from the boundary of $[a, b]$ and $[a, b]$ is simultaneously in the bulk of ρ_1 and ρ_2. The existence of such an interval is always guaranteed.

As already mentioned above, we now dissect the evaluation of (8.4.7) in several parts. As the first step, we show that the second line of (8.4.7) is in fact negligible. The proofs of Lemma 8.4.7 and all the other Lemmas 8.4.8–8.4.11 is given in Section 8.6.

Lemma 8.4.7 (The second line is negligible)**.** *Under the assumptions of Theorem 8.2.4 (ii) it holds that*

$$I_{E_0}^{(2)} := \frac{1}{4\pi}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-z_2)}\,\langle M(E_0 + i\eta_0, z_2)\rangle\,\mathrm{d}z_2 = \mathcal{O}\left(\frac{1}{t}\right).$$

For the remaining first line of (8.4.7), we then find that the main contribution of the γ_2 integral comes from the interval $[a, b] + \mathrm{i}\eta_2$, i.e. we can cut away the tails.

Lemma 8.4.8 (Cutting tails)**.** *Under the assumptions of Theorem 8.2.4 (ii) it holds that*

$$\begin{aligned} I_{E_0}^{(1)} &:= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1}\oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(z_1-z_2)}\frac{\eta_0}{(z_1-E_0)^2+\eta_0^2}\langle M(z_1,z_2)\rangle \mathrm{d}z_1\mathrm{d}z_2 \\ &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \oint_{\gamma_1}\int_a^b \mathrm{e}^{\mathrm{i}t(z_1-E_2-i\eta_2)}\frac{\eta_0}{(z_1-E_0)^2+\eta_0^2}\langle M(z_1,E_2+\mathrm{i}\eta_2)\rangle \mathrm{d}z_1\mathrm{d}E_2 + \mathcal{O}\left(\frac{1}{t}+\frac{\eta_0}{\Delta}\right). \end{aligned}$$

The following lemma formally implements inside the integral from Lemma 8.4.8 the approximation

$$\langle M(z_1, E_2 + \mathrm{i}\eta_2)\rangle = \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{z_1 - (E_2+\mathrm{i}\eta_2) - \mathfrak{s}(z_1, E_2+\mathrm{i}\eta_2)} \approx \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{z_1 - (E_2+\mathrm{i}\eta_2) - \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)},$$

which is valid in the main contributing regime $E_1 \approx E_2$. This is our *first replacement* $\mathfrak{s}(z_1, E_2+\mathrm{i}\eta_2) \to \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)$.

Lemma 8.4.9 (First replacement). *Denote* $\mathfrak{d} := \min_{E_2\in[a,b]} |\eta_1 + \eta_2 + \operatorname{Im} \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|$. *Then, under the assumptions of Theorem 8.2.4 (ii), it holds that*

$$\begin{aligned}
&\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \oint_{\gamma_1} \mathrm{d}z_1 \int_a^b \mathrm{d}E_2\, \mathrm{e}^{\mathrm{i}t(z_1-E_2-\mathrm{i}\eta_2)} \frac{\eta_0}{(z_1-E_0)^2+\eta_0^2} \langle M(z_1, E_2+\mathrm{i}\eta_2)\rangle \\
=&\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \oint_{\gamma_1} \mathrm{d}z_1 \int_a^b \mathrm{d}E_2\, \mathrm{e}^{\mathrm{i}t(z_1-E_2-\mathrm{i}\eta_2)} \frac{\eta_0}{(z_1-E_0)^2+\eta_0^2} \cdot \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{z_1-(E_2+\mathrm{i}\eta_2)-\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)} \\
&\quad + \mathcal{O}\left(\eta_0 + \Delta|\log\Delta| + \Delta|\log\mathfrak{d}|\right).
\end{aligned} \tag{8.4.18}$$

Next, plugging in the Stieltjes representation $\langle M_1(z_1)\rangle = \int_{\mathbf{R}} \rho_1(x)(x-z_1)^{-1}\mathrm{d}x$, the γ_1 integral in Lemma 8.4.9 can be explicitly computed using residue calculus. The "unwanted" residue contributions arising in this way can be estimated using the oscillatory factor and integration by parts (see the proof of Lemma 8.4.10 in Section 8.6).

Lemma 8.4.10 (Residue computation after the first replacement). *Denote* $\mathfrak{a} := \min_{E_2\in[a,b]} |\eta_0 - \eta_2 - \operatorname{Im} \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|$ *and suppose that*

$$\eta_1 + \eta_2 + \operatorname{Im} \mathfrak{s}_0(E_2) > 0, \quad \forall E_2 \in [a,b]. \tag{8.4.19}$$

Then, again under the assumptions of Theorem 8.2.4 (ii), it holds that

$$\begin{aligned}
&\left(\frac{1}{2\pi \mathrm{i}}\right)^2 \oint_{\gamma_1} \mathrm{d}z_1 \int_a^b \mathrm{d}E_2 \mathrm{e}^{\mathrm{i}t(z_1-E_2-i\eta_2)} \frac{\eta_0}{(z_1-E_0)^2+\eta_0^2} \cdot \frac{\langle M_1(z_1)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{z_1-(E_2+\mathrm{i}\eta_2)-\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)} \\
=&-\frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} \mathrm{d}x \int_a^b \mathrm{d}E_2 \mathrm{e}^{\mathrm{i}t(x-E_2-\mathrm{i}\eta_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)}{x-(E_2+\mathrm{i}\eta_2)-\mathfrak{s}_0^{\eta_1,\eta_2}(E_2)} \\
&\quad + \mathcal{O}\left(\frac{|\log\mathfrak{a}|}{t} + \frac{\Delta+t^{-1}}{t\mathfrak{a}} + \eta_0|\log\mathfrak{a}| + \frac{\eta_0(\Delta+t^{-1})}{\mathfrak{a}}\right).
\end{aligned} \tag{8.4.20}$$

In the following lemma, we (i) complete the integral $\int_a^b$ to a full contour integral $\oint_{\gamma_2}$, i.e. put back the tails that were cut away in Lemma 8.4.8, and (ii) implement the *second replacement*

$$\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) \to \mathfrak{s}_0 := \mathfrak{s}_0^{\eta_1,\eta_2}\left(\left(f^{\eta_1,\eta_2}\right)^{-1}(E_0)\right) \tag{8.4.21}$$

inside the integral from Lemma 8.4.10. This replacement leads to a small error comparing to the leading term since $\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) \approx \mathfrak{s}_0$ in the relevant regime $E_2 \approx E_0$.

Lemma 8.4.11 (Second replacement). *Let* $\mathfrak{b} := \min_{E_2\in[a,b]} |\eta_2 + \operatorname{Im} \mathfrak{s}_0^{\eta_1,\eta_2}(E_2)|$ *and* $\mathfrak{s}_0$ *as in* (8.4.21). *Then, again under the assumptions of Theorem 8.2.4 (ii), it holds that*

$$\begin{aligned}
&-\frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} \mathrm{d}x \int_a^b \mathrm{d}E_2\, \mathrm{e}^{\mathrm{i}t(x-E_2-\mathrm{i}\eta_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)}{x-(E_2+i\eta_2+\mathfrak{s}_0^{\eta_1,\eta_2}(E_2))} \\
=&-\frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} \mathrm{d}x \oint_{\gamma_2} \mathrm{d}z_2 \mathrm{e}^{\mathrm{i}t(x-z_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)}{x-(z_2+\mathfrak{s}_0)} \\
&\quad + \mathcal{O}\left(\frac{\eta_0+\mathfrak{b}}{\mathfrak{b}}\Delta|\log(\eta_0+\mathfrak{b})| + \eta_0|\log\mathfrak{b}| + \frac{1}{t}\right).
\end{aligned} \tag{8.4.22}$$

Armed with Lemmas 8.4.7–8.4.11, we can finally give the proof of Theorem 8.2.4 (ii).

Proof of Theorem 8.2.4 (ii). Combining Lemmas 8.4.7 - 8.4.11 we find that

$$I_{E_0,\eta_0}(t) = -\frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} \oint_{\gamma_2} \mathrm{e}^{\mathrm{i}t(x-z_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)\mathrm{d}x}{x-(z_2+\mathfrak{s}_0)} \mathrm{d}z_2 + \mathcal{O}\big(\widehat{\mathcal{E}}(t)\big), \tag{8.4.23}$$

where we collected all the error terms in

$$\widehat{\mathcal{E}}(t) := \frac{\eta_0}{\Delta} + \Delta|\log\Delta| + \Delta|\log\mathfrak{d}| + \frac{|\log\mathfrak{a}|}{t} + \frac{\Delta + t^{-1}}{t\mathfrak{a}} + \eta_0|\log\mathfrak{a}| + \frac{\eta_0(\Delta + t^{-1})}{\mathfrak{a}} + \frac{\eta_0 + \mathfrak{b}}{\mathfrak{b}}\Delta|\log(\eta_0+\mathfrak{b})| + \eta_0|\log\mathfrak{b}|.$$

We shall now estimate $\widehat{\mathcal{E}}(t)$ in different time regimes. First note that Lemmas 8.4.4 and 8.4.6 imply the existence of positive constants $\{c_j\}_{j=1}^4$ such that

$$\begin{aligned} |\mathfrak{s}(z_1,z_2)| \le c_1\Delta, \quad &\text{for all} \quad |z_1| \le 2R,\, E_2 \in [a,b],\, \eta_2 \in [0,1], \quad \text{and} \\ c_2\Delta^2 \le \operatorname{Im}\mathfrak{s}_0^{\eta_1,\eta_2}(E_2) \le c_3\Delta^2, \quad &\text{for all} \quad E_2 \in [a,b],\, \eta_j \in [0, c_4\Delta], j = 1,2. \end{aligned} \tag{8.4.24}$$

First regime: For $1 \le t \le 4Kc_3/(c_4\Delta)$ we take $\eta_2 := 8Kc_1c_3/(c_4t)$. Then, for any $E_2 \in [a,b]$ it holds that

$$\eta_2 + \operatorname{Im}\mathfrak{s}_0(E_2) \ge 8Kc_1c_3/(c_1t) - c_1\Delta \ge 4Kc_1c_3/(c_1t) > 0.$$

In particular, the parameters $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{d}$ from Lemmas 8.4.10, 8.4.11, and 8.4.9, respectively, are all of order $1/t$ and $\widehat{\mathcal{E}}(t)$ is bounded as

$$\widehat{\mathcal{E}}(t) \lesssim \frac{1+\log t}{t} + \frac{\eta_0}{\Delta} + \Delta|\log\Delta| + \Delta\log t, \quad \text{for} \quad 1 \le t \le \frac{4Kc_3}{c_4}\cdot\frac{1}{\Delta}. \tag{8.4.25}$$

Second regime: For $4Kc_3/(c_4\Delta) \le t \le 2Kc_3/\eta_0$, we take $\eta_2 := \frac{4Kc_3}{t}$. In this regime, $\eta_2 \le c_4\Delta$, so the positivity of $\eta_2 + \operatorname{Im}\mathfrak{s}_0(E_2)$ follows from (8.4.24). We also have that $\eta_2 \ge 2\eta_0$ and again $\mathfrak{a} \sim \mathfrak{b} \sim \mathfrak{d} \sim 1/t$. Therefore, (8.4.25) holds in the whole regime $1 \le t \le 2Kc_3/\eta_0$.

Third regime: It remains to study the regime $2Kc_3/\eta_0 \le t \le K/\Delta^2$. If $\eta_0 \le 2c_3\Delta^2$ it is in fact empty, hence we may assume $\eta_0 \ge 2c_3\Delta^2$. In this case, we take $\eta_2 := \min\{\eta_0/4, c_4\Delta, 1/t\}$ and find that $\mathfrak{a} \sim \eta_0$, $\mathfrak{b} \gtrsim \Delta^2$, $\mathfrak{d} \gtrsim \Delta^2$. Moreover, the error term $\widehat{\mathcal{E}}(t)$ is bounded as

$$\widehat{\mathcal{E}}(t) \lesssim \frac{1+\log t}{t} + \Delta|\log\Delta| + \frac{\eta_0|\log\Delta|}{\Delta}, \quad \text{for} \quad 2Kc_3/\eta_0 \le t \le K/\Delta^2.$$

After having chosen η_2 in all time regimes explicitly, we can perform z_2-integration in (8.4.23). Note that in all time regimes η_2 was chosen in such a way that $\eta_2 + \operatorname{Im}\mathfrak{s}_0 > 0$, which guarantees that γ_2 encircles the point $x - \mathfrak{s}_0$ for $x \in \operatorname{supp}(\rho_1)$. So, (8.4.23) evaluates to

$$I_{E_0,\eta_0}(t) = \mathrm{e}^{\mathrm{i}t\mathfrak{s}_0}\int_{\mathbf{R}} \frac{\eta_0}{(x-E_0)^2+\eta_0^2}\rho_1(x)\mathrm{d}x + \mathcal{O}\big(\widehat{\mathcal{E}}(t)\big) = \mathrm{e}^{\mathrm{i}t\mathfrak{s}_0}\operatorname{Im}\langle M_1(E_0+\mathrm{i}\eta_0)\rangle + \mathcal{O}\big(\widehat{\mathcal{E}}(t)\big). \tag{8.4.26}$$

After dividing by $\operatorname{Im}\langle M_1(E_0+\mathrm{i}\eta_0)\rangle$ and taking the absolute value square, it is left to notice that, setting

$$\Gamma := 2\operatorname{Im}\mathfrak{s}_0^{0,0}\left(\left(f^{0,0}\right)^{-1}(E_0)\right), \tag{8.4.27}$$

it holds that

$$\operatorname{Im}\mathfrak{s}_0 = \operatorname{Im}\mathfrak{s}_0^{\eta_1,\eta_2}\left(\left(f^{\eta_1,\eta_2}\right)^{-1}(E_0)\right) = \Gamma/2 + \mathcal{O}(\Delta(\eta_1+\eta_2)) = \Gamma/2 + \mathcal{O}(\Delta/t).$$

Here we used (8.4.16) from Lemma 8.4.5 and (8.4.14) from Lemma 8.4.4 together with the bound $\eta_j \lesssim 1/t$, $j = 1,2$. By Lemma 8.4.6, we finally see that the implicit constants in $\Gamma \sim \Delta^2$ only depend on κ and L. This finishes the proof of Theorem 8.2.4 (ii). □

8.5 Stability operator and shift: Proofs for Section 8.4.2

8.5.1 Bound on the stability operator: Proof of Proposition 8.4.2

Throughout the proof, we will use the shorthand notations $E_j := \operatorname{Re} z_j$, $\eta_j := |\operatorname{Im} z_j|$, $\rho_j := \frac{1}{\pi}|\langle \operatorname{Im} M_j(z_j)\rangle|$ and $\omega_j := z_j + \langle M_j(z_j)\rangle$, for $j \in [2]$.

We will conclude Proposition 8.4.2 from the following lemma.

Lemma 8.5.1. *Under the assumptions of Proposition 8.4.2 and using the notations from above, we have that:*

$$|1 - \langle M_1 M_2\rangle|^{-1} \lesssim (\eta_1/\rho_1 + \eta_1/\rho_2)^{-1} \vee 1. \tag{8.5.1}$$

$$|1 - \langle M_1 M_2\rangle|^{-1} \lesssim (\Delta^2 + |\omega_1 - \overline{\omega}_2|^2)^{-1}. \tag{8.5.2}$$

$$|1 - \langle M_1 M_2\rangle|^{-1} \lesssim |z_1 - z_2|^{-2} \tag{8.5.3}$$

Combining (8.5.1)–(8.5.3) with the simple observation $|\omega_1 - \overline{\omega}_2| \geq |\langle \operatorname{Im} M_1\rangle + \langle \operatorname{Im} M_2\rangle|$, we conclude (8.4.9), i.e. the proof of Proposition 8.4.2. □

Proof of Lemma 8.5.1. For (8.5.1), it is sufficient to check that for some $c \in (0,1)$ we have $|\langle M_1 M_2\rangle| \leq (1 - c(\eta_1/\rho_1 + \eta_2/\rho_2)) \vee (1-c)$. This follows from a simple Cauchy-Schwarz inequality $|\langle M_1 M_2\rangle| \leq \langle |M_1|^2\rangle^{1/2}\langle |M_2|^2\rangle^{1/2}$ together with the estimate

$$\langle |M_j|^2\rangle^{1/2} = \left(\frac{\langle \operatorname{Im} M_j\rangle}{\operatorname{Im} z_j + \langle \operatorname{Im} M_j\rangle}\right)^{1/2} \lesssim \left(\frac{\rho_j}{\eta_j + \rho_j}\right)^{1/2} \leq \left(1 - \frac{1}{2}\cdot\frac{\eta_j}{\rho_j}\right) \vee (1-c), \quad j \in [2]$$

where the first step follows by taking the imaginary part of the MDE (8.4.6).

For (8.5.2), we note that it is sufficient to show

$$\operatorname{Re}\langle M_1 M_2\rangle \leq 1 - c\left(\left\langle (D_1 - D_2)^2\right\rangle + |\omega_1 - \bar{\omega}_2|^2\right) \quad \text{for some} \quad c > 0\,. \tag{8.5.4}$$

The idea for proving (8.5.4) is to translate it to a question for the spectral measures of D_1 and D_2.

In order to do so, for $j \in [2]$, denote the eigenvalues and eigenvectors of D_j by $\{\lambda_k^{(j)}\}_{k=1}^N$ and $\{\boldsymbol{u}_k^{(j)}\}_{k=1}^N$, respectively, and the normalized spectral measure by $\mu_j := N^{-1}\sum_{k=1}^N \delta_{\lambda_k^{(j)}}$. By the MDE (8.4.6), we immediately see that ω_j solves the equation $\omega_j - z_j = m_{\mu_j}(\omega_j)$, where $m_\mu(z) := \int_{\mathbf{R}} \mathrm{d}\mu(x)(x-z)^{-1}$ is the Stieltjes transform of the probability measure μ. By taking the imaginary part and estimating $|\operatorname{Im}\omega_j| > |\operatorname{Im}\omega_j - \operatorname{Im} z_j|$ we hence find

$$\int \frac{\mathrm{d}\mu_j(x)}{|x - \omega_j|^2} < 1\,. \tag{8.5.5}$$

Using the above notations, we further see that M_j can be written as $M_j = \sum_{k=1}^N (\lambda_k^{(j)} - \omega_j)^{-1} |\boldsymbol{u}_k^{(j)}\rangle\langle \boldsymbol{u}_k^{(j)}|$ and thus

$$\langle M_1 M_2\rangle = \frac{1}{N^2}\sum_{a,b=1}^N \frac{1}{\lambda_a^{(1)} - \omega_1}\cdot\frac{1}{\lambda_b^{(2)} - \omega_2} f\big(\lambda_a^{(1)}, \lambda_b^{(2)}\big), \quad \text{with} \quad f\big(\lambda_a^{(1)}, \lambda_b^{(2)}\big) := N\big|\big\langle \boldsymbol{u}_a^{(1)}, \boldsymbol{u}_b^{(2)}\big\rangle\big|^2\,.$$

Extending $f(x,y)$ to $\mathbf{R}^2$ by zero, we immediately see the following properties of f:

1. $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.
2. $\int f(x,y)\mathrm{d}\mu_2(y) = \mathbf{1}_{\operatorname{supp}\mu_1}(x)$ and $\int f(x,y)\mathrm{d}\mu_1(x) = \mathbf{1}_{\operatorname{supp}\mu_2}(y)$.

3. On $\mathbf{R}^2$, $\mathrm{d}\nu(x,y) \coloneqq f(x,y)\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y)$ is a probability measure with marginals μ_1 and μ_2.

In this way, the desired inequality (8.5.4) can equivalently be rewritten as

$$\mathrm{Re}\iint \frac{1}{x-\omega_1}\cdot\frac{1}{y-\omega_2}\mathrm{d}\nu(x,y) \le 1 - c\left(\iint (x-y)^2\mathrm{d}\nu(x,y) + |\omega_1-\bar\omega_2|^2\right). \tag{8.5.6}$$

In this form, using (8.5.5), we begin by estimating the lhs. of (8.5.6) as

$$\mathrm{Re}\iint \frac{1}{x-\omega_1}\cdot\frac{1}{y-\omega_2}\mathrm{d}\nu(x,y) < 1 - \frac{1}{2}\iint \left|\frac{1}{x-\omega_1} - \frac{1}{y-\bar\omega_2}\right|^2 \mathrm{d}\nu(x,y),$$

Thus, in order to arrive at (8.5.6), it suffices to bound

$$\begin{aligned}\iint \left|\frac{1}{x-\omega_1} - \frac{1}{y-\bar\omega_2}\right|^2 \mathrm{d}\nu(x,y) &\gtrsim \iint |(x-y)-(\omega_1-\bar\omega_2)|^2\,\mathrm{d}\nu(x,y)\\ &= \iint (x-y)^2\mathrm{d}\nu(x,y) - 2\mathrm{Re}\,(\omega_1-\bar\omega_2)\iint (x-y)\mathrm{d}\nu(x,y) + |\omega_1-\bar\omega_2|^2\\ &= \iint (x-y)^2\mathrm{d}\nu(x,y) + |\omega_1-\bar\omega_2|^2.\end{aligned}$$

where in the first step we employed $|x-\omega_1| \lesssim \|D_1\| + |\omega_1| \lesssim 1$ (and analogously for $|y-\bar\omega_2|$), while in the last step we used that fact that D_1 and D_2 are traceless. This finishes the proof of (8.5.2).

Finally, for (8.5.3), we use (8.5.2) and (8.4.11) to get that

$$\begin{aligned}|z_1-z_2|^2 &= |\langle M_2(D_1-D_2)M_1\rangle + (1-\langle M_1M_2\rangle)(z_1-z_2+\langle M_1\rangle - \langle M_2\rangle)|^2\\ &\lesssim |\langle M_1(D_1-D_2)M_2\rangle|^2 + |1-\langle M_1M_2\rangle|\\ &\lesssim \left\langle (D_1-D_2)^2\right\rangle + |1-\langle M_1M_2\rangle| \lesssim |1-\langle M_1M_2\rangle|.\end{aligned}$$

□

8.5.2 Properties of the shift: Proof of Lemmas 8.4.4–8.4.6

We finally prove the properties of the shift from Lemmas 8.4.4–8.4.6.

Proof of Lemma 8.4.4. The proof is split in two parts in the statement of the lemma.

Part (1): Given $|\langle M_1M_2\rangle| \sim 1$, note that the bound (8.4.13) immediately follows since, if, say, z_1 is such that $\rho_1(z_1) \ge \kappa/2$, then $\|M_1\| \lesssim 1$ and $\langle |M_2|^2\rangle^{1/2} \le 1$. Both of these estimates easily follow by taking the imaginary part of the respective MDEs (8.4.6).

It is hence left to prove $|\langle M_1M_2\rangle| \sim 1$. The upper bound $|\langle M_1M_2\rangle| \le 1$ is a consequence of the Cauchy-Schwarz inequality and $\langle |M_j|^2\rangle^{1/2} \le 1$. In order to prove the lower bound, we may assume w.l.o.g. that $|\langle M_1M_2\rangle| \le 1/2$, in which case $|1-\langle M_1M_2\rangle| \sim 1$. Now, the numerator in the rhs. of the M-resolvent identity (8.4.11) is of order one, since $1 \gtrsim |\langle M_1\rangle - \langle M_2\rangle| \gtrsim |\langle \mathrm{Im}\,M_1\rangle| + |\langle \mathrm{Im}\,M_2\rangle| \gtrsim 1$. Thus, by (8.4.11) again, we find that $|(z_1-z_2)\langle M_1M_2\rangle - \langle M_1(D_1-D_2)M_2\rangle| \sim 1$, so, in particular,

$$1 \lesssim |(z_1-z_2)\langle M_1M_2\rangle - \langle M_1(D_1-D_2)M_2\rangle| \lesssim |\langle M_1M_2\rangle| + \Delta.$$

Therefore, for some constant $c>0$ which depends only on L and κ we have

$$|\langle M_1M_2\rangle| \ge c - \Delta \gtrsim 1,$$

i.e. we get the desired lower bound for $|\langle M_1M_2\rangle|$.

Part (2): Assume w.l.o.g. that $\rho_1(z_1) \ge \kappa/2$. The derivative $\partial_{z_1}\mathfrak{s}(z_1,z_2)$ can be computed explicitly as

$$\partial_{z_1}\mathfrak{s}(z_1,z_2) = \frac{\langle M_1^2(D_1-D_2)M_2\rangle\langle M_1M_2\rangle - \langle M_1(D_1-D_2)M_2\rangle\langle M_1^2M_2\rangle}{\langle M_1M_2\rangle^2(1-\langle M_1^2\rangle)}$$

and we note that, by analogous reasoning as in part (1), the numerator is bounded from above by Δ. Since $|\langle M_1 M_2\rangle| \sim 1$, from part (1), it holds that

$$|\partial_{z_1}\mathfrak{s}(z_1,z_2)| \lesssim \frac{\Delta}{|1-\langle M_1^2\rangle|} \lesssim \Delta,$$

where in the last step we used the bound $|1-\langle M_1^2\rangle| \gtrsim \rho_1(z_1)^2$ with the aid of Proposition 8.4.2. □

Proof of Lemma 8.4.5. The argument is split in two parts: First, we prove existence and uniqueness of the energy renormalization function f. Second, we estimate the partial derivatives (8.4.16) of f and the renormalized (one point) shift $\mathfrak{s}_0$.

Part (1): Existence and uniqueness of f. First, from Lemma 8.4.5, we have that, for z_1 with $|z_1| \le L$ and $\operatorname{Im} z_1 < 0$, it holds that $|\mathfrak{s}(z_1,z_2)| \le C\Delta$ for some $C > 0$. For fixed $z_2 = E_2 + \mathrm{i}\eta_2$, we introduce the auxiliary (differentiable) function

$$h(E_1) \coloneqq E_1 - E_2 - \operatorname{Re}\mathfrak{s}(E_1 - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2)\,,$$

which has the property that $h(E_1) < 0$ for $E_1 < E_2 - C\Delta$, and $h(E_2) > 0$ for $E_1 > E_2 + C\Delta$. Hence, $h(E_1) = 0$ has a solution in $\mathcal{I} \coloneqq [E_2 - C\Delta, E_2 + C\Delta]$. To see uniqueness, we differentiate h and find that $h'(E_1) \ge 1 - c\Delta$ for $E_1 \in \mathcal{I}$ and some $c > 0$ by means of (8.4.14) from Lemma 8.4.4. Thus h has a unique zero on $\mathcal{I}$ (and hence in $(-L, L)$) which we denote by $f(E_2) = f^{\eta_1,\eta_2}(E_2)$ – the desired energy renormalization function. Differentiability of f easily follows from the implicit function theorem.

Part (2): Bounds on derivatives. Differentiating the identity $h(f^{\eta_1,\eta_2}(E_2)) = 0$ in E_2, we find that

$$\partial_{E_2} f^{\eta_1,\eta_2}(E_2) = \frac{1 + \operatorname{Re}\partial_2\mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2)}{1 - \operatorname{Re}\partial_1\mathfrak{s}(f(E_2) - \mathrm{i}\eta_1, E_2 + \mathrm{i}\eta_2))} = 1 + \mathcal{O}(\Delta),$$

by means of (8.4.14) from Lemma 8.4.4. Here, $\partial_j\mathfrak{s}$ denotes the partial derivative of $\mathfrak{s}$ w.r.t. its j^{th} argument. Similarly,

$$\partial_{\eta_1} f^{\eta_1,\eta_2}(E_2) = -\frac{\operatorname{Re}\left[\mathrm{i}\partial_1\mathfrak{s}\right]}{1 - \operatorname{Re}\left[\partial_1\mathfrak{s}\right]}, \quad \partial_{\eta_2} f^{\eta_1,\eta_2}(E_2) = \frac{\operatorname{Re}\left[\mathrm{i}\partial_1\mathfrak{s}\right]}{1 - \operatorname{Re}\left[\partial_2\mathfrak{s}\right]}\,,$$

where $\mathfrak{s}$ has arguments $f(E_2) - \mathrm{i}\eta_1$ and $E_2 + \mathrm{i}\eta_2$. This concludes the bound $\left|\partial_{\eta_j} f^{\eta_1,\eta_2}(E_2)\right| \lesssim \Delta$ for $j = 1, 2$. The bound on $|\partial_{E_2}\mathfrak{s}_0(E_2)|$ is obtained in a similar fashion and thus left to the reader. □

Proof of Lemma 8.4.6. The proof is divided in two parts: In the first part, we prove (8.4.17) for $\eta_1 = \eta_2 = +0$. In the second part of the argument, we treat the the general case as a perturbation thereof.

Part (1): Proof on the real line. Applying the M-resolvent identity (8.4.11) for $z_1 \coloneqq f(E) - \mathrm{i}0$ and $z_2 \coloneqq E + \mathrm{i}0$ and using Proposition 8.4.2, we find that

$$\left|\frac{\langle M_1(z_1)\rangle - \langle M_2(z_2)\rangle}{f(E) - E - \mathfrak{s}_0(E)}\right| \lesssim \frac{1}{\Delta^2}\,.$$

Since the numerator on the lhs. is of order one and the real part of the denominator vanishes by definition of $f(E)$, we deduce that

$$\Delta^2 \lesssim |\operatorname{Im}\left[f(E) - E - \mathfrak{s}_0(E)\right]| = |\operatorname{Im}\mathfrak{s}_0(E)|\,,$$

i.e. we have a lower bound on the *modulus* of $\operatorname{Im}\mathfrak{s}_0(E)$. To turn this into a lower bound on $\operatorname{Im}\mathfrak{s}_0(E)$ itself, we need to show that it is positive.

This will be done via a proof by contraction: Suppose that $\operatorname{Im}\mathfrak{s}_0(E) < 0$. By (8.4.11) for $z_1 := f(E) - \mathrm{i}0$, $z_2 := E + \mathrm{i}0$ we get

$$\frac{\langle M_1 M_2\rangle}{1 - \langle M_1 M_2\rangle} = \frac{\langle M_1\rangle - \langle M_2\rangle}{-\mathrm{i}\operatorname{Im}\mathfrak{s}_0(E)}\,. \tag{8.5.7}$$

Since $\operatorname{Im}\left[\langle M_1\rangle - \langle M_2\rangle\right] = -c$ for some $c > 0$ and $|\operatorname{Re}\left[\langle M_1\rangle - \langle M_2\rangle\right]| \lesssim \Delta$, we obtain, using our assumption $\operatorname{Im}\mathfrak{s}_0(E) < 0$,

$$\langle M_1\rangle - \langle M_2\rangle = |\langle M_1\rangle - \langle M_2\rangle| \mathrm{e}^{-\frac{\mathrm{i}\pi}{2} + \mathrm{i}\mathcal{O}(\Delta)} \quad \text{and} \quad \frac{\langle M_1\rangle - \langle M_2\rangle}{-\mathrm{i}\operatorname{Im}\mathfrak{s}_0(E)} = \left|\frac{\langle M_1\rangle - \langle M_2\rangle}{-\mathrm{i}\operatorname{Im}\mathfrak{s}_0(E)}\right| \mathrm{e}^{\mathrm{i}\pi + \mathrm{i}\mathcal{O}(\Delta)}\,,$$

where here and in the following $\mathcal{O}(\Delta)$ is real-valued. In a similar way, we find that $\langle M_1 M_2\rangle = 1 + \mathcal{O}(\Delta) + \mathrm{i}\mathcal{O}(\Delta)$ and $\langle M_1 M_2\rangle = |\langle M_1 M_2\rangle| \mathrm{e}^{\mathrm{i}\mathcal{O}(\Delta)}$. Hence, (8.5.7) implies

$$1 - \langle M_1 M_2\rangle = \left|\frac{-\mathrm{i}\operatorname{Im}\mathfrak{s}_0(E)}{\langle M_1\rangle - \langle M_2\rangle}\langle M_1 M_2\rangle\right| \mathrm{e}^{\mathrm{i}\pi + \mathcal{O}(\Delta)}\,,$$

i.e., in particular, $\operatorname{Re}\left[1 - \langle M_1 M_2\rangle\right] < 0$. On the other hand, it holds that $\operatorname{Re}\left[1 - \langle M_1 M_2\rangle\right] \geq 1 - |\langle M_1 M_2\rangle| \geq 0$, so we arrived at a *contradiction* and thus $\operatorname{Im}\mathfrak{s}_0(E) > 0$ and $\operatorname{Im}\mathfrak{s}_0(E) \gtrsim \Delta^2$.

For part (1), we are now left to prove $|\operatorname{Im}\mathfrak{s}_0(E)| \lesssim \Delta^2$, which is done via a perturbative argument in Appendix 8.A. This concludes part (1), i.e. $\operatorname{Im}\mathfrak{s}_0^{0,0}(E) \sim \Delta^2$.

Part (2): Extension away from the real line. By (8.4.16) and the fundamental theorem of calculus, we have

$$\left|\operatorname{Im}\mathfrak{s}_0^{\eta_1,\eta_2}(E) - \operatorname{Im}\mathfrak{s}_0^{0,0}(E)\right| \leq \left|\int_0^{\eta_1} \partial_{\zeta_1}\mathfrak{s}_0^{\zeta_1,\eta_2}(E)\mathrm{d}\zeta_1\right| + \left|\int_0^{\eta_2} \partial_{\zeta_2}\mathfrak{s}_0^{0,\zeta_2}(E)\mathrm{d}\zeta_2\right| \lesssim \Delta(\eta_1 + \eta_2)\,.$$

Hence, if $0 < \eta_j \leq c_2\Delta$ for some $c_2 > 0$ small enough, we obtain $\operatorname{Im}\mathfrak{s}_0^{\eta_1,\eta_2}(E) \sim \operatorname{Im}\mathfrak{s}_0^{0,0}(E) \sim \Delta^2$. □

8.6 Contour integration: Proof of technical lemmas from Section 8.4.3

The goal of this section is to give the proofs of the technical lemmas from Section 8.4.3, for which we recall the construction of the contours γ_1, γ_2 from Section 8.4, in particular (8.4.2)–(8.4.3) and Figure 8.1.1, and the definition of the $[a, b]$ interval from the beginning of Section 8.4.3.

In all of the estimates below, we will frequently use the following simple tools:

- To gain $1/t$-factors from the oscillatory $\mathrm{e}^{\mathrm{i}t(z_1 - z_2)}$, we integrate by parts.
- When pulling absolute values inside an integral, we bound $|\mathrm{e}^{\mathrm{i}t(z_1 - z_2)}| \lesssim 1$ (recall $|\operatorname{Im} z_j| \lesssim 1/t$).
- The convolution of two Cauchy kernels yields another Cauchy kernel: For $\eta_j > 0$ and $E_j \in \mathbf{R}$, $j \in [2]$ it holds that

$$\int_{\mathbf{R}} \frac{\eta_1}{(x - E_1)^2 + \eta_1^2}\frac{\eta_2}{(x - E_2)^2 + \eta_2^2}\mathrm{d}x \lesssim \frac{\eta_1 + \eta_2}{(E_1 - E_2)^2 + (\eta_1 + \eta_2)^2}\,. \tag{8.6.1}$$

We now turn to the proofs of the lemmas from Section 8.4.3.

8.6.1 The second line of (8.4.23) is negligible: Proof of Lemma 8.4.7

We discuss the contributions from the flat and semicircular part of γ_2 separately (recall (8.4.3)).

First, the smallness of the integral over $\gamma_2^{(2)}$ (the semicircular part) is granted by the factor $\mathrm{e}^{t\mathrm{Im}\, z_2}$ (note that $\mathrm{Im}\, z_2 \in [-R+\eta_2, \eta_2]$) and the estimate $|\langle M(E_0+i\eta_0, z_2)\rangle| \lesssim 1$, which follows from (8.4.9). More precisely, we have that

$$\left| \oint_{\gamma_2^{(2)}} \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle \, \mathrm{d}z_2 \right| \lesssim R \int_\pi^{2\pi} \mathrm{e}^{tR\sin\theta} \mathrm{d}\theta \lesssim \frac{1}{t}\,. \tag{8.6.2}$$

Next, we bound the integral over $\gamma_2^{(1)}$ – the flat part. As a first step, integration by parts yields

$$\left| \int_{\gamma_2^{(1)}} \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle \, \mathrm{d}z_2 \right| \lesssim \frac{1}{t} + \left| \frac{1}{\mathrm{i}t} \int_{-R}^{R} \mathrm{e}^{-\mathrm{i}tE_2} \partial_{E_2} \langle M(E_0+\mathrm{i}\eta_0, E_2+\mathrm{i}\eta_2)\rangle \, \mathrm{d}E_2 \right|.$$

The derivative can be explicitly computed as

$$\partial_{z_2} \langle M(z_1, z_2)\rangle = \frac{\langle M_1 M_2^2\rangle}{(1-\langle M_2^2\rangle)(1-\langle M_1 M_2\rangle)^2}. \tag{8.6.3}$$

Since E_0 is in the bulk of ρ_1 and $z_0 := E_0 + \mathrm{i}\eta_0$ and z_2 are in the same half-plane we infer $|1-\langle M_1 M_2\rangle| \gtrsim 1$ and thus

$$\left| \oint_{\gamma_2^{(1)}} \mathrm{e}^{\mathrm{i}t(E_0+\mathrm{i}\eta_0-z_2)} \langle M(E_0+\mathrm{i}\eta_0, z_2)\rangle \, \mathrm{d}z_2 \right| \lesssim \frac{1}{t} + \frac{1}{t} \int_{-R}^{R} \left| \frac{1}{1-\langle M_2(E_2+\mathrm{i}\eta_2)^2\rangle} \right| \mathrm{d}E_2.$$

In order to conclude the proof of Lemma 8.4.7, we finally use that the one-body stability operator $|1-\langle M_2(E_2+\mathrm{i}\eta_2)^2\rangle|^{-1}$ is locally integrable, see Lemma 8.A.1 in Appendix 8.A. □

8.6.2 Cutting tails in the first line of (8.4.23): Proof of Lemma 8.4.8

For cutting the tails, we focus on the more critical regime, where both parameters are on the horizontal part of the contours, $z_j \in \gamma_j^{(1)}$ for $j \in [2]$ (recall (8.4.2)–(8.4.3)). Indeed, if this is not the case, a simple computation using Proposition 8.4.2 and arguing similarly to (8.6.2) yields $(1+\eta_0/\Delta)/t \lesssim 1/t$ as an upper bound for the corresponding integrals.

In the critical regime $z_j \in \gamma_j^{(1)}$ for $j \in [2]$ we carry out only the case $E_2 = \mathrm{Re}\, z_2 \in [b, R]$; for $E_2 \in [-R, a]$ the argument is identical. Let $\delta := (b-E_0)/2$ and split the region of the $E_1 = \mathrm{Re}\, z_1$-integration into the two parts, $[b-\delta, 2R]$ and $[-2R, b-\delta]$. In the first regime, using $|E_1 - E_0| \gtrsim 1$ and, from Proposition 8.4.2, $|\langle M(E_1-\mathrm{i}\eta_1, E_2+\mathrm{i}\eta_2)\rangle| \lesssim \left((E_1-E_2)^2+\Delta^2\right)^{-1}$, we find that

$$\int_{b-\delta}^{2R} \int_b^R \left| \frac{\eta_0}{(E_1-\mathrm{i}\eta_1-E_0)^2+\eta_0^2} \langle M(E_1-\mathrm{i}\eta_1, E_2+\mathrm{i}\eta_2)\rangle \right| \mathrm{d}E_1 \mathrm{d}E_2 \lesssim \frac{\eta_0}{\Delta}.$$

For $E_1 \in [-2R, b-\delta]$, by Proposition 8.4.2 again, we have $|\langle M(z_1, z_2)\rangle| \lesssim 1$, since $|E_1 - E_2| \sim 1$. Using this and integration by parts in E_2, similarly to the proof of Lemma 8.4.7, in combination with (8.6.3) and Lemma 8.A.1, we find that

$$\left| \int_{-2R}^{b-\delta} \int_b^R \mathrm{e}^{\mathrm{i}t(z_1-E_2-\mathrm{i}\eta_2)} \frac{\eta_0}{(E_1-\mathrm{i}\eta_1-E_0)^2+\eta_0^2} \langle M(E_1-\mathrm{i}\eta_1, E_2+\mathrm{i}\eta_2)\rangle \mathrm{d}E_1 \mathrm{d}E_2 \right| \lesssim \frac{1}{t}\,.$$

This finishes the proof of Lemma 8.4.8. □

8.6.3 First replacement: Proof of Lemma 8.4.9

Let $\delta > 0$ be such that $[a-\delta, b+\delta]$ is in the bulk of ρ_1. We now compare the two integrals on the lhs. and rhs. of (8.4.18) by taking their difference. Using integration by parts, the contribution from $(z_1, z_2) \in \gamma_1^{(2)} \times [a,b]$ is bounded by η_0/t. Analogously to the proof of Lemma 8.4.8, we also find that the contribution from $([-R,R] \setminus [a-\delta, b+\delta]) \times [a,b]$ is bounded by η_0, since in this regime $|\langle M(z_1, I, z_2)\rangle| \lesssim 1$ and $|z_1 - z_2 - \mathfrak{s}_0(z_2)|^{-1} \gtrsim 1$.

We are hence left to estimate the contribution from the region $[a-\delta, b+\delta] \times [a,b]$. Using that $|\mathfrak{s}(z_1,z_2) - \mathfrak{s}_0(E_2)| \lesssim \Delta |E_1 - \operatorname{Re} f(z_2)|$ by means of Lemma 8.4.4, we find that this can be bounded by

$$\mathcal{E} := \int_{a-\delta}^{b+\delta} \int_a^b \frac{\eta_0}{(E_1-E_0)^2+\eta_0^2} \cdot \frac{\Delta |E_1 - \operatorname{Re} f(z_2)|}{|z_1 - z_2 - \mathfrak{s}(z_1,z_2)| \cdot |z_1 - z_2 - \mathfrak{s}_0(E_2)|} \mathrm{d}E_1 \mathrm{d}E_2 \,.$$

To have better control on $\mathcal{E}$, we now bound the denominators in the second factor from below. First, using the definition of $\mathfrak{d}$ from the formulation of Lemma 8.4.9, we get

$$|z_1 - z_2 - \mathfrak{s}_0(E_2)|^2 = (E_1 - E_2 - \operatorname{Re}\mathfrak{s}_0(E_2))^2 + (\eta_1 + \eta_2 + \operatorname{Im}\mathfrak{s}_0(E_2))^2 \gtrsim (E_1 - f(E_2))^2 + \mathfrak{d}^2 \,. \tag{8.6.4}$$

Next, using that $|z_1 - z_2 - \mathfrak{s}(z_1,z_2)| \gtrsim \Delta^2$, as simple consequence of the stability bound (8.4.9), we infer

$$|z_1 - z_2 - \mathfrak{s}(z_1,z_2)|^2 \sim |z_1 - z_2 - \mathfrak{s}(z_1,z_2)|^2 + \Delta^4 \gtrsim (E_1 - E_2 - \operatorname{Re}\mathfrak{s}(z_1,z_2))^2 + \Delta^4 \,. \tag{8.6.5}$$

Finally, using the defining properties of the renormalization function f given in Lemma 8.4.5, (8.4.14) from Lemma 8.4.4, and the triangle inequality, one easily sees that

$$|E_1 - E_2 - \operatorname{Re}\mathfrak{s}(z_1,z_2)| \sim |E_1 - f(E_2)|. \tag{8.6.6}$$

Hence, combining (8.6.4) and (8.6.5)–(8.6.6) we find that

$$\begin{aligned} \mathcal{E} &\lesssim \int_{a-\delta}^{b+\delta} \int_a^b \frac{\eta_0}{(E_1-E_0)^2+\eta_0^2} \cdot \frac{\Delta |E_1 - f(E_2)|}{|E_1 - f(E_2)|^2 + (\min\{\mathfrak{d}, \Delta^2\})^2} \mathrm{d}E_1 \mathrm{d}E_2 \\ &\lesssim \int_{a-\delta}^{b+\delta} \frac{\eta_0 \Delta (|\log\Delta| + |\log\mathfrak{d}|)}{(E_1-E)^2+\eta_0^2} \mathrm{d}E_1 \lesssim \Delta(|\log\Delta| + |\log\mathfrak{d}|). \end{aligned}$$

where in the second step we changed the integration variable from E_2 to $f(E_2)$ and employed (8.4.16) from Lemma 8.4.5. This concludes the proof of Lemma 8.4.9. □

8.6.4 Residue computation after the first replacement: Proof of Lemma 8.4.10

Using the integral representation $\langle M_1(z_1)\rangle = \int_{\mathbf{R}} \rho_1(x)(x-z_1)^{-1}\mathrm{d}x$ and carrying out the residue computation (note that (8.4.19) ensures $z_2 + \mathfrak{s}_0(E_2)$ is encircled by the contour γ_1), we find the lhs. of (8.4.20) to equal

$$-\frac{1}{2\pi\mathrm{i}} \int_a^b \mathrm{d}E_2 \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}t(x - E_2 - \mathrm{i}\eta_2)} \frac{\eta_0}{(x-E_0)^2+\eta_0^2} \cdot \frac{\rho_1(x)\mathrm{d}x}{x - (E_2 + \mathrm{i}\eta_2 + \mathfrak{s}_0(E_2))} + \mathcal{E}_1 + \mathcal{E}_2 \,,$$

where we introduced the shorthand notations

$$\begin{aligned} \mathcal{E}_1 &:= -\frac{1}{4\pi} \int_a^b \mathrm{e}^{\mathrm{i}t(E_0 + \mathrm{i}\eta_0 - E_2 - \mathrm{i}\eta_2)} \frac{\langle M_1(E_0+\mathrm{i}\eta_0)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle}{E_0 + \mathrm{i}\eta_0 - (E_2 + \mathrm{i}\eta_2 + \mathfrak{s}_0(E_2))} \mathrm{d}E_2, \\ \mathcal{E}_2 &:= \frac{1}{2\pi\mathrm{i}} \int_a^b \mathrm{e}^{\mathrm{i}t\mathfrak{s}_0(E_2)} \frac{\eta_0 \left(\langle M_1(E_0+\mathrm{i}\eta_0)\rangle - \langle M_2(E_2+\mathrm{i}\eta_2)\rangle\right)}{(E_2 + \mathrm{i}\eta_2 + \mathfrak{s}_0(E_2) - E_0)^2 + \eta_0^2} \mathrm{d}E_2 \,. \end{aligned}$$

Moreover, we shall abbreviate $z_0 := E_0 + \mathrm{i}\eta_0$, $z_2 := E_2 + \mathrm{i}\eta_2$. Then, to estimate $\mathcal{E}_1$, we employ integration by parts and find that, since $|\partial_{E_2}\langle M_2(z_2)\rangle| \lesssim 1$ as $\rho_2(z_2) \gtrsim 1$, using (8.4.16) from Lemma 8.4.5, and recalling the definition of $\mathfrak{a}$ from the formulation of Lemma 8.4.10,

$$\left|\partial_{E_2}\frac{\langle M_1(z_0)\rangle - \langle M_2(z_2)\rangle}{z_0 - (z_2 + \mathfrak{s}_0(E_2))}\right| \lesssim \frac{1}{|E_0 - f(E_2)| + \mathfrak{a}} + \frac{|\langle M_1(z_0)\rangle - \langle M_2(z_2)\rangle|}{|E_0 - f(E_2)|^2 + \mathfrak{a}^2}.$$

Applying the M-resolvent identity (8.4.11) to z_0 and z_2 we infer, by application of the stability bound from Proposition 8.4.2 together with (8.4.13) and $\eta_2 \lesssim 1/t$, $\eta_0 \lesssim \Delta$, that $|\langle M_1(z_0)\rangle - \langle M_2(z_2)\rangle| \lesssim |E_0 - f(E_2)| + \Delta + 1/t$, and hence

$$|\mathcal{E}_1| \lesssim \frac{1}{t} + \frac{1}{t}\int_a^b \left(\frac{1}{|E_0 - f(E_2)| + \mathfrak{a}} + \frac{|E_0 - f(E_2)| + \Delta + t^{-1}}{|E_0 - f(E_2)|^2 + \mathfrak{a}^2}\right)\mathrm{d}E_2 \lesssim \frac{|\log\mathfrak{a}|}{t} + \frac{\Delta + t^{-1}}{t\mathfrak{a}}.$$

Similarly, $\mathcal{E}_2$ admits the bound $|\mathcal{E}_2| \lesssim \eta_0|\log\mathfrak{a}| + \eta_0\mathfrak{a}^{-1}(\Delta + t^{-1})$. This finishes the proof of Lemma 8.4.10. □

8.6.5 Second replacement: Proof of Lemma 8.4.11

The argument is split in two parts. First, we estimate the error of the second replacement within the interval $[a, b]$. Then, we put back the tails to complete the full contour integral.

For the first part, using $|\mathfrak{s}_0(E_2) - \mathfrak{s}_0| \lesssim \Delta|f(E_2) - E_0|$ as a consequence of (8.4.16), we find the error to be bounded by (a constant times) $\mathcal{E}_1 + \mathcal{E}_2$, where

$$\mathcal{E}_1 := \int_{\mathbf{R}}\mathrm{d}x\int_a^b \frac{\eta_0}{(x - E_0)^2 + \eta_0^2}\cdot\frac{\Delta|f(E_2) - E_0|}{|x - (z_2 + \mathfrak{s}_0(E_2))|^2}\mathrm{d}E_2 \tag{8.6.7}$$

and $\mathcal{E}_2$ is the same integral as $\mathcal{E}_1$, but with $\mathfrak{s}_0(E_2)$ being replaced by $\mathfrak{s}_0$. Next, convolving Cauchy kernels (8.6.1) in the x-variable and using (8.4.16) together with the definition of $\mathfrak{b}$ we arrive at

$$\mathcal{E}_1 \lesssim \frac{\eta_0 + \mathfrak{b}}{\mathfrak{b}}\int_a^b \frac{\Delta|f(E_2) - E_0|}{(f(E_2) - E_0)^2 + (\eta_0 + \mathfrak{b})^2}\mathrm{d}E_2 \lesssim \frac{\eta_0 + \mathfrak{b}}{\mathfrak{b}}\Delta|\log(\eta_0 + \mathfrak{b})|\,.$$

For $\mathcal{E}_2$, the argument is similar: We simply replace $f(E_2) - E_0$ in the denominator by $E_0 - E_2 - \mathrm{Re}\,\mathfrak{s}_0$ and estimate $|E_0 - f(E_2)| \lesssim |E_0 - E_2 - \mathrm{Re}\,\mathfrak{s}_0|$ in the numerator. This shows that the error for the first bound is bounded by $(\eta_0 + \mathfrak{b})\mathfrak{b}^{-1}\Delta|\log(\eta_0 + \mathfrak{b})|$.

In the second part, we estimate the tails on the rhs. of (8.4.22). In the regime when $z_2 \in \gamma_2^{(2)}$ we find the bound $1/t$, similarly to (8.6.2). If instead $z_2 \in \gamma_2^{(1)} \setminus ([a, b] + \mathrm{i}\eta_2)$, say, $E_2 = \mathrm{Re}\,z_2 \in [b, R]$ for concreteness, we have that $|E_0 - E_2 - \mathrm{Re}\,\mathfrak{s}_0| \sim 1$, so the singularities in x on the rhs. of (8.4.22) are separated from each other. Now, pick $\delta \sim 1$ such that $[E_0 - \delta, E_0 + \delta] \subset [a, b]$ and $|x - E_2 - \mathrm{Re}\,\mathfrak{s}_0| \sim 1$ for any $E_2 \in [b, R]$, $x \in [E_0 - \delta, E_0 + \delta]$. Then, for $|x - E_0| \geq \delta$, it holds that

$$\left|\int_{|x - E_0|\geq\delta}\mathrm{d}x\int_b^R e^{it(x - E_2 - i\eta_2)}\frac{\eta_0}{(x - E_0)^2 + \eta_0^2}\cdot\frac{\rho_1(x)\mathrm{d}x}{x - (E_2 + i\eta_2 + \mathfrak{s}_0)}\mathrm{d}E_2\right| \lesssim \eta_0|\log\mathfrak{b}|\,,$$

where, in order to get $\mathfrak{b}$, we employed Lemma 8.4.6 and (8.4.16). Finally, for $|x - E_0| \leq \delta$, we employ integration by parts in E_2 and use $|x - E_2 - \mathrm{Re}\,\mathfrak{s}_0| \sim 1$ for any $E_2 \in [b, R]$, $x \in [E_0 - \delta, E_0 + \delta]$ to get

$$\left|\int_{E_0 - \delta}^{E_0 + \delta}\mathrm{d}x\mathrm{e}^{\mathrm{i}t(x - \mathrm{i}\eta_2)}\frac{\eta_0\,\rho_1(x)}{(x - E_0)^2 + \eta_0^2}\int_b^R \mathrm{e}^{-\mathrm{i}tE_2}\frac{\mathrm{d}E_2}{x - (E_2 + i\eta_2 + \mathfrak{s}_0)}\right| \lesssim \frac{1}{t}\,.$$

This finishes the justification of the replacement (8.4.22) and thus the proof of Lemma 8.4.11. □

8.7 Second echo protocol: Proof of Theorem 8.2.10

The argument for part (i) is very similar to that for the proof of Theorem 8.2.4 (i). The only two differences are the following: First, the formerly algebraic cancellations $\langle \widetilde{P}[H_1, H_2]\rangle = 0$ below (8.3.3) and (8.3.5) are replaced by the estimate $|\langle \psi, W\phi\rangle| \prec \|\psi\|\,\|\phi\| N^{-1/2}$ for deterministic $\phi, \psi \in \mathbf{C}^N$. This follows by residue calculus and using an isotropic global law for the Wigner matrix W together with the fact that the first moment of the semicircular density vanishes, $\int_{\mathbf{R}} x\rho_{\mathrm{sc}}(x)\mathrm{d}x = 0$, by symmetry. More precisely, using $\|W\| \le 2+\epsilon$ with very high probability,

$$\begin{aligned} |\langle \psi, W\phi\rangle| &= \left| \frac{1}{2\pi \mathrm{i}} \oint_{|z|=3} z\langle \psi, (W-z)^{-1}\phi\rangle \mathrm{d}z \right| \\ &\lesssim \|\psi\|\,\|\phi\| \left| \frac{1}{2\pi\mathrm{i}} \oint_{|z|=3} z m_{\mathrm{sc}}(z)\mathrm{d}z \right| + \|\psi\|\,\|\phi\| \mathcal{O}_{\prec}(N^{-1/2}) \\ &\lesssim \|\psi\|\,\|\phi\| \left| \int_{\mathbf{R}} x\rho_{\mathrm{sc}}(x)\mathrm{d}x \right| + \|\psi\|\,\|\phi\| \mathcal{O}_{\prec}(N^{-1/2}) \prec \|\psi\|\,\|\phi\| N^{-1/2} \end{aligned} \tag{8.7.1}$$

where, to go to the last line, we used the Stieltjes representation $m_{\mathrm{sc}}(z) = \int_{\mathbf{R}} (x-z)^{-1}\rho_{\mathrm{sc}}(x)\mathrm{d}x$ and simple residue calculus. Second, in the analog of (8.3.6) it suffices to estimate all the λW simply by operator norm, recalling $\|W\| \le 2+\epsilon$ with very high probability. The rest of the argument goes along the same lines as in the proof of Theorem 8.2.4 (i) with straightforward modifications.

Part (ii) may be derived from Theorem 7.2.4, but here we give a direct proof relying just on the argument given in Section 7.3.2.1. First, by means of the single resolvent global law, we have that

$$\begin{aligned} \langle \psi_0, \mathrm{e}^{\mathrm{i}tH_\lambda}\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle &= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle \psi_0, G_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle \mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle \psi_0, M_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle \mathrm{d}z + \mathcal{O}_{\prec}\big(C(t,\lambda)/\sqrt{N}\big) \end{aligned} \tag{8.7.2}$$

for some constant $C(t,\lambda) > 0$ depending only on time t and coupling λ. Next, we approximate $\langle M_\lambda(z)\rangle \approx \overline{m_0(E_0)}$, leading to

$$M_\lambda(z) \approx \frac{1}{H_0 - z - \lambda^2 \overline{m_0(E_0)}}\,. \tag{8.7.3}$$

Plugging the approximation (8.7.3) into (8.7.2), we find

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}\big\langle \psi_0, \big(H_0 - z - \lambda^2\overline{m_0(E_0)}\big)^{-1}\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\big\rangle \mathrm{d}z = \mathrm{e}^{-\mathrm{i}\overline{m_0(E_0)}\lambda^2 t} \tag{8.7.4}$$

from simple residue calculus for $\lambda > 0$ small enough, using that $|m_0(E_0)| \lesssim 1$ (as follows from ρ_0 being $C^{1,1}$ around E_0; recall (8.2.18)) and γ encircles the spectrum of H_0. We have thus extracted the main term in (8.7.2), and it remains to estimate the errors resulting from the replacements in (8.7.3).

Denoting the spectral decomposition of H_0 by $H_0 = \sum_j \mu_j |\boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j|$ and using Assumption 8.2.9, we have that

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle \psi_0, M_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle \mathrm{d}z = \sum_{\mu_j \in I_\Delta} \langle \psi_0, \boldsymbol{u}_j\rangle\langle \boldsymbol{u}_j, \psi_t\rangle \widetilde{\vartheta}(j)\,, \tag{8.7.5}$$

where we denoted $\psi_t := \mathrm{e}^{-\mathrm{i}tH_0}\psi_0$ and

$$\widetilde{\vartheta}(j) := \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{e}^{\mathrm{i}tz}}{\mu_j - z - \lambda^2\langle M_\lambda(z)\rangle}\mathrm{d}z\,. \tag{8.7.6}$$

The key to approximating (8.7.5) is the following lemma, the proof of which is identical to that of Lemma 7.3.3 and so omitted.

Lemma 8.7.1 (cf. Lemma 7.3.3)**.** *Under the above assumptions and notations, for every $j \in [N]$ such that $\mu_j \in I_\Delta$, denote $\vartheta(j) := (2\pi\mathrm{i})^{-1} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}(\mu_j - z - \lambda^2\overline{m_0(E_0)})^{-1}\mathrm{d}z$. Then it holds that*

$$\sup_{\mu_j \in I_\Delta} \left|\widetilde{\vartheta}(j) - \vartheta(j)\right| \lesssim \mathcal{E} \tag{8.7.7}$$

for sufficiently small $\lambda > 0$ and N large enough (dependent on λ, cf. Lemma 7.A.1). Here, recalling (8.2.16) *for the definition of $\epsilon_0 = \epsilon_0(N)$, we denoted*

$$\mathcal{E} = \mathcal{E}(\lambda, t, \Delta, N) := \lambda^2 t\,\Delta + \lambda\,(1 + \lambda^2 t) + \frac{\lambda}{\Delta}\left(1 + \frac{\lambda}{\Delta}\right) + \lambda^2 t\,\epsilon_0\,. \tag{8.7.8}$$

Therefore, by means of Lemma 8.7.1, employing a Hölder inequality in (8.7.5), and using (8.7.4), we find that

$$\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{e}^{\mathrm{i}tz}\langle\psi_0, M_\lambda(z)\mathrm{e}^{-\mathrm{i}tH_0}\psi_0\rangle\mathrm{d}z = \mathrm{e}^{-\mathrm{i}\overline{m_0(E_0)}\lambda^2 t} + \mathcal{O}(\mathcal{E})\,. \tag{8.7.9}$$

Combining with (8.7.2) and taking the absolute value square of (8.7.9), we arrive at (8.2.21). This concludes the proof of Theorem 8.2.10. □

8.A Additional proofs

8.A.1 Upper bound on the renormalized shift: Perturbation argument for Lemma 8.4.6

The goal of this section is to prove the upper bound $|\mathrm{Im}\,\mathfrak{s}_0^{0,0}(E)| \lesssim \Delta^2$, as claimed at the end of part (1) of the proof of Lemma 8.4.6 in Section 8.5.

This is done via a perturbative calculation, which we carry out in a slightly more general setting: Consider two spectral parameters $z_1 = E_1 - \mathrm{i}0$, $z_2 = E_2 + \mathrm{i}0$, such that E_j is in the bulk of ρ_j, $j \in [2]$. Introducing the averaged and relative coordinates

$$D := (D_1 + D_2)/2, \quad z := (E_1 + E_2)/2 + i0, \quad \Theta := (D_2 - D_1)/2 - (E_2 - E_1)/2\,,$$

we find that $D_1 - z_1 = D - z - \Theta$ and $D_2 - z_2 = D - z + \Theta$. Let M be the solution of the MDE with the averaged coordinates, i.e.

$$-\frac{1}{M} = z - D + \langle M\rangle.$$

Using the identity $MM^* = \mathrm{Im}\,M/\langle\mathrm{Im}\,M\rangle$, it is easy to compute by Taylor expansion

$$M_2M_1 = \frac{1}{\langle\mathrm{Im}\,M\rangle}\left(\mathrm{Im}\,M + 2\mathrm{i}\mathrm{Im}\,[\mathrm{Im}\,M\Theta M] + 2\mathrm{i}\mathrm{Im}\left[\frac{\langle\Theta M^2\rangle}{1 - \langle M^2\rangle}\mathrm{Im}\,M\cdot M\right] + \mathcal{O}(|\Theta|^2)\right), \tag{8.A.1}$$

where $\mathcal{O}(|\Theta|^2)$ indicates terms containing at least two Θ's. Plugging (8.A.1) in the definition of the shift (8.4.12), we find that

$$\mathfrak{s}(z_1, z_2) + (z_2 - z_1) = -2\frac{\langle\Theta\mathrm{Im}\,M\rangle + 2\mathrm{i}\langle\Theta\mathrm{Im}\,[\mathrm{Im}\,M\Theta M]\rangle + 2\mathrm{i}\left\langle\Theta\mathrm{Im}\left[\frac{\langle\Theta M^2\rangle}{1-\langle M^2\rangle}\mathrm{Im}\,M\cdot M\right]\right\rangle + \mathcal{O}(\langle|\Theta|^3\rangle)}{\langle\mathrm{Im}\,M\rangle + 2\mathrm{i}\langle\mathrm{Im}\,[\mathrm{Im}\,M\Theta M]\rangle + 2\mathrm{i}\left\langle\mathrm{Im}\left[\frac{\langle\Theta M^2\rangle}{1-\langle M^2\rangle}\mathrm{Im}\,M\cdot M\right]\right\rangle + \mathcal{O}(\langle|\Theta|^2\rangle)}.$$

which implies

$$\begin{aligned}\frac{\langle\mathrm{Im}\,M\rangle^2}{2}[\mathfrak{s}(z_1, z_2) + (z_2 - z_1)] = &-\langle\Theta\mathrm{Im}\,M\rangle\langle\mathrm{Im}\,M\rangle - 2\mathrm{i}\langle\mathrm{Im}\,M\rangle\left\langle\left(\Theta - \frac{\langle\Theta\mathrm{Im}\,M\rangle}{\langle\mathrm{Im}\,M\rangle}\right)\mathrm{Im}\,M\Theta\mathrm{Im}\,M\right\rangle \\ &- 2\mathrm{i}\langle\mathrm{Im}\,M\rangle\left\langle\left(\Theta - \frac{\langle\Theta\mathrm{Im}\,M\rangle}{\langle\mathrm{Im}\,M\rangle}\right)\mathrm{Im}\left[\frac{\langle\Theta M^2\rangle}{1 - \langle M^2\rangle}\mathrm{Im}\,M\cdot M\right]\right\rangle + \mathcal{O}(\langle|\Theta|^3\rangle).\end{aligned}$$

Using $\operatorname{Im}[z_2 - z_1] = 0$ and $\operatorname{Im}[\langle \Theta \operatorname{Im} M\rangle \langle \operatorname{Im} M\rangle] = 0$, the imaginary part is given by

$$\frac{\langle \operatorname{Im} M\rangle}{4} \operatorname{Im} \mathfrak{s}(z_1, z_2) = -\left\langle \left(\Theta - \frac{\langle \Theta \operatorname{Im} M\rangle}{\langle \operatorname{Im} M\rangle} \right) \operatorname{Im} M \left(\Theta \operatorname{Im} M + \operatorname{Im}\left[\frac{\langle \Theta M^2\rangle}{1 - \langle M^2\rangle} M \right] \right) \right\rangle + \mathcal{O}(\langle |\Theta|^3 \rangle)$$

and hence

$$|\operatorname{Im} \mathfrak{s}(z_1, z_2)| = \frac{4}{\langle \operatorname{Im} M\rangle} \left| \left\langle \left(\Theta - \frac{\langle \Theta \operatorname{Im} M\rangle}{\langle \operatorname{Im} M\rangle} \right) \operatorname{Im} M \left(\Theta \operatorname{Im} M + \operatorname{Im}\left[\frac{\langle \Theta M^2\rangle}{1 - \langle M^2\rangle} M \right] \right) \right\rangle + \mathcal{O}(\langle |\Theta|^3 \rangle) \right| \lesssim \langle |\Theta|^2 \rangle ,$$

since $\|\Theta\| \lesssim 1$. Specializing to the setting of the Lemma 8.4.6 this result means that

$$|\operatorname{Im} \mathfrak{s}_0^{0,0}(E)| \lesssim \Delta^2 + |f(E) - E|^2 \lesssim \Delta^2 ,$$

where in the last step we used (8.4.13) and Lemma 8.4.5. This concludes the proof of the upper bound in part (1) of Lemma 8.4.6.

8.A.2 The one-body stability operator is locally integrable

In Section 8.6 we frequently use that the one-body stability operator is locally integrable. This is the statement of the following Lemma.

Lemma 8.A.1 (Integral of one-body stability operator)**.** *Fix a (large) positive constant L. Uniformly in $\eta \in [0,1]$ and in D satisfying Assumption 8.2.2 with constant L we have*

$$\int_{-L}^{L} \frac{\mathrm{d}E}{|1 - \langle M^2(E + i\eta)\rangle|} \lesssim 1. \tag{8.A.2}$$

Proof. With the notation (8.2.2), we use the classification of local minima of ρ from [22, Theorem 7.1]. This result addresses the case of a *diagonal* deformation, while D in the formulation of Lemma 8.A.1 does not need to be diagonal. Since the deterministic approximation $M(z)$ to the resolvent $(H - z)^{-1}$ of a random matrix H depends only on the first two joint moments of entries of H, we have that $M(z)$ in (8.A.2) coincides with the deterministic approximation to $(W_{\mathrm{GUE}} + D - z)^{-1}$, where W_{GUE} is a GUE matrix. Let U be a unitary diagonalizing D, i.e. $U^* D U = D_0$, where D_0 is diagonal. Invariance of GUE under unitary conjugations gives that $\widetilde{M}(z) := U^* M(z) U$ is a deterministic approximation to $(W_{\mathrm{GUE}} + D_0 - z)^{-1}$, so $\widetilde{M}(z)$ solves the MDE

$$-\widetilde{M}^{-1}(z) = z - D_0 + \langle \widetilde{M}(z)\rangle, \quad \operatorname{Im} z \operatorname{Im} \widetilde{M}(z) > 0 \quad \text{for} \quad z \in \mathbf{C} \setminus \mathbf{R} .$$

We remark that $\widetilde{M}$ satisfies the assumptions of [22, Theorem 7.1], since $\mathcal{S} = \langle \cdot \rangle$ is flat and by means of Assumption 8.2.2.

Thus [22, Theorem 7.1] applied to $\widetilde{M}$ together with the observation $\langle \widetilde{M}\rangle = \langle M\rangle$ gives that there exist positive constants $\rho_* > 0$ and $\delta_* > 0$ dependent only on L such that for any local minimum τ_0 of ρ with $\rho(\tau_0) < \rho_*$ one of the following possibilities holds:

$$\rho(\tau_0 + \omega) \sim \min\{\Delta^{-1/6} \omega^{1/2}, \omega^{1/3}\}, \qquad \omega \in [0, \delta_*], \qquad \text{(left edge)} \tag{8.A.3a}$$

$$\rho(\tau_0 + \omega) \sim \min\{\Delta^{-1/6} |\omega|^{1/2}, |\omega|^{1/3}\}, \qquad \omega \in [-\delta_*, 0], \qquad \text{(right edge)} \tag{8.A.3b}$$

$$\rho(\tau_0 + \omega) \sim |\omega|^{1/3}, \qquad \omega \in [-\delta_*, \delta_*], \qquad \text{(cusp)} \tag{8.A.3c}$$

$$\rho(\tau_0 + \omega) \sim \tilde{\rho} + \min\{\tilde{\rho}^{-5} \omega^2, |\omega|^{1/3}\}, \qquad \omega \in [-\delta_*, \delta_*], \qquad \text{(internal minimum)} \tag{8.A.3d}$$

where $\tilde{\rho} \sim \rho(\tau_0)$ in (8.A.3d). In (8.A.3a), $\Delta := 1$ if τ_0 is an extreme left edge of $\operatorname{supp} \rho$ and Δ is the length of the gap between the intervals of support which ends at point τ_0 otherwise (see also [22, Lemma 7.16])[12], for the right edge (8.A.3b) Δ is defined similarly.

[12] To be consistent with [22] we use Δ to denote the size of the gap inside of the proof of Lemma 8.A.1. This should not lead to any confusion with the rest of the paper, where Δ is used for the Hilbert-Schmidt norm of $D_1 - D_2$.

As a first preparatory step for (8.A.2), we give a lower bound for $|1-\langle M^2(E+\mathrm{i}\eta)\rangle|$ in terms of $\rho(E)$. In fact, we will show that uniformly in $E\in[-L,L]$ it holds that

$$|1-\langle M^2(E+\mathrm{i}\eta)\rangle| \gtrsim \rho^2(E)\,. \tag{8.A.4}$$

By Lemma 8.5.1 the LHS of (8.A.4) has a lower bound of order $\rho^2(E+\mathrm{i}\eta)+\eta/\rho(E+\mathrm{i}\eta)$. Recall from [22, Proposition 2.4] that ρ is $1/3$-Hölder regular, i.e. there exists a constant C_0 depending only on L such that $|\rho(z_1)-\rho(z_2)|\le C_0|z_1-z_2|^{1/3}$ uniformly in $z_1,z_2\in\mathbf{C}$ with $\operatorname{Im} z_1\operatorname{Im} z_2>0$ and $|z_j|\le 2L$, $j=1,2$. If $\eta\le\rho(E)^3/(2C_0)^3$, then $\rho(E+\mathrm{i}\eta)\sim\rho(E)$, i.e. (8.A.4) holds. In the complementary case, $\eta>\rho(E)^3/(2C_0)^3$, we have $\rho(E+\mathrm{i}\eta)<\rho(E)$ and hence $\eta/\rho(E+\mathrm{i}\eta)\gtrsim\rho^2(E)$, i.e. (8.A.4) again holds.

Now, armed with (8.A.4), we are ready to prove (8.A.2). We split the region of integration into several regimes according to the classification of local minima of ρ. For each local minimum τ_0 with $\rho(\tau_0)<\rho_*$ the integration over $\tau_0+[-\delta_*,\delta_*]\cap\mathcal{D}$ will be considered separately. Here, $\mathcal{D}=\mathbf{R}$ for cusps and internal minima, $\mathcal{D}=[0,+\infty)$ for left edges and $(-\infty,0]$ for right edges. The set $\mathcal{D}$ is chosen in such a way that $\tau_0+[-\delta_*,\delta_*]\cap\mathcal{D}$ covers the part, where ρ is positive and small. The complementary regimes, the *bulk regime* (where $\rho\ge\rho_*$) and the *gap regime* (where $\rho=0$), are treated separately.

Bulk regime: It holds that $\rho(E)\ge\rho_*$, hence desired bound on the lhs. of (8.A.2) in the bulk regime immediately follows from (8.A.4).

Gap regime: Let $\tau_1<\tau_0$ be two edges of $\operatorname{supp}\rho$ such that $\rho(E)=0$ for any $E\in[\tau_1,\tau_0]$; the cases when either τ_1 is an extreme right edge or τ_0 is an extreme left edge are treated similarly. Since $\partial_z M=M^2(1-\langle M^2\rangle)^{-1}$, we have $|1-\langle M^2\rangle|^{-1}\le 1+|\langle M'\rangle|$. Together with (8.A.3a) and (8.A.3b) this gives that

$$|1-\langle M^2(E+i\eta)\rangle| \gtrsim \left((\min\{|E-\tau_1|,|E-\tau_0|\})^2+\eta^2\right)^{1/3},$$

so the integral of $|1-\langle M^2(E+\mathrm{i}\eta)\rangle|^{-1}$ over $E\in[\tau_1,\tau_0]$ is uniformly bounded in $\eta\in[0,1]$.

Internal minimum with $\rho(\tau_0)<\rho_*$: Using (8.A.4) along with (8.A.3d) we find that

$$\int_{-\delta_*}^{\delta_*}\frac{\mathrm{d}\omega}{|1-\langle M^2(\tau_0+\omega+i\eta)\rangle|}\lesssim\int_{-\delta_*}^{\delta_*}\frac{\mathrm{d}\omega}{\rho^2(\tau_0+\omega)}\lesssim\int_0^{\tilde\rho^3}\frac{\mathrm{d}\omega}{\tilde\rho^2}+\int_{\tilde\rho^3}^{\delta_*}\frac{\mathrm{d}\omega}{\omega^{2/3}}\lesssim 1\,.$$

Cusp regime: This works in the exact same way as the internal minimum, using (8.A.3c) instead of (8.A.3d).

Edge regime: Let τ_0 be a left edge of ρ, for the right edge the argument is the same. First, [22, Corollary 5.3] gives that $|1-\langle M^2(z)\rangle|\gtrsim\rho(z)(|\sigma(z)|+\rho(z))$, where $\sigma(z)$ is a $1/3$-Hölder regular function in $\{z\in\mathbf{C}:\operatorname{Im} z>0\}$ (by [22, Lemma 5.5]) and $|\sigma(\tau_0)|\sim\Delta^{1/3}$ (by [22, Theorem 7.7, Lemma 7.16]). Therefore, there exists a (small) positive constant $c\sim 1$ such that for all z with $\operatorname{Re} z\in[\tau_0,\tau_0+c\Delta]$ and $\operatorname{Im} z\in[0,c\Delta]$ it holds that $|\sigma(z)|\sim\Delta^{1/3}$. It is easy to see that the integral of the one-body stability operator over $[\tau_0+c\Delta,\tau_0+\delta_*]$ has an upper bound of order one by means of (8.A.4) and (8.A.3a). In the complementary regime $[\tau_0,\tau_0+c\Delta]$ we distinguish between two cases (i) $\eta\in[0,c\Delta]$ and (ii) $\eta>c\Delta$. In the first case, note that, by the integral representation $\rho(E+i\eta)=\int_{\mathbf{R}}\mathrm{d}x\rho(x)\eta/((x-E)^2+\eta^2)$ and (8.A.3a), it holds that $\rho(E+\mathrm{i}\eta)\gtrsim\rho(E)$ for $E\in[\tau_0,\tau_0+c\Delta]$. Thus

$$\int_0^{c\Delta}\frac{\mathrm{d}\omega}{|1-\langle M^2(\tau_0+\omega+i\eta)\rangle|}\lesssim\Delta^{1/3}\int_0^{c\Delta}\frac{\mathrm{d}\omega}{\omega^{1/2}(\Delta^{1/2}+\omega^{1/2})}\lesssim 1.$$

In the second case, $\eta>c\Delta$, we use (8.A.4) and the bound $|1-\langle M^2(z)\rangle|\gtrsim|\operatorname{Im} z|$ to get

$$\int_0^{c\Delta}\frac{\mathrm{d}\omega}{|1-\langle M^2(\tau_0+\omega+i\eta)\rangle|}\lesssim\int_0^{c\Delta}\frac{\mathrm{d}\omega}{\rho^2(\tau_0+\omega)+\Delta}\sim\Delta^{1/3}\int_0^{c\Delta}\frac{\mathrm{d}\omega}{\omega+\Delta^{4/3}}\lesssim 1\,,$$

which concludes the proof for the regular edge.

A careful examination of the proof shows that all implicit constants in the inequalities above depend only on L. □

Chapter 9

Eigenstate thermalization hypothesis for translation invariant spin systems

This chapter contains the paper [541]:

S. Sugimoto, J. Henheik, V. Riabov, and L. Erdős. Eigenstate thermalisation hypothesis for translation invariant spin systems. *J. Stat. Phys.*, 190(7):128, 2023

Abstract. We prove the Eigenstate Thermalisation Hypothesis (ETH) for local observables in a typical translation invariant system of quantum spins with mean field interaction. This mathematically verifies the observation made in Ref. [512] that ETH may hold for systems with additional translation symmetries for a naturally restricted class of observables. We also present numerical support for the same phenomenon for Hamiltonians with local interaction.

Content

9.1 Introduction

Recent experiments have demonstrated thermalisation of isolated quantum systems under unitary time evolution [569, 397, 174, 363, 469, 549]. In this context, thermalisation means that, after a long time evolution, observables attain their equilibrium (thermal) values determined by statistical mechanics. The primary mechanism behind this thermalisation of isolated quantum systems is an

even stronger concept, the *Eigenstate Thermalisation Hypothesis (ETH)* [581, 221, 535]. Informally, the ETH asserts that (i) *physical* observables A take their thermal value on every eigenstate of a many-body quantum system and (ii) off-diagonal elements of A in the energy eigenbasis are vanishingly small. In particular, the ETH ensures the thermalisation of A for any initial state with a macroscopically definite energy, given no massive degeneracy in the energy spectrum [195, 455, 222]. The ETH has numerically been verified for individual models with several local or few-body observables [504, 502, 503, 86, 512, 536, 69]. On the other hand, recent studies have revealed several classes of systems for which the ETH breaks down: examples include systems with an extensive number of local conserved quantities [505, 139, 320], many-body localisation [49, 465], and quantum many-body scars [573, 128].

As another approach to this question, it has been proven that the ETH holds true for *any* deterministic observable for *almost all* Hamiltonians H [581, 495, 165] sampled from a Wigner matrix ensemble which has no further unitary symmetry (see also [5] and Chapter 2 for ETH for more general mean field ensembles). If the Hamiltonian has some unitary symmetry, the ETH clearly breaks down for conserved quantities related to those symmetries because we can find simultaneous eigenstates of the Hamiltonian and conserved quantities. However, Ref. [512] observed an interesting phenomenon, namely that local quantities still satisfy the ETH even in a system with translational symmetry. Therefore, the question of how *generically* and for what class of observables the ETH holds true in realistic situations has yet to be fully resolved.

In this paper we mathematically rigorously prove an instance of the observation from [512]. More precisely we show that, for the mean-field case of an ensemble with translational symmetry, the ETH typically holds for quantities whose support does not exceed half of the system size with the optimal speed of convergence. The ETH also typically holds for quantities whose support exceeds half the system size but with a slower convergence speed, while it typically breaks down for some observables whose support extends to the entire system. We complement our analytical results for the mean-field case with a numerical simulation for an ensemble of more realistic Hamiltonians with local interactions.

9.2 Setup

We consider a one-dimensional periodic quantum spin system on the $L \in \mathbb{N}$ sites of the standard discrete torus

$$\mathbb{T}_L \coloneqq {}^{\mathbb{Z}}\!/_{L\mathbb{Z}}.$$

On each vertex $j \in \mathbb{T}_L$, the one particle Hilbert space $\mathcal{H}_j$ is given by $\mathbf{C}^2$ and we denote its canonical basis by $\{|\uparrow\rangle, |\downarrow\rangle\}$. The corresponding L-particle Hilbert space

$$\mathcal{H} \coloneqq \bigotimes_{j=1}^{L} \mathbf{C}^2$$

is simply given by the tensor product with dimension 2^L. For simplicity, we restrict ourselves to the spin-$1/2$ case, but our results can straightforwardly be extended to general spin s with one particle Hilbert space being $\mathbf{C}^{2s+1}$.

Next, we introduce the ensemble of Hamiltonians, which is first introduced in Ref. [539] and shall be studied in this article. The main parameter in the definition is a tunable range $\ell \leq L$ of interactions, which allows us to consider how generically the ETH holds in realistic situations.

Definition 9.2.1 (Hamiltonian)**.** *Let $T = T_L$ be the (left) translation operator acting on L spins at the vertices of $\mathbb{T}_L$. We define the ensemble of Hamiltonians with local interactions as*

$$H_L^{(\ell)} \coloneqq \sum_{j=0}^{L-1} T_L^{-j}(h_\ell \otimes I_{L-\ell})T_L^j, \quad h_\ell \coloneqq \sum_{p_1,\ldots,p_\ell=0}^{3} J_{p_1,\ldots,p_\ell}\sigma_1^{(p_1)}\ldots\sigma_\ell^{(p_\ell)} \tag{9.2.1}$$

where $\ell \le L$ is the interaction range, $I_{L-\ell}$ is the identity on the sites $\ell+1,\dots,L$. Here $\sigma_j^{(p)}$ is the p^{th} Pauli matrix $\sigma^{(p)}$ acting on the site $j \in \mathbb{T}_L$, where we recall the standard Pauli matrices,

$$\sigma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^{(2)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.2.2}$$

The 4^ℓ coefficients $J_{p_1,\dots,p_\ell}$ are independent, identically distributed real Gaussian random variables with zero mean, $\mathbb{E}J_{p_1,\dots,p_\ell} = 0$, and variance

$$v_\ell^2 := \mathbb{E}|J_{p_1,\dots,p_\ell}|^2 .$$

The ensemble of Hamiltonians h_ℓ (9.2.1) contains prototypical spin models such as the XYZ model, $h_\ell = \sum_{p=1}^3 J_p \sigma_1^{(p)} \sigma_2^{(p)}$.

Observe that the Hamiltonian $H_L^{(\ell)}$ is a shifted version of the same local Hamiltonian h_ℓ. In particular, $H_L^{(\ell)}$ is translation invariant by construction, i.e., $T_L H_L^{(\ell)} T_L^{-1} = H_L^{(\ell)}$. We impose this structure to study a Hamiltonian with a symmetry. In the sequel we shall exploit this feature of $H_L^{(\ell)}$ by switching from position space to momentum space.

Lemma 9.2.2. *Let*

$$\Pi_k := \frac{1}{L} \sum_{j=1}^{L} e^{2\pi i \frac{kj}{L}} T_L^{-j} \quad \text{for} \quad k = 0,\dots,L-1 \tag{9.2.3}$$

be the projection operator onto the k-momentum space, i.e., $T_L \Pi_k = e^{2\pi i \frac{k}{L}} \Pi_k$. Then $H_L^{(\ell)}$ is block-diagonal in the momentum space representation, i.e. in the eigenbasis of T_L, since we have

$$H_L^{(\ell)} = L \sum_{k=0}^{L-1} \Pi_k (h_\ell \otimes I_{L-\ell}) \Pi_k . \tag{9.2.4}$$

Proof. This follows immediately by substituting the spectral decomposition of T given by $T = \sum_{k=0}^{L-1} e^{2\pi i k/L} \Pi_k$ into (9.2.1). □

As we will show in Lemma 9.3.4, the dimensions of each of the L momentum sectors are almost equal to each other, $\operatorname{Tr}_L \Pi_k \approx 2^L/L$.

In order to present our main result, the ETH in translation-invariant systems (Theorem 9.3.1), in a concise form, we need to introduce the *microcanonical average*. Below, we denote by $|E_\alpha^{(k)}\rangle$ the normalised eigenvector of $H_L^{(\ell)}$ in the k-momentum sector with eigenvalue E_α, i.e. $H_L^{(\ell)} |E_\alpha^{(k)}\rangle = E_\alpha |E_\alpha^{(k)}\rangle$ and $\Pi_k |E_\alpha^{(k)}\rangle = |E_\alpha^{(k)}\rangle$. It is easy to see that the spectrum of H in each momentum sector is simple almost surely.

Definition 9.2.3 (Microcanonical ensemble)**.** *For every energy $E \in \mathbb{R}$ and energy window $\Delta > 0$, we define the* microcanonical energy shell *$\mathcal{H}_{E,\Delta}$ centered at energy E with width 2Δ by*

$$\mathcal{H}_{E,\Delta} := \bigoplus_{k=0}^{L-1} \mathcal{H}_{E,\Delta}^{(k)}, \quad \text{where} \quad \mathcal{H}_{E,\Delta}^{(k)} := \operatorname{span}\{|E_\alpha^{(k)}\rangle : |E_\alpha^{(k)} - E| \le \Delta\} .$$

We denote the dimension of $\mathcal{H}_{E,\Delta}^{(k)}$ by $d_{E,\Delta}^{(k)}$ and that of $\mathcal{H}_{E,\Delta}$ by $d_{E,\Delta} = \sum_{k=0}^{L-1} d_{E,\Delta}^{(k)}$.

Whenever $d_{E,\Delta} > 0$, we define the microcanonical average *of any self-adjoint observable $A \in \mathbf{C}^{N\times N}$ within $\mathcal{H}_{E,\Delta}$ by*

$$\langle A \rangle_\Delta^{(\mathrm{mc})}(E) := \frac{1}{d_{E,\Delta}} \sum_{k=0}^{L-1} \sum_{|E_\alpha^{(k)}\rangle \in \mathcal{H}_{E,\Delta}^{(k)}} \langle E_\alpha^{(k)} | A | E_\alpha^{(k)} \rangle . \tag{9.2.5}$$

Remark 9.2.4. *The microcanonical average mimics the microcanonical ensemble before taking the thermodynamic limit. In order to be physically meaningful, there are two natural requirements on the energy shell $\mathcal{H}_{E,\Delta}$:*

(i) The density of states is approximately constant in the interval $[E-\Delta, E+\Delta]$.

(ii) The microcanonical energy shell contains $\gg 1$ states, i.e. $d_{E,\Delta} \to \infty$ as $L \to \infty$.

Note that for any fixed energy E, (i) corresponds to an upper bound and (ii) corresponds to a lower bound on Δ, both being dependent on E. We point out that very close to the spectral edges with only a few states, it is not guaranteed that both requirements can be satisfied simultaneously.

Indeed, from a physics perspective, viewing $\langle A\rangle_{\Delta}^{(\mathrm{mc})}(E)$ from (9.2.5) *as a finite dimensional approximation of the microcanonical ensemble is meaningless whenever (i) and (ii) are not satisfied. However, we will simply view Definition 9.2.3 for arbitrary Δ as an extension of the standard definition of the microcanonical average from the physics literature. Our main result, Theorem 9.3.1, will even hold with the microcanonical average in this extended sense.*

We set

$$N \coloneqq 2^L = \dim\mathcal{H}$$

for the total Hilbert space dimension. Our analytic results below will always be understood in the limit of large system size, i.e. $L \to \infty$, or, equivalently $N \to \infty$. We shall also use the following common notion (see, e.g., [241]) of stochastic domination.

Definition 9.2.5. *Given two families of non-negative random variables*

$$X \equiv \left(X^{(N)}(u) : N \in \mathbb{N},\ u \in U^{(N)}\right) \quad \text{and} \quad Y \equiv \left(Y^{(N)}(u) : N \in \mathbb{N},\ u \in U^{(N)}\right)$$

indexed by N, we say that X is stochastically dominated by Y, if for all ξ, $D > 0$, we have

$$\sup_{u \in U^{(N)}} \mathbb{P}\Big[X^{(N)}(u) > N^{\xi} Y^{(N)}(u)\Big] \leq N^{-D}$$

for any sufficiently large $N \geq N_0(\xi, D)$ and use the notation $X \prec Y$ or $X = \mathcal{O}_{\prec}(Y)$ in that case.

9.3 Main result in the mean-field case

Throughout the entire section, we are in the mean-field case $\ell = L$. For any $q \leq L$ we also introduce the concept of *q-local observables* for self-adjoint operators of the form $A = A_q \otimes I_{L-q}$, i.e. A_q is self-adjoint and only acts on the first q sites.

Our main result in this setting is the following theorem.

Theorem 9.3.1 (ETH in translation-invariant systems)**.** *Let $\ell = L$ and consider the Hamiltonian $H_L^{(L)}$ from* (9.2.1) *with eigenvalues $E_\alpha^{(k)}$ and associated normalised eigenvectors $|*\rangle\, E_\alpha^{(k)}$. Then, for every $\Delta > 0$ and bounded q-local observable $A = A_q \otimes I_{L-q}$, $\|*\|\, A \lesssim 1$, it holds that*

$$\max_{\alpha,\beta} \max_{k,k'} |\langle E_\alpha^{(k)} | A | E_\beta^{(k')}\rangle - \delta_{\alpha\beta}\delta_{k,k'} \langle A\rangle_{\Delta}^{(\mathrm{mc})}(E_\alpha^{(k)})| \prec \frac{1}{2^{\min\{L/2, L-q\}}}\,, \tag{9.3.1}$$

where the maxima are taken over all indices labeling the eigenvectors of $H_L^{(L)}$. In particular, for $q \leq L/2$ the ETH holds with optimal speed of convergence of order $1/\sqrt{N}$.

An extension of Theorem 9.3.1 to arbitrary dimension $d \geq 2$ is provided in Theorem 9.A.3 in the Appendix.

Remark 9.3.2 (Typicality of ETH)**.** *Theorem 9.3.1 asserts that for any fixed local observable A the ETH in the form* (9.3.1) *holds with a very high probability, i.e. apart from an event of probability $N^{-D} = 2^{-LD}$, for any fixed D, see the precise Definition 9.2.5. This exceptional event may depend on the observable A. However, as long as q is L-independent (in fact some mild logarithmic increase is allowed), it also holds that*

$$\max_{\alpha,\beta} \max_{k,k'} \max_{A} |\langle E_\alpha^{(k)} | A | E_\beta^{(k')} \rangle - \delta_{\alpha\beta}\delta_{k,k'} \langle A \rangle_\Delta^{(\mathrm{mc})}(E_\alpha^{(k)})| \prec \frac{1}{2^{L/2}}\,, \tag{9.3.2}$$

i.e. we may take the supremum over all bounded q-local observables A in (9.3.1)*. This extension is a simple consequence of choosing a sufficiently fine grid in the unit ball of the $4^q \times 4^q$ dimensional space of q-local observables and taking the union bound. The estimate* (9.3.2) *can be viewed as a very strong form of the typicality of ETH within our class of translation invariant mean field operators $H_L^{(L)}$. It asserts that apart from an exceptional set of the coupling constants $J_{p_1,\dots,p_L}$ the Hamiltonian $H_L^{(L)}$ satisfies the ETH with optimal speed of convergence, uniformly in the entire spectrum and tested against all finite range (q-local) observables. The exceptional set has exponentially small measure of order 2^{-LD} for any D if L is sufficiently large.*

In Lemma 9.3.5 we will see that in the mean–field case the Hamiltonian on each momentum sector is a GUE matrix, in particular the density of states of H follows Wigner's semicircle law. An elementary calculation shows that the radius of this semicircle is given by

$$R := 2 \cdot 2^L \sqrt{L} v_L (1 + O(2^{-L}))\,.$$

In light of Remark 9.2.4 we also mention that $\langle A \rangle_\Delta^{(\mathrm{mc})}(E)$ in (9.3.1) can be considered as an approximation of the expectation of A in the microcanonical ensemble at energy $|E| \le R$ if

$$\frac{R}{N^{2/3}} \ll \Delta \ll R - |E|\,. \tag{9.3.3}$$

The upper bound in (9.3.3) comes from requirement (i) in Remark 9.2.4, while the lower bound in (9.3.3) stems from (ii) using that the eigenvalue spacing near the spectral edge for Wigner matrices is of order $R/N^{2/3}$.

For the sequel we introduce the notation

$$\langle A \rangle := \frac{\operatorname{Tr} A}{\operatorname{Tr} I}$$

for the normalised trace of an operator A on any finite-dimensional Hilbert space, where I is the identity on that space. In particular, if $A = A_q \otimes I_{L-q}$ is a q-local observable, then $\langle A \rangle = \langle A_q \rangle$.

The proof of Theorem 9.3.1 crucially relies on the fact that in our mean-field case $\langle E_\alpha^{(k)} | A | E_\beta^{(k')} \rangle$ converges to $\delta_{\alpha\beta}\delta_{k,k'}\langle A \rangle$. In other words, the thermodynamics of the system is trivial; the thermal value of A is always given by its average trace. This is formalised in the following main proposition:

Proposition 9.3.3. *Under the assumptions of Theorem 9.3.1 it holds that*

$$\max_{\alpha,\beta} \max_{k,k'} |\langle E_\alpha^{(k)} | A | E_\beta^{(k')} \rangle - \delta_{\alpha\beta}\delta_{k,k'} \langle A \rangle| \prec \frac{1}{2^{\min\{L/2, L-q\}}}\,. \tag{9.3.4}$$

Having Proposition 9.3.3 at hand, we can readily prove Theorem 9.3.1.

Proof of Theorem 9.3.1. Averaging (9.3.4) for $\alpha = \beta$ and $k = k'$ according to the microcanonical average (9.2.5), we find that

$$\max_{\alpha} \max_{k} |\langle A \rangle_\Delta^{(\mathrm{mc})}(E_\alpha^{(k)}) - \langle A \rangle| \prec \frac{1}{2^{\min\{L/2, L-q\}}}\,.$$

Combining this with (9.3.4), the claim immediately follows. □

The rest of this section is devoted to the proof of Proposition 9.3.3, which is conducted in four steps.

1. The momentum sectors are all of the same size with very high precision (Lemma 9.3.4).
2. In each momentum sector the mean-field Hamiltonian $H_L^{(L)}$, represented in the eigenbasis of the translation operator T, is a GUE matrix (Lemma 9.3.5).
3. The ETH holds within each momentum sector separately (Lemma 9.3.6).
4. The averaged trace on each momentum sector and the total averaged trace are close to each other – at least for local observables (Lemma 9.3.7).

We shall first formulate all the four lemmas precisely and afterwards conclude the proof of Proposition 9.3.3.

Lemma 9.3.4 (Step 1: Dimensions of momentum sectors)**.** *The dimension $\operatorname{Tr}_L \Pi_k$ of the k-momentum sectors $(k = 0, \dots, L-1)$ is almost equal to each other in the sense that we have*

$$\operatorname{Tr}_L \Pi_k = \frac{2^L}{L} + \mathcal{O}(L^{1/2} 2^{L/2}) .$$

The proof is given in Section 9.3.1

Lemma 9.3.5 (Step 2: GUE in momentum blocks)**.** *Each momentum-block of the mean-field Hamiltonian $H_L^{(L)}$, represented in an eigenbasis of T, is an i.i.d. complex Gaussian Wigner matrix (GUE), whose entries have mean zero and variance $2^L L^2 v_L^2$. Recall that $v_L^2 = \mathbb{E}|J_{p_1,\dots,p_L}|^2$ from Definition 9.2.1.*

Proof. In the mean-field case $\ell = L$, a simple direct calculation of all first and second moments of the matrix elements shows that the interaction matrix h_ℓ is a complex Gaussian Wigner matrix whose entries have variance $2^L v_L^2$. Since the transformation from the standard basis to an eigenbasis of T is unitary, and the Gaussian distribution is invariant under unitary transformation, h_ℓ represented in an eigenbasis of T is again a Gaussian Wigner matrix. Finally, the projection operators Π_k in (9.2.4) set the off-diagonal blocks to zero. Incorporating the additional factor L in (9.2.4) into the variance proves Lemma 9.3.5. □

As the next step, we show that the ETH holds within each momentum sector.

Lemma 9.3.6 (Step 3: ETH within each momentum sector)**.** *For an arbitrary deterministic observable A with $\|*\| A \lesssim 1$ it holds that*

$$\max_{\alpha,\beta} \max_k |\langle E_\alpha^{(k)} | A | E_\beta^{(k)} \rangle - \delta_{\alpha\beta} \frac{\operatorname{Tr}_L(\Pi_k A \Pi_k)}{\operatorname{Tr}_L \Pi_k}| \prec \frac{1}{2^{L/2}} . \tag{9.3.5}$$

Proof. For any fixed k, Lemma 9.3.5 asserts that $\Pi_k H_L^{(L)} \Pi_k$ is a standard GUE matrix (up to normalisation by v_L). Using [165, Theorem 2.2], therefore its eigenvectors $|*\rangle E_\alpha = |*\rangle E_\alpha^{(k)}$ satisfy ETH in the form that $\langle E_\alpha | A | E_\beta \rangle$ is approximately given by the normalised trace of A in the k-momentum sector

$$\langle A \rangle_k := \frac{\operatorname{Tr}_L(\Pi_k A \Pi_k)}{\operatorname{Tr}_L \Pi_k}$$

with very high probability and with an error given by the square root of the inverse of the dimension of the k-momentum sector, $1/\sqrt{\operatorname{Tr}_L \Pi_k}$. This holds in the sense of stochastic domination given in Definition 9.2.5. Using that $\operatorname{Tr}_L \Pi_k \approx 2^L/L$ from Lemma 9.3.4, we obtain that (9.3.5) holds for

each fixed k, uniformly in all eigenvectors. Finally, the very high probability control in the stochastic domination allows us to take the maximum over $k = 1, 2, \ldots, L$ by a simple union bound. This completes the proof of (9.3.5). □

We remark that the essential ingredient of this proof, the Theorem 2.2 from [165], applies not only for the Gaussian ensemble but for arbitrary Wigner matrices with i.i.d. entries (with some moment condition on their entry distribution) and its proof is quite involved. However, ETH for GUE, as needed in Lemma 9.3.6, can also be proven with much more elementary methods using that the eigenvectors are columns of a Haar unitary matrix. Namely, moments of $\langle E_\alpha|A|E_\beta\rangle$ can be directly computed using Weingarten calculus [179]. Since in (9.3.5) we aim at a control with very high probability, this would require to compute arbitrary high moments of $\langle E_\alpha|A|E_\beta\rangle - \delta_{\alpha,\beta}\langle A\rangle_k$. The Weingarten formalism gives the exact answer but it is somewhat complicated for high moments, so identifying their leading order (given by the "ladder" diagrams) requires some elementary efforts. For brevity, we therefore relied on [165, Theorem 2.2] in the proof of Lemma 9.3.6 above.

Finally, we formulate the fourth and last step of the proof of Proposition 9.3.3 in the following lemma, the proof of which is given in Section 9.3.2.

Lemma 9.3.7 (Step 4: Traces within momentum sectors)**.** *Let $A = A_q \otimes I_{L-q}$ be an arbitrary q-local observable with $\|*\|A \lesssim 1$. Then it holds that*

$$\max_k \left| \frac{\mathrm{Tr}_L(\Pi_k A \Pi_k)}{\mathrm{Tr}_L \Pi_k} - \langle A\rangle \right| \leq \mathcal{O}\left(\frac{L}{2^{\min(L-q,L/2)}}\right). \tag{9.3.6}$$

Moreover, for $q > L/2 + 1$ this bound is optimal (up to the factor L).

Armed with these four lemmas, we can now turn to the proof of Proposition 9.3.3.

Proof of Proposition 9.3.3. First, for any q-local observable $A = A_q \otimes I_{L-q}$, we conclude from Lemma 9.3.6 and Lemma 9.3.7 that

$$\max_{\alpha,\beta} \max_k \left| \langle E_\alpha^{(k)}|A|E_\beta^{(k)}\rangle - \delta_{\alpha\beta} \frac{\mathrm{Tr}_L(\Pi_k A \Pi_k)}{\mathrm{Tr}_L \Pi_k} \right| \prec \frac{1}{2^{\min(L-q,L/2)}}. \tag{9.3.7}$$

For the element $\langle E_\alpha^{(k)}|A|E_\beta^{(k')}\rangle$ with $k \neq k'$, i.e. in off-diagonal blocks, $|*\rangle\, E_\alpha^{(k)}$ and $|*\rangle\, E_\beta^{(k')}$ are normalised Gaussian vectors independent of each other. Therefore standard concentration estimate shows that

$$\max_{k\neq k'} |\langle E_\alpha^{(k)}|A|E_\beta^{(k')}\rangle| \prec \frac{1}{2^{L/2}}. \tag{9.3.8}$$

Combining (9.3.7) with (9.3.8), we have proven Proposition 9.3.3. □

9.3.1 Dimensions of momentum sectors: Proof of Lemma 9.3.4

In this section we prove Lemma 9.3.4, and establish that the sizes of the momentum sectors are almost equal. To this end, we show that the leading term in the size of each of the momentum blocks is given by the number of aperiodic elements in the product basis of $\mathcal{H}$.

We present the proof using group theory notation, which is not strictly necessary for the one-dimensional case under consideration since the translation group of the torus $\mathbb{T}_L$ is cyclic. Nevertheless, we do it to allow for a more straightforward generalisation to the d-dimensional case (cf. Lemma 9.A.4).

Proof. We introduce the following objects. Let $\mathfrak{s}$ denote the canonical product basis of $\mathcal{H}$,

$$\mathfrak{S}(L) \coloneqq \{\sigma : \mathbb{T}_L \to \{|\uparrow\rangle, |\downarrow\rangle\}\}, \tag{9.3.9}$$

and let $\mathcal{G}$ be the group of translations of $\mathbb{T}_L$ generated by $T = T_L$. Note that $\mathcal{G}$ is a finite cyclic group of size $|\mathcal{G}| = L$. The action of $\mathcal{G}$ on $\mathfrak{S}(L)$ is defined by

$$(g\sigma)(x) \coloneqq \sigma(g^{-1}(x)), \quad x \in \mathbb{T}_L, \quad \sigma \in \mathfrak{S}(L), \quad g \in \mathcal{G}. \tag{9.3.10}$$

In particular, the set $\mathfrak{S}(L)$ is a disjoint union of sets $\mathfrak{S}_b(L)$ defined by

$$\mathfrak{S}_b(L) \coloneqq \{\sigma \in \mathfrak{S}(L) : |\mathcal{G}_\sigma| = b\}, \quad b = 1, 2, \ldots, L,$$

where $\mathcal{G}_\sigma \subset \mathfrak{S}(L)$ is the stabilizer of σ under the action (9.3.10). By the orbit-stabilizer theorem, $\mathfrak{S}_b(L) = \varnothing$ for all b that do not divide L. Since the group $\mathcal{G}$ is cyclic, it has a unique subgroup of size b for all $b|L$, given explicitly by

$$\mathcal{G}^{(b)} \coloneqq \{T^{L/b}, T^{2L/b}, \ldots, T^L\}.$$

Observe that each $\sigma \in \mathfrak{S}_b(L)$ corresponds to a unique map $\widetilde{\sigma}$ on a reduced torus $\mathfrak{S}(L/b) \coloneqq \mathbb{T}_L / \mathcal{G}^{(b)}$, which is defined by

$$\widetilde{\sigma}([x]) \coloneqq \sigma(x), \quad [x] \in \mathfrak{S}(L/b). \tag{9.3.11}$$

Since σ is stabilised by $\mathcal{G}^{(b)}$, the map $\sigma \mapsto \widetilde{\sigma}$ in (9.3.11) is well-defined and injective. In particular, $|\mathfrak{S}_b(L)| \le 2^{L/b}$, and hence

$$2^L = \sum_{b|L} |\mathfrak{S}_b(L)| = M(L) + \sum_{b|L, b\ge 2} |\mathfrak{S}_b(L)| \le M(L) + \mathcal{O}(L^{1/2} 2^{L/2}), \tag{9.3.12}$$

where $M(L) \coloneqq |\mathfrak{S}_1(L)|$ denotes the number of elements in $\mathfrak{S}(L)$ with a trivial stabilizer. The last inequality follows from the fact that L has at most $\mathcal{O}(L^{1/2})$ divisors.

Since $M(L) \le 2^L$, we conclude from (9.3.12) that

$$M(L) = 2^L + \mathcal{O}(L^{1/2} 2^{L/2}). \tag{9.3.13}$$

For any $k \in \{0, \ldots, L-1\}$, we can construct an eigenvector of T corresponding to the eigenvalue $e^{2\pi i k/L}$ by defining

$$\boldsymbol{v}(\sigma, k) \coloneqq \Pi_k \sigma = \frac{1}{L} \sum_{j=0}^{L-1} e^{2\pi i \frac{kj}{L}} T^{-j} \sigma, \quad \sigma \in \mathfrak{S}_1(L). \tag{9.3.14}$$

Since the orbit of σ under T consists of L distinct basis elements, the vector $\boldsymbol{v}(\sigma, k)$ is non-zero. Furthermore, the vectors $\boldsymbol{v}(\sigma, k)$ and $\boldsymbol{v}(\sigma', k)$ corresponding to σ and σ' in disjoint orbits are linearly independent because they share no basis element. Therefore, the dimension of the k-th momentum space is bounded from below by the number of disjoint orbits in $\mathfrak{S}_1(L)$, that is

$$\operatorname{Tr}_L \Pi_k \ge \frac{2^L}{L} + \mathcal{O}(L^{-1/2} 2^{L/2}), \tag{9.3.15}$$

where we used inequality (9.3.13) and the fact that all orbits in $\mathfrak{S}_1(L)$ have size L. By means of (9.3.15), we obtain the following chain of inequalities

$$\operatorname{Tr}_L \Pi_k = 2^L - \sum_{j \ne k} \operatorname{Tr}_L \Pi_j \le \frac{2^L}{L} + \mathcal{O}(L^{1/2} 2^{L/2}), \tag{9.3.16}$$

which, together with (9.3.15) concludes the proof of Lemma 9.3.4. □

9.3.2 Traces within momentum sectors: Proof of Lemma 9.3.7

In this section, we give a proof of Lemma 9.3.7, which evaluates the difference of the noramalised trace $\mathrm{Tr}_L(\Pi_k A \Pi_k)/\mathrm{Tr}_L\,\Pi_k$ on a momentum sector and the full normalised trace $\langle A\rangle$ for a q-local observable $A = A_q \otimes I_{L-q}$. We separate A into the tracial part $\langle A\rangle I$ and the traceless part $\mathring{A} \coloneqq A - \langle A\rangle I$.

Proof of Lemma 9.3.7. Substituting $\Pi_k \coloneqq \frac{1}{L}\sum_{j=1}^{L} e^{2\pi i \frac{kj}{L}} T_L^{-j}$, we obtain

$$\begin{aligned}\mathrm{Tr}_L(\Pi_k A \Pi_k) &= \langle A\rangle \,\mathrm{Tr}_L\,\Pi_k + \mathrm{Tr}_L(\Pi_k \mathring{A}) \\ &= \langle A\rangle \,\mathrm{Tr}_L\,\Pi_k + \frac{1}{L}\sum_{j=1}^{L-1} e^{2\pi i \frac{kj}{L}}\,\mathrm{Tr}_L(T_L^{-j}\mathring{A})\,. \end{aligned} \tag{9.3.17}$$

Then, the task is to evaluate the size of the quantity $\mathrm{Tr}_L(T_L^{-j}\mathring{A})$.

Lemma 9.3.8. *Let $A \coloneqq A_q \otimes I_{L-q}$ be a q-local observable with $\|*\|\,A \lesssim 1$. Then, for any $j = 1,\dots,L-1$, we have*

$$|\mathrm{Tr}_L(T_L^{-j}A)| \lesssim 2^{\max(q,\gcd(j,L))}\,, \tag{9.3.18}$$

where $\gcd$ *stands for the greatest common divisor.*

Combining (9.3.18) with $\gcd(j,L) \le L/2$ for $j = 1,\dots,L-1$ and Lemma 9.3.4 gives the bound (9.3.6). The optimality of (9.3.6) for $q > L/2+1$ is proven in Lemma 9.3.9 below. □

It remains to give the proof of Lemma 9.3.8.

Proof of Lemma 9.3.8. We choose a product basis $\{|s_1\dots s_L\rangle \mid s_j \in \{\uparrow,\downarrow\}\}$ to calculate the trace. Then, we obtain

$$\begin{aligned}|\mathrm{Tr}_L(T_L^{-j}A)| &= |\sum_{s_1\dots s_L} \langle s_{1+j}\dots s_{q+j}|A_q|s_1\dots s_q\rangle \prod_{m=q+1}^{L}\delta_{s_m s_{m+j}}| \\ &\lesssim \sum_{s_1\dots s_L}\prod_{m=q+1}^{L}\delta_{s_m s_{m+j}}\,. \end{aligned} \tag{9.3.19}$$

Because of the product $\prod_{m=q+1}^{L}\delta_{s_m s_{m+j}}$ of Kronecker deltas, not all of the summation variables $s_1,\dots,s_L$ are independent.

To count the number of independent summations in the right-hand side of (9.3.19) and obtain an upper bound for $\mathrm{Tr}_L(T_L^{-j}A)$ with $j = 1,\dots,L-1$, we count the number of independent deltas in the product

$$\mathcal{G}_{q,j}^{(L)} \coloneqq \prod_{m=q+1}^{L}\delta_{s_m s_{m+j}}\,. \tag{9.3.20}$$

Here, not all of the delta functions in $\mathcal{G}_{q,j}^{(L)}$ are independent in the sense that we may express $\mathcal{G}_{q,j}^{(L)}$ with a fewer number of deltas. For example, we have $\mathcal{G}_{1,2}^{(4)} = \delta_{s_2 s_4}\delta_{s_3 s_1}\delta_{s_4 s_2} = \delta_{s_3 s_1}\delta_{s_4 s_2}$.

To obtain an expression of $\mathcal{G}_{q,j}^{(L)}$ with the minimal number of deltas, we graphically represent the product $\prod_{m=q+1}^{L}\delta_{s_m s_{m+j}}$ by arranging the sites on a circle and representing the $\delta_{s_m s_{m+j}}$'s with a line connecting the site m and $m+j$ (Figure 9.3.1). A minimal representation of $\mathcal{G}_{q,j}^{(L)}$ is obtained by removing exactly one delta for every occurrence of a loop in the graph of $\prod_{m=q+1}^{L}\delta_{s_m s_{m+j}}$.

The graph of $\prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$ can be obtained in two steps: First, in step (i), drawing the graph of $\prod_{m=1}^{L} \delta_{s_m s_{m+j}}$ and second, in step (ii), removing the lines corresponding to the delta functions $\delta_{s_m s_{m+j}}$ $(m = 1, \dots, q)$, which are depicted with red dashed lines in Figure 9.3.1.

In the first step (i), there are exactly $\gcd(j, L)$ loops each starting from the sites $1, \dots, \gcd(j, L)$. If $q > \gcd(j, L)$, there is no loop remaining after the second step (ii). Thus, we obtain a minimal representation of $\mathcal{G}_{q,j}^{(L)}$ as $\mathcal{G}_{q,j}^{(L)} = \prod_{m=q+1}^{L} \delta_{s_m s_{m+j}}$. If $q \leq \gcd(j, L)$, the loops starting from the sites $q+1, \dots, \gcd(j, L)$ remain after the second step (ii), for each of which we remove one delta to obtain a minimal representation of $\mathcal{G}_{q,j}^{(L)}$ as $\mathcal{G}_{q,j}^{(L)} = \prod_{m=\gcd(j,L)+1}^{L} \delta_{s_m s_{m+j}}$.

In summary, we obtain a minimal representation of $\mathcal{G}_{q,j}^{(L)}$ as

$$\mathcal{G}_{q,j}^{(L)} = \prod_{m=\max(q,\gcd(j,L))+1}^{L} \delta_{s_m s_{m+j}} \,. \tag{9.3.21}$$

By substituting (9.3.21) into (9.3.19) we obtain

$$|\mathrm{Tr}_L(T_L^{-j} A)| \lesssim \sum_{s_1 \dots s_L} \prod_{m=\max(q,\gcd(j,L))+1}^{L} \delta_{s_m s_{m+j}} = 2^{\max(q,\gcd(j,L))} \,. \qquad \square$$

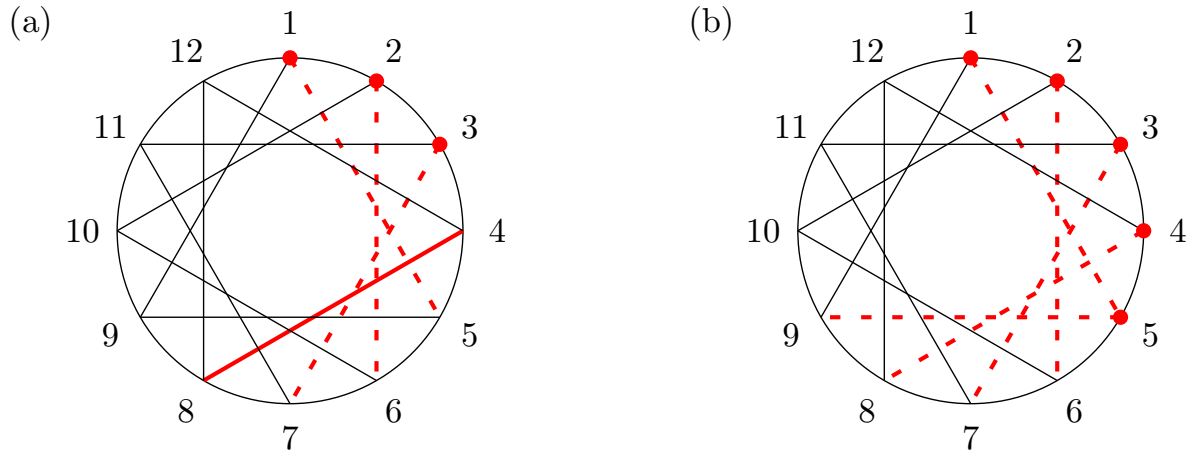

Figure 9.3.1: Graphical representation of the product $\prod_{m=q+1}^{L} \delta_{s_{m+j} s_m}$ for (a) $L = 12$, $q = 3$, $j = 4$ and (b) $L = 12$, $q = 5$, $j = 4$. For the first case (a) where $q < \gcd(j, L)$, there is a loop 4-8-12-4 remaining after the step (ii), which contains exactly one redundant delta function $\delta_{s_4 s_8}$ depicted with a solid red line. In general, exactly one redundant delta function appears for every occurrence of a loop in the graph of $\prod_{m=q+1}^{L} \delta_{s_{m+j} s_m}$.

Finally, we prove the optimality of (9.3.6) in the regime $q > L/2 + 1$.

Lemma 9.3.9. *Let $B_q \coloneqq T_q + T_q^{-1} - 2^{2-q} I_q$, where T_q is the (left) translation operator acting only on the first q spins arranged on the torus $\mathbb{T}_q$. Observe that B_q is Hermitian and traceless. Then, for $q > L/2 + 1$, the normalised trace of $B \coloneqq B_q \otimes I_{L-q}$ within the k-momentum sector is given by*

$$\frac{\mathrm{Tr}_L(\Pi_k B \Pi_k)}{\mathrm{Tr}_L \Pi_k} = \frac{2}{2^{L-q}} \cos\left(\frac{2\pi k}{L}\right) + \mathcal{O}\left(\frac{L}{2^{L/2}}\right). \tag{9.3.22}$$

This shows that the q-local observable $B_q \coloneqq T_q + T_q^{-1} - 2^{2-q} I_q$ saturates the bound (9.3.6) when $q > L/2 + 1$. It also shows that the deviation of the normalised trace within a momentum sector, $\mathrm{Tr}_L(\Pi_k B \Pi_k)/\mathrm{Tr}_L \Pi_k$, from $\langle B \rangle = 0$, which is of order $2^{-(L-q)}$, becomes the dominant source of error in the ETH whenever $q > L/2 + 1$.

Proof of Lemma 9.3.9. We first reduce the range of the summation over j in the generally valid expression (9.3.17) applied to B_q. To do so, we introduce the parity operator P_L defined by $P_L|*\rangle s_1 s_2 \dots s_L \coloneqq |*\rangle s_L \dots s_2 s_1$. It satisfies $P_L T_L P_L = T_L^{-1}$ and $P_L A P_L = I_{L-q} \otimes (P_q A_q P_q)$ for any $A = A_q \otimes I_{L-q}$. Since B_q is invariant under the parity transformation, we have

$$\mathrm{Tr}_L(T_L^{-j} B) = \mathrm{Tr}_L[T_L^{+j}(I_{L-q} \otimes B_q)] = \mathrm{Tr}_L(T_L^{-(L-j)} B)\,,$$

Therefore, we can rewrite (9.3.17) with the aid of (9.2.3) as

$$\mathrm{Tr}_L(\Pi_k B \Pi_k) = \frac{2}{L} \sum_{j=1}^{\lfloor \frac{L}{2} \rfloor} \mathrm{Tr}_L(T_L^{-j} \mathring{B}) \cos\left(\frac{2\pi k j}{L}\right) + \begin{cases} \frac{(-1)^k}{L} \mathrm{Tr}_L(T_L^{-\frac{L}{2}} \mathring{B}) & L \text{ even} \\ 0 & L \text{ odd}\,. \end{cases} \tag{9.3.23}$$

When $q > L/2 + 1$, we have $j < q$ and cannot skip over the region $1, \dots, j$ when going along the lines in the graph of $\mathcal{G}_{q,j}^{(L)}$ (recall (9.3.20)). Therefore, each line starting at one of the sites $p \in \{q+1, \dots, q+j\}$ passes through a point in $\{1, \dots, j\}$. Moreover, the correspondence between p and the first intersection of the line starting at p with $\{1, \dots, j\}$ is one-to-one. Therefore, there exists a permutation τ_j on $1, \dots, j$ such that $s_{q+i} = s_{\tau(i)}$ for $i = 1, \dots, j$ due to $\mathcal{G}_{q,j}^{(L)}$. With this permutation τ, we obtain

$$\begin{aligned} \mathrm{Tr}_L(T_L^{-j} B) &= \sum_{s_1 \dots s_L} \langle s_{q+1} \dots s_q s_{q+1} s_{q+j} | B_q | s_1 \dots s_q \rangle \, \mathcal{G}_{q,j}^{(L)} \\ &= \sum_{s_1 \dots s_q} \langle s_{q+1} \dots s_q s_{\tau(1)} s_{\tau(j)} | B_q | s_1 \dots s_q \rangle \\ &= \mathrm{Tr}_q(\tau_j^\dagger T_q^{-j} B_q) \\ &= \mathrm{Tr}_q(\tau_j^\dagger T_q^{-(j-1)}) + \mathrm{Tr}_q(\tau_j^\dagger T_q^{-(j+1)}) - 2^{2-q} \mathrm{Tr}_q(\tau_j^\dagger T_q^{-j})\,. \end{aligned} \tag{9.3.24}$$

Because τ_j is a j-local operator (not necessarily self-adjoint) on the q-site chain, we can apply Lemma 9.3.8 to each term in (9.3.24). Combined with $j < q - 1$ and $\gcd(j, q) \le j$, we obtain

$$\mathrm{Tr}_L(T_L^{-j} B) = \delta_{j1} 2^q + \mathcal{O}(2^j) = \delta_{j1} 2^q + \mathcal{O}(2^{L/2})\,.$$

Substituting this result into (9.3.23) and employing $\mathrm{Tr}_L \Pi_k = \frac{2^L}{L} + \mathcal{O}(L^{1/2} 2^{L/2})$ from Lemma 9.3.4, we obtain the result (9.3.22). □

9.4 Numerical verification of Theorem 9.3.1 for $\ell = \mathcal{O}(1)$

In this section, we numerically demonstrate that Theorem 9.3.1 also holds for the non-mean-field case of $\ell = 2$. For that purpose, we adopt the following measure of the ETH used in Refs [539, 540]. For any self-adjoint operator A we define

$$\Lambda = \Lambda(A) \coloneqq \mathbf{E} \max_k {\max_\alpha}' \frac{|\langle E_\alpha^{(k)} | A | E_\alpha^{(k)} \rangle - \langle A \rangle_\Delta^{(\mathrm{mc})}(E_\alpha^{(k)})|}{a_{\max} - a_{\min}}\,, \tag{9.4.1}$$

where $a_{\max(\min)}$ is the maximum (minimum) eigenvalue of A. Here, $\mathbf{E}$ denotes the average over the realisations of the Hamiltonian (9.2.1), and $\max_\alpha{}'$ denotes the maximum over the eigenstates $|E_\alpha^{(k)}\rangle$ in the energy shell at the center of the spectrum, i.e. those α for which

$$|E_\alpha^{(k)} - \langle H \rangle| \le \Delta\,.$$

The width Δ of the energy interval is set to be $\Delta = 0.4/L$ such that it satisfies the two physical requirements mentioned in Remark 9.2.4 for $L \ge 6$. With this choice of Δ, the microcanonical energy

shell $\mathcal{H}_{\langle H\rangle,\Delta}$ defined by (9.4.1) typically contains more than 10 states, while the density of states does not change too much within $\mathcal{H}_{\langle H\rangle,\Delta}$.

As the observable, we choose $A = B_q \otimes I_{L-q}$ with $B_q \coloneqq T_q + T_q^{-1} - 2^{2-q} I_q$ for $q = 2, \dots, L$, which saturates the upper bound in (9.3.6) and thus also saturates that of (9.3.2). With this choice we have $a_{\max} - a_{\min} \simeq 4$ for any L and q. Therefore, the ETH measure Λ is essentially the same as the diagonal part of the left-hand side of (9.3.2) in Theorem 9.3.1 – except that the maximum over α is now taken only at the center of the spectrum (and we do not take maximum over all A). This is because the eigenstate expectation value $\langle E_\alpha^{(k)}|A|E_\alpha^{(k)}\rangle$ of a local observable $A = A_q \otimes I_{L-q}$ with $q \ll L$ typically acquires an energy dependence when $\ell \ll L$ [321], and the number of states becomes not enough to calculate the microcanonical average near the edges for the computationally accessible system size. The ETH measure Λ satisfies reasonable thermodynamical properties. It is (i) invariant under the linear transformation $A \mapsto aA + b$, (ii) dimensionless, and (iii) thermodynamically intensive for additive observables A [539].

Figures 9.4.1(a)-(c) depict the L-dependence of the ETH measure Λ for different values of the parameter q. In particular, Figure 9.4.1(b) illustrates that, whenever L is approximately equal to q so that $L - q < L/2$, the ETH measure Λ decays as $\propto 2^{-L}$. The rate of this decay is slower for smaller values of q, but approaches 2^{-L} as q becomes larger. In Figure 9.4.1(c), we take a closer look at the L-dependence of Λ for $q = 6$. The data indicates that for $L - q \ll L/2$, Λ decays as $\propto 1.8^{-L}$, whereas for $L \gtrsim 2q$, Λ decays as $\propto 1.8^{-L/2}$. These numerical observations are in agreement with our analytical results for the mean-field case in Theorem 9.3.1, which predicts that the exponent of the exponential decrease of Λ in L should be twice as large in the region $L - q \ll L/2$ compared to the region $L/2 \gtrsim L - q$. This fact suggests that the theorem remains qualitatively valid for $\ell = \mathcal{O}(1)$ in the bulk of the spectrum as long as the energy shell width is appropriately chosen.

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9.A Extension to higher dimensions

In this appendix, we extend our main result, Theorem 9.3.1, to the d-dimensional case.

9.A.1 Multidimensional setup

Let $\boldsymbol{L} \coloneqq (L_1, \dots, L_d)$ be a vector of positive integers and set $V \coloneqq \prod_{s=1}^d L_s$. We consider a d-dimensional system with V quantum spins at the vertices of the classical discrete torus

$$\mathbb{T}_{\boldsymbol{L}} \coloneqq \bigtimes_{s=1}^{d} \mathbb{Z}/L_s\mathbb{Z}\,.$$

As before, on each vertex, the one particle Hilbert space is given by $\mathbf{C}^2$ with canonical basis $\{|\uparrow\rangle, |\uparrow\rangle\}$. The corresponding V-particle Hilbert space is given by

$$\mathcal{H} \coloneqq \bigotimes_{s=1}^{V} \mathbf{C}^2 \quad \text{with dimension} \quad \dim\mathcal{H} = 2^V\,.$$

For a vector $\boldsymbol{q} = (q_1, \dots, q_d) \in \mathbb{T}_{\boldsymbol{L}}$, we introduce a rectangular subregion $\mathcal{R}_{\boldsymbol{q}} \subset \mathbb{T}_{\boldsymbol{L}}$ by

$$\mathcal{R}_{\boldsymbol{q}} \coloneqq \{\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{T}_{\boldsymbol{L}} : 1 \le x_s \le q_s\,,\ s = 1, \dots, d\}\,.$$

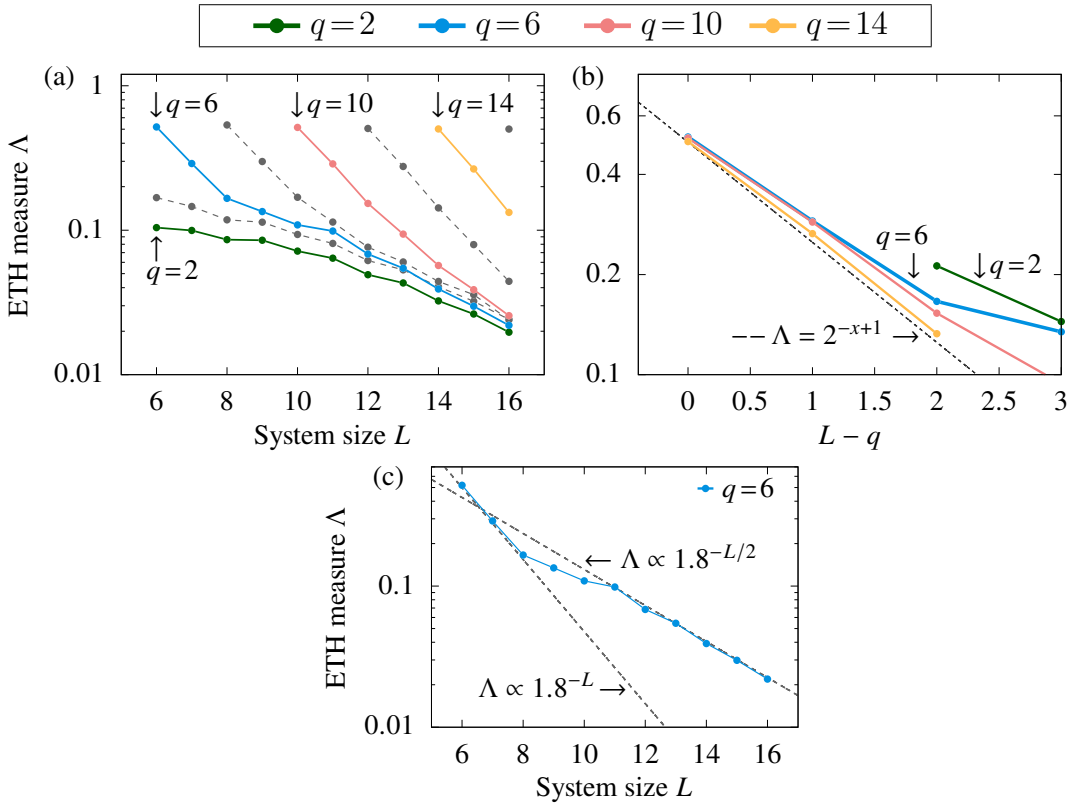

Figure 9.4.1: (a) System-size dependence of the ETH measure Λ for the observable $A = B_q \otimes I_{L-q}$ with $B_q \coloneqq T_q + T_q^{-1} - 2^{2-q} I_q$. Grey curves between colored curves show intermediate values of q, i.e., $q = 4, 8, 12$. (b) The same data as the panel (a) for $q = 2, 6, 10$ and 14 plotted against $L - q$. When $L \simeq q$ so that $L - q < L/2$, Λ decreases as $\propto 2^{-L}$. (c) The same data as the panel (a) for $q = 6$. It decreases as $\propto 1.8^{-L}$ when $L - q \ll L/2$. When $L/2 \gtrsim L - q$, the decrease in Λ becomes slower and follows a different exponential decay with a base of $1.8^{-1/2}$, instead of 1.8^{-1}. Aside from the value of the base, this behavior is consistent with (9.3.2), which predicts that the exponent of the exponential decrease of Λ in L should be twice as large in the region $L - q \ll L/2$ compared to the region $L/2 \gtrsim L - q$. The standard errors are smaller than the size of the data points. The number of samples lies between 1000 and 10000 for each datum.

A self-adjoint operator of the form $A = A_{\boldsymbol{q}} \otimes I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}}$ is called a ***q*-local observable**, where $A_{\boldsymbol{q}}$ is self-adjoint and acts on the Hilbert space of the spins in $\mathcal{R}_{\boldsymbol{q}}$, and $I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}}$ is the identity on $\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}$.

Finally, let T_s be the (left) translation operator along the s-th coordinate acting on $\mathbb{T}_{\boldsymbol{L}}$. For a vector $\boldsymbol{j} \coloneqq (j_1, \dots, j_d) \in \mathbb{T}_{\boldsymbol{L}}$, we introduce $T^{\boldsymbol{j}} \coloneqq \prod_{s=1}^{d} T_s^{j_s}$.

The d-dimensional version of our model in Definition 9.2.1 is given as follows.

Definition 9.A.1. *Set the vector $\boldsymbol{\ell} \coloneqq (\ell_1, \dots, \ell_d) \in \mathbb{T}_{\boldsymbol{L}}$ that determines the interaction range in each coordinate direction. We define the ensemble of Hamiltonians with local interactions as*

$$H_{\boldsymbol{L}}^{(\boldsymbol{\ell})} \coloneqq \sum_{\boldsymbol{j} \in \mathbb{T}_{\boldsymbol{L}}} T^{-\boldsymbol{j}} (h_{\boldsymbol{\ell}} \otimes I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{\ell}}}) T^{\boldsymbol{j}} \quad \text{with} \quad h_{\boldsymbol{\ell}} \coloneqq \sum_{p_{\boldsymbol{1}}, \dots, p_{\boldsymbol{\ell}}=0}^{3} J_{p_{\boldsymbol{1}}, \dots, p_{\boldsymbol{\ell}}} \sigma_{\boldsymbol{1}}^{(p_{\boldsymbol{1}})} \dots \sigma_{\boldsymbol{\ell}}^{(p_{\boldsymbol{\ell}})} \tag{9.A.1}$$

where the symbols $\boldsymbol{1}$, $\boldsymbol{2}$, …, $\boldsymbol{\ell}$ label the elements of $\mathcal{R}_{\boldsymbol{\ell}}$ in an arbitrary order. As in (9.2.1), *$\sigma^{(p)}$ for $p \in \{0, 1, 2, 3\}$ are the Pauli matrices* (9.2.2).

The $4^{|\mathcal{R}_{\boldsymbol{\ell}}|}$ coefficients $J_{p_{\boldsymbol{1}}, \dots, p_{\boldsymbol{\ell}}}$ are i.i.d. real Gaussian random variables with zero mean, $\mathbb{E} J_{p_{\boldsymbol{1}}, \dots, p_{\boldsymbol{\ell}}} = 0$, and variance $v_{\boldsymbol{\ell}}^2 \coloneqq \mathbb{E} |J_{p_{\boldsymbol{1}}, \dots, p_{\boldsymbol{\ell}}}|^2$.

We have the following multidimensional analog of Lemma 9.2.2,

Lemma 9.A.2. *Let*

$$\Pi_{\boldsymbol{k}} \coloneqq \frac{1}{V} \sum_{\boldsymbol{j} \in \mathbb{T}_{\boldsymbol{L}}} e^{2\pi i \sum_{s=1}^{d} \frac{k_s j_s}{L_s}} T^{-\boldsymbol{j}} \quad \text{for} \quad \boldsymbol{k} \in \mathbb{T}_{\boldsymbol{L}}$$

be the projection operator onto the $\boldsymbol{k}$-momentum space, i.e. $T_s \Pi_{\boldsymbol{k}} = e^{2\pi i \frac{k_s}{L_s}} \Pi_{\boldsymbol{k}}$ for all $s = 1, \dots, d$. Then we have

$$H_{\boldsymbol{L}}^{(\ell)} = V \sum_{\boldsymbol{k} \in \mathbb{T}_{\boldsymbol{L}}} \Pi_{\boldsymbol{k}} (h_{\ell} \otimes I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\ell}}) \Pi_{\boldsymbol{k}} \,.$$

Proof. This follows by Lemma 9.2.2 coordinatewise. □

Denoting by $|E_\alpha^{(\boldsymbol{k})}\rangle$ the normalised eigenvector of $H_{\boldsymbol{L}}^{(\ell)}$ belonging to an eigenvalue E_α and the $\boldsymbol{k}$-momentum sector, i.e. $H_{\boldsymbol{L}}^{(\ell)} |E_\alpha^{(\boldsymbol{k})}\rangle = E_\alpha |*\rangle\, E_\alpha^{(\boldsymbol{k})}$ and $\Pi_{\boldsymbol{k}} |*\rangle\, E_\alpha^{(\boldsymbol{k})} = |*\rangle\, E_\alpha^{(\boldsymbol{k})}$, the definition of the *microcanocical average* is completely analogous to Definition 9.2.3.

Moreover, whenever we use the notation $\prec$ for stochastic domination (Definition 9.2.5), it is always understood with $N \coloneqq 2^V$.

9.A.2 Multidimensional version of the main result

The d-dimensional version of Theorem 9.3.1 is then given as follows.

Theorem 9.A.3 (ETH in d-dimensional translation-invariant systems)**.** *Let $\ell = \boldsymbol{L}$ and consider the the Hamiltonian $H_{\boldsymbol{L}}^{(\boldsymbol{L})}$ from* (9.A.1) *with eigenvalues $E_\alpha^{(\boldsymbol{k})}$ and normalised eigenvectors $|*\rangle\, E_\alpha^{(\boldsymbol{k})}$. Then, for every $\Delta > 0$ and bounded $\boldsymbol{q}$-local observable $A = A_{\boldsymbol{q}} \otimes I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}}$ with $q_s \le L_s/2$ for all $s = 1, \dots, d$, it holds that*

$$\max_{\alpha,\beta} \max_{\boldsymbol{k},\boldsymbol{k}'} \left| \langle E_\alpha^{(\boldsymbol{k})} | A | E_\beta^{(\boldsymbol{q}\boldsymbol{k}')} \rangle - \delta_{\alpha\beta} \delta_{\boldsymbol{k},\boldsymbol{k}'} \langle A \rangle_\Delta^{(\mathrm{mc})} (E_\alpha^{(\boldsymbol{k})}) \right| \prec \frac{1}{2^{V/2}} \,. \tag{9.A.2}$$

That is, the ETH holds with optimal speed of convergence.

The principal strategy for proving Theorem 9.A.3 is exactly the same as for Theorem 9.3.1, which has been outlined right below Proposition 9.3.3. We shall hence only discuss the differences compared to the proof in Section 9.3, which consist solely of Step 1 (generalizing Lemma 9.3.4, cf. Lemma 9.A.4) and Step 4 (generalizing Lemma 9.3.7, cf. Lemma 9.A.4).

Lemma 9.A.4 (Step 1: Dimensions of momentum sectors)**.** *The dimension $\mathrm{Tr}_{\boldsymbol{L}} \Pi_{\boldsymbol{k}}$ of the $\boldsymbol{k}$-momentum sectors for $\boldsymbol{k} \in \mathbb{T}_{\boldsymbol{L}}$ is almost equal to each other in the sense that we have*

$$\mathrm{Tr}_{\boldsymbol{L}}\, \Pi_{\boldsymbol{k}} = \frac{2^V}{V} + \mathcal{O}\big(2^{V/2 + (\log_2 V)^2}\big) \,.$$

Proof. Let $\mathfrak{S} = \mathfrak{S}(\boldsymbol{L})$ denote the canonical product basis of $\mathcal{H}$, as in (9.3.9), and let $\mathcal{G}$ be the commutative group generated by the translation operators $\{T_s\}_{s=1}^d$. The action of $\mathcal{G}$ on $\mathfrak{S}$ is defined by (9.3.10).

In general, the group $\mathcal{G}$ is not cyclic, hence the subgroups of $\mathcal{G}$ are not uniquely determined by their size. However, $\mathfrak{S}$ can be decomposed into a disjoint union of sets $\mathfrak{S}_{\mathcal{K}} = \mathfrak{S}_{\mathcal{K}}(\boldsymbol{L})$ defined by

$$\mathfrak{S}_{\mathcal{K}} \coloneqq \{\sigma \in \mathfrak{S} : \mathcal{G}_\sigma = \mathcal{K}\} \,,$$

where $\mathcal{G}_\sigma \subset \mathcal{G}$ is the stabilizer of σ under the action (9.3.10), and $\mathcal{K} \leq \mathcal{G}$ is a subgroup of $\mathcal{G}$. Similarly to (9.3.11), for any subgroup $\mathcal{K}$ of $\mathcal{G}$, we define the map

$$\varphi_{\mathcal{K}} : \mathfrak{S}_{\mathcal{K}} \to \left(\mathbb{T}_{\boldsymbol{L}} \diagup \mathcal{K} \to \{\uparrow, \downarrow\} \right), \quad \big(\varphi_{\mathcal{K}}(\sigma)\big)([x]) \coloneqq \sigma(x), \quad [x] \in {}^{\mathbb{T}_{\boldsymbol{L}}} \diagup_{\mathcal{K}},$$

which is easily seen to be an injection and hence and injection, $|\mathfrak{S}_{\mathcal{K}}| \leq 2^{V/|\mathcal{K}|}$. Therefore, denoting the number of elements in $\mathfrak{S}$ with a trivial stabilizer by $M(\boldsymbol{L})$, we obtain

$$2^V = \sum_{\mathcal{K} \leq G} |\mathfrak{S}_{\mathcal{K}}| = M(\boldsymbol{L}) + \sum_{K \leq G, |\mathcal{K}| \geq 2} |\mathfrak{S}_{\mathcal{K}}| \leq M(\boldsymbol{L}) + s(\mathcal{G}) 2^{V/2},$$

where $s(\mathcal{G})$ denotes the number of subgroups of $\mathcal{G}$. Combining this with the following well-known bound[1]

$$s(\mathcal{G}) \leq |\mathcal{G}|^{\log_2 |\mathcal{G}|} \tag{9.A.3}$$

and the trivial estimate $M(\boldsymbol{L}) \leq 2^V$, we conclude that

$$M(\boldsymbol{L}) = 2^V + \mathcal{O}\big(2^{V/2 + (\log_2 V)^2}\big). \tag{9.A.4}$$

The construction of a linearly independent vectors with a fixed momentum $\boldsymbol{k} \in \mathbb{T}_{\boldsymbol{L}}$ for each disjoint orbit with a trivial stabilizer is analogous to (9.3.14). Using estimates analogous to (9.3.15) and (9.3.16) together with (9.A.4) concludes the proof of Lemma 9.A.4. □

Finally, we discuss the generalisation of Step 4, i.e. Lemma 9.3.7.

Lemma 9.A.5 (Step 4: Traces within momentum sectors)**.** *Let $A = A_{\boldsymbol{q}} \otimes I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}}$ be a bounded $\boldsymbol{q}$-local observable with $q_s \leq L_s/2$ for all $s = 1, \dots, d$. Then it holds that*

$$\max_{\boldsymbol{k}} \left| \frac{\operatorname{Tr}_{\boldsymbol{L}}(\Pi_{\boldsymbol{k}} A \Pi_{\boldsymbol{k}})}{\operatorname{Tr}_{\boldsymbol{L}} \Pi_{\boldsymbol{k}}} - \langle A \rangle \right| \leq \mathcal{O}\left(\frac{V}{2^{V/2}} \right). \tag{9.A.5}$$

Proof. Substituting $\Pi_{\boldsymbol{k}} \coloneqq \frac{1}{V} \sum_{\boldsymbol{j} \in \mathbb{T}_{\boldsymbol{L}}} e^{2\pi i \sum_{s=1}^d \frac{k_s j_s}{L_s}} T^{-\boldsymbol{j}}$ for $\boldsymbol{k} \in \mathbb{T}_{\boldsymbol{L}}$, we obtain

$$\operatorname{Tr}_{\boldsymbol{L}}(\Pi_{\boldsymbol{k}} A \Pi_{\boldsymbol{k}}) = \langle A \rangle \operatorname{Tr}_{\boldsymbol{L}} \Pi_{\boldsymbol{k}} + \frac{1}{V} \sum_{\boldsymbol{j} \in \mathbb{T}_{\boldsymbol{L}} \setminus \{\boldsymbol{0}\}} e^{2\pi i \sum_{s=1}^d \frac{k_s j_s}{L_s}} \operatorname{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}} \mathring{A}).$$

Then, the task is to evaluate the size of the quantity $\operatorname{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}} \mathring{A})$.

Lemma 9.A.6. *Let $A \coloneqq A_{\boldsymbol{q}} \otimes I_{\mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}}$ be a $\boldsymbol{q}$-local observable with $q_s \leq L_s/2$ for all $s = 1, \dots, d$ and $\|A\| \lesssim 1$. Then, for any $\boldsymbol{j} \in \mathbb{T}_{\boldsymbol{L}} \setminus \{\boldsymbol{0}\}$, we have that*

$$|\operatorname{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}} A)| \lesssim 2^{V/2}. \tag{9.A.6}$$

Combining (9.A.6) with Lemma 9.A.4 gives the bound (9.A.5). □

It remains to prove Lemma 9.A.6.

[1] More precisely, in order to see that (9.A.3) holds, observe that for any subgroup $\mathcal{K}$ of $\mathcal{G}$ and any $g \in \mathcal{G} \backslash \mathcal{K}$, the size of the subgroup generated by $\mathcal{K}$ and g is at least $2|\mathcal{K}|$. Therefore, any subgroup $\mathcal{K}$ is generated by at most $\log_2 |\mathcal{G}|$ elements, hence the set of all subgroups of $\mathcal{G}$ can be injectively mapped to $\mathcal{G}^{\log_2 |\mathcal{G}|}$.

Proof of Lemma 9.A.6. The $d = 1$ case is proven in Lemma 9.3.8. Thus, we assume $d \geq 2$ in the following. We choose an orthonormal basis of the Hilbert space on $\mathbb{T}_{\boldsymbol{L}}$ as $\{|s\rangle \mid s\colon \mathbb{T}_{\boldsymbol{L}} \to \{\uparrow,\downarrow\}\}$ to calculate the trace. Then, similarly to (9.3.19), we obtain

$$|\mathrm{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}}A)| \lesssim \sum_{s:\mathbb{T}_{\boldsymbol{L}}\to\{\uparrow,\downarrow\}} \prod_{\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{L}}\setminus\mathcal{R}_{\boldsymbol{q}}} \delta_{s(\boldsymbol{x}),s(\boldsymbol{x}+\boldsymbol{j})}\,. \tag{9.A.7}$$

Next, analogously to (9.3.20), we count the number of independent summations on the right-hand side of (9.A.7). To do so, we consider a graph $\mathcal{G}_{\boldsymbol{L},\boldsymbol{q},\boldsymbol{j}} = (V, E)$, whose vertices and edges are given by $V \coloneqq \mathbb{T}_{\boldsymbol{L}}$ and $E \coloneqq \{(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{j})\colon \boldsymbol{x} \in \mathbb{T}_{\boldsymbol{L}} \setminus \mathcal{R}_{\boldsymbol{q}}\}$, respectively. Exactly one redundant delta function appears in the product $\prod_{\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{L}}\setminus\mathcal{R}_{\boldsymbol{q}}} \delta_{s(\boldsymbol{x}),s(\boldsymbol{x}+\boldsymbol{j})}$ for every occurrence of a loop in $\mathcal{G}_{\boldsymbol{L},\boldsymbol{q},\boldsymbol{j}}$. Thus, by denoting the number of loops in $\mathcal{G}_{\boldsymbol{L},\boldsymbol{q},\boldsymbol{j}}$ by $N(\boldsymbol{L},\boldsymbol{q},\boldsymbol{j})$, we obtain

$$|\mathrm{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}}\mathring{A})| \lesssim 2^{|\mathcal{R}_{\boldsymbol{q}}|+N(\boldsymbol{L},\boldsymbol{q},\boldsymbol{j})}\,. \tag{9.A.8}$$

As in the one-dimensional case, Lemma 9.3.8, the graph $\mathcal{G}_{\boldsymbol{L},\boldsymbol{q},\boldsymbol{j}}$ is obtained from $\mathcal{G}_{\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}}$ by removing the edge $(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{j})$ for all $\boldsymbol{x} \in \mathcal{R}_{\boldsymbol{q}}$. Therefore, we have $N(\boldsymbol{L},\boldsymbol{q},\boldsymbol{j}) \leq N(\boldsymbol{L},\boldsymbol{0},\boldsymbol{j})$ for all $\boldsymbol{q}$. Now, the number of loops in $\mathcal{G}_{\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}}$ can be counted by considering the orbits of the cyclic group $\langle T^{\boldsymbol{j}}\rangle$ on $\mathbb{T}_{\boldsymbol{L}}$. It is clear that the size of each orbit is equal to one another. Denoting it by $g(\boldsymbol{j})$, the number of loops in $\mathcal{G}_{\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}}$ is given by

$$N(\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}) = \frac{V}{g(\boldsymbol{j})}\,.$$

Note that, since $\boldsymbol{j} \neq \boldsymbol{0}$ by assumption, we have $g(\boldsymbol{j}) \geq 2$.

If $g(\boldsymbol{j}) \geq 4$, the bound (9.A.6) is already proven because

$$|\mathrm{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}}\mathring{A})| \lesssim 2^{|\mathcal{R}_{\boldsymbol{q}}|+N(\boldsymbol{L},\boldsymbol{q},\boldsymbol{j})} \leq 2^{\frac{V}{4}+\frac{V}{4}} = 2^{\frac{V}{2}}\,,$$

where we used $|\mathcal{R}_{\boldsymbol{q}}| \leq V/2^d$ and $d \geq 2$ by assumption.

If $g(\boldsymbol{j}) = 2$ or $g(\boldsymbol{j}) = 3$, we must have $g(\boldsymbol{j})j_s \equiv 0 \pmod{L_s}$ for all s. Using again that $\boldsymbol{j} \neq \boldsymbol{0}$, there exists a non-zero component j_t. For such a coordinate direction $t \in \{1, ..., d\}$, we must have $g(\boldsymbol{j}) \mid L_t$ because $g(\boldsymbol{j}) \in \{2, 3\}$ is prime. We hence have a decomposition

$$\mathbb{T}_{\boldsymbol{L}} = \mathcal{A}_t \sqcup T^{\boldsymbol{j}}\mathcal{A}_t \sqcup T^{2\boldsymbol{j}}\mathcal{A}_t\mathcal{A}_t \coloneqq \{\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{L}}\colon 1 \leq x_t \leq \frac{L_t}{g(\boldsymbol{j})}\}\,.$$

Every loop in $\mathcal{G}_{\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}}$ can be considered to start from a site in $\mathcal{A}_t$. Therefore, removing the edge $(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{j})$ for all $\boldsymbol{x} \in \mathcal{R}_{\boldsymbol{q}}$ from $\mathcal{G}_{\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}}$ decreases the number of loops at least by $|\mathcal{R}_{\boldsymbol{q}} \cap \mathcal{A}_t|$, which implies

$$N(\boldsymbol{L},\boldsymbol{q},\boldsymbol{j}) \leq N(\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}) - |\mathcal{R}_{\boldsymbol{q}} \cap \mathcal{A}_t|.$$

Thus, from (9.A.8), we obtain

$$|\mathrm{Tr}_{\boldsymbol{L}}(T^{-\boldsymbol{j}}\mathring{A})| \lesssim 2^{|\mathcal{R}_{\boldsymbol{q}}|-|\mathcal{R}_{\boldsymbol{q}}\cap\mathcal{A}_t|+N(\boldsymbol{L},\boldsymbol{0},\boldsymbol{j})} = 2^{|\mathcal{R}_{\boldsymbol{q}}\setminus\mathcal{A}_t|+N(\boldsymbol{L},\boldsymbol{0},\boldsymbol{j})}\,.$$

Finally, we have

$$|\mathcal{R}_{\boldsymbol{q}} \setminus \mathcal{A}_t| + N(\boldsymbol{L},\boldsymbol{0},\boldsymbol{j}) \leq (\frac{L_t}{2} - \frac{L_t}{g(\boldsymbol{j})}) \prod_{s(\neq t)} \frac{L_s}{2} + \frac{V}{g(\boldsymbol{j})} = \frac{V}{2^d}(1 - \frac{2}{g(\boldsymbol{j})}) + \frac{V}{g(\boldsymbol{j})} \leq \frac{V}{2}\,,$$

which completes the proof of Lemma 9.A.6. □

Part II

BCS Theory

Chapter 10

The BCS critical temperature at high density

This chapter contains the paper [333]:

J. Henheik. The BCS Critical Temperature at High Density. *Math. Phys. Anal. Geom.*, 25(1), 2022

Abstract. We investigate the BCS critical temperature T_c in the high–density limit and derive an asymptotic formula, which strongly depends on the behavior of the interaction potential V on the Fermi–surface. Our results include a rigorous confirmation for the behavior of T_c at high densities proposed by Langmann, Triola, and Balatsky (Phys. Rev. Lett. 122, 2019) and identify precise conditions under which superconducting domes arise in BCS theory.

Content

10.1 Introduction

The Bardeen–Cooper–Schrieffer (BCS) gap equation [48]

$$\Delta(p) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} \hat{V}(p-q) \frac{\Delta(q)}{E_{\Delta,\mu}(q)} \tanh\left(\frac{E_{\Delta,\mu}(q)}{2T}\right) \mathrm{d}q\,, \tag{10.1.1}$$

with dispersion relation $E_{\Delta,\mu}(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$, has played an important role in physics since its introduction. The function Δ is interpreted as the order parameter describing paired fermions (Cooper pairs) interacting via the local pair potential $2V$, which we assume to be integrable, i.e. $V \in L^1(\mathbf{R}^3)$. In this case, $\hat{V}(p) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} V(x) \mathrm{e}^{-\mathrm{i}p\cdot x} \mathrm{d}x$ denotes its Fourier transform. The positive parameters T and μ are the temperature and the chemical potential, respectively, where the latter controls the density of fermions. Whenever the temperature T is below a certain critical temperature T_c (see Definition 10.2.1), the gap equation (10.1.1) admits non-trivial solutions, above it does not. Physically, this corresponds to the the system being in a superconducting state ($T < T_c$) or a normal state ($T \geq T_c$).

BCS theory has previously been studied in the weak–coupling limit [312, 269] and low–density limit [311, 403]. In the weak–coupling limit one considers a potential λV for a fixed potential V for small coupling constant $\lambda \to 0$. In this limit, it was shown by Hainzl and Seiringer [312] that the critical temperature satisfies $T_c \sim A\exp(-B/\lambda)$ for explicit constants $A, B > 0$. In the low–density limit, $\mu \to 0$, it is shown, again by Hainzl and Seiringer [311], that $T_c \sim \mu A \exp(-B/\sqrt{\mu})$ for some (different) explicit constants $A, B > 0$ (see Equation (10.2.8)).

In this paper we are interested in the critical temperature for the existence of non-trivial solutions of the BCS gap equation (10.1.1) in the high–density limit, i.e. $\mu \to \infty$. Studying the high–density limit of the critical temperature is especially relevant for explaining superconducting domes [376, 194, 522, 444, 593, 137], i.e. a non-monotonic $T_c(\mu)$ exhibiting a maximum value at finite μ and going to zero for large μ. In a recent paper [400], the authors claim the *ubiquity of superconducting domes in BCS theory*, but only for pure s–wave superconductivity (i.e. angular momentum $\ell = 0$, see Remark 10.2.3). Their result disproves the conventional wisdom, that the presence of a superconducting dome necessarily indicates some kind of exotic superconductivity, e.g. resulting from competing orders. BCS theory containing a non–monotonic behavior of $T_c(\mu)$ is in particular relevant for understanding superconducting domes in doped band insulators [593] and magic–angle graphene [137], where no competing orders occur, and thus a more conventional explanation is desirable.

There is a simple physical picture arising from an interplay of length scales, that explains the ubiquitous appearance of superconducting domes (see [400]). If the effective range ξ of the interaction is much smaller than the mean interparticle distance $\mu^{-1/2}$, i.e. $\xi \ll \mu^{-1/2}$, the critical temperature T_c increases by increasing μ as predicted by standard BCS theory [48] and rigorously justified in [311]. At high densities, i.e. if $\xi \gg \mu^{-1/2}$, the pairing of electrons near the Fermi surface (with approximately opposite momenta), which is responsible for the superconducting behavior, becomes weaker with increasing μ due to the decay of the interaction in Fourier space, suppressing T_c towards zero. Therefore, at some intermediate density, where $\xi \sim \mu^{-1/2}$, a superconducting dome arises. This simple argument is reflected in our results by the presence of the operator $\mathcal{V}_\mu$, defined in Equation (10.2.1), acting on functions on the (rescaled) Fermi surface.

Our results in Section 10.2 are threefold: first, we confirm a proposed asymptotic formula from [400] for the critical temperature at high densities for s–wave superconductivity (to leading order) by proving a more general result for radially symmetric interaction potentials V (Theorem 10.2.2); second, we provide a counterexample, showing that the assumptions on V from [400] are not quite sufficient to conclude a non–monotonic behavior of T_c and need to be slightly strengthened (Proposition 10.2.4); third, we use these strengthened assumptions to improve the asymptotics obtained in Theorem 10.2.2 to second order with the aid of perturbation theory, and obtain an analogous formula to the ones proven in the weak–coupling and low–density limit (Theorem 10.2.7). All proofs are given in Section 10.3.

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10.2 Main Results

10.2.1 Preliminaries

It was proven in [309] (see also [316] for a more recent review) that the critical temperature for the existence of non–trivial solutions of the BCS gap equation (10.1.1) can be characterized as follows.

Definition 10.2.1. (Critical Temperature)
Let $\mu > 0$, $V \in L^{3/2}(\mathbf{R}^3)$ be real–valued and $K_{T,\mu}$ denote the multiplication operator in momentum space

$$K_{T,\mu}(p) = \frac{|p^2 - \mu|}{\tanh\left(\frac{|p^2-\mu|}{2T}\right)} .$$

The critical temperature for the BCS gap equation (10.1.1) *is given by*

$$T_c = \inf \{T > 0 \mid K_{T,\mu}(p) + V(x) \geq 0\} .$$

One might think of the operator $K_{T,\mu}(p) + V(x)$ as the Hessian in the BCS functional of superconductivity at a normal state (see [316]), where the positivity corresponds to the "stability" of this normal state, which is directly related to the existence of a non–trivial solution of the BCS gap equation (10.1.1). Note that the continuous spectrum of $K_{T,\mu}$ starts at $2T$ and thus T_c is well defined by Sobolev's inequality [418, Thm. 8.3] since $K_{T,\mu} \sim p^2$ for large $|p|$.

Moreover, note that $K_{T,\mu}$ takes its minimum value $2T$ on the $\mathrm{codim}{-}1$ submanifold $\{p^2 = \mu\}$. Thus, similarly to the weak coupling situation [269] and as pointed out by Laptev, Safronov, and Weidl [401] (see also [314]), the spectrum of $K_{T,\mu} + V$ is mainly determined by the behavior of V near $\{p^2 = \mu\}$, i.e. the Fermi sphere. More precisely, as emphasized in the introduction, a crucial role for the investigation of T_c in the high–density limit is played by the (rescaled) operator $\mathcal{V}_\mu : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ where

$$\left(\mathcal{V}_\mu u\right)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{S}^2} \hat{V}(\sqrt{\mu}(p-q)) u(q) \, \mathrm{d}\omega(q) . \tag{10.2.1}$$

Here $\mathrm{d}\omega$ denotes the uniform (Lebesgue) measure on the unit sphere $\mathbb{S}^2$. The pointwise evaluation of $\hat{V}$ (and thus also on a $\mathrm{codim}{-}1$ submanifold) is well defined since $\hat{V}$ is continuous for $V \in L^1(\mathbf{R}^3)$. See Remark 10.2.9 for a discussion of the assumption $V \in L^1(\mathbf{R}^3)$ (cf. also [186]). The lowest eigenvalue of $\mathcal{V}_\mu$, which we denote by

$$e_\mu = \inf \mathrm{spec}\, \mathcal{V}_\mu$$

will be of particular importance. Note, that $\mathcal{V}_\mu$ is a compact operator (so $e_\mu \leq 0$), which is in fact trace class (see the argument above Equation (3.2) in [269]) with

$$\mathrm{tr}(\mathcal{V}_\mu) = \frac{1}{2\pi^2} \int_{\mathbf{R}^3} V(x) \mathrm{d}x .$$

The case $e_\mu < 0$ will be important for our main results as it corresponds to an attractive interaction between (some) electrons on the Fermi sphere. Since $\mathcal{V}_\mu$ is trace class, a sufficient condition for $e_\mu < 0$ is that the trace of $\mathcal{V}_\mu$ is negative, i.e. $\int V < 0$. Moreover, by considering a trial function that is concentrated on two small sets on the rescaled Fermi sphere $\mathbb{S}^2$ separated by a distance $|p| < 2$, one can easily see that $e_\mu < 0$ if $|\hat{V}(p)| > \hat{V}(0)$ for some $|p| < 2\sqrt{\mu}$.

In this work, we restrict ourselves to the special case of radial potentials V depending only on $|x|$, where the spectrum of $\mathcal{V}_\mu$ can be determined more explicitly (see, e.g., Section 2.1 in [269]). Indeed, if V is radially symmetric, the eigenfunctions of $\mathcal{V}_\mu$ are spherical harmonics and the corresponding eigenvalues are

$$\frac{1}{2\pi^2} \int_{\mathbf{R}^3} V(x) \left(j_\ell(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x \,, \tag{10.2.2}$$

with $\ell \in \mathbb{N}_0$ and where j_ℓ denotes the ℓ^{th}–order spherical Bessel function. A few important properties of the spherical Bessel functions used in our proofs are collected in Proposition 10.3.7. By Equation (10.2.2), the lowest eigenvalue e_μ is thus given by

$$e_\mu = \frac{1}{2\pi^2} \inf_{\ell \in \mathbb{N}_0} \int_{\mathbf{R}^3} V(x) \left(j_\ell(\sqrt{\mu}|x|)\right)^2 \mathrm{d}x \,.$$

If additionally $\hat{V} \leq 0$, the minimal eigenvalue is attained for the constant eigenfunction (i.e. the spherical harmonic with $\ell = 0$) by the Perron–Frobenius Theorem and we thus have the more concrete expression

$$e_\mu = \frac{1}{2\pi^2} \int_{\mathbf{R}^3} V(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \,. \tag{10.2.3}$$

We refer to Remark 10.2.8 for a discussion of the radiality assumption on V.

10.2.2 Results

As desribed in the introduction, our results are threefold: First, we show an asymptotic formula for radial potentials V (Theorem 10.2.2), including the rigorous confirmation of the result from [400] to leading order. Afterwards, we provide a counterexample showing that the assumptions made in [400] are not quite sufficient to conclude a non–monotonic behavior of T_c, i.e. a superconducting dome (Proposition 10.2.4). Finally, by slightly strengthening the assumptions on V, we provide an asymptotic formula for the critical temperature valid to second order (Theorem 10.2.7). All proofs are given in Section 10.3.

Theorem 10.2.2. *Let $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ be real–valued and radially symmetric. Assume that there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ we have $e_\mu < 0$. Then $T_c(\mu) > 0$ for all sufficiently large μ and*

$$\lim_{\mu \to \infty} \sqrt{\mu}\, e_\mu \log \frac{\mu}{T_c} = -1 \,. \tag{10.2.4}$$

Or in other words, we have the asymptotic behavior

$$T_c = \mu\, \mathrm{e}^{(1+o(1))/(\sqrt{\mu} e_\mu)} \tag{10.2.5}$$

in the limit of large μ. Note, that the right hand side is the same formula as in the weak–coupling case [269, 312] but we have coupling parameter $\lambda = 1$.

Remark 10.2.3. (Connection to the result from [400])
Assume that $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ is real–valued, radially symmetric and additionally satisfies $\hat{V} \leq 0$ and $\hat{V}(0) < 0$ (the latter implies that $e_\mu < 0$ for all $\mu > 0$). Note that these conditions,

which are identical to the ones required in [400], are included in the more general conditions of Theorem 10.2.2. Then we have, using the notation from [400], that

$$\sqrt{\mu}e_\mu = \frac{\sqrt{\mu}}{2\pi^2}\int_{\mathbf{R}^3} V(x)\frac{\sin^2(\sqrt{\mu}|x|)}{\mu|x|^2}\mathrm{d}x = \frac{1}{4\pi^2}\frac{f_{-2V}(4\mu)}{4\sqrt{\mu}} =: -\lambda\,,$$

where the first equality follows by Equation (10.2.3) *and after inserting the definition of the function f_{-2V} from [400], the second equality is a simple computation using Fubini. By means of Theorem 10.2.2, we thus confirmed the validity of Equation (6) in [400] in the high–density limit to leading order, i.e.*

$$T_c = \mu\,\mathrm{e}^{-(1+o(1))/\lambda}\,.$$

In full generality, the asymptotic formula proposed in Equation (6) in [400] reads

$$T_c = \frac{2\mathrm{e}^\gamma}{\pi}\mu\exp\left(-\frac{1}{\lambda}+\sum_{n=0}^{\infty}a_n\lambda^n\right),$$

where $\gamma \approx 0.577$ denotes the Euler–Mascheroni constant and $(a_n)_{n\geq 0}$ is a sequence of explicit constants determined by an iterative procedure. The quantity λ is understood as an intrinsic small parameter which encodes either a weak–coupling, low–density, or high–density limit, or an appropriate combination.

In order to obtain a meaningful asymptotic formula of the critical temperature at high densities in a rigorous way, the question to be addressed now is the behavior of $\sqrt{\mu}e_\mu$ in the limit $\mu\to\infty$. In the following proposition we present a special family of interaction potentials (V_α) showing that the conditions of Theorem 10.2.2 (which include the more restricted conditions from [400]) not necessarily lead to a non–monotonic behavior of T_c as claimed in [400], since $|\sqrt{\mu}e_\mu| \gg \log(\mu)^{-1}$ in the limit of large μ for this family of potentials. More precisely, the $L^{3/2}(\mathbf{R}^3)$–condition, which essentially concerns the behavior of the interaction potential near the origin, is not quite sufficient to obtain a dome–shaped behavior of $T_c(\mu)$. Since the potentials (V_α) are perfectly well behaved away from the origin and decay rapidly at infinity, they illustrate the significance of the behavior of interaction potentials near the origin for the asymptotics of the critical temperature. It is natural that the critical temperature is sensitive to the short range behavior of the interaction potential, since the interparticle distance as the physically relevant length scale that depends on the particle density tends to zero in the high–density limit.

Proposition 10.2.4. *Let $\alpha \in (1/3, 1/2)$ and set*

$$V_\alpha(x) = -\frac{\exp(-|x|)}{|x|^2\left(\log^2(|x|)+1\right)^\alpha}\,.$$

Then the critical temperature T_c associated with $K_{T,\mu}+V_\alpha$ approaches infinity as $\mu\to\infty$.

Our observations from Proposition 10.2.4 lead to the following definition of "admissible potentials", that are slightly better behaved at the origin and, in particular, allow for an analysis of e_μ (and also all the other eigenvalues of $\mathcal{V}_\mu$) by requiring certain definiteness conditions of V (cf. Lemma 10.3.3 and Lemma 10.3.4).

Definition 10.2.5. (Admissible potentials) *Let $V \in L^1(\mathbf{R}^3)\cap L^{3/2}(\mathbf{R}^3)$ be a real–valued radial function and define*

$$s_\pm^* := \sup\left\{s\geq 0 : |\cdot|^{-s}V_\pm \in L^1(\mathbf{R}^3)\right\} \qquad s^* := \min(s_+^*, s_-^*)\,, \tag{10.2.6}$$

where $V_\pm = \max\{\pm V, 0\}$ are the positive and negative parts of V. We call V an admissible potential *if the following is satisfied:*

(a) There exists $a > 0$ such that

$$\sup\left\{r \geq 0 : \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm(x)\mathrm{d}x = 0\right\} = \sup\left\{r \geq 0 : \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm|^*_{B_a}(x)\,\mathrm{d}x = 0\right\},$$

where $V_\pm|^*_{B_a}$ denotes the symmetric decreasing rearrangement of $V_\pm|_{B_a}$, the restriction of $V_\pm$ to the ball of radius a around 0,

(b) if $|\cdot|^{-2}V \notin L^1(\mathbf{R}^2)$, we have $s^* = s^*_- < s^*_+$, if $|\cdot|^{-2}V \in L^1(\mathbf{R}^2)$, we have $\int_{\mathbf{R}^3} \frac{V(x)}{|x|^2}\mathrm{d}x < 0$,

(c) $s^* > 1$, and

(d) if $s^* \geq 53/27$, we have $V \in L^p(\mathbf{R}^3)$ for some $p > 5/3$.

Condition (d) can be dropped, whenever we have control on the ground state space of $\mathcal{V}_\mu$ in the following sense: There exists $\mu_0 > 0$ and $\mathcal{L} \subset \mathbb{N}_0$ with $|\mathcal{L}| < \infty$, such that for all $\mu \geq \mu_0$, the ground state space of $\mathcal{V}_\mu$ is contained in the subspace of $L^2(\mathbb{S}^2)$ spanned by the spherical harmonics with angular momentum $\ell \in \mathcal{L}$.

In a nutshell, an admissible potential is a radial potential $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$, which satisfies the following:

(i) There exists some $a > 0$ such that both, positive and negative part, have their strongest singularity in B_a at the origin.

(ii) It has a dominating attractive part (for short distances), i.e. $s^*_- < s^*_+$ resp. $\int_{\mathbf{R}^3} \frac{V(x)}{|x|^2}\mathrm{d}x < 0$.

(iii) It is slightly less divergent at the origin than allowed by the $L^{3/2}(\mathbf{R}^3)$-assumption, i.e. $s^* > 1$.

The most relevant examples for admissible potentials are the attractive Yukawa and Gaussian potential, i.e.

$$V_{\mathrm{Yukawa}}(x) = -\frac{1}{4\pi|x|}\mathrm{e}^{-|x|} \qquad \text{and} \qquad V_{\mathrm{Gauss}}(x) = -(2\pi)^{-3/2}\mathrm{e}^{-|x|^2/2}\,.$$

Remark 10.2.6. (On condition (d) for admissible potentials)
The additional $L^p(\mathbf{R}^3)$-assumption with $p > 5/3$ for $s^* \geq 53/27$ in condition (d) is due to technical reasons and will we be explained during the proof of Theorem 10.2.7, which is formulated below. Note that, since $s^* \geq 53/27$ and $V \in L^1(\mathbf{R}^3)$, this condition is essentially about regularity away from 0 and infinity. However, our proof would work without change if we only had $p > f(s^*)$, where f has some complicated (explicit) expression (see Lemma 10.3.3 and Equation (10.3.13)*) and is strictly monotonically increasing between $53/27$ and 2, and satisfies $f(53/27) = 3/2$ and $f(s^*) = 5/3$ for all $s^* \geq 2$. We do not state Theorem 10.2.7 with this slight generalization for simplicity. Whenever we have some control on the ground state space of $\mathcal{V}_\mu$, the $L^p(\mathbf{R}^3)$-assumption is not necessary. For example, in the special case $\hat{V} \leq 0$, one can choose $\mathcal{L} = \{0\}$ by means of Equation* (10.2.3) *and completely drop condition (d).*

We will show in Lemma 10.3.4 that for any admissible potential $e_\mu < 0$ for μ large enough. Moreover, for any radial potential $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ with $e_\mu < 0$ and $s^* > 1$ (in particular any admissible potential), by application of Theorem 10.2.2, the critical temperature decays exponentially fast as $\mu \to \infty$ since

$$|\sqrt{\mu}e_\mu| \leq \frac{1}{2\pi^2} \left\| \frac{V}{|\cdot|^s} \right\|_{L^1} \sup_{\ell\in\mathbb{N}_0} \left\| |\cdot|^{s/2} j_\ell \right\|^2_{L^\infty} \mu^{\frac{1-s}{2}} \tag{10.2.7}$$

for $s \in (1, s^*)$ and the term involving j_ℓ is finite as long as $s \leq 5/3$ by uniform decay of spherical Bessel functions (see Proposition 10.3.7 (iii)). A slightly different bound as given in Lemma 10.3.3

allows to improve this threshold. Note that the class of interaction potentials from Proposition 10.2.4 is not admissible since $s^* = 1$ for these potentials.

The existence of a maximal critical temperature at some intermediate density (*superconducting dome*), can now be obtained by combining the decay of T_c in the high–density limit from Equation (10.2.5) and Equation (10.2.7) for admissible potentials in the sense of Definition 10.2.5 to the decay of T_c in the low–density limit, where

$$T_c = \mu \left(\frac{8}{\pi} \mathrm{e}^{\gamma-2} + o(1) \right) \mathrm{e}^{\pi/(2\sqrt{\mu}a)} \tag{10.2.8}$$

as shown in [312]. This result was obtained for (not necessarily radially symmetric) real valued interaction potentials V, with $V(x)(1+|x|) \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$, negative scattering length a, and in the absence of bound states. Thus, we rigorously confirmed the *ubiquity of superconducting domes in BCS theory* for a general class of interaction potentials, as claimed in [400].

As our next result, we shall derive the second order correction to Equation (10.2.5), i.e. we shall compute the constant in front of the exponential for admissible potentials. For this purpose we define the operator $\mathcal{W}_\mu^{(\kappa)}$ on $u \in L^2(\mathbb{S}^2)$ via its quadratic form

$$\begin{aligned} \left\langle u | \mathcal{W}_\mu^{(\kappa)} | u \right\rangle = \sqrt{\mu} \int_0^\infty \mathrm{d}|p| \Bigg(& \frac{|p|^2}{||p|^2-1|} \left[\int_{\mathbb{S}^2} \mathrm{d}\omega(p) \left(|\hat{\varphi}(\sqrt{\mu}p)|^2 - |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2 \right) \right] \\ & + \frac{|p|^2}{|p|^2+\kappa^2} \int_{\mathbb{S}^2} \mathrm{d}\omega(p) |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2 \Bigg) \end{aligned} \tag{10.2.9}$$

for fixed $\kappa \geq 0$ (cf. Equation (13) in [311] for an analogous definition in the weak coupling case with $\kappa = 0$). Here, we denote $\hat{\varphi}(p) = (2\pi)^{-3/2} \int_{\mathbb{S}^2} \hat{V}(p - \sqrt{\mu}q) u(q) \mathrm{d}\omega(q)$, and $(|p|, \omega(p)) \in (0,\infty) \times \mathbb{S}^2$ are spherical coordinates for $p \in \mathbb{R}^3$. Since $V \in L^1(\mathbf{R}^3)$, the map $|p| \mapsto \int_{\mathbb{S}^2} \mathrm{d}\omega(p) |\hat{\varphi}(p)|^2$ is Lipschitz continuous for any $u \in L^2(\mathbb{S}^2)$, such that the radial integral in Equation (10.2.9) is well defined even in the vicinity of $|p| \sim 1$. For large $|p|$ the integral converges since $V \in L^{3/2}(\mathbf{R}^3)$. Although we formulate our result in Theorem 10.2.7 only for $\kappa = 0$, the case of a positive parameter $\kappa > 0$ is crucial in the proof of this statement, as it ensures, e.g., that the first term in the decomposition of the Birman–Schwinger operator associated with $K_{T,\mu} + V$ is small (cf. Equation (10.3.2)). Whenever it does not lead to confusion, we refer to some κ–dependent quantity at $\kappa = 0$ by simply dropping the (κ)–superscript.

Now, we define the operator

$$\mathcal{B}_\mu^{(\kappa)} = \frac{\pi}{2} \left(\mathcal{V}_\mu - \mathcal{W}_\mu^{(\kappa)} \right), \tag{10.2.10}$$

which measures the strength of the interaction potential near the Fermi surface up to second order and let $b_\mu^{(\kappa)}$ denote its lowest eigenvalue,

$$b_\mu^{(\kappa)} = \inf \operatorname{spec} \mathcal{B}_\mu^{(\kappa)}. \tag{10.2.11}$$

We introduced the factor $\pi/2$ in Equation (10.2.10) since $b_\mu = b_\mu^{(0)}$ has the interpretation of an effective scattering length, which is best illustrated in the case of small μ (see Proposition 1 in [311]). Moreover, we will see in the proof of Theorem 10.2.7 that if $e_\mu < 0$ then also $b_\mu^{(\kappa)} < 0$ for large enough μ. With the aid of $b_\mu^{(\kappa)}$ we can now state our second main result concerning the asymptotic formula for the critical temperature valid up to second order.

Theorem 10.2.7. *Let V be an admissible potential. Then the critical temperature T_c is positive and satisfies*

$$\lim_{\mu\to\infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu} \right) = 2 - \gamma - \log(8/\pi).$$

In other words,

$$T_c = \mu\left(\frac{8}{\pi}\mathrm{e}^{\gamma-2} + o(1)\right)\mathrm{e}^{\pi/(2\sqrt{\mu}b_\mu)}$$

in the limit $\mu \to \infty$. Similarly to Theorem 10.2.2, this formula is in complete analogy to the weak–coupling case [312] (replace $V \to \lambda V$ and take the limit $\lambda \to 0$) but we have coupling parameter $\lambda = 1$ here. As discussed in the introduction, this analogy is not entirely surprising. In physical terms, only those fermions with momenta close to the Fermi surface $\{p^2 = \mu\}$ contribute to the superconductivity. Therefore, due to the decay of the interaction $\hat{V}$ in Fourier space, the high–density limit, $\mu \to \infty$, is effectively a weak–coupling limit.

The constant in front of the exponential is in particular relevant for obtaining the universality of the ratio of the critical temperature and the energy gap, which is achieved in Chapter 11, where a similar asymptotic formula for the energy gap is proven.

Remark 10.2.8. (Radiality)
The assumption of the interaction potential being radially symmetric enters the proofs of our main theorems in a crucial way. On the one hand, the radial symmetry allows an additional averaging over the sphere $\mathbb{S}^2$ in position space in the proof of Theorem 10.2.2, which leads to a "decoupling" of the position variables x and y (cf. Equation (10.3.6)*) as the arguments of integral kernels of operators that appear after employing the Birman–Schwinger principle [309, 269, 316]. Without this averaging the supposed error terms in Equation* (10.3.2) *could not be concluded to be small. On the other hand, the radial symmetry enables us to obtain useful bounds on the quantity e_μ (cf. Lemma 10.3.2, Lemma 10.3.3, and Lemma 10.3.4), which naturally appears in the obtained asymptotics in Theorem 10.2.2 and Theorem 10.2.7. Although the assumption of a radial potential is a loss of generality compared to the weak coupling [269, 312] and low density [311] situation, the case of an isotropic interactions seems physically the most relevant and natural.*

Remark 10.2.9. (Potentials with slow decay at infinity)
The recent work [186] by Cuenin and Merz indicates how to generalize our results to interaction potentials with slow decay at infinity, i.e. which fail to satisfy $V \in L^1(\mathbf{R}^3)$. The main idea is to employ the Tomas–Stein Theorem to define the Fourier transform of the potential on the $\mathrm{codim}{-}1$ *submanifold $\mathbb{S}^2 \subset \mathbf{R}^3$ having non–vanishing curvature. Moreover, by using the methods from [291], where Gontier, Hainzl, and Lewin originally studied a lower bound on the Hartree–Fock energy of the electron gas, one can see that*

$$T_c \leq C_1\,\mu \exp\left(-C_2\,\mu^{1/4}\right) \tag{10.2.12}$$

for any real–valued potential V satisfying $|\cdot|V \in L^\infty(\mathbf{R}^3)$. A detailed proof of this estimate is given in Section 10.3. Note that for an admissible potential V that satisfies $|\cdot|V \in L^\infty(\mathbf{R}^3)$, we have $s^* \geq 2$ and infer by Equation (10.2.7) *that the bound provided by Equation* (10.2.12) *is not optimal. Although these results indicate that it is mathematically possible to deal with slow decay at infinity, it seems physically natural to assume fast decay at infinity, at least in the high–density limit for an effective interaction potential, when the phenomenon of screening plays an important role.*

10.3 Proofs

The most important tool for our proofs will be the Birman–Schwinger principle (see [309, 269, 316]). According to this principle, T_c is determined by the fact that for $T = T_c$ the smallest eigenvalue of

$$B_{T,\mu} = V^{1/2}\frac{1}{K_{T,\mu}}|V|^{1/2}$$

equals -1. Here, we used the notation $V(x)^{1/2} = \mathrm{sgn}(V(x))|V(x)|^{1/2}$. The main simplification is that the study of the spectrum of the unbounded operator $K_{T,\mu} + V$ reduces to identifying the

singular part of the compact Birman–Schwinger operator. With this in mind, our proofs will build on a convenient decomposition of $B_{T,\mu}$ in a dominant singular term and other error terms.

Let $\mathfrak{F}_\mu : L^1(\mathbf{R}^3) \to L^2(\mathbb{S}^2)$ denote the rescaled Fourier transform restricted to $\mathbb{S}^2$ with

$$(\mathfrak{F}_\mu \psi)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} \mathrm{e}^{-\mathrm{i}\sqrt{\mu} p \cdot x} \psi(x) \mathrm{d}x \,,$$

which is well–defined by the Riemann–Lebesgue Lemma. Since $V \in L^1(\mathbf{R}^3)$, the multiplication with $|V|^{1/2}$ is a bounded operator from $L^2(\mathbf{R}^3)$ to $L^1(\mathbf{R}^3)$, and hence $\mathfrak{F}_\mu |V|^{1/2}$ is a bounded operator from $L^2(\mathbf{R}^3)$ to $L^2(\mathbb{S}^2)$. A further important ingredient in our proofs is to study the asymptotic behavior of

$$m_\mu^{(\kappa)}(T) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2 + \kappa^2 \mu} \right) \mathrm{d}p$$

for fixed $\kappa > 0$, which was done in a similar way for the low–density and weak–coupling limit of the critical temperature and the energy gap (see [312, 311, 316, 403]). Indeed, using Lemma 1 from [312] one can easily see that

$$m_\mu^{(\kappa)}(T) = \sqrt{\mu} \left(\log \frac{\mu}{T} + \gamma - 2 + \log \frac{8}{\pi} + \kappa \frac{\pi}{2} + o(1) \right) \tag{10.3.1}$$

as long as $T/\mu \to 0$. Using the definitions above, we arrive at our convenient decomposition, which we define as

$$B_{T,\mu} = V^{1/2} \frac{1}{p^2 + \kappa^2 \mu} |V|^{1/2} + m_\mu^{(\kappa)}(T)\, V^{1/2} \mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2} + A_{T,\mu}^{(\kappa)} \tag{10.3.2}$$

for $\kappa > 0$, where $A_{T,\mu}^{(\kappa)}$ is implicitly defined. For the first term to be small, we need that $\kappa > 0$. For the second term, note that

$$V^{1/2} \mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2}$$

is isospectral to $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu{}^\dagger$. In fact, the spectra agree at first except possibly at 0, but 0 is in both spectra as the operators are compact on an infinite dimensional space.

This second term will be the dominant term, which is how the quantity e_μ appears in the asymptotic formulas in our main theorems, whereas the first and third term are negligible error terms in the limit $\mu \to \infty$. Showing this, is the objective of the proofs of Theorem 10.2.2 and Theorem 10.2.7

A priori, it is not clear, how T_c behaves at high densities. Therefore, before we go to the proofs of Theorem 10.2.2 and Theorem 10.2.7, let us fix the following

Lemma 10.3.1. *$T_c \leq O(\mu)$ as $\mu \to \infty$.*

Proof. Since $\tanh(t) \leq \min(1, t)$ for $t \geq 0$, we have

$$\begin{aligned} K_{T,\mu} + V &\geq \frac{1}{2} \left(|p^2 - \mu| + 2T \right) + V \\ &\geq \frac{1}{2} \left(p^2 + \mu + 2V \right) + (T - \mu) \,. \end{aligned}$$

The first term is non–negative for sufficiently large μ by application of Sobolev's inequality [418, Thm. 8.3] using $V \in L^{3/2}(\mathbf{R}^3)$. Thus, by Definition 10.2.1, we obtain $T_c \leq \mu$. □

In the proof of Theorem 10.2.2 below, we will in fact show that $T_c = o(\mu)$, so Equation (10.3.1) gives the correct asymptotic behavior.

10.3.1 Proof of Theorem 10.2.2

Proof of Theorem 10.2.2. Fix $\kappa > 0$. As outlined above, the strategy of the proof is to show that the first and the third term in the decomposition (10.3.2) vanish in operator norm in the high–density limit and thus the asymptotic behavior is entirely determined by the spectrum of the operator in the second term. We discuss this in detail now.

For the first term, note that the Fourier transform of $\frac{1}{p^2+\kappa^2\mu}$ is given by $\frac{\mathrm{e}^{-\kappa\sqrt{\mu}|x|}}{|x|}$, up to a constant. Thus the Hilbert–Schmidt norm $\|\cdot\|_{\mathrm{HS}}$, which is always an upper bound for the operator norm $\|\cdot\|_{\mathrm{op}}$, is given by

$$\left\| V^{1/2} \frac{1}{p^2+\kappa^2\mu} |V|^{1/2} \right\|_{\mathrm{HS}}^2 = C \int_{\mathbf{R}^3} \mathrm{d}x \int_{\mathbf{R}^3} \mathrm{d}y \, |V(x)| \frac{\mathrm{e}^{-2\kappa\sqrt{\mu}|x-y|}}{|x-y|^2} |V(y)| \,.$$

This vanishes as $\mu \to \infty$ by an application of the dominated convergence theorem in combination with the Hardy–Littlewood–Sobolev inequality [418, Thm. 4.3]. Here and in the following, we shall use the notation c and C for generic positive small resp. large constants, possible having a different value in each appearance. If we want to emphasize the dependence on a certain parameter, we add a subscript, e.g. by writing c_δ or $C_{p,a}$.

For the third term, we will heavily use the radiality of V. In fact, since V is radially symmetric, every eigenfunction of $K_{T,\mu}$ and thus B_T will have definite angular momentum and we can focus on $f, g \in L^2(\mathbf{R}^3)$ of the form $f(x) = f(|x|)Y_\ell^m(\hat{x})$ resp. $g(x) = g(|x|)Y_{\ell'}^{m'}(\hat{x})$, with a slight abuse of notation, where $\hat{x} = x/|x|$ denotes the unit vector in direction x. Now we aim to bound $\langle f | A_{T,\mu}^{(\kappa)} | g \rangle$, uniformly in (ℓ, ℓ') (and (m, m')). As $A_{T,\mu}^{(\kappa)}$ has integral kernel

$$A_{T,\mu}^{(\kappa)}(x,y) = C V^{1/2}(x) |V(y)|^{1/2} \int_{\mathbf{R}^3} \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\kappa^2\mu} \right) \left(\mathrm{e}^{\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}\sqrt{\mu}\hat{p}\cdot(x-y)} \right) \mathrm{d}p \,,$$

and using the radial symmetry of V we arrive at

$$\begin{aligned} \left\langle f | A_{T,\mu}^{(\kappa)} | g \right\rangle &= C \int_0^\infty \mathrm{d}|x| \, |x|^2 \int_0^\infty \mathrm{d}|y| \, |y|^2 \bar{f}(|x|) V^{1/2}(|x|) |V(|y|)|^{1/2} g(|y|) && (10.3.3) \\ &\times \int_{\mathbf{R}^3} \mathrm{d}p \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\kappa^2\mu} \right) && (10.3.4) \\ &\times \int_{\mathbb{S}^2} \mathrm{d}\omega(x) \int_{\mathbb{S}^2} \mathrm{d}\omega(y) \overline{Y}_\ell^m(\hat{x}) Y_{\ell'}^{m'}(\hat{y}) \left(\mathrm{e}^{\mathrm{i}p\cdot(x-y)} - \mathrm{e}^{\mathrm{i}\sqrt{\mu}\hat{p}\cdot(x-y)} \right) . && (10.3.5) \end{aligned}$$

Note that the angular integrals of x and y can be performed first only by the radial symmetry of V. If V were not radially symmetric, we would have had to compute the angular integral of p first and would have ended up with completely different integrals to estimate. Now, using the plane wave expansion $\mathrm{e}^{\mathrm{i}p\cdot x} = 4\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \mathrm{i}^\ell j_\ell(|p||x|) Y_\ell^m(\hat{p}) \overline{Y}_\ell^m(\hat{x})$, the last line (10.3.5) evaluates to

$$16\pi^2 (-\mathrm{i})^{\ell+\ell'} \left(j_\ell(|p||x|) j_{\ell'}(|p||y|) - j_\ell(\sqrt{\mu}|x|) j_{\ell'}(\sqrt{\mu}|y|) \right) \overline{Y}_\ell^m(\hat{p}) Y_{\ell'}^{m'}(\hat{p}) \,. \tag{10.3.6}$$

In order to get a non-zero angular p-integral from the second line (10.3.4), we need to have $\ell = \ell'$ and $m = m'$, by orthogonality of spherical harmonics. We can hence focus on that case and write x, y, and p instead of $|x|$, $|y|$, and $|p|$, respectively, such that (10.3.4) and (10.3.5) combine to (a constant times)

$$\int_0^\infty \mathrm{d}p \, p^2 \left(\frac{1}{K_{T,\mu}(p)} - \frac{1}{p^2+\kappa^2\mu} \right) \left(j_\ell(px) j_\ell(py) - j_\ell(\sqrt{\mu}x) j_\ell(\sqrt{\mu}y) \right) . \tag{10.3.7}$$

After changing the integration variable $p \to p/\sqrt{\mu}$ and inserting $\pm j_\ell(p\sqrt{\mu}x) j_\ell(\sqrt{\mu}y)$, we use the uniform Lipschitz continuity and the uniform decay of spherical Bessel functions (Proposition 10.3.7

(ii) and Proposition 10.3.7 (iii)) to obtain

$$|(10.3.7)| \leq C\mu^{1/2}\int_0^\infty \mathrm{d}p\, p^2 \left|\frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+\kappa^2}\right| |p-1|^\varepsilon \left(\frac{1}{p^\varepsilon}+1\right) \\ \times \Big(|j_\ell(p\sqrt{\mu}x)|^{1-11\varepsilon/5} + |j_\ell(\sqrt{\mu}x)|^{1-11\varepsilon/5}\Big)\Big(|j_\ell(p\sqrt{\mu}y)|^{1-11\varepsilon/5} + |j_\ell(\sqrt{\mu}y)|^{1-11\varepsilon/5}\Big)\,,$$

for any $\varepsilon \in (0, 2/11)$. Using that and by employing Hölder for the integrals over x and y in Equation (10.3.3), we get

$$\left|\left\langle f\middle|A^{(\kappa)}_{T,\mu}\middle|g\right\rangle\right| \leq C\,\|f\|_{L^2}\,\|g\|_{L^2}\int_0^\infty \mathrm{d}p \left|\frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+\kappa^2}\right| |p-1|^\varepsilon \left(\frac{1}{p^\varepsilon}+1\right)p^2 \tag{10.3.8}$$

$$\times\, \mu^{1/2}\int_{\mathbf{R}^3}\mathrm{d}x\,|V(x)|\left(|j_\ell(p\sqrt{\mu}|x|)|^{2-22\varepsilon/5} + |j_\ell(\sqrt{\mu}|x|)|^{2-22\varepsilon/5}\right). \tag{10.3.9}$$

In Lemma 10.3.2 below (as $\varepsilon < 2/11$ we have $2 - 22\varepsilon/5 > 6/5$), we show that the last line (10.3.9) can be estimated by

$$(10.3.9) \leq \left(\frac{1}{p}+1\right)|o(1)|\,,$$

where $o(1)$ is some function of μ that vanishes as $\mu \to \infty$. Thus, we arrive at

$$\left|\left\langle f\middle|A^{(\kappa)}_{T,\mu}\middle|g\right\rangle\right| \leq C\,|o(1)|\,\|f\|_{L^2}\,\|g\|_{L^2}\int_0^\infty \mathrm{d}p \left|\frac{1}{K_{T/\mu,1}(p)} - \frac{1}{p^2+\kappa^2}\right| |p-1|^\varepsilon p^{1-\varepsilon}(1+p^{1+\varepsilon})\,,$$

where the integral is uniformly bounded (since $\kappa > 0$) as long as $T \leq C\mu$ and we conclude

$$\limsup_{\mu\to\infty}\sup_{0<T\leq C\mu}\left\|A^{(\kappa)}_{T,\mu}\right\|_{\mathrm{op}} = 0\,,$$

since all bounds above are uniform in ℓ. Therefore, as long as $T_c = O(\mu)$, the spectrum of the Birman–Schwinger operator approaches the spectrum of the operator in the second term in Equation (10.3.2) as $\mu \to \infty$.

Summarizing our considerations above, we get that, since by assumption $e_\mu < 0$ for $\mu \geq \mu_0$, $T_c > 0$ for all sufficiently large μ. This is because the second term in the decomposition (10.3.2) can be made arbitrarily negative by taking $T \to 0$, whereas the first and the third term are bounded uniformly in $T \leq C\mu$. Thus we get with the aid of Lemma 10.3.1 that

$$-1 = \lim_{\mu\to\infty} m^{(\kappa)}_\mu(T_c)\, e_\mu\,.$$

In order to obtain Equation (10.2.4) by means of Equation (10.3.1), it remains to show that $T_c = o(\mu)$. Since it is already shown in Lemma 10.3.1 that $T_c = O(\mu)$, we assume that $T_c = \Theta(\mu)$, i.e. there exist $0 < c < C$ such that $c\mu \leq T_c \leq C\mu$. This means that $m^{(\kappa)}_\mu(T_c)$ is of order $\sqrt{\mu}$, which leads to a contradiction since $\sqrt{\mu}e_\mu = o(1)$ by Lemma 10.3.2 below. So, Equation (10.3.1) implies Equation (10.2.4) as desired. □

Lemma 10.3.2. *Let $V \in L^{3/2}(\mathbf{R}^3)$ and $\alpha > 6/5$. Then*

$$\limsup_{\mu\to\infty}\sqrt{\mu}\sup_{\ell\in\mathbb{N}_0}\int_{\mathbf{R}^3}\mathrm{d}x\,|V(x)|\,|j_\ell(\sqrt{\mu}|x|)|^\alpha = 0\,.$$

Proof. We estimate

$$\sqrt{\mu}\sup_{\ell\in\mathbb{N}_0}\int_{\mathbf{R}^3}\mathrm{d}x\,|V(x)|\,|j_\ell(\sqrt{\mu}|x|)|^\alpha \leq C\sqrt{\mu}\int_{\mathbf{R}^3}\mathrm{d}x\,|V(x)|\frac{1}{\left(\sqrt{\mu}|x|\right)^{5\alpha/6}+1}\,, \tag{10.3.10}$$

where the inequality follows from the uniform decay of spherical Bessel functions (see Proposition 10.3.7 (iii)). By using Hölder, we can further bound

$$(10.3.10) \leq C\|V-\phi\|_{L^{3/2}} \left\| \frac{1}{|\cdot|^{5\alpha/6}+1} \right\|_{L^3} + C\sqrt{\mu} \int_{\mathbf{R}^3} \mathrm{d}x\, |\phi(x)| \frac{1}{\left(\sqrt{\mu}|x|\right)^{5\alpha/6}+1}$$

for any $\phi \in C_0^\infty(\mathbf{R}^3)$. Since $\alpha > 6/5$, the second term vanishes as $\mu \to \infty$ by dominated convergence using $\phi \in C_0^\infty(\mathbf{R}^3)$, and the first term can be made arbitrarily small as $C_0^\infty(\mathbf{R}^3)$ is dense in $L^{3/2}(\mathbf{R}^3)$. Thus, we have proven the claim. □

10.3.2 Proof of Proposition 10.2.4

Proof of Proposition 10.2.4. We check that V_α satisfies the assumptions of Theorem 10.2.2. First, V_α is radial and clearly satisfies $V_\alpha \in L^1(\mathbf{R}^3)$. $V_\alpha \in L^{3/2}(\mathbf{R}^3)$ follows since $\alpha > 1/3$. Next, we calculate the derivative of $|x||V_\alpha(x)|$ w.r.t. $|x|$ as

$$\left(\frac{(\log(|x|)+\alpha)^2}{|x|(\log^2(|x|)+1)} + \frac{1-\alpha^2}{|x|(\log^2(|x|)+1)} + 1 \right) \frac{\exp(-|x|)}{|x|(\log^2(|x|)+1)}$$

and conclude that $|x|V_\alpha(x)$ is monotonically increasing in $|x|$, since $\alpha < 1/2$. Therefore, by using the radiality of V_α and the argument from Equation (4) in [570], we find that $\hat{V}_\alpha \leq 0$ and infer

$$e_\mu = \frac{1}{2\pi^2} \int_{\mathbf{R}^3} V_\alpha(x) \left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|} \right)^2 \mathrm{d}x < 0$$

by Equation (10.2.3). Thus, V_α satisfies all conditions of Theorem 10.2.2. In order to obtain a lower bound on $T_c(\mu)$ we estimate

$$|\sqrt{\mu} e_\mu| \geq c \int_0^{\sqrt{\mu}/2} \frac{\sin(x)^2}{x^2 |\log(x/\sqrt{\mu})|^{2\alpha}} \mathrm{d}x \geq c \frac{1}{|\log(\mu)|^{2\alpha}}$$

for some $c > 0$ and μ large enough. Using Theorem 10.2.2, this yields

$$T_c \gtrsim \mu \exp(-\log(\mu)^{2\alpha}/c) \to \infty$$

as $\mu \to \infty$ since $\alpha < 1/2$. □

10.3.3 Proof of Theorem 10.2.7

The proof of Theorem 10.2.7 is based on the following two Lemmas providing the necessary estimates for a perturbation theoretic argument yielding the next order correction to the asymptotics obtained in Theorem 10.2.2. While Lemma 10.3.3 improves the upper bounds on integrals of the interaction potential against spherical Bessel functions obtained in Lemma 10.3.2 and Equation (10.2.7), in particular for $s^* > 5/3$, Lemma 10.3.4 provides a lower bound on e_μ for admissible potentials. We postpone the proofs of Lemma 10.3.3 and Lemma 10.3.4 until the end of this Section.

Lemma 10.3.3. *Let $V \in L^1(\mathbf{R}^3) \cap L^p(\mathbf{R}^3)$ for some $p \in [3/2, 9/4]$ with dual $q = \frac{p}{p-1} \in [9/5, 3]$. Assume that $s^* > 1$, with s^* as in Definition 10.2.5 and set*

$$\beta_p^* = \begin{cases} \frac{s^*}{2} & \text{for} \quad s^* \in (1, 5/3] \\ \min\left(\frac{(q+1)s^*-4}{3qs^*-7} + \frac{1}{2}, \frac{10q-11}{12q-14} \right) & \text{for} \quad s^* > 5/3\,. \end{cases} \tag{10.3.11}$$

Note that β_p^* depends continuously on s^* and is (strictly) monotonically increasing (between 1 and 2), and $\beta_p^* \leq \min(s^*, 2)/2$ for any $s^* > 1$. Then for any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ we have

$$\limsup_{\mu\to\infty} \mu^{\beta_p^*-\delta} \sup_{\ell\in\mathbb{N}_0} \int_{\mathbf{R}^3} \mathrm{d}x |V(x)|\, |j_\ell(\sqrt{\mu}|x|)|^{2-\varepsilon} = 0\,.$$

For admissible potentials that satisfy the $L^p(\mathbf{R}^3)$-assumption in condition (d) from Definition 10.2.5, we will use this Lemma with

$$\beta^* = \begin{cases} \frac{s^*}{2} & \text{for} \quad s^* \in (1, 5/3] \\ \frac{4s^*-4}{9s^*-7} + \frac{1}{2} & \text{for} \quad s^* \in (5/3, 53/27) \\ \min\left(\frac{7s^*-8}{15s^*-14} + \frac{1}{2},\, \frac{7}{8}\right) + \delta_p & \text{for} \quad s^* \geq 53/27\,, \end{cases}$$

for some $\delta_p > 0$ since $p > 5/3$. For our perturbation theoretic argument to work in the general case, where we have no control on the ground state space of $\mathcal{V}_\mu$, it turns out to be necessary that

$$4\beta^* - \frac{3\min(s^*, 2)}{2} - \frac{1}{2} > 0\,, \tag{10.3.12}$$

which is why we need the $L^p(\mathbf{R}^3)$-assumption in Definition 10.2.5 for $s^* \geq 53/27$. The function $f(s^*)$ from Remark 10.2.6 can explicitly be determined by requiring that

$$4\beta^*_{f(s^*)} - \frac{3\min(s^*, 2)}{2} - \frac{1}{2} = 0\,. \tag{10.3.13}$$

Lemma 10.3.4. *Let V be an admissible potential (cf. Definition 10.2.5, condition (d) can be dropped). Then for any $\delta > 0$ there exists $c_\delta > 0$ such that*

$$\liminf_{\mu\to\infty} |\mu^{\min(s^*+\delta,2)/2}\, e_\mu| \geq c_\delta\,.$$

In particular, for admissible V, we have $e_\mu < 0$ for μ large enough.

The proof of Theorem 10.2.7 is divided in two parts. In the first part, Lemma 10.3.5, we provide an asymptotic formula for T_c with parameter $\kappa > 0$. In the second part, Lemma 10.3.6, we asymptotically compare $1/(\sqrt{\mu} b_\mu^{(\kappa)})$ with $1/(\sqrt{\mu} b_\mu)$. By combining these formulas, we obtain Theorem 10.2.7.

Lemma 10.3.5. *Let V be an admissible potential and fix $\kappa > 0$. Then the critical temperature T_c is positive and satisfies*

$$\lim_{\mu\to\infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right) = 2 - \gamma - \kappa\, \frac{\pi}{2} - \log(8/\pi)\,. \tag{10.3.14}$$

Lemma 10.3.6. *Let V be an admissible potential and $\kappa > 0$. Then*

$$\lim_{\mu\to\infty} \left(\frac{\pi}{2\sqrt{\mu} b_\mu} - \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right) = \kappa\, \frac{\pi}{2}\,. \tag{10.3.15}$$

Proof of Theorem 10.2.7. By combining Lemma 10.3.5 with Lemma 10.3.6 we obtain

$$\begin{aligned} \lim_{\mu\to\infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu} \right) &= \lim_{\mu\to\infty} \left(\log \frac{\mu}{T_c} + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right) + \lim_{\mu\to\infty} \left(\frac{\pi}{2\sqrt{\mu} b_\mu} - \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} \right) \\ &= 2 - \gamma - \kappa\, \frac{\pi}{2} - \log(8/\pi) + \kappa\, \frac{\pi}{2} = 2 - \gamma - \log(8/\pi)\,. \end{aligned}$$

□

The rest of this Section is devoted to the proofs of the four Lemmas above. We begin with Lemma 10.3.5 and Lemma 10.3.6.

Proof of Lemma 10.3.5. Fix $\kappa > 0$. We first assume condition (d) from Definition 10.2.5 and discuss the changes for the special case afterwards. Similarly to the proof of Theorem 10.2.2, we show that the first and the third term in the decomposition (10.3.2) vanish in operator norm.

For the first term, we need to improve the estimate from Theorem 10.2.2, where we employed the easily accessible Hilbert–Schmidt norm as an upper bound to the operator norm. Indeed, using the radial symmetry of V, similarly to the bound of $A_{T,\mu}^{(\kappa)}$ in Equation (10.3.9), the operator norm of the compact operator $L_\mu^{(\kappa)} := V^{1/2}(p^2+\kappa^2\mu)^{-1}|V|^{1/2}$ can be estimated as

$$\left\|L_\mu^{(\kappa)}\right\|_{\mathrm{op}} \leq C\,\mu^{1/2}\int_0^\infty \mathrm{d}p \frac{p^2}{p^2+\kappa^2}\sup_{\ell\in\mathbb{N}_0}\int_{\mathbf{R}^3}\mathrm{d}x|V(x)|\,|j_\ell(\sqrt{\mu}p|x|)|^2\,, \tag{10.3.16}$$

which is bounded by $\mu^{-\beta^*+1/2+\delta}$ for any $\delta>0$ by means of Lemma 10.3.3 (note that the p–integral is finite since $s^*>1$). Recall from the prove of Theorem 10.2.2 (in particular Equation 10.3.9) that

$$\begin{aligned}\left\|A_{T,\mu}^{(\kappa)}\right\|_{\mathrm{op}} &\leq C\int_0^\infty \mathrm{d}p\left|\frac{1}{K_{T/\mu,1}(p)}-\frac{1}{p^2+\kappa^2}\right||p-1|^\varepsilon\left(\frac{1}{p^\varepsilon}+1\right)p^2 \\ &\quad\times\mu^{1/2}\sup_{\ell\in\mathbf{N}_0}\int_{\mathbf{R}^3}\mathrm{d}x\,|V(x)|\left(|j_\ell(p\sqrt{\mu}|x|)|^{2-22\varepsilon/5}+|j_\ell(\sqrt{\mu}|x|)|^{2-22\varepsilon/5}\right).\end{aligned} \tag{10.3.17}$$

for any $\varepsilon\in(0,5/11)$. Again by Lemma 10.3.3 we may bound the x–integral by $\mu^{-\beta^*+\delta}(1+p^{-\beta^*+\delta})$ for any $\delta>0$ and the p–integral is finite as long as $0<T\leq C\mu$. We now define, analogously to Equation (28) in [312],

$$V^{1/2}M_{T,\mu}^{(\kappa)}|V|^{1/2} := V^{1/2}K_{T,\mu}^{-1}|V|^{1/2} - m_\mu^{(\kappa)}(T)V^{1/2}\mathfrak{F}_\mu^\dagger\mathfrak{F}_\mu|V|^{1/2} = L_\mu^{(\kappa)} + A_{T,\mu}^{(\kappa)}$$

and combine the bounds (10.3.16) and (10.3.17) from above to obtain

$$\limsup_{\mu\to\infty}\mu^{\beta^*-1/2-\delta}\sup_{0<T\leq C\mu}\left\|V^{1/2}M_{T,\mu}^{(\kappa)}|V|^{1/2}\right\|_{\mathrm{op}} = 0 \tag{10.3.18}$$

for any $\delta>0$. Also, since $V^{1/2}\mathfrak{F}_\mu^\dagger\mathfrak{F}_\mu|V|^{1/2}$ is isospectral to $\mathcal{V}_\mu$, so its eigenvalues are given by Equation (10.2.2), one can easily see, using Lemma 10.3.3 again, that

$$\limsup_{\mu\to\infty}\mu^{\beta^*-\delta}\left\|V^{1/2}\mathfrak{F}_\mu^\dagger\mathfrak{F}_\mu|V|^{1/2}\right\|_{\mathrm{op}} = 0 \tag{10.3.19}$$

for any $\delta>0$. In particular, since $s^*>1$, the bound (10.3.18) implies that $1+V^{1/2}M_{T,\mu}^{(\kappa)}|V|^{1/2}$ is invertible for any $0<T\leq C\mu$ and μ large enough.

We can thus, following the argument around Equation (30) in [312], conclude that the Birman-Schwinger operator $B_{T,\mu}$ having an eigenvalue -1 is equivalent to the selfadjoint operator

$$\mathfrak{F}_\mu|V|^{1/2}\frac{m_\mu^{(\kappa)}(T)}{1+V^{1/2}M_{T,\mu}^{(\kappa)}|V|^{1/2}}V^{1/2}\mathfrak{F}_\mu^\dagger\,. \tag{10.3.20}$$

acting on $L^2(\mathbb{S}^2)$ having an eigenvalue -1. At $T=T_c$, -1 is the smallest eigenvalue of $B_{T,\mu}$, hence (10.3.20) has an eigenvalue -1 for this value of T. By continuity and monotonicity of $m_\mu^{(\kappa)}(T)$ one can in fact show that -1 is the smallest eigenvalue of (10.3.20) in this case (cf. the discussion below Equation (31) in [312]).

Since $\mathfrak{F}_\mu V\mathfrak{F}_\mu^\dagger = \mathcal{V}_\mu$ (see Equation (10.2.1)) and $e_\mu = \inf\operatorname{spec}\mathcal{V}_\mu<0$ by Lemma 10.3.4, it immediately follows that

$$-1 = \lim_{\mu\to\infty}\inf\operatorname{spec}\mathcal{V}_\mu\, m_\mu^{(\kappa)}(T_c) = \lim_{\mu\to\infty}e_\mu m_\mu^{(\kappa)}(T_c)\,,$$

which, in combination with the asymptotics (10.3.1) and the argument for $T_c = o(\mu)$ from the proof of Theorem 10.2.2, reproves (10.2.4) resp. (10.2.5), i.e. the asymptotic formula for T_c to leading order. To obtain the next order, we employ first order perturbation theory as in the proof of Theorem 1 in [312] (in particular, see Equation (32)).

Indeed, using Equation (10.3.18) and Equation (10.3.19), we can expand the geometric series in Equation (10.3.20) to first order and employ first order perturbation theory to arrive at

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) = \frac{-1}{\sqrt{\mu} e_\mu - \sqrt{\mu}\langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u\rangle + O(\mu^{-4\beta^* + \min(s^*,2)/2+3/2+\delta})}, \tag{10.3.21}$$

for any $\delta > 0$. Here, u is the normalized eigenfunction corresponding to the lowest negative eigenvalue e_μ of the compact operator $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu^\dagger$ (see Lemma 10.3.4). In case of (finite!) degeneracy, one has to choose the ground state u of $\mathcal{V}_\mu$ that minimizes the second term in the denominator of (10.3.21). The error term in Equation (10.3.21) is twofold. The first part comes from expanding the geometric series. The second part comes from first order perturbation theory, where we made use of the fact that

$$|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)/2+1/2} \quad \text{and} \quad \sup_{0<T\leq C\mu} \sqrt{\mu} \big\| \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger \big\|_{\mathrm{op}} \leq C_\delta \mu^{-2\beta^*+1+\delta} \tag{10.3.22}$$

for any $\delta > 0$ by Lemma 10.3.4 resp. Equation (10.3.19) and Equation (10.3.18) (recall $T_c = o(\mu)$ from above). The error from the series expansion is of order $O(\mu^{-3\beta^*+3/2+\delta})$ and the error from the perturbation argument is of order $O(\mu^{-4\beta^*+\min(s^*,2)/2+3/2+\delta})$ and hence dominates, since $\beta^* \leq \min(s^*,2)/2$.

Equation (10.3.21) is an implicit equation for T_c. By the second estimate in Equation (10.3.22) and since $T_c \to 0$ as $\mu \to \infty$, we need to evaluate the limit of $\langle u | \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u\rangle$ as $T \to 0$. By the same arguments as used in Equation (35) in [312] (dominated convergence and Lipschitz continuity of the angular integrals), this yields

$$\lim_{T\to 0} \langle u | \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u\rangle = \langle u | \mathcal{W}_\mu^{(\kappa)} | u\rangle ,$$

uniformly in $u \in L^2(\mathbb{S}^2)$ with $\|u\|_{L^2(\mathbb{S}^2)} = 1$, where $\mathcal{W}_\mu^{(\kappa)}$ was defined in (10.2.9). Using that T_c is exponentially small (in some positive power of μ) as $\mu \to \infty$ by application of Theorem 10.2.2 in combination with Equation (10.2.7), we obtain

$$\big| \langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u\rangle - \langle u | \mathcal{W}_\mu^{(\kappa)} | u\rangle \big| \leq C_D \mu^{-D} \tag{10.3.23}$$

for any $D > 0$, uniformly in $u \in L^2(\mathbb{S}^2)$ with $\|u\|_{L^2(\mathbb{S}^2)} = 1$. Combining the second estimate in Equation (10.3.22) with Equation (10.3.23) we thus get

$$\big\| \mathcal{W}_\mu^{(\kappa)} \big\|_{\mathrm{op}} \leq C_\delta \mu^{-2\beta^*+1/2+\delta} \tag{10.3.24}$$

for any $\delta > 0$. Since $|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)+1/2}$, we have that whenever $e_\mu < 0$ also $b_\mu^{(\kappa)} < 0$ for large enough μ (recall Equation (10.2.10) and Equation (10.2.11)). In particular, combining Equations (10.3.21), (10.3.23) and (10.3.24), we have shown that

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} = O(\mu^{-4\beta^*+3\min(s^*,2)/2+1/2+\delta}) ,$$

for any $\delta > 0$. Since $4\beta^* - 3\min(s^*,2)/2 - 1/2 > 0$ (see Equation (10.3.12)), we conclude Equation (10.3.14) by employing the asymptotics (10.3.1).

In case that there exists $\mu_0 > 0$ and $\mathcal{L} \subset \mathbb{N}_0$ with $|\mathcal{L}| < \infty$, such that for all $\mu \geq \mu_0$, the ground state space of $\mathcal{V}_\mu$ is contained in the finite–dimensional subspace

$$\mathcal{I}_\mathcal{L} := \operatorname{span}\{Y_\ell^m : \ell \in \mathcal{L}, |m| \leq \ell\}$$

of $L^2(\mathbb{S}^2)$, spanned by the spherical harmonics with angular momentum $\ell \in \mathcal{L}$, we can drop condition (d) from Definition 10.2.5. In order to see this, take Y_ℓ^m with $\ell \in \mathcal{L}$ and $|m| \leq \ell$ and estimate

$$\begin{aligned}\left\| |V|^{1/2} \mathbf{F}_\mu^\dagger Y_\ell^m \right\|_{L^2}^2 &= C \int_{\mathbf{R}^3} |V(x)| \left| \int_{\mathbb{S}^2} e^{i\sqrt{\mu} p \cdot x} Y_\ell^m(p) \mathrm{d}\omega(p) \right|^2 \mathrm{d}x \\ &= C \int_{\mathbf{R}^3} |V(x)| (j_\ell(\sqrt{\mu}|x|))^2 \mathrm{d}x \leq C_{\ell,\delta}\, \mu^{-\min(s^*,2)+\delta},\end{aligned}$$

for any $\delta > 0$. The second equality follows by the radiality of V and the final estimate by the decay of spherical Bessel functions (see the first bound in Proposition 10.3.7 (iii)). Using Equation (10.3.18) with $\beta_{3/2}^*$ instead of β^* by means of Lemma 10.3.3, this implies that

$$\sup_{u \in \mathcal{I}_\mathcal{L},\, \|u\|_{L^2}=1} \left| \sqrt{\mu} \left\langle u | \mathfrak{F}_\mu V M_{T,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \right\rangle \right| \leq C_{\mathcal{L},\delta}\, \mu^{-\beta_{3/2}^* - \min(s^*,2)/2+1+\delta} \tag{10.3.25}$$

for any $\delta > 0$ (and μ large enough). Therefore, since $\beta_{3/2}^* \leq \min(s^*, 2)/2$, the error from the geometric expansion dominates the error from the perturbation theory in Equation (10.3.21) and we get

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) = \frac{-1}{\sqrt{\mu} e_\mu - \sqrt{\mu} \left\langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \right\rangle + O(\mu^{-3\beta_{3/2}^* + 3/2 + \delta})}, \tag{10.3.26}$$

for any $\delta > 0$, instead. Moreover, using Equation (10.3.25) and Equation (10.3.23), we get

$$\left\| \mathcal{W}_\mu^{(\kappa)} \big|_{\mathcal{I}_\mathcal{L}} \right\|_{\mathrm{op}} \leq C_{\mathcal{L},\delta} \mu^{-\beta_{3/2}^* - \min(s^*,2)/2+1/2+\delta} \tag{10.3.27}$$

for any $\delta > 0$. By combining Equations (10.3.26), (10.3.23) and (10.3.27) with $|\sqrt{\mu} e_\mu| \geq c_\delta\, \mu^{-\min(s^*+\delta,2)+1/2}$, we find

$$\frac{1}{\sqrt{\mu}} m_\mu^{(\kappa)}(T_c) + \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} = O(\mu^{-3\beta_{3/2}^* + \min(s^*,2)+1/2+\delta})$$

for any $\delta > 0$, and the proof comes to an end in the same way as above by realizing that $3\beta_{3/2}^* - \min(s^*, 2) - 1/2 > 0$. □

Proof of Lemma 10.3.6. The proof follows a similar perturbation theoretic argument as in the proof of Lemma 10.3.5. We first assume condition (d) from Definition 10.2.5 and discuss the changes for the special case afterwards. To begin with, we derive a bound on $\|\mathcal{W}_\mu\|_{\mathrm{op}}$. For this purpose, we take a normalized $u \in L^2(\mathbb{S}^2)$ and estimate

$$\begin{aligned}\left| \left\langle u | \mathcal{W}_\mu | u \right\rangle \right| \leq\ & \left| \left\langle u | \mathcal{W}_\mu | u \right\rangle - \left\langle u | \mathcal{W}_\mu^{(\kappa)} | u \right\rangle \right| \\ &+ \left| \left\langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \right\rangle - \left\langle u | \mathcal{W}_\mu^{(\kappa)} | u \right\rangle \right| + \left| \left\langle u | \mathfrak{F}_\mu V M_{T_c,\mu}^{(\kappa)} V \mathfrak{F}_\mu^\dagger | u \right\rangle \right|.\end{aligned}$$

The second term is smaller than any inverse power of μ by Equation (10.3.23). Using Equation (10.3.18) and Equation (10.3.19), the third term is bounded by $\mu^{-2\beta^*+1/2+\delta}$ for any $\delta > 0$, uniformly in $u \in L^2(\mathbb{S}^2)$. Since

$$\left\langle u | \mathcal{W}_\mu | u \right\rangle - \left\langle u | \mathcal{W}_\mu^{(\kappa)} | u \right\rangle = \sqrt{\mu} \int_0^\infty \mathrm{d}|p| \left(1 - \frac{|p|^2}{|p|^2 + \kappa^2} \right) \|\mathcal{V}_\mu u\|_{L^2}^2 = \kappa \frac{\pi}{2} \sqrt{\mu} \|\mathcal{V}_\mu u\|_{L^2}^2, \tag{10.3.28}$$

we infer by means of Equation (10.3.18) that also the first term is bounded by $\mu^{-2\beta^*+1/2+\delta}$, uniformly in $u \in L^2(\mathbb{S}^2)$, and we thus have

$$\|\mathcal{W}_\mu\|_{\mathrm{op}} \leq C_\delta \mu^{-2\beta^*+1/2+\delta} \tag{10.3.29}$$

for any $\delta > 0$. In particular, since $|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)+1/2}$ for any $\delta > 0$, this shows that, whenever $e_\mu < 0$ also $b_\mu < 0$ for large enough μ. Moreover, using $|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)+1/2}$ together with Equation (10.3.24) and Equation (10.3.29), first order perturbation theory implies

$$\begin{aligned} \frac{\pi}{2\sqrt{\mu} b_\mu} - \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} &= \frac{\pi}{2} \frac{b_\mu^{(\kappa)} - b_\mu}{\sqrt{\mu} \, b_\mu^{(\kappa)} b_\mu} \\ &= \frac{\big(e_\mu - \big\langle u \big| \mathcal{W}_\mu^{(\kappa)} \big| u \big\rangle\big) - \big(e_\mu - \big\langle u' \big| \mathcal{W}_\mu \big| u' \big\rangle\big) + O(\mu^{-4\beta^*+\min(s^*,2)/2+3/2+\delta})}{\sqrt{\mu} e_\mu^2 + O(\mu^{-2\beta^*+1+\delta})} \\ &= \kappa \, \frac{\pi}{2} + O(\mu^{-4\beta^*+3\min(s^*,2)/2+1/2+\delta}) \, . \end{aligned} \tag{10.3.30}$$

As in the proof of Lemma 10.3.5, u resp. u' is a (the) normalized eigenfunction corresponding to the lowest eigenvalue e_μ of $\mathcal{V}_\mu$. In case of (finite!) degeneracy, one has to choose the ground state u resp. u' of $\mathcal{V}_\mu$ that minimizes the second term in each bracket $(\cdots)$ in Equation (10.3.30). *A priori*, u and u' could be different. But, by application of Equation (10.3.28) we get that $\mathcal{W}_\mu$ and $\mathcal{W}_\mu^{(\kappa)}$ differ only by the constant $(\kappa\pi\sqrt{\mu} e_\mu^2)/2$ on the ground state space of $\mathcal{V}_\mu$. Therefore, $u = u'$ and the last equality in Equation (10.3.30) follows by Equation (10.3.28) in combination with $|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)+1/2}$. Since $4\beta^* - 3\min(s^*,2)/2 - 1/2 > 0$ (see Equation (10.3.12)), Equation (10.3.30) implies Equation (10.3.15).

In case that there exists $\mu_0 > 0$ and $\mathcal{L} \subset \mathbb{N}_0$ with $|\mathcal{L}| < \infty$, such that for all $\mu \geq \mu_0$, the ground state space of $\mathcal{V}_\mu$ is contained in the finite–dimensional subspace

$$\mathcal{I}_\mathcal{L} \coloneqq \operatorname{span} \{ Y_\ell^m : \ell \in \mathcal{L}, \, |m| \leq \ell \}$$

of $L^2(\mathbb{S}^2)$, spanned by the spherical harmonics with angular momentum $\ell \in \mathcal{L}$, we can drop condition (d) from Definition 10.2.5. In order to see this, we use Equation (10.2.2) and estimate

$$\Big\| \mathcal{V}_\mu \big|_{\mathcal{I}_\mathcal{L}} \Big\|_{\mathrm{op}} = \sup_{\ell \in \mathcal{L}} \left| \frac{1}{2\pi^2} \int_{\mathbf{R}^3} V(x) \, (j_\ell(\sqrt{\mu}|x|))^2 \, \mathrm{d}x \right| \leq C_{\mathcal{L},\delta} \, \mu^{-\min(s^*,2)+\delta}$$

for any $\delta > 0$ (and μ large enough) by means of Proposition 10.3.7 (iii). Combining this with Equation (10.3.25) and using $\beta_{3/2}^* \leq \min(s^*,2)/2$, we get by the same argument as above that

$$\Big\| \mathcal{W}_\mu \big|_{\mathcal{I}_\mathcal{L}} \Big\|_{\mathrm{op}} \leq C_{\mathcal{L},\delta} \mu^{-\beta_{3/2}^* - \min(s^*,2)/2+1/2+\delta} \tag{10.3.31}$$

for any $\delta > 0$. Using first order perturbation theory, Equation (10.3.31) and Equation (10.3.28) together with $|\sqrt{\mu} e_\mu| \geq c_\delta \, \mu^{-\min(s^*+\delta,2)+1/2}$ imply

$$\frac{\pi}{2\sqrt{\mu} b_\mu} - \frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} = \kappa \, \frac{\pi}{2} + O(\mu^{-3\beta_{3/2}^* + \min(s^*,2)+1/2+\delta})$$

for any $\delta > 0$. Since $3\beta_{3/2}^* - \min(s^*,2) - 1/2 > 0$ we conclude the desired. □

Finally, we give the proofs of Lemma 10.3.3 and Lemma 10.3.4.

Proof of Lemma 10.3.3. For $s^* \in (1, 5/3]$ the statement easily follows from the uniform decay of spherical Bessel functions (see Proposition 10.3.7 (iii)). For $s^* > 5/3$ choose

$$\alpha = \max\left(\frac{5q-7}{3qs^*-7}, \frac{5q-7}{6q-7} \right) \in (0,1) \tag{10.3.32}$$

and for (small) $\delta > 0$ set $s \coloneqq \min(s^*,2) - \delta/\alpha$. Recall that $q = p/(p-1)$ denotes the dual of $p \in [3/2, 9/4]$. We now employ Hölder's inequality to obtain

$$\sup_{\ell \in \mathbb{N}_0} \int_{\mathbf{R}^3} \mathrm{d}x |V(x)| |j_\ell(\sqrt{\mu}|x|)|^{2-\varepsilon}$$

$$\leq C \left\| \frac{V}{|\cdot|^s} \right\|_{L^1}^{\alpha} \|V\|_{L^p}^{1-\alpha} \sup_{\ell \in \mathbb{N}_0} \left(\int_0^\infty \mathrm{d}x \, x^{\frac{q\alpha s}{1-\alpha}+2} \, |j_\ell(\sqrt{\mu}x)|^{\frac{q}{1-\alpha}(2-\varepsilon)} \right)^{\frac{1-\alpha}{q}}$$

$$\leq C\mu^{-\frac{\alpha s+3(1-\alpha)/q}{2}} \left\| \frac{V}{|\cdot|^s} \right\|_{L^1}^{\alpha} \|V\|_{L^p}^{1-\alpha} \sup_{\ell \in \mathbb{N}_0} \left(\int_0^\infty \mathrm{d}x \, x^{\frac{q\alpha s}{1-\alpha}+2} \, |j_\ell(x)|^{\frac{q}{1-\alpha}(2-\varepsilon)} \right)^{\frac{1-\alpha}{q}} .$$

For $\varepsilon(\delta) > 0$ small enough, the integral is finite by the uniform L^p–integrability of spherical Bessel functions (see Proposition 10.3.7 (iv)) since $\alpha < (5q-7)/(3qs-7)$ and thus the claim follows since $\frac{\alpha s+3(1-\alpha)/q}{2} = \beta_p^* - \delta/2$ (cf. Equation (10.3.11) for the definition of β_p^*, and Equation (10.3.32)). □

Proof of Lemma 10.3.4. To begin with the proof, we have two important observations.

First, recall the definition of $s_\pm^*$ from Equation (10.2.6). We aim to prove that $r_\pm^* = s_\pm^*$, where

$$r_\pm^* := \sup \left\{ r \geq 0 : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm(x) \mathrm{d}x = 0 \right\} .$$

For this purpose, we define

$$s_\pm^*(a) := \sup \left\{ s \geq 0 : |\cdot|^{-s} V_\pm|_{B_a}^* \in L^1(\mathbf{R}^3) \right\}$$

and

$$r_\pm^*(a) := \sup \left\{ r \geq 0 : \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^r} \int_{B_\varepsilon} V_\pm|_{B_a}^*(x) \, \mathrm{d}x = 0 \right\}$$

for the same $a > 0$, for which we assumed that $r_\pm^* = r_\pm^*(a)$ in Definition 10.2.5.

Note that $r_\pm^* \geq s_\pm^*$ by definition. Using that $|\cdot|^{-s}$ is equal to its symmetric decreasing rearrangement, we can employ the basic rearrangement inequality [418, Thm. 3.4] to obtain $s_\pm^* \geq s_\pm^*(a)$. Therefore, since $r_\pm^* = r_\pm^*(a)$ by assumption, we have

$$r_\pm^*(a) = r_\pm^* \geq s_\pm^* \geq s_\pm^*(a) .$$

In order to see $r_\pm^* = s_\pm^*$ it is sufficient to prove that $s_\pm^*(a) \geq r_\pm^*(a)$. Assume the contrary, i.e. $s_\pm^*(a) < r_\pm^*(a)$, and let $r, r+\delta \in (s_\pm^*(a), r_\pm^*(a))$ for some $\delta > 0$. We denote $V_{\pm,a}^* \equiv V_\pm|_{B_a}^*$ for short. By definition of $s_\pm^*$ and $r_\pm^*$, we thus have

$$\int_{B_\varepsilon} \frac{V_{\pm,a}^*(x)}{|x|^r} \mathrm{d}x \geq c \qquad \text{and} \qquad \int_{B_\varepsilon} V_{\pm,a}^*(x) \mathrm{d}x = o(\varepsilon^{r+\delta}) . \tag{10.3.33}$$

The first integral actually equals infinity, but we only need that it is uniformly bounded from below by some $c > 0$. Since $|V_{\pm,a}|^*$ is symmetric–decreasing and thus one–sided limits exist, the auxiliary quantity

$$t_\pm^*(a) := \inf \left\{ t \geq 0 : \lim_{|x| \to 0} |x|^t \, V_{\pm,a}^*(x) = 0 \right\}$$

is well defined. By definition of $t_\pm^*(a)$ we thus get

$$\frac{c_\nu}{|x|^{t_\pm^*(a)-\nu}} \leq V_{\pm,a}^*(x) \leq \frac{C_\nu}{|x|^{t_\pm^*(a)+\nu}}$$

for any $\nu > 0$ and $|x|$ small enough. Inserting this in Equation (10.3.33) we arrive at

$$\varepsilon^{3-t_\pm^*(a)-r-\nu} \geq c_\nu \qquad \text{and} \qquad \varepsilon^{3-t_\pm^*(a)-r-\delta+\nu} \leq C_\nu$$

which yields a contradiction by choosing $\nu \in (0, \delta/2)$. Therefore, $r_\pm^*(a) = s_\pm^*(a)$, which proves that $r_\pm^* = s_\pm^*$.

Second, note that for any $f \in L^1(\mathbf{R}^3)$ we have

$$\int_{\mathbf{R}^3} f(x)(\sin(n|x|))^2 \mathrm{d}x = \frac{1}{2}\int_{\mathbf{R}^3} f(x)(1-\cos(2n|x|))\mathrm{d}x \longrightarrow \frac{1}{2}\int_{\mathbf{R}^3} f(x)\mathrm{d}x$$

as $n \to \infty$ by the Riemann–Lebesgue Lemma.

In order to prove Lemma 10.3.4, we study the asymptotic behavior of the integral

$$v_\mu := \int_{\mathbf{R}^3} V(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x$$

in three different cases.

Case 1. If $|\cdot|^{-2}V \in L^1(\mathbf{R}^3)$, we get by our second observation that

$$v_\mu = \mu^{-1}\left(\frac{1}{2}\int_{\mathbf{R}^3} \frac{V(x)}{|x|^2}\,\mathrm{d}x + o(1)\right) \leq -c\mu^{-1},$$

which immediately proves the claim.

Case 2. If $|\cdot|^{-2}V \notin L^1(\mathbf{R}^3)$ and $s^* < 2$ we take some $r \in (0, 1/2)$ and estimate

$$\begin{aligned}\int_{\mathbf{R}^3} V(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \\ = \int_{B_r} V_+(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x - \int_{B_r} V_-(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x + O(\mu^{-1}).\end{aligned}$$

The first term can be bounded by $\mu^{-s_+^*/2+\delta}$ for any $\delta > 0$. The second term can be estimated from below as

$$\begin{aligned}\int_{B_r} V_-(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \geq \int_{B_{\frac{r}{\sqrt{\mu}}}} V_-(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \\ \geq c_r \int_{B_{\frac{r}{\sqrt{\mu}}}} V_-(x)\mathrm{d}x \geq c_{r,\delta}\mu^{-s_-^*/2-\delta}\end{aligned}$$

for any $\delta > 0$. Since $s^* = s_-^* < s_+^*$, we get that $v_\mu \leq -c_\delta \mu^{-\min(s^*+\delta,2)/2}$ for any $\delta > 0$.

Case 3. If $|\cdot|^{-2}V \notin L^1(\mathbf{R}^3)$ and $s^* = 2$ we have that $|\cdot|^{-2}V_+ \in L^1(\mathbf{R}^3)$ but $|\cdot|^{-2}V_- \notin L^1(\mathbf{R}^3)$ since $s^* = s_-^* < s_+^*$. On the one hand, this implies that

$$\int_{\mathbf{R}^3} V_+(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \leq K\mu^{-1}$$

for some $K > 0$ by means of our second observation. On the other hand, let $r > 0$ and estimate

$$\mu\int_{\mathbf{R}^3} V_-(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \geq \mu\int_{B_r^c} V_-(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2 \mathrm{d}x \overset{\mu\to\infty}{\longrightarrow} \frac{1}{2}\int_{B_r^c} \frac{V_-(x)}{|x|^2}\mathrm{d}x.$$

By taking $r \to 0$ the right hand side can be made arbitrarily large, in particular greater than K. This implies that $v_\mu \leq -C\mu^{-1}$ for any $C > 0$. □

10.3.4 Properties of spherical Bessel functions

Proposition 10.3.7. (Properties of spherical Bessel functions [3, 391, 537])
The spherical Bessel functions $(j_\ell)_{\ell\in\mathbb{N}_0}$ satisfy the following properties:

(i) uniform boundedness, i.e. $\sup_{\ell\in\mathbb{N}_0}\sup_{x\geq 0}|j_\ell(x)|\leq 1$,

(ii) uniform Lipschitz continuity, i.e. $\sup_{\ell\in\mathbb{N}_0}\sup_{x\geq 0}|j_\ell'(x)|\leq 1$,

(iii) (uniform) decay, i.e. for every $\ell\in\mathbb{N}_0$, we have $\sup_{x\geq 0}|x\, j_\ell(x)|\leq C_\ell$ *for some* $C_\ell>0$, *and* $\sup_{\ell\in\mathbb{N}_0}\sup_{x\geq 0}|x^{5/6}j_\ell(x)|\leq C$ *for some universal* $C>0$,

(iv) uniform L^p–integrability, i.e. for $p\in(0,\infty)$ and $a\in(-1,p-1)$ if $p\in(0,4]$ or $a\in(-1,5p/6-1/3)$ if $p\in(4,\infty)$, we have

$$\sup_{\ell\in\mathbb{N}_0}\int_0^\infty |j_\ell(x)|^p x^a \mathrm{d}x\leq C_{p,a}$$

for some universal constant $C_{p,a}>0$.

Proof. The first statement (i) is an elementary property of the spherical Bessel functions. The second statement (ii) follows from the uniform boundedness in (i) and the recursion relation [3, Eq. 10.1.20]

$$j_\ell' = \frac{1}{2\ell+1}\left(\ell j_{\ell-1}-(\ell+1)j_{\ell+1}\right).$$

By noticing that $j_\ell(x)=\sqrt{\pi/(2x)}J_{\ell+1/2}(x)$, the third (iii) and the fourth statement (iv) are easy consequence of [3, Eq. 9.2.1], [391, Eq. 1], and [537, Eq. 3], respectively, where analogous estimates for the standard Bessel functions J_ν with $\nu\geq 0$ are proven. □

10.3.5 Proof of Equation (10.2.12)

Proof of Equation (10.2.12). We note that $K_{T,\mu}(p)+V(x)\geq 0$ is equivalent to $K_{T/\mu,1}(p)+\frac{1}{\mu}V(x/\sqrt{\mu})\geq 0$ and estimate

$$\begin{aligned}
K_{T/\mu,1}(p)+\frac{1}{\mu}V(x/\sqrt{\mu}) &\geq \frac{1}{2}\left(|p^2-1|+\frac{2T}{\mu}\right)-\frac{1}{\mu}V_-(x/\sqrt{\mu})\\
&\geq \frac{1}{2}\left(|p^2-1|+\frac{2T}{\mu}-\frac{2}{\mu}V_-(x/\sqrt{\mu})\left(\mathrm{e}^{-m|x|}+m|x|\right)\right)\\
&\geq \frac{1}{2}\left(|p^2-1|+\frac{2T}{\mu}-\frac{2}{\sqrt{\mu}}\||\cdot|V\|_{L^\infty}\left(\frac{\mathrm{e}^{-m|x|}}{|x|}+m\right)\right)
\end{aligned}$$

for any $m>0$. By definition of T_c, we have the bound

$$T_c\leq -\frac{\mu}{2}\inf\operatorname{spec}\left(|p^2-1|-\frac{2}{\sqrt{\mu}}\||\cdot|V\|_{L^\infty}\left(\frac{\mathrm{e}^{-m|x|}}{|x|}+m\right)\right).$$

After taking $m=(\mathrm{const.})\mu^{1/4}\mathrm{e}^{-\sqrt{\pi/(2\||\cdot|V\|_{L^\infty})}\,\mu^{1/4}}$ and using the estimate above Equation (15) in [291], we get

$$T_c\lesssim \mu\exp\left(-\sqrt{\frac{\pi}{2\||\cdot|V\|_{L^\infty}}}\mu^{1/4}\right).$$ □

Chapter 11

The BCS energy gap at high density

This chapter contains the paper [336]:

J. Henheik and A. B. Lauritsen. The BCS Energy Gap at High Density. *J. Stat. Phys.*, 189(5), 2022

Abstract. We study the BCS energy gap Ξ in the high–density limit and derive an asymptotic formula, which strongly depends on the strength of the interaction potential V on the Fermi surface. In combination with the recent result by one of us (Math. Phys. Anal. Geom. 25, 2022; see Chapter 10) on the critical temperature T_c at high densities, we prove the universality of the ratio of the energy gap and the critical temperature.

Content

11.1 Introduction and Main Results

The Bardeen–Cooper–Schrieffer (BCS) theory [48] (see [316] for a review of recent rigorous mathematical work) has been an important theory of superconductivity since its conception. More recently, it has also gained attraction for describing the phenomenon of superfluidity in ultra cold fermionic gases, see [87, 145] for reviews. In either context, BCS theory is often formulated in terms of the BCS gap equation (at zero temperature)

$$\Delta(p) = -\frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} \hat{V}(p-q) \frac{\Delta(q)}{E_{\Delta,\mu}(q)} \mathrm{d}q \,, \tag{11.1.1}$$

where $E_{\Delta,\mu}(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$. At finite temperature $T > 0$ one replaces $E_{\Delta,\mu}$ by $E_{\Delta,\mu}/\tanh(E_{\Delta,\mu}/2T)$. The function Δ is interpreted as the order parameter describing the Cooper

pairs (paired fermions). The interaction is local and given by the potential V, which we will assume satisfies $V \in L^1(\mathbf{R}^3)$, in which case it has a Fourier transform given by $(\mathbf{F}V)(p) = \hat{V}(p) = (2\pi)^{-3/2} \int_{\mathbf{R}^3} V(x) e^{-ip\cdot x} \mathrm{d}x$.

The chemical potential μ controls the density of the fermions, and we investigate the high–density limit, i.e. $\mu \to \infty$, here. Recently this limit was studied by one of us [333], where an asymptotic formula for the critical temperature T_c was found. For temperatures T below the critical temperature, $T < T_c$, the gap equation at temperature T (Equation (11.1.1) with $E_{\Delta,\mu}$ replaced as prescribed) admits a non–trivial solution, for $T \geq T_c$ it does not. The critical temperature may equivalently be characterized by the existence of a negative eigenvalue of a certain linear operator, see [309]. Physically, a system at temperature T is superconducting/–fluid if $T < T_c$, if $T \geq T_c$ it is not.

In this paper we study the energy gap (at zero temperature)

$$\Xi = \inf_p E_{\Delta,\mu}(p) = \inf_p \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}\,. \tag{11.1.2}$$

The function $E_{\Delta,\mu}$ has the interpretation of the dispersion relation for the corresponding BCS Hamiltonian, and so Ξ is indeed an energy gap (see Appendix A in [309]). We show that, in the high–density limit, $\mu \to \infty$, the ratio of the energy gap and the critical temperature tends to a universal constant independent of the interaction potential,

$$\frac{\Xi}{T_c} \approx \frac{\pi}{e^\gamma}\,, \tag{11.1.3}$$

where $\gamma \approx 0.577$ denotes the Euler–Mascheroni constant. This universality is well–known in the physics literature, see, e.g., [295], and was rigorously verified in the weak–coupling limit by Hainzl and Seiringer [312] and in the low–density limit, $\mu \to 0$, by one of us [403] building on a work by Hainzl and Seiringer [311]. The general strategy for proving the universality in these limits has been to establish sufficiently good asymptotic formulas for both, T_c and Ξ, and compare them afterwards.

The weak–coupling limit is studied in [312, 261], where one considers a potential λV for V fixed and a small coupling constant $\lambda \to 0$. In this limit, Hainzl and Seiringer [312] have shown that the critical temperature and energy gap satisfies $T_c \sim A\exp(-B/\lambda)$ and $\Xi \sim C\exp(-B/\lambda)$ respectively for explicit constants $A, B, C > 0$ depending on the interaction potential V and the chemical potential μ. This limit exhibits the same universality and the ratio $C/A = \pi e^{-\gamma}$ is independent of the interaction potential V and the chemical potential μ.

The low–density limit $\mu \to 0$ is studied in [311, 403]. In this limit Hainzl and Seiringer [311] have shown that the critical temperature satisfies $T_c \sim \mu A\exp(-B/\sqrt{\mu})$ and one of us [403] has shown that the energy gap satisfies $\Xi \sim \mu C\exp(-B/\sqrt{\mu})$, for some (different) explicit constants $A, B, C > 0$ depending on the interaction potential V. Also in this limit we have the same universality and the ratio $C/A = \pi e^{-\gamma}$ is independent of the interaction potential V. These results together with the present paper thus show that the universality (11.1.3) holds in both, the low– and high–density limit, as well as in the weak–coupling limit.

To show the universality, we prove in Theorem 11.1.3 an asymptotic formula for the energy gap Ξ in the high–density limit, similar to the corresponding formula for the critical temperature given in Theorem 10.2.7. This formula, as well as the one given in Theorem 11.1.3, depends strongly on the strength of the interaction potential V on the Fermi sphere $\{p^2 = \mu\}$, which becomes weak due to the decay of $\hat{V}$ in momentum space. Together with the formula for the critical temperature in Chapter 10 we prove the universality (11.1.3) in Corollary 11.1.5. All proofs are given in Section 11.2. We now introduce some technical constructions and give the precise statements of our results.

11.1.1 Preliminaries

We will work with the formulation of BCS theory of [311, 312, 269, 309, 316, 333, 403]. There one considers minimizers of the BCS functional (at zero temperature)

$$\mathcal{F}(\alpha) = \frac{1}{2}\int_{\mathbf{R}^3} |p^2-\mu|\left(1-\sqrt{1-4|\hat{\alpha}(p)|^2}\right)\mathrm{d}p + \int_{\mathbf{R}^3} V(x)|\alpha(x)|^2\mathrm{d}x\,. \tag{11.1.4}$$

If α is a minimizer of this, then $\Delta = -2\widehat{V\alpha}$ satisfies the BCS gap equation (11.1.1). As discussed in [312] the minimizer α is in general not necessarily unique, hence also Δ and Ξ are not necessarily unique. However, since we will assume that the interaction V has non–positive Fourier transform, α and thus Ξ is unique (see Lemma 2 in [312]).

A crucial role for the investigation of the energy gap (11.1.2) in the high–density limit is played by the (rescaled) operator $\mathcal{V}_\mu : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ measuring the strength of the interaction potential $\hat{V}$ on the Fermi surface. It is defined as

$$\left(\mathcal{V}_\mu u\right)(p) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{S}^2} \hat{V}(\sqrt{\mu}(p-q))u(q)\,\mathrm{d}\omega(q)\,, \tag{11.1.5}$$

where $\mathrm{d}\omega$ denotes the uniform (Lebesgue) measure on the unit sphere $\mathbb{S}^2$. The pointwise evaluation of $\hat{V}$ (and in particular on a $\mathrm{codim}{-}1$ submanifold) is well defined since $V \in L^1(\mathbf{R}^3)$. The condition that $V \in L^1(\mathbf{R}^3)$ could potentially be relaxed, see [186] and Remark 10.2.9. The lowest eigenvalue of $\mathcal{V}_\mu$, which we denote by

$$e_\mu = \inf \mathrm{spec}\,\mathcal{V}_\mu$$

will be of particular importance. Note, that $\mathcal{V}_\mu$ is a trace–class operator (see the argument above Equation (3.2) in [269]) with

$$\mathrm{tr}(\mathcal{V}_\mu) = \frac{1}{2\pi^2}\int_{\mathbf{R}^3} V(x)\mathrm{d}x = \sqrt{\frac{2}{\pi}}\,\hat{V}(0)\,.$$

We will assume that $\hat{V}(0) < 0$ in which case $e_\mu < 0$. This corresponds to an attractive interaction between (some) electrons on the Fermi sphere.

In this work, we restrict ourselves to the special case of radial potentials V, where the spectrum of $\mathcal{V}_\mu$ can be determined more explicitly (see, e.g., Section 2.1 in [269]). Indeed, for radial V, the eigenfunctions of $\mathcal{V}_\mu$ are spherical harmonics and the corresponding eigenvalues are

$$\frac{1}{2\pi^2}\int_{\mathbf{R}^3} V(x)\left(j_\ell(\sqrt{\mu}|x|)\right)^2\mathrm{d}x\,. \tag{11.1.6}$$

The lowest eigenvalue e_μ is thus given by

$$e_\mu = \frac{1}{2\pi^2}\inf_{\ell\in\mathbb{N}_0}\int_{\mathbf{R}^3} V(x)\left(j_\ell(\sqrt{\mu}|x|)\right)^2\mathrm{d}x\,.$$

Here, j_ℓ denotes the spherical Bessel function of order $\ell \in \mathbb{N}_0$. Additionally, in case that $\hat{V} \le 0$, we have, by the Perron–Frobenius theorem, that the minimal eigenvalue is attained for the constant eigenfunction (i.e. with $\ell = 0$). Thus

$$e_\mu = \frac{1}{2\pi^2}\int_{\mathbf{R}^3} V(x)\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2\mathrm{d}x\,. \tag{11.1.7}$$

For further discussions of the radiality assumption on V, see Remark 10.2.8.

In order to obtain an asymptotic formula for the energy gap that is valid up to second order (see [312, 333]), we define the operator $\mathcal{W}_\mu^{(\kappa)}$ on $u \in L^2(\mathbb{S}^2)$ via its quadratic form

$$\left\langle u|\mathcal{W}_\mu^{(\kappa)}|u\right\rangle = \sqrt{\mu}\int_0^\infty \mathrm{d}|p|\left(\frac{|p|^2}{||p|^2-1|}\left[\int_{\mathbb{S}^2}\mathrm{d}\omega(p)\left(|\hat{\varphi}(\sqrt{\mu}p)|^2 - |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2\right)\right]\right.$$

$$+ \frac{|p|^2}{|p|^2+\kappa^2} \int_{\mathbb{S}^2} \mathrm{d}\omega(p) |\hat{\varphi}(\sqrt{\mu}p/|p|)|^2 \Big) \tag{11.1.8}$$

for any fixed $\kappa \geq 0$ (cf. (10.2.10) resp. Equation (13) in [312] for an analogous definition with $\kappa = 0$). Here $\hat{\varphi}(p) = (2\pi)^{-3/2} \int_{\mathbb{S}^2} \hat{V}(p - \sqrt{\mu}q) u(q) \mathrm{d}\omega(q)$, and $(|p|, \omega(p)) \in (0,\infty) \times \mathbb{S}^2$ denote spherical coordinates for $p \in \mathbb{R}^3$. To see that this operator is well–defined note that the map $|p| \mapsto \int_{\mathbb{S}^2} \mathrm{d}\omega(p) |\hat{\varphi}(p)|^2$ is Lipschitz continuous for any $u \in L^2(\mathbb{S}^2)$ since $V \in L^1(\mathbf{R}^3)$. Hence the radial integral in Equation (11.1.8) is well defined for $|p| \sim 1$. We will further assume that $V \in L^{3/2}(\mathbf{R}^3)$, in which case the integral is well–defined for large $|p|$ as well. We formulate our result in Theorem 11.1.3 only for $\kappa = 0$, but the case of a positive parameter $\kappa > 0$ is crucial in the proof of this statement. For example, $\kappa > 0$ ensures that the second term in the decomposition of the Birman–Schwinger operator associated with $E_{\Delta,\mu} + V$ is small (cf. Equation (11.2.2)). Whenever it does not lead to confusion, we will refer to some κ–dependent quantity at $\kappa = 0$ by simply dropping the (κ)–superscript.

We now define the operator

$$\mathcal{B}_\mu^{(\kappa)} = \frac{\pi}{2} \left(\mathcal{V}_\mu - \mathcal{W}_\mu^{(\kappa)} \right), \tag{11.1.9}$$

which captures the strength of the interaction potential near the Fermi surface to second order and denote its lowest eigenvalue by

$$b_\mu^{(\kappa)} = \inf \operatorname{spec} \mathcal{B}_\mu^{(\kappa)}. \tag{11.1.10}$$

The factor $\pi/2$ is introduced in Equation (11.1.9) since for this scaling, the eigenvalue $b_\mu^{(\kappa)}$ has the interpretation of an effective scattering length in the case of small μ (see Proposition 1 in [312]). Moreover, it was shown during the proof of Theorem 10.2.7 that if $e_\mu < 0$ then also $b_\mu^{(\kappa)} < 0$ for μ large enough. This will also follow from Equation (11.2.17) in the proof below.

11.1.2 Results

The following definition characterizes the class of interaction potentials for which our asymptotic formula will hold.

Definition 11.1.1 (Admissible potentials)**.** *Let $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ be a radial real–valued function with non–positive Fourier transform $\hat{V} \leq 0$ and $\hat{V}(0) < 0$. Denote*

$$s_\pm^* := \sup \left\{ s \geq 0 : |\cdot|^{-s} V_\pm \in L^1(\mathbf{R}^3) \right\}, \qquad s^* := \min\{s_+^*, s_-^*\}, \tag{11.1.11}$$

where $V_\pm = \max\{\pm V, 0\}$ are the positive and negative parts of V. We say that V is admissible *if the following is satisfied:*

(a) There exists $a > 0$ such that

$$\sup \left\{ r \geq 0 : \lim_{\epsilon \to 0} \frac{1}{\epsilon^r} \int_{B_\epsilon} V_\pm(x) \mathrm{d}x = 0 \right\} = \sup \left\{ r \geq 0 : \lim_{\epsilon \to 0} \frac{1}{\epsilon^r} \int_{B_\epsilon} V_\pm|_{B_a}^*(x) \mathrm{d}x = 0 \right\},$$

where $V_\pm|_{B_a}^*$ denotes the symmetric decreasing rearrangement of $V_\pm|_{B_a}$, the restriction of $V_\pm$ to the ball of radius a around 0,

(b) if $|\cdot|^{-2} V \notin L^1(\mathbf{R}^3)$, we have $s^* = s_-^* < s_+^*$, and

(c) $|\cdot| V \in L^2(\mathbf{R}^3)$ and $s^* > 7/5$.

As discussed around Equation (11.1.4), the definiteness of the Fourier transform is needed for ensuring uniqueness of the energy gap Ξ. Intuitively, the other criteria may be though as follows: Assumption (a) captures that the strongest singularity of V near the origin is in fact at the origin,

assumption (b) captures that V is predominantly attractive, and assumption (c) captures that V is slightly less divergent at the origin, than allowed by the $L^{3/2}(\mathbf{R}^3)$-assumption. In view of assumption (a), we remark that it is natural that the system is sensitive to the short range behavior of the interaction potential, since the interparticle distance as the physically relevant length scale that depends on the particle density tends to zero in the high–density limit. Furthermore, note that for $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$, the condition $|\cdot|V \in L^2(\mathbf{R}^3)$ is mainly about regularity away from 0 and infinity.

The most important examples of allowed interaction potentials include the cases of attractive Gaussian, Lorentzian and Yukawa potentials, also discussed in [400]. That is

$$V_{\text{Gauss}}(x) = -(2\pi)^{-3/2} e^{-x^2/2}\,, \quad V_{\text{Lorentz}}(x) = -\frac{1}{\pi^2(1+x^2)^2}\,, \quad V_{\text{Yukawa}}(x) = -\frac{1}{4\pi|x|} e^{-|x|}\,.$$

Remark 11.1.2. *The proof of our main result formulated in Theorem 11.1.3 works without change if we assume $|\cdot|V \in L^r(\mathbf{R}^3)$ for some $2 \geq r > f(s^*)$ instead of $|\cdot|V \in L^2(\mathbf{R}^3)$, where f is some complicated (explicit) expression, see the proof of Proposition 11.2.9. We do not state the theorem with this slight generalization for simplicity. We will however give the proof under this more general assumption for the purpose of illuminating where the assumption on $r = 2$ comes from. Additionally, to further illuminate where the conditions are used, all propositions and lemmas are stated with only the conditions needed on V for that specific statement. (Beyond the conditions that $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ is real–valued, radial and has $\hat{V} \leq 0, \hat{V}(0) < 0$, which is always assumed.)*

We can now state our main result for admissible interaction potentials.

Theorem 11.1.3. *Let V be an admissible potential. Then the energy gap Ξ is positive and satisfies*

$$\lim_{\mu\to\infty} \left(\log \frac{\mu}{\Xi} + \frac{\pi}{2\sqrt{\mu} b_\mu} \right) = 2 - \log(8)\,. \tag{11.1.12}$$

In other words,

$$\Xi = \mu \left(8\, e^{-2} + o(1) \right) \exp\left(\frac{\pi}{2\sqrt{\mu} b_\mu} \right)$$

in the limit $\mu \to \infty$. Similarly as for the critical temperature (see Chapter 10), this asymptotic formula is completely analogous to the weak–coupling case [312] (replace $V \to \lambda V$ and take the limit $\lambda \to 0$) but we have coupling parameter $\lambda = 1$ here. This similarity is not entirely surprising. From a physical perspective, only those fermions with momenta close to the Fermi surface $\{p^2 = \mu\}$ contribute to the superconductivity/–fluidity. Thus, by the decay of the interaction $\hat{V}$ in Fourier space, the high–density limit, $\mu \to \infty$, is effectively a weak–coupling limit.

In order to deduce universality as in Equation (11.1.3) in the high–density limit, we show that every admissible potential in the sense of Definition 11.1.1 satisfies the imposed conditions for the proof of an analogous formula for the critical temperature. These conditions were formulated in Definition 10.2.5.

Proposition 11.1.4. *Every admissible potential satisfies the conditions of Definition 10.2.5.*

Proof. By comparing the two definitions, the statement is trivial apart from the following two points. First, the additional requirement $\int_{\mathbf{R}^3} \frac{V(x)}{|x|^2} \mathrm{d}x < 0$ from Definition 10.2.5 in the case $|\cdot|^{-2} V \in L^1(\mathbf{R}^3)$ is automatically fulfilled, since

$$-\Delta_p \widehat{\frac{V}{|\cdot|^2}}(p) = \hat{V}(p) \leq 0\,.$$

That is, the radial function $\widehat{\frac{V}{|\cdot|^2}}$ is subharmonic and approaches 0 as $|p| \to \infty$ (by the Riemann–Lebesgue Lemma), and thus by the maximum principle assumes a strictly negative value at 0. Second, since $\hat{V} \leq 0$ and by application of the Perron–Frobenius Theorem, the constant spherical harmonic is the unique normalized ground state of $\mathcal{V}_\mu$ and thus condition (d) from Definition 10.2.5 can be dropped. □

Therefore, by means of Theorem 10.2.7, the critical temperature T_c satisfies

$$T_c = \mu\left(\frac{8}{\pi}e^{\gamma-2} + o(1)\right)\exp\left(\frac{\pi}{2\sqrt{\mu}b_\mu}\right)$$

for any admissible potential. Here $\gamma \approx 0.577$ is the Euler–Mascheroni constant. Together with Theorem 11.1.3, this immediately proves the following.

Corollary 11.1.5. *Let V be an admissible potential. Then*

$$\lim_{\mu\to\infty}\frac{\Xi}{T_c} = \frac{\pi}{e^\gamma} \approx 1.764\,.$$

This universality of the ratio between the energy gap and the critical temperature is well known in the physics literature (see, e.g., [295]) and has been previously established rigorously in the weak–coupling and low–density limits (see [312] resp. [403]).

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11.2 Proofs

As in the analysis of the critical temperature in Chapter 10 we introduce the parameter $\kappa > 0$. We have the following comparison of $b_\mu^{(\kappa)}$ with the $\kappa = 0$ quantity.

Lemma 11.2.1 (see Lemma 10.3.6)**.** *Let V be admissible and $\kappa > 0$. In the limit of high density, $\mu \to \infty$, we have*

$$\frac{\pi}{2\sqrt{\mu}b_\mu} = \frac{\pi}{2\sqrt{\mu}b_\mu^{(\kappa)}} + \kappa\,\frac{\pi}{2} + o(1)\,.$$

Proof. This is immediate from Lemma 10.3.6 by invoking Proposition 11.1.4. □

Now, one important ingredient in our proof is the asymptotic behavior of

$$m_\mu^{(\kappa)}(\Delta) = \frac{1}{4\pi}\int_{\mathbf{R}^3}\left(\frac{1}{E_{\Delta,\mu}(p)} - \frac{1}{p^2+\kappa^2\mu}\right)\mathrm{d}p$$

for fixed $\kappa > 0$ (recall that $E_{\Delta,\mu}(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$). This is similar to the strategy for the weak–coupling, low–density, and high–density limits of the critical temperature (see [312, 311, 333]), and for the weak–coupling and low–density limits of the energy gap (see [312, 403]).

Lemma 11.2.2. *Let V be admissible and $\kappa > 0$. In the limit of high density, $\mu \to \infty$, we have*

$$\begin{aligned}
\Xi &= \Delta(\sqrt{\mu})(1 + o(1))\,, \\
m_\mu^{(\kappa)}(\Delta) &= \sqrt{\mu}\left(\log\frac{\mu}{\Delta(\sqrt{\mu})} - 2 + \kappa\,\frac{\pi}{2} + \log(8) + o(1)\right), \\
\frac{m_\mu^{(\kappa)}(\Delta)}{\sqrt{\mu}} &= -\frac{\pi}{2\sqrt{\mu}\,b_\mu^{(\kappa)}} + o(1)\,.
\end{aligned}$$

These three asymptotic equalities are proven in Propositions 11.2.5, 11.2.9, and 11.2.10 respectively.

Proof of Theorem 11.1.3. By Lemma 11.2.2 and Lemma 11.2.1 we get

$$\begin{aligned}
\lim_{\mu\to\infty}\left(\log\frac{\mu}{\Xi} + \frac{\pi}{2\sqrt{\mu}\,b_\mu}\right) &= \lim_{\mu\to\infty}\left(\log\frac{\mu}{\Delta(\sqrt{\mu})} + \frac{\pi}{2\sqrt{\mu}\,b_\mu}\right) \\
&= \lim_{\mu\to\infty}\left(\log\frac{\mu}{\Delta(\sqrt{\mu})} + \frac{\pi}{2\sqrt{\mu}\,b_\mu^{(\kappa)}}\right) + \kappa\,\frac{\pi}{2} = \lim_{\mu\to\infty}\left(\log\frac{\mu}{\Delta(\sqrt{\mu})} - \frac{m_\mu^{(\kappa)}(\Delta)}{\sqrt{\mu}}\right) + \kappa\,\frac{\pi}{2} \\
&= 2 - \kappa\,\frac{\pi}{2} + \log(8) + \kappa\,\frac{\pi}{2} = 2 - \log(8)\,,
\end{aligned}$$

which yields (11.1.12) and we have proven Theorem 11.1.3. □

The rest of this paper is devoted to the proof of Lemma 11.2.2.

11.2.1 Proof of Lemma 11.2.2

As remarked, a key idea is to study the integral $m_\mu^{(\kappa)}(\Delta)$. As in [312, 403] we first need some control of Δ in the form of a Lipschitz–like bound (given in Lemma 11.2.4) and a bound controlling $\Delta(p)$ in terms of $\Delta(\sqrt{\mu}q)$ for $q \in \mathbb{S}^2$ (given in Equation (11.2.10)). First, we recall some properties (from [312]) of the minimizer α of the BCS functional at zero temperature

$$\mathcal{F}(\alpha) = \frac{1}{2}\int_{\mathbf{R}^3} |p^2 - \mu|\left(1 - \sqrt{1 - 4|\hat{\alpha}(p)|^2}\right)\mathrm{d}p + \int_{\mathbf{R}^3} V(x)|\alpha(x)|^2\mathrm{d}x\,. \tag{11.2.1}$$

In [312, Lemma 2] it is shown that for potentials V with non-positive Fourier transform there exists a unique minimizer α with (strictly) positive Fourier transform. Moreover, for radial V the BCS functional is invariant under rotations. Hence α and thus also $\Delta = -2\widehat{V\alpha}$ are radial functions. Therefore, with a slight abuse of notation, we will write $\Delta(|p|)$ and mean $\Delta(p)$ for some (any) vector p. (In general for any radial function f, we will write $f(|p|)$ for the value of $f(p)$.) Additionally, since $\hat{V} \le 0$ we have that $\Delta \ge 0$. In fact, by the BCS gap equation (11.1.1), we even have $\Delta > 0$, see Lemma 2 in [312]. Now, we give some *a priori* bounds on the minimizer α. The proofs of Lemma 11.2.3 and Lemma 11.2.4 are given in Section 11.2.2.

Lemma 11.2.3. *Let α be the minimizer of the BCS functional* (11.2.1)*. Then for large μ*

$$\|\alpha\|_{L^2} \le C\mu^{7/20} \quad \textit{and} \quad \|\alpha\|_{H^1} \le C\mu^{3/4}.$$

These estimates on the minimizer α now translate to bounds on $\Delta = -2\widehat{V\alpha}$.

Lemma 11.2.4. *Suppose $V \in L^r(\mathbf{R}^3)$ for some $6/5 \le r \le 2$. Define $\delta_r = \frac{3}{4} - \frac{6}{5r}$. Then for sufficiently large μ we have*

$$\|\Delta\|_{L^\infty} \le C\mu^{\frac{24-5r}{20r}} = C\mu^{\frac{1}{2}-\delta_r}.$$

Similarly, if $|\cdot|V \in L^r(\mathbf{R}^3)$ then

$$|\Delta(p) - \Delta(q)| \le C\mu^{\frac{24-5r}{20r}}||p| - |q|| = C\mu^{\frac{1}{2}-\delta_r}||p| - |q||$$

for all p, q. In particular, if $r > 8/5$ then $\delta_r > 0$ and thus $1/2 - \delta_r < 1/2$.

We will use the first bound as $\|\Delta\|_{L^\infty} \le C\mu^{11/20} = o(\mu)$ for $r = 3/2$, and the second bound as $|\Delta(p) - \Delta(q)| \le C\mu^{7/20}||p| - |q||$ for $r = 2$.

Armed with these *a* priori bounds on Δ, we can now prove the asymptotic formulas in Lemma 11.2.2 and start with the first one.

Proposition 11.2.5. *Suppose $|\cdot|V \in L^r(\mathbf{R}^3)$ for $r > 8/5$. Then $\Xi = \Delta(\sqrt{\mu})(1 + o(1))$.*

Proof. Clearly $\Xi = \inf\sqrt{|p^2 - \mu|^2 + |\Delta(p)|^2} \le \Delta(\sqrt{\mu})$. Take now p with $|p^2 - \mu| \le \Xi \le \Delta(\sqrt{\mu})$. Then

$$|\Delta(p) - \Delta(\sqrt{\mu})| \le C\mu^{1/2-\delta_r}||p| - \sqrt{\mu}| \le C\mu^{1/2-\delta_r}\frac{\Delta(\sqrt{\mu})}{|p| + \sqrt{\mu}} \le C\mu^{-\delta_r}\Delta(\sqrt{\mu})$$

where $\delta_r > 0$ by assumption. Hence, $\Delta(p) = \Delta(\sqrt{\mu})(1 + o(1))$ for any such p and we conclude the desired. □

The proofs of the second and third equality (Proposition 11.2.9 and Proposition 11.2.10, respectively) heavily use Lemma 11.2.6 and Lemma 11.2.7, which we import from Chapter 10. Lemma 11.2.6 provides an upper bound for integrals of the potential against spherical Bessel functions j_ℓ, uniformly in $\ell \in \mathbb{N}_0$. These naturally arise by the spherical symmetry of V (cf. Equation (11.1.6)).

Lemma 11.2.6 (see Lemma 10.3.3)**.** *Let $V \in L^1(\mathbf{R}^3) \cap L^{3/2}(\mathbf{R}^3)$ and assume that $s^* > 1$, with s^* as in Definition 11.1.1. Set*

$$\beta^* = \begin{cases} \frac{s^*}{2} & \text{for} \quad s^* \in (1, 5/3] \\ \min\left(\frac{4s^*-4}{9s^*-7} + \frac{1}{2}, \frac{19}{22}\right) & \text{for} \quad s^* > 5/3\,. \end{cases}$$

Note that β^* depends continuously on s^* and is (strictly) monotonically increasing (between 1 and 2), and $\beta^* \le \min(s^*, 2)/2$ for any $s^* > 1$. Then for any $\delta > 0$ there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$ we have

$$\limsup_{\mu\to\infty} \mu^{\beta^*-\delta} \sup_{\ell\in\mathbb{N}_0} \int_{\mathbf{R}^3} \mathrm{d}x|V(x)||j_\ell(\sqrt{\mu}|x|)|^{2-\varepsilon} = 0\,.$$

Lemma 11.2.7 gives a lower bound on the quantity e_μ that measures the strength of the interaction potential on the Fermi surface (see Equation (11.1.7)).

Lemma 11.2.7 (see Lemma 10.3.4)**.** *Let V be an admissible potential (cf. Definition 11.1.1). Then for any $\delta > 0$ there exists $c_\delta > 0$ such that*

$$\liminf_{\mu\to\infty} |\mu^{\min(s^*+\delta,2)/2}\, e_\mu| \ge c_\delta\,.$$

Proof. This is immediate from Lemma 10.3.4 by invoking Proposition 11.1.4. □

An upper bound is trivially obtained as $|e_\mu| \leq C_\delta \mu^{-\min(s^*-\delta,2)/2}$ for any $\delta > 0$ by definition of s^* in Equation (11.1.11) (see also Equation (11.2.9)). Note that both, upper and lower bound, remain true if we replace the exponent with $\min(s^*,2)/2 \pm \delta$, i.e. $c_\delta \mu^{-\min(s^*,2)/2-\delta} \leq |e_\mu| \leq C_\delta \mu^{-\min(s^*,2)/2+\delta}$. This is the formulation we will use.

Beside these two Lemmas, we will use the following observation: It can easily be checked (see Lemma 3 in [312]) that the operator $E_{\Delta,\mu}(p) + V(x)$ has 0 as its lowest eigenvalue, and that α is the (unique) eigenvector with this eigenvalue. By employing the Birman–Schwinger principle (see [269, 309, 316]), this is equivalent to the fact that the Birman–Schwinger operator

$$B_{\Delta,\mu} = V^{1/2} \frac{1}{E_{\Delta,\mu}} |V|^{1/2}$$

has -1 as its lowest eigenvalue with $V^{1/2}\alpha$ being the corresponding (unique) eigenvector. Here we used the notation $V(x)^{1/2} = \operatorname{sgn}(V(x))|V(x)|^{1/2}$. In the following we need a convenient decomposition of $B_{\Delta,\mu}$ in a dominant singular term and other error terms. For this purpose we let $\mathfrak{F}_\mu : L^1(\mathbf{R}^3) \to L^2(\mathbb{S}^2)$ denote the (rescaled) Fourier transform restricted to $\mathbb{S}^2$ with

$$(\mathfrak{F}_\mu \psi)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} e^{-\mathrm{i}\sqrt{\mu} p \cdot x} \psi(x) \mathrm{d}x \,,$$

which is well–defined by the Riemann–Lebesgue Lemma. Now, we decompose the Birman–Schwinger operator as

$$B_{\Delta,\mu} = m_\mu^{(\kappa)}(\Delta)\, V^{1/2} \mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2} + V^{1/2} M_{\Delta,\mu}^{(\kappa)} |V|^{1/2} \,, \tag{11.2.2}$$

where $M_{\Delta,\mu}^{(\kappa)}$ is such that this holds. For the first term, note that $V^{1/2}\mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2}$ is isospectral to $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu{}^\dagger$. In fact, the spectra agree at first except possibly at 0, but 0 is in both spectra as the operators are compact on an infinite dimensional space. This first term in the decomposition (11.2.2) will be the dominant term, which is how the third equality in Lemma 11.2.2 will arise.

Analogously to the proof of Lemma 10.3.5 and the proof of Theorem 1 in [312], we further decompose

$$V^{1/2} M_{\Delta,\mu}^{(\kappa)} |V|^{1/2} = V^{1/2} \frac{1}{p^2 + \kappa^2 \mu} |V|^{1/2} + A_{\Delta,\mu}^{(\kappa)} =: L_\mu^{(\kappa)} + A_{\Delta,\mu}^{(\kappa)} \,, \tag{11.2.3}$$

where now $A_{\Delta,\mu}^{(\kappa)}$ is such that this holds. During the proof of Lemma 10.3.5 (see (10.3.16)) it was shown that

$$\left\| L_\mu^{(\kappa)} \right\|_{\mathrm{op}} \leq C\, \mu^{1/2} \int_0^\infty \mathrm{d}p \frac{p^2}{p^2 + \kappa^2} \sup_{\ell \in \mathbf{N}_0} \int_{\mathbf{R}^3} \mathrm{d}x |V(x)|\, |j_\ell(\sqrt{\mu} p |x|)|^2 \,,$$

which may be bounded by $\mu^{-\beta^* + 1/2 + \delta}$ for any $\delta > 0$ by means of Lemma 11.2.6. We continue with a bound on the operator norm of $A_{\Delta,\mu}^{(\kappa)}$ by estimating the matrix elements $\langle f | A_{\Delta,\mu}^{(\kappa)} | g \rangle$ for functions $f, g \in L^2(\mathbf{R}^3)$. This computation is analogous to the computation in the proof of Theorem 10.2.2. We give it here for completeness.

Note that, since V is radial, it is enough to restrict to functions of definite angular momentum. That is, with a slight abuse of notation, functions of the form $f(x) = Y_\ell^m(\hat{x}) f(|x|)$, where Y_ℓ^m denotes the spherical harmonics and we write $\hat{x} = x/|x|$. The operator $A_{\Delta,\mu}^{(\kappa)}$ is indeed block–diagonal in the angular momentum as will follow from the computations below. Since functions of definite angular momentum span $L^2(\mathbf{R}^3)$ [319, Sections 17.6-17.7] it is thus enough to bound $\langle f | A_{\Delta,\mu}^{(\kappa)} | g \rangle$ for f, g of the form $f(x) = Y_\ell^m(\hat{x}) f(|x|)$, $g(x) = Y_{\ell'}^{m'}(\hat{x}) g(|x|)$.

Now, $A_{\Delta,\mu}^{(\kappa)}$ has integral kernel

$$A_{\Delta,\mu}^{(\kappa)}(x,y) = C V^{1/2}(x) |V(y)|^{1/2} \int_{\mathbf{R}^3} \left(\frac{1}{E_{\Delta,\mu}(p)} - \frac{1}{p^2 + \kappa^2 \mu} \right) \left(e^{ip\cdot(x-y)} - e^{i\sqrt{\mu}\hat{p}\cdot(x-y)} \right) \mathrm{d}p \,.$$

Thus, by the radiality of V we get

$$\begin{aligned}\langle f|A^{(\kappa)}_{\Delta,\mu}|g\rangle = C\int_0^\infty \mathrm{d}|x|\,|x|^2 V^{1/2}(|x|)\overline{f(|x|)}\int_0^\infty \mathrm{d}|y|\,|y|^2|V(|y|)|^{1/2}g(|y|)\\ \times\int_0^\infty \mathrm{d}|p|\,|p|^2\left(\frac{1}{E_{\Delta,\mu}(|p|)}-\frac{1}{|p|^2+\kappa^2\mu}\right)\int_{\mathbb{S}^2}\mathrm{d}\omega(\hat{p})\\ \times\int_{\mathbb{S}^2}\mathrm{d}\omega(\hat{x})\int_{\mathbb{S}^2}\mathrm{d}\omega(\hat{y})\,\overline{Y^m_\ell(\hat{x})}Y^{m'}_{\ell'}(\hat{y})\left(e^{-ip\cdot(x-y)}-e^{-i\sqrt{\mu}\hat{p}\cdot(x-y)}\right).\end{aligned} \tag{11.2.4}$$

Now, using the plane–wave expansion $e^{ip\cdot x}=4\pi\sum_{\ell=0}^\infty\sum_{m=-\ell}^\ell i^\ell j_\ell(|p||x|)Y^m_\ell(\hat{p})\overline{Y^m_\ell(\hat{x})}$, the spherical integrations in x and y may be evaluated as

$$16\pi^2(-i)^{\ell+\ell'}\left(j_\ell(|p||x|)j_{\ell'}(|p||y|)-j_\ell(\sqrt{\mu}|x|)j_{\ell'}(\sqrt{\mu}|y|)\right)\overline{Y^m_\ell(\hat{p})}Y^{m'}_{\ell'}(\hat{p})$$

using the orthogonality of the spherical harmonics. The spherical p–integral of this gives a factor $\delta_{\ell\ell'}\delta_{mm'}$ again by orthogonality of the spherical harmonics. (This shows that $A^{(\kappa)}_{\Delta,\mu}$ is block–diagonal in the angular momentum as claimed.) We may thus restrict to the case of $\ell=\ell'$ and $m=m'$. Hereinafter, we will write x, y, and p instead of $|x|$, $|y|$, and $|p|$.

Recall the following bounds on spherical Bessel functions

$$\sup_{\ell\in\mathbf{N}_0}\sup_{x\geq 0}|j_\ell(x)|\leq 1\,,\qquad \sup_{\ell\in\mathbf{N}_0}\sup_{x\geq 0}|j'_\ell(x)|\leq 1\,,\qquad \sup_{\ell\in\mathbf{N}_0}\sup_{x\geq 0}x^{5/6}|j_\ell(x)|\leq C\,,$$

where the first one is elementary, the second one follows from [3, Eq. 10.1.20], and the third one may be found in [391, Eq. 1] (see also Proposition 10.3.7). Adding $\pm j_\ell(px)j_\ell(\sqrt{\mu}y)$ and using these bounds we may estimate for any $0<\epsilon<5/11$

$$\begin{aligned}&|j_\ell(px)j_\ell(py)-j_\ell(\sqrt{\mu}x)j_\ell(\sqrt{\mu}y)|\\ &\quad\leq C|p-\sqrt{\mu}|^\epsilon\left(p^{-\epsilon}+(\sqrt{\mu})^{-\epsilon}\right)\left(|j_\ell(px)|^{1-11\epsilon/5}+|j_\ell(\sqrt{\mu}x)|^{1-11\epsilon/5}\right)\\ &\quad\times\left(|j_\ell(py)|^{1-11\epsilon/5}+|j_\ell(\sqrt{\mu}y)|^{1-11\epsilon/5}\right).\end{aligned} \tag{11.2.5}$$

The radial p–integral in Equation (11.2.4) is then (a constant times)

$$\int_0^\infty \mathrm{d}p\left(\frac{1}{E_{\Delta,\mu}(p)}-\frac{1}{p^2+\kappa^2\mu}\right)\left(j_\ell(px)j_\ell(py)-j_\ell(\sqrt{\mu}x)j_\ell(\sqrt{\mu}y)\right) \tag{11.2.6}$$

Using Equation (11.2.5) and changing integration variable $p\to\sqrt{\mu}p$ we get

$$\begin{aligned}|(11.2.6)|\leq C\mu^{1/2}\int_0^\infty \mathrm{d}p\,p^2\left|\frac{1}{\sqrt{(p^2-1)^2+|\Delta(\sqrt{\mu}p)/\mu|^2}}-\frac{1}{p^2+\kappa^2}\right||p-1|^\epsilon\left(\frac{1}{p^\epsilon}+1\right)\\ \times\left(|j_\ell(\sqrt{\mu}px)|^{1-11\epsilon/5}+|j_\ell(\sqrt{\mu}x)|^{1-11\epsilon/5}\right)\\ \times\left(|j_\ell(\sqrt{\mu}py)|^{1-11\epsilon/5}+|j_\ell(\sqrt{\mu}y)|^{1-11\epsilon/5}\right).\end{aligned}$$

Plugging this into Equation (11.2.4) and using Hölder for the x– and y–integrations we thus get

$$\begin{aligned}&\left|\langle f|A^{(\kappa)}_{\Delta,\mu}|g\rangle\right|\\ &\quad\leq C\mu^{1/2}\int_0^\infty \mathrm{d}p\,p^2\left|\frac{1}{\sqrt{(p^2-1)^2+|\Delta(\sqrt{\mu}p)/\mu|^2}}-\frac{1}{p^2+\kappa^2}\right||p-1|^\epsilon\left(\frac{1}{p^\epsilon}+1\right)\\ &\quad\times\int_{\mathbf{R}^3}\mathrm{d}x\,|V(x)|\left(|j_\ell(\sqrt{\mu}p|x|)|^{2-22\epsilon/5}+|j_\ell(\sqrt{\mu}|x|)|^{2-22\epsilon/5}\right),\end{aligned}$$

where we changed back to x denoting a vector in $\mathbf{R}^3$. By Lemma 11.2.6 we may bound the x–integral by $\mu^{-\beta^*+\delta}(1+p^{-\beta^*+\delta})$ for any $\delta>0$. Also, $\|\Delta\|_{L^\infty}=o(\mu)$ by Lemma 11.2.4. Hence the p–integral will be finite uniformly in μ for μ large enough. We conclude that

$$\left\|A^{(\kappa)}_{\Delta,\mu}\right\|_{\mathrm{op}} \leq C\mu^{-\beta^*+1/2+\delta}$$

for any $\delta>0$ and for μ large enough. Combining this with the bound on $\|L^{(\kappa)}_\mu\|_{\mathrm{op}}$ from above, we get

$$\limsup_{\mu\to\infty}\mu^{\beta^*-1/2-\delta}\left\|V^{1/2}M^{(\kappa)}_{\Delta,\mu}|V|^{1/2}\right\|_{\mathrm{op}}=0 \tag{11.2.7}$$

for any $\delta>0$. Also, since $V^{1/2}\mathfrak{F}^\dagger_\mu\mathfrak{F}_\mu|V|^{1/2}$ is isospectral to $\mathcal{V}_\mu$, so its eigenvalues are given by Equation (11.1.6), one can easily see, using Lemma 11.2.6 again, that

$$\limsup_{\mu\to\infty}\mu^{\beta^*-\delta}\left\|V^{1/2}\mathfrak{F}^\dagger_\mu\mathfrak{F}_\mu|V|^{1/2}\right\|_{\mathrm{op}}=0\,, \tag{11.2.8}$$

for any $\delta>0$. Finally, by definition of s^* (see Equation (11.1.11)), we get for any $\delta>0$ that

$$\limsup_{\mu\to\infty}\mu^{\min(s^*,2)/2-\delta}\int_{\mathbf{R}^3}|V(x)|\left(\frac{\sin(\sqrt{\mu}|x|)}{\sqrt{\mu}|x|}\right)^2\mathrm{d}x=0\,. \tag{11.2.9}$$

As the last ingredient we need the following Lemma, which provides a bound controlling $\Delta(p)$ in terms of $\Delta(\sqrt{\mu})$. Its proof is given in Section 11.2.2.

Lemma 11.2.8. *Suppose $s^*>1$ and let $u(p)=(4\pi)^{-1/2}$ be the constant function on the sphere $\mathbb{S}^2$ and let*

$$\hat\varphi(p)=\sqrt{4\pi}\mathfrak{F}V\mathfrak{F}^\dagger_\mu u(p)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{S}^2}\hat V(p-\sqrt{\mu}q)\mathrm{d}\omega(q)\,,$$

where $\mathfrak{F}$ denotes the usual Fourier transform. Then

$$\Delta(p)=f(\mu)\left[\hat\varphi(p)+\eta_\mu(p)\right],$$

for some function $f(\mu)$. The function η_μ satisfies

$$\limsup_{\mu\to\infty}\mu^{\beta^*+\min(s^*,2)/4-1/2-\delta}\|\eta_\mu\|_{L^\infty}=0 \quad \textit{and} \quad \limsup_{\mu\to\infty}\mu^{\beta^*+\min(s^*,2)/2-1/2-\delta}|\eta_\mu(\sqrt{\mu})|=0$$

for any $\delta>0$.

Note that $\hat\varphi(\sqrt{\mu})=\sqrt{4\pi}\mathfrak{F}_\mu V\mathfrak{F}^\dagger_\mu u(1)=e_\mu$. Now, combining this with Lemmas 11.2.6 and 11.2.7, we see that $\Delta(\sqrt{\mu})=f(\mu)e_\mu(1+o(1))$, from which we conclude that

$$\Delta(p)=\frac{\hat\varphi(p)+\eta_\mu(p)}{e_\mu+\eta_\mu(\sqrt{\mu})}\Delta(\sqrt{\mu})=\left[1+\frac{\hat\varphi(p)-\hat\varphi(\sqrt{\mu})}{e_\mu}+\frac{\eta_\mu(p)}{e_\mu}\right](1+o(1))\Delta(\sqrt{\mu})\,.$$

Now, it is an easy computation to see $|\hat\varphi(p)-\hat\varphi(q)|\leq C\mu^{-1/2}|p-q|$ for all p,q. Thus

$$|\Delta(p)|\leq C\left(1+\mu^{\min(s^*,2)/2-1/2+\delta}|p-\sqrt{\mu}|+\mu^{\min(s^*,2)/4-\beta^*+1/2+\delta}\right)\Delta(\sqrt{\mu}) \tag{11.2.10}$$

for any $\delta>0$, again by means of Lemma 11.2.6 and Lemma 11.2.7, assuming that V is admissible. So, we get the desired control on $\Delta(p)$ in terms of $\Delta(\sqrt{\mu})$.

The bound on $\eta_\mu(\sqrt{\mu})$ is effectively a bound on $\left\langle u\middle|\mathfrak{F}^\dagger_\mu V M^{(\kappa)}_{\Delta,\mu}V\mathfrak{F}_\mu\middle|u\right\rangle$. (This will be clear from the proof.) For sufficiently large μ we have

$$\left|\left\langle u\middle|\mathfrak{F}^\dagger_\mu V M^{(\kappa)}_{\Delta,\mu}V\mathfrak{F}_\mu\middle|u\right\rangle\right|\leq C_\delta\mu^{-\beta^*-\min(s^*,2)/2+1/2+\delta} \tag{11.2.11}$$

for any $\delta>0$. This will be of importance in the perturbation argument in Proposition 11.2.10.

We are now able to prove the second and third equality in Lemma 11.2.2.

Proposition 11.2.9. *Let V be an admissible potential. Then we have*

$$m_\mu^{(\kappa)}(\Delta) = \sqrt{\mu}\left(\log\frac{\mu}{\Delta(\sqrt{\mu})} - 2 + \kappa\,\frac{\pi}{2} + \log(8) + o(1)\right)$$

in the limit $\mu \to \infty$.

Proof. Computing the angular integral, and substituting $s = \pm\frac{p^2-\mu}{\mu}$ we get

$$\begin{aligned} m_\mu^{(\kappa)}(\Delta) = \frac{\sqrt{\mu}}{2}\Bigg[&\int_0^1 \left(\frac{\sqrt{1-s}-1}{\sqrt{s^2+x_-(s)^2}} + \frac{\sqrt{1+s}-1}{\sqrt{s^2+x_+(s)^2}} - \frac{\sqrt{1-s}}{1-s+\kappa^2} - \frac{\sqrt{1+s}}{1+s+\kappa^2}\right)\mathrm{d}s \\ &+ \int_0^1 \left(\frac{1}{\sqrt{s^2+x_+(s)^2}} + \frac{1}{\sqrt{s^2+x_-(s)^2}}\right)\mathrm{d}s \\ &+ \int_1^\infty \left(\frac{\sqrt{1+s}}{\sqrt{s^2+x_+(s)^2}} - \frac{\sqrt{1+s}}{1+s+\kappa^2}\right)\mathrm{d}s\Bigg], \end{aligned}$$

where $x_\pm(s) = \frac{\Delta(\sqrt{\mu}\sqrt{1\pm s})}{\mu}$. Now, using dominated convergence and $\|\Delta\|_{L^\infty} = o(\mu)$, it is easy to see that the first and last integrals converge to

$$\int_0^1 \left(\frac{\sqrt{1-s}-1}{s} + \frac{\sqrt{1+s}-1}{s} - \frac{\sqrt{1-s}}{1-s+\kappa^2} - \frac{\sqrt{1+s}}{1+s+\kappa^2}\right)\mathrm{d}s$$

and

$$\int_1^\infty \left(\frac{\sqrt{1+s}}{s} - \frac{\sqrt{1+s}}{1+s+\kappa^2}\right)\mathrm{d}s\,,$$

respectively, in the limit $\mu \to \infty$. For the middle integral we claim that

$$\int_0^1 \left(\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right)\mathrm{d}s \to 0 \quad \text{as} \quad \mu\to\infty\,. \tag{11.2.12}$$

As in [312, 403] this is where we need both the Lipschitz–like bound on Δ (Lemma 11.2.4) and the bound controlling $\Delta(p)$ in terms of $\Delta(\sqrt{\mu})$ (Equation (11.2.10)). In terms of $x_\pm$, Lemma 11.2.4 reads

$$|x_\pm(s) - x_\pm(0)| \le C\mu^{-\delta_r} s\,. \tag{11.2.13}$$

In terms of $x_\pm$, Equation (11.2.10) reads

$$x_\pm(s) \le C(1 + \mu^{\min(s^*,2)/2+\delta} s + \mu^{\min(s^*,2)/4-\beta^*+1/2+\delta})x_\pm(0)\,. \tag{11.2.14}$$

Now, the integrand in Equation (11.2.12) is bounded by

$$\frac{|x_\pm(s)^2 - x_\pm(0)^2|}{\sqrt{s^2+x_\pm(s)^2}\sqrt{s^2+x_\pm(0)^2}\left(\sqrt{s^2+x_\pm(s)^2}+\sqrt{s^2+x_\pm(0)^2}\right)}\,.$$

We introduce a cutoff $\rho \in (0,1)$ and compute the integrals $\int_\rho^1$ and $\int_0^\rho$. For the first integral we have

$$\begin{aligned} &\int_\rho^1 \frac{|x_\pm(s)^2 - x_\pm(0)^2|}{\sqrt{s^2+x_\pm(s)^2}\sqrt{s^2+x_\pm(0)^2}\left(\sqrt{s^2+x_\pm(s)^2}+\sqrt{s^2+x_\pm(0)^2}\right)}\mathrm{d}s \\ &\le C\mu^{-\delta_r}\int_\rho^1 \frac{1}{s}\frac{x_\pm(s)+x_\pm(0)}{\sqrt{s^2+x_\pm(s)^2}+\sqrt{s^2+x_\pm(0)^2}}\mathrm{d}s \\ &\le C\mu^{-\delta_r}|\log\rho|\,. \end{aligned}$$

which vanishes for any $\rho \gg \exp\left(-\mu^{\delta_r}\right)$, in particular for $\rho = \mu^{-N}$ for suitable $N > 0$, which we choose here. For the second integral we have

$$\begin{aligned}
&\int_0^\rho \frac{|x_\pm(s)^2 - x_\pm(0)^2|}{\sqrt{s^2 + x_\pm(s)^2}\sqrt{s^2 + x_\pm(0)^2}\left(\sqrt{s^2 + x_\pm(s)^2} + \sqrt{s^2 + x_\pm(0)^2}\right)} \mathrm{d}s \\
&\quad \leq C \int_0^\rho \mu^{-\delta_r} \left(1 + \mu^{\min(s^*,2)/4 - \beta^* + 1/2 + \delta} + \mu^{\min(s^*,2)/2 + \delta} s\right) \\
&\qquad \times \frac{x_\pm(0)}{\sqrt{x_\pm(0)^2 + s^2}\left(s + \sqrt{x_\pm(0)^2 + s^2}\right)} \mathrm{d}s \\
&\quad \leq C\mu^{\min(s^*,2)/4 - \beta^* - \delta_r + 1/2 + \delta} \int_0^\rho \frac{x_\pm(0)}{\sqrt{x_\pm(0)^2 + s^2}\left(s + \sqrt{x_\pm(0)^2 + s^2}\right)} \mathrm{d}s \\
&\quad \leq C\mu^{\min(s^*,2)/4 - \beta^* - \delta_r + 1/2 + \delta} .
\end{aligned}$$

Note that for $r = 2$, we have $\delta_{r=2} = 3/20$ and thus $\beta^* - \min(s^*, 2)/4 - 1/2 + 3/20 > 0$ for any $s^* > 7/5$ (see Remark 11.1.2). Also, optimizing this expression in the allowed r's gives the assumption $r > f(s^*)$ given in Remark 11.1.2. Therefore, also this second integral vanishes as desired by choosing $0 < \delta < \beta^* - \min(s^*, 2)/4 - 7/20$. We conclude that

$$\begin{aligned}
m_\mu^{(\kappa)}(\Delta) = \frac{\sqrt{\mu}}{2}\Bigg[&\int_0^1 \left(\frac{\sqrt{1-s}-1}{s} + \frac{\sqrt{1+s}-1}{s} - \frac{\sqrt{1-s}}{1-s+\kappa^2} - \frac{\sqrt{1+s}}{1+s+\kappa^2} \right) \mathrm{d}s \\
&+ \int_0^1 \frac{2}{\sqrt{s^2 + \left(\frac{\Delta(\sqrt{\mu})}{\mu}\right)^2}} \mathrm{d}s + \int_1^\infty \left(\frac{\sqrt{1+s}}{s} - \frac{\sqrt{1+s}}{1+s+\kappa^2} \right) \mathrm{d}s + o(1) \Bigg].
\end{aligned}$$

This may be computed (perhaps most easily by adding and subtracting the corresponding integral with $\kappa = 0$) as

$$m_\mu^{(\kappa)} = \sqrt{\mu}\left(\log \frac{\mu}{\Delta(\sqrt{\mu})} - 2 + \log(8) + \kappa\frac{\pi}{2} + o(1) \right). \qquad \square$$

We conclude by showing the third equality of Lemma 11.2.2.

Proposition 11.2.10. *Let V be an admissible potential. Then*

$$\frac{m_\mu^{(\kappa)}(\Delta)}{\sqrt{\mu}} = -\frac{\pi}{2\sqrt{\mu} b_\mu^{(\kappa)}} + o(1) .$$

Proof. Recall that, by the Birman–Schwinger principle the lowest eigenvalue of $B_{\Delta,\mu}$ is -1. Using the decomposition in Equation (11.2.2) and the bound in Equation (11.2.7) we get that

$$-1 = \lim_{\mu\to\infty} m_\mu^{(\kappa)}(\Delta) \inf \mathrm{spec}\left(V^{1/2} \mathfrak{F}_\mu{}^\dagger \mathfrak{F}_\mu |V|^{1/2} \right) = \lim_{\mu\to\infty} m_\mu^{(\kappa)}(\Delta) e_\mu .$$

Now, since $s^* > 7/5$ we have that $|\sqrt{\mu} e_\mu| \leq C\mu^{-2/5}$ by Lemma 11.2.6 (recall Equation (11.1.6) and Equation (11.2.9)). Thus, by Proposition 11.2.9 we conclude that $\Delta(\sqrt{\mu})$ is exponentially small (in some positive power of μ) as $\mu \to \infty$.

To obtain the next order in the expansion of $m_\mu(\Delta)$, we note that $1 + V^{1/2} M_{\Delta,\mu}^{(\kappa)} |V|^{1/2}$ is invertible for μ large enough by means of Equation (11.2.7). We can thus factorize the Birman–Schwinger operator (11.2.2) as

$$1 + B_{\Delta,\mu} = \left(1 + V^{1/2} M_{\Delta,\mu}^{(\kappa)} |V|^{1/2}\right) \left(1 + \frac{m_\mu^{(\kappa)}(\Delta)}{1 + V^{1/2} M_{\Delta,\mu}^{(\kappa)} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2} \right) .$$

Because $B_{\Delta,\mu}$ has -1 as its lowest eigenvalue by the Birman–Schwinger principle, we conclude that, for μ large enough, the self–adjoint operator

$$T_{\Delta,\mu} \coloneqq m_\mu^{(\kappa)}(\Delta)\mathfrak{F}_\mu|V|^{1/2}\frac{1}{1+V^{1/2}M_{\Delta,\mu}^{(\kappa)}|V|^{1/2}}V^{1/2}\mathfrak{F}_\mu^\dagger$$

acting on $L^2(\mathbb{S}^2)$ has -1 as its lowest eigenvalue since it is isospectral to the right–most operator above. (This follows from the fact that for operators A, B the operators AB and BA have the same spectrum apart from possibly at 0. See also the argument around (10.3.20) as well as around Equation (30) and Equation (47) in [312].)

To highest order $T_{\Delta,\mu}$ is proportional to $\mathcal{V}_\mu$. Since the constant function $u(p) = (4\pi)^{-1/2}$ on $\mathbb{S}^2$ is the unique eigenvector of $\mathcal{V}_\mu$ with lowest eigenvalue, this is true also for $T_{\Delta,\mu}$ whenever μ is large enough.

To find the lowest eigenvalue (which is -1) we expand the geometric series to first order and employ first order perturbation theory. This is completely analogous to the arguments in [312] and (10.3.21). We obtain

$$\frac{1}{\sqrt{\mu}}m_\mu^{(\kappa)}(\Delta) = \frac{-1}{\mu^{1/2}e_\mu - \mu^{1/2}\big\langle u\big|\mathfrak{F}_\mu V M_{\Delta,\mu}^{(\kappa)}V\mathfrak{F}_\mu^\dagger\big|u\big\rangle + O(\mu^{-3\beta^*+3/2+\delta})} \tag{11.2.15}$$

for any $\delta > 0$ (recall Equations (11.2.7), (11.2.8) and (11.2.11)). The error term in Equation (11.2.15) is twofold. The first part comes from the expansion of the geometric series. The second part comes from first order perturbation theory using the bounds

$$|\sqrt{\mu}e_\mu| \geq c_\delta\mu^{-\min(s^*,2)/2+1/2-\delta} \quad \text{and} \quad \big|\mu^{1/2}\big\langle u\big|\mathfrak{F}_\mu V M_{\Delta,\mu}^{(\kappa)}V\mathfrak{F}_\mu^\dagger\big|u\big\rangle\big| \leq C_\delta\mu^{-\beta^*-\min(s^*,2)/2+1+\delta}$$

for any $\delta > 0$ from Lemma 11.2.7 and Equation (11.2.11). The error from the series expansion is of order $O(\mu^{-3\beta^*+3/2+\delta})$ and the error from the perturbation argument is of order $O(\mu^{-2\beta^*-\min(s^*,2)/2+3/2+\delta})$ and is hence dominated by the expansion of the geometric series, since $\beta^* \leq \min(s^*,2)/2$.

Now, we need to show that $\mathfrak{F}_\mu V M_{\Delta,\mu}^{(\kappa)}V\mathfrak{F}_\mu^\dagger$ is close to $\mathcal{W}_\mu^{(\kappa)}$, when evaluated in $\langle u|\cdots|u\rangle$. Therefore, considering their difference, we split the involved radial p–integral according to $|p| \leq \mu^N$ and $|p| > \mu^N$ for some large $N > 0$. The second part is clearly bounded by, e.g., $C\mu^{-N/2}$. For the first part, we have $\Delta(p) \leq C\mu^N\Delta(\sqrt{\mu})$ by Equation (11.2.10). Using this in combination with the fact that $\Delta(\sqrt{\mu})$ is exponentially small, we find, by dominated convergence and Lipschitz continuity of the involved angular integrals (cf. Equation (35) in [312] and (10.3.23)), that this part is bounded by $C_D\mu^{-D}$ for any $D > 0$. Since $N > 0$ was arbitrary, we conclude that

$$\big|\big\langle u\big|\mathfrak{F}_\mu V M_{\Delta,\mu}^{(\kappa)}V\mathfrak{F}_\mu^\dagger - \mathcal{W}_\mu^{(\kappa)}\big|u\big\rangle\big| \leq C_D\mu^{-D} \tag{11.2.16}$$

for any $D > 0$. Thus, by combining Equation (11.2.11) and Equation (11.2.16) (recall Equation (11.1.9) and Equation (11.1.10)) we get

$$\big|\big\langle u\big|\mathcal{W}_\mu^{(\kappa)}\big|u\big\rangle\big| \leq C_\delta\mu^{-\beta^*-\min(s^*,2)/2+1/2+\delta} \tag{11.2.17}$$

for any $\delta > 0$. (In particular $b_\mu^{(\kappa)} < 0$ for large μ. This was also shown in Chapter 10.)

In particular, combining Equations (11.2.15), (11.2.16) and (11.2.17), we get again by a perturbation theory argument that

$$\frac{1}{\sqrt{\mu}}m_\mu^{(\kappa)}(\Delta) = -\frac{\pi}{2\sqrt{\mu}b_\mu^{(\kappa)}} + O(\mu^{-3\beta^*+\min(s^*,2)+1/2+\delta}),$$

for any $\delta > 0$. Since $3\beta^* - \min(s^*,2) - 1/2 > 0$ we conclude the desired. □

11.2.2 Proofs of Auxiliary Lemmas

In this Subsection, we prove the auxiliary Lemmas 11.2.3, 11.2.4, and 11.2.8.

Proof of Lemma 11.2.3. First we show

$$\|\alpha\|_{H^1}^2 \leq C\,\|\alpha\|_{L^2}^2 + C\mu^{3/2}. \tag{11.2.18}$$

Since $V \in L^{3/2}(\mathbf{R}^3)$ we have by Sobolev's inequality [418, Thm. 8.3] $\inf \operatorname{spec}\left(\frac{p^2}{2} + V\right) > -\infty$. Thus, using $\sqrt{1-4x^2} \leq 1-2x^2$ and $\hat{\alpha} \leq 1/2$ we get

$$\begin{aligned}
\mathcal{F}(\alpha) &= \frac{1}{2}\int_{\mathbf{R}^3} |p^2-\mu|\left(1-\sqrt{1-4\hat{\alpha}(p)^2}\right)\mathrm{d}p + \int_{\mathbf{R}^3} V(x)|\alpha(x)|^2\mathrm{d}x \\
&\geq \int_{\mathbf{R}^3} (p^2-\mu)\hat{\alpha}(p)^2\mathrm{d}p + \int_{\mathbf{R}^3} V(x)|\alpha(x)|^2\mathrm{d}x \\
&= \left\langle \alpha \middle| \frac{p^2}{2} + V \middle| \alpha \right\rangle + \int_{\mathbf{R}^3} \left(\frac{p^2}{2}-\mu\right)\hat{\alpha}(p)^2\mathrm{d}p \\
&\geq \frac{1}{4}\,\|\alpha\|_{H^1}^2 - C\,\|\alpha\|_{L^2}^2 + \int_{\mathbf{R}^3} \left(\frac{p^2}{4}-\mu-\frac{1}{4}\right)\hat{\alpha}(p)^2\mathrm{d}p \\
&\geq \frac{1}{4}\,\|\alpha\|_{H^1}^2 - C\,\|\alpha\|_{L^2}^2 - \frac{1}{4}\int_{\mathbf{R}^3} \left[\frac{p^2}{4}-\mu-\frac{1}{4}\right]_- \mathrm{d}p \\
&\geq \frac{1}{4}\,\|\alpha\|_{H^1}^2 - C\,\|\alpha\|_{L^2}^2 - C\mu^{3/2},
\end{aligned}$$

which gives the desired. Now we show that

$$\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2} \leq C\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2} + C\mu^{2\delta-1}\,\|\alpha\|_{H^1}^2,$$

for $t = \mu^\delta$ and $0 < \delta < 1/2$.

To see this, we split the integrals in the functional $\mathcal{F}$ according to small or large momentum p and compute

$$\begin{aligned}
\mathcal{F}(\alpha) &= \frac{1}{2}\int_{\mathbf{R}^3} |p^2-\mu|\left(1-\sqrt{1-4\hat{\alpha}(p)^2}\right)\mathrm{d}p + \int_{\mathbf{R}^3} V(x)|\alpha(x)|^2\mathrm{d}x \\
&\geq \int_{|p|<t} |p^2-\mu|\hat{\alpha}(p)^2\mathrm{d}p + \int_{|p|>t} |p^2-\mu|\hat{\alpha}(p)^2\mathrm{d}p \\
&\qquad + \frac{1}{(2\pi)^{3/2}}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\mathrm{d}p\mathrm{d}q \\
&\geq \mu\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}^2 - \left\|p^2\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}^2 + \left\langle \hat{\alpha}\mathbf{1}_{\{|p|>t\}} \middle| p^2+V \middle| \hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\rangle - \mu\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2}^2 \\
&\quad + \frac{1}{(2\pi)^{3/2}}\left[\iint_{|p|,|q|<t} \hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\mathrm{d}p\mathrm{d}q + 2\iint_{|p|<t,|q|>t} \hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\mathrm{d}p\mathrm{d}q\right].
\end{aligned}$$

Note that, again by Sobolev's inequality [418, Thm. 8.3], we have

$$\left\langle \hat{\alpha}\mathbf{1}_{\{|p|>t\}} \middle| p^2+V \middle| \hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\rangle \geq -C\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2}^2.$$

Moreover, by application of Young's inequality [418, Thm. 4.2] we obtain

$$\begin{aligned}
\int_{|p|<t}\int_{|q|<t} \hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\mathrm{d}p\mathrm{d}q &\geq -\left\|\hat{V}\right\|_{L^3}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^{6/5}}^2 \\
&\geq -C\left(t^3\right)^{2\cdot(5/6-1/2)}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}^2 \\
&= -C\mu^{2\delta}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}\int_{|p|<t}\int_{|q|>t}\hat{\alpha}(p)\hat{V}(p-q)\hat{\alpha}(q)\mathrm{d}p\mathrm{d}q &\geq -\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^1}\left\|\hat{V}\right\|_{L^3}\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^{3/2}}\\ &\geq -Ct^{3/2}\left\|\alpha\right\|_{H^1}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}\\ &= -C\mu^{3\delta/2}\left\|\alpha\right\|_{H^1}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}\,,\end{aligned}$$

where we used that $\|\hat{g}\|_{L^{3/2}} \leq C\|g\|_{H^1}$. Thus we arrive at

$$\mathcal{F}(\alpha) \geq c\mu\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}^2 - C_1\mu^{3\delta/2}\left\|\alpha\right\|_{H^1}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2} - C_2\mu\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2}^2,$$

where we absorbed all non-leading terms in these. This is a second degree polynomial in $\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}$ and thus the value of $\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}$ lies between the roots, i.e.

$$\begin{aligned}\left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2} &\leq \frac{C_1\mu^{3\delta/2}\left\|\alpha\right\|_{H^1} + \sqrt{C_1^2\mu^{3\delta}\left\|\alpha\right\|_{H^1}^2 + 4cC_2\mu^2\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2}^2}}{2c\mu}\\ &\leq C\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2} + C\mu^{3\delta/2-1}\left\|\alpha\right\|_{H^1}.\end{aligned}$$

From the estimate

$$\left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2}^2 = \int_{|p|>t}\hat{\alpha}(p)^2\mathrm{d}p \leq \int_{|p|>t}\hat{\alpha}(p)^2\frac{1+p^2}{1+t^2}\mathrm{d}p \leq \frac{1}{1+t^2}\left\|\alpha\right\|_{H^1}^2 \leq C\mu^{-2\delta}\left\|\alpha\right\|_{H^1}^2\,,$$

we conclude that

$$\left\|\alpha\right\|_{L^2}^2 = \left\|\hat{\alpha}\mathbf{1}_{\{|p|<t\}}\right\|_{L^2}^2 + \left\|\hat{\alpha}\mathbf{1}_{\{|p|>t\}}\right\|_{L^2}^2 \leq C\left(\mu^{-2\delta}+\mu^{3\delta-2}\right)\left\|\alpha\right\|_{H^1}^2.$$

Choosing the optimal $\delta = 2/5$ we get $\|\alpha\|_{L^2} \leq C\mu^{-2/5}\|\alpha\|_{H^1}$, which, in combination with Equation (11.2.18), yields

$$\left\|\alpha\right\|_{H^1}^2 \leq C\mu^{-4/5}\left\|\alpha\right\|_{H^1}^2 + \mu^{3/2}.$$

Hence $\|\alpha\|_{H^1} \leq C\mu^{3/4}$ and thus also $\|\alpha\|_{L^2} \leq C\mu^{-2/5}\|\alpha\|_{H^1} \leq C\mu^{7/20}$ for sufficiently large μ. □

We now turn to the proof of Lemma 11.2.4.

Proof of Lemma 11.2.4. Let $t = \frac{5}{2} - \frac{3}{r}$. Then we have

$$\left\|\Delta\right\|_{L^\infty} \leq C\left\|V\alpha\right\|_{L^1} \leq C\left\|V\right\|_{L^r}\left\|\alpha\right\|_{L^{r'}} \leq C\left\|\alpha\right\|_{L^2}^t\left\|\alpha\right\|_{L^6}^{1-t} \leq C\mu^{\frac{15-8t}{20}} = C\mu^{\frac{24-5r}{20r}}$$

by Sobolev's inequality [418, Thm. 8.3]. For the difference note that $\Delta(p) - \Delta(q)$ is (proportional to) the Fourier transform of $V(x)\left(1-e^{i(p-q)\cdot x}\right)\alpha(x)$. Then

$$\left\|V(x)\left(1-e^{i(p-q)x}\right)\right\|_{L^r}^r = \int_{\mathbf{R}^3}|V(x)|^r|1-e^{i(p-q)\cdot x}|^r\mathrm{d}x \leq C\int_{\mathbf{R}^3}|V(x)|^r|p-q|^r|x|^r\mathrm{d}x.$$

Using radiality of Δ, the same argument as before gives the desired. □

Finally, we give the proof of Lemma 11.2.8.

Proof of Lemma 11.2.8. Recall from the factorization of the Birman–Schwinger operator in the proof of Proposition 11.2.10, that the self–adjoint operator

$$m_\mu^{(\kappa)}(\Delta)\mathfrak{F}_\mu|V|^{1/2}\frac{1}{1+V^{1/2}M_{\Delta,\mu}^{(\kappa)}|V|^{1/2}}V^{1/2}\mathfrak{F}_\mu^\dagger$$

acting on $L^2(\mathbb{S}^2)$ has -1 as its lowest eigenvalue and $u(p) = (4\pi)^{-1/2}$ is the unique eigenvector with lowest eigenvalue for μ large enough. Hence, one can easily see that

$$\frac{1}{1 + V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2}} V^{1/2} \mathfrak{F}^\dagger_\mu u$$

is an eigenvector of $B_{\Delta,\mu}$ for the lowest eigenvalue and thus proportional to $V^{1/2}\alpha$. By expanding $\frac{1}{1+x} = 1 - \frac{x}{1+x}$ we conclude that $\Delta = f(\mu)[\hat{\varphi} + \eta_\mu]$, where

$$\eta_\mu = -\sqrt{4\pi}\mathfrak{F}|V|^{1/2} \frac{V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2}}{1 + V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2}} V^{1/2} \mathfrak{F}^\dagger_\mu u \,,$$

which can easily be bounded as

$$\|\eta_\mu\|_{L^\infty} \leq C \|V\|_{L^1}^{1/2} \left\| V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2} \right\|_{\mathrm{op}} \left\| V^{1/2} \mathfrak{F}^\dagger_\mu u \right\|_{L^2} .$$

For $|p| = \sqrt{\mu}$, we first note that $\hat{\varphi}(\sqrt{\mu}) = \sqrt{4\pi}\mathfrak{F}_\mu V \mathfrak{F}^\dagger_\mu u(1) = e_\mu$. Similarly, since η_μ is radial, we have that

$$\eta_\mu(\sqrt{\mu}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \eta_\mu(\sqrt{\mu} q) \mathrm{d}\omega(q) = -\left\langle u \middle| \mathfrak{F}_\mu |V|^{1/2} \frac{V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2}}{1 + V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2}} V^{1/2} \mathfrak{F}^\dagger_\mu \middle| u \right\rangle$$

and we can thus bound

$$|\eta_\mu(\sqrt{\mu})| \leq C \left\| V^{1/2} M^{(\kappa)}_{\Delta,\mu} |V|^{1/2} \right\|_{\mathrm{op}} \left\| |V|^{1/2} \mathfrak{F}^\dagger_\mu u \right\|^2_{L^2} .$$

It remains to check that

$$\begin{aligned} \left\| |V|^{1/2} \mathfrak{F}^\dagger_\mu u \right\|^2_{L^2} &= C \int_{\mathbf{R}^3} |V(x)| \left| \int_{\mathbb{S}^2} e^{i\sqrt{\mu} p \cdot x} \frac{1}{\sqrt{4\pi}} \mathrm{d}\omega(p) \right|^2 \mathrm{d}x \\ &= C \int_{\mathbf{R}^3} |V(x)| \left(\frac{\sin \sqrt{\mu}|x|}{\sqrt{\mu}|x|} \right)^2 \mathrm{d}x \,. \end{aligned}$$

Now the claim follows by application of Equations (11.2.7) and (11.2.9). □

Chapter 12

Universality in low-dimensional BCS theory

This chapter contains the paper [338]:

J. Henheik, A. B. Lauritsen, and B. Roos. Universality in low-dimensional BCS theory. *Rev. Math. Phys.*, page 2360005, 2023

Abstract. It is a remarkable property of BCS theory that the ratio of the energy gap at zero temperature Ξ and the critical temperature T_c is (approximately) given by a universal constant, independent of the microscopic details of the fermionic interaction. This universality has rigorously been proven quite recently in three spatial dimensions and three different limiting regimes: weak coupling, low density, and high density. The goal of this short note is to extend the universal behavior to lower dimensions $d = 1, 2$ and give an exemplary proof in the weak coupling limit.

Content

12.1 Introduction

The Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity [48] is governed by the *BCS gap equation*. For translation invariant systems without external fields the BCS gap equation is

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{E_\Delta(p)} \tanh\left(\frac{E_\Delta(p)}{2T}\right) \mathrm{d}q \tag{12.1.1}$$

with dispersion relation $E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$. Here, $T \geq 0$ denotes the temperature and $\mu > 0$ the chemical potential. We consider dimensions $d \in \{1,2,3\}$. The Fourier transform of the potential $V \in L^1(\mathbf{R}^d) \cap L^{p_V}(\mathbf{R}^d)$ (with a d-dependent $p_V \geq 1$ to be specified below), modeling their effective interaction, is denoted by $\hat{V}(p) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} V(x) e^{-ip\cdot x} \mathrm{d}x$.

According to BCS theory, a system is in a superconducting state, if there exists a non-zero solution Δ to the gap equation (12.1.1). The question of existence of such a non-trivial solution Δ hinges, in particular, on the temperature T. It turns out, there exists a critical temperature $T_c \geq 0$ such that for $T < T_c$ there exists a non-trivial solution, and for $T \geq T_c$ it does not [309, Theorem 12.1.3 and Definition 12.1.4]. This critical temperature is one of the key (physically measurable) quantities of the theory and its asymptotic behavior, in three spatial dimensions, has been studied in three physically rather different limiting regimes: In a weak-coupling limit (i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$) [312, 269], in a low-density limit (i.e. $\mu \to 0$) [311], and in a high-density limit (i.e. $\mu \to \infty$) [333].

As already indicated above, at zero temperature, the function E_Δ may be interpreted as the dispersion relation of a certain 'approximate' Hamiltonian of the quantum many-body system, see [309, Appendix A]. In particular

$$\Xi := \inf_{p \in \mathbf{R}^d} E_\Delta(p) \tag{12.1.2}$$

has the interpretation of an energy gap associated with the approximate BCS Hamiltonian and as such represents a second key quantity of the theory. Analogously to the critical temperature, the asymptotic behavior of this energy gap, again in three spatial dimensions, has been studied in the same three different limiting regimes: In a weak coupling limit [312], in a low density limit [403], and in a high density limit (see Chapter 11).

In this paper, we focus on a remarkable feature of BCS theory, which is well known in the physics literature [48, 477, 400]: The ratio of the energy gap Ξ and critical temperature T_c tends to a *universal constant, independent of the microscopic details* of the interaction between the fermions, i.e. the potential V. More precisely, in three spatial dimension, it holds that

$$\frac{\Xi}{T_c} \approx \frac{\pi}{e^\gamma} \approx 1.76\,, \tag{12.1.3}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, in each of the three physically very different limits mentioned above. This result follows as a limiting equality by combining asymptotic formulas for the critical temperature T_c (see [269, 312, 311, 333]) and the energy gap Ξ (see [312, 403] and Chapter 11) in the three different regimes. Although these scenarios (weak coupling, low density, and high density) are physically rather different, they all have in common that 'superconductivity is weak' and one can hence derive an asymptotic formula for T_c and Ξ as they depart from being zero (in the extreme cases $\lambda = 0$, $\mu = 0$, $\mu = \infty$, respectively). However, all the asymptotic expressions are *not perturbative*, as they depend exponentially on the natural dimensionless small parameter in the respective limit. We refer to the above mentioned original works for details.

The goal of this note is to prove the *same* universal behavior (12.1.3), which has already been established in three spatial dimension, also in dimensions $d = 1, 2$ in the weak coupling limit (i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$). This situation serves as a showcase for the methods involved in the proofs of the various limits in three dimensions (see Remark 12.3.6 and Remark 12.3.9 below). Apart from the mathematical curiosity in $d = 1, 2$, there have been recent studies in lower-dimensional superconductors in the physics literature, out of which we mention one-dimensional superconducting nanowires [467] and two-dimensional 'magic angle' graphene [137].

In the remainder of this introduction, we briefly recall the mathematical formulation of BCS theory, which has been developed mostly by Hainzl and Seiringer, but also other co-authors [309, 269, 316]. Apart from the universality discussed here, also many other properties of BCS theory have been shown using this formulation: Most prominently, Ginzburg-Landau theory, as an effective theory

describing superconductors close to the critical temperature, has been derived from BCS theory [270, 272, 220, 219]. More recently, it has been shown that the effect of boundary superconductivity occurs in the BCS model [310]. We refer to [316] for a more comprehensive review of developments in the mathematical formulation of BCS theory. The universal behavior in the weak coupling limit for lower dimensions $d = 1, 2$ is presented in Section 12.2. Finally, in Section 12.3, we provide the proofs of the statements from Section 12.2.

12.1.1 Mathematical formulation of BCS theory

We will now briefly recall the mathematical formulation [309, 316] of BCS theory [48], which is an effective theory developed for describing superconductivity of a fermionic gas. In the following, we consider these fermions in $\mathbf{R}^d$, $d = 1, 2$, at temperature $T \geq 0$ and chemical potential $\mu \in \mathbf{R}$, interacting via a two-body potential V, for which we assume the following.

Assumption 12.1.1. *We have that V is real-valued, reflection symmetric, i.e. $V(x) = V(-x)$ for all $x \in \mathbf{R}^d$, and it satisfies $V \in L^{p_V}(\mathbf{R}^d)$, where $p_V = 1$ if $d = 1$, $p_V \in (1, \infty)$ if $d = 2$.*

Moreover, we neglect external fields, in which case the system is translation invariant.

The central object in the mathematical formulation of the theory is the BCS functional, which can naturally be viewed as a function of BCS states Γ. These states are given by a pair of functions (γ, α) and can be conveniently represented as a 2×2 matrix valued Fourier multiplier on $L^2(\mathbf{R}^d) \oplus L^2(\mathbf{R}^d)$ of the form

$$\hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1 - \hat{\gamma}(p) \end{pmatrix} \tag{12.1.4}$$

for all $p \in \mathbf{R}^d$. In (12.1.4), $\hat{\gamma}(p)$ denotes the Fourier transform of the one particle density matrix and $\hat{\alpha}(p)$ is the Fourier transform of the Cooper pair wave function. We require reflection symmetry of $\hat{\alpha}$, i.e. $\hat{\alpha}(-p) = \hat{\alpha}(p)$, as well as $0 \leq \hat{\Gamma}(p) \leq 1$ as a matrix.

The *BCS free energy functional* takes the form

$$\mathcal{F}_T[\Gamma] := \int_{\mathbf{R}^d} (p^2 - \mu)\hat{\gamma}(p)\mathrm{d}p - TS[\Gamma] + \int_{\mathbf{R}^d} V(x)|\alpha(x)|^2 \mathrm{d}x\,, \qquad \Gamma \in \mathcal{D}, \tag{12.1.5}$$

$$\mathcal{D} := \left\{ \hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1 - \hat{\gamma}(p) \end{pmatrix} : 0 \leq \hat{\Gamma} \leq 1\,,\ \hat{\gamma} \in L^1(\mathbf{R}^d, (1 + p^2)\mathrm{d}p)\,,\ \alpha \in H^1_{\mathsf{sym}}(\mathbf{R}^d) \right\},$$

where the *entropy density* is defined as

$$S[\Gamma] = -\int_{\mathbf{R}^d} \mathrm{Tr}_{\mathbf{C}^2} \left[\hat{\Gamma}(p) \log \hat{\Gamma}(p) \right] \mathrm{d}p\,.$$

The minimization problem associated with (12.1.5) is well defined. In fact, the following result has only been proven for $d = 3$ and $V \in L^{3/2}(\mathbf{R}^3)$, but its extension to $d = 1, 2$ is straightforward.

Proposition 12.1.2 ([309], see also [316])**.** *Under Assumption 12.1.1 on V, the BCS free energy is bounded below on $\mathcal{D}$ and attains its minimum.*

The BCS gap equation (12.1.1) arises as the Euler–Lagrange equations of this functional [309]. Namely by defining $\Delta = -2\widehat{V\alpha}$, the Euler–Lagrange equation for α takes the form of the BCS gap equation (12.1.1). Additionally, one has the following linear criterion for the BCS gap equation to have non-trivial solutions. Again, so far, a proof has only been given in spatial dimension $d = 3$ and for $V \in L^{3/2}(\mathbf{R}^3)$, but its extension to $d = 1, 2$ is straightforward.

Theorem 12.1.3 ([309, Thm. 1]). *Let V satisfy Assumption 12.1.1 and let $\mu \in \mathbf{R}$ as well as $T \geq 0$. Then, writing $\mathcal{F}_T[\Gamma] \equiv \mathcal{F}_T(\gamma, \alpha)$, the following are equivalent.*

1. *The minimizer of $\mathcal{F}_T$ is not attained with $\alpha = 0$, i.e.*

$$\inf_{(\gamma,\alpha)\in\mathcal{D}} \mathcal{F}_T(\gamma,\alpha) < \inf_{(\gamma,0)\in\mathcal{D}} \mathcal{F}_T(\gamma,0),$$

2. *There exists a pair $(\gamma, \alpha) \in \mathcal{D}$ with $\alpha \neq 0$ such that $\Delta = -2\widehat{V\alpha}$ satisfies the BCS gap equation* (12.1.1),

3. *The linear operator $K_T + V$, where $K_T(p) = \frac{p^2-\mu}{\tanh((p^2-\mu)/(2T))}$ has at least one negative eigenvalue.*

The third item immediately leads to the following definition of the *critical temperature* T_c for the existence of non-trivial solutions of the BCS gap equation (12.1.1).

Definition 12.1.4 (Critical temperature, see [269, Def. 1]). *For V satisfying Assumption 12.1.1, we define the critical temperature $T_c \geq 0$ as*

$$T_c := \inf\{T > 0 : K_T + V \geq 0\}. \tag{12.1.6}$$

By $K_T(p) \geq 2T$ and the asymptotic behavior $K_T(p) \sim p^2$ for $|p| \to \infty$, Sobolev's inequality [418, Thm. 8.3] implies that the critical temperature is well defined.

The other object we study is the energy gap Ξ defined in (12.1.2). The energy gap depends on the solution Δ of the gap equation (12.1.1) at $T = 0$. A priori, Δ may not be unique. However, for potentials with non-positive Fourier transform, this possibility can be ruled out.

Proposition 12.1.5 (see [312, (21)-(22) and Lemma 2]). *Let V satisfy Assumption 12.1.1 (and additionally $V \in L^1(\mathbf{R}^2)$ in case that $d = 2$). Moreover, we assume that $\hat{V} \leq 0$ and $\hat{V}(0) < 0$. Then, there exists a unique minimizer Γ of $\mathcal{F}_0$ (up to a constant phase in α). One can choose the phase such that α has strictly positive Fourier transform $\hat{\alpha} > 0$.*

In particular, we conclude that Δ is strictly positive. Moreover, by means of the gap equation (12.1.1), Δ is continuous and thus $\Xi > 0$.

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12.2 Main Results

As explained in the introduction, our main result in this short note is the extension of the universality (12.1.3) from $d = 3$ to lower spatial dimensions $d = 1, 2$ in the limit of weak coupling (i.e., replacing $V \to \lambda V$ and taking $\lambda \to 0$). We assume the following properties for the interaction potential V.

Assumption 12.2.1. *Let $d \in \{1, 2\}$ and assume that V satisfies Assumption 12.1.1 as well as $\hat{V} \leq 0$, $\hat{V}(0) < 0$. Moreover, for $d = 1$ we assume that $(1 + |\cdot|^\varepsilon)V \in L^1(\mathbf{R}^1)$ for some $\varepsilon > 0$. Finally, in case that $d = 2$, we suppose that $V \in L^1(\mathbf{R}^2)$ is radial.*

By Proposition 12.1.5, this means that, in particular, the minimizer of $\mathcal{F}_0$ is unique (up to a phase) and the associated energy gap at zero temperature (12.1.2) is strictly positive, $\Xi > 0$. We are now ready to state our main result.

Theorem 12.2.2 (BCS Universality in one and two dimensions)**.** *Let V be as in Assumption 12.2.1. Then the critical temperature $T_c(\lambda)$ (defined in* (12.1.6)*) and the energy gap $\Xi(\lambda)$ (defined in* (12.1.2)*) are strictly positive for all $\lambda > 0$ and it holds that*

$$\lim_{\lambda\to 0} \frac{\Xi(\lambda)}{T_c(\lambda)} = \frac{\pi}{e^\gamma},$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

To prove the universality, we separately establish asymptotic formulas for T_c (see Theorem 12.2.5) and Ξ (see Theorem 12.2.7), valid to second order, and compare them by taking their ratio. The asymptotic formula for T_c is valid under weaker conditions on V than Assumption 12.2.1, because we do not need uniqueness of Δ. To obtain the asymptotic formulas, we first introduce two self-adjoint operators $\mathcal{V}_\mu^{(d)}$ and $\mathcal{W}_\mu^{(d)}$ mapping $L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ and as such measuring the strength of the interaction $\hat{V}$ on the (rescaled) Fermi surface (see [312, 333] and Chapter 11). To assure that $\mathcal{V}_\mu^{(d)}$ and $\mathcal{W}_\mu^{(d)}$ will be well-defined and compact, we assume the following.

Assumption 12.2.3. *Let V satisfy Assumption 12.1.1. Additionally, assume that for $d = 1$, $\big(1 + (\ln(1 + |\cdot|))^2\big)V \in L^1(\mathbf{R}^1)$ and for $d = 2$, $V \in L^1(\mathbf{R}^2)$.*

First, in order to capture the strength to leading order, we define $\mathcal{V}_\mu^{(d)}$ via

$$(\mathcal{V}_\mu^{(d)} u)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(\sqrt{\mu}(p-q)) u(q) \mathrm{d}\omega(q),$$

where $\mathrm{d}\omega$ is the Lebesgue measure on $\mathbb{S}^{d-1}$. Since $V \in L^1(\mathbf{R}^d)$, we have that $\hat{V}$ is a bounded continuous function and hence $\mathcal{V}_\mu^{(d)}$ is a Hilbert-Schmidt operator (in fact, trace class with trace being equal to $(2\pi)^{-d}|\mathbb{S}^{d-1}| \int_{\mathbf{R}^d} V(x)\mathrm{d}x$). Therefore, its lowest eigenvalue $e_\mu^{(d)} := \inf \operatorname{spec} \mathcal{V}_\mu^{(d)}$ satisfies $e_\mu^{(d)} \leq 0$ and it is strictly negative if e.g. $\int V < 0$ as in Assumption 12.2.1.

Second, in order to capture the strength of $\hat{V}$ to next to leading order, we define the operator $\mathcal{W}_\mu^{(d)}$ via its quadratic form

$$\begin{aligned}&\left\langle u \middle| \mathcal{W}_\mu^{(d)} \middle| u \right\rangle \\ &\quad = \mu^{d/2-1} \left[\int_{|p|<\sqrt{2}} \frac{1}{|p^2-1|} \left(|\psi(\sqrt{\mu}p)|^2 - |\psi(\sqrt{\mu}p/|p|)|^2 \right) \mathrm{d}p + \int_{|p|>\sqrt{2}} \frac{1}{|p^2-1|} |\psi(\sqrt{\mu}p)|^2 \mathrm{d}p \right],\end{aligned}$$

where $\psi(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(p - \sqrt{\mu}q) u(q) \mathrm{d}\omega(q)$ and $u \in L^2(\mathbb{S}^{d-1})$. The proof of the following proposition shall be given in Section 12.3.3.

Proposition 12.2.4. *Let $d \in \{1, 2\}$ and let V satisfy Assumption 12.2.3. The operator $\mathcal{W}_\mu^{(d)}$ is well-defined and Hilbert-Schmidt.*

Next, we define the self-adjoint Hilbert-Schmidt operator

$$\mathcal{B}_\mu^{(d)}(\lambda) := \frac{\pi}{2} \left(\lambda \mathcal{V}_\mu^{(d)} - \lambda^2 \mathcal{W}_\mu^{(d)} \right)$$

on $L^2(\mathbb{S}^{d-1})$ and its ground state energy

$$b_\mu^{(d)}(\lambda) := \inf \operatorname{spec} \left(\mathcal{B}_\mu^{(d)}(\lambda) \right). \tag{12.2.1}$$

Note that if $e_\mu^{(d)} < 0$, then also $b_\mu^{(d)}(\lambda) < 0$ for small enough λ. After these preparatory definitions, we are ready to state the separate asymptotic formulas for the critical temperature and the energy gap in one and two dimensions, which immediately imply Theorem 12.2.2.

Theorem 12.2.5 (Critical Temperature for $d = 1, 2$)**.** *Let $\mu > 0$. Let V satisfy Assumption 12.2.3 and additionally $e_\mu^{(d)} < 0$. Then the critical temperature T_c, given in Definition 12.1.4, is strictly positive and satisfies*

$$\lim_{\lambda\to 0}\left(\ln\left(\frac{\mu}{T_c(\lambda)}\right) + \frac{\pi}{2\,\mu^{d/2-1}\,b_\mu^{(d)}(\lambda)}\right) = -\gamma - \ln\left(\frac{2c_d}{\pi}\right),$$

where γ denotes the Euler-Mascheroni constant and $c_1 = \frac{4}{1+\sqrt{2}}$ and $c_2 = 1$.

Here, the Assumptions on V are weaker than Assumption 12.2.1, since $\hat{V}(0) < 0$ implies that $e_\mu^{(d)} < 0$. We thus have the asymptotic behavior

$$T_c(\lambda) = 2c_d\,\frac{e^\gamma}{\pi}\,\big(1 + o(1)\big)\mu\, e^{\pi/(2\mu^{d/2-1}b_\mu^{(d)}(\lambda))}$$

in the limit of small λ.

Remark 12.2.6. *Theorem 12.2.5 is essentially a special case of [314, Theorem 2]. We give the proof here for two main reasons.*

(i) There is still some work required to translate the statement of [314, Theorem 2] into a form in which it is comparable to that of Theorem 12.2.7 (in order to prove Theorem 12.2.2). The main difficulty is that the operator $\mathcal{W}_\mu^{(d)}$ in [314] is only defined via a limit, [314, Equation (2.10)].

(ii) The goal of this paper is to give an exemplary proof of Theorem 12.2.5 in order to compare it to the proofs of the similar statements in the literature concerning the asymptotic behavior of the critical temperature in various limits [311, 312, 333].

Theorem 12.2.5 is complemented by the following asymptotics for the energy gap.

Theorem 12.2.7 (Energy Gap for $d = 1, 2$)**.** *Let V satisfy Assumption 12.2.1 and let $\mu > 0$. Then there exists a unique radially symmetric minimizer (up to a constant phase) of the BCS functional* (12.1.5) *at temperature $T = 0$. The associated energy gap Ξ, given in* (12.1.2)*, is strictly positive and satisfies*

$$\lim_{\lambda\to 0}\left(\ln\left(\frac{\mu}{\Xi}\right) + \frac{\pi}{2\,\mu^{d/2-1}\,b_\mu^{(d)}(\lambda)}\right) = -\ln(2c_d),$$

where $b_\mu^{(d)}$ is defined in (12.2.1) *and $c_1 = \frac{4}{1+\sqrt{2}}$ and $c_2 = 1$.*

In other words, we have the asymptotic behavior

$$\Xi(\lambda) = 2c_d\,\big(1 + o(1)\big)\mu\, e^{\pi/(2\mu^{d/2-1}b_\mu^{(d)}(\lambda))}$$

in the limit of small λ. Now, Theorem 12.2.2 follows immediately from Theorems 12.2.5 and 12.2.7.

Remark 12.2.8 (Other limits in dimensions $d = 1, 2$)**.** *Similarly to the presented results, one could also consider the limits of low and high density. We expect that also here the universality* $\frac{\Xi}{T_c} \approx \frac{\pi}{e^\gamma}$ *holds. As mentioned in the introduction, this has already been shown in three spatial dimensions; see [311, 403, 333] and Chapters 10–11. We expect that one could generalize the arguments of [311, 403] and Chapters 10–11 to lower dimensions, but that there will be some non-trivial technical difficulties in doing so. Also for the weak coupling limit presented here, the overall structure and ideas of the proof are the same in lower dimensions as in three dimensions [312], but with non-trivial technical differences, see Remark 12.3.3.*

The following is an example of such a non-trivial difference in the low-density limit. In three spatial dimensions [311, 403] the asymptotic formulas for T_c *and* Ξ *were obtained for attractive potentials* V *not creating bound states of* $-\nabla^2 + V$*. This latter condition ensures that the low-density limit is given by* $\mu \to 0$*. However, in spatial dimensions one and two, attractive potentials, no matter how weak, always give rise to bound states of* $-\nabla^2 + V$*, see [525]. This means that one should not take the limit* $\mu \to 0$*, but rather the limit* $\mu \to -E_b$*, with* $-E_b < 0$ *the energy of the (lowest energy) bound state, see [315]. We will not deal with the low- and high-density limits here.*

The rest of the paper is devoted to proving Theorem 12.2.5 and Theorem 12.2.7.

12.3 Proofs

The overall structure of our proofs is as follows: First, we argue that the Schrödinger type operators $K_{T_c} + \lambda V$ and $E_\Delta + \lambda V$ have lowest eigenvalue zero. The second step is to study the corresponding Birman-Schwinger operators

$$B_T^{(d)} := \lambda V^{1/2} K_T^{-1} |V|^{1/2} \quad \text{and} \quad B_\Delta^{(d)} := \lambda V^{1/2} E_\Delta^{-1} |V|^{1/2},$$

where $V(x)^{1/2} = \operatorname{sgn}(V(x))|V(x)|^{1/2}$. According to the Birman-Schwinger principle, the lowest eigenvalue of $B_{T_c}^{(d)}$ and $B_\Delta^{(d)}$ is -1. It turns out, that for $X \in \{T, \Delta\}$ one can decompose

$$B_X^{(d)} = \lambda m_\mu^{(d)}(X) V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2} + \lambda V^{1/2} M_X^{(d)} |V|^{1/2}, \tag{12.3.1}$$

where $V^{1/2} M_X^{(d)} |V|^{1/2}$ are bounded operators,

$$m_\mu^{(d)}(T) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{|p|<\sqrt{2\mu}} \frac{1}{K_T(p)} \mathrm{d}p, \tag{12.3.2}$$

$$m_\mu^{(d)}(\Delta) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{|p|<\sqrt{2\mu}} \frac{1}{E_\Delta(p)} \mathrm{d}p, \tag{12.3.3}$$

and $\mathfrak{F}_\mu^{(d)} : L^1(\mathbf{R}^d) \to L^2(\mathbb{S}^{d-1})$ is the (scaled) Fourier transform restricted to the (rescaled) Fermi sphere,

$$\left(\mathfrak{F}_\mu^{(d)} \psi\right)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \psi(x) e^{-i\sqrt{\mu} p \cdot x} \mathrm{d}x\,.$$

Note that for an L^1-function, pointwise values of its Fourier transform are well-defined by the Riemann–Lebesgue lemma. (In particular the restriction to a co–dimension 1 manifold of a sphere is well-defined.)

To satisfy the constraint that the lowest eigenvalue of the Birman-Schwinger operators is -1, the functions $m_\mu^{(d)}$ must diverge as $\lambda \to 0$. It turns out that this is only possible if T and Δ go to zero. For each $m_\mu^{(d)}$ one then derives the asymptotics up to second order in two ways, once from the constraint that $B_X^{(d)}$ has lowest eigenvalue -1 and once by directly computing the asymptotics of $m_\mu^{(d)}$ for T and Δ going to zero.

Indeed, for the critical temperature we obtain the following asymptotics, which, by combining them, immediately prove Theorem 12.2.5.

Proposition 12.3.1. *Let $\mu > 0$. Let V satisfy Assumption 12.2.3 and additionally $e_\mu^{(d)} < 0$. Then, the critical temperature T_c is positive and, as $\lambda \to 0$, we have that*

$$m_\mu^{(d)}(T_c) = -\frac{\pi}{2b_\mu^{(d)}(\lambda)} + o(1)\,,$$
$$m_\mu^{(d)}(T_c) = \mu^{d/2-1}\left(\ln\left(\frac{\mu}{T_c}\right) + \gamma + \ln\left(\frac{2c_d}{\pi}\right) + o(1)\right).$$

For the energy gap we obtain the following asymptotics, which, again by combining them, immediately prove Theorem 12.2.7.

Proposition 12.3.2. *Let V satisfy Assumption 12.2.1 and let $\mu > 0$. Then (by Proposition 12.1.5) we have a strictly positive radially symmetric gap function Δ and associated energy gap Ξ, which, as $\lambda \to 0$, satisfy the asymptotics*

$$\Xi = \Delta(\sqrt{\mu})\big(1 + o(1)\big)$$
$$m_\mu^{(d)}(\Delta) = -\frac{\pi}{2b_\mu^{(d)}(\lambda)} + o(1)$$
$$m_\mu^{(d)}(\Delta) = \mu^{d/2-1}\left(\ln\left(\frac{\mu}{\Delta(\sqrt{\mu})}\right) + \ln(2c_d) + o(1)\right)$$

With a slight abuse of notation, using radiality of Δ, we wrote $\Delta(\sqrt{\mu})$ instead of $\Delta(\sqrt{\mu}\hat{p})$ for some $\hat{p} \in \mathbb{S}^{d-1}$.

Remark 12.3.3. *The main technical differences between $d \in \{1, 2\}$ considered here and the proof for $d = 3$ in [312] arise when bounding $V^{1/2}M_X^{(d)}|V|^{1/2}$. The underlying reason is that the Fourier transform of the constant function on the sphere $j_d(x) = (2\pi)^{-d/2}\int_{\mathbb{S}^{d-1}} e^{ip\cdot x}\mathrm{d}\omega(p)$ decays like $1/|x|$ for large $|x|$ in three dimensions, but only like $|x|^{-1/2}$ in two dimensions and does not decay for $d = 1$.*

Remark 12.3.4. *In [186], Cuenin and Merz use the Tomas-Stein theorem to define $\mathfrak{F}_\mu^{(d)}$ on a larger space than $L^1(\mathbf{R}^d)$. With this they are able to prove a general version of Theorem 12.2.5 under slightly weaker conditions on V. However, we do not pursue this here, see Remark 12.2.6.*

12.3.1 Proof of Proposition 12.3.1

Proof of Proposition 12.3.1. The argument is divided into several steps.

1. A priori spectral information on $K_{T_c} + \lambda V$. First note that, due to Theorem 12.1.3 and Definition 12.1.4, the critical temperature T_c is determined by the lowest eigenvalue of $K_T + \lambda V$ being 0 exactly for $T = T_c$.

2. Birman-Schwinger principle. Next, we employ the Birman-Schwinger principle, which says that the compact Birman-Schwinger operator $B_T^{(d)} = \lambda V^{1/2}K_T^{-1}|V|^{1/2}$ has -1 as its lowest eigenvalue exactly for $T = T_c$, see [269, 312].

Using the notation for the Fourier transform restricted to the rescaled Fermi sphere introduced above, we now decompose the Birman-Schwinger operator as in (12.3.1), where $M_T^{(d)}$ is defined through the integral kernel

$$M_T^{(d)}(x,y) = \frac{1}{(2\pi)^d}\left[\int_{|p|<\sqrt{2\mu}} \frac{1}{K_T(p)}\left(e^{ip\cdot(x-y)} - e^{i\sqrt{\mu}p/|p|\cdot(x-y)}\right)\mathrm{d}p + \int_{|p|>\sqrt{2\mu}} \frac{1}{K_T} e^{ip\cdot(x-y)}\mathrm{d}p\right]. \tag{12.3.4}$$

We claim that $V^{1/2}M_T^{(d)}|V|^{1/2}$ is uniformly bounded.

Lemma 12.3.5. *Let $\mu > 0$. Let V satisfy Assumption 12.2.3. Then we have for all $T \geq 0$*

$$\left\| V^{1/2}M_T^{(d)}|V|^{1/2} \right\|_{\mathsf{HS}} \leq C\,,$$

where $C > 0$ denotes some positive constant and $\|\cdot\|_{\mathsf{HS}}$ is the Hilbert-Schmidt norm.

Armed with this bound, we have that for sufficiently small λ that $1 + \lambda V^{1/2}M_T^{(d)}|V|^{1/2}$ is invertible, and hence

$$1 + B_T^{(d)} = (1 + \lambda V^{1/2}M_T^{(d)}|V|^{1/2})\left(1 + \frac{\lambda m_\mu^{(d)}(T)}{1 + \lambda V^{1/2}M_T^{(d)}|V|^{1/2}} V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)}|V|^{1/2}\right).$$

Thus, the fact that $B_T^{(d)}$ has lowest eigenvalue -1 at $T = T_c$ is equivalent to

$$\lambda m_\mu^{(d)}(T)\mathfrak{F}_\mu^{(d)}|V|^{1/2}\frac{1}{1 + \lambda V^{1/2}M_T^{(d)}|V|^{1/2}}V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger \tag{12.3.5}$$

having lowest eigenvalue -1, again at $T = T_c$, as it is isospectral to the rightmost operator on the right-hand-side above. (Recall that for bounded operators A, B, the operators AB and BA have the same spectrum apart from possibly at 0. However, in our case, both operators are compact on an infinite dimensional space and hence 0 is in both spectra.)

We now prove Lemma 12.3.5.

Proof of Lemma 12.3.5. We want to bound the integral kernel (12.3.4) of $M_T^{(d)}$ uniformly in T. Hence, we will bound $K_T \geq |p^2 - \mu|$. The computation is slightly different in $d = 1$ and $d = 2$, so we do them separately.

$\underline{d = 1.}$ The second integral in (12.3.4) is bounded by

$$2\int_{|p|>\sqrt{2\mu}} \frac{1}{|p^2 - \mu|}\mathrm{d}p = \frac{2\mathrm{arcoth}\sqrt{2}}{\sqrt{\mu}}.$$

For the first integral, we use that $|e^{ix} - e^{iy}| \leq \min\{|x - y|, 2\}$, $|p^2 - \mu| \geq \sqrt{\mu}||p| - \sqrt{\mu}|$, and increase the domain of integration to obtain the bound

$$\begin{aligned}\frac{2}{\sqrt{\mu}}\int_0^{2\sqrt{\mu}} \frac{\min\left\{||p - \sqrt{\mu}||x - y|, 2\right\}}{|p - \sqrt{\mu}|}\mathrm{d}p &= \frac{8}{\sqrt{\mu}}\left[1 + \ln\left(\max\left\{\frac{|x - y|\sqrt{\mu}}{2}, 1\right\}\right)\right] \\ &\leq \frac{8}{\sqrt{\mu}}(1 + \ln(1 + \sqrt{\mu}\max\{|x|, |y|\}).\end{aligned}$$

We conclude that $|M_T^{(1)}(x, y)| \lesssim \frac{1}{\sqrt{\mu}}(1 + \ln(1 + \sqrt{\mu}\max\{|x|, |y|\}))$. Hence,

$$\left\| V^{1/2}M_T^{(1)}|V|^{1/2}\right\|_{\mathsf{HS}}^2 \lesssim \frac{1}{\mu}\left(\|V\|_{L^1(\mathbf{R})}^2 + \|V\|_{L^1(\mathbf{R})}\int_{\mathbf{R}} |V(x)|(1 + \ln(1 + \sqrt{\mu}|x|))^2\mathrm{d}x\right).$$

$\underline{d = 2}$. We first compute the angular integral. Note that $\int_{\mathbb{S}^1} e^{ipx}\mathrm{d}\omega(p) = 2\pi J_0(|x|)$, where J_0 is the zeroth order Bessel function. For the second integral in (12.3.4) we may bound $|p^2 - \mu| \geq cp^2$. Up to some finite factor, the second integral is hence bounded by

$$\int_{\sqrt{2\mu}}^\infty \frac{1}{p}|J_0(p|x - y|)|\mathrm{d}p \leq C\int_{\sqrt{2\mu}}^\infty \frac{1}{p^{1+\lambda}}|x - y|^{-\lambda}\mathrm{d}p \leq C_\lambda|x - y|^{-\lambda},$$

for any $0 < \lambda \le 1/2$ since $|J_0(x)| \le C$ and $\sqrt{x}J_0(x) \le C$, see e.g. [122, (9.55f), (9.57a)]. For the first integral we get the bound

$$\int_0^{\sqrt{2\mu}} \frac{p}{|p^2-\mu|}|J_0(p|x-y|) - J_0(\sqrt{\mu}|x-y|)|\mathrm{d}p.$$

Here we use that J_0 is Lipschitz, since its derivative J_{-1} is bounded (see e.g. [122, (9.55a),(9.55f)]), so that

$$|J_0(x) - J_0(y)| \le C|x-y|^{1/3}(|J_0(x)| + |J_0(y)|)^{2/3} \le C|x-y|^{1/3}\left(x^{-1/3} + y^{-1/3}\right).$$

That is

$$|J_0(p|x-y|) - J_0(\sqrt{\mu}|x-y|)| \le C\frac{|p-\sqrt{\mu}|^{1/3}}{p^{1/3} + \sqrt{\mu}^{1/3}}.$$

This shows that the first integral is bounded. We conclude that $|M_T^{(2)}(x,y)| \lesssim 1 + \frac{1}{|x-y|^\lambda}$ for any $0 < \lambda \le 1/2$. Then, by the Hardy–Littlewood–Sobolev inequality [418, Theorem 4.3] we have that

$$\left\|V^{1/2}M_T^{(2)}|V|^{1/2}\right\|_{\mathsf{HS}}^2 = \iint |V(x)||M_T^{(2)}(x,y)||V(y)|\mathrm{d}x\mathrm{d}y \lesssim \|V\|_{L^1(\mathbf{R}^2)}^2 + \|V\|_{L^p(\mathbf{R}^2)}^2$$

for any $1 < p \le 4/3$. □

3. First order. Evaluating (12.3.5) at $T = T_c$ and expanding the geometric series to first order we get

$$\begin{aligned} -1 &= \lambda m_\mu^{(d)}(T_c) \inf\operatorname{spec}\left(\mathfrak{F}_\mu^{(d)}|V|^{1/2}\frac{1}{1+\lambda V^{1/2}M_{T_c}^{(d)}|V|^{1/2}}V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger\right) \\ &= \lambda\, m_\mu^{(d)}(T_c)\, \inf\operatorname{spec}\mathcal{V}_\mu^{(d)}(1+O(\lambda)) = \lambda\, m_\mu^{(d)}(T_c)\, e_\mu^{(d)}(1+O(\lambda)) \end{aligned}$$

where we used $\mathcal{V}_\mu^{(d)} = \mathfrak{F}_\mu^{(d)}V(\mathfrak{F}_\mu^{(d)})^\dagger$. Since by assumption $e_\mu^{(d)} < 0$, this shows that $m_\mu^{(d)}(T_c) \to \infty$ as $\lambda \to 0$.

4. A priori bounds on T_c. By (12.3.2), the divergence of $m_\mu^{(d)}$ as $\lambda \to 0$ in particular shows that $T_c/\mu \to 0$ in the limit $\lambda \to 0$.

5. Calculation of the integral $m_\mu^{(d)}(T_c)$. This step is very similar to [312, Lemma 1] and [310, Lemma 3.5], where the asymptotics have been computed for slightly different definitions of $m_\mu^{(d)}$ in three and one spatial dimension, respectively. Integrating over the angular variable and substituting $s = \left|\frac{|p|^2}{\mu} - 1\right|$, we get

$$m_\mu^{(d)}(T_c) = \mu^{d/2-1}\int_0^1 \tanh\left(\frac{s}{2(T_c/\mu)}\right)\frac{(1+s)^{d/2-1} + (1-s)^{d/2-1}}{2s}\mathrm{d}s.$$

According to [312, Lemma 1],

$$\lim_{T_c\downarrow 0}\left(\int_0^1 \frac{\tanh\left(\frac{s}{2(T_c/\mu)}\right)}{s}\mathrm{d}s - \ln\frac{\mu}{T_c}\right) = \gamma - \ln\frac{\pi}{2}.$$

By monotone convergence, it follows that

$$m_\mu^{(d)}(T_c) = \mu^{d/2-1}\left[\ln\frac{\mu}{T_c} + \gamma - \ln\frac{\pi}{2} + \int_0^1 \frac{(1-s)^{d/2-1} + (1+s)^{d/2-1} - 2}{2s}\mathrm{d}s + o(1)\right].$$

The remaining integral equals $\ln c_d$ and we have thus proven the second item in Proposition 12.3.1.

Combining this with the third step, one immediately sees that the critical temperature vanishes exponentially fast, $T_c \sim e^{1/\lambda e_\mu}$, as $\lambda \to 0$, recalling that $e_\mu^{(d)} < 0$ by assumption.

6. Second order. Now, to show the universality, we need to compute the next order correction. To do so, we expand the geometric series in (12.3.5) and employ first order perturbation theory, yielding that

$$m_\mu^{(d)}(T_c) = \frac{-1}{\lambda \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle - \lambda^2 \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V M_{T_c}^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle + O(\lambda^3)}, \tag{12.3.6}$$

where u is the (normalized) ground state (eigenstate of lowest eigenvalue) of $\mathfrak{F}_\mu^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger$. (In case of a degenerate ground state, u is the ground state minimizing the second order term.)

This second order term in the denominator of (12.3.6) is close to $\mathcal{W}_\mu^{(d)}$. More precisely, it holds that

$$\lim_{\lambda \to 0} \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V M_{T_c}^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle = \left\langle u \middle| \mathcal{W}_\mu^{(d)} \middle| u \right\rangle , \tag{12.3.7}$$

which easily follows from dominated convergence, noting that $\frac{1}{K_T}$ increases to $\frac{1}{|p^2-\mu|}$ as $T \to 0$. We then conclude that

$$\lim_{\lambda \to 0} \left(m_\mu^{(d)}(T_c) + \frac{\pi}{2 b_\mu^{(d)}(\lambda)} \right) = 0 ,$$

since $\left\langle u \middle| \lambda \mathcal{V}_\mu^{(d)} - \lambda^2 \mathcal{W}_\mu^{(d)} \middle| u \right\rangle = \inf \operatorname{spec}(\lambda \mathcal{V}_\mu^{(d)} - \lambda^2 \mathcal{W}_\mu^{(d)}) + O(\lambda^3) = \frac{\pi}{2} b_\mu^{(d)}(\lambda) + O(\lambda^3)$, again by first-order perturbation theory. This concludes the proof of Proposition 12.3.1. □

We conclude this subsection with several remarks, comparing our proof with those of similar results from the literature.

Remark 12.3.6 (Structure here vs. in earlier papers on T_c)**.** *We compare the structure of our proof to that of the different limits in three dimensions [312, 311] and Chapter 10:*

- ***Weak coupling:*** *The structure of the proof we gave here is quite similar to that of [312], only they do Steps 5 and 6 in the opposite order. Also the leading term for T_c was shown already in [269], where a computation somewhat similar to Steps 1–4 is given.*
- ***High density:*** *For $\mu \to \infty$, the structure of the proof in Chapter 10 is slightly different compared to the one given here. This is basically due to the facts that (i) the necessary a priori bound $T_c = o(\mu)$ already requires the Birman-Schwinger decomposition and (ii) the second order requires strengthened assumptions compared to the first order. To conclude, the order of steps in Chapter 10 can be thought of as: 1, 5, 4 (establishing $T_c = O(\mu)$), 2, 3, 4 (establishing $T_c = o(\mu)$), 2 (again), 6. Here the final step is much more involved than in the other limits considered.*
- ***Low density:*** *As above, for the proof of the low density limit in [311] the structure is slightly different. One first needs the a priori bound $T_c = o(\mu)$ on the critical temperature before one uses the Birman-Schwinger principle and decomposes the Birman-Schwinger operator.*[1] *Also, the decomposition of the Birman-Schwinger operator is again different. For the full decomposition and analysis of the Birman-Schwinger operator one needs also the first-order analysis, that is Step 2, which is done in two parts. The order of the steps in [311] can then mostly be though of as: 1, 4, 5, 2, 3, 2 (again), 6.*

[1] Strictly speaking, in [311], it is only proven that $T_c = O(\mu)$ (which is sufficient for applying the Birman-Schwinger principle), while the full $T_c = o(\mu)$ itself requires the Birman-Schwinger decomposition (see [402, Remark 4.12] for details).

12.3.2 Proof of Proposition 12.3.2

Proof of Proposition 12.3.2. The structure of the proof is parallel to that of Proposition 12.3.1 for the critical temperature.

1. A priori spectral information on $E_\Delta + \lambda V$. First, it is proven in [312, Lemma 2] that $\mathcal{F}_0$ has a unique minimizer α which has strictly positive Fourier transform. Using radiality of V, it immediately follows that this minimizer is rotationally symmetric (since otherwise rotating α would give a different minimizer) and hence also $\Delta = -2\lambda\hat{V} \star \hat{\alpha}$ is rotation invariant. It directly follows from [312, (43) and Lemma 3] that that $E_\Delta + \lambda V$ has lowest eigenvalue 0, and that the minimizer α is the corresponding eigenfunction.

2. Birman-Schwinger principle. This implies, by means of the Birman-Schwinger principle, that the Birman-Schwinger operator $B_\Delta^{(d)} = \lambda V^{1/2} E_\Delta^{-1} |V|^{1/2}$ has -1 as its lowest eigenvalue. As in the proof of Proposition 12.3.1, we decompose it as described in (12.3.1) and prove the second summand to be uniformly bounded.

Lemma 12.3.7. *Let $\mu > 0$. Let V satisfy Assumption 12.2.3. Then, uniformly in small λ, we have*

$$\left\| V^{1/2} M_\Delta^{(d)} |V|^{1/2} \right\|_{\mathsf{HS}} \leq C\,.$$

With this one may similarly factor

$$1 + B_\Delta^{(d)} = (1 + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}) \left(1 + \frac{\lambda m_\mu^{(d)}(\Delta)}{1 + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2} \right) \tag{12.3.8}$$

and conclude that

$$T_\Delta^{(d)} := \lambda m_\mu^{(d)}(\Delta) \mathfrak{F}_\mu^{(d)} |V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \tag{12.3.9}$$

has lowest eigenvalue -1.

Proof of Lemma 12.3.7. Note that M_Δ has kernel

$$\begin{aligned} M_\Delta(x,y) = \frac{1}{(2\pi)^d} \Bigg[& \int_{|p|<\sqrt{2\mu}} \frac{1}{E_\Delta(p)} \left(e^{ip\cdot(x-y)} - e^{i\sqrt{\mu}p/|p|\cdot(x-y)} \right) \mathrm{d}p \\ & + \int_{|p|>\sqrt{2\mu}} \frac{1}{E_\Delta(p)} e^{ip\cdot(x-y)} \mathrm{d}p \Bigg]. \end{aligned}$$

We may bound this exactly as in the proof of Lemma 12.3.5 using that $E_\Delta(p) \geq |p^2 - \mu|$. □

3. First order. Expanding the geometric series in (12.3.9) to first order, we see that

$$\begin{aligned} -1 &= \lambda m_\mu^{(d)}(\Delta) \inf \operatorname{spec} \left(\mathfrak{F}_\mu^{(d)} |V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \right) \\ &= \lambda m_\mu^{(d)}(\Delta) \inf \operatorname{spec} \mathcal{V}_\mu^{(d)} (1 + O(\lambda)) = \lambda e_\mu^{(d)} m_\mu^{(d)}(\Delta)(1 + O(\lambda)). \end{aligned}$$

Hence, in particular, $m_\mu^{(d)}(\Delta) \sim -\frac{1}{\lambda e_\mu^{(d)}} \to \infty$ as $\lambda \to 0$.

4. A priori bounds on Δ. We now prepare for the computation of the integral $m_\mu^{(d)}(\Delta)$ in terms of $\Delta(\sqrt{\mu})$. This requires two types of bounds on Δ: One bound estimating the gap function $\Delta(p)$ at general momentum $p \in \mathbf{R}^d$ in terms of $\Delta(\sqrt{\mu})$ (see (12.3.10)), and one bound controlling the difference $|\Delta(p) - \Delta(q)|$ in some kind of Hölder-continuity estimate (see (12.3.11)).

Lemma 12.3.8. *Suppose that V is as in Assumption 12.2.1. Then for λ small enough*

$$\Delta(p) = f(\lambda)\left(\int_{\mathbb{S}^{d-1}} \hat{V}(p-\sqrt{\mu}q)\mathrm{d}\omega(q) + \lambda\eta_\lambda(p)\right),$$

where f is some function of λ and $\|\eta_\lambda\|_{L^\infty(\mathbf{R}^d)}$ is bounded uniformly in λ.

Proof. Recall that α is the eigenfunction of $E_\Delta + \lambda V$ with lowest eigenvalue 0. Then, by the Birman-Schwinger principle, $\phi = V^{1/2}\alpha$ satisfies

$$B_\Delta\phi = \lambda V^{1/2}\frac{1}{E_\Delta}|V|^{1/2}V^{1/2}\alpha = -\phi.$$

With the decomposition Equation (12.3.8) then ϕ is an eigenfunction of

$$\frac{\lambda m_\mu^{(d)}(\Delta)}{1+\lambda V^{1/2}M_\Delta^{(d)}|V|^{1/2}}V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger\mathfrak{F}_\mu^{(d)}|V|^{1/2}$$

of eigenvalue -1. Thus, $\mathfrak{F}_\mu^{(d)}|V|^{1/2}\phi$ is an eigenfunction of $T_\Delta^{(d)}$ of (lowest) eigenvalue -1. Now $u = |\mathbb{S}^{d-1}|^{-1/2}$ is the unique eigenfunction corresponding to the lowest eigenvalue of $\mathcal{V}_\mu^{(d)}$ by radiality of V and the assumption $\hat{V} \le 0$ (see e.g. [269]). Hence, for λ small enough, u is the unique eigenfunction of $T_\Delta^{(d)}$ of smallest eigenvalue. Thus,

$$\phi = f(\lambda)\frac{1}{1+\lambda V^{1/2}M_\Delta^{(d)}|V|^{1/2}}V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger u = f(\lambda)\left(V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger u + \lambda\xi_\lambda\right)$$

for some number $f(\lambda)$. The function ξ_λ satisfies $\|\xi_\lambda\|_{L^2(\mathbf{R}^d)} \le C$ by Lemma 12.3.7. Noting that $\Delta = -2\widehat{|V|^{1/2}}\phi$ and bounding $\left\||\widehat{V|^{1/2}}\xi_\lambda\right\|_{L^\infty} \le \|V\|_{L^1}^{1/2}\|\xi_\lambda\|_{L^2}$ we get the desired. □

Evaluating the formula in Lemma 12.3.8 at $p = \sqrt{\mu}$ we get $|f(\lambda)| \le C\Delta(\sqrt{\mu})$ for λ small enough. This in turn implies that

$$\Delta(p) \le C\Delta(\sqrt{\mu})\,. \tag{12.3.10}$$

For the Hölder-continuity, we have by rotation invariance

$$\begin{aligned}
\left|\int \hat{V}(p-\sqrt{\mu}r) - \hat{V}(q-\sqrt{\mu}r)\mathrm{d}\omega(r)\right| &= \left|\int \hat{V}(|p|e_1-\sqrt{\mu}r) - \hat{V}(|q|e_1-\sqrt{\mu}r)\mathrm{d}\omega(r)\right| \\
&= \left|\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d}\mathrm{d}x\left(V(x)\left(e^{i|p|x_1}-e^{i|q|x_1}\right)\int_{\mathbb{S}^{d-1}}e^{-i\sqrt{\mu}x\cdot r}\mathrm{d}\omega(r)\right)\right| \\
&\le C_\epsilon\mu^{-\epsilon/2}||p|-|q||^\epsilon\int\mathrm{d}x\left(|V(x)|(\sqrt{\mu}|x|)^\epsilon\left|\int_{\mathbb{S}^{d-1}}e^{-i\sqrt{\mu}x\cdot r}\mathrm{d}\omega(r)\right|\right),
\end{aligned}$$

for any $0 < \epsilon \le 1$. For $d = 2$ we have $V \in L^1(\mathbf{R}^2)$ and

$$\left|\int_{\mathbb{S}^{d-1}}e^{-i\sqrt{\mu}xr}\mathrm{d}\omega(r)\right| = |J_0(\sqrt{\mu}|x|)| \le (\sqrt{\mu}|x|)^{-1/2}.$$

For $d = 1$ we have $|x|^\epsilon V \in L^1(\mathbf{R})$ for some $\epsilon > 0$ and

$$\left|\int_{\mathbb{S}^{d-1}}e^{-i\sqrt{\mu}x\cdot r}\mathrm{d}\omega(r)\right| = 2|\cos(\sqrt{\mu}|x|)| \le 2.$$

We conclude that with $\epsilon = 1/2$ for $d = 2$ and small enough $\epsilon > 0$ for $d = 1$

$$|\Delta(p)-\Delta(q)| \le C|f(\lambda)|\left(\mu^{-\epsilon/2}||p|-|q||^\epsilon+\lambda\right) \le C|\Delta(\sqrt{\mu})|\left(\mu^{-\epsilon/2}||p|-|q||^\epsilon+\lambda\right). \tag{12.3.11}$$

Additionally, since $m_\mu^{(d)}(\Delta) \to \infty$ we have that $\Delta(p) \to 0$ at least for some $p \in \mathbf{R}^d$ by (12.3.3). Then it follows from Lemma 12.3.8 that $f(\lambda) \to 0$, i.e. that $\Delta(p) \to 0$ for all p.

5. Calculation of the integral $m_\mu^{(d)}(\Delta)$. Armed with the a priori bounds (12.3.10) and (12.3.11), we can now compute the integral $m_\mu^{(d)}(\Delta)$. Carrying out the angular integration and substituting $s = \left|\frac{|p|^2-\mu}{\mu}\right|$ we have

$$m_\mu^{(d)}(\Delta) = \frac{\mu^{d/2-1}}{2}\left[\int_0^1 \left(\frac{(1-s)^{d/2-1}-1}{\sqrt{s^2+x_-(s)^2}} + \frac{(1+s)^{d/2-1}-1}{\sqrt{s^2+x_+(s)^2}}\right)\mathrm{d}s + \int_0^1 \left(\frac{1}{\sqrt{s^2+x_-(s)^2}} + \frac{1}{\sqrt{s^2+x_+(s)^2}}\right)\mathrm{d}s\right],$$

where $x_\pm(s) = \frac{\Delta(\sqrt{\mu}\sqrt{1\pm s})}{\mu}$. By dominated convergence, using that $x_\pm(s) \to 0$, the first integral is easily seen to converge to

$$\int_0^1 \left(\frac{(1-s)^{d/2-1}-1}{s} + \frac{(1+s)^{d/2-1}-1}{s}\right)\mathrm{d}s = 2\ln c_d$$

for $\lambda \to 0$. For the second integral, we will now show that

$$\int_0^1 \left(\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right)\mathrm{d}s \to 0\,.$$

In fact, the integrand is bounded by

$$\begin{aligned}&\left|\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right| \\ &= \frac{\left|x_\pm(0)^2 - x_\pm(s)^2\right|}{\sqrt{s^2+x_\pm(s)^2}\sqrt{s^2+x_\pm(0)^2}\left(\sqrt{s^2+x_\pm(s)^2}+\sqrt{s^2+x_\pm(0)^2}\right)} \\ &\le \frac{C x_\pm(0)(s^\epsilon+\lambda)}{\sqrt{s^2+x_\pm(s)^2}\sqrt{s^2+x_\pm(0)^2}},\end{aligned}$$

using the Hölder continuity from (12.3.11). By continuity of $\hat{V}$ there exists some s_0 (independent of λ) such that for $s < s_0$ we have $x_\pm(s) \ge c x_\pm(0)$. We now split the integration into $\int_0^{s_0}$ and $\int_{s_0}^1$. For the first we have

$$\begin{aligned}\int_0^{s_0}\left|\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right|\mathrm{d}s &\le C\int_0^{s_0}\frac{x_\pm(0)}{s^2+x_\pm(0)^2}(s^\epsilon+\lambda)\mathrm{d}s \\ &= O(x_\pm(0)^\epsilon+\lambda)\,.\end{aligned}$$

For the second we have

$$\int_{s_0}^1\left|\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right|\mathrm{d}s \le C\int_{s_0}^1 x_\pm(0)\frac{s^\epsilon+\lambda}{s^2}\mathrm{d}s = O(x_\pm(0))\,.$$

Collecting all the estimates, we have thus shown that $m_\mu^{(d)}(\Delta)$ equals

$$\begin{aligned}&\mu^{d/2-1}\left(\ln c_d + \int_0^1 \frac{1}{\sqrt{s^2+\Delta(\sqrt{\mu})^2/\mu^2}}\mathrm{d}s + o(1)\right) \\ &= \mu^{d/2-1}\left(\ln c_d + \ln\left(\frac{\mu+\sqrt{\mu^2+\Delta(\sqrt{\mu})^2}}{|\Delta(\sqrt{\mu})|}\right) + o(1)\right) = \mu^{d/2-1}\ln\left(\frac{2\mu c_d}{|\Delta(\sqrt{\mu})|} + o(1)\right).\end{aligned}$$

This proves the third inequality in Proposition 12.3.2.

Combining this with the third step, one immediately sees that the gap function evaluated on the Fermi sphere vanishes exponentially fast, $\Delta(\sqrt{\mu}) \sim e^{1/\lambda e_\mu}$, as $\lambda \to 0$, recalling that $e_\mu^{(d)} < 0$ by assumption.

6. Second order. To obtain the next order, we recall that $T_\Delta^{(d)}$ has lowest eigenvalue -1 (see (12.3.9)), and hence, by first-order perturbation theory,

$$m_\mu^{(d)}(\Delta) = \frac{-1}{\lambda \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle - \lambda^2 \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V M_\Delta^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle + O(\lambda^3)}, \tag{12.3.12}$$

where $u(p) = |\mathbb{S}^{d-1}|^{-1/2}$ is the constant function on the sphere. Recall that u is the unique ground state of $\mathcal{V}_\mu^{(d)}$.

In the second order term we have that

$$\lim_{\lambda \to 0} \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V M_\Delta^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle = \left\langle u \middle| \mathcal{W}_\mu^{(d)} \middle| u \right\rangle ,$$

which follows from a simple dominated convergence argument as for T_c, noting that $\Delta(p) \to 0$ pointwise.

By again employing first–order perturbation theory, similarly to the last step in the proof of Proposition 12.3.1, we conclude the second equality in Proposition 12.3.2.

7. Comparing $\Delta(\sqrt{\mu})$ to Ξ. To prove the first equality in Proposition 12.3.2 we separately prove upper and lower bounds. The upper bound is immediate from

$$\Xi = \inf_{p \in \mathbf{R}^d} E_\Delta(p) = \inf_{p \in \mathbf{R}^d} \sqrt{|p^2 - \mu| + \Delta(p)^2} \leq \Delta(\sqrt{\mu}) .$$

Hence, for the lower bound, take $p \in \mathbf{R}^d$ with $\sqrt{|p^2 - \mu|} \leq \Xi \leq \Delta(\sqrt{\mu})$. Then by (12.3.11)

$$\begin{aligned} \Delta(p) \geq \Delta(\sqrt{\mu}) - |\Delta(p) - \Delta(\sqrt{\mu})| &\geq \Delta(\sqrt{\mu}) - C\Delta(\sqrt{\mu}) \left(||p| - \sqrt{\mu}|^\epsilon + \lambda \right) \\ &\geq \Delta(\sqrt{\mu})(1 + o(1)). \end{aligned}$$

In combination with the upper bound, we have thus shown that $\Xi = \Delta(\sqrt{\mu})(1 + o(1))$ as desired. This concludes the proof of Proposition 12.3.2. □

We conclude this subsection with several remarks, comparing our proof with those of similar results from the literature.

Remark 12.3.9 (Structure here vs. in earlier papers on Ξ)**.** *We now compare the proof above to the proofs of the three different limits in 3 dimensions [312, 403] and Chapter 11:*

- ***Weak coupling:*** *The structure of our proof here is very similar to that of [312]. Essentially, only the technical details in Lemma 12.3.7 and the calculation of $m_\mu^{(d)}(\Delta)$ in Step 5 are different.*
- ***High density:*** *For the high-density limit in Chapter 11, we needed some additional a priori bounds on Δ before we could employ the Birman-Schwinger argument. Apart from that, in Chapter 11 the comparison of $\Delta(\sqrt{\mu})$ and Ξ are done right after these a priori bounds. Additionally, since one starts with finding a priori bounds on Δ, one does not need the first-order analysis in Step 3. One may think of the structure in Chapter 11 as being ordered in the above steps as follows: 4, 7, 1, 2, 4 (again), 5, 6.*

- ***Low density:*** *For the low-density limit in [403] the structure is quite different. Again, one first needs some a priori bounds on Δ before one can use the Birman-Schwinger argument. One then improves these bounds on Δ using the Birman-Schwinger argument, which in turn can be used to get better bounds on the error term in the decomposition of the Birman–Schwinger operator. In this sense, the Steps 2–4 are too interwoven to be meaningfully separated. Also, Step 5 is done in two parts.*

12.3.3 Proof of Proposition 12.2.4

Note that $\mathcal{W}_\mu^{(d)} = \mathfrak{F}_\mu^{(d)} V M_0^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger$, where $M_0^{(d)}$ is defined in (12.3.4). By Lemma 12.3.5, $V^{1/2} M_0^{(d)} V^{1/2}$ is Hilbert-Schmidt. The integral kernel of $\mathcal{W}_\mu^{(d)}$ is bounded by

$$|\mathcal{W}_\mu^{(d)}(p,q)| \leq \frac{1}{(2\pi)^d} \int_{\mathbf{R}^{2d}} |V(x)||M_0^{(d)}(x,y)||V(y)| \mathrm{d}x\mathrm{d}y \leq \frac{1}{(2\pi)^d} \|V\|_1 \|V^{1/2} M_0^{(d)} V^{1/2}\|_{\mathsf{HS}}. \tag{12.3.13}$$

It follows that $\|\mathcal{W}_\mu^{(d)}\|_{\mathsf{HS}} \leq \frac{|\mathbb{S}^{d-1}|}{(2\pi)^d} \|V\|_1 \|V^{1/2} M_0^{(d)} V^{1/2}\|_{\mathsf{HS}}$. □

Chapter 13

Universal behavior of the BCS energy gap

This chapter contains the paper [337]:

J. Henheik and A. B. Lauritsen. Universal behavior of the BCS energy gap. *J. Spectr. Theory*, 2025. (online first)

Abstract. We consider the BCS energy gap $\Xi(T)$ (essentially given by $\Xi(T) \approx \Delta(T, \sqrt{\mu})$, the BCS order parameter) at all temperatures $0 \leq T \leq T_c$ up to the critical one, T_c, and show that, in the limit of weak coupling, the ratio $\Xi(T)/T_c$ is given by a universal function of the relative temperature T/T_c. On the one hand, this recovers a recent result by Langmann and Triola (Phys. Rev. B 108.10 (2023)) on three-dimensional s-wave superconductors for temperatures bounded uniformly away from T_c. On the other hand, our result lifts these restrictions, as we consider arbitrary spatial dimensions $d \in \{1, 2, 3\}$, discuss superconductors with non-zero angular momentum (primarily in two dimensions), and treat the perhaps physically most interesting (due to the occurrence of the superconducting phase transition) regime of temperatures close to T_c.

Content

13.1 Introduction

The Bardeen–Cooper–Schrieffer (BCS) [48] theory of superconductivity exhibits many interesting features. Notably it predicts, for s-wave superconductors (i.e. those for which the gap function has angular momentum $\ell = 0$, i.e. is radially symmetric), that the superconducting energy gap Ξ (essentially given by $\Xi \approx \Delta(\sqrt{\mu})$, the BCS order parameter) is proportional to the critical temperature T_c with a universal proportionality constant independent of the microscopic details of the electronic interactions, i.e. the specific superconductor. At zero temperature, the claimed universality is the (approximate) formula $\Xi/T_c \approx \pi e^{-\gamma} \approx 1.76$ with $\gamma \approx 0.57$ the Euler–Mascheroni constant, a property which is well-known in the physics literature [477, 48]. More recently, based on the variational formulation of BCS theory first introduced by Leggett [411] and later developed on mostly by Hainzl and Seiringer with others [309, 269, 313, 316], it has been put on rigorous grounds in various (physically quite different) limiting regimes [269, 312, 311, 403, 333, 336, 338] (see Section 13.1.2.1 for details). The general picture in all these works is that the universal behavior appears in a limit where "superconductivity is weak", meaning that T_c is small.

The predicted universality at *positive* temperature is notably less studied. It is expected that the ratio $\Xi(T)/T_c$ is given by some universal function of the relative temperature T/T_c [412, 399],

see Figure 13.1.1. For three-dimensional superconductors,[1] this has recently been shown in [399] (building on ideas of [400]) for temperatures uniformly in an interval $[0, (1-\varepsilon)T_c]$ for any $\varepsilon > 0$ in an appropriate limit where T_c is small. The perhaps most interesting regime of temperatures, however, are those close to the critical temperature, due to the phase transition occurring there. For such temperatures one expects[2] the behavior [561, Eq. (3.54)]

$$\Xi(T)/T_c \approx C_{\text{univ}}\sqrt{1-T/T_c}, \qquad C_{\text{univ}} = \sqrt{\frac{8\pi^2}{7\zeta(3)}} \approx 3.06. \tag{13.1.1}$$

Notably, the critical exponent $1/2$ (i.e. the order parameter $\Delta(\sqrt{\mu}) \approx \Xi$ vanishing as a square root) agrees with the prediction from the phenomenological Landau theory [390] for second order phase transitions (not to be confused with *Ginzburg*-Landau theory of superconductivity [282, 294, 201]) in mean-field systems.

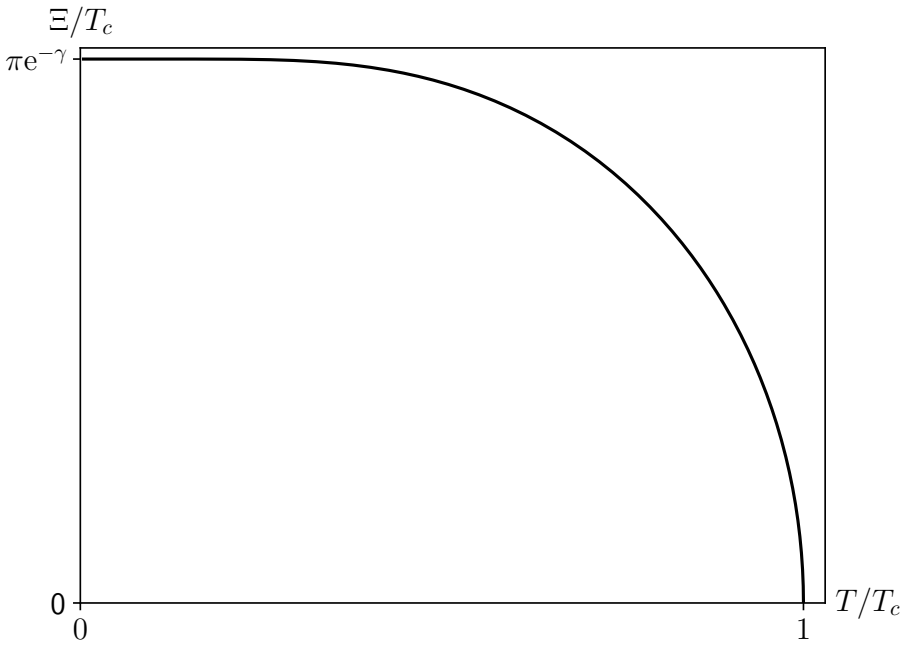

Figure 13.1.1: The ratio of the BCS energy gap and the critical temperature, Ξ/T_c, is well approximated by a *universal function* of the relative temperature T/T_c, which is given by $\mathsf{f}_{\mathsf{BCS}}(\sqrt{1-T/T_c})$ with $\mathsf{f}_{\mathsf{BCS}}$ defined in (13.2.9) below. At $T = 0$, it approaches the well-known constant $\pi e^{-\gamma} \approx 1.76$, with $\gamma \approx 0.57$ being the Euler-Mascheroni constant.

In this paper we extend the previously shown universality in three important directions: Firstly, we consider all spatial dimensions $d \in \{1, 2, 3\}$. Secondly, we treat the full range of temperatures $0 \leq T \leq T_c$. Thirdly, we extend the result to the case of non-zero angular momentum in two dimensions, in particular proving the formula in (13.1.1). Interestingly the case of non-zero angular momentum in two dimensions has the exact same universal behavior as s-wave superconductors in any dimensions: Independently of the angular momentum we find the same universal function describing the ratio $\Xi(T)/T_c$. This is substantially different from the three-dimensional case, where one still expects some sort of universal behavior to occur, only the universal function strongly depends on the angular momentum, see, e.g., [492] and Remark 13.2.15 below.

[1] In [399], only the three-dimensional case is considered explicitly. However, their arguments seem to be easily extendable to handle also the cases of one- and two-dimensional superconductors.

[2] Historically, the first article suggesting the square root behavior near T_c is by Buckingham [127]. In [48, Eq. (3.31)], BCS verified this in their original model, however, with the numerical constant given by 3.2 instead of $C_{\text{univ}} \approx 3.06$.

One of the central ideas in the analysis of temperatures close to the critical temperature is the use of Ginzburg-Landau (GL) theory. In the physics literature it is well-known that for temperatures close to the critical BCS theory is well-approximated by GL theory [294]. This correspondence has been studied, and put on rigorous grounds, quite recently in the mathematical physics literature [270, 272, 220, 219]. See Section 13.1.2.2 for more details.

13.1.1 Mathematical formulation of BCS theory

We consider a gas of fermions in $\mathbf{R}^d$ for $d = 1, 2, 3$ at temperature $T > 0$ and chemical potential $\mu > 0$. The interaction is described by a two-body, real-valued and reflection-symmetric potential $V \in L^1(\mathbf{R}^d)$, for which we assume the following.

Assumption 13.1.1. *We have that $V \in L^{p_V}(\mathbf{R}^d)$ for $p_V = 1$ if $d = 1$, $p_V \in (1, \infty)$ if $d = 2$, or $p_V = 3/2$ if $d = 3$.*

A BCS state Γ is given by a pair of functions (γ, α) and can be conveniently represented as a 2×2 matrix valued Fourier multiplier on $L^2(\mathbf{R}^d) \oplus L^2(\mathbf{R}^d)$ of the form

$$\hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1 - \hat{\gamma}(p) \end{pmatrix} \tag{13.1.2}$$

for all $p \in \mathbf{R}^d$. Here, $\hat{\gamma}(p)$ denotes the Fourier transform of the one particle density matrix and $\hat{\alpha}(p)$ is the Fourier transform of the Cooper pair wave function. We require reflection symmetry of $\hat{\alpha}$, i.e. $\hat{\alpha}(-p) = \hat{\alpha}(p)$, as well as $0 \leq \hat{\Gamma}(p) \leq 1$ as a matrix. Recall the definition of the BCS free energy functional [411, 309], which is given by

$$\mathcal{F}_T[\Gamma] := \int_{\mathbf{R}^d} (p^2 - \mu)\hat{\gamma}(p)\mathrm{d}p - TS[\Gamma] + \int_{\mathbf{R}^d} V(x)|\alpha(x)|^2 \mathrm{d}x\,, \tag{13.1.3}$$

where the entropy per unit volume is defined as

$$S[\Gamma] = -\int_{\mathbf{R}^d} \mathrm{Tr}_{\mathbf{C}^2}\left[\hat{\Gamma}(p)\log\hat{\Gamma}(p)\right]\mathrm{d}p\,.$$

The variational problem associated with the BCS functional is studied on

$$\mathcal{D} := \left\{\Gamma \text{ as in (13.1.2)} : 0 \leq \hat{\Gamma} \leq 1\,,\ \hat{\gamma} \in L^1(\mathbf{R}^d, (1+p^2)\mathrm{d}p)\,,\ \alpha \in H^1_{\mathrm{sym}}(\mathbf{R}^d)\right\}\,.$$

The following proposition provides the foundation for studying this problem.

Proposition 13.1.2 ([309], see also [316])**.** *Under Assumption 13.1.1 on V, the BCS free energy is bounded below on $\mathcal{D}$ and attains its minimum.*

However, in general, the minimizer is not necessarily unique. This potential non-uniqueness shall not bother us at this stage but will be of importance later on (see Sections 13.1.1.1 and 13.2.3). The Euler–Lagrange equation for α associated with the minimization problem is the celebrated *BCS gap equation*

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} \hat{V}(p-q)\frac{\Delta(q)}{K_T^\Delta(q)}\mathrm{d}q\,, \tag{13.1.4}$$

satisfied by $\Delta(p) = -2\,(2\pi)^{-d/2}(\hat{V} \star \hat{\alpha})(p)$, where α is the off–diagonal entry of a minimizing $\Gamma \in \mathcal{D}$ of (13.1.3), see [309, 316]. Here, $\hat{V}(p) = (2\pi)^{-d/2}\int_{\mathbf{R}^d} V(x)\,\mathbf{E}^{-\mathrm{i}px}\,\mathrm{d}x$ denotes the Fourier transform of V, and we have introduced the notation

$$K_T^\Delta(p) = \frac{E^\Delta(p)}{\tanh\left(\frac{E_\Delta(p)}{2T}\right)} \qquad \text{with} \qquad E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}\,.$$

The gap equation can equivalently be written as

$$(K_T^{\Delta} + V)\alpha = 0\,, \tag{13.1.5}$$

where $K_T^{\Delta}(p)$ is understood as a multiplication operator in momentum space and $V(x)$ is understood as a multiplication operator in position space. The Euler–Lagrange equation for γ (see [309, 316]) is given by

$$\hat{\gamma}(p) = \frac{1}{2} - \frac{p^2 - \mu}{2K_T^{\Delta}(p)}\,. \tag{13.1.6}$$

Remark 13.1.3 (Log-divergence)**.** *For $T = \Delta = 0$ we have $K_{T=0}^{\Delta=0}(p) = |p^2 - \mu|$. This gives rise to a logarithmic divergence in Equation* (13.1.4)*. Understanding how to treat this* $\log$*-divergence was one of the key insights of Langmann and Triola [399].*

13.1.1.1 Critical temperature and energy gap

The system described by $\mathcal{F}_T$ is *superconducting* if and only if any minimizer Γ of $\mathcal{F}_T$ has off–diagonal entry $\alpha = \Gamma_{12} \not\equiv 0$ (or, equivalently, (13.1.4) has a solution $\Delta \not\equiv 0$). The question, whether a system is superconducting or not can be reduced to a linear criterion involving the pseudo–differential operator with symbol

$$K_T(p) \equiv K_T^0(p) = \frac{p^2 - \mu}{\tanh\left(\frac{p^2-\mu}{2T}\right)}\,.$$

In fact, as shown in [309], the system is superconducting if and only if the operator $K_T + V$ has at least one negative eigenvalue. Moreover, there exists a unique *critical temperature* $T_c \geq 0$ being defined as

$$\boxed{T_c := \inf\{T > 0 : K_T + V \geq 0\}}\,, \tag{13.1.7}$$

for which $K_{T_c} + V \geq 0$ and $\inf \operatorname{spec}(K_T + V) < 0$ for all $T < T_c$. By Assumption 13.1.1 and the asymptotic behavior $K_{T_c}(p) \sim p^2$ for $|p| \to \infty$ the critical temperature is well-defined by Sobolev's inequality [418, Thm. 8.3]. Note that, for $T \geq T_c$ the BCS functional (13.1.3) is uniquely minimized by the normal state $\Gamma_{\mathrm{FD}} \equiv (\gamma_{\mathrm{FD}}, 0)$, where

$$\hat{\gamma}_{\mathrm{FD}}(p) = \frac{1}{1 + \mathbf{E}^{\frac{1}{T}(p^2-\mu)}} \tag{13.1.8}$$

is the usual Fermi-Dirac distribution. In contrast, for temperatures $0 \leq T < T_c$ strictly below the critical temperature, the normal state Γ_{FD} is not a minimizer of (13.1.3) and it is a priori not clear whether or not the minimizer of (13.1.3) is unique.

In this paper we deal with two different cases. In the case of s-wave superconductivity we will assume properties of V such that the minimizer is unique and in the case of 2-dimensional non-zero angular momentum we will assume properties of V such that there are at most 2 minimizers, see Section 13.2.3.

For the s-wave case we assume the following.

Assumption 13.1.4. *Let the (real valued) interaction potential $V \in L^1(\mathbf{R}^d)$ be* radially symmetric *and assume that V is of* negative type*, i.e. $\hat{V} \leq 0$ and $\hat{V}(0) < 0$.*

As shown in [312], Assumption 13.1.4 implies that, in particular, the critical temperature is non-zero, i.e. $T_c > 0$.[3] Moreover, as already indicated above, it ensures that the minimizer of (13.1.3) is unique. While this fact is already known at zero temperature [312, Lemma 2], we are not aware of any place in the literature where the extension to positive temperature is given. As we will need this extension, we formulate it in the following proposition and give a proof in Section 13.A.1.

[3] To be precise, the arguments in [312] cover only the case $d = 3$, but, as already noted in [270], they are immediately transferable to the cases $d = 1, 2$.

Proposition 13.1.5 (Uniqueness of minimizers for potentials of negative type)**.** *Let V satisfy Assumptions 13.1.1 and 13.1.4, and consider the BCS functional* (13.1.3)*. Then we have the following:*

(i) For $0 \leq T < T_c$, let $\Gamma \equiv (\gamma, \alpha)$ be a minimizer of the BCS functional (13.1.3) *(which exists by means of Proposition 13.1.2). Then the operator $K_T^{\Delta} + V$ from* (13.1.5) *is non–negative and α is its unique ground state with eigenvalue zero 0.*

(ii) The minimizer $\Gamma =: \Gamma_* \equiv (\gamma_*, \alpha_*)$ of (13.1.3) *is unique up to a phase of α_* and can be chosen to have strictly positive Fourier transform $\hat{\alpha}_*$. Moreover, both γ_* and α_* are radial functions.*

In particular, under Assumption 13.1.4, we have that the *energy gap*

$$\boxed{\Xi(T) := \inf_{p \in \mathbf{R}^d} \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2}}, \tag{13.1.9}$$

for Δ being the (up to multiplication by a constant phase) unique non-zero solution of (13.1.4) and temperatures $0 \leq T < T_c$, is well-defined.

In case there is more than one solution Δ of the BCS gap equation (13.1.4) (i.e. more than one minimizer of the BCS functional) we may for each such Δ define the energy gap Ξ as in Equation (13.1.9). In the case of two dimensions with (definite) non-zero angular momentum we shall prove that there exist exactly two (up to multiplication of either by a constant phase) such functions, $\Delta_{\pm}$. They however satisfy $|\Delta_+| = |\Delta_-|$ and so the energy gap Ξ is also here uniquely defined. For the details see Section 13.2.3.

Remark 13.1.6. *The energy gap is essentially the same as the order parameter $\left|\Delta(\sqrt{\mu})\right|$ as we show in Equations* (13.3.18) *and* (13.3.29) *below. In particular, one may replace Ξ with $\left|\Delta(\sqrt{\mu})\right|$ in our main results, Proposition 13.2.1 and Theorems 13.2.4 and 13.2.11.*

13.1.1.2 Weak coupling

We consider here the weak–coupling limit where the interaction is of the form λV for a $\lambda > 0$ and we consider the limit $\lambda \to 0$. In the weak–coupling limit an important role is played by the (rescaled) operator $\mathcal{V}_\mu : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ [312, 338, 186, 314]. This operator, which is defined as

$$(\mathcal{V}_\mu u)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(\sqrt{\mu}(p-q)) u(q) \, \mathrm{d}\omega(q) \,, \tag{13.1.10}$$

where $\mathrm{d}\omega$ denotes the uniform (Lebesgue) measure on the unit sphere $\mathbb{S}^{d-1}$, measures the strength of the interaction potential $\hat{V}$ on the Fermi surface. The pointwise evaluation of $\hat{V}$ (and in particular on a $\mathrm{codim}{-}1$ submanifold) is well-defined since we assume that $V \in L^1(\mathbf{R}^d)$.

The lowest eigenvalue $e_\mu = \inf \mathrm{spec}\, \mathcal{V}_\mu$ is of particular importance. Note, that $\mathcal{V}_\mu$ is a trace-class operator (see the argument above [269, Equation (3.2)]) with

$$\mathrm{tr}(\mathcal{V}_\mu) = \frac{|\mathbb{S}^{d-1}|}{(2\pi)^d} \int_{\mathbf{R}^d} V(x) \mathrm{d}x = \frac{|\mathbb{S}^{d-1}|}{(2\pi)^{d/2}} \hat{V}(0) \,.$$

For radial potentials one sees that the eigenfunctions of $\mathcal{V}_\mu$ are the spherical harmonics.

For potentials of negative type we have $\hat{V}(0) < 0$ and so $e_\mu < 0$. This corresponds to an attractive interaction between (some) electrons on the Fermi sphere. Further, one easily sees that the constant function $u(p) = (|\mathbb{S}^{d-1}|)^{-1/2}$ is an eigenfunction of $\mathcal{V}_\mu$, which, since $\hat{V} \leq 0$ by Assumption 13.1.4, is in fact the ground state by the Perron–Frobenius theorem, i.e.

$$e_\mu = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(\sqrt{\mu} - q\sqrt{\mu}) \, \mathrm{d}\omega(q) \,. \tag{13.1.11}$$

In two dimensions the spherical harmonics take the form $u_{\pm\ell}(p) = (2\pi)^{-1/2} e^{\pm i\ell\varphi}$ with φ denoting the angle of $p \in \mathbf{R}^2$ in polar coordinates. In this case the ground state space of $\mathcal{V}_\mu$ is spanned by $\{u_{\pm\ell}\}_{\ell\in\mathcal{L}}$ for some set of angular momenta $\mathcal{L}$. If $\ell_0 \neq 0$ for some $\ell_0 \in \mathcal{L}$ then the ground state is at least twice degenerate, since then both $u_{\pm\ell_0}$ are eigenfunctions with this lowest eigenvalue.

13.1.2 Previous mathematical results

So far, all mathematical results on solutions of the BCS gap equation (13.1.4) focused either on zero temperature, $T = 0$, or the regime close to the critical one, $T \approx T_c$, where the transition from superconducting to normal behavior is described by Ginzburg-Landau theory.

13.1.2.1 BCS theory in limiting regimes: Universality at $T = 0$

At zero temperature it is expected, that the ratio of the energy gap and the critical temperature is given by a universal constant,

$$\frac{\Xi(T=0)}{T_c} \approx \pi e^{-\gamma}, \tag{13.1.12}$$

with $\gamma \approx 0.577$ the Euler–Mascheroni constant in a limiting regime where "superconductivity is weak", meaning that T_c is small.

In the literature three such limits have been studied: Historically, the first regime, which has been considered is the *weak coupling limit* in three spatial dimensions [312, 269], which we recently extended to one and two dimensions in Chapter 12. The critical temperature in the *low density limit* in three dimensions was studied in [311] and later complemented by a study of the energy gap by one of us in [403], thus, in combination, yielding the above-mentioned universal behavior. Finally, we considered the *high density limit*, again in three dimension, in Chapters 10–11 and proved (13.1.12) in this regime.

13.1.2.2 Superconductors close to T_c: Ginzburg-Landau theory

For temperatures close to the critical BCS theory is well-approximated by Ginzburg-Landau (GL) theory. In contrast to the microscopic BCS model, GL theory is a phenomenological model, which describes the superconductor on a macroscopic scale. Moreover, as suggested by Equation (13.1.1) a natural parameter measuring "closeness to T_c" is the parameter $h = \sqrt{1 - T/T_c}$. A rigorous analysis of various aspects of BCS theory in the limit $h \to 0$ was then studied in [270, 272, 271], very recently also allowing for general external fields [220, 219]. Of particular interest to us is the fact that any minimizer of the BCS functional (γ, α) has $\alpha \approx h\psi\mathfrak{a}_0$ with $\mathfrak{a}_0 \in \ker(K_{T_c} + \lambda V)$ fixed and $\psi \in \mathbf{C}$ a minimizer of the corresponding GL functional, see [272, Theorem 2.10].

13.1.3 Outline of the paper

The rest of this paper is structured as follows. In Section 13.2 we present our main result, starting with the prototypical universality in the original BCS model (Section 13.2.1). Afterwards, in Sections 13.2.2 and 13.2.3 we describe our results on universality for s-wave superconductors in arbitrary dimension $d \in \{1, 2, 3\}$, and for two-dimensional superconductors having pure angular momentum, respectively. The proofs of these results are given in Section 13.3, while several additional proofs are deferred to Appendix 13.A.

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13.2 Main Result

We next describe the main results of the paper. We first consider the example of an interaction as considered by BCS [48]. The reason for doing this is twofold:

1. It highlights, how the universal function $\Xi(h)/T_c \approx \mathsf{f}_{\mathsf{BCS}}(h)$ appears.
2. A central idea in the proof of removing the $\log$-divergence is already present in the BCS gap equation (13.1.4). (Recall Remark 13.1.3.)

13.2.1 Energy gap in the original BCS approximation [48]

In their seminal work [48], Bardeen–Cooper–Schrieffer modeled the interaction by a so called *separable potential* $V(x,y)$ (i.e. factorizing and depending not only on the relative coordinate $x-y$), whose Fourier transform $\hat{V}(p,q)$ is a product of two *radial single variable* functions, that are compactly supported in the shell

$$\mathcal{S}_\mu(T_D) := \{p \in \mathbf{R}^d : |p^2 - \mu| \le T_D\} \tag{13.2.1}$$

around the Fermi surface $\{p \in \mathbf{R}^d : p^2 = \mu\}$, the only (material dependent) parameter being the so-called *Debye temperature* $0 < T_D < \mu$. Switching from momentum p to energy $\epsilon = p^2 - \mu$, the just mentioned single variable functions are chosen in such a way, that[4]

$$\hat{V}(\epsilon,\epsilon')N(\epsilon') = -\lambda_{\mathrm{BCS}}\,\theta(1-|\epsilon/T_D|)\theta(1-|\epsilon'/T_D|)\,, \qquad \lambda_{\mathrm{BCS}} > 0\,, \tag{13.2.2}$$

where the electronic density of states (DOS) is denoted by $N(\epsilon) \sim (\epsilon+\mu)^{(d-2)/2}$ and θ is the Heaviside function. ($\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ otherwise.)

In this case, the (unique non-negative) solution to the BCS gap equation (13.1.4) is given by

$$\Delta(\epsilon) = \Delta \cdot \theta(1-|\epsilon/T_D|) \tag{13.2.3}$$

for some temperature dependent constant $\Delta \ge 0$, which is determined by the scalar gap equation (cf. [48, Eq. (3.27)])

$$\frac{1}{\lambda_{\mathrm{BCS}}} = \int_0^{T_D} \frac{\tanh\left(\frac{\sqrt{\epsilon^2+\Delta^2}}{2T}\right)}{\sqrt{\epsilon^2+\Delta^2}}\mathrm{d}\epsilon \tag{13.2.4}$$

for any temperature $0 \le T < T_c$. In turn, the critical temperature $T_c > 0$ is determined by (13.2.4) with $\Delta = 0$, i.e.

$$\frac{1}{\lambda_{\mathrm{BCS}}} = \int_0^{T_D} \frac{\tanh\left(\frac{\epsilon}{2T_c}\right)}{\epsilon}\mathrm{d}\epsilon\,. \tag{13.2.5}$$

In case of a small BCS coupling parameter, $\lambda_{\mathrm{BCS}} \ll 1$,[5] it holds that T_c is exponentially small in λ_{BCS}, i.e. $T_c \sim \mathrm{e}^{-1/\lambda_{\mathrm{BCS}}}$ (see [48, Eq. (3.29)]). Moreover, it is easily checked that Δ as a function of temperature is monotonically decreasing in the interval $[0,T_c]$ and satisfies $\Delta(T=0) \sim \mathrm{e}^{-1/\lambda_{\mathrm{BCS}}}$, similarly to the critical temperature.

[4] Assuming that $\hat{V}$ is constant throughout the energy shell (13.2.1) (as done in [48]), the BCS coupling parameter emerges as $\lambda_{\mathrm{BCS}} = -\hat{V}(0,0)N(0)$.

[5] This can happen for various reasons. One example is that V itself is scaled by a coupling parameter $\lambda > 0$, i.e. $V \to \lambda V$, and one considers the limit $\lambda \to 0$, as done in Sections 13.2.2–13.2.3.

Next, changing variables as $x := \epsilon/T_c$ and setting $\delta := \Delta/T_c$ as well as[6]

$$h := \sqrt{1 - \frac{T}{T_c}} \quad \text{for} \quad 0 \leq T \leq T_c\,, \tag{13.2.6}$$

we can subtract (13.2.4) and (13.2.5) to find

$$\int_0^{T_D/T_c} \left\{ \frac{\tanh\left(\frac{\sqrt{x^2+\delta^2}}{2(1-h^2)}\right)}{\sqrt{x^2+\delta^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{x} \right\} \mathrm{d}x = 0\,. \tag{13.2.7}$$

Note that this difference formula (13.2.7) removes the divergences of (13.2.4)–(13.2.5) as $\lambda_{\mathrm{BCS}} \to 0$.

The proof of the following proposition is given in Section 13.3.1. (In the statement of Proposition 13.2.1, one may replace Ξ by the order parameter $\Delta(\sqrt{\mu})$, see Remark 13.1.6 above.)

Proposition 13.2.1 (Energy gap in the original BCS model [48])**.** *Let $\mu > 0$, fix a Debye temperature $0 < T_D < \mu$ and let $\lambda_{\mathrm{BCS}} > 0$ be the BCS coupling parameter as above. Let the critical temperature T_c and the gap function $\Delta(p)$ be defined via* (13.2.3)–(13.2.5)*.*

Then the energy gap Ξ (defined in (13.1.9)*) as a function of $h = \sqrt{1 - T/T_c}$ for $0 \leq T \leq T_c$ (recall* (13.2.6)*) is given by*

$$\Xi(h) = T_c\, \mathsf{f}_{\mathsf{BCS}}(h) \left(1 + O(\mathrm{e}^{-1/\lambda_{\mathrm{BCS}}})\right) \tag{13.2.8}$$

uniformly in $h \in [0,1]$, where the function $\mathsf{f}_{\mathsf{BCS}} : [0,1] \to [0,\infty)$ is implicitly defined via

$$\int_{\mathsf{R}} \left\{ \frac{\tanh \frac{\sqrt{s^2+\mathsf{f}_{\mathsf{BCS}}(h)^2}}{2(1-h^2)}}{\sqrt{s^2+\mathsf{f}_{\mathsf{BCS}}(h)^2}} - \frac{\tanh \frac{s}{2}}{s} \right\} \mathrm{d}s = 0 \tag{13.2.9}$$

and plotted in Figure 13.2.1.

This means that, independent of the material dependent Debye temperature $T_D > 0$ and the chemical potential $\mu > 0$, the energy gap Ξ within the original BCS approximation [48], follows a universal curve, described by (13.2.8), in the limit of weak BCS coupling. A similar formula for $\mathsf{f}_{\mathsf{BCS}}$ like (13.2.9) (but as a function of $x := 1 - h^2$) also appeared in the monograph of Leggett [412, Eq. (5.5.21)]. We now list a few basic properties of $\mathsf{f}_{\mathsf{BCS}}$, whose proofs we omit, as they can be obtained by means of the implicit function theorem and further elementary tools (see also [399, Lemma 1] as well as Lemmas 13.3.1 and 13.3.13 below). Almost all of these properties become apparent from Figure 13.2.1.

Lemma 13.2.2 (Properties of $\mathsf{f}_{\mathsf{BCS}}$)**.** *There exists a unique implicitly defined solution function $\mathsf{f}_{\mathsf{BCS}} : [0,1] \to [0,\infty)$ of* (13.2.9)*. Moreover, $\mathsf{f}_{\mathsf{BCS}}$ has the following properties:*

(i) It is strictly monotonically increasing in $[0,1]$.

(ii) It is C^1 in $(0,1)$ and has continuous one-sided derivatives at the boundaries 0 and 1.

(iii) It has the boundary values $\mathsf{f}_{\mathsf{BCS}}(0) = 0$, $\mathsf{f}'_{\mathsf{BCS}}(0) = C_{\mathrm{univ}}$ and $\mathsf{f}_{\mathsf{BCS}}(1) = \pi \mathrm{e}^{-\gamma} \approx 1.76$, $\mathsf{f}'_{\mathsf{BCS}}(1) = 0$. Here, $\gamma \approx 0.57$ is the Euler-Mascheroni constant and

$$C_{\mathrm{univ}} := \sqrt{\frac{8\pi^2}{7\zeta(3)}} \approx 3.06\,, \tag{13.2.10}$$

where $\zeta(s)$ denotes Riemann's ζ-function.

[6] As mentioned above, the parameter h is commonly used (see, e.g., [270, 272]) in the context of Ginzburg-Landau theory, where it served as a 'semiclassical' small parameter in the derivation this theory.

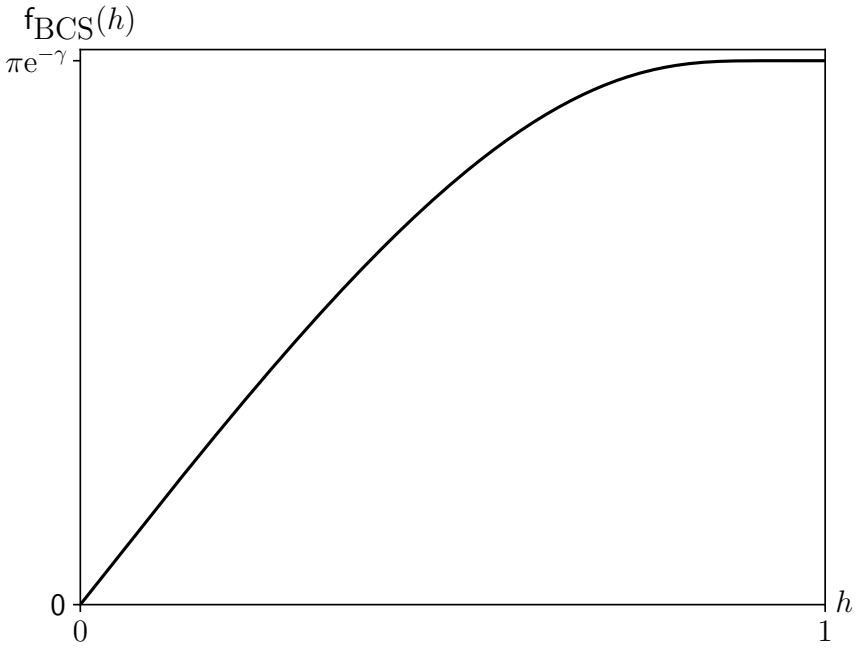

Figure 13.2.1: Sketch of the function $\mathsf{f}_{\mathrm{BCS}}$ obtained via the implicit relation (13.2.9).

Remark 13.2.3 (Contact interactions)**.** *Our proof of Proposition 13.2.1 can easily be generalized to all BCS models, in which the energy gap is constant (at least near the Fermi surface).*

(a) In case of a delta potential, $V(x) = -\delta(x)$ in one spatial dimension, $d = 1$, the gap function solving (13.1.4) *is given by a constant (simply because here $\hat{V}$ is constant). This setting can be analyzed similarly (in a weak coupling limit, i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$) as done in Proposition 13.2.1 for the original BCS model [48].*

(b) Also for contact interactions in three spatial dimensions, $d = 3$, the situation is similar. This setting is studied in [125, 126], where it is shown that for a suitable sequence of potentials V_ℓ converging to a point interaction with scattering length $a < 0$, the gap function Δ_ℓ converges (uniformly on compact sets, see [125, Eq. (14)]) to a constant Δ solving the gap equation

$$-\frac{1}{4\pi a} = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \left(\frac{1}{K_T^\Delta(p)} - \frac{1}{p^2} \right) \mathrm{d}p \,.$$

Replacing the limit of weak coupling by a small scattering length limit, $a \to 0$, one can obtain a result similar to Proposition 13.2.1.

13.2.2 Universal behavior of the s-wave BCS energy gap

After having discussed the prototypical universality in the seminal BCS paper [48], we can now formulate our main result on general s-wave superconductors with local interactions. The proof of Theorem 13.2.4 is presented in Sections 13.3.2–13.3.4, while the main ideas are briefly described in Remark 13.2.7 below. (We remark that, in the statement of Theorem 13.2.4, one may replace Ξ by the order parameter $\Delta(\sqrt{\mu})$, see Remark 13.1.6 above.)

Theorem 13.2.4 (BCS energy gap for s-wave superconductors)**.** *Let $d \in \{1, 2, 3\}$, $\mu > 0$, $\lambda > 0$ and let V satisfy Assumptions 13.1.1 and 13.1.4. Let T_c and Ξ be as in* (13.1.7) *and* (13.1.9) *with interaction λV and Δ the unique non-zero solution the BCS gap equation* (13.1.4) *with interaction λV.*

Then, with $\mathsf{f}_{\mathsf{BCS}}(h), h = \sqrt{1 - T/T_c}$ *being the function defined via* (13.2.9)*, we have the following:*

(a) Assuming additionally that $|\cdot|^2 V \in L^1(\mathbf{R}^d)$*, it holds that*

$$\Xi(h) = T_c \, \mathsf{f}_{\mathsf{BCS}}(h) \left(1 + O(h^{-1} \mathrm{e}^{-c/\lambda})\right) \tag{13.2.11}$$

for some constant $c > 0$ *independent of* λ *and* h*.*

(b) Assuming additionally that $(1 + |\cdot|)V \in L^2(\mathbf{R}^d)$*, it holds that*

$$\Xi(h) = T_c \, \mathsf{f}_{\mathsf{BCS}}(h) \left(1 + O(h \mathrm{e}^{c'/\lambda}) + o_{\lambda \to 0}(1)\right) \tag{13.2.12}$$

for some constant $c' > 0$ *independent of* λ *and* h *and where* $o_{\lambda \to 0}(1)$ *vanishes as* $\lambda \to 0$ *uniformly in* h*.*

For the special case $h = 1$, i.e. $T = 0$, (13.2.11) reproduces the results from [312] (for $d = 3$) and Chapter 12 (for $d = 1, 2$), which state the universality

$$\lim_{\lambda \to 0} \frac{\Xi(T = 0)}{T_c} = \frac{\pi}{\mathrm{e}^{\gamma}}$$

at $T = 0$. Moreover, by (13.2.11) again, we find that, uniformly in temperatures bounded away from T_c, i.e. $h \in [\varepsilon, 1]$ for some fixed $\varepsilon > 0$,

$$\lim_{\lambda \to 0} \frac{\Xi(h)}{T_c} = \mathsf{f}_{\mathsf{BCS}}(h) \,,$$

recovering the universality result in [399] (for $d = 3$), with an exponential speed $O(\mathrm{e}^{-c/\lambda})$ of convergence. In the complementary case, for temperatures very close to the critical temperature, $T \approx T_c$, the question of universality is (i) physically more interesting due to the phase transition from superconducting to normal behavior and (ii) mathematically more delicate than in the previous scenarios. This is because now there are *two* small parameters λ and h, instead of λ only, and the error term in (13.2.11) might actually be large compared to one. However, now involving both, (13.2.11) *and* (13.2.12), we find that

$$\lim_{\substack{\lambda, h \to 0 \\ \mathrm{e}^{-c/\lambda} \ll h}} \frac{\Xi(h)}{T_c \, h} = C_{\mathrm{univ}} \quad \text{and} \quad \lim_{\substack{\lambda, h \to 0 \\ h \ll \mathrm{e}^{-c'/\lambda}}} \frac{\Xi(h)}{T_c \, h} = C_{\mathrm{univ}} \tag{13.2.13}$$

with the aid of Lemma 13.2.2 (iii). In particular, the ratio $\Xi(h)/(T_c h)$ converges to the *same* universal constant C_{univ} (recall (13.2.10)) in both orders of limits, $\lim_{\lambda \to 0} \lim_{h \to 0}$ and $\lim_{h \to 0} \lim_{\lambda \to 0}$.

Remark 13.2.5 (Joint limit)**.** *A careful inspection of the proof reveals that the constants* c, c' *satisfy* $c < c'$*. In particular, the proof does* not *allow the two regimes considered in* (13.2.13) *to be overlapping and we cannot prove that* $\lim_{\lambda, h \to 0} \frac{\Xi(h)}{T_c h} = C_{\mathrm{univ}}$ *in any joint limit. We expect this to hold in any joint limit, however, as we saw for the particular example from [48] in Proposition 13.2.1*

Remark 13.2.6 (Comparison of assumptions with [399])**.** *Compared to the similar result in [399] our assumptions hold for a slightly different class of potentials. The assumptions of [399] are essentially on the smoothness of the interaction* V *(formulated via some regularity/decay assumption on the Fourier transform* $\hat{V}$*). Our assumptions on the other hand are on the regularity/decay of* V*. In particular, our assumptions cover the examples of [399, Table I] which are* not *covered by the assumptions of [399]. These are (in three dimensions)*

$$V_{\mathsf{Yukawa}}(x) = \frac{e^{-|x|}}{4\pi|x|}, \qquad V_{a\mathsf{Y}+b\mathsf{E}}(x) = \frac{(2a + b|x|)e^{-|x|}}{8\pi|x|}, \qquad V_{x\text{-box}}(x) = \frac{3\theta(1 - |x|)}{4\pi} \,.$$

Remark 13.2.7 (On the proof)**.** *The main ideas in the proof of Theorem 13.2.4 are the following.*

(a) For part (a), we crucially use that both, $K_T^\Delta + \lambda V$ and $K_{T_c} + \lambda V$, have lowest eigenvalue zero. We then consider their corresponding Birman-Schwinger (BS) operators and use that, for λ small enough, two naturally associated operators on the Fermi sphere both have the same *ground state. Evaluating the difference of these two associated operators in this common ground state, we find that a* difference *of two logarithmically divergent integrals, similarly to* (13.2.9)*, vanishes up to exponentially small errors $O(\mathrm{e}^{-c/\lambda})$.*

The removal of the $\log$-divergence in this way (which – in a similar fashion – was the major insight in [399]) is the key idea to (i) access also non-zero temperatures and (ii) obtain extremely precise error estimates (compared to all the previous results mentioned in Section 13.1.2.1).

(b) For part (b), we employ Ginzburg-Landau (GL). The principal realm of GL theory is to describe superconductors and superfluids close to their critical temperature T_c. In this regime, when superconductivity is weak, the main idea is that the prime competitor for developing a small off-diagonal component $\hat{\alpha}$ for a BCS minimizer, is the normal state $\Gamma_{\mathrm{FD}} = (\gamma_{\mathrm{FD}}, 0)$*, with γ_{FD} given by the usual Fermi-Dirac distribution (recall* (13.1.8)*). Moreover, to leading order, the off-diagonal component $\hat{\alpha}$ lies in the kernel (which agrees with the ground state space) of the operator $K_{T_c} + \lambda V$.*

The main input, which we use, is that every minimizer (γ, α) of the BCS functional has $\alpha \approx h\psi\mathfrak{a}_0$ with $\mathfrak{a}_0 \in \ker(K_{T_c} + \lambda V)$ fixed and $\psi \in \mathbf{C}$ minimizing the corresponding GL functional [272, Theorem 2.10]. Taking the convolution of $\hat{\mathfrak{a}}_0$ with $\hat{V}$, we find the universal constant (13.2.10) *appearing in $\Xi/(T_c h) \approx C_{\mathrm{univ}}$.*

Moreover, the "additional assumptions" in Theorem 13.2.4 are not quite rigid, meaning that they can be weakened in the following sense.

(a) In case that $|\cdot|^{2\alpha} V \in L^1$ for a $0 < \alpha \leq 1$ the error term in Equation (13.2.11) *should instead be $O(h^{-1} e^{-c\alpha/\lambda})$ with the constant c then being independent also of α.*

(b) In case that $(1 + |\cdot|)V \in L^p(\mathbf{R}^d)$ for $p < 2$, the factor h in the first error term in (13.2.12) *would not appear raised to the first power but with exponent*

$$(3p-4)/p \quad \text{for} \quad d = 1\,, \quad (3p-4)/p - \epsilon \quad \text{for} \quad d = 2\,, \quad \text{and} \quad (4p-6)/p \quad \text{for} \quad d = 3\,.$$

Remark 13.2.8 (Other limits)**.** *Although, in this paper, we considered only the weak coupling limit, we expect the relation $\Xi(h) \approx T_c \mathsf{f}_{\mathrm{BCS}}(h)$ to hold also in other limiting regimes in which "superconductivity is weak", that is, e.g., the low-*[7] *and high-density limit, that were studied in [311, 403] and Chapters 10–11, respectively. This idea is already contained in [399], where the authors considered a "universal" parameter λ in [399, Eq. (7)], which can be small for various physical situations.*

Remark 13.2.9 (Non-universality)**.** *We recover also the formula [399, Equation (16)]*

$$\frac{\Delta(p)}{\Delta(\sqrt{\mu})} = F(p) + O(e^{-c/\lambda}) \tag{13.2.14}$$

for some function F not depending on the temperature and some constant $c > 0$. The function F depends on the interaction V however. For this reason (13.2.14) *is called a "non-universal" feature in [399]. The proof of* (13.2.14) *is given in Section 13.3.3.*

[7] For dimensions $d = 1, 2$, the same caveats mentioned in Remark 12.2.8 apply.

13.2.3 The case of pure angular momentum for $d = 2$

In this section, we generalize Theorem 13.2.4 from s-wave superconductors to two-dimensional systems which have a definite (or *pure*) angular momentum $\ell_0 \in \mathbf{N}_0$, which can differ from 0.

Assumption 13.2.10 (Pure angular momentum)**.** *Let $V \in L^1(\mathbf{R}^2)$ be radially symmetric and* attractive on the Fermi sphere*, i.e. the lowest eigenvalue e_μ of $\mathcal{V}_\mu$ is strictly negative (recall* (13.1.10)–(13.1.11)*). Moreover, suppose that for all $\lambda > 0$ small enough the lowest eigenvalue of $K_{T_c} + \lambda V$ is at most twice degenerate, i.e. $\dim\ker(K_{T_c} + \lambda V) \in \{1, 2\}$.*

Since K_{T_c} commutes with the Laplacian, Assumption 13.2.10 ensures the ground state of $K_{T_c} + \lambda V$ to have definite angular momentum. More precisely, it holds that

$$\ker(K_{T_c} + \lambda V) = \operatorname{span}\{\rho\} \otimes \mathcal{S}_{\ell_0} \quad \text{with} \quad \mathcal{S}_\ell := \operatorname{span}\left\{\mathrm{e}^{\pm \mathrm{i}\ell\varphi}\right\} \subset L^2(\mathbb{S}^1) \quad \text{for some} \quad \ell_0 \in \mathbf{N}_0\,, \tag{13.2.15}$$

where $\rho \in L^2((0,\infty); r\mathrm{d}r)$ is a (λ-dependent) radial function.[8]

We can now formulate our main result in the case of pure angular momentum for $d = 2$. (We again remark that, in the statement of Theorem 13.2.11, one may replace Ξ by the order parameter $|\Delta(\sqrt{\mu})|$, see Remark 13.1.6 above.)

Theorem 13.2.11 (BCS energy gap for $2d$ pure angular momentum)**.** *Let $d = 2$, $\mu > 0$ and let V satisfy Assumptions 13.1.1 and 13.2.10. Define the critical temperature T_c and energy gap Ξ as in* (13.1.7) *and* (13.1.9) *with interaction λV for a $\lambda > 0$ and Δ being* any (arbitrary!) *non-zero solution the BCS gap equation* (13.1.4) *with interaction λV.*

Then, with $\mathsf{f}_{\mathsf{BCS}}(h), h = \sqrt{1 - T/T_c}$ being the function defined via (13.2.9)*, we have the following:*

(a) Assume additionally that $V \in L^2(\mathbf{R}^2)$, $\hat{V} \in L^r(\mathbf{R}^2)$ for some $1 \le r < 2$ and that $|\cdot|^2 V \in L^1(\mathbf{R}^d)$. Then there exists $0 \le \tilde{T} < T_c$ with $\tilde{T}/T_c \le \mathrm{e}^{-\mathfrak{c}/\lambda}$ for some $\mathfrak{c} > 0$, such that for all temperatures $T \in (\tilde{T}, T_c)$ it holds that

$$\Xi(h) = T_c\, \mathsf{f}_{\mathsf{BCS}}(h) \left(1 + O(h^{-1}\mathrm{e}^{-c/\lambda})\right) \tag{13.2.16}$$

for some constant $c > 0$ independent of λ and h.

(b) Assuming additionally that $(1 + |\cdot|)V \in L^2(\mathbf{R}^d)$, it holds that

$$\Xi(h) = T_c\, \mathsf{f}_{\mathsf{BCS}}(h) \left(1 + O(h\mathrm{e}^{c'/\lambda}) + o_{\lambda\to 0}(1)\right) \tag{13.2.17}$$

for some constant $c' > 0$ independent of λ and h and where $o_{\lambda\to 0}(1)$ vanishes as $\lambda \to 0$ uniformly in h.

The proof of Theorem 13.2.11 is given in Section 13.3.5.

Remark 13.2.12 (On the assumptions)**.** *The additional assumptions in part (a) here compared to Theorem 13.2.4 (namely $V \in L^2$ and $\hat{V} \in L^r$) are those of [218, Theorem 2.1]. The proof of Equation* (13.2.16) *centrally uses this result. As discussed in [218, Remark 2.3] these additional assumptions are expected to be of a technical nature.*

[8] In fact, the angular momentum of the kernel of $K_{T_c} + \lambda V$ must be *even*, i.e. $\ell_0 \in 2\mathbf{N}_0$. This is because BCS theory is formulated for reflection symmetric α, whence $K_{T_c} + \lambda V$ is naturally defined on the space of reflection symmetric functions only.

Remark 13.2.13 (The temperature $\tilde{T}$)**.** *The presence of the temperature $\tilde{T}$ in Theorem 13.2.11 (a) arises from the first excited eigenvalue of $K_{T_c} + \lambda V$, see [218, Remark 2.2]. As discussed in the proof, the temperatures $T_c, \tilde{T}$ are given by $T_c = T_c(\ell_0)$ and $\tilde{T} = T_c(\ell_1)$, the critical temperatures restricted to angular momenta ℓ_0 and ℓ_1, for some angular momenta $\ell_0 \neq \ell_1$, see also [218, Remark 2.2]. For temperatures $T \in (\tilde{T}, T_c)$ the BCS minimizer(s) then have angular momentum ℓ_0 [218, Theorem 2.1]. For temperatures $T < \tilde{T}$ however, we do not in general know whether the BCS minimizer(s) have angular momentum ℓ_0. The proof crucially uses that the minimizer(s) have a definite angular momentum. If we however know a priori, that the BCS minimizer(s) have angular momentum ℓ_0 for some larger ranger of temperatures (T_1, T_c), then the formula in Equation* (13.2.16) *holds in this larger range of temperatures.*

Remark 13.2.14 (Nodes of the gap function)**.** *As already mentioned in Section 13.1.1.1, we establish during the proof, that any solution Δ of the BCS gap equation* (13.1.4) *has a radially symmetric absolute value, $|\Delta(p)|$, which is, moreover, independent of the particular solution Δ. In particular, every solution Δ of the BCS gap equation* (13.1.4) *does* not *have nodes on the Fermi surface. This contrasts many examples of d-wave superconductors in the physics literature, where a (necessarily) non-radial interaction V leads to a gap function Δ with nodes on the Fermi surface, see, e.g., [103, 429, 261, 601].*

Remark 13.2.15 (Non-extension to three dimensions)**.** *The formula $\Xi(h) \approx T_c \mathsf{f}_{\mathrm{BCS}}(h)$ is* not *expected to hold in three dimensions for non-zero angular momentum, see for instance [492, Figure 14.6]. More precisely, we have the following:*

(i) For non-zero angular momentum in three dimensions, our method of proving Theorem 13.2.11 (a) breaks down. In fact, we crucially use that $K_T^{\Delta} + \lambda V \geq 0$ for $\Delta = -2\lambda \widehat{V\alpha}$ with α a minimizer of the BCS functional. However, as shown in [218, Proposition 2.11] this implies that $|\hat{\alpha}|$ is a radial function. In particular, in three dimensions, α (and therefore also Δ) cannot have a definite non-zero angular momentum.

(ii) Assume *that we know* a priori *that a solution of the BCS gap equation* (13.1.4) *(in spherical coordinates) satisfies $\Delta(p, \omega) = \Delta_0(p) Y_\ell^m(\omega)$ — at least to leading order.*[9] *Here, Y_ℓ^m is the usual L^2-normalized (complex) spherical harmonic with $\ell \in \mathbf{N}_0$ and $m \in \{-\ell, ..., \ell\}$. Then, by application of [272, Theorem 2.10], following very similar arguments to Sections 13.3.4 and 13.3.5.2, we find that the radial part of the gap function is given by*

$$|\Delta_0(\sqrt{\mu})| \approx c_{\ell,m} C_{\mathrm{univ}} h T_c \tag{13.2.18}$$

on the Fermi sphere $\{p^2 = \mu\}$. Here C_{univ} was defined in (13.2.10) *and we denoted*

$$\begin{aligned} c_{\ell,m} &:= \left(\int_{\mathbb{S}^2} |Y_\ell^m(\omega)|^4 \mathrm{d}\omega \right)^{-1/2} \\ &= \left(\sum_{L=0}^{2\ell} \frac{(2\ell+1)^2}{4\pi(2L+1)} |\langle \ell, \ell; 0, 0 | L; 0 \rangle|^2 \, |\langle \ell, \ell; m, m | L; 2m \rangle|^2 \right)^{-1/2} \end{aligned} \tag{13.2.19}$$

with $\langle \ell_1, \ell_2; m_1, m_2 | L; M \rangle$ being the well tabulated Clebsch-Gordan coefficients (see, e.g., [176, p. 1046]). The relation (13.2.19) *shows that, in particular, even in the subspace of fixed angular momentum $\ell \neq 0$, the behavior* (13.2.18) *is non-universal due to a non-trivial dependence on $m \in \{-\ell, ..., \ell\}$, as, for example (see [272, Eq. (6.8)]),*

$$c_{2,0} = \sqrt{\frac{28\pi}{15}} \quad \text{and} \quad c_{2,\pm 1} = c_{2,\pm 2} = \sqrt{\frac{14\pi}{5}} \, .$$

[9] If one models the interaction V by a rank one projection $V = |\chi\rangle\langle\chi|$ (similarly to (13.2.2)), instead of a multiplication operator, such a form of Δ can easily be enforced by taking $\chi(p, \omega) = \chi_0(p) Y_\ell^m(\omega)$.

For temperatures $0 \le T \le T_c$ and $h := \sqrt{1 - T/T_c}$, we expect (13.2.18) *to generalize to*

$$|\Delta_0(\sqrt{\mu})| \approx T_c \mathsf{f}_{\mathsf{BCS}}^{(\ell,m)}(h)$$

with $\mathsf{f}_{\mathsf{BCS}}^{(\ell,m)} : [0,1] \to [0,\infty)$ being implicitly defined via

$$\int_0^\infty \mathrm{d}s \int_{\mathbb{S}^2} \mathrm{d}\omega \left\{ \frac{\tanh \frac{\sqrt{s^2 + \left(\mathsf{f}_{\mathsf{BCS}}^{(\ell,m)}(h)\right)^2 |Y_\ell^m(\omega)|^2}}{2(1-h^2)}}{\sqrt{s^2 + \left(\mathsf{f}_{\mathsf{BCS}}^{(\ell,m)}(h)\right)^2 |Y_\ell^m(\omega)|^2}} - \frac{\tanh \frac{s}{2}}{s} \right\} |Y_\ell^m(\omega)|^2 = 0\,, \tag{13.2.20}$$

similarly to [492, Eq. (14.33)]. For $\ell = m = 0$, (13.2.20) *yields that $\mathsf{f}_{\mathsf{BCS}}^{(0,0)} = (4\pi)^{1/2} \mathsf{f}_{\mathsf{BCS}}$ with $\mathsf{f}_{\mathsf{BCS}}$ from* (13.2.9) *due to the L^2-normalization of the spherical harmonics (recall $\Delta(p,\omega) = \Delta_0(p) Y_\ell^m(\omega)$).*

A detailed analysis of the three-dimensional case with non-zero angular momentum is deferred to future work.

13.3 Proofs of the main results

This section contains the proofs of our main results formulated in Section 13.2.

13.3.1 Proof of Proposition 13.2.1

For ease of notation, we shall henceforth write λ instead of λ_{BCS}. From the explicit form (13.2.3) it is clear that $\Xi = \Delta$ and $\delta(h) \equiv \delta = \Delta/T_c$ is determined through (13.2.7). Hence, the goal is to show that $\delta(h)/\mathsf{f}_{\mathsf{BCS}}(h) = 1 + O(\mathrm{e}^{-1/\lambda})$ uniformly in $h \in [0,1]$. The proof of this is conducted in three steps.

13.3.1.1 A priori bound on δ

We shall prove the following lemma.

Lemma 13.3.1. *For $\delta = \delta(h)$ defined through* (13.2.7) *and $\lambda > 0$ small enough, it holds that*

$$\delta(h) \le Ch\,. \tag{13.3.1}$$

Proof. First, we note that $\delta(h) \le C$ uniformly for $h \in [0,1]$. This easily follows from observing that $\delta(h)$ is strictly monotonically increasing (as follows from elementary monotonicity properties of the integrand in (13.2.7)) and $\delta(1)$ is necessarily bounded.

In order to show (13.3.1), we employ the implicit function theorem to derive an asymptotic ODE for $\delta(h)$. For this purpose, we now introduce the function (recalling $T_c \sim \mathrm{e}^{-1/\lambda}$)

$$G_\lambda : [0,1] \times [0,\infty) \to \mathbf{R}\,, (h,\delta) \mapsto \int_0^{T_D/T_c} \left\{ \frac{\tanh\left(\frac{\sqrt{x^2+\delta^2}}{2(1-h^2)}\right)}{\sqrt{x^2+\delta^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{x} \right\} \mathrm{d}x$$

and trivially note that (13.2.7) is equivalent to $G_\lambda(h, \delta(h)) = 0$. Since G_λ is C^1 (away from the boundary) in δ and h (this easily follows from dominated convergence), we can apply the implicit function theorem to obtain the differential equation

$$\frac{\partial \delta(h)}{\partial h} = \frac{(1-h^2)h}{\delta(h)} \left(\int_0^{T_D/T_c} \frac{1}{\cosh^2\left(\frac{\sqrt{x^2+\delta^2}}{2(1-h^2)}\right)} \mathrm{d}x \Big/ \int_0^{T_D/T_c} \frac{g_1\left(\frac{\sqrt{x^2+\delta^2}}{1-h^2}\right)}{\frac{\sqrt{x^2+\delta^2}}{1-h^2}} \mathrm{d}x \right), \tag{13.3.2}$$

where we introduced the auxiliary functions

$$g_0(z) \coloneqq \frac{\tanh(z/2)}{z}\,, \qquad g_1(z) \coloneqq -g_0'(z) = z^{-1} g_0(z) - \frac{1}{2} z^{-1} \frac{1}{\cosh^2(z/2)}\,. \tag{13.3.3}$$

It is elementary to check that the even function $z \mapsto g_1(z)/z$ is (strictly) positive and (strictly) decreasing for $z \in [0, \infty)$. In combination with $\delta(h) \leq C$ and $T_c \sim \mathrm{e}^{-1/\lambda}$, one can thus bound the denominator on the r.h.s. of (13.3.2) from below. Together with an upper bound on the integral in the numerator (obtained by using elementary monotonicity properties of the hyperbolic cosine), we find that

$$\frac{\partial \delta(h)}{\partial h} \leq C' \frac{h}{\delta(h)} \left(\int_0^\infty \frac{1}{\cosh^2(x)} \mathrm{d}x \Big/ \int_0^C \frac{g_1\left(\sqrt{x^2+C^2}\right)}{\sqrt{x^2+C^2}} \mathrm{d}x \right) \leq C'' \frac{h}{\delta(h)} \tag{13.3.4}$$

for $h > 0$ and $\lambda > 0$ small enough (to ensure $T_D/T_c \geq C$).

Finally, the differential inequality (13.3.4) can be integrated using the boundary condition $\delta(0) = 0$ to conclude the desired.[10] □

13.3.1.2 Uniform error estimate

Having Lemma 13.3.1 as an input, we shall now prove the following.

Lemma 13.3.2. *For $\delta = \delta(h)$ defined through* (13.2.7)*, it holds that*

$$\int_{T_D/T_c}^\infty \left| \frac{\tanh\left(\frac{\sqrt{x^2+\delta^2}}{2(1-h^2)}\right)}{\sqrt{x^2+\delta^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{x} \right| \mathrm{d}x \leq C\, h^2\, \mathrm{e}^{-2/\lambda}\,. \tag{13.3.5}$$

Proof. First, we add and subtract $\tanh(x/2)/\sqrt{x^2+\delta^2}$ in (13.3.5). Then, we employ $T_c \sim \mathrm{e}^{-1/\lambda}$ and Lemma 13.3.1 to estimate

$$\int_{T_D/T_c}^\infty \left| \frac{\tanh\left(\frac{\sqrt{x^2+\delta^2}}{2(1-h^2)}\right)}{\sqrt{x^2+\delta^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{\sqrt{x^2+\delta^2}} \right| \mathrm{d}x \leq C\, h^2 \int_{T_D/T_c}^\infty \frac{1}{\cosh^2(x/2)} \mathrm{d}x \leq C\, h^2\, \mathrm{e}^{-2/\lambda}$$

and

$$\int_{T_D/T_c}^\infty \left| \frac{\tanh\left(\frac{x}{2}\right)}{\sqrt{x^2+\delta^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{x} \right| \mathrm{d}x \leq C\, h^2 \int_{T_D/T_c}^\infty \frac{1}{x^3} \mathrm{d}x \leq C\, h^2\, \mathrm{e}^{-2/\lambda}\,.$$

Combining these bounds yields the claim by means of the triangle inequality. □

From Lemma 13.3.2 and Equation (13.2.7), we immediately conclude that

$$\int_{\mathbf{R}} \left\{ \frac{\tanh\left(\frac{\sqrt{x^2+\delta^2}}{2(1-h^2)}\right)}{\sqrt{x^2+\delta^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{x} \right\} \mathrm{d}x = O(h^2\, \mathrm{e}^{-2/\lambda})\,. \tag{13.3.6}$$

[10] Strictly speaking, this requires to extend the function $\delta(h)$ in $(0,1)$, obtained via the implicit function theorem for G_λ, to the boundary points 0. In order to do so, note that, for $h \in (0, 1/2)$, (13.3.2) yields

$$\frac{\partial \delta(h)}{\partial h} \sim \frac{h}{\delta(h)}\,,$$

from which we immediately conclude that $|\partial_h \delta(h)| \leq C$, uniformly in $(0, 1/2)$. Hence, $\delta(h)$ continuously extends to 0. The same is true for its derivative by means of (13.3.2) again. We remark that by a similar argument, $\delta(h)$ can be extended to 1 as well.

13.3.1.3 Comparison with $\mathsf{f}_{\mathsf{BCS}}$

Given (13.3.6), the remaining task is to show that, because δ approximately *solves* the defining equation of $\mathsf{f}_{\mathsf{BCS}}$, it is *actually* close to $\mathsf{f}_{\mathsf{BCS}}$. This is the content of the following lemma.

Lemma 13.3.3. *Fix $h \in [0,1]$. If $\phi \in [0,\infty)$ satisfies*[11]

$$\int_{\mathbf{R}} \left\{ \frac{\tanh\left(\frac{\sqrt{x^2+\phi^2}}{2(1-h^2)}\right)}{\sqrt{x^2+\phi^2}} - \frac{\tanh\left(\frac{x}{2}\right)}{x} \right\} \mathrm{d}x = R \tag{13.3.7}$$

for some $|R| \leq C$, then

$$\phi = \mathsf{f}_{\mathsf{BCS}}(h) + O(|R|^{1/2}) \tag{13.3.8}$$

with $\mathsf{f}_{\mathsf{BCS}}$ defined in (13.2.9).

Hence, combining (13.3.6) with (13.3.8) and invoking Lemma 13.2.2 (iii), we find that

$$\delta(h) = \mathsf{f}_{\mathsf{BCS}}(h) + O(h\mathrm{e}^{-1/\lambda}) = \mathsf{f}_{\mathsf{BCS}}(h)\left(1 + O(\mathrm{e}^{-1/\lambda})\right).$$

This concludes the proof of Proposition 13.2.1.

Proof of Lemma 13.3.3. First, we note that, given $h \in [0,1]$, (13.3.7) has a solution $\phi \in [0,\infty)$ if and only if $R \leq \log\left(1/(1-h^2)\right)$. Then, as in the proof of [399, Lemma 6] (see [399, Equation (C50)]) we find that

$$\phi = \mathrm{e}^{-R}\mathsf{f}_{\mathsf{BCS}}\left(\sqrt{h^2 + (1-h^2)(1-e^R)}\right). \tag{13.3.9}$$

Taylor expanding in R around 0, using regularity of $\mathsf{f}_{\mathsf{BCS}}$ from Lemma 13.2.2, we get

$$\mathrm{e}^R\phi = \mathsf{f}_{\mathsf{BCS}}(h) + \int_0^1 \mathsf{f}'_{\mathsf{BCS}}\left(\sqrt{h^2 + (1-h^2)(1-\mathrm{e}^{tR})}\right) \frac{-(1-h^2)R\mathrm{e}^{tR}}{2\sqrt{h^2 + (1-h^2)(1-\mathrm{e}^{tR})}}\mathrm{d}t.$$

To bound the integral we change variables to $s = h^2 + (1-h^2)(1-e^{tR})$ and bound $|\mathsf{f}'_{\mathsf{BCS}}(h)| \leq C$ by Lemma 13.2.2. We split into cases depending on the sign of R.

$\underline{R > 0:}$ For $R > 0$ the integral is bounded by

$$\int_{h^2-(1-h^2)(\mathrm{e}^R-1)}^{h^2} \frac{\mathrm{d}s}{2\sqrt{s}} = h - \sqrt{h^2 - (1-h^2)(\mathrm{e}^R-1)} = \frac{(1-h^2)(\mathrm{e}^R-1)}{\sqrt{h^2 - (1-h^2)(\mathrm{e}^R-1)} + h}.$$

Noting that $\frac{\delta}{\sqrt{\varepsilon-\delta}+\sqrt{\varepsilon}} \leq \sqrt{\delta}$ and that $(1-h^2)(\mathrm{e}^R-1) \leq CR$ we find that the integral is bounded by $\sqrt{R}$.

$\underline{R < 0:}$ For $R < 0$ the integral is similarly bounded by

$$\int_{h^2}^{h^2+(1-h^2)(1-\mathrm{e}^R)} \frac{\mathrm{d}s}{2\sqrt{s}} = \sqrt{h^2 + (1-h^2)(1-\mathrm{e}^R)} - h \leq C\sqrt{|R|}.$$

Plugging these bounds into (13.3.9), we conclude the desired. □

13.3.2 Proof of Theorem 13.2.4(a)

We give here the proof of Theorem 13.2.4(a). The argument is divided into several steps.

[11] We follow the convention that, if $h = 1$, we replace $\tanh(...)$ by the constant function 1.

13.3.2.1 A priori spectral information

For any temperature T we have by Proposition 13.1.5 that there exists a unique (up to a constant global phase) minimizer (γ_*, α_*) of the BCS functional. The function α_* is radial and has $\hat{\alpha}_* > 0$. Moreover, the operator $K_T^\Delta + \lambda V$ has lowest eigenvalue 0 and α_* is the unique eigenfunction with this eigenvalue.

13.3.2.2 Weak a priori bound on Δ

From the proof of Proposition 13.1.2 in [316] we have the following bound on the minimising (γ_*, α_*) of the BCS functional [316, Eqn. 3.12]

$$\int (1+p^2)(|\hat{\alpha}_*(p)|^2 + \hat{\gamma}_*(p))\mathrm{d}p \\ \leq 8T \int \log\left(1 + e^{-(p^2-\mu)/4T}\right)\mathrm{d}p + 8\int \left[p^2/4 - 1 + C_2(\lambda)\right]_- \mathrm{d}p \leq C_\lambda$$

with $C_2(\lambda) = \inf \operatorname{spec}(p^2/4 + \lambda V) \leq 0$ and thus C_λ uniformly bounded for λ small enough. In particular $\|\alpha_*\|_{H^1} \leq C$ uniformly for λ small enough. By Sobolev's inequality [418, Thm 8.3] we then have

$$\|\alpha\|_{L^\infty}^2 \leq C \|\nabla\alpha\|_{L^2} \|\alpha\|_{L^2} \leq C \qquad (d=1)$$
$$\|\alpha\|_{L^q}^2 \leq C \|\nabla\alpha\|_{L^2} \|\alpha\|_{L^2} \leq C \qquad (d=2)$$
$$\|\alpha\|_{L^6}^2 \leq C \|\nabla\alpha\|_{L^2}^2 \leq C \qquad (d=3)$$

for any $2 \leq q < \infty$. Thus,

$$\|\Delta\|_{L^\infty} \leq 2\lambda \|V\alpha\|_{L^1} \leq \begin{cases} 2\lambda \|V\|_{L^1} \|\alpha\|_{L^\infty} & d=1 \\ 2\lambda \|V\|_{L^p} \|\alpha\|_{L^2}^{(2pq-2q-2p)/(pq-2p)} \|\alpha\|_{L^q}^{(2q-pq)/(pq-2p)} & d=2 \\ 2\lambda \|V\|_{L^{3/2}} \|\alpha\|_{L^2}^{1/2} \|\alpha\|_{L^6}^{1/2} & d=3 \end{cases} \leq C\lambda \tag{13.3.10}$$

uniformly for λ small enough (for any $1 < p < \min\{2, p_V\}$ by choosing q large enough in dimension $d=2$). In particular we see that $\Delta(p) \to 0$ as $\lambda \to 0$.

13.3.2.3 Birman–Schwinger principle

Next, by the Birman–Schwinger principle [269, 309, 316] the fact that $K_T^\Delta + \lambda V$ has lowest eigenvalue 0 with α_* being the unique eigenvector is equivalent to the Birman–Schwinger operator

$$B_{T,\Delta} := \lambda V^{1/2} \frac{1}{K_T^\Delta} |V|^{1/2}$$

having -1 as its lowest eigenvalue and $\phi = V^{1/2}\alpha_*$ being the unique eigenfunction corresponding to this eigenvalue. Here we use the convention $V^{1/2}(x) = \operatorname{sgn}(V(x))|V(x)|^{1/2}$. We decompose the Birman–Schwinger operator into a dominant singular term and an error term. For this purpose we define the (rescaled) Fourier transform restricted to the sphere $\mathfrak{F}_\mu : L^1(\mathbf{R}^d) \to L^2(\S^{d-1})$ by

$$\mathfrak{F}_\mu \psi(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \psi(x) e^{-\mathrm{i}\sqrt{\mu} p\cdot x} \mathrm{d}x,$$

which is well-defined by the Riemann–Lebesgue Lemma. Define then

$$m(T,\Delta) = \frac{1}{\left|\S^{d-1}\right|} \int_{|p| \leq \sqrt{2\mu}} \frac{1}{K_T^\Delta(p)} \mathrm{d}p \tag{13.3.11}$$

and decompose

$$B_{T,\Delta} = \lambda m(T,\Delta) V^{1/2} \mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2} + \lambda V^{1/2} M_{T,\Delta} |V|^{1/2},$$

with $M_{T,\Delta}$ defined such that this holds. Analogously to [269, Lemma 2] and Lemma 12.3.5 we have the following lemma, the proof of which (it is analogous to the one of Lemma 12.3.5) is given in Section 13.A.2.1.

Lemma 13.3.4. *We have $\left\| V^{1/2} M_{T,\Delta} |V|^{1/2} \right\|_{\mathrm{HS}} \leq C$ for small λ uniformly in T and Δ, where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of an operator.*

We conclude that $1 + \lambda V^{1/2} M_{T,\Delta} |V|^{1/2}$ is invertible for sufficiently small λ and thus, analogously to [312, Lemma 4] and Lemma 11.2.8 and Proposition 11.2.10 the fact that $B_{T,\Delta}$ has lowest eigenvalue -1 is equivalent to the fact that the operator

$$S_{T,\Delta} := \lambda m(T,\Delta) \mathfrak{F}_\mu |V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \tag{13.3.12}$$

acting on $L^2(\mathbb{S}^{d-1})$ has lowest eigenvalue -1. Moreover, the function $\mathfrak{F}_\mu |V|^{1/2} \phi = \mathfrak{F}_\mu V \alpha_*$ is the unique eigenfunction of $S_{T,\Delta}$ corresponding to the eigenvalue -1.

13.3.2.4 A priori bounds on Δ

For the analysis of the integral $m(T,\Delta)$ we need some a priori bounds on Δ. Analogously as in [312] and Chapter 12 we need some control of $\Delta(p)$ in terms of $\Delta(\sqrt{\mu})$ and some type of Lipschitz-continuity of Δ. These are collected in the following lemma.

Lemma 13.3.5. *The function Δ satisfies the bounds*

$$\Delta(p) \leq C \Delta(\sqrt{\mu}), \tag{13.3.13}$$

$$|\Delta(p) - \Delta(q)| \leq C \Delta(\sqrt{\mu}) |p - q| \tag{13.3.14}$$

for sufficiently small λ.

Proof. As noted above, the function $\mathfrak{F}_\mu |V|^{1/2} \phi = \mathfrak{F}_\mu V \alpha$ is the eigenfunction of $S_{T,\Delta}$ corresponding to the lowest eigenvalue -1.

Further, to leading order, $S_{T,\Delta}$ is proportional to $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu^\dagger$. Since the constant function $u = \left|\mathbb{S}^{d-1}\right|^{-1/2} \in L^2(\mathbb{S}^{d-1})$ is the ground state of $\mathcal{V}_\mu$ (see the argument around (13.1.11)), the same is also true for $S_{T,\Delta}$ whenever λ is small enough. Hence, one can easily see that

$$\frac{1}{1 + \lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger u$$

is an eigenvector of $B_{T,\Delta}$ corresponding to the eigenvalue -1 and thus proportional to $\phi = V^{1/2} \alpha_*$. Thus, with $\mathfrak{F}$ denoting the usual Fourier transform, by expanding $\frac{1}{1+x} = 1 - \frac{x}{1+x}$ we have

$$\begin{aligned} \Delta &= -2\lambda \mathfrak{F} |V|^{1/2} \phi = f(\lambda) \mathfrak{F} |V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger u \\ &= f(\lambda) \left(\int_{\mathbb{S}^{d-1}} \hat{V}(p - \sqrt{\mu} q) \mathrm{d}\omega(q) + \lambda \eta_\lambda(p) \right) \end{aligned}$$

for some constant $f(\lambda)$ (where we absorbed the factor $|\mathbb{S}^{d-1}|^{-1/2} (2\pi)^{-d/2}$ into $f(\lambda)$). One easily verifies that

$$\eta_\lambda = -(2\pi)^{d/2} \mathfrak{F} |V|^{1/2} \frac{V^{1/2} M_{T,\Delta} |V|^{1/2}}{1 + \lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger u$$

has $\|\eta_\lambda\|_\infty \leq C$ uniformly in $\lambda < \lambda_0$ for some λ_0 by Lemma 13.3.4. By evaluating at $p = \sqrt{\mu}$ we find $|f(\lambda)| \leq C\Delta(\sqrt{\mu})$ for small λ and thus the global bound (13.3.13). Moreover, we have the Lipschitz-bound:

$$\begin{aligned}|\Delta(p) - \Delta(q)| &\leq |f(\lambda)||p-q| \left\| |x||V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger u \right\|_{L^1} \\ &\leq C\Delta(\sqrt{\mu})|p-q| \left\| |x|^2 V \right\|_{L^1}^{1/2} \frac{1}{1-\lambda \left\| V^{1/2} M_{T,\Delta} |V|^{1/2} \right\|} \left\| V^{1/2} \mathfrak{F}_\mu^\dagger u \right\|_{L^2} \\ &\leq C\Delta(\sqrt{\mu})|p-q|\end{aligned}$$

for sufficiently small λ. □

13.3.2.5 First order

Expanding the resolvent in Equation (13.3.12) to first order in a geometric series we see that $S_{T,\Delta}$ to leading order is proportional to the operator $\mathcal{V}_\mu$ (defined in (13.1.10) above). Moreover, we have

$$\begin{aligned}-1 &= \lambda m(T,\Delta) \inf \operatorname{spec} \left(\mathfrak{F}_\mu |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \right) \\ &= \lambda m(T,\Delta) \inf \operatorname{spec} \mathcal{V}_\mu (1 + O(\lambda)) = \lambda e_\mu m(T,\Delta)(1+O(\lambda)).\end{aligned}$$

In particular $m(T,\Delta) = \frac{-1}{\lambda e_\mu}(1+O(\lambda)) \to \infty$ as $\lambda \to 0$.

13.3.2.6 Exponential vanishing of Δ

Pointwise we may bound $K_T^\Delta \geq E_\Delta$. Thus, by the first–order analysis above, we have

$$\frac{-1}{\lambda e_\mu}(1+O(\lambda)) = m(T,\Delta) = \frac{1}{|\S^{d-1}|} \int_{|p|<\sqrt{2\mu}} \frac{1}{K_T^\Delta} \mathrm{d}p \leq \frac{1}{|\S^{d-1}|} \int_{|p|<\sqrt{2\mu}} \frac{1}{\sqrt{|p^2-\mu|^2 + |\Delta(p)|^2}} \mathrm{d}p$$

The latter integral is calculated in [312] and Chapter 12. The same calculation is valid here by the bounds (13.3.13) and (13.3.14) and the fact that $\Delta(p) \to 0$ by (13.3.10). That is

$$\frac{-1}{\lambda e_\mu} \leq \mu^{d/2-1} \left(\log \frac{\mu}{\Delta(\sqrt{\mu})} + O(1) \right)$$

in the limit $\lambda \to 0$. We conclude that $\Delta(\sqrt{\mu}) \lesssim e^{-c/\lambda}$ as $\lambda \to 0$ (with $c = -1/e_\mu \mu^{d/2-1}$).

The constant c shall henceforth be used generically and its precise value might change from line to line.

13.3.2.7 Infinite order

Recall that, for small λ, the unique eigenfunction of $S_{T,\Delta}$ corresponding to the eigenvalue -1 is given by the constant function u. Thus, for small λ we have

$$-1 = \lambda m(T,\Delta) \left\langle u \middle| \mathfrak{F}_\mu |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \middle| u \right\rangle.$$

Combining this for the temperatures T and T_c we find

$$\begin{aligned}&m(T,\Delta) - m(T_c,0) \\ &= \frac{-1}{\lambda} \left[\frac{1}{\left\langle u \middle| \mathfrak{F}_\mu |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{T,\Delta} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \middle| u \right\rangle} - \frac{1}{\left\langle u \middle| \mathfrak{F}_\mu |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{T_c,0} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger \middle| u \right\rangle} \right]\end{aligned}$$

$$= \frac{-1}{\lambda e_\mu^2}(1+O(\lambda))\left\langle u\middle|\mathfrak{F}_\mu|V|^{1/2}\left(\frac{1}{1+\lambda V^{1/2}M_{T,\Delta}|V|^{1/2}}-\frac{1}{1+\lambda V^{1/2}M_{T_c,0}|V|^{1/2}}\right)V^{1/2}\mathfrak{F}_\mu^\dagger\middle|u\right\rangle \tag{13.3.15}$$

for small enough λ by expanding to first order in the denominator and noting that $\inf\operatorname{spec}\mathcal{V}_\mu = e_\mu$. The proof of the following lemma (which is somewhat analogous to the proof of Lemma 13.3.4) is given in Section 13.A.2.2.

Lemma 13.3.6. *There exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$*

$$\left\|V^{1/2}(M_{T,\Delta}-M_{T_c,0})|V|^{1/2}\right\|_{\mathsf{HS}} \le Ce^{-c/\lambda}$$

for some constants $C, c > 0$ uniformly in λ.

Having Lemma 13.3.6 at hand, we write the difference as a telescoping sum

$$\begin{aligned}
&\frac{1}{1+\lambda V^{1/2}M_{T,\Delta}|V|^{1/2}}-\frac{1}{1+\lambda V^{1/2}M_{T_c,0}|V|^{1/2}}\\
&\quad=\sum_{k=1}^{\infty}(-\lambda)^k\left[\left(V^{1/2}M_{T,\Delta}|V|^{1/2}\right)^k-\left(V^{1/2}M_{T_c,0}|V|^{1/2}\right)^k\right]\\
&\quad=\sum_{k=1}^{\infty}(-\lambda)^k\sum_{\ell=0}^{k-1}\left(V^{1/2}M_{T,\Delta}|V|^{1/2}\right)^{k-1-\ell}V^{1/2}(M_{T,\Delta}-M_{T_c,0})|V|^{1/2}\left(V^{1/2}M_{T_c,0}|V|^{1/2}\right)^{\ell}.
\end{aligned}$$

Thus, by Lemmas 13.3.4 and 13.3.6 we have

$$\begin{aligned}
\left\|\frac{1}{1+\lambda V^{1/2}M_{T,\Delta}|V|^{1/2}}-\frac{1}{1+\lambda V^{1/2}M_{T_c,0}|V|^{1/2}}\right\|_{\mathsf{HS}} &\le \sum_{k=1}^{\infty}\lambda^k\sum_{\ell=0}^{k-1}C^{k-1-\ell}\times Ce^{-c/\lambda}\times C^{\ell}\\
&\le \sum_{k=1}^{\infty}k\lambda^kC^ke^{-c/\lambda}\le C\lambda e^{-c/\lambda}.
\end{aligned}$$

We conclude that

$$|m(T,\Delta)-m(T_c,0)| \le Ce^{-c/\lambda}. \tag{13.3.16}$$

13.3.2.8 Calculation of the integral $m(T,\Delta)-m(T_c,0)$

To extract the asymptotics in (13.2.11) from the bound in (13.3.16) we calculate the difference $m(T,\Delta)-m(T_c,0)$ and show that it is essentially the left-hand-side of (13.2.9). The argument is essentially given in [399, Appendix C.4]. For completeness, we give the argument here.

By changing variables to $s = (p^2-\mu)/\mu$ and defining $x(s) = \Delta(\sqrt{\mu(1+s)})/\mu$ we get

$$\begin{aligned}
m(T,\Delta)-m(T_c,0) &= \int_0^{\sqrt{2\mu}}\left(\frac{1}{K_T^\Delta}-\frac{1}{K_{T_c}}\right)p^{d-1}\mathrm{d}p\\
&= \frac{\mu^{d/2-1}}{2}\int_{-1}^{1}\left(\frac{\tanh\left(\frac{\sqrt{s^2+x(s)^2}}{2T/\mu}\right)}{\sqrt{s^2+x(s)^2}}-\frac{\tanh\frac{s}{2T_c/\mu}}{s}\right)(1+s)^{d/2-1}\mathrm{d}s.
\end{aligned}$$

This is of the form where we can use [399, Lemma 5].

Lemma 13.3.7 ([399, Lemma 5])**.** *Let g, G be functions with $g(0) = G(0) = 1$ and $g \in L^\infty$ and let $\tau, \tau_c, \delta > 0$. Assume that $\tilde{g}(s) := (g(s)-1)/s$ and $\tilde{G}(s) := (G(s)-1)/s$ satisfy $\tilde{g}, \tilde{G} \in L^\infty(\mathbf{R})$. Let*

$s_1 > 0$ such that $g(s) > 1/2$ for $|s| < s_1$ and define

$$J_{\tau,\delta,\tau_c}(g,G) = \int_{\mathbf{R}} \left\{ \frac{\tanh \frac{\sqrt{s^2+g(s)^2\delta^2}}{2\tau}}{\sqrt{s^2+g(s)^2\delta^2}} - \frac{\tanh \frac{s}{2\tau_c}}{s} \right\} G(s)\mathrm{d}s,$$

$$J^{(0)}_{\tau,\delta,\tau_c} = J_{\tau,\delta,\tau_c}(1,1) = \int_{\mathbf{R}} \left\{ \frac{\tanh \frac{\sqrt{s^2+\delta^2}}{2\tau}}{\sqrt{s^2+\delta^2}} - \frac{\tanh \frac{s}{2\tau_c}}{s} \right\} \mathrm{d}s.$$

Then

$$\left| J_{\tau,\delta,\tau_c}(g,G) - J^{(0)}_{\tau,\delta,\tau_c} \right| \leq \left\| \tilde{G} \right\|_{L^\infty} (4\tau + 4\tau_c + \pi\delta \|g\|_{L^\infty}) + 4\delta \|\tilde{g}\|_{L^\infty} (1 + \|g\|_{L^\infty}) \left(1 + \frac{\delta}{2s_1}\right).$$

To apply this lemma we write

$$x(s) = \frac{\Delta(\sqrt{\mu(1+s)})}{\Delta(\sqrt{\mu})} \frac{\Delta(\sqrt{\mu})}{\mu} = g(s)\delta.$$

Then $g \in L^\infty$ uniformly in λ by (13.3.13) and $\tilde{g} \in L^\infty$ uniformly in λ by (13.3.14). Finally, clearly $G(s) = (1+s)^{d/2-1}\chi_{|s|\leq 1}$ has $\tilde{G} \in L^\infty$. We conclude that

$$m(T,\Delta) - m(T_c,0) = \frac{\mu^{d/2-1}}{2} \int_{\mathbf{R}} \left\{ \frac{\tanh \frac{\sqrt{s^2+(\Delta(\sqrt{\mu})/\mu)^2}}{2T/\mu}}{\sqrt{s^2+(\Delta(\sqrt{\mu})/\mu)^2}} - \frac{\tanh \frac{s}{2T_c/\mu}}{s} \right\} \mathrm{d}s + O(e^{-c/\lambda}).$$

Writing $T = T_c(1-h^2)$, recalling the bound in (13.3.16) and changing variables we find

$$\int_{\mathbf{R}} \left\{ \frac{\tanh \frac{\sqrt{s^2+(\Delta(\sqrt{\mu})/T_c)^2}}{2(1-h^2)}}{\sqrt{s^2+(\Delta(\sqrt{\mu})/T_c)^2}} - \frac{\tanh \frac{s}{2}}{s} \right\} \mathrm{d}s = R,$$

with $R = O(e^{-c/\lambda})$. Hence, by Lemma 13.3.3, we find that

$$\frac{\Delta(\sqrt{\mu})}{T_c} = \mathsf{f}_{\mathsf{BCS}}(h) + O(|R|^{1/2}) = \mathsf{f}_{\mathsf{BCS}}(h)(1 + O(h^{-1}e^{-c/\lambda})) \tag{13.3.17}$$

since $\mathsf{f}_{\mathsf{BCS}}(h) \sim h$ for small h by Lemma 13.2.2.

13.3.2.9 Comparing $\Delta(\sqrt{\mu})$ and Ξ

We finally prove that Ξ is essentially given by $\Delta(\sqrt{\mu})$.

Clearly $\Xi = \inf_p E_\Delta(p) \leq E_\Delta(\sqrt{\mu}) = \Delta(\sqrt{\mu})$. To show a corresponding lower bound consider p with $|p^2 - \mu| \leq \Xi \leq \Delta(\sqrt{\mu})$. Then by Equation (13.3.14) and the bound $\Delta(\sqrt{\mu}) = O(e^{-c/\lambda})$ we have

$$\Delta(p) \geq \Delta(\sqrt{\mu}) - C\Delta(\sqrt{\mu})|p - \sqrt{\mu}| \geq \Delta(\sqrt{\mu})\,(1 - C\Delta(\sqrt{\mu})) \geq \Delta(\sqrt{\mu})(1 + O(e^{-c/\lambda})).$$

We conclude that

$$\Xi = \Delta(\sqrt{\mu}) \left(1 + O(e^{-c/\lambda})\right). \tag{13.3.18}$$

Together with (13.3.17) this concludes the proof of Theorem 13.2.4(a).

13.3.3 Non-universal property of Δ: Proof of Equation (13.2.14)

From the Birman–Schwinger argument (Section 13.3.2.3) we have that $\phi = V^{1/2}\alpha_*$ is the (unique) eigenvector of

$$\frac{\lambda m(T,\Delta)}{1+\lambda V^{1/2} M_{T,\Delta}|V|^{1/2}} V^{1/2}\mathfrak{F}_\mu^\dagger \mathfrak{F}_\mu |V|^{1/2}$$

corresponding to the eigenvalue -1. Recalling that $\Delta = -2\lambda \mathfrak{F} V \alpha_*$ we thus get the equation

$$\Delta = -\mathfrak{F}|V|^{1/2}\frac{\lambda m(T,\Delta)}{1+\lambda V^{1/2} M_{T,\Delta}|V|^{1/2}} V^{1/2}\mathfrak{F}_\mu^\dagger \Delta(\sqrt{\mu}\,\cdot)$$

with $\Delta(\sqrt{\mu}\,\cdot)$ being the constant function on the unit sphere of value $\Delta(\sqrt{\mu})$. Recall that $-1 = \lambda m(T,\Delta)\left\langle u \middle| \mathfrak{F}_\mu |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{T,\Delta}|V|^{1/2}} V^{1/2}\mathfrak{F}_\mu^\dagger \middle| u \right\rangle$ for small enough λ. By the same argument as in Section 13.3.2.7 we may replace $M_{T,\Delta}$ by $M_{0,0}$, its corresponding value at $T = \Delta = 0$, up to errors of order $e^{-c/\lambda}$. (Concretely one can define $M_{0,0}$ via the representation of its kernel as given in Equations (13.A.3) to (13.A.5), (13.A.8) and (13.A.9), setting $T = \Delta = 0$.) Hence, for sufficiently small λ,

$$\frac{\Delta}{\Delta(\sqrt{\mu})} = \frac{|\S^{d-1}|^{1/2}\mathfrak{F}|V|^{1/2}\frac{1}{1+\lambda V^{1/2} M_{0,0}|V|^{1/2}} V^{1/2}\mathfrak{F}_\mu^\dagger u}{\left\langle u \middle| \mathfrak{F}_\mu |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_{0,0}|V|^{1/2}} V^{1/2}\mathfrak{F}_\mu^\dagger \middle| u \right\rangle} + O(e^{-c/\lambda}) = F + O(e^{-c/\lambda}).$$

Clearly, the function F does not depend on the temperature T.

13.3.4 Ginzburg-Landau theory: Proof of Theorem 13.2.4(b)

As mentioned above, the proof of Theorem 13.2.4(b) builds on Ginzburg–Landau (GL) theory. For convenience of the reader, we recall the main input from GL theory for the purpose of the present paper in Proposition 13.3.9 below. More general and detailed statements can be found in the original papers [270, 272, 220, 219]. In particular, these works allow for external fields or ground state degeneracy (cf. Lemma 13.3.8 below), respectively.

As a preparation for Proposition 13.3.9, we have the following lemma.

Lemma 13.3.8 (Ground state of $K_{T_c} + \lambda V$)**.** *Let V satisfy Assumptions 13.1.1 and 13.1.4. Then $K_{T_c} + \lambda V$ has 0 as a non-degenerate ground state eigenvalue and its $L^2(\mathbf{R}^d)$-normalized ground state $\mathfrak{a}_0$ can be chosen to have strictly positive Fourier transform. Moreover, it holds that $\hat{\mathfrak{a}}_0 \in L^\infty(\mathbf{R}^d)$.*

Proof. Since $T_c > 0$ (recall the discussion below Assumption 13.1.4), we first note that the Fourier multiplier K_{T_c} is strictly positive. Then using $\hat{V} \le 0$, the claim follows from a Perron-Frobenius type argument (see also [312] and [270, Assumption 2]). The fact that $\hat{\mathfrak{a}}_0 \in L^\infty(\mathbf{R}^d)$ follows from [270, Proposition 2] by invoking Assumption 13.1.1. □

We can now formulate the main results from GL theory, needed for the present paper.

Proposition 13.3.9 (Ginzburg-Landau theory, see [272, Theorem 2.10])**.** *Let V be a function satisfying Assumptions 13.1.1 and 13.1.4 and suppose that $0 \le T < T_c$. Then, using the notations from Proposition 13.1.5 and Lemma 13.3.8, we have that*

$$\mathcal{F}_T[\Gamma_*] - \mathcal{F}_T[\Gamma_{\mathrm{FD}}] = h^4 \mathcal{E}_{\mathrm{GL}}(\psi_{\mathrm{GL}}) + \mathcal{O}(h^6) \quad \text{as} \quad h \to 0\,,$$

where $\psi_{\mathrm{GL}} \neq 0$ minimizes the Ginzburg-Landau "functional" $\mathcal{E}_{\mathrm{GL}} : \mathbf{C} \to \mathbf{R}$,

$$\begin{aligned}\mathcal{E}_{\mathrm{GL}}(\psi) =& |\psi|^4 \left[\frac{1}{T_c^3} \int_{\mathfrak{R}^d} \frac{g_1((p^2-\mu)/T_c)}{(p^2-\mu)/T_c} |K_{T_c}(p)|^4 |\hat{\mathfrak{a}}_0(p)|^4 \mathrm{d}p \right] \\ & - |\psi|^2 \left[\frac{1}{2T_c} \int_{\mathfrak{R}^d} \frac{1}{\cosh^2\left((p^2-\mu)/(2T_c)\right)} |K_{T_c}(p)|^2 |\hat{\mathfrak{a}}_0(p)|^2 \mathrm{d}p \right].\end{aligned} \tag{13.3.19}$$

Here we used the auxiliary function g_1 from (13.3.3)*. Moreover, we can decompose the off–diagonal element $\hat{\alpha}_*$ of Γ_* as*

$$\hat{\alpha}_* = h|\psi_0|\hat{\mathfrak{a}}_0 + \hat{\xi} \tag{13.3.20}$$

where $\|\xi\|_{L^2} = \mathcal{O}(h^2)$ and $\psi_0 \neq 0$ approximately minimizes (13.3.19)*, i.e.*

$$\mathcal{E}_{\mathrm{GL}}(\psi_0) \leq \mathcal{E}_{\mathrm{GL}}(\psi_{\mathrm{GL}}) + \mathcal{O}(h^2)\,. \tag{13.3.21}$$

Remark 13.3.10. *We emphasize that all error terms in the above proposition (and also the implicit constants hidden in $\hat{\mathfrak{a}}_0 \in L^\infty(\mathbf{R}^d)$) are* not *uniform in λ. This crucially limits the applicability of our GL theory based method for temperatures slightly away from the critical one with $\mathrm{e}^{-c'/\lambda} \ll h \ll 1$ (cf. the error bound in* (13.2.12)*). Indeed, a careful examination of the proofs in [270, 272] reveals that the hidden dependencies on the critical temperature T_c are at most inverse polynomially and hence exponential in λ, i.e. $\mathrm{e}^{c/\lambda}$ for some $c > 0$ (independent of λ and h).*

13.3.4.1 Minimizing the Ginzburg-Landau functional

Given the inputs from GL theory, Theorem 13.2.4(b) is based on the following Proposition 13.3.11, the proof of which we postpone after finishing the proof of Theorem 13.2.4(b).

Proposition 13.3.11. *The (up to a phase unique) minimizer ψ_{GL} of the GL functional* (13.3.19) *satisfies*

$$|\psi_{\mathrm{GL}}| = C_{\mathrm{univ}}\,\frac{T_c}{\Delta_0(\sqrt{\mu})}\big(1 + o_{\lambda\to 0}(1)\big)\,, \tag{13.3.22}$$

where $\Delta_0 := -2(2\pi)^{-d/2}\lambda\hat{V} \star \hat{\mathfrak{a}}_0$,[12] *the constant C_{univ} is given in* (13.2.10)*, and the error is uniform in h.*

The fact that $|\psi_0| > 0$ approximately minimizes (13.3.19) (see (13.3.21)), implies that (recalling Remark 13.3.10)

$$|\psi_0| = |\psi_{\mathrm{GL}}| + O\big(h\mathrm{e}^{c/\lambda}\big)\,.$$

Therefore, by means of (13.3.20) in combination with Proposition 13.3.11, we infer

$$\hat{\alpha}_* = C_{\mathrm{univ}}\,T_c\,h\,\frac{\hat{\mathfrak{a}}_0}{\Delta_0(\sqrt{\mu})}\big(1 + o_{\lambda\to 0}(1) + O\big(h\mathrm{e}^{c/\lambda}\big)\big) + \hat{\xi}\,.$$

Thus, after taking the convolution with $\lambda\hat{V}$,

$$\Delta = C_{\mathrm{univ}}\,T_c\,h\,\frac{\Delta_0}{\Delta_0(\sqrt{\mu})}\big(1 + o_{\lambda\to 0}(1) + O\big(h\mathrm{e}^{c/\lambda}\big)\big) + \Delta_\xi\,, \tag{13.3.23}$$

where $\Delta > 0$ is the unique solution of the BCS gap equation (13.1.4) (see Proposition 13.1.5) and we denoted $\Delta_\xi := -2(2\pi)^{-d/2}\lambda\hat{V} \star \hat{\xi}$.

13.3.4.2 A priori bounds on Δ_0

For the proof of Theorem 13.2.4(b) we need some a priori bounds on Δ_0 analogously to those of Section 13.3.2.4. The bounds follow from the following lemma, the proof of which is analogous to the argument of Sections 13.3.2.3 and 13.3.2.4 and given in Section 13.A.2.3.

[12] Note that Δ_0 was denoted by t in [270, 272]. We also remark that $\Delta_0(\sqrt{\mu}) \neq 0$ as follows from $\hat{\mathfrak{a}}_0 > 0$ (by Lemma 13.3.8) and $\hat{V} \leq 0$ (by Assumption 13.1.4).

Lemma 13.3.12 (c.f. [312, Lemma 4] and Lemma 11.2.8)**.** *Let $\mathfrak{a}_0 \in H^1(\mathbf{R}^d)$ with $\hat{\mathfrak{a}}_0 > 0$ be the unique $H^1(\mathbf{R}^d)$-normalized ground state of $K_{T_c} + \lambda V$ from Lemma 13.3.8. Moreover, let $u(p) = (|\mathbb{S}^{d-1}|)^{-1/2}$ be the constant function on the sphere $\mathbb{S}^{d-1}$ and let*

$$\hat{\varphi}(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(p - \sqrt{\mu} q)\, \mathrm{d}\omega(q)\,. \tag{13.3.24}$$

Then $\Delta_0 = -2(2\pi)^{-d/2} \lambda \hat{V} \star \hat{\mathfrak{a}}_0$ can be expanded as

$$\Delta_0(p) = f(\lambda)[\hat{\varphi}(p) + \lambda \eta_\lambda(p)] \tag{13.3.25}$$

for some positive function $f(\lambda)$ and $\|\eta_\lambda\|_{L^\infty(\mathbf{R}^d)}$ bounded uniformly in $\lambda > 0$.

After realizing $\hat{\varphi}(\sqrt{\mu}) = -e_\mu$ by (13.1.11), we conclude for small enough $\lambda > 0$ that

$$\Delta_0(p) - \Delta_0(\sqrt{\mu}) = \frac{[\hat{\varphi}(p) - \hat{\varphi}(\sqrt{\mu})] + \lambda[\eta_\lambda(p) - \eta_\lambda(\sqrt{\mu})]}{-e_\mu + \lambda \eta_\lambda(\sqrt{\mu})} \Delta_0(\sqrt{\mu})\,.$$

Now it is an easy computation to see $|\hat{\varphi}(p) - \hat{\varphi}(q)| \leq C \min\{||p| - |q||, 1\}$ for all $p, q \in \mathbf{R}^d$. Thus,

$$\big|\Delta_0(p) - \Delta_0(\sqrt{\mu})\big| \leq C \left(\min\big\{\big||p| - \sqrt{\mu}\big|, 1\big\} + \lambda\right) \Delta_0(\sqrt{\mu})\,. \tag{13.3.26}$$

13.3.4.3 A priori bounds on Δ_ξ

For the following arguments, we need two estimates on $\Delta_\xi = -2(2\pi)^{-d/2} \lambda \hat{V} \star \hat{\xi}$.

- First, it is a simple consequence of Young's inequality and $\|\hat{\xi}\|_{L^2} = \|\xi\|_{L^2} = O\big(h^2 \mathrm{e}^{c/\lambda}\big)$, that

$$\|\Delta_\xi\|_{L^\infty} = \|V\|_{L^2}\, O\big(h^2 \mathrm{e}^{c/\lambda}\big)\,. \tag{13.3.27}$$

- Second, we note that $\Delta_\xi(p) - \Delta_\xi(q)$ is (proportional to) the Fourier transform of $V(x)\big(1 - \mathbf{E}^{\mathrm{i}(p-q)\cdot x}\big)\xi(x)$, and thus

$$\Big\|V(x)\big(1 - \mathbf{E}^{\mathrm{i}(p-q)\cdot x}\big)\Big\|_{L^2}^2 = \int_{\mathbf{R}^d} |V(x)|^2 \Big|1 - \mathbf{E}^{\mathrm{i}(p-q)\cdot x}\Big|^2 \mathrm{d}x \leq C|p-q|^2 \int_{\mathbf{R}^d} |V(x)|^2 |x|^2 \mathrm{d}x\,.$$

  Using radiality of Δ and Δ_0, we conclude the radiality of Δ_ξ and therefore

$$|\Delta_\xi(p) - \Delta_\xi(q)| \leq \big||p| - |q|\big|\, \||\cdot| V\|_{L^2}\, O\big(h^2 \mathrm{e}^{c/\lambda}\big)\,. \tag{13.3.28}$$

Recall that $\|(1 + |\cdot|)V\|_{L^2} < \infty$ by assumption.

13.3.4.4 Comparing $\Delta(\sqrt{\mu})$ and Ξ

We aim at proving

$$\Xi = \Delta(\sqrt{\mu}) \Big(1 + O\big(\lambda + h^2 \mathrm{e}^{c/\lambda}\big)\Big)\,. \tag{13.3.29}$$

In order to see this, we note that clearly $\Xi = \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2} \leq \Delta(\sqrt{\mu})$. For the reverse inequality, let $p \in \mathbf{R}^d$ with $|p^2 - \mu| \leq \Xi \leq \Delta(\sqrt{\mu})$. Then

$$|\Delta(p) - \Delta(\sqrt{\mu})| \leq C\, T_c\, h\big(1 + o_{\lambda \to 0}(1) + O(h \mathrm{e}^{c/\lambda})\big) \cdot \big(\big||p| - \sqrt{\mu}\big| + \lambda\big) + C\big||p| - \sqrt{\mu}\big|\, h^2 \mathrm{e}^{c/\lambda}$$

by application of (13.3.26) and (13.3.28). Using that $\Delta(\sqrt{\mu}) \sim T_c h$, as a consequence of (13.3.23) for h small enough (meaning $h\mathrm{e}^{c/\lambda} \ll 1$), we then conclude

$$|\Delta(p) - \Delta(\sqrt{\mu})| \leq C \left(\lambda + h^2 \mathrm{e}^{c/\lambda}\right) \Delta(\sqrt{\mu})$$

In combination with the upper bound, this proves (13.3.29).

13.3.4.5 Conclusion: Proof of Theorem 13.2.4(b)

We evaluate (13.3.23) at $p = \sqrt{\mu}$, such that we find

$$\Xi = \Big[C_{\mathrm{univ}}\, T_c\, h\,\big(1 + o_{\lambda\to 0}(1) + O\big(h\mathrm{e}^{c/\lambda}\big)\big) + O\big(h^2\mathrm{e}^{c/\lambda}\big)\Big]\cdot\big(1 + O\big(\lambda + h^2\mathrm{e}^{c/\lambda}\big)\big)$$

with the aid of (13.3.27) and (13.3.29). Collecting all the error terms leaves us with

$$\Xi = C_{\mathrm{univ}}\, T_c\, h\big(1 + o_{\lambda\to 0}(1) + O\big(h\mathrm{e}^{c/\lambda}\big)\big)\,. \tag{13.3.30}$$

Hence, using $\mathsf{f}_{\mathsf{BCS}}(h) = C_{\mathrm{univ}}h + O(h^2)$, by Lemma 13.2.2, we arrive at Theorem 13.2.4(b). □

13.3.4.6 Proof of Proposition 13.3.11

In the following estimates, we use the shorthand notations (recall the definition of the auxiliary function g_1 from (13.3.3))

$$f_4(p) := \frac{g_1((p^2-\mu)/T_c)}{(p^2-\mu)/T_c} \qquad f_2(p) := \frac{1}{\cosh^2\left((p^2-\mu)/(2T_c)\right)}\,, \tag{13.3.31}$$

such that the absolute value of the minimizer ψ_{GL} of (13.3.19) is given by

$$|\psi_{\mathrm{GL}}| = T_c\left(\frac{\int_{\mathbf{R}^d} f_2(p)|K_{T_c}(p)|^2|\hat{\mathfrak{a}}_0(p)|^2\mathrm{d}p}{4\int_{\mathbf{R}^d} f_4(p)|K_{T_c}(p)|^4|\hat{\mathfrak{a}}_0(p)|^4\mathrm{d}p}\right)^{1/2} = T_c\left(\frac{\int_{\mathbf{R}^d} f_2(p)|\Delta_0(p)|^2\mathrm{d}p}{\int_{\mathbf{R}^d} f_4(p)|\Delta_0(p)|^4\mathrm{d}p}\right)^{1/2}\,.$$

We denoted $\Delta_0 = -2(2\pi)^{-d/2}\lambda\hat{V}\star\hat{\mathfrak{a}}_0$ (as in Proposition 13.3.11) and used that $\mathfrak{a}_0 \in \ker\big(K_{T_c}+\lambda V\big)$. Note that, $\Delta_0 = |\Delta_0|$ by means of Proposition 13.1.5 and $\hat{V}\le 0$ from Assumption 13.1.4.

Next, we add and subtract $|\Delta_0(\sqrt{\mu})|^2$ (resp. $|\Delta_0(\sqrt{\mu})|^4$) in the integral in the numerator (resp. denominator). The terms involving $|\Delta_0(\sqrt{\mu})|^j$ are evaluated as follows.

Lemma 13.3.13 (Emergence of C_{univ} in GL theory)**.** *Let $\mu > 0$. In the limit $T_c/\mu \to 0$ we have that (recall C_{univ} from* (13.2.10)*)*

$$\left(\int_{\mathbf{R}^d}\frac{1}{\cosh^2\left(\frac{p^2-\mu}{2T_c}\right)}\mathrm{d}p\Big/\int_{\mathbf{R}^d}\frac{g_1((p^2-\mu)/T_c)}{(p^2-\mu)/T_c}\mathrm{d}p\right)^{1/2} \longrightarrow C_{\mathrm{univ}} \tag{13.3.32}$$

for all $d = 1, 2, 3$.

Proof. Since the integrands are both radial, we switch to spherical coordinates and neglect the common $|\mathbb{S}^{d-1}|$–factor in numerator and denominator. By splitting the remaining radial integration according to $p^2 \le \mu$ and $p^2 \ge \mu$ and changing the integration variables from $(p^2-\mu)/2T_c$ to $-t$ resp. t we find the numerator of (13.3.32) being equal to

$$2T_c\mu^{(d-2)/2}\left[\int_0^{\mu/2T_c}\big(1-2\tfrac{T_c}{\mu}t\big)^{(d-2)/2} + \int_0^{\infty}\big(1+2\tfrac{T_c}{\mu}t\big)^{(d-2)/2}\right]\left(\frac{1}{\cosh^2(t)}\right)\mathrm{d}t\,. \tag{13.3.33}$$

Similarly, we find the denominator of (13.3.32) to equal

$$\frac{2T_c\mu^{(d-2)/2}}{8}\left[\int_0^{\mu/2T_c}\big(1-2\tfrac{T_c}{\mu}t\big)^{(d-2)/2} + \int_0^{\infty}\big(1+2\tfrac{T_c}{\mu}t\big)^{(d-2)/2}\right]\left(\frac{\tanh(t)}{t^3} - \frac{1}{t^2\cosh^2(t)}\right)\mathrm{d}t\,. \tag{13.3.34}$$

We now take the ratio of (13.3.33) and (13.3.34) and send $T_c/\mu \to 0$. By means of the dominated convergence theorem (note that the integrand in (13.3.34) behaves as t^{-3} for large t) we thus find the limit being given as the ratio of

$$\int_0^\infty \frac{1}{\cosh^2(t)}\mathrm{d}t \qquad \text{and} \qquad \frac{1}{8}\int_0^\infty \left(\frac{\tanh(t)}{t^3} - \frac{1}{t^2\cosh^2(t)}\right)\mathrm{d}t\,.$$

While the former is easily evaluated as being equal to one, the latter is given by $\frac{7\zeta(3)}{8\pi^2}$ (see, e.g., [298, 3.333.3]). This proves the claim. □

With the aid of (13.3.26) and noting $f_j > 0$, the resulting differences (from adding and subtracting $|\Delta_0(\sqrt{\mu})|^j$) can be estimated as

$$\left|\int_{\mathbf{R}^d} f_j(p)\big(|\Delta_0(p)|^j - |\Delta_0(\sqrt{\mu})|^j\big)\mathrm{d}p\right| \leq C\,|\Delta_0(\sqrt{\mu})|^j \int_{\mathbf{R}^d} f_j(p)\left(\min\left(\big||p|-\sqrt{\mu}\big|,1\right)+\lambda\right)\mathrm{d}p$$

for $j = 2, 4$. These integrals can be treated analogously to (13.3.33) and (13.3.34) in Lemma 13.3.13 (for the $\big||p|-\sqrt{\mu}\big|$-term, note that f_j essentially concentrates around $|p| \approx \sqrt{\mu}$) and we find them to be smaller than the corresponding leading term $\int_{\mathbf{R}^d} f_j(p)|\Delta_0(\sqrt{\mu})|^j\mathrm{d}p$ in the limit $\lambda \to 0$ (and hence $T_c \to 0$). Therefore,

$$|\psi_{\mathrm{GL}}| = \frac{T_c}{\Delta_0(\sqrt{\mu})}\left(\frac{\int_{\mathbf{R}^d} f_2(p)\mathrm{d}p}{\int_{\mathbf{R}^d} f_4(p)\mathrm{d}p}\right)^{1/2}\cdot\big(1+o_{\lambda\to 0}(1)\big) = C_{\mathrm{univ}}\,\frac{T_c}{\Delta_0(\sqrt{\mu})}\cdot\big(1+o_{\lambda\to 0}(1)\big)\,,$$

where we used Lemma 13.3.13 in the last step. As the GL functional (13.3.19) is entirely independent of the relative difference to the critical temperature $(T_c - T)/T_c = h^2$, it is clear that all the errors here hold uniform in the parameter h. This finishes the proof of Proposition 13.3.11. □

13.3.5 Pure angular momentum for $d = 2$: Proof of Theorem 13.2.11

13.3.5.1 Part (a)

The proof of Theorem 13.2.11 (a) is mostly the same as that of Theorem 13.2.4 (a). We sketch the argument here, highlighting the few differences.

The operator $\mathcal{V}_\mu$. Using the Birman–Schwinger principle on the operator $K_{T_c} + \lambda V$ (as is done in [269, 312]) we find that, for sufficiently small λ, the lowest eigenvalue e_μ of $\mathcal{V}_\mu$ (recall (13.1.10)) is an eigenvalue for angular momentum ℓ_0, since this is the angular momentum of the ground state(s) of $K_{T_c} + \lambda V$ by assumption. Further, since V is radial, the eigenfunctions of $\mathcal{V}_\mu$ all have a definite angular momentum. In particular the first excited state has some angular momentum $\ell_1 \neq \ell_0$:

$$e_\mu^{(1)} = \inf_{u\perp u_{\pm\ell_0}} \langle u|\mathcal{V}_\mu|u\rangle = \langle u_{\pm\ell_1}|\mathcal{V}_\mu|u_{\pm\ell_1}\rangle\,,$$

with $u_{\pm\ell}(p) = (2\pi)^{-1/2}e^{\pm i\ell\varphi}$ the eigenfunctions of angular momentum ℓ. Here φ denotes the angle of $p \in \mathbf{R}^2$ in polar coordinates. Note that $e_\mu^{(1)} \leq 0$ since $\mathcal{V}_\mu$ is a compact operator on an infinite-dimensional space.

A priori spectral information. It is proved in [218, Theorem 2.1] that there exists a temperature $\tilde{T}$ such that for temperatures $\tilde{T} < T < T_c$ the minimizers of the BCS functional are given by

$$\hat{\alpha}_\pm(p) = e^{\pm i\ell_0\varphi}\hat{\alpha}_0(p)$$

where φ denotes the angle of $p \in \mathbf{R}^2$ in polar coordinates, $\hat{\alpha}_0$ is a radial function, and ℓ_0 is the angular momentum given by Equation (13.2.15). The BCS gap functions are then $\Delta_\pm(p) = \Delta_0(p)e^{\pm i\ell_0\varphi}$, with Δ_0 a radial function.[13] Further, we have $K_T^{\Delta_0} + \lambda V \geq 0$ for temperatures $T \in (\tilde{T}, T_c)$ [218, Proposition 4.3] and $\ker(K_T^{\Delta_0} + \lambda V) = \mathrm{span}\{\alpha_+, \alpha_-\}$.

[13] This should not be confused with the function Δ_0 used in Section 13.3.4.

The temperature $\tilde{T}$. As discussed in [218, Remark 2.6] the temperature $\tilde{T}$ is given by $\tilde{T} = T_c(\ell_1)$, the critical temperature when restricted to angular momentum ℓ_1. Following the argument in [269] (see also [314, Theorem 1]) we find

$$\tilde{T} \leq \begin{cases} Ce^{1/\lambda e_\mu^{(1)}} & e_\mu^{(1)} < 0, \\ Ce^{-c/\lambda^2} & e_\mu^{(1)} = 0. \end{cases}$$

Recalling that $T_c \sim e^{1/\lambda e_\mu}$ and that $e_\mu < e_\mu^{(1)} \leq 0$ then clearly $\tilde{T}/T_c \leq Ce^{-\mathfrak{c}/\lambda}$ for some $\mathfrak{c} > 0$.

Weak a priori bound on $\Delta_\pm$. Exactly as in Equation (13.3.10) we have $\|\Delta_\pm\|_{L^\infty} \leq C\lambda$.

Birman–Schwinger principle. Analogously to Section 13.3.2.3 we have by the Birman–Schwinger principle that

$$B_{T,\Delta_0} = V^{1/2} \frac{1}{K_T^{\Delta_0}} |V|^{1/2}$$

has -1 as its lowest eigenvalue, only the eigenspace is spanned by the two vectors $\phi_\pm = V^{1/2}\alpha_\pm$. By a completely analogous argument is in Section 13.3.2.3 we find that

$$S_{T,\Delta_0} = \lambda m(T,\Delta_0)\mathfrak{F}_\mu |V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M_{T,\Delta_0} |V|^{1/2}} V^{1/2} \mathfrak{F}_\mu^\dagger$$

has -1 as its lowest eigenvalue with corresponding eigenspace spanned by $\mathfrak{F}_\mu V \alpha_\pm$.

A priori bounds on $\Delta_\pm$. Analogously to Lemma 13.3.5 we claim

Lemma 13.3.14. *The functions $\Delta_\pm$ satisfy the bounds (with slight abuse of notation, recall that Δ_0 is a radial function)*

$$|\Delta_\pm(p)| \leq C|\Delta_0(\sqrt{\mu})|, \qquad |\Delta_\pm(p) - \Delta_\pm(q)| \leq C|\Delta_0(\sqrt{\mu})||p-q|\,.$$

Proof. The proof is analogous to that of Lemma 13.3.5. First we note that $\mathcal{V}_\mu = \mathfrak{F}_\mu V \mathfrak{F}_\mu^\dagger$ has eigenfunctions of lowest eigenvalue $u_{\pm\ell_0}(p) = (2\pi)^{-1/2} e^{\pm i\ell_0\varphi}$ and that the operator S_{T,Δ_0} preserves the angular momentum. Analogously to the proof of Lemma 13.3.5 we find

$$\Delta_\pm(p) = f_\pm(\lambda)\left(\int_{\mathbb{S}^1} \hat{V}(p - \sqrt{\mu}q) u_{\pm\ell_0}(q) \mathrm{d}\omega(q) + \lambda \eta_{\lambda\pm}(p) \right)$$

with $\|\eta_{\lambda\pm}\|_{L^\infty} \leq C$ uniformly in λ. Evaluating on the Fermi surface $\{p^2 = \mu\}$ we get (recall that $\inf \operatorname{spec} \mathcal{V}_\mu = e_\mu$)

$$\Delta_\pm(\sqrt{\mu}p/|p|) = f_\pm(\lambda)\left(e_\mu u_{\pm\ell_0}(p/|p|) + \lambda \eta_{\lambda\pm}(\sqrt{\mu}p/|p|)\right).$$

In particular, we conclude that $\left|\Delta_0(\sqrt{\mu})\right| = \left|\Delta_\pm(\sqrt{\mu}p/|p|)\right| > 0$ for λ small enough and that $|f_\pm(\lambda)| \leq C\left|\Delta_0(\sqrt{\mu})\right|$. We conclude the rest of the proof exactly as for Lemma 13.3.5. □

The remaining parts of the argument (first order analysis of m, the exponential vanishing of $\Delta_\pm$, infinite order analysis of m, calculation of the integral $m(T,\Delta_0) - m(T_c,0)$ and comparing $\Delta_\pm$ on the Fermi surface with Ξ) are exactly as in Sections 13.3.2.5 to 13.3.2.9 only replacing Δ and u by $\Delta_\pm$ and $u_{\pm\ell_0}$, respectively. This concludes the proof of Theorem 13.2.11 (a).

13.3.5.2 Part (b)

Again, we highlight only the main differences compared to the proof of Theorem 13.2.4 (b).

Ginzburg-Landau functional Since every function $\hat{\mathfrak{a}}_0$ in kernel of $K_{T_c} + \lambda V$ can be written (in polar coordinates) as

$$\hat{\mathfrak{a}}_0(p,\varphi) = \hat{\rho}(p)\big[\psi_+ \mathrm{e}^{\mathrm{i}\ell_0\varphi} + \psi_- \mathrm{e}^{-\mathrm{i}\ell_0\varphi}\big]$$

for an appropriate normalized $\hat{\rho} \in L^2((0,\infty); p\mathrm{d}p)$ and $\psi_\pm \in \mathbf{C}$ by Assumption 13.2.10 (cf. (13.2.15)), the analog of the Ginzburg-Landau functional (13.3.19) becomes [272, Theorems 2.10 and 3.5]

$$\begin{aligned}\mathcal{E}_{\mathrm{GL}}(\psi_+,\psi_-) = \big[|\psi_+|^4 + |\psi_-|^4 + 4|\psi_+|^2|\psi_-|^2\big] \times \left[\frac{2\pi}{T_c^3}\int_0^\infty \frac{g_1((p^2-\mu)/T_c)}{(p^2-\mu)/T_c}|K_{T_c}(p)|^4|\hat{\rho}(p)|^4\, p\,\mathrm{d}p\right] \\ - \big[|\psi_+|^2 + |\psi_-|^2\big] \times \left[\frac{\pi}{T_c}\int_0^\infty \frac{1}{\cosh^2\left((p^2-\mu)/(2T_c)\right)}|K_{T_c}(p)|^2|\hat{\rho}(p)|^2\, p\,\mathrm{d}p\right].\end{aligned} \tag{13.3.35}$$

Minimizers of the GL functional In contrast to (13.3.19), the functional (13.3.35) now has *two* (up to a phase unique) minimizers. This follows from observing that $\mathcal{E}_{\mathrm{GL}}(\psi_+,\psi_-) = \mathcal{E}_{\mathrm{GL}}(\psi_-,\psi_+)$ and that one of the $\psi_\pm$ is necessarily zero for any minimizer of (13.3.35). In fact, these minimizers are

$$(|\psi_{\mathrm{GL}}|, 0) \quad \text{and} \quad (0, |\psi_{\mathrm{GL}}|)$$

with $|\psi_{\mathrm{GL}}|$ given in (13.3.22) but with $\Delta_0 := -2(2\pi)^{-d/2}\lambda\hat{V} \star \hat{\rho}$, where $\hat{\rho}$ is understood as a radial function in $L^2(\mathbf{R}^2)$.[14] Hence, using the notation from Proposition 13.3.9 (see also [272, Theorem 2.10], which provides a general analog of (13.3.20)–(13.3.21), valid also for the concrete functional (13.3.35)) and (13.3.23), we find that (up to a constant phase) *every* non-zero solution of the BCS gap equation (13.1.4) can be written as

$$\Delta_\pm(p,\varphi) = C_{\mathrm{univ}}\, T_c\, h\, \frac{\Delta_0(p)}{\Delta_0(\sqrt{\mu})}\mathrm{e}^{\pm\mathrm{i}\ell_0\varphi}\big(1 + o_{\lambda\to 0}(1) + O\big(h\mathrm{e}^{c/\lambda}\big)\big) + \Delta_\xi(p,\varphi)\,.$$

The rest of the argument (a priori bounds on $\Delta_0(p)\mathrm{e}^{\pm\mathrm{i}\ell_0\varphi}$ and Δ_ξ, comparison of $|\Delta_\pm|$ on the Fermi surface with Ξ) works completely analogously to Section 13.3.4 with similar adjustments as explained in Section 13.3.5.1. This concludes the proof of Theorem 13.2.11 (b). □

13.A Additional proofs

13.A.1 Uniqueness of the minimizer: Proof of Proposition 13.1.5

Finally, we present the proof of Proposition 13.1.5.

Proof of Proposition 13.1.5. We remark that the argument has already partly been sketched in [316, 218]. The key observation for our proof is, that, if $\hat{V} \le 0$, then

$$\langle\hat{\alpha}|\hat{V}\star\hat{\alpha}\rangle \ge \langle|\hat{\alpha}||\hat{V}\star|\hat{\alpha}|\rangle\,. \tag{13.A.1}$$

Let $(\hat{\gamma},\hat{\alpha})$ minimize the BCS functional (13.1.3). Then, by means of (13.A.1), we have $\mathcal{F}_T[(\hat{\gamma},\hat{\alpha})] \ge \mathcal{F}_T[(\hat{\gamma},|\hat{\alpha}|)]$, hence also $(\hat{\gamma},|\hat{\alpha}|)$ is a minimizer. Consequently, the (inverse) Fourier transform of $|\hat{\alpha}|$ is an eigenvector of $K_T^\Delta + V$ with $\Delta = -2(2\pi)^{-d/2}\hat{V}\star|\hat{\alpha}|$ with the eigenvalue zero. Note that, using continuity of Δ and the BCS gap equation (13.1.4), we not only have $|\hat{\alpha}| \ge 0$ but also $|\hat{\alpha}| > 0$ everywhere (see [312, Lemma 2.1]). By the observation (13.A.1) again, we find that for any ground state $\hat{\alpha}_{\mathrm{GS}}$ of $K_T^\Delta + V$ also $|\hat{\alpha}_{\mathrm{GS}}|$ is a ground state. But $|\hat{\alpha}_{\mathrm{GS}}|$ is non–orthogonal to $|\hat{\alpha}|$, which implies that zero has to be the lowest eigenvalue of $K_T^\Delta + V$, i.e.

$$K_T^\Delta + V \ge 0\,. \tag{13.A.2}$$

[14] The fact that $\Delta_0(\sqrt{\mu}) \neq 0$ can be seen in a similar way as in the proof of Lemma 13.3.14.

By writing out (13.A.1), we see that the inequality is actually an application of Cauchy-Schwarz and thus becomes strict, unless $\hat{\alpha}(p) = \mathbf{E}^{\mathrm{i}\phi}|\hat{\alpha}(p)|$ for some fixed $\phi \in \mathbf{R}$. Therefore, by repeating the above arguments, we find that the ground state of (13.A.2) is non–degenerate and we have proven item (i).

In order to prove item (ii), let $\Gamma_i \equiv (\gamma_i, \alpha_i)$, $i = 1, 2$, be two (non–trivial) minimizers of the BCS functional (13.1.3) and denote the corresponding gap functions by Δ_1 resp. Δ_2. We now apply the relative entropy identity (see [270] and [272, Prop. 5.2]) and a simple trace inequality (see [270, Lemma 3] and [272, Lemma 5.7]) to find that

$$\mathcal{F}_T[\Gamma_1] - \mathcal{F}_T[\Gamma_2] \geq \Big\langle (\alpha_1 - \alpha_2) \Big| K_T^{\Delta_1} + V \Big| (\alpha_1 - \alpha_2) \Big\rangle \geq 0$$

(and the same inequality with indices 1 and 2 interchanged) by means of (13.A.2). Since $\mathcal{F}_T[\Gamma_1] = \mathcal{F}_T[\Gamma_2]$, this implies $(\alpha_1 - \alpha_2) \in \ker(K_T^{\Delta_1} + V)$ and thus $\alpha_2 = \psi_{21}\alpha_1$ for some $\psi_{21} \in \mathbf{C} \setminus \{0\}$ (recall from (i) that $\ker(K_T^{\Delta_1} + V)$ is one–dimensional). From this we conclude

$$\big(K_T^{\psi_{21}\Delta_1} + V\big)\alpha_1 = 0\,.$$

Now, the pointwise strict monotonicity of $|\psi_{21}| \mapsto K_T^{\psi_{21}\Delta_1}(p)$ together with the fact that one can choose $|\hat{\alpha}_1|$ to be strictly positive, implies that $|\psi_{21}| = 1$ and we have shown uniqueness of minimizers up to a constant phase, which can be chosen in such a way that it ensures strict positivity of $\hat{\alpha}$. Finally, it is shown in [218, Proposition 2.9] that if α is not radial, then (13.A.2) is violated. Radiality of the corresponding γ follows from (13.1.6). This finishes the proof. □

13.A.2 Proofs of technical lemmas within the proof of Theorem 13.2.4

This section contains the proofs of Lemmas 13.3.4, 13.3.6, and 13.3.12.

13.A.2.1 Proof of Lemma 13.3.4

The argument is slightly different in dimensions $d = 1, 2, 3$. The case $d = 3$ is similar to [312, 269] and the case $d = 1, 2$ is similar to Chapter 12.

The case $d = 3$: We write

$$V^{1/2} M_{T,\Delta} |V|^{1/2} = V^{1/2} \frac{1}{p^2} |V|^{1/2} + V^{1/2} \left(M_{T,\Delta} - \frac{1}{p^2} \chi_{|p| > \sqrt{2\mu}} \right) |V|^{1/2} - V^{1/2} \frac{1}{p^2} \chi_{|p| \leq \sqrt{2\mu}} |V|^{1/2}. \tag{13.A.3}$$

The first term in Equation (13.A.3) has kernel (proportional to)

$$V(x)^{1/2} \frac{1}{|x-y|} |V(y)|^{1/2} \in L^2(\mathbf{R}^3 \times \mathbf{R}^3) \tag{13.A.4}$$

by the Hardy–Littlewood–Sobolev inequality [418, Theorem 4.3]. The kernel of the second term in Equation (13.A.3) is given by

$$\begin{aligned} V(x)^{1/2} |V(y)|^{1/2} \frac{1}{(2\pi)^3} \Bigg[& \int_{|p| < \sqrt{2\mu}} \frac{1}{K_T^{\Delta}(p)} \left(e^{ip(x-y)} - e^{i\sqrt{\mu} p/|p|(x-y)} \right) \mathrm{d}p \\ & + \int_{|p| > \sqrt{2\mu}} \left(\frac{1}{K_T^{\Delta}(p)} - \frac{1}{p^2} \right) e^{ip(x-y)} \mathrm{d}p \Bigg]. \end{aligned} \tag{13.A.5}$$

We compute the angular integral first. In the first term the integral is

$$4\pi \int_0^{\sqrt{2\mu}} \frac{1}{K_T^{\Delta}(p)} \left[\frac{\sin|p||x-y|}{|p||x-y|} - \frac{\sin\sqrt{\mu}|x-y|}{\sqrt{\mu}|x-y|} \right] |p|^2 \mathrm{d}|p|. \tag{13.A.6}$$

Here we bound $\left|\frac{\sin a}{a} - \frac{\sin b}{b}\right| \le C\frac{|a-b|}{a+b}$ for $a, b > 0$. Thus we get the bound

$$|(13.A.6)| \le C\int_0^{\sqrt{2\mu}} \frac{1}{K_T^\Delta(p)} \frac{\left|p-\sqrt{\mu}\right|}{p+\sqrt{\mu}} p^2 \mathrm{d}p \le C\int_0^{\sqrt{2\mu}} \frac{1}{|p^2-\mu|} \frac{\left|p-\sqrt{\mu}\right|}{p+\sqrt{\mu}} p^2 \mathrm{d}p \le C.$$

In the second term the integral is (bounded by)

$$4\pi\int_{\sqrt{2\mu}}^\infty \left|\frac{1}{K_T^\Delta(p)} - \frac{1}{p^2}\right| \frac{|\sin|p||x-y||}{|p||x-y|} |p|^2 \mathrm{d}|p| \le \frac{4\pi}{|x-y|}\int_{\sqrt{2\mu}}^\infty \left|\frac{1}{K_T^\Delta(p)} - \frac{1}{p^2}\right| p\mathrm{d}p.$$

To bound the remaining integral we bound

$$\left|\frac{1}{K_T^\Delta(p)} - \frac{1}{p^2}\right| \le \frac{\left|\tanh\frac{\sqrt{|p^2-\mu|^2+\Delta(p)^2}}{2T} - 1\right|}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}} + \left|\frac{1}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}} - \frac{1}{|p^2-\mu|}\right| + \left|\frac{1}{|p^2-\mu|} - \frac{1}{p^2}\right|.$$

Note first that $|\tanh x - 1| \le 2e^{-2x}$. Thus, we have

$$\begin{aligned}\int_{\sqrt{2\mu}}^\infty \frac{\left|\tanh\frac{\sqrt{|p^2-\mu|^2+\Delta(p)^2}}{2T} - 1\right|}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}} p\mathrm{d}p &\le 2\int_{\sqrt{2\mu}}^\infty e^{-\sqrt{|p^2-\mu|^2+\Delta(p)^2}/T} \frac{1}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}} p\mathrm{d}p \\ &\le 2\int_{\sqrt{2\mu}}^\infty e^{-|p^2-\mu|/T} \frac{p}{|p^2-\mu|}\mathrm{d}p \le CT.\end{aligned}$$

Next, we estimate

$$\begin{aligned}&\left|\frac{1}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}} - \frac{1}{|p^2-\mu|}\right| \\ &\quad= \frac{1}{|p^2-\mu|} \frac{\Delta(p)^2}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}\left(|p^2-\mu| + \sqrt{|p^2-\mu|^2+\Delta(p)^2}\right)} \\ &\quad\le \frac{1}{|p^2-\mu|} \frac{\|\Delta\|_{L^\infty}^2}{\sqrt{|p^2-\mu|^2+\|\Delta\|_{L^\infty}^2}\left(|p^2-\mu| + \sqrt{|p^2-\mu|^2+\|\Delta\|_{L^\infty}^2}\right)} \le \frac{\|\Delta\|_{L^\infty}^2}{2|p^2-\mu|^3},\end{aligned} \tag{13.A.7}$$

using pointwise monotonicity in $\Delta(p)$. Thus, changing variables to $u = p^2 - \mu$ we have

$$\int_{\sqrt{2\mu}}^\infty \left|\frac{1}{\sqrt{|p^2-\mu|^2+\Delta(p)^2}} - \frac{1}{|p^2-\mu|}\right| p\mathrm{d}p \le \frac{1}{4}\|\Delta\|_{L^\infty}^2 \int_\mu^\infty \frac{1}{u^3}\mathrm{d}u \le C\|\Delta\|_{L^\infty}^2.$$

Finally,

$$\int_{\sqrt{2\mu}}^\infty \left|\frac{1}{|p^2-\mu|} - \frac{1}{p^2}\right| p\mathrm{d}p \le C.$$

We conclude that the kernel of the second term in Equation (13.A.3) is bounded by

$$|V(x)|^{1/2}\left(\frac{1+T+\|\Delta\|_{L^\infty}^2}{|x-y|} + 1\right)|V(y)|^{1/2} \in L^2(\mathbf{R}^3 \times \mathbf{R}^3).$$

Finally, the last term of Equation (13.A.3) has kernel

$$4\pi V(x)^{1/2}|V(y)|^{1/2}\int_0^{\sqrt{2\mu}} \frac{\sin p|x-y|}{p|x-y|}\mathrm{d}p \in L^2\left(\mathbf{R}^3 \times \mathbf{R}^3\right) \tag{13.A.8}$$

since the integral is bounded by $\sqrt{2\mu}$.

The cases $d = 1$ and $d = 2$: The kernel of $M_{T,\Delta}$ is given by

$$M_{T,\Delta}(x,y) = \frac{1}{(2\pi)^d}\left[\int_{|p|<\sqrt{2\mu}} \frac{1}{K_T^\Delta}\left(e^{ip(x-y)} - e^{i\sqrt{\mu}p/|p|(x-y)}\right) + \int_{|p|>\sqrt{2\mu}} \frac{1}{K_T^\Delta} e^{ip(x-y)}\right] \tag{13.A.9}$$

Now, one may bound $K_T^\Delta \geq |p^2 - \mu|$ uniformly in T, Δ. Then we may bound $M_{T,\Delta}$ exactly as in Lemma 12.3.5. That is, we have the bounds

$$\left\|V^{1/2} M_{T,\Delta} |V|^{1/2}\right\|_{\mathsf{HS}}^2 \lesssim \begin{cases} \|V\|_{L^1}^2 + \|V\|_{L^1} \int_{\mathbf{R}} |V(x)| \left[1 + \log(1 + \sqrt{\mu}|x|)\right]^2 \mathrm{d}x & d = 1, \\ \|V\|_{L^1}^2 + \|V\|_{L^p}^2 & d = 2 \end{cases}$$

for any $1 < p \leq 4/3$. This concludes the proof of Lemma 13.3.4. □

13.A.2.2 Proof of Lemma 13.3.6

To bound the difference, first note that by computing the angular integrals we have

$$\begin{aligned}\left[M_{T,\Delta} - M_{T_c,0}\right](x,y) = \frac{|\S^{d-1}|}{(2\pi)^d}\Bigg[&\int_0^{\sqrt{2\mu}} \left(\frac{1}{K_T^\Delta} - \frac{1}{K_{T_c}}\right)\left(j_d(p|x-y|) - j_d(\sqrt{\mu}|x-y|)\right) p^{d-1}\mathrm{d}p \\ &+ \int_{\sqrt{2\mu}}^\infty \left(\frac{1}{K_T^\Delta} - \frac{1}{K_{T_c}}\right) j_d(p|x-y|) p^{d-1}\mathrm{d}p\Bigg]\end{aligned} \tag{13.A.10}$$

where

$$j_d(x) = \frac{1}{|\S^{d-1}|}\int_{\S^{d-1}} e^{ix\omega}\mathrm{d}\omega = \begin{cases} \cos x & d = 1 \\ J_0(|x|) & d = 2 \\ \frac{\sin|x|}{|x|} & d = 3 \end{cases}$$

with J_0 the zero'th Bessel function.

Bounding Equation (13.A.10) is similar in spirit to the proof of Lemma 13.3.4 above. We bound

$$\left|\frac{1}{K_T^\Delta} - \frac{1}{K_{T_c}}\right| \leq \frac{\left|\tanh\frac{E_\Delta}{2T} - 1\right|}{E_\Delta} + \left|\frac{1}{E_\Delta} - \frac{1}{|p^2-\mu|}\right| + \frac{\left|1 - \tanh\frac{|p^2-\mu|}{2T_c}\right|}{|p^2-\mu|}.$$

We bound the first term as follows

$$\frac{\left|\tanh\frac{E_\Delta}{2T} - 1\right|}{E_\Delta} \leq 2e^{-E_\Delta/T}\frac{1}{E_\Delta} \leq 2e^{-|p^2-\mu|/T}\frac{1}{|p^2-\mu|} \leq 2e^{-|p^2-\mu|/T_c}\frac{1}{|p^2-\mu|}.$$

Similarly,

$$\frac{\left|1 - \tanh\frac{|p^2-\mu|}{2T_c}\right|}{|p^2-\mu|} \leq 2e^{-|p^2-\mu|/T_c}\frac{1}{|p^2-\mu|}.$$

Finally, we estimate, exactly as in (13.A.7) in the course of proving Lemma 13.3.4,

$$\begin{aligned}\left|\frac{1}{E_\Delta} - \frac{1}{|p^2-\mu|}\right| &\leq \frac{1}{|p^2-\mu|}\frac{\|\Delta\|_{L^\infty}^2}{\sqrt{|p^2-\mu|^2 + \|\Delta\|_{L^\infty}^2}\left(|p^2-\mu| + \sqrt{|p^2-\mu|^2 + \|\Delta\|_{L^\infty}^2}\right)} \\ &\leq \frac{\|\Delta\|_{L^\infty}^2}{2|p^2-\mu|^3}.\end{aligned}$$

We will use the first bound for the first integral in Equation (13.3.13) and the second bound for the second integral in Equation (13.3.13). Note further that $\frac{1}{\sqrt{x^2+A^2}\left(x+\sqrt{x^2+A^2}\right)}$ is decreasing in x and $|p^2-\mu| \geq \sqrt{\mu}|p-\sqrt{\mu}|$. That is, we have the bound

$$\left|\frac{1}{K_T^\Delta}-\frac{1}{K_{T_c}}\right| \le 4e^{-|p^2-\mu|/T_c}\frac{1}{|p^2-\mu|}+\chi_{|p|>\sqrt{2\mu}}\frac{\|\Delta\|_{L^\infty}^2}{2|p^2-\mu|^3}$$
$$+\chi_{|p|<\sqrt{2\mu}}\frac{1}{\sqrt{\mu}|p-\sqrt{\mu}|}\frac{\|\Delta\|_{L^\infty}^2}{\sqrt{\mu|p-\sqrt{\mu}|^2+\|\Delta\|_{L^\infty}^2}\left(\sqrt{\mu}|p-\sqrt{\mu}|+\sqrt{\mu|p-\sqrt{\mu}|^2+\|\Delta\|_{L^\infty}^2}\right)}. \quad (13.A.11)$$

In the first integral in Equation (13.A.10) we bound $|j_d(a)-j_d(b)|\le C|a-b|$. Then the contribution of the first term of Equation (13.A.11) to the first integral in Equation (13.A.10) is bounded by (changing variables to $s=\sqrt{\mu}|p-\sqrt{\mu}|/T_c$)

$$\int_0^{\sqrt{2\mu}} e^{-|p^2-\mu|/T_c}\frac{1}{|p^2-\mu|}|p-\sqrt{\mu}||x-y|p^{d-1}\mathrm{d}p \le C|x-y|\int_0^{\sqrt{2\mu}} e^{-\sqrt{\mu}|p-\sqrt{\mu}|/T_c}\mathrm{d}p$$
$$\le T_c|x-y|\int_0^{\mu/T_c} e^{-s}\mathrm{d}s \le CT_c|x-y|.$$

Next, the contribution of the last term of Equation (13.A.11) is bounded by (changing variables to $s=\sqrt{\mu}|p-\sqrt{\mu}|/\|\Delta\|_{L^\infty}$)

$$\int_0^{\sqrt{2\mu}} \frac{\|\Delta\|_{L^\infty}^2}{\sqrt{\mu|p-\sqrt{\mu}|^2+\|\Delta\|_{L^\infty}^2}\left(|p-\sqrt{\mu}|\sqrt{\mu}+\sqrt{\mu|p-\sqrt{\mu}|^2+\|\Delta\|_{L^\infty}^2}\right)}\frac{|p-\sqrt{\mu}||x-y|p^{d-1}}{|p-\sqrt{\mu}|\sqrt{\mu}}\mathrm{d}p$$
$$\le C\|\Delta\|_{L^\infty}|x-y|\int_0^{\mu/\|\Delta\|_{L^\infty}}\frac{1}{\sqrt{s^2+1}(s+\sqrt{s^2-1})}\mathrm{d}s \le C\|\Delta\|_{L^\infty}|x-y|.$$

Next we estimate the last integral of Equation (13.A.10). Here we note that $|j_d(a)|\le C$. Then the contributions of the first and second term in Equation (13.A.11) to Equation (13.A.10) is bounded by

$$\int_{\sqrt{2\mu}}^\infty e^{-|p^2-\mu|/T_c}\frac{1}{|p^2-\mu|}p^{d-1}\mathrm{d}p \le C\int_{\sqrt{2\mu}}^\infty e^{-|p^2-\mu|/T_c}|p^2-\mu|^{d/2-2}p\mathrm{d}p \le CT_ce^{-\mu/T_c}\le CT_c$$
$$\text{and}\qquad \int_{\sqrt{2\mu}}^\infty \frac{\|\Delta\|_{L^\infty}^2}{|p^2-\mu|^3}p^{d-1}\mathrm{d}p \le C\|\Delta\|_{L^\infty}^2 .$$

We conclude that (using $\|\Delta\|_{L^\infty}\le CT_c$)

$$\left\|V^{1/2}(M_{T,\Delta}-M_{T_c,0})|V|^{1/2}\right\|_{\mathrm{HS}}^2 \le CT_c^2\iint |V(x)||V(y)|(\mu|x-y|^2+1)\mathrm{d}x\mathrm{d}y$$
$$\le CT_c^2\left(\mu\left\||\cdot|^2V\right\|_1\|V\|_1+\|V\|_1^2\right)\le Ce^{-c/\lambda}$$

by assumption on V. This finishes the proof of Lemma 13.3.6. □

13.A.2.3 Proof of Lemma 13.3.12

The proof is very similar to the ones of [312, Lemma 4] and Lemma 11.2.8 and follows from a Birman–Schwinger argument analogously to Sections 13.3.2.3 and 13.3.2.4.

First of all, recall from Proposition 13.1.5 (ii), that $K_{T_c}+\lambda V$ has 0 as a (non-degenerate) ground state eigenvalue, which, by the Birman–Schwinger principle, is equivalent to the fact that the Birman-Schwinger operator $B_{T_c}:=\lambda V^{1/2}K_{T_c}^{-1}|V|^{1/2}$ has -1 as its (non-degenerate) ground state eigenvalue. As in Section 13.3.2.3, defining $m(T_c):=m(T_c,0)$ (recall (13.3.11)), we decompose B_{T_c} as

$$B_{T_c}=\lambda m(T_c)V^{1/2}\mathfrak{F}_\mu^\dagger\mathfrak{F}_\mu|V|^{1/2}+\lambda V^{1/2}M_{T_c}|V|^{1/2},$$

where M_{T_c} is such that this holds. It has been shown in [269, Lemma 2] (for $d=3$) and Lemma 12.3.5 (for $d=1,2$), that the Hilbert-Schmidt norm $\|V^{1/2}M_{T_c}|V|^{1/2}\|_{\mathrm{HS}}$ of the second term is uniformly bounded for small T_c (i.e. small λ).

Then, by an argument completely analogous to the one in the proof of Lemma 13.3.5 in Section 13.3.2.4 we find that $\Delta_0 = f(\lambda)\big[\hat{\varphi} + \lambda\eta_\lambda\big]$ with $\hat{\varphi}$ defined in (13.3.24) and η_λ has $\|\eta_\lambda\|_{L^\infty} \leq C$ uniformly in small λ (cf. (13.3.25)). This concludes the proof of Lemma 13.3.12. □

Chapter 14

Multi-band superconductors have enhanced critical temperatures

This chapter contains the paper [335]:

J. Henheik, E. Langmann, and A. B. Lauritsen. Multi-band superconductors have enhanced critical temperatures. *arXiv:2409.17297*, 2024

Abstract. We introduce a multi-band BCS free energy functional and prove that for a multi-band superconductor the effect of inter-band coupling can only increase the critical temperature, irrespective of its attractive or repulsive nature and its strength. Further, for weak coupling and weaker inter-band coupling, we prove that the dependence of the increase in critical temperature on the inter-band coupling is (1) linear, if there are two or more equally strongly superconducting bands, or (2) quadratic, if there is only one dominating band.

Content

14.1 Introduction and main results

Shortly after the development of the celebrated Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [48], Suhl, Matthias and Walker [542], and independently Moskalenko [457], introduced an extension of BCS theory allowing for more complex electronic band structures. These models for *multi-band superconductors* were subsequently theoretically studied, e.g. by Kondo [375] and Leggett [410] in the 1960s.

Despite these quite early modifications of BCS theory, it took around two decades until the first experimental realization [82] of multi-band superconductivity in Nb doped $\mathrm{SrTiO_3}$. Still, (probably) due to the relatively low critical temperature T_c (below which the material becomes superconducting — see (14.1.8) below for a mathematical definition), interest in multi-band superconductivity remained small for another two decades. This changed with proposals that high-temperature superconductivity in cuprates [53] exhibit multi-band structure [383, 458]. The most flourishing period of research on multi-band superconductivity was kicked off by the discovery [464] of a relatively high $T_c \approx 39\mathrm{K}$ in the conventional superconductor $\mathrm{MgB_2}$, whose characteristic feature is the interaction of two different electronic bands [532]. After MgB_2, other materials, such as $\mathrm{NbSe_2}$ [351], or iron based high-temperature superconductors [501], were found to be multi-band superconductors.

One of the most important features of multi-band superconductors is a strong T_c-enhancing effect of the inter-band interactions. This was already pointed out by Kondo [375] and further studied in the physics literature [135, 132, 134, 133]. In particular either linear [134] or quadratic [132, 133, 135] T_c-enhancements are expected depending on the multi-band superconductor. The aim of this paper is to put these predictions on rigorous ground. We restrict ourselves to continuum models where the Brillouin zone is $\mathbf{R}^d$; we believe that our methods would be applicable also to models with other Brillouin zones, see Remark 14.A.1 for further details. We also mention that the results in the present paper are restricted to the leading order in the weak coupling parameter λ. It would be interesting to extend our results to higher order, in generalization of such results in the one-band case [312, 400, 399], but this is beyond the scope of the present paper.

Multi-band BCS theory has not been studied much in the mathematical physics literature with the exception being the work of Yang [589]. In [589], however, rather restrictive assumptions on the interaction are imposed. In this paper, we give a much more general mathematical formulation of multi-band BCS theory, similarly to the single-band setting by Hainzl, Seiringer and others [309, 316] and study the effect of multi-band interactions on the critical temperature of the system. As our main results, we prove that:

Proposition 14.1.4. Inter-band couplings can only increase the critical temperature T_c, irrespective of its attractive or repulsive nature and its strength.

Theorem 14.1.6. For weak coupling and weaker inter-band coupling, T_c depends either (1) linearly or (2) quadratically on the inter-band coupling for (1) two or more equally strongly superconducting bands or (2) a single dominating band.

Structure of the paper. In Section 14.1.1, we provide the mathematical formulation of multi-band BCS theory. Afterwards, in Section 14.1.2, we formulate our main results, whose proofs are given in Section 14.2. In Section 14.A we give a heuristic derivation of the multi-band BCS functional (14.1.4) from a many-body Hamiltonian on a physics level of rigor, and in Section 14.B we collect some basic notation used in the paper.

14.1.1 Multi-band functional, gap equation, and critical temperature

We consider a gas of fermions in $\mathbf{R}^d$ for $d = 1, 2, 3$ at temperature $T > 0$. The particles are assumed to occupy $n \in \mathbf{N}$ *bands* (alternatively, they come in n different species), characterized by different dispersion relations $\epsilon_a(p)$ (i.e. a relation between momentum p and energy ϵ) for $a = 1, ..., n$, which we assume to satisfy the following (cf. [314]).

Assumption 14.1.1 (On the dispersion relation)**.** *For every $a \in \{1, ..., n\}$, we have that the zero set of $\epsilon_a(p)$ is a manifold (a* generalized Fermi surface*)*

$$S_a := \{p \in \mathbf{R}^d : \epsilon_a(p) = 0\} \subset \mathbf{R}^d \tag{14.1.1}$$

of codimension one, which is not necessarily connected but consists of finitely many components. Moreover, there exists some $\sigma > 0$ and a compact neighborhood $\Omega \subset \mathbf{R}^d$ of S_a containing S_a, such that $\operatorname{dist}(S_a, \Omega^c) \geq \sigma$ (Ω^c is the complement of Ω in $\mathbf{R}^d$). For ϵ_a we further assume that

(i) it is locally bounded, measurable, reflection-symmetric ($\epsilon_a(-p) = \epsilon_a(p)$) and satisfies $\epsilon_a \in C^2(\Omega)$;

(ii) its gradient $\nabla\epsilon_a(p)$ does not vanish in Ω;

(iii) there exist constants $c, C > 0$ such that $\epsilon_a(p) \geq cp^2 - C$ for all $a \in \{1, ..., n\}$.

Here and in the following we use the convention that C denotes any positive constant and its value may change line by line.

Assumption 14.1.1 is satisfied by all relevant (non-relativistic) dispersion relations, in particular the *Sommerfeld band dispersion relation*

$$\epsilon_a(p) := \frac{p^2}{2m_a} - \mu_a \tag{14.1.2}$$

with *effective mass* $m_a > 0$ and *effective chemical potential* $\mu_a > 0$. In previous works on superconductivity in the single-band case (see, e.g., [309]), the authors always (with the exception of [314]) chose (14.1.2) with mass being set to $1/2$ by simple scaling. However, since, even if we restrict to Sommerfeld dispersion relations of the form (14.1.2) only, effective masses and effective chemical potentials can and will be different in different bands, this cannot be achieved in general in our multiband setting. Therefore, we keep the most general form of ϵ_a as specified in Assumption 14.1.1. This also allows for non-spherical Fermi surfaces S_a. In the physics literature, many multi-band models arise from one-band models with a non-spherical Fermi surface, and the Fermi surfaces in the emerging (multiple) bands are then also possibly non-spherical; see e.g. [437, 20, 398] (in the context of the Eliashberg theory).

The interaction between fermions in bands a and b is described by a two-body potential V_{ab}, for which we assume the following.

Assumption 14.1.2 (On the interaction potential)**.** *For any $a, b \in \{1, \dots, n\}$ the interaction $V_{ab} = V_{ba} \in L^1(\mathbf{R}^d) \cap L^{p_V}(\mathbf{R}^d)$ is real-valued and reflection-symmetric (meaning $V_{ab}(-x) = V_{ab}(x)$) with $p_V = 1$ if $d = 1$, $p_V \in (1, \infty)$ if $d = 2$, or $p_V = 3/2$ if $d = 3$.*

We stress that the class of models we consider is large. It includes multi-orbital models obtained from a one-band model where the dispersion relation is rotation invariant but the two-body interaction potential is not; in such a case, a can be identified with the angular momentum ℓ (which is $0, 2, \dots$ for s-, d-wave, and higher even angular momenta, respectively), and V_{ab} for $a \neq b$ are interactions between different angular momentum channels. Another example are multiband models with Sommerfeld dispersion relations and rotation invariant interactions within and inbetween different bands; see Example 14.1.9 for details.

A multi-band BCS state Γ is given by n pairs of functions $(\gamma_a, \alpha_a)_{a=1}^n$ and can be conveniently represented as a $2n \times 2n$ matrix valued Fourier multiplier on $L^2(\mathbf{R}^d; \mathbf{C}^n) \oplus L^2(\mathbf{R}^d; \mathbf{C}^n)$ of the form

$$\hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & \mathbf{1} - \hat{\gamma}(p) \end{pmatrix}, \qquad \gamma = \operatorname{diag}[\gamma_a]_{a=1}^n, \qquad \alpha = \operatorname{diag}[\alpha_a]_{a=1}^n \tag{14.1.3}$$

for all $p \in \mathbf{R}^d$ (the bar indicates complex conjutation). Here, for every band $a = 1, ..., n$, $\hat{\gamma}_a(p)$ denotes the Fourier transform of the one particle density matrix and $\hat{\alpha}_a(p)$ is the Fourier transform of the

Cooper pair wave function, both in band a. We require reflection symmetry of $\hat{\alpha}_a$, i.e. $\hat{\alpha}_a(-p) = \hat{\alpha}_a(p)$, as well as $0 \leq \hat{\Gamma}(p) \leq 1$ as a matrix. In this paper, we study the following multi-band version of the standard BCS free energy functional [411, 309] (see also [313, 316, 312, 311, 403] and Chapters 10–11), which we will derive from a many-body Hamiltonian [542, 457, 375, 410] in Appendix 14.A on a physics level of rigor. It is given by

$$\mathcal{F}_T[\Gamma] := \int_{\mathbf{R}^d} \sum_{a=1}^{n} \epsilon_a(p)\hat{\gamma}_a(p)\mathrm{d}p - TS[\Gamma] + \int_{\mathbf{R}^d} \sum_{a,b=1}^{n} V_{ab}(x)\overline{\alpha_a(x)}\alpha_b(x)\mathrm{d}x\,, \tag{14.1.4}$$

where entropy per unit volume is defined as

$$S[\Gamma] = -\int_{\mathbf{R}^d} \mathrm{Tr}_{\mathbf{C}^{2n}}\left[\hat{\Gamma}(p)\log\hat{\Gamma}(p)\right]\mathrm{d}p\,. \tag{14.1.5}$$

The variational problem associated with the BCS functional (14.1.4) is studied on

$$\mathcal{D} := \left\{\Gamma \text{ as in (14.1.3)} : 0 \leq \hat{\Gamma} \leq 1,\ \hat{\gamma}_a \in L^1(\mathbf{R}^d, (1+p^2)\mathrm{d}p),\ \alpha_a \in H^1_{\mathrm{sym}}(\mathbf{R}^d, \mathrm{d}x),\ a = 1, ..., n\right\}.$$

Here H^1_{sym} denotes the set of reflection-symmetric H^1-functions. The following proposition, whose proof is completely analogous to those in [309, 316], and so omitted, provides the foundation for studying this problem.

Proposition 14.1.3. *Under Assumption 14.1.2 on V, the BCS free energy is bounded below on $\mathcal{D}$ and attains its minimum.*

The associated Euler-Lagrange equation is easily found to be

$$(K_T^{\Delta} + V)\alpha = 0, \qquad K_T^{\Delta} = \mathrm{diag}\left[K_{T,a}^{\Delta_a}\right]_{a=1}^{n} \tag{14.1.6}$$

where

$$K_{T,a}^{\Delta_a}(p) = \frac{\sqrt{\epsilon_a(p)^2 + |\Delta_a(p)|^2}}{\tanh\left(\frac{\sqrt{\epsilon_a(p)^2+|\Delta_a(p)|^2}}{2T}\right)}\,.$$

Here, $V = (V_{ab})_{a,b=1}^n$ is the matrix of interactions, and we denoted the vector of gaps by $\Delta(p) = -2\,(2\pi)^{-d/2}(\hat{V} \star \hat{\alpha})(p)$ with $(\hat{V} \star \hat{\alpha})(p) := \int_{\mathbf{R}^d} \hat{V}(p-q)\hat{\alpha}(q)\mathrm{d}q$ the convolution. Further, we define $E_a(p) = \sqrt{\epsilon_a(p)^2 + |\Delta_a(p)|^2}$, the modified dispersion relation(s) arising from the gap function(s) Δ_a.

An equivalent form of (14.1.6) is the following natural analog of the celebrated (standard single-band, see [309]) *BCS gap equation*, given by

$$\Delta_a(p) = -\frac{1}{(2\pi)^{d/2}} \sum_{b=1}^{n} \int_{\mathbf{R}^d} \hat{V}_{ab}(p-q)\frac{\tanh\left(\frac{E_b(q)}{2T}\right)}{E_b(q)}\Delta_b(q)\mathrm{d}q\,, \qquad a = 1, \ldots, n. \tag{14.1.7}$$

Written without the indices, the gap equation takes the following form, where the relevant objects are matrix-valued, $\Delta(p) = -(2\pi)^{-d/2}\int_{\mathbf{R}^d} \hat{V}(p-q)K_T^{\Delta}(q)^{-1}\Delta(q)\mathrm{d}q$.

The system described by the functional $\mathcal{F}_T$ is *superconducting* if and only if any minimizer Γ of $\mathcal{F}_T$ has a non-vanishing vector of off-diagonal entries, $\alpha \not\equiv 0$ (or, equivalently, (14.1.7) has a solution $\Delta \not\equiv 0$). The (a priori highly non-linear) question, whether a system is superconducting or not can be reduced to a *linear* criterion involving the matrix-valued pseudo-differential operator with symbol $K_T(p) \equiv K_T^0(p)$. In fact, as can be shown completely analogously to [309], the system is superconducting if and only if the (matrix-valued) operator $K_T + V$ has at least one negative eigenvalue. Moreover, there exists a unique *critical temperature* $T_c \geq 0$ being defined as

$$\boxed{T_c := \inf\{T > 0 : K_T + V \geq 0\}}\,, \tag{14.1.8}$$

for which $K_{T_c} + V \geq 0$ and $\inf\mathrm{spec}(K_T + V) < 0$ for all $T < T_c$. It can easily be seen that, by Assumption 14.1.2 and the asymptotic behavior $K_{T_c}(p) \gtrsim p^2$ for $|p| \to \infty$, the critical temperature is well-defined by invoking Sobolev's inequality [418, Thm. 8.3].

14.1.2 Main results

In this paper, we study the effect of the interband coupling due to V_{ab} for $a \neq b$ on the critical temperature T_c. More concretely, we rescale the original interaction matrix V as

$$V \to \lambda V^{\mathrm{d}} + \kappa\lambda V^{\mathrm{od}} \qquad \text{with} \qquad \lambda > 0 \quad \text{and} \quad \kappa \in \mathbf{R}\,, \tag{14.1.9}$$

where V^{d} denotes the diagonal part of V and V^{od} the off-diagonal part. We will in particular consider the scaling in (14.1.9) in the limit of weak coupling, $\lambda \ll 1$. The parameter $\kappa \in \mathbf{R}$ regulates the relative strength between the intra-band coupling V^{d} and inter-band coupling V^{od}. We point out that κ does not have a sign, which means that the inter-band coupling can be either attractive or repulsive. To indicate the dependence on the parameters λ and κ, we shall henceforth write $T_c = T_c(\lambda, \kappa)$.

Similar to previous works [186, 312, 314] and Chapter 12, in the weak coupling limit, a special role is played by the self-adjoint trace-class operator $\mathcal{V} : \bigoplus_{a=1}^n L^2(S_a) \to \bigoplus_{a=1}^n L^2(S_a)$, measuring the strength of the interaction matrix V on the Fermi surfaces S_a and S_b whose action is defined as

$$(\mathcal{V}u)_a(p) := \sum_{b=1}^n (\mathcal{V}_{ab}u_b)(p) := \sum_{b=1}^n \frac{1}{(2\pi)^{d/2}} \frac{2}{\sqrt{|\nabla\epsilon_a(p)|}} \int_{S_b} \frac{\hat{V}_{ab}(p-q)}{\sqrt{|\nabla\epsilon_b(q)|}} u_b(q)\mathrm{d}\omega(q)\,. \tag{14.1.10}$$

Here, $\mathcal{V}_{ab}$ maps $L^2(S_b) \to L^2(S_a)$ and we note that $\mathcal{V}_{ba} = \mathcal{V}_{ab}^*$. Moreover, corresponding to the decomposition of V into diagonal and off-diagonal part in (14.1.9), we shall also write $\mathcal{V} = \mathcal{V}^{\mathrm{d}} + \kappa\mathcal{V}^{\mathrm{od}}$. Finally, note that the pointwise evaluation of $\hat{V}_{ab}$ (and in particular on a nice codim–1 submanifold – recall Assumption 14.1.1) is well-defined since we assume $V \in L^1(\mathbf{R}^d)$.

Our first result is that the effect of the inter-band coupling can only increase the critical temperature:

Proposition 14.1.4 (Increase of critical temperature)**.** *Let $d \in \{1,2,3\}$ and let the dispersion relations ϵ_a satisfy Assumption 14.1.1 and the interaction matrix $V = (V_{ab})_{a,b=1}^n$ satisfy Assumption 14.1.2. Assume in addition that $V^{\mathrm{od}} \not\equiv 0$.*

Then, for any $\lambda > 0$ there exists $\kappa_c^{\pm} \in [0,\infty)$ such that

- *For $\kappa \in [-\kappa_c^-, \kappa_c^+]$ we have $T_c(\lambda,\kappa) = T_c(\lambda,0)$, and*
- *For $\kappa \notin [-\kappa_c^-, \kappa_c^+]$ we have $T_c(\lambda,\kappa) > T_c(\lambda,0)$.*

The proof of Proposition 14.1.4 is given in Section 14.2.

Remark 14.1.5. *In particular we note that $T_c(\lambda,\kappa) > 0$ for large $|\kappa|$* even if *$T_c(\lambda,0) = 0$. As an example we can consider a system with $V_{ab} \geq 0$ for all a,b. For $\kappa \geq 0$ then all intra- and inter-band interactions are repulsive. At no inter-band coupling we have $T_c(\lambda,0) = 0$, since a repulsive single-band system is never superconducting. However, for κ large enough, the system becomes superconducting,* even though also the inter-band interactions are repulsive.

For our second (main) result we will assume that at least one of the intra-band interactions V_{aa} has an attractive part on the Fermi surface, meaning that

$$\mathfrak{e}_a := \inf \operatorname{spec} \mathcal{V}_{aa} \tag{14.1.11}$$

is strictly negative for at least one $a = 1,\dots,n$. Since the trace of $\mathcal{V}_{aa}$ can be computed as $\operatorname{Tr}(\mathcal{V}_{aa}) = 2(2\pi)^{-d/2}\hat{V}_{aa}(0)\int_{S_a}|\nabla\epsilon_a(p)|^{-1}\mathrm{d}\omega(p)$, a sufficient condition for $\mathfrak{e}_a < 0$ is that $\int V_{aa} < 0$. Finally, for every $a \in \{1,\dots,n\}$, we denote the ground state space of $\mathcal{V}_{aa}$ by

$$\mathcal{L}_a := \operatorname{span}\left\{u \in L^2(S_a) : (\mathcal{V}_{aa} - \mathfrak{e}_a)u = 0\right\}\,. \tag{14.1.12}$$

We can now formulate our main result, the proof of which is given in Section 14.2.

Theorem 14.1.6 (Weak coupling). *Let $d \in \{1,2,3\}$ and assume that ϵ and V satisfy Assumptions Assumption 14.1.2 and 14.1.1. Assume in addition that $\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |V_{ab}(x)||x-y|^2|V_{a'b'}(y)|\mathrm{d}x\mathrm{d}y < \infty$ for all $a,b,a',b' \in \{1,...,n\}$ and that $\min_{a\in\{1,...,n\}} \mathfrak{e}_a < 0$.*

Then we have the following:

(i) There exist constants $A_1^{\pm} \in [0,\infty)$ such that for small λ and $|\kappa|$ we have

$$\log\left(\frac{T_c(\lambda,\kappa)}{T_c(\lambda,0)}\right) = A_1^{\mathrm{sgn}(\kappa)}\,|\kappa|\lambda^{-1} + O(\kappa^2\lambda^{-1}) + O(\kappa). \tag{14.1.13}$$

We have $A_1^{\pm} > 0$ if and only if there exist a least two minima $\hat{a}_1, \hat{a}_2 \in \{1,...,n\}$ of $a \mapsto \mathfrak{e}_a$ and functions $u_{\hat{a}_i} \in \mathcal{L}_{\hat{a}_i}$ for $i = 1,2$ such that the quadratic form $\langle u_{\hat{a}_1}, \mathcal{V}_{\hat{a}_1\hat{a}_2}u_{\hat{a}_2}\rangle_{L^2(S_{\hat{a}_1})} = \langle \mathcal{V}_{\hat{a}_2\hat{a}_1}u_{\hat{a}_1}, u_{\hat{a}_2}\rangle_{L^2(S_{\hat{a}_2})} \neq 0$ does not vanish.

If $\hat{a}_1, \hat{a}_2$ are the only minima of $a \mapsto \mathfrak{e}_a$ and $\dim \mathcal{L}_{\hat{a}_i} = 1$ for at least one $i = 1,2$, it holds that $A_1^+ = A_1^-$.

(ii) Suppose that the minimizer $\hat{a}$ of $a \mapsto \mathfrak{e}_a$ is unique. Then there exists a constant $A_2 \in [0,\infty)$ such that for small λ and $|\kappa|$

$$\log\left(\frac{T_c(\lambda,\kappa)}{T_c(\lambda,0)}\right) = A_2\kappa^2\lambda^{-1} + O(\kappa^3\lambda^{-1}) + O(\kappa^2)\,. \tag{14.1.14}$$

We have $A_2 > 0$ if and only if $\mathcal{V}_{a\hat{a}}\big|_{\mathcal{L}_{\hat{a}}} \not\equiv 0$ for some $a \neq \hat{a}$.

We now informally interpret Proposition 14.1.4 and Theorem 14.1.6 in the following Remark 14.1.7.

Remark 14.1.7 (Qualitative interpretation of our main results). *Proposition 14.1.4 says, in particular, that $T_c(\lambda,\kappa) \geq T_c(\lambda,0)$ with no assumptions on the coupling strengths λ and κ. They may be order one or even large. Thus, it may be understood as the statement:*

(1) In a multi-band superconductor, the critical temperature increases when invoking inter-band couplings, irrespective of both its attractive/repulsive nature and its strength.

Part (i) of Theorem 14.1.6 describes a degenerate case of (at least) two bands giving rise to (approximately) the same critical temperature. Assuming further that these (at least) two bands couple non-trivially then $A_1^{\pm} > 0$. Rewriting (14.1.13) *in this case, we have*

$$T_c(\lambda,\kappa) = T_c(\lambda,0)\exp\left[\frac{A_1^{\mathrm{sgn}(\kappa)}}{\lambda^2}|\kappa\lambda|\,[1 + O(\lambda) + O(\kappa)]\right] \approx T_c(\lambda,0)\exp\left[\frac{A_1^{\mathrm{sgn}(\kappa)}}{\lambda^2}|\kappa\lambda|\right]. \tag{14.1.15}$$

Thus, part (i) may be understood as

(2) In the degenerate case of (at least) two equally strong bands, this increase in the critical temperature is linear *for small inter-band coupling strengths $\kappa\lambda$.*

This effect has been previously observed in the physics literature [134], in contrast with the usual quadratic enhancement. Moreover, for three (or more) bands that give rise to the same critical temperature in the absence of interband couplings, the slope of the enhancement will generically depend on the sign of κ, i.e., $A_1^+ \neq A_1^-$; see e.g. [594].

Part (ii) of Theorem 14.1.6 describes the generic case of a unique band $\hat{a}$ being the strongest. Further, one usually also has that $\mathcal{V}_{a\hat{a}}\big|_{\mathcal{L}_{\hat{a}}} \not\equiv 0$ (meaning this band couples non-trivially to the rest).

Thus, Theorem 14.1.6 (ii) says that the constant A_2 generically takes value $A_2 > 0$. Hence, rewriting (14.1.14) *in that generic case, we have*

$$T_c(\lambda,\kappa) = T_c(\lambda,0)\exp\left[\frac{A_2}{\lambda^3}(\kappa\lambda)^2\left[1+O(\lambda)+O(\kappa)\right]\right] \approx T_c(\lambda,0)\exp\left[\frac{A_2}{\lambda^3}(\kappa\lambda)^2\right] \tag{14.1.16}$$

Thus, part (ii) may be understood as follows:

(3) In the generic case of one dominating band, this increase in the critical temperature is quadratic *for small inter-band coupling strengths $\kappa\lambda$.*

This effect has been previously observed in the physics literature [135, 132, 133]. In our approach, the quadratic enhancement of T_c eventually stems from second order perturbation theory for the Birman–Schwinger operator associated with $K_{T_c}+V$ from (14.1.8).

Remark 14.1.8. *As an illustrative simple example of where the 'linear' versus 'quadratic' increase arises, one may consider the eigenvalues of the 2×2 matrices*

$$\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}, \quad \lambda_{\max} = 1+\kappa, \qquad \text{and} \qquad \begin{bmatrix} 1 & \kappa \\ \kappa & 0 \end{bmatrix}, \quad \lambda_{\max} = \frac{1+\sqrt{1+4\kappa^2}}{2} = 1+\kappa^2+O(\kappa^4).$$

For the first matrix the dependence of the largest eigenvalue on κ is linear, while for the second it is quadratic for small κ; see also Example 14.1.10 below. This is the underlying effect distinguishing the two different cases in Theorem 14.1.6.

We conclude this section by providing an explicit expression for the constant A_2 in a simple example and assuming the generic case of a unique strongest band.

Example 14.1.9 (Explicit formula for A_2)**.** *Assuming Sommerfeld dispersion relations*[1]

$$\epsilon_a(p) = \frac{p^2}{2m_a} - \mu_a \tag{14.1.17}$$

with $m_a, \mu_a > 0$ for all $a \in \{1,...,n\}$ and radial interaction potentials V_{ab}, one can easily find an explicit expression for the constant A_2 in (14.1.14).

Assume that $\hat{a}$ is the unique minimum of $a \mapsto \mathfrak{e}_a$ and $\dim \mathcal{L}_{\hat{a}} = 1$, which corresponds to s-wave superconductivity (for $\dim \mathcal{L}_{\hat{a}} > 1$ the formulas below are similar). Let $e_1 \in \mathbf{R}^d$ be the unit vector in 1-direction[2] *and denote*

$$j_d(y) := \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} \mathrm{e}^{\mathrm{i}yp\cdot e_1}\mathrm{d}\omega(p) \quad \text{for} \quad y \in \mathbf{R}\,. \tag{14.1.18}$$

Moreover, for $a,b \in \{1,...,n\}$, let

$$\mathfrak{v}_{ab} := \frac{|\mathbb{S}^{d-1}|}{(2\pi)^d}(4m_am_b)^{d/4}(\mu_a\mu_b)^{\frac{d-2}{4}}\int_{\mathbf{R}^d} V_{ab}(x)j_d(\sqrt{2m_a\mu_a}|x|)j_d(\sqrt{2m_b\mu_b}|x|)\mathrm{d}x\,. \tag{14.1.19}$$

Note that $\mathfrak{v}_{\hat{a}\hat{a}} = \mathfrak{e}_{\hat{a}}$ and $\mathfrak{v}_{aa} \in \operatorname{spec}\mathcal{V}_{aa}$ for all $a \in \{1,...,n\}$, showing that $|\mathfrak{v}_{\hat{a}\hat{a}} - \mathfrak{v}_{aa}| \gtrsim 1-\delta_{\hat{a}a}$, with δ_{ab} the Kronecker delta.

[1] In case that all the dispersion relations are radially symmetric, the additional condition $\int_{\mathbf{R}^d}\int_{\mathbf{R}^d}|V_{ab}(x)||x-y|^2|V_{a'b'}(y)|\mathrm{d}x\mathrm{d}y < \infty$ from Theorem 14.1.6 can be relaxed, cf. [314, Theorem 2.1].

[2] By rotational invariance, one could have chosen any unit vector in $\mathbf{R}^d$. This shows that, in particular, j_d is real-valued.

With these notations (14.1.18)–(14.1.19)*, in Section 14.2, we prove the constant* A_2 *to be given by*

$$A_2 = \sum_{a \neq \hat{a}} \frac{|\mathfrak{v}_{a\hat{a}}|^2}{|\mathfrak{v}_{\hat{a}\hat{a}}|^2 |\mathfrak{v}_{\hat{a}\hat{a}} - \mathfrak{v}_{aa}|} \,. \tag{14.1.20}$$

Armed with (14.1.20)*, we find that* $A_2 > 0$ *if and only if* $\mathfrak{v}_{\hat{a}a} \neq 0$ *for some* $a \neq \hat{a}$*. On the one hand, as* $\mathfrak{v}_{\hat{a}a}$ *is essentially a Fourier transform, there surely exist ("ungeneric") potentials* $V_{\hat{a}a}$ *such that* $\mathfrak{v}_{\hat{a}a}$ *vanishes for whole intervals of* m*'s and* μ*'s (by compact support in the Fourier-type space). On the other hand, given* $V_{\hat{a}a}$ *with exponential decay at infinity, it is an elementary exercise, invoking analyticity in the* m *and* μ *parameters, to show that the set of such values for which* $\mathfrak{v}_{\hat{a}a} = 0$ *is isolated. Hence,* $A_2 > 0$ *is the generic scenario.*

Example 14.1.10 (Explicit formula for T_c for a two-band model)**.** *Consider the setting of radial interactions and Sommerfeld dispersions as in Example 14.1.9 above with* $\dim \mathcal{L}_a = 1$ *for* $a = 1, 2$ *(but not necessarily assuming that* $\inf \mathfrak{e}_a < 0$*). We further restrict to a two-band case and assume that for all* $\kappa \in \mathbf{R}$ *the ground state space of* $\mathcal{V} = \mathcal{V}^{\mathrm{d}} + \kappa \mathcal{V}^{\mathrm{od}}$ *is contained in* $\mathcal{L}_1 \oplus \mathcal{L}_2$*. (Physically this means that we have two coupled* s*-wave bands.) Then,* T_c *can be computed analytically in* κ*: In case that* $\mathfrak{v}_{\min}(\kappa) := \frac{\mathfrak{v}_{11}+\mathfrak{v}_{22}}{2} - \sqrt{\left(\frac{\mathfrak{v}_{11}-\mathfrak{v}_{22}}{2}\right)^2 + \kappa^2 |\mathfrak{v}_{12}|^2} < 0$ *(which happens, e.g., if* $\mathfrak{v}_{11} < 0$ *or* $\mathfrak{v}_{22} < 0$*), we claim that*

$$T_c(\lambda, \kappa) = T_0 \exp\left[\frac{1}{\lambda \mathfrak{v}_{\min}(\kappa) + O_\kappa(\lambda^2)} \right] \tag{14.1.21}$$

for some fixed temperature scale T_0*, and where the implicit constant in* $O_\kappa(\lambda^2)$ *depends on* κ*. Note that this expression for the critical temperature in a two-band superconductor already appeared in the seminal paper by Suhl–Matthias–Walker [542]. We point out that, even if* $\mathfrak{v}_{ab} > 0$ *for all* $a, b = 1, 2$*, i.e., all interactions are repulsive, we have* $\mathfrak{v}_{\min}(\kappa) < 0$ *for* $|\kappa|$ *large enough; recall also Remark 14.1.5. Further, we point out that the leading-order* κ*-dependence in* (14.1.21) *is linear respectively quadratic in case* $\mathfrak{v}_{11} = \mathfrak{v}_{22}$ *respectively* $\mathfrak{v}_{11} \neq \mathfrak{v}_{22}$*, matching the two different settings in Theorem 14.1.6; recall also Remark 14.1.8. Finally, on the other hand, if* $\mathfrak{v}_{\min}(\kappa) > 0$*, then* $T_c(\lambda, \kappa) = 0$ *for* λ *small enough.*

The formula (14.1.21) *above follows from the proof of Theorem 14.1.6. More concretely, inspecting the proof of Theorem 14.1.6 below and expanding the formula* $-1 = \inf \operatorname{spec} S_{T_c(\lambda,\kappa)}$ *to first order in* λ*, the formula follows.*

In the following Section 14.2 we prove Proposition 14.1.4, Theorem 14.1.6 and the explicit formula (14.1.20) for A_2.

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14.2 Proofs

We first give the

Proof of Proposition 14.1.4. Consider (14.1.8). At $\kappa = 0$ and $T = T_c(\lambda, 0)$ we thus find for any $\lambda > 0$ that $K_{T_c(\lambda,0)} + \lambda V^{\mathrm{d}} \geq 0$. Define then the function by the variational principle

$$f(\kappa) = \inf \operatorname{spec}(K_{T_c(\lambda,0)} + \lambda V^{\mathrm{d}} + \lambda \kappa V^{\mathrm{od}}) = \inf_{\psi : \|\psi\|_{L^2} = 1} \left\langle \psi \middle| K_{T_c(\lambda,0)} + \lambda V^{\mathrm{d}} + \lambda \kappa V^{\mathrm{od}} \middle| \psi \right\rangle .$$

Being an infimum over affine functions, f is concave in κ. Moreover, since V^{od} is off-diagonal f assumes a local maximum at $\kappa = 0$, again by the variational principle. It follows that $f(\kappa) \leq 0$ for all κ and with equality only on some (possibly infinite) interval $[-\kappa_c^-, \kappa_c^+]$. Since K_T is a monotone increasing function of T then $T_c(\lambda, \kappa) \geq T_c(\lambda, 0)$ with strict inequality outside the interval $[-\kappa_c^-, \kappa_c^+]$.

To see that the interval $[-\kappa_c^-, \kappa_c^+]$ is finite we note that, since V^{od} is off-diagonal, we can find a function $\psi \in H^1$ (of finite kinetic energy) with $\langle\psi|\lambda V^{\mathrm{od}}|\psi\rangle \leq -e < 0$ for some $e > 0$. Then, by Sobolev's inequality [418, Theorem 8.3], we have $\langle\psi|K_{T_c(0)} + \lambda V^{\mathrm{d}}|\psi\rangle \leq C$ and so, by the variational principle

$$f(\kappa) \leq \langle\psi|K_{T_c(\lambda,0)} + \lambda V^{\mathrm{d}} + \lambda\kappa V^{\mathrm{od}})|\psi\rangle \leq C - \kappa e < 0, \tag{14.2.1}$$

for $\kappa > C/e$. Thus $\kappa_c^+ < C/e$. Similarly κ_c^- is finite. By (14.1.8), this concludes the proof. □

Next, we give the

Proof of Theorem 14.1.6. First, we note that $\kappa = 0$ correspond to decoupled one-band models. Thus, $T_c(\lambda, 0) > 0$ by [269]. In particular then by Proposition 14.1.4 we have $T_c(\lambda, \kappa) \geq T_c(\lambda, 0) > 0$.

Next, analogously to the single-band case [269, 312, 311, 311] and Chapters 10 and 12, we use the Birman–Schwinger principle to relate spectral properties of the unbounded operator $K_T + \lambda V$ to the compact Birman–Schwinger operator

$$B_T := \lambda V^{1/2} K_T^{-1} |V|^{1/2}. \tag{14.2.2}$$

In (14.2.2), we used a polar decomposition $V = U|V|$ for the self-adjoint interaction matrix V and denoted $V^{1/2} := U|V|^{1/2}$. Note that B_T has real spectrum: Indeed, B_T is isospectral to[3] the self-adjoint operator $\lambda K_T^{-1/2}|V|^{1/2}V^{1/2}K_T^{-1/2} = \lambda K_T^{-1/2} V K_T^{-1/2}$ since U and $|V|$ commute. The Birman-Schwinger principle then says that -1 is the lowest eigenvalue of B_T exactly for $T = T_c$, see [269].

Further, we decompose the Birman–Schwinger operator into a dominant singular and a bounded error term as

$$B_T = \lambda \log\left(\frac{T_0}{T}\right) V^{1/2}\mathfrak{F}^\dagger\mathfrak{F}|V|^{1/2} + \lambda V^{1/2} M_T |V|^{1/2} \tag{14.2.3}$$

with all the operators in (14.2.3) being matrices and $T_0 > 0$ a fixed reference temperature. More precisely, we introduced the rescaled restricted Fourier transforms $\mathfrak{F} := \mathrm{diag}(\mathfrak{F}_a)_{a=1}^n$ with

$$\mathfrak{F}_a : L^1(\mathbf{R}^d) \to L^2(S_a), \quad \text{where} \quad (\mathfrak{F}_a\psi)(p) := \frac{1}{(2\pi)^{d/2}} \frac{\sqrt{2}}{\sqrt{|\nabla\epsilon_a(p)|}} \int_{\mathbf{R}^d} \mathrm{e}^{-\mathrm{i}p\cdot x}\psi(x)\mathrm{d}x\Big|_{p\in S_a}$$

and $M_T := \mathrm{diag}(M_{T,a})_{a=1}^n$ is such that (14.2.3) holds.

The boundedness of the second summand in (14.2.3) is the content of the following lemma, which is rather standard in the context of BCS theory. We give the proof below.

Lemma 14.2.1 (cf. [269, Lemma 2], Lemma 12.3.5, and [314, Lemma 3.2])**.**
Under the conditions of Theorem 14.1.6, we have that, uniformly in $T > 0$, $V^{1/2}M_T|V|^{1/2}$ is a bounded operator, $\sup_{T>0} \|V^{1/2}M_T|V|^{1/2}\| \leq C$.

Using Lemma 14.2.1, we find, by a similar argument as in Chapter 13 (see also [338, 312]), that

$$S_T := \lambda \log\left(\frac{T_0}{T}\right) \mathfrak{F}|V|^{1/2} \frac{1}{1 + \lambda V^{1/2} M_T |V|^{1/2}} V^{1/2}\mathfrak{F}^\dagger$$

[3] This follows from the general fact that $\mathrm{spec}(AB) \setminus \{0\} = \mathrm{spec}(BA) \setminus \{0\}$ for any bounded operators A, B. Moreover, in our case, 0 is in both spectra, since AB and BA are compact operators on an infinite-dimensional space.

has -1 as its lowest eigenvalue exactly for $T = T_c$. Similarly to Chapter 13 we wish to use this fact for the two settings with or without interband coupling. To do this we first note that, to leading order in λ, S_T is proportional to $\mathfrak{F} V \mathfrak{F}^\dagger = \mathcal{V}$. Thus, the ground states of S_T are among those of $\mathcal{V}$ for λ small enough. (The ground states of $\mathcal{V}$ may have different S_T-expectations to higher order in λ. The ground states of S_T are those with smallest higher-order terms.) As $\mathcal{V}$ depends on κ so does the ground state space of S_T. Denote this space by $\mathcal{L}^\kappa$.

Next, consider degenerate perturbation theory (in κ) for the operator S_T. From degenerate first order perturbation theory for the ground states, we find that any (normalized) ground state $u_\kappa \in \mathcal{L}^\kappa$ of $S_{T_c(\lambda,\kappa)}$ can be written as

$$u_\kappa = \frac{u^{(0)} + \kappa u^{(1)}}{\sqrt{1 + \kappa^2 \left\langle u^{(1)} \middle| u^{(1)} \right\rangle}}, \quad u^{(0)} \in \mathcal{L}^0, \quad u^{(1)} \perp \mathcal{L}^0, \quad \left\langle u^{(0)} \middle| u^{(0)} \right\rangle = 1, \quad \left\langle u^{(1)} \middle| u^{(1)} \right\rangle \leq C.$$

Then, for small enough λ, we have $-1 = \left\langle u_\kappa \middle| S_{T_c(\lambda,\kappa)} \middle| u_\kappa \right\rangle$ and $-1 = \left\langle u^{(0)} \middle| S_{T_c(\lambda,0)} \middle| u^{(0)} \right\rangle$. Next, writing $\frac{1}{1+x} = 1 - \frac{x}{1+x}$ we decompose S_T as

$$S_T = \lambda \log\left(\frac{T_0}{T}\right) \left[\mathcal{V} - \lambda \mathfrak{F} |V|^{1/2} \frac{V^{1/2} M_T |V|^{1/2}}{1 + \lambda V^{1/2} M_T |V|^{1/2}} V^{1/2} \mathfrak{F}^\dagger \right].$$

At the critical temperature $T_c(\lambda, \kappa)$ we write the second term in $[\ldots]$ as $\lambda \mathcal{W}(\lambda, \kappa)$. Combining then for both with and without inter-band couplings we get (noting that u_κ and $u^{(0)}$ are ground states of $\mathcal{V}^{\mathrm{d}} + \kappa \mathcal{V}^{\mathrm{od}}$ and $\mathcal{V}^{\mathrm{d}}$ respectively)

$$\begin{aligned} &\lambda \log\left(\frac{T_c(\lambda,\kappa)}{T_c(\lambda,0)}\right) \\ &= \frac{1}{\inf \operatorname{spec}\left(\mathcal{V}^{\mathrm{d}} + \kappa \mathcal{V}^{\mathrm{od}}\right) - \lambda \left\langle u_\kappa | \mathcal{W}(\lambda,\kappa) | u_\kappa \right\rangle} - \frac{1}{\inf \operatorname{spec}\left(\mathcal{V}^{\mathrm{d}}\right) - \lambda \left\langle u^{(0)} \middle| \mathcal{W}(\lambda,0) \middle| u^{(0)} \right\rangle}. \end{aligned}$$

To evaluate this, we first note that, by simple perturbation theory for the compact self-adjoint operator $\mathcal{V}^{\mathrm{d}} + \kappa \mathcal{V}^{\mathrm{od}}$ up to second order

$$\inf \operatorname{spec}\left(\mathcal{V}^{\mathrm{d}} + \kappa \mathcal{V}^{\mathrm{od}}\right) = \mathfrak{e}_{\hat{a}} - U_1^{\operatorname{sgn}(\kappa)} |\kappa| - U_2^{\operatorname{sgn}(\kappa)} \kappa^2 + O(\kappa^3) \tag{14.2.4}$$

for some constants $U_1^\pm, U_2^\pm \geq 0$. (The signs of $U_1^\pm$ follows from the the fact that $T_c(\lambda,\kappa) \geq T_c(\lambda,0)$. The signs of $U_2^\pm$ is a general feature of second order perturbation theory.) Second, expanding $\langle u_\kappa | \mathcal{W}(\lambda,\kappa) | u_\kappa \rangle$ in powers of κ, we have by Lemma 14.2.1 (and using that $\mathfrak{F}$ and $\mathfrak{F}^\dagger$ as well as multiplication by $|V|^{1/2}$ and $V^{1/2}$ are bounded operators with the appropriate (co)domains)

$$\begin{aligned} &\langle u_\kappa | \mathcal{W}(\lambda,\kappa) | u_\kappa \rangle \\ &= \left\langle u^{(0)} \middle| \mathcal{W}(\lambda,0) \middle| u^{(0)} \right\rangle + \kappa \left[\left\langle u^{(0)} \middle| \partial_\kappa \mathcal{W}(\lambda,0) \middle| u^{(0)} \right\rangle + 2\mathrm{Re} \left\langle u^{(0)} \middle| \mathcal{W}(\lambda,0) \middle| u^{(1)} \right\rangle \right] + O(\kappa^2). \end{aligned} \tag{14.2.5}$$

To prove Theorem 14.1.6 (i) we use (14.2.4) and (14.2.5) to order κ. Bounding further $\|\mathcal{W}(\lambda,0)\| \leq C$ by Lemma 14.2.1 again it follows that

$$\lambda \log\left(\frac{T_c(\lambda,\kappa)}{T_c(\lambda,0)}\right) = \frac{U_1^{\operatorname{sgn}(\kappa)}}{\mathfrak{e}_{\hat{a}}^2} |\kappa| + O(\kappa^2) + O(|\kappa|\lambda).$$

To prove the second part of Theorem 14.1.6 (i) we note that $A_1^\pm = U_1^\pm / \mathfrak{e}_{\hat{a}}^2 > 0$ happens when first order perturbation theory in (14.2.4) does not vanish. This is the case, precisely if there exist a least two minima $\hat{a}_1, \hat{a}_2 \in \{1, \ldots, n\}$ of $a \mapsto \mathfrak{e}_a$ and functions $u_{\hat{a}_i} \in \mathcal{L}_{\hat{a}_i}$ for $i = 1, 2$ such that the quadratic form $\langle u_{\hat{a}_1}, \mathcal{V}_{\hat{a}_1 \hat{a}_2} u_{\hat{a}_2} \rangle_{L^2(S_{\hat{a}_1})} = \langle \mathcal{V}_{\hat{a}_2 \hat{a}_1} u_{\hat{a}_1}, u_{\hat{a}_2} \rangle_{L^2(S_{\hat{a}_2})} \neq 0$ does not vanish. Finally, one can easily

check that, in case of exactly two minima $\hat{a}_1, \hat{a}_2$ of $a \mapsto \mathfrak{e}_a$ and $\dim \mathcal{L}_{a_i} = 1$ for at least one $i = 1, 2$, the constants $U_1^{\pm}$ in (14.2.4) and hence $A_1^{\pm}$agree.

Next, to prove Theorem 14.1.6 (ii) we use (14.2.4) and (14.2.5) to order κ^2. By assumption, the order κ term in (14.2.4) vanishes using the argument from above. Further, in this case also $U_2 \coloneqq U_2^+ = U_2^-$ agree. We claim that also the order κ terms in (14.2.5) vanish. This follows from the fact that $\mathcal{W}(\lambda, 0)$ is diagonal and the perturbation is off-diagonal. More precisely, since $\partial_\kappa \mathcal{W}(\lambda, 0)$ is off-diagonal we conclude that $\langle u^{(0)} | \partial_\kappa \mathcal{W}(\lambda, 0) | u^{(0)} \rangle = 0$. Second, $\mathcal{W}(\lambda, 0) u^{(0)} \in \mathcal{L}^0$ since $\mathcal{L}^0$ is a subset of the ground state space of $\mathcal{V}$ and is an eigenspace for S_T. Thus, $\langle u^{(0)} | \mathcal{W}(\lambda, 0) | u^{(1)} \rangle = 0$. Combining $\|\mathcal{W}(\lambda, 0)\| \leq C$ (by Lemma 14.2.1 again) with (14.2.4), we conclude that

$$\lambda \log \left(\frac{T_c(\lambda, \kappa)}{T_c(\lambda, 0)} \right) = \frac{U_2}{\mathfrak{e}_{\hat{a}}^2} \kappa^2 + O(\kappa^3) + O(\lambda \kappa^2) \,,$$

which immediately shows the first part of Theorem 14.1.6 (ii). Lastly, $A_2 = U_2 / \mathfrak{e}_{\hat{a}}^2 = 0$ happens when both first and second order perturbation theory vanish in (14.2.4). This is precisely the case if $\mathcal{V}_{a\hat{a}}\big|_{\mathcal{L}_{\hat{a}}} \equiv 0$ for all $a \neq \hat{a}$.

This concludes the proof of Theorem 14.1.6. □

It remains to give the

Proof of Lemma 14.2.1. The argument is very similar to [314, Lemma 3.2] for dimensions $d = 2, 3$ (the adjustments to $d = 1$ are straightforward), hence we will be very brief.

First, multiplying by the unitary U^*, we see that $|V|^{1/2} M_T |V|^{1/2}$ is self-adjoint and satisfies $\||V|^{1/2} M_T |V|^{1/2}\| = \|V^{1/2} M_T |V|^{1/2}\|$. Then, for $\psi \in L^2(\mathbf{R}^d, \mathbf{C}^n)$ setting $\varphi \coloneqq |V|^{1/2} \psi$, it holds that

$$\left\langle \psi \middle| |V|^{1/2} M_T |V|^{1/2} \middle| \psi \right\rangle = \sum_{a=1}^{n} \left[\int_{\mathbf{R}^d} \frac{|\hat{\varphi}_a(p)|^2}{K_{T,a}(p)} \mathrm{d}p - 2 \log \left(\frac{T_0}{T} \right) \int_{S_a} \frac{|\hat{\varphi}_a(p)|^2}{|\nabla \epsilon_a(p)|} \mathrm{d}\omega(p) \right] \tag{14.2.6}$$

since $\mathfrak{F}^\dagger \mathfrak{F}$ and K_T^{-1} are diagonal, and we denoted $K_{T,a}(p) \coloneqq \epsilon_a(p) / \tanh\left(\frac{\epsilon_a(p)}{2T}\right)$. The expression in (14.2.6) is the analog of [314, Eq. (3.25)] with the identifications $T(p) + e \to (K_{T,a}(p) - 2T) + 2T$, $f(e) \to 2 \log(T_0/T)$, and $\nabla P \to \nabla \epsilon_a$. Now, we estimate every summand in (14.2.6) separately, following the arguments in [314, pp. 496–498] for $r = 1$[4] with the following modifications: The bound [314, Eq. (3.33)] is replaced by

$$\begin{aligned} |\hat{\varphi}_a(p)|^2 &\lesssim \sum_{b,b'=1}^{n} \iint (|V|^{1/2})_{ab}(x) (|V|^{1/2})_{ab'}(y) \overline{\psi_b(x)} \psi_{b'}(y) \mathrm{e}^{\mathrm{i}p \cdot (x-y)} \mathrm{d}x \mathrm{d}y \\ &\lesssim \max_{a,b=1}^{n} \|V_{ab}\|_{L^1(\mathbf{R}^d)} \|\psi\|_{L^2(\mathbf{R}^d, \mathbf{C}^n)}^2 \lesssim \|\psi\|_{L^2(\mathbf{R}^d, \mathbf{C}^n)}^2 \end{aligned}$$

and, in a similar fashion, the bound [314, Eq. (3.34)] is replaced by

$$\begin{aligned} \left| \nabla |\hat{\varphi}_a(p)|^2 \right| &\lesssim \left(\max_{b,b'=1}^{n} \iint |V|_{ab}(x) |x-y|^2 |V|_{ab'}(y) \mathrm{d}x \mathrm{d}y \right) \|\psi\|_{L^2(\mathbf{R}^d, \mathbf{C}^n)}^2 \\ &\lesssim \left(\max_{a,b,a',b'=1}^{n} \iint |V_{ab}|(x) |x-y|^2 |V_{a'b'}|(y) \mathrm{d}x \mathrm{d}y \right) \|\psi\|_{L^2(\mathbf{R}^d, \mathbf{C}^n)}^2 \lesssim \|\psi\|_{L^2(\mathbf{R}^d, \mathbf{C}^n)}^2 \,. \end{aligned}$$

In both estimates, the second step follows from elementary linear algebra. This concludes the proof of Lemma 14.2.1. □

[4] Note that the second to last line in [314, p. 497] contains a misprint: The estimate $\int_0^\tau t(t^2 + e)^{-1} \mathrm{d}t \leq \mathrm{const}\, g(e)$ should rather be $\int_0^\tau t(t^r + e)^{-1} \mathrm{d}t \leq \mathrm{const}\, g(e)$.

Finally, we give the

Proof of Equation (14.1.20). From the proof of Theorem 14.1.6 we have $A_2 = U_2/\mathfrak{e}_{\hat{a}}^2$.

Since V_{aa} is radial by assumption, all eigenfunctions of $\mathcal{V}_{aa}$ are given by appropriately normalized d-dimensional spherical harmonics $(2m_a\mu_a)^{-(d-1)/4}Y_\ell$ and their eigenvalues are denoted $\mathfrak{e}_a(\ell)$. Here, we abused the common notation Y from three spatial dimensions for all $d = 1, 2, 3$ and regard ℓ as an angular momentum multi-index. The Y_ℓ form an orthonormal basis of $L^2(\mathbb{S}^{d-1})$, i.e. $\langle Y_\ell, Y_{\ell'}\rangle_{L^2(\mathbb{S}^{d-1})} = \delta_{\ell\ell'}$, and Y_0 is always understood to be the constant spherical harmonic. Moreover, we shall also write $u_a(\ell) := (0, ..., 0, (2m_a\mu_a)^{-(d-1)/4}Y_\ell, 0, ..., 0) \in \bigoplus_{a=1}^n L^2(S_a)$, where the only non-zero entry is at position $a \in \{1, ..., n\}$.

With these notations, using $\dim \mathcal{L}_{\hat{a}} = 1$, it follows that $u_{\hat{a}}(0)$ is the unique (normalized) ground state vector of $\mathcal{V}^{\mathrm{d}}$. Hence, second order perturbation theory shows that U_2 can be evaluated as

$$U_2 = \sum_{(a,\ell)\neq(\hat{a},0)} \frac{|\langle u_a(\ell), \mathcal{V}^{\mathrm{od}} u_{\hat{a}}(0)\rangle|^2}{\mathfrak{e}_a(\ell) - \mathfrak{e}_{\hat{a}}} \,. \tag{14.2.7}$$

Since all the V_{ab} are radial by assumption and using orthogonality of spherical harmonics, the large sum in (14.2.7) (over all bands and angular momenta) further collapses to a sum only over the bands, i.e.

$$U_2 = \sum_{a\neq\hat{a}} \frac{|\langle u_a(0), \mathcal{V}^{\mathrm{od}} u_{\hat{a}}(0)\rangle|^2}{|\mathfrak{e}_a(0) - \mathfrak{e}_{\hat{a}}|} \,,$$

where we put the absolute value in the denominator for better comparability with (14.1.20). Indeed, in order to arrive at (14.1.20), we simply note that $\mathfrak{e}_{\hat{a}} = \mathfrak{v}_{\hat{a}\hat{a}}$, $\mathfrak{e}_a(0) = \mathfrak{v}_{aa}$, as well as $\langle u_a(0), \mathcal{V}^{\mathrm{od}} u_{\hat{a}}(0)\rangle = \mathfrak{v}_{a\hat{a}}$, where $\mathfrak{v}_{ab}$ was defined in (14.1.19). □

14.A Derivation of the multi-band BCS functional

We give here a heuristic derivation of the functional in (14.1.4), see also [411] and [309, Appendix A]. It arises as a formal infinite volume limit of the (negative) pressure functional restricted to a certain class of states.

To start, we consider a spin-$\frac{1}{2}$ Fermi gas localized to some box $\Lambda = [0, L]^d$ so that Fourier space is $\Lambda^* = \frac{2\pi}{L}\mathbf{Z}^d$ (we use periodic boundary conditions for the fermion operators). The fermions come in n different species — these are the *bands*. We consider the most general interactions carrying 4 band indices. Thus, the Hamiltonian is given by

$$H = \sum_{k,\sigma}\sum_a \epsilon_a^{(0)}(k) a^*_{k,\sigma,a} a_{k,\sigma,a} + \frac{1}{2L^d} \sum_{p,k,k',\sigma,\tau} \sum_{a,a',b,b'} \hat{V}_{aa',bb'}(p) a^*_{k+p,\sigma,a} a^*_{k'-p,\tau,a'} a_{k',\tau,b'} a_{k,\sigma,b}$$

with (bare) dispersion relations $\epsilon_a^{(0)}$ and interactions $V_{aa',bb'}$ (we abuse notation slightly and use the same symbol a both for fermin operators and a band index). Here $a^*_{k,\sigma,a}$ and $a_{k,\sigma,a}$ denote the creation and annihilation operators of a particle of momentum $k \in \Lambda^*$, spin $\sigma \in \{\uparrow, \downarrow\}$ and in band a. Next, we restrict H to quasi-free states with no fixed particle number. These are the states relevant for BCS theory [309, 411, 48]. Quasi-free states obey the Wick rule

$$\left\langle a_1^\# a_2^\# a_3^\# a_4^\# \right\rangle = \left\langle a_1^\# a_4^\# \right\rangle \left\langle a_2^\# a_3^\# \right\rangle - \left\langle a_1^\# a_3^\# \right\rangle \left\langle a_2^\# a_4^\# \right\rangle + \left\langle a_1^\# a_2^\# \right\rangle \left\langle a_3^\# a_4^\# \right\rangle ,$$

with each $a_i^\#$ being either $a_{k,\sigma,a}$ or $a^*_{k,\sigma,a}$. The three terms are the direct, exchange and pairing term respectively. Then, the expectation $\langle H\rangle$ can be written in terms of

$$\hat{\gamma}_{(a,\sigma),(b,\tau)}(k,k') = \left\langle a^*_{k,\sigma,a} a_{k',\tau,b}\right\rangle \quad \text{and} \quad \hat{\alpha}_{(a,\sigma),(b,\tau)}(k,k') = \left\langle a_{k',\tau,b} a_{k,\sigma,a}\right\rangle;$$

note that the functions $\alpha_{(a,\sigma),(b,\tau)}(k,k')$ are odd under the exchange $(k,\sigma,a) \leftrightarrow (k',\tau,b)$. Next, we make the following three simplifying assumptions:

(a) We consider only translation invariant states. This means that for the one-particle density matrix γ that $k = k'$ for the non-zero entries and for the pairing function α that $k = -k'$ for the non-zero entries.

(b) We consider only spin rotation invariant states and, in particular, only spin singlet superconducting states. (In the mathematical physics literature, this is referred to as $SU(2)$-invariance.)

(c) We consider only pairing functions with one band index, meaning that for $\alpha_{(a,\sigma),(b,\tau)}$ only terms with $a = b$ are non-zero.

Points (a) and (b) are discussed at length in [316]; the simplification in (c) is common in the multi-band physics literature [542, 457, 375, 410].

With these assumptions, the expectation $\langle H \rangle$ can be written in terms of the simpler functions

$$\hat{\gamma}_{ab}(k) = \left\langle a^*_{k,\uparrow,a} a_{k,\uparrow,b} \right\rangle = \left\langle a^*_{k,\downarrow,a} a_{k,\downarrow,b} \right\rangle \quad \text{and} \quad \hat{\alpha}_a(k) = \left\langle a_{-k,\uparrow,a} a_{k,\downarrow,a} \right\rangle = -\left\langle a_{-k,\downarrow,a} a_{k,\uparrow,a} \right\rangle;$$

note that the latter condition means that we only consider spin-singlet superconducting states, and this condition and the canonical anti-commutator relations imply that $\hat{\alpha}_a(k)$ are even functions of k. The matrix-valued functions $\hat{\gamma} = [\hat{\gamma}_{ab}]_{a,b=1}^n$ and $\hat{\alpha} = \mathrm{diag}[\hat{\alpha}_a]_{a=1}^n$ can be conveniently grouped together in the generalized reduced one-particle density matrix Γ given by

$$\hat{\Gamma}(p) = \begin{bmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & \mathbf{1} - \hat{\gamma}(-p) \end{bmatrix}$$

As a matrix this satisfies $0 \leq \hat{\Gamma}(p) \leq \mathbf{1}$ pointwise. The expectation of H in the state with generalized reduced one-particle density matrix Γ is then

$$\begin{aligned} \langle H \rangle_\Gamma &= 2 \sum_{k,a} \epsilon_a(k)^{(0)} \hat{\gamma}_{aa}(k) + \frac{2}{L^d} \sum_{a,a',b,b'} \hat{V}_{aa',bb'}(0) \left[\sum_k \hat{\gamma}_{ab}(k) \right] \left[\sum_k \hat{\gamma}_{a'b'}(k) \right] \\ &\quad - \frac{1}{L^d} \sum_{k,p,a,a',b,b'} \hat{V}_{aa'bb'}(p) \hat{\gamma}_{ab'}(p-k) \hat{\gamma}_{a'b}(p) + \frac{1}{L^d} \sum_{k,p,a,b} \hat{V}_{aa,bb}(p) \overline{\hat{\alpha}_a(k-p)} \hat{\alpha}_b(k) \\ &= 2 \sum_{k,a} \epsilon_a(k)^{(0)} \hat{\gamma}_{aa}(k) + \iint \sum_{a,b} V_{aa,bb}(x-y) \overline{\alpha_a(x-y)} \alpha_b(x-y) \mathrm{d}x \mathrm{d}y \,, \\ &\quad + \iint \sum_{a,a',b,b'} V_{aa',bb'}(x-y) \left[2\gamma_{ab}(0) \overline{\gamma_{a'b'}(0)} - \gamma_{a'b}(x-y) \overline{\gamma_{ab'}(x-y)} \right] \mathrm{d}x \mathrm{d}y \end{aligned}$$

where the factors of two arise from the spin degrees of freedom. The pressure functional of the state (with generalized reduced one-particle density matrix) Γ is then given by

$$\mathcal{P}[\Gamma] = \frac{1}{L^d} \left[2TS(\Gamma) - \langle H \rangle_\Gamma \right], \qquad S(\Gamma) = -\sum_p \mathrm{Tr}_{\mathbf{C}^{2n}} \left[\hat{\Gamma}(p) \ln \hat{\Gamma}(p) \right],$$

where the factor two in front of the entropy again comes from the spin degrees of freedom. Taking a formal infinite volume limit of $-\frac{1}{2}\mathcal{P}$ and replacing $V \to 2V$ we find the functional

$$\begin{aligned} \mathcal{F}^{(0)}[\Gamma] &= \int \sum_a \epsilon_a^{(0)}(p) \hat{\gamma}_{aa}(p) \mathrm{d}p - TS[\Gamma] + \int_{\mathbf{R}^d} \sum_{a,b} \overline{\alpha_a(x)} V_{aa,bb}(x) \alpha_b(x) \mathrm{d}x \\ &\quad + \int_{\mathbf{R}^d} \sum_{a,a',b,b'} V_{aa',bb'}(x) \left[2\gamma_{ab}(0) \overline{\gamma_{a'b'}(0)} - \gamma_{a'b}(x) \overline{\gamma_{ab'}(x)} \right] \mathrm{d}x, \end{aligned}$$

with the entropy per unit volume $S[\Gamma]$ as in (14.1.5). The claim is then that, up to corrections which are negligible, the direct and exchange terms effectively only serve to renormalize the dispersions $\epsilon_a^{(0)}$, see [411]. Since in the interaction only terms with $a = a'$ and $b = b'$ appear, we define the matrix $V_{ab} = V_{aa,bb}$. Finally, by concavity of the entropy, the minimizer has γ diagonal, and we may thus restrict to diagonal matrices γ. We then recover the functional in (14.1.4).

Remark 14.A.1 (Brillouin zones)**.** *It would be natural to consider also models with other Brillouin zones, meaning that the k-integrals are not over $\mathbf{R}^d$ but over some different domain. For instance, a Brillouin zone of a (cubic) torus of sidelengths $2\pi/a$ corresponds to a model of fermions on a lattice $a\mathbf{Z}^d$ where $a > 0$ is the lattice spacing.*

It would be interesting to extend our results to such lattice fermion models with compact Brillouin zones, but this is beyond the scope of the present paper. We expect that the main results stated in the beginning of the introduction are true also for such models, and we believe that the methods of this paper would be applicable also to these other settings.

14.B Notation

For the convenience of the reader, we collect some notations used throughout this paper.

- For $f(x)$ a (suitable) function of $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, we denote its Fourier transform by $\hat{f}(p) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) \mathrm{e}^{-\mathrm{i}p\cdot x} \mathrm{d}x$, (here $p \in \mathbf{R}^d$).
- For $\epsilon_a(p)$ a dispersion relation determining a Fermi surface S_a as explained in Assumption 14.1.1, we denote as $\mathrm{d}\omega$ the (Lebesgue) measure on S_a. In particular, letting δ be Dirac delta, for any function f the Lebesgue measure satisfies

$$\int_{S_a} \frac{1}{|\nabla \epsilon_a(p)|} f(p) \mathrm{d}\omega(p) = \int_{\mathbf{R}^d} \delta(\epsilon_a(p)) f(p) \mathrm{d}p \,.$$

- The unit sphere is denoted $\S^{d-1} = \{p \in \mathbf{R}^d : |p| = 1\}$ and the (Lebesgue) measure on $\S^{d-1}$ is denoted $\mathrm{d}\omega$. In particular, $\int_{\S^{d-1}} f(p) \mathrm{d}\omega(p) = \int_{\mathbf{R}^d} f(p) \delta(|p| - 1) \mathrm{d}p$ for any f and $|\mathbb{S}^{d-1}| = \int_{\mathbf{R}^d} \delta(|p| - 1) \mathrm{d}p$.
- For a general measure μ on $\mathbf{R}^d$ we denote by $L^p(\mathbf{R}^d, \mathrm{d}\mu)$ the space of all $\mathbf{C}$-valued functions for which $\int |f|^p \mathrm{d}\mu < \infty$. For Lebesgue measure μ we simply write $L^p(\mathbf{R}^d, \mathrm{d}x) = L^p(\mathbf{R}^d)$.
- The space H^1_{sym} of reflection-symmetric H^1 functions is more concretely given as

$$H^1_{\mathsf{sym}}(\mathbf{R}^d, \mathrm{d}x) = \left\{ f \in L^2(\mathbf{R}^d, \mathrm{d}x) : \hat{f} \in L^2(\mathbf{R}^d, (1 + p^2)\mathrm{d}p), \quad f(x) = f(-x) \ \forall x \right\}.$$

Part III

Quantum Lattice Systems

Chapter 15

Local stability of ground states in locally gapped and weakly interacting quantum spin systems

This chapter includes the paper [343]:

J. Henheik, S. Teufel, and T. Wessel. Local stability of ground states in locally gapped and weakly interacting quantum spin systems. *Lett. Math. Phys.*, 112(1):9, 2022

Abstract. Based on a result by Yarotsky (J. Stat. Phys. 118, 2005), we prove that localized but otherwise arbitrary perturbations of weakly interacting quantum spin systems with uniformly gapped on-site terms change the ground state of such a system only locally, even if they close the spectral gap. We call this a *strong version* of the *local perturbations perturb locally* (LPPL) principle which is known to hold for much more general gapped systems, but only for perturbations that do not close the spectral gap of the Hamiltonian. We also extend this strong LPPL-principle to Hamiltonians that have the appropriate structure of gapped on-site terms and weak interactions only locally in some region of space.

While our results are technically corollaries to a theorem of Yarotsky, we expect that the paradigm of systems with a locally gapped ground state that is completely insensitive to the form of the Hamiltonian elsewhere extends to other situations and has important physical consequences.

Content

15.1 Introduction

We consider weakly interacting quantum spin systems on finite subsets Λ of the lattice $\mathbf{Z}^\nu$, $\nu \in \mathbf{N}$, described by a self-adjoint Hamiltonian

$$H = H_0 + H_{\text{int}}\,, \tag{15.1.1}$$

which is composed of a non-interacting part H_0 and an interacting part H_{int}. The non-interacting Hamiltonian H_0 is a sum of non-negative on-site Hamiltonians h_x, $x \in \Lambda$. Each h_x is assumed to have a non-degenerate ground state with ground state energy 0 and spectral gap of size at least g above the ground state. The interaction Hamiltonian H_{int} is a sum of interaction terms Φ_x of finite range R and of small uniformly bounded norm $\|\Phi_x\|$. We show that for such Hamiltonians a strong version of the *local perturbations perturb locally* (LPPL) principle holds: For any self-adjoint perturbation P, supported in a region $X \subset \Lambda$, any ground state ρ_P of the perturbed Hamiltonian $H + P$ agrees with the ground state ρ of the unperturbed Hamiltonian H when tested against observables A supported in a region $Y \subset \Lambda$ up to an error that is exponentially small in the distance $\operatorname{dist}(Y, X)$. More precisely, Theorem 15.2.3 states that there are positive constants $c, c_1, c_2 > 0$ depending only on R and g, but not on Λ, A, H or P, such that whenever $\|\Phi_x\| \leq c$ for all $x \in \Lambda$, it holds that

$$\left|\operatorname{tr}\big((\rho_P - \rho)A\big)\right| \leq \mathrm{e}^{c_1|Y|}\, \|A\|\, \mathrm{e}^{-c_2 \operatorname{dist}(Y,X)}\,. \tag{15.1.2}$$

Note that the uniformity of the error estimate with respect to the system size $|\Lambda|$ is one key aspect which makes this estimate non-trivial. Note also, that the bound on $\|\Phi_x\|$ implies that H has a gap above its unique ground state ρ as we discuss below. However, for our result we neither require nor actually have any uniform lower bound on the gap above the possibly non-unique ground state ρ_P of the perturbed Hamiltonian $H + P$.

As a corollary of our main theorem, we show that a bound of the form (15.1.2) also holds for systems that have the appropriate structure of gapped on-site terms and weak interactions only locally in some region of space. In particular, this shows that the notion of a locally gapped ground state, which is completely insensitive to the form of the Hamiltonian elsewhere, is perfectly valid in this setup.

The LPPL-principle was coined by Bachmann, Michalakis, Nachtergaele, and Sims in [39], where a similar estimate with subexponential decay was proven. While their result covers much more general interacting quantum spin systems, it requires the gap above the ground state to remain open also for the perturbed Hamiltonian $H + P$. More precisely, it relies on connecting $H(0) := H$ with $H(1) := H + P$ by a continuous path $[0, 1] \ni t \mapsto H(t)$ in the space of Hamiltonians, such that the gap above the ground state of $H(t)$ remains open uniformly along the whole path. Then the locality of the quasi-adiabatic evolution introduced by Hastings and Wen in [326] can be used to prove the result. Their subexponential bound was improved to exponential precision for finite-range interactions by De Roeck and Schütz in [207]. See also [461, 462] for recent developments.

While we prove the strong version of the LPPL-principle only for weakly interacting spin systems, we expect it to hold somewhat more generally. For example, we expect it to hold for fermions on the lattice with weak finite range interactions, a physical setup where the strong LPPL-principle would have important consequences. It would imply that a gapped ground state for such a system with periodic boundary conditions remains unchanged in the bulk when introducing open boundary conditions that may close the global gap due to the emergence of edge states. And as a consequence, it would also explain why the adiabatic response to external fields in the bulk of such systems is not affected by edge states that close the gap, see [36, 452, 556, 342, 341] for related results. However, it is known that the strong LPPL-principle cannot hold in general, but requires further conditions on the unperturbed ground state sector such as local topological quantum order (LTQO) [448, 463].

Shortly before resubmitting the final version of this article, Bachmann et al. published a preprint containing a closely related result. In [35] they prove an LPPL-bound as in (15.1.2), but with

subexponential decay, assuming LTQO for a unique frustration-free gapped ground state of the unperturbed Hamiltonian which has no long-range entanglement.

Our result is a corollary of a result by Yarotsky [591] (see Theorem 15.3.2 below), which provides a bound on the difference of so-called finite volume ground states in quantum spin systems described by Hamiltonians of the form (15.1.1). His aim and main result in that work was to show the uniqueness of the ground state of such systems in the thermodynamic limit. In a different work Yarotsky [590] has shown that Hamiltonians of the form (15.1.1) with $\|\Phi\| < c$ indeed have a unique ground state separated by a gap $\tilde{g} > 0$ from the rest of the spectrum, with $\tilde{g}$ independent of Λ (see [206, 273, 324] for similar results). Closely related to the stability of the gap is the stability of phase diagrams at low temperatures, see [102, 197, 198].

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15.2 Main results

Consider the lattice $\mathbb{Z}^\nu$ for fixed $\nu \in \mathbb{N}$ equipped with the ℓ^1-metric $d\colon \mathbf{Z}^\nu \times \mathbf{Z}^\nu \to \mathbb{N}_0$ and define $\mathcal{P}_0(\mathbf{Z}^\nu) = \{\, \Lambda \subset \mathbf{Z}^\nu | \Lambda| < \infty \,\}$, where $|\Lambda|$ denotes the cardinality of Λ. With each site $x \in \mathbf{Z}^\nu$ one associates a (possibly infinite dimensional) Hilbert space $\mathcal{H}_x$. For $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ set $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and denote the algebra of bounded linear operators on $\mathcal{H}_\Lambda$ by $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. Due to the tensor product structure, we have $\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x)$. Hence, for $\Lambda' \subset \Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$, any $A \in \mathcal{A}_{\Lambda'}$ can be viewed as an element of $\mathcal{A}_\Lambda$ by identifying A with $A \otimes \mathbf{1}_{\Lambda\setminus\Lambda'} \in \mathcal{A}_\Lambda$, where $\mathbf{1}_{\Lambda\setminus\Lambda'}$ denotes the identity in $\mathcal{A}_{\Lambda\setminus\Lambda'}$. Note that

$$[A, B] = 0 \quad \text{for all} \quad A \in \mathcal{A}_\Lambda\,,\ B \in \mathcal{A}_{\Lambda'} \quad \text{with} \quad \Lambda \cap \Lambda' = \varnothing\,.$$

Similarly, we will also denote by K the closure of $\mathbf{1}_{\Lambda\setminus\Lambda'} \otimes K$ on $\mathcal{H}_{\Lambda\setminus\Lambda'} \otimes D(K)$ for any self-adjoint operator K on $\mathcal{H}_{\Lambda'}$. Here and in the following, $D(K)$ denotes the domain of the operator K.

Our main result will be formulated for a Hamiltonian

$$H = H_0 + H_{\mathrm{int}} \in \mathcal{A}_\Lambda$$

that is composed of a non-interacting part H_0 and an interacting part H_{int}. The non-interacting part H_0 is assumed to be of the form

$$H_0 = \sum_{x\in\Lambda} h_x\,,$$

where each h_x is a non-negative self-adjoint (possibly unbounded) operator on $\mathcal{H}_x$ with a unique gapped ground state $\psi_x \in D(h_x)$ satisfying

$$h_x\psi_x = 0 \quad \text{and} \quad h_x\big|_{D(h_x)\ominus\psi_x} \geq g\,, \tag{15.2.1}$$

for some fixed $g > 0$. The latter means that $\langle \varphi_x, (h_x - g\mathbf{1}_x)\varphi_x\rangle \geq 0$ for all $\varphi_x \in D(h_x)$ with $\langle \psi_x, \varphi_x\rangle = 0$. In other words, all Hamiltonians h_x have a spectral gap of size at least g above the bottom of their spectrum. The interacting part is of the form

$$H_{\mathrm{int}} = \sum_{x\in\Lambda} \Phi_x\,,$$

with $\Phi_x \in \mathcal{A}_{b_x(R)}$ self-adjoint for each $x \in \Lambda$ and some fixed $R \in \mathbf{N}$. Here $b_x(R) := \{\, y \in \Lambda d(x, y) \leq R \,\}$ denotes the ℓ^1-ball with radius R centered at $x \in \Lambda$. We set

$$\|\Phi\| := \sup_{x\in\Lambda} \|\Phi_x\|\,.$$

Definition 15.2.1 (Weakly interacting spin system)**.** *For any $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ we call a Hamiltonian $H = H_0 + H_{\rm int}$ on $\mathcal{H}_\Lambda$ with H_0 and $H_{\rm int}$ satisfying the above conditions a* weakly interacting spin system on Λ *with on-site gap g, interaction range R and interaction strength $\|\Phi\|$.*

We use the following definition for ground states and briefly explain how it is connected to the standard definition in Appendix 15.A.

Definition 15.2.2. *Let $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and K be a self-adjoint and bounded below operator on $\mathcal{H}_\Lambda$. We say that $[K, A]$ is a bounded operator $B \in \mathcal{A}_\Lambda$, whenever A leaves $D(K)$ invariant and $[K, A] = B$ on $D(K)$.*

A state $\rho \in \mathcal{A}_\Lambda$, i.e. a positive semi-definite bounded operator with trace equal to one, is called a ground state of K, if

$$\mathrm{tr}(A^*[K, A]\,\rho) \geq 0 \quad \textit{for all } A \in \mathcal{A}_\Lambda \textit{ such that } [K, A] \textit{ is bounded.}$$

Our first main result is the following.

Theorem 15.2.3 (The strong LPPL-principle)**.** *Let $R \in \mathbf{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$, such that for any $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and any weakly interacting spin system $H = H_0 + H_{\rm int}$ on Λ with on-site gap at least g, interaction range R, and interaction strength $\|\Phi\| \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and P be a symmetric operator on $\mathcal{H}_X$ such that P is relatively bounded with respect to H with H-bound less than one. Set $H_P = H + P$. Then for any ground state ρ of H, any ground state ρ_P of H_P, and all $A \in \mathcal{A}_Y$ with $Y \subset \Lambda$ it holds that

$$\big|\mathrm{tr}\big((\rho_P - \rho)A\big)\big| \leq \mathrm{e}^{c_1|Y|}\,\|A\|\,\mathrm{e}^{-c_2\,\mathrm{dist}(Y,X)}\,. \tag{15.2.2}$$

Under the assumptions of the theorem, Yarotsky has proven in [590] that H has a unique ground state ρ, whenever $c > 0$ is small enough.[1] In the following we will assume that this is the case.

For X at the edge of Λ, the perturbation P can be employed to realize all kinds of boundary conditions, e.g. if $\Lambda = \{-M, \ldots, M\}^\nu$ is a box, periodic boundary conditions can be modeled by some P connecting opposite sites in Λ. Therefore, if X is at the edge, one can take the thermodynamic limit $\Lambda \nearrow \mathbf{Z}^\nu$ in (15.2.2) and conclude that there exists a unique ground state ρ, i.e. a normalized positive functional, on the C^*-algebra of quasi-local observables $\mathcal{A} = \overline{\mathcal{A}_{\rm loc}}^{\|\cdot\|}$, independent of the imposed boundary conditions for the finite systems. This uniqueness of ground states for the infinite system was the main result of [591] and has been shown by Yarotsky based on Theorem 15.3.2, which we quote below.

As mentioned in the introduction, we expect a similar strong LPPL-principle to hold also for fermionic lattice systems with weak finite range interactions. As discussed in [342, 341], this would have important consequences for linear response and adiabatic theorems for systems with a gap only in the bulk.

Our second main result is a local version of Theorem 15.2.3, where we assume the on-site gap and the weak interaction only locally.

[1] Note that the systems for which Yarotsky proves existence and uniqueness of the ground state in [590] differ slightly from our definition of weakly interacting spin systems in the treatment of interaction terms near the boundary of the domain. To obtain the same result with our definition, one extends the Hamiltonian H to $\Omega \supset \Lambda$ as in the proof of Theorem 15.2.3, applies the result from [590] and restricts the resulting ground state to Λ by taking the partial trace.

Definition 15.2.4 (Locally weakly interacting spin system)**.** *For any $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and $\Lambda' \subset \Lambda$ we say that a self-adjoint operator H on $\mathcal{H}_\Lambda$ is* weakly interacting in the region Λ' *with on-site gap g, range R and strength s, if and only if there exists a weakly interacting spin system $\tilde{H} = \tilde{H}_0 + \tilde{H}_{\mathrm{int}}$ on Λ with on-site gap g, range R and strength $\|\Phi\| = s$ such that $H - \tilde{H} = \mathbf{1}_{\mathcal{H}_{\Lambda'}} \otimes Q$ with Q a possibly unbounded symmetric operator on $\mathcal{H}_{\Lambda \setminus \Lambda'}$ such that Q is infinitesimally $\tilde{H}$-bounded.*

Corollary 15.2.5 (The strong LPPL-principle for local gaps)**.** *Let $R \in \mathbf{N}$, $g > 0$, and $c, c_1, c_2 > 0$ be the constants from Theorem 15.2.3. Then for any $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$, $\Lambda' \subset \Lambda$, and any self-adjoint operator H on $\mathcal{H}_\Lambda$ which is weakly interacting in the region Λ' with on-site gap at least g, range R and strength $s \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and P be a symmetric operator on $\mathcal{H}_X$ such that P is relatively bounded with respect to H with H-bound less than one. Set $H_P = H + P$ (see Figure 15.2.1). Then for any ground state ρ of H, any ground state ρ_P of H_P, and all $A \in \mathcal{A}_Y$ with $Y \subset \Lambda'$ it holds that

$$\left|\mathrm{tr}\big((\rho_P - \rho)A\big)\right| \leq 2\,\mathrm{e}^{c_1|Y|}\,\|A\|\,\mathrm{e}^{-c_2 \min\{\mathrm{dist}(Y,X),\mathrm{dist}(Y,\Lambda\setminus\Lambda')\}}.$$

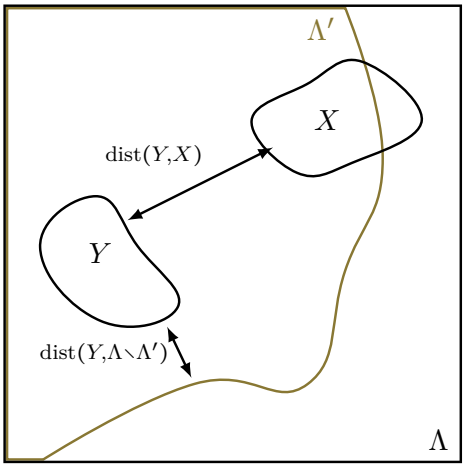

Figure 15.2.1: Depicted is the setting from Corollary 15.2.5. The system H defined on Λ is assumed to be weakly interacting and to have an on-site gap in $\Lambda' \subset \Lambda$. For any perturbation P acting on $X \subset \Lambda$, ground states of H and $H + P$ agree in regions Y away from X and $\Lambda \setminus \Lambda'$.

Proof. Let $\tilde{H}$ and Q be as in Definition 15.2.4. Then $\tilde{H}$, $\tilde{H} + Q$ and $\tilde{H} + Q + P$ are self-adjoint. For the latter this follows, because also $Q + P$ is relatively bounded with respect to $\tilde{H}$ with $\tilde{H}$-bound less than one. This is not obvious, but the proof is a straightforward calculation that we skip.

Let $\tilde{\rho}$ be a ground state of $\tilde{H}$, see the comment after Theorem 15.2.3 for existence. Then the triangle inequality and two applications of Theorem 15.2.3 yield

$$\begin{aligned}\left|\mathrm{tr}\big((\rho_P - \rho)A\big)\right| &\leq \left|\mathrm{tr}\big((\rho_P - \tilde{\rho})A\big)\right| + \left|\mathrm{tr}\big((\rho - \tilde{\rho})A\big)\right| \\ &\leq \mathrm{e}^{c_1|Y|}\,\|A\|\,\Big(\mathrm{e}^{-c_2\,\mathrm{dist}(Y,X\cup(\Lambda\setminus\Lambda'))} + \mathrm{e}^{-c_2\,\mathrm{dist}(Y,\Lambda\setminus\Lambda')}\Big).\end{aligned}$$

□

15.3 Proof

The proof of Theorem 15.2.3 is essentially a reinterpretation of a result by Yarotsky [591]. Since we only deal with finite volumes, we modify Yarotsky's notion of *finite volume ground states* to *ground states in the bulk*. To make the arguments as transparent as possible, we will add superscripts to Hamiltonians and states indicating on which subset of $\mathbf{Z}^\nu$ they are defined. These superscripts are also used to distinguish different operators and states. From now on let

$$\mathcal{D}_\Lambda := \big\{\, A \in \mathcal{A}_\Lambda \,[H_0^\Lambda, A] \text{ is bounded} \,\big\}$$

and note, that also $\{ A \in \mathcal{A}_\Lambda [H_0^\Lambda + K, A] \text{ is bounded} \} = \mathcal{D}_\Lambda$ for all bounded operators $K \in \mathcal{A}_\Lambda$.

Definition 15.3.1 (Ground states in the bulk)**.** *Let $R \in \mathbb{N}$, $\Lambda_* \subset \Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and $H^{\Lambda_*} = H_0^{\Lambda_*} + H_{\mathrm{int}}^{\Lambda_*} \in \mathcal{A}_{\Lambda_*}$ be a weakly interacting spin system on Λ_* with range R. Then we call*

$$\Lambda_*^\circ := \{ x \in \Lambda_* \operatorname{dist}(x, \mathbf{Z}^\nu \setminus \Lambda_*) > 2R \}$$

the bulk *of the Hamiltonian H^{Λ_*} and any state $\rho^\Lambda \in \mathcal{A}_\Lambda$ satisfying*

$$\operatorname{tr}\big(\rho^\Lambda A^* [H^{\Lambda_*}, A]\big) \geq 0 \quad \textit{for all} \quad A \in \mathcal{D}_{\Lambda_*^\circ}$$

a ground state in the bulk of H^{Λ_*}.

Our proof is based on the following theorem due to Yarotsky [591].

Theorem 15.3.2 (Theorem 2 in [591])**.** *Let $R \in \mathbf{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$ such that for any $\Lambda_* \in \mathcal{P}_0(\mathbf{Z}^\nu)$, and any weakly interacting spin system $H^{\Lambda_*} = H_0^{\Lambda_*} + H_{\mathrm{int}}^{\Lambda_*}$ on Λ_* with on-site gap at least g, range R and interaction strength $\|\Phi\| \leq c$ the following holds:*

Let $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ be such that $\Lambda_* \subset \Lambda$. Then for any two ground states ρ_1^Λ and $\rho_2^\Lambda \in \mathcal{A}_\Lambda$ in the bulk of H^{Λ_*} in the sense of Definition 15.3.1, $Y \subset \Lambda_*$, and $A \in \mathcal{A}_Y$ it holds that

$$\big|\operatorname{tr}\big((\rho_1^\Lambda - \rho_2^\Lambda)A\big)\big| \leq \mathrm{e}^{c_1|Y|} \, \|A\| \, \mathrm{e}^{-c_2 \operatorname{dist}(Y, \mathbf{Z}^\nu \setminus \Lambda_*^\circ)} .$$

Note that the set denoted by Λ in [591, Theorem 2] corresponds to our set Λ_*. Note, moreover, that any ground state ρ^Λ in the bulk of H^{Λ_*} trivially defines a finite-volume ground state $A \mapsto \operatorname{tr}(\rho^\Lambda (A \otimes \mathbf{1}_{\Lambda \setminus \Lambda_*}))$ of H^{Λ_*} in the sense of [591, Definition 2]. Allowing an arbitrary on-site gap $g > 0$ instead of $g = 1$, as in [591], is achieved by simple scaling.

Lemma 15.3.3. *Let $R \in \mathbb{N}$, $\Lambda_* \subset \Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and $H^\Lambda = H_0^\Lambda + H_{\mathrm{int}}^\Lambda \in \mathcal{A}_\Lambda$ be a weakly interacting spin system. Then the canonical restriction of H^Λ to Λ_* defined by*

$$H^\Lambda|_{\Lambda_*} = H_0^\Lambda|_{\Lambda_*} + H_{\mathrm{int}}^\Lambda|_{\Lambda_*} := \sum_{x \in \Lambda_*} h_x + \sum_{\substack{x \in \Lambda_* : \\ \operatorname{dist}(x, \Lambda \setminus \Lambda_*) > R}} \Phi_x$$

is a weakly interacting spin system on Λ_* with the same on-site gap, range and strength and has the following property: For any symmetric operator Q on $\mathcal{H}_{\Lambda \setminus \Lambda_*^\circ}$ such that Q is relatively bounded with respect to H^Λ with H^Λ-bound less than one, any ground state of $H^\Lambda + Q$ is also a ground state in the bulk of $H^\Lambda|_{\Lambda_*}$.

Proof. It is clear that $H^\Lambda|_{\Lambda_*}$ is a weakly interacting spin system on Λ_*. A simple calculation shows, that Q is also relatively bounded with respect to $H_0^{\Lambda \setminus \Lambda_*^\circ} = \sum_{x \in \Lambda \setminus \Lambda_*^\circ} h_x$ with $H_0^{\Lambda \setminus \Lambda_*^\circ}$-bound less than one. Hence, $K := (H^\Lambda - H^\Lambda|_{\Lambda_*} + Q)$ is a self-adjoint operator on $\mathcal{H}_{\Lambda \setminus \Lambda_*^\circ}$. Moreover, any $A \in \mathcal{A}_{\Lambda_*^\circ}$ leaves invariant the domain of $\mathbf{1}_{\Lambda_*^\circ} \otimes K$ and satisfies

$$\big[\mathbf{1}_{\Lambda_*^\circ} \otimes K, A \otimes \mathbf{1}_{\Lambda \setminus \Lambda_*^\circ}\big] = 0.$$

Similarly, for all $A \in \mathcal{D}_{\Lambda_*^\circ}$, $[H^\Lambda + Q, A]$ is bounded and satisfies

$$[H^\Lambda + Q, A] = [H^\Lambda|_{\Lambda_*}, A].$$

Therefore, any ground state of $H^\Lambda + Q$ is also a ground state in the bulk of $H^\Lambda|_{\Lambda_*}$. □

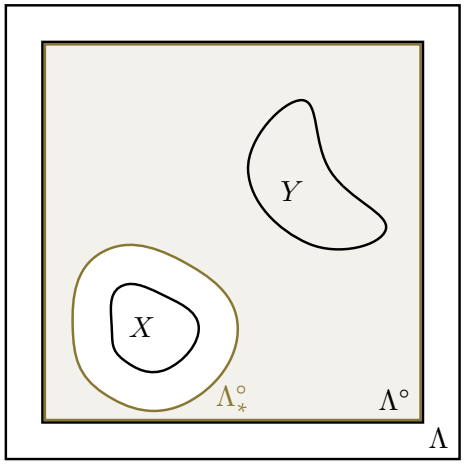

Figure 15.3.1: Depicted is the setting from the proof of Proposition 15.3.4. The subset $X \subset \Lambda$ is the region where the perturbation P acts, and we choose $\Lambda_* = \Lambda \setminus X$. The shaded region Λ_*° is the bulk of H^{Λ_*}. $Y \subset \Lambda_*^\circ$ is the support of the observable A. This indicates why (15.3.1) holds.

Before we prove Theorem 15.2.3, let us give an intermediate result, which follows rather directly from Theorem 15.3.2 and Lemma 15.3.3.

Proposition 15.3.4. *Let $R \in \mathbf{N}$ and $g > 0$. There exist constants $c, c_1, c_2 > 0$ such that for any $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and any weakly interacting spin system $H^\Lambda = H_0^\Lambda + H_{\mathrm{int}}^\Lambda$ on Λ with on-site gap at least g, interaction range R, and interaction strength $\|\Phi\| \leq c$ the following holds:*

Let $X \subset \Lambda$ be non-empty and P be a symmetric operator on $\mathcal{H}_X$ such that P is relatively bounded with respect to H^Λ with H^Λ-bound less than one. Set $H_P^\Lambda = H^\Lambda + P$. Then for any ground state ρ^Λ of H^Λ, any ground state ρ_P^Λ of H_P^Λ, and all $A \in \mathcal{A}_Y$ with $Y \subset \Lambda$ it holds that

$$\left|\mathrm{tr}\big((\rho_P^\Lambda - \rho^\Lambda)A\big)\right| \leq \mathrm{e}^{c_1|Y|}\,\|A\|\,\mathrm{e}^{-c_2\min\{\mathrm{dist}(Y,\mathbf{Z}^\nu\setminus\Lambda^\circ),\,\mathrm{dist}(Y,X)-2R\}}\,.$$

Proof. Assume w.l.o.g. that $Y \subset \Lambda^\circ$. Otherwise, the statement in Proposition 15.3.4 is trivially satisfied after a possible adjustment of c_1.

Let $\Lambda_* = \Lambda \setminus X$, and let $H^\Lambda|_{\Lambda_*}$ be the canonical restriction of H^Λ to Λ_* as defined in Lemma 15.3.3. Then $\Lambda_*^\circ \cap X = \varnothing$. We can assume w.l.o.g. that $\mathrm{dist}(X,Y) > 2R$ since otherwise the statement in Proposition 15.3.4 is trivially satisfied after a possible adjustment of c_1. Then also $Y \subset \Lambda_*^\circ$ (compare Figure 15.3.1). By application of Lemma 15.3.3 with $Q = P$ and $Q = 0$ we find that both, ρ_P^Λ and ρ^Λ, are ground states in the bulk of $H^\Lambda|_{\Lambda_*}$. Hence, Theorem 15.3.2 implies that

$$\left|\mathrm{tr}\big((\rho_P^\Lambda - \rho^\Lambda)A\big)\right| \leq \mathrm{e}^{c_1|Y|}\,\|A\|\,\mathrm{e}^{-c_2\,\mathrm{dist}(Y,\mathbf{Z}^\nu\setminus\Lambda_*^\circ)}\,.$$

From

$$\mathbf{Z}^\nu \setminus \Lambda_*^\circ = (\mathbf{Z}^\nu \setminus \Lambda^\circ) \cup \{\, x \in \mathbf{Z}^\nu \,\mathrm{dist}(x,X) \leq 2R \,\}$$

we immediately conclude that

$$\mathrm{dist}(Y, \mathbf{Z}^\nu \setminus \Lambda_*^\circ) = \min\{\mathrm{dist}(Y, \mathbf{Z}^\nu \setminus \Lambda^\circ),\, \mathrm{dist}(Y,X) - 2R\}\,, \tag{15.3.1}$$

which yields the claim. □

We now extend this result to obtain Theorem 15.2.3.

Proof of Theorem 15.2.3. In the following, we add superscripts Λ to the Hamiltonians and states from the statement of Theorem 15.2.3.

Let $\Omega \in \mathcal{P}_0(\mathbf{Z}^\nu)$ be such that $\Lambda \subset \Omega$. For each $x \in \Omega \setminus \Lambda$ let $h_x \in \mathcal{A}_{\{x\}}$ be a self-adjoint operator with gap at least g and non-degenerate ground state ψ_x satisfying (15.2.1). Then $\rho^{\Omega\setminus\Lambda} = \bigotimes_{x\in\Omega\setminus\Lambda} |\psi_x\rangle\langle\psi_x|$ is the ground state of

$$H_0^{\Omega\setminus\Lambda} := \sum_{x\in\Omega\setminus\Lambda} h_x \,.$$

Moreover, $\rho^\Omega := \rho^\Lambda \otimes \rho^{\Omega\setminus\Lambda}$ is a ground state of $H^\Omega := H^\Lambda + H_0^{\Omega\setminus\Lambda}$ which is a weakly interacting spin system on Ω with on-site gap at least g, range R, and interaction strength $\|\Phi\|$. And also $\rho_P^\Omega := \rho_P^\Lambda \otimes \rho^{\Omega\setminus\Lambda}$ is a ground state of $H_P^\Omega := H_P^\Lambda + H_0^{\Omega\setminus\Lambda} = H^\Omega + P$.

According to Proposition 15.3.4 we have

$$\big|\mathrm{tr}\big((\rho_P^\Omega - \rho^\Omega)A\big)\big| \le \mathrm{e}^{c_1|Y|}\,\|A\|\,\mathrm{e}^{-c_2\min\{\mathrm{dist}(Y,\mathbf{Z}^\nu\setminus\Omega^\circ),\,\mathrm{dist}(Y,X)-2R\}}$$

for all $A \in \mathcal{A}_Y$ and $Y \subset \Omega$. By requiring $Y \subset \Lambda$ we obtain

$$\big|\mathrm{tr}\big((\rho_P^\Lambda - \rho^\Lambda)A\big)\big| = \big|\mathrm{tr}\big((\rho_P^\Omega - \rho^\Omega)A\big)\big| \le \mathrm{e}^{c_1|Y|}\,\|A\|\,\mathrm{e}^{-c_2\min\{\mathrm{dist}(\Lambda,\mathbf{Z}^\nu\setminus\Omega^\circ),\,\mathrm{dist}(Y,X)-2R\}}.$$

Since this bound is independent of Ω, we can choose Ω sufficiently large such that $\mathrm{dist}(\Lambda, \mathbf{Z}^\nu \setminus \Omega^\circ) > \mathrm{dist}(Y,X) - 2R$. Absorbing e^{2c_2R} in c_1 yields the claim. □

15.A Characterization of ground states

In the following lemma we show that every ground state in the usual sense, i.e. every minimizer of the energy functional, is also a ground state according to Definition 15.2.2. While Definition 15.2.2 is often used as a characterization of ground states in the context of extended quantum lattice systems, we could not find any reference in the literature, which covers the statement of the following lemma also for unbounded operators.

Lemma 15.A.1. *Let $\Lambda \in \mathcal{P}_0(\mathbf{Z}^\nu)$ and K be a self-adjoint and bounded below operator on $\mathcal{H}_\Lambda$. A state $\rho \in \mathcal{A}_\Lambda$ is a ground state in the usual sense, i.e.*

$$\mathrm{tr}(K\rho) \le \mathrm{tr}(K\tilde\rho) \quad \textit{for all states } \tilde\rho \in \mathcal{A}_\Lambda\,, \tag{15.A.1}$$

if and only if $\mathrm{ran}(\rho) \subset D(K)$ and

$$\mathrm{tr}(A^*[K,A]\rho) \ge 0 \quad \textit{for all } A \in \mathcal{A}_\Lambda \textit{ such that } [K,A] \textit{ is bounded.} \tag{15.A.2}$$

Here we adopt the convention that the trace of an operator that is not trace class is $+\infty$.

For bounded K this implies that for any state $\rho \in \mathcal{A}_\Lambda$ the conditions (15.A.1) and (15.A.2) are equivalent. And for unbounded K, any ground state in the usual sense is a ground state according to our Definition 15.2.2. It could be that equivalence extends to unbounded operators, i.e. that (15.A.2) implies $\mathrm{ran}(\rho) \subset D(K)$. However, we could not find a proof for this.

Proof of Lemma 15.A.1. Let $E_0 := \inf\sigma(K)$ and let (ϕ_n) be a Weyl sequence for E_0, i.e. $\phi_n \in D(K)$, $\|\phi_n\| = 1$ and $\|(K - E_0)\,\phi_n\| \le 1/n$ for all $n \in \mathbf{N}$.

Assume that ρ satisfies (15.A.1). Since $\mathrm{tr}(K\,|\phi_n\rangle\langle\phi_n|) \le E_0 + 1/n$, it follows that $\mathrm{tr}(K\rho) = E_0$. Hence, E_0 is an eigenvalue of K and the range of ρ is contained in the ground state eigenspace. Let $A \in \mathcal{A}_\Lambda$ such that $[K,A]$ is bounded. Then the operator $A^*(K - E_0)A$ is non-negative and

$$\mathrm{tr}\big(A^*[K,A]\rho\big) = \mathrm{tr}\big(\rho^{\frac12}A^*[K,A]\rho^{\frac12}\big) = \mathrm{tr}\big(\rho^{\frac12}A^*(K-E_0)A\rho^{\frac12}\big) \ge 0$$

follows.

Now assume that ρ is a ground state in the sense of Definition 15.2.2, denote by $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ a spectral decomposition of ρ with ψ_i normalized. Since $\mathrm{ran}(\rho) \subset D(K)$, also $\psi_i \in D(K)$. The operator $A_{n,j} := |\phi_n\rangle\langle\psi_j|$ then has a bounded commutator with K and inequality (15.A.2) yields

$$0 \leq \mathrm{tr}\big(A_{n,j}^*[K, A_{n,j}]\,\rho\big) = \lambda_j \,\langle[\rangle\,]\phi_n, [K, A_{n,j}]\,\psi_j \leq \lambda_j \left(E_0 + \tfrac{1}{n} - \langle\psi_j, K\psi_j\rangle\right).$$

Thus, $\langle\psi_j, K\psi_j\rangle \leq E_0$ for all j. Hence, $\mathrm{tr}(K\rho) = E_0$ and ρ is indeed a ground state of K. □

Chapter 16

On adiabatic theory for gapped fermionic lattice systems

This chapter includes the paper [346]:

J. Henheik and T. Wessel. On adiabatic theory for extended fermionic lattice systems. *J. Math. Phys.*, 63(12):121101, 2022

Abstract. We review recent results on adiabatic theory for ground states of extended gapped fermionic lattice systems under several different assumptions. More precisely, we present generalized super-adiabatic theorems for extended but finite and infinite systems, assuming either a *uniform gap* or a *gap in the bulk* above the unperturbed ground state. The goal of this note is to provide an overview of these adiabatic theorems and briefly outline the main ideas and techniques required in their proofs.

Content

16.1 Introduction

In this article, we review four recent results on adiabatic theory for ground states of extended finite and infinite fermionic lattice systems at zero temperature [556, 342, 341]. These results are *generalized super-adiabatic theorems* (see Section 16.1.2) and concern Hamiltonians of the form

$$H^\varepsilon = H_0 + \varepsilon V\,,$$

where the unperturbed Hamiltonian H_0 is a sum-of-local-terms (SLT) operator describing short-range interacting fermions and is assumed to have a spectral gap above its ground state. This gap might be closed by the (small) perturbation εV, which is given by a short-range Hamiltonian, a Lipschitz potential, or a sum of both. Consequently, the results presented in this article are adiabatic theorems for resonances of H^ε (cf. [2, 235]).

The most important corollary and main motivation for proving such theorems in the context of extended fermionic lattice systems is the rigorous justification of linear response theory [556, 340] and the Kubo formula [386] for (topological) insulators [432], such as quantum Hall systems [560], where the prototypical relevant perturbation is a linear external potential modeling a constant electric field closing the gap of H_0 for every $\varepsilon \neq 0$ (see Figure 16.1.1 on page 550).

In the remainder of this introduction, we first briefly discuss the connection between linear response and adiabatic theory in Section 16.1.1 (see also [556, 340]). Furthermore, we point out the key ingredients and developments, which allowed to prove the four adiabatic theorems reviewed in this paper. Afterwards, in Section 16.1.2, we explain the notion of generalized super-adiabatic theorems and thereby introduce *(super-adiabatic) non-equilibrium almost-stationary states* (NEASSs) [556] as the above mentioned resonances of H^ε. A first brief but somewhat precise statement and overview of the results is given in Section 16.1.3.

16.1.1 Linear response and adiabatic theory

The formalism of linear response theory [386] has been widely used in physics to calculate the response of a system in thermal equilibrium to external perturbations. Put briefly, linear response theory provides an answer to the following question: What is the response of a system described by a Hamiltonian H_0, that is initially in an equilibrium state ρ_0, to a small static perturbation εV? Or, in somewhat more mathematical terms: What is the change[1]

$$\rho_\varepsilon(A) - \rho_0(A) = \varepsilon\,\sigma_A + o(\varepsilon)$$

of the expectation value of an observable A induced by the perturbation εV to leading order in its strength $0 < \varepsilon \ll 1$? Here, ρ_ε denotes the state of the system after the perturbation has been (adiabatically) turned on and σ_A denotes the linear response coefficient.

The answer to this fundamental question of linear response clearly hinges on the problem of determining ρ_ε. Although in few particular situations one expects ρ_ε to remain an equilibrium state

[1] To be consistent with the rest of the paper, we view states as linear functionals on the algebra of observables (see Section 16.2).

for the perturbed Hamiltonian $H^\varepsilon = H_0 + \varepsilon V$, the original linear response theory [386] was developed for situations where the system is driven out of equilibrium, i.e. ρ_ε being a resonance state. As prominently formulated by Simon [526] in his 'Fifteen problems in mathematical physics' from 1984, the latter non-equilibrium situation causes the main challenges in a rigorous mathematical treatment. However, in either case, the linear response coefficient σ_A is customarily expected to be given by the celebrated *Kubo formula* [386], and rigorously justifying it was formulated as one of the problems by Simon [526]. For a more detailed recent review on the (mathematical) problem of proving Kubo's formula and its relevance in the context of quantum Hall systems, we refer to [340].

In a nutshell, the problem of justifying linear response theory and proving Kubo's formula is thus to verify that a system, initially in an equilibrium state ρ_0, is adiabatically driven by a small perturbation εV into a non-equilibrium state $\rho_\varepsilon \approx \rho_0$. Since the perturbation acts over a very long (macroscopic) time, this problem clearly supersedes standard perturbation theory: The change of the state being small is *not* a trivial consequence of the smallness of the perturbation εV. Instead, verifying that the two states, ρ_ε and ρ_0, are close to each other requires an adiabatic type theorem.

However, even in our rather simple setting (zero temperature, assuming that ρ_0 is the gapped ground state of H_0 describing an extended fermionic lattice system, the perturbation εV might close the gap), the problem of justifying the linear response formalism also goes beyond standard adiabatic theory. In fact, the applicability of the standard adiabatic theorem of quantum mechanics is rather restrictive for the following three reasons:

(i) The standard adiabatic theorem requires the perturbation εV to *not close the spectral gap*. In that scenario, it asserts that ρ_ε is (close to) the gapped ground state of $H^\varepsilon = H_0 + \varepsilon V$ and as such a (nearly) equilibrium state.

(ii) Even if we neglect the first issue, the usual adiabatic theorem estimates the difference between ρ_ε and the ground state of the perturbed Hamiltonian H^ε in *operator norm*, leaving the translation to local differences in expectation values as an additional and potentially non-trivial step.

(iii) In general, extended systems are plagued by the *orthogonality catastrophe*: Whenever for single-particle states $\psi, \tilde{\psi}$ we have $\|\psi - \tilde{\psi}\| \sim \varepsilon$, the non-interacting many-particle states $\otimes_{x\in\Lambda}\psi_x$ and $\otimes_{x\in\Lambda}\tilde{\psi}_x$ satisfy $\| \otimes_{x\in\Lambda} \psi_x - \otimes_{x\in\Lambda}\tilde{\psi}_x\| \sim \varepsilon|\Lambda|$, i.e. the norm-estimate deteriorates when $|\Lambda| \to \infty$. This means that the approximation error in the standard adiabatic theorem grows with the systems size, and it is thus not applicable for macroscopic systems.

A major breakthrough in overcoming these obstacles has recently been achieved by Bachmann, De Roeck and Fraas [36] (see also their introductory lecture notes [37]). They proved the first adiabatic theorem for extended (but finite) lattice systems with short-range interactions, thereby solving the second and third problem in the list above. More precisely, their result concerns differences in expectation values and provides error estimates, which are uniform in the system size.

For these lattice systems with short-range interactions, well known *Lieb-Robinson bounds* [419, 460, 461] ensure a finite speed of correlation and prevent build-up of long-range entanglement. Having Lieb-Robinson bounds at hand allowed Bachmann et al. [39] to prove that the generator of the *spectral flow*, introduced by Hastings and Wen [326], is an SLT operator and thus preserves good locality properties. The general spectral flow technique can then be used to prove automorphic equivalence of two gapped ground states ρ_0 and ρ_1 of Hamiltonians $H(0)$ and $H(1)$, respectively: Given a smooth path $s \mapsto H(s)$ of (uniformly) gapped SLT Hamiltonians, their ground states are automorphically equivalent (equal up to a conjugation by unitaries) with the generator of the automorphism being an SLT operator [39]. This automorphic equivalence allowed Bachmann et al. [36] to prove a *super-adiabatic theorem* (see Section 16.1.2 for an explanation of this notion) for such systems,

however, still requiring the spectral gap not only for H_0 but also for H^ε, i.e. the gap must remain open.[2]

The four theorems presented in this article also solved the last remaining problem given under item (i) in the above list, i.e. they allow the perturbation εV to close the spectral gap of H_0. The main idea for establishing this generalization is that a spatially local gap should suffice for an adiabatic theorem to hold. This underlies the space-time adiabatic perturbation theory originally developed for non-interacting fermions by Panati, Spohn and Teufel [484, 483], where one utilizes a gap that exists locally in space (and time) but does not exist globally. It also underlies the recent results by De Roeck, Elgart and Fraas [204], where an adiabatic theorem holds even if the 'spectral gap' is filled with eigenvalues, whose corresponding eigenvectors are spatially localized, leaving a gap (with smaller size) locally open. Finally, this is also the idea behind the Theorems 16.3.7 and especially 16.3.9 presented below, where one still has an adiabatic type theorem although the gap closes at the boundary of the lattices.

Combining the ideas from the space-time adiabatic perturbation theory with the methods invented in [36], the first of the four theorems presented in this article was proven by Teufel [556]. It concerns extended but finite systems and requires a spectral gap for H_0, uniformly in the system size (see Assumption ($\mathrm{GAP}_{\mathrm{unif}}$)). The precise statement is formulated in Theorem 16.3.2 below. In order to extend this result from finite lattices to an infinite system, Henheik and Teufel [342] adapted ideas from Nachtergaele, Sims and Young [461] on controlling the thermodynamic limit of automorphisms with SLT generators. This result is formulated in Theorem 16.3.5 below.

So far, all mentioned results were obtained under the assumption of a (uniform) spectral gap for the finite systems (which also implies a gap for the infinite system). However, the recent result on automorphic equivalence with a *gap only in the bulk* (via the GNS construction) by Moon and Ogata [454], opened the door for a new class of adiabatic theorems, where the unperturbed Hamiltonian H_0 is no longer required to have a uniform spectral gap. Instead, Theorem 16.3.7, originally proven by Henheik and Teufel [341], is a result for the infinite volume states and requires a gap in the bulk. This technically means a gap for the infinite system (cf. Assumption ($\mathrm{GAP}_{\mathrm{bulk}}$)) but can be understood as requiring a local gap in the interior of the finite lattices (cf. Remark 16.3.6).

Moreover, by employing strong locality estimates from [461, 454] and assuming fast convergence of ground states, Theorem 16.3.7 can be traced back to extended but finite systems which only have a gap in the bulk. This was also proven in [341] and is formulated in Theorem 16.3.9 below.

16.1.2 Non-equilibrium almost-stationary states

For the results presented in this paper, we consider time-dependent families

$$H^\varepsilon(t) = H_0(t) + \varepsilon V(t)\,, \quad t \in I \subset \mathbf{R}\,, \tag{16.1.1}$$

of many-body Hamiltonians for lattice fermions in $\Gamma \subset \mathbf{Z}^d$ with short-range interactions. Here, Γ will either be a finite box Λ or the whole of $\mathbf{Z}^d$. For each $t \in I$, we denote by $\rho_0(t)$ the instantaneous ground state of $H_0(t)$ on the (quasi-local) algebra of observables $\mathcal{A}_\Gamma$. For simplicity of the presentation, we shall assume that the ground state is unique.[3] Moreover, we assume that the ground state is separated by a gap from the rest of the spectrum (see Assumptions ($\mathrm{GAP}_{\mathrm{unif}}$) and ($\mathrm{GAP}_{\mathrm{bulk}}$) in Section 16.3 for the precise formulation). The perturbation $V(t)$ can be a Hamiltonian with

[2] A slight generalization of their result can be found in [452], where the authors used an alternative gauge with a time-dependent vector potential for a quantum Hall model.

[3] We refer to the original papers [556, 342, 341] for the most general assumptions. However, note that the results from [341], corresponding to our Theorems 16.3.7 and 16.3.9, are only formulated for a unique ground state, although the underlying result on automorphic equivalence of gapped phases [454] can easily be generalized to any gapped pure state (see [454, Remark 1.4]). In general, allowing for a degenerate ground state (or even a gapped spectral patch) requires understanding an enhanced modification of the spectral flow.

short-range interactions or a possibly unbounded external Lipschitz potential or a sum of both (see Section 16.2 and the Assumptions (INT_1)–(INT_4) in Section 16.3).

As mentioned above, the main results presented in this article are so-called *generalized super-adiabatic theorems* for $\rho_0(t)$, which we briefly explain in the following. For $\varepsilon = 0$, the results are *'standard' super-adiabatic theorems* and establish the existence of super-adiabatic states $\rho_0^\eta(t)$ on $\mathcal{A}_\Gamma$ close to $\rho_0(t)$, i.e.

$$\left|\rho_0^\eta(t)(A) - \rho_0(t)(A)\right| = \mathcal{O}(\eta)\,,$$

such that the adiabatic time-evolution $\mathfrak{U}_{t,t_0}^\eta$ on $\mathcal{A}_\Gamma$ generated by $\frac{1}{\eta}H_0(\cdot)$ intertwines the super-adiabatic states to all orders in the adiabatic parameter $\eta > 0$, i.e.

$$\left|\rho_0^\eta(t_0)(\mathfrak{U}_{t,t_0}^\eta[\![A]\!]) - \rho_0^\eta(t)(A)\right| = \mathcal{O}(\eta^\infty) \tag{16.1.2}$$

for all A in a dense subspace $\mathcal{D} \subset \mathcal{A}_\Gamma$. Throughout the entire paper, we shall study our system in the *Heisenberg picture*, meaning that the observable A evolves in time, not the state $\rho_0^\eta(t_0)$ (see also Proposition 16.3.4). Note that the comparison state $\rho_0^\eta(t)$ does *not* involve any time evolution but simply depends on the Hamiltonian at time t (see Definition 16.3.1 for details). Here and in the following, we write the arguments of (densely defined) linear operators on $\mathcal{A}_\Gamma$ inside the brackets $[\![\cdot]\!]$ for better readability.

For $\varepsilon > 0$, the scope of the adiabatic theorem (16.1.2) extends considerably since the perturbation $\varepsilon V(t)$ might close the spectral gap and turn the ground state $\rho_0(t)$ of $H_0(t)$ into an instantaneous resonance state $\Pi^\varepsilon(t)$ for $H^\varepsilon(t)$. These states have a life-time of order $\mathcal{O}(\varepsilon^{-\infty})$ for the dynamics $s \mapsto \mathrm{e}^{\mathrm{i}s\mathcal{L}_{H^\varepsilon(t)}}$ with $\mathcal{L}_{H^\varepsilon(t)}[\![\cdot]\!] := [H^\varepsilon(t), \cdot\,]$ (formally) denoting the derivation associated to $H^\varepsilon(t)$. That is, for all $n \in \mathbf{N}$ and fixed t, it holds that

$$\left|\Pi^\varepsilon(t)(\mathrm{e}^{\mathrm{i}s\mathcal{L}_{H^\varepsilon(t)}}[\![A]\!]) - \Pi^\varepsilon(t)(A)\right| = \mathcal{O}\Big(\varepsilon^n\left(1 + |s|^{d+1}\right)\Big)\,,$$

which is why they were called *non-equilibrium almost-stationary states* (NEASSs) in this context by Teufel [556]. The generalized super-adiabatic theorems then establish the existence of a *super-adiabatic NEASS* $\Pi^{\varepsilon,\eta}(t)$ on $\mathcal{A}_\Gamma$ close to $\Pi^\varepsilon(t)$ such that the adiabatic time-evolution $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}$ generated by $\frac{1}{\eta}H^\varepsilon(\cdot)$ approximately intertwines the super-adiabatic NEASSs in the following sense: for any $n > d$ and for all $A \in \mathcal{D} \subset \mathcal{A}_\Gamma$, we have

$$\left|\Pi^{\varepsilon,\eta}(t_0)(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]) - \Pi^{\varepsilon,\eta}(t)(A)\right| = \mathcal{O}\left(\eta^{n-d} + \frac{\varepsilon^{n+1}}{\eta^{d+1}}\right) \tag{16.1.3}$$

uniformly for t in compact sets, which we call a *generalized super-adiabatic theorem*.

In our setting of gapped Hamiltonians H_0 describing insulating materials, there is indeed a clear and simple physical picture suggesting the existence of NEASSs for H^ε as observed in [556, 340] (see Figure 16.1.1). For simplicity, assume that H_0 is a periodic one-body operator in one spatial dimension and that the Fermi energy μ (chemical potential) lies in a gap of size g. For the perturbation, we consider the potential of a small constant electric field ε. In the initial state ρ_0, before the perturbation is turned on, all one-body states with energy smaller than μ are occupied. After the voltage has been applied, the energy of an electron located at position x_0 gets substantially shifted by εx_0, but is only subject to small force of order ε. As indicated in Figure 16.1.1, in order to make a transition, such an electron must either overcome the gap of size g or tunnel a macroscopic distance of order g/ε. Thus, although ρ_0 is neither close to the ground state nor any other equilibrium state of the perturbed Hamiltonian $H^\varepsilon = H_0 + \varepsilon V$, it is still almost stationary for H^ε. This heuristic picture remains valid if short-range interactions between the electrons are taken into account.

While for $\varepsilon = 0$ the generalized super-adiabatic theorem (16.1.3) reduces to the standard one (16.1.2), for $0 < \varepsilon \ll 1$ the right-hand side of (16.1.3) is small if and only if also η is small, but not too

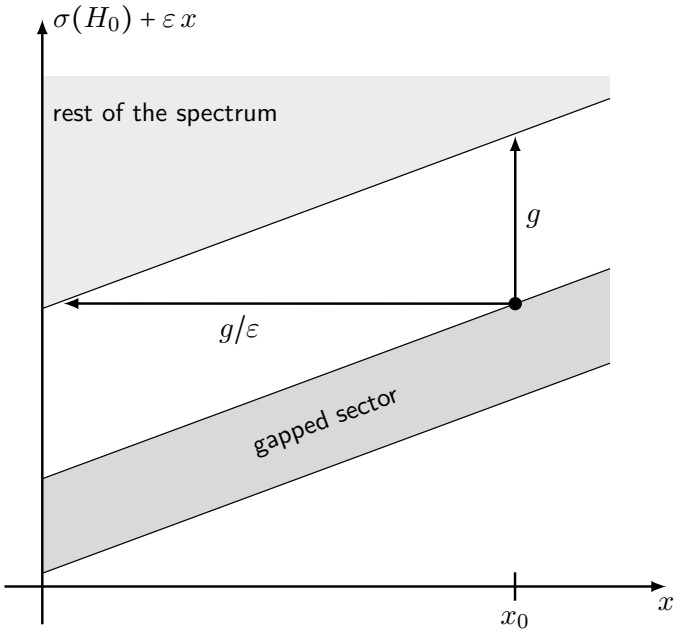

Figure 16.1.1: Let H_0 be a Hamiltonian with a gapped sector and a gap g. Perturbing with a Lipschitz potential $v(x) = \varepsilon\, x$, the gap gets closed (for large enough lattices). But, as indicated in the figure, a local gap persists and an electron at location x_0 would either need to overcome the gap (vertical arrow) or tunnel along the distance g/ε (horizontal arrow) in order to make a transition from the gapped sector. [556, 340]

small compared to ε, i.e. $\varepsilon^{\frac{n+1}{d+1}} \ll \eta \ll 1$ for some $n \in \mathbf{N}$. Physically, this simply means that the adiabatic approximation breaks down when the adiabatic switching occurs at times that exceed the lifetime of the NEASS, an effect that has been observed in adiabatic theory for resonances before, see, e.g., [2, 235]. It can also be heuristically understood from the tunneling picture given in Figure 16.1.1.

Moreover, in view of the linear response problem discussed in Section 16.1.1, let us only mention here that a statement like (16.1.3), in fact, yields a solution to this problem after expanding the state $\Pi^{\varepsilon,\eta}(t)$ in powers of ε, where the linear term (eventually stemming from the first order operator A_1 given in (16.4.6)) does, in fact, constitute Kubo's formula. See [556, 340, 342] for details.

16.1.3 Brief statement of the results

We shall establish the existence of super-adiabatic NEASSs in four generally quite different situations, the main differences are also summarized in Table 16.1:

(I) On finite systems $\Lambda_k \Subset \mathbf{Z}^d$ with suitable boundary conditions, assuming that the unperturbed Hamiltonians $H_0^{\Lambda_k}(t)$ have a gapped ground state uniformly in Λ_k, there exists NEASSs on $\mathcal{A}_{\Lambda_k}$ such that the constants in (16.1.3) are independent of Λ_k. See Theorem 16.3.2 and [556].

(II) Additionally assuming convergence of the Hamiltonians (they have a thermodynamic limit, cf. Definition 16.3.3) and ground states, there also exists a super-adiabatic NEASS on $\mathcal{A}_{\mathbf{Z}^d}$ after taking the thermodynamic limit $\Lambda_k \nearrow \mathbf{Z}^d$. See Theorem 16.3.5 and [342].

(III) For the infinite system $\mathbf{Z}^d$, assuming that the unperturbed Hamiltonian H_0 has a unique gapped ground state (via the GNS construction), there exists a NEASS on $\mathcal{A}_{\mathbf{Z}^d}$, while a (uniform) spectral gap for finite sub-systems is not required. See Theorem 16.3.7 and [341].

Table 16.1: Overview of the adiabatic theorems and the original papers.

	Finite volume	Infinite volume
Uniform gap	Theorem 16.3.2; see [556]	Theorem 16.3.5; see [342]
Gap in the bulk	Theorem 16.3.9; see [341]	Theorem 16.3.7; see [341]

(IV) Additionally assuming a quantitative control on the convergence of the finite volume Hamiltonians $H^{\Lambda_k}(t)$ (they have a rapid thermodynamic limit, cf. Definition 16.3.8) and the *unperturbed* ground states in the thermodynamic limit, there also exist NEASSs on $\mathcal{A}_{\Lambda_k}$ (again with a uniform constant) up to an error vanishing faster than any inverse polynomial in the distance to the boundary. See Theorem 16.3.9 and [341].

A typical example of a physically relevant class of Hamiltonians [452, 556, 340], to which the above generalized super-adiabatic theorems apply, is given by

$$\begin{aligned} H_0^{\Lambda_k} = & \sum_{x,y\in\Lambda_k} a_x^* T(x-y)\, a_y \;+\; \sum_{x\in\Lambda_k} a_x^*\, \phi(x)\, a_x \\ & + \sum_{x,y\in\Lambda_k} a_x^* a_x\, W\big(d^{\Lambda_k}(x,y)\big)\, a_y^* a_y \;-\; \mu\, N_{\Lambda_k}\,, \end{aligned} \tag{16.1.4}$$

modeling Chern or topological insulators. In agreement with the precise locality assumptions (INT_1)–(INT_4) in Section 16.3, we suppose that the kinetic term $T\colon \mathbf{Z}^d \to \mathcal{L}(\mathbb{C}^r)$ is an exponentially decaying function with $T(-x) = T(x)^*$, the potential term $\phi\colon \mathbf{Z}^d \to \mathcal{L}(\mathbb{C}^r)$ is a bounded function taking values in the self-adjoint matrices, and the two-body interaction $W\colon [0,\infty) \to \mathcal{L}(\mathbb{C}^r)$ is exponentially decaying and also takes values in the self-adjoint matrices. Note, that $x-y$ in the kinetic term refers to the difference modulo the imposed boundary condition on Λ_k. In the first line of (16.1.4), a_x is the column vector of the annihilation operators $a_{x,i}$ (i labels internal degrees of freedom such as spin) and a_x^* the row vector of the creation operators $a_{x,i}^*$ (see Section 16.2). And with a slight abuse of notation in the second line of (16.1.4), we wrote $a_x^* a_x$ for the *row* vector with entries $a_{x,i}^* a_{x,i}$ and $a_y^* a_y$ for the *column* vector with entries $a_{y,i}^* a_{y,i}$.

It is well known that non-interacting Hamiltonians H_0, i.e. with $W \equiv 0$, of the type (16.1.4) on a *torus* (periodic boundary condition) have a *uniform spectral gap* (see Assumption ($\mathsf{GAP}_{\mathsf{unif}}$)) whenever the chemical potential μ multiplying the number operator lies in a gap of the spectrum of the corresponding one-body operator on the infinite domain. It was recently shown [324, 206], that the spectral gap remains open when perturbing by sufficiently small short-range interactions $W \neq 0$. On the other hand, the Hamiltonian H_0 on a *cube* with open boundary condition has, in general, no longer a spectral gap because of the appearance of edge states. However, away from the boundary, a *gap in the bulk* (see Assumption ($\mathsf{GAP}_{\mathsf{bulk}}$)) is still present. While also *uniqueness* of the ground state is expected to hold for such models, to our knowledge it has been shown only for certain types of quantum spin systems, cf. [591, 273, 462, 463] and Chapter 15. For further details, we refer to the original papers [556, 342, 341]. Finally, it is an interesting program to extend Table 16.1 by further rows representing different notions of a spectral gap for H_0, e.g. a *local gap* as in Chapter 15 or even only a *mobility gap* (see [204] for a first result in this direction).

After a brief introduction to the relevant mathematical framework in Section 16.2, we formulate the four main theorems in Section 16.3. Ideas of their proofs are provided in Section 16.4.

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16.2 Mathematical Framework

In this section, we briefly introduce the (standard) mathematical framework used in the formulation of the adiabatic theorems. More explanations and details are provided in [556, 342, 341].

16.2.1 Algebra of observables

We consider fermions with r spin or other internal degrees of freedom on the lattice $\mathbf{Z}^d$. Let $\{X \Subset \mathbf{Z}^d\} := \{X \subset \mathbf{Z}^d : |X| < \infty\}$ denote the set of finite subsets of $\mathbf{Z}^d$, where $|X|$ is the number of elements in X. For each $X \Subset \mathbf{Z}^d$ let $\mathfrak{F}_X$ be the fermionic Fock space built up from the one-body space $\ell^2(X, \mathbb{C}^r)$. The C^*-algebra of bounded operators $\mathcal{A}_X := \mathcal{L}(\mathfrak{F}_X)$ is generated by the identity element $\mathbf{1}_{\mathcal{A}_X}$ and the creation and annihilation operators $a^*_{x,i}$, $a_{x,i}$ for $x \in X$ and $1 \leq i \leq r$, which satisfy the canonical anti-commutation relations (CAR). Whenever $X \subset X'$, then $\mathcal{A}_X$ is naturally embedded as a subalgebra of $\mathcal{A}_{X'}$. For infinite systems, the *algebra of local observables* is defined as the inductive limit

$$\mathcal{A}_{\mathrm{loc}} := \bigcup_{X \Subset \mathbf{Z}^d} \mathcal{A}_X\,, \qquad \text{and its completion} \qquad \mathcal{A}_{\mathbf{Z}^d} := \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

with respect to the operator norm $\|\cdot\|$ is a C^*-algebra, called the *quasi-local algebra*. The even elements $\mathcal{A}^+_{\mathbf{Z}^d} \subset \mathcal{A}_{\mathbf{Z}^d}$ form a C^*-subalgebra. Also, note that for any $X \Subset \mathbf{Z}^d$ the set of elements $\mathcal{A}^N_X$ commuting with the number operator $N_X := \sum_{x \in X} a^*_x a_x := \sum_{x \in X} \sum_{i=1}^r a^*_{x,i} a_{x,i}$ forms a subalgebra of the even subalgebra, i.e. $\mathcal{A}^N_X \subset \mathcal{A}^+_X \subset \mathcal{A}_X$. As only even observables will be relevant to our considerations, we will drop the superscript $^+$ from now on and redefine $\mathcal{A}_{\mathbf{Z}^d} := \mathcal{A}^+_{\mathbf{Z}^d}$.

Since a very similar construction is common for quantum spin systems (see, e.g., [461]), all the results immediately translate to this setting.

16.2.2 Interactions and operator families

We shall consider sequences of Hamiltonians defined on centered boxes $\Lambda_k := \{-k, \ldots, +k\}^d$ of size $2k$ with metric $d^{\Lambda_k}(\cdot,\cdot)$. This *metric* may differ from the standard ℓ^1-distance $d(\cdot,\cdot)$ on $\mathbf{Z}^d$ restricted to Λ_k if one considers discrete tube or torus geometries, but satisfies the bulk-compatibility condition

$$\forall k \in \mathbb{N}\ \forall x, y \in \Lambda_k : d^{\Lambda_k}(x,y) \leq d(x,y) \quad \text{and} \quad d^{\Lambda_k}(x,y) = d(x,y) \text{ whenever } d(x,y) \leq k\,.$$

An *interaction on a domain* Λ_k is a map

$$\Phi^{\Lambda_k} : \{X \subset \Lambda_k\} \to \mathcal{A}^N_{\Lambda_k}\,,\ X \mapsto \Phi^{\Lambda_k}(X) \in \mathcal{A}^N_X$$

with values in the self-adjoint operators. Note that the maps Φ^{Λ_k} can be extended to maps on the whole $\{X \Subset \mathbf{Z}^d\}$ or restricted to a smaller Λ_l, trivially. In order to describe fermionic systems on the lattice $\mathbf{Z}^d$ in the thermodynamic limit, one considers sequences $\Phi = \left(\Phi^{\Lambda_k}\right)_{k \in \mathbb{N}}$ of interactions on domains Λ_k and calls the whole sequence an *interaction*.

An *infinite volume interaction* is a map

$$\Psi : \{X \Subset \mathbf{Z}^d\} \to \mathcal{A}^N_{\mathrm{loc}}\,,\ X \mapsto \Psi(X) \in \mathcal{A}^N_X\,,$$

again with values in the self-adjoint operators. Such an infinite volume interaction defines a general interaction $\Psi = \left(\Psi^{\Lambda_k}\right)_{k \in \mathbb{N}}$ by restriction, i.e. by setting $\Psi^{\Lambda_k} := \Psi|_{\{X \subset \Lambda_k\}}$.[4] With any interaction Φ, one associates an *operator family*, which is a sequence $A = (A^{\Lambda_k})_{k \in \mathbb{N}}$ of self-adjoint operators

$$A^{\Lambda_k} := A^{\Lambda_k}(\Phi) := \sum_{X \subset \Lambda_k} \Phi^{\Lambda_k}(X) \in \mathcal{A}^N_{\Lambda_k}.$$

[4] We will use the convention that Φ denotes general interactions and Ψ infinite volume interactions.

For any $a > 0$ and $n \in \mathbf{N}_0$, we define the norm

$$\|\Phi\|_{a,n} := \sup_{k\in\mathbf{N}} \sup_{x,y\in\mathbf{Z}^d} \sum_{\substack{X\subset\Lambda_k:\\ x,y\in X}} d^{\Lambda_k}\text{-diam}(X)^n \, \mathrm{e}^{a\cdot d^{\Lambda_k}(x,y)} \, \|\Phi^{\Lambda_k}(X)\| \tag{16.2.1}$$

on the space of interactions.[5] Note that these norms depend on the sequence of metrics d^{Λ_k} on the cubes Λ_k, i.e. on the boundary conditions.

Similar constructions for interactions and interaction norms are long known. More commonly, the norms are independent of the particular lattice Λ_k and the interaction $\left(\Phi^{\Lambda_k}\right)_{k\in\mathbf{N}}$ is given by restrictions of a single infinite volume interaction. Moreover, in earlier works [506, 511] the authors did not require additional decay properties, which were only added later (see, e.g., [527, 325, 461]). The use of interactions and corresponding norms, which are *not* simply restrictions of an infinite volume interaction, originates in [452] to incorporate non-trivial boundary conditions. In order to control commutators with Lipschitz potentials (see Section 16.2.3), the dependence on the diameter $d^{\Lambda_k}\text{-diam}(X)$ was added in [556]. Finally, to ensure the existence of the thermodynamic limit, it is necessary to require the bulk-compatibility condition [342, 341]. Yet another variant of defining interaction norms is to replace $\mathrm{dist}(x,y)$ with $\mathrm{diam}(X)$ in (16.2.1) (see, e.g., [325, 35]).

In order to quantify the difference of interactions in the bulk (see Section 16.3.2), we also introduce for any interaction Φ^{Λ_l} on the domain Λ_l and any $\Lambda_M \subset \Lambda_l$ the quantity

$$\|\Phi^{\Lambda_l}\|_{a,n,\Lambda_M} := \sup_{x,y\in\Lambda_M} \sum_{\substack{X\subset\Lambda_M:\\ x,y\in X}} \mathrm{diam}(X)^n \, \mathrm{e}^{a\cdot d(x,y)} \, \|\Phi^{\Lambda_l}(X)\| \,,$$

where d and diam now refer to the ℓ^1-distance on $\mathbf{Z}^d$.

Let $\mathcal{B}_{a,n}$ be the *Banach space of interactions* with finite $\|\cdot\|_{a,n}$-norm and define the space of *exponentially localized interactions* as the intersection $\mathcal{B}_{a,\infty} := \bigcap_{n\in\mathbb{N}_0} \mathcal{B}_{a,n}$. In the literature, the vector spaces of operator families, which can be written in terms of such interactions, are denoted by $\mathcal{L}_{a,n}$ and $\mathcal{L}_{a,\infty}$. Moreover, we will be a bit sloppy in the following terminology and call the elements A^{Λ_k} of an operator sequence A *sum-of-local-terms* (SLT) operators, whenever its interaction Φ_A has a finite interaction norm similar to (16.2.1), but with the exponential replaced by a function growing faster than any polynomial. This will allow us to formulate the results and the ideas of the proofs without too many details. For the precise conditions see, e.g., [342, Section 2.2].

Now, let $I \subset \mathbb{R}$ be an open interval. We say that a map $\Phi\colon I \to \mathcal{B}_{a,n}$ is *smooth and bounded* whenever it is (i) term- and point-wise smooth in $t \in I$, i.e. $t \mapsto \Phi^{\Lambda_k}(t,X)$ are C^∞-functions for all $k \in \mathbf{N}$ and $X \subset \Lambda_k$, and (ii) $\sup_{t\in I} \|\frac{\mathrm{d}^i}{\mathrm{d}t^i}\Phi(t)\|_{a,n} < \infty$ for all $i \in \mathbf{N}_0$. The corresponding spaces of smooth and bounded time-dependent interactions and operator families are denoted by $\mathcal{B}_{I,a,n}$ and $\mathcal{L}_{I,a,n}$ and are equipped with the norm $\|\Phi\|_{I,a,n} := \sup_{t\in I} \|\Phi(t)\|_{a,n}$. We say that $\Phi\colon I \to \mathcal{B}_{a,\infty}$ is smooth and bounded, if $\Phi\colon I \to \mathcal{B}_{a,n}$ is smooth and bounded for all $n \in \mathbf{N}_0$, and we write $\mathcal{B}_{I,a,\infty}$ and $\mathcal{L}_{I,a,\infty}$ for the corresponding spaces of *time-dependent exponentially localized interactions* and *operator families* respectively.

For (time-dependent) *infinite volume interactions* Ψ, we add a superscript $^\circ$ to the norms and to the normed spaces defined above, emphasizing in particular the use of open boundary conditions, i.e. $d^{\Lambda_k} \equiv d$. Note that the compatibility condition for the metrics d^{Λ_k} implies that $\|\Psi\|_{a,n} \leq \|\Psi\|^\circ_{a,n}$.

[5] One should be aware that the norm definition (16.2.1) is slightly modified compared to the original works [556, 342, 341] for the sake of simplicity in presentation. For more general and precise statements of the theorems we refer the reader to the original works.

16.2.3 Lipschitz potentials

For the perturbation we will allow external potentials $v = \big(v^{\Lambda_k}\colon \Lambda_k \to \mathbb{R}\big)_{k\in\mathbb{N}}$ that satisfy the Lipschitz condition

$$\mathcal{C}_v := \sup_{k\in\mathbb{N}} \sup_{\substack{x,y\in\Lambda_k:\\ x\neq y}} \frac{\big|v^{\Lambda_k}(x) - v^{\Lambda_k}(y)\big|}{d^{\Lambda_k}(x,y)} < \infty\,, \tag{16.2.2}$$

and call them for short *Lipschitz potentials*.[6] With a Lipschitz potential v we associate the corresponding operator-sequence $V_v = (V_v^{\Lambda_k})_{k\in\mathbb{N}}$ defined by

$$V_v^{\Lambda_k} := \sum_{x\in\Lambda_k} v^{\Lambda_k}(x)\, a_x^* a_x$$

and denote the space of Lipschitz potentials by $\mathcal{V}$. We emphasize that, since $\sup_{k\in\mathbb{N}} \sup_{x\in\Lambda_k} |v^{\Lambda_k}(x)|$ might be infinite, V_v is in general no SLT operator. However, this is still more restrictive than general onsite potentials, because it only varies slowly in space. Moreover, we say that the map $v\colon I \to \mathcal{V}$ is smooth and bounded whenever (i) $v^{\Lambda_k}(x,\cdot)$ are C^∞-functions for all $k \in \mathbf{N}$ and $x \in \Lambda_k$, and (ii) satisfies $\sup_{t\in I} C_{\frac{\mathrm{d}^i}{\mathrm{d}t^i} v(t)} < \infty$ for all $i \in \mathbb{N}_0$. The space of smooth and bounded *time-dependent Lipschitz potentials* is denoted by $\mathcal{V}_I$.

As above, we also introduce *infinite volume Lipschitz potentials* $v_\infty\colon \mathbf{Z}^d \to \mathbf{R}$, which, again by restriction and invoking the compatibility condition for the metrics d^{Λ_k}, can be viewed as a Lipschitz potential with $d^{\Lambda_k} \equiv d$ in (16.2.2). And analogously to Section 16.2.2, for (time-dependent) *infinite volume Lipschitz potentials*, we add a superscript $^\circ$ to the constant from (16.2.2) and to the spaces, emphasizing the use of open boundary conditions. Note that the compatibility condition for the metrics d^{Λ_k} implies that $\mathcal{C}_v \geq \mathcal{C}_v^\circ$.

16.3 Adiabatic theorems for gapped quantum systems

As mentioned in the introduction, we shall distinguish two generally quite different settings regarding the presence of a spectral gap of the unperturbed Hamiltonian H_0 grouped as Theorem 16.3.2 and Theorem 16.3.5 in Section 16.3.1 as well as Theorem 16.3.7 and Theorem 16.3.9 in Section 16.3.2. First, in Section 16.3.1, we will work under the assumption that there exists a sequence of subsystems $(\Lambda_k)_{k\in\mathbf{N}}$ equipped with an appropriate metric (reflecting, e.g., periodic boundary conditions), ensuring that $H_0^{\Lambda_k}$ have a *uniform gap* above their ground state, which is made precise in Assumption ($\mathrm{GAP}_{\mathsf{unif}}$) below. Then, in Section 16.3.2, however, we drop this assumption and solely assume that H_0 has a *gap in the bulk*, meaning that the GNS Hamiltonian, describing the system in the thermodynamic limit, has a spectral gap above its ground state eigenvalue zero (see Assumption ($\mathrm{GAP}_{\mathsf{bulk}}$)). Note that the second group of results is more general than the first group with regard to the gap condition, since a uniform gap for finite systems guarantees a spectral gap for the GNS Hamiltonian describing the infinite system (see Proposition 5.4 in [38]). Therefore, the second row in Table 16.1 somewhat improves the results in the first row since finding a suitable geometry for which one already has a spectral gap for finite systems is no longer necessary.

In the precise formulation of the adiabatic theorems, we shall frequently use the abbreviating phrase that a state $\Pi^{\varepsilon,\eta}(t)$ *is a super-adiabatic NEASS* (see Section 16.1.2), which we generally define as follows, reminiscent of [556, 342, 341].

Definition 16.3.1 (Super-adiabatic non-equilibrium almost-stationary states)**.** *We assume to be in the following general setting, which is made precise in concrete situations: For (small) $\varepsilon > 0$, define*

[6] Teufel [556] instead allowed slightly more general *slowly-varying potentials*. And while the phrase captures the idea very well, the technical definition is less transparent and slightly complicates the presentation of the proofs. Hence, we here, as in [342, 341], restrict to the subclass of Lipschitz potentials.

the time-dependent Hamiltonian

$$H^{\varepsilon}(t) = H_0(t) + \varepsilon V(t)\,,\ t \in I\,, \quad \text{on} \quad \Gamma \subset \mathbf{Z}^d$$

and let $\rho_0(t)$ be (close to)[7] *the ground state of $H_0(t)$. Moreover, denote the Heisenberg time-evolution on the algebra of (quasi-local) observables $\mathcal{A}_\Gamma$ generated by $\frac{1}{\eta}H^{\varepsilon}(t)$ as $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$, where $t, t_0 \in I$ for some open interval $I \subset \mathbf{R}$ and $\eta > 0$ is a (small) adiabatic parameter.*

Then, we say that a state $\Pi^{\varepsilon,\eta}(t)$ on $\mathcal{A}_\Gamma$ is a super-adiabatic non-equilibrium almost-stationary state *for the state $\rho_0(t)$ and the time-evolution $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$ on $\mathcal{A}_\Gamma$ if it satisfies the following properties:*

1. $\Pi^{\varepsilon,\eta}$ *almost* **intertwines the time evolution**: *For any $n \in \mathbb{N}$, there exists a constant C_n such that for any $t, t_0 \in I$ and for all $X \Subset \Gamma$ and $A \in \mathcal{A}_X \subset \mathcal{A}_\Gamma$ we have*

$$\left|\Pi^{\varepsilon,\eta}(t_0)(\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}[\![A]\!]) - \Pi^{\varepsilon,\eta}(t)(A)\right| \leq\ C_n\ \frac{\varepsilon^{n+1}+\eta^{n+1}}{\eta^{d+1}}\ \left(1+|t-t_0|^{d+1}\right)\|A\|\,|X|^2. \qquad (16.3.1)$$

2. $\Pi^{\varepsilon,\eta}$ *is* **local in time**: *$\Pi^{\varepsilon,\eta}(t)$ only depends on H_0 and V and their time derivatives at time t.*

3. $\Pi^{\varepsilon,\eta}$ *is* ***stationary*** *whenever the Hamiltonian is stationary: If for some fixed $t \in I$ all time-derivatives of H_0 and V vanish at time t, then $\Pi^{\varepsilon,\eta}(t)$ equals the NEASS*[8] *$\Pi^{\varepsilon}(t)$ for the instantaneous ground state $\rho_0(t)$ and the time-evolution $s \mapsto \mathrm{e}^{\mathrm{i}s\mathcal{L}_{H^{\varepsilon}(t)}}$ generated by the time-independent Hamiltonian $H^{\varepsilon}(t)$.*

4. $\Pi^{\varepsilon,\eta}$ *equals the (approximate) ground state ρ_0 of H_0 whenever the perturbation vanishes and the Hamiltonian is stationary: If for some $t \in I$ all time-derivatives of H_0 and V vanish at time t and $V(t) = 0$, then $\Pi^{\varepsilon,\eta}(t) = \Pi^{\varepsilon,0}(t) = \rho_0(t)$.*

We could have written bound (16.3.1) in a more general form as indicated by (16.1.3). For example, we could allow $(1+|t-t_0|^{d+1})$ to be replaced by a constant $C_K < \infty$, depending only on a compact subset $K \subset I$ of times, or, similarly, $|X|^2$ to be replaced by a constant $C_X < \infty$, depending only on the support $X \Subset \mathbf{Z}^d$ of the observable A. Also, the power of η in the denominator could be allowed to be more general, e.g. some constant $C_d < \infty$ instead of $d+1$. However, the concrete form of (16.3.1) indeed matches the precise bounds of the results in Section 16.3.

16.3.1 Systems with a uniform gap

Throughout this section, we assume that H_0 has a uniformly gapped unique ground state in the following sense.

(GAP$_{\text{unif}}$) Assumptions on the ground state of H_0.

Let $\Phi_{H_0} = \left(\Phi^{\Lambda_k}_{H_0}\right)_{k\in\mathbf{N}}$ be an interaction. There exists $L \in \mathbb{N}$ such that for all $t \in I$, $k \geq L$ and corresponding Λ_k the operator $H^{\Lambda_k}_0(t)$ has a simple gapped ground state eigenvalue $E^{\Lambda_k}_0(t) = \inf\sigma(H^{\Lambda_k}_0(t))$, i.e. there exists $g > 0$ such that $\operatorname{dist}\left(E^{\Lambda_k}_0(t), \sigma(H^{\Lambda_k}_0(t)) \setminus \{E^{\Lambda_k}_0(t)\}\right) \geq g$, for all $t \in I$, $k \geq L$. We denote the spectral projection of $H^{\Lambda_k}_0(t)$ corresponding to $E^{\Lambda_k}_0(t)$ by $P^{\Lambda_k}_0(t)$ and write $\rho^{\Lambda_k}_0(t)(\cdot) := \operatorname{tr}\left(P^{\Lambda_k}_0(t)\,\cdot\,\right)$ for the canonically associated state on $\mathcal{A}_{\Lambda_k}$.

A physically relevant class of Hamiltonians satisfying this assumption (possibly up to the uniqueness, which we require for simplicity of the presentation) was given in (16.1.4) in Section 16.1.3. In the following, we shall present adiabatic theorems for extended but finite systems (Theorem 16.3.2) as well as for infinite systems (Theorem 16.3.5) under Assumption (GAP$_{\text{unif}}$).

[7] See the comment below Assumption (S$_{\text{bulk}}$) on page 561 for a precise definition.

[8] It follows from the construction sketched in Section 16.4.1.1 that $\Pi^{\varepsilon}(t) = \Pi^{\varepsilon,0}(t)$. Moreover, $\Pi^{\varepsilon}(t)$ is almost stationary with a bound as in (16.3.1) where the fraction is replaced by ε^{n+1}.

16.3.1.1 Extended but finite systems

The basic assumption on the Hamiltonian says that it is composed of exponentially localized interactions and/or a Lipschitz potential.

(INT$_1$) Assumptions on the interactions.

Let H_0, H_1 be the Hamiltonians of two time-dependent exponentially localized interactions, i.e. $\Phi_{H_0}, \Phi_{H_1} \in \mathcal{B}_{I,a,\infty}$ for some $a > 0$, and $v \in \mathcal{V}_I$ be a time-dependent Lipschitz potential.

The following results due to Teufel [556] marks the starting point for generalized super-adiabatic theorems for extended fermionic lattice systems.

Theorem 16.3.2 (Adiabatic theorem for finite systems with a uniform gap, see Theorem 5.1 in [556])**.** *Under Assumptions* (GAP$_{\text{unif}}$) *and* (INT$_1$)*, there exists a sequence of near-identity*[9] *automorphisms* $\beta^{\varepsilon,\eta,\Lambda_k}(t) = \mathrm{e}^{\mathrm{i}\varepsilon\mathcal{L}^{\Lambda_k}_{S^{\varepsilon,\eta}(t)}}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon, \eta \in (0,1]$ *and* $t \in I$ *such that the states*

$$\Pi^{\varepsilon,\eta,\Lambda_k}(t) := \rho_0^{\Lambda_k}(t) \circ \beta^{\varepsilon,\eta,\Lambda_k}(t) \tag{16.3.2}$$

are super-adiabatic NEASSs for the Heisenberg time-evolution $\mathfrak{U}^{\varepsilon,\eta,\Lambda_k}_{t,t_0}$ *on* $\mathcal{A}_{\Lambda_k}$ *generated by* $\frac{1}{\eta} H^{\varepsilon,\Lambda_k}(\cdot)$ *with*

$$\tfrac{1}{\eta} H^{\varepsilon,\Lambda_k}(t) := \tfrac{1}{\eta}\left(H_0^{\Lambda_k}(t) + \varepsilon\left(V_v^{\Lambda_k}(t) + H_1^{\Lambda_k}(t)\right)\right)$$

uniformly in $k \geq L$*. That is, for every* $n \in \mathbf{N}$*, there exists a constant* C_n*, such that for any* $A \in \mathcal{A}_X$*,* $\varepsilon, \eta \in (0,1]$ *and all* $t, t_0 \in I$ *it holds that*

$$\sup_{k \geq L} \left| \Pi^{\varepsilon,\eta,\Lambda_k}(t_0)(\mathfrak{U}^{\varepsilon,\eta,\Lambda_k}_{t,t_0}[\![A]\!]) - \Pi^{\varepsilon,\eta,\Lambda_k}(t)(A) \right| \leq C_n \, \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{d+1}} \left(1 + |t - t_0|^{d+1}\right) \|A\| \, |X|^2 \, .$$

The proof of this result fundamentally builds on space-time adiabatic perturbation theory [484, 483] and technical estimates originally derived in [36]. The latter show that the operations necessary for the construction of the generator of the near-identity automorphism in the definition of the NEASS in (16.3.2) (almost) preserve exponential localization required for the Hamiltonian (see Section 16.4). As already mentioned in the introduction, although the adiabatic theorem in [36] is at first sight quite similar to the one above, it requires the perturbation to *not* close the spectral gap of the Hamiltonian H_0 and is thus not generalized in the sense explained in Section 16.1.2.

16.3.1.2 Infinite systems

The next result is obtained from Theorem 16.3.2 by taking $\Lambda_k \nearrow \mathbf{Z}^d$. This requires the interactions and the Lipschitz potential composing the Hamiltonian (16.1.1) to *have a thermodynamic limit* [342] in the following sense.

Definition 16.3.3 (Thermodynamic limit of interactions and potentials)**.** *(a) An exponentially localized time-dependent interaction* $\Phi \in \mathcal{B}_{I,a,\infty}$ *is said to have a thermodynamic limit (have a TDL) if there exists an infinite volume interaction* $\Psi \in \mathcal{B}^{\circ}_{I,a,\infty}$ *such that*

$$\forall\, n \in \mathbf{N},\, i \in \mathbb{N}_0,\, M \in \mathbb{N} : \lim_{k\to\infty} \sup_{t\in I} \left\| \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\Psi - \Phi^{\Lambda_k}\right)(t) \right\|_{a,n,\Lambda_M} = 0 \, ,$$

and we write $\Phi \xrightarrow{\text{td}} \Psi$ *in this case.*

An operator family is said to have a TDL if and only if the corresponding interaction does.

For more general (non-exponentially localized) SLT operators, the definition is completely analogous.

[9] Indeed, $\sup_{k\in\mathbf{N}} \|A - \beta^{\varepsilon,\eta,\Lambda_k}(t)[\![A]\!]\| \leq (\varepsilon + \eta)\, C \, \|A\| \, |X| \, |t|$ for $A \in \mathcal{A}_X$ and t in a bounded interval by [461, Theorem 3.4(i)].

(b) A Lipschitz potential $v \in \mathcal{V}_I$ is said to have a TDL if there exists an infinite volume Lipschitz potential $v_\infty \in \mathcal{V}_I^\circ$ such that

$$\forall M \in \mathbb{N} \quad \exists K \geq M \quad \forall k \geq K,\, t \in I : v^{\Lambda_k}(t,\cdot)|_{\Lambda_M} = v_\infty(t,\cdot)|_{\Lambda_M}\,.$$

Again, we write $v \xrightarrow{\mathrm{td}} v_\infty$ in this case.

Note that, whenever $\Phi = \Psi$ for some infinite-volume interaction Ψ, or $v = v_\infty$ for some infinite volume Lipschitz potential v_∞, both Φ and v trivially have a TDL.

The following proposition is a standard consequence of Lieb-Robinson bounds and shows that the property of having a TDL for interactions and Lipschitz potentials guarantees the existence of the thermodynamic limit for the corresponding evolution operators [461, 124]. We remark that it remains true under less restrictive assumptions on the localization quality of the interaction (see, e.g., Proposition 2.2 in [342]).

Proposition 16.3.4 (Thermodynamic limit of evolution operators)**.** *Let $K_0 \in \mathcal{L}_{I,a,\infty}$ and $w \in \mathcal{V}_I$ both have a thermodynamic limit, i.e. $\Phi_{K_0} \xrightarrow{\mathrm{td}} \Psi_{K_0}$ and $w \xrightarrow{\mathrm{td}} w_\infty$ for some $\Psi_{K_0} \in \mathcal{B}^\circ_{I,a,\infty}$ and $w_\infty \in \mathcal{V}_I^\circ$. Set $K = K_0 + V_w$ and let $U^{\eta,\Lambda_k}(t,t_0)$ denote the evolution family generated by $K^{\Lambda_k}(t)$ in scaled time with $\eta > 0$, i.e. the solution to the Schrödinger equation*

$$\mathrm{i}\,\eta\,\frac{\mathrm{d}}{\mathrm{d}t}\,U^{\eta,\Lambda_k}(t,t_0) = K^{\Lambda_k}(t)\,U^{\eta,\Lambda_k}(t,t_0)$$

with $U^{\eta,\Lambda_k}(t_0,t_0) = \mathrm{id}$. Then, there exists a co-cycle of automorphisms $\mathfrak{U}^\eta_{t,t_0} : \mathcal{A}^d_{\mathbf{Z}} \to \mathcal{A}_{\mathbf{Z}^d}$ such that for all $A \in \mathcal{A}_{\mathrm{loc}}$,

$$\mathfrak{U}^\eta_{t,t_0}[\![A]\!] = \lim_{k\to\infty} \mathfrak{U}^{\eta,\Lambda_k}_{t,t_0}[\![A]\!] := \lim_{k\to\infty} U^{\eta,\Lambda_k}(t,t_0)^*\,A\,U^{\eta,\Lambda_k}(t,t_0)\,.$$

The co-cycle $\mathfrak{U}^\eta_{t,t_0}$ only depends on Ψ_{K_0} and w_∞ and is generated by the time-dependent (closed) derivation $(\mathcal{L}_{K(t)}, D(\mathcal{L}_{K(t)}))$ associated with $K(t)$.

As mentioned above, since the following Theorem 16.3.5 is deduced from Theorem 16.3.2 by taking $\Lambda_k \nearrow \mathbf{Z}^d$, we will need to assume the existence of a thermodynamic limit for the building blocks of the Hamiltonian (16.1.1).

($\mathbf{INT_2}$) Assumptions on the interactions.

For $\Psi_{H_0}, \Psi_{H_1} \in \mathcal{B}^\circ_{I,a,\infty}$ for some $a > 0$ and $v_\infty \in \mathcal{V}_I^\circ$ there exist $\Phi_{H_0}, \Phi_{H_1} \in \mathcal{B}_{I,a,\infty}$ and $v \in \mathcal{V}_I$ with appropriate boundary conditions (encoded in the definition of the norms defining the spaces $\mathcal{L}$ and the Lipschitz condition) all having a TDL with the respective object as the limit, i.e. $\Phi_{H_0} \xrightarrow{\mathrm{td}} \Psi_{H_0}$, $\Phi_{H_1} \xrightarrow{\mathrm{td}} \Psi_{H_1}$ and $v \xrightarrow{\mathrm{td}} v_\infty$.

We also assume the convergence of ground states, by means of the Banach-Alaoglu Theorem (the unit sphere in $\mathcal{A}^*_{\mathbf{Z}^d}$ is weak*-compact), essentially only in order to avoid the extraction of a subsequence.

($\mathbf{S_{unif}}$) Assumptions on the convergence of states.

Assume that for every $t \in I$ the sequence $\big(\rho_0^{\Lambda_k}(t)\big)_{k\in\mathbb{N}}$ of ground states (naturally extended to the whole of $\mathcal{A}_{\mathbf{Z}^d}$) converges in the weak*-topology to a state $\rho_0(t)$ on $\mathcal{A}_{\mathbf{Z}^d}$, which we call the *gapped limit ground state at $t \in I$*.

We can now formulate the second generalized super-adiabatic theorem concerning infinite systems with a uniform gap [342].

Theorem 16.3.5 (Adiabatic theorem for infinite systems with a uniform gap, see Theorems 3.2 and 3.5 in [342]). *Under the Assumptions* ($\mathsf{GAP}_{\mathsf{unif}}$), ($\mathsf{INT}_2$) *and* ($\mathsf{S}_{\mathsf{unif}}$), *there exists a near-identity automorphism* $\beta^{\varepsilon,\eta}(t) = \mathrm{e}^{\mathrm{i}\varepsilon\mathcal{L}_{S^{\varepsilon,\eta}(t)}}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon, \eta \in (0,1]$ *and* $t \in I$, *such that the state*

$$\Pi^{\varepsilon,\eta}(t) := \rho_0(t) \circ \beta^{\varepsilon,\eta}(t)$$

is a super-adiabatic NEASS for the Heisenberg time-evolution on $\mathcal{A}_{\mathbf{Z}^d}$ *generated by* $\frac{1}{\eta}\Psi_{H^\varepsilon(\cdot)}$ *with*

$$\Psi_{H^\varepsilon} := \Psi_{H_0} + \varepsilon\left(V_{v_\infty} + \Psi_{H_1}\right).$$

The crucial point in the proof of Theorem 16.3.5 in [342] is to show that the property of having a TDL is designed in such a way that it is preserved under all necessary operations for the construction of the NEASS (see Section 16.4). Therefore, also the near-identity automorphism from (16.3.2) converges as $\Lambda_k \nearrow \mathbf{Z}^d$ by means of Proposition 16.3.4.

16.3.2 Systems with a gap in the bulk

In this section, we drop Assumption ($\mathsf{GAP}_{\mathsf{unif}}$) of a uniform gap for finite systems, but merely work under the condition of a *gap in the bulk*, which is formulated via the Gelfand-Naimark-Segal (GNS) construction in Assumption ($\mathsf{GAP}_{\mathsf{bulk}}$) below: Let $\Psi_{H_0} \in \mathcal{B}^\circ_{a,0}$ be an infinite volume interaction and $\mathcal{L}_{H_0}$ denote the induced derivation on (a dense subset of) $\mathcal{A}_{\mathbf{Z}^d}$. A state ω on $\mathcal{A}_{\mathbf{Z}^d}$ is called an $\mathcal{L}_{H_0}$-ground state, if and only if $\omega(A^*\mathcal{L}_{H_0}(A)) \geq 0$ for all $A \in D(\mathcal{L}_{H_0})$. Let ω be an $\mathcal{L}_{H_0}$-ground state and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the corresponding GNS triple ($\mathcal{H}_\omega$ a Hilbert space, $\pi_\omega\colon \mathcal{A} \to \mathcal{L}(\mathcal{H}_\omega)$ a representation and $\Omega_\omega \in \mathcal{H}_\omega$ a cyclic vector). Then, there exists a unique densely defined, self-adjoint positive operator $H_{0,\omega} \geq 0$ on $\mathcal{H}_\omega$ satisfying

$$\pi_\omega(\mathrm{e}^{\mathrm{i}t\mathcal{L}_{H_0}}[\![A]\!]) = \mathrm{e}^{\mathrm{i}tH_{0,\omega}}\,\pi_\omega(A)\,\mathrm{e}^{-\mathrm{i}tH_{0,\omega}} \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}tH_{0,\omega}}\,\Omega_\omega = \Omega_\omega \tag{16.3.3}$$

for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{0,\omega}$ the *bulk Hamiltonian* (or *GNS Hamiltonian*) associated with Ψ_{H_0} and ω. See [118] for the general theory.

We assume that Ψ_{H_0} has a unique gapped ground state in the following sense (cf. [454, 341]):

($\mathsf{GAP}_{\mathsf{bulk}}$) Assumptions on the ground state of Ψ_{H_0}.

(i) **Uniqueness.** For each $t \in I$, there exists a unique $\mathcal{L}_{H_0(t)}$-ground state $\rho_0(t)$.

(ii) **Gap.** There exists $g > 0$ such that $\sigma\big(H_{0,\rho_0(t)}(t)\big) \setminus \{0\} \subset [g, \infty)$ for all $t \in I$.

(iii) **Regularity.** For any strictly positive $f \in \mathcal{S}(\mathbf{R})$ (Schwarz functions), define $\mathcal{D}_f$ as the set of observables $A \in \mathcal{A}_{\mathbf{Z}^d}$ for which $\|A\|_f := \|A\| + \sup_{k\in\mathbb{N}}\big(\|(1-\mathbb{E}_{\Lambda_k})[\![A]\!]\|/f(k)\big) < \infty$, where $\mathbb{E}_{\Lambda_k}[\![\cdot]\!]$ denotes the conditional expectation (see [342, Appendix C]). Then, for any $A \in \mathcal{D}_f$, $t \mapsto \rho_0(t)(A)$ is differentiable and there exists a constant C_f such that

$$\sup_{t\in I} |\dot\rho_0(t)(A)| \leq C_f\,\|A\|_f\,.$$

The smoothness of expectation values of (almost) exponentially localized observables as under item (iii) is a rather technical condition and a consequence of a uniform gap as in Assumption ($\mathsf{GAP}_{\mathsf{unif}}$) (see Remark 4.15 in [454] and Lemma 6.0.1 in [453]). Although uniqueness of the ground state in item (i), which we required throughout the paper, is expected to hold for the physically relevant type of Hamiltonian (16.1.4), it has been shown, to our present knowledge, only in very specific quantum spin systems. These include (a) weak perturbations of non-interacting gapped frustration-free systems [591] (see also Chapter 15), and (b) short-range interacting frustration-free models fulfilling *local topological quantum order* (LTQO) [463, 35].

Remark 16.3.6. *As mentioned in the beginning of Section 16.3, item (ii) holds, in particular, if one has a uniform gap for finite systems as spelled out in Assumption* ($\mathsf{GAP_{unif}}$)*, since it cannot close abruptly in the thermodynamic limit for the GNS Hamiltonian (see Proposition 5.4 in [38]). However, we observe that a considerably* weaker *sufficient condition for having a gap for the GNS Hamiltonian as in Assumption* ($\mathsf{GAP_{bulk}}$) *(ii) is to have a* gap in the bulk for the finite systems Λ_k *in the following sense: There exists $g > 0$ such that for all $k \in \mathbf{N}$ there exists some $l = l(k) \in \mathbf{N}$ with $l(k) \to \infty$ as $k \to \infty$, and we have*

$$\rho_0^{\Lambda_k}(t)\big(A^* \mathcal{L}_{H_0(t)}^{\Lambda_k}[\![A]\!]\big) \geq g\left(\rho_0^{\Lambda_k}(t)(A^*A) - \left|\rho_0^{\Lambda_k}(t)(A)\right|^2\right) \tag{16.3.4}$$

for all $A \in \mathcal{A}_{\Lambda_l}$ and all $t \in I$, where $\rho_0^{\Lambda_k}(t)$ denotes a suitable ground state of $H_0^{\Lambda_k}(t)$. Indeed, assuming that $\rho_0^{\Lambda_k}(t) \rightharpoonup \rho_0(t)$ for every $t \in I$,[10] *this simply follows after taking the limit $k \to \infty$ on both sides of* (16.3.4) *and realizing that, as $k \to \infty$, the set of admissible observables $A \in \mathcal{A}_{\Lambda_{l(k)}}$ exhausts $\mathcal{A}_{\mathrm{loc}}$, which is dense in $\mathcal{A}_{\mathbf{Z}^d}$ by definition. The resulting inequality immediately yields the desired spectral gap for the GNS Hamiltonian (cf. [118, Proposition 5.3.19] and [462, Section 7]).*

In the following, we shall present adiabatic theorems for infinite systems (Theorem 16.3.7) as well as for extended but finite systems (Theorem 16.3.9) under Assumption ($\mathsf{GAP_{bulk}}$).

16.3.2.1 Infinite systems

Analogously to Section 16.3.1, the basic assumptions on the Hamiltonian say that it is composed of exponentially localized interactions and/or a Lipschitz potential. In addition, the Hamiltonian H_0 satisfies a technical regularity assumption in t, for which we recall that $I \subset \mathbb{R}$ denotes an open time interval.

($\mathsf{INT_3}$) Assumptions on the interactions.

(i) Let $\Psi_{H_0}, \Psi_{H_1} \in \mathcal{B}^\circ_{I,a,\infty}$ be time-dependent infinite volume interactions and $v_\infty \in \mathcal{V}_I^\circ$ a time-dependent infinite volume Lipschitz potential.

(ii) Assume that the map $I \to \mathcal{B}^\circ_{a,\infty}$, $t \mapsto \Psi_{H_0(t)}$ is continuously differentiable.[11]

We can now formulate the third generalized super-adiabatic theorem concerning infinite systems with a gap in the bulk [341].

Theorem 16.3.7 (Adiabatic theorem for infinite systems with a gap in the bulk, see Theorem 3.4 in [341])**.** *Under Assumptions* ($\mathsf{GAP_{bulk}}$) *and* ($\mathsf{INT_3}$)*, there exists a near-identity automorphism $\beta^{\varepsilon,\eta}(t) = \mathrm{e}^{\mathrm{i}\varepsilon\mathcal{L}_{S^{\varepsilon,\eta}(t)}}$ on $\mathcal{A}_{\mathbf{Z}^d}$ with SLT generators $S^{\varepsilon,\eta}$ for any $\varepsilon, \eta \in (0,1]$ and $t \in I$ such that the state*

$$\Pi^{\varepsilon,\eta}(t) := \rho_0(t) \circ \beta^{\varepsilon,\eta}(t)$$

is a super-adiabatic NEASS for $\rho_0(t)$ and the Heisenberg time-evolution on $\mathcal{A}_{\mathbf{Z}^d}$ generated by $\frac{1}{\eta}\Psi_{H^\varepsilon(\cdot)}$ with

$$\Psi_{H^\varepsilon} := \Psi_{H_0} + \varepsilon\left(V_{v_\infty} + \Psi_{H_1}\right).$$

[10] Note that the sequence $(\rho_0^{\Lambda_k}(t))_{k\in\mathbf{N}}$ is compact for every fixed $t \in I$ (Banach-Alaoglu Theorem). Moreover, it is shown in Proposition 5.3.25 in [118] that every limit point of a sequence of ground states associated to a converging sequence of derivations $\mathcal{L}_{H_0(t)}^{\Lambda_k} \to \mathcal{L}_{H_0(t)}$ is a ground state of the limiting derivation.

[11] Note that this technical assumption does not follow from $\Psi_{H_0} \in \mathcal{B}^\circ_{I,a,\infty}$, as the spaces of smooth and bounded interactions are defined via term-wise and point-wise time derivatives (cf. Section 16.2.2).

The key role of the spectral gap condition is that it allows to construct an inverse of the Liouvillian $\mathcal{L}_{H_0(t)}$, appearing in the construction of the NEASS, which maps SLT operators to SLT operators with slightly deteriorated locality properties. Hence, the inverse of $\mathcal{L}_{H_0(t)}$ is called the *quasi-local inverse of the Liouvillian*.[12] Assuming a gap only in the bulk, as done in ($\mathsf{GAP}_{\mathsf{bulk}}$), means that the action of the Liouvillian can only be *inverted in the bulk* (see Section 16.4).

16.3.2.2 Extended but finite systems

Contrary to the results in Section 16.3.1, the adiabatic theorem describing an infinite system with a gap in the bulk did *not* require any notion of having a TDL in its formulation. Instead, in order to derive a finite-volume analogue from Theorem 16.3.7 (with qualitative additional error terms, see Theorem 16.3.9 below), we need to introduce the stronger notion of *having a rapid thermodynamic limit* for the exponentially localized interactions and the Lipschitz potential. We refer to [341] for a detailed discussion of this property.

Definition 16.3.8 (Rapid thermodynamic limit of interactions and potentials)**.** *(a) An exponentially localized time-dependent interaction $\Phi \in \mathcal{B}_{I,a,\infty}$ is said to* have a rapid thermodynamic limit with exponent $\gamma \in (0,1)$ *(have a $RTDL_\gamma$) if there exists an infinite volume interaction $\Psi \in \mathcal{B}^\circ_{I,a,\infty}$ such that*

$$\forall\, n \in \mathbf{N},\, i \in \mathbb{N}_0 \quad \exists \lambda, C > 0 \quad \forall M \in \mathbf{N} \quad \forall\, k \geq M + \lambda M^\gamma : \\ \sup_{t\in I} \left\| \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\Psi - \Phi^{\Lambda_k} \right)(t) \right\|_{a,n,\Lambda_M} \leq C\, \mathrm{e}^{-aM^\gamma}\,, \tag{16.3.5}$$

and we write $\Phi \xrightarrow{\mathrm{rtd}} \Psi$ in this case.

A family of operators is said to have a RTDL *if and only if the corresponding interaction does.*

For more general (non-exponentially localized) SLT operators, the definition is completely analogous.

(b) A Lipschitz potential $v \in \mathcal{V}_I$ is said to have a RTDL_γ *if it is eventually independent of k, i.e. if there exists an infinite volume Lipschitz potential $v_\infty \in \mathcal{V}_I^\circ$ such that*

$$\exists \lambda > 0 \quad \forall M \in \mathbf{N} \quad \forall\, k \geq M + \lambda M^\gamma, t \in I : \quad v_\infty(t,\cdot)|_{\Lambda_M} = v^{\Lambda_k}(t,\cdot)|_{\Lambda_M}\,.$$

Again, we write $v \xrightarrow{\mathrm{rtd}} v_\infty$ in this case.

In a nutshell, having a RTDL_γ means that the interaction (or the Lipschitz potential) essentially agrees with a corresponding infinite volume object, up to terms located on a thin shell with relative size of order $k^{\gamma-1}$ right at the boundary of Λ_k. Note that, whenever $\Phi = \Psi$ for some infinite-volume interaction Ψ, or $v = v_\infty$ for some infinite volume Lipschitz potential v_∞, both Φ and v trivially have a RTDL_γ (with any exponent $\gamma \in (0,1)$).

The following Theorem 16.3.9 is deduced from Theorem 16.3.7 by comparing the time evolution $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$ and the near identity automorphism $\beta^{\varepsilon,\eta}$ in the definition of the NEASS on the infinite system $\mathbf{Z}^d$ with the same objects for large (but finite) systems Λ_k. Therefore, we will need to assume the existence of a rapid thermodynamic limit for the building blocks of the Hamiltonian (16.1.1).

[12] This particular phrase was used in [556]. Others call it *local inverse* [36, 452] or just *inverse* [340, 342, 461]. In [341], it was called *SLT inverse*. To avoid confusion, we want to reserve the SLT prefix for operators, i.e. SLT operator or SLT generator, but not for maps between SLT operators.

(INT$_4$) Assumptions on the interactions.

The interactions $\Phi_{H_0}, \Phi_{H_1} \in \mathcal{B}_{I,a,\infty}$ and the Lipschitz potential $v \in \mathcal{V}_I$ all have a RTDL, i.e. $\Phi_{H_0} \xrightarrow{\mathrm{rtd}} \Psi_{H_0}$, $\Phi_{H_1} \xrightarrow{\mathrm{rtd}} \Psi_{H_1}$ and $v \xrightarrow{\mathrm{rtd}} v_\infty$. The limiting objects Ψ_{H_0}, Ψ_{H_1} and v_∞ satisfy Assumption (INT$_3$).

In Theorem 16.3.9 we shall consider finite volume states $\rho_0^{\Lambda_k}(t)$, which are close to the infinite volume ground state $\rho_0(t)$ away from the boundary in following sense.

(S$_{\mathrm{bulk}}$) Assumption on the convergence of states.

The sequence $\big(\rho_0^{\Lambda_k}(t)\big)_{k\in\mathbf{N}}$ of states on $\mathcal{A}_{\Lambda_k}$ converges rapidly to $\rho_0(t)$ in the bulk: there exist $C \in \mathbf{R}$, $m \in \mathbb{N}$ and $h \in \mathcal{S}$ such that for any finite $X \subset \mathbf{Z}^d$, $A \in \mathcal{A}_X$ and $\Lambda_k \supset X$

$$\sup_{t\in I} \Big|\rho_0(t)(A) - \rho_0^{\Lambda_k}(t)(A)\Big| \leq C\,\|A\|\,\mathrm{diam}(X)^m\, h\big(\,\mathrm{dist}(X, \mathbf{Z}^d \setminus \Lambda_k)\big)\,.$$

While the sequence $\rho_0^{\Lambda_k}(t) \equiv \rho_0(t)|_{\mathcal{A}_{\Lambda_k}}$ of simple restrictions satisfies Assumption (S$_{\mathrm{bulk}}$) trivially, the adiabatic theorem ensures the existence of a super-adiabatic NEASS constructed for *any* such sequence.[13] Most interesting for physical application would be a sequence of ground states $\rho_0^{\Lambda_k}(t)$ of the finite volume Hamiltonians $H_0^{\Lambda_k}(t)$. While the above assumption is expected to hold for any sequence of finite volume ground states for Hamiltonians modeling Chern or topological insulators like in (16.1.4), the only result we are aware of indeed *proving* such a statement is again (see the discussion below Assumption (GAP$_{\mathrm{bulk}}$)) for weakly interacting spin systems [591]. In spirit, assuming (S$_{\mathrm{bulk}}$) is very similar to supposing that the system satisfies *local topological quantum order* (LTQO) [448, 463] or a strong *local perturbations perturb locally* (LPPL) principle for perturbations acting at the boundary of the system [35] (see also Chapter 15).

We can now formulate the fourth and last generalized super-adiabatic theorem concerning finite systems with a gap in the bulk [341].

Theorem 16.3.9 (Adiabatic theorem for finite systems with gap in the bulk, see Theorem 4.1 in [341])**.** *Under the Assumptions* (GAP$_{\mathrm{bulk}}$), (INT$_4$) *and* (S$_{\mathrm{bulk}}$)*, there exists a sequence of near-identity automorphisms* $\beta^{\varepsilon,\eta,\Lambda_k}(t) = \mathrm{e}^{\mathrm{i}\varepsilon\mathcal{L}^{\Lambda_k}_{S^{\varepsilon,\eta}(t)}}$ *with SLT generators* $S^{\varepsilon,\eta}$ *for any* $\varepsilon, \eta \in (0,1]$ *and* $t \in I$*, such that the states*

$$\Pi^{\varepsilon,\eta,\Lambda_k}(t) := \rho_0^{\Lambda_k}(t) \circ \beta^{\varepsilon,\eta,\Lambda_k}(t)$$

are super-adiabatic NEASSs for the Heisenberg time-evolution $\mathfrak{U}^{\varepsilon,\eta,\Lambda_k}_{t,t_0}$ *on* $\mathcal{A}_{\Lambda_k}$ *generated by* $\frac{1}{\eta}\,H^{\varepsilon,\Lambda_k}(\cdot)$ *with*

$$\tfrac{1}{\eta}\,H^{\varepsilon,\Lambda_k}(t) := \tfrac{1}{\eta}\Big(H_0^{\Lambda_k}(t) + \varepsilon\,\big(V_v^{\Lambda_k}(t) + H_1^{\Lambda_k}(t)\big)\Big)\,,$$

up to an error vanishing faster than any inverse polynomial in the distance to the boundary. That is, for any $n \in \mathbb{N}$ *there exists a constant* C_n *and for any compact* $K \subset I$ *and* $m \in \mathbf{N}$ *there exists a constant* $\tilde{C}_{n,m,K}$ *such that for all* $k \in \mathbf{N}$*, all* $X \subset \Lambda_k$*, all* $A \in \mathcal{A}_X$ *and all* $t, t_0 \in K$

$$\begin{aligned}\Big|\Pi^{\varepsilon,\eta,\Lambda_k}(t_0)(\mathfrak{U}^{\varepsilon,\eta,\Lambda_k}_{t,t_0}[\![A]\!]) - \Pi^{\varepsilon,\eta,\Lambda_k}(t)(A)\Big| \\ \leq\; C_n\,\frac{\varepsilon^{n+1}+\eta^{n+1}}{\eta^{d+1}}\,\big(1+|t-t_0|^{d+1}\big)\;\|A\|\,|X|^2 \\ +\; \tilde{C}_{n,m,K}\,\big(1+\eta\,\mathrm{dist}(X,\mathbf{Z}^d\setminus\Lambda_k)\big)^{-m}\,\|A\|\,\mathrm{diam}(X)^{2d}\,.\end{aligned} \tag{16.3.6}$$

[13] This is why we wrote *'(close to) a ground state'* in Definition 16.3.1.

The above theorem asserts that by assuming ($\mathrm{GAP}_{\mathrm{bulk}}$), one obtains similar adiabatic bounds also for states of finite systems (without a spectral gap!) which are close to the infinite volume ground state in the bulk as formulated in Assumption ($\mathrm{S}_{\mathrm{bulk}}$). Since adiabaticity potentially breaks at the boundaries of the finite systems, non-adiabatic effects arising close to the boundary may propagate into the bulk. Therefore, an additional error term appears, but it decays faster than any polynomial in the size of the finite system for any fixed η. The actual form of the additional error term in the last line of (16.3.6) coming out of the proof in [341, Section 5] is slightly better but more complicated, which is why we refrain from stating it here.

The main points in the proof of Theorem 16.3.9, which we discuss in Section 16.4, are to show that (i) the property of having a RTDL_γ is preserved under all necessary operations for the construction of the NEASS (similarly as for Theorem 16.3.5) and (ii) having a RTDL_γ for an interaction provides an explicit rate of convergence for the associated evolution family as in Proposition 16.3.4.

16.4 Idea of the proofs

The goal of the present section is to convey the main ideas relevant for proving the individual theorems from Section 16.3, where we already glimpsed the key steps required in their proofs. For many technical details we refer the reader to the original works [556, 342, 341].

16.4.1 Systems with a uniform gap

The fundamental conceptual idea behind the proof for all four variants of the generalized super-adiabatic theorems is a perturbative scheme, which was called *space-time adiabatic perturbation theory* in [484, 483]. The basic structure of this computation is most easily presented for finite systems, where no further technical difficulties arise since all appearing operators are in fact matrices and thus bounded. However, it is still necessary to show that all estimates are uniform in the size of the system Λ_k.

16.4.1.1 Extended but finite systems: Proof of Theorem 16.3.2

The form in which we presented Theorem 16.3.2 differs slightly from the original result [556, Theorem 5.1]. The original statement concerns a sequence $\Pi_n^{\varepsilon,\eta,\Lambda_k}(t) := \rho_0^{\Lambda_k}(t) \circ \beta_n^{\varepsilon,\eta,\Lambda_k}(t)$ of states on Λ_k (indexed by $n \in \mathbf{N}$), where

$$\beta_n^{\varepsilon,\eta,\Lambda_k}(t)[\![A]\!] := \mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{L}^{\Lambda_k}_{S_n^{\varepsilon,\eta}(t)}}[\![A]\!] \quad \text{and} \quad \varepsilon\mathcal{L}^{\Lambda_k}_{S_n^{\varepsilon,\eta}(t)} := \sum_{j=1}^{n} \varepsilon^j \mathcal{L}^{\Lambda_k}_{A_j^{\varepsilon,\eta}(t)}\,.$$

From this, Theorem 16.3.2 (and similarly all other three theorems) follows by a simple *resummation* of the $\varepsilon^j \mathcal{L}^{\Lambda_k}_{A_j^{\varepsilon,\eta}(t)}$, which will be discussed in Section 16.4.3 below.

The main idea of the proof is to choose each operator $A_j^{\varepsilon,\eta,\Lambda_k}(t)$, $j = 1, \ldots, n$, in such a way that the j^{th}-order term in the perturbative scheme vanishes. For the n-dependent result (i.e. prior to resummation), we apply the fundamental theorem of calculus to get

$$\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A) = -\int_{t_0}^{t} \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s}\, \rho_0^{\Lambda_k}(s)\Big(\beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big) \quad (16.4.1)$$

and then aim to bound the integrand. Calculating the derivative by using the chain rule and Duhamel's formula leaves us with

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \rho_0^{\Lambda_k}(s)\Big(\beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big) = -\frac{\mathrm{i}}{\eta}\, \rho_0^{\Lambda_k}(s)\Big(\Big[Q_n^{\varepsilon,\eta,\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\Big), \quad (16.4.2)$$

where $Q_n^{\varepsilon,\eta,\Lambda_k}(s)$ is a shorthand notation for

$$\begin{aligned}\eta\,\mathcal{I}_s^{\Lambda_k}\big(\dot H_0^{\Lambda_k}(s)\big) + \eta\int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)}\,\varepsilon\dot S_n^{\varepsilon,\eta,\Lambda_k}(s)\,\mathrm{e}^{\mathrm{i}\lambda\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)} \\ + \mathrm{e}^{-\mathrm{i}\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)}\big(H_0^{\Lambda_k}(s)+\varepsilon V^{\Lambda_k}(s)\big)\mathrm{e}^{\mathrm{i}\varepsilon S_n^{\varepsilon,\eta,\Lambda_k}(s)} \\ =: H_0^{\Lambda_k}(s) + \sum_{j=1}^n \varepsilon^j R_j^{\varepsilon,\eta,\Lambda_k}(s) + \varepsilon^{n+1} R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s)\,, \end{aligned} \tag{16.4.3}$$

and $V^{\Lambda_k} = V_v^{\Lambda_k} + H_1^{\Lambda_k}$. Here, $\mathcal{I}_s^{\Lambda_k}\big(\dot H_0^{\Lambda_k}(s)\big)$ is the SLT generator of the parallel transport within the vector-bundle $\Xi_{0,I}^{\Lambda_k}$ over I defined by $t \mapsto \rho_0^{\Lambda_k}(t)$.[14] This parallel transport is also known as the *spectral flow*, which plays a fundamental role in proving automorphic equivalence of gapped ground state phases (see e.g. [39, 36]). Moreover, the operator $\mathcal{I}_s^{\Lambda_k}\colon \mathcal{A}_{\Lambda_k} \to \mathcal{A}_{\Lambda_k}$ is called the *quasi-local inverse of the Liouvillian*[12] $\mathcal{L}_{H_0(s)}^{\Lambda_k}$, since it satisfies [36, 556]

$$\rho_0^{\Lambda_k}(s)\Big(\big[\mathcal{L}_{H_0(s)}^{\Lambda_k}\circ\mathcal{I}_s^{\Lambda_k}[\![B_1]\!] - \mathrm{i}\,B_1, B_2\big]\Big) = 0 \quad \text{for all} \quad B_1, B_2 \in \mathcal{A}_{\Lambda_k}\,, \quad s\in I\,, \tag{16.4.4}$$

and also preserves good localization of its argument (in particular, it maps SLT operators to SLT operators). This combined property of $\mathcal{I}_s^{\Lambda_k}$ heavily relies on the ground state $\rho_0^{\Lambda_k}(s)$ being gapped [36, 556, 461] and will be of fundamental importance in the following.

In the last line of (16.4.3), we expanded in powers of ε and η in the sense that $R_j^{\varepsilon,\eta,\Lambda_k}(s)$, for $j \le n$, are polynomials in η/ε of order (at most) j with ε- and η-independent SLT operators as coefficients. A more detailed step-by-step calculation can be found in the proof of Proposition 5.1 in [556]. Let us here only report the general structure

$$R_j^{\varepsilon,\eta,\Lambda_k}(s) = -\mathrm{i}\,\mathcal{L}_{H_0(s)}^{\Lambda_k}\big(A_j^{\varepsilon,\eta,\Lambda_k}(s)\big) + \tilde R_j^{\varepsilon,\eta,\Lambda_k}(s), \tag{16.4.5}$$

where the first remainder term is given by

$$\tilde R_1^{\varepsilon,\eta,\Lambda_k}(s) = \tfrac{\eta}{\varepsilon}\,\mathcal{I}_s^{\Lambda_k}\big(\dot H_0^{\Lambda_k}(s)\big) - V^{\Lambda_k}(s)$$

and all other $\tilde R_j^{\varepsilon,\eta,\Lambda_k}(s)$ are composed of iterated commutators of the operators $A_i^{\varepsilon,\eta,\Lambda_k}(s)$ and $\dot A_i^{\varepsilon,\eta,\Lambda_k}(s)$, for $i < j \le n$, with $H_0^{\Lambda_k}(s)$ and $V^{\Lambda_k}(s)$. In contrast to general onsite potentials, the commutator of a Lipschitz potential with an SLT operator is an SLT operator itself [556, Lemma 2.1]. For the commutator of SLT operators, this is easy to see.

We now consider individual terms from (16.4.3) when plugged into (16.4.2). The zero-order term vanishes, because $\rho_0^{\Lambda_k}(s)$ is the ground state of $H_0^{\Lambda_k}(s)$. By application of (16.4.4) we can iteratively choose

$$A_j^{\varepsilon,\eta,\Lambda_k}(s) = -\mathcal{I}_s^{\Lambda_k}\big(\tilde R_j^{\varepsilon,\eta,\Lambda_k}(s)\big) \tag{16.4.6}$$

such that (16.4.2) vanishes up to

$$-\mathrm{i}\,\frac{\varepsilon^{n+1}}{\eta}\,\rho_0^{\Lambda_k}(s)\Big(\Big[R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k}(s)\circ\mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\Big). \tag{16.4.7}$$

Moreover, all the operations involved in calculating the $A_j^{\varepsilon,\eta,\Lambda_k}(s)$, i.e. taking commutators and applying the quasi-local inverse of the Liouvillian preserve the locality properties of the operators as shown in the appendices of [556, 452], which are heavily based on [36]. Hence, also all $A_j^{\varepsilon,\eta,\Lambda_k}$ are SLT operators.

[14] Since we assumed uniqueness of the ground state $\rho_0^{\Lambda_k}(t)$, the vector-bundle $\Xi_{0,I}^{\Lambda_k}$ is one-dimensional. If this were not the case, one had to include further terms generating the internal dynamic in $\Xi_{0,I}^{\Lambda_k}$ (see [556, 342]).

It turns out that also $R_{n+1}^{\varepsilon,\eta,\Lambda_k}$ is a polynomial in η/ε of order at most $n+1$ and its coefficients, as we just explained, are SLT operators [556, Proof of Proposition 6.1]. Thus, the absolute value of (16.4.1) is bounded by

$$\begin{aligned}
&\frac{\varepsilon^{n+1}}{\eta}\left|\int_{t_0}^{t}\mathrm{d}s\,\rho_0^{\Lambda_k}(s)\Big(\Big[R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s),\beta_n^{\varepsilon,\eta,\Lambda_k}(s)\circ\mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\Big)\right|\\
&\leq\frac{\varepsilon^{n+1}}{\eta}\,|t-t_0|\sup_{s\in[t_0,t]}\left\|\Big[\big(\beta_n^{\varepsilon,\eta,\Lambda_k}\big)^{-1}(s)\big(R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s)\big),\mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\Big]\right\|\\
&\leq C_n\,\frac{\varepsilon^{n+1}}{\eta}\,|t-t_0|\left(1+\Big(\frac{\eta}{\varepsilon}\Big)^{n+1}\right)\left(1+\Big(\frac{|t-t_0|}{\eta}\Big)^{d}\right)\|A\|\,|X|^2 \qquad (16.4.8)\\
&\leq C_n\,\frac{\varepsilon^{n+1}+\eta^{n+1}}{\eta^{d+1}}\,|t-t_0|\left(1+|t-t_0|^d\right)\|A\|\,|X|^2\,,
\end{aligned}$$

where we essentially used a generalized Lieb-Robinson bound [556, Lemma B.5] to estimate the commutator. Note that the $(1+\big(|t-t_0|/\eta\big)^d)$-factor comes from the Lieb-Robinson bound and the adiabatic $1/\eta$-scaling of the time evolution $\mathfrak{U}_{t,s}$. The $(1+(\eta/\varepsilon)^{n+1})$-factor comes from bounding the interaction norm of $R_{n+1}^{\varepsilon,\eta,\Lambda_k}(s)$ by separating the polynomial dependence on η/ε such that C_n is independent of Λ_k, ε and η. We have thus shown that the NEASS almost intertwines the time evolution, i.e. item 1 of Definition 16.3.1.

We are left with discussing the remaining three characterizing properties of the NEASS given in Definition 16.3.1: By construction, all $A_j^{\varepsilon,\eta,\Lambda_k}(t)$ depend only on $H_0^{\Lambda_k}(t)$ and $V^{\Lambda_k}(t)$ and their j^{th} derivatives at time t. This shows that the NEASS is local in time, i.e. item 2. Moreover, if all time derivatives of H_0 and V vanish for some $t\in I$, then all non-constant (i.e. in front of some positive power of η/ε) coefficients in $R_j^{\varepsilon,\eta,\Lambda_k}$ vanish and $\Pi_n^{\varepsilon,\eta,\Lambda_k}(t)=\Pi_n^{\varepsilon,0,\Lambda_k}(t)$. This shows that the NEASS is stationary whenever the Hamiltonian is stationary, i.e. item 3. If, for some $t\in I$, $\dot{H}_0^{\Lambda_k}(t)$ and $V^{\Lambda_k}(t)$ vanish, then $\tilde{R}_1^{\varepsilon,\eta,\Lambda_k}$ and thus $A_1^{\varepsilon,\eta,\Lambda_k}$ vanish. If additionally all derivatives of $H_0^{\Lambda_k}$ and V^{Λ_k} at t vanish, also $\tilde{R}_j^{\Lambda_k}(t)$ and thus $A_j^{\varepsilon,\eta,\Lambda_k}(t)$ vanish. Hence, $\beta_n^{\varepsilon,\eta,\Lambda_k}(t)=\mathbf{1}_{\Lambda_k}$ and the NEASS equals the ground state, i.e. item 4 holds.

The above listed general arguments immediately translate to the other three theorems.

16.4.1.2 Infinite systems: Proof of Theorem 16.3.5

Without any further assumptions, the sequence Hamiltonian $H^{\varepsilon,\Lambda_k}$ and its constituents $H_0^{\Lambda_k}$ and V^{Λ_k} could have nothing in common for different lattice sizes k (they might even describe different physical systems), so taking the limit $\Lambda_k\nearrow\mathbf{Z}^d$ might not be well-defined. In order to avoid this somewhat meaningless situation, we assumed that the building blocks of the Hamiltonian *have a TDL* (see Definition 16.3.3 and Assumption (INT_2)) and also the sequence of ground states $\big(\rho_0^{\Lambda_k}(t)\big)_{k\in\mathbf{N}}$ converges (Assumption ($\mathsf{S}_{\mathsf{unif}}$)). Since the property of having a TDL guarantees the existence of the thermodynamic limit for the corresponding evolution operators (see Proposition 16.3.4 and [461]), it remains to show that the operator sequences $\big(A_j^{\varepsilon,\eta,\Lambda_k}(t)\big)_{k\in\mathbf{N}}$, $j=1,\dots,n$, constructed in Section 16.4.1.1 have a TDL. More precisely, one needs to show that taking time-derivatives, sums of commutators with the building blocks of H^ε (and $\dot{H}_0$), and the inverse of the Liouvillian (see (16.4.5)) leaves the property of having a TDL for SLT operators invariant, which is in fact the main point of the proof in [342]. It is then straightforward to show that compositions of states and automorphisms, all having a thermodynamic limit, converge as $\Lambda_k\nearrow\mathbf{Z}^d$. Since the constant C_n from (16.4.8) is uniformly bounded in k, the (sketch of a) proof of Theorem 16.3.5 is complete.

16.4.2 Systems with a gap in the bulk

For systems having a spectral gap only in the bulk (i.e. for the GNS Hamiltonian), the characteristic (16.4.4) of $\mathcal{I}_s^{\Lambda_k}$, that it essentially inverts the Liouvillian $\mathcal{L}_{H_0(s)}^{\Lambda_k}$ (and still maps SLT operators to SLT operators), is now only fulfilled for certain B_1 and B_2 in a dense domain $\mathcal{D} \subset \mathcal{A}_{\mathbf{Z}^d}$ after taking the limit $\Lambda_k \nearrow \mathbf{Z}^d$ (see [341, Proposition 3.3]). Presuming that the limit actually exists, this point is the main challenge in proving an adiabatic theorem under the less restrictive gap Assumption ($\mathsf{GAP}_{\mathsf{bulk}}$).

16.4.2.1 Infinite systems: Proof of Theorem 16.3.7

As just explained, the main difficulty in proving Theorem 16.3.7 is that (16.4.4) only holds if $H_0^{\Lambda_k}$ is gapped. On top of that, we cannot handle the limit $\Lambda_k \nearrow \mathbf{Z}^d$ of the $\tilde{R}_j^{\varepsilon,\eta,\Lambda_k}$ directly nor could they be used in the infinite volume version of (16.4.4) because it only holds for $B_1, B_2 \in \mathcal{D} \subset \mathcal{A}_{\mathbf{Z}^d}$. However, the rest of the construction from Section 16.4.1.1 is still valid, but the lower order terms in (16.4.1) have a non-vanishing contribution in finite domains. We thus repeat this construction but take coefficients $A_j^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)$, which are built up from $H_0^{\Lambda_k}(t)$ but restricting the perturbations $\dot{H}_0(t)$ and $V(t)$ to Λ_l with $l < k$. In this way, one can take the limit $\Lambda_k \nearrow \mathbf{Z}^d$ in (16.4.4) with $B_1 = \lim_{k\to\infty} \tilde{R}_j^{\varepsilon,\eta,\Lambda_k,\Lambda_l} \in \mathcal{A}_{\mathbf{Z}}^d$ (see (16.4.10) and (16.4.11) and the comment thereafter for technical obstructions in taking the limit). Using this notational convention, we introduce the states

$$\Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t) = \rho_0(t) \circ \beta^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)\,,$$

where $\rho_0(t)$ is the infinite volume ground state, and compare them to the actual objects in infinite volume while estimating

$$\begin{aligned}\Big|\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta}(t)(A)\Big| \leq{}& \Big|\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big)\Big| \\ &+ \Big|\Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big) - \Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)(A)\Big| \\ &+ \Big|\Pi_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(t)(A) - \Pi_n(t)(A)\Big|\end{aligned} \tag{16.4.9}$$

by means of the triangle inequality. Since all the interactions (and the Lipschitz potential) have a TDL, one can prove [341, Section 5.1(b)] that the first and last summand in (16.4.9) can be made arbitrarily small for $k, l \in \mathbf{N}$ large enough, and we can thus focus on the second summand. However, since (16.4.4) only holds in the limit $\Lambda_k \nearrow \mathbf{Z}^d$ and also $\rho_0(t)$ is not necessarily a ground state of $H_0^{\Lambda_k}(t)$, the lower order terms in the analogues of (16.4.2) and (16.4.3) do not vanish for finite k and l. Instead, only

$$\lim_{k\to\infty} \rho_0(s)\Big(\big[H_0^{\Lambda_k}(s), \beta_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big]\Big) = 0 \tag{16.4.10}$$

and

$$\lim_{k\to\infty} \rho_0(s)\Big(\big[R_j^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(s), \beta_n^{\varepsilon,\eta,\Lambda_k,\Lambda_l}(s) \circ \mathfrak{U}_{t,s}^{\varepsilon,\eta,\Lambda_k,\Lambda_l}[\![A]\!]\big]\Big) = 0 \tag{16.4.11}$$

for all $l \in \mathbf{N}$ and uniformly for s and t in compacts. These statements require a careful analysis of deteriorating localization properties along the expansion as well as convergence estimates in norms measuring the quality of localization (cf. the norm $\|\cdot\|_f$ introduced in Assumption ($\mathsf{GAP}_{\mathsf{bulk}}$) (iii)), such that the limits really converge to the infinite volume version of (16.4.4) with B_1 and B_2 in a dense domain $\mathcal{D} \subset \mathcal{A}_{\mathbf{Z}^d}$. For further details, we refer to Proposition 3.2 and the statements in Appendix B of [341], which are adaptions of technical estimates that were originally established for the proof of automorphic equivalence with a gap only in the bulk [454]. Now, combining (16.4.10) and (16.4.11) with the estimates on the first and third summand in (16.4.9), we conclude that all the lower order terms vanish in the limit $k \to \infty$ followed by $l \to \infty$, which finishes our sketch of the proof of Theorem 16.3.7.

16.4.2.2 Extended but finite systems: Proof of Theorem 16.3.9

Let us briefly explain the strategy to prove Theorem 16.3.9. In order to show (16.3.6), we first estimate

$$\begin{aligned}\left|\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big)-\Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A)\right| &\leq \left|\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big)-\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big)\right| \\ &\quad+\left|\Pi_n^{\varepsilon,\eta}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}[\![A]\!]\big)-\Pi_n^{\varepsilon,\eta}(t)(A)\right| \\ &\quad+\left|\Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A)-\Pi_n^{\varepsilon,\eta}(t)(A)\right|\end{aligned} \tag{16.4.12}$$

and treat the three summands separately. The second summand corresponds to the infinite system and can be estimated by means of Theorem 16.3.7, such that it accounts for the first contribution on the RHS of (16.3.6). We are left with bounding the remaining two summands in (16.4.12). These contribute the additional error term on the RHS of (16.3.6). To estimate them, we need explicit control on the speed of convergence (it must be faster than any inverse polynomial) for the states (see Assumption ($\mathsf{S}_{\mathsf{bulk}}$)) and automorphisms $\beta_n^{\varepsilon,\eta,\Lambda_k}$ and $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}$. For the time evolution $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}$, the rapid convergence to $\mathfrak{U}_{t,t_0}^{\varepsilon,\eta}$ is ensured by supposing that the building blocks of H^ε have a RTDL (see Definition 16.3.8 and Assumption (INT_4)). This was carried out in [341, Appendix B], building on estimates from [461, Section 3]. We remark that the adiabatic $1/\eta$-scaling of the time evolution is responsible for the factor η appearing in the additional error term in (16.3.6). In order to show that also $\beta_n^{\varepsilon,\eta,\Lambda_k}\to\beta_n^{\varepsilon,\eta}$ sufficiently fast, we need to show that all $A_j^{\varepsilon,\eta}$ have a RTDL, i.e. the operations involved in constructing the generator of $\beta_n^{\varepsilon,\eta}$ leave the property of having a RTDL (essentially) invariant (see [341, Appendix C]). This finishes the sketch of the proof of Theorem 16.3.9 and we refer to [341, Section 5.2] for further details.

16.4.3 Resummation of the NEASS

As mentioned in the beginning of Section 16.4.1.1, the statements formulated in Section 16.3 require a resummation, which we explain in the following. First, note that the generator $\varepsilon S_n^{\varepsilon,\eta}$ of $\beta_n^{\varepsilon,\eta}$ constructed above can be rewritten as $\varepsilon S_n^{\varepsilon,\eta}=\sum_{j=1}^n\sum_{i=0}^j\varepsilon^i\eta^{j-1}A_{j,i}$, where the coefficients $A_{j,i}$ are time-dependent SLT operators and independent of ε and η. Now, it is easy to show (see [342, Lemma E.1]) that there exists a sequence $\delta_j\to 0$ and constants C_n such that the resummed generator

$$\varepsilon S^{\varepsilon,\eta}=\sum_{j=1}^{\infty}\chi_{[0,1]}(\varepsilon/\delta_j)\,\chi_{[0,1]}(\eta/\delta_j)\sum_{i=0}^{j}\varepsilon^i\eta^{j-i}A_{j,i} \tag{16.4.13}$$

satisfies

$$\left\|\varepsilon S^{\varepsilon,\eta}-\varepsilon S_n^{\varepsilon,\eta}\right\|_{\mathrm{SLT}}\leq C_n\left(\varepsilon^n+\eta^n\right),$$

where $\|\cdot\|_{\mathrm{SLT}}$ denotes an interaction norm similar to (16.2.1). Resummations of this type are standard, e.g., in microlocal analysis [438] and the above estimate immediately leads to the bounds (cf. [342, Lemmata E.3, E.4])

$$\sup_{t\in I}\left|\Pi^{\varepsilon,\eta,\Lambda_k}(t)(A)-\Pi_n^{\varepsilon,\eta,\Lambda_k}(t)(A)\right|\leq C_n'\left(\varepsilon^n+\eta^n\right)\|A\|\,|X|^2\quad\text{and} \tag{16.4.14}$$

$$\left|\Pi^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big)-\Pi_n^{\varepsilon,\eta,\Lambda_k}(t_0)\big(\mathfrak{U}_{t,t_0}^{\varepsilon,\eta,\Lambda_k}[\![A]\!]\big)\right|\leq C_n''\,\frac{\varepsilon^n+\eta^n}{\eta^{d+1}}\left(1+|t-t_0|\right)^{d+1}\|A\|\,|X|^2, \tag{16.4.15}$$

uniformly in the size of the system Λ_k. In the context of Theorem 16.3.5 and Theorem 16.3.7, corresponding estimates hold in infinite volume, i.e. without the subscript Λ_k.

Next, since the sum in (16.4.13) is finite for every fixed $\varepsilon>0$, also the resummed generator $S^{\varepsilon,\eta,\Lambda_k}$ has a TDL as soon as $S_n^{\varepsilon,\eta,\Lambda_k}$ has a TDL. Therefore, the states $\Pi^{\varepsilon,\eta,\Lambda_k}$ constructed using the $S^{\varepsilon,\eta,\Lambda_k}$

instead of the $S_n^{\varepsilon,\eta,\Lambda_k}$ have a well-defined thermodynamic limit $\Pi^{\varepsilon,\eta}$ (see [342, Lemma E.2]) and since the bounds (16.4.14) and (16.4.15) are independent of Λ_k, they also hold for the respective objects in the thermodynamic limit. Hence, the results formulated in Section 16.3 can be concluded by combining the n-dependent statements discussed earlier in this section with the bounds (16.4.14) and (16.4.15) (or their infinite volume correspondents).

Chapter 17

Response theory for locally gapped systems

This chapter includes the paper [347]:

J. Henheik and T. Wessel. Response theory for locally gapped systems. *arXiv:2410.10809*, 2024

Abstract. We introduce a notion of a *local gap* for interacting many-body quantum lattice systems and prove the validity of response theory and Kubo's formula for localized perturbations in such settings. On a high level, our result shows that the usual spectral gap condition, concerning the system as a whole, is not a necessary condition for understanding local properties of the system.

More precisely, we say that an equilibrium state ρ_0 of a Hamiltonian H_0 is locally gapped in $\Lambda^{\mathrm{gap}} \subset \Lambda$, whenever the Liouvillian $-\mathrm{i}\,[H_0, \cdot\,]$ is almost invertible on local observables supported in Λ^{gap} when tested in ρ_0. To put this into context, we provide other alternative notions of a local gap and discuss their relations.

The validity of response theory is based on the construction of *non-equilibrium almost stationary states* (NEASSs). By controlling locality properties of the NEASS construction, we show that response theory holds to any order, whenever the perturbation ϵV acts in a region which is further than $|\log \epsilon|$ away from the non-gapped region $\Lambda \setminus \Lambda^{\mathrm{gap}}$.

Content

17.1 Introduction

Spectral gaps lie at the heart of many areas of physics and mathematics, and their existence has several profound consequences. Classical mathematical examples include (i) Cheeger's inequality [143, 451], relating the spectral gap of the (discrete) Laplacian on a Riemannian manifold (a graph) to geometric properties of the underlying space, (ii) exponential heat kernel estimates [65], (iii) classical Poincaré or Nash inequalities [418], or (iv) the logarithmic Sobolev inequality and hypercontractivity for Markov semigroups [302, 301, 42].

In physics, questions concerning the spectral gap lie at the center of several of the most challenging problems, e.g. the Berry-Tabor [67] and Bohigas-Giannoni-Schmit [89] conjectures in quantum chaos, Haldane's conjecture on integer valued antiferromagnetic Heisenberg chains [317, 318], or the Yang-Mills mass gap problem. The great interest in spectral gaps, far beyond these famous conjectures, is rooted in the fact that its existence has tremendous effects on fundamental properties of the system: It is textbook material in condensed matter theory, that the (non)-existence of a band gap determines the isolating (conducting) behavior of a material, and in superconductivity, the existence of a non-zero solution of the BCS gap equation [48] decides whether a system is superconducting.

Moreover, in quantum many-body physics, the existence of a spectral gap above the ground state eigenvalue has far-reaching consequences for entanglement properties [459, 323] and ground state correlations [325]. On the other hand, the *closing* of a spectral gap is related to the occurrence of a (topological) quantum phase transition [39, 461]. Finally, the assumption of a spectral gap played a crucial role in recent proofs of adiabatic theory and linear response for many-body systems (see Section 17.1.1 for a detailed discussion).

However, an overall drawback in all of the above examples is that the desired spectral property is of *global* nature, i.e. involving the studied system as a whole, and thus seldomly compatible with a notion of *locality* in an underlying physical space. Therefore, a natural question to ask is:

> How can one express that a system is *locally* gapped, and which consequences that one has for globally gapped systems persist?

In this paper, we study this question in the setting of locally interacting many-body quantum spin lattice systems; see Section 17.2 for precise definitions. More precisely, in the above spirit of our guiding question, this paper has two main goals:

(i) We propose a notion of a *local gap* via an (equivalent) dynamical characterization and exemplary prove that local perturbations of Hamiltonians with a frustration free product ground state satisfy this condition. Moreover, we study possible alternative notions of local gaps and their relations among each other (see Sections 17.1.2 and 17.4).

(ii) As an application to a physically relevant problem, we show that for Hamiltonians with a local gap, response theory approximately holds to any order and thus justify *Kubo's formula* (see Sections 17.1.3 and 17.3.1).

There are only few works in the literature studying many-body quantum systems under a non-standard gap condition, i.e. one differing from the clean separation of eigenvalues: In [204], the authors derive Kubo's formula for two-dimensional disordered systems having only a *mobility gap* (cf. also the recent paper [205] dealing with spectral gaps in presence of disorder). Together with Teufel, one of us in [341, Theorem 4.1], proved that finite systems, whose analog in the thermodynamic limit has a usual spectral gap, approximately obey an adiabatic theorem. In another recent work [596], the authors developed a theory of metastable states, characterized by the requirement that local operators raise the energy of such a state by a certain minimal amount (their condition is similar to an alternative notion of a local gap given in (17.4.8) below). Finally, we remark that, in the context of Lie group theory, the notion of a "local gap" has recently been introduced [117] and proved itself to have profound consequences [117, 116].

Next, in Section 17.1.1, we discuss the problem of justifying linear (and higher order) response theory and Kubo's formula based on adiabatic theory. Afterwards, in Section 17.1.2, we introduce our *local dynamical gap condition* ($\mathrm{LDG}_{\mathrm{intro}}$). Finally, in Section 17.1.3, we discuss our main result on response theory.

17.1.1 Response theory in many-body quantum systems

The purpose of response theory is to express how quantum expectation values change, after a small perturbation is slowly turned on. More precisely, one considers an unperturbed Hamiltonian H_0 with equilibrium state (usually a ground state) ρ_0 and slowly turns on a small additive perturbation εV. Denoting by ρ_ε the state of the system after εV has been turned on, one aims to understand, how the expectation value of an observable B changes, i.e. determine

$$\langle B\rangle_{\rho_\varepsilon} - \langle B\rangle_{\rho_0} = \varepsilon\,\sigma_B + o(\varepsilon) \tag{17.1.1}$$

at least to leading order in the strength ε.

A central piece of response theory is *Kubo's formula* [386], which provides a simple expression for the so-called linear response coefficient σ_B in (17.1.1). Despite the simplicity and empirical success of Kubo's formula, the problem of justifying it in a very general framework has so far escaped rigorous treatment. The fundamental difficulty lies in the fact that, in general, the state ρ_ε is no longer an

equilibrium state, and therefore determining it is outside the powerful realm of equilibrium statistical mechanics. This problem has prominently been pointed out by Simon [526] in 1984 in his "Fifteen problems in mathematical physics", containing a rigorous justification of Kubo's formula from first principles as problem (4B).

However, in the particular setting of many-body lattice systems with a spectral gap at zero temperature, it has recently been possible [36, 452, 556, 340] to actually prove Kubo's formula and justify the applicability of linear response theory to compute the change in expectation values (17.1.1). The more general underlying results establish *generalized super-adiabatic theorems*[1] for short range interacting Hamiltonians, which can be written as a sum of local terms, and are hence called *SLT operators* [342, 341] (see also Chapter 16).

The recent breakthrough, which paved the way for these results, was achieved by Bachmann, De Roeck, and Fraas [36] (see [452] for an adaptation to fermionic systems), who proved the first adiabatic theorem for extended (but finite) quantum lattice systems. One key difficulty is that, for macroscopic systems, typical operator norm bounds in adiabatic theory deteriorate due to the *orthogonality catastrophe* and one instead has to formulate the result in a weaker topology by testing against local observables. On a high level, the main ingredient for their proof are the well-known *Lieb-Robinson bounds* (LRBs) [419], which ensure a finite speed of correlation and prevent build-up of long-range entanglement. These LRBs, enabled to prove that the generator of the *spectral flow*, introduced by [326], is in fact an SLT operator and hence maintains good locality properties [326, 39] (showing so-called *automorphic equivalence* of ground states).

However, the work [36] had the limitation that the spectral gap of H_0 is assumed to remain open after adding the perturbation εV. To allow εV to close the gap, [556] combined ideas from space-time adiabatic perturbation theory [484, 483] with locality estimates from [36]. The underlying perturbative scheme is an iterative application of locality preserving Schrieffer-Wolff transformations (a.k.a. Lie-Schwinger block diagonalization [273]), which proved to be a powerful approach in several rather recent works in that direction [36, 556, 342, 341, 595, 205, 596]. In this paper, we carefully exploit locality properties of the operations involved in the perturbative scheme, which allows to deal with *locally* gapped systems (see Section 17.1.2 below).

The rough physical picture underlying [556, 484, 483] is that a gap that is locally intact after adding perturbation should be sufficient for adiabatic theory to be valid (cf. [340, Figure 1]). The results from [36, 452, 556] for large but finite systems were subsequently extended to the thermodynamic limit [342, 341], building on an extension of the spectral flow techniques to infinite systems by [454]. We point out that, contrary to [342], the papers [454, 341] assumed a spectral gap *only* for the GNS Hamiltonian of the infinite system (a *gap in the bulk*). More comprehensive reviews of the developments discussed in this section are given in [340, 346].

In view of the linear response problem and the second of our principal goals, the contribution of this paper is to extend the previous results for uniformly or bulk gapped systems to systems where H_0 is locally gapped. This important extension allows to rigorously treat systems with impurities of gap-closing edge modes (see the discussions in Sections 17.1.2–17.1.3 below). Technically, our contribution is to control operations on SLT operators, which are localized on a subregion of the whole lattice; see Section 17.2.2 and Appendix 17.A.

Lastly, we remark that we only consider finite-dimensional spaces and bounded operators for simplicity of the presentation.

[1] This term describes adiabatic theorems for time-dependent Hamiltonians of the form $H_\varepsilon(t) = H_0(t) + \varepsilon V(t)$, where $H_0(t)$ is assumed to have a spectral gap. Now "super-adiabatic" means that for $\varepsilon = 0$, there exists a state $\rho_0^\eta(t)$ close to the instantaneous ground state $\rho_0(t)$ of $H_0(t)$, such that the time-evolution generated by $\eta^{-1}H_0(\,\cdot\,)$ intertwines $\rho_0^\eta(t_0)$ and $\rho_0^\eta(t)$ to any order in η. The term "generalized" means that, even for a gap-closing perturbation V, there exist super-adiabatic non-equilibrium almost-stationary states (NEASSs) $\Pi^{\varepsilon,\eta}(t)$, which are intertwined by the time evolution generated by $\eta^{-1}H_\varepsilon(\,\cdot\,)$ to any order in ε and η. We refer to [452, 556, 342, 341] and Chapter 16 for details (see also Sections 17.3.2 and 17.5.2).

17.1.2 A local dynamical gap condition

All the results on linear response and adiabatic theory mentioned in Section 17.1.1 above, heavily rely on the range of the equilibrium state $\rho \equiv \rho_0$ being contained in a gapped part of the spectrum of the unperturbed Hamiltonian $H \equiv H_0$.[2] More precisely, assume

$$\operatorname{spec}(H) = \sigma_1 \cup \sigma_2 \qquad \text{with} \qquad d(\sigma_1, \sigma_2) \geq g \tag{17.1.2}$$

for some gap size $g > 0$. Then, denoting the spectral projection[3] associated to H onto σ_1 by P, we have that $P\rho P = \rho$.

On a technical level, in all of the works [36, 452, 556, 342, 341], the crucial importance of the gap of H lies in the *local invertibility of the Liouvillian* $\mathcal{L}_H[\![\,\cdot\,]\!] := -\mathrm{i}\,[H, \cdot\,]$. That is, there exists a locality preserving (usually called *quasi-local*) map $\mathcal{I}_{H,g} = \mathcal{I}_{H,g}[\![\,\cdot\,]\!]$ on the observable algebra, depending on the SLT Hamiltonian H and the gap size g, which inverts the Liouvillian in the projection $\langle\,\cdot\,\rangle_P = \operatorname{tr}\big(P\,\cdot\big)$ onto the gapped part of H. More precisely (see Proposition 17.1.1), for all local observables A, B it holds that

$$\big\langle [\mathcal{L}_H \circ \mathcal{I}_{H,g}[\![A]\!] - A, B] \big\rangle_P = 0. \tag{17.1.3}$$

Note that such a map cannot be uniquely characterized as a "weak" right inverse of $\mathcal{L}_H$, and is thus clearly not unique. However, mostly for concreteness, we will always work with an *explicitly* constructed [326, 39] variant (see Remarks 17.A.10 and 17.A.12 for relaxed, rather abstract conditions on $\mathcal{I}$), denoted by

$$\mathcal{I}_{H,g}[\![A]\!] := \int_{\mathbf{R}} \mathrm{d}t\, w_g(t) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Hs}\, A\, \mathrm{e}^{-\mathrm{i}Hs}, \tag{17.1.4}$$

and henceforth called the *quasi-local inverse of the Liouvillian*. The positive weight function $w_g \in L^1(\mathbf{R})$, normalized to $\int w_g = 1$, is required to have Fourier transform[4] $\widehat{w_g}$ with support

$$\operatorname{supp}(\widehat{w_g}) \subset [-g, g]. \tag{17.1.5}$$

Moreover, for the explicitly constructed w_g (see Lemma 17.A.9 in Appendix 17.A.3), we additionally have the bound

$$|w_g(t)| \leq C\, \mathrm{e}^{-|t|^q} \quad \text{for all} \quad q < 1. \tag{17.1.6}$$

This estimate (17.1.6) together with classical Lieb-Robinson bounds [419] for the dynamics generated by H ensure that $\mathcal{I}_{H,g}$ acts as a quasi-local operator. In Appendix 17.A.3 we will briefly recall the construction of $\mathcal{I}_{H,g}$ and report on its properties in more detail.

17.1.2.1 Dynamical characterization of a spectral gap

Interestingly, the Hamiltonian H having a spectral gap is actually *equivalent* to the invertibility of the Liouvillian. In particular, the spectral property (17.1.2) can be *dynamically* characterized via (17.1.3). A short proof of this fact is given in Section 17.6.1.

Proposition 17.1.1 (Dynamical characterization of a spectral gap)**.** *Let H be a self-adjoint operator on a finite dimensional Hilbert space $\mathcal{H}$. Let $g > 0$, $w_g \in L^1(\mathbf{R})$ be positive, normalized to $\int w_g = 1$ and satisfy* (17.1.5) *with $\widehat{w_g}|_{(-g,g)} > 0$. Decompose the spectrum of H as $\operatorname{spec}(H) = \sigma_1 \cup \sigma_2$ and let P be the spectral projection onto σ_1. Then, denoting $\mathcal{L}_H[\![\,\cdot\,]\!] = -\mathrm{i}\,[H, \cdot\,]$ and $\mathcal{I}_{H,g}$ as in* (17.1.4)*, we have that*

$$\big\langle [\mathcal{L}_H \circ \mathcal{I}_{H,g}[\![A]\!] - A, B] \big\rangle_P = 0 \quad \forall A, B \in \mathcal{B}(\mathcal{H}) \qquad \Longleftrightarrow \qquad d(\sigma_1, \sigma_2) \geq g. \tag{17.1.7}$$

[2] For ease of notation and since there will be no perturbation εV in the current Section 17.1.2, we will drop the subscript 0 here.

[3] We will follow the convention that *(orthogonal) projections* will be denoted by P (i.e. satisfying $P^2 = P$ and $P^* = P$), while *states* are denoted by ρ (i.e. satisfying $\rho = \rho^*$ and $0 \leq \rho \leq 1$ with $\operatorname{tr}\rho = 1$). Clearly, if P is an orthogonal projection, then $\rho := P/\dim\operatorname{rank} P$ is a state.

[4] We use the convention that $\widehat{f}(p) := (2\pi)^{-1/2} \int_{\mathbf{R}} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}px} f(x)$ for the Fourier transform.

The goal of this article is to relax the requirement of a globally spectrally gapped Hamiltonian H and instead work with a so-called *local dynamical gap condition* ($\mathrm{LDG}_{\mathrm{intro}}$). This condition roughly asserts that, the Hamiltonian H behaves as if it had a gap in a spatially localized region, i.e. that the Liouvillian can (almost) be locally inverted in that region. A more formal version of ($\mathrm{LDG}_{\mathrm{intro}}$) is formulated in Assumption ($\mathrm{LDG}_{\mathrm{main}}$).

($\mathrm{LDG}_{\mathrm{intro}}$) Local dynamical gap condition – informal version.

Let H be an SLT Hamiltonian and ρ an equilibrium state of H, i.e. $[H,\rho]=0$. We say that ρ is *locally dynamically gapped* of size at least $g>0$ in a region $\Lambda^{\mathrm{gap}}\subset\Lambda$ if and only if for all observables[5] $A\in\mathcal{A}_X$ and $B\in\mathcal{A}_Y$ localized in $X\subset\Lambda$ and $Y\subset\Lambda$, it holds that

$$\begin{aligned}\Big|\big\langle\big[\mathcal{L}_H\circ\mathcal{I}_{H,g}[\![A]\!]-A,B\big]\big\rangle_\rho\Big| \leq C\,\|A\|\,\|B\|\,\big(\mathrm{diam}(X)+\mathrm{diam}(Y)\big)^{\ell} \\ \times\exp\Big(-\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})+d(Y,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)^{q}\Big)\end{aligned} \tag{17.1.8}$$

for some fixed $\ell\in\mathbf{N}_0$ and constants C, $q>0$, independent of the sizes $|\Lambda|$ and $|\Lambda^{\mathrm{gap}}|$.

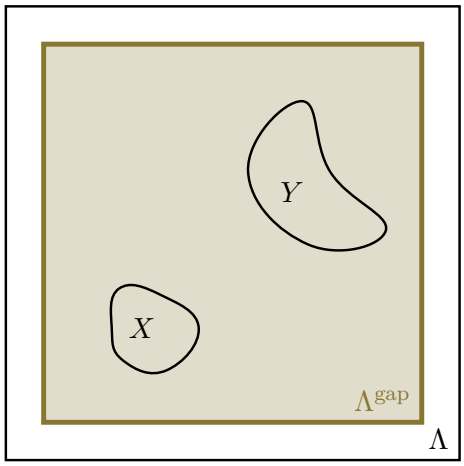

Figure 17.1.1: Illustrated is the local dynamical gap condition ($\mathrm{LDG}_{\mathrm{intro}}$) in the case where the system is gapped in the bulk of Λ, e.g., due to gap closing edge modes. If the observables $A\in\mathcal{A}_X$ and $B\in\mathcal{A}_Y$ are supported well inside $\Lambda\setminus\Lambda^{\mathrm{gap}}$, the rhs. of (17.1.8) is small, i.e. the Liouvillian is locally almost invertible.

In a nutshell, this means that, within Λ^{gap}, the Hamiltonian H approximately behaves as if it were spectrally gapped – up to an error vanishing (stretched) exponentially fast in the distance to $\Lambda\setminus\Lambda^{\mathrm{gap}}$. On the physical level, one might think of $\Lambda\setminus\Lambda^{\mathrm{gap}}$ as some impurity region causing the global spectral gap to close, or the boundary of Λ and hence allowing for gap-closing edge modes (see Figure 17.1.1). Indeed, as we will show in Section 17.4.4, the local gap condition ($\mathrm{LDG}_{\mathrm{intro}}$) is satisfied for ground states of locally in $\Lambda\setminus\Lambda^{\mathrm{gap}}$ (but arbitrarily strongly) perturbed Hamiltonians of certain quantum spin systems, which have a globally gapped ground state.

17.1.2.2 Verifying the local dynamical gap condition

Despite the supportive examples above, our concrete formulation of a local gap condition ($\mathrm{LDG}_{\mathrm{intro}}$) might still seem a bit *ad hoc* at the moment. Therefore, we will outline several alternative ways to formulate such a condition and discuss their respective features and relations in Section 17.4. In particular, in Proposition 17.4.1 we show the following (the constants q,C,ℓ have the same meaning as in (17.1.8) but might take different values):

[5] Throughout this paper, $\mathcal{A}_X$ denotes the algebra of observables with support in X.

(1) Let the SLT Hamiltonian H (with equilibrium state ρ) be obtained from a globally gapped SLT Hamiltonian H_* (with equilibrium state ρ_*) by an SLT perturbation J localized in $\Lambda \setminus \Lambda^{\mathrm{gap}}$, i.e. $H = H_* + J$. Then, if the locally tested difference $\rho - \rho_*$ is small in trace norm, i.e.

$$\|(\rho-\rho_*)\,A\|_{\mathrm{tr}} + \|A\,(\rho-\rho_*)\|_{\mathrm{tr}} \leq C\,\|A\|\,\mathrm{diam}(X)^{\ell}\exp\big(-d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})^{q}\big),$$

then ρ is locally dynamically gapped (cf. Proposition 17.4.1 (i) and (vi)).

(2) In the same setting as in (1), it holds that, whenever there exists a norm-preserving automorphism τ on the observable algebra, i.e. $\langle\cdot\rangle_\rho = \langle\tau[\![\cdot]\!]\rangle_{\rho_*}$, which satisfies

$$\big\|(\tau-\mathbf{1})[\![A]\!]\big\| \leq C\,\|A\|\,\mathrm{diam}(X)^{\ell}\,\exp\big(-d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})^{q}\big),$$

then ρ is locally dynamically gapped (cf. Proposition 17.4.1 (vii)).

(3) Let H be an SLT Hamiltonian and $\rho = |\psi\rangle\langle\psi|$ its pure *product ground state*. Then, if one has an effective gap well inside Λ^{gap} of the form[6]

$$\mathrm{i}\big\langle A^*\mathcal{L}[\![A]\!]\big\rangle_\rho \geq g\Big(1 - C\,\mathrm{diam}(X)^{\ell}\,\exp\big(-d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})^{q}\big)\Big)\Big(\langle A^*A\rangle_\rho - \big|\langle A\rangle_\rho\big|^2\Big),$$

for all $A \in \mathcal{A}_X$, then ρ is locally dynamically gapped (cf. Proposition 17.4.1 (viii)).

Items (1) and (2) will be used in Section 17.4.4.2, to show that ground states of perturbations of gapped frustration free Hamiltonians have a local dynamical gap.

17.1.3 Discussion of our main result

We can now formulate an informal version of our main result as a showcase application of our local gap condition ($\mathrm{LDG}_{\mathrm{intro}}$) to a physically relevant problem – the validity of response theory. In a nutshell, it says that, even after relaxing the usual condition of a global gap to ($\mathrm{LDG}_{\mathrm{intro}}$), we have *response theory to all orders* for a perturbation localized in Λ^{pert} – provided that $d(\Lambda^{\mathrm{pert}}, \Lambda\setminus\Lambda^{\mathrm{gap}})$ is sufficiently large compared to $|\log(\varepsilon)|^{1/q}$, where $\varepsilon > 0$ is the strength of the perturbation and $q \in (0,1)$ is some small constant (see (17.1.14) and Figure 17.1.2).

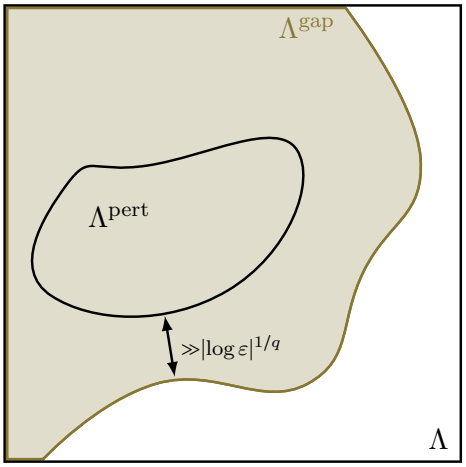

Figure 17.1.2: Let H_0 be locally dynamically gapped in $\Lambda\setminus\Lambda^{\mathrm{gap}}$ and consider a small Λ^{pert}-localized perturbation εV, which is adiabatically turned on. Then, if the distance between Λ^{pert} and $\Lambda\setminus\Lambda^{\mathrm{gap}}$ is large compared to $|\log\varepsilon|^{1/q}$, response theory (17.1.12) holds to any order.

More precisely, let H_0 be an SLT Hamiltonian and ρ_0 an equilibrium state of H_0 that is locally dynamically gapped in Λ^{gap} (according to Assumption ($\mathrm{LDG}_{\mathrm{intro}}$)). Let V be a Λ^{pert}-localized

[6] For $\Lambda^{\mathrm{gap}} = \Lambda$, this condition for all observables, is actually equivalent to the usual spectral gap condition.

perturbation by an SLT Hamiltonian (see (17.2.7) and Lemma 17.2.1 for details), $f : \mathbf{R} \to [0,1]$ a smooth switching function with $f(t) = 0$ for $t \leq -1$ and $f(t) = 1$ for $t \geq 0$, and define

$$H_\varepsilon(t) := H_0 + \varepsilon f(t) V. \tag{17.1.9}$$

Moreover, let $\rho^{\varepsilon,\eta,f}(t)$ be the solution of the time-dependent adiabatic Schrödinger equation

$$\mathrm{i}\eta \frac{\mathrm{d}}{\mathrm{d}t} \rho^{\varepsilon,\eta,f}(t) = \big[H_\varepsilon(t), \rho^{\varepsilon,\eta,f}(t)\big] \tag{17.1.10}$$

with adiabatic parameter $\eta \in (0,1]$ and initial datum $\rho^{\varepsilon,\eta,f}(t) = \rho_0$ for all $t \leq -1$. Finally, for an observable $B \in \mathcal{A}_Y$ define the response to the perturbation as

$$\Sigma_B^{\varepsilon,\eta,f}(t) := \langle B \rangle_{\rho^{\varepsilon,\eta,f}(t)} - \langle B \rangle_{\rho_0}. \tag{17.1.11}$$

Main Result (see Theorem 17.3.1)**.** *For every $j \in \mathbf{N}$ there exists a response coefficient $\sigma_{B,j}$, independent of ε, η and f, such that the following holds: There exists a constant $q \in (0,1)$ and for every $n, m \in \mathbf{N}$ and $T > 0$, there exists a constant $C > 0$, independent of ε, such that for every $t \in [0,T]$ we have that*

$$\sup_{\eta \in \left[\varepsilon^m, \varepsilon^{\frac{1}{m}}\right]} \left| \Sigma_B^{\varepsilon,\eta,f}(t) - \sum_{j=1}^{n} \varepsilon^j \sigma_{B,j} \right| \leq C \,|\!|\!| B |\!|\!|\, \varepsilon^{n+1} \Big(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})^q - C \log(\varepsilon)} \Big), \tag{17.1.12}$$

where $|\!|\!| B |\!|\!|$ measures the norm of B and its support Y.

The first order coefficient is given by Kubo's formula

$$\sigma_{B,1} = -\mathrm{i} \left\langle \big[\mathcal{I}_{H_0,g}[\![V]\!], B \big] \right\rangle_{\rho_0}. \tag{17.1.13}$$

When $\Lambda^{\mathrm{gap}} = \Lambda$, the equilibrium state ρ_0 of H_0 is *globally* dynamically gapped and the exponential in (17.1.12) is absent since $d(\Lambda^{\mathrm{pert}}, \varnothing) := \infty$. This special case of our result (when $\rho_0 = P_0 / \dim \operatorname{rank} P_0$ and P_0 projects onto a gapped spectral patch) has already been proven in [556] with extensions to infinite systems in [342, 341]; see also Remark 17.3.2 below. Moreover, observe that, whenever

$$d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg |\log(\varepsilon)|^{1/q}, \tag{17.1.14}$$

with q from our main result, the exponential in (17.1.12) is small compared to 1. In particular, this is the case, if $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg \varepsilon^{-\delta}$ for some (arbitrarily small) $\delta > 0$. Hence, our main result gives an effective condition on the distance from the perturbation region, Λ^{pert}, to the non-gapped region, $\Lambda \setminus \Lambda^{\mathrm{gap}}$, in comparison to the perturbation strength ε, which ensures that response theory to all orders is valid (see Figure 17.1.2).

It is interesting to compare our main result to the adiabatic theorem [341, Theorem 4.1] for finite systems with a gap in the bulk, which – in some sense – are *locally gapped* systems. By routine arguments (see [556, Proof of Theorem 4.1]), [341, Theorem 4.1] yields the analog of (17.1.12) again with an *additional error term*: Instead of the exponential in (17.1.12), one obtains[7] $\mathcal{O}\big(\big(\varepsilon^C d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}}) \big)^{-\infty} \big)$. Therefore, in order to have response theory to all orders in this setting, one needs that

$$d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}}) \gg \varepsilon^{-C}. \tag{17.1.15}$$

We point out the following two differences between the conditions in (17.1.14) and (17.1.15): First, in (17.1.14), the relevant distance is between 'where the perturbation V acts', i.e. Λ^{pert}, and 'where we do not have a gap', i.e. $\Lambda \setminus \Lambda^{\mathrm{gap}}$. In contrast to that, the relevant distance in (17.1.15) is between 'where the observable B tests', i.e. Y, and 'where we do not have a gap', i.e. $\Lambda \setminus \Lambda^{\mathrm{gap}}$. Second, while in (17.1.14) the distance must be much bigger than a power of the *logarithm* of the perturbation strength ε, in (17.1.15), the distance must be much bigger than a large *inverse power* of ε.

[7] The bound in [341, eq. (4.1)] essentially means that the unperturbed system on $\Lambda \equiv \Lambda_k$ is gapped within $\Lambda^{\mathrm{gap}} \equiv \Lambda_{\lfloor k(1-o(1)) \rfloor}$, where Λ_l denotes the box of side length $2l+1$ in $\mathbf{Z}^d$ centered at zero.

17.1.4 Structure of the paper

The rest of this paper is structured as follows. After introducing the mathematical framework, in particular the underlying space and the concept of locality of SLT operators, in Section 17.2, we precisely formulate our main result in Theorem 17.3.1 in Section 17.3.1 and also give its proof based on the NEASS construction. In Section 17.4 we discuss the problem of formulating a local gap condition, formulate different variants and explain their connections, and, moreover, show certain exemplary systems to have a local dynamical gap. Afterwards, in Section 17.5, we perform the NEASS construction under Assumption ($\mathrm{LDG}_{\mathrm{intro}}$) and prove the necessary inputs for Theorem 17.3.1. Additional proofs concerning the formulation of the local gap condition are given in Section 17.6; several technical lemmata and auxiliary results needed for the arguments in Section 17.5 are deferred to Appendix 17.A.

Acknowledgments.

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17.2 Mathematical framework

In this section, we briefly introduce the (standard) mathematical framework used in the formulation of the adiabatic theorems. For similar setups see [556, 342, 341, 36].

17.2.1 Spatial structure and algebra of observables

We consider a quantum spin system on a finite graph Λ equipped with the graph distance $d(\cdot,\cdot)$. Let $B_r(x) := \{ y \in \Lambda : d(x,y) \leq r \}$ be the ball of radius r centered at $x \in \Lambda$. The graph is assumed have dimension (at most) $d > 0$, i.e. there exists a constant $C_{\mathrm{vol}} > 0$ such that

$$\sup_{x\in\Lambda} |B_r(x)| \leq 1 + C_{\mathrm{vol}}\, r^d, \tag{17.2.1}$$

where $|X|$ denotes the number of sites in $X \subset \Lambda$. The set of all such graphs Λ is denoted by

$$\mathcal{G}(d, C_{\mathrm{vol}}) := \{ * \}\, \Lambda \text{ finite graph} : \sup_{x\in\Lambda} |B_r(x)| \leq 1 + C_{\mathrm{vol}}\, r^d \text{ for all } r > 0. \tag{17.2.2}$$

To each vertex $x \in \Lambda$, we associate a single-particle Hilbert space $\mathcal{H}_x$, which we assume to be of finite dimension, $\sup_{x\in\Lambda} \dim \mathcal{H}_x < \infty$. Moreover, for each $X \subset \Lambda$ let $\mathcal{H}_X := \otimes_{x\in X} \mathcal{H}_x$ be the many-particle Hilbert space on X and denote the associated C^*-algebra of observables by $\mathcal{A}_X := \mathcal{L}(\mathcal{H}_X)$. Whenever $X \subset X'$, then $\mathcal{A}_X$ is naturally embedded as a subalgebra of $\mathcal{A}_{X'}$ and we set $\mathcal{A} := \mathcal{A}_\Lambda$.

Since a very similar construction is common for fermionic lattice systems (see, e.g., [460] and Chapter 16), all the results almost immediately translate to this setting.

17.2.2 Interactions and SLT operators

An *interaction* is a map

$$\Phi : \{ X \subset \Lambda \} \to \mathcal{A}^N, \; X \mapsto \Phi(X) = \Phi(X)^* \in \mathcal{A}_X. \tag{17.2.3}$$

With any interaction, one associates a *sum of local terms (SLT) operator* A via

$$A := \sum_{X\subset\Lambda} \Phi(X) \in \mathcal{A}. \tag{17.2.4}$$

Note that, while every interaction defines a unique operator, there are multiple interactions realizing the same operator, i.e. the assignment $\Phi \mapsto A(\Phi)$ is not invertible. Note that all interactions and SLT operators are by definition always self-adjoint.

For any $b > 0$ and $p \in (0,1]$, we consider the stretched exponential function

$$\chi_{b,p} : [0,\infty) \to (0,1]\,,\ x \mapsto \mathrm{e}^{-bx^p} \tag{17.2.5}$$

as *localization functions* and define the associated *SLT interaction norm*

$$\|\Phi\|_{b,p} := \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)} \tag{17.2.6}$$

on interactions. The quality of the localization for an interaction will be expressed by the finiteness of a norm $\|\Phi\|_{b,p}$, *independent* of the (size of the) graph Λ. Operators with this property will often be referred to as (b,p)-*localized SLT operators*.

Moreover, in order to further quantify, how well an interaction is localized in a region $\Omega \subset \Lambda$, we introduce the *localized SLT interaction norm*

$$\|\Phi\|_{b,p;\Omega} := \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} \tag{17.2.7}$$

and refer to operators with the property that $\|\Phi\|_{b,p;\Omega} \leq C$ as (b,p,Ω)-*localized SLT operators*. Whenever it is clear from the context, or irrelevant for the discussion, we will often also omit the arguments (b,p), and simply refer to Ω-*localized SLT operators*. Finally, observe that $\|\Phi\|_{b,p;\Lambda} = \|\Phi\|_{b,p}$.

The following simple lemma, whose proof is given in Appendix 17.A.4, relates conceptually easier notions of locality of SLT operators to boundedness of the norm (17.2.7).

Lemma 17.2.1. *Let A be an SLT operator stemming from an interaction Φ, for which we assume that $\|\Phi\|_{b,p} \leq C$ for some constant $C > 0$, and let $\Omega \subset \Lambda$.*

(i) Let A be strictly Ω-localized*, i.e. $\Phi(Z) = 0$ whenever $Z \cap (\Lambda \setminus \Omega) \neq \varnothing$. Then it holds that $\|\Phi\|_{b,p;\Omega} \leq C$*

(ii) Let A be strongly Ω-localized*, i.e. $\Phi(Z) = 0$ whenever $Z \cap \Omega = \varnothing$. Then it holds that $\|\Phi\|_{b/2,p;\Omega} \leq C$.*

17.3 Main result: Response theory for locally gapped systems

In this section, we formulate our main result, the validity of response theory to any order, in Theorem 17.3.1 in Section 17.3.1. Its proof, based on the NEASS construction, is given in Section 17.3.2.

17.3.1 Response theory

We recall the assumption ($\mathrm{LDG}_{\mathsf{intro}}$) from Section 17.1 of a local gap, now formulated preparatory of our main result, Theorem 17.3.1 below.

($\mathrm{LDG}_{\mathsf{main}}$) Local dynamical gap condition – formal version.

We say that an equilibrium state ρ_0 of the SLT-operator H_0, i.e. with $[H_0,\rho_0] = 0$, is *locally dynamically gapped* of size at least $g > 0$ in a region $\Lambda^{\mathrm{gap}} \subset \Lambda$ with respect to C_{gap}, b, $p > 0$ and $\ell \in \mathbf{N}_0$, if and only if for all $X, Y \subset \Lambda$ and $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, it holds that

$$\begin{aligned}\Big|\big\langle [\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0}[\![A]\!] - A, B]\big\rangle_{\rho_0}\Big| \leq C_{\mathrm{gap}}\,\|A\|\,\|B\|\,\big[\,\mathrm{diam}(X) + \mathrm{diam}(Y)\big]^{\ell} \\ \times \chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}}) + d(Y,\Lambda\setminus\Lambda^{\mathrm{gap}})\big).\end{aligned} \tag{17.3.1}$$

We can now formulate our main result. In a nutshell, it says the following: We have validity of *response theory to all orders* under the assumption of ($\mathsf{LDG}_{\mathsf{main}}$) for a perturbation localized in Λ^{pert} – provided that the distance to $\Lambda \setminus \Lambda^{\mathrm{gap}}$ is sufficiently large compared to $|\log(\varepsilon)|^{1/q}$, where $\varepsilon > 0$ is the strength of the perturbation and $q > 0$ is some small constant.

The proof of Theorem 17.3.1 is given at the end of Section 17.3.2.

Theorem 17.3.1 (Response theory to all orders)**.** *Fix $n, m \in \mathbf{N}$ and let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$, $C_{\mathrm{int}} > 0$ and $g > 0$, $C_{\mathrm{gap}} > 0$, $\ell \in \mathbf{N}_0$, and $C_{\mathrm{switch}} > 0$. Take any $q \in (0,p)$. Then there exist a constant $C_{n,m} > 0$ (in particular depending on n and m) such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ (recall (17.2.2)), subsets $\Lambda^{\mathrm{pert}} \subset \Lambda$ and SLT-operators H_0 and V, with corresponding interactions that satisfy $\|\Phi_{H_0}\|_{b,p} < C_{\mathrm{int}}$ and $\|\Phi_V\|_{b,p;\Lambda^{\mathrm{pert}}} < C_{\mathrm{int}}$, respectively, the following holds:*

Assume that the equilibrium state ρ_0 of H_0 is locally dynamically gapped in Λ^{gap} of size at least g and with respect to C_{gap}, b, p and ℓ according to Assumption ($\mathsf{LDG}_{\mathsf{main}}$)*. Let $Y \subset \Lambda$ and $B \in \mathcal{A}_Y$. Then there exist response coefficients $\sigma_{B,j}$ in the following sense: For $\varepsilon > 0$ and smooth switching function $f\colon \mathbf{R} \to [0,1]$ satisfying $f(t) = 0$ for $t \leq -1$, $f(t) = 1$ for $t \geq 0$, and $\|f\|_{C^{\mathfrak{C}_{n,m}}(\mathbf{R})} < C_{\mathrm{switch}}$ with $\mathfrak{C}_{n,m} := \lceil m(n+1+(2d+\ell)/p) \rceil$, consider the time-dependent Hamiltonian*

$$H_\varepsilon(t) := H_0 + \varepsilon\, f(t)\, V. \tag{17.3.2}$$

Let $\rho^{\varepsilon,\eta,f}(t)$ be the solution of the time-dependent adiabatic Schrödinger equation

$$\mathrm{i}\,\eta\, \frac{\mathrm{d}}{\mathrm{d}t} \rho^{\varepsilon,\eta,f}(t) = \big[H_\varepsilon(t), \rho^{\varepsilon,\eta,f}(t)\big] \tag{17.3.3}$$

with adiabatic parameter $\eta \in (0,1]$ and initial datum $\rho^{\varepsilon,\eta,f}(t) = \rho_0$ for all $t \leq -1$.

Then, the response to the perturbation, $\Sigma_B^{\varepsilon,\eta,f}(t) := \langle B \rangle_{\rho^{\varepsilon,\eta,f}(t)} - \langle B \rangle_{\rho_0}$, satisfies

$$\begin{aligned} \sup_{\eta \in \left[\varepsilon^m, \varepsilon^{1/m}\right]} \left| \Sigma_B^{\varepsilon,\eta,f}(t) - \sum_{j=1}^{n} \varepsilon^j \sigma_{B,j} \right| &\leq C_{n,m}\, \|B\|\, \mathrm{diam}(Y)^{(3+n)d+\ell} \big(1+t\big)^{(2d+\ell)/p+1}\, \varepsilon^{n+1} \\ &\quad \times \Big(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})^q - (\mathfrak{C}_{n,m}+1)\log(\varepsilon)}\Big), \end{aligned} \tag{17.3.4}$$

for all $t \geq 0$. The first order coefficient is given by Kubo's formula (17.1.13).

Our result can also be extended to infinite systems.

Remark 17.3.2 (Extension to infinite systems)**.** *Following the arguments from [342], it is reasonably straightforward to extend our result to the case of infinite systems. More precisely, in order to do so, we need to*

- *consider Λ to be part of a sequence of graphs exhausting an infinite graph Γ, e.g. $\Lambda \equiv \Lambda_k := \{-k, \ldots, k\}^d \subset \mathbf{Z}^d$ and $\Gamma := \mathbf{Z}^d$;*
- *assume that the interactions associated to H_0 and V* have a thermodynamic limit *(see [342, Definition 2.1]);*
- *require that the sequence of equilibrium states $\rho_0 \equiv \rho_0^\Lambda$ satisfies the local dynamical gap condition* ($\mathsf{LDG}_{\mathsf{main}}$) *with constants independent of Λ and converges (in the weak* sense in the dual to the algebra of quasi-local observables, see [342, Section 2.5]) as $\Lambda \nearrow \Gamma$;*
- *and suppose that also the perturbation region Λ^{pert} as well as the gapped region Λ^{gap}* converge *(in a suitable sense) to some $\Gamma^{\mathrm{pert}}, \Gamma^{\mathrm{gap}} \subset \Gamma$, respectively, at least ensuring that $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}}) \to d(\Gamma^{\mathrm{pert}}, \Gamma \setminus \Gamma^{\mathrm{gap}})$ as $\Lambda \nearrow \Gamma$.*

Then, (17.3.4) *also holds for the infinite system but with* $d(\Gamma^{\mathrm{pert}}, \Gamma \setminus \Gamma^{\mathrm{gap}})$ *in the exponential.*

In the special case, where the gapped region exhausts the entire graph, i.e. $\Lambda^{\mathrm{gap}} \nearrow \Gamma$*, we find* (17.3.4) without *the additional exponential error term. This corresponds to* H_0 *having a* gap in the bulk *(see [341], Chapter 16 and also the discussion around* (17.4.8) *in Section 17.4), where in the finite systems* H_0 *could have gap closing edge modes (see Figure 17.1.1).*

17.3.2 Non-equilibrium almost stationary states and proof of Theorem 17.3.1

The main underlying idea of our justification of linear response theory is the construction of so-called *non-equilibrium almost-stationary states (NEASS)* [452, 556, 340, 342, 341] for the dynamics of the perturbed Hamiltonian

$$H_\varepsilon := H_0 + \varepsilon V. \tag{17.3.5}$$

More precisely, for every $n \in \mathbf{N}$ there exists[8] a state Π_n^ε, which is obtained from the equilibrium state ρ_0 of H_0 by a unitary transformation with a small SLT generator, in such a way, that it is almost stationary under the dynamics generated by (17.3.5).

Similarly to [556, Theorem 3.1], one can obtain Proposition 17.3.3 by following the proof of Proposition 17.3.4, given in Section 17.5.2, and combining it with Proposition 17.3.5. As the arguments in [556, Theorem 3.1] are rather standard, we omit the details for brevity.

Proposition 17.3.3 (Non-equilibrium almost-stationary states)**.** *Fix* $n \in \mathbf{N}$ *and let* $d \in \mathbf{N}$*,* $C_{\mathrm{vol}} > 0$*,* $b > 0$*,* $p \in (0,1]$*,* $C_{\mathrm{int}} > 0$ *and* $g > 0$*,* $C_{\mathrm{gap}} > 0$*,* $\ell \in \mathbf{N}_0$*. Take any* $q \in (0,p)$*. Then there exist a constant* $C_n > 0$ *(in particular depending on* n*) such that for all lattices* $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ *(recall* (17.2.2)*), subsets* $\Lambda^{\mathrm{pert}} \subset \Lambda$ *and SLT-operators* H_0 *and* V*, with corresponding interactions that satisfy* $\|\Phi_{H_0}\|_{b,p} < C_{\mathrm{int}}$ *and* $\|\Phi_V\|_{b,p;\Lambda^{\mathrm{pert}}} < C_{\mathrm{int}}$*, respectively, the following holds:*

Assume that the equilibrium state ρ_0 *of* H_0 *is locally dynamically gapped in* Λ^{gap} *of size at least* $g > 0$ *and with respect to* C_{gap}*,* b*,* p *and* ℓ *according to Assumption* ($\mathrm{LDG}_{\mathsf{main}}$)*. Then, there exists a sequence* $(A_\mu)_{\mu \in \mathbf{N}}$ *of SLT operators, which are* $(1, p', \Lambda^{\mathrm{pert}})$*-localized for any* $p' < p$*, such that the state*

$$\Pi_n^\varepsilon := \mathrm{e}^{\mathrm{i}\varepsilon S_n^\varepsilon} \rho_0 \, \mathrm{e}^{-\mathrm{i}\varepsilon S_n^\varepsilon} \quad \textit{with} \quad S_n^\varepsilon := \sum_{\mu=1}^{n} \varepsilon^{\mu-1} A_\mu \tag{17.3.6}$$

is almost-stationary for the dynamics generated by $H_\varepsilon = H_0 + \varepsilon V$ *in the following sense: Let* $\rho^\varepsilon(t)$ *be the solution to the Schrödinger equation*

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\rho^\varepsilon(t) = \big[H_\varepsilon, \rho^\varepsilon(t)\big] \quad \textit{with} \quad \rho^\varepsilon(0) = \Pi_n^\varepsilon. \tag{17.3.7}$$

Under these conditions, for all $B \in \mathcal{A}_Y$ *with* $Y \subset \Lambda$ *and* $t \geq 0$*, it holds that*

$$\begin{aligned} \big|\langle B\rangle_{\rho^\varepsilon(t)} - \langle B\rangle_{\Pi_n^\varepsilon}\big| \leq C_{n,m} \, \|B\| \, \mathrm{diam}(Y)^{3d+\ell} \, |t| \big(1 + |t|^{(2d+\ell)/p}\big) \, \varepsilon^{n+1} \\ \times \Big(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})^q - (n+1)\log(\varepsilon)}\Big). \end{aligned} \tag{17.3.8}$$

We point out that, similarly to [556, Theorem 3.1], one can improve the bound in (17.3.8) by rescaling every t by ε^m for some $m \in \mathbf{N}$, at the cost of increasing the constant in front of $\log(\varepsilon)$.

From a technical perspective, our proof of response theory for locally gapped quantum spin systems, Theorem 17.3.1, rests on the following two propositions, the proof of which shall be given in Section 17.5. The first, Proposition 17.3.4, states that the NEASS constructed in Proposition 17.3.3

[8] Following the *resummation procedure* in [342, Appendix E], one could even construct a single (i.e. n independent) state Π^ε, which, for every fixed n, has the same properties as Π_n^ε. We will, however, refrain from doing so for brevity of the presentation.

is also almost-stationary under the dynamics generated by the time-dependent Hamiltonian (17.3.2) whose perturbation V is turned on by the switching function f on the adiabatic time scale $1/\eta$ (see, e.g., [556, Prop. 3.2]). The proof of Proposition 17.3.4 is given in Section 17.5.2.

Proposition 17.3.4 (Adiabatic switching and the NEASS)**.** *Under the assumptions of Theorem 17.3.1 (in particular recalling* (17.3.3)*) it holds that*

$$\left|\langle B\rangle_{\rho^{\varepsilon,\eta,f}(t)} - \langle B\rangle_{\Pi_n^\varepsilon}\right| \leq C\,\|B\|\,\mathrm{diam}(Y)^{3d+\ell}\big(1+t\big)^{(2d+\ell)/p+1}\,\frac{\varepsilon^{n+1}+\eta^{n+1}}{\eta^{(2d+\ell)/p+1}} \times\left(1+\mathrm{e}^{-d(\Lambda^{\mathrm{pert}},\Lambda\setminus\Lambda^{\mathrm{gap}})^q-(n+1)\log(\varepsilon)}\right), \tag{17.3.9}$$

for $t\geq 0$, where Π_n^ε is the NEASS (17.3.6) *constructed in Proposition 17.3.3.*

For our application to response theory, it is important to have an explicit expansion of expectation values in the NEASS in powers of ε with coefficients given by expectations in the unperturbed equilibrium state, the linear term constituting the celebrated *Kubo formula*. This is the content of the following proposition, whose proof works in the exact same way as in [452, Theorem 3.3] or [556, Proposition 5.2] and is hence omitted.

Proposition 17.3.5 (Asymptotic expansion of the NEASS)**.** *Under the assumptions of Proposition 17.3.3, there exist linear maps $\mathcal{K}_j\colon\mathcal{A}\to\mathcal{A}$, $j\in\mathbf{N}$, given by nested commutators with the A_μ's in* (17.3.6)*, such that for $n\geq m$ it holds that*

$$\left|\langle B\rangle_{\Pi_n^\varepsilon} - \sum_{j=1}^{m}\varepsilon^j\langle\mathcal{K}_j[\![B]\!]\rangle_{\rho_0}\right| \leq C_{n,m}\,\varepsilon^{m+1}\,\|B\|\,|Y|^m\,\chi_{b',p}\big(d(\Lambda^{\mathrm{pert}},Y)\big), \tag{17.3.10}$$

with b',p the parameters of the localization of the A_μ given in Proposition 17.3.3. The first two orders of the expansion (17.3.10) *are explicitly given by*

$$\langle\mathcal{K}_0[\![B]\!]\rangle_{\rho_0} = \langle B\rangle_{\rho_0} \quad \textit{and} \quad \langle\mathcal{K}_1[\![B]\!]\rangle_{\rho_0} = -\mathrm{i}\big\langle\big[\mathcal{I}_{H_0,g}[\![V]\!],B\big]\big\rangle_{\rho_0}, \tag{17.3.11}$$

where $\mathcal{I}_{H_0,g}[\![\cdot]\!]$ is the inverse Liouvillian from (17.1.4).

We can finally give the proof of Theorem 17.3.1.

Proof of Theorem 17.3.1. Armed with Propositions 17.3.4–17.3.5, the proof of Theorem 17.3.1 follows by a simple application of the triangle inequality when applying Proposition 17.3.4 with $n\to\tilde{n}:=\lceil m(n+1+(2d+\ell)/p)\rceil$ and Proposition 17.3.5 for $n\to\tilde{n}$ and $m\to n$; cf. [556, Theorem 4.1]. □

17.4 How to formulate a local gap condition?

Formulating a (i) physically *and* mathematically transparent, (ii) practically applicable, and (iii) reasonably restrictive condition of a *local gap* for a Hamiltonian H is a non-trivial task. In this section, we discuss several different possible approaches to do so. In principle, there are two main ways: Either we compare H to a globally gapped reference Hamiltonian H_* (*extrinsic formulation*, see Section 17.4.1), or the condition involves the original Hamiltonian H alone (*intrinsic formulation*, see Section 17.4.2). We shall first list all the different variants and afterwards compare their respective features in view of the above listed three requirements and explain their relations in Section 17.4.3. Finally, in Section 17.4.4 we discuss two exemplary systems, which we prove to have a local dynamical gap in the sense of Assumption ($\mathrm{LDG}_{\mathsf{main}}$).

Throughout the entire section (except differently stated explicitly), we will use C,ℓ,b,p as generic constants satisfying $C>0$, $\ell\geq 0$, $b>0$, and $p\in(0,1]$. Their precise value might change from line to

line and it only depends on the model parameters, i.e. the interaction norms (17.2.6)–(17.2.7) of the involved SLT Hamiltonians, the lattice parameters in (17.2.2), or the gap size $g > 0$ of a reference Hamiltonian.

17.4.1 Extrinsic local gap conditions

In this section, we describe several ways of *extrinsically* expressing that an equilibrium state ρ (e.g. the ground state) of a Hamiltonian H is locally gapped. We call these formulations of a local gap condition *extrinsic*, since the Hamiltonian H and equilibrium state P of interest are compared to another reference Hamiltonian H_*, called the *parent Hamiltonian*, with equilibrium state ρ_*, which is globally gapped. Throughout this section, for simplicity of the presentation, we will assume that both ρ and ρ_* have rank one.

The common core of all the different ways to extrinsically formulate a local gap conditions, is to assume that H and H_* differ only locally: That is, H_* and H are related as

$$H = H_* + J \tag{17.4.1}$$

where J is some SLT operator, which is localized in $\Lambda \setminus \Lambda^{\mathrm{gap}}$ (e.g., in one of the senses mentioned in Lemma 17.2.1). This form (17.4.1) of H and H_* differing only locally is then passed on to their equilibrium states, ρ and ρ_*, respectively. More precisely, we will assume that local perturbations of H_* perturb its equilibrium state ρ_* only locally. Whenever this holds, one (usually) says that H_* satisfies the *local perturbations perturb locally (LPPL)* principle, which has been shown to be the case for *ground states* in several contexts [207, 35] (see also Chapter 15).

Local gap via 'classical' LPPL The simplest way to formulate an *extrinsic* local gap condition for H given by (17.4.1) with ground state ρ, is to assume that the globally parent gapped Hamiltonian H_* with ground state ρ_* satisfies a strong form[9] of the LPPL principle. This means that, for $A \in \mathcal{A}_X$, we have

$$\big|\langle A\rangle_\rho - \langle A\rangle_{\rho_*}\big| \leq C\, \|A\|\, \mathrm{diam}(X)^\ell\, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big). \tag{17.4.2}$$

An estimate of the form (17.4.2) has been shown to hold for weakly interacting spin systems in Chapter 15 (based on ideas of [591]) and – in a similar way – in [35]. Moreover, in [341, Theorem 4.1], such type of assumption was used to formulate an adiabatic theorem for large but finite systems with a gap in the bulk. In this case (cf. [341, eq. (4.1)]), the additional error term (compared to the case of having a global gap) in the response theory expansion (17.1.12) becomes $\mathcal{O}\Big(\big(\varepsilon^C\, d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)^{-\infty}\Big)$. Moreover, using that H_* and J are SLT operators, (17.4.2) implies that ($\mathrm{LDG}_{\mathsf{main}}$) holds *but* with the argument of $\chi_{b,p}$ in (17.3.1) being the *minimum* of the distances (or – up to changing constants – the distance of the *union* of the two supports to $\Lambda \setminus \Lambda^{\mathrm{gap}}$) instead of their sum (i.e. their maximum). Hence, the decay in $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})$, emerging in the course of proving Proposition 17.3.4 in Section 17.5, will eventually be lost.

In conclusion, although this way (17.4.2) of saying that H has a local gap, is quite simple, the resulting error terms are considerably bad and, moreover, it does not quite allow for tracking $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})$-decays.

Trace norm LPPL One could even further strengthen the LPPL assumption (17.4.2) to a *trace norm LPPL*. This means, that ρ and ρ_* are not only close in the weak* sense (17.4.2), but we have

$$\|(\rho - \rho_*)\, A\|_{\mathrm{tr}} + \|A\, (\rho - \rho_*)\|_{\mathrm{tr}} \leq C\, \|A\|\, \mathrm{diam}(X)^\ell\, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{17.4.3}$$

[9] According to the terminology introduced in Chapter 15, calling an LPPL principle *strong*, simply means, that the perturbation J is *not* required to leave the spectral gap of H_* open – contrary to earlier works [39, 207] on LPPL.

where $\|\cdot\|_{\mathrm{tr}} := \mathrm{tr}(|\cdot|)$ denotes the trace norm. We remark, that surely (17.4.3) implies (17.4.2). A somewhat weakened version of (17.4.3) is to ask that only the trace norm of the *commutator* $[\rho - \rho_*, A]$ is small like in (17.4.3), i.e.

$$\|[\rho - \rho_*, A]\|_{\mathrm{tr}} \leq C\,\|A\|\,\mathrm{diam}(X)^{\ell}\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{17.4.4}$$

As we show in Proposition 17.4.1 in Section 17.4.3 (see also Figure 17.4.1), both these ways, (17.4.3) and (17.4.4), of saying that H has a local gap are sufficient to conditions for H having a local gap in the sense of Assumption ($\mathrm{LDG}_{\mathsf{main}}$).[10] However, it is a quite restrictive condition that ρ and ρ_* are close in such strong topology as (17.4.3) or (17.4.4).

Intertwining automorphism Another, compared to (17.4.2) strengthened, way of saying that ρ and ρ_* are close to each other, is to assume the following: There exists a intertwining norm-preserving $*$-automorphism τ on $\mathcal{A}$, i.e. $\langle\cdot\rangle_\rho = \langle\tau[\![\cdot]\!]\rangle_{\rho_*}$, which satisfies

$$\big\|(\tau - \mathbf{1})[\![A]\!]\big\| \leq C\,\|A\|\,\mathrm{diam}(X)^{\ell}\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{17.4.5}$$

where $\mathbf{1}$ is the identity map. Similarly to (17.4.3), we surely have that (17.4.5) implies (17.4.2). Moreover, (17.4.5) is a sufficient condition for having ($\mathrm{LDG}_{\mathsf{main}}$) (see Proposition 17.4.1 and Figure 17.4.1 below), as will be used for proving that both of the examples studied in Section 17.4.4 below satisfy ($\mathrm{LDG}_{\mathsf{main}}$) (see Section 17.6).

17.4.2 Intrinsic local gap conditions

In this section, we describe two ways of *intrinsically* expressing that an equilibrium state ρ of a Hamiltonian H is locally gapped. In contrast to the previous extrinsic formulation, they are called *intrinsic*, as they do not refer to another (parent) Hamiltonian.

Local spectral gap conditions *only* for ground states In case of ρ being the (unique) *ground state*, a very natural way to connect the notion of locality with spectral analysis is to require that a variational condition characterizing the spectral gap is tested only locally. Generally speaking, assume that ψ_0 is the unique ground state with eigenvalue E_0 of a local Hamiltonian H in some underlying physical space Λ. Then, a spectral gap above E_0 of size (at least) $g > 0$ is characterized by

$$\inf_{\psi \perp \psi_0} \frac{\langle\psi, (H - E_0)\psi\rangle}{\langle\psi, \psi\rangle} \geq g. \tag{17.4.6}$$

If the minimization in (17.4.6) is restricted a smaller set of ψ's, i.e. those which are localized to a region, say, $\Lambda^{\mathrm{gap}} \subset \Lambda$ in some appropriate sense, one could say that H is locally gapped in Λ^{gap}. Alternatively, for every fixed ψ, the gap size g could be assumed to be non-constant but dependent on the distance of the support of ψ to the region Λ^{gap}.

For quantum spin systems on the graph Λ, it can easily be checked that the analog of (17.4.6) for an SLT Hamiltonian H with unique ground state ρ is

$$\mathrm{i}\,\langle A^* \mathcal{L}[\![A]\!]\rangle_\rho \geq g\,\Big(\langle A^* A\rangle_\rho - \big|\langle A\rangle_\rho\big|^2\Big), \tag{17.4.7}$$

for all observables $A \in \mathcal{A}$, where $\mathcal{L}[\![\cdot]\!] := -\mathrm{i}\,[H, \cdot\,]$ denotes the Liouvillian. We now give two options to turn (17.4.7) into a local gap condition.

As a first option, one could require that the *gap size* g *decays* as the support X of the observable $A \in \mathcal{A}_X$ approaches the complement of Λ^{gap}, e.g. as

$$\mathrm{i}\,\langle A^* \mathcal{L}[\![A]\!]\rangle_\rho \geq g\,\big(1 - C\,\mathrm{diam}(X)^{\ell}\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)\big)\,\Big(\langle A^* A\rangle_\rho - \big|\langle A\rangle_\rho\big|^2\Big). \tag{17.4.8}$$

[10] This fact is used for proving ($\mathrm{LDG}_{\mathsf{main}}$) (which is done in Section 17.6) for one of the examples discussed in Section 17.4.4 below.

In case that $\rho = |\psi\rangle\langle\psi|$ with ψ being a *product state*, variants of Proposition 14 and Lemma 15 in [596] can be used to show the following: The gap decay condition (17.4.8) implies a slightly weakened version (see (17.4.12) below) of our local dynamical gap condition; see Proposition 17.4.1 (viii).

We further remark that, if one is interested in *taking the thermodynamic limit*, $\Lambda \nearrow \Gamma$ for some infinite graph Γ, and $\Lambda^{\mathrm{gap}} \nearrow \Gamma$ in this limit (e.g. in the scenario of gap-closing edge modes), then (17.4.8) yields a *gap in the bulk* of the naturally associated infinite system (see Remark 16.3.6).[11]

Another, compared to (17.4.8) weaker, option is to include a separate *additive* error term on the lhs. of (17.4.7), e.g. as

$$\mathrm{i}\,\langle A^*\mathcal{L}[\![A]\!]\rangle_\rho \geq g\left(\langle A^*A\rangle_\rho - \left|\langle A\rangle_\rho\right|^2\right) - C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big), \tag{17.4.9}$$

which we call *defective coercivity* for the following reason.

The global gap characterization can equivalently be rewritten as

$$\mathrm{i}\,\langle \tilde{A}^*\mathcal{L}[\![\tilde{A}]\!]\rangle_\rho \geq g\,\langle \tilde{A}^*\tilde{A}\rangle_\rho \quad \text{with} \quad \tilde{A} := A - \langle A\rangle_\rho,$$

which means that on $\mathcal{A}^\perp := \{\,\rho\,\}^\perp = \left\{\,\tilde{A} : A \in \mathcal{A}\,\right\} \subset \mathcal{A}$ the bounded sesquilinear form

$$\mathcal{B}\colon \mathcal{A}^\perp \times \mathcal{A}^\perp \to \mathbf{C}\,,\,(A,B) \mapsto \mathrm{i}\,\langle A^*\mathcal{L}[\![B]\!]\rangle_P$$

is *coercive* with respect to the semi-norm $|\!|\!|A|\!|\!| := \sqrt{\langle A^*A\rangle_\rho}$ on $\mathcal{A}^\perp$. Hence, (17.4.9) expresses some defect in the original coercivity of (17.4.7). We remark that, at least morally, one could use the Lax-Milgram theorem to deduce existence of an inverse of the Liouvillian $\mathcal{L}$ given such a coercivity estimate. However, the problem is that the inverse obtained in this way does not necessarily have any nice locality properties, which are crucially used for practical purposes.

Finally, we point out that the defective coercivity (17.4.9) is implied by the 'classical' strong LPPL (17.4.2). More precisely, assume there exists a parent Hamiltonian H_* which is related to H like in (17.4.1). Then, if the globally gapped ground state ρ_* of H_* satisfies the strong form (17.4.2) of LPPL, then (17.4.9) holds – modulo adjusting the constants b, p, ℓ, and C; see Proposition 17.4.1 and Figure 17.4.1 below.

Local dynamical gap condition The – at least in view of our application – most important property of an SLT Hamiltonian with a gapped part of its spectrum, is that (the offdiagonal part of) its Liouvillian can be *locally* inverted; see, e.g., (17.1.3) in Section 17.1.2 and also [556, Appendix C]. In these applications, roughly said, the local invertibility of the Liouvillian guarantees that the effect of perturbations remains local. More precisely, it allowed to prove automorphic equivalence of gapped ground states [39, 454] and adiabatic theorems in cases where the perturbation is not allowed to close the gap [36, 452] and later also for gap closing perturbations [556, 340, 342, 341].

As already noted in Section 17.1.2, the local invertibility of the Liouvillian is in fact *equivalent* to the Hamiltonian having a gapped part of its spectrum (see Proposition 17.1.1), which yields a *dynamical* characterization of a *spectral* property. A local version of this feature is formulated in our *local dynamical gap condition* in Assumptions ($\mathrm{LDG}_{\mathsf{intro}}$) and ($\mathrm{LDG}_{\mathsf{main}}$): Let $\mathcal{I} \equiv \mathcal{I}_{H,g}$ denote the inverse Liouvillian from (17.1.4) with gap size $g > 0$. Then, an equilibrium state ρ (*not* necessarily the ground state!) of H is locally dynamically gapped, if for all observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ localized in $X \subset \Lambda$ and $Y \subset \Lambda$, it holds that

$$\begin{aligned}\left|\big\langle[\mathcal{L}\circ\mathcal{I}[\![A]\!] - A, B]\big\rangle_\rho\right| \leq C\,\|A\|\,\|B\|\,\big[\,\mathrm{diam}(X) + \mathrm{diam}(Y)\big]^\ell \\ \times\,\chi_{b,p}(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}}) + d(Y,\Lambda\setminus\Lambda^{\mathrm{gap}}))\,.\end{aligned} \tag{17.4.10}$$

[11] More precisely, this also requires that an interaction $\Phi = \Phi_H$, associated to the SLT Hamiltonian H, *has a thermodynamic limit* in a suitable sense (see [342, Definition 2.1], [341, Definition 3.1], and Definitions 16.3.3 and 16.3.8). In this case, the linear functional $A \mapsto \mathrm{tr}(PA)$ converges in weak* sense to a *state* on the C^*-algebra of quasi-local observables on Γ. A *gap in the bulk* then means, that the naturally associated GNS Hamiltonian of the infinite system has a spectral gap above zero.

In the special case of $\ell = 0$, by taking a supremum over all observables B with $\|B\| \leq 1$ in (17.4.10), we find that – for $\ell = 0$ – our local dynamical gap condition (17.4.10) is actually *equivalent* to

$$\big\|[\mathcal{L} \circ \mathcal{I}[\![A]\!] - A, \rho]\big\|_{\mathrm{tr}} \leq C\, \|A\|\, \chi_{b,p}(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})).$$

Here, we additionally used cyclicity of the trace together with $\sup_{\|D\| \leq 1} |\mathrm{tr}(CD)| = \|C\|_{\mathrm{tr}}$.

Moreover, we point out that, while the rhs. of (17.4.10) is obviously symmetric in A and B, the lhs. is as well. This follows by rewriting (recall (17.1.4))

$$\mathcal{L}_H \circ \mathcal{I}_{H,g}[\![A]\!] - A = \int_{\mathbf{R}} \mathrm{d}t\, w_g(t)\, \mathrm{e}^{\mathrm{i}Ht}\, A\, \mathrm{e}^{-\mathrm{i}Ht} =: \mathcal{J}_{H,g}[\![A]\!] = \mathcal{J}[\![A]\!] \tag{17.4.11}$$

and using $[H, \rho] = 0$ together with the symmetry $w_g(t) = w_g(-t)$.

A slightly weakened (asymmetric) version of (17.4.10) would be to require that for all $X \subset \Lambda$ satisfying $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^{\beta}$ (for some $\beta > 0$) and observables $A \in \mathcal{A}_X$, and $Y \subset \Lambda$ and observables $B \in \mathcal{A}_Y$, it holds that

$$\Big|\big\langle [\mathcal{L} \circ \mathcal{I}[\![A]\!] - A, B]\big\rangle_{\rho}\Big| \leq C\, \|A\|\, \|B\|\, \mathrm{diam}(Y)^{\ell}\, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big). \tag{17.4.12}$$

This version will be introduced as the *weakened local gap condition* ($\mathrm{LDG}_{\mathrm{weak}}$) in Section 17.5.1 below. Such an assumption will in fact be sufficient for proving our main result in Theorem 17.3.1.

Finally, we remark that another symmetric (in A and B) bound on the lhs. of (17.4.10) would be to replace the sum of the distances in the exponent in (17.4.10) by $d(X \cup Y, \Lambda \setminus \Lambda^{\mathrm{gap}})$ (i.e. take the minimum of the distances to $\Lambda \setminus \Lambda^{\mathrm{gap}}$ instead of their maximum). Using quasi-locality estimates for $\mathcal{J}[\![\cdot]\!]$ defined in (17.4.11) (see, e.g., [461, Lemma 5.1]), this bound could actually be proven as a consequence of the LPPL principle (17.4.2), provided that there is a globally spectrally gapped parent Hamiltonian H_* for H. Therefore, similarly to the paragraph below (17.4.2), using this assumption, the decay in $d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})$, emerging in the course of proving Proposition 17.3.4 in Section 17.5, would eventually be lost.

17.4.3 Summary and comparison

In the previous two Sections 17.4.1–17.4.2 we described several different ways of expressing that a Hamiltonian is locally gapped, distinguishing between *extrinsic* (Section 17.4.1) and *intrinsic* (Section 17.4.2) formulations.

While the extrinsic conditions are easy to formulate, they rely – by definition – on a reference (parent) Hamiltonian with a global gap satisfying some form of LPPL principle (cf. (17.4.2)–(17.4.5)). Since for a system of interest, it is not guaranteed to have such a well-understood parent Hamiltonian available, it is conceptually more desirable to formulate a local gap condition in an intrinsic way. Or, in other words, saying that a system is (or behaves as if it were) *locally* gapped should not only make sense relative to another *globally* gapped system.

In the intrinsic category, we formulated two local *spectral* gap conditions (17.4.8)–(17.4.9), which, however, are (i) only meaningful for (non-degenerate) ground states and (ii) although mathematically clean, hardly applicable in physical problems in a direct way (apart from the connection in Proposition 17.4.1 (viii)). These two issues are then resolved by our local dynamical gap condition (LDG) in (17.4.10) and its weakened version in (17.4.12).

The following proposition formulates the implications shown in Figure 17.4.1 precisely.

Proposition 17.4.1 (Relations among the local gap conditions)**.** *Fix $g > 0$, $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $\tilde{b} > 0$, $\tilde{p} \in (0, 1]$, $C_{\mathrm{int}} > 0$. Let $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, $\Lambda^{\mathrm{gap}} \subset \Lambda$ and let H_* and J be SLT-operators with corresponding interactions satisfying $\|\Phi_{H_*}\|_{\tilde{b},\tilde{p}} < C_{\mathrm{int}}$ and $\|\Phi_J\|_{\tilde{b},\tilde{p};\Lambda\setminus\Lambda^{\mathrm{gap}}} < C_{\mathrm{int}}$. Let ρ_* be an equilibrium state of H_* and ρ and equilibrium state of $H := H_* + J$.*

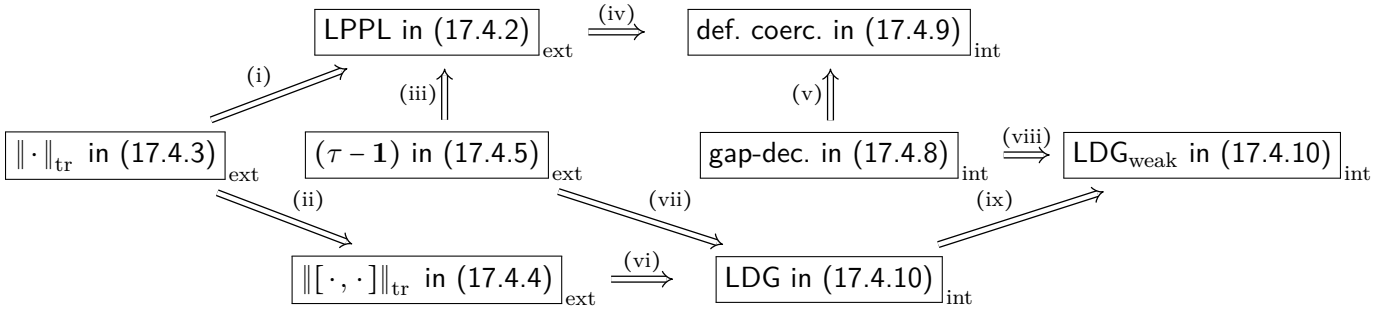

Figure 17.4.1: Implications among the various local gap conditions from Sections 17.4.1–17.4.2. The numbering refers to the precise statements in Proposition 17.4.1.

Then the following implications hold (modulo adjusting the constants C, ℓ, b, and p in a way which only depends on $g, d, C_{\mathrm{vol}}, \tilde{b}, \tilde{p}$, and C_{int}):

(i) Trace norm LPPL implies commutator trace norm LPPL, i.e. we have (17.4.3) $\implies$ (17.4.4).

(ii) Trace norm LPPL implies the usual LPPL, i.e. we have (17.4.3) $\implies$ (17.4.2).

(iii) An intertwining automorphism implies the usual LPPL, i.e. we have (17.4.5) $\implies$ (17.4.2).

(iv) Additionally assuming that ρ_* is the unique ground states of H_* with a spectral gap of size at least $g > 0$ above its ground state (see (17.4.7)*), the usual LPPL implies defective coercivity, i.e.* (17.4.2) $\implies$ (17.4.9).

(v) A decaying gap size implies defective coercivity, i.e. (17.4.8) $\implies$ (17.4.9).

(vi) Additionally assuming that ρ_* has a LDG (17.4.10) *(e.g. if ρ_* is a normalized projection onto a gapped spectral patch of H_*; see Proposition 17.1.1), the commutator trace norm LPPL implies LDG for ρ, i.e.* (17.4.4) $\implies$ (17.4.10).

(vii) Additionally assuming that $\langle[\mathcal{L}_{H_*} \circ \mathcal{I}_{H_*,g}[\![A]\!] - A, B]\rangle_{\rho_*} = 0$ for all $A, B \in \mathcal{A}$ (e.g. if ρ_* is a normalized projection onto a gapped spectral patch of H_*; see Proposition 17.1.1), an intertwining automorphism implies LDG, i.e. (17.4.5) $\implies$ (17.4.10).

(viii) Assume that $\rho = |\psi\rangle\langle\psi|$ and ψ is a product state. Then, a decaying gap size implies a weak LDG, i.e. (17.4.8) $\implies$ (17.4.12).

(ix) A local dynamical gap implies a weakened local dynamical gap, i.e. (17.4.10) $\implies$ (17.4.12).

17.4.4 Two exemplary systems with a local dynamical gap

In this section, we will discuss two exemplary systems, which we will show to satisfy the local gap condition (17.1.8) by means of Proposition 17.4.1 (vi) and (vii). The first example in Section 17.4.4.1 is concerned with a (local perturbation of a) classical Ising model. It is contained in a whole class of examples considered in Section 17.4.4.2, which are studied based on [35]. We nevertheless discuss it separately, as proving it to satisfy the local dynamical gap condition ($\mathrm{LDG}_{\mathsf{main}}$) is elementary, in particular not relying on [35]. The actual proofs for the two examples are given in Section 17.6.3.

17.4.4.1 Perturbations of the classical Ising model with weak interaction

As the first example we consider the classical Ising model on $\Lambda \subset \mathbf{Z}^d$

$$H_* = \sum_{x\in\Lambda} \sigma_x^3 + \frac{1}{2}\sum_{x,y\in\Lambda} \lambda(x-y)\,\sigma_x^3\sigma_y^3, \tag{17.4.13}$$

where σ_x^i is the i^{th} Pauli matrix σ_i acting only on the spin on site $x \in \Lambda$. More precisely,

$$\sigma_x^i = \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes \underbrace{\sigma^i}_{\text{site } x} \otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1} \in \mathcal{B}\big(\otimes_{x\in\Lambda}\mathcal{H}_x\big), \tag{17.4.14a}$$

where, as usual,

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17.4.14b}$$

The Hamiltonian H_* features a magnetic field of unit strength in 3-direction and (small) symmetric coupling function $\lambda : \mathbf{Z}^d \to \mathbf{R}$ of finite range, i.e. $\|\lambda\|_1 := \sum_x |\lambda(x)| < 2$ and there exists some $R > 0$ such that $\lambda(x) = 0$ for $|x| > R$. In particular, for any $p \in (0,1]$ and $b > 0$, there exists a constant $C_* > 0$, such that $\|\Phi_{H_*}\|_{b,p} \le C_*$, uniformly in Λ, where Φ_{H_*} is the canonical interaction realizing H_*.

As we show in Section 17.6.3.1, **every ground state ρ for every perturbation of the form $H = H_* + J$ has a local dynamical gap** in the sense of Assumption ($\mathrm{LDG}_{\mathrm{main}}$). Here, J is a *strictly* $\Lambda \setminus \Lambda^{\mathrm{gap}}$-localized SLT Hamiltonian, i.e. there exists a constant $C_J > 0$ such that for some $p \in (0,1]$ and $b > 0$, we have $\|\Phi_J\|_{b,p} \le C_J$ for some interaction Φ_J realizing J, for which also $\Phi_J(X) = 0$ whenever $X \cap \Lambda^{\mathrm{gap}} \neq \varnothing$ (recall Lemma 17.2.1 (i)).

The example (17.4.13) can immediately be generalized to an arbitrary graph $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$. Moreover, our above assertions remain valid for any Hamiltonian with gapped on-site terms and sufficiently weak mutually commuting finite range interactions that can be simultaneously diagonalized with the on-site terms (i.e. one has a *classical* system).

17.4.4.2 Perturbations of frustration free product states

As the second basic example, we consider an SLT Hamiltonian H_* of the form

$$H_* = \sum_{Z\subset\Lambda} \Phi(Z), \tag{17.4.15}$$

for which there exist $p \in (0,1]$ and $b > 0$ and a constant $C_* > 0$ such that $\|\Phi\|_{b,p} \le C_*$. Moreover $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ in (17.4.15) is a finite graph as described in Section 17.2. Apart from the locality of the interaction Φ, we will impose the following further conditions (cf. [35]).

(A1) Frustration-free ground state.

All terms in the SLT Hamiltonian (17.4.15) are non-negative, i.e. $\Phi(Z) \ge 0$ for all $Z \subset \Lambda$. There exists a unique (up to a phase) normalized vector $\psi_* \in \mathcal{H}$ such that $\psi_* \in \ker \Phi(Z)$ for all $Z \subset \Lambda$. The corresponding ground state (projection) is denoted by $\rho_* = |\psi_*\rangle\langle\psi_*|$.

We note that the frustration free assumption depends on the explicit way (17.4.15) the Hamiltonian is written, i.e. on the interaction Φ.

(A2) Product property and regularity.

The vector $|\psi_*\rangle \in \mathcal{H}$ factorizes as $|\psi_*\rangle = \otimes_{z\in\Lambda} |\psi_{*,z}\rangle$ and for every $\Omega \subset \Lambda$, the unique ground state vector of

$$H_*\big|_\Omega = \sum_{Z\subset\Omega} \Phi(Z) \tag{17.4.16}$$

is given by $|\psi_*|_\Omega\rangle = \otimes_{z\in\Omega} |\psi_{*,z}\rangle$.

The latter condition can be thought of as a strong variant of the common *local topological quantum order (LTQO)*. Moreover, it is possible to relax both, the product property of $|\psi_*\rangle$ and the strong LTQO condition, in the following way: Instead of the product property, we could only assume that $|\psi_*\rangle$ (possibly upon adjoining an auxiliary state after doubling the Hilbert space, cf. Assumption 4 in [35]) is unitarily conjugate – with SLT-generator – to a product state; instead of the strong LTQO property, we could only assume that, upon adjoining a suitable state $|\psi_*|_{\Lambda\setminus\Omega}\rangle \in \otimes_{z\in\Lambda\setminus\Omega}\mathcal{H}_x$, the unique ground state $|\psi_*|_\Omega\rangle$ of (17.4.16) is unitarily conjugate – with SLT-generator localized at the boundary of Ω – to $|\psi_*\rangle$, i.e. $|\psi_*|_\Omega\rangle \otimes |\psi_*|_{\Lambda\setminus\Omega}\rangle = \mathrm{e}^{\mathrm{i}F}|\psi_*\rangle$, where F is an SLT operator localized at the boundary of Ω. However, we refrain from going into this further generalization of our exemplary system (17.4.15) for simplicity.

The final assumption on (17.4.15) concerns the spectral gaps

$$\gamma := \inf\big(\mathrm{spec}(H_*)\setminus\{0\}\big) \qquad \text{and} \qquad \gamma(\Omega) := \inf\big(\mathrm{spec}(H_*\big|_\Omega)\setminus\{0\}\big)$$

of H_* and its restrictions $H_*\big|_\Omega$ to some $\Omega \subset \Lambda$, respectively.

(A3) Gap condition.

The SLT Hamiltonian H_* from (17.4.15) has a spectral gap, i.e. $\gamma > 0$. Moreover, the gap of restrictions (17.4.16) of H_* to *balls* $\Omega = B_r(x) = \{\, y\in\Lambda : d(x,y)\le r \,\}$ in Λ shrink at most polynomially with the radius, i.e. there exist $C_\gamma, d_\gamma > 0$ such that for every $x\in\Lambda$ it holds that

$$\frac{1}{\gamma(B_r(x))} \le C_\gamma r^{d_\gamma}. \tag{17.4.17}$$

An exemplary system satisfying all of the above assumptions (up to a constant energy shift) is given by the Heisenberg XXZ model for small enough nearest neighbor interactions $|\lambda_1|$, $|\lambda_3|$ (depending on the dimension d and the constant C_{vol}, cf. (17.2.1) and (17.2.2)). The corresponding Hamiltonian is given by

$$H_* = \sum_{x\in\Lambda}\sigma_x^3 + \sum_{(x,y)\in E(\Lambda)} \lambda_1\,\sigma_x^1\sigma_y^1 + \lambda_1\,\sigma_x^2\sigma_y^2 + \lambda_3\,\sigma_x^3\sigma_y^3,$$

where $E(\Lambda)$ denote the edges of Λ and we recall the notations (17.4.14).

Back to the general setting, as we show in Section 17.6.3.2 building on [35], **every ground state ρ for every perturbation of the form $H = H_* + J$ has a local dynamical gap** in a slightly weakened sense of ($\mathrm{LDG}_{\mathrm{main}}$); see Assumption ($\mathrm{LDG}_{\mathrm{weak}}$) in Section 17.5, which is, however, sufficient for obtaining our main result, Theorem 17.3.1. Here, J is a *strongly* $\Lambda\setminus\Lambda^{\mathrm{gap}}$-localized SLT Hamiltonian, which means that there exists a constant $C_J > 0$ such that for some $p\in(0,1]$ and $b > 0$, we have $\|\Phi_J\|_{b,p} \le C_J$ for some interaction Φ_J realizing J, for which also $\Phi_J(Z) = 0$ whenever $Z\cap\Lambda^{\mathrm{gap}} \neq \varnothing$ (recall Lemma 17.2.1 (ii)).

17.5 Construction of the NEASS: Proofs of Propositions 17.3.3–17.3.5

The fundamental conceptual idea behind the proof of Proposition 17.3.4 is a perturbative scheme, which was called *space-time adiabatic perturbation theory* in [484, 483]. Before going into this expansion in Section 17.5.2, we show that a weakened version of our local dynamical gap condition carries over to SLT operators (see Lemma 17.5.1 in Section 17.5.1). The main technical input for carrying out the space-time adiabatic perturbation scheme, is to show that all the operations involved in the expansion preserve localization of SLT operators as required for Lemma 17.5.1 to apply. This is the content of several auxiliary technical results in Appendix 17.A.

17.5.1 Weakened version of the local dynamical gap condition for SLT operators

Throughout the proof, we will work under the following weakened version of the original local dynamical gap condition (see Assumptions ($\mathrm{LDG}_{\mathrm{intro}}$) and ($\mathrm{LDG}_{\mathrm{main}}$)).

($\mathrm{LDG}_{\mathrm{weak}}$) Local dynamical gap condition – weakened version.

We say that an equilibrium state ρ_0 of an SLT operator H_0, i.e. with $[H_0, \rho_0] = 0$, is *weakly locally dynamically gapped* of size at least $g > 0$ in a region $\Lambda^{\mathrm{gap}} \subset \Lambda$ with respect to C_{gap}, b, p, $\beta > 0$ and $\ell \in \mathbf{N}_0$, if and only if the following holds: For all $X \subset \Lambda$ satisfying $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$ and observables $A \in \mathcal{A}_X$, and $Y \subset \Lambda$ and observables $B \in \mathcal{A}_Y$, it holds that

$$\Big|\big\langle [\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![A]\!] - A, B]\big\rangle_{\rho_0}\Big| \leq C_{\mathrm{gap}}\, \|A\|\, \|B\|\, \mathrm{diam}(Y)^\ell\, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big). \tag{17.5.1}$$

As mentioned above, we start with the following basic lemma which will heavily be used in our proof. It says that the local dynamical gap condition naturally carries over to SLT operators.

Lemma 17.5.1. *Let $b > 0$, $p \in (0,1)$ and H_0 be a $\chi_{b,p}$-SLT operator. Assume that the equilibrium state ρ_0 of H_0 satisfies the weakened local gap condition, Assumption ($\mathrm{LDG}_{\mathrm{weak}}$) above, with gap size at least $g > 0$ and with respect to C_{gap}, b, p, β and ℓ.*

Then, there exists a constant C, such that for any (b, p, Ω)-localized SLT operator A and observable $B \in \mathcal{A}_Y$, we have that

$$\Big|\big\langle [\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![A]\!] - A, B]\big\rangle_{\rho_0}\Big| \leq C\, \mathrm{diam}(Y)^{\ell+d}\, \|B\|\, \|\Phi_A\|_{b,p;\Omega}\, \chi_{b/2,p\min\{\beta,1\}}\big(d(\Omega, \Lambda \setminus \Lambda^{\mathrm{gap}})\big). \tag{17.5.2}$$

The proof of Lemma 17.5.1 is presented in Appendix 17.A.5. The principal idea is to write $A = \sum_{Z \subset \Lambda} \Phi_A(Z)$ and then estimate only the contribution of 'small' Z's (i.e. those with $\mathrm{diam}(Z) \leq d(Z, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$) by (17.5.1). Large Z's (i.e. those with $\mathrm{diam}(Z) > d(Z, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$) are treated using Lieb-Robinson bounds and the smallness of $\|\Phi_A(Z)\|$ (by definition (17.2.7)).

17.5.2 The adiabatic perturbation scheme

For the proof of Proposition 17.3.4 we use the same strategy as in [556]. However, since we only have a local gap, the lower order terms do not vanish exactly, but can be bounded using Assumption ($\mathrm{LDG}_{\mathrm{weak}}$).

The statements in Propositions 17.3.4, will be deduced from a time-dependent NEASS, which is part of the next Theorem. In contrast to the previous works, it will not include a time-dependent unperturbed Hamiltonian $H_0(t)$, because there is no spectral flow available. That means, we construct a time-dependent NEASS $\Pi_n^\varepsilon(t)$ specifically for the switching Hamiltonian given in (17.3.2). For times $t \geq 0$ it will turn out to be time-independent.

In order to formulate the result, we introduce time-dependent interactions

$$\Phi\colon I \times \{X \subset \Lambda\} \to \mathcal{A}^N, \quad (t, X) \mapsto \Phi(t, X) = \Phi(t, X)^* \in \mathcal{A}_X$$

for $I \subset \mathbf{R}$. We will assume that $t \mapsto \Phi(t, X)$ is smooth for every $X \subset \Lambda$ and we denote the term-wise time derivatives by $\Phi^{(k)}$, i.e. $\Phi^{(k)}(t, X) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Phi(t, X)$ for every $X \subset \Lambda$. Moreover, we identify $\Phi(t, X) = \big(\Phi(t)\big)(X)$, such that for every fixed $t \in I$, $\Phi(t)$ can be viewed as a time-independent interaction. The notion of SLT operators naturally translates to the time-dependent setting.

Theorem 17.5.2 (Time-dependent NEASS). *Fix $n \in \mathbf{N}$ and let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$, $C_{\mathrm{int}} > 0$ and $g > 0$, $C_{\mathrm{gap}} > 0$, $\beta > 0$, $\ell \in \mathbf{N}_0$. Take any $q \in (0, p\min\{1, \beta\})$. Then there exist a constant $C_n > 0$ (in particular depending on n) such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ (recall (17.2.2)), subsets*

$\Lambda^{\mathrm{pert}} \subset \Lambda$, *intervals* $I \subset \mathbf{R}$ *and SLT-operators* H_0 *and* $V(t)$, *with corresponding interactions that satisfy* $\|\Phi_{H_0}\|_{b,p} < C_{\mathrm{int}}$ *and* $\sup_{t\in I} \left\|\Phi_V^{(k)}(t)\right\|_{b,p;\Lambda^{\mathrm{pert}}} < C_{\mathrm{int}}$ *for all* $k \le n$, *respectively, the following holds:*

Assume that the equilibrium state ρ_0 *of* H_0 *is locally dynamically gapped in* Λ^{gap} *of size at least* $g > 0$ *and with respect to* C_{gap}, b, p *and* ℓ *according to Assumption* (LDG$_{\mathsf{weak}}$). *And let*

$$H_\varepsilon(t) := H_0 + \varepsilon V(t) \tag{17.5.3}$$

be the perturbed Hamiltonian.

Then, there exists a sequence $(A_\mu)_{\mu\in\mathbf{N}}$ *of time-dependent SLT operators, which are* $(1, p', \Lambda^{\mathrm{pert}})$*-localized for any* $p' < p$, *such that the state*

$$\Pi_n^{\varepsilon,\eta}(t) := \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(t)}\, \rho_0\, \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(t)} \quad \textit{with} \quad S_n^{\varepsilon,\eta}(t) := \sum_{\mu=1}^{n} \varepsilon^\mu A_\mu(t), \tag{17.5.4}$$

is almost-stationary for the dynamics generated by $H_\varepsilon(t)$ *in the following sense: Let* $t_0 \in \mathbf{R}$ *and let* $\rho^{\varepsilon,\eta,f}(t)$ *be the solution of the time-dependent adiabatic Schrödinger equation*

$$\mathrm{i}\eta \frac{\mathrm{d}}{\mathrm{d}t}\rho^{\varepsilon,\eta,f}(t) = \left[H_\varepsilon(t), \rho^{\varepsilon,\eta,f}(t)\right] \quad \textit{with} \quad \rho^{\varepsilon,\eta,f}(t_0) = \Pi_n^{\varepsilon,\eta}(t_0) \tag{17.5.5}$$

with adiabatic parameter $\eta \in (0, 1]$.

Under these conditions, for all $B \in \mathcal{A}_Y$ *with* $Y \subset \Lambda$ *and* $t \in \mathbf{R}$, *it holds that*

$$\begin{aligned}\left|\langle B\rangle_{\rho_{t_0}^{\varepsilon,\eta}(t)} - \langle B\rangle_{\Pi_n^{\varepsilon,\eta}(t)}\right| \le C_n\, \|B\|\, \mathrm{diam}(Y)^{3d+\ell} \left(1 + \frac{\eta^{n+1}}{\varepsilon^{n+1}}\right)\varepsilon^{n+1} \\ \times \frac{|t-t_0|}{\eta}\left(1 + \frac{v\,|t-t_0|}{\eta}\right)^{(\ell+2d)/p} \\ \times \left(1 + \mathrm{e}^{-d(\Lambda^{\mathrm{pert}}, \Lambda\setminus\Lambda^{\mathrm{gap}})^q - (n+1)\log(\varepsilon)}\right).\end{aligned} \tag{17.5.6}$$

Moreover, the operators $A_\mu(t)$ *at time* t *depend only on* V *and its first* μ *derivatives at time* t.

Before we prove Theorem 17.5.2, let us deduce the results from Section 17.3, which will follow by taking $\beta = 1$ in Theorem 17.5.2.

Proof of Proposition 17.3.3. We choose $V(t) = V$, which implies that also all A_μ are time-independent (i.e. the time-dependent interaction is constant). Moreover, since there is no adiabatic timescale in (17.3.7), we choose $\eta = 1$. To obtain the correct scaling, we inspect the proof of Theorem 17.5.2: In equation (17.5.9), we expand in powers of ε and $R_j^{\varepsilon,\eta}$ are polynomials in η/ε. Here, since $\dot S_n^{\varepsilon,\eta} = 0$, these polynomials are just constants and there is no η in (17.5.9) or any of the later expressions. Hence, also the norm estimates $\left\|\Phi_{\tilde R_j^{\varepsilon,\eta}(s)}\right\| < C\,(1 + \eta^j/\varepsilon^j)$ used in the end of the proof simplify to $\left\|\Phi_{\tilde R_j^{\varepsilon}}\right\| < C$ uniformly in ε. Hence, $(1+\eta^{n+1}/\varepsilon^{n+1})$ is replaced by 1 in (17.5.6). All other η in (17.5.6) come from the adiabatic timescale and are thus replaced by 1. □

Proof of Proposition 17.3.4. We choose $V(t) = f(t)\,V$ and since $V(t)$ constant for $t \le 0$, we obtain $\Pi_n^{\varepsilon,\eta}(t) = \Pi_n^{\varepsilon}$ for all $t \ge 0$. To compare with the solution of (17.3.3), we choose $t_0 = -1$ such that $\Pi_n^{\varepsilon,\eta}(t_0) = \rho_0$. Then $|t - t_0| = 1 + t$ and (17.5.6) gives the statement. □

Proof of Proposition 17.3.5. To prove the asymptotic expansion, we first expand (17.3.6) and obtain

$$\operatorname{tr}\big(\Pi_n^\varepsilon B\big) = \operatorname{tr}\big(\rho_0\, \mathrm{e}^{\varepsilon \mathcal{L}_{S_n^\varepsilon}} [\![B]\!]\big) = \sum_{k=0}^{m} \frac{1}{k!} \big\langle \mathcal{L}_{S_n^\varepsilon}^k [\![B]\!] \big\rangle_{\rho_0} + \frac{1}{(m+1)!} \operatorname{tr}\big(\rho_0\, \mathrm{e}^{\tilde\varepsilon \mathcal{L}_{S_n^\varepsilon}} \circ \mathcal{L}_{S_n^\varepsilon}^{m+1} [\![B]\!]\big)$$

for some $\tilde\varepsilon \in [0,\varepsilon]$. Since the A_μ and thus also the S_n^ε are Λ^{pert}-localized, we use Lemma 17.A.4 to bound the remainder by

$$\frac{1}{(m+1)!} \big\| \mathcal{L}_{S_n^\varepsilon}^{m+1} [\![B]\!] \big\| \le C\, \|B\|\, |Y|^{m+1}\, \chi_{b',p}\big(d(Y,\Lambda^{\mathrm{pert}})\big)\, \|S_n^\varepsilon\|_{b',p,\Lambda^{\mathrm{pert}}}^{m+1},$$

where $\|S_n^\varepsilon\|_{b',p;\Lambda^{\mathrm{pert}}} \le \varepsilon \tilde C$. It is clear from the proof of Theorem 17.5.2, that $\tilde C$ depends only on $n, d, C_{\mathrm{vol}}, b, p, C_{\mathrm{int}}$, and g. We now expand S_n^ε in the first term and group the terms according to the powers in ε. The zero order term clearly is $\langle B\rangle_{\rho_0}$. In first order, we obtain $\varepsilon \langle \mathcal{L}_{A_1}\rangle_{\rho_0} = -\varepsilon\, \mathrm{i} \big\langle \big[\mathcal{I}_{H_0,g}[\![V]\!], B\big]\big\rangle_{\rho_0}$ as can be read of from (17.5.12). All $\mathcal{K}_j$ for $j \le m$ are constructed in this way. In the end, some higher order terms are left. They all come with multi-commutators of Λ^{pert}-localized A_μ with B and can be bounded as the remainder above using Lemma 17.A.4. □

We now prove the time-dependent NEASS from Theorem 17.5.2.

Proof of Theorem 17.5.2. For the proof we first assume the form of (17.5.4) and then iteratively choose $A_\mu(t)$ so that the statement holds. Therefore, let $U_{t,t_0}^{\varepsilon,\eta}$ be the solution of

$$\mathrm{i}\,\eta \frac{\mathrm{d}}{\mathrm{d}t} U_{t,t_0}^{\varepsilon,\eta} = H_\varepsilon(t)\, U_{t,t_0}^{\varepsilon,\eta} \qquad \text{with} \qquad U_{t_0,t_0}^{\varepsilon,\eta} = \mathbf{1} \quad \text{for all} \quad t, t_0 \in I, \tag{17.5.7}$$

with H_ε given in (17.3.2). Then, $\rho_{t_0}^{\varepsilon,\eta}(t) := U_{t,t_0}^{\varepsilon,\eta}\, \Pi_n^{\varepsilon,\eta}(t_0)\, U_{t_0,t}^{\varepsilon,\eta}$ is the solution of (17.5.5). To obtain (17.5.6) we use the fundamental theorem of calculus, and get

$$\langle B\rangle_{\rho_{t_0}^{\varepsilon,\eta}(t)} - \langle B\rangle_{\Pi_n^{\varepsilon,\eta}(t)} = -\int_{t_0}^{t} \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s} \Big\langle \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)}\, U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} \Big\rangle_{\rho_0}. \tag{17.5.8}$$

By product rule and Duhamel's formula, the derivative evaluates as

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)}\, U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} = -\frac{\mathrm{i}}{\eta} \Big[Q_n^{\varepsilon,\eta}(s), \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)}\, U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} \Big],$$

where

$$\begin{aligned} Q_n^{\varepsilon,\eta}(s) &= \eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{-\mathrm{i}\lambda S_n^{\varepsilon,\eta}(s)}\, \dot S_n^{\varepsilon,\eta}(s)\, \mathrm{e}^{\mathrm{i}\lambda S_n^{\varepsilon,\eta}(s)} + \mathrm{e}^{-\mathrm{i}S_n^{\varepsilon,\eta}(s)} \big(H_0 + \varepsilon V(s)\big)\, \mathrm{e}^{\mathrm{i}S_n^{\varepsilon,\eta}(s)} \\ &= \eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\lambda \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} [\![\dot S_n^{\varepsilon,\eta}(s)]\!] + \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} [\![H_0 + \varepsilon V(s)]\!] \\ &= H_0 + \sum_{j=1}^{n} \varepsilon^j R_j^{\varepsilon,\eta}(s) + \varepsilon^{n+1} R_{n+1}^{\varepsilon,\eta}(s). \end{aligned} \tag{17.5.9}$$

In the last line we expanded in powers of ε and η such that $R_j^{\varepsilon,\eta}(s)$ are polynomials in η/ε of order j with ε- and η-independent SLT operators as coefficients. In this way, the joint power of ε and η in front of the SLT operators collected in $R_j^{\varepsilon,\eta}$ is j. By Taylor formula with mean-value form of the remainder, there exist $\theta \in [0,1]$ such that

$$\begin{aligned} \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} [\![H_0 + \varepsilon V(s)]\!] = \sum_{k=0}^{n} \frac{1}{k!} \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}^k [\![H_0 + \varepsilon V(s)]\!] \\ + \frac{1}{(n+1)!}\, \mathrm{e}^{\theta \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}^{n+1} [\![H_0 + \varepsilon V(s)]\!]. \end{aligned} \tag{17.5.10}$$

Similarly, for the first term in (17.5.9) we expand the integrand using the integral form of the remainder and obtain

$$\begin{aligned}
&\eta \int_0^1 \mathrm{d}\lambda\, \mathrm{e}^{\lambda \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] \\
&\quad = \eta \sum_{k=0}^{n-2} \frac{\int_0^1 \mathrm{d}\lambda\, \lambda^k}{k!} \mathcal{L}^k_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] \\
&\qquad + \eta \int_0^1 \mathrm{d}\lambda \int_0^\lambda \mathrm{d}\mu\, \frac{(\lambda-\mu)^{n-2}}{(n-2)!} \mathrm{e}^{\mu \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \mathcal{L}^{n-1}_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!], \\
&\quad = \eta \sum_{k=0}^{n-2} \frac{1}{(k+1)!} \mathcal{L}^k_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!] \\
&\qquad + \eta \int_0^1 \mathrm{d}\mu\, \frac{(1-\mu)^{n-1}}{(n-1)!} \mathrm{e}^{\mu \mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \mathcal{L}^{n-1}_{S_n^{\varepsilon,\eta}(s)} [\![\dot{S}_n^{\varepsilon,\eta}(s)]\!],
\end{aligned} \tag{17.5.11}$$

From this expansion and (17.5.4), we can read of

$$R_1^{\varepsilon,\eta}(s) = -\mathcal{L}_{H_0} [\![A_1^{\varepsilon,\eta}(s)]\!] + V(s), \tag{17.5.12}$$

$$R_2^{\varepsilon,\eta}(s) = -\mathcal{L}_{H_0} [\![A_2^{\varepsilon,\eta}(s)]\!] + \mathcal{L}_{A_1^{\varepsilon,\eta}(s)} [\![V(s)]\!] + \tfrac{\eta}{\varepsilon} \dot{A}_1^{\varepsilon,\eta}(s), \tag{17.5.13}$$

and more generally

$$R_j^{\varepsilon,\eta}(s) = -\mathcal{L}_{H_0} [\![A_j^{\varepsilon,\eta}(s)]\!] + \tilde{R}_j^{\varepsilon,\eta}(s), \tag{17.5.14}$$

where the $\tilde{R}_j^{\varepsilon,\eta}(s)$ are sums of iterated commutators of the operators $A_i^{\varepsilon,\eta}(s)$ and $\dot{A}_i^{\varepsilon,\eta}(s)$ for $i < j \leq n$ and $V(s)$. We can now iteratively choose

$$A_j^{\varepsilon,\eta}(s) = \mathcal{I}_{H_0,g} [\![\tilde{R}_j^{\varepsilon,\eta}(s)]\!].$$

Clearly, for all $p' < p$, it holds that $\tilde{R}_1^{\varepsilon,\eta}(s) = V(s)$ is a $(b, p', \Lambda^{\mathrm{pert}})$-localized SLT operator. Hence, by Lemma 17.A.11, $A_1^{\varepsilon,\eta}(s)$ is $(b', p', \Lambda^{\mathrm{pert}})$-localized for any $b' < b$. This step only works for $p' < p$, because Lemma 17.A.11 requires a slightly better localization of the Hamiltonian H_0 compared to the argument of the inverse Liouvillian. Finally, by Lemma 17.A.5 also $\mathcal{L}_{H_0} [\![A_1^{\varepsilon,\eta}(s)]\!]$ and thus $R_1^{\varepsilon,\eta}(s)$ are $(b', p', \Lambda^{\mathrm{pert}})$-localized SLT operator for any slightly smaller b'. The same arguments hold for the higher orders $R_j^{\varepsilon,\eta}(s)$ with $j \leq n$ as well, at each step lowering b'. We point out that clearly the smallest b' can be chosen independently of n by choosing all intermediate b' in an n-dependent equidistant way.

With this expansion it is also immediate, that the $A_\mu(s)$ only depend on V and its derivatives at time s.

Putting everything back together and denoting $\tau_{t,s} [\![B]\!] := U_{s,t}^{\varepsilon,\eta}\, B\, U_{t,s}^{\varepsilon,\eta}$, we find

$$\begin{aligned}
&\left| \langle B \rangle_{\rho_{t_0}^{\varepsilon,\eta}(t)} - \langle B \rangle_{\Pi_n^{\varepsilon,\eta}(t)} \right| \\
&\quad \leq \frac{|t-t_0|}{\eta} \sup_{s \in [t_0,t]} \left(\begin{aligned} &\sum_{j=1}^n \varepsilon^j \left| \left\langle \left[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g} [\![\tilde{R}_j^{\varepsilon,\eta}(s)]\!] - \tilde{R}_j^{\varepsilon,\eta}(s), \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s} [\![B]\!] \right] \right\rangle_{\rho_0} \right| \\ &+ \varepsilon^{n+1} \left| \left\langle \left[R_{n+1}^{\varepsilon,\eta}(s), \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s} [\![B]\!] \right] \right\rangle_{\rho_0} \right|. \end{aligned} \right)
\end{aligned} \tag{17.5.15}$$

The remainder $R_{n+1}^{\varepsilon,\eta}(s)$ collects all the remaining terms which are (a) the higher order terms from the first lines and (b) the remainder terms from the second lines of (17.5.10) and (17.5.11). The former are local by the previous arguments. The latter additionally include an evolution by the local operator $S_n^{\varepsilon,\eta}(s)$ and are local by Lemma 17.A.8.

To apply the local gap assumption in the form given in Lemma 17.5.1 to the lower order terms, we need to decompose the second entry of the commutator into strictly local operators. Therefore, we use the same decomposition as in the proof of Lemma 17.A.7

$$\tau_{t,s}[\![B]\!] = \sum_{k=0}^{\infty} \Delta_k,$$

where

$$\begin{aligned} &\Delta_0 \in \mathcal{A}_{Y_\delta} \subset \mathcal{A}_{Y_{\delta^{1/p}}}, \quad \|\Delta_0\| \leq \|B\|, \\ \text{and for } k \in \mathbf{N} \qquad &\Delta_k \in \mathcal{A}_{Y_{(\delta+k)^{1/p}}}, \qquad \|\Delta_k\| \leq C_{\mathrm{LR}}\, \mathbf{E}\, \|B\|\, |Y|\, \mathrm{e}^{-bk}. \end{aligned}$$

For better readability we abbreviate $\delta = 1+v\,|t-s|/\eta$ here. The extra $1/\eta$ is due to the scaling in (17.5.7). For the outer automorphism, we can use the decomposition from the proof of Lemma 17.A.8, just replace $O = \Delta_k$ there, to obtain, for each k, the decomposition

$$\mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}}[\![\Delta_k]\!] = \sum_{l=0}^{\infty} \Delta_{k,l}$$

with

$$\begin{aligned} &\Delta_{k,0} \in \mathcal{A}_{Y_{(\delta+k)^{1/p}}}, \quad \big\|\Delta_{k,0}\big\| \leq \|\Delta_k\|, \quad \text{and} \\ &\Delta_{k,l} \in \mathcal{A}_{Y_{l+(\delta+k)^{1/p}}}, \quad \big\|\Delta_{k,l}\big\| \leq C_{\mathrm{LR}}\, \|\Delta_k\|\, \big|Y_{(\delta+k)^{1/p}}\big| \big(\mathrm{e}^{c\,\|\Phi_S\|_{d,q}} - 1\big)\, \chi_{d',j}(l) \end{aligned}$$

for $l \in \mathbf{N}$. Thus, in total we have

$$\mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s}[\![B]\!] = \sum_{k,l=0}^{\infty} \Delta_{k,l},$$

where the sum is actually finite on finite lattices as discussed in Lemmata 17.A.7 and 17.A.8. Then we use the triangle inequality and Lemma 17.5.1 to apply (17.5.2) and bound

$$\begin{aligned} &\Big|\Big\langle\Big[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![\tilde{R}_j^{\varepsilon,\eta}(s)]\!] - \tilde{R}_j^{\varepsilon,\eta}(s), \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s}[\![B]\!]\Big]\Big\rangle_{\rho_0}\Big| \\ &\quad \leq C\, \Big\|\Phi_{\tilde{R}_j^{\varepsilon,\eta}(s)}\Big\|_{b',p';\Lambda^{\mathrm{pert}}} \sum_{k,l=0}^{\infty} \mathrm{diam}(Y_{l+(\delta+k)^{1/p}})^{\ell+d}\, \big\|\Delta_{k,l}\big\|\, \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \\ &\quad \leq C\, \Big\|\Phi_{\tilde{R}_j^{\varepsilon,\eta}(s)}\Big\|_{b',p';\Lambda^{\mathrm{pert}}} \sum_{k=0}^{\infty} \mathrm{diam}(Y_{(\delta+k)^{1/p}})^{\ell+2d}\, \|\Delta_k\| \\ &\qquad \times \Bigg(1 + \big(\mathrm{e}^{c\,\|\Phi_S\|_{d,q}} - 1\big) \sum_{l=1}^{\infty} l^{\ell+d}\, \chi_{d',j}(l)\Bigg) \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \\ &\quad \leq C\, \Big\|\Phi_{\tilde{R}_j^{\varepsilon,\eta}(s)}\Big\|_{b',p';\Lambda^{\mathrm{pert}}}\, \mathrm{e}^{c\,\|\Phi_S\|_{d,q}}\, \mathrm{diam}(Y)^{\ell+3d}\, \|B\|\, \delta^{(\ell+2d)/p} \\ &\qquad \times \sum_{k=0}^{\infty} k^{(\ell+2d)/p}\, \mathrm{e}^{-bk}\, \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \\ &\quad \leq C\, \Big\|\Phi_{\tilde{R}_j^{\varepsilon,\eta}(s)}\Big\|_{b',p';\Lambda^{\mathrm{pert}}}\, \mathrm{e}^{c\,\|\Phi_S\|_{d,q}}\, \mathrm{diam}(Y)^{\ell+3d}\, \|B\|\, \delta^{(\ell+2d)/p}\, \chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \end{aligned}$$

where we abbreviated $q' := p' \min\{\beta, 1\}$ (according to Lemma 17.5.1) and b' is the smallest of the b' such that all $\Phi_{\tilde{R}_j^{\varepsilon,\eta}}(s)$ are $(b', p', \Lambda^{\mathrm{pert}})$-localized. Finally, for any $q < q'$ we bound

$$\chi_{b'/2,q'}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \leq C\, \chi_{1,q}\big(d(\Lambda^{\mathrm{pert}}, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \tag{17.5.16}$$

for some constant $C > 0$ depending only on b', q', and q. Since $p' < p$ was arbitrary, this q can be chosen arbitrarily in the interval $(0, p \min\{\beta, 1\})$.

With the same local decomposition argument as above and using Lemma 17.A.4 we can also bound the remainder

$$\begin{aligned}&\left|\left\langle\left[R_{n+1}^{\varepsilon,\eta}(s), \mathrm{e}^{\mathcal{L}_{S_n^{\varepsilon,\eta}(s)}} \circ \tau_{t,s}[\![B]\!]\right]\right\rangle_{\rho_0}\right| \\ &\qquad \leq C \left\|\Phi_{\tilde{R}_{n+1}^{\varepsilon,\eta}}(s)\right\|_{b',p';\Lambda^{\mathrm{pert}}} \mathrm{e}^{c\,\|\Phi_S\|_{d,q}}\, \mathrm{diam}(Y)^{3d}\, \|B\|\, \delta^{2d/p}.\end{aligned}$$

We recall, that like the $R_j^{\varepsilon,\eta}(s)$, also the $\tilde{R}_j^{\varepsilon,\eta}(s)$ are polynomials of order j in η/ε with Λ^{pert}-localized SLT operators as coefficients. Hence, the interaction norms in the above equations can be bounded by a constant depending on the interaction norms of H and $V(t)$ times $1 + (\eta/\varepsilon)^j$. With this observation, we can insert these bounds into (17.5.15) and conclude (17.5.6).

To reduce the number of constants in the formulation of the statement, we actually do the proof for $(p+p')/2$ instead of p'. In the end we then estimate $\chi_{b',(p+p')/2} \leq C\,\chi_{1,p'}$ for the locality of the operators $A_\mu(t)$, similarly to (17.5.16). □

17.6 Local gap conditions: Proofs for Proposition 17.1.1 and Section 17.4

This section collects several proofs concerning the local dynamical gap condition ($\mathrm{LDG}_{\mathsf{intro}}$) (or ($\mathrm{LDG}_{\mathsf{main}}$) for the formal version), which were skipped in earlier sections. More precisely, we will prove the dynamical characterization of a spectral gap in Proposition 17.1.1 and the relations among the various local gap conditions formulated in Proposition 17.4.1 in Section 17.4.3. Finally, we show that the examples from Section 17.4.4 satisfy ($\mathrm{LDG}_{\mathsf{main}}$) by means of Proposition 17.4.1.

17.6.1 Dynamical characterization of a spectral gap: Proof of Proposition 17.1.1

Deriving the lhs. from the rhs. is standard material; see, e.g., [461, Lemma 6.8 and Proposition 6.9]. Therefore, dropping the subscripts H and g for ease of notation, suppose that for all $A, B \in \mathcal{B}(\mathcal{H})$ we have (recall (17.4.11))

$$0 = \left\langle\left[\mathcal{L} \circ \mathcal{I}[\![A]\!] - A, B\right]\right\rangle_P = \left\langle\left[\mathcal{J}[\![A]\!], B\right]\right\rangle_P = -\,\mathrm{tr}\big(\left[\mathcal{J}[\![A]\!], P\right] B\big).$$

Since B is arbitrary, this means $[\mathcal{J}[\![A]\!], P] = 0$. Moreover, inserting the spectral decomposition $H = \sum_n E_n\, P_n$, this can be written as

$$0 = \left[\mathcal{J}[\![A]\!], P\right] = \sqrt{2\pi}\left(\sum_{\substack{E_n\in\sigma_2\\E_m\in\sigma_1}} \widehat{w}(E_m - E_n)\, P_n\, A\, P_m - \sum_{\substack{E_n\in\sigma_1\\E_m\in\sigma_2}} \widehat{w}(E_m - E_n)\, P_n\, A\, P_m\right). \tag{17.6.1}$$

For *contradiction*, we now assume that $d(\sigma_1, \sigma_2) < g$. Then, since $\widehat{w}|_{(-g,g)} > 0$, one can easily construct an observable A, which violates (17.6.1), e.g. $A = P_{n_*} P_{m_*}$ with $E_{n_*} \in \sigma$, $E_{m_*} \in \sigma_2$ satisfying $|E_{n_*} - E_{m_*}| = d(\sigma_1, \sigma_2) < g$.

17.6.2 Relations among local gap conditions: Proof of Proposition 17.4.1

We prove the seven implications gathered in Proposition 17.4.1 one by one. Unless differently stated, we will use the constants C, ℓ, b and p from the formulation of Proposition 17.4.1 generically, i.e. their precise value might change from line to line. Some technical details are kept brief in this section, more detailed proofs using similar arguments are given in Appendix 17.A.

Proofs of (i)–(iii): All of these are obvious, by application of the estimates

$$\begin{aligned}\|[\rho-\rho_*,A]\|_{\mathrm{tr}} &\leq \|(\rho-\rho_*)\,A\|_{\mathrm{tr}} + \|A\,(\rho-\rho_*)\|_{\mathrm{tr}}\,,\\ |\mathrm{tr}((\rho-\rho_*)A)| &\leq \|(\rho-\rho_*)A\|_{\mathrm{tr}}\,,\end{aligned}$$

and

$$\big|\mathrm{tr}\big(\rho_*\big(\tau-\mathbf{1}\big)[\![A]\!]\big)\big| \leq \big\|\big(\tau-\mathbf{1}\big)[\![A]\!]\big\|,$$

for (i), (ii), and (iii), respectively.

Proof of (iv): By assumption, it holds that

$$\mathrm{i}\,\langle A^*\mathcal{L}_{H_*}[\![A]\!]\rangle_{\rho_*} \geq g\left[\langle A^*A\rangle_{\rho_*} - \big|\langle A\rangle_{\rho_*}\big|^2\right] \tag{17.6.2}$$

for all observables $A\in\mathcal{A}$, where $\mathcal{L}_{H_*}[\![\,\cdot\,]\!] := -\mathrm{i}\,[H_*,\,\cdot\,]$ denotes the Liouvillian of H_*. The idea is now to replace $H_*\to H$ and $\rho_*\to\rho$ in (17.6.2) and estimate the resulting error in such a way that we arrive at (17.4.9).

First, by application of (17.4.2), we replace $[\langle A^*A\rangle_{\rho_*} - |\langle A\rangle_{\rho_*}|^2]$ by $[\langle A^*A\rangle_{\rho} - |\langle A\rangle_{\rho}|^2]$ on the rhs. of (17.6.2) at the cost of an error bounded by $C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)$.

For the lhs. of (17.6.2), we estimate

$$\begin{aligned}&|\langle A^*\mathcal{L}_{H_*}[\![A]\!]\rangle_{\rho_*} - \langle A^*\mathcal{L}_{H}[\![A]\!]\rangle_{\rho}|\\ &\qquad\leq \Big|\big\langle A^*\big(\mathcal{L}_{H_*}[\![A]\!]-\mathcal{L}_{H}[\![A]\!]\big)\big\rangle_{\rho_*}\Big| + \big|\mathrm{tr}\big((\rho_*-\rho)A^*\mathcal{L}_H[\![A]\!]\big)\big|\end{aligned} \tag{17.6.3}$$

by means of the triangle inequality. The first term on the rhs. of (17.6.3) can now be bounded as (recall that $H=H_*+J$, J is $\Lambda\setminus\Lambda^{\mathrm{gap}}$-localized, and $A\in\mathcal{A}_X$)

$$\Big|\big\langle A^*\big(\mathcal{L}_{H_*}[\![A]\!]-\mathcal{L}_{H}[\![A]\!]\big)\big\rangle_{\rho_*}\Big| \leq 2\,\|\Phi_J\|_{b,p;\Lambda\setminus\Lambda^{\mathrm{gap}}}\,\|A\|^2\,\mathrm{diam}(X)^d\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big) \tag{17.6.4}$$

by application of (17.A.2) in Lemma 17.A.4.

For the second term on the rhs. of (17.6.3), we write $\mathcal{L}_H[\![A]\!] = \mathbb{E}_{X_n}\mathcal{L}_H[\![A]\!] + (\mathbf{1}-\mathbb{E}_{X_n})\mathcal{L}_H[\![A]\!]$ for some n to be chosen below, where $X_n := \{\,x\in\Lambda : d(x,X)\leq n\,\}$ denotes the n-fattening of the set $X\subset\Lambda$. We now estimate the two terms separately. For the first term, we employ (17.4.2) to bound

$$\big|\mathrm{tr}\big((\rho_*-\rho)A^*\mathbb{E}_{X_n}\mathcal{L}_H[\![A]\!]\big)\big| \leq C\,\|A\|^2\,\big(\mathrm{diam}(X)+n\big)^\ell\chi_{b,p}\big(d(X_n,\Lambda\setminus\Lambda^{\mathrm{gap}})\big) \tag{17.6.5}$$

where we used that $\mathbb{E}_{X_n}\mathcal{L}_H[\![A]\!]\in\mathcal{A}_{X_n}$ (by definition) and $\|\mathbb{E}_{X_n}\mathcal{L}_H[\![A]\!]\|\leq C|X|\,\|A\|$ (by (17.A.2) from Lemma 17.A.4 and Lemma 17.A.6 (c)). For the second term, we use (17.A.3) from Lemma 17.A.4 and Lemma 17.A.6 (e) for estimating the difference $\|(\mathbf{1}-\mathbb{E}_{X_n})\mathcal{L}_H[\![A]\!]\|$ to get

$$\big|\mathrm{tr}\big((\rho_*-\rho)A^*(\mathbf{1}-\mathbb{E}_{X_n})\mathcal{L}_H[\![A]\!]\big)\big| \leq C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b,p}(n) \tag{17.6.6}$$

Using $d(X_n,\Lambda\setminus\Lambda^{\mathrm{gap}})\geq d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})-n$ for (17.6.5), we can pick $n=d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})/2$, say, to estimate

$$(17.6.5)+(17.6.6) \leq C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b/2,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big).$$

Combining this with (17.6.4), we estimate (17.6.3) by $C\,\|A\|^2\,\mathrm{diam}(X)^\ell\,\chi_{b,p}\big(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)$ and we thus arrive at (17.4.9).

Proof of (v): This is obvious, because $\big|\langle A^*A\rangle_\rho - \big|\langle A\rangle_\rho\big|^2\big| \leq 2\,\|A\|^2$.

Proof of (vi): By assumption (17.4.10), it holds that, for all observables $A\in\mathcal{A}_X$ and $B\in\mathcal{A}_Y$ localized in $X\subset\Lambda$ and $Y\subset\Lambda$, it holds that (recall (17.4.11) for the definition of $\mathcal{J}$)

$$\begin{aligned}\Big|\big\langle[\mathcal{J}_{H_*}[\![A]\!],B]\big\rangle_{\rho_*}\Big| &\leq C\,\|A\|\,\|B\|\,\big[\,\mathrm{diam}(X)+\mathrm{diam}(Y)\big]^\ell\\ &\qquad\times\chi_{b,p}(d(X,\Lambda\setminus\Lambda^{\mathrm{gap}})+d(Y,\Lambda\setminus\Lambda^{\mathrm{gap}}))\,.\end{aligned} \tag{17.6.7}$$

Similarly to the proof of (iv), the idea is now to replace $H_* \to H$ and $\rho_* \to \rho$ on the lhs. (17.6.7) at the price of an error that is bounded in terms of the rhs. of (17.6.7). In order to do so, we will heavily exploit the symmetry of (17.6.7) in A and B (recall the discussion around (17.4.11)). That is, we will prove the bound first only with $d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})$ in the argument of $\chi_{b,p}$ and later obtain their sum (like on the rhs. of (17.6.7)) by symmetry (modulo changing $b \to b/2$).

To begin with, by the triangle inequality, we have

$$\begin{aligned} &\Big|\big\langle[\mathcal{J}_{H_*}[\![A]\!], B]\big\rangle_{\rho_*} - \big\langle[\mathcal{J}_H[\![A]\!], B]\big\rangle_{\rho}\Big| \\ &\qquad \le \Big|\big\langle\big[\big(\mathcal{J}_{H_*} - \mathcal{J}_H\big)[\![A]\!], B\big]\big\rangle_{\rho}\Big| + \big|\mathrm{tr}\big((\rho - \rho_*)\big[B, \mathcal{J}_{H_*}[\![A]\!]\big]\big)\big|\,. \end{aligned} \tag{17.6.8}$$

For the first term on the rhs. of (17.6.8), we estimate

$$\|*\|\,\big(\mathcal{J}_{H_*} - \mathcal{J}_H\big)[\![A]\!] \le \int_{\mathbf{R}} \mathrm{d}t\, w_g(t)\; \|*\|\,\mathrm{e}^{\mathrm{i}tH_*} A \mathrm{e}^{-\mathrm{i}tH_*} - \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}. \tag{17.6.9}$$

Recalling $H = H_* + J$, the difference between the two time evolutions can be written as

$$\mathrm{e}^{\mathrm{i}tH_*}\, A\, \mathrm{e}^{-\mathrm{i}tH_*} - \mathrm{e}^{\mathrm{i}tH}\, A\, \mathrm{e}^{-\mathrm{i}tH} = -\mathrm{i} \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}tH} \big[J, \mathrm{e}^{\mathrm{i}sH_*}\, A\, \mathrm{e}^{-\mathrm{i}sH_*}\big]\, \mathrm{e}^{-\mathrm{i}tH}.$$

We thus find that

$$\begin{aligned} (17.6.9) &\le \int_{\mathbf{R}} \mathrm{d}t\, w_g(t)\, |t| \sup_{s\in[0,t]} \big\|\big[J, \mathrm{e}^{\mathrm{i}sH_*}\, A\, \mathrm{e}^{-\mathrm{i}sH_*}\big]\big\| \\ &\le C\, \|A\|\, \mathrm{diam}(X)^{2d} \bigg(\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \int_I \mathrm{d}t\, w_g(t)\, |t| \big(1 + |t|\big)^{d/p} \\ &\qquad\qquad + \int_{\mathbf{R}\setminus I} \mathrm{d}t\, w_g(t)\, |t| \big(1 + |t|\big)^{d/p} \bigg) \\ &\le C\, \|A\|\, \mathrm{diam}(X)^{2d}\, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \end{aligned} \tag{17.6.10}$$

where we denoted $I := \{t \in \mathbf{R} : |t| \le \big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})/(2v)\big)^p/2\}$. Here, v is the Lieb-Robinson velocity from Lemma 17.A.7, which we employed in the second step. In the final step, we used the stretched exponential decay of w_g (see (17.1.6) and Lemma 17.A.9 later) and possibly adjusted the constants C, b, and p.

The second term on the rhs. of (17.6.8) can be estimated by means of (17.4.4), since

$$\big|\mathrm{tr}\big((\rho - \rho_*)\big[B, \mathcal{J}_{H_*}[\![A]\!]\big]\big)\big| \le \|B\|\, \big\|\big[\rho - \rho_*, \mathcal{J}_{H_*}[\![A]\!]\big]\big\|_{\mathrm{tr}}.$$

For $\mathcal{J}_{H_*}[\![A]\!]$ we now apply the local decomposition technique, analogously to the arguments around (17.6.5)–(17.6.6). More precisely, taking $A \in \mathcal{A}_X$, we now write $\mathcal{J}_{H_*}[\![A]\!] = \mathbb{E}_{X_n}\mathcal{J}_{H_*}[\![A]\!] + (\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}_{H_*}[\![A]\!]$ for some n to be chosen below, where $X_n := \{\, x \in \Lambda : d(x, X) \le n \,\}$ denotes the n-fattening of the set $X \subset \Lambda$. We now estimate the two terms separately. For the first term, we employ (17.4.4) to bound

$$\big\|\big[\rho - \rho_*, \mathbb{E}_{X_n}\mathcal{J}_{H_*}[\![A]\!]\big]\big\|_{\mathrm{tr}} \le C\, \|A\|\, \big(\mathrm{diam}(X) + n\big)^{\ell}\, \chi_{b,p}\big(d(X_n, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \tag{17.6.11}$$

where we used that $\mathbb{E}_{X_n}\mathcal{J}_{H_*}[\![A]\!] \in \mathcal{A}_{X_n}$ (by definition) and $\|\mathbb{E}_{X_n}\mathcal{J}_{H_*}[\![A]\!]\| \le \|A\|$. For the second term, we simply use Lemma 17.A.6 together with (17.A.11) applied with $\mathcal{I} \to \mathcal{J}$ for estimating the difference $\|(\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}_{H_*}[\![A]\!]\|$ to get

$$\begin{aligned} &\big\|\big[\rho - \rho_*, (\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}_{H_*}[\![A]\!]\big]\big\|_{\mathrm{tr}} \\ &\qquad \le 2\, \|\rho - \rho_*\|_{\mathrm{tr}}\, \|(\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}_{H_*}[\![A]\!]\| \le C\, \|A\|\, \mathrm{diam}(X)^{\ell}\, \chi_{b,p}(n)\,. \end{aligned} \tag{17.6.12}$$

Using $d(X_n, \Lambda \setminus \Lambda^{\mathrm{gap}}) \geq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}}) - n$ for (17.6.11), we can pick $n = d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})/2$, say, to estimate

$$(17.6.11) + (17.6.12) \leq C \, \|A\| \, \mathrm{diam}(X)^{\ell} \, \chi_{b/2,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big).$$

Finally, as mentioned above, interchanging the roles of A and B (by symmetry of the original expression; recall the discussion around (17.4.11)), we thus arrive at a bound on $\big|\big\langle [\mathcal{J}_H[\![A]\!], B]\big\rangle_{\rho}\big|$ of the form given by the rhs. of (17.6.7) (modulo changing $b \to b/2$). Therefore, combining (17.6.7) with (17.6.8), and (17.6.9)–(17.6.10) as well as (17.6.11)–(17.6.12), we conclude the desired.

Proof of (vii): Instead of (17.6.7), we start with (by assumption)

$$\big\langle [\mathcal{J}_{H_*}[\![A]\!], B]\big\rangle_{\rho_*} = 0 \qquad \text{for all} \qquad A, B \in \mathcal{A}. \tag{17.6.13}$$

Apart from this, the idea is identical to (iv) and (vi). Hence, by means of the triangle inequality, we obtain the same two terms from (17.6.8). The first term can be estimated in exactly the same way as in (17.6.9)–(17.6.10). The second term in (17.6.8) has to be treated a bit differently as in (17.6.11)–(17.6.12), since we now assumed (17.4.5) instead of (17.4.4).

In fact, for this term, using $\langle \cdot \rangle_{\rho} = \langle \tau[\![\cdot]\!] \rangle_{\rho_*}$, that τ is a $*$-automorphism, and (17.6.13), we get

$$\big|\mathrm{tr}\big((\rho - \rho_*)\big[B, \mathcal{J}_{H_*}[\![A]\!]\big]\big)\big| = \big|\mathrm{tr}\big(\rho_*\big[\tau[\![B]\!], (\tau - \mathbf{1}) \circ \mathcal{J}_{H_*}[\![A]\!]\big]\big)\big| \leq \|(\tau - \mathbf{1}) \circ \mathcal{J}_{H_*}[\![A]\!]\| \, .$$

This can now be treated exactly as done in the argument around (17.6.11)–(17.6.12) (i.e. taking $A \in \mathcal{A}_X$, writing $\mathcal{J}_{H_*}[\![A]\!] = \mathbb{E}_{X_n} \mathcal{J}_{H_*}[\![A]\!] + (\mathbf{1} - \mathbb{E}_{X_n}) \mathcal{J}_{H_*}[\![A]\!]$, and estimating the two terms separately while optimizing in n).

Proof of (viii): Without loss of generality, we may assume that $X \subset \Lambda$ in (17.4.8) is such that $C \, \mathrm{diam}(X)^{\ell} \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) < 1/2$, say (otherwise there is nothing to prove), and $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^{\beta}$ for some $\beta < 1$. By assumption, for such $X \subset \Lambda$ and $A \in \mathcal{A}_X$, we have

$$\mathrm{i}\langle \psi, A^* \mathcal{L}[\![A]\!] \psi \rangle \geq \frac{g}{2}\Big(\langle \psi, A^* A \psi \rangle - \big|\langle \psi, A \psi \rangle\big|^2\Big) \quad \text{with} \quad |\psi\rangle = \otimes_{x \in \Lambda} |\psi_x\rangle \, , \ \|\psi_x\| = 1. \tag{17.6.14}$$

Our claim will be a consequence of the following lemma.

Lemma 17.6.1 (cf. Proposition 14 in [596])**.** *Take a Hermitian $A \in \mathcal{A}_X$ as above and assume* (17.6.14)*. Denote the n-fattening of X by $X_n := \{ z \in \Lambda : d(z, X) \leq n \}$. Then, there exists a Hermitian operator $\widetilde{A} \in \mathcal{A}_{X_n}$ with $n := \lfloor d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})/2 \rfloor$ such that (i) ψ is an eigenvector of $\widetilde{A}$, i.e. $\widetilde{A}\psi = \widetilde{E}\psi$ for some $\widetilde{E} \in \mathbf{R}$, and (ii) we have the bound*

$$\big\| \mathbb{E}_{X_n} \mathcal{J}[\![A]\!] - \widetilde{A} \big\| \leq C \, \|A\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big) \tag{17.6.15}$$

with $\mathcal{J}$ being defined in (17.4.11)*.*

Armed with Lemma 17.6.1, we now turn to estimating the lhs. of (17.4.12), which is given by $\langle [\mathcal{J}[\![A]\!], B] \rangle_{\rho} = \langle \psi, [\mathcal{J}[\![A]\!], B] \psi \rangle$. By the triangle inequality, we have

$$\begin{aligned} &\big|\big\langle \psi, [\mathcal{J}[\![A]\!], B] \psi \big\rangle\big| \\ &\qquad \leq \big|\big\langle \psi, [(\mathbf{1} - \mathbb{E}_{X_n}) \mathcal{J}[\![A]\!], B] \psi \big\rangle\big| + \big|\big\langle \psi, [\mathbb{E}_{X_n} \mathcal{J}[\![A]\!] - \widetilde{A}, B] \psi \big\rangle\big| + \big|\big\langle \psi, [\widetilde{A}, B] \psi \big\rangle\big| \end{aligned} \tag{17.6.16}$$

and estimate the three terms separately. The first term in (17.6.16) can be treated as in (17.6.12), yielding the bound

$$C \, \|A\| \, \|B\| \, \chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{17.6.17}$$

where we additionally used that $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^{\beta}$. For the second term, we employ (17.6.15), yielding the same bound as for the first term. The third term in (17.6.16) vanishes since $\widetilde{A}\psi = \widetilde{E}\psi$. Therefore, (17.6.16) is bounded by (17.6.17) and we have proven Proposition 17.4.1 (viii).

It thus remains to give the proof of Lemma 17.6.1.

Proof of Lemma 17.6.1. The principal idea is similar to [596, Proposition 14]. To start with, we assume w.l.o.g. that $H\psi = 0$, i.e. ψ is an eigenvector to the eigenvalue zero. Then, we note that, since $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^{\beta}$, the bound in (17.6.12) implies that

$$\big\|(\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}[\![A]\!]\big\| \leq C\,\|A\|\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big). \tag{17.6.18}$$

We continue by decomposing

$$\big(\mathcal{J}[\![A]\!] - \langle\psi, \mathcal{J}[\![A]\!]\psi\rangle\big)\,|\psi\rangle = |\phi_{\leq n}\rangle + |\phi_{>n}\rangle \tag{17.6.19}$$

where we defined

$$|\phi_{\leq n}\rangle = |\phi_{\leq n}\rangle_{X_n} \otimes |\psi\rangle_{X_n^c} \coloneqq \big(\mathbb{E}_{X_n}\mathcal{J}[\![A]\!] - \langle\psi, \mathbb{E}_{X_n}\mathcal{J}[\![A]\!]\psi\rangle\big)\,|\psi\rangle$$

and $|\phi_{>n}\rangle$, which is defined such that (17.6.19) holds, satisfies the bound

$$\begin{aligned}\||\phi_{>n}\rangle\| &\leq \big\|\big((\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}[\![A]\!] - \langle\psi, (\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}[\![A]\!]\psi\rangle\big)\,|\psi\rangle\big\| \\ &\leq \big\|(\mathbf{1} - \mathbb{E}_{X_n})\mathcal{J}[\![A]\!]\big\| \leq C\,\|A\|\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big),\end{aligned}$$

where in the last step we employed (17.6.18). Then, similarly to [596, Lemma 15, eq. (B49)], one can compute $\langle\phi_{\leq n}, H^2\phi_{\leq n}\rangle$ and use the Payley-Zygmund inequality to show that the norm of $|\phi_{\leq n}\rangle$ is essentially bounded by the norm of $|\phi_{>n}\rangle$. That is, in our case, we find

$$\||\phi_{\leq n}\rangle\| \leq C\,\|A\|\,\chi_{b,p}\big(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\big), \tag{17.6.20}$$

as always modulo adjusting the constants C, b and p. Hence, defining the Hermitian operator

$$\widetilde{A} \coloneqq \mathbb{E}_{X_n}\mathcal{J}[\![A]\!] - \big(|\phi_{\leq n}\rangle_{X_n}\langle\psi| + \mathrm{h.c.}\big),$$

supported in X_n we easily see that $\widetilde{A}\,|\psi\rangle = \langle\psi, \mathbb{E}_{X_n}\mathcal{J}[\![A]\!]\psi\rangle\,|\psi\rangle =: \widetilde{E}\,|\psi\rangle$ and the bound (17.6.15) follows from (17.6.20). □

Proof of (ix): This is obvious from the definitions (17.4.10) and (17.4.12).

This concludes the proof Proposition 17.4.1. □

17.6.3 Local dynamical gap for the examples in Section 17.4.4

In this section, we prove the systems considered in Section 17.4.4 to have a local dynamical gap.

17.6.3.1 Perturbations of the classical Ising model with weak interactions

In this section, we prove the claim of a local dynamical gap from Section 17.4.4.1, where we considered perturbations of the classical Ising model with weak interactions. First, the (unique) ground state vector of (17.4.13) and the associated ground state energy is easily found as

$$\psi_* = \otimes_{x\in\Lambda}|\downarrow\rangle \quad \text{satisfying} \quad H_*\psi_* = \left(-|\Lambda| + \frac{1}{2}\sum_{x,y}\lambda(x-y)\right)\psi_*,$$

and the associated spectral projection (ground state) is simply given by $\rho_* = P_* = |\psi_*\rangle\langle\psi_*|$. We note that this is a globally gapped eigenstate of H_*, since the ground state energy corresponding to ψ_* is separated by a spectral gap $g \geq 2 - \|\lambda\|_1 > 0$ from the first excited state.

For the following argument, it is important to observe that, for any given $\Lambda^{\mathrm{gap}} \subset \Lambda$, the ground state projection factorizes, i.e.

$$\rho_* = \left(\bigotimes_{x\in\Lambda^{\mathrm{gap}}}|\downarrow\rangle\langle\downarrow|\right) \otimes \left(\bigotimes_{x\in\Lambda\setminus\Lambda^{\mathrm{gap}}}|\downarrow\rangle\langle\downarrow|\right) =: \rho_*^{\Lambda^{\mathrm{gap}}} \otimes \rho_*^{\Lambda\setminus\Lambda^{\mathrm{gap}}}. \tag{17.6.21}$$

Indeed, since $\|\lambda\|_1 < 2$, every ground state ρ of $H = H_* + J$, where J is a *strictly* $\Lambda \setminus \Lambda^{\mathrm{gap}}$-localized SLT Hamiltonian, as described in Section 17.4.4.1, also factorizes as

$$\rho = \rho_*^{\Lambda^{\mathrm{gap}}} \otimes \rho^{\Lambda \setminus \Lambda^{\mathrm{gap}}}. \tag{17.6.22}$$

In order to see this, first note that there exists an eigenbasis[12] of H for which every eigenvector ψ of H can be written as a linear combination $\sum_j c_j e_j \otimes \varphi_j$, where $e_j \in \mathcal{H}_{\Lambda^{\mathrm{gap}}}$ are eigenvectors of

$$H\big|_{\Lambda^{\mathrm{gap}}} \coloneqq \sum_{x \in \Lambda^{\mathrm{gap}}} \sigma_x^3 + \frac{1}{2} \sum_{x,y \in \Lambda^{\mathrm{gap}}} \lambda(x-y)\, \sigma_x^3 \sigma_y^3$$

to a *common* eigenvalue and $\varphi_j \in \mathcal{H}_{\Lambda \setminus \Lambda^{\mathrm{gap}}}$. Then, to see (17.6.22), it suffices to realize that, for every $x \in \Lambda^{\mathrm{gap}}$, starting from the *unique* ground state vector $\psi_*|_{\Lambda^{\mathrm{gap}}} = \otimes_{x \in \Lambda^{\mathrm{gap}}} |{\downarrow}\rangle$ of $H\big|_{\Lambda^{\mathrm{gap}}}$, the energy cost for flipping the spin $|{\downarrow}\rangle$ to $|{\uparrow}\rangle$ in the first term of (17.4.13) is two, whereas the potential gain stemming from the second summand in (17.4.13) is bounded by $\|\lambda\|_1 < 2$, yielding (17.6.22). In particular, any *overall* pure ground state of H can be obtained by tensorizing the unique separate ground state vector of $H|_{\Lambda^{\mathrm{gap}}}$, i.e. $\psi|_{\Lambda^{\mathrm{gap}}}$, with an appropriate (not necessarily unique) minimizer $\varphi^{\Lambda \setminus \Lambda^{\mathrm{gap}}}$ of

$$\min_{\substack{\varphi \in \mathcal{H}_{\Lambda \setminus \Lambda^{\mathrm{gap}}} \\ \|\varphi\|=1}} \langle [\rangle \Big] \psi_*|_{\Lambda^{\mathrm{gap}}} \otimes \varphi, \left(H - H|_{\Lambda^{\mathrm{gap}}}\right) \psi_*|_{\Lambda^{\mathrm{gap}}} \otimes \varphi,$$

i.e. by conditioning on the first factor $\psi_*|_{\Lambda^{\mathrm{gap}}}$. The pure ground state is then obtained as $|\psi_*|_{\Lambda^{\mathrm{gap}}}\rangle \langle \psi_*|_{\Lambda^{\mathrm{gap}}}| \otimes |\varphi^{\Lambda \setminus \Lambda^{\mathrm{gap}}}\rangle \langle \varphi^{\Lambda \setminus \Lambda^{\mathrm{gap}}}|$.

Therefore, combining (17.6.21) and (17.6.22), we have the following: For ρ being a *pure state*, i.e. $\rho^{\Lambda \setminus \Lambda^{\mathrm{gap}}}$ from (17.6.22) can be written as $\rho^{\Lambda \setminus \Lambda^{\mathrm{gap}}} = |\varphi^{\Lambda \setminus \Lambda^{\mathrm{gap}}}\rangle \langle \varphi^{\Lambda \setminus \Lambda^{\mathrm{gap}}}|$, there exists a unitary $U \equiv U^{\Lambda \setminus \Lambda^{\mathrm{gap}}} \in \mathcal{A}_{\Lambda \setminus \Lambda^{\mathrm{gap}}}$ such that $|\varphi^{\Lambda \setminus \Lambda^{\mathrm{gap}}}\rangle = U \otimes_{x \in \Lambda \setminus \Lambda^{\mathrm{gap}}} |{\downarrow}\rangle$. In particular, $\rho = U \rho_* U^*$ and hence we have a norm-preserving $*$-automorphism $\tau[\![A]\!] \coloneqq U^* A U$ on $A \in \mathcal{A}$, which intertwines the ground states, i.e. $\langle \cdot \rangle_\rho = \langle \tau[\![\cdot]\!] \rangle_{\rho_*}$, and satisfies (17.4.5). By means of Proposition 17.4.1 (vii) (note that, since ρ_* is spectrally gapped, it fulfills the additional assumption of 17.4.1 (vii) by means of Proposition 17.1.1), we thus find that ρ satisfies Assumption ($\mathrm{LDG}_{\mathsf{main}}$). Finally, for a general (*mixed*) state ρ, we conclude the desired after noticing that Assumption ($\mathrm{LDG}_{\mathsf{main}}$) is invariant under taking convex combinations.

17.6.3.2 Perturbations of gapped frustration free product states

In this section, we prove the claim of a local dynamical gap from Section 17.4.4.2, where we considered perturbations of gapped frustration free Hamiltonians with a product ground state.

Similarly to Section 17.6.3.1, one can easily verify that H_* from (17.4.15) is globally gapped with its ground state vector being given by $\otimes_{x \in \Lambda} |\psi_{*,x}\rangle$. The same is true for all restrictions $H_*|_{\Lambda'}$.

Moreover, for $\Lambda^{\mathrm{gap}} \subset \Lambda$ and a fixed exponent $\beta > 0$, consider $X \subset \Lambda$ satisfying $\mathrm{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$. Under Assumptions (A1), (A2), and (A3), the authors of [35] have proven the following: Let $|\psi\rangle$ be a ground state vector of $H = H_* + J$, where J is a *strongly* $\Lambda \setminus \Lambda^{\mathrm{gap}}$-localized SLT Hamiltonian, as described in Section 17.4.4.2, and $P_{*,X}$ denote the projection onto the ground state vector $|\psi_*|_X\rangle$ of $H_*|_X$. Then it holds that

$$\|\rho - \tilde{\rho}\|_{\mathrm{tr}} \leq C \exp\!\left(-\left(d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})\right)^q\right) \tag{17.6.23}$$

for some $C, q > 0$ and $\|\cdot\|_{\mathrm{tr}}$ being the trace norm. Here, $\rho \equiv P$ and $\tilde{\rho} \equiv \tilde{P}$ denote the orthogonal projections on $|\psi\rangle$ and $P_{*,X}|\psi\rangle$, respectively, i.e. they are *pure states*.

[12] This is simply a common eigenbasis of H and $H\big|_{\Lambda^{\mathrm{gap}}} \otimes \mathbf{1}_{\mathcal{A}_{\Lambda \setminus \Lambda^{\mathrm{gap}}}}$, which commute.

Due to the product structure of the ground state vector of H_* and its restrictions, we easily see that $\tilde{\rho}$ can be written as

$$\tilde{\rho} = |\psi_*|_X\rangle\langle\psi_*|_X| \otimes \tilde{\rho}^{\Lambda\setminus X}$$

for some state $\tilde{\rho}^{\Lambda\setminus X}$ on $\Lambda \setminus X$. This means that, analogously to Section 17.6.3.1, $\langle\cdot\rangle_{\rho_*}$ and $\langle\cdot\rangle_{\tilde{\rho}}$ can be intertwined by a norm preserving $*$-automorphism τ satisfying (17.4.5). In particular, by means of Proposition 17.4.1 (vii) (note that, since ρ_* is spectrally gapped, it fulfills the additional assumption of 17.4.1 (vii) by means of Proposition 17.1.1), we thus find that $\tilde{\rho}$ satisfies Assumption ($\mathsf{LDG}_{\mathsf{main}}$) – but only for observables supported in $X \subset \Lambda$ with $\operatorname{diam}(X) \leq d(X, \Lambda \setminus \Lambda^{\mathrm{gap}})^{\beta}$ and without $d(Y, \Lambda \setminus \Lambda^{\mathrm{gap}})$ in the argument of $\chi_{b,p}$; that is, Assumption ($\mathsf{LDG}_{\mathsf{weak}}$). This implies, by means of Proposition 17.4.1 (i)+(vi) (trivially modified to the setting of ($\mathsf{LDG}_{\mathsf{weak}}$)) and (17.6.23), that ρ satisfies Assumption ($\mathsf{LDG}_{\mathsf{weak}}$). Finally, for a general (*mixed*) state ρ, we conclude the desired by taking convex combinations (as at the end of the argument in Section 17.6.3.1).

17.A Technical lemmata

In this section, we prove the technical lemmata required for the construction of the NEASS. We begin with some general properties of the functions $\chi_{b,p}$ and Lieb-Robinson bounds for (b,p)-localized SLT-operators in Appendix 17.A.1. In Appendix 17.A.2, we prove that the various operations used in the construction of the NEASS preserve locality. In Appendix 17.A.3 we recall the construction of the quasi-local inverse of the Liouvillian and prove that it also preserves locality of SLT-operators. Finally, Appendices 17.A.4 and 17.A.5 are concerned with the proofs of Lemmata 17.2.1 and 17.5.1, respectively.

In all proofs, $C > 0$ is a generic constant that might change within the computations. It can in particular depend on all the parameters chosen in the statements, but it is uniform in the chosen lattice and the operators appearing.

17.A.1 Properties of the decay function $\chi_{b,p}$

Let us first collect some properties of the decay function $\chi_{b,p}$ we use in the definition of the interaction norm. From [428, Lemma 7.2.3] we have the following Lemma, where we simplified the statements.

Lemma 17.A.1. *For any $b \geq 0$ and $s \in (0,1]$, the function $\chi_{b,p}$ satisfies the following properties:*

(a) *$\chi_{b,p}$ is logarithmically superadditive, i.e. $\chi_{b,p}(x+y) \geq \chi_{b,p}(x)\,\chi_{b,p}(y)$ for all $x, y \geq 0$.*

(b) *For every $b > 0$ and $k \geq 0$ there exists a constant $C > 0$ such that*

$$\sup_{x\geq 0} x^k\, \chi_{b,p}(x) = C.$$

As a direct consequence of Lemma 17.A.1 (b), we get the following Lemma, which we write out to fix the constant and recall it in later proofs.

Lemma 17.A.2. *Let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$ and $k \in \mathbf{N}$. Then there exists a constant $\mathcal{C}_{\mathrm{vol},b,p,k} > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$ and sets $Z \subset \Lambda$*

$$|Z|^k\, \chi_{b,p}\big(\operatorname{diam}(Z)\big) \leq \mathcal{C}_{\mathrm{vol},b,p,k}.$$

Before we state the Lieb-Robinson bound, which is a crucial ingredient in the proof, let us briefly recall the time-dependent Heisenberg evolution. For a time-dependent interaction defined on an interval $I \subset \mathbf{R}$ with corresponding SLT operator $H(t)$, let $\tau_{t,s}$ be the unique solution of

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\tau_{t,s}(A) = \tau_{t,s}\big([H(t), A]\big) \qquad \text{and} \qquad \tau_{s,s} = \mathrm{Id} \qquad \text{for all } s,\, t \in I.$$

This was already used with a different time scaling for the Hamiltonian $H_\varepsilon(t)$ in the proof of Theorem 17.5.2. Under locality assumptions on the Hamiltonian, one finds the following Lieb-Robinson bound.

Lemma 17.A.3 (Lieb-Robinson bound [428, Theorem 7.3.3])**.** *Let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $b' > b > 0$, $p \in (0,1]$ and $k \in \mathbf{N}$. There exists constants C and $c > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, intervals $I \subset \mathbf{R}$, time-dependent interactions Φ, disjoint subsets X, $Y \subset \Lambda$, observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, and s, $t \in I$ it holds that*

$$\big\|\big[\tau_{t,s}(A), B\big]\big\| \leq C \, \|A\| \, \|B\| \, \big(\mathrm{e}^{b\, v\, |t-s|} - 1\big) \, D(X,Y), \tag{17.A.1}$$

where $v = c \, \|\Phi\|_{b',p} / b$ is the Lieb-Robinson velocity and

$$\begin{aligned} D(X,Y) &:= \min\left\{ \sum_{x\in X} \chi_{b,p}\big(d(x,Y)\big), \sum_{y\in Y} \chi_{b,p}\big(d(y,X)\big) \right\} \\ &\leq \min\big\{|X|, |Y|\big\} \, \chi_{b,p}\big(d(X,Y)\big). \end{aligned}$$

The Lieb-Robinson velocity is defined including the $1/b$ because (17.A.1) can be bounded by

$$C \, \|A\| \, \|B\| \, \min\big\{|X|, |Y|\big\} \, \mathrm{e}^{b\left(v\,|t-s| - d(X,Y)^p\right)}.$$

17.A.2 Commutators and dynamics of localized SLT-operators

Lemma 17.A.4 (Commutator with local observable)**.** *Let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $b > 0$, $p \in (0,1]$ and $k \in \mathbf{N}$. There exists constants C and $C_k > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, subsets Ω, $X \subset \Lambda$, SLT operators A_1 and observables $O \in \mathcal{A}_X$ it holds that*

$$\big\|[A_1, O]\big\| \leq 2 \, \|O\| \, |X| \, \chi_{b,p}\big(d(X,\Omega)\big) \, \|\Phi_{A_1}\|_{b,p;\Omega}. \tag{17.A.2}$$

For a second observable $\tilde{O} \in \mathcal{A}_Y$, it holds that

$$\big\|[[A_1, O], \tilde{O}]\big\| \leq 4 \, \|O\| \, \|\tilde{O}\| \, |X| \, \chi_{b,p}\big(d(X,Y)\big) \, \chi_{b,p}\big(d(X,\Omega)\big) \, \|\Phi_{A_1}\|_{b,p;\Omega}. \tag{17.A.3}$$

Finally, if additionally also $A_2, \ldots, A_k$ are SLT operators, then

$$\big\|\mathrm{ad}_{A_k} \cdots \mathrm{ad}_{A_1}(O)\big\| \leq C_k \, \|O\| \, |X|^k \, \chi_{b,p}\big(d(X,\Omega)\big) \, \|\Phi_{A_1}\|_{b,p;\Omega} \prod_{j=2}^{k} \big\|\Phi_{A_j}\big\|_{b,p}. \tag{17.A.4}$$

All three bounds, in particular, also hold for $\Omega = \Lambda$, where $\|\cdot\|_{b,p;\Omega} = \|\cdot\|_{b,p}$ and $d(X,\Omega) = 0$.

Proof. We begin with the first statement and write $A = A_1$. Since $[\Phi_A(Z), O]$ vanishes whenever $Z \cap X = \varnothing$, we find

$$\begin{aligned} \big\|[A,O]\big\| &\leq \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\varnothing}} 2 \big\|\Phi_A(Z)\big\| \, \|O\| \\ &\leq 2 \, \|O\| \sum_{z\in X} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\big\|\Phi_A(Z)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big) \, \chi_{b,p}\big(d(z,\Omega)\big)} \, \chi_{b,p}\big(d(z,\Omega)\big) \\ &\leq 2 \, \|O\| \sum_{z\in X} \chi_{b,p}\big(d(z,\Omega)\big) \, \|\Phi_A\|_{b,p;\Omega} \\ &\leq 2 \, \|O\| \, |X| \, \chi_{b,p}\big(d(X,\Omega)\big) \, \|\Phi_A\|_{b,p;\Omega}, \end{aligned}$$

where we just overcount in the second inequality. Clearly, the same statement also holds with $\Omega = \Lambda$. The proof of (17.A.3) is analogous to the proof of (17.A.2) and so omitted.

We conclude by proving (17.A.4) using induction. Note that the outer operators are all SLT-operators on Λ. The $k = 1$ case is given in (17.A.2). We now assume (17.A.4) for some fixed k and with $\Omega = \Lambda$. Then, we add a further commutator with A_0 to conclude

$$\begin{aligned}
&\big\|\mathrm{ad}_{A_k}\cdots\mathrm{ad}_{A_1}\,\mathrm{ad}_{A_0}(O)\big\| \\
&\quad\le \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} \big\|\mathrm{ad}_{A_k}\cdots\mathrm{ad}_{A_1}[\Phi_{A_0}(Z),O]\big\| \\
&\quad\le \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} C_k\,\big\|\big[\Phi_{A_0}(Z),O\big]\big\|\,|X\cup Z|^k\prod_{j=1}^{k}\big\|\Phi_{A_j}\big\|_{b,p} \\
&\quad\le C_k\,2^k\,\|O\|\,|X|^k\sum_{z\in X}\chi_{b,p}\big(d(z,\Omega)\big)\prod_{j=1}^{k}\big\|\Phi_{A_j}\big\|_{b,p} \\
&\qquad\times\sum_{\substack{Z\subset\Lambda:\\ z\in Z}}\frac{\big\|\Phi_{A_0}(Z)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\,\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,|Z|^k \\
&\quad\le \mathcal{C}_{\mathrm{vol},b,p,k}\,C_k\,2^k\,\|O\|\,|X|^{k+1}\,\chi_{b,p}\big(d(X,\Omega)\big)\,\|\Phi_{A_0}\|_{b,p;\Omega}\prod_{j=1}^{k}\big\|\Phi_{A_j}\big\|_{b,p},
\end{aligned}$$

where we used Lemma 17.A.2 in the last step. This finishes the induction. □

Lemma 17.A.5 (Multi-commutators)**.** *Let $d\in\mathbf{N}$, $C_{\mathrm{vol}}>0$, $b>0$, $p\in(0,1]$, $\varepsilon>0$ and $k\in\mathbf{N}$. There exists a constant $C>0$ such that for all lattices $\Lambda\in\mathcal{G}(d,C_{\mathrm{vol}})$, subsets $\Omega\subset\Lambda$ and SLT operators $A_0,\dots,A_k$ it holds that*

$$\big\|\Phi_{\mathrm{ad}_{A_k}\cdots\mathrm{ad}_{A_1}(A_0)}\big\|_{b,p;\Omega}\le C\,\|\Phi_{A_0}\|_{b+\varepsilon,p;\Omega}\prod_{j=1}^{k}\big\|\Phi_{A_j}\big\|_{2b+\varepsilon,p}. \tag{17.A.5}$$

Proof. For the proof we first need to constructed an interaction for the commutator of two SLT operators A and B. It turns out that it can be given as

$$\Phi_{[A,B]}(Z)=\sum_{\substack{X,Y\subset\Lambda:\\ X\cup Y=Z\\ X\cap Y\neq\emptyset}}\big[\Phi_A(X),\Phi_B(Y)\big].$$

Then,

$$\begin{aligned}
&\sum_{\substack{Z\subset\Lambda:\\ z\in Z}}\frac{\big\|\Phi_{[A,B]}(Z)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} \\
&\qquad\le\sum_{\substack{Z\subset\Lambda:\\ z\in Z}}\sum_{\substack{X,Y\subset\Lambda:\\ X\cup Y=Z\\ X\cap Y\neq\emptyset}}\frac{2\big\|\Phi_A(X)\big\|\,\big\|\Phi_B(Y)\big\|}{\chi_{b,p}\big(\mathrm{diam}(X)\big)\,\chi_{b,p}\big(\mathrm{diam}(Y)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)},
\end{aligned}$$

where we used $\mathrm{diam}(Z)\le\mathrm{diam}(X)+\mathrm{diam}(Y)$ and the properties of $\chi_{b,p}$. The above sum can be bounded by the sum of the terms where $z\in X$ or $z\in Y$. The latter can be upper bounded by

$$2\sum_{\substack{Y\subset\Lambda:\\ z\in Y}}\frac{\big\|\Phi_B(Y)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Y)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\sum_{x\in Y}\sum_{\substack{X\subset\Lambda:\\ x\in X}}\frac{\big\|\Phi_A(X)\big\|}{\chi_{b,p}\big(\mathrm{diam}(X)\big)}$$

$$
\begin{aligned}
&\leq 2 \sum_{\substack{Y \subset \Lambda: \\ z \in Y}} \frac{\|\Phi_B(Y)\|}{\chi_{b+\varepsilon,p}\big(\operatorname{diam}(Y)\big)\, \chi_{b+\varepsilon,p}\big(d(z,\Omega)\big)}\, \chi_{\varepsilon,p}\big(\operatorname{diam}(Y)\big)\, |Y|\, \|\Phi_A\|_{b,p} \\
&\leq C\, \mathcal{C}_{\mathrm{vol},\varepsilon,p,1} \sum_{\substack{Y \subset \Lambda: \\ z \in Y}} \frac{\|\Phi_B(Y)\|}{\chi_{b+\varepsilon,p}\big(\operatorname{diam}(Y)\big)\, \chi_{b+\varepsilon,p}\big(d(z,\Omega)\big)}\, \|\Phi_A\|_{b,p} \\
&\leq C\, \|\Phi_B\|_{b+\varepsilon,p;\Omega}\, \|\Phi_A\|_{b,p},
\end{aligned}
$$

where we used Lemma 17.A.2 in the third step. Using $\chi_{b,p}\big(d(z,\Omega)\big) \geq \chi_{b,p}\big(\operatorname{diam}(X)\big)\, \chi_{b,p}\big(d(x,\Omega)\big)$ for all $z, x \in X$, the part of the sum where $z \in X$ can be bounded by

$$
\begin{aligned}
&2 \sum_{\substack{X \subset \Lambda: \\ z \in X}} \frac{\|\Phi_A(X)\|}{\chi_{2b,p}\big(\operatorname{diam}(X)\big)} \sum_{y \in X} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{1}{\chi_{b,p}\big(d(y,\Omega)\big)}\, \frac{\|\Phi_B(Y)\|}{\chi_{b,p}\big(\operatorname{diam}(Y)\big)} \\
&\qquad \leq 2 \sum_{\substack{X \subset \Lambda: \\ z \in X}} \frac{\|\Phi_A(X)\|}{\chi_{2b+\varepsilon,p}\big(\operatorname{diam}(X)\big)}\, \chi_{\varepsilon,p}\big(\operatorname{diam}(X)\big)\, |X|\, \|\Phi_B\|_{b,p;\Omega} \\
&\qquad \leq C\, \|\Phi_A\|_{2b+\varepsilon,p}\, \|\Phi_B\|_{b,p;\Omega}.
\end{aligned}
$$

Both bounds together prove the claim for $k = 1$. To proceed by induction we assume that the statement holds for some fixed k. Then, applying first the statement for $k = 1$ and then $k = k$ both with $\varepsilon/2$ we obtain

$$
\begin{aligned}
\big\|\Phi_{\mathrm{ad}_{A_{k+1}} \cdots \mathrm{ad}_{A_1}(A_0)}\big\|_{b,p;\Omega} &\leq C\, \|\Phi_{\mathrm{ad}_{A_k} \cdots \mathrm{ad}_{A_1}(A_0)}\|_{b+\varepsilon/2,p;\Omega}\, \|\Phi_{A_{k+1}}\|_{2b+\varepsilon/2,p} \\
&\leq C\, \|\Phi_{A_0}\|_{b+\varepsilon,p;\Omega} \prod_{j=1}^{k+1} \|\Phi_{A_j}\|_{2b+\varepsilon,p}\,.
\end{aligned}
$$

□

For the following statements we need to approximate the time evolution of local operators, which in principle live on the whole lattice. This can be done by a so called conditional expectation, which is just the partial trace in our case of finite spin systems. We collect its properties in the following lemma.

Lemma 17.A.6 ([461, Lemma 4.1])**.** *Let Λ be a lattice and $X \subset \Lambda$. Then there exists a unit-preserving, completely positive linear map $\mathbb{E}_X : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ satisfying*

(a) $\mathbb{E}_X(\mathcal{A}_\Lambda) \subset \mathcal{A}_X$*;*

(b) $\mathbb{E}_X(ABC) = A\, \mathbb{E}_X(B)\, C$ *for all $B \in \mathcal{A}_\Lambda$ and $A, C \in \mathcal{A}_X$; This in particular implies $\mathbb{E}_X(A) = A$ for all $A \in \mathcal{A}_X$;*

(c) $\|\mathbb{E}_X\| = 1$*;*

(d) $\mathbb{E}_X \circ \mathbb{E}_Y = \mathbb{E}_{X \cap Y}$*, for $X, Y \subset \Lambda$;*

(e) *If $A \in \mathcal{A}_\Lambda$ satisfies*

$$
\big\|[A,B]\big\| \leq \eta\, \|A\|\, \|B\| \quad \text{for all} \quad B \in \mathcal{A}_{\Lambda \setminus X}, \tag{17.A.6}
$$

for some $\eta > 0$, then

$$
\|A - \mathbb{E}_X(A)\| \leq \eta\, \|A\|. \tag{17.A.7}
$$

Together with the Lieb-Robinson bound we can now obtain the following.

Lemma 17.A.7 (Dynamics). *Let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $b, b' > 0$, and $p, p' \in (0,1]$ satisfying $p' < 1$ or $b' < b$. There exists constants C and $c > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, the following holds: Let $I \subset \mathbf{R}$ an interval, the interaction Φ generate the dynamics $\tau_{t,s}$ with Lieb-Robinson velocity $v = c\,\|\Phi\|_{b,p}/b$. For every $\chi_{b',p'}$-SLT operator A, subsets Ω, $X \subset \Lambda$, observables $O \in \mathcal{A}_X$, and $t, s \in I$ it holds that*

$$\big\|\big[A, \tau_{t,s}(O)\big]\big\| \leq C\,\|O\|\,|X|\,|X_{(2v|t-s|)^{1/p}}|\,\|\Phi_A\|_{b',p';\Omega}\,\chi_{b',p'}\big(d(X_{(2v|t-s|)^{1/p}}, \Omega)\big)$$

Proof. We use the local decomposition technique similar to [461, Section 5]. Therefore, let

$$\Delta_0 := \mathbb{E}_{X_{v|t-s|}}\big(\tau_{t,s}(O)\big)$$

and

$$\Delta_k := \mathbb{E}_{X_{(v|t-s|+k)^{1/p}}}\big(\tau_{t,s}(O)\big) - \mathbb{E}_{X_{(v|t-s|+k-1)^{1/p}}}\big(\tau_{t,s}(O)\big),$$

so that $\tau_{t,s}(O) = \sum_{k=0}^{\infty} \Delta_k$, where the sum is finite since eventually $X_{(v|t-s|+k)^{1/p}} = \Lambda$. By the properties of the conditional expectation

$$\|\Delta_0\| \leq \|\tau_{t,s}(O)\| = \|O\|$$

and for $k \geq 1$ it holds that

$$\Delta_k = \mathbb{E}_{X_{(v|t-s|+k)^{1/p}}}\Big(\big(1 - \mathbb{E}_{X_{(v|t-s|+k-1)^{1/p}}}\big)\,\tau_{t,s}(O)\Big)$$

and thus

$$\|\Delta_k\| \leq \big\|\big(1 - \mathbb{E}_{X_{(v|t-s|+k-1)^{1/p}}}\big)\,\tau_{t,s}(O)\big\|.$$

Furthermore, for all $B \in \mathcal{A}_{\Lambda \setminus X_{(v|t-s|+k-1)^{1/p}}}$ by the Lieb-Robinson bound (Lemma 17.A.3)

$$\big\|[\tau_{t,s}(O), B]\big\| \leq C_{\mathrm{LR}}\,\|O\|\,\|B\|\,|X|\,\mathrm{e}^{b\left(v|t-s| - (v|t-s|+k-1)\right)} = C_{\mathrm{LR}}\,\mathbf{E}\,\|O\|\,\|B\|\,|X|\,\mathrm{e}^{-bk}$$

because $d(X, \Lambda \setminus X_{v|t-s|+k-1}) \geq v|t-s| + k - 1$, and thus by Lemma 17.A.6

$$\|\Delta_k\| \leq C_{\mathrm{LR}}\,\mathbf{E}\,\|O\|\,|X|\,\mathrm{e}^{-bk}.$$

Now we can apply Lemma 17.A.4 to each of the summands in the decomposition

$$\begin{aligned}
\big\|\big[A, \tau_{t,s}(O)\big]\big\| &\leq \sum_{k=0}^{\infty} \big\|\big[A, \Delta_k\big]\big\| \\
&\leq \sum_{k=0}^{\infty} 2\,\|\Delta_k\|\,\big|X_{(v|t-s|+k)^{1/p}}\big|\,\chi_{b',p'}\big(d(X_{(v|t-s|+k)^{1/p}}, \Omega)\big)\,\|\Phi_A\|_{b',p';\Omega} \\
&\leq \tilde{C}\,\|O\|\,|X|\,\|\Phi_A\|_{b',p';\Omega} \sum_{k=0}^{\infty} \big|X_{(v|t-s|+k)^{1/p}}\big|\,\chi_{b',p'}\big(d(X_{(v|t-s|+k)^{1/p}}, \Omega)\big)\,\mathrm{e}^{-bk}. \\
&\leq \tilde{C}\,\|O\|\,|X|\,|X_{(2v|t-s|)^{1/p}}|\,\|\Phi_A\|_{b',p';\Omega}\,\chi_{b',p'}\big(d(X_{(2v|t-s|)^{1/p}}, \Omega)\big) \\
&\quad \times \sum_{k=0}^{\infty} (1 + C_{\mathrm{vol}}\,(2k)^{d/p})\,\chi^{-1}_{b',p'}(k)\,\mathrm{e}^{-bk},
\end{aligned}$$

where we abbreviated $\tilde{C} = 2\max\{1, C_{\mathrm{LR}}\,\mathrm{e}\}$ and used $(v|t-s|+k)^{1/p} \leq (2v|t-s|)^{1/p} + (2k)^{1/p}$. To conclude the result, we observe, that the series is bounded for $p' < 1$ or $b' < b$ if $p' = 1$. □

Lemma 17.A.8 (Conjugation with unitaries). *Let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $\varepsilon > 0$, a, $b > 0$, and $p, q \in (0,1]$ satisfying $p < q$ or $p = q$ and $a > (2^p + 1)\,b$. There exists constants C and $c > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, the following holds: For SLT operators D and S it holds that $A := \mathrm{e}^{\mathrm{i}S}\,D\,\mathrm{e}^{-\mathrm{i}S}$ is an SLT operator as well. More precisely, there exists an interaction Φ_A such that*

$$\|\Phi_A\|_{b,p;\Omega} \leq C\,\mathrm{e}^{c\,\|\Phi_S\|_{a,q}}\,\big\|\Phi_D\big\|_{b+\varepsilon,p;\Omega}$$

Proof. The proof uses the same technique as the proof of Lemma 17.A.7.

First fix $X \subset \Lambda$ and $O \in \mathcal{A}_X$ and denote $\tau(O) = \mathrm{e}^{\mathrm{i}S} O \mathrm{e}^{-\mathrm{i}S}$. Then define

$$\Delta_0(O) := \mathbb{E}_X\big(\tau(O)\big)$$

and

$$\Delta_k(O) := \mathbb{E}_{X_k}\big(\tau(O)\big) - \mathbb{E}_{X_{k-1}}\big(\tau(O)\big) = \mathbb{E}_{X_k}\Big(\big(\mathbf{1} - \mathbb{E}_{X_{k-1}}\big)\,\tau(O)\Big).$$

By properties of the conditional expectation, Lemma 17.A.6, and the Lieb-Robinson bound, Lemma 17.A.3, one can bound

$$\|\Delta_0(O)\| \leq \|O\|$$

and

$$\|\Delta_k(O)\| \leq C_{\mathrm{LR}}\, \|O\|\, |X| \big(\mathrm{e}^{c\,\|\Phi_S\|_{a,q}} - 1\big)\, \chi_{a',q}(k) \tag{17.A.8}$$

for $k \geq 1$ because $d(X, \Lambda \setminus X_{k-1}) = k$ in our geometry, where we chose $a' < a$.

We now construct an interaction for A. First note, that

$$A = \tau(A) = \sum_{Z \subset \Lambda} \tau\big(\Phi_D(Z)\big) = \sum_{Z \subset \Lambda} \sum_{k=0}^{\infty} \Delta_k\big(\Phi_D(Z)\big)$$

where $\Delta_k\big(\Phi_A(Z)\big) \in \mathcal{A}_{Z_k}$ and the sum is actually finite. For any function $f\colon \{\, \Omega \subset \Lambda \,\} \to \mathcal{A}_\Lambda$ and $k \geq 0$ it holds that

$$\sum_{Y \subset \Lambda} f(Y) = \sum_{Y \subset \Lambda} \Big(\sum_{Z \subset \Lambda} \mathbf{1}_{Z = Y_k} \Big) f(Y) = \sum_{Z \subset \Lambda} \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} f(Y).$$

Applying this with $f\colon Z \mapsto \Delta_k(\Phi_D(Z))$ for each k we find

$$A = \sum_{k=0}^{\infty} \sum_{Z \subset \Lambda} \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \Delta_k\big(\Phi_D(Y)\big) = \sum_{Z \subset \Lambda} \Phi_A(Z) \quad \text{with} \quad \Phi_A(Z) := \sum_{k=0}^{\infty} \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \Delta_k\big(\Phi_D(Y)\big).$$

With this interaction for A and any $z \in \Lambda$ we bound

$$\begin{aligned} \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \frac{\|\Phi_A(Z)\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\, \chi_{b,p}\big(d(z,\Omega)\big)} &\leq \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \sum_{k=0}^{\infty} \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \frac{\big\|\Delta_k\big(\Phi_D(Y)\big)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\, \chi_{b,p}\big(d(z,\Omega)\big)} \\ &= \sum_{k=0}^{\infty} \sum_{Y \subset \Lambda} \mathbf{1}_{z \in Y_k} \frac{\big\|\Delta_k\big(\Phi_D(Y)\big)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Y_k)\big)\, \chi_{b,p}\big(d(z,\Omega)\big)}. \end{aligned}$$

The $k = 0$ term is bounded by $\|\Phi_D\|_{p,b;\Omega}$. For $k \geq 0$ we use (17.A.8). Moreover, $\mathrm{diam}(Y_k) \leq \mathrm{diam}(Y) + 2k$ and, since $z \in Y_k$, there exists $y \in B_z(k) \cap Y$, such that $d(z,\Omega) \leq k + d(y,\Omega)$. Hence, the remaining sum is bounded by

$$\begin{aligned} &C_{\mathrm{LR}} \big(\mathrm{e}^{c\,\|\Phi_S\|_{a,q}} - 1\big) \sum_{k=1}^{\infty} \sum_{y \in B_z(k)} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{|Y|\, \big\|\Phi_D(Y)\big\|}{\chi_{b,p}\big(\mathrm{diam}(Y)\big)\, \chi_{b,p}\big(d(y,\Omega)\big)} \frac{\chi_{a',q}(k)}{\chi_{b,p}(2k)\, \chi_{b,p}(k)} \\ &\quad = C_{\mathrm{LR}} \big(\mathrm{e}^{c\,\|\Phi_S\|_{a,q}} - 1\big)\, C \big\|\Phi_D\big\|_{b+\varepsilon,p;\Omega} \sum_{k=1}^{\infty} \big(1 + C_{\mathrm{vol}}\, k^d\big) \frac{\chi_{a',q}(k)}{\chi_{(2^p+1)b,p}(k)}, \end{aligned}$$

for some $C > 0$. The remaining sum is bounded if $q > p$, or $q = p$ and $a' > (2^p + 1)\, b$. The last condition is equivalent to $a > (2^p + 1)\, b$, by our choice of a'. To conclude the statement, we choose the total constant larger than 1 and add the $k = 0$ term $\big\|\Phi_D\big\|_{b,p;\Omega} \leq \big\|\Phi_D\big\|_{b+\varepsilon,p;\Omega}$. □

17.A.3 Quasi-local inverse of the Liouvillian

In this section, we briefly recall the construction of the quasi-local inverse of the Liouvillian $\mathcal{I}$ (see (17.1.4) and (17.A.10)) and a related operator $\mathcal{J}$ used in (17.4.11). Both of them use use certain properties (recall (17.1.5)–(17.1.6)) of a *weight function* w_g, which one can construct explicitly.

Lemma 17.A.9 (Explicit weight function, cf. Lemma 2.3 from [39])**.** *Let $g > 0$ and consider the sequence $(a_n)_{n\geq 1}$ of positive numbers, defined as $a_n = a_1(n(\log n)^2)^{-1}$ for $n \geq 2$ and a_1 chosen such that $\sum_{n\geq 1} a_n = \gamma/2$. Then, the positive function $w_g \in L^1(\mathbf{R})$ defined via the infinite product*

$$w_g(t) := c_g \prod_{n=1}^{\infty} \left(\frac{\sin(a_n t)}{a_n t} \right)^2 \tag{17.A.9}$$

and $c_g > 0$ chosen such that $\int_{\mathbf{R}} \mathrm{d}t\, w_g(t) = 1$, has Fourier transform $\widehat{w_g}$ with compact support $\mathrm{supp}(\widehat{w_g}) \subset [-g, g]$ (cf. (17.1.5)) and satisfies the bound $|w_g(t)| \leq C\mathrm{e}^{-|t|^q}$ for every $q < 1$ (cf. (17.1.6)).

Given the explicit weight function (17.A.9), the quasi-local inverse of the Liouvillian $\mathcal{I}_{H,g}[\![\,\cdot\,]\!] : \mathcal{A} \to \mathcal{A}$ of the Hamiltonian H with gap parameter $g > 0$, acting on $A \in \mathcal{A}$, is then defined as

$$\mathcal{I}_{H,g}[\![A]\!] := \int_{\mathbf{R}} \mathrm{d}t\, w_g(t) \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}Hs} A\, \mathrm{e}^{-\mathrm{i}Hs}. \tag{17.A.10}$$

Remark 17.A.10 (On the weight function)**.** *We point out, that, in principle and unless additional conditions are given, any map $\mathcal{I}_{H,g}$ with the properties (17.1.4)–(17.1.6) would work for all of our proofs in this paper, in particular including the statements from Section 17.4.*

Together with the map $\mathcal{J}_{H,g}: \mathcal{A} \to \mathcal{A}$, again depending on the Hamiltonian H and gap parameter $g > 0$, with action on $A \in \mathcal{A}$ defined as

$$\mathcal{J}_{H,g}[\![A]\!] := \int_{\mathbf{R}} \mathrm{d}t\, w_g(t)\, \mathrm{e}^{\mathrm{i}Ht} A\, \mathrm{e}^{-\mathrm{i}Ht},$$

one then has (recalling the Liouvillian $\mathcal{L}_H[\![\,\cdot\,]\!] = -\mathrm{i}\,[H, \,\cdot\,]$) the identity $\mathcal{L}_H \circ \mathcal{I}_{H,g}[\![A]\!] - A = \mathcal{J}_{H,g}[\![A]\!]$ for all $A \in \mathcal{A}$; see (17.4.11).

The inverse Liouvillian $\mathcal{I}_{H,g}$ is called *quasi-local*, since, if H satisfies the Lieb-Robinson bound from Lemma 17.A.3, then it holds that, for $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ (see, e.g., [461, Example 5.7])

$$\|*\|\,[\mathcal{I}_{H,g}[\![A]\!], B] \leq C\,\|A\|\,\|B\|\,\min\{|X|, |Y|\}\chi_{\tilde{b},\tilde{p}}\big(d(X,Y)\big) \tag{17.A.11}$$

for some $\tilde{b} > 0$ (depending on the Lieb-Robinson velocity v from Lemma 17.A.3) and $\tilde{p} \in (0,1)$, which can be chosen as p from (17.A.1) if $p < 1$. The estimate (17.A.11) holds verbatim with $\mathcal{I}$ replaced by $\mathcal{J}$.

Beside the classical quasi-locality estimate (17.A.11), the inverse Liouvillian even preserves locality of SLT operators. This is the content of the following lemma, the special case for $p = 1$ without localization (i.e. for Λ-localized SLT operators) already appeared in [428, Theorem 7.5.6] and is based on [39, Theorem 4.8].

Lemma 17.A.11 (Quasi-local inverse of the Liouvillian on SLT operators)**.** *Let $d \in \mathbf{N}$, $C_{\mathrm{vol}} > 0$, $\varepsilon > 0$, a, $b > 0$, p, $q \in (0,1]$ satisfying $q < p$, and $C_{\mathrm{int}} > 0$. There exist a constant $C > 0$ such that for all lattices $\Lambda \in \mathcal{G}(d, C_{\mathrm{vol}})$, the following holds: For SLT operators H and D it holds that $\mathcal{I}_{H,g}(D)$ is an SLT operator as well. More precisely if $\|\Phi_H\|_{b,p} \leq C_{\mathrm{int}}$, then there exists an interaction $\Phi_{\mathcal{I}_{H,g}(D)}$ such that*

$$\|\Phi_{\mathcal{I}_{H,g}(D)}\|_{a,q;\Omega} \leq C\,\big\|\Phi_D\big\|_{a+\varepsilon,q;\Omega}.$$

The statement holds verbatim when replacing $\mathcal{I}$ by $\mathcal{J}$ from (17.4.11).

In the proof of the Lemma we use the equality

$$\mathcal{I}_{H,g}[\![A]\!] = \int_{\mathbf{R}} \mathrm{d}t\, \mathcal{W}_g(t)\, \mathrm{e}^{\mathrm{i}Ht} A \mathrm{e}^{-\mathrm{i}Ht},$$

where the function $\mathcal{W}_g$ is given by $\mathcal{W}_g(t) = -\int_{-\infty}^{t} \mathrm{d}s\, w_g(s) + \mathbf{1}_{[0,\infty)}(t)$ with $\mathbf{1}_{[0,\infty)}$ being the characteristic function of $[0,\infty)$. In many works, e.g. [39, 36, 452, 556, 461], this is used as a definition for the inverse Liouvillian. It can easily be checked that, by Lemma 17.A.9, also $\mathcal{W}_g$ satisfies $|\mathcal{W}_g(t)| \leq C\mathrm{e}^{-|t|^q}$.

Proof. The proof uses the same technique as the proofs of Lemma 17.A.7 and 17.A.8.

First fix $X \subset \Lambda$ and $O \in \mathcal{A}_X$ and denote $\tau(O) = \mathrm{e}^{\mathrm{i}H} O \mathrm{e}^{-\mathrm{i}H}$. Then, define

$$\Delta_0(O) = \mathrm{i} \int_{\mathbf{R}} \mathrm{d}s\, \mathcal{W}_g(s)\, \mathbb{E}_X\big(\tau_s(O)\big)$$

and

$$\begin{aligned}\Delta_k(O) &= \mathrm{i} \int_{\mathbf{R}} \mathrm{d}s\, \mathcal{W}_g(s) \Big(\mathbb{E}_{X_k}\big(\tau_s(O)\big) - \mathbb{E}_{X_{k-1}}\big(\tau_s(O)\big)\Big) \\ &= \mathrm{i} \int_{\mathbf{R}} \mathrm{d}s\, \mathcal{W}_g(s)\, \mathbb{E}_{X_k}\Big(\big(\mathbf{1} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big)\end{aligned}$$

for $k \geq 1$. Then $\mathcal{I}_{H,g}(O) = \sum_{k\in\mathbf{N}} \Delta_k(O)$ where the sum is eventually finite.

For $k = 0$ we have

$$\|\Delta_0(O)\| \leq \|\mathcal{W}_g\|_{L^1}\, \|O\|.$$

For $b' \in (0,b)$, $k \geq 1$ and some $T > 0$ to be chosen, Lemma 17.A.3 and the properties of the conditional expectation, yield

$$\begin{aligned}&\Big\|\mathrm{i} \int_{-T}^{T} \mathrm{d}s\, \mathcal{W}_g(s)\, \mathbb{E}_{X_k}\Big(\big(\mathbf{1} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big)\Big\| \\ &\quad\leq \|\mathcal{W}_g\|_{L^\infty} \int_{-T}^{T} \mathrm{d}s\, \Big\|\big(\mathbf{1} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big\| \\ &\quad\leq C_{\mathrm{LR}}\, \|O\|\, |X|\, \|\mathcal{W}_g\|_{L^\infty} \int_{-T}^{T} \mathrm{d}s\, (\mathrm{e}^{b' v |s|} - 1)\, \chi_{b',p}(k) \\ &\quad= 2\, C_{\mathrm{LR}}\, \|O\|\, |X|\, \|\mathcal{W}_g\|_{L^\infty} \frac{\mathrm{e}^{b' v T} - b' v T - 1}{b' v}\, \chi_{b',p}(k) \\ &\quad\leq 2\, C_{\mathrm{LR}}\, \|O\|\, |X|\, \|\mathcal{W}_g\|_{L^\infty}\, \chi_{b'/2,p}(k)/(b' v)\end{aligned}$$

where we chose $T = k^p/(2v)$ for the last step. Furthermore, by Lemma 17.A.9 and after integrating twice, for any $0 < \tilde{p} < 1$ there exists C and $\tilde{b} > 0$ such that

$$\begin{aligned}&\Big\|\mathrm{i} \int_{|s|\geq T} \mathrm{d}s\, \mathcal{W}_g(s)\, \mathbb{E}_{X_k}\Big(\big(\mathbf{1} - \mathbb{E}_{X_{k-1}}\big)\big(\tau_s(O)\big)\Big)\Big\| \\ &\quad\leq 2\, \|O\| \int_{|s|\geq T} \mathrm{d}s\, |\mathcal{W}_g(s)| \\ &\quad\leq C\, \chi_{\tilde{b},\tilde{p}}(T) \\ &\quad\leq C\, \chi_{\tilde{b}',\tilde{p}p}(k),\end{aligned}$$

where $\tilde{b}' = \big(1/(2v)\big)^{\tilde{p}}\, \tilde{b}$. Then, for all $p' \in (0,p)$ we can choose $\tilde{p} = p'/p < 1$ and combine the two bounds. Then, there exists C and $\eta > 0$ such that

$$\|\Delta_k(O)\| \leq C\, |X|\, \|O\|\, \chi_{\eta,p'}(k) \qquad \text{for all} \qquad k \geq 0. \tag{17.A.12}$$

An interaction for $A := \mathcal{I}_{H,g}(D)$ is given by

$$\Phi_A(Z) = \sum_{k=0}^{\infty} \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \Delta_k(\Phi_D(Y)).$$

It follows that

$$\begin{aligned}\sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \frac{\|\Phi_A(Z)\|}{\chi_{a,q}\big(\operatorname{diam}(Z)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)} &\leq \sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \sum_{k=0}^{\infty} \sum_{\substack{Y \subset \Lambda: \\ Y_k = Z}} \frac{\|\Delta_k\big(\Phi_D(Y)\big)\|}{\chi_{a,q}\big(\operatorname{diam}(Z)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)} \\ &= \sum_{k=0}^{\infty} \sum_{Y \subset \Lambda} \mathbf{1}_{z \in Y_k} \frac{\|\Delta_k\big(\Phi_D(Y)\big)\|}{\chi_{a,q}\big(\operatorname{diam}(Y_k)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)}.\end{aligned}$$

The $k = 0$ term is bounded by $\|\mathcal{W}_g\|_{L^1}\,\|\Phi_D\|_{a,q;\Omega}$. For $k \geq 1$ and $z \in Y_k$ there exists $y \in B_z^\Lambda(k) \cap Y$ such that $d(z,\Omega) \leq k + d(y,\Omega)$. Furthermore, $\operatorname{diam}(Y_k) \leq 2k + \operatorname{diam}(Y)$. Hence, using (17.A.12) the rest of the sum is bounded by

$$\begin{aligned}&C \sum_{k=1}^{\infty} \chi_{\eta,p'}(k) \sum_{y \in B_z^\Lambda(k)} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{\|\Phi_D(Y)\|\,|Y|}{\chi_{a,q}\big(\operatorname{diam}(Y_k)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)} \\ &\leq C \sum_{k=1}^{\infty} \frac{\chi_{\eta,p'}(k)}{\chi_{a,q}(2k)\,\chi_{a,q}(k)} \sum_{y \in B_z^\Lambda(k)} \mathcal{C}_{\mathrm{vol},\varepsilon,q,1} \sum_{\substack{Y \subset \Lambda: \\ y \in Y}} \frac{\|\Phi_D(Y)\|}{\chi_{a+\varepsilon,q}\big(\operatorname{diam}(Y)\big)\,\chi_{a,q}\big(d(y,\Omega)\big)} \\ &\leq C\,\mathcal{C}_{\mathrm{vol},\varepsilon,q,1}\,\|\Phi_D\|_{a+\varepsilon,q;\Omega} \sum_{k=1}^{\infty} \frac{\chi_{\eta,p'}(k)}{\chi_{(2^q+1)a,q}(k)} \big(1 + C_{\mathrm{vol}}\,k^d\big).\end{aligned}$$

The remaining sum is bounded if $p' > q$, which we can ensure if $p > q$ by choosing $p' \in (q,p)$. Thus, there exists C such that

$$\sum_{\substack{Z \subset \Lambda: \\ z \in Z}} \frac{\|\Phi_A(Z)\|}{\chi_{a,q}\big(\operatorname{diam}(Z)\big)\,\chi_{a,q}\big(d(z,\Omega)\big)} \leq C\,\|\Phi_D\|_{a+\varepsilon,q;\Omega}$$

for some constant C depending on $\|\Phi_H\|_{p,b}$, p, b, q, a, ε and d which finishes the proof. □

Remark 17.A.12 (Abstract properties of $\mathcal{I}$ needed in the proof of our main result)**.** *For the purpose of proving our main result, Theorem 17.3.1, it is not necessary to work with the explicit $\mathcal{I}_{H,g}$ from* (17.A.10)*. In fact, by inspecting the proof of Proposition 17.3.4 in Section 17.5, which is the key input for our main result, we realize the following: For Theorem 17.3.1 being valid (up to minor adjustments of constants), one only needs that there exists some operator $\widetilde{\mathcal{I}} : \mathcal{A} \to \mathcal{A}$ for which Assumption* ($\mathrm{LDG}_{\mathrm{intro}}$) *is satisfied and such that*

(i) for $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ it holds that (cf. (17.A.15)*)*

$$\Big|\big\langle\big[\mathcal{L}_H \circ \widetilde{\mathcal{I}}[\![A]\!] - A, B\big]\big\rangle_{\rho_0}\Big| \leq C\,\|A\|\,\|B\|\,\operatorname{diam}(X)^\ell \exp(-d(X,Y)^q)$$

for some positive constants $C, q, \ell > 0$, i.e. the composition $\mathcal{L}_H \circ \widetilde{\mathcal{I}}$ behaves as a quasi-local operator if tested in the above way;

(ii) Lemma 17.A.11 holds, i.e. $\widetilde{\mathcal{I}}$ maps localized SLT operators to localized SLT operators.

These relaxed abstract conditions are, however, not sufficient for showing the relations among the various gap conditions outlined in Section 17.4.

17.A.4 Localized SLT operators: Proof of Lemma 17.2.1

For (i), it suffices to realize that, since for strictly Ω-localized Φ it holds that $\Phi(Z) = 0$ whenever $Z \cap (\Lambda \setminus \Omega) \neq \varnothing$, we have

$$\|\Phi\|_{b,p;\Omega} = \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Omega:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\operatorname{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} \leq \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\operatorname{diam}(Z)\big)} = \|\Phi\|_{b,p} \leq C.$$

Next, for (ii) and strongly Ω-localized Φ, we have that

$$\begin{aligned}\|\Phi\|_{b/2,p;\Omega} &= \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ Z\cap\Omega\neq\varnothing,\, z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b/2,p}\big(\operatorname{diam}(Z)\big)\,\chi_{b/2,p}\big(d(z,\Omega)\big)} \\ &\leq \sup_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \frac{\|\Phi(Z)\|}{\chi_{b,p}\big(\operatorname{diam}(Z)\big)} = \|\Phi\|_{b,p} \leq C\end{aligned}$$

since $d(z,\Omega) \leq \operatorname{diam}(Z)$ for $z \in Z$ and $Z \cap \Omega \neq \varnothing$, together with monotonicity of $\chi_{b/2,p}$ and using $(\chi_{b/2,p})^2 = \chi_{b,p}$. This concludes the proof of Lemma 17.2.1. □

17.A.5 Assumption (LDG$_{\text{weak}}$) and SLT operators: Proof of Lemma 17.5.1

We write $A = \sum_{Z\subset\Lambda} \Phi_A(Z)$ (cf. (17.2.4)) and estimate

$$\begin{aligned}&\Big|\big\langle[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![A]\!] - A, B]\big\rangle_{\rho_0}\Big| \\ &\qquad \leq \sum_{Z\subset\Lambda} \Big|\Big\langle\big[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![\Phi_A(Z)]\!] - \Phi_A(Z), B\big]\Big\rangle_{\rho_0}\Big|.\end{aligned} \tag{17.A.13}$$

For the 'small' $Z \subset \Lambda$ satisfying $\operatorname{diam}(Z) \leq d(Z, \Lambda \setminus \Lambda^{\mathrm{gap}})^\beta$, we bound

$$\begin{aligned}&\Big|\Big\langle\big[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![\Phi_A(Z)]\!] - \Phi_A(Z), B\big]\Big\rangle_{\rho_0}\Big| \\ &\qquad \leq C_{\mathrm{gap}}\, \|\Phi_A(Z)\|\, \|B\|\, \operatorname{diam}(Y)^\ell\, \chi_{b,p}\big(d(Z, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)\end{aligned} \tag{17.A.14}$$

by means of Assumption (LDG$_{\text{weak}}$). Additionally, we need the following alternative estimate on (17.A.14) (recall (17.4.11)):

$$\begin{aligned}&\Big|\Big\langle\big[\mathcal{L}_{H_0} \circ \mathcal{I}_{H_0,g}[\![\Phi_A(Z)]\!] - \Phi_A(Z), B\big]\Big\rangle_{\rho_0}\Big| \\ &\quad \leq \int_{\mathbf{R}} \mathrm{d}t\, w_g(t)\, \big\|\big[\mathrm{e}^{\mathrm{i}tH_0}\, \Phi_A(Z)\, \mathrm{e}^{-\mathrm{i}tH_0}, B\big]\big\| \\ &\quad \leq C\, \|\Phi_A(Z)\|\, \|B\|\, \operatorname{diam}(Z)^d \Big(\chi_{b/2,p}\big(d(Z,Y)\big) \int_I \mathrm{d}t\, w_g(t) + \int_{\mathbf{R}\setminus I} \mathrm{d}t\, w_g(t)\Big) \\ &\quad \leq C\, \|\Phi_A(Z)\|\, \|B\|\, \operatorname{diam}(Z)^d\, \chi_{b/2,p}\big(d(Z,Y)\big)\end{aligned} \tag{17.A.15}$$

where we denoted $I := \{t \in \mathbf{R} : v|t| \leq d(Z,Y)^p/3\}$. Here, v is the Lieb-Robinson velocity from Lemma 17.A.3, which we employed in the second step with $b \to 3b/4$. In the final step, we used the stretched exponential decay of w_g (see (17.1.6) and Lemma 17.A.9). Note that (17.A.14) and (17.A.15) track two different relevant distances, namely those of Z to $\Lambda \setminus \Lambda^{\mathrm{gap}}$ and Y, respectively.

In fact, a weighted geometric mean of (17.A.14) and (17.A.15), that combines these two effects, can now be summed up as (neglecting the factor $C\, \|B\|\, \operatorname{diam}(Y)^\ell$, which will be put back in (17.A.17))

$$\sum_{\substack{Z\subset\Lambda:\\ \operatorname{diam}(Z)\leq d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta}} \operatorname{diam}(Z)^d\, \chi_{b-\varepsilon,p}\big(d(Z, \Lambda \setminus \Lambda^{\mathrm{gap}})\big)\, \chi_{\varepsilon/2,p}\big(d(Z,Y)\big)\, \|\Phi_A(Z)\|$$

$$
\begin{aligned}
&\leq \sum_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \|\Phi_A(Z)\| \, \frac{\chi_{b-\varepsilon,p}\big(\mathrm{diam}(Z) + d(z,\Omega) + d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)} \, \chi_{\varepsilon/2,p}\big(d(z,Y)\big) \\
&\leq \|\Phi_A\|_{b,p;\Omega}\ \chi_{b-\varepsilon,p}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big) \sum_{z\in\Lambda} \chi_{\varepsilon/2,p}\big(d(z,Y)\big) \\
&\leq C\ \mathrm{diam}(Y)^d\ \|\Phi_A\|_{b,p;\Omega}\ \chi_{b-\varepsilon,p}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big). \qquad (17.A.16)
\end{aligned}
$$

For the first bound, we used logarithmic superadditivity of $\chi_{b,p}$ together with elementary monotonicity properties from Lemma 17.A.1 (a) and estimated $\mathrm{diam}(Z)^d$ by $1/\chi_{\varepsilon,p}\big(\mathrm{diam}(Z)\big)$. For the second bound, we used the definition of $\|\Phi_A\|_{b,p;\Omega}$ from (17.2.7) and the fact that, for $z \in Z$, we have $\mathrm{diam}(Z) + d(z,\Omega) + d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}}) \geq d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})$. In the final step, we employed summability of $\chi_{\varepsilon/2,p}\big(d(z,Y)\big)$.

Therefore, combining (17.A.13) with (17.A.16), the contribution of those $Z \subset \Lambda$, for which $\mathrm{diam}(Z) \leq d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$, to (17.A.13) is bounded by

$$
C\ \mathrm{diam}(Y)^{d+\ell}\ \|B\|\ \|\Phi_A\|_{b,p;\Omega}\ \chi_{b-\varepsilon,p}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big). \qquad (17.A.17)
$$

For the 'large' $Z \subset \Lambda$ that satisfy $\mathrm{diam}(Z) > d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$, we simply use the estimate from (17.A.15), which we can sum up as

$$
\begin{aligned}
&\sum_{\substack{Z\subset\Lambda:\\ \mathrm{diam}(Z)>d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta}} \mathrm{diam}(Z)^d\, \chi_{b/2,p}\big(d(Z,Y)\big)\ \|\Phi_A(Z)\| \\
&\leq \sum_{z\in\Lambda} \sum_{\substack{Z\subset\Lambda:\\ z\in Z}} \|\Phi_A(Z)\|\ \frac{\chi_{b/2,p\beta}\big(\mathrm{diam}(Z) + d(z,\Omega) + d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)}{\chi_{b,p}\big(\mathrm{diam}(Z)\big)\,\chi_{b,p}\big(d(z,\Omega)\big)}\ \chi_{b/2,p}\big(d(z,Y)\big) \qquad (17.A.18) \\
&\leq C\ \mathrm{diam}(Y)^d\ \|\Phi_A\|_{b,p;\Omega}\ \chi_{b/2,p\beta}\big(d(\Omega,\Lambda\setminus\Lambda^{\mathrm{gap}})\big)
\end{aligned}
$$

analogously to (17.A.16). In the second step we used that $\mathrm{diam}(Z) > d(Z,\Lambda\setminus\Lambda^{\mathrm{gap}})^\beta$ and elementary monotonicity properties of $\chi_{b,p}$ in b, p.

Therefore, by means of (17.A.18), also the large Z's contribute only in a way that is controlled in terms of (17.A.17) (but with worse constants $b/2$ and $p\min\{\beta,1\}$). This concludes the proof of Lemma 17.5.1. □

Appendix: Miscellaneous Results

Chapter A

Deformational rigidity of Liouville metrics on the torus

This chapter includes (the extended arXiv:2210.02961 version of) the paper [334]:

J. Henheik. Deformational rigidity of integrable metrics on the torus. *Ergod. Theory Dyn. Syst.*, 45:467–503, 2025

Abstract. It is conjectured that the only integrable metrics on the two-dimensional torus are Liouville metrics. In this paper, we study a deformative version of this conjecture: We consider integrable deformations of a non-flat Liouville metric in a conformal class and show that for a fairly large class of such deformations the deformed metric is again Liouville. The principal idea of the argument is that the preservation of rational invariant tori in the foliation of the phase space forces a linear combination on the Fourier coefficients of the deformation to vanish. Showing that the resulting linear system is non-degenerate will then yield the claim. Since our method of proof immediately carries over to higher dimensional tori, we obtain analogous statements in this more general case. In order to put our results in perspective, we review existing results about integrable metrics on the torus.

Content

A.1 Introduction

Let $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ be the two-dimensional torus being equipped with a C^2-smooth global Liouville metric g, i.e. having line element

$$\mathrm{d}s^2 = \big(f_1(x^1) + f_2(x^2)\big)\big((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\big), \tag{A.1.1}$$

where $(x^1, x^2) \in \mathbf{T}^2$ are the standard periodic coordinates and $f_1, f_2 \in C^2(\mathbf{T})$ are positive Morse functions[1] or positive constants and thus 'non-degenerate'. The corresponding geodesic flow (see Section A.1.1) is well known to be integrable and a longstanding folklore conjecture says that *Liouville metrics are the only integrable metrics on* $\mathbf{T}^2$. We emphasize that, in this context, integrability always allows for singularities in the foliation of the phase space of the naturally associated Hamiltonian system, which is made precise in Definition A.1.3 below.

Although the validity of the folklore conjecture appeared conceivable for a long time, there is strong indication for it being false in its very general form, as shown in [182]: Here, the authors constructed a Hamiltonian counterexample which is *locally* integrable in a p-cone in the cotangent bundle. This means that, on a fixed energy level, there exists an analytic change of variables, transforming the Hamiltonian with non-Liouville potential to the standard form $(p_1^2 + p_2^2)/2$ but only for p_i's in a certain cone in $\mathbf{R}^2$ (see also Theorem A.3.7 below for a more precise statement). However, despite this delicate example, certain suitably weakened conjectures are still believed to be true, which is supported by a variety of partial results obtained in this direction, starting from classical ones by Dini [225], Darboux [196], and Birkhoff [84] and further developed in [34, 367, 374]. In particular, several works by Bialy, Mironov [72, 77, 78, 76], Denisova, Kozlov, Treshev [212, 213, 214, 382, 215], Mironov [449], and others [34, 374, 12, 547], strongly indicate the validity of the following (yet unproven) conjecture:[2] *Every polynomially integrable metric g on $\mathbf{T}^2$ is of Liouville type.* We refer to Section A.3 for details.

In this paper, we are concerned with a *perturbative* version of the folklore conjecture: Let $(g_\varepsilon)_{|\varepsilon|\le\varepsilon_0}$ for some small $\varepsilon_0 > 0$ be a family of perturbations of $g \equiv g_0$ in the same conformal class[3] having line-element

$$\mathrm{d}s_\varepsilon^2 = \big(f_1(x^1) + f_2(x^2) + \varepsilon\lambda(x^1, x^2)\big)\big((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\big), \tag{A.1.2}$$

where $\lambda \in C^2(\mathbf{T}^2)$ is assumed to be a Morse function (or constant) and have an absolutely convergent Fourier series. We will assume that the perturbed family g_ε *remains integrable*, meaning that within the foliation of the phase space for the *unperturbed* Liouville metric (A.1.1), the deformation (A.1.2)

[1] Recall that Morse functions on a manifold M are characterized by having no degenerate critical points. They form a dense and open set in $C^2(M)$ and are thus 'generic'.

[2] See [131, 92] for recent surveys on open problems and questions concerning geodesics and integrability of finite-dimensional systems.

[3] Note that on the torus there exist global isothermal coordinates [94, Chapter 11].

preserves sufficiently many *rational invariant tori* (see Assumption (P) below for a precise formulation of our requirement on the preservation of these tori). Then we obtain that λ is necessarily *separable* in a sum of two single-valued functions, i.e.

$$\lambda(x^1, x^2) = \lambda_1(x^1) + \lambda_2(x^2)$$

for some $\lambda_1, \lambda_2 \in C^2(\mathbf{T})$. Therefore, our main results formulated (somewhat informally) below assert the following:

> *The class of Liouville metrics is deformationally rigid under a fairly wide class of integrable conformal perturbations.*

To the best of our knowledge, this is the first instance of a rigidity result for (not necessarily analytically) integrable dynamical systems allowing for singularities in the invariant foliation of the unperturbed system. The precise statements of our main results are given in Theorem A.I, Theorem A.II and Theorem A.III in Section A.2.

Main Result. *Let g be a non-degenerate Liouville metric on $\mathbf{T}^2$ as in* (A.1.1) *and assume that the family $(g_\varepsilon)_{|\varepsilon|\leq\varepsilon_0}$ of perturbations defined in* (A.1.2) *remains integrable. Then we have the following:*

(i) In case that $f_1, f_2 \equiv \mathrm{const.}$, then λ is separable.

(ii) In case that $f_1 \equiv \mathrm{const.}$, λ is a trigonometric polynomial in x^2 and the relative difference μ_2 between f_2 and its mean $\int_{\mathbf{T}} f_2$, i.e. $\mu_2 := \|f_2 - \int_{\mathbf{T}} f_2\|_{C^0} / \int_{\mathbf{T}} f_2$ is small, then λ is separable. If, additionally, f_2 is analytic, we have that λ is separable, irrespective of the size μ_2 of the fluctuations of f_2 (but only for μ_2 outside of an exceptional (Lebesgue) null-set).

(iii) In general, if λ is a trigonometric polynomial and the relative differences μ_i, $i = 1, 2$, between the f_i's and their means $\int_{\mathbf{T}} f_i$, i.e. the $\mu_i := \|f_i - \int_{\mathbf{T}} f_i\|_{C^0} / \int_{\mathbf{T}} f_i$ are small, then λ is separable. If, additionally, f_i is analytic (for one or both $i = 1, 2$), we have that λ is separable, irrespective of the size μ_i of the fluctuations of f_i (outside of an exceptional null-set).

It is straightforward to generalize our results to higher dimensional tori $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$. In order to ease notation and make the presentation clearer, we only mention it here and postpone a more detailed discussion to Appendix A.A.

Remark A.1.1. (Generalization to higher dimensions)
Analogously to (A.1.1), *let $\mathbf{T}^d$ be equipped with a C^2-smooth global Liouville metric g having line element*

$$\mathrm{d}s^2 = \big(f_1(x^1) + \ldots + f_d(x^d)\big)\big((\mathrm{d}x^1)^2 + \ldots + (\mathrm{d}x^d)^2\big), \tag{A.1.3}$$

where $x = (x^1, ..., x^d) \in \mathbf{T}^d$ are standard periodic coordinates and $f_i \in C^2(\mathbf{T})$ for $1 \leq i \leq d$ are positive Morse functions or constants. Again, it is easy to see that the geodesic flow is integrable. Just as in (A.1.2), *we now perturb* (A.1.3) *in the same conformal class by some $\lambda \in C^2(\mathbf{T}^d)$ having an absolutely convergent Fourier series.*

Under the assumption that the family of perturbed metrics $(g_\varepsilon)_{|\varepsilon|\leq\varepsilon_0}$ remains integrable, we have the following (somewhat informal) rigidity result:

Let $f_i \equiv \mathrm{const.}$ for the first $0 \leq d_{\mathrm{flat}} \leq d$ indices, and f_j be analytic for the last $0 \leq d_{\mathrm{anlyt}} \leq d - d_{\mathrm{flat}}$ indices. Then, if λ is a trigonometric polynomial in x^k for $k \in \{d_{\mathrm{flat}} + 1, ..., d\}$, and the relative differences between $f_{d_{\mathrm{flat}}+1}, ..., f_{d-d_{\mathrm{anlyt}}}$ and their mean values are small, we have that λ is separable, irrespective of size μ_j of the fluctuations of f_j (outside of a null-set).

This results unifies and generalizes the three separate statements given above. A precise formulation is given in Theorem A.IV in Appendix A.A.

The present paper is not the first study on *rigidity* of important integrable systems: In [33, 362], Avila-Kaloshin-de Simoi and Kaloshin-Sorrentino recently solved both, a deformative and a perturbative version of the famous *Birkhoff conjecture* concerning integrable billiards in two dimensions. In a nutshell, their result says that *a strictly convex domain with integrable billiard dynamics sufficiently close to an ellipse is necessarily an ellipse*. This can be viewed as an analogue of the perturbative version of the folklore conjecture formulated above [361]. More precisely, our main results concerning general $f_i \in C^2(\mathbf{T})$ are similar – in spirit – to the deformational rigidity for ellipses of *small* eccentricity (cf. f_1, f_2 in (A.1.1) having small fluctuations), which has been shown first in [33], later extended by Huang-Kaloshin-Sorrentino [352] to a local notion of integrability and finally significantly improved in [378]. The overall strategy pursued in [33, 362, 352] also inspired the arguments employed in the present paper.

In a more recent work, Arnaud-Massetti-Sorrentino [26] (replacing the earlier preprint [440]) studied the rigidity of integrable symplectic twist maps on the $2d$-dimensional annulus $\mathbf{T}^d \times \mathbf{R}^d$. More precisely, they consider one-parameter families $(f_\varepsilon)_{\varepsilon \in \mathbf{R}}$ of symplectic twist maps $f_\varepsilon(x,p) = f_0(x, p + \varepsilon \nabla G(x))$ and prove two main rigidity results: First, in the analytic category for f_0 and the perturbation G, if a *single* rational invariant Lagrangian graph of f_ε exists for infinitely many values of ε (e.g. an interval around zero), then G must necessarily be constant. Second, if f_0 is analytic and *completely integrable* (i.e. not plagued with singularities in the invariant foliation of the phase space, see [75, 543]), G is of class C^2, and sufficiently (infinitely) many rational invariant Lagrangian graphs of f_ε persist for small $\varepsilon \neq 0$, then G must necessarily be constant. Note that in this second result, the entire phase space is foliated by invariant tori, and the perturbation solely depends on the angle variables of the dynamical system. In this sense, Theorem A.I can – morally – be viewed as a special case of the second result in [26] (see also [440, Theorem 2]), but Theorem A.II and Theorem A.III generalize this statement to more general functional dependencies of the perturbation. Apart from this, our general results (i.e. those not concerning analytic functions f_i) do not require any regularity beyond the standard C^2.

As mentioned above, by assuming that the family of metrics $(g_\varepsilon)_{|\varepsilon| \leq \varepsilon_0}$ remains integrable, we mean that, in particular, sufficiently many rational invariant tori in an isoenergy manifold of the Hamiltonians associated to the metric by the *Maupertuis principle* (see Section A.1.3) are preserved. This will be made precise in Assumption (P) below. As we will show, the preservation of an (n,m)-rational invariant torus 'annihilates' the Fourier coefficients λ_{k_1,k_2} with indices $(k_1,k_2) \in \{(n,m)\}^\perp$ of

$$\lambda(x,y) = \sum_{(k_1,k_2) \in \mathbf{Z}^2} \lambda_{k_1,k_2} \mathrm{e}^{2\pi \mathrm{i}(k_1 x + k_2 y)},$$

or of the corresponding perturbing mechanical potential, denoted by U later on. We already noted that, contrary to items (ii) and (iii), the unperturbed metric in our first result is guaranteed to be *completely integrable*. Moreover, the perturbation λ depends solely on the *angular* but not the *action* coordinates of the unperturbed problem (see Theorem A.1.6). Although the analog of this result for symplectic twist maps in this peculiar setting has already been shown in [26, 440] by methods similar to ours, we reprove it by pursuing an only slightly different but original strategy, which is suitable for certain inevitable modifications for the proofs of the more general statements under item (ii) and (iii). These two cases (corresponding to surfaces of revolution and general Liouville metrics, see Section A.3) build on perturbative estimates for (possibly infinitely many) systems of linear equations for the Fourier coefficients. These are obtained from the first order term of an expansion in ε, somewhat similar to the *(subharmonic) Melnikov potential* in the Poincaré-Melnikov method [303, 568, 30]. Establishing this expansion as well as proving that the resulting systems of linear equations are of full rank requires perturbative estimates on action-angle coordinates and certain basic objects from weak KAM theory [530]. Finally, the extension of our results for analytic functions f_i beyond the perturbative regime are proven by exploiting the analytic dependence of the linear system on the size μ_i of the fluctuations of f_i (see Appendix A.C).

In the remainder of this introduction, we recall basic notions in geometry and dynamical systems, which are frequently used in this paper, and introduce the problem of classifying integrable metrics on Riemannian manifolds, in particular the torus $\mathbf{T}^2$, as formulated in Questions (Q1) and (Q2) below. In Section A.2 we formulate our main results in Theorem A.I, A.II, and A.III. In Section A.3 we present related existing results and known partial answers on the classification problem for integrable metrics on the torus $\mathbf{T}^2$ (a few of which have already been mentioned above) in order to put our results into context. In Section A.4 we give the proofs of our main results, and, finally, comment on possible generalizations, different approaches and a list of open problems in Section A.5. As already mentioned above, the precise formulation of our result for higher dimensions is given in Theorem A.IV in Appendix A.A. A fundamental perturbation theoretic lemma on action-angle coordinates, a concise study on important analyticity properties of these, and a brief overview of the relevant aspects of weak KAM theory are presented in three further appendices.

A.1.1 Geodesic flow on Riemannian manifolds

Let (M, g) be a (compact) C^2-smooth n-dimensional connected Riemannian manifold without boundary equipped with the Riemannian metric $g = (g_{ij}(x))_{ij}$. Geodesics of the given metric g are defined as smooth parameterized curves $\gamma(t) = (x^1(t), \dots, x^n(t))$ that are solutions to the system of differential equations

$$\nabla_{\dot\gamma}\dot\gamma = 0\,, \tag{A.1.4}$$

where $\dot\gamma = \frac{\mathrm{d}\gamma}{\mathrm{d}t}$ denotes the velocity vector of the curve γ, and ∇ is the covariant derivative operator related to the Levi-Civita connection associated with the metric g.

It is well known that for every point $x \in M$ and for every tangent vector $v \in T_xM$ there exists a unique geodesic γ with $\gamma(0) = x$ and $\dot\gamma(0) = v$, which allows to define the *geodesic flow* as a local $\mathbf{R}$-action on the tangent bundle TM via

$$\mathbf{R} \ni t \mapsto \Psi_t(V) = \dot\gamma_V(t) \in TM\,,$$

where γ_V denotes the geodesic with initial data $\dot\gamma_V(0) = V \in TM$.

The geodesic equation (A.1.4) can also be viewed as a Hamiltonian system on the cotangent bundle T^*M, and the geodesics γ themselves can be regarded as projections of trajectories of the Hamiltonian system onto M. Therefore, let x and p be natural coordinates on the cotangent bundle T^*M, where $x = (x^1, \dots, x^n)$ are the coordinates of a point in M (position space), and $p = (p_1, \dots, p_n)$ are the coordinates of a covector from the cotangent space T^*_xM (momentum space) in the basis $\mathrm{d}x^1, \dots, \mathrm{d}x^n$. Let $\omega = \mathrm{d}x \wedge \mathrm{d}p$ on T^*M denote the standard symplectic structure and define the Hamiltonian function $H \in C^2(T^*M)$ as

$$H(x,p) = \frac{1}{2}\sum_{ij} g^{ij}(x)p_ip_j = \frac{1}{2}|p|^2_g\,. \tag{A.1.5}$$

The related Hamiltonian vector field X_H, defined via $\omega(X_H, \cdot) = \mathrm{d}H$, governs the associated *Hamiltonian flow* $\Phi_t^{X_H}$ as a local $\mathbf{R}$-action on the cotangent bundle T^*M. A trajectory $(x(t), p(t))$ is an integral curve for the Hamiltonian vector field, if and only if the Hamiltonian system of differential equations

$$\begin{cases} \dot p_i = & -\frac{\partial H}{\partial x^i} \\ \dot x^i = & \frac{\partial H}{\partial p_i} \end{cases}\,, \tag{A.1.6}$$

written in local coordinates, is satisfied. The Hamiltonian flow is also called a *cogeodesic flow* for this special case of a Hamiltonian function (A.1.5), and the geodesic flow and the cogeodesic flow are equivalent in the following sense.

Proposition A.1.2. (Geodesic flow and cogeodesic flow, Prop. 11.1 in [94])

(a) If $(x(t), p(t))$ is an integral curve for X_H on T^*M, then the curve $x(t)$ in M is a geodesic and its velocity vector $\dot{x}(t)$ satisfies $\dot{x}^i(t) = \sum_j g^{ij}(x)p_j(t)$.

(b) Conversely, if $x(t)$ is a geodesic in M, then the trajectory $(x(t), p(t))$, where $p_i(t) = \sum_j g_{ij}\dot{x}^j(t)$, is an integral curve for X_H on T^*M.

A.1.2 Integrable Hamiltonian systems

It is natural to ask for a classification of Riemannian manifolds (M, g), for which the geodesic equations (A.1.4) can be solved explicitly. In the language of integrability of Hamiltonian systems and using the equivalence between geodesic flow and cogeodesic flow from Proposition A.1.2, we can formulate the following questions:

(Q1) On which manifolds do there exist Riemannian metrics whose (co-)geodesic flow is integrable?

(Q2) Given such a manifold, how to characterize the class of metrics with integrable geodesic flow?

Clearly, the answers and their complexity hinge on the notion of integrability for the Hamiltonian system (see Section A.3). In this paper we will be concerned with the standard notion, that is *Liouville integrability*, which we recall for the readers convenience.

Definition A.1.3. *The geodesic flow on (M, g) is called Liouville integrable, if there exist n functions $F_1, ..., F_n \in C^2(T^*M)$ (called first integrals), that are*

(i) functionally independent on T^*M, i.e. the vector fields $X_{F_1}(x, p), ..., X_{F_n}(x, p)$ are linear independent in $T_{(x,p)}(T^*M)$ for all $(x, p) \in \mathcal{M} \subset T^*M$, where $\mathcal{M}$ is some open and everywhere dense set of full measure (cf. the restriction to Morse functions);

(ii) pairwise in involution, i.e.

$$\{F_k, F_l\} := \omega(X_{F_k}, X_{F_l}) = \sum_i \left(\frac{\partial F_k}{\partial x^i}\frac{\partial F_l}{\partial p_i} - \frac{\partial F_k}{\partial p_i}\frac{\partial F_l}{\partial x^i} \right) = 0\,.$$

Whenever the geodesic flow on (M, g) is Liouville integrable, we call g an integrable metric on M. Moreover, we call the Hamiltonian system (A.1.6) *(or the corresponding Hamiltonian* (A.1.5) *itself) integrable, whenever the associated metric g is integrable on M.*

Remark A.1.4. *Whenever the first integrals $F_1, ..., F_n$ can be chosen to be functions that are polynomially in the momentum variables, the metric is often called polynomially integrable or algebraically integrable. If we aim at indicating the order of the polynomial, we speak of linearly/quadratically/... integrable metrics.*

Remark A.1.5. *Note that, since one can always choose $H = F_1$ as a first integral for the geodesic flow, the question of integrability for one-dimensional manifolds is completely answered. Therefore, the simplest manifolds, for which the answers to* (Q1) *and* (Q2) *are non-trivial, are two-dimensional.*

In this work, we are mainly concerned with a characterization of integrable metrics in the sense of Question (Q2) for the two-dimensional torus $\mathbf{T}^2$. In this case, the largest known class of such metrics g are so called *Liouville metrics*, where the line element takes the form

$$\mathrm{d}s^2 = (f_1(x^1) + f_2(x^2))\left((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right)\,, \tag{A.1.7}$$

in appropriate global coordinates (x^1, x^2) and where f_1 and f_2 are sufficiently regular positive periodic functions. See Section A.3.2 for more details.

The most important result about integrable Hamiltonian systems is the following well known theorem, establishing the existence of so-called *action-angle coordinates*, which shall be employed in our proofs in Section A.4.

Theorem A.1.6. (Liouville-Arnold Theorem [29])
Let H be a Liouville integrable Hamiltonian on T^*M and let

$$T_{\boldsymbol{f}} = \{(x,p) \in T^*M : F_i(x) = f_i,\, i = 1,...,n\}$$

be a regular level surface of the first integrals $F_1,...,F_n$. Then we have the following:

(a) The level set $T_{\boldsymbol{f}} \subset T^*M$ is a smooth submanifold of dimension n that is invariant under the geodesic flow. Any compact connected component of $T_{\boldsymbol{f}}$ (again denoted by $T_{\boldsymbol{f}}$) is diffeomorphic to an n-dimensional torus $\mathbf{T}^n$, called a Liouville torus.

(b) There exists a neighborhood U of $T_{\boldsymbol{f}}$ and a coordinate system $(\theta, I) : U \to \mathbf{T}^n \times \mathbf{R}^n$ with $\omega = \sum_{i=1}^n \mathrm{d}\theta^i \wedge \mathrm{d}I_i$, called action-angle variables, *such that $T_{\boldsymbol{f}} = I^{-1}(0)$ is a level set of the action variables and $F_i = F_i(I)$. Therefore, the Hamiltonian equations* (A.1.6) *take the form*

$$\begin{cases} \dot{I}_i = & 0 \\ \dot{\theta}^i = & \omega_i(I_1,...,I_n)\,. \end{cases} \tag{A.1.8}$$

A.1.3 Maupertuis principle

In order to approach the questions (Q1) and (Q2), we will utilize the *Maupertuis Principle* (see, e.g., [96]): For a compact Riemannian manifold, (M,g), let

$$H(x,p) = \frac{1}{2}\sum_{ij} g^{ij}(x)p_ip_j - V(x) \tag{A.1.9}$$

be a natural mechanical Hamiltonian function on T^*M, where $V \in C^2(M)$ denotes some potential function. Moreover, let $T_h = \{H(x,p) = h\}$ be an isoenergy submanifold for some $h > -\min_x V(x)$ and note that T_h is also an isoenergy submanifold for another system with Hamiltonian function

$$\widetilde{H}(x,p) = \frac{1}{2}\sum_{ij} \frac{g^{ij}(x)}{h+V(x)}p_ip_j\,,$$

i.e. $T_h = \{\widetilde{H}(x,p) = 1\}$. Now, the *Maupertuis principle* states that the integral curves for the Hamiltonian vector fields X_H and $X_{\widetilde{H}}$ on the fixed isoenergy submanifold T_h coincide. Moreover, if there exists an additional first integral F for H on T_h, then there also exists a first integral $\tilde{F}$ for $\widetilde{H}$ on the *whole* of T^*M (except, potentially, at the zero section). Finally, note that the vector field $X_{\widetilde{H}}$ gives rise to the geodesic flow of the Riemannian metric $\widetilde{g}$ with

$$\widetilde{g}_{ij}(x) = (h+V(x))g_{ij}(x)\,, \tag{A.1.10}$$

which is the correspondence between Hamiltonian systems and geodesic flows we will use.

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A.2 Main results

The main results of this paper are rigidity results in the sense of Question (Q2) for classes of integrable metrics on the two-torus $\mathbb{T}^2 = \mathbf{R}^2/\Gamma$, initially equipped with the flat metric, and hence obtained by a Hamiltonian defined on $T^*\mathbf{T}^2$ by means of the Maupertuis principle. In general, $\Gamma \subset \mathbf{R}^2$ is an arbitrary lattice, but we focus on the case $\Gamma = \mathbf{Z}^2$ here. We define the Hamiltonian function

$$H_0(x,p) = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \mu_1 V_1(x^1) - \mu_2 V_2(x^2) \tag{A.2.1}$$

on $T^*\mathbb{T}^2$, where $\mu_i \in [0,\infty)$ are parameters, and $V_i \in C^2(\mathbb{T})$ with $V_i \geq 0$ and $\|V_i\|_{C^0} \leq \mathcal{C}_i$ are Morse functions (or constant). We may assume w.l.o.g. that $\min_{x^i} V_i(x^i) = 0$. This includes, e.g., the situation of two pendulums, i.e. $V_i(x^i) = 1 - \cos(2\pi x^i)$. The torus coordinates are denoted by $x = (x^1, x^2) \in \mathbb{T}^2$ and the conjugate coordinate pairs are (x^1, p_1) and (x^2, p_2). By the Maupertuis principle, for fixed $e > 0$, the Hamiltonian flow on the isoenergy manifold $T_e = \{H_0 = e\}$ coincides with the geodesic flow on $\mathbb{T}^2$ with the Liouville metric g_e (see eq. (A.1.7) and Section A.3.2 for more details) having line element

$$\mathrm{d}s_e^2 = \big(e + \mu_1 V_1(x^1) + \mu_2 V_2(x^2)\big)\big((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\big)\,.$$

The system with Hamiltonian function (A.2.1) is clearly integrable in the sense of Definition A.1.3, since an additional conserved quantity can easily be found as

$$F_1(x,p) = \frac{p_1^2}{2} - \mu_1 V_1(x^1)\,. \tag{A.2.2}$$

The Liouville foliation of T_e has the following qualitative structure, that is similar to the phase portrait of the pendulum. The common level surface

$$T_{(e,f)} = \{H_0 = e,\ F_1 = f\}$$

differs in shape, depending on the values of e and f. Recall that $e > 0$ and $V_i \geq 0$. If (i) $f \in (-\mu_1 \max_{x^1} V_1(x^1), 0)$ and $e - f > 0$, $T_{(e,f)}$ is an *annulus*; if (ii) $f > 0$ and $e - f > 0$, $T_{(e,f)}$ is a *torus*; if (iii) $f > 0$ and $e - f \in (-\mu_2 \max_{x^2} V_2(x^2), 0)$, $T_{(e,f)}$ is an *annulus*. Therefore, if V_1 and V_2 are both non-constant, the foliation qualitatively exhibits a pendulum-like phase portrait (see Figure A.1).

A.2.1 Definitions and assumptions

Our main results concern perturbations of the Hamiltonian function (A.2.1) in the class of mechanical systems as

$$H_\varepsilon(x,p) = H_0(x,p) + \varepsilon U(x)\,, \tag{A.2.3}$$

where $\varepsilon \in \mathbf{R}$ and $U \in C^2(\mathbb{T}^2)$ denotes a perturbing potential, which is assumed to be a Morse function (or a constant) and have an absolutely convergent Fourier series[4]

$$U(x) = \sum_{k_1 \in \mathbf{Z}} U_{k_1}(x^2)\, \mathrm{e}^{\mathrm{i}2\pi k_1 x^1} = \sum_{(k_1,k_2)\in\mathbf{Z}^2} U_{k_1,k_2}\, \mathrm{e}^{\mathrm{i}2\pi(k_1 x^1 + k_2 x^2)}\,.$$

In the following, we introduce several subsets of $\mathbf{Z}^2$ in such a way, that their definitions immediately carry over in arbitrary dimension $d \in \mathbf{N}$ (see Remark A.1.1). First, we define the *spectrum* of U, i.e. the set of non-vanishing Fourier coefficients, as

$$\mathcal{S}_U := \big\{\boldsymbol{k} = (k_1,k_2) \in \mathbf{Z}^2 : U_{\boldsymbol{k}} \neq 0\big\}\,, \tag{A.2.4}$$

[4] Note that in two dimensions, C^2-regularity is not sufficient for ensuring an absolutely convergent Fourier series, although in one dimension it is.

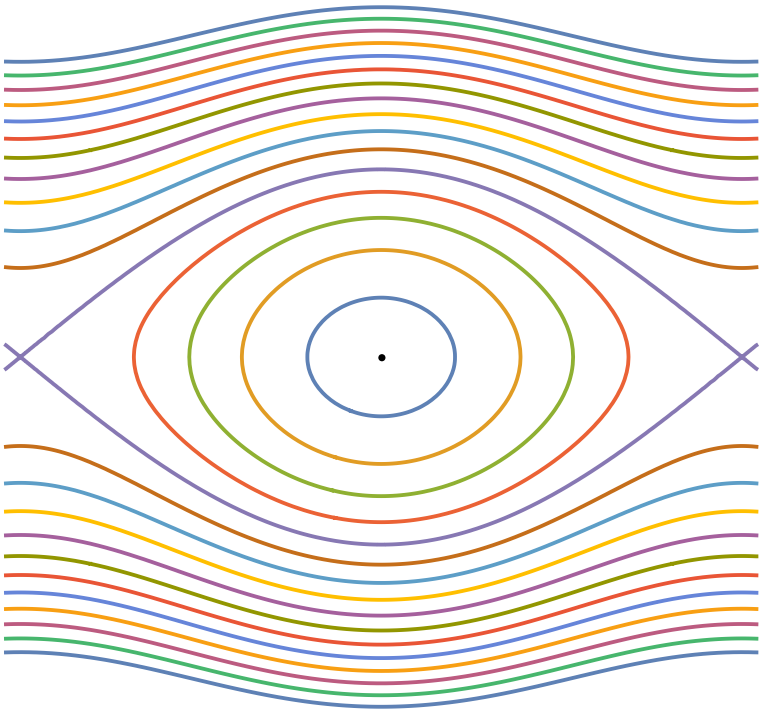

Figure A.1: Schematic picture of the Liouville foliation of the phase space $T^*\mathbf{T} \cong \mathbf{T} \times \mathbf{R}$ for the classical one-dimensional pendulum system described by the Hamiltonian function

$$H(x,p) = \frac{p^2}{2} - \big(1 - \cos(2\pi x)\big)\,.$$

The horizontal direction covers slightly more than one period of length one.

while the *non-singular* spectrum is denoted by

$$\mathcal{S}_{U,0} := \{\boldsymbol{k} \in \mathcal{S}_U : \exists i \neq j \text{ s.th. } k_i \cdot k_j \neq 0\}\,. \tag{A.2.5}$$

Moreover, we define the *coprime set of the orthogonal complement of* $\mathcal{S}_U$ as well as its *non-singular subset* via

$$\mathcal{B}(\mathcal{S}_U^\perp) := \big\{\boldsymbol{b} \in \mathcal{S}_U^\perp : \boldsymbol{b} \text{ coprime}\big\} \quad \text{and} \quad \mathcal{B}_0(\mathcal{S}_U^\perp) := \left\{\boldsymbol{b} \in \mathcal{B}(\mathcal{S}_U^\perp) : \prod_i b_i \neq 0\right\}, \tag{A.2.6}$$

respectively. Note that the orthogonal complement is taken within $\mathbf{Z}^2$. For the proofs in Section A.4 and the generalization in Appendix A.A it is important to observe that for every $\boldsymbol{k} \in \mathcal{S}_{U,0}$ exists some $\boldsymbol{b} \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ such that $\boldsymbol{b} \cdot \boldsymbol{k} = 0$.

Our main results will be formulated under the following assumptions.

1. Assumptions on the perturbed Hamiltonian function (A.2.3)**.**
Let $H_0 \in C^2(T^*\mathbf{T}^2)$ denote the Hamiltonian function from (A.2.1) with $\min V_i = 0$, $\|V_i\|_{C^0} \leq \mathcal{C}_i$ and $\mu_i \in [0, \tilde{\mu}_i]$ for some $\tilde{\mu}_i \in [0,\infty)$, $i \in \{1,2\}$, and U be a perturbing potential as in (A.2.3), which satisfies one of the following assumptions.

(A1) If $\tilde{\mu}_1 = \tilde{\mu}_2 = 0$, we have $U \in C^2(\mathbb{T}^2)$.

(A2) If, w.l.o.g., $\tilde{\mu}_1 = 0$ and $\tilde{\mu}_2 > 0$, we have $U \in C^2(\mathbb{T}^2)$ and there exists $d^{(2)} \geq 0$ such that

$$\mathcal{S}_U \subset \mathbf{Z} \times \big[-d^{(2)}, d^{(2)}\big]\,, \tag{A.2.7}$$

i.e. U is a trigonometric polynomial in the second variable x^2.

(A3) If $\tilde{\mu}_1, \tilde{\mu}_2 > 0$, we have $U \in C^2(\mathbb{T}^2)$ and there exist $d^{(1)}, d^{(2)} \geq 0$ such that

$$\mathcal{S}_U \subset \left[-d^{(1)}, d^{(1)}\right] \times \left[-d^{(2)}, d^{(2)}\right], \tag{A.2.8}$$

i.e. U is a trigonometric polynomial.

We denote the minimum over all $d^{(i)}$ such that (A.2.7) resp. (A.2.8) holds as $\deg_U^{(i)}$ and call it the *i-degree of U*. Whenever we refer to one of the Assumptions (A1), (A2), or (A3), we implicitly assume that $H_0 \in C^2(T^*\mathbf{T}^2)$ is of the form (A.2.1).

Note that the assumption on the spectrum (A.2.4) of U is more restrictive when we include more general potentials $\mu_1 V_1$ and $\mu_2 V_2$ in the unperturbed Hamiltonian H_0 (A.2.1).

The following basic proposition is fundamental for the precise formulation of our assumptions concerning preservation of integrability. It rephrases certain aspects of the Liouville-Arnold Theorem A.1.6 in our concrete setting using standard notions from weak KAM theory (see Appendix A.D).

Proposition A.2.1. (Liouville-Arnold Theorem and weak KAM theory [530])
Let $H_0 \in C^2(T^*\mathbf{T}^2)$ be the Hamiltonian function from (A.2.1)*.*

(a) In the region of phase space, where $f > 0$ as well as $e - f > 0$, each of the two connected component of a Liouville torus $T_{(e,f)}$ (again denoted by $T_{(e,f)}$) is a Lipschitz[5] Lagrangian graph, i.e.

$$T_{(e,f)} = \left\{(x, \boldsymbol{c} + \nabla_x u_{\boldsymbol{c}}) : x \in \mathbf{T}^2\right\}$$

for a unique cohomology class $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R}) \cong \mathbf{R}^2$ with $|c_i| > \sqrt{\mu_i}\,\mathfrak{c}(V_i)$ and $u_{\boldsymbol{c}} \in C^{1,1}(\mathbf{T}^2)$,[6] so we may equivalently write $T_{(e,f)} \equiv T_{\boldsymbol{c}}$. The function $u_{\boldsymbol{c}} \in C^{1,1}(\mathbf{T}^2)$ is a classical solution of the Hamilton-Jacobi equation

$$\alpha(\boldsymbol{c}) = H_0(x, \boldsymbol{c} + \nabla_x u_{\boldsymbol{c}}(x))\,,$$

where the lhs. is Mather's α-function (see Appendix A.D).

(b) The Hamiltonian flow on $T_{\boldsymbol{c}}$ is conjugated to a rotation on $\mathbf{T}^2$, i.e. there exists a diffeomorphism $\varphi : \mathbf{T}^2 \to T_{\boldsymbol{c}}$ such that $\varphi^{-1} \circ \Phi_t^{X_H} \circ \varphi = R_t^{\boldsymbol{\omega}}$, $\forall t \in \mathbf{R}$, where $R_t^{\boldsymbol{\omega}} : \mathbf{T}^2 \to \mathbf{T}^2$, $x \mapsto (x + \boldsymbol{\omega} t \mod \mathbf{Z}^2)$ for some rotation vector $\boldsymbol{\omega} \in \mathbf{R}^2$.

An invariant Liouville torus $T_{\boldsymbol{c}}$ is called *irrational* or *non-resonant*, if $\boldsymbol{k} \cdot \boldsymbol{\omega} \neq 0$ for all $\boldsymbol{k} \in \mathbf{Z}^2 \setminus \{0\}$. If this is not the case, the invariant torus is *rational* or *resonant*. For two-dimensional manifolds (and if $\omega_2 \neq 0$), this can be phrased as a distinction between $\omega_1/\omega_2 \notin \mathbb{Q}$ and $\omega_1/\omega_2 \in \mathbb{Q}$.

2. Assumptions on the preserved integrability of (A.2.3)**.**
Let $H_0 \in C^2(T^*\mathbf{T}^2)$ denote the Hamiltonian function from (A.2.1) satisfying one of the Assumptions (A1) - (A3), and U a perturbing potential as in (A.2.3) such that the following statement concerning the perturbed Hamilton-Jacobi equation (HJE)

$$\alpha_\varepsilon(\boldsymbol{c}) = H_\varepsilon(x, \boldsymbol{c} + \nabla_x u_{\varepsilon,\boldsymbol{c}}(x)) \tag{A.2.9}$$

as well as the preserved integrability of H_ε is satisfied.

[5] We will see in Appendix A.D that $u_{\boldsymbol{c}} \in C^3(\mathbf{T}^2)$, so the regularity of $T_{(e,f)}$ is in fact C^2.

[6] Here, $\mathfrak{c}(V_i) := \int_0^1 \sqrt{2\,V_i(x^i)}\,\mathrm{d}x^i$ (see Appendix A.D) and $C^{1,1}$ denotes the functions in C^1 with Lipschitz derivative.

(P) There exists an energy $e > 0$, such that for every $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ (recall (A.2.6)) and $\mu_i \in [0, \tilde{\mu}_i]$, $i \in \{1, 2\}$, there exists a sequence $(\varepsilon_k)_{k\in\mathbf{N}}$ with $\varepsilon_k \neq 0$ but $\varepsilon_k \to 0$ such that we have the following:

(i) The resonant torus from Proposition A.2.1 characterized by $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R})$ with

$$|c_i| > \sqrt{\mu_i}\, \mathfrak{c}(V_i) \tag{A.2.10}$$

in the isoenergy submanifold T_e having rotation vector proportional to (n, m) is preserved under the sequence of deformations $(H_{\varepsilon_k})_{k\in\mathbf{N}}$.

(ii) For $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R})$ satisfying (A.2.10), Mather's α-function and a solution $u_{\varepsilon,\boldsymbol{c}} : \mathbf{T}^2 \to \mathbb{R}$ of the HJE (A.2.9) can be expanded to first order in ε, i.e.

$$u_{\varepsilon,\boldsymbol{c}} = u_{\boldsymbol{c}}^{(0)} + \varepsilon u_{\boldsymbol{c}}^{(1)} + \mathcal{O}_{\boldsymbol{c}}(\varepsilon^2) \quad \text{and} \quad \alpha_\varepsilon = \alpha^{(0)} + \varepsilon\alpha^{(1)} + \mathcal{O}(\varepsilon^2)\,, \tag{A.2.11}$$

where $u_{\boldsymbol{c}}^{(0)}$, $u_{\boldsymbol{c}}^{(1)} \in C^{1,1}(\mathbf{T}^2)$ and $O_{\boldsymbol{c}}(\varepsilon^2)$ is understood in $C^{1,1}$-sense.[7]

We comment on the validity of assuming (P) in Remark A.D.8 in Appendix A.D. Moreover, we shall also discuss an alternative to (A.2.11) in Remark A.D.10. Finally, one can easily see from the proofs given in Section A.4, that the condition on a *fixed* isoenergy manifold $\{H_\varepsilon = e\}$ can be relaxed to having preservation of invariant tori in isoenergy manifolds characterized by energies $e \geq e_0$ for some fixed $e_0 > 0$.

Note that the rational invariant tori are the most 'fragile' objects of an integrable system as the KAM Theorem [373, 27, 456] predicts that general (non-integrable) perturbations preserve only 'sufficiently irrational' (*Diophantine*) invariant tori.

A.2.2 Results

As mentioned above, our main results in Theorem A.I, Theorem A.II, and Theorem A.III concern rigidity of certain deformations of integrable metrics (in the sense of Question (Q2)), which, by means of the Maupertuis principle, correspond to perturbations of the form (A.2.3). More precisely, under the assumptions formulated above, our results show that the perturbed Hamiltonian function (A.2.3) has to be of the same general form as the unperturbed Hamiltonian function (A.2.1). This means, that the potential U is *separable*, i.e. there exist $U_1, U_2 \in C^2(\mathbf{T}^2)$ such that

$$U(x) = U_1(x^1) + U_2(x^2)\,.$$

Theorem A.I. *Let H_ε from* (A.2.3) *satisfy Assumption* (A1) *and Assumption* (P) *for some energy $e > 0$. Then U is separable in a sum of two single-valued functions.*

Put briefly, in view of of the Maupertuis principle, this means that integrable deformations in the same conformal class of a flat metric are Liouville metrics. Now, Theorem A.II generalizes Theorem A.I to Hamiltonian functions which depend on one toral position variable via a mechanical potential.

Theorem A.II. *Let H_ε from* (A.2.3) *satisfy Assumption* (A2) *and Assumption* (P) *for some energy $e > 0$. Then the following holds:*

[7] Having C^1-regularity here would be sufficient for our proofs in Section A.4. However, we chose $C^{1,1}$-regularity for the formulation of Assumption (P) to be in agreement with the statement from Proposition A.2.1 (b). More precisely, $C^{1,1}$-regularity is kind of a compromise between the true C^3-regularity of $u_{\boldsymbol{c}}$ and the required C^1-regularity of $u_{\varepsilon,\boldsymbol{c}}$. In addition, $C^{1,1}$ is the optimal regularity for *sub*solutions of (A.2.9), which exist, even if the Hamiltonian H_ε is *not* integrable (see [256, 66]).

(a) *If $\tilde{\mu}_2 = \tilde{\mu}_2(\mathcal{C}_2, \deg_U^{(2)}, e) > 0$ is small enough (see Lemma A.4.2), we have that U is separable in a sum of two single-valued functions.*

(b) *If, additionally, V_2 is analytic, then U is separable, irrespective of $\tilde{\mu}_2 > 0$, but only for $\mu_2 \in [0, \tilde{\mu}_2]$ outside of an exceptional null-set.*

Therefore, by means of the Maupertuis principle, we infer that integrable deformations in the same conformal class of metrics realizing surfaces of revolution (see Section A.3.2) are Liouville metrics. Finally, Theorem A.III generalizes the above results to Hamiltonian functions, which correspond to arbitrary Liouville metrics by means of the Maupertuis principle.

Theorem A.III. *Let H_ε from* (A.2.3) *satisfy Assumption* (A3) *and Assumption* (P) *for some energy $e > 0$. Then the following holds:*

(a) *If $\tilde{\mu}_1 = \tilde{\mu}_1(\mathcal{C}_1, \deg_U^{(1)}, \deg_U^{(2)}, e) > 0$ and $\tilde{\mu}_2 = \tilde{\mu}_2(\mathcal{C}_2, \deg_U^{(1)}, \deg_U^{(2)}, e) > 0$ are small enough (see Lemma A.4.3), we have that U is separable in a sum of two single-valued functions.*

(b) *If, additionally, V_2 is analytic and $\tilde{\mu}_1 = \tilde{\mu}_1(\mathcal{C}_2, \deg_U^{(1)}, \deg_U^{(2)}, e) > 0$ is small enough, then U is separable, irrespective of $\tilde{\mu}_2 > 0$, but only for $\mu_2 \in [0, \tilde{\mu}_2]$ outside of an exceptional one-dimensional null-set (depending on $\mu_1 \in [0, \tilde{\mu}_1]$).*

(c) *If both, V_i for $i = 1, 2$, are analytic, then U is separable, irrespective of $\tilde{\mu}_1, \tilde{\mu}_2 > 0$, but only for $(\mu_1, \mu_2) \in [0, \tilde{\mu}_1] \times [0, \tilde{\mu}_2]$ outside of an exceptional two-dimensional null-set.*

Our results formulated in Theorem A.I, Theorem A.II, and Theorem A.III can each be viewed as a verification of a special case of the following conjecture, saying that *'(nice) integrable deformations of Liouville metrics are Liouville metrics'*.

Conjecture: Deformational rigidity of Liouville metrics.
Let g be a Liouville metric on $\mathbf{T}^2$ and let $(g_t)_{t\in[0,1]}$ with $g_0 = g$ be a deformation that preserves all rational invariant tori (except finitely many). Then g_t is a Liouville metric for all $t \in [0, 1]$.

This conjecture is in strong analogy to the perturbative Birkhoff conjecture for integrable billiards, which is discussed in Section A.3.4 below.

A.3 Literature review: Integrable metrics on the torus

As pointed out in Section A.1.2, integrability of metrics on one-dimensional manifolds is not questionable and the first non-trivial examples occur whenever M has dimension two. Recall from Definition A.1.3 that integrability of metrics on two-dimensional manifolds requires the existence of only one additional first integral (beside the Hamiltonian).

A.3.1 Topological obstructions

The following Theorem due to Kozlov [380, 381] (see [74] for a strengthened version of this result) categorizes two-dimensional compact manifolds regarding the possibility to endow them with an integrable metric (see Question (Q1)).

Theorem A.3.1. (Kozlov [380, 381])
Let M be a two-dimensional compact and real-analytic manifold that is endowed with a real-analytic Riemannian metric g. If the Euler characteristic χ_M of M is negative, then there exists no other non-trivial real-analytic first integral.

A result similar to Theorem A.3.1 holds for polynomially integrable geodesic flows.

Theorem A.3.2. (Kolokoltsov [374])
There exist no polynomially integrable geodesic flow on a closed two-dimensional Riemannian manifold M with negative Euler characteristic χ_M.

Recall that any two-dimensional compact manifold M can be represented either as the sphere with handles or the sphere with Möbius strips, in the orientable and non-orientable case, respectively. The Euler characteristic χ_M can be computed as

$$\chi_M = 2 - 2g \qquad \text{resp.} \qquad \chi_M = 2 - m\,,$$

where g is the number of handles (the genus) and m is the number of Möbius strips. In order to have integrability, the above theorem imposes the condition $\chi_M \geq 0$ on M and we thus know that the number of handles is at most 1 and the number of Möbius strips is not greater than 2. Therefore, any real-analytic two-dimensional compact Riemannian manifold (M, g) with real-analytic (or polynomial) additional integral is either the sphere $\mathbb{S}^2$ or the torus $\mathbf{T}^2$ (in the orientable case), or the projective plane $\mathbf{R}\mathbb{P}^2$ or the Klein bottle $\mathbb{K}^2$ (in the non-orientable case).[8]

In this work, we focus on integrable metrics on the torus $\mathbf{T}^2$ and refer to works by Bolsinov, Fomenko, Matveev, Kolokoltsov and others [95, 263, 374, 472] for studies on integrable metrics on the sphere, the projective plane, and the Klein bottle. See [131, 92] for recent surveys on open problems and questions concerning geodesics and integrability of finite-dimensional systems in general.

A.3.2 Linearly and quadratically integrable metrics

The first non-trivial class of integrable metrics on the torus $\mathbf{T}^2$ are *surfaces of revolution*. Consider a two-dimensional surface $M \subset \mathbf{R}^3$ given by the equation $r = r(z)$ in standard cylindrical coordinates $(r, \varphi, z) \in (0, \infty) \times [0, 2\pi) \times \mathbf{R}$. As local coordinates on M we take z and φ. In case that $r(z)$ is L-periodic and we identify 0 and L, then M is diffeomorphic to the torus $\mathbf{T}^2$ and the Riemannian metric induced on M by the Euclidean metric on $\mathbf{R}^3$ has line element

$$\mathrm{d}s^2 = (1 + r'(z)^2)\mathrm{d}z^2 + r(z)^2\mathrm{d}\varphi^2\,. \tag{A.3.1}$$

Since the corresponding Hamiltonian function (A.1.5) is independent of φ, its associated momentum variable p_φ is an additional first integral and thus the metric (A.3.1) is integrable. Note that the additional first integral is linear in the momentum variables.

As discussed earlier, a Riemannian metric g on $\mathbf{T}^2$ is called a *Liouville metric*, whenever its line element can be written in the form (A.1.1) in appropriate global coordinates (x^1, x^2) and where f_1 and f_2 are smooth positive periodic functions. The corresponding Hamiltonian function (A.1.5) is given by

$$H(x^1, x^2, p_1, p_2) = \frac{p_1^2 + p_1^2}{2\,(f_1(x^1) + f_2(x^2))}$$

and an additional first integral can easily be obtained as

$$F(x^1, x^2, p_1, p_2) = p_1^2 - f_1(x^1)\,H(x^1, x^2, p_1, p_2)\,.$$

Therefore, clearly, also Liouville metrics are integrable. Note that the additional first integral F is quadratic in the momentum variables. It is not hard to see that a surface of revolution is just a particular case of a Liouville metric, where one can choose, e.g., $f_2 \equiv 0$, by employing a simple change of variables.

[8] In [97], Bolsinov and Taimanov give a striking example of a real-analytic Riemannian manifold of dimension three, whose geodesic flow has the peculiar property, that it is smoothly (but not analytically) integrable although it has positive topological entropy [11]. The problem of proving (non-)existence of smoothly (but not analytically) integrable geodesic flows on compact surfaces of genus $g > 1$ is widely open (see [131]).

The following proposition also provides the converse to the observation that surfaces of revolution and Liouville metrics admit additional first integrals which are linear and quadratic in the momenta, respectively. It collects several statements that have been proven in early works by Dini [225], Darboux [196], and Birkhoff [84], and were further developed by Babenko and Nekhoroshev [34], Kiyohara [367], Kolokoltsov [374], and others.

Proposition A.3.3. (Linear and quadratic first integrals [225, 196, 84, 34, 367, 374])

(a) Let the metric g on $\mathbb{T}^2$ possess an additional first integral F that is linear in the momenta. Then there exist global periodic coordinates (x^1, x^2) on the torus such that the line element of g takes the form

$$\mathrm{d}s^2 = f(x^1)\left(a\,(\mathrm{d}x^1)^2 + c\,\mathrm{d}x^1\,\mathrm{d}x^2 + b\,(\mathrm{d}x^2)^2\right),$$

where f is some positive periodic function and $a, b, c \in \mathbf{R}$ such that the quadratic form $a\,(\mathrm{d}x^1)^2 + c\,\mathrm{d}x^1\,\mathrm{d}x^2 + b\,(\mathrm{d}x^2)^2$ is positive definite.

Conversely, any such metric on the torus $\mathbf{T}^2$ admits an additional first integral that is linear in the momentum variables.

In case a linear in momenta F exists locally near a point $q \in \mathbf{T}^2$, then there exists local coordinates (x^1, x^2) near q such that the line element of g reads

$$\mathrm{d}s^2 = f(x^1)\left((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right).$$

(b) A metric g on $\mathbf{T}^2$ possess an additional first integral F that is quadratic in the momenta if and only if there exists a finite-sheeted covering $\pi : \widetilde{\mathbf{T}}^2 \to \mathbf{T}^2$ by another torus, such that the lifted metric $\widetilde{g} = \pi^* g$ is globally Liouville, i.e. there exist global periodic coordinates (x^1, x^2) on $\widetilde{\mathbf{T}}^2$ and smooth positive periodic functions f_1 and f_2 such that the line element of $\widetilde{g}$ takes the form (A.1.1)*.*

There exist Riemannian metrics g on $\mathbf{T}^2$ which are not globally Liouville but have an additional first integral that is quadratic in the momentum variables.

In case a quadratic in momenta F exists locally near a point $q \in \mathbf{T}^2$, then there exist local coordinates (x^1, x^2) near q such that the line element of g takes the form (A.1.1)*.*

This classical result completely characterizes the integrable metrics g on $\mathbf{T}^2$ that admit an additional first integral that is linear or quadratic in the momentum variables. Similar results hold for Riemannian metrics on general two-dimensional manifolds [95, 263, 374, 472].

A.3.3 Polynomially integrable metrics of higher degree

In the case of a sphere $\mathbb{S}^2$, one can easily construct examples of metrics which admit an additional first integral that is cubic resp. quartic in the momentum variables. Using the Maupertuis principle, these can be obtained from the metrics constructed from Goryachev-Chaplygin [296, 142] and Kovaleskaya [379] in the situation of the dynamics of a rigid body. Therefore, let $h > 1$ be large enough (cf. (A.1.10)) and define the metrics g_3 and g_4 on $\mathbf{R}^3$ via their respective line elements

$$\mathrm{d}s_3^2 = \frac{h - x^1}{4}\,\frac{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + 4(\mathrm{d}x^3)^2}{(x^1)^2 + (x^2)^2 + (x^3)^2/4}, \quad \mathrm{d}s_4^2 = \frac{h - x^1}{2}\,\frac{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + 2(\mathrm{d}x^3)^2}{(x^1)^2 + (x^2)^2 + (x^3)^2/2}.$$

By restriction of g_3 and g_4 to the unit sphere $\mathbb{S}^2 \subset \mathbf{R}^3$, the resulting metrics admit an additional first integral that is cubic resp. quartic in the momentum variables. It was shown by Bolsinov, Fomenko and Kozlov [93, 96] that these cannot be reduced to first integrals that are polynomially in the momentum variables of a lower degree, i.e. they are not linearly or quadratically integrable. Since

all attempts to construct such examples for the case of the torus have failed so far, the following folklore conjecture emerged.

Folklore Conjecture. *Liouville metrics are the only integrable metrics on* $\mathbf{T}^2$.

In this general form, there is strong indication for conjecture being false, as to be shown below (see Theorem A.3.7). We will, however, provide existing results, which indicate that a certain weaker version of this conjecture, also formulated below, is indeed true.

It was proven by Korn and Lichtenstein [377, 417] that on every point on a two-dimensional Riemannian manifold (M, g) there exist *locally* isothermal coordinates, that is, locally, the line element takes the form

$$\mathrm{d}s^2 = \lambda(x^1, x^2)\left((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right), \tag{A.3.2}$$

where λ is some smooth positive function. In the case of a torus, it can be shown (by virtue of the uniformization theorem) that there exist *global* isothermal coordinates (not necessarily periodic), so the metric g is conformal equivalent to the Euclidean metric g_{eucl}. In particular, assuming that (x^1, x^2) are just the angular coordinates on the torus $\mathbf{T}^2$ and in the special case of λ being a trigonometric polynomial,[9] we have the following result due to Denisova and Kozlov.

Theorem A.3.4. (Denisova-Kozlov [212])
Let λ *from* (A.3.2) *be a trigonometric polynomial and assume that the geodesic flow on* $\mathbf{T}^2$ *is polynomially integrable. Then there exists an additional polynomial first integral of degree at most two.*

Note that by Weierstrass's Theorem, any conformal factor λ can be approximated as closely as required by a trigonometric polynomial. However, in the case of a general conformal factor λ, there is the following Theorem, again due to Denisova and Kozlov [213].

Theorem A.3.5. (Denisova-Kozlov [213])
Assume that the geodesic flow on $(\mathbf{T}^2, g)$ *is polynomially integrable with first integral* F *of degree* n *such that*

(a) if n *is even, then* F *is an even function of* p_1 *and* p_2,

(b) if n *is odd, then* F *is an even function of* p_1 *(or* p_2*) and an odd function of* p_2 *(or* p_1*).*

Then there exists an polynomial first integral of degree at most two.

In the following Theorem we collect several results from Bialy [72], Denisova, Kozlov [214], and Treshchev [215], Agapov and Aleksandrov [12], and Mironov [449].

Theorem A.3.6. *Let* H *be a natural mechanical Hamiltonian (see* (A.1.9)*) on the torus* $\mathbf{T}^2$ *equipped with the flat metric* g_{eucl}. *Assume that* H *is polynomially integrable of degree* n. *If* $n = 3, 4$, *there exists another polynomial first integral of degree at most two. Whenever* H *is a real-analytic Hamiltonian, this is also true for* $n = 5$.

Kozlov and Treshchev [382] considered the problem from yet another point of view. They investigated the case of a mechanical Hamiltonian

$$H = \frac{1}{2}\sum_{ij} a_{ij}p_ip_j + V(x^1, ..., x^n),$$

[9] This means that the spectrum $\mathcal{S}_\lambda$ defined in (A.2.4) is bounded.

where $A = (a_{ij})_{ij}$ is a positive definite matrix and V is a trigonometric polynomial of $(x^1, ..., x^n) \in \mathbf{T}^n$. On the one hand, they show that there exist n polynomial first integrals if and only if the spectrum $\mathcal{S}_V$ of V is contained in $m \leq n$ mutually orthogonal lines meeting at the origin. On the other hand, they showed that whenever there exist n polynomial integrals with independent forms of highest degree, then there exist n independent involutive polynomial first integrals of degree at most two. In case that $a_{ij} = \delta_{i,j}$ (which can be achieved by diagonalization and scaling), Combot [180] improved the first result from the assumption of *polynomial* integrability to *rational* integrability, i.e. the additional first integrals being rational functions of p_i and $\mathrm{e}^{\mathrm{i}2\pi x^i}$. More recently [518, 331, 519], the problem was rephrased in the language of Killing tensor fields on $\mathbf{T}^2$, where the *order* of an additional (polynomial) first integral is replaced by the *rank* of a Killing tensor filed.

The results of Theorems A.3.4, A.3.5 and A.3.6 support the validity of the following weaker version of the folklore conjecture formulated by Denisova and Kozlov [212].

Conjecture. [212] *If g is a metric on $\mathbf{T}^2$ that is polynomially integrable, then there exists an additional polynomial first integral of degree at most two.*

By Proposition A.3.3 this means that polynomially integrable metrics on $\mathbf{T}^2$ are Liouville metrics. However, beside the partial results given above, a proof of this conjecture is still open. The numerous attempts on proving it used methods of complex analysis [84, 34] and the theory of PDEs [77, 78]. More precisely, it is shown by Kolokoltsov [374] that there exists an additional first integral quadratic in the momenta if and only if there exists a holomorphic function $R(z) = R_1(z) + \mathrm{i}R_2(z)$, with real valued R_1 and R_2 and $z = x^1 + \mathrm{i}x^2$, which solves

$$R_2(\partial^2_{x^2}\lambda - \partial^2_{x^1}\lambda) + R_1(\partial_{x^1}\partial_{x^2}\lambda) - 3(\partial_{x^1}R_2)(\partial_{x^1}\lambda) + 3(\partial_{x^2}R_2)(\partial_{x^2}\lambda) + 2(\partial^2_{x^2}R_2)\lambda = 0\,, \quad \text{(A.3.3)}$$

where λ denotes the conformal factor from (A.3.2). Note that the second term in (A.3.3) disappears whenever λ is the conformal factor of a Liouville metric. In this situation, the linear PDE (A.3.3) always has a holomorphic solution $R = R_1 + \mathrm{i}R_2$. The existence of first integrals of higher degree turns out to be equivalent to delicate questions about non-linear PDEs of hydrodynamic type [77, 76, 78]. The PDE-approach has also successfully been applied to generate new examples of integrable *magnetic* geodesic flows as analytic deformations of Liouville metrics on $\mathbf{T}^2$ without magnetic field (see [13]).In fact, the examples from [13] disprove the folklore conjecture when understood in the larger class of magnetic geodesic flows.

However, even for the *original* folklore conjecture stated above, there is a result due to Corsi and Kaloshin [182], which indicates it being false in the following (considerably weaker) sense.

Theorem A.3.7. (Corsi-Kaloshin [182])
There exists a real-analytic mechanical Hamiltonian

$$H_\varepsilon(x^1, x^2, p_1, p_2) = \frac{p_1^2 + p_2^2}{2} + U(x^1, x^2; \varepsilon)$$

with a non-separable[10] *potential U and an analytic change of variables Φ such that $H_\varepsilon \circ \Phi = (p_1^2 + p_2^2)/2$ on the energy surface $\{H_\varepsilon = 1/2\}$ and $p \in \mathcal{P}$, where $\mathcal{P}$ denotes a certain cone in the action space.*

If one assumes that the *whole* phase space $T^*\mathbf{T}^2$ is foliated by two-dimensional invariant Liouville tori (which is often called C^0-integrability or *complete integrability*), then it follows from Hopf conjecture [350, 130] that the associated metric must be flat.[11] This notion of integrability is thus too strong for a meaningful characterization of integrable metrics on $\mathbf{T}^2$.

[10] The function U is called non-separable whenever it cannot be written as a sum of two single-valued functions.

[11] Similar results have been shown for geodesic flows of more general Finsler metrics on $\mathbf{T}^2$ preserving a sufficiently regular foliation of the phase space [290, 289]

A.3.4 Analogy to integrable billiards

The fundamental question (Q2) of characterizing integrable metrics on the torus $\mathbf{T}^2$ can be thought of as an analogue of identifying the class of integrable billiards [361]. For billiards, integrability is understood in a similar way as for the geodesic flow (see Definition A.1.3). More precisely, integrability is characterized either through the existence of an integral of motion (near the boundary of the billiard table) for the so called billiard ball map, or the existence of a foliation of the phase space (globally, or near the boundary), consisting of invariant curves. The *classical Birkhoff conjecture* [85, 493] states that the boundary of a strictly convex integrable billiard table is necessarily an ellipse. This corresponds to the folklore conjecture formulated above. Remarkably, while the Birkhoff conjecture is believed to be true, and there is strong evidence that this indeed the case [79, 285, 33, 362],[12] the folklore conjecture in its general form was shown to be false by Theorem A.3.7.

However, recall that, if one assumes C^0-integrability of a metric on $\mathbf{T}^2$, the metric is actually flat [350, 130]. This corresponds to the following result from Bialy in the case of billiards.

Theorem A.3.8. (Bialy [73])
If the phase space of the billiard ball map is completely foliated by continuous invariant curves which are all not null-homotopic, then the boundary of the billiard table is a circle.

Following a similar strategy leading to Theorem A.3.8, Bialy and Mironov [80] proved the Birkhoff conjecture for centrally symmetric billiards, assuming only *local* C^0-integrability, i.e. the foliation of a suitable open *proper* subset of the phase space. Beside this, the weakened version of the folklore conjecture (polynomial integrals can be reduced to integrals of degree at most two) corresponds to the so called *algebraic Birkhoff conjecture*, which has recently been proven [79, 285].

The main results of this paper in Theorem A.I, Theorem A.II, and Theorem A.III prove special cases of our conjecture that integrable deformations of Liouville metrics which preserve all (but finitely many) rational invariant tori are again Liouville metrics. This is related to the following conjecture in the case of billiards.

Perturbative Birkhoff conjecture. [361] *A smooth strictly convex domain that is sufficiently close to an ellipse and whose corresponding billiard ball map is integrable, is necessarily an ellipse.*

A first result in this direction was obtained by Delshams and Ramírez-Ros [211]. More recently, Avila, De Simoi, and Kaloshin [33] proved the conjecture for domains which are sufficiently close to a circle. The complete proof for domains sufficiently close to an ellipse of any eccentricity is given by Kaloshin and Sorrentino in [362]. Both works require the preservation of rational caustics[13] which can be thought of as an analogue for the preservation of rational invariant tori as a fundamental assumption of our main results from Section A.2. The result in [33] was later extended by Huang, Kaloshin, and Sorrentino [352] to the case of *local integrability* close to the boundary and finally significantly improved by Koval [378].

Finally, as shown by Vedyushkina and Fomenko [580], linearly and quadratically integrable geodesic flows on orientable two-dimensional Riemannian manifolds are Liouville equivalent to topological billiards, glued from planar billiards bounded by concentric circles and arcs of confocal quadrics, respectively.

[12] On the opposite side, Treschev constructed a non-elliptic billiard table which is *formally* integrable close to a two-periodic orbit [565, 566, 567]. This formal power series has recently been shown to be of Gervey class of order $\sigma > 9/4$ [584].

[13] A curve Γ is a caustic for the billiard in the domain Ω if every time a trajectory is tangent to it, then it remains tangent after every reflection according to the billiard ball map.

A.4 Proofs

In this Section we prove our main result as formulated in Theorem A.I, Theorem A.II, and Theorem A.III. All proofs will, in general, follow the same three step strategy.

(i) Transform the unperturbed system H_0 in action-angle coordinates (cf. Theorem A.1.6, in particular eq. (A.1.8)).

(ii) Derive a first-order harmonic equation (i.e. concerning the Fourier coefficients) for the perturbation by Assumption (P).

(iii) Annihilate sufficiently many Fourier coefficients of the perturbing potential by proving a certain *full-rank* condition for a naturally associated linear system for each of the three theorems separately (cf. Lemmas A.4.1, A.4.2, and A.4.3). Finally, for analytic potentials V_i, the extensions of our results beyond the perturbative regime are proven by exploiting the analytic dependence of the linear system on μ_i (see Appendix A.C).

A.4.1 Proof of Theorem A.I

Step (i). Fix an energy $e > 0$. Since the Hamiltonian is already in action-angle coordinates (cf. (A.1.8)), we simply change notation and write $(x^i, p_i) = (\theta^i, I_i)$ for $i = 1, 2$ as well as $\theta = (\theta^1, \theta^2)$ and $I = (I_1, I_2)$, such that the perturbed Hamiltonian function H_ε takes the form

$$H_\varepsilon(\theta, I) = \frac{I_1^2}{2} + \frac{I_2^2}{2} + \varepsilon U(\theta)\,.$$

Step (ii). By Assumption (P), for any $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ (recall (A.2.6)), we can find (in the isoenergy manifold T_{e_ε} with energy $e = e_\varepsilon$ and $\varepsilon = \varepsilon_k$ for some $k \in \mathbf{N}$) a rational invariant invariant Liouville torus with rotation vector $\boldsymbol{\omega} = (\omega_1, \omega_2)$ which satisfies

$$\frac{\omega_1}{\omega_2} = \frac{n}{m} \in \mathbb{Q}\,. \tag{A.4.1}$$

Moreover, we fix $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R}) \cong \mathbf{R}^2$ to be given by $\boldsymbol{c} = (\omega_1, \omega_2)$. We make this choice to cancel the average over a trajectory (A.4.3) of the first term on the rhs. of (A.4.2) (cf. also (A.4.6)–(A.4.8) below).

Using Assumption (P) again, we can expand the Hamilton-Jacobi equation (A.2.9) as

$$\begin{aligned}
\alpha_\varepsilon(\boldsymbol{c}) &= H_\varepsilon(\theta, \boldsymbol{c} + \nabla u_{\varepsilon,\boldsymbol{c}}(\theta)) \\
&= \frac{|\partial_{\theta^1} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_1|^2}{2} + \frac{|\partial_{\theta^2} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_2|^2}{2} + \varepsilon U(\theta) \\
&= \frac{c_1^2}{2} + \frac{c_2^2}{2} + \langle \boldsymbol{c}, \nabla u_{\varepsilon,\boldsymbol{c}}(\theta)\rangle + \varepsilon U(\theta) + \frac{\left(\partial_{\theta^1} u_{\varepsilon,\boldsymbol{c}}(\theta)\right)^2}{2} + \frac{\left(\partial_{\theta^2} u_{\varepsilon,\boldsymbol{c}}(\theta)\right)^2}{2}\,.
\end{aligned}$$

and it holds that

$$u_{\varepsilon,\boldsymbol{c}} = u_{\boldsymbol{c}}^{(0)} + \varepsilon u_{\boldsymbol{c}}^{(1)} + \mathcal{O}_{\boldsymbol{c}}(\varepsilon^2)$$

with $u_{\boldsymbol{c}}^{(0)} = u_{0,\boldsymbol{c}}$. Since $H_0(\theta, I)$ is integrable (and written in action-angle coordinates), one can choose $u_{0,\boldsymbol{c}} \equiv 0$. By (A.D.8) in Proposition A.D.9 (see also [288]) we have $\alpha^{(1)}(\boldsymbol{c}) = [U]_0$, where

$$[U]_0 = \int_{\mathbf{T}^2} U(x^1, x^2)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\,.$$

Since the sequence $(\varepsilon_k)_{k\in\mathbf{N}}$ from Assumption (P) converges to zero, we compare coefficients and establish the first order equation

$$[U]_0 = \alpha^{(1)}(\boldsymbol{c}) = \langle \boldsymbol{c}, \nabla u_{\boldsymbol{c}}^{(1)}(\theta)\rangle + U(\theta)\,. \tag{A.4.2}$$

Averaging (A.4.2) over the trajectory $\theta(t) = \theta_0 + \boldsymbol{\omega} t \in \mathbf{T}^2$, with initial position $\theta_0 \in \mathbf{T}^2$ and where $\boldsymbol{\omega} = \boldsymbol{c}$ is chosen according to (A.4.1), such that the period $T_{\boldsymbol{\omega}}$ satisfies $T_{\boldsymbol{\omega}} \cdot \boldsymbol{\omega} = (n, m)$, we get

$$[U]_0 = \frac{1}{T_{\boldsymbol{\omega}}} \int_0^{T_{\boldsymbol{\omega}}} \frac{\mathrm{d}}{\mathrm{d}t} u^{(1)}_{\varepsilon,\boldsymbol{c}}(\theta(t))\,\mathrm{d}t + \frac{1}{T_{\boldsymbol{\omega}}} \int_0^{T_{\boldsymbol{\omega}}} U(\theta(t))\,\mathrm{d}t\,. \tag{A.4.3}$$

The first integral vanishes since $\theta(0) = \theta(T_{\boldsymbol{\omega}})$ such that we are left with

$$\int_0^1 \big(U(\theta_0^1 + nt, \theta_0^2 + mt) - [U]_0\big)\,\mathrm{d}t = 0 \tag{A.4.4}$$

for all $\theta_0 = (\theta_0^1, \theta_0^2) \in \mathbf{T}^2$, which easily follows from (A.4.3) after a change of variables.

Before continuing with the third and final step, we have two important observation: First, by replacing $U \to U - [U]_0$, we can assume w.l.o.g. that $[U]_0 = 0$. Second, we define the *separable part*, U_{sep}, of U as

$$U_{\mathrm{sep}}(x^1, x^2) := \sum_{(k_1,k_2)\in\mathcal{S}_U\setminus\mathcal{S}_{U,0}} U_{k_1,k_2} \mathrm{e}^{\mathrm{i}2\pi k_1 x^1} \mathrm{e}^{\mathrm{i}2\pi k_2 x^2} \tag{A.4.5}$$

(recall the definition of the spectrum and the non-singular spectrum in (A.2.4) and (A.2.5)). Then, after a simple computation, we find that

$$\int_0^1 U_{\mathrm{sep}}(\theta_0^1 + nt, \theta_0^2 + mt)\mathrm{d}t = [U_{\mathrm{sep}}]_0\,, \qquad \forall (\theta_0^1, \theta_0^2) \in \mathbf{T}^2\,,$$

holds *generally* (i.e. independent of the first order relation (A.4.2)) by means of (A.D.8) in Proposition A.D.9 (see also Remark A.D.8). We can thus split off the separable part and assume that $\mathcal{S}_U = \mathcal{S}_{U,0}$ in the following. Hence, the third step consists of showing that $\mathcal{S}_U = \mathcal{S}_{U,0} = \varnothing$.

Step (iii). The goal of this final step is to establish the following lemma.

Lemma A.4.1. *Let* $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ *as in* (A.4.1) *from* **Step (ii)**. *Then* $U_{jm,-jn} = 0$ *for all* $j \in \mathbf{Z} \setminus \{0\}$.

Since $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ were arbitrary, this proves that

$$\mathcal{S}_U \subset (\{0\} \times \mathbf{Z}) \cup (\mathbf{Z} \times \{0\})\,,$$

or equivalently $\mathcal{S}_{U,0} = \varnothing$ and we have shown Theorem A.I. It remains to prove Lemma A.4.1.

Proof of Lemma A.4.1. Starting from (A.4.4) we perform a Fourier decomposition to infer

$$\sum_{k_1,k_2\neq 0} \Big[U_{k_1,k_2} \int_0^1 \mathrm{e}^{\mathrm{i}2\pi k_1 nt} \mathrm{e}^{\mathrm{i}2\pi k_2 mt} \mathrm{d}t\Big] \mathrm{e}^{\mathrm{i}2\pi k_1 \theta_0^1} \mathrm{e}^{\mathrm{i}2\pi k_1 \theta_0^2} = 0 \qquad \forall (\theta_0^1, \theta_0^2) \in \mathbf{T}^2\,,$$

which implies that

$$U_{k_1,k_2} \cdot \delta_{k_1 n + k_2 m, 0} = 0\,. \qquad \square$$

Applying Lemma A.4.1 for every $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$, we find that $\mathcal{S}_{U,0} = \varnothing$, which finishes the proof of Theorem A.I. $\square$

A.4.2 Proof of Theorem A.II

For notational simplicity, we write $\mu \equiv \mu_2 > 0$ and $V \equiv V_2 \in C^2(\mathbf{T})$.

Step (i). We fix an energy $e > 0$ and consider the region of the phase space, where the subsystem in the second pair of coordinates is rotating, i.e.

$$\frac{p_2^2}{2} - \mu V(x^2) = e^{(2)} > 0$$

and for $\frac{p_1^2}{2} = e^{(1)} > 0$ we have $e = e^{(1)} + e^{(2)}$. In a neighborhood of each of the two Liouville tori characterized by $H_0 = e$ and $\frac{p_1^2}{2} = e^{(1)}$ we can find a change of variables $(x^2, p_2) = \Phi_\mu^{(2)}(\theta^2, I_2)$ (and we denote $(x^1, p_1) = (\theta^1, I_1)$) such that the Hamiltonian function H_0 gets transformed in action-angle coordinates (see (A.1.8)), i.e.

$$H_0(\theta^1, I_1, \Phi_\mu^{(2)}(\theta^2, I_2)) = \frac{I_1^2}{2} + h_\mu^{(2)}(I_2)$$

for some smooth function $h_\mu^{(2)}$ agreeing with Mather's α-function for the one-dimensional subsystem described by the Hamiltonian $\frac{p_2^2}{2} - \mu V(x^2)$ (see Appendix A.D). The change in the order of the four arguments of H_0 should not lead to confusion. Now, the perturbed Hamiltonian takes the form

$$H_\varepsilon(\theta^1, I_1, \Phi_\mu^{(2)}(\theta^2, I_2)) = \frac{I_1^2}{2} + h_\mu^{(2)}(I_2) + \varepsilon U(\theta^1, x^2(\theta^2, I_2, \mu)),$$

where we write $x_\mu^2(\theta^2, I_2)$ for the first component of $\Phi_\mu^{(2)}(\theta^2, I_2)$.

Step (ii). Assume w.l.o.g. that the 2-degree $\deg_U^{(2)}$ of U is at least 1 (recall (A.2.7)), as otherwise we had $U(x) = U_1(x^1)$ and Theorem A.II was proven. Then, for any $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$, in particular with $|n| \le \deg_U^{(2)}$, we can find (in the isoenergy manifold T_{e_ε} with energy $e = e_\varepsilon$ and $\varepsilon = \varepsilon_k$ for some $k \in \mathbf{N}$) a rational invariant Liouville torus with rotation vector $\boldsymbol{\omega} = (\omega_1, \omega_2)$, which satisfies

$$\frac{\omega_1}{\omega_2} = \frac{n}{m} \in \mathbb{Q} \quad \text{and} \quad \boldsymbol{\omega} = (c_1, \nabla h_\mu^{(2)}(c_2)) \tag{A.4.6}$$

for some $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R}) \cong \mathbf{R}^2$ with $c_1 = \omega_1$ (as around (A.4.1)) and $|c_2| > \gamma + \sqrt{\mu}\mathfrak{c}(V)$ for some $\gamma = \gamma(e, \deg_U^{(2)}) > 0$, which we fix now. This new parameter γ quantifies a safe distance (depending on the total energy $e > 0$ and the degree of the trigonometric polynomial) to the region, *opposite* to where (i) the change of variables $\Phi_\mu^{(2)}$ has bounded derivative (cf. (A.4.7)) and (ii) the function $h_\mu^{(2)}$ is bounded from below (cf. (A.4.12)). In Section A.4.3 we will have two such parameters, γ_1, γ_2, for both coordinates directions which get transformed by some Φ.

By Assumption (P) we have

$$u_{\varepsilon,\boldsymbol{c}} = u_{\boldsymbol{c}}^{(0)} + \varepsilon u_{\boldsymbol{c}}^{(1)} + \mathcal{O}_{\boldsymbol{c}}(\varepsilon^2)$$

with $u_{\boldsymbol{c}}^{(0)} = u_{0,\boldsymbol{c}}$ and since $H_0(\theta, I)$ is integrable (and written in action-angle coordinates), one can choose $u_{0,\boldsymbol{c}} \equiv 0$. Therefore, by Assumption (P) again, we expand the Hamilton Jacobi equation (A.2.9) as

$$\begin{aligned}
\alpha_\varepsilon(\boldsymbol{c}) &= H_\varepsilon(\theta, \boldsymbol{c} + \nabla u_{\varepsilon,\boldsymbol{c}}(\theta)) \qquad (A.4.7)\\
&= \frac{|\partial_{\theta^1} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_1|^2}{2} + h_\mu^{(2)}(\partial_{\theta^2} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_2) + \varepsilon U(\theta^1, x_\mu^2(\theta^2, \partial_{\theta^2} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_2))\\
&= \frac{c_1^2}{2} + h_\mu^{(2)}(c_2) + \varepsilon \left\langle \left(c_1, \nabla h_\mu^{(2)}(c_2)\right), \nabla u_{\varepsilon,\boldsymbol{c}}^{(1)}(\theta) \right\rangle + \varepsilon U(\theta^1, x_\mu^2(\theta^2, c_2))\\
&\qquad + \mathcal{O}\left(\|(\nabla^2 h_\mu^{(2)})|_{\{|c_2| > \gamma + \sqrt{\mu}\mathfrak{c}(V)\}}\|_{C^0} \varepsilon^2 \right) + \mathcal{O}\left(\|(\partial_{I_2} \Phi_\mu^{(2)})|_{\{|c_2| > \gamma + \sqrt{\mu}\mathfrak{c}(V)\}}\|_{C^0} \varepsilon^2 \right)
\end{aligned}$$

Since $|c_2| > \gamma + \sqrt{\mu}\mathfrak{c}(V)$, both error terms are of the order $\mathcal{O}_\gamma(\varepsilon^2)$.

Analogously to the proof of Theorem A.I we thus obtain the first order equation

$$[U]_0 = \left\langle \left(c_1, \nabla h_\mu^{(2)}(c_2)\right), \nabla u_{\varepsilon,\mathfrak{c}}^{(1)}(\theta)\right\rangle + U(\theta^1, x_\mu^2(\theta^2, c_2)) \tag{A.4.8}$$

where the constant $\alpha^{(1)} \equiv [U]_0$ is given in (A.D.8) in Proposition A.D.9 (see also [288]). Just as in the proof of Theorem A.I, after averaging (A.4.8) over the trajectory $\theta(t) = \theta_0 + \boldsymbol{\omega} t \in \mathbf{T}^2$, with initial position $\theta_0 \in \mathbf{T}^2$ and where $\boldsymbol{\omega}$ is chosen according to (A.4.6), such that the period $T_{\boldsymbol{\omega}}$ satisfies $T_{\boldsymbol{\omega}} \cdot \boldsymbol{\omega} = (n, m)$, we find

$$\int_0^1 \left(U(\theta_0^1 + nt,\, x_\mu^2(\theta_0^2 + mt, c_2)) - [U]_0\right) \mathrm{d}t = 0 \tag{A.4.9}$$

for all $\theta_0 = (\theta_0^1, \theta_0^2) \in \mathbf{T}^2$.

Finally, analogously to Section A.4.1, we may assume w.l.o.g. $[U]_0 = 0$ and observe that

$$\int_0^1 U_{\mathrm{sep}}(\theta_0^1 + nt, x_\mu^2(\theta_0^2 + mt, c_2))\mathrm{d}t = [U_{\mathrm{sep}}]_0 \qquad \forall (\theta_0^1, \theta_0^2) \in \mathbf{T}^2$$

holds *generally* (i.e. independent of the first order relation (A.4.8)) by a simple calculation based on (A.D.8) in Proposition A.D.9 (see also Remark A.D.8). We can thus split off the separable part U_{sep} of U defined in (A.4.5) and assume that $\mathcal{S}_U = \mathcal{S}_{U,0}$ in the following. Hence, the third step consists of showing that $\mathcal{S}_U = \mathcal{S}_{U,0} = \varnothing$.

Step (iii). We begin this final step with performing a Fourier decomposition in (A.4.9), such that we obtain

$$\sum_{k_1 \neq 0} \left[\sum_{0 \neq |k_2| \leq \deg_U^{(2)}} U_{k_1,k_2} \int_0^1 \mathrm{e}^{\mathrm{i}2\pi k_1 nt} \mathrm{e}^{\mathrm{i}2\pi k_2 x_\mu^2(\theta_0^2 + mt, c_2)} \mathrm{d}t \right] \mathrm{e}^{\mathrm{i}2\pi k_1 \theta_0^1} = 0\,, \qquad \forall (\theta_0^1, \theta_0^2) \in \mathbf{T}^2\,,$$

which implies that $\left[\cdots\right] = 0$ for every $k_1 \in \mathbf{Z} \setminus \{0\}$ and $\theta_0^2 \in \mathbf{T}$.

After having eliminated $\theta_0^1 \in \mathbf{T}$, we now fix some $k_1 \in \mathbf{Z} \setminus \{0\}$ and consider the family of functions $\left(f_{k_2}^{(k_1,\mu)}\right)_{0 \neq |k_2| \leq \deg_U^{(2)}}$ in the Hilbert space $L^2(\mathbf{T})$, where

$$f_{k_2}^{(k_1,\mu)} : \mathbf{T} \to \mathbf{C}\,, \quad \theta_0^2 \mapsto \sum_{\substack{(n,m) \in \mathcal{B}_0(\mathcal{S}_U^\perp) \\ \exists 0 \neq |\tilde{k}_2| \leq \deg_U^{(2)} : k_1 n + \tilde{k}_2 m = 0}} \int_0^1 \mathrm{e}^{\mathrm{i}2\pi k_1 nt} \mathrm{e}^{\mathrm{i}2\pi k_2 x_\mu^2(\theta_0^2 + mt, c_2)} \mathrm{d}t\,. \tag{A.4.10}$$

Note that the sum in (A.4.10) is finite by Assumption (A2) (more precisely, it ranges over at most $2 \cdot \deg_U^{(2)}$ elements from $\mathcal{B}_0(\mathcal{S}_U^\perp)$) and we suppressed the dependence of $|c_2| > \gamma + \sqrt{\mu}\mathfrak{c}(V)$ on $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ from the notation (recall (A.4.6)).

In this way, the problem of proving Theorem A.II, i.e. justifying $\mathcal{S}_{U,0} = \varnothing$, reduces to a question about linear independence for the family of functions (A.4.10) in the Hilbert space $L^2(\mathbf{T})$. Recall that the family $\left(f_{k_2}^{(k_1,\mu)}\right)_{0 \neq |k_2| \leq \deg_U^{(2)}}$ being linearly independent is equivalent to the *Gram matrix*

$$G^{(k_1,\mu)} = \left(G_{k_2,k_2'}^{(k_1,\mu)}\right)_{0 \neq |k_2|, |k_2'| \leq \deg_U^{(2)}} \quad \text{with} \quad G_{k_2,k_2'}^{(k_1,\mu)} := \left\langle f_{k_2}^{(k_1,\mu)}, f_{k_2'}^{(k_1,\mu)} \right\rangle_{L^2(\mathbf{T})} \tag{A.4.11}$$

being of full rank, where $\langle g, h \rangle_{L^2(\mathbf{T})}$ denotes the standard inner product of $g, h \in L^2(\mathbf{T})$.

Lemma A.4.2. *There exists $\tilde{\mu} = \tilde{\mu}(\mathcal{C}_2, \deg_U^{(2)}, e) > 0$ such that for all $\mu \in [0, \tilde{\mu}]$ the Gram matrix $G^{(k_1,\mu)}$ from* (A.4.11) *is of full rank.*

Proof. Using the version of Lemma A.B.1 for the inverse function, we find that

$$\left\| \mathrm{e}^{\mathrm{i}2\pi k_2 x_\mu^2(\,\cdot\,,c_2)} - \mathrm{e}^{\mathrm{i}2\pi k_2 \cdot} \right\|_{C^0} = \mathcal{O}\left(\deg_U^{(2)} \frac{\mu \|V\|_{C^0}}{h_\mu(\gamma + \sqrt{\mu}\mathfrak{c}(V))} \right) =: \mathcal{O}\big(\mu\big) \tag{A.4.12}$$

uniformly in $|k_2| \leq \deg_U^{(2)}$ and $(n,m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$.

With a slight abuse of notation for the error term, the elements $G^{(k_1,\mu)}_{k_2,k_2'}$ of the Gram matrix can thus be computed as

$$\int_0^1 \mathrm{d}\theta_0^2 \left(\left[\sum_{(n,m)} \int_0^1 \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}2\pi k_1 n t} \left(\mathrm{e}^{-\mathrm{i}2\pi k_2 m t} + \mathcal{O}(\mu) \right) \right] \mathrm{e}^{-\mathrm{i}2\pi k_2 \theta_0^2} \times \right.$$
$$\left. \times\, \mathrm{e}^{\mathrm{i}2\pi k_2' \theta_0^2} \left[\sum_{(n',m')} \int_0^1 \mathrm{d}t' \mathrm{e}^{\mathrm{i}2\pi k_1 n' t'} \left(\mathrm{e}^{\mathrm{i}2\pi k_2' m' t'} + \mathcal{O}(\mu) \right) \right] \right),$$

where the summations over (n,m) and (n',m') are understood as in (A.4.10). Using that, for every $(k_1,k_2) \in \mathcal{S}_{U,0}$ there exist exactly two elements from $\mathcal{B}_0(\mathcal{S}_U^\perp)$ (differing by a sign), we can evaluate both brackets $\big[\cdots\big]$ being equal to $2 + \mathcal{O}\big(\deg_U^{(2)} \mu\big)$.

From this we conclude that

$$G^{(k_1,\mu)}_{k_2,k_2'} = \int_0^1 \mathrm{d}\theta_0^2 \Big[2 + \mathcal{O}\big(\deg_U^{(2)} \mu\big)\Big] \mathrm{e}^{\mathrm{i}2\pi(k_2' - k_2)\theta_0^2} \Big[2 + \mathcal{O}\big(\deg_U^{(2)} \mu\big)\Big] = 4\,\delta_{k_2,k_2'} + \mathcal{O}\big(\deg_U^{(2)} \mu\big)\,.$$

Therefore, going back to (A.4.12), we infer the existence of $\tilde{\mu} = \tilde{\mu}(\mathcal{C}_2, \deg_U^{(2)}, e) > 0$ such that for all $\mu \in [0,\tilde{\mu}]$ the Gram matrix $G^{(k_1,\mu)}$ from (A.4.11) is of full rank. □

Since $k_1 \in \mathbf{Z} \setminus \{0\}$ was arbitrary and Lemma A.4.2 is independent of k_1, this concludes the proof of Theorem A.II (a).

For part (b), we note that $\mathrm{e}^{\mathrm{i}2\pi k_2 x_\mu^2(\theta_0^2 + mt, c_2)}$ from (A.4.10) depends analytically on μ (see Appendix A.C). Therefore, the function $\mu \mapsto G^{(k_1,\mu)}$ mapping to the Gram matrix (A.4.11), for every fixed $k_1 \in \mathbf{Z} \setminus \{0\}$, is also analytic.[14] This in turn implies that $\det(G^{(k_1,\mu)})$ is analytic in μ and thus, since $\det(G^{(k_1,\mu)}) \neq 0$ for $\mu \in (0,\tilde{\mu})$ (see Lemma A.4.2), we find that the zero set

$$\mathcal{E}_0^{(k_1)} := \{\mu \in (0,\infty) \mid \det(G^{(k_1,\mu)}) = 0\} \subset (\tilde{\mu},\infty)$$

of $\mu \mapsto \det(G^{(k_1,\mu)})$ is at most countable (finite in every compact subset), i.e. in particular a set of zero measure. Finally, setting

$$\mathcal{E}_0 := \bigcup_{k_1 \in \mathbf{Z}\setminus\{0\}} \mathcal{E}_0^{(k_1)}\,,$$

we constructed the exceptional null set, for which the conclusion $\mathcal{S}_{U,0} = \varnothing$ is not valid.

This finishes the proof of Theorem A.II (b). □

A.4.3 Proof of Theorem A.III

Step (i). We fix an energy $e > 0$ and consider the region of the phase space, where both one-dimensional subsystems are rotating, i.e.

$$\frac{p_1^2}{2} - \mu_1 V_1(x^1) = e^{(1)} > 0 \quad \text{and} \quad \frac{p_2^2}{2} - \mu_2 V_2(x^2) = e^{(2)} > 0\,,$$

[14] Using joint continuity of $(u,\mu) \mapsto \mathrm{e}^{\mathrm{i}2\pi k_2 x_\mu^2(u,c_2)}$, it is an elementary exercise to show that the integrals over t and θ_0^2 do not disturb the analyticity in μ.

such that we have $e = e^{(1)} + e^{(2)}$. In a neighborhood of each of the two Liouville tori characterized by $H_0 = e$ and $\frac{p_1^2}{2} - \mu_1 V_1(x^1) = e^{(1)}$, we can find two changes of variables $(x^1, p_1) = \Phi_{\mu_1}^{(1)}(\theta^1, I_1)$ and $(x^2, p_2) = \Phi_{\mu_2}^{(2)}(\theta^2, I_2)$ such that the Hamiltonian function H_0 gets transformed in action-angle coordinates (see (A.1.8)), i.e.

$$H_0(\Phi_{\mu_1}^{(1)}(\theta^1, I_1), \Phi_{\mu_2}^{(2)}(\theta^2, I_2)) = h_{\mu_1}^{(1)}(I_1) + h_{\mu_2}^{(2)}(I_2)$$

for some smooth functions $h_{\mu_1}^{(1)}$ and $h_{\mu_2}^{(2)}$, which agree with Mather's α-functions for the one-dimensional subsystem described by the Hamiltonians $\frac{p_1^2}{2} - \mu_1 V(x^1)$ resp. $\frac{p_2^2}{2} - \mu_2 V(x^2)$ (see Appendix A.D). As in the proof of Theorem A.II, the change in the order of the four arguments of H_0 should not lead to confusion.

Now, the perturbed Hamiltonian takes the form

$$H_\varepsilon(\Phi_{\mu_1}^{(1)}(\theta^1, I_1), \Phi_{\mu_2}^{(2)}(\theta^2, I_2)) = h_{\mu_1}^{(1)}(I_1) + h_{\mu_2}^{(2)}(I_2) + \varepsilon U(x_{\mu_1}^1(\theta^1, I_1), x_{\mu_2}^2(\theta^2, I_2)),$$

where we write $x_{\mu_i}^i(\theta^i, I_i)$ for the first component of $\Phi_{\mu_i}^{(i)}(\theta^i, I_i)$, $i \in \{1, 2\}$.

Step (ii). Analogously to the proof of Theorem A.II, we assume w.l.o.g. that the 1- and 2-degree $\deg_U^{(1)}$ and $\deg_U^{(2)}$ of U are at least 1 (recall (A.2.8)), as otherwise we had $U(x) = U_2(x^2)$ or $U(x) = U_1(x^1)$ and Theorem A.III was proven. Then, for any $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$, in particular with $|m| \le \deg_U^{(1)}$ and $|n| \le \deg_U^{(2)}$, we can find (in the isoenergy manifold T_{e_ε} with energy $e = e_\varepsilon$ and $\varepsilon = \varepsilon_k$ for some $k \in \mathbf{N}$) a rational invariant Liouville torus with rotation vector $\boldsymbol{\omega} = (\omega_1, \omega_2)$ which satisfies

$$\frac{\omega_1}{\omega_2} = \frac{n}{m} \in \mathbb{Q} \quad \text{and} \quad \boldsymbol{\omega} = (\nabla h_{\mu_1}^{(1)}(c_1), \nabla h_{\mu_2}^{(2)}(c_2)) \tag{A.4.13}$$

for some $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R}) \cong \mathbf{R}^2$ with $|c_1| > \gamma_1 + \sqrt{\mu_1}\mathfrak{c}(V_1)$ and $|c_2| > \gamma_2 + \sqrt{\mu_2}\mathfrak{c}(V_2)$ for some $\gamma_1 = \gamma_1(e, \deg_U^{(1)}) > 0$ resp. $\gamma_2 = \gamma_2(e, \deg_U^{(2)}) > 0$, which we fix now (see the paragraph below (A.4.6) for a discussion of the γ parameters).

By Assumption (P) we have

$$u_{\varepsilon,\boldsymbol{c}} = u_{\boldsymbol{c}}^{(0)} + \varepsilon u_{\boldsymbol{c}}^{(1)} + O_{\boldsymbol{c}}(\varepsilon^2)$$

with $u_{\boldsymbol{c}}^{(0)} = u_{0,\boldsymbol{c}}$ and since $H_0(\theta, I)$ is integrable (and written in action-angle coordinates), one can choose $u_{0,\boldsymbol{c}} \equiv 0$. Therefore, by Assumption (P) again, we expand the Hamilton Jacobi equation (A.2.9) as

$$\begin{aligned}
\alpha_\varepsilon(\boldsymbol{c}) &= H_\varepsilon(\theta, \boldsymbol{c} + \nabla u_{\varepsilon,\boldsymbol{c}}(\theta)) \\
&= h_{\mu_1}^{(1)}(\partial_{\theta^1} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_1) + h_{\mu_2}^{(2)}(\partial_{\theta^2} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_2) \\
&\qquad + \varepsilon U(x_{\mu_1}^1(\theta^1, \partial_{\theta^1} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_1), x_{\mu_2}^2(\theta^2, \partial_{\theta^2} u_{\varepsilon,\boldsymbol{c}}(\theta) + c_2)) \\
&= \sum_{i=1}^2 h_{\mu_i}^{(i)}(c_i) + \varepsilon \left\langle \left(\nabla h_{\mu_1}^{(1)}(c_1), \nabla h_{\mu_2}^{(2)}(c_2)\right), \nabla u_{\varepsilon,\boldsymbol{c}}^{(1)}(\theta) \right\rangle + \varepsilon U(x_{\mu_1}^1(\theta^1, c_1), x_{\mu_2}^2(\theta^2, c_2)) \\
&\qquad + \mathcal{O}\Bigg(\sum_{i=1}^2 \big(\|(\nabla^2 h_{\mu_i}^{(i)})|_{\{|c_i| > \gamma_i + \sqrt{\mu_i}\mathfrak{c}(V_i)\}}\|_{C^0} + \|(\partial_{I_i} \Phi_{\mu_i}^{(i)})|_{\{|c_i| > \gamma_i + \sqrt{\mu_i}\mathfrak{c}(V_i)\}}\|_{C^0} \big) \varepsilon^2 \Bigg)
\end{aligned}$$

Since $|c_i| > \gamma_i + \sqrt{\mu_i}\mathfrak{c}(V_i)$, the error term is of order $\mathcal{O}_{\gamma_i}(\varepsilon^2)$.

Analogously to the proofs of Theorem A.I and Theorem A.II we thus obtain the first order equation

$$[U]_0 = \left\langle \left(\nabla h_{\mu_1}^{(1)}(c_1), \nabla h_{\mu_2}^{(2)}(c_2)\right), \nabla u_{\varepsilon,\boldsymbol{c}}^{(1)}(\theta) \right\rangle + U(x_{\mu_1}^1(\theta^1, c_1), x_{\mu_2}^2(\theta^2, c_2)) \tag{A.4.14}$$

where the constant $\alpha^{(1)} \equiv [U]_0$ is again given by (A.D.8) in Proposition A.D.9 (see also [288]). Just as in the proof of Theorem A.I and Theorem A.II, after averaging (A.4.14) over the trajectory

$\theta(t) = \theta_0 + \boldsymbol{\omega} t \in \mathbf{T}^2$, with initial position $\theta_0 \in \mathbf{T}^2$ and where $\boldsymbol{\omega}$ is chosen according to (A.4.13), such that the period $T_{\boldsymbol{\omega}}$ satisfies $T_{\boldsymbol{\omega}} \cdot \boldsymbol{\omega} = (n, m)$, we find

$$\int_0^1 \big(U(x^1_{\mu_1}(\theta^1_0 + nt, c_1),\, x^2_{\mu_2}(\theta^2_0 + mt, c_2)) - [U]_0\big)\,\mathrm{d}t = 0 \tag{A.4.15}$$

for all $\theta_0 = (\theta^1_0, \theta^2_0) \in \mathbf{T}^2$.

Finally, analogously to Section A.4.1 and Section A.4.2, we may assume w.l.o.g. $[U]_0 = 0$ and observe that

$$\int_0^1 U_{\mathrm{sep}}(x^1_{\mu_1}(\theta^1_0 + nt, c_1),\, x^2_{\mu_2}(\theta^2_0 + mt, c_2))\mathrm{d}t = [U_{\mathrm{sep}}]_0 \qquad \forall (\theta^1_0, \theta^2_0) \in \mathbf{T}^2$$

holds *generally* (i.e. independent of the first order relation (A.4.14)) by a simple calculation based on (A.D.8) in Proposition A.D.9 (see also Remark A.D.8). We can thus split off the separable part U_{sep} of U defined in (A.4.5) and assume that $\mathcal{S}_U = \mathcal{S}_{U,0}$ in the following. Hence, the third step consists of showing that $\mathcal{S}_U = \mathcal{S}_{U,0} = \varnothing$.

Step (iii). We begin this final step with performing a Fourier decomposition in (A.4.15), such that we obtain

$$\sum_{\substack{0 \neq |k_1| \leq \deg_U^{(1)} \\ 0 \neq |k_2| \leq \deg_U^{(2)}}} U_{k_1,k_2} \int_0^1 \mathrm{e}^{\mathrm{i}2\pi k_1 x^1_{\mu_1}(\theta^1_0 + nt, c_1)} \mathrm{e}^{\mathrm{i}2\pi k_2 x^2_{\mu_2}(\theta^2_0 + mt, c_2)} \mathrm{d}t = 0\,, \qquad \forall (\theta^1_0, \theta^2_0) \in \mathbf{T}^2\,.$$

Analogously to the proof of Theorem A.II, we now consider the family of functions

$$\big(f^{(\mu_1,\mu_2)}_{k_1,k_2}\big)_{0 \neq |k_1| \leq \deg_U^{(1)}, 0 \neq |k_2| \leq \deg_U^{(2)}}$$

in the Hilbert space $L^2(\mathbf{T}^2)$, where

$$f^{(\mu_1,\mu_2)}_{k_1,k_2} : \mathbf{T}^2 \to \mathbf{C}\,, \quad (\theta^1_0, \theta^2_0) \mapsto \sum_{(n,m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)} \int_0^1 \mathrm{e}^{\mathrm{i}2\pi k_1 x^1_{\mu_1}(\theta^1_0 + nt, c_1)} \mathrm{e}^{\mathrm{i}2\pi k_2 x^2_{\mu}(\theta^2_0 + mt, c_2)} \mathrm{d}t\,. \tag{A.4.16}$$

Note that the sum in (A.4.16) is finite by Assumption (A3) (more precisely, it ranges over the at most $(2\deg_U^{(1)}) \cdot (2\deg_U^{(2)})$ elements from $\mathcal{B}_0(\mathcal{S}_U^\perp)$) and we suppressed the dependence of $|c_i| > \gamma_i + \sqrt{\mu_i}\mathfrak{c}(V_i)$ on $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ from the notation (recall (A.4.13)).

In this way, the problem of proving Theorem A.III, i.e. justifying $\mathcal{S}_{U,0} = \varnothing$, reduces to a question about linear independence for the family of functions (A.4.16) in the Hilbert space $L^2(\mathbf{T}^2)$. Recall that the family $\big(f^{(\mu)}_{k_1,k_2}\big)_{(k_1,k_2)}$ being linearly independent is equivalent to the *Gram matrix* $G^{(\mu)}$ with entries

$$G^{(\mu_1,\mu_2)}_{(k_1,k_2),(k_1',k_2')} := \big\langle f^{(\mu_1,\mu_2)}_{k_1,k_2}, f^{(\mu_1,\mu_2)}_{k_1',k_2'} \big\rangle_{L^2(\mathbf{T}^2)} \quad \text{for} \quad 0 \neq |k_i|, |k_i'| \leq \deg_U^{(i)}\,,\ i \in \{1,2\}\,, \tag{A.4.17}$$

being of full rank, where $\langle g, h \rangle_{L^2(\mathbf{T}^2)}$ denotes the standard inner product of $g, h \in L^2(\mathbf{T}^2)$.

Lemma A.4.3. *There exist $\tilde{\mu}_i = \tilde{\mu}(\mathcal{C}_i, \deg_U^{(1)}, \deg_U^{(2)}, e) > 0$ such that for all $\mu_i \in [0, \tilde{\mu}_i]$, $i \in \{1,2\}$, the Gram matrix $G^{(\mu_1,\mu_2)}$ from* (A.4.17) *is of full rank.*

Proof. Using the version of Lemma A.B.1 for the inverse function, we find that

$$\Big\| \mathrm{e}^{\mathrm{i}2\pi k_i x^i_{\mu_i}(\,\cdot\,, c_i)} - \mathrm{e}^{\mathrm{i}2\pi k_i \cdot} \Big\|_{C^0} = \mathcal{O}\bigg(\deg_U^{(i)} \frac{\mu_i \|V_i\|_{C^0}}{h^{(i)}_{\mu_i}(\gamma_i + \sqrt{\mu_i}\mathfrak{c}(V_i))} \bigg) =: \mathcal{O}\big(\mu_i\big) \tag{A.4.18}$$

uniformly in $|k_i| \leq \deg_U^{(i)}$ and $(n, m) \in \mathcal{B}_0(\mathcal{S}_U^\perp)$.

Similarly to Lemma A.4.2, with a slight abuse of notation for the error term, the elements $G^{(\mu_1,\mu_2)}_{(k_1,k_2),(k_1',k_2')}$ of the Gram matrix can thus be computed as

$$\int_0^1 \mathrm{d}\theta_0^1 \int_0^1 \mathrm{d}\theta_0^2 \Bigg(\Bigg[\sum_{(n,m)} \int_0^1 \mathrm{d}t \, \big(\mathrm{e}^{-\mathrm{i}2\pi k_1 n t} + \mathcal{O}(\mu_1)\big)\big(\mathrm{e}^{-\mathrm{i}2\pi k_2 m t} + \mathcal{O}(\mu_2)\big) \Bigg] \mathrm{e}^{-\mathrm{i}2\pi k_1 \theta_0^1} \mathrm{e}^{-\mathrm{i}2\pi k_2 \theta_0^2} \times$$
$$\times \, \mathrm{e}^{\mathrm{i}2\pi k_1' \theta_0^1} \mathrm{e}^{\mathrm{i}2\pi k_2' \theta_0^2} \Bigg[\sum_{(n',m')} \int_0^1 \mathrm{d}t' \, \Big(\mathrm{e}^{\mathrm{i}2\pi k_1' n' t'} + \mathcal{O}(\mu_1)\Big)\Big(\mathrm{e}^{\mathrm{i}2\pi k_2' m' t'} + \mathcal{O}(\mu_2)\Big) \Bigg] \Bigg) ,$$

where the summations over (n,m) and (n',m') are understood as in (A.4.16). Using that for every $(k_1,k_2) \in \mathcal{S}_{U,0}$ there exist exactly two elements from $\mathcal{B}_0(\mathcal{S}_U^\perp)$ (differing by a sign), we can evaluate both brackets $\big[\cdots\big]$ being given by

$$2 + \mathcal{O}\big(\deg_U^{(1)} \deg_U^{(2)} \mu_1\big) + \mathcal{O}\big(\deg_U^{(1)} \deg_U^{(2)} \mu_2\big) =: 2 + \mathcal{O}\big(\deg_U^{(1)} \deg_U^{(2)} (\mu_1 + \mu_2)\big) ,$$

after absorption of the second order error in the first order ones.

From this we conclude that

$$\begin{aligned} G^{(\mu_1,\mu_2)}_{(k_1,k_2),(k_1',k_2')} &= \int_0^1 \mathrm{d}\theta_0^1 \int_0^1 \mathrm{d}\theta_0^2 \Big(\Big[2 + \mathcal{O}\big(\deg_U^{(1)} \deg_U^{(2)} (\mu_1+\mu_2)\big)\Big] \mathrm{e}^{\mathrm{i}2\pi(k_1'-k_1)\theta_0^1} \times \\ &\qquad \times \;\; \mathrm{e}^{\mathrm{i}2\pi(k_2'-k_2)\theta_0^2} \Big[2 + \mathcal{O}\big(\deg_U^{(1)} \deg_U^{(2)} (\mu_1+\mu_2)\big)\Big] \Big) \\ &= 4\,\delta_{k_1,k_1'}\,\delta_{k_2,k_2'} + \mathcal{O}\big(\deg_U^{(1)} \deg_U^{(2)} (\mu_1+\mu_2)\big) . \end{aligned}$$

Therefore, going back to (A.4.18), we infer the existence of $\tilde{\mu}_i = \tilde{\mu}(\mathcal{C}_i, \deg_U^{(1)}, \deg_U^{(2)}, e) > 0$, $i \in \{1,2\}$, such that for all $\mu_i \in [0,\tilde{\mu}_i]$ the Gram matrix $G^{(\mu_1,\mu_2)}$ from (A.4.17) is of full rank. □

This finishes the proof of Theorem A.III (a). For part (b), similarly to the proof of Theorem A.II (b), we observe that for every fixed $\mu_1 \in [0,\tilde{\mu}_1]$ the function $\mu_2 \mapsto \det\big(G^{(\mu_1,\mu_2)}\big)$ is analytic. Since $\det(G^{(\mu_1,\mu_2)}) \neq 0$ for $\mu_2 \in (0,\tilde{\mu}_2)$ (see Lemma A.4.3), we find that the zero set

$$\mathcal{E}_0^{(\mu_1)} := \{\mu_2 \in (0,\infty) \mid \det(G^{(\mu_1,\mu_2)}) = 0\} \subset (\tilde{\mu}_2, \infty)$$

of $\mu_2 \mapsto \det(G^{(\mu_1,\mu_2)})$ is at most countable (finite in every compact subset), i.e. in particular a (one-dimensional) set of zero measure.

Finally, for part (c), we note that, similarly to the proof of Theorem A.II (b) and by means of Hartogs's theorem on separate analyticity [322] (a separately analytic function is jointly analytic), the function $(\mu_1,\mu_2) \mapsto \det\big(G^{(\mu_1,\mu_2)}\big)$ is (jointly) analytic. Since $\det(G^{(\mu_1,\mu_2)}) \neq 0$ for $(\mu_1,\mu_2) \in (0,\tilde{\mu}_1)\times(0,\tilde{\mu}_2)$ (see Lemma A.4.3), we find that the zero set

$$\mathcal{E}_0 := \{(\mu_1,\mu_2) \in (0,\infty)\times(0,\infty) \mid \det(G^{(\mu_1,\mu_2)}) = 0\} \subset (\tilde{\mu}_1,\infty)\times(\tilde{\mu}_2,\infty)$$

of $(\mu_1,\mu_2) \mapsto \det(G^{(\mu_1,\mu_2)})$ is a (two-dimensional) set of zero measure.

This concludes the proof of Theorem A.III (c). □

A.5 Concluding remarks and outlook

We have shown that integrable deformations of Liouville metrics on $\mathbf{T}^2$ are Liouville metrics – at least when more restrictive conditions on the unperturbed metric are balanced with more general conditions on the perturbation. Removing this balancing, i.e. showing that *arbitrary* integrable deformations of *arbitrary* Liouville metrics remain of Liouville type, is an interesting problem for future investigations resolving the conjecture proposed at the end of Section A.2. This would require stronger versions of Lemmas A.4.2 and A.4.3 in two senses:

(a) Allow for possibly *infinitely many non-zero Fourier coefficients* and refrain from restricting to trigonometric polynomials. A resolution of this issue has been found in the context of the *perturbative Birkhoff conjecture* [33, 362] concerning integrable billiards. Here, the authors studied the matrix of correlations between the standard basis $(\mathrm{e}^{\mathrm{i}2\pi kx})_{k\in\mathbf{Z}}$ of $L^2(\mathbf{T})$ and certain deformed dynamical modes (given as some kind of Jacobi elliptic function, see Appendix A.C), corresponding to $\mathrm{e}^{\mathrm{i}2\pi k_i x^i_{\mu_i}(\,\cdot\,,c_i)}$ in Lemma A.4.2 and Lemma A.4.3. Exponential estimates for the entries of this matrix (obtained from considering the maximal width of a strip of analyticity around the real axis for the dynamical modes), allowed to prove a suitable full-rank lemmas, also for infinitely many coefficients.

(b) Allow *arbitrary* $\tilde{\mu_i} > 0$ and refrain from restricting to small ones. Also for this issue, a potential resolution might be found by analytically extending action-angle coordinates to the complex plane and exploiting their singularities away from the real axis. However, this requires the potentials V_i in the unperturbed Hamiltonian to be restrictions of holomorphic functions and as such way more special than generic $V_i \in C^2(\mathbf{T})$.

Moreover, we note that, in [362] the authors also outlined a potential strategy for proving the classical (non-perturbative) Birkhoff conjecture, which might possibly be adapted for proving a suitably weakened version of the folklore conjecture given in Section A.3.

We end this section with a brief list of open problems being related to the main results of the present paper:

(i) As described above, it is an natural follow-up problem to extend our results to the situation, where *arbitrary* integrable deformations of *arbitrary* Liouville metrics remain of Liouville type, i.e. remove the restricting assumptions from (A1) - (A3) and prove the conjecture formulated at the end of Section A.2.

(ii) In particular, starting with (the time-independent version of) Arnold's example [28] for diffusion,

$$H_0(x,p) = \frac{p_1^2}{2} + \frac{p_2^2}{2} - \mu\big(1-\cos(2\pi x^2)\big)\,,$$

is it possible do deduce rigidity, similarly to Theorem A.II, but without restricting to the perturbation being a trigonometric polynomial in x^2 and any smallness condition on $\mu \in [0,1]$? In this case, the *full rank lemma* might be obtained by proving non-degeneracy of certain infinite-dimensional matrices, which have Fourier coefficients of powers of Jacobi elliptic functions (see Appendix A.C) as their entries.

(iii) In view of the non-trivial examples of magnetic geodesic flows found in [13] and the potential counterexample constructed in [182], it is a major task to completely settle the *Folklore Conjecture* mentioned in Sections A.1 and A.3, i.e. clarify which part is only 'folklore' and which part is 'real'.

(iv) In particular, the main result of [182], which we stated in Theorem A.3.7, should be extended to showing that the system is really integrable on an open set in the phase space and not only on an isoenergy manifold. Furthermore, it remains open, whether the PDEs underlying the examples in [13] can be solved with zero magnetic fields or not and thus potentially disproves the folklore conjecture.

(v) For our main results, we assumed the preservation of rational invariant tori 'outside the eye of the pendulum' (cf. Figure A.1). Can one obtain the same result, if only tori 'inside the eye' are preserved?

(vi) An alternative approach to the one chosen here, could be to study perturbations of the additional first integral (A.2.2), i.e. write $F_\varepsilon = F_0 + \varepsilon F_1 + \mathcal{O}(\varepsilon^2)$ and use the vanishing of the Poisson bracket $\{H_\varepsilon, F_\varepsilon\} = 0$ with $H_\varepsilon = H_0 + \varepsilon U$ to obtain the first-order equation

$$\{H_0, F_1\} + \{U, F_0\} = 0$$

for the perturbing potential U.

(vii) Does there exist a Riemannian metric g on $\mathbf{T}^2$, such that its geodesic flow admits hyperbolic periodic orbits of *at least three* different homotopy types? If yes, does there exist a Liouville metric with this property?[15]

A.A Generalization to higher dimensions

Our results from Section A.2 immediately generalize to higher dimensions $d \geq 3$. In this setting, we define the Hamiltonian function

$$H_0(x,p) = \sum_{i=1}^{d} \left(\frac{p_i^2}{2} - \mu_i V_i(x^i) \right) \tag{A.A.1}$$

on $T^*\mathbb{T}^d$, where $\mu_i \in [0,\infty)$ are parameters, and $V_i \in C^2(\mathbb{T})$ with $\|V_i\| \leq \mathcal{C}_i$, $V_i \geq 0$ are Morse functions (or constant). We may assume w.l.o.g. that $\min_{x^i} V_i(x^i) = 0$. The system (A.A.1) is clearly integrable, since additional first integrals can easily be found as

$$F_i(x,p) = \frac{p_i^2}{2} - \mu_i V_i(x^i), \qquad i \in \{1, ..., d-1\}.$$

Completely analogous to Section A.2, we perturb the integrable system (A.A.1) as $H_\varepsilon = H_0 + \varepsilon U$ with $\varepsilon \in \mathbf{R}$ by an additive potential $U \in C^2(\mathbf{T}^d)$, which we assume to have an absolutely convergent Fourier series.

Now, the analogs of the assumptions in Section A.2 read as follows.

1. Assumptions on the perturbed Hamiltonian function H_ε.
Let $H_0 \in C^2(T^*\mathbf{T}^d)$ denote the Hamiltonian function from (A.A.1) with $\|V_i\| \leq \mathcal{C}_i$ and $\mu_i \in [0, \tilde{\mu}_i]$ for some $\tilde{\mu}_i \in [0,\infty)$, $i \in \{1, ..., d\}$, and $U \in C^2(\mathbf{T}^d)$ be a perturbing potential, which satisfies the following assumption.

(A4) If $\tilde{\mu}_i = 0$ for the first $0 \leq d_{\text{flat}} \leq d$ indices, there exist $d^{(k)} \geq 0$ for $k \in \{d_{\text{flat}} + 1, ..., d\}$ such that

$$\mathcal{S}_U \subset \mathbf{Z}^{d_{\text{flat}}} \times \left([-d^{(d_{\text{flat}}+1)}, d^{(d_{\text{flat}}+1)}] \times \cdots \times [-d^{(d)}, d^{(d)}] \right) \tag{A.A.2}$$

i.e. $U \in C^2(\mathbb{T}^d)$ is a trigonometric polynomial in the last $(d - d_{\text{flat}})$ variables.

As in Section A.2, we denote the minimum over all $d^{(i)}$ such that (A.A.2) holds as $\deg_U^{(i)}$ and call it the *i-degree of U*.

Note that Proposition A.2.1 immediately generalizes to higher dimensions, such that we can formulate the analog of Assumption (P) as follows.

2. Assumptions on the preserved integrability of H_ε.
Let $H_0 \in C^2(T^*\mathbf{T}^d)$ denote the Hamiltonian function from (A.A.1) satisfying Assumption (A4), and

[15] These questions were suggested by Vadim Kaloshin.

U a perturbing potential, such that the following statement concerning the perturbed Hamilton-Jacobi equation (HJE)

$$\alpha_\varepsilon(\boldsymbol{c}) = H_\varepsilon(x, \boldsymbol{c} + \nabla_x u_{\varepsilon,\boldsymbol{c}}(x)) \tag{A.A.3}$$

as well as the preserved integrability of H_ε is satisfied.

(P') There exists an energy $e > 0$, such that for every $\boldsymbol{b} \in \mathcal{B}_0(\mathcal{S}_U^\perp)$ (recall (A.2.6)) there exists a sequence $(\varepsilon_k)_{k\in\mathbf{N}}$ with $\varepsilon_k \neq 0$ but $\varepsilon_k \to 0$ such that for any $\mu_i \in [0, \tilde{\mu}_i]$ we have the following:

(i) The $\boldsymbol{b}$-torus from (the analog of) Proposition A.2.1 characterized by $\boldsymbol{c} \in H^1(\mathbf{T}^d, \mathbf{R}) \cong \mathbf{R}^d$ with

$$|c_i| > \sqrt{\mu_i}\, \mathfrak{c}(V_i) \tag{A.A.4}$$

in the isoenergy submanifold T_e is preserved under the sequence of deformations $(H_{\varepsilon_k})_{k\in\mathbf{N}}$, where $\mathfrak{c}(V_i)$ is defined in (A.D.4).

(ii) For $\boldsymbol{c} \in H^1(\mathbf{T}^d, \mathbf{R})$ satisfying (A.A.4), Mather's α-function and a solution $u_{\varepsilon,\boldsymbol{c}} : \mathbf{T}^d \to \mathbb{R}$ of the HJE (A.A.3) can be expanded to first order in ε, i.e.

$$u_{\varepsilon,\boldsymbol{c}} = u_{\boldsymbol{c}}^{(0)} + \varepsilon u_{\boldsymbol{c}}^{(1)} + \mathcal{O}_{\boldsymbol{c}}(\varepsilon^2) \quad \text{and} \quad \alpha_\varepsilon = \alpha^{(0)} + \varepsilon\alpha^{(1)} + \mathcal{O}(\varepsilon^2)\,,$$

where $u_{\boldsymbol{c}}^{(0)}, u_{\boldsymbol{c}}^{(1)} \in C^{1,1}(\mathbf{T}^d)$ and $O_{\boldsymbol{c}}(\varepsilon^2)$ is understood in $C^{1,1}$-sense.

We can now formulate our generalized main result.

Theorem A.IV. *Let H_ε satisfy Assumption* (A4) *and Assumption* (P') *for some energy $e > 0$. If V_j is analytic for $j \in \{ d - d_{\mathrm{anlyt}} + 1, ..., d \}$, where $0 \leq d_{\mathrm{anlyt}} \leq d - d_{\mathrm{flat}}$, and $\tilde{\mu}_k = \tilde{\mu}_k(\mathcal{C}_k, \deg_U^{(d_{\mathrm{flat}}+1)}, ..., \deg_U^{(d)}, e) > 0$ for $k \in \{ d_{\mathrm{flat}} + 1, ..., d - d_{\mathrm{anlyt}} \}$ are small enough, then U is separable, i.e. there exist $U_1, ..., U_d \in C^2(\mathbf{T})$ such that*

$$U(x^1, ..., x^d) = U_1(x^1) + ... + U_d(x^d) \qquad \forall (x^1, ..., x^d) \in \mathbf{T}^d\,.$$

This is irrespective of $\tilde{\mu}_j > 0$ for $j \in \{ d - d_{\mathrm{anlyt}} + 1, ..., d \}$, but only for

$$(\mu_{d-d_{\mathrm{anlyt}}+1}, ..., \mu_d) \in [0, \tilde{\mu}_{d-d_{\mathrm{anlyt}}+1}] \times ... \times [0, \tilde{\mu}_d]$$

outside of an exceptional d_{anlyt}-dimensional null-set (depending on $(\mu_{d_{\mathrm{flat}}+1}, ..., \mu_{d-d_{\mathrm{anylt}}})$).

A.B Basic perturbation lemma

In this appendix, we state and prove a basic perturbation lemma, which is instrumental in the continuity arguments required for the proofs of Lemma A.4.2 and Lemma A.4.3.

Lemma A.B.1. *Let $V \in C^1(\mathbf{T})$ be a non-negative function with $\min V = 0$, $\mu \in [0, 1]$, and define the Hamiltonian function*

$$H_\mu(p, x) = \frac{p^2}{2} - \mu V(x) \tag{A.B.1}$$

on the cotangent bundle $T^*\mathbf{T}$. In the neighborhood of a fixed energy $E > 0$, we can find action-angle coordinates (I, θ) *of* (A.B.1) *as*

$$I = \pm \int_0^1 \sqrt{2(E + \mu V(x))}\, \mathrm{d}x\,, \qquad \theta = \pm \frac{\int_0^x \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}{\int_0^1 \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}\,. \tag{A.B.2}$$

Regarding $\theta = \theta(x)$ as a function on $\mathbf{T}$, we have $\theta \in C^1(\mathbf{T})$ and

$$\|\theta \mp x\|_{C^1} = \mathcal{O}\left(\frac{\mu \|V\|_{C^0}}{E}\right) \quad \text{as} \quad \mu \to 0\,. \tag{A.B.3}$$

The same holds true if we regard $x = x(\theta)$ as a function on $\mathbf{T}$.

Proof. In the whole proof, we focus on the first sign choice in (A.B.2) and (A.B.3), the second one is completely analogous and hence omitted.

The time-independent Hamiltonian (A.B.1) is a conserved quantity along the Hamiltonian flow. By restricting to the neighborhood of an isoenergy manifold with $E > 0$, which is topologically $\mathbf{T}$ and puts us in the rotating phase of the system (A.B.1), the local action-angle coordinates can be found in the following way. The action coordinate is obtained by integrating the momentum p as a solution of

$$\frac{p^2}{2} - \mu V(x) = E$$

over a full rotation, i.e.

$$I_\mu = \int_0^1 \sqrt{2(E + \mu V(x))}\,\mathrm{d}x\,.$$

This quantity is preserved along the Hamiltonian flow and one can express E in terms of I by the implicit function theorem. This allows us to calculate the time-derivative of the conjugate coordinate θ of I as

$$\dot\theta = \frac{\partial E}{\partial I} = \left(\frac{\partial I}{\partial E}\right)^{-1} = \left(\int_0^1 \frac{\mathrm{d}x}{\sqrt{2(E + \mu V(x))}}\right)^{-1} =: \omega\,. \tag{A.B.4}$$

Integrating the equation of motion $\dot x = p = \sqrt{2(E + \mu V(x))}$ and using $\theta = \omega t$ (obtained by integrating (A.B.4) w.r.t. t starting from $t = 0$), we find $\theta = \theta(x)$ as a function of x being given by

$$\theta = \omega \int_0^x \frac{\mathrm{d}x'}{\sqrt{2(E + \mu V(x'))}} = \frac{\int_0^x \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}{\int_0^1 \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}\,.$$

From this, the approximation (A.B.3) can now easily be derived by expanding the square roots using $\sqrt{1+y} = 1 + \frac{y}{2} + \mathcal{O}(y^2)$ and $\frac{1}{1+y} = 1 - y + \mathcal{O}(y^2)$ for $|y| \to 0$. The reversed statement for $x = x(\theta)$ is a simple consequence. This finishes the proof of Lemma A.B.1. □

A.C Action-angle coordinates and analyticity

This appendix is concerned with analyticity properties of action-angle coordinates for one-dimensional Hamiltonian system

$$H_\mu(p, x) = \frac{p^2}{2} - \mu V(x) \tag{A.C.1}$$

being defined on the cotangent bundle $T^*\mathbf{T}$, where μ is a positive parameter and $V \geq 0$ an analytic function. Just as in Appendix A.B, in the neighborhood of a fixed energy $E > 0$, we can find action-angle coordinates (I, θ) of (A.C.1) as

$$I = \pm \int_0^1 \sqrt{2(E + \mu V(x))}\,\mathrm{d}x\,, \qquad \theta = \pm \frac{\int_0^x \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}{\int_0^1 \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}\,. \tag{A.C.2}$$

From now on, we shall restrict to the first sign choice in (A.C.2).

In our proofs of the analyticity cases in Theorem A.II and Theorem A.III, we shall exploit the fact that the function

$$\theta : (x, \mu) \mapsto \frac{\int_0^x \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}}{\int_0^1 \frac{\mathrm{d}x'}{\sqrt{1+\mu V(x')/E}}} \tag{A.C.3}$$

is analytic in both variables. (Note that the further implicit dependence on μ via $E = E(I)$ is also analytic.) Now, for every fixed $\mu > 0$, the function $x \mapsto \theta(x, \mu)$ is analytic and invertible,

and we denote its analytic inverse by $\theta \mapsto x_\mu(\theta)$ (cf. **Step (i)** in the proofs of Theorem A.II and Theorem A.III). Moreover, most importantly, also the function

$$(\theta, \mu) \mapsto x_\mu(\theta)$$

is analytic in μ, as shown in the following simple lemma applied to $f(z,w) \equiv \theta(x,\mu)$ in (A.C.3).

Lemma A.C.1. *Let $D_z, D_w \subset \mathbf{R}$ be open sets and*

$$f : D_z \times D_w \to \mathbf{R}\,, \quad (z,w) \mapsto f(z,w) \tag{A.C.4}$$

an analytic function. Moreover, assume that the one-variable restriction $f(\cdot,w) : D_z \to \mathbf{R}$ is invertible and satisfies $f(D_z,w) = D$ for every fixed $w \in D_w$ and some open $D \subset \mathbf{R}$, such that we can write its analytic inverse function as

$$f^{-1}(\cdot,w) : D \to D_z\,, \quad \zeta \mapsto f^{-1}(\zeta,w)\,.$$

Then it holds that, with a slight abuse of notation, also

$$f^{-1} : D \times D_w \to D_z\,, \quad (\zeta,w) \mapsto f^{-1}(\zeta,w)$$

is an analytic function.

Proof. Since $f^{-1}(\cdot,w) : D \to D_z$ is analytic for every fixed $w \in D_w$, it can be represented as

$$f^{-1}(\zeta,w) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{z \cdot (\partial_z f)(z,w)}{f(z,w) - \zeta} \mathrm{d}z \tag{A.C.5}$$

by Cauchy's integral formula, where C is a closed contour for which $|f(z,w) - \zeta| \geq \rho > 0$. In this form, since f from (A.C.4) is itself analytic and by involving Hartogs's theorem [322] (a separately analytic function is jointly analytic), the rhs. of (A.C.5) defines (locally) a jointly analytic function in both variables (ζ, w). □

We note that, although θ from (A.C.3) is always analytic in μ, the lower regularity in x for a general $V \in C^2(\mathbf{T})$ prevents the analyticity in μ to carry over to the inverse function.

We conclude this appendix, by showing analyticity for the important special case of a pendulum, i.e. $V(x) = 1 - \cos(2\pi x)$, in a more explicit way. In this particular situation, θ can be represented as

$$\theta = \frac{1}{2} - \frac{F\big(\pi(\frac{1}{2} - x)\,|\,m_\mu\big)}{2K(m_\mu)}\,, \tag{A.C.6}$$

where we introduced the shorthand notation $m_\mu = \frac{2\mu}{E+2\mu}$. Here, $F(\varphi\,|\,m)$ (resp. $K(m)$) for $k \in [0,1)$ denotes the *incomplete (resp. complete) elliptic integral of the first kind*, i.e.

$$F(\varphi\,|\,m) = \int_0^\varphi \frac{\mathrm{d}\vartheta}{\sqrt{1 - m\sin^2(\vartheta)}} \qquad \text{and} \qquad K(m) = F(\tfrac{\pi}{2}\,|\,m)\,. \tag{A.C.7}$$

The quantity m is called the *parameter*, φ the *amplitude*.

Now, the so-called *Jacobi elliptic function* are obtained by inverting the incomplete elliptic integral (A.C.7). More precisely, if $u := F(\varphi\,|\,m)$ denotes the *argument*, and u and φ are related in this way (we also write $\varphi = \mathrm{am}(u\,|\,m)$ for the amplitude), then we define the *Jacobi elliptic functions* as

$$\mathrm{sn}(u\,|\,m) := \sin\big(\mathrm{am}(u\,|\,m)\big)\,, \qquad \mathrm{cn}(u\,|\,m) := \cos\big(\mathrm{am}(u\,|\,m)\big)\,,$$

which are called the *elliptic sine* and *elliptic cosine*, respectively. Moreover, using the notation introduced above, we can invert the relation (A.C.6) to find that

$$x = \frac{1}{2} - \frac{1}{\pi}\,\mathrm{am}\big(K(m_\mu)(1-2\theta_\mu)\,|\,m_\mu\big)\,. \tag{A.C.8}$$

Most commonly, the elliptic sine and cosine are considered for fixed parameter m as functions of u, in which way they in fact behave as *elliptic functions*, i.e. doubly-periodic meromorphic function on the complex plane. However, as a function of the parameter parameter m (see [583]), we have that

$$m \mapsto \mathrm{sn}(K(m)u\,|\,m) \quad \text{and} \quad m \mapsto \mathrm{cn}(K(m)u\,|\,m) \tag{A.C.9}$$

are analytic for $m \in \mathbf{C} \setminus [1,\infty)$ and fixed $u \in \mathbf{R}$. This easily follows by representing $\mathrm{sn}(K(m)u\,|\,m)$ and $\mathrm{sn}(K(m)u\,|\,m)$ as ratios of *Jacobi theta functions* [4, Eq. 16.36.3] (see also [524, Eq. 5]), whose zeros are known explicitly [4, Eq. 16.36.2].

A.D Weak KAM theory

In this appendix, we provide an overview on basic concepts and results of weak KAM theory and Aubry-Mather theory, which are relevant in the proofs of our main results. In particular, we discuss separable Hamiltonian systems on $T^*\mathbf{T}^2$, i.e. sums of two independent systems on $T^*\mathbf{T}$. The presentation partly follows lecture notes from Sorrentino [530], which build on seminal works from Mather [441, 442, 443], Aubry [32], Mañé [445], Fathi [254, 255] and others.

In the following, let (M,g) be a compact and connected smooth Riemannian manifold without boundary, e.g. the torus $\mathbf{T}^2$. As in Section A.1.1, TM denotes its tangent bundle and T^*M its cotangent bundle. While a point in TM is denoted by (x,v), where $x \in M$ and $v \in T_xM$, a point in T^*M is denoted by (x,p), where $x \in M$ and $p \in T_x^*M$ is a linear functional on T_xM. The Riemannian metric g induces a metric d on M as well as a norm $\|\cdot\|_x$ on T_xM. We shall use the same notation for the norm induced on T_x^*M. The standard assumptions on a Hamiltonian $H : T^*M \to \mathbf{R}$ are summarized as follows.

Definition A.D.1. (Tonelli Hamiltonians)
A function $H : T^*M \to \mathbf{R}$ is called a Tonelli Hamiltonian if and only if H is (i) of class C^2; (ii) strictly convex in each fiber in C^2-sense, i.e. the quadratic form $(\partial^2 H/\partial p^2)(x,p)$ is positive definite for any $(x,p) \in T^*M$; (iii) superlinear in each fiber, i.e. $\lim_{\|p\|_x \to \infty} \frac{H(x,p)}{\|p\|_x} = \infty$.

A Hamiltonian $H : T^*M \to \mathbf{R}$ is canonically associated to a Lagrangian $L : TM \to \mathbf{R}$ as being each others Fenchel-Legendre transforms $(x,p) = \mathcal{L}(x,v)$, i.e.

$$\begin{aligned} H : T^*M \to \mathbf{R}, &\quad (x,p) \mapsto \sup_{v \in T_xM} \big[\langle p, v\rangle_x - L(x,v)\big]\,, \\ L : TM \to \mathbf{R}, &\quad (x,v) \mapsto \sup_{p \in T_x^*M} \big[\langle p, v\rangle_x - H(x,p)\big]\,. \end{aligned}$$

It is easy to check that the Lagrangian associated to a Tonelli Hamiltonian is also of Tonelli type (defined analogously to Definition A.D.1).

Piecewise C^1 curves $\gamma : [0,1] \to M$, which minimize the action functional

$$A_L(\gamma) := \int_0^1 L(\gamma(t), \dot\gamma(t))\mathrm{d}t\,,$$

satisfy the associated Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v}(\gamma(t),\dot\gamma(t)) = \frac{\partial L}{\partial x}(\gamma(t),\dot\gamma(t))\,, \qquad t \in [0,1]\,.$$

In case that $\det \frac{\partial^2 L}{\partial v^2} \neq 0$ (Legendre condition), the Euler-Lagrange equation is equivalent to an equation, which can be solved for $\ddot{\gamma}(t)$, which allows to define a vector field X_L on TM, such that the solutions of $\ddot{\gamma}(t) = X_L(\gamma(t), \dot{\gamma}(t))$ precisely satisfy the Euler-Lagrange equation. The associated flow Φ_t^L is called the *Euler-Lagrange flow*, which is C^1 for L of class C^2.

A.D.1 Basic notions of Aubry-Mather theory

The central objects of study in Aubry-Mather theory are invariant probability measures on TM having finite average action,

$$\mathbf{M}(L) := \left\{ \nu \text{ prob. meas. on } TM \text{ with } \nu \circ \Phi_t^L = \nu\,,\ \forall t \in \mathbf{R}\,,\ \int_{TM} L\,\mathrm{d}\nu < \infty \right\}\,,$$

which shall be endowed with the vague topology, i.e. the weak* topology induced by the continuous functions $f : TM \to \mathbf{R}$ having at most linear growth.

Proposition A.D.2. *Every non-empty energy level $\{H \circ \mathcal{L}(x,v) = E\}$ contains at least one invariant probability measure of Φ_t^L, i.e. $\mathfrak{M}(L) \neq \varnothing$.*

For every $\nu \in \mathfrak{M}(L)$ we now define its average action

$$A_L(\nu) := \int_{TM} L\mathrm{d}\nu\,.$$

Since $A_L : \mathfrak{M}(L) \to \mathbf{R}$ is lower semicontinuous w.r.t. the vague topology on $\mathfrak{M}(L)$, we have the following.

Proposition A.D.3. *There exists $\nu \in \mathfrak{M}(L)$, which minimizes A_L over $\mathfrak{M}(L)$.*

A measure $\nu \in \mathfrak{M}(L)$ minimizing A_L is called an action-minimizing measure of L. The principal goal in Aubry-Mather theory is to characterize invariant sets of the dynamics via action minimizing measures. Since – at least for integrable systems – the phase space is foliated by invariant tori (cf. Theorem A.1.6) and minimizing a single functional will not be sufficient to characterize all of them, one considers certain modifications of the Lagrangian: Let η be a 1-form on M and interpret it as a functional on the tangent space as

$$\hat{\eta} : TM \to \mathbf{R}\,,\ (x,v) \mapsto \langle \eta(x), v \rangle_x\,.$$

One can easily verify that, if η is closed (i.e. $\mathrm{d}\eta = 0$), then L and $L_\eta := L - \hat{\eta}$ have the same Euler-Lagrange flow. Moreover, if $\eta = \mathrm{d}f$ is an exact 1-form, then $\int \widehat{\mathrm{d}f}\mathrm{d}\nu = 0$ for any $\nu \in \mathfrak{M}(L)$. Therefore, for fixed L, the minimizing measures of L_η will depend only on the de Rham cohomology class $\boldsymbol{c} = [\eta] \in H^1(M, \mathbf{R})$. Hereafter, η_c shall denote a closed 1-form with cohomology class $\boldsymbol{c} \in H^1(M, \mathbf{R})$.

Definition A.D.4. *We define Mather measures, Mather's α-function, and Mather sets as follows:*

- *If $\nu \in \mathfrak{M}(L)$ minimizes $A_{L_{\eta_{\boldsymbol{c}}}}$, we call ν a Mather measure with cohomology $\boldsymbol{c}$.*
- *The map*

$$\alpha : H^1(M, \mathfrak{R}) \to \mathfrak{R}\,,\ \boldsymbol{c} \mapsto -\min_{\nu \in \mathfrak{M}(L)} A_{L_{\eta_{\boldsymbol{c}}}}(\nu) \tag{A.D.1}$$

  is called Mather's α-function. It is well defined and easily seen to be convex.

- *For $\boldsymbol{c} \in H^1(M, \mathbf{R})$, we define the Mather set of cohomology class $\boldsymbol{c}$ as*

$$\widetilde{\mathcal{M}}_{\boldsymbol{c}} := \bigcup_{\nu \in \mathfrak{M}_{\boldsymbol{c}}(L)} \operatorname{supp} \nu \subset TM\,,$$

where we denoted $\mathfrak{M}_{\boldsymbol{c}}(L) := \{\nu \in \mathfrak{M}_L \,:\, A_{L_{\eta_{\boldsymbol{c}}}}(\nu) = -\alpha(\boldsymbol{c})\}$. The projection $\mathcal{M}_{\boldsymbol{c}} = \pi\big(\widetilde{\mathcal{M}}_{\boldsymbol{c}}\big) \subset M$ on the base manifold is called the projected Mather set with cohomology class $\boldsymbol{c}$.

By duality, one can also define Mather's β-function: Since $\int_{TM} \hat{\eta} \mathrm{d}\nu = 0$ for exact 1-forms η, the linear functional

$$H^1(M, \mathbf{R}) \to \mathbf{R}\,,\ \boldsymbol{c} \mapsto \int_{TM} \hat{\eta_{\boldsymbol{c}}} \mathrm{d}\nu$$

is well defined. By duality, there exists $\boldsymbol{\omega}(\nu) \in H_1(M, \mathbf{R})$ such that

$$\int_{TM} \hat{\eta}_{\boldsymbol{c}} \mathrm{d}\nu = \langle \boldsymbol{c}, \boldsymbol{\omega}(\nu) \rangle \qquad \forall \boldsymbol{c} \in H^1(M, \mathbf{R})\,,$$

where we call $\boldsymbol{\omega}(\nu)$ the rotation vector of ν, which will turn to out to be matching the earlier definition in Proposition A.2.1. The map $\boldsymbol{\omega} : \mathfrak{M}_L \to H_1(M, \mathbf{R})$ is continuous, affine linear and surjective. In combination with the lower semicontinuity of A_L, this shows that Mather's β-function

$$\beta : H_1(M, \mathfrak{R}) \to \mathfrak{R}\,,\ \boldsymbol{h} \mapsto \min_{\nu \in \mathfrak{M}(L) : \boldsymbol{\omega}(\nu) = \boldsymbol{h}} A_L(\nu) \tag{A.D.2}$$

is well defined. It can easily seen to be the convex and, in fact, being the convex conjugate (Fenchel transform) of the α-function, showing that both, α and β, have superlinear growth.

We will see below, that the Liouville tori $T_{\boldsymbol{c}}$ with $|c_i| > \sqrt{\mu_i}\mathfrak{c}(V_i)$ from Proposition A.2.1 agree with the Mather set of cohomology class $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R}) \cong \mathbf{R}^2$, i.e. $\widetilde{\mathcal{M}}_{\boldsymbol{c}} = \mathcal{L}^{-1}(T_{\boldsymbol{c}})$. Basically, this will be concluded from the following two fundamental results.

Theorem A.D.5. (Mather's graph Theorem [442])
The Mather set $\widetilde{\mathcal{M}}_{\boldsymbol{c}}$ is compact, invariant under the Euler-Lagrange flow and $\pi|_{\widetilde{\mathcal{M}}_{\boldsymbol{c}}} : \widetilde{\mathcal{M}}_{\boldsymbol{c}} \to M$ is an injective map, whose inverse $\pi^{-1} : \mathcal{M}_{\boldsymbol{c}} \to \widetilde{\mathcal{M}}_{\boldsymbol{c}}$ is Lipschitz.

Theorem A.D.6. (Carneiro [138])
The Mather set $\widetilde{\mathcal{M}}_{\boldsymbol{c}}$ is contained in the energy level $\{H \circ \mathcal{L}(x, v) = \alpha(\boldsymbol{c})\}$.

A.D.2 Aubry-Mather theory in one dimension

In the following, we discuss the basic objects introduced above for the one-dimensional example of a mechanical Hamiltonian on $M = \mathbf{T}$. Note that the unperturbed Hamiltonian (A.2.1) in the formulation of our main results is a sum of two such one-dimensional systems. Let $V \in C^2(\mathbf{T})$ be a non-negative Morse function with $\min_{x \in \mathbf{T}} V(x) = 0$, $\mu \in (0, 1]$, and consider the Hamiltonian

$$H : T^*\mathbf{T} \to \mathbf{R}\,,\ (x, p) \mapsto \frac{p^2}{2} - \mu V(x)\,, \tag{A.D.3}$$

whose corresponding Lagrangian can easily be obtained as $L(x, v) = \frac{v^2}{2} + \mu V(x)$. We note that

$$T\mathbf{T} \cong T^*\mathbf{T} \cong \mathbf{T} \times \mathbf{R} \qquad \text{and} \qquad H_1(\mathbf{T}, \mathbf{R}) \cong H^1(\mathbf{T}, \mathbf{R}) \cong \mathbf{R}\,.$$

First of all, we study invariant probability measures of the system (A.D.3).

- Since V is a Morse functions, the sets of local (isolated) minima and maxima, $\mathfrak{X}_{\min}$ and $\mathfrak{X}_{\max}$, respectively, contain only finitely many elements. This shows that each of the measures

$$\left(\delta_{(x_*,0)}\right)_{x_*\in\mathfrak{X}_{\min}}, \qquad \left(\delta_{(x^*,0)}\right)_{x^*\in\mathfrak{X}_{\max}}$$

  are invariant probability measures of the system, all having zero rotation vector. They correspond to unstable and stable fixed points with respective energies $H(x_*,0) = -\mu V(x_*)$ for $x_* \in \mathfrak{X}_{\min}$ and $H(x^*,0) = -\mu V(x^*)$ for $x^* \in \mathfrak{X}_{\max}$.

- For $E > 0$, the energy level $\{H(x,p) = E\}$ consists of two homotopically non-trivial periodic orbits

$$\mathcal{P}_E^{\pm} := \left\{(x,p) : p = \pm\sqrt{2(E+\mu V(x))},\ x \in \mathbf{T}\right\}.$$

  The probability measures evenly distributed along these orbits – denoted by $\nu_E^{\pm}$ – are invariant probability measures of the system. If we denote by

$$T(E) := \int_0^1 \frac{1}{\sqrt{2(E+\mu V(x))}}\mathrm{d}x$$

  the period of such an orbit, one can easily see that $\boldsymbol{\omega}(\nu_E^{\pm}) = \pm\frac{1}{T(E)}$. Moreover, we have that $T : (0,\infty) \to (0,\infty)$ is continuous, strictly decreasing, and $T(E) \to \infty$ as $E \to 0$, i.e. $\boldsymbol{\omega}(\nu_E^{\pm}) \to 0$ as $E \to 0$.

- For every $E \in (-\mu\max_{x\in\mathbf{T}} V(x), 0) \setminus ((-\mu V(\mathfrak{X}_{\max})) \cup (-\mu V(\mathfrak{X}_{\min})))$, the energy level $\{H(x,p) = E\}$ consists of $N_E < \infty$ disjoint *contractible* periodic orbits. A probability measure $\nu_E^{(k)}$, $k \in \{1,...,N_E\}$, evenly distributed along such an orbit, is invariant for the system. Since the orbit is contractible, the rotation vector of $\nu_E^{(k)}$ is zero, $\boldsymbol{\omega}(\nu_E^{(k)}) = 0$.

The support of the measures $\nu_E^{(k)}$ for $E \in (-\mu\max_{x\in\mathbf{T}} V(x), 0) \setminus ((-\mu V(\mathfrak{X}_{\max})) \cup (-\mu V(\mathfrak{X}_{\min})))$ is not a graph over $\mathbf{T}$. Therefore, by means of Mather's graph Theorem A.D.5, they cannot be action minimizing. Moreover, we also have that the α-function is even, $\alpha(c) = \alpha(-c)$ for all $c \in \mathbf{R}$, which follows by the symmetry $H(x,p) = H(x,-p)$ of the system (A.D.3). In combination with the convexity of α, this shows that $\min_{\mathbf{R}} \alpha(c) = \alpha(0)$. Since $V \geq 0$, we have $A_L(\nu) \geq 0$ for all $\nu \in \mathfrak{M}_L$ and thus $\alpha(c) \geq 0$ for all $c \in \mathbf{R}$. By taking $x_* \in \mathfrak{X}_{\min}$ with $V(x_*) = 0$ (a global minimum), we have $A_L(\delta_{(x_*,0)}) = 0$, which shows that $\min_{\mathbf{R}} \alpha(c) = \alpha(0) = 0$. It follows from Theorem A.D.6, that only energy levels with $E \geq 0$ are capable of containing a Mather set. The Mather set of cohomology $c = 0$ is contained in the energy level with $E = 0$ and we have $\widetilde{\mathcal{M}}_0 = \{V = 0\} \times \{0\}$.

For cohomology classes different from zero, a first observation is that, since α is superlinear and continuous, all energy levels with $E > 0$ must contain some Mather set. Let $E > 0$ and consider the periodic orbit $\mathcal{P}_E^+$ with the invariant probability measure ν_E^+ evenly distributed. The graph of this orbit can be viewed as the graph of the closed 1-form $\eta_E^+ := \sqrt{2(E+\mu V(x))}\mathrm{d}x$, having cohomology class

$$c^+(E) = [\eta_E^+] = \int_0^1 \sqrt{2(E+\mu V(x))}\mathrm{d}x\,.$$

This function is continuous, strictly increasing for $E > 0$ and we have

$$c^+(E) \longrightarrow \sqrt{\mu}\int_0^1 \sqrt{2V(x)}\mathrm{d}x =: \sqrt{\mu}\,\mathfrak{c}(V)\,, \quad \text{as} \quad E \to 0\,. \tag{A.D.4}$$

Therefore, this defines an invertible function $c^+ : (0,\infty) \to (\sqrt{\mu}\,\mathfrak{c}(V), \infty)$, whose inverse we denote by $E(c)$. Using Mather's graph Theorem A.D.5 and the Fenchel-Legendre inequality, $\langle v,p\rangle_x \leq L(x,v) + H(x,p)$, one can show that ν_E^+ is the unique $c^+(E)$-action minimizing measure and we thus have

$$\widetilde{\mathcal{M}}_{c^+(E)} = \mathcal{P}_E^+\,, \quad \text{and similarly} \quad \widetilde{\mathcal{M}}_{c^-(E)} = \mathcal{P}_E^-\,,$$

where $c^-(E) = -c^+(E)$ is the cohomology class of $\eta_E^- = -\eta_E^+$.

It remains to study the non-zero cohomology classes in $[-\sqrt{\mu}\mathfrak{c}(V), \sqrt{\mu}\mathfrak{c}(V)]$. Observe that, by Theorem A.D.6, we have $\alpha(c^\pm(E)) = E$ and thus, using continuity of α, it follows that $\alpha(\pm\sqrt{\mu}\mathfrak{c}(V)) = 0$. By convexity of α and $\min_{\mathbf{R}} \alpha(c) = \alpha(0) = 0$, this implies $\alpha(c) \equiv 0$ for $c \in [-\sqrt{\mu}\mathfrak{c}(V), \sqrt{\mu}\mathfrak{c}(V)]$. Consequently, the corresponding Mather measures lie in the zero energy level, such that we have

$$\widetilde{\mathcal{M}}_c = \{\, V = 0 \,\} \times \{\, 0 \,\} \qquad \text{for all} \quad c \in [-\sqrt{\mu}\mathfrak{c}(V), \sqrt{\mu}\mathfrak{c}(V)]\,.$$

Summarizing the above considerations, we have shown that

$$\widetilde{\mathcal{M}}_c = \begin{cases} \{V = 0\} \times \{0\} & \text{if } |c| \le \sqrt{\mu}\mathfrak{c}(V) \\ \mathcal{P}^{\mathrm{sgn}(c)}_{E(|c|)} & \text{if } |c| > \sqrt{\mu}\mathfrak{c}(V) \end{cases}, \quad \alpha(c) = \begin{cases} 0 & \text{if } |c| \le \sqrt{\mu}\mathfrak{c}(V) \\ E(|c|) & \text{if } |c| > \sqrt{\mu}\mathfrak{c}(V) \end{cases}, \tag{A.D.5}$$

where $\mathrm{sgn}(c)$ denotes the sign of c.

Remark A.D.7. *We note that α from* (A.D.5) *is globally C^1 (which follows from strict convexity of its Fenchel transform β defined in* (A.D.2)*) and C^∞, even analytic, for $|c| > \sqrt{\mu}\mathfrak{c}(V)$ (which follows from the implicit function theorem as $E(\cdot) = (c^+)^{-1}(\cdot)$). Also, α is symmetric around 0 and (strictly) increasing for $c \ge 0$ (for $c \ge \sqrt{\mu}\mathfrak{c}(V)$).*

Remark A.D.8. *By arguing as above for the two independent dimensions of* (A.2.1)*, this demonstrates the connection between part (a) of Theorem A.1.6 and part (a) of Proposition A.2.1. More precisely, the graph property follows from Theorem A.D.6 and the results in* (A.D.5)*. The remaining part of the statement follows after realizing that $\alpha(\boldsymbol{c}) = \alpha_1(c_1) + \alpha_2(c_2)$, where α_i is the α-function of the one-dimensional system with coordinates labeled by i, and taking $u_{\boldsymbol{c}} \in C^3(\mathbf{T}^2)$ with $|c_i| > \sqrt{\mu_i}\mathfrak{c}(V_i)$ according to*

$$\nabla_x u_{\boldsymbol{c}}(x) = -\boldsymbol{c} \pm \begin{pmatrix} \sqrt{2(\alpha_1(c_1) + \mu_1 V_1(x^1))} \\ \sqrt{2(\alpha_2(c_2) + \mu_2 V_2(x^2))} \end{pmatrix},$$

(recall $V_i \in C^2(\mathbf{T})$ is a non-negative Morse function and $\alpha_i(c_i) > 0$) such that the Hamilton-Jacobi equation

$$\alpha(\boldsymbol{c}) = H_0(x, \boldsymbol{c} + \nabla_x u_{\boldsymbol{c}}(x))$$

is satisfied. Moreover, in case that U as in (A.2.3) is *actually separable, one can employ the explicit forms for $c^+(E)$ as the inverse of the α-function and $\nabla u_{\boldsymbol{c}}$ to* prove *the validity of Assumption* (P)*, simply by using the same expansions leading to the proof of Lemma A.B.1. This means that separable systems satisfy Assumption* (P)*, which shows consistency with our main results.*

A.D.3 Fathi's weak KAM theory and perturbations

For concreteness, we specialize to $M = \mathbf{T}^2$, in which case $H^1(\mathbf{T}^2, \mathbf{R}) \cong T^*_x\mathbf{T}^2 \cong \mathbf{R}^2$ for every $x \in \mathbf{T}^2$, such that we can identify $\boldsymbol{c} \in H^1(\mathbf{T}^2, \mathbf{R})$ with a closed 1-form of cohomology class $\boldsymbol{c}$. The central object of investigation in Fathi's weak KAM theory [254, 255] (with important contributions from Siconolfi [256, 257], Bernard [66] and others [184, 426]) is the Hamilton-Jacobi equation (HJE)

$$H(x, \boldsymbol{c} + \nabla_x u) = k\,, \quad k \in \mathbf{R}\,, \tag{A.D.6}$$

where H is a Tonelli Hamiltonian on $T^*\mathbf{T}^2$ with associated Tonelli Lagrangian L.

For classical solutions, i.e. C^1-functions $u : \mathbf{T}^2 \to \mathbf{R}$ solving (A.D.6), it is immediate to check, that there is at most one value $k \in \mathbf{R}$, for which such a C^1-solution may exist. In fact, this value agrees with Mather's α-function defined in (A.D.1). The primary goal of the theory is to define a

weaker notion of (sub)solution (so called *weak KAM solutions*), whose existence is always guaranteed [255, 254], even if the Tonelli Hamiltonian H is *not* integrable. See [184, 426, 257] for approaches to the problem from the theory of partial differential equations.

The following proposition contains perturbative properties of weak KAM solutions u_ε and Mather's α-function α_ε for systems of the form

$$H_\varepsilon(x,p) = H_0(x,p) + \varepsilon H_1(x,p)\,.$$

Proposition A.D.9. (Gomes [288])
Let $H_0 : T^*\mathbf{T}^2 \to \mathbf{R}$ be an integrable Hamiltonian and $u^{(0)}$ a (classical) C^1-solution of the HJE $H_0(x, \boldsymbol{c} + \nabla_x u^{(0)}) = \alpha^{(0)}(\boldsymbol{c})$. Moreover, let $\nu^{(0)}$ denote the projection of a Mather measure with cohomology $\boldsymbol{c}$. Suppose there exists a function $u^{(1)} \in C^1(\mathbf{T}^2)$ and a number $\alpha^{(1)}(\boldsymbol{c})$ such that

$$\alpha^{(1)}(\boldsymbol{c}) = \langle (\nabla_p H_0)(x, \boldsymbol{c} + \nabla_x u^{(0)}), \nabla_x u^{(1)} \rangle + H_1(x, \boldsymbol{c} + \nabla_x u^{(0)})\,, \quad \forall x \in \mathbf{T}^2\,. \tag{A.D.7}$$

Then

$$\alpha^{(1)}(\boldsymbol{c}) = \int_{\mathbf{T}^2} H_1(x, \boldsymbol{c} + \nabla_x u^{(0)})\, \mathrm{d}\nu^{(0)} \quad \textit{and} \quad \alpha_\varepsilon(\boldsymbol{c}) = \alpha^{(0)}(\boldsymbol{c}) + \varepsilon\alpha^{(1)}(\boldsymbol{c}) + \mathcal{O}_{\boldsymbol{c}}(\varepsilon^2)\,. \tag{A.D.8}$$

Remark A.D.10. *The above proposition provides a converse to* (A.2.11) *in Assumption* (P)*. In fact, the transport-type equation* (A.D.7) *for the unknown $u^{(1)}$ (with so far unspecified constant $\alpha^{(1)}(\boldsymbol{c})$) is exactly the first-order expansion obtained in* (A.4.2), (A.4.8), *and* (A.4.14) *in Section A.4 and also fixes $\alpha^{(1)}(\boldsymbol{c})$ to be given by* (A.D.8)*. Moreover, the equation* (A.D.7) *coincides with the relation, which the correction term $u^{(1)}$ of an* approximate solution *$\tilde{u}_\varepsilon = u^{(0)} + \varepsilon\, u^{(1)}$ to the HJE*

$$H_\varepsilon(x, \boldsymbol{c} + \nabla_x u_\varepsilon) = k$$

of order one has to satisfy (see [288]). The approximate solution $\tilde{u}_\varepsilon = u^{(0)} + \varepsilon\, u^{(1)}$ also coincides with the first order truncation of the so-called Lindstedt series *[30, 287], a not necessarily convergent perturbative expansion similar to the ones in KAM theory [373, 27, 456] or the Poincaré-Melnikov method [30, 303, 568]. Finally, it is interesting to note that, if $H_1(x,p) = W(x)$ is independent of the x-variables, then $\alpha_\varepsilon(\boldsymbol{c})$ is a convex function of ε and thus a.e. twice differentiable – yielding the expansion* (A.D.8) *at every such point.*

Chapter B

Creation rate of Dirac particles at a point source

This chapter contains the paper [345]:

J. Henheik and R. Tumulka. Creation Rate of Dirac Particles at a Point Source. *J. Phys. A: Math. Theor.*, 56(44):445201, 2023

Abstract. Only recently has it been possible to construct a self-adjoint Hamiltonian that involves the creation of Dirac particles at a point source in 3d space. Its definition makes use of an interior-boundary condition. Here, we develop for this Hamiltonian a corresponding theory of the Bohmian configuration. That is, we (non-rigorously) construct a Markov jump process $(Q_t)_{t\in\mathbf{R}}$ in the configuration space of a variable number of particles that is $|\psi_t|^2$-distributed at every time t and follows Bohmian trajectories between the jumps. The jumps correspond to particle creation or annihilation events and occur either to or from a configuration with a particle located at the source. The process is the natural analog of Bell's jump process, and a central piece in its construction is the determination of the rate of particle creation. The construction requires an analysis of the asymptotic behavior of the Bohmian trajectories near the source. We find that the particle reaches the source with radial speed 0, but orbits around the source infinitely many times in finite time before absorption (or after emission).

Content

B.1 Introduction

It is notoriously difficult to construct quantum Hamiltonians with particle creation and annihilation at a point source. Sometimes, such Hamiltonians can be obtained through renormalization [470, 216]. A more recent approach is based on interior-boundary conditions (IBCs) [558, 557], which are mathematically related to point interactions [19, 68]. Here, we are concerned with a particular family of self-adjoint Hamiltonians H that we constructed in [344] using IBCs.

Another ingredient in this work is Bell's jump process [58, 230], which is an extension of Bohmian mechanics [90, 232, 231] to quantum theories with particle creation and annihilation. These processes have been developed for theories on a lattice [58], with UV cut-off [230], and with IBCs [229]. However, the processes in [229] were devised for non-relativistic Schrödinger operators (based on the Laplacian operator) or codimension-1 boundaries (such as a surface in $\mathbf{R}^3$), whereas our H is based on the Dirac operator and involves a codimension-3 boundary (corresponding to a point source in $\mathbf{R}^3$). Here, we (non-rigorously) construct an analog of Bell's jump process for H; its construction is somewhat more involved than the cases analyzed in [229], and it has some curious features that we report below and that are absent in the non-relativistic case.

Generally speaking, the advantage of the Bohmian approach is that it allows for an observer-independent "realist" formulation of quantum theories. While it agrees with all of the standard predictions for observations, it avoids the inconsistencies that arise from the orthodox formulation in the analysis of the measurement process and the vagueness in the orthodox definition of the theory. In fact, it does so in a remarkably simple and natural way, essentially by following the thought that what we usually call "particles" actually are particles in the literal sense. Since the problems just mentioned with the orthodox version persist in quantum field theories, it is very relevant to extend Bohmian mechanics also to this realm, and in this paper we take some further steps in this direction. The key element of defining a Bohm-style theory with particle trajectories is to specify the laws governing the particles' motion, that is, to uniquely define a deterministic or stochastic process in configuration space. For non-relativistic quantum mechanics, the deterministic motion defined by Bohm's ODE has proven most convincing, and a variant of it is known also for the Dirac equation [91]. Other prior work [230] has made it plausible that particle creation events correspond to stochastic jumps of the configuration, but still the explicit models studied so far were almost exclusively non-relativistic. Since, in setups with particle creation, cut-offs (i.e., extended sources) disturb the relativistic symmetry and IBCs are key to treating point sources, the Dirac equation with IBCs forms the natural setting for such a model; however, such Hamiltonians have been rigorously available only recently [344].

Our Hamiltonian H is devised for a model of creation and annihilation of Dirac particles in 3 space dimensions by a point source fixed at the origin $\mathbf{0} \in \mathbf{R}^3$. For simplicity, our Hilbert space $\mathscr{H}$ is a mini-Fock space with only two sectors, corresponding to 0 or 1 particles,

$$\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)} = \mathbf{C} \oplus L^2(\mathbf{R}^3, \mathbf{C}^4)\,. \tag{B.1.1}$$

Correspondingly, the configuration space $\mathcal{Q}$ also consists of two sectors,

$$\mathcal{Q} = \mathcal{Q}^{(0)} \cup \mathcal{Q}^{(1)} = \{\varnothing\} \cup (\mathbf{R}^3 \setminus \{\mathbf{0}\})\,. \tag{B.1.2}$$

The process $(Q_t)_{t\in\mathbf{R}}$ that we construct moves in $\mathcal{Q}$. In the upper sector, it moves along a Bohmian trajectory until it hits the origin, at which time it jumps to the empty configuration $\varnothing$, where it remains for a random time and then jumps back to the upper sector, where it follows a Bohmian trajectory starting from $\mathbf{0}$ until it reaches $\mathbf{0}$ again, and so on. In particular, the process is piecewise deterministic (because the Bohmian trajectories are deterministic), and the only stochastic elements are the jumps between $\varnothing \in \mathcal{Q}^{(0)}$ and $\mathbf{0} \in \partial\mathcal{Q}^{(1)}$. More precisely, while the absorption events (jumps to $\varnothing$) are deterministic and occur whenever Q_t reaches $\mathbf{0} \in \partial\mathcal{Q}^{(1)}$, the emission events (jumps from

$\varnothing$) are stochastic in two ways: (i) when they occur and (ii) onto which trajectory the process jumps (because there can be several trajectories starting from $\mathbf{0}$ at the same time).

The trajectories here are the solutions of Bohm's equation of motion for the Dirac equation [91],

$$\frac{\mathrm{d}\boldsymbol{Q}(t)}{\mathrm{d}t} = \frac{\boldsymbol{j}}{\rho}(\boldsymbol{Q}(t)) \tag{B.1.3}$$

(boldface symbols denoting 3d vectors) with probability current

$$\boldsymbol{j} = (\psi^{(1)})^\dagger \boldsymbol{\alpha} \psi^{(1)}\,, \tag{B.1.4}$$

where $\psi^{(1)}$ is the $\mathscr{H}^{(1)}$-component of a wave function $\psi = (\psi^{(0)}, \psi^{(1)})$ in $\mathscr{H}$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ denotes the vector of the standard Dirac α-matrices (see (B.3.8)), and density

$$\rho = j^0 = (\psi^{(1)})^\dagger \psi^{(1)} = \left|\psi^{(1)}\right|^2. \tag{B.1.5}$$

As mentioned, the process jumps to $\varnothing$ when it reaches $\mathbf{0}$. The other law needed to define the process (see Section B.5) specifies the jump rate that applies whenever $Q_t = \varnothing$. The process is designed so that

$$Q_t \text{ is } |\psi_t|^2\text{-distributed} \tag{B.1.6}$$

at every time t. We will see in Section B.5 that the jump rate is in fact uniquely determined by the wish that (B.1.6) holds for all t.[1]

Away from the origin in $\mathbf{R}^3$, H acts like the Dirac operator with a Coulomb potential of strength q,

$$(H\psi)^{(1)}(\boldsymbol{x}) = \Big(-\mathrm{i}c\hbar\boldsymbol{\alpha}\cdot\nabla + mc^2\beta + c\hbar\frac{q}{|\boldsymbol{x}|}\Big)\psi^{(1)}(\boldsymbol{x}) \qquad (\boldsymbol{x} \neq \mathbf{0})\,, \tag{B.1.7}$$

where $\beta = \mathrm{diag}(1,1,-1,-1)$ denotes the standard Dirac β-matrix. On the other hand, H couples the two sectors of $\mathscr{H}$, i.e., none of them stays invariant under the evolution generated by H. We assume that

$$\sqrt{3}/2 < |q| < 1\,. \tag{B.1.8}$$

For $|q| \leq \sqrt{3}/2$, there is no self-adjoint operator that couples the sectors and obeys (B.1.7), and the case $|q| \geq 1$ was not studied in [344]. We will give a full description of H, and write down the IBC, in Section B.2. IBCs for Dirac operators on codimension-1 boundaries (as opposed to codimension 3 considered here) were studied in [420, 513].

The construction of a Bell-type jump process for a similar model in curved space-time was outlined by one of us in [571]. While that construction is very analogous in spirit to the one presented here, a relevant difference is that for the present model, a rigorously defined self-adjoint Hamiltonian H is known, which allows for a precise and detailed description of the process that was not possible in [571].

It is of interest to compare (see Table B.1 below) our model with a non-relativistic variant [389], in which (B.1.1) is replaced by $\mathbf{C} \oplus L^2(\mathbf{R}^3, \mathbf{C})$, H with another operator H_{nr} (where the subscript nr stands for "non-relativistic"), and (B.1.7) by

$$(H_{\mathrm{nr}}\psi)^{(1)} = -\tfrac{\hbar^2}{2m}\Delta\,. \tag{B.1.9}$$

The natural variant of Bell's jump process for H_{nr} is described in [229]. For ψ from the domain of H_{nr}, the probability current

$$\boldsymbol{j}(\boldsymbol{x}) = \tfrac{\hbar}{m}\mathrm{Im}\ \psi^{(1)}(\boldsymbol{x})^* \nabla \psi^{(1)}(\boldsymbol{x}) \tag{B.1.10}$$

[1] However, our considerations make use of the (plausible and common [229]) approximation that the Bohmian velocity field $\boldsymbol{j}/\rho$ varies slowly in time, more specifically that the short-time asymptotics of the Bohmian trajectories coincide with those one would obtain from a time-independent velocity field (as in (B.4.2) below). Further justification is outlined in Remark B.4.2 below.

is, for every unit vector $\boldsymbol{\omega} \in \mathbf{R}^3$, of the form

$$\boldsymbol{j}(r\boldsymbol{\omega}) = j_{\mathrm{rad}}\, \boldsymbol{\omega} + o(r^0) \tag{B.1.11}$$

as $r \searrow 0$ (i.e., to 0 from the right) with a constant j_{rad} independent of r and $\boldsymbol{\omega}$. Put differently, the angular components of $\boldsymbol{j}$ (perpendicular to $\boldsymbol{\omega}$) converge to 0 as $r \searrow 0$, while the radial component (along $\boldsymbol{\omega}$) converges to a generally nonzero value. As a consequence, the Bohmian trajectory, when drawn in spherical coordinates as in Figure B.1, hits the $r = 0$ surface perpendicularly at nonzero speed.

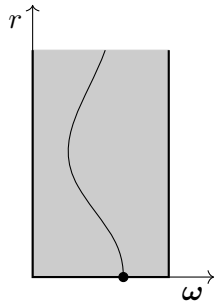

Figure B.1: For the non-relativistic case, the trajectory in $\mathbf{R}^3$ before absorption is shown, represented in spherical coordinates, with only one of the two angles of $\boldsymbol{\omega} = (\varphi, \vartheta)$ drawn (shaded region = admissible values $r > 0$, $0 \le \varphi < 2\pi$, $0 \le \vartheta \le \pi$). The trajectory ends at $r = 0$ at a particular value of $\boldsymbol{\omega}$; the corresponding point $(0, \boldsymbol{\omega})$ in the diagram is marked.

Certain features are different in the relativistic case of our H. Let

$$B := \sqrt{1 - q^2}\,, \tag{B.1.12}$$

where q is the strength of the Coulomb potential as in (B.1.7); note that, due to (B.1.8), $0 < B < \frac{1}{2}$. We will argue that for ψ from a certain subspace of $\mathscr{H}$, a Bohmian trajectory $t \mapsto \boldsymbol{Q}(t) \in \mathbf{R}^3$ that reaches $r = 0$ does so at radial velocity 0 and only after orbiting the z axis infinitely many times.[2] In fact, as depicted in Figure B.2, almost surely,

$$|\boldsymbol{Q}(t)| \sim (\mathrm{const.})\, |t - t_0|^{1/(1-2B)} \tag{B.1.13}$$

as $t \nearrow t_0$, where t_0 is the time it reaches $r = 0$ and $\sim$ means asymptotically equal, i.e.

$$f(t) \sim g(t) \quad :\Leftrightarrow \quad \frac{f(t)}{g(t)} \to 1 \quad \Leftrightarrow \quad f(t) = g(t) + o(g(t)). \tag{B.1.14}$$

Since $1/(1-2B) > 1$, one would expect (and it is the case) that the curve, as a function of t, touches $r = 0$ at t_0 with

$$\frac{\mathrm{d}r}{\mathrm{d}t}(t_0) = 0\,. \tag{B.1.15}$$

Moreover, the polar angle becomes constant at leading order in the limit $r \searrow 0$,

$$\vartheta(r) \sim (\mathrm{const.})\,, \tag{B.1.16}$$

while the dependence $\varphi(r)$ of the azimuthal angle on the radius is asymptotically of the form

$$\varphi(r) \sim (\mathrm{const.})\, r^{-2B} \tag{B.1.17}$$

as $r \searrow 0$, see Figure B.3.

[2] Note that H is not rotationally invariant; it commutes with the z component J_z of angular momentum but not with other components. In fact, the model cannot be rotationally invariant, given that the source is fixed at the origin, and the emission of a spin-$\frac{1}{2}$ particle by a spinless source cannot conserve angular momentum (see [344, Sec 2.D and Ref. 17] for more details).

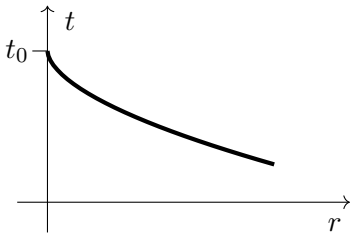

Figure B.2: Asymptotic dependence $r(t)$ of a Bohmian trajectory before absorption, drawn here for $q = \sqrt{187/196}$, i.e., $1/(1-2B) = 7/4$

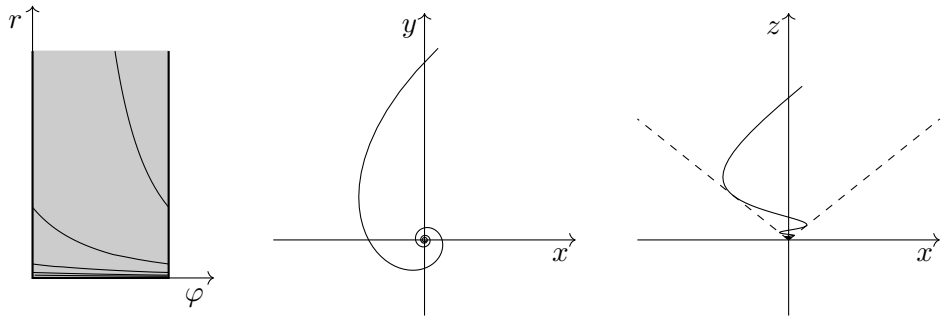

Figure B.3: For the relativistic case, an asymptotic trajectory before absorption is shown for the same q value as in Figure B.2. LEFT: Drawn in spherical coordinates; of the two angles of $\boldsymbol{\omega} = (\vartheta, \varphi)$, only the azimuthal angle φ is shown. Its dependence on r is given by (B.1.17). MIDDLE: Drawn in Cartesian coordinates, seen along the z axis. RIGHT: Drawn in Cartesian coordinates, seen along the y axis (dashed $=$ outline of the cone containing the curve).

As a consequence of (B.1.16), the asymptotic trajectory lies on a cone with (random) opening angle $\vartheta(t_0)$, and φ increases by an infinite amount before $r = 0$ is reached, so it circles the z axis infinitely often; see Figure B.4. In particular, the trajectory does not have a limiting point on the 2-sphere $\{r = 0\}$. Moreover, for each Hamiltonian H from our family (i.e., for each choice of the parameters described in Section B.2), there is a fixed sense of circling the z axis: either, for all ψ, all trajectories asymptotically circle clockwise, or, for all ψ, all trajectories circle counter-clockwise. Likewise, the "speed" of orbiting, meaning here the exponent of r^{-2B}, is fixed by the choice of H and does not depend on ψ. The time dependence $\varphi(t)$ can be obtained by inserting (B.1.13) in (B.1.17), which yields that

$$\varphi(t) \sim (\text{const.})\, |t - t_0|^{-2B/(1-2B)} \tag{B.1.18}$$

as $t \nearrow t_0$; see Figure B.5.

The reverse trajectories that emerge from $r = 0$ display the same behavior, i.e., (B.1.17) (with the reverse orientiation of the trajectory) and (B.1.13) as $t \searrow t_0$. (If the ingoing trajectories circle clockwise, then so do the outgoing ones.)

This behavior, in particular the absence of a limit point on $\{r = 0\}$, creates the following difficulty for the definition of a Bell-type jump process for this Hamiltonian. In the non-relativistic case, we could define a rate for jumping to the point $(0, \boldsymbol{\omega})$, and then there is either a unique trajectory starting from there or a unique trajectory ending there. The rate was set to 0 when a trajectory ends there. Now, in the relativistic case, the trajectories emerging from $r = 0$ do not possess a starting (limiting) point. We will be able to define a Bell-type jump process nevertheless by defining the rate for jumping onto a particular trajectory. In fact, the different trajectories can be characterized by their limiting $\vartheta(r = 0) \equiv \vartheta_0$ values and their offsets (differences) φ_0 in the azimuthal angle. It turns

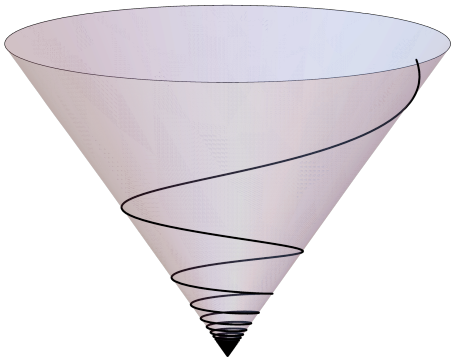

Figure B.4: The same curve as in Figure B.3 is shown as a curve in $\mathbf{R}^3$, seen in a perspective view. The curve lies on a cone of constant ϑ (not related to the light cone).

	non-rel.	rel.
$\frac{\mathrm{d}r}{\mathrm{d}t}(t_0)$	$\neq 0$	0
$\vartheta(t_0)$	const.	const.
$\varphi(t_0)$	const.	$\to \pm\infty$

Table B.1: Comparison between the processes in the non-relativistic and the relativistic case; t_0 is the time of absorption or emission.

out that the jump rate will be uniform over φ_0, so all trajectories with a given ϑ_0 starting from $r = 0$ at a given time are equally probable.

We will only consider wave functions ψ from a certain subspace $\widehat{D} \subset \mathscr{H}$ that is invariant under the time evolution; $\widehat{D}$ is the part of the domain D of H for which the component $\psi^{(1)}$ of $\psi = (\psi^{(0)}, \psi^{(1)})$ in the upper sector lies in a certain angular momentum eigenspace (see Section B.2 for details). In fact, as we will see, the coupling between $\mathscr{H}^{(0)}$ and $\mathscr{H}^{(1)}$ happens only within $\widehat{D}$, so $\widehat{D}$ is the most relevant or interesting part of D. By focusing on $\widehat{D}$, we avoid unnecessarily tedious computation for extracting the qualitative behavior, which we believe will not change much for $\psi \in D \setminus \widehat{D}$. Finally, we remark that both, $\widehat{D}$ and D, contain wavefunctions of negative energy.

The remainder of this paper is organized as follows. In Section B.2, we report the relevant properties of H. In Section B.3, we derive the asymptotic behavior of the current for $\psi \in \widehat{D}$. In Section B.4, we derive from that the (approximate) asymptotics of the Bohmian trajectories and justify the statements made above. In Section B.5, we define the Bell-type jump process and justify the claim that it is equivariant. In Section B.6, we conclude.

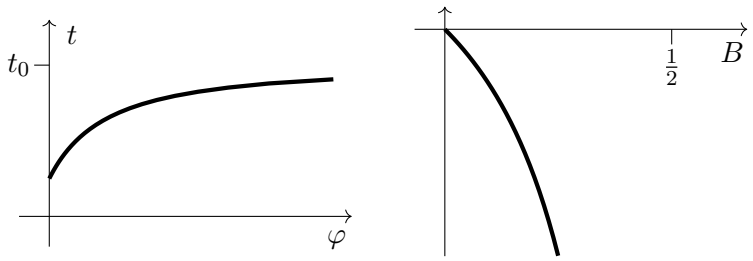

Figure B.5: LEFT: Asymptotic dependence $\varphi(t)$ as in (B.1.18), drawn for the same value of B as in Figures B.2–B.4. RIGHT: The exponent in (B.1.18), $-2B/(1-2B)$, as a function of B.

Acknowledgments.

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B.2 The Hamiltonian

Let $\mathbf{S}^2$ denote the unit sphere in $\mathbf{R}^3$. We will make use of a widely used orthonormal basis of $L^2(\mathbf{S}^2, \mathbf{C}^4)$, traditionally denoted $\Phi^{\pm}_{m_j,\kappa_j}$, for which we have

$$L^2(\mathbf{S}^2, \mathbf{C}^4, d\Omega) = \bigoplus_{j\in\mathbb{N}_0+\frac{1}{2}} \overset{j}{\bigoplus_{m_j=-j}} \bigoplus_{\kappa_j=\pm(j+\frac{1}{2})} \mathscr{K}_{m_j\kappa_j} \tag{B.2.1}$$

with

$$\mathscr{K}_{m_j\kappa_j} = \operatorname{span}(\Phi^{+}_{m_j\kappa_j}, \Phi^{-}_{m_j\kappa_j}) \,. \tag{B.2.2}$$

The $\Phi^{\pm}_{m_j,\kappa_j}$ are simultaneous eigenvectors of $\underline{J}^2, K, J_3$ with $\underline{J} = \underline{L} + \boldsymbol{S}$ the total angular momentum and $K = \beta(2\boldsymbol{S}\cdot\underline{L} + 1)$ the "spin-orbit operator." In the standard representation of Dirac spin space, they are explicitly given by [559, (4.111)]

$$\Phi^{+}_{m_j,\mp(j+1/2)} = \begin{pmatrix} \mathrm{i}\Psi^{m_j}_{j\mp1/2} \\ 0 \end{pmatrix}, \qquad \Phi^{-}_{m_j,\mp(j+1/2)} = \begin{pmatrix} 0 \\ \Psi^{m_j}_{j\pm1/2} \end{pmatrix} \tag{B.2.3}$$

with

$$\Psi^{m_j}_{j-1/2} = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j}\, Y^{m_j-1/2}_{j-1/2} \\ \sqrt{j-m_j}\, Y^{m_j+1/2}_{j-1/2} \end{pmatrix} \tag{B.2.4a}$$

$$\Psi^{m_j}_{j+1/2} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j}\, Y^{m_j-1/2}_{j+1/2} \\ -\sqrt{j+1+m_j}\, Y^{m_j+1/2}_{j+1/2} \end{pmatrix} \tag{B.2.4b}$$

and Y^m_l the usual spherical harmonics (defined for $l \in \mathbb{N}_0$ and $m \in \{-l, \dots, l\}$), given by

$$Y^m_l(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\, P^m_l(\cos(\vartheta))\, \mathrm{e}^{\mathrm{i}m\varphi}, \tag{B.2.5}$$

where

$$P^m_l(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l \tag{B.2.6}$$

are the associated Legendre polynomials.

The Hamiltonian H depends on parameters $g \in \mathbf{C} \setminus \{0\}$, $a_1, a_2, a_3, a_4 \in \mathbf{R}$ with

$$a_1 a_4 - a_2 a_3 = 4B(1+q)\,, \tag{B.2.7}$$

and a fixed

$$(\tilde{m}_j, \tilde{\kappa}_j) \in \mathscr{A} := \Big\{(-\tfrac{1}{2}, -1),\ (-\tfrac{1}{2}, 1),\ (\tfrac{1}{2}, -1),\ (\tfrac{1}{2}, 1)\Big\}\,. \tag{B.2.8}$$

As established in [344] (using in particular results of [348, 275, 277] about Dirac operators with Coulomb potential), the Hamiltonian H and its domain D have the following properties (which characterize the pair (H, D) uniquely):

- For every $\psi \in D$, the upper sector is of the form

$$\psi^{(1)}(\boldsymbol{x}) = c_- \, f^-_{\tilde{m}_j \tilde{\kappa}_j}\big(\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\big)\, |\boldsymbol{x}|^{-1-B} + \sum_{(m_j, \kappa_j) \in \mathscr{A}} c_{+m_j \kappa_j}\, f^+_{m_j \kappa_j}\big(\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\big)\, |\boldsymbol{x}|^{-1+B} + o(|\boldsymbol{x}|^{-1/2}) \tag{B.2.9}$$

  as $\boldsymbol{x} \to \mathbf{0}$ with (uniquely defined) short distance coefficients $c_-, c_{+m_j\kappa_j} \in \mathbf{C}$ and particular functions $f^\pm_{m_j\kappa_j} : \mathbf{S}^2 \to \mathbf{C}^4$ given by

$$f^+_{m_j\kappa_j} = (1+q-B)\Phi^+_{m_j\kappa_j} - (1+q+B)\Phi^-_{m_j\kappa_j} \tag{B.2.10a}$$

$$f^-_{m_j\kappa_j} = (1+q+B)\Phi^+_{m_j\kappa_j} - (1+q-B)\Phi^-_{m_j\kappa_j}\,. \tag{B.2.10b}$$

- Every $\psi \in D$ obeys the IBC

$$a_1\, c_- + a_2\, c_{+\tilde{m}_j\tilde{\kappa}_j} = g\, \psi^{(0)}\,, \tag{B.2.11}$$

  and H acts on $\psi \in D$ according to (B.1.7) and

$$(H\psi)^{(0)} = g^* \, (a_3\, c_- + a_4\, c_{+\tilde{m}_j\tilde{\kappa}_j})\,. \tag{B.2.12}$$

We note that by rotational invariance of the Dirac operator with Coulomb potential, H is block diagonal relative to the sum decomposition

$$\mathscr{H} \cong \widetilde{\mathscr{H}} \oplus \bigoplus_{(j,m_j,\kappa_j) \neq (\tilde{\jmath}, \tilde{m}_j, \tilde{\kappa}_j)} L^2((0,\infty)) \otimes \mathscr{K}_{m_j\kappa_j} \tag{B.2.13}$$

(recall (B.2.2) and note that j is determined by κ_j through $j = |\kappa_j| - \frac{1}{2}$), but, by means of the coupling in (B.2.11) and (B.2.12), not relative to

$$\widetilde{\mathscr{H}} = \mathscr{H}^{(0)} \oplus L^2((0,\infty)) \otimes \mathscr{K}_{\tilde{m}_j\tilde{\kappa}_j}\,. \tag{B.2.14}$$

Therefore, the subspace

$$\widehat{D} := D \cap \widetilde{\mathscr{H}} \tag{B.2.15}$$

is invariant under the time evolution generated by H. Henceforth, we will only consider ψ's from this set. Since the coupling between $\mathscr{H}^{(0)}$ and $\mathscr{H}^{(1)}$ essentially happens within $\widehat{D}$ (it is independent of $c_{+m_j\kappa_j}$ for $(m_j, \kappa_j) \neq (\tilde{m}_j, \tilde{\kappa}_j)$), we expect that the trajectories for other ψ's will be qualitatively similar; although the formulas (B.1.13), (B.1.16), (B.1.17), (B.1.18) may not apply literally, slight modifications of them should.

For $\psi \in \widehat{D}$, we can simplify and refine (B.2.9) as follows:

$$\begin{aligned}\psi^{(1)}(\boldsymbol{x}) = c_- \, f^-_{\tilde{m}_j\tilde{\kappa}_j}\big(\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\big)\, |\boldsymbol{x}|^{-1-B} + c_{+\tilde{m}_j\tilde{\kappa}_j}\, f^+_{\tilde{m}_j\tilde{\kappa}_j}\big(\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\big)\, |\boldsymbol{x}|^{-1+B} \\ + f^-_{\tilde{m}_j\tilde{\kappa}_j}\big(\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\big)\, o(|\boldsymbol{x}|^{-1/2}) + f^+_{\tilde{m}_j\tilde{\kappa}_j}\big(\tfrac{\boldsymbol{x}}{|\boldsymbol{x}|}\big)\, o(|\boldsymbol{x}|^{-1/2})\,.\end{aligned} \tag{B.2.16}$$

That is, apart from the fact that $c_{+m_j\kappa_j} = 0$ for $(m_j, \kappa_j) \neq (\tilde{m}_j, \tilde{\kappa}_j)$, also the error terms must lie in $\mathscr{K}_{\tilde{m}_j\tilde{\kappa}_j}$. Indeed, this follows from (B.2.9) by projecting to $\widetilde{\mathscr{H}}$. In the following sections, we use (B.2.16) instead of (B.2.9).

B.3 The Current

Our goal in this section and the next is to compute the asymptotic behavior of the solutions of the equation of motion (B.1.3) in $\mathcal{Q}^{(1)} = \mathbf{R}^3 \setminus \{\mathbf{0}\}$ that either reach $\mathbf{0}$ or come out of $\mathbf{0}$ at some time t_0. That is, we consider t near t_0 and r near 0. To this end, we replace ψ_t by ψ_{t_0} and determine the asymptotics of the solutions $\boldsymbol{Q}(t)$ of (B.1.3) for fixed $\psi^{(1)} = (\psi_{t_0})^{(1)}$. We first need to establish the asymptotic behavior of the probability current

$$\boldsymbol{j}(\boldsymbol{x}) = \psi^{(1)}(\boldsymbol{x})^\dagger \boldsymbol{\alpha} \psi^{(1)}(\boldsymbol{x}) \tag{B.3.1}$$

from the short-distance asymptotics of $\psi^{(1)}$ given in (B.2.16). We already noted in the previous section that the coupling between the 0–particle sector and the 1–particle sector described by (B.2.11) and (B.2.12) is independent of $c_{+m_j\kappa_j}$ for $(m_j, \kappa_j) \neq (\tilde{m}_j, \tilde{\kappa}_j)$. Since we assume $\psi \in \widehat{D}$, we henceforth write c_+ instead of $c_{+\tilde{m}_j\tilde{\kappa}_j}$ for ease of notation.

Proposition B.3.1. *For $\psi \in \widehat{D}$, the components of the probability current in spherical coordinates obey the following asymptotics as $\boldsymbol{x} \to \mathbf{0}$:*

$$j_r(r\boldsymbol{\omega}) = C_{\mathrm{rad}}\, r^{-2} + o(r^{-3/2-B}) \tag{B.3.2a}$$

$$j_\vartheta(r\boldsymbol{\omega}) = o(r^{-3/2-B}) \tag{B.3.2b}$$

$$j_\varphi(r\boldsymbol{\omega}) = C_{\mathrm{az}} \sin\vartheta\, r^{-2-2B} + \sin\vartheta\, O(r^{-2}) \tag{B.3.2c}$$

where $\boldsymbol{\omega} \in \mathbf{S}^2$, C_{rad} and C_{az} are real constants (that depend on ψ but not on $r\boldsymbol{\omega}$), $j_k := \boldsymbol{e}_k \cdot \boldsymbol{j}$ ($k = r, \vartheta, \varphi$), and $\boldsymbol{e}_k$ is the unit vector in the k direction,

$$\boldsymbol{e}_r = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \boldsymbol{\omega} \tag{B.3.3a}$$

$$\boldsymbol{e}_\vartheta = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) \tag{B.3.3b}$$

$$\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)\,. \tag{B.3.3c}$$

More explicitly, we have that

$$j_r(r\boldsymbol{\omega}) = \frac{2(1+q)B}{\pi} \operatorname{Im}[c_-^* c_+]\, r^{-2} + o(r^{-3/2-B}) \tag{B.3.4a}$$

$$j_\vartheta(r\boldsymbol{\omega}) = o(r^{-3/2-B}) \tag{B.3.4b}$$

$$\begin{aligned} j_\varphi(r\boldsymbol{\omega}) = {} & -\frac{q(1+q)}{\pi} |c_-|^2 \operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j) \sin\vartheta\, r^{-2-2B} \\ & - \frac{2(1+q)}{\pi} \operatorname{Re}[c_-^* c_+] \operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j) \sin\vartheta\, r^{-2} \\ & - \frac{q(1+q)}{\pi} |c_+|^2 \operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j) \sin\vartheta\, r^{-2+2B} + \sin\vartheta\, o(r^{-3/2-B})\,. \end{aligned} \tag{B.3.4c}$$

Proof. From (B.2.16) and (B.3.1), using $c_{+m_j\kappa_j} = 0$ for $(m_j, \kappa_j) \neq (\tilde{m}_j, \tilde{\kappa}_j)$,

$$\boldsymbol{j}(r\boldsymbol{\omega}) = \psi^{(1)}(r\boldsymbol{\omega})^\dagger\, \boldsymbol{\alpha}\, \psi^{(1)}(r\boldsymbol{\omega}) \tag{B.3.5a}$$

$$\begin{aligned} = {} & |c_-|^2 \left\langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \boldsymbol{\alpha} f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}) \right\rangle_{\mathbf{C}^4} r^{-2-2B} \\ & + 2\operatorname{Re}\left[c_-^* c_+ \left\langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \boldsymbol{\alpha} f^+_{m_j\kappa_j}(\boldsymbol{\omega}) \right\rangle_{\mathbf{C}^4} \right] r^{-2} \\ & + |c_+|^2 \left\langle f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \boldsymbol{\alpha} f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}) \right\rangle_{\mathbf{C}^4} r^{-2+2B} \end{aligned} \tag{B.3.5b}$$

$$+ \sum_{\nu,\pi=\pm} \langle f^{\nu}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \boldsymbol{\alpha} f^{\pi}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4}\ o(r^{-3/2-B})\,.$$

In Lemma B.3.2 below, we evaluate the coefficients of r^{-2-2B} and r^{-2+2B} and in particular show that they vanish in the r and ϑ components. Afterwards, in Lemma B.3.3, we evaluate the coefficient of r^{-2} and in particular show that it is independent of $\boldsymbol{\omega}$ in the r component and vanishes in the ϑ component. Lemmas B.3.2 and B.3.3 also show that all terms of the φ component of (B.3.5b) contain a factor of $\sin\vartheta$. This yields (B.3.2). Inserting the precise results for the coefficients in Lemma B.3.2 and Lemma B.3.3 we arrive at (B.3.4). We remark about the last two lines of (B.3.5b) that it depends on B which of the exponents $-2+2B$ and $-3/2-B$ is greater; for $B > 1/6$, $-2+2B$ is greater, so $r^{-2+2B} < r^{-3/2-B}$, and the r^{-2+2B} term could be included in the $o(r^{-3/2-B})$. □

Lemma B.3.2. *For every $\boldsymbol{\omega} \in \mathbf{S}^2$, we have that*

$$\langle f^{\mp}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_r f^{\mp}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0, \tag{B.3.6a}$$

$$\langle f^{\mp}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_\vartheta f^{\mp}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0, \tag{B.3.6b}$$

$$\langle f^{\mp}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_\varphi f^{\mp}_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = -\frac{q(1+q)}{\pi}\operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j)\sin\vartheta\,, \tag{B.3.6c}$$

where the $f^{\pm}_{\tilde{m}_j,\tilde{\kappa}_j}$ were defined in (B.2.10) *and $\alpha_k := \boldsymbol{e}_k \cdot \boldsymbol{\alpha}$ for $k = r, \vartheta, \varphi$.*

Proof. We omit the subscript $\tilde{m}_j, \tilde{\kappa}_j$ for ease of notation. By (B.2.10b) (using that all components of $\boldsymbol{\alpha}$ are self-adjoint),

$$\begin{aligned}\langle f^{\mp}(\boldsymbol{\omega}), \boldsymbol{\alpha} f^{\mp}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} &= (1+q+B)^2 \langle \Phi^{\pm}(\boldsymbol{\omega}), \boldsymbol{\alpha}\Phi^{\pm}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} \\ &\quad - 2(1+q+B)(1+q-B)\operatorname{Re}\langle \Phi^{+}(\boldsymbol{\omega}), \boldsymbol{\alpha}\Phi^{-}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} \\ &\quad + (1+q-B)^2 \langle \Phi^{\mp}(\boldsymbol{\omega}), \boldsymbol{\alpha}\Phi^{\mp}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4}\,.\end{aligned} \tag{B.3.7}$$

Since in the standard representation

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{B.3.8}$$

with $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices, we can read off from the form (B.2) that

$$\langle \Phi^{\pm}(\boldsymbol{\omega}), \boldsymbol{\alpha}\Phi^{\pm}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = \mathbf{0} \tag{B.3.9}$$

for every $\boldsymbol{\omega} \in \mathbf{S}^2$. Thus, the first and the third line of (B.3.7) vanish identically.

We will now compute

$$\langle \Phi^{+}(\boldsymbol{\omega}), \alpha_k \Phi^{-}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = -\mathrm{i}\,\langle \Psi^{m_j}_{j\mp1/2}(\boldsymbol{\omega}), \boldsymbol{e}_k\cdot\boldsymbol{\sigma}\Psi^{m_j}_{j\pm1/2}(\boldsymbol{\omega})\rangle_{\mathbf{C}^2}\,. \tag{B.3.10}$$

For us $j = 1/2$, so we recall that the first few spherical harmonics are

$$Y^0_0(\vartheta,\varphi) = \frac{1}{\sqrt{4\pi}}\,,\quad Y^0_1(\vartheta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\vartheta\,,\quad Y^{\pm1}_1(\vartheta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\vartheta\,\mathrm{e}^{\pm\mathrm{i}\varphi} \tag{B.3.11}$$

and verify:

$$\langle \Psi^{\pm1/2}_1(\boldsymbol{\omega}), \sigma_1\Psi^{\pm1/2}_0(\boldsymbol{\omega})\rangle^*_{\mathbf{C}^2} = \langle \Psi^{\pm1/2}_0(\boldsymbol{\omega}), \sigma_1\Psi^{\pm1/2}_1(\boldsymbol{\omega})\rangle_{\mathbf{C}^2} = \frac{1}{4\pi}\sin\vartheta\,\mathrm{e}^{\pm\mathrm{i}\varphi} \tag{B.3.12a}$$

$$\langle \Psi^{\pm1/2}_1(\boldsymbol{\omega}), \sigma_2\Psi^{\pm1/2}_0(\boldsymbol{\omega})\rangle^*_{\mathbf{C}^2} = \langle \Psi^{\pm1/2}_0(\boldsymbol{\omega}), \sigma_2\Psi^{\pm1/2}_1(\boldsymbol{\omega})\rangle_{\mathbf{C}^2} = \mp\frac{\mathrm{i}}{4\pi}\sin\vartheta\,\mathrm{e}^{\pm\mathrm{i}\varphi} \tag{B.3.12b}$$

$$\langle \Psi^{\pm1/2}_1(\boldsymbol{\omega}), \sigma_3\Psi^{\pm1/2}_0(\boldsymbol{\omega})\rangle^*_{\mathbf{C}^2} = \langle \Psi^{\pm1/2}_0(\boldsymbol{\omega}), \sigma_3\Psi^{\pm1/2}_1(\boldsymbol{\omega})\rangle_{\mathbf{C}^2} = \frac{1}{4\pi}\cos\vartheta\,. \tag{B.3.12c}$$

Thus, we arrive at

$$\langle \Phi^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_r \Phi^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = -\frac{\mathrm{i}}{4\pi} \tag{B.3.13a}$$

$$\langle \Phi^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_\vartheta \Phi^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0 \tag{B.3.13b}$$

$$\langle \Phi^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_\varphi \Phi^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = \operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j)\frac{1}{4\pi}\sin\vartheta\,. \tag{B.3.13c}$$

Now (B.3.6b) follows from (B.3.13b), (B.3.6a) follows from the fact that (B.3.13a) has vanishing real part, and (B.3.6c) is obtained from the middle row of (B.3.7) and (B.3.13c). □

We can also read off from Lemma B.3.2 that the leading order coefficient of j_φ in (B.3.4c) is given by

$$-|c_-|^2 \operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j)\frac{q(1+q)}{\pi}\sin\vartheta\,, \tag{B.3.14}$$

showing that the sign of j_φ near $r = 0$ is fixed for fixed parameters $q, \tilde{m}_j, \tilde{\kappa}_j$.

Lemma B.3.3. *For every $\boldsymbol{\omega} \in \mathbf{S}^2$, we have that*

$$\langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_r f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = -\mathrm{i}\frac{(1+q)B}{\pi}\,, \tag{B.3.15a}$$

$$\langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_\vartheta f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0\,, \tag{B.3.15b}$$

$$\langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), \alpha_\varphi f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = -\frac{1+q}{\pi}\operatorname{sgn}(\tilde{m}_j\tilde{\kappa}_j)\sin\vartheta\,, \tag{B.3.15c}$$

where the $f^\pm_{\tilde{m}_j,\tilde{\kappa}_j}$ were defined in (B.2.10) *and $\alpha_k = \boldsymbol{e}_k \cdot \boldsymbol{\alpha}$ for $k = r, \vartheta, \varphi$.*

Proof. We omit the subscript $\tilde{m}_j, \tilde{\kappa}_j$ again and argue exactly as in the proof of Lemma B.3.2 to find that $\langle f^-(\boldsymbol{\omega}), \boldsymbol{\alpha} f^+(\boldsymbol{\omega})\rangle_{\mathbf{C}^4}$ equals

$$-(1+q+B)^2\langle \Phi^+(\boldsymbol{\omega}), \boldsymbol{\alpha}\Phi^-(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} - (1+q-B)^2\langle \Phi^+(\boldsymbol{\omega}), \boldsymbol{\alpha}\Phi^-(\boldsymbol{\omega})\rangle^*_{\mathbf{C}^4}\,. \tag{B.3.16}$$

Now, using (B.3.13) and $B = \sqrt{1-q^2}$, the claim follows. □

B.4 The Trajectories

From the asymptotic behavior (B.3.2) resp. (B.3.4) of the current and the fact that the probability density

$$\rho(\boldsymbol{x}) = \psi^{(1)}(\boldsymbol{x})^\dagger \psi^{(1)}(\boldsymbol{x}) \tag{B.4.1}$$

is asymptotically proportional to $|\boldsymbol{x}|^{-2-2B}$, we will now draw conclusions about the asymptotic Bohmian trajectories.

To this end, we study approximate solutions of (B.1.3) by neglecting the time dependence of the velocity field $\boldsymbol{j}/\rho$ on the right-hand side of (B.1.3). This means, if $t \mapsto \psi_t = \mathrm{e}^{-\mathrm{i}Ht}\psi_0$ denotes the (strongly differentiable) time–evolution of $\psi_0 \in D$ governed by our Hamiltonian H, we make the simplifying assumption that $\boldsymbol{Q}(t)$ is guided by a *constant* velocity field; that is, we approximate $\psi_t \approx \psi_{t_0}$ and solve the differential equation

$$\frac{\mathrm{d}\boldsymbol{Q}(t)}{\mathrm{d}t} = \frac{\boldsymbol{j}}{\rho}\Big|_{t=t_0}(\boldsymbol{Q}(t)) \tag{B.4.2}$$

instead of (B.1.3) for times t close to t_0. This approximation has already been employed in prior studies of Bohmian trajectories in the context of IBCs [229]; see Remark B.4.2 below for a possible general strategy of rigorously justifying it.

Proposition B.4.1. *Let $\psi_0 \in \widehat{D}$ and $t_0 \in \mathbf{R}$ be any time for which*

$$\mathrm{Im}\,[c_-^*(t_0)c_+(t_0)] \neq 0 \tag{B.4.3}$$

(in particular, $c_-(t_0) \neq 0$). By simple time shifts, we may assume without loss of generality that $t_0 = 0$ and drop the argument t_0 in (B.4.3) *from now on.*

Then the trajectories $t \mapsto \boldsymbol{Q}(t)$ solving (B.4.2) *(as an approximation of* (B.1.3)*) and reaching $r = 0$ at time $t_0 = 0$ (or emanating from $r = 0$ at $t_0 = 0$) can occur only if $\mathrm{Im}\,[c_-^* c_+] < 0$ (resp., $\mathrm{Im}\,[c_-^* c_+] > 0$) and obey for $t < 0$ (resp., $t > 0$) in spherical coordinates the asymptotics*

$$r(t) = \left[2B\,(1-2B)\,\frac{\left|\mathrm{Im}\,[c_-^* c_+]\right|}{|c_-|^2}\right]^{\frac{1}{1-2B}} |t|^{\frac{1}{1-2B}} + O\Big(|t|^{\min\{\frac{1+2B}{1-2B}, \frac{3/2-B}{1-2B}\}}\Big) \tag{B.4.4a}$$

$$\vartheta(t) = \vartheta_0 + o\big(|t|^{\frac{1}{2}}\big) \tag{B.4.4b}$$

$$\varphi(t) = \varphi_0 - q\,\mathrm{sgn}(\tilde{m}_j\tilde{\kappa}_j)\,4B^2\left[\left[\frac{2B}{1-2B}\right]^{2B}\frac{|c_-|^2}{\left|\mathrm{Im}\,[c_-^* c_+]\right|}\right]^{\frac{1}{1-2B}} |t|^{\frac{-2B}{1-2B}} + C_{H,c_\pm}\log|t| + O\Big(|t|^{\min\{\frac{2B}{1-2B}, \frac{1}{2}\}}\Big) \tag{B.4.4c}$$

as $t \to 0$ with some (unique) constants $0 \leq \varphi_0 < 2\pi$ and $0 \leq \vartheta_0 \leq \pi$. Here, $C_{H,c_\pm}$ denotes a constant depending on the chosen Hamiltonian H (i.e., on $q, \tilde{m}_j, \tilde{\kappa}_j$) and the short–distance coefficients $c_\pm$ of $\psi_0 \in \widehat{D}$. Moreover,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = O\Big(|t|^{\frac{2B}{1-2B}}\Big) \xrightarrow{t\to 0} 0\,. \tag{B.4.5}$$

Recall that $\sqrt{3}/2 < |q| < 1$ and thus $B = \sqrt{1-q^2} \in (0, 1/2)$ as defined in (B.1.12). It follows that for every B, the error term in (B.4.4a) has exponent greater than $1/(1-2B) > 0$ and thus is smaller than the explicitly given first term. Likewise in (B.4.4c), the error term is actually smaller than the terms before because $2B/(1-2B)$ is always positive.

The condition (B.4.3) can be thought of as ensuring non-degeneracy of the Bohmian dynamics. Since $\widehat{D}$ is invariant under the time evolution generated by H, all the other short–distance coefficients apart from $c_\pm$ remain zero for all times.

Moreover, observe that, by plugging (B.4.4a) into (B.4.4b) and (B.4.4c), we arrive at (B.1.16) and (B.1.17), respectively. Since the leading order coefficient of (B.4.4c) is given by $-q\,\mathrm{sgn}(\tilde{m}_j\tilde{\kappa}_j)$ times a positive factor depending also on $c_\pm$, we see that the sense of circling the z–axis depends on the choice of the Hamiltonian (viz., on $q, \tilde{m}_j, \tilde{\kappa}_j$) but not on ψ, while the speed of circulation (meaning not just the exponent of $|t|$ but also the prefactor) depends on ψ but is the same for all trajectories.

Remark B.4.2. (On the approximation by a constant velocity field)
The approximate form (B.4.2) *of the equation of motion* (B.1.3) *has already been used in the derivation of Bohmian trajectories for the non-relativistic case in [229]. Although the simplified ODE* (B.4.2) *(and its non-relativistic analog in [229]) most likely yield the correct leading order behavior of Bohmian trajectories shortly after (before) particle creation (annihilation), both [229] and the present work are lacking a rigorous justification of this approximation. In the following, we shall thus briefly outline a potential general strategy of how one could prove the validity of approximating the full guiding equation* (B.1.3) *by the one with a constant velocity field* (B.4.2)*. We will focus on the present relativistic setting, but the principal argument can immediately be translated to the non-relativistic setting [229].*

The basic idea to make the approximation rigorous is to show that for $\psi_0 \in \widehat{D}$, the three terms in the asymptotic expansion for the 1–particle component of the time–evolved wave function

$$\psi_t^{(1)}(r\boldsymbol{\omega}) = c_-(t)\,f^-_{\tilde{m}_j\tilde{\kappa}_j}(\boldsymbol{\omega})\,r^{-1-B} + c_+(t)\,f^+_{m_j\kappa_j}(\boldsymbol{\omega})\,r^{-1+B} + o_t(r^{-1/2}) \tag{B.4.6}$$

are well–behaved in t. More precisely, one needs to show that (i) $c_-(t)$ is a C^1–function of time, (ii) $c_+(t)$ is a C^1–function of time, and (iii) the implicit constant in $o_t(|\boldsymbol{x}|^{-1/2})$ is uniformly bounded for small enough times. First, assuming that we have $a_1 = 1$, $a_4 = 4B(1+q)$, and $a_2 = a_3 = 0$ in (B.2.7)*, the IBC* (B.2.11) *yields that $c_- \in C^1$ since $\psi_t^{(0)}$ is C^1 in time and we have proven (i). Note that, if we had chosen different $a_1, ..., a_4$, we could have drawn the same conclusion for a certain linear combination of c_- and c_+. For (ii) we propose to take the scalar product of $\psi_t^{(1)}$ with $g_t(r\boldsymbol{\omega}) = f^+_{m_j\kappa_j}(\boldsymbol{\omega})x(t)^{-1/2}\mathbf{1}_{\{r<x(t)\}}$ with $x(t) \to 0$ as $t \to 0$. Using that $\big|\langle(\psi_t^{(1)} - \psi_0^{(1)}), g_t\rangle\big| \leq C|t|$ in combination with c_- being C^1, one should be able to deduce the same regularity for c_+ by taking $x(t) \to 0$ as $t \to 0$ arbitrarily slow. For (iii) we note that the $o_t(r^{-1/2})$–error in* (B.4.6) *originates from integrating a $H^1_0((0,\infty))$ function from 0 to r by the fundamental theorem of calculus [277] and dividing by r afterwards. Therefore, in order to show the error term to be bounded uniformly in short times, one could employ Sobolev–to–Sobolev estimates showing that the time evolution $\mathrm{e}^{-\mathrm{i}Ht}$ is a bounded operator from one Sobolev space to another, uniformly for times t in compact intervals (see, e.g., [439]).*

It remains to give the proof of Proposition B.4.1.

Proof of Proposition B.4.1. By the short–distance asymptotics (B.2.16) or (B.2.9), we have that

$$\rho(r\boldsymbol{\omega}) = |c_-|^2\big|f^-_{\tilde{m}_j\tilde{\kappa}_j}(\boldsymbol{\omega})\big|^2 r^{-2-2B} + 2\mathrm{Re}\big[c_-^* c_+ \langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle\big] r^{-2} + O\big(r^{\min\{-2+2B,-3/2-B\}}\big)\,. \quad \text{(B.4.7)}$$

An easy computation yields that

$$\langle \Phi^\pm_{\tilde{m}_j\tilde{\kappa}_j}(\boldsymbol{\omega}), \Phi^\mp_{\tilde{m}_j\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0 \qquad \text{and} \qquad \big|\Phi^\pm_{\tilde{m}_j\tilde{\kappa}_j}(\boldsymbol{\omega})\big|^2 = \frac{1}{4\pi}\,, \quad \text{(B.4.8)}$$

which in particular shows that the r^{-2} term in (B.4.7) is independent of $\boldsymbol{\omega}$, and allows us to infer that

$$\big|f^-_{\tilde{m}_j\tilde{\kappa}_j}(\boldsymbol{\omega})\big|^2 = \frac{1+q}{\pi} \qquad \text{and} \qquad \langle f^-_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega}), f^+_{\tilde{m}_j,\tilde{\kappa}_j}(\boldsymbol{\omega})\rangle = \frac{q(1+q)}{\pi}\,. \quad \text{(B.4.9)}$$

In this way we arrive at

$$\rho(r\boldsymbol{\omega}) = |c_-|^2\frac{1+q}{\pi}r^{-2-2B} + 2\mathrm{Re}\,[c_-^*c_+]\frac{q(1+q)}{\pi}r^{-2} + O\big(r^{\min\{-2+2B,-3/2-B\}}\big)\,, \quad \text{(B.4.10)}$$

where the explicit terms are independent of $\boldsymbol{\omega}$. Combining the asymptote (B.4.10) with Proposition B.3.1 (and using that

$$\frac{1}{A+\varepsilon} = \frac{1}{A} - \frac{\varepsilon}{A^2} + o(\varepsilon) = \frac{1}{A} + O(\varepsilon) \quad \text{(B.4.11)}$$

as $\varepsilon \to 0$ for $A \neq 0$ independent of ε), we obtain from the simplified equation of motion (B.4.2) the following asymptotic system of ODEs for the spherical coordinates $(r(t), \vartheta(t), \varphi(t))$ of $\boldsymbol{Q}(t)$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 2B\frac{\mathrm{Im}\,[c_-^*c_+]}{|c_-|^2}r^{2B} + O\big(r^{\min\{4B,1/2+B\}}\big) \quad \text{(B.4.12a)}$$

$$r\,\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = o(r^{1/2+B}) \quad \text{(B.4.12b)}$$

$$r\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -q\,\mathrm{sgn}(\tilde{m}_j\tilde{\kappa}_j) + \tilde{C}_{H,c_\pm}r^{2B} + O\big(r^{\min\{4B,1/2+B\}}\big)\,. \quad \text{(B.4.12c)}$$

As in (B.4.4c), $\tilde{C}_{H,c_\pm}$ denotes a constant depending on the choice of Hamiltonian H (i.e., on q, $\tilde{m}_j$, and $\tilde{\kappa}_j$) and the short–distance coefficients $c_\pm$. In the last equation, we have already divided by $\sin\vartheta$.

We are now left with the task of solving the system (B.4.12). Using the initial condition $r(0) = 0$, the first equation (B.4.12a) can be integrated by separation of variables, leaving us with

$$r(t)^{1-2B}\left(1 + O\big(r(t)^{\min\{2B,1/2-B\}}\big)\right) = \left[2B\,(1-2B)\,\frac{\big|\mathrm{Im}\,[c_-^* c_+]\big|}{|c_-|^2}\right]|t| \tag{B.4.13}$$

for $\mathrm{sgn}(t) = \mathrm{sgn}(\mathrm{Im}\,[c_-^* c_+])$. Generally, from a relation of the form $t = cr^\alpha + O(r^\beta)$ with $0 < \alpha < \beta$ and $c, t, r > 0$, we can conclude that every $O(r^\gamma)$ is an $O(t^{\gamma/\alpha})$ and vice versa for every $\gamma > 0$. Thus, $r^\alpha = c^{-1}t + O(t^{\beta/\alpha})$ and $r = c^{-1/\alpha}t^{1/\alpha} + O(t^{1/\alpha - 1 + \beta/\alpha})$, which yields (B.4.4a). For (B.4.12b), we make the change of variables $t \to r(t)$, insert the differential (B.4.12a) to obtain that $\mathrm{d}\vartheta/\mathrm{d}r = o(r^{-1/2-B})$, and again integrate by separation of variables, where we now use the initial condition $\vartheta(0) = \vartheta_0$. In this way, we arrive at (B.4.4b) after inverting the change of variables with the aid of (B.4.4a). In order to get (B.4.4c) from (B.4.12c), we pursue the same strategy, i.e., replace $t \to r(t)$ and integrate by separation of variables. However, this time we need to choose the initial condition according to $\varphi(r = r_0) = \tilde{\varphi}_0$ for some sufficiently small but fixed $r_0 > 0$ and $\tilde{\varphi}_0 \in \mathbf{R}$. Absorbing $\tilde{\varphi}_0$ and all terms depending only on r_0 into a new constant $\varphi_0 \in \mathbf{R}$, we arrive at (B.4.4c), again after inverting the change of variables with the aid of (B.4.4a). □

B.5 The Jump Process

B.5.1 Definition

We define the process (Q_t) for $t \geq \tau$ for some time τ regarded as the initial time. Given that, as we will argue in Section B.5.2, the process is equivariant (i.e., $|\psi_t|^2$ distributed at every t), it follows that the processes defined for τ_1 and $\tau_2 > \tau_1$ are equal in distribution on $[\tau_2, \infty)$, so (by the Kolmogorov extension theorem) the processes for all τ's can be combined into a single process $(Q_t)_{t \in \mathbf{R}}$ defined on the whole time axis.

Here is the definition of the process. We assume that the initial wave function ψ_τ lies in $\widehat{D}$; it follows that $\psi_t \in \widehat{D}$ for all t. The initial configuration Q_τ is chosen to be $|\psi_\tau|^2$ distributed. Once $Q_t \in \mathcal{Q}^{(1)} = \mathbf{R}^3 \setminus \{\mathbf{0}\}$, it follows the Bohmian trajectory, i.e., the equation of motion (B.1.3). If the trajectory reaches $\mathbf{0}$ at some time t_0, the process jumps to

$$Q_{t_0} := \varnothing \in \mathcal{Q}^{(0)}\,. \tag{B.5.1}$$

The process is required to be a Markov process, so it only remains to specify the jump rate $\sigma_{t_0}(\vartheta_0, \varphi_0)\,\mathrm{d}\vartheta_0\,\mathrm{d}\varphi_0$ from the 0-particle configuration $\varnothing \in \mathcal{Q}^{(0)}$ to the trajectory in $\mathcal{Q}^{(1)}$ emanating at any given time t_0 from $\mathbf{0}$ with parameters ϑ_0 and φ_0. As we will explain, the natural choice analogous to Bell's jump rate formula [58] (and to the jump rates in the non-relativistic case [229]) is

$$\sigma_{t_0}(\vartheta_0, \varphi_0) = \frac{2(1+q)B}{\pi}\,\frac{\max\big\{0, \mathrm{Im}\,[c_-^*(t_0)\,c_+(t_0)]\big\}}{|\psi_{t_0}^{(0)}|^2}\sin\vartheta_0\,. \tag{B.5.2}$$

Here, it is relevant to observe from (B.4.12a) that if $\mathrm{Im}\,[c_-^* c_+] < 0$, then (according to Proposition B.4.1) all trajectories are ingoing, and if $\mathrm{Im}\,[c_-^* c_+] > 0$, then all are outgoing. In the former case, it is not possible to jump onto an outgoing trajectory because there is no outgoing trajectory, and indeed $\sigma_{t_0} = 0$. In the latter case, there is a 2-parameter family of outgoing trajectories parameterized by ϑ_0 and φ_0. The total jump rate (i.e., the rate of leaving $\varnothing$) is

$$\int_0^\pi \mathrm{d}\vartheta_0 \int_0^{2\pi} \mathrm{d}\varphi_0\,\sigma_{t_0}(\vartheta_0, \varphi_0) = 8(1+q)B\,\frac{\max\big\{0, \mathrm{Im}\,[c_-^*(t_0)\,c_+(t_0)]\big\}}{|\psi_{t_0}^{(0)}|^2}\,. \tag{B.5.3}$$

Given that a jump occurs at t_0, the distribution of the chosen values of ϑ_0 and φ_0 (i.e., of which trajectory to jump to) has density $(4\pi)^{-1}\sin\vartheta_0$, which means that if we think of ϑ_0 and φ_0 as coordinates on a sphere, then the distribution is uniform over the sphere. For definiteness, we set that at the time t_0 of the jump, $Q_{t_0} := \varnothing$. This completes the definition of the process.

B.5.2 Equivariance and Uniqueness of the Rate

We now give a non-rigorous justification of the claim that Q_t will be $|\psi_t|^2$ distributed at every t. Since in $\mathcal{Q}^{(1)}$ (away from $\mathbf{0}$),

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{j}\,, \tag{B.5.4}$$

no ρ is gained or lost there. It follows, first, that away from $\mathbf{0}$ probability gets transported by Q_t so as to maintain the density ρ (as usual in Bohmian mechanics [91, 232]), and second, that the only place in $\mathcal{Q}^{(1)}$ where ρ is gained or lost is $\mathbf{0}$. We now want to express the rate at which ρ is gained or lost there; for simplicity, we write $\psi_{t_0} = \psi$. As before, we neglect how ψ changes near t_0. Consider first the flux of probability through the surface element $\mathrm{d}^2\boldsymbol{\omega}$ of the sphere around $\mathbf{0}$ of small but nonzero radius r: it is

$$j_r(r\boldsymbol{\omega})\, r^2\, \mathrm{d}^2\boldsymbol{\omega}\,. \tag{B.5.5}$$

From Proposition B.3.1, we obtain that for small r, this is equal to

$$(C_{\mathrm{rad}} + o(r^{1/2-B}))\, \mathrm{d}^2\boldsymbol{\omega}\,, \tag{B.5.6}$$

which for $r \to 0$ converges to $C_{\mathrm{rad}}\, d^2\boldsymbol{\omega}$. Since $C_{\mathrm{rad}} = 2\pi^{-1}(1+q)B\,\mathrm{Im}\,[c_-^* c_+]$ is independent of $\boldsymbol{\omega}$, the rate of gain (positive or negative) of ρ at $\mathbf{0}$ is given by $4\pi C_{\mathrm{rad}}$.

This agrees with the rate of gain (positive or negative) of probability at $\mathbf{0}$ of Q_t: Indeed, if $C_{\mathrm{rad}} > 0$ then no trajectories end at $\mathbf{0}$ at t_0 (so no probability is lost there), and the amount transported by jumps from $\varnothing$ to trajectories emanating from $\mathbf{0}$ at t_0 is the probability at $\varnothing$ times the total jump rate from $\varnothing$, or

$$|\psi^{(0)}|^2 \int_0^\pi \mathrm{d}\vartheta_0 \int_0^{2\pi} \mathrm{d}\varphi_0\, \sigma_{t_0}(\vartheta_0,\varphi_0) \overset{(B.5.3)}{=} 8(1+q)B\,\mathrm{Im}\,[c_-^*\, c_+] = 4\pi C_{\mathrm{rad}}\,. \tag{B.5.7}$$

If, however, $C_{\mathrm{rad}} < 0$ then no upward jumps occur (so no probability is gained at $\mathbf{0}$), while the amount lost automatically agrees (since Q_t is $|\psi_t|^2$ distributed) with the flux across the sphere in the limit $r \to 0$.

Finally, to ensure preservation of the $|\psi|^2$ distribution, it remains to verify that the distribution of Q_t over the emanating trajectories agrees with that required for $|\psi_t|^2$, i.e., yields the flux (B.5.5) through $r\,\mathrm{d}^2\boldsymbol{\omega}$ in the limit $r \to 0$: Indeed, using that (i) the leading terms in the radial velocity (B.4.12a) and the azimuthal velocity (B.4.12c) are independent of $\boldsymbol{\omega}$, (ii) the polar velocity (B.4.12b) is essentially 0, and (iii) the distribution defined by $\sigma_{t_0}(\vartheta_0,\varphi_0)$ over the sphere with coordinates ϑ_0 and φ_0 is uniform as remarked after (B.5.3), we obtain that the distribution of Q_t over the r-sphere is uniform to leading order as $r \to 0$. Using again that the leading term in the radial velocity (B.4.12a) is independent of $\boldsymbol{\omega}$, we obtain that the radial current of Q_t is independent of $\boldsymbol{\omega}$ in the limit $r \to 0$. Since the total current agrees with $4\pi C_{\mathrm{rad}}$, the flux of Q_t through $r\,\mathrm{d}^2\boldsymbol{\omega}$ agrees with (B.5.5) in the limit $r \to 0$, as desired. This completes the argument for equivariance.

As a byproduct of this reasoning, we see that conversely, the formula (B.5.2) is uniquely determined by the demand for equivariance (and Markovianity): Whenever $C_{\mathrm{rad}} < 0$, σ_{t_0} must vanish because there are no outgoing trajectories, and whenever $C_{\mathrm{rad}} \geq 0$, σ_{t_0} must be given by $C_{\mathrm{rad}}|\psi^{(0)}|^{-2}\mathrm{d}^2\boldsymbol{\omega}$ in order to feed the correct probability distribution into the Bohmian flow.

A further observation is that (B.5.2) is analogous to the jump rate formula determined in [229, Sec. 3.1 and 7.2] for the non-relativistic case; in fact, both formulas can be expressed in a common form if we write $\sigma_{t_0}(r,\boldsymbol{\omega})\mathrm{d}^2\boldsymbol{\omega}$ for the rate, at time t_0, for jumping from $\varnothing$ to a trajectory that at radius r will have position in $r\mathrm{d}^2\boldsymbol{\omega}$:

$$\lim_{r\to 0} \sigma_{t_0}(r,\boldsymbol{\omega}) = \lim_{r\to 0} \frac{r^2 \max\{0, \psi^{(1)}(r\boldsymbol{\omega})^\dagger\, \alpha_r\, \psi^{(1)}(r\boldsymbol{\omega})\}}{|\psi^{(0)}|^2} \tag{B.5.8}$$

with $\psi = \psi_{t_0}$. It also becomes evident that the jump rate formula (B.5.2) is analogous to Bell's jump rate formula [58, 230]. Presumably, it also arises as a limit of Bell's rate if we can obtain the IBC Hamiltonian as a limit of Hamiltonians with UV cut-off.

B.6 Conclusions

We have studied a model of creation and annihilation of a Dirac particle at a point source at the origin $\mathbf{0}$ in $\mathbf{R}^3$ and constructed, in a non-rigorous way, a Markov process $(Q_t)_{t\in\mathbf{R}}$ in the configuration space $\mathcal{Q}^{(0)} \cup \mathcal{Q}^{(1)} = \{\varnothing\} \cup (\mathbf{R}^3 \setminus \{\mathbf{0}\})$ that is $|\psi_t|^2$ distributed at every time t. Since a UV cutoff has the unphysical consequence that a particle can be created at non-zero distance from the source [230, 229], we have used instead an interior-boundary condition (IBC), which has the reasonable consequence that particles can only be created and annihilated directly at the point source. The key element of the definition of the process $(Q_t)_{t\in\mathbf{R}}$ was the law (B.5.2) specifying the creation rate. It is analogous to Bell's jump rate formula [58, 230]. This process is the first example of a configuration process for a Dirac Hamiltonian with IBC; non-relativistic versions were described in [229]. We believe that this work might contribute to the extension of Bohmian mechanics to relativistic quantum field theory.

The Hamiltonian H we use was recently constructed rigorously in [344] based on prior work in [348, 275, 277]. Some of our considerations here were not rigorous, although all Propositions and Lemmas were proven rigorously. But even the non-rigorous conclusions have benefited from the rigorous construction of H; in fact, certain features and details of the process $(Q_t)_{t\in\mathbf{R}}$ (such as the fact that a newly created particle circles the z axis infinitely often) have only become accessible due to the detailed information about H (such as the near-$\mathbf{0}$ asymptotics of the functions in the domain) provided by its rigorous construction. We have also outlined where we see the biggest hurdle for a full rigorous treatment, and which strategies could be applied to overcome it.

Further questions that would be of interest for future research include whether other models based on Dirac Hamiltonians and IBCs, such as the model of [571] in curved space-time, could also be defined rigorously, whether other Dirac Hamiltonians (such as the model of [365]) would allow for IBCs, what the corresponding Bell-type jump processes look like, and whether there are examples in which the process is qualitatively different from the one described here; in particular, whether there are models for which the jump rate is angle dependent.

Chapter C

How a Space-Time Singularity Helps Remove the Ultraviolet Divergence Problem

This chapter contains the paper [339]:

J. Henheik, B. Poudyal, and R. Tumulka. How a Space-Time Singularity Helps Remove the Ultraviolet Divergence Problem. *arXiv:2409.00677*, 2024

Abstract. Particle creation terms in quantum Hamiltonians are usually ultraviolet divergent and thus mathematically ill defined. A rather novel way of solving this problem is based on imposing so-called interior-boundary conditions on the wave function. Previous papers showed that this approach works in the non-relativistic regime, but particle creation is mostly relevant in the relativistic case after all. In flat relativistic space-time (that is, neglecting gravity), the approach was previously found to work only for certain somewhat artificial cases. Here, as a way of taking gravity into account, we consider curved space-time, specifically the super-critical Reissner-Nordström space-time, which features a naked timelike singularity. We find that the interior-boundary approach works fully in this setting; in particular, we prove rigorously the existence of well-defined, self-adjoint Hamiltonians with particle creation at the singularity, based on interior-boundary conditions. We also non-rigorously analyze the asymptotic behavior of the Bohmian trajectories and construct the corresponding Bohm-Bell process of particle creation, motion, and annihilation. The upshot is that in quantum physics, a naked space-time singularity need not lead to a breakdown of physical laws, but on the contrary allows for boundary conditions governing what comes out of the singularity and thereby removing the ultraviolet divergence.

Content

C.1 Introduction

It is a notoriously difficult problem [478, 470, 284] (and still active [494, 422]) to rigorously implement particle creation and annihilation in quantum Hamiltonians at point sources, as they are usually plagued by ultraviolet (UV) divergences. The traditional way of resolving this issue is to employ so-called UV cut-offs (e.g., [470], see also [572, Sec. 6.2.5]), corresponding to smearing out the source of particle creation to a positive volume, and (if possible) defining a renormalized Hamiltonian [577, 470, 216, 300] in a limiting procedure removing the cut-off. A different, rather novel approach to this problem is based on *interior boundary conditions* (IBCs) [558, 557]: These relate the wave function ψ, defined on a configuration space of a variable number of particles, at the *interior* of the n-particle sector to the *boundary* (i.e., where creation/annihilation occurs) of the $n+1$ particle sector.

The IBC approach has previously successfully been applied in the non-relativistic setting [389, 388], i.e., for the Schrödinger equation involving the Laplacian. However, since particle creation is mostly relevant in the relativistic case, it is of particular importance to study the IBC approach in that setting, for example for the Dirac equation. In flat relativistic space-time (i.e., neglecting gravity), two of us have shown the following *no-go* result (see [344, Theorem 1]): In three spatial dimensions, there exists no self-adjoint Hamiltonian on Fock space that involves particle creation and annihilation

at the origin but otherwise acts like the free Dirac Hamiltonian. As a positive, but somewhat artificial result [344, Theorem 6], it was shown that IBC Hamiltonians with particle creation at the origin *can* in fact be implemented in that setting upon adding a sufficiently strong Coulomb potential at the origin. Here, we obtain an IBC Hamiltonian without coupling to a Coulomb potential; we do so by relying only on gravity in a general-relativistic way. In fact, the presence of a space-time singularity makes the IBC approach work without the assumption of a strong Coulomb potential. For further works on IBCs, see [364, 420, 513, 494, 83].

In another recent work [345], some of us studied the corresponding Bohmian trajectories and (non-rigorously) constructed a $|\Psi|^2$-distributed Markov jump process (the *Bohm-Bell process* [58, 230]) in the configuration space of a variable number of particles. Here, we provide the analogous construction with gravity (see Sections C.1.1.2 and C.2.4).

In this paper, as a way of taking gravity into account, we consider curved space-time, specifically the super-critical Reissner-Nordström (sRN) space-time [475, 497, 585, 476, 327], which is the static curved space-time surrounding a single charged point mass, a solution of the Einstein-Maxwell equations with mass $M \geq 0$, charge $Q \in \mathbf{R}$, and angular momentum 0, where "super-critical" means

$$|Q| > M\,. \tag{C.1.1}$$

More precisely, the general Reissner-Nordström space-time is given by the manifold $\mathscr{M} = \mathbf{R} \times (\mathbf{R}^3 \setminus \{\mathbf{0}\}) \cong \mathbf{R} \times (0,\infty) \times \mathbf{S}^2$, where $\mathbf{0}$ denotes the origin of $\mathbf{R}^3$, equipped with the Lorentzian metric g with line element

$$\mathrm{d}s^2 = A^2(r)\,\mathrm{d}t^2 - \frac{1}{A^2(r)}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\boldsymbol{\omega}^2 \tag{C.1.2}$$

in spherical coordinates $(t \in \mathbf{R}; r \in (0,\infty); \boldsymbol{\omega} \in \mathbf{S}^2)$. Here, $\mathrm{d}\boldsymbol{\omega}^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$ in terms of the polar angle ϑ and the azimuthal angle φ, and we used natural units $\hbar = c = G = 1$ and the abbreviation

$$A^2(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{C.1.3}$$

with parameters M and Q representing the mass and charge at $r = 0$. Finally, the electromagnetic field is denoted by

$$A_\mu = (Q/r, 0, 0, 0)\,, \tag{C.1.4}$$

not to be confused with the scalar A function introduced in (C.1.3). In the super-critical regime (C.1.1), where $A^2(r) > 0$ for all r, the singularity is timelike and naked (i.e., not surrounded by a horizon), and the metric is static and asymptotically flat. We also take $A(r) > 0$. The singularity will be regarded here as the boundary of $\mathscr{M}$, i.e., $\partial\mathscr{M} = \{r = 0\} = \mathbf{R} \times \{0\} \times \mathbf{S}^2$.

We note in passing that the charge and mass value of an electron satisfy the super-criticality condition (C.1.1), so the classical space-time surrounding an electron would be sRN, provided that the electron spin does not contribute to the angular momentum of the space-time. While it is not known whether real electrons contain space-time singularities, we are in part motivated by the possibility that they might (see Section C.2.1 for more discussion).

The basic physical picture, illustrated in Figure C.1 and underlying the entire paper, is that a relativistic quantum-mechanical spin-$1/2$ particle of mass $m \geq 0$ and charge $q \in \mathbf{R}$ can be emitted and absorbed at the singularity $\partial\mathscr{M}$. In our setting, this can naturally be associated with a "source particle" of mass $M \geq 0$ and charge $Q \in \mathbf{R}$ obeying (C.1.1) (see Section C.2.5 for further details). Away from the singularity, the wave function of the quantum particle is governed by the Dirac equation on sRN space-time with Hamiltonian H_1 explicitly given in (C.2.15) below.

As our results, briefly described in Section C.1.1 below, we (i) rigorously construct a self-adjoint Hamiltonian H with particle creation, based on IBCs (see Theorem C.3.1 in Section C.3.1), and (ii) non-rigorously analyze the asymptotic behavior of the Bohmian trajectories close to the space-time

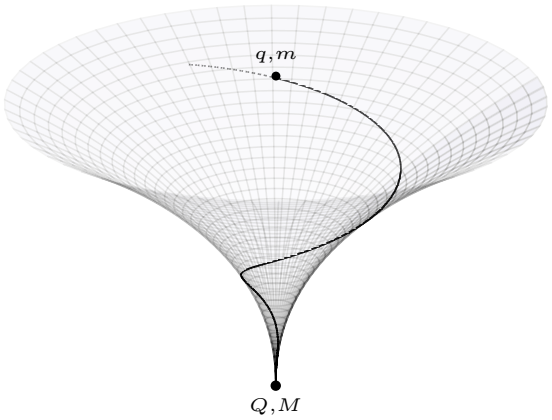

Figure C.1: Qualitative depiction of the setup in this paper: A relativistic quantum mechanical spin-1/2 particle of charge q and mass m moves in a curved space representing the gravitational field of a "source particle" with charge Q and mass M (and fixed location, which then is a curvature singularity). The quantum particle can be absorbed and emitted by the source particle. The trajectory shown is a Bohmian trajectory of the quantum particle shortly before absorption or after emission by the source particle.

singularity in sRN and construct the corresponding Bohm-Bell process (see Section C.3.2, in particular Proposition C.3.7) for a particular choice of H and "nice" wave functions.

It follows that the quantum particle has nonzero probability to hit the singularity, although the latter could be thought of as a 0d set in 3d space, and the probability of hitting a generic 0d or 1d set vanishes. (The reason for this kind of effective attraction to the singularity $\partial\mathscr{M}$ is that at $\partial\mathscr{M}$, the arriving wave function will be transported to the 0-particle sector of Fock space, thereby effectively exerting a kind of suction on the nearby wave function.)

C.1.1 Description of Our Main Results

In this section, we briefly describe the main results of the present paper and provide some comments on them. Full details are given in Section C.3.

C.1.1.1 IBC Hamiltonian with Particle Creation

As our first main result, Theorem C.3.1, we devise a certain Hamiltonian H with particle creation and annihilation, and prove that it is self-adjoint. As mentioned above, emission/absorption of a particle occurs at a single point in space (or world line in space-time), the naked singularity in sRN space-time (C.1.2). Thus, on the one hand, the present work rigorously extends the IBC approach to curved space-time (with fixed background metric), and on the other hand, our treatment deals with (and gives physical meaning to) the well-known fact [175] that the 1-particle Dirac Hamiltonian H_1 on the sRN space-time is not essentially self-adjoint, and thus does not uniquely define a unitary time evolution. Our Hamiltonian H is based on H_1 but is defined on a version of Fock space, as appropriate for particle creation. For simplicity, we consider only the 0-particle and 1-particle sectors of Fock space (but our approach could be extended to the full Fock space along the lines of [389]). It is common to exclude wave functions of negative energy as unphysical, but we will not exclude them in our model. Our proofs make particular use of mathematical results of Cohen and Powers

[175] about the domain of the adjoint of H_1, in particular described by the asymptotic behavior of wave functions near the singularity. These asymptotics are then exploited to devise an IBC, coupling the 1-particle to the 0-particle sector and thus constituting the Hamiltonian H.

We finally remark that, in [571, Eq. (52)], one of us already conjectured an IBC for this case; the IBC investigated here is similar but not identical, and we leave open the question whether a self-adjoint Hamiltonian can be devised for the IBC of [571]. For a comparison of the two IBCs, see Remark C.3.3 in Section C.3.1.

C.1.1.2 Bohmian Trajectories and Bohm-Bell Jump Process

As our second main result (see Section C.3.2, in particular Proposition C.3.7), in addition to the Hamiltonian H, we construct the Bohm-Bell process [58, 230] for a particular choice of H (viz., $\widetilde{\kappa}_j = \pm 1$ in the notation of Section C.3.1) and an initial wave function Ψ_0 from a suitable subspace of Hilbert space. It is a piecewise-deterministic Markovian jump process in the configuration space of a variable number of particles that is $|\Psi_t|^2$ distributed at every coordinate time t, and its jumps correspond to the creation or annihilation events. Similar processes were devised in [229] for non-relativistic space-time and in [345] for Minkowski space-time with a Coulomb field. While we do not rigorously prove the existence of the process, we can specify what its defining equations must be, in particular the law (C.3.25) for the rate of particle creation at the singularity. This rate depends on the wave function and thus on time, while the direction of emission is uniformly distributed over all directions. A similar law had been conjectured in [571].

Here is a comparison between the non-relativistic [389], the special-relativistic [345], and the present general-relativistic case (summarized in Table C.1). While the special-relativistic process circles the origin infinitely many times before hitting it, our process does not, and thus is similar in this respect to the non-relativistic process. Another such similarity concerns the radial speed with which the quantum particle hits the origin: while it does with speed 0 in the special-relativistic case, it does with nonzero speed in our and the non-relativistic case. (Note that the geometrically appropriate way of measuring this speed is $\mathrm{d}R/\mathrm{d}t$, where R denotes the radial metric distance from the singularity, which is not the same as r, as the latter is defined so that $4\pi r^2$ is the surface area of the sphere with coordinate r, see Section C.4.2.)

C.1.2 Structure of the Paper

The remainder of this paper is organized as follows. In Section C.2, we put the results into context and provide relevant background information. In Section C.3, we state our main results. In Section C.4, we review the known facts about the Dirac equation in the sRN space-time. In Section C.5, we prove our theorem about the existence of the IBC Hamiltonian. In Section C.6, we give the details about the construction of the associated Bohm-Bell process. In Section C.7, we conclude. In Appendix C.A, we provide the explicit form of the angular momentum eigenfunctions in a spinor basis corresponding to spherical coordinates.

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C.2 Motivation, Significance, and Background

In this section, we further motivate our paper, connect our results to existing literature, and provide additional background information.

	non-rel. [389]	SR [345]	GR (here)
$\frac{\mathrm{d}r}{\mathrm{d}t}(t_0)$	$\neq 0$	0	$\frac{\mathrm{d}R}{\mathrm{d}t}(t_0) \neq 0$
$\vartheta(t_0)$	const.	const.	const.
$\varphi(t_0)$	const.	$\to \pm\infty$	const.

Table C.1: Comparison between the Bohm-Bell processes in the non-relativistic, the special-relativistic, and the general-relativistic case; t_0 is the time of absorption or emission, and R means the "tortoise coordinate" (radial metric distance) defined in (C.4.8); see (C.4.17) for the relation $R(t)$ and (C.3.21) for $\vartheta(t)$ and $\varphi(t)$.

C.2.1 Space-Time Singularities

One motivation for this research concerns the status of space-time singularities (i.e., of points of infinite space-time curvature): It would seem that the laws of physics break down at singularities, as anything could come out of a singularity if it is timelike (as it is for the sRN space-time). However, in the model considered here, certain laws of nature (the IBC and the law determining the creation rate) govern what comes out of the singularity. That is, the singularity does not lead to a breakdown of physical laws, it provides just the room needed for imposing laws for particle creation and annihilation; this point is discussed further in [571].

Here is how that is related to Roger Penrose's cosmic censorship conjecture [489], which states that naked singularities do not form according to general relativity and classical evolution through gravitational collapse from non-singular initial data. Even if that is true, it leaves open whether elementary particles might contain naked singularities, and whether naked singularities might occur in the quantum world. Anyway, we find the possibility of naked singularities worthy of study, in part *because* our results show that they need not entail a breakdown of physical laws, but rather a gap in the physical laws that can be filled by the laws studied here.

C.2.2 Ultraviolet Divergence

Another aspect concerns the problem of ultraviolet infinities. Terms in a Hamiltonian representing particle creation and annihilation at a point source usually diverge, which keeps the Hamiltonian from being rigorously defined. Sometimes, renormalization can provide a way of rigorously defining a Hamiltonian [470, 216] by means of a limiting procedure. Here, we follow a different approach based on IBCs [558, 557], which allow us to directly characterize the Hamiltonian and its domain without a limiting procedure; IBCs are mathematically related to point interactions [19, 68]. We limit ourselves to the (easier) case in which the source (i.e., the emitting and absorbing particle) is fixed at some point (taken to be the coordinate origin). This case was studied for non-relativistic Hamiltonians (based on the Laplacian) in [389].

For the question of whether IBCs can be relevant to realistic quantum field theories, it matters whether they can be applied in a relativistic setting. As a test case, we assume that the particles created are spin-$\frac{1}{2}$ particles governed by the Dirac equation. (Photons would be even more interesting, but no general formula is known for their probability current [572, Sec. 7.3.9], which is why we prefer the Dirac equation.) It has been shown [344] that in Minkowski space-time, IBCs can work in the (somewhat artificial) setting of the Dirac particles feeling a sufficiently strong Coulomb potential around the source, but not if the Coulomb potential is absent or too weak. That sounds not very encouraging; it sounds as if IBCs often failed to work, and as if we should not expect that IBCs could one day be found to work for uncharged relativistic particles such as photons.

However, the picture changes a lot with the results of the present paper. Basically, the gravitational field of the source (which would also apply to photons) makes the IBC approach work in a similar way as for a strong Coulomb field, regardless of how big the charge q and the mass m of the Dirac particles are. In particular, it also works for uncharged and/or massless particles.[1] That is, the present paper provides support for expecting the applicability of the IBC approach in more realistic models.

C.2.3 Self-adjoint Extensions on Fock Space

Mathematically, our problem can be expressed in terms of self-adjoint extensions. This is because, apart from particle creation and annihilation (which happens only at certain places), the Hamiltonian H acts as the Dirac Hamiltonian H_1 and we thus devise a self-adjoint extension of H_1 to an enlarged Hilbert space, a (truncated) Fock space. (Note, however, that unlike usual self-adjoint extensions, which start from a densely defined operator, H_1 in our case is not densely defined, see below.)

In curved space-time, a 1-particle wave function ψ is a cross-section of a smooth complex vector bundle S over $\mathscr{M}$ (called the spinor bundle) with fibers S_x (called the Dirac spin space) for $x \in \mathscr{M}$ that are 4-dimensional complex vector spaces.

For the construction of our H, we are building on previous work on the Dirac Hamiltonian on sRN space-time [175, 55, 56, 260, 474, 365], particularly on [175]. A crucial difference to these prior works is that, since we consider a mini-Fock space consisting of merely the 0-particle and 1-particle sector, our Hilbert space is 1 dimension larger than what was considered in the prior works: If Σ is a $t = \mathrm{const.}$ surface for the Reissner-Nordström time coordinate t, then the prior works considered the 1-particle Hilbert space $\mathscr{H}^{(1)}$ of functions $\psi : \Sigma \to S$ that are cross-sections (i.e., $\psi(x) \in S_x$) with $\langle \psi, \psi \rangle < \infty$ for the inner product

$$\langle \psi, \phi \rangle = \int_\Sigma V(\mathrm{d}^3 x)\, \overline{\psi}(x)\, n_\mu(x)\, \gamma_x^\mu\, \phi(x)\,, \tag{C.2.1}$$

where V is the Riemann volume measure arising from the 3-metric on Σ and $n_\mu(x)$ the future unit normal vector to Σ at x (see [572, Sec. 7.3.4] for why this is a Hilbert space).

In contrast, since we consider particle creation, our Hilbert space is the orthogonal sum

$$\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)} \tag{C.2.2}$$

of the 0-particle space $\mathscr{H}^{(0)}$ and the 1-particle space $\mathscr{H}^{(1)}$ and thus constitutes a truncated Fock space. The 0-particle space $\mathscr{H}^{(0)} = \mathbf{C}$ is 1-dimensional (because it is spanned by the vacuum state). The Dirac Hamiltonian is at first defined as a differential operator H_1^0 on a dense domain

[1] On the other hand, we use here that the source has sufficiently large charge, $|Q| > M$, but that is, first, actually satisfied for the charge and mass of an electron (as we often think of the sRN space-time as the gravitational field of an elementary particle), and second, it is not so much an issue of the IBC approach as one of the Einstein equation, as the Reissner-Nordström space-time for $0 < |Q| \leq M$ has a complicated structure with infinitely many singularities, wormholes, and asymptotically flat regions [327, Fig.s 25 and 26(i)], while for $Q = 0$ it becomes the Schwarzschild space-time, for which the singularity becomes spacelike and thus not at all like the world line of a particle.

D_1^0 in $\mathscr{H}^{(1)}$; while the prior works were studying self-adjoint extensions in $\mathscr{H}^{(1)}$, we are looking at self-adjoint extensions in $\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$; in particular, the operator H_1^0 we extend is densely defined in $\mathscr{H}^{(1)}$ but not in $\mathscr{H}$. If H_1^0 were *essentially self-adjoint* in $\mathscr{H}^{(1)}$, it would have a unique self-adjoint extension in $\mathscr{H}^{(1)}$, and that would be bad for our purpose because it would entail [344, Theorem 1] that all self-adjoint extensions in $\mathscr{H}$ are block diagonal, which means that no transitions between $\mathscr{H}^{(0)}$ and $\mathscr{H}^{(1)}$ ever occur, and thus no particle creation or annihilation. However, as found in [175], H_1^0 is not essentially self-adjoint in $\mathscr{H}^{(1)}$, which gives us room to impose an IBC to obtain a self-adjoint extension H in $\mathscr{H}$.

The situation here is different from that in Minkowski space-time: In the latter case, for an uncharged particle ($q = 0$) on Euclidean 3-space with one point (say, the origin $\mathbf{0}$) removed, the Dirac Hamiltonian is essentially self-adjoint [545]. This roughly means that no probability can flow into or out of the point $\mathbf{0}$ and has the consequence [344] that no IBC Hamiltonian with particle creation exists. As mentioned in the introduction, that changes when a sufficiently strong Coulomb field is added to the Hamiltonian: then the Dirac Hamiltonian is not essentially self-adjoint, and IBC Hamiltonians exist [344]. In the present paper, the action of a Coulomb field on the quantum particle is not necessary (i.e., we can allow $q = 0$), as the graviational field alone already lifts the essential self-adjointness of the Dirac Hamiltonian. In fact, we can even allow $m = 0$, and the gravitational field created by Q, M is still sufficient to ensure that the Dirac Hamiltonian is not essentially self-adjoint, and an IBC Hamiltonian exists.

We do not aim here at identifying all possible IBC Hamiltonians on the sRN space-time; we limit ourselves to a few examples.

For Reissner-Nordström space-times in the subcritical regime $|Q| < M$ or the critical regime $|Q| = M$, an IBC should be implementable as well because they have neighborhoods of the singularities that look qualitatively similar to the sRN space-time; however, due to wormholes and several asymptotically flat regions, they are more complicated (and less natural as a model of a point source).

C.2.4 Trajectories

We also introduce the natural analog of the Bohm-Bell process for our Hamiltonian H (see Section C.3.2). The Bohm-Bell process [58, 230] is the natural extension of Bohmian mechanics [90, 232, 572] to include particle creation and annihilation. The process is a Markov process in configuration space; the creation and annihilation events correspond to jumps, as the number of particles changes at these events. Between the jumps, the process is deterministic and follows the Bohmian equation of motion.

The value of Bohmian mechanics lies in the fact that it provides a realist version of quantum theory [572] while its empirical predictions agree with the standard ones. In fact, Bohmian mechanics resolves the paradoxes and inconsistencies of orthodox quantum mechanics and introduces precision where orthodox quantum mechanics is vague, specifically in the theory of measurement. The Bohm-Bell process that we develop here contributes a further step towards a convincing extension of Bohmian mechanics to quantum field theory.

C.2.5 On the Structure of the Model

Here is how our model fits into a wider class of models. It involves two kinds of particles, let us call them x-particles and y-particles. The x-particles can emit and absorb y-particles as in the scheme $x \leftrightarrows x + y$. The x-particles have mass M and charge Q, the y-particles mass m and charge q. We treat the y-particles quantum-mechanically, whereas the x-particles (the sources of emission and absorption) are treated here as non-dynamical and just sit at fixed positions. We limit ourselves to the case of a single x-particle and include the classical, general-relativistic gravitational and electromagnetic fields generated by x, which is the Reissner-Nordström space-time (C.1.2),

considered here for $|Q| > M$ (sRN); recall (C.1.1). The metric is singular at the location of the x-particle, which is why the x-particle can be identified with the sRN singularity. If we wanted to treat the x-particles quantum-mechanically, while including their general-relativistic gravitational fields, we would presumably need a quantum gravity theory. A non-relativistic IBC-model with quantum-mechanical x-particles was defined in [388].

We remark that our model breaks rotational invariance (which would imply conservation of angular momentum) because under the simplifying assumptions made here, that x has spin 0 and y spin $1/2$, local conservation of angular momentum during creation or annihilation events is impossible (already in flat space-time [344, Sec. 2.4]). We expect that IBC Hamiltonians will respect rotational symmetry in more realistic models. The model also violates interaction locality, i.e., the condition that the Hamiltonian contains no interaction between spacelike separated regions. The simple reason is that we allow only 0 or 1 y-particles, so once a y-particle has been created, and perhaps traveled far away, another creation event at the origin is not possible. We expect that the corresponding model on a full Fock space, allowing all $n \in \mathbf{N} \cup \{0\}$ for the number of y-particles, will respect interaction locality.

C.2.6 Dirac Equation in Curved Space-Time

There is a standard way of defining the 1-particle Dirac equation in a curved space-time $(\mathscr{M}, g)$ (see, e.g., [262, 355, 149, 490, 404]), which we recall here for convenience of the reader.

C.2.6.1 Coordinate-free Form

As mentioned already in Section C.2.3, the 1-particle wave function ψ is a cross-section of a vector bundle S over $\mathscr{M}$ whose fibers S_x are the Dirac spin spaces. The vector bundle S is equipped with an irreducible representation of the (complexified) Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(T_x\mathscr{M}, g)$ on the spin spaces, $\mathrm{Cl}_{\mathbf{C}}(T_x\mathscr{M}, g) \to \mathrm{End}(S_x)$, where $T_x\mathscr{M}$ is the tangent space at $x \in \mathscr{M}$; since $T_x\mathscr{M}$ is itself embedded in the Clifford algebra, the representation includes a linear mapping $\gamma_x : T_x\mathscr{M} \to \mathrm{End}(S_x)$, called the general-relativistic gamma matrices and subject to the Clifford relation

$$\gamma_x^\mu \gamma_x^\nu + \gamma_x^\nu \gamma_x^\mu = 2g^{\mu\nu}(x)\, I_x\,, \tag{C.2.3}$$

where $\gamma_x^\mu = \gamma_x(e^\mu)$ for any basis e^0, e^1, e^2, e^3 of $T_x\mathscr{M}$ and I_x denotes the identity operator in S_x. If $(\mathscr{M}, g)$ is orientable, time-orientable, and possesses spin structure [490, (1.5.3)], which sRN does [490, (1.5.6)], then the bundle S and the above-mentioned representation exist; if $\mathscr{M}$ is simply connected, which the sRN manifold is, then they are unique up to isomorphism [490, p. 54]. We also need the appropriate connection on S or covariant derivative

$$\nabla : \Gamma(S) \to \Gamma(T^*\mathscr{M} \otimes S)\,, \tag{C.2.4}$$

where $\Gamma(S)$ denotes the set of smooth cross-sections of the bundle S; ∇ is uniquely defined by the metric [490, Sec. 4.4]. The 1-particle Dirac equation in $(\mathscr{M}, g)$ is then

$$\big(\mathrm{i}\gamma_x^\mu \nabla_\mu - q\gamma_x^\mu A_\mu(x) - m\big)\psi(x) = 0\,, \tag{C.2.5}$$

where $m \geq 0$ is called the mass of the particle and $q \in \mathbf{R}$ its charge. Finally, S_x is equipped with an "overbar" operation $\psi \mapsto \overline{\psi}$, a conjugate-linear mapping from S_x to its dual space S_x^*, and the Born distribution ("$|\psi|^2$ distribution") on a spacelike surface Σ is the measure given by

$$n_\mu(x)\, j^\mu(x)\, V(\mathrm{d}^3x) \tag{C.2.6}$$

with the probability current 4-vector field

$$j^\mu(x) = \overline{\psi}(x)\, \gamma_x^\mu\, \psi(x)\,. \tag{C.2.7}$$

C.2.6.2 Expression in Spherical Coordinates

The Dirac equation in the sRN space-time has been studied before in many works, e.g., [175, 55, 260, 56, 474, 365]. We adopt a widely used basis b_x in spin space S_x defined as follows: From the coordinate basis[2] $(\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi)$ of $T_x\mathscr{M}$, we obtain an orthonormal basis (Lorentz frame) e_x by normalizing the vectors,

$$e_x = (e_x^0, e_x^1, e_x^2, e_x^3) = \left(A^{-1}\partial_t, A\partial_r, r^{-1}\partial_\vartheta, (r\sin\vartheta)^{-1}\partial_\varphi\right). \tag{C.2.8}$$

To this orthonormal basis there corresponds a basis b_x of S_x; the correspondence is canonical up to an overall sign which we choose continuously in x; b_x is an orthonormal basis relative to the scalar product $\overline{\psi}\,\gamma^\mu(x)\,g_{\mu\nu}(x)\,n^\nu(x)\,\phi$ in S_x associated with the surface $\{t = \text{const.}\}$ or its future unit normal vector $n(x) = A^{-1}\partial_t$. Relative to the bases e_x and b_x, the gamma matrices have their standard entries [559],

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{C.2.9}$$

with σ_i the i-th Pauli matrix. That is, in these bases the general-relativistic gamma matrices γ_x^μ reduce to the *special-relativistic* gamma matrices, to which the symbol γ^μ will henceforth refer. Likewise, in the basis b_x, the overbar operation is represented in the same way as in any Lorentz frame,

$$\overline{\psi} = \psi^\dagger\gamma^0. \tag{C.2.10}$$

The Hilbert space $\mathscr{H}^{(1)}$ of 1-particle wave functions on $\Sigma = \{t = 0\}$ can therefore be represented in coordinates as

$$\mathscr{H}^{(1)} = L^2\Big((0,\infty)\times\mathbf{S}^2, \mathbf{C}^4, A^{-1}r^2\,\mathrm{d}r\,\mathrm{d}^2\boldsymbol{\omega}\Big) \tag{C.2.11}$$

with $\mathrm{d}^2\boldsymbol{\omega} = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$ and inner product

$$\langle\psi,\phi\rangle = \int_0^\infty \mathrm{d}r \int_{\mathbf{S}^2} \mathrm{d}^2\boldsymbol{\omega}\, A^{-1}\, r^2\, \psi(r,\boldsymbol{\omega})^\dagger\, \phi(r,\boldsymbol{\omega}). \tag{C.2.12}$$

(Note for comparison that L^2 of 3d Euclidean space is equivalent to $L^2((0,\infty)\times\mathbf{S}^2, r^2\,\mathrm{d}r\,\mathrm{d}^2\boldsymbol{\omega})$ in spherical coordinates.) Indeed, (C.2.12) follows from the general expression (C.2.3), as the Riemannian volume measure V on Σ has in general density $\left|\det{}^3 g\right|^{1/2}$ in coordinates and is in this case given by $V(\mathrm{d}r\times\mathrm{d}^2\boldsymbol{\omega}) = A^{-1}r^2\,\mathrm{d}r\,\mathrm{d}^2\boldsymbol{\omega}$, while $n_\mu = (1,0,0,0)$ in the basis e_x.

Correspondingly, the Born distribution is given in coordinates by

$$|\psi(r,\boldsymbol{\omega})|^2\, A^{-1}\, r^2\,\mathrm{d}r\,\mathrm{d}^2\boldsymbol{\omega}, \tag{C.2.13}$$

where $|\psi|^2$ means $\psi^\dagger\psi$ or, equivalently, the sum of the absolute squares of the four complex components of ψ.

The Dirac equation on sRN space-time then reads in coordinates

$$\mathrm{i}\partial_t\psi = H_1\psi \tag{C.2.14}$$

with Hamiltonian [175, (2.10)][3]

$$\begin{aligned} H_1 = {} & -\mathrm{i}\alpha^1 A^2(\partial_r + r^{-1} + \tfrac{1}{2}A^{-1}A') - \mathrm{i}\alpha^2 r^{-1}A\partial_\vartheta \\ & - \mathrm{i}\alpha^3(r\sin\vartheta)^{-1}A\partial_\varphi + mA\beta + qQr^{-1}, \end{aligned} \tag{C.2.15}$$

where $A' = \partial_r A$ is the derivative of A and, as usual, $\beta = \gamma^0$ and $\alpha^i = \gamma^0\gamma^i$. This operator is defined on $C_c^\infty((0,\infty)\times\mathbf{S}^2, \mathbf{C}^4)$, the space of smooth functions with compact support, which is a dense subspace of $\mathscr{H}^{(1)}$.

[2] We follow here the convention of identifying a tangent vector with the directional derivative operator in that direction.

[3] Cohen and Powers [175] write γ^μ for our $-\mathrm{i}\gamma^\mu$, q for our $-qQ$, and χ for our ψ.

C.3 Main Results

In this section, we formulate our main results. Recall that the Hilbert space of our model is the mini-Fock space $\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$ corresponding to 0 or 1 y-particles (see Section C.2.5 for the terminology of x- and y-particles) with $\mathscr{H}^{(0)} = \mathbf{C}$ and $\mathscr{H}^{(1)}$ given by (C.2.11). The corresponding configuration space is

$$\mathcal{Q} = \mathcal{Q}^{(0)} \cup \mathcal{Q}^{(1)} = \{\varnothing\} \cup \Sigma\,. \tag{C.3.1}$$

Here, $\varnothing$ is the 0-particle configuration and Σ is any one of the $\{t = \text{const.}\}$ surfaces; these surfaces can be identified with each other in a canonical way (as the mapping connecting points with equal (r, ϑ, φ) coordinates is an isometry) and represented in coordinates $(r, \boldsymbol{\omega})$ as the Riemannian 3-manifold

$$\Sigma = (0, \infty) \times \mathbf{S}^2 \tag{C.3.2}$$

with the metric

$$\mathrm{d}s^2 = A^{-2}\mathrm{d}r^2 + r^2\mathrm{d}\boldsymbol{\omega}^2\,. \tag{C.3.3}$$

The Born distribution on $\mathcal{Q}$ for $\Psi \in \mathscr{H}$ is the measure $\mathbf{P}_\Psi$ with

$$\mathbf{P}_\Psi(\{\varnothing\}) = |\Psi^{(0)}|^2 \tag{C.3.4a}$$

$$\mathbf{P}_\Psi(\mathrm{d}r \times \mathrm{d}^2\boldsymbol{\omega}) = |\Psi^{(1)}(r, \boldsymbol{\omega})|^2\, A^{-1}\, r^2\, \mathrm{d}r\, \mathrm{d}^2\boldsymbol{\omega}\,. \tag{C.3.4b}$$

C.3.1 IBC Hamiltonian with Particle Creation

In order to formulate our first main result, the existence of the Hamiltonian, we use a certain orthonormal basis of $L^2(\mathbf{S}^2, \mathbf{C}^4, \mathrm{d}^2\boldsymbol{\omega})$ traditionally denoted $\Phi^\pm_{m_j,\kappa_j}$, where (m_j, κ_j) varies in the set

$$\mathscr{A} := \Big\{(m_j, \kappa_j) : \kappa_j \in \mathbf{Z} \setminus \{0\},\ m_j + \tfrac{1}{2} \in \mathbf{Z},\ |m_j| \le |\kappa_j| - \tfrac{1}{2}\Big\}\,. \tag{C.3.5}$$

Without going into details (see Appendix C.A or [559, Sec. 4.6.4] for the definition), we remark that the $\Phi^\pm_{m_j\kappa_j}$ are the joint eigenfunctions of $\boldsymbol{J}^2, J_z, K$, and β, viz.,

$$\boldsymbol{J}^2\Phi^\pm_{m_j\kappa_j} = j(j+1)\Phi^\pm_{m_j\kappa_j} \tag{C.3.6a}$$

$$J_3\Phi^\pm_{m_j\kappa_j} = m_j\Phi^\pm_{m_j\kappa_j} \tag{C.3.6b}$$

$$K\Phi^\pm_{m_j\kappa_j} = \kappa_j\Phi^\pm_{m_j\kappa_j} \tag{C.3.6c}$$

$$\beta\Phi^\pm_{m_j\kappa_j} = \pm\Phi^\pm_{m_j\kappa_j} \tag{C.3.6d}$$

with $j = |\kappa_j| - \frac{1}{2}$, where (again without going into details) $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is the triple of angular momentum operators, $\boldsymbol{L}$ the orbital angular momentum, $\boldsymbol{S}$ the spin angular momentum, and $K = \beta(2\boldsymbol{S}\cdot\boldsymbol{L} + 1)$ the spin-orbit operator.

We also note for use in the IBC (C.3.7) that since the β matrix has eigenvalues ± 1, the projection to the eigenspace with eigenvalue -1 is $\frac{1}{2}(I - \beta)$.

We define a Hamiltonian H in $\mathscr{H}$ for every choice of $(\widetilde{m}_j, \widetilde{\kappa}_j) \in \mathscr{A}$ and of a coupling constant $g \in \mathbf{C} \setminus \{0\}$; H acts on wave functions subject to the interior-boundary condition

$$\lim_{r \searrow 0} \tfrac{1}{2}(I - \beta) r^{1/2}\, \Psi^{(1)}(r, \boldsymbol{\omega}) = g\, |Q|^{-1/2} \Phi^-_{\widetilde{m}_j\widetilde{\kappa}_j}(\boldsymbol{\omega})\, \Psi^{(0)} \qquad \forall \boldsymbol{\omega} \in \mathbf{S}^2 \tag{C.3.7}$$

according to

$$(H\Psi)^{(0)} = g^*\, |Q|^{1/2} \lim_{r \searrow 0} \int_{\mathbf{S}^2} \mathrm{d}^2\boldsymbol{\omega}\, \Phi^+_{\widetilde{m}_j\widetilde{\kappa}_j}(\boldsymbol{\omega})^\dagger\, r^{1/2} \Psi^{(1)}(r, \boldsymbol{\omega}) \tag{C.3.8a}$$

$$(H\Psi)^{(1)}(r, \boldsymbol{\omega}) = H_1 \Psi^{(1)}(r, \boldsymbol{\omega}) \quad (r > 0) \tag{C.3.8b}$$

with H_1 the Dirac Hamiltonian as in (C.2.15).

Here is the precise statement about the IBC Hamiltonian H:

Theorem C.3.1 (IBC Hamiltonian with particle creation)**.** *For every choice of the parameters $(\widetilde{m}_j, \widetilde{\kappa}_j) \in \mathscr{A}$ and $g \in \mathbf{C} \setminus \{0\}$, there is a self-adjoint operator H with domain $D \subset \mathscr{H}$ such that*

1. *For every $\Psi \in D$, the upper sector is of the form*

$$\Psi^{(1)}(r, \boldsymbol{\omega}) = f(\boldsymbol{\omega})\, r^{-1/2} + \mathcal{O}(r^{1/2}) \tag{C.3.9}$$

   as $r \to 0$ for some (uniquely determined, Ψ-dependent) $f \in L^2(\mathbf{S}^2, \mathbf{C}^4, \mathrm{d}^2\boldsymbol{\omega})$. In particular, the limit on the left-hand side of (C.3.7) *exists and is the part of f in the eigenspace of β with eigenvalue -1.*

2. *Every $\Psi \in D$ satisfies the IBC* (C.3.7)*.*

3. *For every $\Psi^{(1)} \in C_c^\infty((0,\infty) \times \mathbf{S}^2, \mathbf{C}^4)$, $(0, \Psi^{(1)}) \in D$, and $H(0, \Psi^{(1)}) = (0, H_1 \Psi^{(1)})$ with H_1 as in* (C.2.15)*. Put differently, (H, D) is a self-adjoint extension of (H_1^0, D^0) with $D^0 = \{0\} \oplus C_c^\infty((0,\infty) \times \mathbf{S}^2, \mathbf{C}^4)$ and $H_1^0(0, \psi) = (0, H_1 \psi)$.*

4. *The 0-particle action of H is given by* (C.3.8a)*, which holds in the precise sense that*

$$(H\Psi)^{(0)} = g^* |Q|^{1/2} \langle \Phi^+_{\widetilde{m}_j \widetilde{\kappa}_j}, f \rangle_{L^2(\mathbf{S}^2, \mathbf{C}^4, \mathrm{d}^2\boldsymbol{\omega})}\,. \tag{C.3.10}$$

5. *Particle creation occurs, i.e., H is not block diagonal in the decomposition $\mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$.*

Theorem C.3.1 will follow as a special case of the slightly reformulated and more general Theorem C.5.1, formulated in Section C.5.1. We give the proof of Theorem C.5.1 in Section C.5.2.

Remark C.3.2 (Boundary conditions for the Dirac equation)**.** *While a boundary condition for the Laplacian usually specifies the value of ψ on the boundary (as in a Dirichlet boundary condition) or its normal derivative (as in a Neumann boundary condition), boundary conditions for the Dirac equation usually specify two of the four components of the wave function on the boundary, leaving the other two unspecified (e.g., [259]). Likewise, except for the scaling factor r (which has to do with how to extend the bundle S to the boundary [571, Sec. 5.3]),* (C.3.7) *specifies two of the four components of $\Psi^{(1)}$ at $r = 0$ (those in the eigenspace of β with eigenvalue -1), leaving the other two unspecified (those in the eigenspace of β with eigenvalue $+1$).*

Remark C.3.3 (Comparison to [571])**.** *In [571], one of us conjectured what a Hamiltonian on a sRN space-time with an IBC at the singularity and the corresponding Bohm-Bell process might look like. The description there was based on plausibility rather than rigorous analysis, but gets qualitatively confirmed by Theorem C.3.1 above. Since our proof technique for Theorem C.3.1 makes use of the angular momentum eigenspaces $\mathscr{K}_{m_j \kappa_j}$ spanned by $\Phi^+_{m_j \kappa_j}$ and $\Phi^-_{m_j \kappa_j}$, while the IBC and Hamiltonian in [571] were not related to these subspaces, the H provided by Theorem C.3.1 is not the same as the one described in [571], and we cannot answer whether the equations in [571] for the IBC and the action of the Hamiltonian do or do not define a self-adjoint operator. But the H of Theorem C.3.1 is similar to the one described in [571] in that (i) the IBC* (C.3.7)*, just as [571, (52)], concerns two components of the limiting values of $\Psi^{(1)}$ on the singularity, rescaled by $r^{1/2}$, and requires them to be $\Psi^{(0)}$ times a certain spinor function of $\boldsymbol{\omega}$; (ii) the expression* (C.3.8a) *for $(H\Psi)^{(0)}$, just as [571, last line of (53)], is the inner product over $\mathbf{S}^2$ of the rescaled $\Psi^{(1)}$ at the singularity with another spinor function of $\boldsymbol{\omega}$; and (iii) H acts like the Dirac Hamiltonian away from the singularity.*

Remark C.3.4 (Comparison to [344])**.** *In [344, Thm. 6], two of us proved the existence of a self-adjoint IBC Hamiltonian in flat Minkowski space-time under the assumption of a sufficiently strong Coulomb potential acting on the quantum particle. Some elements of the construction and the proofs were similar; some differences are that the asymptote* (C.3.9) *of $\Psi^{(1)}$ as $r \to 0$ had a different form involving different powers of r, thus requiring a different power of r in the IBC; that only few choices of $\widetilde{m}_j, \widetilde{\kappa}_j$ worked; and the IBC involved a different spinor field instead of $\Phi^-_{\widetilde{m}_j \widetilde{\kappa}_j}$.*

Remark C.3.5 (Full Fock space)**.** *Along the lines of [389], our construction could be extended to full Fock space $\mathscr{F}$ with an arbitrary number $n \in \{0, 1, 2, 3, \ldots\}$ of y-particles. For each value $\tau \in \mathbf{R}$ of the time coordinate t, let $\Sigma_\tau := \{t = \tau\}$ and the configuration space be $\mathcal{Q}_\tau := \bigcup_{n=0}^{\infty} \Sigma_\tau^n$. The boundary of configuration space consists of those configurations with at least one y-particle at $r = 0$, and the IBC will relate the n-particle sector $\psi^{(n)}$ of $\psi \in \mathscr{F}$ to the values of $\psi^{(n+1)}$ on the boundary.*

Remark C.3.6 (Multi-time wave functions)**.** *In relativistic space-time, it is usually possible and of interest to extend the domain of definition of wave functions so as to make them* multi-time *wave functions [421, 420], i.e., defined not only for simultaneous n-particle configurations but for any spacelike n-particle configuration or even any n-particle configuration at all. This is also possible for the present model, including states of arbitrary particle number n as in Remark C.3.5, but the x-particle, serving as the source at the singularity, needs to be taken into account: while it cannot occupy other locations than the origin $r = 0$, it should be given its own time variable t_x in a multi-time approach, leading to wave functions of the form*

$$\psi^{(n)}(t_x, t_1, r_1, \vartheta_1, \varphi_1, \ldots, t_n, r_n, \vartheta_n, \varphi_n)\,, \tag{C.3.11}$$

where $t_j, r_j, \vartheta_j, \varphi_j$ are the coordinates of the j-th y-particle. Since for multi-time wave functions, the space-time points of two interacting particles need to be spacelike separated, each $(t_j, r_j, \vartheta_j, \varphi_j)$ is constrained to the region spacelike from $(t_x, r = 0)$ (shaded in Figure C.1). In fact, for any n points in this region, the function (C.3.11) *is uniquely determined from the n-particle wave function on Σ_{t_x} (provided by the single-time evolution) as the solution of the free Dirac equation in each $(t_j, r_j, \vartheta_j, \varphi_j)$ away from the singularity.*

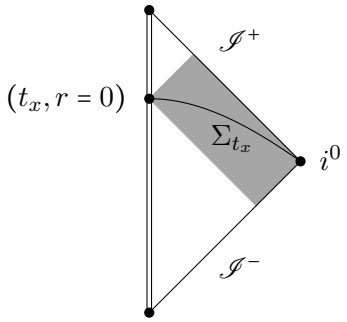

Figure C.1: Penrose conformal diagram of sRN space-time $\mathscr{M}$, shown with the spacelike coordinate surface $\Sigma_{t_x} = \{t = t_x\}$ bordering on the point $(t_x, r = 0)$ on the singularity $\partial\mathscr{M} = \{r = 0\}$ (shown as the vertical double line); the value of t_x was chosen arbitrarily; $\mathscr{I}^\pm$ is the future (past) null infinity, i^0 is the spacelike infinity, and the shaded region comprises the points spacelike separated from $(t_x, r = 0)$.

C.3.2 Bohmian Trajectories and Bohm-Bell Jump Process

In Theorem C.3.1, we constructed a self-adjoint Hamiltonian involving the creation of Dirac particles at the sRN singularity using an IBC (C.3.7). In this section, we construct a Markov process Q_t (the "Bohm-Bell process") in the configuration space $\mathcal{Q}$ as in (C.3.1) that is Born ("$|\Psi_t|^2$") distributed at every $t \in \mathbf{R}$. Our approach is analogous to "Bell-type quantum field theory" [58] in which motion of the configuration along deterministic trajectories are interrupted by stochastic jumps. That is, Q_t follows Bohmian trajectories between the jumps, and the latter correspond to the creation/annihilation of particles. A key element of this construction is to determine the rate of particle creation that ensures equivariance of the process (i.e., preservation of the Born distribution), and to this end it is relevant to determine the asymptotic Bohmian trajectories near the singularity for this Hamiltonian.

C.3.2.1 Bohmian Trajectories

We now review the definition of the Bohmian trajectories and determine the coordinate form of their equation for our setup.

The Bohmian trajectories $X : \mathbf{R} \to \mathscr{M}$ are solutions to Bohm's equation of motion [90] for the Dirac equation [91], given by

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}s} \propto j^\mu(X(s))\,, \tag{C.3.12}$$

where s is any curve parameter and j^μ given by (C.2.7). In words, the world line is everywhere tangent to the vector field j^μ.

We now want to express the equation of motion in coordinates. A subtle point is that there are two relevant bases in the tangent space $T_x\mathscr{M}$ in which $j(x)$ can be expressed: the coordinate basis $(\partial_t, \partial_r, \partial_\vartheta, \partial_\varphi)$ and the basis e_x of (C.2.8) (the normalized coordinate basis). We write $(j^t, j^r, j^\vartheta, j^\varphi)$ for the components of $j(x)$ relative to the former and (j^0, j^1, j^2, j^3) for those relative to the latter,

$$j(x) = j^t\partial_t + j^r\partial_r + j^\vartheta\partial_\vartheta + j^\varphi\partial_\varphi \tag{C.3.13a}$$

$$j(x) = j^0 e_x^0 + j^1 e_x^1 + j^2 e_x^2 + j^3 e_x^3\,. \tag{C.3.13b}$$

One can read off from (C.2.8) that $j^t = j^0 A^{-1}$, $j^r = j^1 A$, $j^\vartheta = j^2 r^{-1}$, and $j^\varphi = j^3(r\sin\vartheta)^{-1}$. Since the world line X is tangent to the vector field j on $\mathscr{M}$, the image of the world line in coordinate space with axes t, r, ϑ, φ is tangent to the image of j, which has components $(j^t, j^r, j^\vartheta, j^\varphi)$. Therefore, (C.3.12) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r(t) \\ \vartheta(t) \\ \varphi(t) \end{pmatrix} = \boldsymbol{v}\big(t, r(t), \vartheta(t), \varphi(t)\big) \tag{C.3.14}$$

with

$$v^1 = \frac{j^r}{j^t} = \frac{j^1 A}{j^0 A^{-1}} = A^2 \frac{(\Psi^{(1)})^\dagger \alpha^1 \Psi^{(1)}}{|\Psi^{(1)}|^2} \tag{C.3.15a}$$

$$v^2 = \frac{j^\vartheta}{j^t} = \frac{j^2 r^{-1}}{j^0 A^{-1}} = \frac{A}{r}\frac{(\Psi^{(1)})^\dagger \alpha^2 \Psi^{(1)}}{|\Psi^{(1)}|^2} \tag{C.3.15b}$$

$$v^3 = \frac{j^\varphi}{j^t} = \frac{j^3(r\sin\vartheta)^{-1}}{j^0 A^{-1}} = \frac{A}{r\sin\vartheta}\frac{(\Psi^{(1)})^\dagger \alpha^3 \Psi^{(1)}}{|\Psi^{(1)}|^2}\,. \tag{C.3.15c}$$

C.3.2.2 Asymptotics of Bohmian Trajectories

We now determine the asymptotic form of the trajectories just before reaching (or after emanating from) the singularity. In the following, we assume that

$$\widetilde{\kappa}_j = \pm 1\,, \tag{C.3.16}$$

and we will only consider wave functions Ψ lying in a certain subspace $\widehat{D} \subset \mathscr{H}$ which is invariant under the time evolution generated by H. More precisely, we take $\widehat{D}$ to be the part of the domain D of H whose 1-particle component $\Psi^{(1)}$ has angular momentum corresponding to the chosen $(\widetilde{m}_j, \widetilde{\kappa}_j) \in \mathscr{A}$. In the notation of Sections C.4.2 and C.5.2.1,

$$\widehat{D} := (1 \oplus U^{-1})\widehat{D}_{\widetilde{m}_j\widetilde{\kappa}_j} \tag{C.3.17}$$

involving the unitary transformation U as in (C.4.13) and the subspace $\widehat{D}_{\widetilde{m}_j\widetilde{\kappa}_j}$ as in (C.5.13). As becomes apparent from the proof of Theorem C.3.1 in Section C.5, the coupling between $\mathscr{H}^{(0)}$

and $\mathscr{H}^{(1)}$ happens only within $\widehat{D}$, hence $\widehat{D}$ is the most relevant or interesting part of D. Thus, by focusing on $\widehat{D}$, we avoid unnecessarily tedious computation for extracting the qualitative behavior, which we believe will not change much for general $\Psi \in D \setminus \widehat{D}$; cf. [345].

Moreover, in the following asymptotic analysis of the Bohmian trajectories, we will also make use of a (plausible and common [229, 345]) approximation for Bohm's equation of motion: We assume that the Bohmian velocity field $\boldsymbol{v}$ as in (C.3.15) varies slowly in time. More specifically, we assume that for times t close to the reference time $t_0 \in \mathbf{R}$, the asymptotics of the true Bohmian trajectories as solutions of (C.3.14) coincide (to leading order) with those one would obtain from a time-independent velocity field $\boldsymbol{v}(t_0, \cdot)$, i.e., with solutions of

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r(t) \\ \vartheta(t) \\ \varphi(t) \end{pmatrix} = \boldsymbol{v}\big(t_0, r(t), \vartheta(t), \varphi(t)\big)\,. \tag{C.3.18}$$

This approximation corresponds to approximating $\Psi_t \approx \Psi_{t_0}$ in a suitable topology; see [345, Remark 1] for a possible general strategy of rigorously justifying it.

This is our main result on the asymptotics of the Bohmian trajectories.

Proposition C.3.7 (Asymptotics of Bohmian trajectories)**.** *Let $\widetilde{\kappa}_j = \pm 1$, $\Psi_0 \in \widehat{D}$, denote the time-evolved state by $\Psi_t := \mathrm{e}^{-\mathrm{i}Ht}\Psi_0 \in \widehat{D}$ and write*

$$c_\pm(t) = |Q|^{1/2}\langle \Phi^\pm_{\widetilde{m}_j\widetilde{\kappa}_j}, f_t\rangle = |Q|^{1/2}\lim_{r\searrow 0}\int_{\mathbf{S}^2} \mathrm{d}^2\boldsymbol{\omega}\, \Phi^\pm_{\widetilde{m}_j\widetilde{\kappa}_j}(\boldsymbol{\omega})^\dagger\, r^{1/2}\, \Psi_t^{(1)}(r,\boldsymbol{\omega})\,, \tag{C.3.19}$$

where f_t is the analog of f from (C.3.9) *obtained from Ψ_t. Let $t_0 \in \mathbf{R}$ be any time for which*

$$\mathrm{Im}\,[c_-^*(t_0)\, c_+(t_0)] \neq 0 \tag{C.3.20}$$

and abbreviate $c_\pm := c_\pm(t_0)$.

Then the solution to (C.3.18) *with $r(t_0) = 0$, i.e., the trajectories emanating from/reaching the singularity at time t_0, occur only if $\mathrm{Im}\,[c_-^* c_+] < 0$ (resp. $\mathrm{Im}\,[c_-^* c_+] > 0$) and they obey for $t > t_0$ (resp. $t < t_0$) the following asymptotics as $t \to t_0$:*

$$r(t) = C_{\mathrm{rad}}\,|t-t_0|^{1/3} + \mathcal{O}\Big(|t-t_0|^{2/3}\Big) \tag{C.3.21a}$$

$$\vartheta(t) = \vartheta_0 + \mathcal{O}\Big(|t-t_0|^{2/3}\Big) \tag{C.3.21b}$$

$$\varphi(t) = \varphi_0 + \mathrm{sgn}(t-t_0)\, C_{\mathrm{az}}\,|t-t_0|^{1/3} + \mathcal{O}\Big(|t-t_0|^{2/3}\Big) \tag{C.3.21c}$$

for some constants $\vartheta_0 \in [0,\pi]$ and $\varphi_0 \in [0, 2\pi)$ and with coefficients

$$C_{\mathrm{rad}} = \left(\frac{6Q^2\,|\mathrm{Im}\,[c_-^* c_+]|}{|c_+|^2 + |c_-|^2}\right)^{1/3} \tag{C.3.22a}$$

$$C_{\mathrm{az}} = \frac{6^{1/3}\mathrm{sgn}(Q\widetilde{m}_j\widetilde{\kappa}_j)\,\mathrm{Re}\,[c_-^* c_+]}{|Q|^{1/3}(|c_+|^2+|c_-|^2)^{1/3}|\mathrm{Im}\,[c_-^* c_+]|^{2/3}}\,. \tag{C.3.22b}$$

Moreover,

$$\varphi(r) = \varphi_0 - \frac{\mathrm{sgn}(\widetilde{m}_j\widetilde{\kappa}_j)}{Q}\,\frac{\mathrm{Re}\,[c_-^* c_+]}{\mathrm{Im}\,[c_-^* c_+]}\,r + \mathcal{O}(r^2) \tag{C.3.23}$$

as $r \to 0$.

The proof is given in Section C.6.2. Note that the denominators in (C.3.22) and (C.3.23) are nonzero by (C.3.20) and (C.1.1).

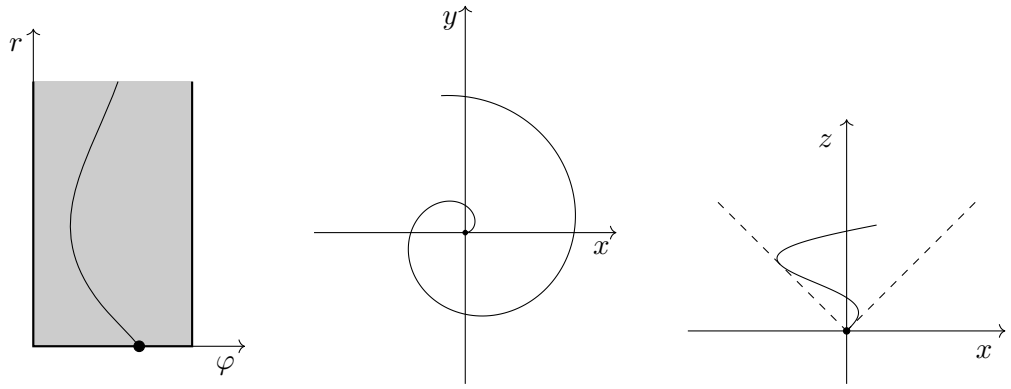

Figure C.2: Illustrated is a Bohmian trajectory shortly before/after absorption/emission, asymptotically obeying (C.3.21); the figure is analogous to [345, Figure 3] but shows quite different behavior. LEFT: Drawn in spherical coordinates, with only the azimuthal angle φ shown; to leading order near $r = 0$, $\varphi(r) = \varphi_0 + Cr$ as in (C.3.23); the dot marks $(r = 0, \varphi_0)$. MIDDLE: The curve $\varphi(r) = \varphi_0 + Cr$ drawn in 2d cartesian coordinates. RIGHT: The curve $\varphi(r) = \varphi_0 + Cr$, $\vartheta = \vartheta_0$ drawn in 3d cartesian coordinates, seen along the y-axis. Dashed is the cone $\{\vartheta = \vartheta_0\}$.

C.3.2.3 Bohm-Bell Jump Process

We now give the definition of the Bohm-Bell jump process $(Q_t)_{t\geq 0}$ in $\mathcal{Q}$ assuming $\widetilde{\kappa}_j = \pm 1$ and $\Psi_0 \in \widehat{D}$. It is a Markov process with the following structure (similar to the ones considered in [571, 229, 345]):

Initial Distribution. The initial configuration Q_0 has probability distribution given by the Born distribution $\mathbf{P}_{\Psi_0}$ as in (C.3.4).

Deterministic Evolution by Bohm's Equation of Motion. At any time t at which Q_t lies in the upper sector $\mathcal{Q}^{(1)}$, it moves according to Bohm's equation of motion (C.3.14); that is, the world line is tangent to the 4-vector field j^μ.

Deterministic Jumps. When Q_t reaches the singularity $r = 0$ at time t_0, it jumps to the lower sector, $Q_{t_{0^+}} = \varnothing$, and stays there for some time interval.

Stochastic Jumps. When Q_t sits in the lower sector, it jumps to a trajectory leaving the singularity with a certain jump rate. The general formula for the rate of jumping at time t, given that $Q_{t-} = q'$, to anywhere in an infinitesimal set $\mathrm{d}q$ can be derived [281] to be

$$\sigma_t(q' \to \mathrm{d}q) = \frac{\max\{0, J^\perp_{\Psi_t}(q)\}}{\rho_{\Psi_t}(q')} \nu(\mathrm{d}q, q')\,, \tag{C.3.24}$$

where $J^\perp$ is the component of probability current in coordinates orthogonal to the boundary of configuration space (in our case, the radial component), ρ is the probability density and ν the surface area measure on the part of the boundary allowed for jumps from q'. In our case, only $q' = \varnothing$ can occur, $\rho_{\Psi_t}(q') = |\Psi_t^{(0)}|^2$, and $\nu(\cdot, \varnothing)$ is the surface area measure on $\mathbb{S}^2$ [571]. The trajectory onto which to jump gets characterized by the boundary point q at which it starts; in our case, q lies on the boundary $\{0\} \times \mathbf{S}^2$ of $[0, \infty) \times \mathbf{S}^2$ and thus represents the *direction of emission*. As we will show in Section C.6.3, (C.3.24) asserts in our case that the rate of jumping to a point q in the surface element $\{0\} \times \mathrm{d}^2\boldsymbol{\omega}$ is

$$\sigma_t(\varnothing \to \mathrm{d}^2\boldsymbol{\omega}) = \frac{\max\{0, -\mathrm{Im}[c_-^*(t)c_+(t)]\}}{2\pi|Q|\left|\Psi_t^{(0)}\right|^2} \mathrm{d}^2\boldsymbol{\omega} \tag{C.3.25}$$

with $c_\pm(t)$ from (C.3.19). The total jump rate (or the rate of leaving $\varnothing$) at t is thus given by

$$\sigma_t(\varnothing \to \mathbf{S}^2) = \int_{\boldsymbol{\omega}\in\mathbf{S}^2} \sigma_t(\varnothing \to \mathrm{d}^2\boldsymbol{\omega}) = 2\frac{\max\{0, -\mathrm{Im}[c_-^*(t)c_+(t)]\}}{|Q|\left|\Psi_t^{(0)}\right|^2}\,. \tag{C.3.26}$$

As we elucidate in Section C.6.3, the rate (C.3.25) ensures equivariance. Since the fraction in (C.3.25) does not depend on $\boldsymbol{\omega}$, the probability distribution of the direction of emission, given that a jump occurs at t, is uniform over the sphere.

This completes the definition of the process. We briefly note that the description just given agrees with what was conjectured in [571] about the form of the Bohm-Bell process (except that the IBC considered there is not the same as our (C.3.7)). We conclude this section with two remarks.

Remark C.3.8 (Negative times)**.** *The definition can be extended to provide a process $(Q_t)_{t\in\mathbf{R}}$ also for negative times by choosing the initial time to be any t_0 instead of 0, noting that different choices of t_0 are compatible with each other (in the sense that the two processes are equal in distribution after the later of the two choices of t_0), and letting $t_0 \to -\infty$.*

Remark C.3.9 (Foliation)**.** *We define the process relative to the foliation given by the Reissner-Nordström time coordinate, but the random path in space-time is actually indendent of the choice of the foliation. The situation will be different for more than 1 y-particle [228].*

C.3.3 Structure of the Following Sections

The rest of the paper is devoted to proving Theorem C.3.1 and justifying our claims on the trajectories and the jump process from Section C.3.2, in particular proving Proposition C.3.7. In order to do so, we first recall some preliminaries in Section C.4. Afterwards, in Section C.5 we construct the IBC Hamiltonian and thus prove Theorem C.3.1.The following Section C.6 deals with the trajectories and the jump process. The ultimate Section C.7 contains some concluding remarks.

C.4 Preparation of Proofs: Symmetries and Transformations

In this section we gather some preliminary information regarding the Dirac Hamiltonian in the sRN background. The principal goal of this section is to transform the Hamiltonian in a simple form, thereby exploiting its built-in spherical symmetry (see Section C.4.1 and [175, 559, 56, 365]) and a convenient scalar change of variables (see Section C.4.2). We follow mostly Cohen and Powers [175] and Thaller [559].

C.4.1 Radial Symmetry: Hilbert Space Decomposition

We write the Hilbert space $\mathscr{H}^{(1)}$, given by (C.2.11), in the form

$$\mathscr{H}^{(1)} = L^2\big((0,\infty), \mathbf{C}, A^{-1}r^2\mathrm{d}r\big) \otimes L^2\big(\mathbf{S}^2, \mathbf{C}^4, \mathrm{d}^2\boldsymbol{\omega}\big)\,. \tag{C.4.1}$$

As a consequence of its rotational symmetry, H_1 leaves angular momentum eigenspaces invariant; in particular, it leaves the specific eigenspaces $L^2((0,\infty), \mathbf{C}, A^{-1}r^2\mathrm{d}r) \otimes \mathscr{K}_{m_j\kappa_j}$ invariant, where

$$\mathscr{K}_{m_j,\kappa_j} = \left\{c^+\Phi^+_{m_j,\kappa_j} + c^-\Phi^-_{m_j,\kappa_j} \;:\; c^\pm \in \mathbf{C}\right\} \tag{C.4.2}$$

and the $\Phi^\pm_{m_j\kappa_j}$ form an ONB of $L^2(\mathbf{S}^2, \mathbf{C}^4, \mathrm{d}^2\boldsymbol{\omega})$ given explicitly in Appendix C.A. As a consequence, with respect to the decomposition

$$L^2(\mathbf{S}^2, \mathbf{C}^4, \mathrm{d}^2\boldsymbol{\omega}) = \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{\kappa_j=\pm(j+\frac{1}{2})} \mathscr{K}_{m_j,\kappa_j}\,, \tag{C.4.3}$$

H_1 is block diagonal,

$$H_1 = \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{\kappa_j=\pm(j+\frac{1}{2})} H^{\text{red}}_{1 m_j \kappa_j} . \tag{C.4.4}$$

We consider each block $H^{\text{red}}_{1 m_j \kappa_j}$ individually. Relative to the basis $\{\Phi^+_{m_j\kappa_j}, \Phi^-_{m_j\kappa_j}\}$, it can be written as a 2×2 matrix whose entries are operators acting on the radial Hilbert space $L^2((0,\infty), \mathbf{C}, A^{-1} r^2 \mathrm{d}r)$, in fact

$$H^{\text{red}}_{1 m_j \kappa_j} = \begin{bmatrix} qQr^{-1} + mA & -A^2(\partial_r + \frac{1}{r}) - \dfrac{AA'}{2} + \dfrac{\kappa_j A}{r} \\ A^2(\partial_r + \frac{1}{r}) + \dfrac{AA'}{2} + \dfrac{\kappa_j A}{r} & qQr^{-1} - mA \end{bmatrix} . \tag{C.4.5}$$

(Recall that A is a function of r and A' its derivative.) To see this, we note first that the operators α^1 (which in our spinor basis b_x is the α associated with the radial direction) and β (and thus also $\gamma^1 = \beta\alpha^1$) leave the subspaces $\mathscr{K}_{m_j\kappa_j}$ invariant; with respect to the basis $\{\Phi^+_{m_j\kappa_j}, \Phi^-_{m_j\kappa_j}\}$, they take the form

$$\alpha^1 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} . \tag{C.4.6}$$

Second, we note that [175, (2.9)]

$$\gamma^1 K = -\alpha^2 \partial_\vartheta - \alpha^3 (\sin\vartheta)^{-1} \partial_\varphi . \tag{C.4.7}$$

With these relations and (C.3.6), (C.4.5) follows from (C.2.15).

C.4.2 The R Coordinate

It turns out useful to change coordinates from r to the "tortoise coordinate" $R(r)$, defined for $r \geq 0$ to be the solution to

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{1}{A(r)^2} \quad \text{with} \quad R(r=0) = 0 \tag{C.4.8}$$

and A from (C.1.3) (see Figure C.1); R is called x in [175, 55, 56] and r^* in [327]. The physical meaning of R of a space-time point x is the length of the radial spacelike curve along $\{t = \text{const.}\}$ connecting x to $\{r = 0\}$, or the metric distance of x from the singularity (while the r coordinate is set up so that $4\pi r^2$ is the area of the sphere through x obtained by varying ϑ and φ). Although we do not need the explicit form of the solution, we mention that it is given by [55, 56]

$$R(r) = r + M \log\left(\frac{r^2 - 2Mr + Q^2}{Q^2}\right) + \frac{2M^2 - Q^2}{\sqrt{Q^2 - M^2}} \arctan\left(\frac{r - M}{\sqrt{Q^2 - M^2}}\right) + C \tag{C.4.9}$$

with suitable integration constant C.[4]

Lemma C.4.1 (The R-coordinate transformation)**.** *Let $R, r > 0$ be related by* (C.4.8)*. Then*

$$\lim_{R\to 0} R^{-1/3}\, r(R) = (3Q^2)^{1/3} \tag{C.4.10a}$$

$$\lim_{R\to 0} r^2(R)\, A^2(r(R)) = Q^2 . \tag{C.4.10b}$$

In particular, as $R \to 0$ (or, equivalently, $r \to 0$),

$$R \sim r^3 \tag{C.4.11}$$

and

$$A^2(r(R)) \sim R^{-2/3}. \tag{C.4.12}$$

[4] The expression given in [327, p. 157] has incorrect constant prefactors.

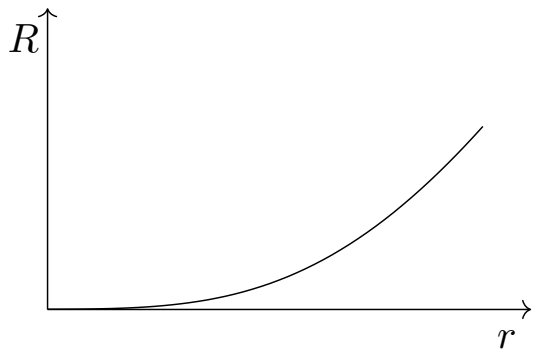

Figure C.1: Graph of the function $R(r)$ defined in (C.4.8) and given explicitly in (C.4.9) for $M = 1$ and $Q = 2$; in this case, $C \approx -0.601$.

Proof. This follows from (C.4.8) and the definition of $A(r)$. □

Next, we make use of the R coordinate to define a unitary transformation U of the radial Hilbert space as in [175]:

$$\begin{aligned} U : L^2((0,\infty), \mathbf{C}, A^{-1} r^2 \mathrm{d}r) &\to L^2((0,\infty), \mathbf{C}, \mathrm{d}R) \\ \Psi(r) &\mapsto \phi(R) := r(R)\, A(r(R))^{1/2}\, \Psi(r(R)) \end{aligned} \tag{C.4.13}$$

It is unitary because $\mathrm{d}R = A^{-2}\,\mathrm{d}r$, so $|\phi|^2 \mathrm{d}R = r^2 A |\Psi|^2 A^{-2} \mathrm{d}r = |\Psi|^2 A^{-1} r^2 \mathrm{d}r$.

The main advantage of introducing the R coordinate is that it removes the A^2-factor in front of the differential operator in (C.4.5):

Lemma C.4.2 (Transformed Hamiltonian)**.** *Under the unitary transformation U defined by* (C.4.13)*, the reduced Hamiltonian acting on each subspace is given by $h_{m_j,\kappa_j} = U H^{\mathrm{red}}_{1 m_j \kappa_j} U^{-1}$ with*

$$h_{m_j,\kappa_j} = \begin{bmatrix} qQr(R)^{-1} + mA(r(R)) & -\partial_R + \kappa_j A(r(R)) r(R)^{-1} \\ \\ \partial_R + \kappa_j A(r(R)) r(R)^{-1} & qQr(R)^{-1} - mA(r(R)) \end{bmatrix}, \tag{C.4.14}$$

which is well defined on the domain

$$D(h_{m_j,\kappa_j}) = C_0^\infty\big((0,\infty), \mathscr{K}_{m_j,\kappa_j}\big) \cong C_0^\infty\big((0,\infty), \mathbf{C}^2\big)\,. \tag{C.4.15}$$

Proof. This follows from

$$\partial_R(U(\Psi)) = U\bigg(\Big(A^2 \partial_r + \frac{A^2}{r} + \frac{AA'}{2}\Big)\Psi\bigg), \tag{C.4.16}$$

which can be easily verified using $\partial_R = A^2 \partial_r$. □

Remark C.4.3. *From* (C.3.21a) *and* (C.4.10a)*, it follows that*

$$R(t) = \frac{C_{\mathrm{rad}}^3}{3Q^2}\,|t - t_0| + \mathcal{O}\Big(|t - t_0|^{4/3}\Big)\,. \tag{C.4.17}$$

C.5 Constructing an IBC: Proof of Theorem C.3.1

In this section, we construct an IBC Hamiltonian with particle creation and thereby prove Theorem C.3.1.

C.5.1 A Family of IBC Hamiltonians: Proof of Theorem C.3.1

Our main result, Theorem C.3.1, will be directly obtained from the following slightly reformulated and generalized version of it. Recall that, the Hilbert space of our model is the mini-Fock space $\mathscr{H} = \mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$ with $\mathscr{H}^{(0)} = \mathbf{C}$ and $\mathscr{H}^{(1)}$ given by (C.2.11).

Theorem C.5.1 (Generalized reformulation of Theorem C.3.1)**.** *For every $(\widetilde{m}_j, \widetilde{\kappa}_j) \in \mathscr{A}$, $g \in \mathbf{C} \setminus \{0\}$, and real numbers $a_1, ..., a_4 \in \mathbf{R}$ satisfying $a_1 a_4 - a_2 a_3 = 1$, there is a self-adjoint operator H with domain $D \subset \mathscr{H}$ such that*

1. *For every $\Psi \in D$, the upper sector is of the form*

$$\Psi^{(1)}(r, \omega) = \frac{c_-}{|Q|^{1/2}} r^{-1/2} \Phi^-_{\widetilde{m}_j, \widetilde{\kappa}_j}(\omega) + \sum_{(m_j, \kappa_j) \in \mathscr{A}} \frac{c_{+m_j \kappa_j}}{|Q|^{1/2}} r^{-1/2} \Phi^+_{m_j, \kappa_j}(\omega) + \mathcal{O}(r^{1/2}) \quad \text{(C.5.1)}$$

   as $r \to 0$ for some (uniquely determined) short-distance coefficients *$c_-, c_{+m_j \kappa_j} \in \mathbf{C}$ and $\Phi^\pm$ from* (C.A.5)*.*

2. *Every $\Psi \in D$ satisfies the IBC*

$$a_1 c_- + a_2 c_{+\widetilde{m}_j \widetilde{\kappa}_j} = g \Psi^{(0)} \quad \text{(C.5.2)}$$

3. *For every $\Psi^{(1)} \in C_c^\infty((0, \infty) \times \mathbf{S}^2, \mathbf{C}^4)$, $(0, \Psi^{(1)}) \in D$, and $H(0, \Psi^{(1)}) = (0, H_1 \Psi^{(1)})$ with H_1 as in* (C.2.15)*. Put differently, (H, D) is a self-adjoint extension of (H_1^0, D^0) with $D^0 = \{0\} \oplus C_c^\infty((0, \infty) \times \mathbf{S}^2, \mathbf{C}^4)$ and $H_1^0(0, \psi) = (0, H_1 \psi)$.*

4. *The 0-particle action of H is given by*

$$(H\Psi)^{(0)} = g^*(a_3 c_- + a_4 c_{+\widetilde{m}_j \widetilde{\kappa}_j}) \quad \text{(C.5.3)}$$

5. *Particle creation occurs, i.e., H is not block diagonal in the decomposition $\mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$.*

Proof of Theorem C.3.1. Theorem C.3.1 follows from Theorem C.5.1 by taking $a_1, a_4 = 1$, $a_2, a_3 = 0$, and invoking the particular form of $\Phi^\pm$ from (C.A.5) and $\beta = \mathrm{diag}(1, 1, -1, -1)$. In particular, the IBC Hamiltonian presented in Theorem C.3.1 is in fact part of an entire family of Hamiltonians described by the four real parameters $a_1, ..., a_4$ under the constraint $a_1 a_4 - a_2 a_3 = 1$. This concludes the proof of Theorem C.3.1. □

The rest of this section is devoted to proving Theorem C.5.1.

Remark C.5.2 (Outline of the proof of Theorem C.5.1)**.** *In constructing the self-adjoint H in Theorem C.5.1, we will decompose the domain D^0 into fixed angular momentum sectors $\mathscr{K}_{m_j, \kappa_j}$ as in Section C.4. That is, we will exploit that D^0 is unitarily equivalent to*

$$\{0\} \oplus \bigoplus_{j, m_j, \kappa_j} C_c^\infty((0, \infty), \mathbf{C}, \mathrm{d}R) \otimes \mathscr{K}_{m_j, \kappa_j} \,. \quad \text{(C.5.4)}$$

The construction of H now proceeds separately for each sector. We couple one chosen angular momentum sector $\mathscr{K}_{\widetilde{m}_j, \widetilde{\kappa}_j}$ to the 0-particle sector $\mathscr{H}^{(0)}$ of $\mathscr{H}$ while all the other angular momentum sectors do not couple to the 0-particle part. In particular, H is block diagonal relative to the decomposition

$$\mathscr{H} \cong \widehat{\mathscr{H}} \oplus \bigoplus_{(m_j, \kappa_j) \neq (\widetilde{m}_j, \widetilde{\kappa}_j)} L^2((0, \infty), \mathbf{C}, \mathrm{d}R) \otimes \mathscr{K}_{m_j, \kappa_j} \,, \quad \text{(C.5.5)}$$

but not relative to

$$\widehat{\mathscr{H}} = \mathscr{H}^{(0)} \oplus L^2((0, \infty), \mathbf{C}, \mathrm{d}R) \otimes \mathscr{K}_{\widetilde{m}_j, \widetilde{\kappa}_j} \quad \text{(C.5.6)}$$

In the proof, we construct a self-adjoint $\widehat{H}$ acting on $\widehat{\mathscr{H}}$ using interior boundary conditions. This is done by connecting the near-origin behavior of functions in the adjoint *domain of $C_c^\infty((0,\infty),\mathbf{C},\mathrm{d}R)\otimes \mathscr{K}_{\widetilde{m}_j,\widetilde{\kappa}_j}$ (see [175, Theorem 5.2]) to the 0-particle sector $\mathscr{H}^{(0)}$. In a similar way, we will choose self-adjoint extensions of h_{m_j,κ_j} on $C_c^\infty((0,\infty),\mathbf{C},\mathrm{d}R)\otimes\mathscr{K}_{m_j,\kappa_j}$ for $(m_j,\kappa_j)\neq(\widetilde{m}_j,\widetilde{\kappa}_j)$ which do not couple to $\mathscr{H}^{(0)}$. This completes the construction of a self-adjoint H.*

C.5.2 Proof of Theorem C.5.1

Throughout the entire proof of Theorem C.5.1, we will heavily use the change of variables from Section C.4.2, i.e., use the coordinate R instead of the usual radial variable r, which amounts to the unitary transformation in (C.4.13). Moreover, as a preparation of our proof, we state and prove the following lemma concerning the asymptotic behavior of wave functions ϕ in the adjoint domain $D(h^*_{m_j\kappa_j})$ of $D(h_{m_j\kappa_j})$ from (C.4.14)–(C.4.15)

Lemma C.5.3. *Let $\phi=(\phi_+,\phi_-)\in D(h^*_{m_j\kappa_j})$. Then ϕ is continuous at $R=0$, i.e., $\lim_{R\to 0}\phi_\pm(R)=\phi_\pm(0)$ exists, and*

$$\phi_\pm(R)=\phi_\pm(0)+\mathcal{O}(R^{1/3})\quad \text{as}\quad R\to 0\,. \tag{C.5.7}$$

Proof. In [175, Lemma 5.1], Cohen and Powers prove that the functions in $D(h^*_{m_j\kappa_j})$ are continuous at $R=0$. Here, we obtain more precise information on their asymptotics. From [175, Eq. (5.3)], $\phi_\pm(R)$ can be expressed as

$$\phi_+(R)=\mathrm{e}^{\eta(R)}\Big(\phi_+(0)-\int_0^R \mathrm{e}^{-\eta(y)}\Big(h_2(y)-\big(m-v_2(y)\big)\phi_-(y)\Big)\mathrm{d}y\Big) \tag{C.5.8a}$$

$$\phi_-(R)=\mathrm{e}^{-\eta(R)}\Big(\phi_-(0)+\int_0^R \mathrm{e}^{\eta(y)}\Big(h_1(y)-\big(m+v_1(y)\big)\phi_+(y)\Big)\mathrm{d}y\Big)\,, \tag{C.5.8b}$$

where we denoted $h^*_{m_j\kappa_j}\phi=(h_1,h_2)$ and

$$u(R)=\frac{\kappa_j A(r(R))}{r(R)} \tag{C.5.9a}$$

$$\eta(R)=\int_0^R u(y)\mathrm{d}y \tag{C.5.9b}$$

$$v_1(R)=qQr(R)^{-1}+mA(r(R))-m \tag{C.5.9c}$$

$$v_2(R)=qQr(R)^{-1}-mA(r(R))+m \tag{C.5.9d}$$

By Lemma C.4.1, we have, asymptotically as $R\to 0^+$, $v_i(R)\sim R^{-1/3}$ and $u(R)\sim R^{-2/3}$. Hence, $\eta(R)=\mathcal{O}(R^{1/3})$ and further $\mathrm{e}^{\eta(R)}=1+\mathcal{O}(R^{1/3})$. Now we show that the integral term in (C.5.8a) contributes only $\mathcal{O}(R^{1/2})$ by estimating the three summands in the integral in (C.5.8a) separately.

First, note that since $\phi_-(y)$ is a continuous function on the compact interval $[0,R]$, it is bounded. Moreover, since η is also bounded, we find that $\int_0^R \mathrm{d}y|\mathrm{e}^{-\eta(y)}m\phi_-(y)|=\mathcal{O}(R)$. Next, as $v_2(R)=\mathcal{O}(R^{-1/3})$, we obtain $\int_0^R \mathrm{d}y|\mathrm{e}^{-\eta(y)}\,v_2(y)\,\phi_-(y)|=\mathcal{O}(R^{2/3})$. It thus remains to estimate $\int_0^R \mathrm{d}y\,\mathrm{e}^{-\eta(y)}\,h_2(y)$.

Instead of the previous L^∞-bounds on the other integrands, we now apply the Cauchy-Schwarz inequality to get

$$\left|\int_0^R \mathrm{e}^{-\eta(y)}h_2(y)\mathrm{d}y\right|\le \|\mathrm{e}^{-\eta}\|_{L^2[0,R]}\|h_2\|_{L^2[0,R]}=\mathcal{O}(R^{1/2})\,, \tag{C.5.10}$$

where we used that $\|\mathrm{e}^{-\eta}\|_{L^2[0,R]}=\mathcal{O}(R^{1/2})$, since η is bounded, and $h_2\in L^2[0,\infty)$, which certainly implies $\|h_2\|_{L^2[0,R]}=\mathcal{O}(1)$.[5]

[5] By the dominated convergence theorem, this can in fact be strengthened to $\|h_2\|_{L^2[0,R]}=o(1)$ as $R\to 0$, but we do not follow this improvement for simplicity.

Combining all the estimates above, we finally conclude that

$$\phi_+(R) = \phi_+(0) + \mathcal{O}(R^{1/3}) \quad \text{as} \quad R \to 0 \tag{C.5.11}$$

as desired. Similarly, we also get $\phi_-(R) = \phi_-(0) + \mathcal{O}(R^{1/3})$ as $R \to 0$. □

Armed with Lemma C.5.3, we can now turn to the actual proof of Theorem C.5.1. This is divided in three steps:

(i) First, in Section C.5.2.1, we define the domain D of H and show that every $\Psi \in D$ satisfies the asymptotics in (C.5.1) and obeys the IBC (C.5.2).

(ii) In Section C.5.2.2, we then proceed to show that H acting as in items 3 and 4 of Theorem C.5.1 is in fact self-adjoint on D.

(iii) Finally, in Section C.5.2.3, we prove that particle creation occurs with the so defined Hamiltonian, i.e., it is not block diagonal in the decomposition $\mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$.

C.5.2.1 Definition of the Domain D

We define the domain $D \subset \mathscr{H}$ of our Hamiltonian H (to be devised) as

$$D := \big((1 \oplus U^{-1})\widehat{D}_{\widetilde{m}_j\widetilde{\kappa}_j}\big) \oplus \bigoplus_{\substack{j,m_j,\kappa_j \\ (m_j,\kappa_j)\neq(\widetilde{m}_j,\widetilde{\kappa}_j)}} U^{-1} D^{\theta=0}_{m_j\kappa_j}\,, \tag{C.5.12}$$

where we denoted (recall (C.5.6) for the definition of $\widehat{\mathscr{H}}$)

$$\widehat{D}_{\widetilde{m}_j\widetilde{\kappa}_j} := \Big\{(\Psi^{(0)}, \phi^{(1)}) \in \widehat{\mathscr{H}} : \phi^{(1)} \in D(h^*_{\widetilde{m}_j,\widetilde{\kappa}_j}) \text{ and } a_1\phi^{(1)}_-(0) + a_2\phi^{(1)}_+(0) = g\Psi^{(0)}\Big\}. \tag{C.5.13}$$

Moreover, for $\theta \in [0, 2\pi)$, we denoted

$$D^\theta_{m_j\kappa_j} := \Big\{\phi = (\phi_+, \phi_-) \in D(h^*_{m_j,\kappa_j}) \;:\; \phi_+(0)\sin\theta + \phi_-(0)\cos\theta = 0\Big\}. \tag{C.5.14}$$

This means that, for $\theta = 0$ and $(m_j, \kappa_j) \neq (\widetilde{m}_j, \widetilde{\kappa}_j)$, $c_{-m_j\kappa_j} := \phi_-(0) = 0$ and $c_{+m_j\kappa_j} := \phi_+(0) \in \mathbf{C}$ is free. We also denote $c_{\pm\widetilde{m}_j,\widetilde{\kappa}_j} := \phi^{(1)}_\pm(0)$ for $(\widetilde{m}_j, \widetilde{\kappa}_j)$ as in (C.5.13) and abbreviate $c_- \equiv c_{-\widetilde{m}_j,\widetilde{\kappa}_j}$. Therefore, inverting the unitary transform U from (C.4.13) in (C.5.12) and invoking Lemma C.4.1, we find that for every $\Psi \in D$, the upper sector part $\Psi^{(1)}$ obeys the asymptotics given in (C.5.1). Moreover, inverting U again, we also find that, by definition of $\widehat{D}_{\widetilde{m}_j\widetilde{\kappa}_j}$, every $\Psi \in D$ obeys the IBC from (C.5.2). This proves items 1 and 2 of Theorem C.5.1.

C.5.2.2 Self-adjointness of H on D

First, we have that h_{m_j,κ_j} on $D(h_{m_j\kappa_j})$ from (C.4.15) has self-adjoint extensions parametrized by $\theta \in [0, 2\pi)$ as [175, Theorem 5.2]

$$h^\theta_{m_j,\kappa_j} = h^*_{m_j,\kappa_j}\Big|_{D^\theta_{m_j\kappa_j}}\,, \tag{C.5.15}$$

where $D^\theta_{m_j\kappa_j}$ is defined in (C.5.14). Therefore, since H leaves the decomposition into angular momentum subspaces invariant, the task of proving self-adjointness of H on D immediately simplifies: It reduces to proving that the Hamiltonian $\widehat{H} \equiv \widehat{H}_{\widetilde{m}_j\widetilde{\kappa}_j}$ acting on $\phi = (\phi^{(0)}, \phi^{(1)}) \in \widehat{D} \equiv \widehat{D}_{\widetilde{m}_j\widetilde{\kappa}_j}$ from (C.5.13) with $\phi^{(0)} \equiv \Psi^{(0)}$ as (recall the notation below (C.5.14))

$$(\widehat{H}\phi)^{(0)} = g^*[a_3c_- + a_4c_+] \tag{C.5.16a}$$

$$(\widehat{H}\phi)^{(1)} = h^*\phi^{(1)}\,, \tag{C.5.16b}$$

is self-adjoint. Here and in the following, to ease notation, we denote $h \equiv h_{\widetilde{m}_j,\widetilde{\kappa}_j}$ as well as $\mathscr{K} = \mathscr{K}_{\widetilde{m}_j,\widetilde{\kappa}_j}$

The proof of $(\widehat{H},\widehat{D})$ being self-adjoint is very similar to [344, p. 12–13], hence we will be quite brief. First, the fact that $\widehat{D} \subset \widehat{\mathscr{H}}$ is dense, can be seen in exactly the same way as in [344].

Next, in order to show that $\widehat{H}$ is symmetric on $\widehat{D}$, we take, completely analogously to [344, Eqs. (73)–(86)], some $\phi,\eta \in \widehat{D}$ and compute the difference $\langle \phi, \widehat{H}\eta\,\rangle_{\widehat{\mathscr{H}}} - \langle \widehat{H}\phi, \eta\,\rangle_{\widehat{\mathscr{H}}}$. Denoting $c_\pm = \eta_\pm(0)$ and $d_\pm = \phi_\pm(0)$, and using that $a_1a_4 - a_2a_3 = 1$, we find

$$\begin{aligned}\langle \phi, \widehat{H}\eta\,\rangle_{\widehat{\mathscr{H}}} - \langle \widehat{H}\phi, \eta\,\rangle_{\widehat{\mathscr{H}}} \\ = \langle \phi^{(1)}, h^*\eta^{(1)}\rangle_{L^2((0,\infty),\mathscr{K})} - \langle h^*\phi^{(1)}, \eta^{(1)}\rangle_{L^2((0,\infty),\mathscr{K})} - [d_+^*c_- - d_-^*c_+]\,,\end{aligned} \tag{C.5.17}$$

just as in [344]. To se that $\langle \phi, \widehat{H}\eta\,\rangle_{\widehat{\mathscr{H}}} = \langle \widehat{H}\phi, \eta\,\rangle_{\widehat{\mathscr{H}}}$, we are now left to compute

$$\begin{aligned}&\langle \phi^{(1)}, h^*\eta^{(1)}\rangle_{L^2((0,\infty),\mathscr{K})} - \langle h^*\phi^{(1)}, \eta^{(1)}\rangle_{L^2((0,\infty),\mathscr{K})} \\ &= \int_0^\infty \mathrm{d}R\ \partial_R\Big[\phi_-^{(1)}(R)^\dagger\,\eta_+^{(1)}(R) - \phi_+^{(1)}(R)^\dagger\,\eta_-^{(1)}(R)\Big] \\ &= \Big[\phi_-^{(1)}(R)^\dagger\,\eta_+^{(1)}(R) - \phi_+^{(1)}(R)^\dagger\,\eta_-^{(1)}(R)\Big]_0^\infty \\ &= \lim_{R\searrow 0}\Big[\phi_+^{(1)}(R)^\dagger\,\eta_-^{(1)}(R) - \phi_-^{(1)}(R)^\dagger\,\eta_+^{(1)}(R)\Big] \\ &= d_+^*c_- - d_-^*c_+\,,\end{aligned}$$

where in the first step we employed that all the terms not involving the derivative ∂_R cancel (cf. [344, Eqs. (75)–(79)]). In the penultimate step, we used that $\phi_\pm^{(1)}, \eta_\pm^{(1)}$ vanish at infinity (as follows from them being continuous and in L^2). Finally, in the last step we used the IBC in the form of (C.5.13).

After having proven that $\widehat{H}$ is symmetric on $\widehat{D}$, it remains to show that $\widehat{D} = D(\widehat{H}^*)$. In order to do so, first note that $\widehat{D} \subseteq D(\widehat{H}^*) \subseteq \mathbf{C} \oplus D(h^*)$. Any given $\phi \in \mathbf{C} \oplus D(h^*)$ lies in $D(\widehat{H}^*)$ if and only if there exists some $\xi \in \widehat{\mathscr{H}}$ such that for every $\eta \in \widehat{D}$, it holds that $\langle \xi, \eta\rangle_{\widehat{\mathscr{H}}} = \langle \phi, \widehat{H}\eta\rangle_{\widehat{\mathscr{H}}}$. The right-hand side can now be computed, completely analogously to [344, Eqs. (89)–(94)], as

$$\begin{aligned}\langle \phi, \widehat{H}\eta\rangle_{\widehat{\mathscr{H}}} = \big[-(d_-a_1+d_+a_2) + g\phi^{(0)}\big]^*(a_3c_- + a_4c_+) \\ + \langle h^*\phi^{(1)}, \eta^{(1)}\rangle_{L^2((0,\infty),\mathscr{K})} + \langle g^*(a_3d_- + a_4d_+), \eta^{(0)}\rangle_{\mathbf{C}}\end{aligned} \tag{C.5.18}$$

where we again abbreviated $c_\pm = \eta_\pm(0)$ and $d_\pm = \phi_\pm(0)$. From (C.5.18) we conclude that $\langle \xi, \eta\rangle_{\widehat{\mathscr{H}}} = \langle \phi, \widehat{H}\eta\rangle_{\widehat{\mathscr{H}}}$ is true for all $\eta \in \widehat{D}$, if and only if

$$\xi^{(0)} = g^*(a_3d_- + a_4d_+) \quad \text{and} \quad \xi^{(1)} = h^*\phi^{(1)}, \tag{C.5.19}$$

and ϕ satisfies the IBC

$$a_1d_- + a_2d_+ = g\phi^{(0)}\,. \tag{C.5.20}$$

This means, $\phi \in \widehat{D}$ and $\xi = \widehat{H}\phi$, i.e., $\widehat{H}$ is self-adjoint on $\widehat{D}$.

C.5.2.3 Particle Creation

Assume that particle creation did *not* occur, i.e., that the Hamiltonian were block diagonal in the decomposition $\mathscr{H}^{(0)} \oplus \mathscr{H}^{(1)}$, say

$$H = \left(\begin{array}{c|c} F_0 & 0 \\ \hline 0 & F_1 \end{array}\right), \tag{C.5.21}$$

where F_0 and F_1 are blocks that act on $\mathscr{H}^0$ and $\mathscr{H}^{(1)}$ respectively. Under this assumption, the domain of H would be the Cartesian product of the domain of F_0 (which must be $\mathscr{H}^{(0)}$) and the domain of F_1 (a dense subspace of $\mathscr{H}^{(1)}$). Thus, for any $\Psi^{(0)} \in \mathbf{C} \setminus \{0\}$, $(\Psi^{(0)}, \Psi^{(1)} \equiv 0)$ is in the domain of a block-diagonal H. On the other hand, wave functions in the domain of H must satisfy the IBC (C.5.2), which implies that, since $\Psi^{(0)} \neq 0$, $\Psi^{(1)}$ cannot be identically equal to zero. This is a contradiction, and hence the IBC forces H to be non block-diagonal and we have thus proven item 5 of Theorem C.5.1.

This concludes the proof of Theorem C.5.1. □

C.6 Creation Rate and Trajectories: Proof of Proposition C.3.7

In this section, we verify the claims from Section C.3.2. To this end, we compute the asymptotics of the probability current j^μ in Proposition C.6.1 in Section C.6.1. Afterwards, in Section C.6.2, we give the proof of Proposition C.3.7, yielding the asymptotic behavior of the trajectories as solutions to the simplified Bohmian equation of motion (C.3.18). Finally, in Section C.6.3, we (non-rigorously) verify that the Bohm-Bell jump process defined in Section C.3.2.3 is equivariant.

As in Section C.3.2.1, we will consider only the Hamiltonian provided by Theorem C.3.1 (i.e., $a_1 = 1 = a_4$, $a_2 = 0 = a_3$ in the notation of Theorem C.5.1) and only wave functions Ψ from $\widehat{D} \subset \mathscr{H}$ as in (C.3.17), an invariant subspace comprising $\mathscr{H}^{(0)}$ and $\mathscr{K}_{\widetilde{m}_j\widetilde{\kappa}_j}$.

C.6.1 Probability Current

In the following proposition, we provide the asymptotic behavior of the probability current j^μ. Recall that

$$j^0 = |\Psi^{(1)}|^2, \quad j^i = \Psi^{(1)\dagger}\alpha^i\Psi^{(1)} \text{ for } i = 1,2,3. \tag{C.6.1}$$

Proposition C.6.1 (Asymptotic behavior of the current)**.** *Let $\Psi \in \widehat{D}$ and let $c_\pm$ be defined as in* (C.3.19)*. Then the components of the probability current j^μ defined in* (C.2.7) *in the basis e_x of* (C.2.8) *obey the asymptotics (as $r \to 0$)*

$$j^0(r,\vartheta,\varphi) = \frac{|c_+|^2 + |c_-|^2}{4\pi|Q|} r^{-1} + \mathcal{O}(r^0) \tag{C.6.2a}$$

$$j^1(r,\vartheta,\varphi) = -\frac{\mathrm{Im}[c_-^* c_+]}{2\pi|Q|} r^{-1} + \mathcal{O}(r^0) \tag{C.6.2b}$$

$$j^2(r,\vartheta,\varphi) = \mathcal{O}(r^0) \tag{C.6.2c}$$

$$j^3(r,\vartheta,\varphi) = \mathrm{sgn}(\widetilde{m}_j\widetilde{\kappa}_j)\sin\vartheta \frac{\mathrm{Re}[c_-^* c_+]}{2\pi|Q|} r^{-1} + \sin\vartheta\,\mathcal{O}(r^0)\,. \tag{C.6.2d}$$

Proof. By (C.5.1) and $\Psi \in \widehat{D}$,

$$\Psi^{(1)}(r,\boldsymbol{\omega}) = \left(\frac{c_-}{|Q|^{1/2}}\Phi^-_{\widetilde{m}_j\widetilde{\kappa}_j}(\boldsymbol{\omega}) + \frac{c_+}{|Q|^{1/2}}\Phi^+_{\widetilde{m}_j\widetilde{\kappa}_j}(\boldsymbol{\omega})\right) r^{-1/2} + \mathcal{O}(r^{1/2})\,. \tag{C.6.3}$$

Eq. (C.6.2a) follows from the facts that

$$\langle \Phi^+_{m_j\kappa_j}(\boldsymbol{\omega}), \Phi^-_{m_j\kappa_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0 \quad \forall \boldsymbol{\omega} \in \mathbf{S}^2\,, \tag{C.6.4}$$

that $\{w_1, w_2\}$ is orthonormal in $\mathbf{C}^2$, and that

$$\left|\Phi^\pm_{\widetilde{m}_j\widetilde{\kappa}_j}(\boldsymbol{\omega})\right|^2 = \frac{1}{4\pi} \quad \forall \boldsymbol{\omega} \in \mathbf{S}^2 \tag{C.6.5}$$

for $\widetilde{\kappa}_j = \pm 1$ (so $j = \frac{1}{2}$), which can be easily verified from the definition (C.A.6) using

$$Y_0^0(\vartheta,\varphi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^{\pm 1}(\vartheta,\varphi) = \pm\sqrt{\frac{3}{8\pi}}\mathrm{e}^{\pm \mathrm{i}\varphi}\sin\vartheta, \quad Y_1^0(\vartheta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\vartheta\,. \tag{C.6.6}$$

We turn to (C.6.2b)–(C.6.2d). Recalling that

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{C.6.7}$$

one sees from (C.A.5) that

$$\langle \Phi^{\pm}_{m_j\kappa_j}(\boldsymbol{\omega}), \alpha^i \Phi^{\pm}_{m_j\kappa_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0 \quad \forall \boldsymbol{\omega} \in \mathbf{S}^2\ \forall i = 1,2,3. \tag{C.6.8}$$

Further calculations show that

$$\langle \Phi^{+}_{\widetilde{m}_j,\widetilde{\kappa}_j}(\boldsymbol{\omega}), \alpha^1 \Phi^{-}_{\widetilde{m}_j,\widetilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = \frac{-\mathrm{i}}{4\pi}\,, \tag{C.6.9a}$$

$$\langle \Phi^{+}_{\widetilde{m}_j,\widetilde{\kappa}_j}(\boldsymbol{\omega}), \alpha^2 \Phi^{-}_{\widetilde{m}_j,\widetilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = 0\,, \tag{C.6.9b}$$

$$\langle \Phi^{+}_{\widetilde{m}_j,\widetilde{\kappa}_j}(\boldsymbol{\omega}), \alpha^3 \Phi^{-}_{\widetilde{m}_j,\widetilde{\kappa}_j}(\boldsymbol{\omega})\rangle_{\mathbf{C}^4} = \mathrm{sgn}(\widetilde{m}_j\widetilde{\kappa}_j)\,\sin\vartheta\frac{1}{4\pi}\,. \tag{C.6.9c}$$

(In fact, this follows from [345, Eq. (47)] and (C.A.4).) From these relations, (C.6.2b)–(C.6.2d) follow. □

C.6.2 Bohmian Trajectories: Proof of Proposition C.3.7

From (C.6.2) at t_0 while assuming (C.3.20) (in particular $c_- \neq 0 \neq c_+$) together with (C.3.15), we obtain for the approximate trajectories (i.e., the solutions to (C.3.18)), analogously to [345, Eq. (60)], that

$$\frac{\mathrm{d}r(t)}{\mathrm{d}t} = -\frac{2Q^2\,\mathrm{Im}\,[c_-^* c_+]}{|c_+|^2 + |c_-|^2}\, r^{-2} + \mathcal{O}(r^{-1}) \tag{C.6.10a}$$

$$\frac{\mathrm{d}\vartheta(t)}{\mathrm{d}t} = \mathcal{O}(r^{-1}) \tag{C.6.10b}$$

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = \mathrm{sgn}(\widetilde{m}_j\widetilde{\kappa}_j)\frac{2Q\,\mathrm{Re}\,[c_-^* c_+]}{|c_+|^2 + |c_-|^2}\, r^{-2} + \mathcal{O}(r^{-1})\,. \tag{C.6.10c}$$

Similarly to the arguments given at the end of Section 4 in [345], the differential equations (C.6.10) can be solved by a simple separation of variables, where one first solves (C.6.10a) and then feeds the result into the other two relations, eventually yielding (C.3.21). □

C.6.3 Equivariance of the Bohm-Bell Process

In this section, we non-rigorously verify that the process Q_t is equivariant.

First, away from the origin, we must have equivariance by conservation of probability expressed through the continuity equation

$$\nabla_\mu j^\mu = 0\,. \tag{C.6.11}$$

Therefore, the only place where probability is gained or lost is at the singularity $r = 0$. Consider the probability flux through the surface element $\mathrm{d}^2\boldsymbol{\omega}$ near $r = 0$ in coordinate space $[0,\infty) \times \mathbf{S}^2$, which

is $v^1(r,\boldsymbol{\omega})\,\rho(r,\boldsymbol{\omega})\,\mathrm{d}^2\boldsymbol{\omega}$ with probability density ρ in coordinate space given by $|\Psi^{(1)}(r,\boldsymbol{\omega})|^2 A^{-1} r^2$ according to (C.3.4b). By (C.3.15), $v^1 = A^2 j^1/j^0$. Thus, the flux is

$$A r^2 \Psi^{(1)}(r,\boldsymbol{\omega})^\dagger \alpha^1\, \Psi^{(1)}(r,\boldsymbol{\omega})\,\mathrm{d}^2\boldsymbol{\omega} \tag{C.6.12}$$

which converges, as $r \to 0$, to

$$J_{\mathrm{rad}}\,\mathrm{d}^2\boldsymbol{\omega} := -\frac{\mathrm{Im}\,[c_-^* c_+]}{2\pi|Q|}\,\mathrm{d}^2\boldsymbol{\omega} \tag{C.6.13}$$

by (C.6.2b). This is the quantity $J^\perp_{\Psi_t}(q)\,\nu(\mathrm{d}q, q')$ of (C.3.24). Thus, the rate of gain (positive or negative) of probability at the singularity is given by $4\pi J_{\mathrm{rad}}$.

This agrees with the rate of gain (positive or negative) of probability at $r = 0$ of Q_t: Indeed, in case that $J_{\mathrm{rad}} > 0$, then no trajectory ends at the origin (so no probability is lost) and the amount transported by jumps from $\varnothing$ to the trajectories emanating at time t_0 is given by the probability at $\varnothing$ times the total jump rate (C.3.26) from $\varnothing$, i.e.,

$$\left|\Psi^{(0)}_{t_0}\right|^2 \sigma_{t_0}(\varnothing \to \mathbf{S}^2) = \frac{2}{|Q|}\max\{0, -\mathrm{Im}[c_-^*(t_0)c_+(t_0)]\} = 4\pi J_{\mathrm{rad}}\,. \tag{C.6.14}$$

In the contrary case, $J_{\mathrm{rad}} < 0$, then no upward jump occurs (and thus no probability is gained at the origin) and the lost amount of probability automatically agrees with the flux across the sphere (since Q_t is $|\Psi_t|^2$-distributed).

Finally, in order to ensure preservation of the $|\Psi|^2$-distribution, it remains to check that the distribution of Q_t over the emanating trajectories yields the flux (C.6.13) through $\mathrm{d}^2\boldsymbol{\omega}$ in the $r \to 0$ limit. This follows from the fact that both the flux (C.6.13) and the jump rate (C.3.25) are uniform over the sphere. This concludes our argument for equivariance.

C.7 Conclusions

In this work, we have considered a model of particle creation and annihilation at the singularity of the sRN space-time that avoids the problem of ultraviolet divergence by using interior-boundary conditions. Furthermore, we constructed the corresponding Bohm-Bell process, an equivariant Markov jump process defined through 2 equations: Bohm's equation of motion (C.3.12) and the formula (C.3.25) which dictates the rate at which particle creation occurs.

For further research, one can consider full Fock space, including particle sectors with more than 1 particle. It would also be of interest to prove the existence of the Bohm-Bell process, and to define it also for $\widetilde{\kappa}_j \neq \pm 1$ and/or wave functions outside the subspace $\widehat{D}$. We expect the process to be qualitatively similar in these other cases. Furthermore, it would be interesting to consider the case of space-time singularities other than that of sRN.

C.A The $\Phi_{m_j\kappa_j}$ in Spherical Coordinates

In Minkowski space-time, let $\tilde{e}$ be an orthonormal basis (Lorentz frame) and $\tilde{b}$ the corresponding basis in 4d Dirac spin space S. Now for spherical coordinates $r \in (0,\infty), \vartheta \in [0,\pi], \varphi \in [0, 2\pi)$, let

$$\boldsymbol{e}_r = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \tag{C.A.1a}$$
$$\boldsymbol{e}_\vartheta = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) \tag{C.A.1b}$$
$$\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0) \tag{C.A.1c}$$

be the orthonormal basis of $\mathbf{R}^3$ whose vectors point in the directions of increasing r, ϑ, φ coordinates. Together with the timelike vector of $\tilde{e}$, they form another, (r,ϑ,φ)-dependent Lorentz frame e; let

b be the corresponding basis of S. Then, for any element of S, its (spherical) b-coefficients are obtained from the (Cartesian) $\tilde{b}$-coefficients through multiplication by

$$\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} \tag{C.A.2}$$

with the unitary 2×2 matrix

$$W := \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i}\mathrm{e}^{\mathrm{i}(\vartheta+\varphi)/2} & \mathrm{e}^{\mathrm{i}(\vartheta-\varphi)/2} \\ \mathrm{i}\mathrm{e}^{\mathrm{i}(-\vartheta+\varphi)/2} & -\mathrm{e}^{\mathrm{i}(-\vartheta-\varphi)/2} \end{bmatrix}, \tag{C.A.3}$$

(whose columns will be denoted by w_1 and w_2). This follows from the easily verifiable facts that, for $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the triple of Pauli matrices,

$$W^{-1}\sigma_1 W = \boldsymbol{e}_r \cdot \boldsymbol{\sigma} \tag{C.A.4a}$$
$$W^{-1}\sigma_2 W = \boldsymbol{e}_\vartheta \cdot \boldsymbol{\sigma} \tag{C.A.4b}$$
$$W^{-1}\sigma_3 W = \boldsymbol{e}_\varphi \cdot \boldsymbol{\sigma}\,, \tag{C.A.4c}$$

which shows that 2-spinors transform according to W, together with the fact that spatial rotations are implemented on 4-spinors as block diagonal 4×4 matrices with 2×2 blocks that are equal to each other and given by the action of the rotation on 2-spinors [559, (2.172) and (1.38)].

Relative to the Cartesian basis $\tilde{b}$ in S, the explicit form of the functions $\Phi^{\pm}_{m_j\kappa_j}$ is given in [559, Sec. 4.6.4]; translated into the spherical basis b, they are given as follows:

$$\Phi^{+}_{m_j,\mp(j+\frac{1}{2})} = \begin{pmatrix} \mathrm{i}\Psi^{m_j}_{j\mp\frac{1}{2}} \\ 0 \end{pmatrix}, \qquad \Phi^{-}_{m_j,\mp(j+\frac{1}{2})} = \begin{pmatrix} 0 \\ \mathrm{i}\Psi^{m_j}_{j\pm\frac{1}{2}} \end{pmatrix}, \tag{C.A.5}$$

where

$$\Psi^{m_j}_{j-\frac{1}{2}} = \sqrt{\frac{j+m_j}{2j}}\, Y^{m_j-1/2}_{j-1/2} w_1 + \sqrt{\frac{j-m_j}{2j}}\, Y^{m_j+1/2}_{j-1/2} w_2 \tag{C.A.6a}$$
$$\Psi^{m_j}_{j+\frac{1}{2}} = \sqrt{\frac{j+1-m_j}{2j+2}}\, Y^{m_j-1/2}_{j+1/2} w_1 - \sqrt{\frac{j+1+m_j}{2j+2}}\, Y^{m_j+1/2}_{j+1/2} w_2 \tag{C.A.6b}$$

with Y^m_ℓ the usual spherical harmonics (e.g., [559, Sec. 4.6.4]), defined for $\ell \in \mathbb{N} \cup \{0\}$ and $m \in \{-\ell, \ldots, \ell\}$ (not to be confused with the mass in the Dirac equation).

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