REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 29, Pages 256–288 (April 10, 2025)
https://doi.org/10.1090/ert/689

INDECOMPOSABLE CHARACTERS OF INDUCTIVE LIMITS OF SYMMETRIC GROUPS

NIKOLAY NESSONOV AND NHOK TKHAI SHON NGO

Abstract. In this paper we obtain a complete description of all indecomposable characters (central positive-definite functions) of inductive limits of the symmetric groups under block diagonal embedding. As a corollary we obtain the full classification of the isomorphism classes of these inductive limits.

1. Introduction

Consider the space $X = [0, 1)$ with the standard Lebesgue measure ν. Denote by $\mathrm{Aut}_0(X, \nu)$ the group of automorphisms of the space (X, ν) which preserve the measure ν. In particular, one can consider finite subgroups of this group which correspond to the so called *rational rearrangements* of X.

Namely, each symmetric group $\mathfrak{S}_N$ regarded as group of bijections of the set $\mathbb{X}_N = \{0, 1, 2, \dots, N-1\}$ can be embedded into $\mathrm{Aut}_0(X, \mu)$ in the following way. For each $\sigma \in \mathfrak{S}_N$ define automorphism $T_N(\sigma) \in \mathrm{Aut}_0(X, \mu)$ via the formula

$$T_N(\sigma)(x) = \frac{Nx - [Nx] + \sigma([Nx])}{N},\ x \in [0, 1).$$

In other words, the map $T_N(\sigma)$ acts on half-closed intervals $[\frac{k-1}{N}, \frac{k}{N})$, $k \in \overline{1, N}$ via the permutation σ. Here we use the notation $\overline{p, q}$ for the set $\{p, p+1, \dots, q\} \subset \mathbb{Z}$. It is easy to verify that T_N is an injective homomorphism.

In order to understand which automorphism in $T_{n_1 n_2}(\mathfrak{S}_{n_1 n_2})$ coincides with $T_{n_1}(\sigma)$, let us represent each element y of $\mathbb{X}_{n_1 n_2}$ as $y = x_1 + n_1 x_2$, where $x_1 \in \mathbb{X}_{n_1}$ and $x_2 \in \mathbb{X}_{n_2}$. Then

$$T_{n_1}(\sigma)\left[\frac{k}{n_1} + \frac{i}{n_1 n_2}, \frac{k}{n_1} + \frac{i+1}{n_1 n_2}\right) = \left[\frac{\sigma(k)}{n_1} + \frac{i}{n_1 n_2}, \frac{\sigma(k)}{n_1} + \frac{i+1}{n_1 n_2}\right),$$

where $k \in \overline{0, n_1 - 1}$, $i \in \overline{0, n_2 - 1}$. It means that $T_{n_1}(\sigma) = T_{n_1 n_2}(\mathfrak{i}(\sigma))$, where the permutation $\mathfrak{i}(\sigma) \in \mathfrak{S}_{n_1 n_2}$ acts as follows:

$$\mathfrak{i}(\sigma)(x_1 + n_1 x_2) = \sigma(x_1) + n_1 x_2,\ \text{where } x_1 \in \mathbb{X}_{n_1}, x_2 \in \mathbb{X}_{n_2}.$$

Received by the editors September 9, 2022, and, in revised form, October 18, 2023, and September 20, 2024.

2020 *Mathematics Subject Classification.* Primary 20C32.

Key words and phrases. Infinite symmetric group, character, factor representation.

The authors were partially supported by the "Long-term program of support of the Ukrainian research teams at the Polish Academy of Sciences carried out in collaboration with the U.S. National Academy of Sciences with the financial support of external partners". The second author was also supported by the Austrian Science Fund (FWF) grant "Geometry of the tip of the global nilpotent cone" no. 10.55776/P35847.

In this way we obtain a natural embedding of the symmetric group $\mathfrak{S}_{n_1}$ into the group $\mathfrak{S}_{n_1 n_2}$. Note that this embedding corresponds to the inclusion $T_{n_1}(\mathfrak{S}_{n_1}) \subset T_{n_1 n_2}(\mathfrak{S}_{n_1 n_2})$.

If we identify $\mathbb{X}_{n_1 n_2}$ with $\mathbb{X}_{n_1} \times \mathbb{X}_{n_2}$ using the correspondence $(x, y) \leftrightarrow x + n_1 y$, then $\mathfrak{i}(\sigma)$ acts as follows:

$$\mathfrak{i}(\sigma)((x, y)) = (\sigma(x), y), x \in \mathbb{X}_{n_1}, y \in \mathbb{X}_{n_2}.$$

Continuing this process and letting $N_k = n_1 n_2 \cdots n_k$ we obtain an infinite chain of subgroups

$$T_{N_1}(\mathfrak{S}_{N_1}) \subset T_{N_2}(\mathfrak{S}_{N_2}) \subset \ldots \subset T_{N_k}(\mathfrak{S}_{N_k}) \subset \ldots \subset \mathrm{Aut}_0(X, \nu).$$

Their union $\bigcup\limits_k T_{N_k}(\mathfrak{S}_{N_k})$ is a countable subgroup in $\mathrm{Aut}_0(X, \nu)$. This subgroup of $\mathrm{Aut}_0(X, \nu)$ is naturally isomorphic to the inductive limit $\varinjlim \mathfrak{S}_{N_k}$ which corresponds to the embedding $\mathfrak{i}(\sigma)$. Note that, in general, different sequences $\{N_k\}$ define non-isomorphic inductive limits. In particular, they might be simple as well as contain a nontrivial normal subgroup. In the present paper we obtain a complete description of pairs of sequences $\{N'_k\}$ and $\{N''_k\}$ for which the corresponding inductive limits are isomorphic (see Proposition 1.3).

For each prime number p denote by $\deg_p(N_k)$ the degree of p in the prime factorization $N_k = \prod_p p^{\deg_p(N_k)}$. In case when for each prime p the sequence $\left\{\deg_p(N_k)\right\}_{k=1}^{\infty}$ is unbounded, i.e. $\lim\limits_{k\to\infty} \deg_p(N_k) = \infty$, the group $\bigcup\limits_k T_{N_k}(\mathfrak{S}_{N_k})$ is called *the group of rational rearrangements of a segment*. Denote this group as $\mathfrak{S}_{\mathbb{Q}}$. In particular, $\mathfrak{S}_{\mathbb{Q}}$ is a simple group. In [5] the full description of *indecomposable characters* on $\mathfrak{S}_{\mathbb{Q}}$ was obtained. Recall that a positive definite function χ on group G is called *central* or *character* if it satisfies the condition

$$\chi(gh) = \chi(hg) \text{ for all } g, h \in G.$$

In the present paper we consider only *normalized* characters, i.e. which equal to 1 on the identity element of G. A character χ is called *indecomposable* if the unitary representation Π_χ of the group G, constructed via χ according to the Gelfand–Naimark–Segal (GNS) construction, is a factor representation. Namely, in this case the operators $\Pi_\chi(G)$ generate a factor of type II_1 [10]. This definition is equivalent to the following property: indecomposable characters are the extreme points of the simplex of all characters.

An important special case of the group $\varinjlim \mathfrak{S}_{N_k}$, where $N_k = 2^{M_k}$, was studied by A. Dudko [2]. This group is also a simple group and in [2] all indecomposable characters of this group were found. In [3] the full description of indecomposable characters was given for more general symmetric groups which act on the paths of the Bratelli's diagram. Besides that, a similar problem was solved by F. Leinen and O. Puglisi in [6] for the inductive limits of alternating groups with diagonal embeddings. However, the results of papers [2], [3] and [5] did not cover the case of an arbitrary sequence $\{N_k\}$. In the present paper we obtain the description of all characters on groups $\varinjlim \mathfrak{S}_{N_k}$ without any additional conditions on the sequences $\{N_k\}$. We obtain this description via the Vershik–Kerov approach, i.e. by studying weak limits of characters of symmetric groups (see [11]), and using the Okounkov–Vershik approach to the representation theory of symmetric groups (see [1] and [8]).

Contents. The paper is organised as follows. In Section 1 we introduce the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ as an inductive limit of symmetric groups and establish the connections between factor representations of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and spherical representations of $\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}$. In Section 2 we give explicit constructions of a factor representation of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and the corresponding representation of the $\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}$. In Section 3 we briefly review the representation theory of symmetric groups (we mainly follow the Okounkov–Vershik approach) and related notions such as characters, Young diagrams and tableaux. In Section 4 we prove the main part of the proof of Theorem 1.2, namely, the approximation theorem for the characters on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. Section 5 is devoted to proofs of several technical lemmas which are used in the proof of Theorem 1.2. In Section 6 we prove Theorem 1.2 and in Section 7 using the main theorem we prove Theorem 1.3 which gives a complete classification of groups of type $\mathfrak{S}_{\widehat{\mathbf{n}}}$ up to isomorphism. Finally, in Section 8 (Appendix) we prove some additional properties of the construction discussed in Section 2. These properties are used only in the second proof of Lemma 6.1.

1.1. **The inductive limit of symmetric groups.** In this subsection we define the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ as an inductive limit of symmetric groups with the diagonal embedding.

We regard the group $\mathfrak{S}_N$ as the group of all bijections (symmetries) of the set $\mathbb{X}_N = \{0, 1, 2, \dots, N-1\}$. We identify $\mathbb{X}_{NM}$ with $\mathbb{X}_N \times \mathbb{X}_M$ via the isomorphism $\mathbb{X}_{NM} \ni x_1 + Nx_2 \leftrightarrow (x_1, x_2) \in \mathbb{X}_N \times \mathbb{X}_M$. Denote by $\mathfrak{i}$ the embedding $\mathfrak{S}_N$ into $\mathfrak{S}_{NM}$ defined as follows:

$$\mathfrak{i}(\sigma)(x_1, x_2) = (\sigma(x_1), x_2). \tag{1.1}$$

In order to define formally the inductive limit of groups $\mathfrak{S}_N$ which correspond to the embedding (1.1), consider a sequence $\widehat{\mathbf{n}} = \{n_k\}_{k=1}^{\infty}$ of positive integers such that $n_k > 1$ for all k.

Suppose that $j > k$. Put $N_k = \prod_{i=1}^{k} n_i$ and $N_{k,j} = \frac{N_j}{N_k}$. Then $\mathbb{X}_{N_j} = \mathbb{X}_{N_k} \times \mathbb{X}_{N_{k,j}}$. Denote by $\mathfrak{i}_{k,j}$ the embedding of the group $\mathfrak{S}_{N_k}$ into $\mathfrak{S}_{N_j}$, defined as in (1.1). Define the group $\mathfrak{S}_{\widehat{\mathbf{n}}} = \bigcup_{k=1}^{\infty} \mathfrak{S}_{N_k}$ by identifying elements $\mathfrak{S}_{N_k}$ with their images under the maps $\mathfrak{i}_{k,j}$. In other words, $\mathfrak{S}_{\widehat{\mathbf{n}}}$ is the inductive limit $\varinjlim \mathfrak{S}_{N_k}$ that corresponds to the embeddings $\mathfrak{i}_{k,j}$. Denote by id the identity element of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

Remark 1.1. The group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ is isomorphic to the subgroup $\bigcup_k T_{N_k}(\mathfrak{S}_{N_k})$ of the group $\mathrm{Aut}_0(X, \nu)$.

Suppose that $\sigma \in \mathfrak{S}_{N_k}$ belongs to the conjugacy class $\mathfrak{C}_{\mathfrak{m}_{|k}}$ in $\mathfrak{S}_{N_k}$ consisting of permutations of the cycle type $\mathfrak{m}_{|k} = (m_{|k,1}, m_{|k,2}, \dots, m_{|k,l})$, where $m_{|k,i}$ is the number of cycles of the length i in the decomposition of σ into disjoint cycles (see more detailed definitions in Subsection 3.1). Then $\mathfrak{i}_{k,j}\left(\mathfrak{C}_{\mathfrak{m}_{|k}}\right) \subset \mathfrak{C}_{\mathfrak{m}_{|j}}$, where

$$\mathfrak{m}_{|j} = \left(m_{|j,1}, m_{|j,2}, \dots, m_{|j,l}\right) = \left(m_{|k,1}\frac{N_j}{N_k}, m_{|k,2}\frac{N_j}{N_k}, \dots, m_{|k,l}\frac{N_j}{N_k}\right). \tag{1.2}$$

For each $\sigma \in \mathfrak{S}_{N_k}$ define $\mathrm{supp}_{N_k}\, \sigma = \{x \in \mathbb{X}_{N_k} : \sigma(x) \neq x\}$. Note that if $\sigma \in \mathfrak{S}_{N_k}$, then

$$\#\left(\mathrm{supp}_{N_j}\, \mathfrak{i}_{k,j}(\sigma)\right) = \#\left(\mathrm{supp}_{N_k}\, \sigma\right) \cdot \frac{N_j}{N_k}. \tag{1.3}$$

Define the multiplicative character $\operatorname{sgn}_{N_k}\colon \mathfrak{S}_{N_k} \to \{-1,1\}$ via the formula $\operatorname{sgn}_{N_k}(\sigma) = (-1)^{\kappa_{N_k}(\sigma)}$. Here by $\kappa_{N_k}(\sigma)$ we denote the minimal number of factors in the decomposition of σ into the product of transpositions. It is known that if $\sigma \in \mathfrak{C}_{\mathfrak{m}_{|k}}$, then $\kappa_{N_k}(\sigma) = N_k - \sum_{p=1}^{l} m_{|k,p}$. Note that

$$\kappa_{N_j}(\sigma) = \frac{N_j}{N_k}\kappa_{N_k}(\sigma).$$

Therefore, $\operatorname{sgn}_{N_j}(\sigma) = \big(\operatorname{sgn}_{N_k}(\sigma)\big)^{N_j/N_k}$. This implies that for each $s \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ there exists $M(s)$ such that

$$\operatorname{sgn}_{N_j}(s) = \operatorname{sgn}_{N_i}(s) \text{ for all } i,j > M(s).$$

Thus, there exists a limit $\operatorname{sgn}_\infty(s) = \lim\limits_{j\to\infty} \operatorname{sgn}_{N_j}(s)$. Clearly, the function $\operatorname{sgn}_\infty$ is a multiplicative character of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

The following statement is immediate.

Proposition 1.1. *Denote by $\mathfrak{A}_{\widehat{\mathbf{n}}}$ the subgroup $\{g \in \mathfrak{S}_{\widehat{\mathbf{n}}} : \operatorname{sgn}_\infty(g) = 1\}$. Then*

(a) *$\mathfrak{A}_{\widehat{\mathbf{n}}}$ is a simple group;*
(b) *if the sequence $\widehat{\mathbf{n}} = \{n_k\}_{k=1}^{\infty}$ contains infinitely many even numbers, then $\mathfrak{S}_{\widehat{\mathbf{n}}}$ is a simple group.*

1.2. **The main result.** Let $\mathbb{M}_N(\mathbb{C})$ be the algebra of the complex $N\times N$ matrices, and let I_N be the identity $N\times N$ matrix. Denote by Tr_N the standard trace on $\mathbb{M}_N(\mathbb{C})$ and put $\operatorname{tr}_N(A) = \frac{1}{N}\operatorname{Tr}_N(A)$ for $A \in \mathbb{M}_N(\mathbb{C})$. Define on $\mathbb{M}_N(\mathbb{C})$ an inner product via the formula $\langle A,B\rangle_N = \operatorname{tr}_N(B^*A)$, $A,B \in \mathbb{M}_N(\mathbb{C})$. The elements σ of the symmetric group $\mathfrak{S}_N$ are realized as $\{0,1\}$-matrices $\Sigma_\sigma = \big[\delta_{i\sigma(j)}\big]$ in $\mathbb{M}_N(\mathbb{C})$, where $\delta_{ij} = \begin{cases} 1, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$ The operators of the left multiplication by Σ_σ define on $\mathbb{M}_N(\mathbb{C})$ a unitary representation $\mathfrak{L}_N$ of the group $\mathfrak{S}_N$:

$$\mathfrak{L}(\sigma)A = \Sigma_\sigma \cdot A, A \in \mathbb{M}_N(\mathbb{C}).$$

Put $\varphi_N(\sigma) = \langle \mathfrak{L}(\sigma)I_N, I_N\rangle = \operatorname{tr}_N(\Sigma_\sigma)$. Clearly, φ_N is a character on $\mathfrak{S}_N$ and

$$\varphi_N(\sigma) = 1 - \frac{\#\operatorname{supp}_N \sigma}{N}. \tag{1.4}$$

Note that if $\sigma \in \mathfrak{S}_{N_k}$, then for $j > k$ from (1.2) we have

$$\varphi_{N_j}(\mathfrak{i}_{k,j}(\sigma)) = \varphi_{N_k}(\sigma).$$

Hence, the sequence $\{\varphi_{N_k}\}$ defines a character χ_{nat} on the inductive limit $\mathfrak{S}_{\widehat{\mathbf{n}}} = \varinjlim \mathfrak{S}_{N_k}$. In other words, for $\sigma \in \mathfrak{S}_{N_k} \subset \mathfrak{S}_{\widehat{\mathbf{n}}}$ we have

$$\chi_{\mathrm{nat}}(\sigma) = \varphi_{N_k}(\sigma) = 1 - \frac{\#\operatorname{supp}_{N_k}\sigma}{N_k}. \tag{1.5}$$

The main result of the present paper is Theorem 1.2.

Theorem 1.2. *For $p \in \mathbb{N}\cup\{0,\infty\}$ define the character χ^p_{nat} by the formula $\chi^p_{\mathrm{nat}}(\sigma) = (\chi_{\mathrm{nat}}(\sigma))^p$ when $p \in \mathbb{N}\cup\{0\}$, and by the formula $\chi^\infty_{\mathrm{nat}}(\sigma) = \begin{cases} 1, & \text{if } \sigma = \mathrm{id},\\ 0, & \text{if } \sigma \neq \mathrm{id},\end{cases}$ when $p = \infty$. If χ is an indecomposable character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$, then there exists $p \in \mathbb{N}\cup\{0,\infty\}$ such that $\chi = \chi^p_{\mathrm{nat}}$ or $\chi = \operatorname{sgn}_\infty \cdot \chi^p_{\mathrm{nat}}$, where $(\operatorname{sgn}_\infty \cdot \chi^p_{\mathrm{nat}})(\sigma) = \operatorname{sgn}_\infty(\sigma)\cdot\chi^p_{\mathrm{nat}}(\sigma)$ for $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$.*

As a corollary, we also obtain a complete classification of the isomorphism classes of groups $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

Theorem 1.3. *Let $\widehat{\mathbf{n}}' = \{n'_k\}_{k=1}^{\infty}$ and $\widehat{\mathbf{n}}'' = \{n''_k\}_{k=1}^{\infty}$, where $n'_k, n''_k > 1$ for all k, be the sequences of positive integers. Put $N'_k = \prod_{i=1}^{k} n'_i$ and $N''_k = \prod_{i=1}^{k} n''_i$. Then, groups $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ and $\mathfrak{S}_{\widehat{\mathbf{n}}''}$ (see Subsection 1.1) are isomorphic if and only if for each prime number p the following condition holds:*

$$\lim_{k\to\infty} \deg_p(N'_k) = \lim_{k\to\infty} \deg_p(N''_k). \tag{1.6}$$

In other words, groups $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ and $\mathfrak{S}_{\widehat{\mathbf{n}}''}$ are isomorphic if and only if for each prime p either both sequences $\{\deg_p(N'_k)\}_{k=1}^{\infty}$ and $\{\deg_p(N''_k)\}_{k=1}^{\infty}$ are unbounded, or there is a non-negative integer d_p such that $\deg_p(N'_k) = \deg_p(N''_k) = d_p$ for all sufficiently large k.

Remark 1.2. Here for a positive integer N and a prime p we define $\deg_p(N)$ to be the exponent of p in the prime factorization of N.

1.3. II_1 **factor representations of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and spherical representations of $\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}$.** In this subsection we establish the connections between factor representations of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and spherical representations of the group $\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}$.

Let χ be an arbitrary character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. Consider the GNS-representation $(\pi_\chi, \mathcal{H}_\chi, \xi_\chi)$ of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ corresponding to χ. Here ξ_χ is the unit cyclic vector for $\pi_\chi(\mathfrak{S}_{\widehat{\mathbf{n}}})$ in the Hilbert space $\mathcal{H}_\chi$ such that $\chi(\sigma) = \langle \pi_\chi(\sigma)\xi_\chi, \xi_\chi\rangle$ for all $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. Denote by M the w^*-algebra generated by the set of operators $\pi_\chi(\mathfrak{S}_{\widehat{\mathbf{n}}})$. Using the character χ we can define a finite normal faithful trace tr on M. Namely, we put

$$\mathrm{tr}\,(\pi_\chi(\sigma)) = \chi(\sigma), \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}.$$

Taking into account the definition of the GNS-construction, we assume that $\mathcal{H}_\chi = L^2(M, \mathrm{tr})$, i.e. $\mathcal{H}_\chi$ is the completion of w^*-algebra M endowed with the inner product $\langle\cdot,\cdot\rangle$ defined as follows:

$$\langle m_1, m_2\rangle = \mathrm{tr}\,(m_2^* m_1), \quad m_1, m_2 \in M,$$

and ξ_χ is the identity operator from M. Finally, we recall that the operators $\pi_\chi(\sigma)$, $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ act on $L^2(M, \mathrm{tr})$ by the left multiplication

$$L^2(M, \mathrm{tr}) \ni v \overset{\pi_\chi(\sigma)}{\longmapsto} \pi_\chi(\sigma)\cdot v \in L^2(M, \mathrm{tr}). \tag{1.7}$$

In particular, M can be also regarded as a subset of bounded linear operators on $\mathcal{H}_\chi$ (elements of M act on $\mathcal{H}_\chi$ by the left multiplication). Denote by $\mathcal{B}(\mathcal{H}_\chi)$ the set of all bounded linear operators on $\mathcal{H}_\chi$ and put

$$M' = \{A \in \mathcal{B}(\mathcal{H}_\chi) : AB = BA \text{ for all } B \in M\}.$$

Since tr is a central state on M, i.e. $\mathrm{tr}(m_1 m_2) = \mathrm{tr}(m_2 m_1)$ for all $m_1, m_2 \in M$, the mapping

$$L^2(M, \mathrm{tr}) \ni v \overset{\pi'_\chi(\sigma)}{\longmapsto} v\cdot \pi_\chi(\sigma^{-1}) \in L^2(M, \mathrm{tr}) \tag{1.8}$$

defines a unitary operator $\pi'_\chi(\sigma) \in M'$ on $\mathcal{H}_\chi$. Denote by J the antilinear isometry of $\mathcal{H}_\chi$ which acts as follows: $L^2(M, \mathrm{tr}) \ni m \overset{J}{\mapsto} m^*$. It is clear that $J^2 = I$.

It follows from the above that

$$JMJ \subset M' \text{ and } M \subset JM'J. \tag{1.9}$$

Let us prove that

$$M = JM'J. \tag{1.10}$$

Consider an arbitrary operator A that belongs to $JM'J$. Since ξ_χ is a cyclic vector for M, i.e. the set $M\xi_\chi$ is norm dense in $L^2(M, \mathrm{tr})$, there exists a sequence $\{A_n\}_{n\in\mathbb{N}} \subset M$ such that

$$\lim_{n\to\infty} \|A\xi_\chi - A_n\xi_\chi\|_{L^2(M,\mathrm{tr})} = 0. \tag{1.11}$$

Applying the centrality of χ gives that

$$\lim_{m,n\to\infty} \|A_m^*\xi_\chi - A_n^*\xi_\chi\|_{L^2(M,\mathrm{tr})} = 0.$$

It follows that the sequence $\{A_n^*\xi_\chi\}$ converges in norm to some element $\eta \in L^2(M, \mathrm{tr})$. Hence, for each $U' \in JMJ$ we obtain the next chain of equalities:

$$\langle (U')^*\xi_\chi, \eta\rangle = \lim_{n\to\infty} \langle (U')^*\xi_\chi, A_n^*\xi_\chi\rangle$$

$$= \lim_{n\to\infty} \langle A_n\xi_\chi, U'\xi_\chi\rangle \overset{(1.11)}{=} \langle A\xi_\chi, U'\xi_\chi\rangle = \langle (U')^*\xi_\chi, A^*\xi_\chi\rangle .$$

Therefore, using the cyclicity of ξ_χ for JMJ, we have

$$A^*\xi_\chi = \eta; \text{ i.e. } \lim_{n\to\infty} \|A_n^*\xi_\chi - A^*\xi_\chi\|_{L^2(M,\mathrm{tr})} = 0. \tag{1.12}$$

Lemma 1.4. *Put $\omega(x) = \langle x\xi_\chi, \xi_\chi\rangle$, where $x \in \mathcal{B}(\mathcal{H}_\chi)$. Then*

(1) *ω is a central state on M' and on $JM'J$; i.e.*

$$\begin{aligned} &\omega(A'B') = \omega(B'A') \text{ for all } A', B' \in M' \text{ and} \\ &\omega(AB) = \omega(BA) \text{ for all } A, B \in JM'J; \end{aligned} \tag{1.13}$$

(2) *$AB' = B'A$ for all $A \in JM'J$ and $B' \in M'$.*

Proof.

Property 1. Since ξ_χ is a cyclic vector for M, there exist two sequences $\{A_n\}$ and $\{B_n\}$ in M such that

$$\lim_{n\to\infty} \|A\xi_\chi - A_n\xi_\chi\|_{L^2(M,\mathrm{tr})} = \lim_{n\to\infty} \|B\xi_\chi - B_n\xi_\chi\|_{L^2(M,\mathrm{tr})} = 0.$$

Hence, using (1.12), we obtain

$$\lim_{n\to\infty} \|A^*\xi_\chi - A_n^*\xi_\chi\|_{L^2(M,\mathrm{tr})} = \lim_{n\to\infty} \|B^*\xi_\chi - B_n^*\xi_\chi\|_{L^2(M,\mathrm{tr})} = 0.$$

Therefore, $\omega(AB) = \lim_{n\to\infty} \omega(A_nB_n) = \lim_{n\to\infty} \mathrm{tr}(A_nB_n) = \lim_{n\to\infty} \mathrm{tr}(B_nA_n) = \lim_{n\to\infty} \omega(B_nA_n) = \omega(BA)$. We leave it to the reader to verify that $\omega(A'B') = \omega(B'A')$ for all $A', B' \in M'$.

Property 2. Since $J^2 = I$, then, by the cyclicity of the vector ξ_χ for M, there exist the sequences $\{A_n\}$, $\{B_n\}$ in M such that

$$\lim_{n\to\infty} \|A\xi_\chi - A_n\xi_\chi\| = 0 \text{ and } \lim_{n\to\infty} \|JB'J\xi_\chi - B_n\xi_\chi\| = 0.$$

Since ω is a central state on M', we have the equality

$$\|B'\xi_\chi - JB_nJ\xi_\chi\| = \|(B')^*\xi_\chi - (JB_nJ)^*\xi_\chi\| .$$

Hence, we obtain that for any $U' \in JMJ$ and $V \in M$ we have

$$\langle AB'U'\xi_\chi, V\xi_\chi\rangle = \lim_{n\to\infty}\langle A_n\, JB_nJ\, U'\xi_\chi, V\xi_\chi\rangle \overset{(1.9)}{=} \lim_{n\to\infty}\langle JB_nJ\, A_n\, U'\xi_\chi, V\xi_\chi\rangle$$
$$= \lim_{n\to\infty}\left\langle A_n\, U'\xi_\chi, V\,(JB_nJ)^*\,\xi_\chi\right\rangle = \langle A\,U'\xi_\chi, V(B')^*\xi_\chi\rangle = \langle B'A\,U'\xi_\chi, V\xi_\chi\rangle\,.$$

Consequently, $AB' = B'A$. In particular, this establishes the equality (1.10). □

Now we define the representation $\pi^{(2)}_\chi$ of the group $\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}$ as follows:

$$\pi^{(2)}_\chi((\sigma_1,\sigma_2)) = \pi_\chi(\sigma_1)\pi'_\chi(\sigma_2),\ (\sigma_1,\sigma_2) \in \mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}. \tag{1.14}$$

Hence, applying (1.7) and (1.8), we obtain

$$\pi^{(2)}_\chi((\sigma,\sigma))\xi_\chi = \xi_\chi \text{ for all } \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}. \tag{1.15}$$

Denote by $\mathcal{C}(M)$ the center of w^*-algebra M.

Proposition 1.5. $\left(\pi^{(2)}_\chi\left(\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}\right)\right)' = \mathcal{C}(M)$.

Proof. It follows from (1.8) and (1.10) that $\left(\pi^{(2)}_\chi\left(\mathfrak{S}_{\widehat{\mathbf{n}}} \times \mathfrak{S}_{\widehat{\mathbf{n}}}\right)\right)' = M \cap M' = \mathcal{C}(M)$. □

Define the orthogonal projection E_k by

$$E_k = \frac{1}{N_k!}\sum_{\sigma\in\mathfrak{S}_{N_k}} \pi^{(2)}_\chi\,(\sigma,\sigma)\,. \tag{1.16}$$

It is clear that $E_k \geq E_{k+1}$. Therefore, there exists the limit $E = \lim\limits_{k\to\infty} E_k$. It follows from (1.16) that

$$E\mathcal{H}_\chi = \left\{\eta \in \mathcal{H}_\chi : \pi^{(2)}_\chi((\sigma,\sigma))\eta = \eta \text{ for all } \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}\right\}. \tag{1.17}$$

Proposition 1.6. *The following three conditions are equivalent:*

(i) π_χ *is a factor representation;*
(ii) *representation* $\pi^{(2)}_\chi$ *is irreducible;*
(iii) $\dim E\mathcal{H}_\chi = 1$.

Proof. The equivalence (i) and (ii) follows from Proposition 1.5.

Let us prove that (i) implies (iii). On the contrary, suppose that $\dim E\mathcal{H}_\chi \geq 2$. Then there exists a unit vector $v \in E\mathcal{H}_\chi$ such that

$$\langle v, \xi_\chi\rangle = 0. \tag{1.18}$$

Since ξ_χ is a cyclic vector for M, then

$$\|v - m\xi_\chi\|_{\mathcal{H}_\chi} < 1 \text{ for some } m \in M. \tag{1.19}$$

Hence, using (1.15) and applying the equality

$$\pi^{(2)}_\chi((\sigma,\sigma))\, m\, \pi^{(2)}_\chi((\sigma^{-1},\sigma^{-1})) = \pi_\chi(\sigma)\, m\pi_\chi(\sigma^{-1}),$$

we obtain

$$\left\|v - \pi_\chi(\sigma)\, m\pi_\chi(\sigma^{-1})\ \xi_\chi\right\|_{\mathcal{H}_\chi} = \|v - m\xi_\chi\|_{\mathcal{H}_\chi} < 1 \text{ for all } \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}. \tag{1.20}$$

Consequently,

$$\left\| v - \frac{1}{N_k!} \sum_{\sigma \in \mathfrak{S}_{N_k}} \pi_\chi(\sigma)\, m \pi_\chi(\sigma^{-1})\, \xi_\chi \right\|_{\mathcal{H}_\chi} \leq \|v - m\xi_\chi\|_{\mathcal{H}_\chi} < 1. \tag{1.21}$$

It is easy to check that the sequence $\left\{ m_k = \frac{1}{N_k!} \sum_{\sigma \in \mathfrak{S}_{N_k}} \pi_\chi(\sigma)\, m \pi_\chi(\sigma^{-1}) \right\} \subset M$ converges in the strong operator topology to an operator $Em \in \mathcal{C}(M)$. Let us emphasize that here we identify $Em \in \mathcal{H}_\chi = L^2(M, \mathrm{tr})$ with the corresponding left multiplication operator from M. In view of (1.21),

$$\|v - Em\xi_\chi\|_{\mathcal{H}_\chi} \leq \|v - m\xi_\chi\|_{\mathcal{H}_\chi} < 1.$$

Now note that if the condition (i) holds, then $\mathcal{C}(M)$ consists only of scalar operators. Thus, $Em = \alpha \cdot I$ for some $\alpha \in \mathbb{C}$. Hence, we obtain

$$\|v - \alpha\xi_\chi\|_{\mathcal{H}} = \sqrt{1 + |\alpha|^2 - 2\,\mathrm{Re}(\alpha \langle v, \xi_\chi \rangle)} \overset{(1.18)}{=} \sqrt{1 + |\alpha|^2} \geq 1, \text{ where } \alpha \in \mathbb{C},$$

and this contradicts the previous inequality.

Finally, let us prove that (iii) implies (i). Suppose the contrary, then there exists an orthogonal projection $Q \in \mathcal{C}(M)$ with the following properties:

$$v = Q\xi_\chi \neq 0, w = (I - Q)\xi_\chi \neq 0.$$

Since v and w are mutually orthogonal vectors from $E\mathcal{H}_\chi$, we have $\dim E\mathcal{H}_\chi \geq 2$ which contradicts the condition (i). □

Remark 1.3. Note that if representation $\pi_\chi^{(2)}$ is irreducible, then due to the condition (iii) it will be also a spherical representation.

2. The realizations of II_1 factor representations

In this section we give the explicit construction of a type II_1 factor representation of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and the corresponding irreducible representation of the group $\overline{\mathfrak{S}_{\widehat{\mathbf{n}}}} \times \overline{\mathfrak{S}_{\widehat{\mathbf{n}}}}$.

2.1. **Preliminaries.** Denote by ν_m the uniform probability measure on the set $\mathbb{X}_m = \{0, 1, \ldots, m-1\}$, i.e. $\nu_m(\{j\}) = \frac{1}{m}$ for all $j \in \mathbb{X}_m$. Let $\mathbb{X}_{\widehat{\mathbf{n}}} = \prod_{k=1}^{\infty} \mathbb{X}_{n_k}$. For $x = (x_1, x_2, \ldots) \in \mathbb{X}_{\widehat{\mathbf{n}}}$, we set $x|j = (x_1, x_2, \ldots, x_j) \in \prod_{k=1}^{j} \mathbb{X}_{n_k}$ and $x|(q, r) = (x_{q+1}, x_{q+2}, \ldots, x_r) \in \prod_{k=q+1}^{r} \mathbb{X}_{n_k}$. By definition, $x|j = x|(0, j)$. Each element $y \in \mathbb{X}_{\widehat{\mathbf{n}}}$ defines a cylinder set

$$\mathbb{A}_{y|q,r} = \{x \in \mathbb{X}_{\widehat{\mathbf{n}}} : x|(q, r) = y|(q, r)\} \subset \mathbb{X}_{\widehat{\mathbf{n}}}. \tag{2.1}$$

For simplicity of notation, we will write $\mathbb{A}_{y|r}$ instead $\mathbb{A}_{y|0,r}$. Now introduce the probability measure $\nu_{\widehat{\mathbf{n}}} = \prod_{k=1}^{\infty} \nu_{n_k}$ on $\mathbb{X}_{\widehat{\mathbf{n}}}$ by the formula

$$\nu_{\widehat{\mathbf{n}}}(\mathbb{A}_{y|j}) = \frac{1}{n_1 n_2 \cdots n_j} = \frac{1}{N_j}.$$

Let $\mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ be the group of automorphisms of the Lebesgue space $\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ which preserve the measure $\nu_{\widehat{\mathbf{n}}}$. It follows from the definition of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ that one can regard $\mathfrak{S}_{\widehat{\mathbf{n}}}$ as a subgroup of $\mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$.

Remark 2.1. Here an element $\sigma \in \mathfrak{S}_{N_k}$ acts on $\mathbb{X}_{\widehat{\mathbf{n}}}$ as follows: σ maps an element $(x, y) \in \mathbb{X}_{N_k} \times \prod_{j=k+1}^{\infty} \mathbb{X}_{n_j}$ to $(\sigma(x), y)$ (see also (1.1)).

Define the action of an automorphism $O \in \mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ on

$$x = (x_1, x_2, \ldots) \in \mathbb{X}_{\widehat{\mathbf{n}}} \setminus (n_1 - 1, n_2 - 1, \ldots, n_k - 1, \ldots)$$

in the following way: $Ox = (y_1, y_2, \ldots)$, where

$$y_p = \begin{cases} x_p + 1 \,(\mathrm{mod}\, n_p), & \text{if } p \leq \min\left\{i : x_i < n_i - 1\right\}, \\ x_p, & \text{if } p > \min\left\{i : x_i < n_i - 1\right\}. \end{cases}$$

Also, define O at $(n_1 - 1, n_2 - 1, \ldots, n_k - 1, \ldots)$ as

$$O(n_1 - 1, n_2 - 1, \ldots, n_k - 1, \ldots) = (0, 0, \ldots, 0, \ldots).$$

The following fact is immediate.

Lemma 2.1. *For a given $x = (x_1, x_2, \ldots) \in \mathbb{X}_{\widehat{\mathbf{n}}}$, we set*

$$O^m x = \left(\left(O^m x\right)_1, \left(O^m x\right)_2, \ldots\right).$$

Then

(a) *the following equalities hold: $x|j = \left(O^{N_j} x\right)|j$, $O^m\left(\mathbb{A}_{x|j}\right) = \mathbb{A}_{(O^m x)|j}$ and*

$$\bigcup_{m=0}^{N_j - 1} \mathbb{A}_{(O^m x)|j} = \mathbb{X}_{\widehat{\mathbf{n}}};$$

(b) *for the map $O_{x|j}$, defined as follows:*

$$O_{x|j}(z) = \begin{cases} Oz, & \text{if } z \in \bigcup_{m=0}^{N_j - 2} O^m\left(\mathbb{A}_{x|j}\right), \\ O^{-(N_j - 1)} z, & \text{if } z \in O^{N_j - 1}\left(\mathbb{A}_{x|j}\right), \end{cases}$$

where $z = (z_1, z_2, \ldots) \in \mathbb{X}_{\widehat{\mathbf{n}}}$, the period of each $z \in \mathbb{X}_{\widehat{\mathbf{n}}}$ equals N_j;

(c) *for the element $\mathbf{0} = (0, 0, \ldots) \in \mathbb{X}_{\widehat{\mathbf{n}}}$ we have*

$$O_{\mathbf{0}|j}(z) = \left(\left(O_{\mathbf{0}|j}(z)\right)_1, \left(O_{\mathbf{0}|j}(z)\right)_2, \ldots, \left(O_{\mathbf{0}|j}(z)\right)_j, z_{j+1}, z_{j+2}, \ldots\right),$$

where

$$\left(O_{\mathbf{0}|j}(z)\right)_p = \begin{cases} z_p + 1 \,(\mathrm{mod}\, n_p), & \text{if } p \leq \min\left\{i : z_i < n_i - 1\right\} \leq j, \\ z_p, & \text{if } \min\left\{i : z_i < n_i - 1\right\} < p \leq j, \\ 0, & \text{if } p \leq j < \min\left\{i : z_i < n_i - 1\right\}. \end{cases} \tag{2.2}$$

Define an invariant metric ρ on the group $\mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ by the formula

$$\rho(\alpha, \beta) = \nu_{\widehat{\mathbf{n}}}\left(x \in \mathbb{X}_{\widehat{\mathbf{n}}} : \alpha(x) \neq \beta(x)\right), \ \alpha, \beta \in \mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right). \tag{2.3}$$

For an automorphism $\alpha \in \mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ denote by $[\alpha]$ the subgroup in $\mathrm{Aut}_0\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ defined as follows: $\beta \in [\alpha]$, if there exists a measurable function $d(\beta, \cdot)$ on $\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right)$ with values in $\mathbb{Z}$ such that the equality

$$\beta(z) = \alpha^{d(\beta, z)}(z) \tag{2.4}$$

holds for almost all $z \in \mathbb{X}_{\widehat{\mathbf{n}}}$. Denote by Σ_j the σ-algebra on $\mathbb{X}_{\widehat{\mathbf{n}}}$ generated by the collection of cylinder subsets $\mathbb{A}_{y|j}$, $y \in \mathbb{X}_{\widehat{\mathbf{n}}}$.

Lemma 2.2. *Let $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$ be the closure of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ with respect to the metric ρ. Then*

(a) $\mathfrak{S}_{N_j} = \{\beta \in [O_{\mathbf{o}|j}] : \beta\Sigma_j = \Sigma_j\}$;
(b) $O \in \overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$;
(c) *the action of automorphism O on $(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}})$ is ergodic;*
(d) *for every $l \in \mathbb{Z} \setminus \{0\}$ the automorphism O^l acts freely on $\mathbb{X}_{\widehat{\mathbf{n}}}$; i.e. if there is an $x \in \mathbb{X}_{\widehat{\mathbf{n}}}$ such that $O^l x = x$, then $l = 0$.*

Proof. The property (a) is a consequence of Lemma 2.1 (c). From the parts (b) and (c) of Lemma 2.1 we have

$$\nu_{\widehat{\mathbf{n}}}\left(x \in \mathbb{X}_{\widehat{\mathbf{n}}} : Ox \neq O_{\mathbf{o}|j}(x)\right) \leq \frac{1}{N_j}.$$

Taking this and part (a) of Lemma 2.1 into account, we obtain the part (b) of Lemma 2.2. Therefore, the ergodicity of the automorphism O is equivalent to the ergodicity of the action of $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$. Finally, the property (d) follows from the definition of the automorphism O. □

Lemma 2.1 (b,c) and Lemma 2.2 (a) imply that the action of each automorphism $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ on $x \in \mathbb{X}_{\widehat{\mathbf{n}}}$ can be expressed as follows:

$$\sigma(x) = O^{d(\sigma,x)}(x), \tag{2.5}$$

where $d(\sigma, x) \in \mathbb{Z}$. The uniqueness of the function $d(\sigma, \cdot)$ in (2.5) is a consequence of Lemma 2.2 (d). Note that if $\gamma, \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$, then

$$d(\gamma\sigma, x) = d(\gamma, \sigma(x)) + d(\sigma, x). \tag{2.6}$$

Remark 2.2. It follows from (2.4) and (2.5) that $[O] = \overline{\mathfrak{S}_{\widehat{\mathbf{n}}}}$. In particular, $d(\sigma, x)$ is defined for all $\sigma \in \overline{\mathfrak{S}_{\widehat{\mathbf{n}}}}$ and $x \in \mathbb{X}_{\widehat{\mathbf{n}}}$.

2.2. **Construction of a II_1 factor representation of the group $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$.** In the Hilbert space $\mathcal{H} = L^2\left(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}\right) \otimes l^2(\mathbb{Z})$ define a unitary operator $\mathcal{F}(\sigma)$, where $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$, as follows:

$$\left(\mathcal{F}(\sigma)\eta\right)(x, m) = \eta\left(\sigma^{-1}(x), m - d(\sigma^{-1}, x)\right) \text{ for all } \eta \in \mathcal{H}. \tag{2.7}$$

Equality (2.6) implies that the map $\sigma \ni \mathfrak{S}_{\widehat{\mathbf{n}}} \mapsto \mathcal{F}(\sigma)$ is a unitary representation of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$, which can be extended by continuity with respect to the metric ρ (see (2.3)) to a representation of the group $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$. Thus, (2.7) defines a representation of the group $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$.

Denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators acting on $\mathcal{H}$. Put

$$\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)' = \left\{A \in \mathcal{B}(\mathcal{H}) : AB = BA \text{ for all } B \in \mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)\right\}.$$

Denote by $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$ the w^*-algebra generated by operators $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)$. Let $\beth$ be the function on $\mathbb{X}_{\widehat{\mathbf{n}}}$ that is identically one on $\mathbb{X}_{\widehat{\mathbf{n}}}$. For each $i \in \mathbb{Z}$ define the function δ_i on $\mathbb{Z}$ as follows: $\delta_i(m) = \begin{cases} 1, & \text{if } m = i, \\ 0, & \text{if } m \neq i. \end{cases}$ Put $\xi_0 = \beth \otimes \delta_0$. It is easy to check that

$$\left(\mathcal{F}(s)\mathcal{F}(\sigma)\xi_0, \xi_0\right) = \left(\mathcal{F}(\sigma)\mathcal{F}(s)\xi_0, \xi_0\right) \text{ for all } \sigma, s \in \mathfrak{S}_{\widehat{\mathbf{n}}}.$$

Thus, the vector state tr on $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$ defined as

$$\operatorname{tr}(A) = (A\xi_0, \xi_0)\,,\ A \in \mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'', \tag{2.8}$$

is central, i.e. the equality $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ holds for all $A, B \in \mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$. In particular, it follows from (1.5) that for any $s \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ we have

$$\chi_{\mathrm{nat}}(s) = \operatorname{tr}\left(\mathcal{F}(s)\right). \tag{2.9}$$

Indeed, Lemma 2.2 (d) and formulas (2.7) and (1.5) imply that for $\sigma \in \mathfrak{S}_{N_k} \subset \mathfrak{S}_{\widehat{\mathbf{n}}}$ we have

$$\operatorname{tr}\left(\mathcal{F}(\sigma)\right) = \nu_{\widehat{\mathbf{n}}}\left(x \in \mathbb{X}_{\widehat{\mathbf{n}}} : \sigma x = x\right) = \frac{\#\left\{x \in \mathbb{X}_{N_k} : \sigma x = x\right\}}{N_k} = \chi_{\mathrm{nat}}(\sigma). \tag{2.10}$$

Remark 2.3. If we regard $\mathfrak{S}_{\widehat{\mathbf{n}}}$ as a subgroup of $\mathrm{Aut}_0(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}})$, then for any $\sigma_1, \sigma_2 \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ we have

$$\chi_{\mathrm{nat}}(\sigma_1\sigma_2^{-1}) = 1 - \rho(\sigma_1, \sigma_2) \quad \text{(see (2.10) and (2.3))}.$$

In particular, χ_{nat} is a continuous function on $\overline{\mathfrak{S}_{\widehat{\mathbf{n}}}}$.

Now consider two families of operators $\left\{\mathfrak{M}'(f)\right\}_{f \in L^\infty(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}})}$ and $\left\{\mathcal{F}'(O^k)\right\}_{k \in \mathbb{Z}}$ which belong to $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'$ and whose action on an element $\eta \in \mathcal{H}$ is defined in the following way:

$$\left(\mathfrak{M}'(f)\eta\right)(x, m) = f\left(O^m x\right)\eta(x, m),\ \left(\mathcal{F}'(O^k)\eta\right)(x, m) = \eta(x, m-k). \tag{2.11}$$

Using (2.7), one can check that $\mathfrak{M}'(f)$ and $\mathcal{F}'(O^k)$ belong to $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'$. Thus, we obtain the following statement:

Lemma 2.3. *Let $\mathcal{N}'$ be the w^*-subalgebra of $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'$, which is generated by the operators $\left\{\mathfrak{M}'(f)\right\}_{f \in L^\infty(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}})}$ and $\mathcal{F}'(O)$. Then, a vector ξ_0 is cyclic for $\mathcal{N}'$, i.e. the closure of the set $\mathcal{N}'\xi_0$ coincides with $\mathcal{H}$.*

Lemma 2.4. *Let $y \in \prod_{k=1}^{j} \mathbb{X}_{n_k}$, $\mathfrak{I}_{\mathbb{A}_y}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{A}_y, \\ 0, & \text{if } x \notin \mathbb{A}_y, \end{cases}$ (see (2.1)) and operator $\mathfrak{M}\left(\mathfrak{I}_{\mathbb{A}_y}\right)$ acts on $\eta \in \mathcal{H}$ as follows:*

$$\left(\mathfrak{M}\left(\mathfrak{I}_{\mathbb{A}_y}\right)\eta\right)(x, m) = \mathfrak{I}_{\mathbb{A}_y}(x)\eta(x, m). \tag{2.12}$$

Then, the operator $\mathfrak{M}\left(\mathfrak{I}_{\mathbb{A}_y}\right)$ belongs to the algebra $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$.

Proof. Applying Lemma 2.1 (a) we obtain that

$$O^{N_k}\mathbb{A}_y = \mathbb{A}_y \text{ for all } k \geq j. \tag{2.13}$$

For $k \geq j$ define an automorphism $D_{k,y}$ as

$$D_{k,y}(x) = \begin{cases} x, & \text{if } x \in \mathbb{A}_y, \\ O^{N_k}x, & \text{if } x \notin \mathbb{A}_y. \end{cases} \tag{2.14}$$

In view of Lemma 2.2 (b) the automorphism $D_{k,y}$ belongs to the group $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$. Since the representation $\mathcal{F}$ of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ can be extended to a representation of $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$ (see (2.7)) it suffices to prove that

$$w - \lim_{k \to \infty} \mathcal{F}\left(D_{k,y}\right) = \mathfrak{M}\left(\mathfrak{I}_{\mathbb{A}_y}\right). \tag{2.15}$$

Here "$w - \lim$" stands for the limit in the weak operator topology.

Using (2.7) and (2.14) we obtain

$$\mathcal{F}\left(D_{k,y}\right)\xi_0 = \mathfrak{I}_{\mathbb{A}_y}\otimes\delta_0 + \left(\mathfrak{1}-\mathfrak{I}_{\mathbb{A}_y}\right)\otimes\delta_{N_k}.$$

Taking into account the weak convergence of the sequence δ_{N_k} to zero in $l^2(\mathbb{Z})$, we obtain that the sequence $\mathcal{F}\left(D_{k,y}\right)\xi_0$ converges weakly in $\mathcal{H}$ to the vector $\mathfrak{I}_{\mathbb{A}_y}\otimes\delta_0 = \mathfrak{M}\left(\mathfrak{I}_{\mathbb{A}_y}\right)\xi_0$. Now (2.15) is a consequence of Lemma 2.3. □

The following statement is a direct corollary of Lemma 2.4.

Corollary 2.5. *The algebra $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$ contains the family of operators $\{\mathfrak{M}\left(f\right)\}_{f\in L^\infty\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)}$, acting on an element $\eta\in\mathcal{H}$ as follows*

$$\left(\mathfrak{M}\left(f\right)\eta\right)(x,m) = f(x)\eta(x,m). \tag{2.16}$$

In particular, the vector ξ_0 is a cyclic vector for the algebra $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$ (see Lemma 2.3) and due to (2.7) the following relations hold:

$$\mathcal{F}(\sigma)\mathfrak{M}(f)\left(\mathcal{F}(\sigma)\right)^{-1} = \mathfrak{M}(\sigma\cdot f),\ \text{where } (\sigma\cdot f)(x) = f\left(\sigma^{-1}(x)\right),\ \sigma\in\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}. \tag{2.17}$$

Proposition 2.6. *The algebra $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$ is a II_1 factor.*

Proof. Suppose that an operator A belongs to $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''\cap\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'$. Since $\mathcal{H} = L^2\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)\otimes l^2(\mathbb{Z})$, we have

$$A\xi_0 = \sum_{i\in\mathbb{Z}} f_i\otimes\delta_i \ \text{ where } f_i\in L^2\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right),\ \delta_i(m) = \begin{cases}1, & \text{if } m=i\\ 0, & \text{if } m\neq i.\end{cases}$$

Recall that $O\in\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$ and the representation $\mathcal{F}$ of $\mathfrak{S}_{\widehat{\mathbf{n}}}$ can be extended by continuity with respect to the metric ρ to a representation of $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$. Therefore, the following equality holds:

$$A\xi_0 = \sum_{i\in\mathbb{Z}}\mathcal{F}\left(O^{-i}\right)\left(\left(O^i\cdot f_i\right)\otimes\delta_0\right). \tag{2.18}$$

Hence, for any $f\in L^\infty\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)$ we have

$$A\mathfrak{M}(f)\xi_0 \overset{(2.16)}{=} \mathfrak{M}(f)\,A\xi_0 = \sum_{i\in\mathbb{Z}} f f_i\otimes\delta_i. \tag{2.19}$$

We thus get

$$\|A\|^2\int\limits_{\mathbb{X}_{\widehat{\mathbf{n}}}}|f(x)|^2\,\mathrm{d}\,\nu_{\widehat{\mathbf{n}}} = \|A\|^2\,\|\mathfrak{M}(f)\xi_0\|^2 \geq \int\limits_{\mathbb{X}_{\widehat{\mathbf{n}}}}|f(x)|^2\left(\sum_{i\in\mathbb{Z}}|f_i(x)|^2\right)\mathrm{d}\,\nu_{\widehat{\mathbf{n}}}.$$

It follows from this that

$$\left\|\sum_{i\in\mathbb{Z}}|f_i|^2\right\|_{L^\infty\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)}\leq\|A\|^2.$$

In particular, $f_i\in L^\infty\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)$ and $\mathfrak{M}(f_i)\in\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''$ for all i (see Corollary 2.5). Hence, using (2.18), we obtain

$$A\xi_0 = \sum_{i\in\mathbb{Z}}\mathcal{F}\left(O^{-i}\right)\mathfrak{M}(O^i\cdot f_i)\xi_0 \overset{(2.17)}{=} \sum_{i\in\mathbb{Z}}\mathfrak{M}\left(f_i\right)\mathcal{F}\left(O^{-i}\right)\xi_0. \tag{2.20}$$

The last equality, Lemma 2.3 and Corollary 2.5 imply that

$$A = \sum_{i\in\mathbb{Z}} \mathfrak{M}(f_i)\,\mathcal{F}\left(O^{-i}\right). \tag{2.21}$$

Therefore, the equality $\mathcal{F}(O)A\xi_0 = A\mathcal{F}(O)\xi_0$ is equivalent to relations $O\cdot f_i = f_i$, where $i\in\mathbb{Z}$. It follows from Lemma 2.2 that function f_i should be constant almost everywhere. In other words, there are constants c_i, $i\in\mathbb{Z}$ such that $f_i \equiv c_i$ almost everywhere and $A\xi_0 = \sum\limits_{i\in\mathbb{Z}} c_i\,\mathcal{F}(O^{-i})\xi_0$.

Finally, note that equality $\mathfrak{M}(f)A\xi_0 = A\mathfrak{M}(f)\xi_0$, $f\in L^\infty\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)$ is equivalent to

$$c_i\cdot f = c_i\cdot(O^{-i}\cdot f), i\in\mathbb{Z}.$$

Since f is arbitrary, Lemma 2.2 implies that $c_i = 0$ for all $i\neq 0$. Therefore, $A\xi_0 = c_0\xi_0$. By Lemma 2.3, A is a scalar operator, so the center $\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''\cap\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'$ is trivial. □

2.3. **Construction of an irreducible representation of the group $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}\times\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$.** For the constructed above II_1 factor representation $\mathcal{F}$ there is a corresponding irreducible representation $\mathcal{F}^{(2)}$ of the group $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}\times\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$ acting in the Hilbert space $\mathcal{H} = L^2\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)\otimes l^2(\mathbb{Z})$ satisfying the following properties:

$$\begin{aligned}
&\mathcal{F}^{(2)}(g,\mathrm{id}) = \mathcal{F}(g),\ \text{where id is the identity of } \overline{\mathfrak{S}}_{\widehat{\mathbf{n}}},\ g\in\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}};\\
&\mathcal{F}^{(2)}(g,g)\xi_0 = \xi_0 \text{ for all } g\in\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}} \text{ and } \mathcal{F}^{(2)}\left(\mathrm{id},\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}\right)\subset\mathcal{F}\left(\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}\right)';\\
&\mathcal{F}^{(2)}\left(\mathrm{id},O^k\right) = \mathcal{F}'(O^k)\ \text{(see (2.11))}.
\end{aligned} \tag{2.22}$$

In order to define $\mathcal{F}^{(2)}$ let us introduce an antiunitary operator $\mathcal{J}$ acting on $\mathcal{H}$ as follows:

$$(\mathcal{J}\eta)(x,m) = \overline{\eta\left(O^m x,-m\right)},\ \eta\in\mathcal{H} = L^2\left(\mathbb{X}_{\widehat{\mathbf{n}}},\nu_{\widehat{\mathbf{n}}}\right)\otimes l^2(\mathbb{Z}). \tag{2.23}$$

Then, direct calculations show that

$$\begin{aligned}
\mathcal{J}\,\mathcal{F}(g)\xi_0 = \mathcal{F}\left(g^{-1}\right)\xi_0 = \left(\mathcal{F}(g)\right)^*\xi_0,\ g\in\mathfrak{S}_{\widehat{\mathbf{n}}};\\
(\mathcal{J}\eta,\mathcal{J}\zeta) = (\zeta,\eta)\ \text{ for all } \zeta,\eta\in\mathcal{H}.
\end{aligned}$$

Combining (2.7), (2.11), (2.16) and (2.23) we obtain

$$\mathcal{J}\mathfrak{M}(f)\mathcal{J} = \mathfrak{M}'(\overline{f}),\ \mathcal{J}\mathcal{F}(O^k)\mathcal{J} = \mathcal{F}'(O^k). \tag{2.24}$$

These equalities, Lemma 2.3 and Corollary 2.5 imply that

$$\mathcal{N}' = \mathcal{J}\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''\mathcal{J}\subset\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'. \tag{2.25}$$

Therefore, the operators $\left\{\mathcal{F}^{(2)}(g,h) = \mathcal{F}(g)\mathcal{J}\mathcal{F}(h)\mathcal{J}\right\}_{g,h\in\mathfrak{S}_{\widehat{\mathbf{n}}}}$ define a representation of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}\times\mathfrak{S}_{\widehat{\mathbf{n}}}$. It is easy to check that $\mathcal{F}^{(2)}$ satisfies the conditions (2.22).

Proposition 2.7. *The representation $\mathcal{F}^{(2)}$ is irreducible.*

Proof. In view of Proposition 2.6 it suffices to show that

$$\mathcal{J}\mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)''\mathcal{J} = \mathcal{F}\left(\mathfrak{S}_{\widehat{\mathbf{n}}}\right)'. \tag{2.26}$$

Denote for convenience the factor $\mathcal{F}(\mathfrak{S}_{\widehat{\mathfrak{n}}})''$ as M. Consider arbitrary bounded operators A' and B such that $A' \in M'$ and $B \in \mathcal{J}M'\mathcal{J}$. In order to prove (2.26) it is enough to check that

$$A'B = BA'. \tag{2.27}$$

According to Corollary 2.5, the vector ξ_0 is a cyclic vector for M. Hence, there is a sequence $\{B_n\}_{n=1}^{\infty} \subset M$ such that

$$\|B\xi_0 - B_n\xi_0\|_{\mathcal{H}} = 0. \tag{2.28}$$

In particular,

$$\lim_{l,m\to\infty} \|B_l\xi_0 - B_m\xi_0\|_{\mathcal{H}} = 0. \tag{2.29}$$

Since $(UV\xi_0, \xi_0) = (VU\xi_0, \xi_0)$ for all $U, V \in M$ we have

$$\|B_l^*\xi_0 - B_m^*\xi_0\|_{\mathcal{H}} = \|B_l\xi_0 - B_m\xi_0\|_{\mathcal{H}}. \tag{2.30}$$

Relations (2.29), (2.30) and equalities $V'B_n = B_nV'$ imply that the sequence $\{B_n^*V'\xi_0\}_{n=1}^{\infty}$ is a Cauchy sequence for any $V' \in M'$. Namely,

$$\lim_{l,m\to\infty} \|B_l^*V'\xi_0 - B_m^*V'\xi_0\|_{\mathcal{H}} = \lim_{l,m\to\infty} \|V'(B_l^*\xi_0 - B_m^*\xi_0)\| = 0.$$

Therefore, we can define the linear operators $\widehat{B}$ and $B^\sharp$ as follows

$$\widehat{B}V'\xi_0 = \lim_{n\to\infty} B_nV'\xi_0,\ B^\sharp V'\xi_0 = \lim_{n\to\infty} B_n^*V'\xi_0,\ \ V' \in M'.$$

Denote by $\mathcal{D}\big(\widehat{B}\big)$ and $\mathcal{D}\big(B^\sharp\big)$ the domains of the operators $\widehat{B}$ and $B^\sharp$ respectively. It is clear that $M'\xi_0 \subset \mathcal{D}\big(\widehat{B}\big)$ and $M'\xi_0 \subset \mathcal{D}\big(B^\sharp\big)$. For any $W' \in \mathcal{J}M\mathcal{J} \subset M'$ we have $W'B = BW'$. Now, it follows from (2.28) that for any $V' \in M'$ we have

$$\begin{aligned}(BW'\xi_0, V'\xi_0) = (W'B\xi_0, V'\xi_0) &\overset{(2.28)}{=} \lim_{n\to\infty} (W'B_n\xi_0, V'\xi_0)\\ &= \lim_{n\to\infty} (B_nW'\xi_0, V'\xi_0) = \lim_{n\to\infty} (W'\xi_0, B_n^*V'\xi_0)\\ &= \big(W'\xi_0, B^\sharp V'\xi_0\big).\end{aligned}$$

Thus, $(W'\xi_0, B^*V'\xi_0) = \big(W'\xi_0, B^\sharp V'\xi_0\big)$ for all $W' \in \mathcal{J}M\mathcal{J} = \mathcal{N}'$ and $V' \in M'$ (see Lemma 2.3). By Lemma 2.3 the set $\{W'\xi_0\}_{W'\in\mathcal{J}M\mathcal{J}}$ is norm dense in $\mathcal{H}$. Therefore,

$$(\eta, B^*V'\xi_0) = \big(\eta, B^\sharp V'\xi_0\big) \tag{2.31}$$

for all $\eta \in \mathcal{H}$ and $V' \in M'$. Hence, for arbitrary $U', V' \in \mathcal{J}M\mathcal{J} = \mathcal{N}'$ we have

$$\begin{aligned}(A'BU'\xi_0, V'\xi_0) = (A'U'B\xi_0, V'\xi_0) &= \lim_{n\to\infty} (A'U'B_n\xi_0, V'\xi_0)\\ &= \lim_{n\to\infty} (B_nA'U'\xi_0, V'\xi_0) = \lim_{n\to\infty} (A'U'\xi_0, B_n^*V'\xi_0)\\ &= \big(A'U'\xi_0, B^\sharp V'\xi_0\big) \overset{(2.31)}{=} (A'U'\xi_0, B^*V'\xi_0) = (BA'U'\xi_0, V'\xi_0).\end{aligned}$$

This proves the equality (2.27) and concludes the proof. □

Remark 2.4. Note that the proof of Proposition 2.7 is similar to the proof of the second part of Lemma 1.4.

3. Preliminaries about the representation theory of the symmetric groups

In this section we recall some notions and facts from the representation theory of the symmetric groups $\mathfrak{S}_N$ which are used in the proof of the main theorem. We are following the Okounkov–Vershik approach to the representation theory of symmetric groups (see [1] and [8] for more details).

3.1. **The minimal element of the conjugacy class.** Consider the symmetric group $\mathfrak{S}_N$. Denote by $\mathfrak{C}_\mathfrak{m}$ the conjugacy class of the group $\mathfrak{S}_N$, which consists of permutations of the cycle type $\mathfrak{m} = (m_1, m_2, \dots, m_N)$, where m_i is the number of independent cycles of length i.

Definition 3.1. The *support* of a permutation $s \in \mathfrak{S}_N$ is defined as $\operatorname{supp}_{_N} s = \{x \in \mathbb{X}_N \mid sx \neq x\}$.

Remark 3.1. Further, we also use the notation $\mathfrak{m} = (m_1, m_2, \dots, m_l)$ for a cycle type $\mathfrak{m}$ instead of $\mathfrak{m} = (m_1, m_2, \dots, m_N)$ if $m_{l+1} = \dots = m_N = 0$ (here $l < N$).

Clearly, for $\sigma \in \mathfrak{C}_\mathfrak{m} \subset \mathfrak{S}_N$ we have $\sum\limits_{i=1}^{N} i \cdot m_i = N$ and $\#\operatorname{supp}_{_N} s = \sum\limits_{i=2}^{N} i \cdot m_i$. For any distinct elements $n_0, n_1, \dots, n_{j-1} \in \mathbb{X}_N$ denote by $(n_0\ n_1\ \dots\ n_{j-1})$ the cyclic permutation $c \in \mathfrak{S}_N$ such that $c(n_i) = n_{i+1 \pmod j}$ and $c(n) = n$ for all $n \in \mathbb{X}_N \setminus \{n_0, \dots, n_{j-1}\}$. Denote by s_i the transposition $(i\ \ i+1)$. The elements $\{s_1, s_2, \dots, s_{N-1}\}$ are also known as the *Coxeter generators* of the symmetric group $\mathfrak{S}_N$.

Definition 3.2. Consider a conjugacy class $\mathfrak{C}_\mathfrak{m}$ of the symmetric group $\mathfrak{S}_N$. Let $\{i_1, i_2, \dots, i_p\}$, where $1 < i_1 < i_2 < \dots < i_p$, be the set of those $i \in \{2, 3, \dots, N\}$ for which $m_i \geq 1$. The *minimal element* of the conjugacy class $\mathfrak{C}_\mathfrak{m}$ is the permutation $\sigma_\mathfrak{m}$ defined as the following product:

$$\begin{aligned}\sigma_\mathfrak{m} = {} & (1\ 2\ \dots\ i_1) \cdots ((m_{i_1} - 1)\, i_1 + 1\ \ (m_{i_1} - 1)\, i_1 + 2\ \ \dots\ \ m_{i_1}\, i_1) \cdots \\ & (m_{i_1}\, i_1 + 1\ \ m_{i_1}\, i_1 + 2\ \ \dots\ \ m_{i_1}\, i_1 + i_2) \cdots \\ & (m_{i_1}\, i_1 + (m_{i_2} - 1)\, i_2 + 1\ \ m_{i_1}\, i_1 + (m_{i_2} - 1)\, i_2 + 2\ \ \dots\ \ m_{i_1}\, i_1 + m_{i_2}\, i_2) \cdots\end{aligned}$$

It is clear that $\sum\limits_{q=1}^{p} i_q m_{i_q} = N - m_1$ and $\operatorname{supp}_{_N} \sigma_\mathfrak{m} = \{1, 2, \dots, N - m_1\} \subset \mathbb{X}_N$. The crucial property of $\sigma_\mathfrak{m}$ is the following decomposition into the product of Coxeter generators:

$$\sigma_\mathfrak{m} = s_{j_1} s_{j_2} \cdots s_{j_r} \tag{3.1}$$

for some elements $j_1 < j_2 < \dots < j_r$ of $\{1, 2, \dots, N\}$ and $r = N - m_1 - \sum\limits_{q=1}^{p} m_{i_q}$. The existence of such a decomposition follows from the identity

$$(i\ i+1\ \dots\ i+j) = s_i s_{i+1} \dots s_{i+j-1}.$$

For any permutation $\sigma \in \mathfrak{S}_N$ denote by $\kappa_{_N}(\sigma)$ the minimal number of factors in the decomposition of σ into the product of transpositions. Define the *sign (or signature)* of permutation σ as $\operatorname{sgn}_N(\sigma) = (-1)^{\kappa_{_N}(\sigma)}$.

3.2. **Young tableaux and diagrams.** Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_j)$, where $\lambda_1 \geq \dots \geq \lambda_j \geq 1$, be a partition of a positive integer $N = |\lambda|$ into positive integer summands $\lambda_1, \lambda_2, \dots, \lambda_j$, i.e. $\sum_{i=1}^{j} \lambda_i = N$. We also write $\lambda \vdash N$ if λ is a partition of N. We identify λ with the corresponding Young diagram consisting of j rows of the length $\lambda_1, \lambda_2, \dots, \lambda_j$ aligned to the left (the i-th row consists of λ_i boxes). The conjugate or transposed Young diagram λ' is obtained from λ by swapping its rows with its columns.

A *standard Young tableau* T of the shape λ is the diagram λ, whose boxes are filled with positive integers from 1 to N such that each number occurs exactly once and $n_{pq} < \min\{n_{p+1\,q}, n_{p\,q+1}\}$ for all p, q (see Figure 1). In this paper we consider only standard Young tableaux.

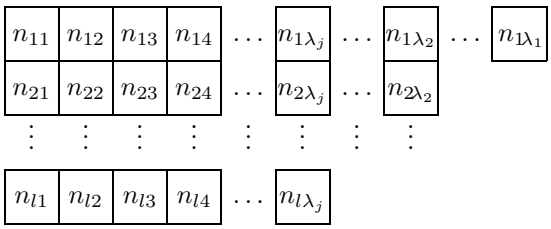

FIGURE 1. A standard Young tableau of the shape λ

Denote the set of all standard Young tableaux of the shape λ by $\mathrm{Tab}(\lambda)$. The number $c(p, q) = q - p$ is called the *content* of the box with coordinates (p, q) of the diagram λ (see [7]) or the content of the element n_{pq} of tableau $T \in \mathrm{Tab}(\lambda)$ (see [1]). We set $a_i = c(p, q)$ for $n_{pq} = i$. For each tableau T define two functions $\mathfrak{R}_T$ and $\mathfrak{C}_T$ (row and column numbers) on the set $\{1, 2, \dots, N = |\lambda|\}$ as follows

$$\mathfrak{R}_T(n_{pq}) = p, \quad \mathfrak{C}_T(n_{pq}) = q. \tag{3.2}$$

3.3. **Realizations of the irreducible representations of $\mathfrak{S}_N$.** In this subsection we recall the explicit constructions of the irreducible representations of the symmetric group $\mathfrak{S}_N$ (we refer the reader to [1] and [8] for more details). Besides that, we prove several technical facts about the characters of irreducible representations of $\mathfrak{S}_N$.

It is known that the irreducible representations of $\mathfrak{S}_N$ are parameterized by the Young diagrams λ with N boxes. Let $\mathcal{R}_\lambda$ be one of those unitary irreducible representations which acts in the vector space V_λ. There exists an orthonormal basis $\{v_T\}_{T \in \mathrm{Tab}(\lambda)}$ in V_λ such that the Coxeter generators $s_i = (i\ i+1)$ act on elements of this basis as follows:

- if $\mathfrak{R}_T(i) = \mathfrak{R}_T(i+1)$, then $\mathcal{R}_\lambda(s_i)v_T = v_T$;
- if $\mathfrak{C}_T(i) = \mathfrak{C}_T(i+1)$, then $\mathcal{R}_\lambda(s_i)v_T = -v_T$;
- if $\mathfrak{R}_T(i) \neq \mathfrak{R}_T(i+1)$ and $\mathfrak{C}_T(i) \neq \mathfrak{C}_T(i+1)$, then swapping the positions of elements i and $i+1$ in T also gives some tableau T'. Then, the operator $\mathcal{R}_\lambda(s_i)$ in the two-dimensional space $\mathrm{span}\{v_T, v_{T'}\}$ in the basis $\{v_T, v_{T'}\}$ has the following matrix:

$$\begin{pmatrix} 1/d & \sqrt{1 - 1/d^2} \\ \sqrt{1 - 1/d^2} & -1/d \end{pmatrix}, \tag{3.3}$$

where $d = a_{i+1} - a_i$ and a_i is the content of the box of T that contains i. As a consequence, we obtain the following statement.

Lemma 3.3. *Let $\sigma \in \mathfrak{S}_N$. Put $\widehat{\mathcal{R}}_\lambda(\sigma) = \mathrm{sgn}(\sigma)\mathcal{R}_\lambda(\sigma)$. Then the representations $\widehat{\mathcal{R}}_\lambda$ and $\mathcal{R}_{\lambda'}$ of the group $\mathfrak{S}_N$ are unitarily equivalent.*

Define the *normalized character* of the irreducible representation $\mathcal{R}_\lambda$ by the formula:

$$\chi_\lambda(\sigma) = \frac{\mathrm{Tr}\,(\mathcal{R}_\lambda(\sigma))}{\mathrm{Tr}\,(\mathcal{R}_\lambda(\mathrm{id}))}, \tag{3.4}$$

where Tr is the standard trace of a finite-dimensional operator, $\sigma \in \mathfrak{S}_N$ and id is the identity element of $\mathfrak{S}_N$. Note that Lemma 3.3 implies the equality

$$\chi_{\lambda'}(\sigma) = \mathrm{sgn}(\sigma)\chi_\lambda(\sigma) \tag{3.5}$$

for all $\sigma \in \mathfrak{S}_N$ and $\lambda \vdash N$.

Lemma 3.4. *Suppose that permutation $\sigma \in \mathfrak{S}_N$ is expressed as a product of distinct Coxeter generators as:*

$$\sigma = s_{i_1} \dots s_{i_r}, \text{ where } i_1 < i_2 < \dots < i_r.$$

Then for any tableau T the following equality holds:

$$|(\mathcal{R}_\lambda(\sigma)v_T, v_T)| = \prod_j \frac{1}{|a_{i_j+1} - a_{i_j}|}, \tag{3.6}$$

where the product is taken over all indices j for which the numbers i_j and i_j+1 are in different rows and different columns of the tableau T, i.e. $\mathfrak{R}_T(i_j) \neq \mathfrak{R}_T(i_j+1)$ and $\mathfrak{C}_T(i_j) \neq \mathfrak{C}_T(i_j+1)$.

Proof. Consider an arbitrary Coxeter generator s_i and a basis vector $v_T \in V_\lambda$, where $T \in \mathrm{Tab}(\lambda)$. If i and $i+1$ are in the same row or in the same column of T, we have $\mathcal{R}_\lambda(s_i)v_T = \pm v_T$. Thus, in this case for any permutation $\tau \in \mathfrak{S}_N$ we have

$$(\mathcal{R}_\lambda(\tau s_i)v_T, v_T) = \pm(\mathcal{R}_\lambda(\tau)v_T, v_T). \tag{3.7}$$

Otherwise, according to (3.3), we have

$$\mathcal{R}_\lambda(s_i)v_T = \pm\frac{1}{a_{i+1} - a_i}v_T + \sqrt{1 - \frac{1}{(a_{i+1} - a_i)^2}}v_{T'}, \tag{3.8}$$

where $T' = s_i(T)$ is the tableau obtained from T by transposition of elements i and $i+1$ (recall that a_i and a_{i+1} are the contents of the boxes of T which contain i and $i+1$ respectively).

Next, we show that for any $\tau \in \mathfrak{S}_N$ whose support $\mathrm{supp}_{_N}\,\sigma$ does not contain $i+1$ we have

$$(\mathcal{R}_\lambda(\tau s_i)v_T, v_T) = \pm\frac{1}{a_{i+1} - a_i}(\mathcal{R}_\lambda(\tau)v_T, v_T). \tag{3.9}$$

Indeed, it suffices to check that the vector $\mathcal{R}_\lambda(\tau)v_{T'}$ is orthogonal to v_T. It follows from the fact that $\mathcal{R}_\lambda(\tau)v_{T'}$ is a linear combination of vectors $v_{T''}$ which correspond to those tableaux T'' in which the number $i+1$ is written in the same box as in T'. In particular, for all such T'' we have $T'' \neq T$ and hence, $(v_{T''}, v_T) = 0$. Then $(\mathcal{R}_\lambda(\tau)v_{T'}, v_T) = 0$, so the required orthogonality is proved.

Thus, for any permutation $\tau \in \mathfrak{S}_N$ whose support does not contain $i+1$ we have

$$(\mathcal{R}_\lambda(\tau s_i)v_T, v_T) = \pm k_i(T) \cdot (\mathcal{R}_\lambda(\tau)v_T, v_T),$$

where

$$k_i(T) = \begin{cases} \frac{1}{a_{i+1}-a_i}, & \text{if } \mathfrak{R}_T(i) \neq \mathfrak{R}_T(i+1) \text{ and } \mathfrak{C}_T(i) \neq \mathfrak{C}_T(i+1), \\ 1, & \text{otherwise.} \end{cases}$$

Now, let us apply this observation to $(\mathcal{R}_\lambda(\sigma)v_T, v_T)$. Note that the sequence $\{i_j\}$ is strictly increasing, so for all j the support of permutation $s_{i_1} \dots s_{i_{j-1}}$ does not contain $i_j + 1$. Therefore,

$$\begin{aligned}(\mathcal{R}_\lambda(\sigma)v_T, v_T) &= (\mathcal{R}_\lambda(s_{i_1} \dots s_{i_r})v_T, v_T) \\ &= \pm k_{i_r}(T)(\mathcal{R}_\lambda(s_{i_1} \dots s_{i_{r-1}})v_T, v_T) = \dots = \pm \prod_{j=1}^{r} k_{i_j}(T)(v_T, v_T).\end{aligned}$$

Thus,

$$|(\mathcal{R}_\lambda(\sigma)v_T, v_T)| = \prod_{j=1}^{r} |k_{i_j}(T)|$$

and we obtain the formula (3.6). Lemma 3.4 is proved. □

3.4. **Upper bound for the characters of the symmetric group.** In the proof of the main theorem we use the following important bound for the characters of the symmetric group (see [9] and also [4]).

Proposition 3.5 (**Roichman**, 1996)**.** *There exist absolute constants $a \in (0,1)$ and $b > 0$ such that for any Young diagram λ with $N = |\lambda| \geq 4$ boxes and for any $\sigma \in \mathfrak{S}_N$ the following inequality holds:*

$$\left|\chi_\lambda(\sigma)\right| \leq \left(\max\left\{\frac{\lambda_1}{N}, \frac{\lambda'_1}{N}, a\right\}\right)^{b \cdot \#\left(\mathrm{supp}_N(\sigma)\right)}. \tag{3.10}$$

Here λ_1 (respectively, λ'_1) is the number of boxes in the first row (respectively, column) of the diagram λ.

4. The approximation theorem for the characters on $\mathfrak{S}_{\widehat{\mathbf{n}}}$

In this section we state and prove the approximation theorem for characters on the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$, which is a crucial part of the proof of Theorem 1.2.

Firstly, let us recall the definition of a character.

Definition 4.1. A function $\chi\colon G \to \mathbb{C}$ on a group G is called a *character* if the following conditions hold:

(a) χ is *central*, i.e. $\chi(gh) = \chi(hg)$ for all $g, h \in G$;
(b) χ is a *positive-definite function*, i.e. for any elements $g_1, \dots, g_k \in G$ the matrix $\left[\chi(g_i g_j^{-1})\right]$ is a Hermitian and positive-semidefinite matrix;
(c) χ is *normalized*, i.e. $\chi(\mathrm{id}) = 1$, where id is the identity of the group G.

If additionally

(d) χ is *indecomposable*, i.e. χ cannot be represented as a sum of two linear independent functions that satisfy (a) and (b),

then χ is called an *indecomposable character*.

Remark 4.1. Let π_χ be the representation of G, obtained from χ via the Gelfand–Naimark–Segal (shortly GNS) construction. Then the property (d) is equivalent to π_χ being a factor representation.

The following fact is an analogue of the approximation theorem from [11] for the characters of the standard infinite symmetric group $\mathfrak{S}_\infty$.

Proposition 4.2. *Each indecomposable character χ on $\mathfrak{S}_{\widehat{\mathbf{n}}}$ is a weak limit of some sequence of normalized irreducible characters of the groups $\mathfrak{S}_{N_k}$. Namely, there exists an increasing sequence $\{k(l)\}_{l=1}^{\infty}$ of positive integers and there exists a sequence of partitions $\{\lambda(k(l)) \vdash N_{k(l)}\}_{l=1}^{\infty}$ such that*

$$\lim_{l\to\infty} \chi_{\lambda(k(l))}(g) = \chi(g) \text{ for all } g \in \mathfrak{S}_{\widehat{\mathbf{n}}}.$$

Example of approximation. Note that Proposition 2.6 together with formulas (2.8) and (2.9) imply that χ_{nat} is an indecomposable character of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$. Then, according to Proposition 4.2 χ_{nat} has to be a weak limit of a sequence of normalized irreducible characters of $\mathfrak{S}_{N_k}$. In fact, this can be seen directly. Namely, denote by $\mathfrak{P}_k$ a representation of $\mathfrak{S}_{N_k}$ that acts on a vector $(v_1, v_2, \dots, v_{N_k})$ from the space $\mathbb{C}^{N_k}$ by the permutations of the coordinates: $\mathfrak{P}_k(s)\,(v_1, v_2, \dots, v_{N_k}) = \left(v_{s^{-1}(1)}, v_{s^{-1}(2)}, \dots, v_{s^{-1}(N_k)}\right)$, where $s \in \mathfrak{S}_{N_k}$. If a projection P_{id} is defined by $P_{id}\,(v_1, v_2, \dots, v_{N_k}) = \frac{1}{N_k}\left(\sum_{j=1}^{N_k} v_j, \sum_{j=1}^{N_k} v_j, \dots, \sum_{j=1}^{N_k} v_j\right)$, then $P_{id}\,\mathfrak{P}_k(s) = P_{id}$ for all $s \in \mathfrak{S}_{N_k}$. Denote by $\mathfrak{P}_k^0$ the restriction of $\mathfrak{P}_k$ to the subspace $(I - P_{id})\,\mathbb{C}^{N_k}$. In the notations of Subsection 3.3 we have $\chi_k = \chi_{\lambda(k)}$, where $\lambda(k)$ is the partition $(N_k - 1, 1)$. Let Tr be the ordinary trace of operators, and let χ_k be the character of the irreducible representation $\mathfrak{P}_k^0$. Then $\chi_k(s) = \frac{1}{N_k}\left(\mathrm{Tr}\,(\mathfrak{P}_k(s)) - \mathrm{Tr}\,(P_{id})\right) = 1 - \frac{\#\,\mathrm{supp}\ s}{N_k} - \frac{1}{N_k}$. According to (1.5), we conclude that $\lim_{k\to\infty} \chi_{\lambda(k)}(s) = \chi_{\mathrm{nat}}(s)$ for all $s \in \mathfrak{S}_{\widehat{\mathbf{n}}}$.

Proof of Proposition 4.2. Consider the GNS-representation $(\pi_\chi, \mathcal{H}_\chi, \xi_\chi)$ of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ acting in the Hilbert space $\mathcal{H}_\chi$ with a cyclic and separating vector ξ_χ such that $(\pi_\chi(g)\xi_\chi, \xi_\chi) = \chi(g)$ for all $g \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. Denote by $\mathcal{B}(\mathcal{H}_\chi)$ the set of all bounded linear operators on $\mathcal{H}_\chi$. For any conjugation-invariant subset $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_\chi)$ define its commutant by

$$\mathcal{S}' = \{A \in \mathcal{B}(\mathcal{H}_\chi) : AB = BA \text{ for all } B \in \mathcal{S}\}, (\mathcal{S}')' = \mathcal{S}''.$$

We denote by $[\mathcal{S}\mathfrak{V}]$ the smallest closed subspace containing $\mathcal{S}\mathfrak{V}$, where $\mathfrak{V}$ is a subset of $\mathcal{H}_\chi$.

By the properties of the GNS-construction applied to a character χ, we have

$$[\pi_\chi(\mathfrak{S}_{\widehat{\mathbf{n}}})\xi_\chi] = [\pi_\chi(\mathfrak{S}_{\widehat{\mathbf{n}}})'\xi_\chi] = \mathcal{H}_\chi. \tag{4.1}$$

For convenience we denote the w^*-algebra $(\pi_\chi(\mathfrak{S}_{\widehat{\mathbf{n}}}))''$ by M. Note that M is a factor of type II_1 and its faithful normal normalized trace tr on M is the vector state defined by ξ_χ. In other words, $\mathrm{tr}(a) = (a\xi_\chi, \xi_\chi)$ for $a \in M$ (see also Subsection 1.3).

Denote by $L^2(M, \mathrm{tr})$ the Hilbert space which is the completion of M with respect to the norm which corresponds to the inner product $\langle a, b\rangle_{\mathrm{tr}} = \mathrm{tr}\,(b^* a)$, where $a, b \in M$.

From now on we suppose that $\mathcal{H}_\chi = L^2(M, \mathrm{tr})$. Then we can assume that the operators $a \in M$ act by the *left multiplication*; i.e.

$$L^2(M, \mathrm{tr}) \ni \eta \overset{a}{\mapsto} a\eta \in L^2(M, \mathrm{tr}).$$

In this case the operators $a' \in M'$ are realized by the operators of the *right multiplication* by the elements of $a \in M$

$$L^2(M, \mathrm{tr}) \ni \eta \overset{a'}{\mapsto} \eta \cdot a \in L^2(M, \mathrm{tr}).$$

Each element $g \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ defines an inner automorphism $\mathrm{Ad}\, g$ of the factor M, defined by the formula

$$\mathrm{Ad}\, g(a) = \pi_\chi(g) a \pi_\chi(g^{-1}), \quad a \in M.$$

The map E_{N_k} given by the formula

$$M \ni a \overset{E_{N_k}}{\mapsto} \frac{1}{N_k!} \sum_{g \in \mathfrak{S}_{N_k}} \mathrm{Ad}\, g(a) \in M \tag{4.2}$$

is the *conditional expectation* (see [10]) which projects M onto the subalgebra

$$M_{N_k} = \{a \in M : \mathrm{Ad}\, g(a) = a \ \text{ for all } g \in \mathfrak{S}_{N_k}\}.$$

For $\lambda \vdash N_k$ define $P_\lambda = \frac{\dim^2 \lambda}{N_k!} \sum\limits_{g \in \mathfrak{S}_{N_k}} \chi_\lambda(g) \pi_\chi(g)$, where $\dim \lambda$ is the dimension of the irreducible representation of $\mathfrak{S}_{N_k}$ that corresponds to λ. Then P_λ belongs to the center of the finite-dimensional algebra $\pi_\chi(\mathfrak{S}_{N_k})''$ and

$$E_{N_k}(\pi_\chi(g)) = \sum_{\lambda \vdash N_k} \chi_\lambda(g)\, P_\lambda \ \text{ for all } \ g \in \mathfrak{S}_{N_k}. \tag{4.3}$$

The conditional expectations E_{N_k} are orthogonal projections in $L^2(M, \mathrm{tr})$. Since $E_{N_k} \geq E_{N_{k+1}}$, we have

$$\lim_{k \to \infty} \|E_{N_k}(a) - E_\infty(a)\|_{L^2(M, \mathrm{tr})} = 0 \ \text{ for any } a \in M, \tag{4.4}$$

where $E_\infty = \lim\limits_{k \to \infty} E_{N_k}$. Since w^*-algebra $M = \pi_\chi(\mathfrak{S}_{\widehat{\mathbf{n}}})''$ is a factor (according to Proposition 1.6), formula (4.2) implies that

$$E_\infty(a) = \mathrm{tr}(a) I \ \text{ for any } a \in M.$$

Thus, combining (4.3) and (4.4) and setting $a = \pi_\chi(g)$ for $g \in \mathfrak{S}_{\widehat{\mathbf{n}}}$, we obtain

$$\lim_{k \to \infty} \sum_{\lambda \vdash N_k} (\chi_\lambda(g) - \chi(g))^2 \,\mathrm{tr}\,(P_\lambda) = 0 \ \text{ for any } g \in \mathfrak{S}_{\widehat{\mathbf{n}}}. \tag{4.5}$$

Consider two sequences of positive reals $\{\epsilon_l\}$ и $\{\delta_l\}$ which satisfy the following conditions:

$$\lim_{l \to \infty} \max\{\epsilon_l, \delta_l\} = 0 \ \text{ и } \lim_{l \to \infty} \frac{\epsilon_l\, N_l!}{\delta_l} = 0. \tag{4.6}$$

Now, using (4.5) we find for each l a positive integer $k(l)$ such that

$$\sum_{\lambda \vdash N_m} (\chi_\lambda(g) - \chi(g))^2 \,\mathrm{tr}\,(P_\lambda) < \epsilon_l \text{ for all } g \in \mathfrak{S}_{N_l} \text{ and } m \geq k(l). \tag{4.7}$$

Put $\Lambda(m,g) = \left\{\lambda \vdash N_m : (\chi_\lambda(g) - \chi(g))^2 > \delta_l\right\}$, where $g \in \mathfrak{S}_{N_l}$, and $\Lambda_l(m) = \bigcup\limits_{g\in\mathfrak{S}_{N_l}} \Lambda(m,g)$. Applying (4.7), we obtain

$$\sum_{\lambda:(\lambda\vdash N_m)\&(\lambda\notin\Lambda_l(m))} \mathrm{tr}(P_\lambda) > 1 - \frac{\epsilon_l\, N_l!}{\delta_l}. \tag{4.8}$$

This inequality and (4.6) imply that for each l there exists a partition $\lambda(k(l)) \vdash N_{k(l)}$ such that

$$\left|\chi_{\lambda(k(l))}(g) - \chi(g)\right| \le \sqrt{\delta_l} \ \text{ for all } \ g \in \mathfrak{S}_{N_l}.$$

Since $\lim\limits_{l\to\infty} \delta_l = 0$ we have $\lim\limits_{l\to\infty} \chi_{\lambda(k(l))}(g) = \chi(g)$ for all $g \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. Thus, the sequences $\{k(l)\}_{l=1}^\infty$ and $\{\lambda(k(l)) \vdash N_{k(l)}\}_{l=1}^\infty$ satisfy the required conditions. Proposition 4.2 is proved. □

5. Technical lemmas

In this section we establish several technical facts which are used in the proof of Theorem 1.2.

First, let us introduce some notations. Let $\Upsilon(n)$ be the set of all partitions of a positive integer n. Denote by (n) the partition in $\Upsilon(n)$ which consists of only one part and which corresponds to the Young diagram with only one row with n boxes. Put $(1^n) = (n)'$, i.e. (1^n) is the partition which consists of n parts equal 1 and whose Young diagram has only one column with n boxes. For an arbitrary partition μ denote $\Upsilon_\mu(n) = \{\lambda \in \Upsilon(n) : \lambda \setminus (\lambda_1) = \mu\}$ and $\widehat{\Upsilon}_\mu(n) = \left\{\lambda \in \Upsilon(n) : \lambda \setminus \left(1^{\lambda'_1}\right) = \mu\right\}$, where $\lambda = (\lambda_1, \dots, \lambda_l)$. It is clear that $\Upsilon_\mu(n) = \left\{\lambda \in \Upsilon(n) : \lambda' \setminus \left(1^{\lambda_1}\right) = \mu'\right\}$ and $\widehat{\Upsilon}_\mu(n) = \{\lambda \in \Upsilon(n) : \lambda' \setminus (\lambda'_1) = \mu'\}$.

Define in the set of all sequences $\{\lambda(k) \vdash N_k\}_{k=1}^\infty$ of partitions the subsets C_∞, $\widehat{C}_\infty$ C_μ and $\widehat{C}_\mu$, where μ is a partition and $|\mu| < \infty$, as follows:

- (C_∞) there is a subsequence $\{k_i\}$ such that
$$\lim_{i\to\infty} |\lambda(k_i) \setminus (\lambda_1(k_i))| = \infty;$$
- $(\widehat{C}_\infty)$ there is a subsequence $\{k_i\}$ such that
$$\lim_{i\to\infty} \left|\lambda(k_i) \setminus \left(1^{\lambda'_1(k_i)}\right)\right| = \infty;$$
- (C_μ) there is a subsequence $\{k_i\}$ such that $\lambda(k_i) \in \Upsilon_\mu(N_{k_i})$, where μ is independent of i;
- $(\widehat{C}_\mu)$ there is a subsequence $\{k_i\}$ such that $\lambda(k_i) \in \widehat{\Upsilon}_\mu(N_{k_i})$, where μ is independent of i.

Remark 5.1. Clearly, $\widehat{C}_\mu$ (respectively, $\widehat{C}_\infty$) is the set of all sequences which can be obtained by transposing elementwise the sequences $\{\lambda(k) \vdash N_k\}$ in C_μ (respectively, C_∞).

The following statement is trivial.

Lemma 5.1. *The union* $(C_\infty \cap \widehat{C}_\infty) \cup \bigcup\limits_\mu C_\mu \cup \bigcup\limits_\mu \widehat{C}_\mu$ *coincides with the set of all sequences of partitions* $\{\lambda(k) \vdash N_k\}_{k=1}^\infty$.

Now consider an arbitrary sequence $\{\lambda(k) \vdash N_k\}_{k=1}^{\infty}$ of partitions. Let $\mathcal{R}_{\lambda(k)}$ be the irreducible representation of $\mathfrak{S}_{N_k}$ which acts in the Hilbert space $V_{\lambda(k)}$. Let $\{v_T\}_{T\in \mathrm{Tab}(\lambda(k))}$ be the orthonormal basis of $V_{\lambda(k)}$, whose vectors are parameterized by the Young tableaux of shape $\lambda(k)$ (see Subsection 3.3). Then

$$\chi_{\lambda(k)}(\sigma) = \frac{\mathrm{Tr}\left(\mathcal{R}_{\lambda(k)}(\sigma)\right)}{\mathrm{Tr}\left(\mathcal{R}_{\lambda(k)}(\mathrm{id})\right)} = \frac{\sum\limits_{T\in \mathrm{Tab}(\lambda(k))} \left(\mathcal{R}_{\lambda(k)}(\sigma)\, v_T, v_T\right)}{\mathrm{Tr}\left(\mathcal{R}_{\lambda(k)}(\mathrm{id})\right)}. \tag{5.1}$$

Remark 5.2. Suppose that σ belongs to the conjugacy class $\mathfrak{C}_{\mathfrak{m}_{|k}}$ of the group $\mathfrak{S}_{N_k}$, which consists of the permutations of the cycle type $\mathfrak{m}_{|k} = (m_{1|k}, m_{2|k}, \dots, m_{l|k})$ (see Subsection 3.1 and Remark 3.1). Recall that $\mathfrak{i}_{k,j}$, where $j > k$, is the embedding of $\mathfrak{S}_{N_k}$ into $\mathfrak{S}_{N_j}$ (see Section 1.1 and also (1.1)). Then $\mathfrak{i}_{k,j}\left(\mathfrak{C}_{\mathfrak{m}_{|k}}\right) \subset \mathfrak{C}_{\mathfrak{m}_{|j}}$, where

$$\mathfrak{m}_{|j} = \left(m_{1|j}, m_{2|j}, \dots, m_{l|j}\right) = \left(m_{1|k}\frac{N_j}{N_k}, m_{2|k}\frac{N_j}{N_k}, \dots, m_{l|k}\frac{N_j}{N_k}\right),\ \frac{N_j}{N_k} = \prod_{i=k+1}^{j} n_i.$$

In particular, if $\sigma \in \mathfrak{S}_{N_k}$, then $\#\left(\mathrm{supp}_{N_j}\ \mathfrak{i}_{k,j}(\sigma)\right) = \#\left(\mathrm{supp}_{N_k}\ \sigma\right)\cdot \frac{N_j}{N_k}$ and $\kappa_{N_j}(\sigma) = \frac{N_j}{N_k}\,\kappa_{N_k}(\sigma)$ (see Proposition 3.5). From now on we identify $g \in \mathfrak{S}_{N_k}$ with its image $\mathfrak{i}_{k,j}(g) \in \mathfrak{S}_{N_j}$ while taking into account the changes of the cycle type.

Recall that sgn_{N_k} is a one-dimensional representation of the group $\mathfrak{S}_{N_k}$ defined as

$$\mathrm{sgn}_{N_k}(s) = \begin{cases} -1, & \text{if } s \in \mathfrak{S}_{N_k} \text{ is an odd permutation,} \\ 1, & \text{if } s \in \mathfrak{S}_{N_k} \text{ is an even permutation.} \end{cases}$$

The following statement is immediate.

Lemma 5.2. *For any $s \in \mathfrak{S}_{\widehat{\mathbf{n}}}$ there exists a positive integer $N(s)$ such that $\mathrm{sgn}_{N_k}(s) = \mathrm{sgn}_{N_j}(s)$ for all $j, k > N(s)$.*

Hence, there exists a one-dimensional representation sgn_{∞} of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ defined as follows

$$\mathrm{sgn}_{\infty}(s) = \lim_{k\to\infty} \mathrm{sgn}_{N_k}(s),\quad s \in \mathfrak{S}_{\widehat{\mathbf{n}}}. \tag{5.2}$$

Remark 5.3. If the sequence $\widehat{\mathbf{n}} = \{n_k\}_{k=1}^{\infty}$ contains infinitely many even numbers, then $\mathrm{sgn}_{\infty}(s) = 1$ for all $s \in \mathfrak{S}_{\widehat{\mathbf{n}}}$.

Remark 5.4. Suppose that there are only finitely many even numbers in the sequence $\widehat{\mathbf{n}} = \{n_k\}_{k=1}^{\infty}$. Then one can choose two sequences $\{s_n\}$ and $\{\sigma_n\}$ in $\mathfrak{S}_{\widehat{\mathbf{n}}}$ such that $\mathrm{sgn}_{\infty}(s_n) = 1$ and $\mathrm{sgn}_{\infty}(\sigma_n) = -1$ for all n but which converge to the same automorphism $s \in \overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$ with respect to the metric ρ (see (2.3))

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sigma_n = s.$$

Therefore, sgn_{∞} cannot be extended by continuity to the closure $\overline{\mathfrak{S}}_{\widehat{\mathbf{n}}}$ of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ with respect to the metric ρ.

Lemma 5.3. *Let χ be a character of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and let $\{\lambda(k) \vdash N_k\}_{k=1}^{\infty}$ be a sequence of partitions such that $\lim\limits_{k\to\infty} \chi_{\lambda(k)}(g) = \chi(g)$ for all $g \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. If $\{\lambda(k)\}_{k=1}^{\infty} \in C_{\infty} \cap \widehat{C}_{\infty}$, then $\chi(g) = \begin{cases} 1, & \text{if } g = \mathrm{id}; \\ 0, & \text{if } g \neq \mathrm{id}\,. \end{cases}$*

Proof. Clearly, $\chi(\mathrm{id}) = 1$. Take any element $g \neq \mathrm{id}$ of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$. Due to Lemma 3.3, it is sufficient to consider the case when $\{\lambda(k)\}_{k=1}^{\infty} \in C_{\infty}$ and $\lambda_1(k) \geq \lambda'_1(k)$ for all k. Fix a subsequence $\{k_i\}_{i=1}^{\infty}$ which satisfies the condition

$$\lim_{i\to\infty} |\lambda(k_i)/\left(\lambda_1(k_i)\right)| = \infty. \tag{5.3}$$

Suppose that $g \in \mathfrak{S}_{N_{k_1}} \subset \mathfrak{S}_{\widehat{\mathbf{n}}}$.

It suffices to consider the case when $k_i = i$ for all i, i.e. when the sequence $\{k_i\}_{i=1}^{\infty}$ coincides with the sequence $\{k\}_{k=1}^{\infty}$ (the proof of the general case is analogous). Then, Proposition 3.5 implies that

$$\left|\chi_{\lambda(k)}(g)\right| \leq \left(\max\left\{\frac{\lambda_1(k)}{N_k}, a\right\}\right)^{b\cdot \#\left(\mathrm{supp}_{N_k}(g)\right)}, \text{ where } a \in (0,1), b > 0.$$

Hence, after passing to the limit when $k \to \infty$ we obtain

$$|\chi(g)| \leq \limsup_{k\to\infty} \left(\frac{\lambda_1(k)}{N_k}\right)^{b\cdot \#\left(\mathrm{supp}_{N_k}(g)\right)} = \limsup_{k\to\infty} \left(\frac{\lambda_1(k)}{N_k}\right)^{bN_k\cdot \#\left(\mathrm{supp}_{N_1}(g)\right)/N_1}.$$

Therefore,

$$\begin{aligned} |\chi(g)| &\leq \limsup_{k\to\infty} \left(1 - \frac{N_k - \lambda_1(k)}{N_k}\right)^{bN_k\cdot \#\left(\mathrm{supp}_{N_1}(g)\right)/N_1} \\ &\leq \limsup_{k\to\infty} \exp\left(-\left|\lambda(k)/\left(\lambda_1(k)\right)\right| \cdot \frac{b}{N_1} \cdot \#\left(\mathrm{supp}_{N_1}(g)\right)\right). \end{aligned}$$

Using (5.3) and the fact that $\#\,\mathrm{supp}_{N_1}(g) > 1$, we obtain that $\chi(g) = 0$. This finishes the proof. □

Lemma 5.4. *Let χ be a character of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ and let $\{\lambda(k) \vdash N_k\}_{k=1}^{\infty}$ be a sequence of partitions such that $\lim_{j\to\infty} \chi_{\lambda(k)}(\sigma) = \chi(\sigma)$ for all $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. If $\{\lambda(k) \vdash N_k\}_{k=1}^{\infty} \in C_{\mu}$ for some partition μ, then $\chi(\sigma) = \chi_{\mathrm{nat}}(\sigma)^{|\mu|}$.*

Proof. As in the proof of Lemma 5.3, it suffices to consider the case when $\lambda(k) \setminus (\lambda_1(k)) = \mu$ for all k. If μ is the empty partition, then $\chi_{\lambda(k)}(\sigma) = 1$ for each $\sigma \in \mathfrak{S}_{N_k}$ and the statement is trivial. Hence, we can now suppose that $|\mu| \geq 1$.

Recall that $\mathrm{Tab}\,(\lambda(k))$ is the set of all (standard) Young tableaux of the shape $\lambda(k)$ filled with the numbers $1, 2, \ldots, N_k$. Let $\mathcal{R}_{\lambda(k)}$ be the irreducible representation of the group $\mathfrak{S}_{N_k}$ defined in Subsection 3.3 and let $\chi_{\lambda(k)}$ be its normalized character (see (3.4)).

If $\mu = (\mu_1, \mu_2, \ldots)$ and $m = \mu_1$ then the hook length formula [7] implies that

$$\begin{aligned} \#\left(\mathrm{Tab}(\lambda(k))\right) = \dim \lambda(k) &= \frac{\dim \mu}{|\mu|!} \cdot \frac{N_k!}{(N_k - |\mu| - m)! \prod_{i=1}^{m} (N_k - |\mu| - i + 1 + \mu'_i)} \\ &= \frac{\dim \mu}{|\mu|!} \cdot \frac{\prod_{i=1}^{|\mu|+m} (N_k - |\mu| - m + i)}{\prod_{i=1}^{m} (N_k - |\mu| - i + 1 + \mu'_i)}, \end{aligned} \tag{5.4}$$

where $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_m)$ is the conjugate partition and $\dim \lambda(k)$ is the dimension of representation $\mathcal{R}_{\lambda(k)}$.

Let $\mathbb{S}^{N_k}_{|\mu|}$ be the family of all $|\mu|$-element subsets $\{i_1 < i_2 < \ldots < i_{|\mu|}\}$ of the set $\{1, 2, \ldots, N_k\}$. Clearly, $\#\mathbb{S}^{N_k}_{|\mu|} = \binom{N_k}{|\mu|}$. Denote by $S(T, \mu)$ the subset of those elements of $\{1, 2, \ldots, N_k\}$ which are located in the boxes of the diagram $\lambda(k) \setminus (\lambda_1(k)) = \mu$ in a tableau $T \in \mathrm{Tab}(\lambda(k))$. Since for each $T \in \mathrm{Tab}(\lambda(k))$ we have $\#S(T, \mu) = |\mu|$ we can regard $S(T, \mu)$ as an element of $\mathbb{S}^{N_k}_{|\mu|}$. Note that a tableau T is defined uniquely by the filling of the diagram $\lambda(k) \setminus (\lambda_1(k)) = \mu$. Denote by $\widehat{\mathrm{Tab}}(\lambda(k))$ the set of all tableaux $T \in \mathrm{Tab}(\lambda(k))$ such that $S(T, \mu)$ consists only of subsets $\{i_1 < i_2 < \ldots < i_{|\mu|}\}$ which satisfy $i_{l+1} - i_l > 1$ for all $l = 1, 2, \ldots, |\mu| - 1$.

Consider an arbitrary element $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. Suppose that σ belongs to the conjugacy class $\mathfrak{C}_{\mathfrak{m}_{|k}}$ of the group $\mathfrak{S}_{N_k}$ that consists of the permutations of the cycle type $\mathfrak{m}_{|k} = (m_{1|k}, m_{2|k}, \ldots, m_{l|k})$ (see Subsection 3.1). For any $j > k$ denote by $\sigma_{\mathfrak{m}_{|j}}$ the minimal element of the conjugacy class $\mathfrak{C}_{\mathfrak{m}_{|j}}$ that contains $\mathrm{i}_{k,j}(\sigma)$. In view of Remark 5.2 there is a rational number α independent of j such that

$$\mathrm{supp}_{N_j} \sigma_{\mathfrak{m}_{|j}} = \{1, 2, \ldots, \alpha N_j\} \text{ for all } j > k. \tag{5.5}$$

Now take an arbitrary positive integer parameter $Q > |\mu|$ which will tend to infinity. Denote by $\mathrm{Tab}_Q(\lambda(j))$ the subset of those tableaux in $\mathrm{Tab}(\lambda(j))$ whose first Q boxes of the first row are filled with numbers $1, 2, \ldots, Q$. It is clear that

$$\begin{aligned} &\#\left(\widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))\right) \\ &\geq \dim \mu \cdot \frac{(N_j - Q)(N_j - Q - 3) \cdots (N_j - Q - 3(|\mu| - 1))}{|\mu|!}. \end{aligned} \tag{5.6}$$

According to (5.1), we have

$$\chi_{\lambda(j)}(\sigma) = \chi_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}}) = \frac{\sum\limits_{T \in \mathrm{Tab}(\lambda(j))} \left(\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}}) v_T, v_T\right)}{\#\left(\mathrm{Tab}(\lambda(j))\right)}.$$

It follows from (5.4) and (5.6) that

$$\lim_{j \to \infty} \frac{\#\left(\widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))\right)}{\#\left(\mathrm{Tab}(\lambda(j))\right)} = 1.$$

Thus, in order to compute the limit $\lim\limits_{j \to \infty} \chi_{\lambda(j)}(\sigma)$ we need to estimate the matrix elements $(\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}}) v_T, v_T)$ for $T \in \widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))$.

Take a tableau $T \in \widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))$ such that at least one element e from $\mathrm{supp}_{N_j} \sigma_{\mathfrak{m}_{|j}}$ (see (5.5)) belongs to the diagram μ. Then either e or $e - 1$ belongs to $\{j_i\}_{i=1}^r$ from (3.1).

Let us first consider the case when the transposition $s_e = (e \;\; e{+}1)$ is contained in the decomposition $\sigma_{\mathfrak{m}_{|j}} = s_{j_1} s_{j_2} \cdots s_{j_r}$, where $j_i < j_{i+1}$ (see (3.1)). Then, according to the definitions of sets $\widehat{\mathrm{Tab}}(\lambda(j))$ and $\mathrm{Tab}_Q(\lambda(j))$, we have

$$e + 1 \in (\lambda_1(j)), \;\; a_{e+1} \geq Q. \tag{5.7}$$

In other words, the number $e + 1$ is contained in the first row of the tableau T. Since $\sigma_{\mathfrak{m}_{|j}} = s_{j_1} s_{j_2} \cdots s_{j_r}$ satisfies the conditions of Lemma 3.4 we have

$$|(\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}}) v_T, v_T)| = \prod_l \frac{1}{|a_{i_l+1} - a_{i_l}|},$$

where the product is taken over all indices l such that elements i_l and i_l+1 are contained in different rows and different columns of the tableau T (recall that a_i is the content of the box of T that contains i). Since e and $e+1$ are in different rows and columns of T, we have

$$\left|(\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}})v_T, v_T)\right| \leq \frac{1}{|a_{e+1}-a_e|}. \tag{5.8}$$

According to our assumption, e is contained in the diagram μ. Therefore,

$$-|\mu| \leq a_e \leq |\mu| - 2.$$

Thus, using (5.7) and (5.8) we obtain

$$\left|(\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}})v_T, v_T)\right| \leq \frac{1}{Q-|\mu|+2}. \tag{5.9}$$

If the transposition $s_{e-1} = (e-1 \;\; e)$ is contained in the decomposition $\sigma_{\mathfrak{m}_{|k}} = s_{j_1}s_{j_2}\cdots s_{j_r}$, where $j_i < j_{i+1}$ (see (3.1)) then we can obtain the estimate (5.9) in a similar way.

Now let us estimate the number of tableaux $T \in \widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))$, whose rows, starting from the second (i.e. rows of the diagram μ), contain only elements of the set

$$\{1,2,\ldots,N_j\} \setminus \left(\mathrm{supp}_{N_j}\,\sigma_{\mathfrak{m}_{|j}}\right) = \{1+\alpha N_j, 2+\alpha N_j, \ldots, N_j\} \;\text{ (see (5.5)).}$$

In other words, we are considering those tableaux T for which $\mathrm{supp}_{N_j}\,\sigma_{\mathfrak{m}_{|j}}$ is contained in the first row of T. Denote the set of all tableaux (not necessarily from $\widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))$) that satisfy this property by $\mathrm{Tab}^{\sigma}_Q(\lambda(j))$. Note that for all sufficiently large j the inequality $|\mu| < Q < \alpha N_j$ holds. Therefore, applying (5.5), we obtain

$$\begin{aligned} &\mathrm{Tab}^{\sigma}_Q(\lambda(j)) = \{T : S(T,\mu) \subset \{1+\alpha N_j, 2+\alpha N_j, \ldots, N_j\}\}, \\ &\#\,\mathrm{Tab}^{\sigma}_Q(\lambda(j)) = \dim\mu \cdot \binom{N_j - \alpha N_j}{|\mu|}. \end{aligned} \tag{5.10}$$

Put

$$\begin{aligned} \Sigma_{j,0} &= \sum_{T\in\mathrm{Tab}^{\sigma}_Q(\lambda(j))} (\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|k}})v_T, v_T), \\ \Sigma_{j,1} &= \sum_{T\in\mathrm{Tab}(\lambda(j))\setminus\mathrm{Tab}^{\sigma}_Q(\lambda(j))} (\mathcal{R}_{\lambda(j)}\left(\sigma_{\mathfrak{m}_{|k}}\right)v_T, v_T). \end{aligned} \tag{5.11}$$

From the properties of $\mathcal{R}_{\lambda(j)}$ (see Subsection 3.3) we have

$$\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|k}})v_T = v_T \text{ for all } T \in \mathrm{Tab}^{\sigma}_Q(\lambda(j)).$$

Hence, using (5.10) we get

$$\Sigma_{j,0} = \dim\mu \cdot \binom{N_j - \alpha N_j}{|\mu|}. \tag{5.12}$$

In order to estimate $\Sigma_{j,1}$ consider two subsets

$$\begin{aligned} \mathrm{Tab}_{j,10} &= \left(\mathrm{Tab}(\lambda(j)) \setminus \mathrm{Tab}^{\sigma}_Q(\lambda(j))\right) \cap \left(\widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))\right), \\ \mathrm{Tab}_{j,11} &= \left(\mathrm{Tab}(\lambda(j)) \setminus \mathrm{Tab}^{\sigma}_Q(\lambda(j))\right) \setminus \left(\widehat{\mathrm{Tab}}(\lambda(j)) \cap \mathrm{Tab}_Q(\lambda(j))\right). \end{aligned}$$

Then, $\mathrm{Tab}(\lambda(j)) \setminus \mathrm{Tab}^{\sigma}_{Q}(\lambda(j)) = \mathrm{Tab}_{j,10} \sqcup \mathrm{Tab}_{j,11}$ and hence,

$$\Sigma_{j,1} = \sum_{T \in \mathrm{Tab}_{j,10}} (\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|k}})v_T, v_T) + \sum_{T \in \mathrm{Tab}_{j,11}} (\mathcal{R}_{\lambda(j)}(\sigma_{\mathfrak{m}_{|k}})v_T, v_T). \tag{5.13}$$

Applying the estimate (5.9) to matrix elements in the first sum we obtain

$$|\Sigma_{j,1}| \leq \frac{\#\,\mathrm{Tab}_{j,10}}{Q - |\mu| + 2} + \#\,\mathrm{Tab}_{j,11}\,. \tag{5.14}$$

Next, combining (5.4) and (5.6) gives

$$\#\,\mathrm{Tab}_{j,11} \leq \frac{\dim \mu}{|\mu|!} \left(\frac{\prod_{i=1}^{|\mu|+m} (N_j - |\mu| - m + i)}{\prod_{i=1}^{m} (N_j - |\mu| - i + 1 + \mu'_i)} - \prod_{i=1}^{|\mu|} (N_j - Q - 3(|\mu| - i)) \right).$$

Therefore, there exists a positive constant C independent of j such that

$$\#\,\mathrm{Tab}_{j,11} \leq C\, N_j^{|\mu|-1} \text{ for all } j.$$

Hence, applying the formula (5.4) and the bound (5.14) yields the inequality

$$\frac{|\Sigma_{j,1}|}{\#\,\mathrm{Tab}(\lambda(j))} \leq \frac{1}{Q - |\mu| + 2} + \frac{C_1}{N_j}, \tag{5.15}$$

where C_1 is a positive constant independent of j.

Finally, let us estimate the expression

$$\left| \chi_{\lambda(j)}(\sigma_{\mathfrak{m}_{|k}}) - \frac{\Sigma_{j,0}}{\#\,\mathrm{Tab}(\lambda(j))} \right| = \left| \chi_{\lambda(j)}(\sigma_{\mathfrak{m}_{|k}}) - \frac{\dim \mu}{\#\,\mathrm{Tab}(\lambda(j))} \cdot \binom{N_j - \alpha N_j}{|\mu|} \right|.$$

Combining (5.1), (5.11) and (5.15) we obtain

$$\left| \chi_{\lambda(j)}(\sigma_{\mathfrak{m}_{|j}}) - \frac{\dim \mu}{\#\,\mathrm{Tab}(\lambda(j))} \cdot \binom{N_j - \alpha N_j}{|\mu|} \right| \leq \frac{|\Sigma_{j,1}|}{\#\,\mathrm{Tab}(\lambda(j))} \leq \frac{1}{Q - |\mu| + 2} + \frac{C_1}{N_j}.$$

Passing to the limit when $j \to \infty$ gives the inequality

$$\left| \chi(\sigma) - (1 - \alpha)^{|\mu|} \right| \leq \frac{1}{Q - |\mu| + 2}.$$

Since Q can be chosen arbitrarily large, we have $\chi(\sigma) = (1 - \alpha)^{|\mu|}$. The statement of Lemma 5.4 now follows from (2.10) and (5.5). □

Corollary 5.5. *Let χ be a character of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$, and let $\{\lambda(k) \vdash N_k\}_{k=1}^{\infty}$ be a sequence of partitions such that $\lim_{j\to\infty} \chi_{\lambda(k)}(\sigma) = \chi(\sigma)$ for all $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. If $\{\lambda(k) \vdash N_k\}_{k=1}^{\infty} \in \widehat{C}_{\mu}$ for some partition μ, then $\chi(\sigma) = \mathrm{sgn}_{\infty}(\sigma) \cdot \chi_{\mathrm{nat}}(\sigma)^{|\mu|}$.*

Proof. The statement follows directly from Lemma 5.4, Remark 5.1 and the formula (3.5). □

Denote by $\mathrm{ex\,Char}\,(\mathfrak{S}_{\widehat{\mathbf{n}}})$ the set of functions on $\mathfrak{S}_{\widehat{\mathbf{n}}}$ which are claimed in Theorem 1.2 to be indecomposable characters. Namely, we put

$$\begin{aligned} &\mathrm{ex\,Char}^{+}\,(\mathfrak{S}_{\widehat{\mathbf{n}}}) = \{\chi^{p}_{\mathrm{nat}} : p \in \mathbb{N} \cup \{0\}\}\,, \\ &\mathrm{ex\,Char}^{-}\,(\mathfrak{S}_{\widehat{\mathbf{n}}}) = \{\mathrm{sgn}_{\infty} \cdot \chi^{p}_{\mathrm{nat}} : p \in \mathbb{N} \cup \{0\}\}\,, \\ &\mathrm{ex\,Char}\,(\mathfrak{S}_{\widehat{\mathbf{n}}}) = \mathrm{ex\,Char}^{+}\,(\mathfrak{S}_{\widehat{\mathbf{n}}}) \cup \mathrm{ex\,Char}^{-}\,(\mathfrak{S}_{\widehat{\mathbf{n}}}) \cup \{\chi^{\infty}_{\mathrm{nat}}\}. \end{aligned} \tag{5.16}$$

Lemma 5.6. $\operatorname{ex}\operatorname{Char}(\mathfrak{S}_{\widehat{\mathbf{n}}})$ *is a subset of the set of all characters on* $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

Proof. It is clear that $\chi^{\infty}_{\mathrm{nat}} = \operatorname{sgn}_{\infty} \cdot \chi^{\infty}_{\mathrm{nat}}$ is a character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. It remains to check that χ^{p}_{nat} and $\operatorname{sgn}_{\infty} \cdot \chi^{p}_{\mathrm{nat}}$ are characters on the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ for $p \in \mathbb{N} \cup \{0\}$.

Recall that $\mathcal{F}$ is a unitary representation of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ which acts in the Hilbert space $\mathcal{H} = L^2(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}}) \otimes l^2(\mathbb{Z})$ (see Subsection 2.2). Moreover, this representation satisfies the following property: for $\xi_0 = \beth \otimes \delta_0 \in \mathcal{H}$ the equality

$$\chi_{\mathrm{nat}}(\sigma) = (\mathcal{F}(\sigma)\xi_0, \xi_0)_{\mathcal{H}} \text{ (see (2.9))}$$

holds for all $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}$. Thus, χ_{nat} is a character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

Now for any $p \in \mathbb{N} \cup \{0\}$ consider the unitary representation $\mathcal{F}^{\otimes p}$ (the p-th tensor power of $\mathcal{F}$) acting on $\mathcal{H}^{\otimes p}$. It is clear that

$$\chi^{p}_{\mathrm{nat}}(\sigma) = (\mathcal{F}^{\otimes p}(\sigma)\xi_0^{\otimes p}, \xi_0^{\otimes p})_{\mathcal{H}^{\otimes p}} \text{ for all } \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}.$$

Since $\tau(B) = (B\xi_0^{\otimes p}, \xi_0^{\otimes p})_{\mathcal{H}^{\otimes p}}$ is a vector state, χ^{p}_{nat} is a character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. Similarly, $\operatorname{sgn}_{\infty} \cdot \mathcal{F}^{\otimes p}$ is also a unitary representation acting on $\mathcal{H}^{\otimes p}$ and the same argument implies that $\operatorname{sgn}_{\infty} \cdot \chi^{p}_{\mathrm{nat}}$ is also a character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. □

6. The proof of Theorem 1.2

In this section we prove the main result of the present paper.

Proof of Theorem 1.2. Suppose that χ is an indecomposable character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. According to Proposition 4.2, there exists a subsequence $\{k(l)\}_{l=1}^{\infty}$ and a sequence $\{\lambda(k(l)) \vdash N_{k(l)}\}_{l=1}^{\infty}$ of partitions such that

$$\chi(g) = \lim_{l \to \infty} \chi_{\lambda(k(l))}(g) \text{ for any } g \in \mathfrak{S}_{\widehat{\mathbf{n}}}. \tag{6.1}$$

Lemma 5.1 implies that three cases are possible:

- The sequence $\{\lambda(k(l)) \vdash N_{k(l)}\}_{l=1}^{\infty}$ belongs to the intersection $C_{\infty} \cap \widehat{C}_{\infty}$. In this case Lemma 5.3 implies that $\chi = \chi^{\infty}_{\mathrm{nat}}$.
- The sequence $\{\lambda(k(l)) \vdash N_{k(l)}\}_{l=1}^{\infty}$ belongs to C_{μ} for some partition μ. In this case Lemma 5.4 implies that $\chi = \chi^{|\mu|}_{\mathrm{nat}}$.
- The sequence $\{\lambda(k(l)) \vdash N_{k(l)}\}_{l=1}^{\infty}$ belongs to $\widehat{C}_{\mu}$ for some partition μ. In this case Corollary 5.5 implies that $\chi = \operatorname{sgn}_{\infty} \cdot \chi^{|\mu|}_{\mathrm{nat}}$.

Thus, the character χ equals either χ^{p}_{nat}, or $\operatorname{sgn}_{\infty} \cdot \chi^{p}_{\mathrm{nat}}$ for some $p \in \mathbb{N} \cup \{0, \infty\}$, i.e. $\chi \in \operatorname{ex}\operatorname{Char}(\mathfrak{S}_{\widehat{\mathbf{n}}})$ (see (5.16)). In other words, we proved that the set of all indecomposable characters on $\mathfrak{S}_{\widehat{\mathbf{n}}}$ is a subset of $\operatorname{ex}\operatorname{Char}(\mathfrak{S}_{\widehat{\mathbf{n}}})$. To conclude the proof it remains to check all these functions are indeed indecomposable characters on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. This is proved in Lemma 6.1. □

Lemma 6.1. *Each character from* $\operatorname{ex}\operatorname{Char}(\mathfrak{S}_{\widehat{\mathbf{n}}})$ *is indecomposable.*

Proof. Denote for brevity $\chi^{-}_{p} = \operatorname{sgn}_{\infty} \cdot \chi^{p}_{\mathrm{nat}}$, $\chi^{+}_{p} = \chi^{p}_{\mathrm{nat}}$ and $\chi_{\infty} = \chi^{\infty}_{\mathrm{nat}}$. It is clear that the character χ^{-}_{p} is indecomposable if and only if the character χ^{+}_{p} is indecomposable. Recall that by Proposition 2.6 the character χ_{nat} is indecomposable.

First proof. We suppose the contrary: i.e. assume that some character χ_m^+ is not an extreme point of the set of all normalized characters. Then there exist real numbers $\alpha_p^+, \alpha_p^- \in [0,1]$ satisfying the following equality:

$$\begin{aligned}\chi_m^+(\sigma) = \alpha_0^+ + \alpha_0^- \operatorname{sgn}_\infty(\sigma) + \sum_{j=1}^{m-1} \alpha_j^+ \chi_j^+(\sigma) + \sum_{j=1}^{m-1} \alpha_j^- \chi_j^-(\sigma) \\ + \sum_{j=m+1}^{\infty} \alpha_j^+ \chi_j^+(\sigma) + \sum_{j=m+1}^{\infty} \alpha_j^- \chi_j^-(\sigma) + \alpha_\infty \chi_\infty(\sigma) \text{ for all } \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}.\end{aligned} \tag{6.2}$$

Recall that $\mathfrak{A}_{\widehat{\mathbf{n}}} = \{\sigma \in \mathfrak{S}_n : \operatorname{sgn}_\infty(\sigma) = 1\}$. Take a sequence $\{\sigma_n\} \subset \mathfrak{A}_{\widehat{\mathbf{n}}}$, satisfying the following conditions:

$$\sigma_n \neq \mathrm{id} \text{ for all } n \text{ and } \lim_{n\to\infty} \chi_{\mathrm{nat}}(\sigma_n) = 0.$$

Plugging $\sigma = \sigma_n$ into (6.2) and passing to the limit when $n \to \infty$ gives

$$\alpha_0^+ + \alpha_0^- = 0 \Rightarrow \alpha_0^+ = 0 \text{ and } \alpha_0^- = 0. \tag{6.3}$$

Since the set $\{\chi_{\mathrm{nat}}(\sigma)\}_{\sigma \in \mathfrak{A}_{\widehat{\mathbf{n}}}}$ is dense in $[0,1)$, it follows from (6.2) that

$$\gamma^m = \sum_{j=1}^{m-1} \left(\alpha_j^+ + \alpha_j^-\right) \gamma^j + \sum_{j=m+1}^{\infty} \left(\alpha_j^+ + \alpha_j^-\right) \gamma^j \text{ for all } \gamma \in [0,1).$$

An easy computation shows that

$$\alpha_j^+ = \alpha_j^- = 0 \text{ for all positive integers } j \neq m.$$

Hence, using (6.2) and (6.3), we obtain that

$$\chi_m^+(\sigma) = \alpha_\infty \chi_\infty(\sigma) \text{ for all } \sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}}.$$

Therefore, $\chi_m^+ = \chi_\infty$. Since the character χ_∞ is indecomposable as a regular character of the ICC group (see [10, Proposition 7.9]), this contradicts the assumption that χ_m^+ is a decomposable character. □

Second proof. Here we use the properties of the construction from Section 2 that are outlined in Appendix. By above we can suppose that $m < \infty$. If a character χ_m^+ is not indecomposable, then the corresponding GNS-representation $(\pi, \xi, \mathcal{H})$, where ξ is a unit cyclic vector such that $\chi_m^+(g) = \langle \pi(g)\xi, \xi \rangle_{\mathcal{H}}$ for all $g \in \mathfrak{S}_{\widehat{\mathbf{n}}}$, is not a factor representation. Therefore, there is a nonzero orthogonal projection E in the center $\mathcal{C}(M)$ of the w^*-algebra M, generated by the operators $\{\pi(s)\}_{s \in \mathfrak{S}_{\widehat{\mathbf{n}}}}$, and $\delta \in (0,1)$ such that

$$\|E\xi\|^2 = \delta > 0. \tag{6.4}$$

Since ξ is a cyclic vector for M, for any $\epsilon > 0$ there exists a finite subset $\{s_i\}_{i=1}^{K_p} \subset \mathfrak{S}_{N_p}$ and a collection $\{\vartheta_i\}$ of the complex numbers with the following property:

$$\left\| E\xi - \sum_{i=1}^{K_p} \vartheta_i \pi(s_i)\xi \right\| < \epsilon. \tag{6.5}$$

Let $q_n, r_n, \Theta_n, \mathfrak{I}_n, \mathfrak{S}_{\widehat{\mathbf{n}}}(q_n, r_n)$ be the objects from the Corollary 8.2. It follows from (1.5) that

$$\chi_m^+\left(s_i \mathfrak{I}_n(s_j)\right) = \chi_m^+\left(s_i\right) \cdot \chi_m^+\left(\mathfrak{I}_n(s_j)\right) \text{ for all } i, j. \tag{6.6}$$

Since χ_m^+ is continuous with respect to the topology defined on $\mathfrak{S}_{\widehat{\mathbf{n}}}$ by metric ρ, we obtain from (6.5)

$$\left\|(I-E)\xi - \left(I - \sum_{i=1}^{K_p} \vartheta_i \pi(\mathfrak{I}_n(s_i))\right)\xi\right\| < \epsilon \text{ for all } n > K. \tag{6.7}$$

Put $A = \sum\limits_{i=1}^{K_p} \vartheta_i \pi(s_i)$ and $A_n = I - \sum\limits_{i=1}^{K_p} \vartheta_i \pi(\mathfrak{I}_n(s_i))$ for brevity. Without loss of generality suppose that $A = A^*$ and $A_n = A_n^*$. Now we obtain the following chain of inequalities

$$\begin{aligned}
\langle E\xi, (I-E)\xi\rangle &\overset{(6.5)}{\geq} \langle A\xi, (I-E)\xi\rangle - \epsilon\|(I-E)\xi\| \\
&\overset{(6.7)}{\geq} \langle A\xi, A_n\xi\rangle - \epsilon\|A\xi\| - \epsilon\|(I-E)\xi\| \\
&\geq \langle A\xi, A_n\xi\rangle - \epsilon(\|E\xi\| + \epsilon) - \epsilon\|(I-E)\xi\| \\
&\geq \langle A\xi, A_n\xi\rangle - 2\epsilon - \epsilon^2.
\end{aligned}$$

Hence, applying (6.6), we get

$$\begin{aligned}
0 = \langle E\xi, (I-E)\xi\rangle &\geq \langle A\xi, \xi\rangle \langle A_n\xi, \xi\rangle - 2\epsilon - \epsilon^2 \\
&\geq (\langle E\xi, \xi\rangle - \epsilon)\,(\langle (I-E)\xi, \xi\rangle - \epsilon) - 2\epsilon - \epsilon^2 \\
&\geq \delta(1-\delta) - 3\epsilon.
\end{aligned}$$

The last inequality is false for $0 < \epsilon < \frac{\delta - \delta^2}{3}$ and this leads to contradiction. □

7. The proof of Theorem 1.3

In this section we prove Theorem 1.3 about the isomorphism classes of groups $\mathfrak{S}_{\widehat{\mathbf{n}}}$.

We need the following simple lemma in the proof of Theorem 1.3.

Lemma 7.1. *Let $\widehat{\mathbf{n}}' = \{n'_k\}_{k=1}^{\infty}$ and $\widehat{\mathbf{n}}'' = \{n''_k\}_{k=1}^{\infty}$, where $n'_k, n''_k > 1$ for all k, be the sequences of positive integers. Put $N'_k = \prod_{i=1}^{k} n'_k$ and $N''_k = \prod_{i=1}^{k} n''_k$. Denote*

$$\mathrm{Div}(\widehat{\mathbf{n}}') = \{N \in \mathbb{N} : N \text{ divides } N'_k \text{ for some } k\}, \tag{7.1}$$

$$\mathrm{Div}(\widehat{\mathbf{n}}'') = \{N \in \mathbb{N} : N \text{ divides } N''_k \text{ for some } k\}. \tag{7.2}$$

Then, the following conditions are equivalent:

(a) *for each prime number p the following equality holds:*

$$\lim_{k\to\infty} \deg_p(N'_k) = \lim_{k\to\infty} \deg_p(N''_k); \tag{7.3}$$

(b) *for any element N'_i of the sequence $\{N'_k\}_{k=1}^{\infty}$ there is an element N''_j of the sequence $\{N''_k\}_{k=1}^{\infty}$ such that N'_i divides N''_j and vice versa;*

(c) $\mathrm{Div}(\widehat{\mathbf{n}}') = \mathrm{Div}(\widehat{\mathbf{n}}'')$.

The following statement is a direct consequence of Theorem 1.2.

Lemma 7.2. *Let χ^0 be the trivial character on $\mathfrak{S}_{\widehat{\mathbf{n}}}$. Put $CH(\mathfrak{S}_{\widehat{\mathbf{n}}}) = \mathrm{ex\,Char}\,(\mathfrak{S}_{\widehat{\mathbf{n}}}) \setminus \{\chi^0, \mathrm{sgn}_\infty\}$. For any element $\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}} \setminus \{\mathrm{id}\}$ we have $\chi_{\mathrm{nat}}(\sigma) = \max\limits_{\chi \in CH(\mathfrak{S}_{\widehat{\mathbf{n}}})} \{|\chi(\sigma)|\}$.*

Proof of Proposition 1.3. Let $\widehat{\mathbf{n}}' = \{n'_k\}_{k=1}^{\infty}$ and $\widehat{\mathbf{n}}'' = \{n''_k\}_{k=1}^{\infty}$, where $n'_k, n''_k > 1$ for all k. Suppose that there exists an isomorphism $\alpha\colon \mathfrak{S}_{\widehat{\mathbf{n}}'} \to \mathfrak{S}_{\widehat{\mathbf{n}}''}$. Then, the map

$$\chi \ni \operatorname{ex}\operatorname{Char}\left(\mathfrak{S}''_{\widehat{\mathbf{n}}}\right) \overset{\alpha^*}{\mapsto} \chi \circ \alpha \in \operatorname{ex}\operatorname{Char}\left(\mathfrak{S}'_{\widehat{\mathbf{n}}}\right) \tag{7.4}$$

is a bijection of the set $\operatorname{ex}\operatorname{Char}\left(\mathfrak{S}''_{\widehat{\mathbf{n}}}\right)$ onto $\operatorname{ex}\operatorname{Char}\left(\mathfrak{S}'_{\widehat{\mathbf{n}}}\right)$. By definition of α^*, we have

$$\alpha^*\left(\operatorname{ex}\operatorname{Char}^+\left(\mathfrak{S}''_{\widehat{\mathbf{n}}}\right)\right) = \operatorname{ex}\operatorname{Char}^+\left(\mathfrak{S}'_{\widehat{\mathbf{n}}}\right). \tag{7.5}$$

Let χ'_{nat} and χ''_{nat} be the natural characters on $\mathfrak{S}'_{\widehat{\mathbf{n}}}$ and $\mathfrak{S}''_{\widehat{\mathbf{n}}}$, respectively (see (1.5), (2.9)). Using the characterization of the natural character from Lemma 7.2, we obtain

$$\alpha^*\left(\chi''_{\mathrm{nat}}\right) = \chi''_{\mathrm{nat}} \circ \alpha = \chi'_{\mathrm{nat}}. \tag{7.6}$$

Thus, it follows from (1.5) that

$$\begin{aligned}\chi''_{\mathrm{nat}}\left(\mathfrak{S}''_{\widehat{\mathbf{n}}}\right) = \chi''_{\mathrm{nat}}\left(\alpha\left(\mathfrak{S}'_{\widehat{\mathbf{n}}}\right)\right) = \chi'_{\mathrm{nat}}\left(\mathfrak{S}'_{\widehat{\mathbf{n}}}\right) &= \left\{\frac{L}{N'_k} : k \in \mathbb{N}, 0 \le L \le N'_k\right\} \\ &= \left\{\frac{p}{q} : q \in \operatorname{Div}(\widehat{\mathbf{n}}'), 0 \le p \le q, \gcd(p,q) = 1\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left\{\frac{p}{q} : q \in \operatorname{Div}(\widehat{\mathbf{n}}'), 0 \le p \le q, \gcd(p,q) = 1\right\} \\ &\qquad = \left\{\frac{p}{q} : q \in \operatorname{Div}(\widehat{\mathbf{n}}''), 0 \le p \le q, \gcd(p,q) = 1\right\}.\end{aligned}$$

Therefore, $\operatorname{Div}(\widehat{\mathbf{n}}') = \operatorname{Div}(\widehat{\mathbf{n}}'')$. Finally, Lemma 7.1 implies that the condition (1.6) holds. Now suppose that (1.6) holds. We prove that in this case groups $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ and $\mathfrak{S}_{\widehat{\mathbf{n}}''}$ are indeed isomorphic. To do this, we check that their images in $\operatorname{Aut}_0(X,\nu)$ coincide. According to Remark 1.1 the group $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ is isomorphic to the subgroup $\bigcup_k T_{N'_k}(\mathfrak{S}_{N'_k})$ of the group $\operatorname{Aut}_0(X,\nu)$ and similarly, $\mathfrak{S}_{\widehat{\mathbf{n}}''}$ is isomorphic to the subgroup $\bigcup_k T_{N''_k}(\mathfrak{S}_{N''_k})$ of the group $\operatorname{Aut}_0(X,\nu)$. The condition (1.6) and Lemma 7.1 (b) imply that every subgroup $T_{N'_i}(\mathfrak{S}_{N''_i})$ is contained in some subgroup $T_{N'_j}(\mathfrak{S}_{N''_j})$ and vice versa (see also Section 1). Hence,

$$\bigcup_k T_{N'_k}(\mathfrak{S}_{N'_k}) = \bigcup_k T_{N''_k}(\mathfrak{S}_{N''_k}) \tag{7.7}$$

and therefore, groups $\mathfrak{S}_{\widehat{\mathbf{n}}'}$ and $\mathfrak{S}_{\widehat{\mathbf{n}}''}$ are isomorphic. □

8. Appendix

In this section we prove some additional properties of the II_1 factor representation of the group $\mathfrak{S}_{\widehat{\mathbf{n}}}$ which was discussed in Section 2. These properties are used only in the second proof of Lemma 6.1.

Consider arbitrary positive integers p, q and r such that $1 < p < q < r$. Recall that $\mathbb{X}_{\widehat{\mathbf{n}}} = \prod_{k=1}^{\infty} \mathbb{X}_{n_k}$. For $x = (x_1, x_2, \ldots) \in \mathbb{X}_{\widehat{\mathbf{n}}}$, we set $x|(q,r) = (x_{q+1}, x_{q+2}, \ldots, x_r) \in \prod_{k=q+1}^{r} \mathbb{X}_{n_k}$. Each element $y \in \mathbb{X}_{\widehat{\mathbf{n}}}$ defines a cylinder set

$$\mathbb{A}_{y|q,r} = \{x \in \mathbb{X}_{\widehat{\mathbf{n}}} : \ x|(q,r) = y|(q,r)\} \subset \mathbb{X}_{\widehat{\mathbf{n}}}. \tag{8.1}$$

Recall that $N_{q,r} = \frac{N_r}{N_q}$.

Let us introduce the subgroup $\mathfrak{S}_{\widehat{\mathbf{n}}}(q,r) \subset \mathfrak{S}_{N_r}$ defined as

$$\mathfrak{S}_{\widehat{\mathbf{n}}}(q,r) = \{\sigma \in \mathfrak{S}_{\widehat{\mathbf{n}}} : (\sigma(x))_i = x_i \text{ for all } x \in \mathbb{X}_{\widehat{\mathbf{n}}} \text{ and } i \notin \{q+1, q+2, \dots, r\}\}.$$

It is clear that $\mathfrak{S}_{\widehat{\mathbf{n}}}(q,r)$ is isomorphic to $\mathfrak{S}_{N_{q,r}}$. For all sufficiently large r we can find non-negative integers m and rem such that

$$N_{q,r} = N_p\, m + rem, \text{ where } N_{q,r} > N_p \text{ and } rem < N_p. \tag{8.2}$$

Denote by T the automorphism O^{N_q}, where O is the automorphism in $\mathrm{Aut}_0(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}})$ defined in Subsection 2.1.

Set $\mathbf{0} = (0,0,\dots,0,\dots) \in \mathbb{X}_{\widehat{\mathbf{n}}}$. Then the subsets in the collection $\left\{\mathbb{A}_{\mathbf{y}_i|q,r}\right\}_{i=0}^{N_{q,r}-1}$ (see (2.1)), where $\mathbf{y}_i = (T^i(\mathbf{0}))|(q,r)$, are pairwise disjoint and $\bigsqcup\limits_{i=0}^{N_{q,r}-1} \mathbb{A}_{\mathbf{y}_i|q,r} = \mathbb{X}_{\widehat{\mathbf{n}}}$. For the simplicity of notations, we denote $\mathbb{A}_{\mathbf{y}_i|q,r}$ by $\mathbb{B}_i$. Now define the periodic automorphism $Q \in \mathfrak{S}_{\widehat{\mathbf{n}}}(q,r)$ as follows:

$$Q(x) = \begin{cases} T(x) = O^{N_q}(x), & \text{if } x \in \bigcup\limits_{j=0}^{N_{q,r}-2} \mathbb{B}_j, \\ T^{(1-N_{q,r})}x = O^{N_q(1-N_{q,r})}x, & \text{if } x \in \mathbb{B}_{(N_{q,r}-1)}. \end{cases} \tag{8.3}$$

Let us introduce the subsets $\mathbb{E}_k \subset \mathbb{X}_{\widehat{\mathbf{n}}}$, where $k \in \{0,1,\dots,N_p-1\}$, by

$$\mathbb{E}_k = \bigcup_{i=km}^{(k+1)m-1} \mathbb{B}_i \quad \text{(see (8.2) for the definition of } m\text{)}. \tag{8.4}$$

Therefore, we can conclude that

$$\nu_{\widehat{\mathbf{n}}}(\mathbb{E}_k) = \frac{m}{N_{q,r}} = \frac{m}{N_p\, m + rem} = \frac{1}{N_p} - \frac{rem}{N_p\,(mN_p + rem)} \tag{8.5}$$

for all $k \in \overline{0, N_p - 1}$.

Consider the automorphism S defined by the formula

$$Sx = \begin{cases} Qx, & \text{if } x \in \bigcup\limits_{j=0}^{N_p-2} \mathbb{E}_j; \\ Q^{m(1-N_p)}x, & \text{if } x \in \mathbb{E}_{N_p-1}; \\ x, & \text{if } x \in \mathbb{X}_{\widehat{\mathbf{n}}} \setminus \left(\bigcup\limits_{j=0}^{N_p-1} \mathbb{E}_j\right). \end{cases} \tag{8.6}$$

It is clear that $S^{N_p}(x) = x$ for all $x \in \mathbb{X}_{\widehat{\mathbf{n}}}$ and

$$S(\mathbb{E}_k) = \mathbb{E}_{k+1 \pmod{N_p}}. \tag{8.7}$$

Since $O^{N_q}\mathbb{A}_y = \mathbb{A}_y$ for all $y \in \prod\limits_{k=1}^{q} \mathbb{X}_{n_k}$ and $p < q$, (8.3) and (8.6) show that

$$S\,\mathbb{A}_y = \mathbb{A}_y \text{ for all } y \in \prod_{k=1}^{p} \mathbb{X}_{n_k}. \tag{8.8}$$

If $\mathbf{0} = (0,\dots,0,\dots)$, then the cylinder sets from the collection

$$\left\{O^i_{\mathbf{o}|p}\,\mathbb{A}_{\mathbf{o}|p} = \mathbb{A}_{\mathbf{y}_i|p}\right\}_{i=0}^{N_p-1},$$

where $\mathbf{y}_i = (O^i_{\mathbf{o}|p}(\mathbf{0}))$, are pairwise disjoint and $\bigsqcup_{i=0}^{N_p-1} \mathbb{A}_{\mathbf{y}_i|p} = \mathbb{X}_{\widehat{\mathbf{n}}}$. Therefore, the formula

$$\Theta(x) = \begin{cases} O^{i-j}_{\mathbf{o}|p}\, S^{j-i} x, & \text{if } x \in \mathbb{E}_i \cap \mathbb{A}_{\mathbf{y}_j|p}; \\ x, & \text{if } x \in \mathbb{X}_{\widehat{\mathbf{n}}} \setminus \left(\bigcup_{j=0}^{N_p-1} \mathbb{E}_j \right) \end{cases} \tag{8.9}$$

defines an automorphism from $\mathfrak{S}_{N_r} \subset \mathrm{Aut}_0(\mathbb{X}_{\widehat{\mathbf{n}}}, \nu_{\widehat{\mathbf{n}}})$. A direct calculation shows that $\Theta^2 x = x$ for all $x \in \mathbb{X}_{\widehat{\mathbf{n}}}$.

Denote by $[S]_{\mathbb{E}}$ the subgroup of $\mathrm{Aut}_0(\mathbb{X}_{\widehat{\mathbf{n}}})$, consisting of the automorphisms $\alpha_r \in \mathfrak{S}_{\widehat{\mathbf{n}}}(q,r)$, acting as follows:

$$\alpha_r(x) = \begin{cases} S^{k_i}(x), & \text{if } x \in \mathbb{E}_i; \\ x, & \text{if } x \in \mathbb{X}_{\widehat{\mathbf{n}}} \setminus \left(\bigcup_{j=0}^{N_p-1} \mathbb{E}_j \right), \end{cases} \tag{8.10}$$

where $\left(k_0, k_1, \ldots, k_{N_p-1}\right)$ is a collection of integers which belong to the set $\{0, 1, \ldots, N_p - 1\}$. Then, using (8.9), we obtain

$$\sigma_r(x) = \Theta^{-1} \alpha_r \Theta\,(x) = \begin{cases} O^{k_i}_{\mathbf{o}|p}(x), & \text{if } x \in \mathbb{A}_{\mathbf{y}_i|p} \cap \left(\bigcup_{j=0}^{N_p-1} \mathbb{E}_j \right), \\ x, & \text{if } x \in \mathbb{X}_{\widehat{\mathbf{n}}} \setminus \left(\bigcup_{j=0}^{N_p-1} \mathbb{E}_j \right). \end{cases} \tag{8.11}$$

Now take an arbitrary automorphism $\sigma \in \mathfrak{S}_{N_p}$. By Lemma 2.2, for every $\sigma \in \mathfrak{S}_{N_p}$ there exists a collection of non-negative integers $\{k_i(\sigma)\}_{i=0}^{N_p-1}$, where $k_i(\sigma) \in \{0, 1, \ldots, N_p - 1\}$, such that $\sigma(x) = O^{k_i(\sigma)}_{\mathbf{o}|p}(x)$ if $x \in \mathbb{A}_{\mathbf{y}_i|p}$. Since $q > p$, the cylinder sets $\mathbb{A}_{\mathbf{y}_i|q,r}$ are invariant under the action of the group $\mathfrak{S}_{N_p}$, i.e. $s\left(\mathbb{A}_{\mathbf{y}_i|q,r}\right) = \mathbb{A}_{\mathbf{y}_i|q,r}$ for all $s \in \mathfrak{S}_{N_p}$ and $i = 0, 1, \ldots, N_{q,r} - 1$. Therefore, the automorphism $\sigma_r \in \mathfrak{S}_{N_p} \cdot \mathfrak{S}_{\widehat{\mathbf{n}}}(q,r)$ from (8.11), where we put $k_i = k_i(\sigma)$, is well defined. If ρ is the metric introduced in (2.3), then, using (8.2), (8.5) and (8.11), we get

$$\rho\,(\sigma, \sigma_r) \le \frac{rem}{mN_p + rem} \le \frac{1}{m}, \text{ where } mN_p = N_{q,r} - rem, rem < N_p. \tag{8.12}$$

Thus, we obtain the following statement.

Proposition 8.1. *Let p, q, r be positive integers satisfying the inequality $1 < p < q < r$ and the condition (8.2). There exists an injective homomorphism $\mathfrak{I} : \mathfrak{S}_{N_p} \to \mathfrak{S}_{\widehat{\mathbf{n}}}(q,r)$ and an automorphism $\Theta \in \mathfrak{S}_{N_p} \cdot \mathfrak{S}_{\widehat{\mathbf{n}}}(q,r)$ such that*

$$\rho\left(\Theta\sigma\Theta^{-1}, \mathfrak{I}(\sigma)\right) \le \frac{1}{m} \text{ for all } \sigma \in \mathfrak{S}_{N_p}. \tag{8.13}$$

Corollary 8.2. *For a fixed $p > 1$ take two sequences of positive integers $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ such that $p \le q_n < r_n$ and $\lim\limits_{n\to\infty} q_n = \lim\limits_{n\to\infty} (r_n - q_n) = \infty$. Let $N_{q_n,r_n} = N_p\, m_n + (rem)_n$, where $m_n > 1$ and $(rem)_n < N_p$. Then for each n there exists an injective homomorphism $\mathfrak{I}_n : \mathfrak{S}_{N_p} \to \mathfrak{S}_{\widehat{\mathbf{n}}}(q_n, r_n)$ and an automorphism $\Theta_n \in \mathfrak{S}_{N_p} \cdot \mathfrak{S}_{\widehat{\mathbf{n}}}(q_n, r_n)$ such that*

$$\rho\left(\Theta_n\sigma\Theta_n^{-1}, \mathfrak{I}_n(\sigma)\right) \le \frac{1}{m_n} \text{ for all } \sigma \in \mathfrak{S}_{N_p}. \tag{8.14}$$

Moreover, since $\lim_{n\to\infty}(r_n - q_n) = \infty$, *we have* $\lim_{n\to\infty} \rho\left(\Theta_n \sigma \Theta_n^{-1}, \mathfrak{I}_n(\sigma)\right) = 0$.

Acknowledgment

The authors are grateful to the anonymous referee for careful reading and insightful comments.

References

[1] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli, *Representation theory of the symmetric groups*, Cambridge Studies in Advanced Mathematics, vol. 121, Cambridge University Press, Cambridge, 2010. The Okounkov-Vershik approach, character formulas, and partition algebras, DOI 10.1017/CBO9781139192361. MR2643487

[2] Artem Dudko, *Characters on the full group of an ergodic hyperfinite equivalence relation*, J. Funct. Anal. **261** (2011), no. 6, 1401–1414, DOI 10.1016/j.jfa.2011.04.017. MR2813475

[3] Artem Dudko and Konstantin Medynets, *On characters of inductive limits of symmetric groups*, J. Funct. Anal. **264** (2013), no. 7, 1565–1598, DOI 10.1016/j.jfa.2013.01.013. MR3019724

[4] Valentin Féray and Piotr Śniady, *Asymptotics of characters of symmetric groups related to Stanley character formula*, Ann. of Math. (2) **173** (2011), no. 2, 887–906, DOI 10.4007/annals.2011.173.2.6. MR2776364

[5] E. E. Goryachko and F. V. Petrov, *Indecomposable characters of the group of rational rearrangements of a segment* (Russian, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **378** (2010), no. Teoriya Predstavleniĭ, Dinamicheskie Sistemy, Kombinatornye Metody. XVIII, 17–31, 228, DOI 10.1007/s10958-011-0274-1; English transl., J. Math. Sci. (N.Y.) **174** (2011), no. 1, 7–14. MR2749292

[6] F. Leinen and O. Puglisi, *Positive definite functions of diagonal limits of finite alternating groups*, J. London Math. Soc. (2) **70** (2004), no. 3, 678–690, DOI 10.1112/S0024610704005691. MR2096870

[7] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications. MR1354144

[8] Andrei Okounkov and Anatoly Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math. (N.S.) **2** (1996), no. 4, 581–605, DOI 10.1007/PL00001384. MR1443185

[9] Yuval Roichman, *Upper bound on the characters of the symmetric groups*, Invent. Math. **125** (1996), no. 3, 451–485, DOI 10.1007/s002220050083. MR1400314

[10] M. Takesaki, *Theory of operator algebras. I*, Encyclopaedia of Mathematical Sciences, vol. 124, Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition; Operator Algebras and Non-commutative Geometry, 5. MR1873025

[11] A. M. Vershik and S. V. Kerov, *Asymptotic theory of the characters of a symmetric group* (Russian), Funktsional. Anal. i Prilozhen. **15** (1981), no. 4, 15–27, 96. MR639197

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv 61103, Ukraine

Email address: n.nessonov@gmail.com

Institute of Science and Technology Austria (ISTA), Am Campus 1, Klosterneuburg 3400, Austria

Email address: nhoktkhaishon.ngo@ista.ac.at