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Virtual bound states of the Pauli operator with an Aharonov–Bohm potential

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A maximal realization of the two-dimensional Pauli operator, subject to Aharonov–Bohm magnetic field, is investigated. Contrary to the case of the Pauli operator with regular magnetic potentials, it is shown that both components of the Pauli operator are critical. Asymptotics of the weakly coupled eigenvalues, generated by electric (not necessarily self-adjoint) perturbations, are derived.

Keywords: Pauli operator; Aharonov–Bohm potential; criticality; virtual bound states; weakly coupled eigenvalues.

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1. Introduction

In non-relativistic quantum mechanics, a free spinless particle in the plane is described by the self-adjoint realization of the Laplacian $-\Delta$ in $L^2(\mathbb{R}^2)$. It is well known that the two-dimensional Laplacian is *critical*, in the sense that its spectrum is unstable under small perturbations. More specifically, $-\Delta+v$ possesses a negative eigenvalue whenever $v \in C_0^\infty(\mathbb{R}^2)$ is attractive (i.e. non-trivial and non-positive). In physical terms, $-\Delta$ admits a *virtual bound state* at zero energy, meaning that,

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while the spectrum of $-\Delta$ is purely absolutely continuous, the singularity of the Green function in the spectral parameter leads to the genuine eigenvalue under the arbitrarily small attractive electric perturbations v. More specifically, the *weakly coupled* asymptotics

$$\inf \sigma(-\Delta + \varepsilon v) \sim -\exp\left(\left[\frac{\varepsilon}{4\pi}\int_{\mathbb{R}^2} v\right]^{-1}\right) \quad \text{as } \varepsilon \to 0^+ \tag{1}$$

holds true. As the amount of literature on the subject is vast, we mention only [37] for the pioneering work, [3, 27] for the earlier papers and [8, 9, 12, 25] for the most recent contributions.

Switching on the magnetic field, the situation changes dramatically because of the *diamagnetic* effect. Mathematically, the magnetic Laplacian $-\Delta_a := (-i\nabla - a)^2$ in $L^2(\mathbb{R}^2)$ becomes *subcritical* for any smooth vector potential $a : \mathbb{R}^2 \to \mathbb{R}^2$ such that the magnetic field $b := \operatorname{curl} a$ is non-trivial. Here, the subcriticality means the existence of a Hardy inequality $-\Delta_a \geq \rho$ with positive $\rho : \mathbb{R}^2 \to \mathbb{R}$, which is equivalent to the fact that the singularity of the Green function disappears. Then $\inf \sigma(-\Delta_a + \varepsilon v) \geq 0$ for all sufficiently small ε, so there are no negative weakly coupled eigenvalues. In fact, this change of game equally applies to the singular Aharonov–Bohm potential

$$a_\alpha(x) := \alpha \, \frac{(x^2, -x^1)}{|x|^2} \quad \text{with } \alpha \in \mathbb{R} \tag{2}$$

whenever $\alpha \notin \mathbb{Z}$. This is particularly spectacular because $b_\alpha := \operatorname{curl} a_\alpha$ vanishes almost everywhere in the plane. In fact, $b_\alpha = -2\pi\alpha\delta$ in the sense of distributions, where δ is the zero-centered Dirac function. Again, we mention only [30] for the pioneering work and [7, 10, 11, 16–19, 33] for the most recent contributions.

For particles with spin, however, a more realistic (yet still non-relativistic) description is through the Pauli operator

$$H_a := \begin{pmatrix} -\Delta_a - b & 0 \\ 0 & -\Delta_a + b \end{pmatrix} \quad \text{in } L^2(\mathbb{R}^2, \mathbb{C}^2), \tag{3}$$

subject to matrix-valued potential perturbations $V : \mathbb{R}^2 \to \mathbb{C}^{2\times 2}$. It turns out that these systems exhibit the *paramagnetic* effect. Indeed, it is known [39] (though perhaps less than in the magnetic-free spinless case above) that H_a with any regular potential a does admit a virtual bound state at zero energy. Contrary to the superfast exponential decay (1), however, the weak coupling is stronger now:

$$\inf \sigma(H_a + \varepsilon v I_{\mathbb{C}^2}) \sim -C\, \varepsilon^{1/|\Phi_b|} \quad \text{as } \varepsilon \to 0^+, \tag{4}$$

where C is a constant depending on v and b, $\Phi_b := \frac{1}{2\pi}\int_{\mathbb{R}^2} b$ is the total flux of b and it is assumed that $|\Phi_b| \in (0, 1)$. The eigenvalue asymptotics (4) are due to [21] and [2, 28] in radial and general cases, respectively. (See also [20] for related Lieb–Thirring inequalities.) Of course, the existence of the one virtual bound state (and thus correspondingly unique weakly coupled eigenvalue) is due to the fact that one (and only one) of the operators $-\Delta_a \mp b$ appearing in (3) is critical (the choice $\mp$ depends on the sign of Φ_b, namely $-\Delta_a - b$ is critical if $\Phi_b > 0$).

The objective of this paper is to analyze the criticality properties and weakly coupled eigenvalues of the Pauli operator (3) in the highly singular situation of the Aharonov–Bohm potential (2), formally acting by

$$H_{a_\alpha} := \begin{pmatrix} -\Delta_{a_\alpha} + 2\pi\alpha\delta & 0 \\ 0 & -\Delta_{a_\alpha} - 2\pi\alpha\delta \end{pmatrix} \quad \text{in } L^2(\mathbb{R}^2, \mathbb{C}^2). \tag{5}$$

Note that $\Phi_{b_\alpha} = -\alpha$ for the choice (2). The operator H_{a_α} is rigorously introduced by considering a self-adjoint realization of the Aharonov–Bohm Laplacian with delta-type interactions $-\Delta_{a_\alpha} \pm 2\pi\alpha\delta$ due to Šťovíček *et al.* [15, 22, 23]. Particularly, we set $H_{a_\alpha} := H^+_{\max} \oplus H^-_{\max}$, where the chosen self-adjoint extensions $H^\pm_{\max}$ of $-\Delta_{a_\alpha}$ initially defined on $C_0^\infty(\mathbb{R}^2\backslash\{0\})$ are in some sense the maximal ones, see (10) below for a precise characterization of its domain.

Our motivation is two-fold. First, contrary to the case of regular potentials considered in [2, 21, 28], the resolvent kernel of the unperturbed Hamiltonian H_{a_α} is known explicitly. Consequently, the standard Birman–Schwinger analysis for the study of weakly coupled eigenvalues is available, so it is possible to avoid the radial hypothesis on v as well as the necessity of advanced resolvent expansions due to [21] and [2, 28], respectively. Second, inspired by a recent interest in quantum Hamiltonians with complex electromagnetic fields, our setting allows for considering complex-valued v (and in fact more general matrix-valued perturbations V, though we do not pursue this research in this paper). In summary, comparing with the precedent papers [2, 21, 28], our choice of the magnetic potential is special, but the electric perturbations are allowed to be more general.

In agreement with the case of regular magnetic potentials, the main result of our analysis shows H_{a_α} is critical whenever $\alpha \notin \mathbb{Z}$ (by a unitary equivalence, one may restrict to $\alpha \in (0,1)$). However, there are always *two* virtual bound states now. Indeed, a variant of our main result can be stated as follows.

Theorem 1. *Let $\alpha \in (0,1)$ and $v \in C_0^\infty(\mathbb{R}^2)$ be non-trivial and non-positive. Then*

$$\begin{aligned} \inf \sigma(H^+_{\max} + \varepsilon v) &\sim -C^+\, \varepsilon^{1/(1-\alpha)}, \\ \inf \sigma(H^-_{\max} + \varepsilon v) &\sim -C^-\, \varepsilon^{1/\alpha}, \end{aligned} \qquad \text{as } \varepsilon \to 0^+, \tag{6}$$

where $C^\pm$ are positive constants depending on v and α.

The asymptotics (6) remain essentially the same for complex-valued v, under suitable hypotheses imposed on integrals involving v (the extension to non-diagonal matrix-valued perturbations V of H_{a_α} is left open in this paper). At the same time, we substantially relax the regularity and sign restrictions on real-valued potentials v below. See Theorem 2 for our main general result.

The existence of two weakly coupled eigenvalues (6), contrary to the merely one in the case of regular potentials considered in [2, 21, 28], might seem controversial at a first glance, arguing that the Aharonov–Bohm potential can be approximated by the regular potentials. However, this approximation on the operator level is known to be delicate [4, 5, 32, 34, 35] (see also [38] for Dirac operators). As a matter of fact, there exist several realizations of the Pauli operator (5) with the Aharonov–Bohm potential (5) (see [6] for a recent complete study), each has its pro and con from the point of view of physical properties, however, no common agreement on the right choice seems to exist in the community. In the language of [6] the extension we explore here is the Friedrichs extension in the (s-wave, spin up) and (p-wave, spin down) components, and the Krein extensions in the (p-wave, spin up) and (s-wave, spin down) components (see also [6, Remark 2.4]). Our paper provides a negative answer to a question of Persson's [34, Sec. 4] about the possibility of the approximation by regular potentials for our self-adjoint realization. On the positive side, comparing the constant C^- from the first of our asymptotics (6) with the constant C of (4) due to [21, 28, 2], these weakly coupled eigenvalues quantitatively match.

The paper is organized as follows. The singular Pauli operator H_{a_α} formally written in (5) is rigorously introduced in Sec. 2 via the method of self-adjoint extensions of symmetric operators. Its Green function is determined in Sec. 3. In Sec. 4, we start to develop the Birman–Schwinger analysis, which enables one to reduce the study of eigenvalues of the differential operator H_{a_α} subject to perturbations to the study of an integral operator. This is completed in Sec. 5 by reducing everything to a matrix eigenvalue problem and further to an implicit equation, obtaining in this way the asymptotics of the weakly coupled eigenvalues.

2. The Pauli Operator via The Extension Theory

The goal of this section is to rigorously introduce the singular Pauli operator (5) and state its basic criticality properties. From now on, we abbreviate $H_\alpha := H_{a_\alpha}$.

Recall that the magnetic field associated with the Aharonov–Bohm potential (2) is the distribution $b_\alpha := \operatorname{curl} a_\alpha = \partial_{x^1} a_\alpha^2 - \partial_{x^2} a_\alpha^1 = -2\pi\alpha\delta$. It is therefore natural to introduce the operator (5) via the methods of extension theory of symmetric operators. We follow the approach of [22, 23], which is based on the factorizations

$$H^+ = T_- T_+ \quad \text{and} \quad H^- = T_+ T_-,$$

considered as operator identities in $\mathcal{D}(\mathbb{R}^2\backslash\{0\}) := C_0^\infty(\mathbb{R}^2\backslash\{0\})$, where

$$T_\pm := -i(\partial_{x^1} \pm i\partial_{x^2}) - (a_\alpha^1 \pm i a_\alpha^2) = e^{\pm i\vartheta}\left(-i\partial_r \pm i\frac{\alpha}{r} \pm \frac{\partial_\vartheta}{r}\right).$$

Here, the second equality follows by the usage of polar coordinates $(r, \vartheta) \in [0, \infty) \times (-\pi, \pi)$ defined by $(x^1, x^2) =: (r\cos\vartheta, r\sin\vartheta)$. We use the same symbols $T_\pm$ for the extensions (denoted by $\tilde{T}_\pm$ in [23]) to the space of distributions $\mathcal{D}'(\mathbb{R}^2\backslash\{0\})$.

The *minimal* realizations $H_{\min}^\pm := T_\pm^* \bar{T}_\pm$ are associated with the closure of the quadratic forms

$$h_{\min}^\pm[\psi] := \|T_\pm \psi\|^2, \quad \mathsf{D}(h_{\min}^\pm) := C_0^\infty(\mathbb{R}^2\backslash\{0\}).$$

It turns out (*cf.* [23, Lemma 5.2]) that $H^+_{\min} = H^-_{\min} = -\Delta_{a_\alpha}$, where $-\Delta_{a_\alpha}$ is the usual magnetic Laplacian with the Aharonov–Bohm potential (i.e. the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R}^2\backslash\{0\})$, see [29]). It is well known [30] that $-\Delta_{a_\alpha}$ is subcritical (i.e. there is a Hardy-type inequality) whenever $\alpha \notin \mathbb{Z}$.

The object of our interest are the *maximal* realizations $H^\pm_{\max} := \bar{T}_\mp T^*_\mp$ associated with the closure of the quadratic forms (*cf.* [23, Lemma 5.4])

$$h^\pm_{\max}[\psi] := \|T_\pm\psi\|^2, \quad \mathsf{D}(h^\pm_{\max}) := \{\psi \in L^2(\mathbb{R}^2) :\ T_\pm\psi \in L^2(\mathbb{R}^2)\}.$$

We then define

$$H_\alpha := H^+_{\max} \oplus H^-_{\max} \quad \text{in } L^2(\mathbb{R}^2, \mathbb{C}^2) \cong L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2). \tag{7}$$

Contrary to the case of regular magnetic fields [28] (when the Pauli operator is essentially self-adjoint), it turns out that *both* $H^\pm_{\max}$ are critical (i.e. they satisfy no Hardy-type inequality). To see it, let us restrict from now on, without loss of generality (by a unitary equivalence, see [23, Sec. 6] and references therein or [34, Proposition 3.1]), to

$$\alpha \in (0, 1).$$

Then the virtual bound states $\varphi^\pm$ of $H^\pm_{\max}$ are given by (note that φ^- is related to Ω_0^- of [21])

$$\varphi^-(x) := r^{-\alpha} \quad \text{and} \quad \varphi^+(x) := r^{-(1-\alpha)}e^{-i\vartheta}. \tag{8}$$

More specifically, it is easy to check that $\varphi^\pm \in L^2_{\text{loc}}(\mathbb{R}^2)$ and $T_\pm\varphi^\pm = 0$ in the sense of distributions. Of course, $\varphi^\pm \notin L^2(\mathbb{R}^2)$, however, the boundedness of $\varphi^\pm$ off the origin enables one to apply the usual approximation procedure.

Lemma 1. *There exists a sequence* $\{\varphi^\pm_n\}_{n\in\mathbb{N}} \subset \mathsf{D}(h^\pm_{\max})$ *converging to* $\varphi^\pm$ *pointwise and satisfying*

$$\lim_{n\to\infty} h^\pm_{\max}[\varphi^\pm_n] = 0.$$

Proof. Let us consider a radial function $\xi \in C^\infty(\mathbb{R}^2)$ satisfying $0 \le \xi \le 1$, $\xi(r) = 1$ if $0 \le r \le 1$ and $\xi(r) = 0$ if $r \ge 2$. Here, with an abuse of notation, we write $\xi(r) = \xi(x)$ when $|x| = r$. Set $\xi_n(r) := \xi(r/n)$ for every $n > 0$. Since, $T^+(\varphi^+\xi_n) = r^{\alpha-1}(-\mathrm{i}\partial_r\xi_n)$ with φ^+ from (8), we have

$$\begin{aligned} h^+_{\max}[\varphi^+\xi_n] &= 2\pi \int_n^{2n} r^{2\alpha-2}|\xi_n'(r)|^2 r\,\mathrm{d}r \\ &= \frac{2\pi}{n^2}\int_n^{2n} r^{2\alpha-1}|\xi'(r/n)|^2\,\mathrm{d}r \le \pi\,\|\xi'\|^2_\infty \frac{4^\alpha - 1}{\alpha}\, n^{2\alpha-2}. \end{aligned}$$

Hence $h^+_{\max}[\varphi^+\xi_n] \to 0$ as $n \to \infty$ whenever $\alpha \in (0,1)$. Similarly, we have $T^-(\varphi^-\xi_n) = \mathrm{e}^{-\mathrm{i}\vartheta} r^{-\alpha}(-\mathrm{i}\partial_r \xi_n)$ and thus

$$\begin{aligned} h^-_{\max}[\varphi^-\xi_n] &= 2\pi \int_n^{2n} r^{-2\alpha} |\xi_n'(r)|^2 r \,\mathrm{d}r \\ &= \frac{2\pi}{n^2} \int_n^{2n} r^{-2\alpha+1} |\xi'(r/n)|^2 \,\mathrm{d}r \le \pi \, \|\xi'\|^2_\infty \frac{4^{1-\alpha}-1}{1-\alpha} \, n^{-2\alpha}. \end{aligned}$$

Hence, again $h^-_{\max}[\varphi^-\xi_n] \to 0$ as $n \to \infty$ whenever $\alpha \in (0,1)$. □

With this auxiliary lemma, we now establish the desired claim.

Proposition 1. *The operators $H^\pm_{\max}$ are critical.*

Proof. By contradiction, let us assume that there exists a non-trivial non-negative function $\rho^\pm \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $H^\pm_{\max} \ge \rho^\pm$ in the sense of forms in $L^2(\mathbb{R}^2)$. Then, for any compact set $K \subset \mathbb{R}^2 \setminus \{0\}$, Lemma 1 implies $\int_K \rho^\pm |\varphi^\pm|^2 = 0$. Since $|\varphi^\pm|$ are positive on arbitrary K, we conclude with the contradiction that $\rho^\pm = 0$ almost everywhere in $\mathbb{R}^2$. □

Let us now characterize the domain of the Pauli operator H_α via boundary conditions at the singularity $r = 0$. The action of the operators $H^\pm_{\max}$ and $-\Delta_{a_\alpha}$ coincide on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. Hence, they are two different self-adjoint extensions of the symmetric operator

$$\tilde{X} := -\Delta_{a_\alpha} = -\partial_r^2 - r^{-1}\partial_r + r^{-2}(-\mathrm{i}\partial_\vartheta + \alpha)^2, \quad \mathsf{D}(\tilde{X}) := C_0^\infty(\mathbb{R}^2 \setminus \{0\}). \tag{9}$$

Due to [22], one has the characterization

$$\mathsf{D}(H^+_{\max}) = \{f \in \mathsf{D}(\tilde{X}^*) : \ \Phi_2^{-1}(f) = \Phi_1^0(f) = 0\}, \tag{10a}$$

$$\mathsf{D}(H^-_{\max}) = \{f \in \mathsf{D}(\tilde{X}^*) : \ \Phi_1^{-1}(f) = \Phi_2^0(f) = 0\}, \tag{10b}$$

where

$$\Phi_1^{-1}(f) := \lim_{r\to 0} r^{1-\alpha} \frac{1}{2\pi} \int_0^{2\pi} f(r,\vartheta)\mathrm{e}^{\mathrm{i}\vartheta} \,\mathrm{d}\vartheta,$$

$$\Phi_2^{-1}(f) := \lim_{r\to 0} r^{-1+\alpha} \left(\frac{1}{2\pi} \int_0^{2\pi} f(r,\vartheta)\mathrm{e}^{\mathrm{i}\vartheta} \,\mathrm{d}\vartheta - r^{-1+\alpha}\Phi_1^{-1}(f) \right),$$

$$\Phi_1^0(f) := \lim_{r\to 0} r^{\alpha} \frac{1}{2\pi} \int_0^{2\pi} f(r,\vartheta) \,\mathrm{d}\vartheta,$$

$$\Phi_2^0(f) := \lim_{r\to 0} r^{-\alpha} \left(\frac{1}{2\pi} \int_0^{2\pi} f(r,\vartheta) \,\mathrm{d}\vartheta - r^{-\alpha}\Phi_1^0(f) \right).$$

It is not difficult to find the spectrum of the Pauli operator.

Proposition 2. $\sigma(H_\alpha) = \sigma_{\mathrm{ess}}(H_\alpha) = [0, +\infty)$.

Proof. Since $H^{\pm}_{\max}$ are non-negative, we immediately have $\sigma(H_\alpha) \subset [0, +\infty)$. To prove the opposite inclusion, we construct a Weyl sequence. Given $\varphi \in C_0^\infty((0,\infty)\times\mathbb{R})$ with $\|\varphi\| = 1$, for every positive n we define

$$\varphi_n(x) := \frac{1}{n}\,\varphi\left(\frac{x^1}{n} - n, \frac{x^2}{n}\right).$$

Note that the normalization factor is chosen in such a way that $\|\varphi_n\| = 1$ for every n. Moreover, the scaling ensures that the derivatives of φ_n vanish as $n \to \infty$, namely

$$\|\nabla\varphi_n\| = n^{-1}\,\|\nabla\varphi\| \quad \text{and} \quad \|\Delta\varphi_n\| = n^{-2}\,\|\Delta\varphi\|. \tag{11}$$

Finally, the shift guarantees that the support of φ_n never intersects the origin where the operator is singular, in fact the support is "localized at infinity" in the sense that $\operatorname{supp}\varphi_n = (n^2, 0) + n\operatorname{supp}\varphi$. Now we define $\psi_n(x) := \varphi_n(x)\mathrm{e}^{\mathrm{i}k\cdot x}$, where $k \in \mathbb{R}^2$. Note that $\psi_n \subset C_0^\infty(\mathbb{R}^2\backslash\{0\}) \subset \mathsf{D}(H^{\pm}_{\max})$ and $\|\psi_n\| = 1$ for every n, while both $H^{\pm}_{\max}$ act on $C_0^\infty(\mathbb{R}^2\backslash\{0\})$ as $\tilde{X}$ introduced in (9). Using that a_α is divergence-free outside the origin, one has

$$\|(\tilde{X} - k^2)\psi_n\| \le \|\Delta\varphi_n\| + 2|k|\|\nabla\varphi_n\| + \||a_\alpha|^2\varphi_n\| + 2\|a_\alpha\cdot\nabla\varphi_n\| \xrightarrow[n\to\infty]{} 0.$$

Here, in addition to (11), we have used that $\|a_\alpha\|_{L^\infty(\operatorname{supp}\varphi_n)} \to 0$ as $n \to \infty$. This argument shows that $\sigma(H^{\pm}_{\max}) = \sigma_{\mathrm{ess}}(H^{\pm}_{\max}) = [0, +\infty)$. The same spectral result for H_α now follows by the fact that the spectrum of H_α is the union of the spectra of $H^{+}_{\max}$ and $H^{-}_{\max}$ due to (7). □

In parallel with the Friedrichs extension $-\Delta_{a_\alpha}$ of $\tilde{X}$, the authors of [40, 22] consider the unitarily equivalent operator

$$H := U_\alpha(-\Delta_{a_\alpha})U_\alpha^{-1}, \quad \text{where } U_\alpha\varphi(r,\vartheta) := \mathrm{e}^{\mathrm{i}\alpha\vartheta}\varphi(r,\vartheta). \tag{12}$$

The operator H acts as the Laplacian in $\mathbb{R}^2\backslash\{0\}$ and functions ψ in its domain satisfy the following boundary conditions at the origin and on the cut $\vartheta = \pi$:

$$\psi(0) = 0, \tag{13a}$$

$$\lim_{\vartheta\to\pi_-}\psi(r,\vartheta) = \mathrm{e}^{2\pi\mathrm{i}\alpha}\lim_{\vartheta\to-\pi_+}\psi(r,\vartheta), \tag{13b}$$

$$\lim_{\vartheta\to\pi_-}\partial_r\psi(r,\vartheta) = \mathrm{e}^{2\pi\mathrm{i}\alpha}\lim_{\vartheta\to-\pi_+}\partial_r\psi(r,\vartheta). \tag{13c}$$

3. The Green Function

The goal of this section is two-fold. First, we apply Krein's formula to the Green function G_z of the operator H from (12) to find the Green function of the Pauli operator H_{a_α}. Second, we study the singularities of the Green function.

3.1. *Krein's formula*

The Green function G_z is presented in [40, Eq. (7)]. Denoting by $z \in \mathbb{C}\backslash[0,\infty)$ the spectral parameter and choosing the branch of the square root so that $\Re\sqrt{-z} > 0$ and recalling that we assume $\alpha \in (0,1)$ (without loss of generality), it reads

$$G_z(r,\vartheta;r_0,\vartheta_0) = \hat{C}(\vartheta-\vartheta_0)K_0(\sqrt{-z}\,|x-x_0|) \tag{14a}$$

$$-\frac{\sin(\pi\alpha)}{\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}K_0(\sqrt{-z}\,R(s))\frac{\mathrm{e}^{-\alpha s+\mathrm{i}\alpha(\vartheta-\vartheta_0)}}{1+\mathrm{e}^{-s+\mathrm{i}(\vartheta-\vartheta_0)}}\,\mathrm{d}s. \tag{14b}$$

Here, K_0 is the zeroth modified Bessel function of the second kind

$$|x-x_0|^2 = r^2 + r_0^2 - 2rr_0\cos(\vartheta-\vartheta_0), \quad R(s)^2 := r^2 + r_0^2 + 2rr_0\cosh(s) \tag{15}$$

and

$$\hat{C}(\vartheta-\vartheta_0) := \frac{1}{2\pi}\begin{cases} 1 & \text{if } \vartheta-\vartheta_0 \in (-\pi,\pi), \\ \mathrm{e}^{-2\pi\mathrm{i}\alpha} & \text{if } \vartheta-\vartheta_0 \in (-2\pi,-\pi), \\ \mathrm{e}^{2\pi\mathrm{i}\alpha} & \text{if } \vartheta-\vartheta_0 \in (\pi,2\pi). \end{cases} \tag{16}$$

Remark 1. Despite the three-fold description, G_z is continuous at $\vartheta-\vartheta_0 = \pm\pi$. Indeed, the continuity of the first line (14a) follows from the equality [24, § 6.791-1]

$$\pi K_0(a+b) = \int_{-\infty}^{\infty} K_{\mathrm{i}\tau}(a)K_{\mathrm{i}\tau}(b)\,\mathrm{d}\tau$$

for $|\operatorname{ph}(a)| + |\operatorname{ph}(b)| \le \pi$. To see the continuity of the second line (14b), we use the identity

$$\int_{-\infty}^{\infty} K_{\mathrm{i}\tau}(a)K_{-\mathrm{i}\tau}(b)\frac{\mathrm{e}^{\varphi\tau}}{\sin(\pi(\alpha+\mathrm{i}\tau))}\,\mathrm{d}\tau$$
$$= \int_{-\infty}^{\infty} K_0(\sqrt{a^2+b^2+2ab\cosh(u)})\frac{\mathrm{e}^{-\alpha(u-\mathrm{i}\varphi)}}{1+\mathrm{e}^{-u+\mathrm{i}\varphi}}\,\mathrm{d}u, \tag{17}$$

for $\operatorname{ph}(a), \operatorname{ph}(b) < \pi$, $\alpha \in (0,1)$ and $|\varphi| < \pi$. Its validity can be checked from formula [24, § 6.792-2]

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}u\tau}K_{\mathrm{i}\tau}(a)K_{\mathrm{i}\tau}(b)\,\mathrm{d}\tau = \pi K_0(\sqrt{a^2+b^2+2ab\cosh(u)}). \tag{18}$$

Using the fact that for $|\varphi| < \pi$ the integral $\oint_\gamma \mathrm{e}^{\mathrm{i}u\tau}\frac{\mathrm{e}^{-\alpha(u-\mathrm{i}\varphi)}}{1+\mathrm{e}^{-u+\mathrm{i}\varphi}}\,\mathrm{d}u$ vanishes along the rectangle $\gamma := (-R,R)\cup(R,R-\mathrm{i}\varphi)\cup(R-\mathrm{i}\varphi,-R-\mathrm{i}\varphi)\cup(-R-\mathrm{i}\varphi,-R)$ for any $R > 0$ by the residue theorem, we further compute

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}u\tau}\frac{\mathrm{e}^{-\alpha(u-\mathrm{i}\varphi)}}{1+\mathrm{e}^{-u+\mathrm{i}\varphi}}\,\mathrm{d}u = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\tau(s+\mathrm{i}\varphi)}\frac{\mathrm{e}^{-\alpha s}}{1+\mathrm{e}^{-s}}\,\mathrm{d}s = \mathrm{e}^{-\tau\varphi}B(\alpha-\mathrm{i}\tau, 1-(\alpha-\mathrm{i}\tau)).$$

For the last equality we have used [24, § 3.313-2]. On the right-hand side the Beta function B can be further evaluated as $B(\alpha-\mathrm{i}\tau,1-(\alpha-\mathrm{i}\tau)) = \frac{\pi}{\sin(\pi(\alpha-\mathrm{i}\tau))}$. Now we apply this result to the left-hand side of the $\frac{e^{-\alpha(u-i\varphi)}}{1+e^{-u+i\varphi}}$ multiple of (18) integrated over $u \in (-\infty, \infty)$ and arrive at (17).

To obtain the Green functions $G_z^\pm$ of the extensions $H_{\max}^\pm$, we mimic the steps in [22, Sec. IV] and use the Krein's formula (*cf.* [14, Eq. (2.6)]). Recalling the unitary transformation U_α from (12), Krein's formula yields

$$e^{i\alpha\vartheta} G_z^\pm(x, x_0) e^{-i\alpha\vartheta_0} = G_z(x, x_0) + \sum_{j,k=1,2} (M_z^\pm)^{j,k} f_z^j(x) \overline{f_{\bar{z}}^k(x_0)}. \tag{19}$$

The coefficient matrices $M_z^\pm$ are determined below. Functions f_z^1, f_z^2 form a basis of the deficiency subspaces ker $(X^* - z)$, where X is the Laplacian on test functions on $\mathbb{R}^2 \setminus \{0\}$. In particular, we set (*cf.* [15])

$$\{f_z^1(r,\vartheta), f_z^2(r,\vartheta)\} := \{K_{1-\alpha}(\sqrt{-z}r) e^{i(\alpha-1)\vartheta}, K_\alpha(\sqrt{-z}r) e^{i\alpha\vartheta}\}$$

with K_ν denoting the νth modified Bessel function of the second kind.

As G_z is the integral kernel of the resolvent of H, the range of the corresponding integral operator is $\mathsf{D}(H)$ determined by the boundary conditions (13). The sum in (19) is the integral kernel of a finite-rank operator. Since upon integration over $x_0 \in \mathbb{R}^2$ the Green function $G_z^\pm(x, x_0)$ has to map square integrable functions to the domain $\mathsf{D}(H_{\max}^\pm)$, we need to choose the matrices $M_z^\pm$ in such a manner that $G_z^\pm(x, x_0)$ satisfy (as functions of $x \in \mathbb{R}^2$) the respective boundary conditions of $H_{\max}^\pm$ given in (10). To check these conditions, we investigate the behavior of $G_z(x, x_0)$ and that of f_z^1, f_z^2 for $|x| = r \to 0$. As for the former, one has [22, Eq. (29)]

$$\begin{aligned} G_z(r,\vartheta,r_0,\vartheta_0) = {} & \frac{\sin(\pi\alpha)}{2\pi^2} \frac{\Gamma(\alpha)}{1-\alpha} \left(\frac{\sqrt{-z}r}{2}\right)^{1-\alpha} \overline{f_{\bar{z}}^1(r_0,\vartheta_0)} e^{-i(1-\alpha)\vartheta} \\ & + \frac{\sin(\pi\alpha)}{2\pi^2} \frac{\Gamma(1-\alpha)}{\alpha} \left(\frac{\sqrt{-z}r}{2}\right)^{\alpha} \overline{f_{\bar{z}}^2(r_0,\vartheta_0)} e^{i\alpha\vartheta} + \mathcal{O}(r) \end{aligned}$$

as $r \to 0$. The asymptotics of f_z^1, f_z^2 follow from the behavior of the Bessel functions

$$K_\nu(w) = \frac{\Gamma(\nu)}{2} \left(\frac{w}{2}\right)^{-\nu} (1 + \mathcal{O}(w^2)) - \frac{\Gamma(1-\nu)}{2\nu} \left(\frac{w}{2}\right)^{\nu} (1 + \mathcal{O}(w^2)), \tag{20}$$

as $|w| \to 0$, where $\Re(\nu) > 0$ and $\mathrm{ph}(w) \neq \pm\pi$. In particular, we arrive at

$$\begin{aligned} & G_z^\pm(r,\vartheta,r_0,\vartheta_0) e^{-i\alpha\vartheta_0} \\ & \quad = \sum_{j=1,2} (M_z^\pm)^{j,1} \overline{f_{\bar{z}}^j(r_0,\vartheta_0)} \\ & \qquad \times \left[\frac{\Gamma(1-\alpha)}{2} \left(\frac{r}{2}\right)^{\alpha-1} (\sqrt{-z})^{\alpha-1} - \frac{\Gamma(\alpha)}{2(1-\alpha)} \left(\frac{r}{2}\right)^{1-\alpha} (\sqrt{-z})^{1-\alpha}\right] e^{-i\vartheta} \\ & \qquad + \sum_{j=1,2} (M_z^\pm)^{j,2} \overline{f_{\bar{z}}^j(r,\vartheta)} \left[\frac{\Gamma(\alpha)}{2} \left(\frac{r}{2}\right)^{-\alpha} (\sqrt{-z})^{-\alpha} - \frac{\Gamma(1-\alpha)}{2\alpha} \left(\frac{r}{2}\right)^{\alpha} (\sqrt{-z})^{\alpha}\right] \end{aligned}$$

$$+ \frac{\sin(\pi\alpha)}{2\pi^2} \frac{\Gamma(\alpha)}{1-\alpha} \left(\frac{r}{2}\right)^{1-\alpha} (\sqrt{-z})^{1-\alpha} \mathrm{e}^{-\mathrm{i}\vartheta} \overline{f^1_{\bar{z}}(r_0,\vartheta_0)}$$

$$+ \frac{\sin(\pi\alpha)}{2\pi^2} \frac{\Gamma(1-\alpha)}{\alpha} \left(\frac{r}{2}\right)^{\alpha} (\sqrt{-z})^{\alpha} \overline{f^2_{\bar{z}}(r_0,\vartheta_0)} + \mathcal{O}(r)$$

as $r \to 0$.

Now we are able to check when $G_z^\pm(r,\vartheta,r_0,\vartheta_0)$ satisfy the boundary conditions (10). We first consider $\mathsf{D}(H^+_{\max})$. The need of the functional $\Phi^0_1(\cdot)$ to vanish means that the coefficients in front of $r^{-\alpha}$ have to vanish. This leads to $(M_z^+)^{j,2} = 0$ for all $j = 1, 2$ (since $f_z^{1,2}$ are linearly independent). From vanishing of the functional $\Phi^{-1}_2(\cdot)$ we conclude that the coefficient in front of the term $r^{1-\alpha}\mathrm{e}^{-\mathrm{i}\vartheta}$ is zero. Therefore

$$(M_z^+)^{2,1} = 0 \quad \text{and} \quad (M_z^+)^{1,1} = \frac{\sin(\pi\alpha)}{\pi^2}.$$

Second, we deal with the conditions in $\mathsf{D}(H^-_{\max})$ and obtain

$$(M_z^-)^{j,1} = 0, \quad j = 1, 2,$$

by vanishing coefficients in front of $r^{\alpha-1}\mathrm{e}^{-\mathrm{i}\vartheta}$ coming from the requirement $\Phi^{-1}_1(\cdot) = 0$, and, finally,

$$(M_z^-)^{1,2} = 0 \quad \text{and} \quad (M_z^-)^{2,2} = \frac{\sin(\pi\alpha)}{\pi^2}$$

as $\Phi^0_2(\cdot) = 0$ implies vanishing coefficients in front of r^α.

We summarize our investigation in the following proposition.

Proposition 3. *Let $\alpha \in (0,1)$. For every $z \in \mathbb{C}\setminus[0,+\infty)$, the resolvent of H_α satisfies*

$$(H_\alpha - z)^{-1} = \begin{pmatrix} (H^+_{\max} - z)^{-1} & 0 \\ 0 & (H^-_{\max} - z)^{-1} \end{pmatrix},$$

where the integral kernels $G_z^\pm$ of the operators $(H^\pm_{\max} - z)^{-1}$ are given by

$$G_z^+(r,\vartheta,r_0,\vartheta_0) = \mathrm{e}^{-\mathrm{i}\alpha\vartheta}\, G_z(r,\vartheta,r_0,\vartheta_0)\mathrm{e}^{\mathrm{i}\alpha\vartheta_0} + \frac{\sin(\pi\alpha)}{\pi^2} K_{1-\alpha}(\sqrt{-z}r)\overline{K_{1-\alpha}(\sqrt{-\bar{z}}r_0)}\mathrm{e}^{-\mathrm{i}(\vartheta-\vartheta_0)},$$

$$G_z^-(r,\vartheta,r_0,\vartheta_0) = \mathrm{e}^{-\mathrm{i}\alpha\vartheta}\, G_z(r,\vartheta,r_0,\vartheta_0)\mathrm{e}^{\mathrm{i}\alpha\vartheta_0} + \frac{\sin(\pi\alpha)}{\pi^2} K_{\alpha}(\sqrt{-z}r)\overline{K_{\alpha}(\sqrt{-\bar{z}}r_0)},$$

with G_z being given in (14).

3.2. *The singularities*

It is well known that the criticality of an operator is related to the singularity of its Green function. Moreover, the weak-coupling asymptotics are determined by the nature of the singularity. To apply the Birman–Schwinger analysis below, we need to have precise information about the singularities of the Green functions $G_z^\pm$ as $z \to 0$.

Note that the functions $z \mapsto G_z^\pm$ are analytic outside of the cut $[0, +\infty)$. Moreover, for any fixed $r, \vartheta, r_0, \vartheta_0$, the limits $G^\pm_{k+\mathrm{i}\varepsilon}(r, \vartheta, r_0, \vartheta_0)$ as $\varepsilon \to 0^\pm$ are well defined for every positive k. Therefore, the singularities indeed occur only as $|z|$ approaches zero.

First of all, we establish a technical identity.

Lemma 2. *Let $\alpha \in (0, 1)$. For every $\varphi \in (-2\pi, -\pi) \cup (-\pi, \pi) \cup (\pi, 2\pi)$, one has*

$$2\pi \hat{C}(\varphi) - \frac{\sin(\pi\alpha)}{\pi} \mathrm{e}^{\mathrm{i}\alpha\varphi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\alpha s}}{1 + \mathrm{e}^{-s+\mathrm{i}\varphi}}\, \mathrm{d}s = 0, \tag{21}$$

where $\hat{C}(\varphi)$ is given in (16).

Proof. Note that the integral converges under the restrictions on φ and α. To compute its value we use the residue theorem. To that aim we consider the regularized integrand of (21) by multiplying it by $\mathrm{e}^{-\mathrm{i}\varepsilon s}$ for some $\varepsilon > 0$. Then we integrate along the oriented contour C_R consisting of the real interval $[-R, R]$ and the arc Γ_R of radius R centered at the origin placed in the top half of the complex plane. In summary, we are considering

$$J_\varepsilon := \oint_{C_R} \mathrm{e}^{\mathrm{i}\varepsilon s} \frac{\mathrm{e}^{-\alpha s}}{1 + \mathrm{e}^{-s+\mathrm{i}\varphi}}\, \mathrm{d}s.$$

This integral can be evaluated by summing its residua $\{\exp[-(\varepsilon + \mathrm{i}\alpha)(\varphi + (2k + 1)\pi)]\}_{k=k_{\min}}^\infty$ corresponding to the simple poles $s_k := \mathrm{i}(\varphi + (2k + 1)\pi)$ in the upper half of the complex plane as follows:

$$J_\varepsilon = \mathrm{e}^{-(\epsilon+\mathrm{i}\alpha)(\varphi+\pi)} \sum_{k=k_{\min}}^{\infty} \mathrm{e}^{-(\epsilon+\mathrm{i}\alpha)2\pi k} \quad \text{with } k_{\min} := \begin{cases} 0 & \text{if } \varphi \in (-\pi, \pi), \\ 1 & \text{if } \varphi \in (-2\pi, -\pi), \\ -1 & \text{if } \varphi \in (\pi, 2\pi). \end{cases}$$

Here $k_{\min}$ is given by the condition that $\Im(s_k) > 0$ for all $k \geq k_{\min}$. Note also that the sum is well defined as for all such k the absolute value of the summands is less than one. The integral over the arc of the loop vanishes in the limit $R \to \infty$ since

$$\begin{aligned} \left| \int_{\Gamma_R} \mathrm{e}^{\mathrm{i}\varepsilon s} \frac{\mathrm{e}^{-\alpha s}}{1 + \mathrm{e}^{-s+\mathrm{i}\varphi}}\, \mathrm{d}s \right| &\leq \int_{\Gamma_R} \left| \mathrm{e}^{\mathrm{i}\varepsilon s} \frac{\mathrm{e}^{-\alpha s}}{1 + \mathrm{e}^{-s+\mathrm{i}\varphi}} \right| \mathrm{d}s \\ &= \int_0^{\pi/2} \left| \frac{\mathrm{e}^{-\alpha R\cos\beta - \varepsilon R \sin\beta}}{1 + \mathrm{e}^{-R(\cos\beta + \mathrm{i}\sin\beta) + \mathrm{i}\varphi}} \right| R\, \mathrm{d}\beta \\ &\quad + \int_{\pi/2}^{\pi} \left| \frac{\mathrm{e}^{(1-\alpha)R\cos\beta - \varepsilon R\sin\beta}}{1 + \mathrm{e}^{R(\cos\beta + \mathrm{i}\sin\beta) - \mathrm{i}\varphi}} \right| R\, \mathrm{d}\beta. \end{aligned}$$

Using the dominated convergence we can thus compute our original integral

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\alpha\varphi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\alpha s}}{1+\mathrm{e}^{-s+\mathrm{i}\varphi}}\,\mathrm{d}s &= \mathrm{e}^{\mathrm{i}\alpha\varphi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\varepsilon s}\frac{\mathrm{e}^{-\alpha s}}{1+\mathrm{e}^{-s+\mathrm{i}\varphi}}\,\mathrm{d}s \\ &= \mathrm{e}^{\mathrm{i}\alpha\varphi}\lim_{\epsilon\to 0}\lim_{R\to\infty}\int_{C_R}\mathrm{e}^{\mathrm{i}\varepsilon s}\frac{\mathrm{e}^{-\alpha s}}{1+\mathrm{e}^{-s+\mathrm{i}\varphi}}\,\mathrm{d}s \\ &= \frac{\pi}{\sin(\pi\alpha)}\left\{\begin{matrix}1\\ \mathrm{e}^{-2\pi\mathrm{i}\alpha}\\ \mathrm{e}^{2\pi\mathrm{i}\alpha}\end{matrix}\right\}.\end{aligned}$$

This is equivalent to (21). □

To study the singularities of $G_z^{\pm}$, we start with the part G_z given by (14). First, recall the asymptotics of the zero'th Bessel function (*cf.* [1, § 9.6.12–13]) $K_0(w) = -\log(w/2) + \mathcal{O}(1)$ as $|w| \to 0$. Second, since the function R introduced in (15) is real-valued, we have $\log(\sqrt{-z}R(s)) = \log(\sqrt{-z}) + \log(R(s))$. Consequently, we see that the logarithmic singularity of G_z cancels out due to Lemma 2, yielding the behavior

$$G_z(r,\vartheta;r_0,\vartheta_0) = d(x,x_0) + \mathcal{O}\big((\sqrt{-z})^2\big) \quad \text{as } |z|\to 0$$

with some function d dependent on the spatial coordinates but independent of z.

Now we focus on the second terms in the formulae for $G_z^{\pm}$ given in Proposition 3. From the asymptotics (20), we compute

$$\begin{aligned}&K_{1-\alpha}(\sqrt{-z}r)\overline{K_{1-\alpha}(\sqrt{-\bar{z}}r_0)} \\ &\quad = (\sqrt{-z})^{2(\alpha-1)}\left(\frac{\Gamma(1-\alpha)}{2}\right)^2\left(\frac{rr_0}{4}\right)^{\alpha-1} - \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{4(1-\alpha)} \\ &\qquad \times\left(\left(\frac{r_0}{r}\right)^{1-\alpha} + \left(\frac{r}{r_0}\right)^{1-\alpha}\right) + \left(\frac{\Gamma(\alpha)}{2(1-\alpha)}\right)^2(\sqrt{-z})^{2(1-\alpha)} \\ &\qquad \times\left(\left(\frac{rr_0}{4}\right)^{1-\alpha}\right) + \mathcal{O}(\sqrt{-z}^{2\min\{\alpha,1-\alpha\}}),\end{aligned} \tag{22a}$$

$$\begin{aligned}&K_{\alpha}(\sqrt{-z}r)\overline{K_{\alpha}(\sqrt{-\bar{z}}r_0)} \\ &\quad = (\sqrt{-z})^{-2\alpha}\left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{rr_0}{4}\right)^{-\alpha} - \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{4\alpha}\left(\left(\frac{r_0}{r}\right)^{\alpha} + \left(\frac{r}{r_0}\right)^{\alpha}\right) \\ &\qquad + \left(\frac{\Gamma(1-\alpha)}{2\alpha}\right)^2\sqrt{-z}^{2\alpha}\left(\frac{rr_0}{4}\right)^{\alpha} + \mathcal{O}(\sqrt{-z}^{2\min\{\alpha,1-\alpha\}}),\end{aligned} \tag{22b}$$

as $|z| \to 0$.

In summary, we have established the following asymptotics.

Proposition 4. *Let $\alpha \in (0,1)$ and $r, \vartheta, r_0, \vartheta_0$ be fixed. For every $z \in \mathbb{C}\backslash[0,+\infty)$, one has*

$$G_z^+(r,\vartheta,r_0,\vartheta_0) = C_\alpha \Gamma(1-\alpha)^2(-zrr_0/4)^{\alpha-1}\mathrm{e}^{-\mathrm{i}(\vartheta-\vartheta_0)} + \mathcal{O}(1),$$
$$G_z^-(r,\vartheta,r_0,\vartheta_0) = C_\alpha \Gamma(\alpha)^2(-zrr_0/4)^{-\alpha} + \mathcal{O}(1),$$

as $|z| \to 0$, where

$$C_\alpha := \frac{\sin(\pi\alpha)}{4\pi^2}.$$

4. The Birman–Schwinger Analysis

Let $V : \mathbb{R}^2 \to \mathbb{C}^{2\times 2}$ be a matrix-valued function. For almost every $x \in \mathbb{R}^2$, we consider the matrix polar decomposition

$$V(x) = B(x)A(x) \quad \text{with} \quad \begin{aligned} A(x) &:= \sqrt[4]{V(x)^*V(x)}, \\ B(x) &:= \mathbb{U}(x)\,\sqrt[4]{V(x)^*V(x)}, \end{aligned} \tag{23}$$

where $\mathbb{U}(x)$ is a unitary matrix. By $|V(x)|$ we denote the operator norm of the matrix $V(x)$ when considered as an operator on $\mathbb{C}^2$. Our standing hypothesis about V is as follows.

Assumption 1. Let $\alpha \in (0,1)$. Suppose

$$\int_{\mathbb{R}^2} |V(x)|(|x|^{2\nu} + |x|^{-2\nu})\,\mathrm{d}x < \infty \quad \text{and} \quad |V| \in L^{1+\delta}(\mathbb{R}^2) \tag{24}$$

for some positive δ and $\nu := \max\{1-\alpha, \alpha\}$.

Note that conditions (24) particularly imply that $|V| \in L^1(\mathbb{R}^2)$.

Let ϵ be a (small) positive number. By the Birman–Schwinger principle [26] (justified under Assumption 1 in Remark 2 below), $z \in \mathbb{C}\backslash[0,+\infty)$ is an eigenvalue of $H_\alpha + \epsilon V$ if, and only if, -1 is an eigenvalue of the integral operator

$$R_{z,\epsilon} := \epsilon A(H_\alpha - z)^{-1}B. \tag{25}$$

Here, A, B are considered as the maximal operators of multiplication by the matrix-valued functions denoted by the same symbols.

To apply this principle to the analysis of the weakly coupled eigenvalues, we decompose

$$R_{z,\epsilon} = \epsilon(L_z + Q_z), \tag{26}$$

where L_z and Q_z are the singular and regular parts of the Birman–Schwinger operator, respectively. More specifically, adopting the convention that the kernel of an integral operator T is distinguished calligraphically by $\mathcal{T}$, we define

$$\begin{aligned}\mathcal{L}_z(r,\vartheta,r_0,\vartheta_0) &:= C_\alpha A(r,\vartheta)\\ &\quad\times\begin{pmatrix}\Gamma(1-\alpha)^2(-z|rr_0|/4)^{\alpha-1}\mathrm{e}^{-\mathrm{i}(\vartheta-\vartheta_0)} & 0\\ 0 & \Gamma(\alpha)^2(-z|rr_0|/4)^{-\alpha}\end{pmatrix}\\ &\quad\times B(r_0,\vartheta_0), \qquad (27a)\\ \mathcal{Q}_z(r,\vartheta,r_0,\vartheta_0) &:= A(r,\vartheta)G_z^{\mathrm{reg}}(r,\vartheta,r_0,\vartheta_0)B(r_0,\vartheta_0) \qquad (27b)\\ &= A(r,\vartheta)\begin{pmatrix}G_z^{\mathrm{reg},+}(r,\vartheta,r_0,\vartheta_0) & 0)\\ 0 & G_z^{\mathrm{reg},-}(r,\vartheta,r_0,\vartheta_0)\end{pmatrix}\\ &\quad\times B(r_0,\vartheta_0) \qquad (27c)\end{aligned}$$

with

$$\begin{aligned}G_z^{\mathrm{reg},\pm}(r,\vartheta,r_0,\vartheta_0) &:= \Bigg[\mathrm{e}^{-\mathrm{i}\alpha\vartheta}\, G_z(r,\vartheta,r_0,\vartheta_0)\mathrm{e}^{\mathrm{i}\alpha\vartheta_0} + C_\alpha\Bigg(4K_{\nu^\pm}(\sqrt{-z}r)\\ &\quad\times K_{\nu^\pm}(\sqrt{-z}r_0) - \Gamma(\nu^\pm)^2\left(\frac{-zrr_0}{4}\right)^{-\nu^\pm}\Bigg)\mathrm{e}^{\mathrm{i}(\mp\nu^\pm-\alpha)(\vartheta-\vartheta_0)}\Bigg]. \qquad (27d)\end{aligned}$$

Here $\nu^+ := 1-\alpha$ and $\nu^- := \alpha$.

To analyze the regular part Q_z, we need the following fact about the zero'th modified Bessel function K_0.

Lemma 3. *There exist functions f, g analytic on $\mathbb{C}\backslash(-\infty,0]$ and continuous at 0 such that*

$$K_0(w) = \log(w)f(w) + g(w).$$

Functions f and g can be chosen bounded in absolute value on $\Re w > 0$ by $C_1\mathrm{e}^{-C_2\Re w}$ with some constants $C_1, C_2 > 0$. Moreover, $f(0) = 1$, $f(w) - f(0) = \mathcal{O}(w^2)$ and $g(w) = \mathcal{O}(1)$ as $|w| \to 0$.

Proof. Recalling the series [1, § 9.6.12.–13]

$$\begin{aligned}K_0(w) &= -(\log(w/2)+\gamma)I_0 + \sum_{k\geq 1}(1+1/2+1/3+\cdots+1/k)\frac{(w/2)^{2k}}{(k!)^2} \quad \text{and}\\ I_0(w) &= \sum_{k\geq 0}\frac{(w/2)^{2k}}{(k!)^2},\end{aligned} \qquad (28)$$

we define $f(w) := -I_0\mathrm{e}^{-2w}$ and $g(w) = K_0(w) - \log(w/2)f(w)$. With this choice f is entire and g is analytic outside of the cut $(-\infty, 0]$. Note that g is continuous at 0 and all the claimed properties follow from (28), definitions of f and g and the asymptotics of I_0, K_0 for large arguments [1, § 9.7.1–2]. □

First of all, we argue that the Birman–Schwinger operator associated with $H = U_\alpha(-\Delta_{a_\alpha})U_\alpha^{-1}$ (recall (12)) is regular, in agreement with its subcriticality.

Lemma 4. *Suppose Assumption* 1. *There exists a positive constant C such that, for all $z \in \mathbb{C}\backslash[0, +\infty)$,*

$$\|A(H-z)^{-1}B\|_{\mathrm{HS}} \leq C.$$

Proof. Recall that the integral kernel of $A(H-z)^{-1}B$ reads $A(x)G_z(x, x_0)B(x_0)$, where the Green function G_z is given in (14). We separate the proof of finiteness of the Hilbert–Schmidt norm

$$\|A(H-z)^{-1}B\|_{\mathrm{HS}} = \int_{\mathbb{R}^2\times\mathbb{R}^2} |V(x)|\,|G_z(x,x_0)|^2\,|V(x_0)|\,\mathrm{d}x\,\mathrm{d}x_0$$

into two parts:

(1) showing that it is bounded uniformly in $|z| \in (0, 1]$,
(2) showing that it is bounded uniformly for any $|z| > 1$.

Ad 1. We start with the bound for small $|z|$, as some of the estimates will be useful in proving the second point too. We add and subtract the logarithmic singularity $\log(\sqrt{-z})f(0)$ in the Green function of the Friedrichs extension G_z given by (14) and using Lemma 2, we divide it in two summands

$$G_z(x, x_0) = \hat{C}F_1(|x-x_0|) + F_2(x, x_0)$$

with

$$F_1(t) := [K_0(\sqrt{-z}t) - \log(\sqrt{-z})f(0)], \quad t \geq 0,$$
$$F_2(x, x_0) := \frac{\sin(\pi\alpha)}{\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}F_1(R(s))\frac{\mathrm{e}^{-\alpha s+\mathrm{i}\alpha(\vartheta-\vartheta_0)}}{1+\mathrm{e}^{-s+\mathrm{i}(\vartheta-\vartheta_0)}}\,\mathrm{d}s.$$

Here, f is as in Lemma 3 and it follows that

$$\begin{aligned}|F_1(t)| &= |\log(\sqrt{-z})(f(\sqrt{-z}t) - f(0)) + g(\sqrt{-z}t) + \log t f(\sqrt{-z}t)| \\ &\leq C_1(|\log(\sqrt{-z})||\sqrt{-z}t|^\beta + \log t + 1), \end{aligned} \tag{29}$$

for some constant $C_1 > 0$ and any $\beta \in (0, 1)$. Let us remark, that for any fixed $t \in (0, \infty)$ this stays bounded uniformly in $|z| \in (0, 1)$.

Turning our attention for a moment to F_2, we remark that for a fixed difference $\vartheta - \vartheta_0 \neq \pm\pi$, we have the exponential decay for large s of the fraction

$$\left|\frac{\mathrm{e}^{-\alpha s+\mathrm{i}\alpha(\vartheta-\vartheta_0)}}{1+\mathrm{e}^{-s+\mathrm{i}(\vartheta-\vartheta_0)}}\right|^2 = \frac{\mathrm{e}^{-2\alpha s}}{1+2\mathrm{e}^{-s}\cos(\vartheta-\vartheta_0)+\mathrm{e}^{-2s}} \leq C\mathrm{e}^{-2|s|\min\{\alpha,1-\alpha\}}$$

with some $C > 0$. We also notice that $|\frac{1}{2}\log(R(s)^2)| \leq |s| + |\log(r + r_0)|$ and that $R(s)^\beta \leq (r + r_0)^\beta e^{\beta|s|}$. Hence, with an arbitrary choice $0 < \beta < \min\{\alpha, 1-\alpha\}$ we conclude by (29) that the integral in F_2 is convergent for every fixed pair r, r_0.

Due to these estimates, it is only left to show the finiteness

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |V(x)|\,|1 + \log(a) + a^\beta|^2\,|V(x_0)|\,\mathrm{d}x\,\mathrm{d}x_0 < \infty \quad \text{for } a \in \{r + r_0, |x - x_0|\}.$$

Since

$$|\log a| \leq C_\beta \begin{cases} (1 + a^{-\beta}) & \text{if } a \in (0, 1], \\ (1 + a^\beta) & \text{if } a \in [1, \infty), \end{cases} \tag{30}$$

where C_β is some constant dependent only on β, it is enough to bound the three integrals

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |V(x)| \begin{Bmatrix} 1 \\ a^{-2\beta} \\ a^{2\beta} \end{Bmatrix} |V(x_0)|\,\mathrm{d}x\,\mathrm{d}x_0 \quad \text{for } a \in \{r + r_0, |x - x_0|\}. \tag{31}$$

Finiteness of the first integral with the constant middle term is a direct consequence of Assumption 1. To treat the positive power, notice that since $\beta \leq \max\{\alpha, 1-\alpha\} = \nu$ we have

$$|x - x_0|^{2\beta} \leq (r + r_0)^{2\beta} \leq 2^\nu(1 + |x|^{2\nu})(1 + |x_0|^{2\nu})$$

and the integrability follows by our Assumption 1.

For the negative power we first estimate $(r+r_0)^{-\beta} \leq |x-x_0|^{-\beta} \leq 1+|x-x_0|^{-\mu}$ for any $\mu \geq \beta$. Then the integrability follows under our assumptions by the Hardy–Littlewood–Sobolev inequality [31, Theorem 4.3] choosing $\beta \leq \mu = \frac{\delta}{1+\delta}$. Here, $\delta > 0$ is as in (24).

To finish the proof we point out that G_z is continuous at $\vartheta - \vartheta_0 = \pm\pi$, see Remark 1.

Ad 2. By the analyticity of $K_0(\sqrt{-z}t)$ on $\Re\sqrt{-z} > 0$ with $t > 0$, by the behavior for large arguments [1, § 9.7.2] and by Lemma 3, we can bound

$$|K_0(\sqrt{-z}t)| \leq C(|\sqrt{-z}t)|^{-\beta} + 1) \quad \text{for all } t \in (0, \infty),\ \Re\sqrt{-z} > 0$$

with some constant $C > 0$ and arbitrary $\beta > 0$. For all $t \in (0, \infty)$ this is uniformly bounded in $|z| > 1$. The statement is then a consequence of the finiteness of the top two integrals in (31), which was shown in the first part of the proof. □

Lemma 5. *Suppose Assumption* 1. *Operators L_z and Q_z are Hilbert–Schmidt for all $z \in \mathbb{C}\backslash[0, +\infty)$.*

Proof. Recall the formulae for L_z and Q_z given in (27). The claim for L_z is obvious. Taking Lemma 4 into account, the statement for Q_z follows directly from definitions (27) by the analyticity of the Bessel functions on the complex half-plane with positive real part and their decay properties (*cf.* [1, Eqs. (9.6.2), (9.6.10)])

$$K_\mu(w) = \mathcal{O}(\mathrm{e}^{-w}/\sqrt{w}) \quad \text{as } |w| \to \infty \tag{32}$$

for $\Re w > 0$ and $\mu \in \mathbb{R}$. □

Proposition 5. *Suppose Assumption* 1. *For negative z with all sufficiently large $|z|$, the difference*

$$(H_\alpha + \epsilon V - z)^{-1} - (H_\alpha - z)^{-1}$$

is a compact operator. Moreover, $A(H_\alpha - z)^{-1/2}$ and $B(H_\alpha - z)^{-1/2}$ are also compact.

Proof. By Lemma 5 and Proposition 3, the Birman–Schwinger operator $R_{z,\epsilon}$ introduced in (25) and decomposed in (26) is a Hilbert–Schmidt operator for all $z \in (-\infty, 0)$. Moreover, its integral kernel $\mathcal{R}_{z,\epsilon}$ tends pointwise to zero as $z \to -\infty$. The second part of the proof of Lemma 4 and (32) justify using the dominated convergence on the integral kernel $\mathcal{R}_{z,\epsilon}$ to deduce $\|R_{z,\epsilon}\|_{\mathrm{HS}} = \|A(H_\alpha - z)^{-1}A\|_{\mathrm{HS}} \to 0$ as $z \to -\infty$. Thus, along the lines of [36, Example 7 of Sec. XIII.4], we can argue that the difference of resolvents

$$\begin{aligned}
&(H_\alpha + \epsilon V - z)^{-1} - (H_\alpha - z)^{-1} \\
&\quad = -\sum_{n=0}^{\infty} \epsilon^{n+1} (H_\alpha - z)^{-1} B(-A(H_\alpha - z)^{-1} B)^n A(H_\alpha - z)^{-1}
\end{aligned}$$

is a compact operator for some negative z with large $|z|$. For this conclusion, we use the above observed fact that P^*P with $P := A(H_\alpha - z)^{-1/2}$ tends to zero in the Hilbert–Schmidt norm as $z \to -\infty$. It then follows that $A(H_\alpha - z)^{-1/2}$ and $B(H_\alpha - z)^{-1/2}$ are also compact operators. □

Remark 2. It follows from Proposition 5 that $A(H_\alpha - z)^{-1/2}$ and $B(H_\alpha - z)^{-1/2}$ are bounded operators for all negative z with sufficiently large $|z|$. It justifies the usage of the Birman–Schwinger principle in the spirit of [26]. Moreover, the perturbation V is relatively form bounded with respect to H_α. By making ϵ small, the relative bound can be made arbitrarily small. This justifies the sum $H_\alpha + \epsilon V$, which should be understood in the sense of forms.

Combining Proposition 5 with [36, Theorem XIII.14], we obtain the stability of the essential spectrum.

Corollary 1. *Suppose Assumption* 1. *Then*

$$\sigma_{\mathrm{ess}}(H_\alpha + \epsilon V) = \sigma_{\mathrm{ess}}(H_\alpha) = [0, +\infty).$$

Finally, we establish a uniform version of Lemma 5.

Lemma 6. *Suppose Assumption* 1. *There exists a positive constant* C *such that, for all* $z \in \mathbb{C}\backslash[0,+\infty)$, $\|Q_z\|_{\mathrm{HS}} \leq C$. *At the same time, given any positive* ε, *there exists a positive constant* C_ε, *such that, for all* $z \in \mathbb{C}\backslash[0,+\infty)$ *with* $|z| \geq \varepsilon$, $\|L_z\|_{\mathrm{HS}} \leq C_\varepsilon$.

Proof. The claim for L_z is obvious from (27a). The uniform boundedness is also clear for the part of Q_z coming from the first line of (27d) due to Lemma 4. At the same time, it follows from the structure of the second line of (27d) and the asymptotic behavior (32) that, given any positive ε, there exists a positive constant C_ε, such that, for all $z \in \mathbb{C}\backslash[0,+\infty)$ with $|z| \geq \varepsilon$, $\|Q_z\|_{\mathrm{HS}} \leq C_\varepsilon$. It remains to analyze the asymptotic behavior of the second line of (27d) as $|z| \to 0$.

Let $\mu \in \{\alpha, 1-\alpha\}$. We establish a convenient notation (*cf.* [1, § 9.6.2, § 9.6.10])

$$K_\mu(w) = w^{-\mu} f_{-\mu}(w) - w^{\mu} f_\mu(w) \quad \text{with } f_\mu(w) = \frac{-\pi 2^{-\mu}}{2\sin(\mu\pi)} \sum_{k=0}^{\infty} \frac{(w/2)^{2k}}{k!\Gamma(k+\mu+1)}, \tag{33}$$

$w \in \mathbb{C}\backslash(-\infty,0)$. On account of Lemma 4, it is enough to bound $K_\mu(\sqrt{-z}|x|)$ $K_\mu(\sqrt{-z}|x_0|) - f_{-\mu}^2(0)| - zxx_0|^{-\mu}$ by a constant multiple of $(|x|^\mu + |x|^{-\mu})(|x_0|^\mu + |x_0|^{-\mu})$. To that end we denote by $\xi = \sqrt{-z}|x|$ and by $\zeta = \sqrt{-z}|x_0|$ and write the exact identity

$$\begin{aligned} K_\mu(\xi)K_\mu(\zeta) - f_{-\mu}^2(0)(\xi\zeta)^{-\mu} &= (K_\mu(\xi) - f_{-\mu}(0)\xi^{-\mu})K_\mu(\zeta) \\ &\quad + \xi^{-\mu} f_{-\mu}(0)(K_\mu(\zeta) - \zeta^{-\mu} f_{-\mu}(0)). \end{aligned}$$

If both $|\zeta|, |\xi| \geq 1$ then the left-hand side is bounded by a constant by analyticity of K_μ and the decay (32). If one of the arguments is small, assume without loss of generality $|\xi| < 1$, we notice that $f_\mu(\xi) - f_\mu(0) = \mathcal{O}(\xi^2)$ as $|\xi| \to 0$ and use the bound

$$|K_\mu(\xi) - \xi^{-\mu} f_{-\mu}(0)| \leq C|\xi|^\mu$$

with some positive constant C. If $\zeta \geq 1$ we have $|K_\mu(\zeta)| \leq C_1 \leq C_1 |\zeta|^\mu$ with $C_1 > 0$, while for $|\zeta| < 1$ by (32) it holds

$$|K_\mu(\zeta) - \zeta^{-\mu} f_{-\mu}(0)| \leq C|\zeta|^\mu \quad \text{and} \quad |K_\mu(\zeta)| \leq C_2|\zeta|^{-\mu},$$

where $C, C_2 > 0$ are some constants. Symmetrically we can find bounds in case $|\zeta| < 1$. For any ζ, ξ with positive real part we can thus estimate

$$|K_\mu(\xi)K_\mu(\zeta) - f_{-\mu}^2(0)(\xi\zeta)^{-\mu}| \leq C_3(|\xi/\zeta|^\mu + |\zeta/\xi|^\mu + |\xi|^\mu + 1),$$

for some constant $C_3 > 0$. In particular this stays bounded for any fixed x, y as $|z| \to 0$. Since $\mu \leq \max\{\alpha, 1-\alpha\} = \nu$ implies $\int_{\mathbb{R}^2} |V(x)|(|x|^{2\mu} + |x|^{-2\mu}) \leq 2\int_{\mathbb{R}^2} |V(x)|(|x|^{2\nu} + |x|^{-2\nu})$ we conclude, taking Lemma 4 into account, that under our assumptions on the potential the operator Q_z is Hilbert–Schmidt as $|z| \to 0$. □

5. The Weakly Coupled Eigenvalues

In this section, we establish Theorem 1 as a consequence of its stronger variant.

First of all, we claim that provided that there exist eigenvalues of the perturbed operator $H_\alpha + \epsilon V$ for all small ϵ, they correspond to the singularities in the unperturbed Green function and therefore necessarily tend to zero as the positive parameter ϵ vanishes. The fact that zero is the only possible accumulation point is not obvious, because we allow $H_\alpha + \epsilon V$ to be non-self-adjoint.

Lemma 7. *Suppose Assumption* 1. *Let* $z_\epsilon \in \sigma_{\mathrm{disc}}(H_\alpha + \epsilon V)$ *for all sufficiently small positive* ϵ. *Then* $|z_\epsilon| \to 0$ *as* $\epsilon \to 0$.

Proof. By the Birman–Schwinger principle, there exists a normalized $\psi_\epsilon \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ such that $R_{z,\epsilon}\psi_\epsilon = -\psi_\epsilon$ for all sufficiently small positive ϵ. Then

$$\begin{aligned} 1 = |\langle \psi_\epsilon, R_{z,\epsilon}\psi_\epsilon\rangle| &\le \|R_{z,\epsilon}\| \le \|R_{z,\epsilon}\|_{\mathrm{HS}} \\ &= \epsilon\, \|L_z + Q_z\|_{\mathrm{HS}} \le \epsilon\, (\|L_z\|_{\mathrm{HS}} + \|Q_z\|_{\mathrm{HS}}). \end{aligned}$$

By contradiction, assume that there is a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ converging to zero and a sequence of eigenvalues $\{z_{\epsilon_j}\}_{j\in\mathbb{N}}$ converging to a positive point k of the essential spectrum $[0, +\infty)$. Then the inequality above together with Lemma 6 implies $1 \le \epsilon_j\,(C_{k/2} + C)$, where $C_{k/2}$ and C are the constants from Lemma 6, independent of j. This is obviously a contradiction for all sufficiently large j. □

Our next step is to reformulate the Birman–Schwinger principle in the usual way using the decomposition (26) of the Birman–Schwinger operator $R_{z,\epsilon}$ into the singular part ϵL_z and the regular part ϵQ_z. The existence of eigenvalue -1 for $R_{z,\epsilon}$ is equivalent to the lack of invertibility of

$$(1 + \epsilon(Q_z + L_z)) = (1 + \epsilon Q_z)(1 + \epsilon(\epsilon Q_z + 1)^{-1} L_z).$$

Here the operator $1 + \epsilon Q_z$ is invertible for all sufficiently small ϵ by Lemma 6. That means that, provided that ϵ is sufficiently small, -1 is an eigenvalue of $R_{z,\epsilon}$ if, and only if, -1 is an eigenvalue of the rank-one operator $\epsilon(\epsilon Q_z + 1)^{-1} L_z$.

To find the form of an eigenvalue $\lambda \neq 0$ of the operator $\epsilon(\epsilon Q_z + 1)^{-1} L_z$, let us denote by ψ the corresponding normalized eigenvector. Then by definition of L_z (recall (27a)) we have (using the complex formalism $w := x^1 + \mathrm{i}x^2$ and $w_0 := x_0^1 + \mathrm{i}x_0^2$)

$$\lambda\psi(w) = \epsilon(\epsilon Q_z + 1)^{-1} A(w)\overline{D(w)} \int_{\mathbb{C}} Y_z D(w_0) B(w_0) \psi(w_0)\, \mathrm{d}w_0. \tag{34}$$

Here we have introduced the decomposition of the integral kernel

$$\mathcal{L}_z(w, w_0) = A(w)\overline{D(w)} Y_z D(w_0) B(w_0)$$

using

$$\begin{aligned} Y_z &:= \begin{pmatrix} (-z)^{\alpha-1} & 0 \\ 0 & (-z)^{-\alpha} \end{pmatrix}, \\ D(w) &:= \sqrt{C_\alpha} \begin{pmatrix} \Gamma(1-\alpha)(|w|/2)^{\alpha-1} \mathrm{e}^{\mathrm{i}\ \mathrm{ph}(w)} & 0 \\ 0 & \Gamma(\alpha)(|w|/2)^{-\alpha} \end{pmatrix}. \end{aligned} \tag{35}$$

We rewrite (34) as

$$\begin{aligned} \lambda \psi(w) &= \epsilon(\epsilon Q_z + 1)^{-1} A(w) \overline{D(w)} b_z \quad \text{with} \\ b_z &:= Y_z \int_{\mathbb{C}} D(w_0) B(w_0) \psi(w_0) \, \mathrm{d}w_0 \in \mathbb{C}^2. \end{aligned} \tag{36}$$

Inserting ψ back into a λ-multiple of (34), we get the equation

$$\lambda \epsilon(\epsilon Q_z + 1)^{-1} A(w) \overline{D(w)} b_z = \epsilon(\epsilon Q_z + 1)^{-1} A(w) \overline{D(w)} \epsilon Y_z \mathbb{W}(\epsilon) b_z \tag{37}$$

with the matrix

$$\mathbb{W}(\epsilon) := \int_{\mathbb{C}} D(w_0) B(w_0) (\epsilon Q_z + 1)^{-1} A(w_0) \overline{D(w_0)} \, \mathrm{d}w_0.$$

Applying to both sides of (37) the invertible operator $\epsilon Q_z + 1$ and dividing by ϵ, we see that any non-zero eigenvalue λ of $\epsilon(\epsilon Q_z + 1)^{-1} L_z$ satisfies

$$A(w) \overline{D(w)} \epsilon Y_z \mathbb{W}(\epsilon) b_z = \lambda A(w) \overline{D(w)} b_z \tag{38}$$

which is a generalized eigenvalue problem in $\mathbb{C}^2$. The following proposition summarizes the above analysis and additionally argues that (38) is equivalent to the usual eigenvalue problem by "dividing by" the matrix-valued function $w \mapsto A(w)\overline{D(w)}$.

Proposition 6. *Suppose Assumption* 1. *For all sufficiently small* ϵ, $z \in \mathbb{C} \setminus [0, +\infty)$ *is an eigenvalue of* $H_\alpha + \epsilon V$ *if, and only if,* -1 *is an eigenvalue of the matrix* $\epsilon Y_z \mathbb{W}(\epsilon)$.

Proof. If a non-zero vector $b_z \in \mathbb{C}^2$ solves the matrix eigenvalue problem $\epsilon Y_z \mathbb{W}(\epsilon) b_z = -b_z$, then it is easy to check that the function ψ defined by the first formula of (36) with $\lambda = -1$ solves $\epsilon(\epsilon Q_z + 1)^{-1} L_z \psi = -\psi$. Assuming $\psi = 0$ implies $A(w)\overline{D(w)} b_z = 0$ for almost every $w \in \mathbb{C}$. But then $\mathbb{W}(\epsilon) b_z = 0$, because of the structure of the matrix $\mathbb{W}(\epsilon)$, which is impossible.

Conversely, assume $\epsilon(\epsilon Q_z + 1)^{-1} L_z \psi = -\psi$ with a non-trivial function ψ. Then (36) holds with $\lambda = -1$ and thus defined vector b_z is necessarily non-zero. Applying the matrix $\epsilon Y_z \mathbb{W}(\epsilon)$ to b_z as defined by the integral formula of (36), it is easy to see that $\epsilon Y_z \mathbb{W}(\epsilon) b_z = -(\epsilon Y_z \mathbb{W}(\epsilon))^2 b_z$. Consequently, either b_z solves $\epsilon Y_z \mathbb{W}(\epsilon) b_z = -b_z$ or $\epsilon Y_z \mathbb{W}(\epsilon) b_z = 0$. Because of (38) with $\lambda = -1$, the latter implies $A(w)\overline{D(w)} b_z = 0$ and subsequently (36) yields $\psi = 0$, a contradiction. □

By virtue of the proposition, the eigenvalue problem for the differential operator H_α is reduced to analyzing the matrix eigenvalue problem

$$-f = \epsilon Y_z \mathbb{W}(\epsilon) f, \tag{39}$$

where $f = (f_1, f_2) \in \mathbb{C}^2$. Using the definition of Y_z, this is equivalent to the coupled equations

$$\begin{aligned} -f_1 &= \epsilon(-z)^{\alpha-1}(\mathbb{W}_{11}(\epsilon) f_1 + \mathbb{W}_{12}(\epsilon) f_2), \\ -f_2 &= \epsilon(-z)^{-\alpha}(\mathbb{W}_{22}(\epsilon) f_2 + \mathbb{W}_{21}(\epsilon) f_1). \end{aligned}$$

This pair of equations has a solution if, and only if, there is a solution z of the problem

$$0 = \epsilon\,[z^{-\alpha}\mathbb{W}_{22}(\epsilon) + (-z)^{\alpha-1}\mathbb{W}_{11}(\epsilon)] + \epsilon^2(-z)^{-1}\det \mathbb{W}(\epsilon) + 1. \tag{40}$$

Following the ideas of [13], we now separate the matrix $\mathbb{W}(\epsilon)$ in two pieces $\mathbb{W}(\epsilon) = U + U_1(\epsilon)$ using

$$\begin{aligned} U &:= \int_{\mathbb{C}} D(w)B(w)A(w)\overline{D(w)}\,\mathrm{d}w = \int_{\mathbb{C}} D(w)V(w)\overline{D(w)}\,\mathrm{d}w, \\ U_1(\epsilon) &:= \int_{\mathbb{C}} D(w)B(w)([(\epsilon Q_z + 1)^{-1} - 1]A\overline{D})(w)\,\mathrm{d}w. \end{aligned} \tag{41}$$

Lemma 8. *Suppose Assumption* 1. *Then* $\|U_1(\epsilon)\| = \mathcal{O}(\epsilon)$ *as* $\epsilon \to 0$.

Proof. Denoting by $\mathcal{U}_1(w)$ the integrand of $U_1(\epsilon)$, the Cauchy–Schwarz inequality on $\mathbb{C}^2$ implies

$$\begin{aligned} \left|\int_{\mathbb{C}} \mathcal{U}_1(w)\,\mathrm{d}w\right| &= \sup\left\{\left|\left\langle \psi, \int_{\mathbb{C}} \mathcal{U}_1(w)\,\mathrm{d}w\,\varphi\right\rangle\right| \,:\, \varphi,\psi \in \mathbb{C}^2, \|\varphi\| = \|\psi\| = 1\right\} \\ &\le \int_{\mathbb{C}} |\mathcal{U}_1(w)|\,\mathrm{d}w. \end{aligned}$$

Then the smallness of the norm of $U_1(\epsilon)$ follows from the upper bound

$$\begin{aligned} &|D(w)B(w)[(1+\epsilon Q_z)^{-1} - 1]A(w)\overline{D(w)}| \\ &\quad \le |D(w)B(w)| \cdot \|(1+\epsilon Q_z)^{-1} - 1\|_{\mathrm{HS}} \cdot |A(w)\overline{D(w)}| \\ &\quad \le \|(1+\epsilon Q_z)^{-1} - 1\|_{\mathrm{HS}} \cdot |D(w)|^2 |V(w)|, \end{aligned}$$

yielding

$$\begin{aligned} \|U_1(\epsilon)\| \le \frac{\epsilon\|Q_z\|_{\mathrm{HS}}}{1-\epsilon\|Q_z\|_{\mathrm{HS}}} &\cdot C_\alpha \max\{\Gamma^2(\alpha), \Gamma^2(1-\alpha)\} \\ &\times \int_{\mathbb{C}} (\max\{|w|^{\alpha-1}, |w|^{-\alpha}\})^2 v(w)\,\mathrm{d}w \end{aligned}$$

$$\leq \frac{\epsilon\|Q_z\|_{\mathrm{HS}}}{1-\epsilon\|Q_z\|_{\mathrm{HS}}} C_\alpha \max\{\Gamma^2(\alpha), \Gamma^2(1-\alpha)\}$$
$$\times \int_{\mathbb{C}} (|w|^{-2\nu} + |w|^{2\nu}) v(w)\,\mathrm{d}w. \qquad \square$$

For conciseness, let us write

$$\begin{aligned} a_\epsilon &:= (U+U_1(\epsilon))_{11}, & a_0 &:= U_{11},\\ b_\epsilon &:= (U+U_1(\epsilon))_{22}, & b_0 &:= U_{22},\\ c_\epsilon &:= (U+U_1(\epsilon))_{11}(U+U_1(\epsilon))_{22} & c_0 &:= U_{11}U_{22} - U_{12}U_{21}.\\ &\quad - (U+U_1(\epsilon))_{12}(U+U_1(\epsilon))_{21}, \end{aligned}$$

By Lemma 8,

$$a_\epsilon = a_0 + \mathcal{O}(\epsilon), \quad b_\epsilon = b_0 + \mathcal{O}(\epsilon), \quad c_\epsilon = c_0 + \mathcal{O}(\epsilon), \tag{42}$$

as $\epsilon \to 0$. Then Eq. (40) reads

$$\epsilon^2 c_\epsilon (-z)^{-1} + \epsilon(a_\epsilon(-z)^{-1+\alpha} + b_\epsilon(-z)^{-\alpha}) + 1 = 0. \tag{43}$$

Let us summarize our findings in the following proposition.

Proposition 7. *Suppose Assumption* 1. *For all sufficiently small* ϵ, $z \in \mathbb{C}\backslash[0,+\infty)$ *is an eigenvalue of* $H_\alpha + \epsilon V$ *if, and only if,* z *is a root of* (43).

In this way, the eigenvalue problem for a differential operator has been reduced to an implicit equation. Since we have not been able to systematically analyze (43) in the general case (particular results can be derived, of course), let us restrict to the case of diagonal potentials $V = \mathrm{diag}(V_{11}, V_{22})$.

Theorem 2. *Suppose Assumption* 1 *and assume that* V *is diagonal.*

(1) *If* $V_{11} \neq 0$, *assume* $a_0 \neq 0$ *and* $\mathrm{ph}\big(-\int_{\mathbb{C}} V_{11}(w)|w^2|^{\alpha-1}\,\mathrm{d}w\big) \in (1-\alpha)(-\pi,\pi)$. *Then the operator* $H_\alpha + \epsilon V$ *possesses for all sufficiently small* $\epsilon > 0$ *a discrete eigenvalue* $z_+(\epsilon)$ *with the asymptotics*

$$z_+(\epsilon) = -(-\epsilon a_\epsilon)^{\frac{1}{1-\alpha}} = -(-\epsilon a_0)^{\frac{1}{1-\alpha}} + \mathcal{O}(\epsilon^{\frac{2-\alpha}{1-\alpha}}) \quad \text{as } \epsilon \to 0.$$

(2) *If* $V_{22} \neq 0$, *assume* $b_0 \neq 0$ *and* $\mathrm{ph}\big(-\int_{\mathbb{C}} V_{22}(w)|w^2|^{\alpha}\,\mathrm{d}w\big) \in \alpha(-\pi,\pi)$. *Then the operator* $H_\alpha + \epsilon V$ *possesses for all sufficiently small* $\epsilon > 0$ *a discrete eigenvalue* $z_-(\epsilon)$ *with the asymptotics*

$$z_-(\epsilon) = -(-\epsilon b_\epsilon)^{\frac{1}{\alpha}} = -(-\epsilon b_0)^{\frac{1}{\alpha}} + \mathcal{O}(\epsilon^{\frac{1+\alpha}{\alpha}}) \quad \text{as } \epsilon \to 0.$$

If both $V_{11} \neq 0$ *and* $V_{22} \neq 0$ *satisfy the assumptions above, then there are no other discrete eigenvalues* $H_\alpha + \epsilon V$ *for all sufficiently small* ϵ. *If* $V_{11} \neq 0$ (*respectively,* $V_{22} \neq 0$) *satisfies the assumptions from item* 1 (*respectively, item* 2) *but* $V_{22} = 0$ (*respectively,* $V_{11} = 0$), *then* $z_+(\epsilon)$ (*respectively,* $z_-(\epsilon)$) *is the unique discrete eigenvalue of* $H_\alpha + \epsilon V$ *for all sufficiently small* ϵ.

Proof. If V is diagonal, then so is $\mathbb{W}$, and $c_\epsilon = a_\epsilon b_\epsilon$. This enables us to factorize the Eq. (43) as

$$0 = (\epsilon a_\epsilon(-z)^{-1+\alpha} + 1)(\epsilon b_\epsilon(-z)^{-\alpha} + 1)$$

and we immediately obtain two solutions $z_\pm$ satisfying

$$(-z_+)^{1-\alpha} = -\epsilon a_\epsilon \quad \text{and} \quad (-z_-)^{\alpha} = -\epsilon b_\epsilon.$$

Under our assumptions on the potential these equations have solutions $z_\pm \in \mathbb{C}\backslash[0, +\infty)$ for all ϵ small enough. The expansions for $\epsilon \to 0$ then follow from the Taylor expansions of $z_\pm$ and using (42). □

Note that the eigenvalues $z_+(\epsilon)$ and $z_-(\epsilon)$ are eigenvalues of $H_\alpha^+ + \epsilon V_{11}$ and $H_\alpha^- + \epsilon V_{22}$, respectively. If both V_{11} and V_{22} are real-valued, non-trivial and non-positive, we obtain Theorem 1 from the introduction.

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