Levels in Arrangements: Linear Relations, the g-Matrix, and Applications to Crossing Numbers

Elizaveta Streltsova
Institute of Science and Technology Austria (ISTA), Klosterneuburg, Austria

Uli Wagner
Institute of Science and Technology Austria (ISTA), Klosterneuburg, Austria

Abstract

A long-standing conjecture of Eckhoff, Linhart, and Welzl, which would generalize McMullen's Upper Bound Theorem for polytopes and refine asymptotic bounds due to Clarkson, asserts that for $k \leqslant \lfloor \frac{n-d-2}{2} \rfloor$, the complexity of the $(\leqslant k)$-level in a simple arrangement of n hemispheres in S^d is maximized for arrangements that are polar duals of neighborly d-polytopes. We prove this conjecture in the case $n = d + 4$. By Gale duality, this implies the following result about crossing numbers: In every *spherical arc drawing* of K_n in S^2 (given by a set $V \subset S^2$ of n unit vectors connected by spherical arcs), the number of crossings is at least $\frac{1}{4}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$. This lower bound is attained if every open linear halfspace contains at least $\lfloor (n-2)/2 \rfloor$ of the vectors in V.

Moreover, we determine the space of all linear and affine relations that hold between the face numbers of levels in simple arrangements of n hemispheres in S^d. This completes a long line of research on such relations, answers a question posed by Andrzejak and Welzl in 2003, and generalizes the classical fact that the Dehn–Sommerville relations generate all linear relations between the face numbers of simple polytopes (which correspond to the 0-level).

To prove these results, we introduce the notion of the *g-matrix*, which encodes the face numbers of levels in an arrangement and generalizes the classical *g-vector* of a polytope.

2012 ACM Subject Classification Theory of computation → Computational geometry; Mathematics of computing → Discrete mathematics

Keywords and phrases Levels in arrangements, k-sets, k-facets, convex polytopes, f-vector, h-vector, g-vector, Dehn–Sommerville relations, Radon partitions, Gale duality, g-matrix

Digital Object Identifier 10.4230/LIPIcs.SoCG.2025.75

Related Version *Full Version*: http://arxiv.org/abs/2504.07752 [40]
Full Version: https://arxiv.org/abs/2504.07770 [41]

1 Introduction

Levels in arrangements (and the dual notions of k-sets and k-facets) play a fundamental role in discrete and computational geometry and are a natural generalization of convex polytopes (which correspond to the 0-level); we refer to [21, 30, 45] for more background.

It is a classical result in polytope theory that the *Euler–Poincaré relation* is the only linear relation between the face numbers of arbitrary d-dimensional convex polytopes, and that the *Dehn–Sommerville relations* (which we will review below) generate all linear relations between the face numbers of *simple* (or, dually, *simplicial*) polytopes [23, Chs. 8–9]. Another central result in polytope theory is McMullen's *Upper Bound Theorem* [31], which asserts that *cyclic polytopes* have the largest possible number of faces among all d-dimensional convex polytopes with a given number of vertices.

Here, we are interested in generalizations of these results to levels and sublevels in arrangements. To state our results formally, it will be convenient to work with *spherical arrangements* in S^d (which can be seen as a "compactification" of arrangements of affine hyperplanes or halfspaces in $\mathbf{R}^d$ that avoids technical issues related to *unbounded* faces).

41st International Symposium on Computational Geometry (SoCG 2025).
Editors: Oswin Aichholzer and Haitao Wang; Article No. 75; pp. 75:1–75:18
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1.1 Levels in Arrangements, Dissection Patterns, and Polytopes

Throughout this paper, let $r = d + 1 \geqslant 1$, and let S^d be the unit sphere in $\mathbf{R}^r$. We denote the standard inner product in $\mathbf{R}^r$ by $\langle \cdot, \cdot \rangle$, and write $\operatorname{sgn}(x) \in \{-1, 0, +1\}$ for the *sign* of a real number $x \in \mathbf{R}$. For a sign vector $F \in \{-1, 0, +1\}^n$, let F_+, F_0, and F_- denote the subsets of coordinates $i \in [n]$ such that $F_i = +1$, $F_i = 0$, and $F_i = -1$, respectively.

Let $V = \{v_1, \dots, v_n\} \subset \mathbf{R}^r$ be a set of $n \geqslant r$ vectors; fixing the labeling of the vectors, we will also view V as a matrix $V = [v_1|\dots|v_n] \in \mathbf{R}^{r \times n}$ with columns v_i. Unless stated otherwise, we assume that V is in general position, i.e., that any r of the vectors are linearly independent. We refer to V as a *vector configuration* of *rank* r.

Every vector $v_i \in V$ defines a great $(d-1)$-sphere $H_i = \{x \in S^d \mid \langle v_i, x \rangle = 0\}$ in S^d and two open hemispheres

$$H_i^+ = \{x \in S^d \mid \langle v_i, x \rangle > 0\}, \qquad H_i^- = \{x \in S^d \mid \langle v_i, x \rangle < 0\}.$$

The resulting *arrangement* $\mathcal{A}(V) = \{H_1^+, \dots, H_n^+\}$ of hemispheres in S^d determines a decomposition of S^d into *faces* of dimensions 0 through d, where two points $u, u' \in S^d$ lie in the relative interior of the same face of $\mathcal{A}(V)$ iff $\operatorname{sgn}(\langle v_i, u \rangle) = \operatorname{sgn}(\langle v_i, u' \rangle)$ for $1 \leqslant i \leqslant n$. Let $\mathcal{F}(V)$ be the set of all sign vectors $(\operatorname{sgn}(\langle v_1, u \rangle), \dots, \operatorname{sgn}(\langle v_n, u \rangle)) \in \{-1, 0, +1\}^n$, where u ranges over all non-zero vectors (equivalently, unit vectors) in $\mathbf{R}^r$; we can identify each face of $\mathcal{A}(V)$ with its *signature* $F \in \mathcal{F}(V)$. By general position, the face with signature F has dimension $d - |F_0|$ (the arrangement is *simple*, i.e., there are no faces with $|F_0| > d$). Moreover, we call $|F_-|$ the *level* of the face. Equivalently, the elements of $\mathcal{F}(V)$ correspond bijectively to the partitions of V by oriented linear hyperplanes, and we will also call them the *dissection patterns* of V. In what follows, we will pass freely back and forth between a vector configuration V and the corresponding arrangement $\mathcal{A}(V)$ and refer to this correspondence as *polar duality* (to distinguish it from *Gale duality*, see below).

▶ **Definition 1** (f-matrix and f-polynomial)**.** *For integers s and t, let*[1]

$$f_{s,t} := f_{s,t}(V) := |\{F \in \mathcal{F}(V) \mid |F_0| = s, |F_-| = t\}|.$$

Thus, $f_{s,t}(V)$ counts the $(d-s)$-dimensional faces of level t in $\mathcal{A}(V)$.

Together, these numbers form the f-matrix $f(V) = [f_{s,t}(V)]$. *Equivalently, we can encode this data into the bivariate* f-polynomial $f_V(x, y) \in \mathbf{Z}[x, y]$ *defined by*

$$f_V(x, y) := \sum_{s,t} f_{s,t}(V)\, x^s y^t = \sum_{F \in \mathcal{F}(V)} x^{|F_0|} y^{|F_-|}.$$

We call a vector configuration V *pointed* if it is contained in an open linear halfspace $\{x \in \mathbf{R}^r : \langle u, x \rangle > 0\}$, for some $u \in S^d$, or equivalently, if $\bigcap_{i=1}^n H_i^+ \neq \emptyset$. The closure of this intersection is then a simple (spherical) polytope P, the 0-level of $\mathcal{A}(V)$. By radial projection onto the tangent hyperplane $\{x \in \mathbf{R}^r : \langle u, x \rangle = 1\}$, every pointed configuration $V \subset \mathbf{R}^r$ corresponds to a point set $S \subset \mathbf{R}^d$, see [30, Sec. 5.6]. The convex hull $P^\circ = \operatorname{conv}(S)$ is a simplicial polytope (the polar dual of P), and the elements of $\mathcal{F}(V)$ correspond to the partitions of S by affine hyperplanes; in particular, $f_{s,0}(V)$ counts the $(s-1)$-dimensional faces of P°, and $f_{0,k}(V)$ counts the *k-sets* of S.

---

[1] By general position, $f_{s,t}(V) = 0$ unless $0 \leqslant s \leqslant d$ and $0 \leqslant t \leqslant n - s$.

1.2 Exact Upper Bounds for Sublevels

The following two special vector configurations will play an important role in this paper.

▶ **Example 2** (Cyclic and Cocyclic Configurations). Let $t_1 < t_2 < \cdots < t_n$ be real numbers and define $v_i := (1, t_i, t_i^2, \ldots, t_i^{r-1}) \in \mathbf{R}^r$. We call $V_{\text{cyclic}}(n, r) := \{v_1, \ldots v_n\}$ and $V_{\text{cocyclic}}(n, r) := \{(-1)^i v_i \colon 1 \leqslant i \leqslant n\}$ the *cyclic* and *cocyclic* configurations of n vectors in $\mathbf{R}^r$, respectively.[2]

Cyclic and cocyclic configurations are examples of neighborly and coneighborly configurations, which we define next. Let $V = \{v_1, \ldots, v_n\} \subseteq \mathbf{R}^r$ be a vector configuration in general position. A subset $W \subseteq V$ is *extremal* if there exists a linear hyperplane H bounding an open halfspace H^+ such that $W \subset H$ and $V \setminus W \subset H^+$. In particular, V is pointed iff $\emptyset \subset V$ is extremal.

▶ **Definition 3** (Neighborly and Coneighborly Configurations). *A vector configuration $V = \{v_1, \ldots, v_n\} \subseteq \mathbf{R}^r$ is* coneighborly *if every open linear halfspace contains at least $\lfloor \frac{n-r+1}{2} \rfloor$ vectors of V. It is* neighborly *if every subset $W \subseteq V$ of size $|W| \leqslant \lfloor \frac{r-1}{2} \rfloor$ is* extremal.

Cyclic configurations are neighborly, cocyclic configurations are coneighborly [48, Cor. 0.8], and these notions are Gale dual to each other (see Section 2). Every neighborly vector configuration $V \subset \mathbf{R}^r$ is pointed, hence corresponds to a point set $S \subset \mathbf{R}^d$, $d = r - 1$, and V being neighborly means the simplicial d-polytope $\operatorname{conv}(S)$ is a *neighborly polytope*, i.e., every subset of S of size at most $\lfloor \frac{d}{2} \rfloor$ forms a face. We note that for $r = 1, 2$ ($d = 0, 1$) neighborliness is the same as being pointed, and for $r = 3, 4$ ($d = 2, 3$) V is neighborly iff the point set S is in convex position. By a celebrated result of McMullen [31] neighborly polytopes maximize the number of faces of any dimension:

▶ **Theorem 4** (Upper Bound Theorem for Convex Polytopes). *Let $V \subset \mathbf{R}^r$ be a configuration of n vectors in general position. Then*

$$f_{s,0}(V) \leqslant f_{s,0}(V_{\text{cyclic}}(n, r)) \qquad (0 \leqslant s \leqslant d = r - 1)$$

with equality if V is neighborly.

Eckhoff [20, Conj. 9.8], Linhart [26], and Welzl [46], independently of one another (and in slightly different forms) conjectured a far-reaching generalization of Theorem 4 for sublevels of arrangements. To state this conjecture, let $f_{s,\leqslant k}(V) := \sum_{t \leqslant k} f_{s,t}(V)$.

▶ **Conjecture 5** (Generalized Upper Bound Conjecture for Sublevels). *Let $V \subset \mathbf{R}^r$ be a configuration of n vectors in general position, and $0 \leqslant k \leqslant \lfloor \frac{n-r-1}{2} \rfloor$. Then*

$$f_{s,\leqslant k}(V) \leqslant f_{s,\leqslant k}(V_{\text{cyclic}}(n, r)) \qquad (0 \leqslant s \leqslant d = r - 1)$$

Equality holds if V is neighborly.

A random sampling argument due to Clarkson (see [18]) shows that Conjecture 5 is true *asymptotically*, for fixed r and $n, k \to \infty$, up to a constant factor depending on r. For the case $s = d$ of vertices at sublevel ($\leqslant k$), Conjecture 5 was proved by Peck [36] and by Alon and Győri [10] for $r \leqslant 3$ and by Welzl [46] for *pointed* vector configurations in rank $r = 4$. The second author [44] proved that it is true up to a factor of 4 for arbitrary rank r. Here, we prove the conjecture for *corank* $n - r = 3$.

[2] The combinatorial types of these configurations are independent of the choice of the parameters t_i.

▶ **Theorem 6.** *Let $V \subset \mathbf{R}^{n-3}$ be a configuration of n vectors in general position. Then*

$$f_{s,\leqslant k}(V) \leqslant f_{s,\leqslant k}(V_{\text{cyclic}}(n,r)) \qquad (0 \leqslant k \leqslant 1, 0 \leqslant s \leqslant d = r-1)$$

Equality holds if V is neighborly.

▶ Remark 7. Bounding the maximum number of faces at level *exactly* k is of a rather different flavor. For coneighborly configurations, all $2\binom{n}{r-1}$ vertices of the dual arrangement are concentrated at one or two consecutive levels $k = \lfloor \frac{n-r+1}{2} \rfloor$ and $k = \lceil \frac{n-r+1}{2} \rceil$. By contrast, determining the maximum number $f_{d,k}$ of vertices at level k for *pointed* vector configurations in $\mathbf{R}^r$ is a difficult open problem, first studied by Lovász [28] and Erdős, Lovász, Simmons, and Straus [22] in the 1970s (see [45] or [30, Ch. 11] for more details and background); even for $r = 3$ (i.e., $d = 2$) there remains a big gap between the best upper and lower bounds to date, which are $O(nk^{1/3})$ and $ne^{\Omega(\sqrt{\log k})}$, respectively (due to Dey [19] and Tóth [43]).

1.3 The Spherical Arc Crossing Number of K_n

By Gale duality (see Section 2), Theorem 6 yields the following result about crossing numbers.

Determining the crossing number $\text{cr}(K_n)$ of the complete graph K_n (the minimum number of crossings in any drawing of K_n in the plane, or equivalently in the sphere S^2, with edges represented by arbitrary Jordan arcs) is one of the foundational unsolved problems in geometric graph theory, first studied by Hill in the 1950's. Hill conjectured the following:

▶ **Conjecture 8** (Hill).

$$\text{cr}(K_n) = X(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8}\binom{n}{4} + O(n^3) \tag{1}$$

This is known to hold for $n \leqslant 12$, but remains open in general (see [37, Sec. 1.3] or [42] for further background and references). There are several families of drawings showing that $\text{cr}(K_n) \leqslant X(n)$ for all n, but the best lower bound to date [14] is $\text{cr}(K_n) \geqslant 0.985 \cdot X(n)$.

We prove Hill's conjecture for the following class of drawings. Let $V = \{v_1, \ldots, v_n\} \subset S^2$ be a configuration of n *unit* vectors in general position. If we connect every pair of vectors in V by the shortest geodesic arc between them in S^2 (which is unique, since no two vectors are antipodal, by general position) we obtain a drawing of the complete graph K_n in S^2, which we call a *spherical arc drawing.* Let $\text{cr}(V)$ denote the number of crossings in this drawing.

▶ **Theorem 9.** *For every configuration $V \subset S^2$ of n unit vectors in general position,*

$$\text{cr}(V) \geqslant \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

Moreover, the lower bound is attained with equality if V is coneighborly.

The fact that coneighborly configurations yield spherical arc drawings of K_n achieving the number $X(n)$ of crossings in Hill's conjecture, and the connection to the Eckhoff–Linhart–Welzl conjecture were first observed by Wagner [44, 45]. It is known [35] that there are at least $n^{\frac{3}{2}(1-o(1))n}$ combinatorially different coneighborly configurations of n vectors in S^2.

Spherical arc drawings generalize the well-studied class of *rectilinear drawings* of K_n (given by n points in general position in $\mathbf{R}^2$ connected by straight-line segments), which correspond to spherical arc drawings given by pointed vector configurations $V \subset S^2$.

Theorem 9 complements earlier results of Lovász, Vesztergombi, Wagner, and Welzl [29] and Ábrego and Fernández-Merchant [6], who showed that the *rectilinear crossing number* $\overline{\text{cr}}(K_n)$ (the minimum number of crossings in any rectilinear drawing of K_n) is at least

$X(n)$; in fact, $\overline{\mathrm{cr}}(K_n) \geqslant (\frac{3}{8} + \varepsilon + o(1))\binom{n}{4}$ for some constant $\varepsilon > 0$ [29]. Thus, unlike the spherical arc crossing number, the rectilinear crossing number $\overline{\mathrm{cr}}(K_n)$ is strictly larger than $X(n) \geqslant \mathrm{cr}(K_n)$ in the asymptotically leading term. We refer to [8] for a detailed survey, including a series of subsequent improvements [15, 9, 7, 4] leading to the currently best bound [5] $\overline{\mathrm{cr}}(K_n) > 277/729\binom{n}{4} + O(n^3) > 0.37997\binom{n}{4} + O(n^3)$. We remark that the arguments in [29, 6] have been generalized to verify Hill's conjecture for other classes of drawings, including 2-page drawings [2], monotone drawings [13], cylindrical, x-bounded, and shellable drawings [3], bishellable drawings [1], and seq-shellable drawings [34]. Currently we do not know how spherical arc drawings relate to these other classes of drawings.

1.4 The g-Matrix

The central notion of this paper is the *g-matrix* $g(V \to W)$ of a pair $V, W \in \mathbf{R}^{r\times n}$ of vector configurations, which encodes the difference $f(W) - f(V)$ between the f-matrices. The geometric definition of the g-matrix is given in Sec. 3, based on how the f-matrix changes by *mutations* in the course of generic *continuous motion* (an idea with a long history, see, e.g., [11]). The g-matrix is characterized by the following properties:

▶ **Theorem 10.** *Let $V, W \in \mathbf{R}^{r\times n}$ be a pair of vector configurations in general position.*

The g-matrix $g(V \to W)$ of the pair is an $(r+1)\times(n-r+1)$-matrix with integer entries $g_{j,k} := g_{j,k}(V \to W)$, $0 \leqslant j \leqslant r$, $0 \leqslant k \leqslant n-r$, which has the following properties:

1. *For $0 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant n-r$, the g-matrix satisfies the skew-symmetries*

$$g_{j,k} = -g_{r-j,k} = -g_{j,n-r-k} = g_{r-j,n-r-k} \tag{2}$$

Thus, the g-matrix is determined by the submatrix $[g_{j,k} \colon 0 \leqslant j \leqslant \lfloor\frac{r-1}{2}\rfloor, 0 \leqslant k \leqslant \lfloor\frac{n-r-1}{2}\rfloor]$, which we call the small g-matrix. *Equivalently, the* g-polynomial *$g(x,y) := g_{V\to W}(x,y) := \sum_{j,k} g_{j,k}x^j y^k \in \mathbf{Z}[x,y]$ satisfies*

$$g(x,y) = -x^r g(\tfrac{1}{x}, y) = -y^{n-r} g(x, \tfrac{1}{y}) = x^r y^{n-r} g(\tfrac{1}{x}, \tfrac{1}{y}) \tag{3}$$

2. *The g-polynomial determines the difference $f_W(x,y) - f_V(x,y)$ of f-polynomials by*

$$f_W(x,y) - f_V(x,y) = (1+x)^r g(\tfrac{x+y}{1+x}, y) = \textstyle\sum_{j=0}^{r}\sum_{k=0}^{n-r} g_{j,k}\cdot(x+y)^j(1+x)^{r-j}y^k \tag{4}$$

Equivalently (by comparing coefficients), for $0 \leqslant s \leqslant d$ and $0 \leqslant t \leqslant n$,

$$f_{s,t}(W) - f_{s,t}(V) = \sum_{j,k}\binom{j}{t-k}\binom{r-j}{s-j+t-k} g_{j,k}(V \to W), \tag{5}$$

3. *$g(W \to V) = -g(V \to W)$, and $g(U \to W) = g(U \to V) + g(V \to W)$*

▶ Remark 11. The system of equations (5) yields a linear transformation $T = T_{n,r}$ through which the g-matrix $g = g(V \to W)$ of the pair determines the difference $\Delta f = f(W) - f(V)$ of f-matrices by $\Delta f = T(g)$. In the presence of the skew-symmetries (2), the transformation T is injective (Lemma 25), i.e., $g(V \to W)$ is uniquely determined by $\Delta f = f(W) - f(V)$. Thus, Theorem 10 could be taken as a formal definition of the g-matrix.

1.5 Linear Relations

Linear relations between face numbers of levels in simple arrangements have been studied extensively [33, 24, 27, 12, 16].

The first set of linear relations is given by the antipodal symmetry $F \leftrightarrow -F$ of $\mathcal{F}(V)$:

▶ **Observation 12.** *Let $V = \{v_1, \ldots, v_n\} \subset \mathbf{R}^r$. Then $f_V(x,y) = y^n f_V(\frac{x}{y}, \frac{1}{y})$; equivalently,*

$$f_{s,t}(V) = f_{s,n-s-t}(V) \quad \text{for all } s \text{ and } t \tag{6}$$

Moreover, it is well-known [23, Sec. 18.1] that the total number of faces of a given dimension $d-s$ (of any level) in a simple arrangement in S^d depends only on n, d, and s:

▶ **Lemma 13.** *Let $\mathcal{A}(V)$ be a simple arrangement of n hemispheres in S^d. Then, for $0 \leqslant s \leqslant d$, the total number of $(d-s)$-dimensional faces (of any level) in $\mathcal{A}(V)$ equals*

$$\sum_t f_{s,t}(V) = 2\binom{n}{s}\sum_{i=0}^{d-s}\binom{n-s-1}{i} = \sum_{i=0}^{d}(1+(-1)^i)\binom{n}{d-i}\binom{d-i}{s} \tag{7}$$

In terms of the f-polynomial, this can be expressed very compactly as

$$f_V(x,1) = \textstyle\sum_{i=0}^d \binom{n}{i}\left(1+(-1)^{d-i}\right)(1+x)^i = 2\left(\binom{n}{d}(x+1)^d + \binom{n}{d-2}(x+1)^{d-2} + \ldots\right) \tag{8}$$

Linhart, Yang, and Philipp [27] proved the following result, which generalizes the classical *Dehn–Sommerville relations* for simple polytopes:

▶ **Theorem 14** (Dehn–Sommerville Relations for Levels in Simple Arrangements)**.** *Let $V \in \mathbf{R}^{r\times n}$ be a vector configuration in general position. Then*

$$f_V(x,y) = (-1)^d f_V(-(x+y+1), y) \tag{9}$$

Equivalently (by comparing coefficients), for $0 \leqslant s \leqslant d$ and $0 \leqslant t \leqslant n$,

$$f_{s,t}(V) = \sum_j \sum_\ell (-1)^{d-j}\binom{j}{s}\binom{j-s}{t-\ell} f_{j,\ell}(V) \tag{10}$$

▶ Remark 15. The Dehn–Sommerville relations for polytopes correspond to the identity $f_V(x,0) = (-1)^d f_V(-(x+1),0)$. The coefficients on the right-hand side of (10) are zero unless $\ell \leqslant t$ (and $j \geqslant s$). This yields, for every k, a linear system of equations among the numbers $f_{s,t}$, $0 \leqslant s \leqslant d$ and $t \leqslant k$, of face numbers of the $(\leqslant k)$-*sublevel* of the arrangement $\mathcal{A}(V)$. An equivalent system of equations (expressed in terms of an h-*matrix* that generalizes the h-vector of a simple polytope) was proved earlier by Mulmuley [33], under the additional assumption that the $(\leqslant k)$-sublevel is contained in an open hemisphere. Related relations have been rediscovered several times (e.g., in the recent work of Biswas et al. [16]).

▶ Remark 16. The skew-symmetry $g(x,y) = -x^r g(\frac{1}{x}, y)$ of the g-matrix reflects the Dehn–Sommerville relation (9), and the symmetry $g(x,y) = x^r y^{n-r} g(\frac{1}{x}, \frac{1}{y})$ reflects the antipodal symmetry (6).

Let $\mathcal{V}_{n,r}$ denote the set of vector configurations $V \in \mathbf{R}^{r\times n}$ in general position. Let

$$\mathfrak{F}_{n,r} := \operatorname{aff}\{f(V) \colon V \in \mathcal{V}_{n,r}\}, \qquad \mathfrak{G}_{n,r} := \operatorname{lin}\{g(V \to W) \colon V, W \in \mathcal{V}_{n,r}\}$$

be the affine space spanned by all f-matrices, and the linear space spanned by all g-matrices of pairs, respectively. Let $\mathcal{V}^0_{n,r} \subset \mathcal{V}_{n,r}$ be the subset of pointed configurations, and let

$$\mathfrak{F}^0_{n,r} := \operatorname{aff}\{f(V) \colon V \in \mathcal{V}^0_{n,r}\}, \qquad \mathfrak{G}^0_{n,r} := \operatorname{lin}\{g(V \to W) \colon V, W \in \mathcal{V}^0_{n,r}\}$$

be the corresponding subspaces of $\mathfrak{F}_{n,r}$ and $\mathfrak{G}_{r,n}$. Answering a question posed by Andrzejak and Welzl [12], we determine these spaces:

▶ **Theorem 17.** $\dim \mathfrak{F}_{n,r} = \dim \mathfrak{G}_{n,r} = \lfloor \frac{r+1}{2} \rfloor \lfloor \frac{n-r+1}{2} \rfloor$. *More precisely,*

$$\mathfrak{G}_{n,r} = \left\{ g \in \mathbf{R}^{(r+1)\times(n-r+1)} \colon \begin{array}{c} g_{j,k} = -g_{r-j,k} = -g_{j,n-r-k} = g_{r-j,n-r-k} \\ \text{for } 0 \leqslant j \leqslant r, 0 \leqslant k \leqslant n-r \end{array} \right\} \tag{11}$$

is the space of all real $(r+1) \times (n-r+1)$*-matrices satisfying the skew-symmetries* (2)*, and*

$$\mathfrak{F}_{n,r} = f(V_0) + T(\mathfrak{G}_{n,r})$$

for any fixed $V_0 \in \mathcal{V}_{n,r}$*, where* $T = T_{n,r}$ *is the injective linear transformation given by* (5)*.*

▶ **Theorem 18.** $\dim \mathfrak{F}^0_{n,r} = \dim \mathfrak{G}^0_{n,r} = \lfloor \frac{r-1}{2} \rfloor \lfloor \frac{n-r+1}{2} \rfloor$. *More precisely,*

$$\mathfrak{G}^0_{n,r} = \{ g \in \mathfrak{G}_{n,r} \colon g_{0,k} = 0, 0 \leqslant k \leqslant n-r \}, \quad \text{and} \quad \mathfrak{F}^0_{n,r} = f(V_0) + T(\mathfrak{G}^0_{n,r})$$

for any $V_0 \in \mathcal{V}^0_{n,r}$*.*

As a specific base configuration V_0 in both theorems, one can take the *cyclic* vector configuration $V_{\text{cyclic}}(n,r)$ (see Example 2), whose f-matrix is known explicitly [12].

2 Gale Duality and Dependency Patterns

Let $V = \{v_1, \ldots, v_n\} \subset \mathbf{R}^r$ be a vector configuration in general position, and let $\mathcal{F}^*(V)$ be the set of all sign vectors $(\operatorname{sgn}(\lambda_1), \ldots, \operatorname{sgn}(\lambda_n)) \in \{-1, 0, +1\}^n$ given by non-trivial linear dependencies $\sum_{i=1}^n \lambda_i v_i = 0$ (with coefficients $\lambda_i \in \mathbf{R}$, not all of them are zero). We call $\mathcal{F}^*(V)$ the *dependency patterns* of V. Both $\mathcal{F}^*(V)$ and $\mathcal{F}(V)$ are invariant under invertible linear transformations of $\mathbf{R}^r$ and under positive rescaling (multiplying each vector v_i by some positive scalar $\alpha_i > 0$).

If V is a pointed configuration corresponding to a point set $S \subset \mathbf{R}^d$, $d = r-1$, then the elements of $\mathcal{F}^*(V)$ encode the sign patterns of affine dependencies of S, hence they correspond bijectively to (ordered) *Radon partitions* $S = S_- \sqcup S_0 \sqcup S_+$, $\operatorname{conv}(S_+) \cap \operatorname{conv}(S_-) \neq \emptyset$.

▶ **Definition 19** (f^*-matrix and f^*-polynomial)**.** *For integers* s *and* t*, define*[3]

$$f^*_{s,t}(V) := |\{F \in \mathcal{F}^*(V) \mid |F_-| = t, |F_+| = s-t\}|$$

Together, these numbers form the f^*-matrix $f^*(V) = [f^*_{s,t}(V)]$*. Equivalently, we can encode this data into the bivariate* f^*-polynomial $f^*_V(x,y) \in \mathbf{Z}[x,y]$ *defined by*

$$f^*_V(x,y) := \sum_{F \in \mathcal{F}^*(V)} x^{|F_0|} y^{|F_-|} = \sum_{s,t} f^*_{s,t}(V)\, x^{n-s} y^t$$

Dependency patterns and dissection patterns are, in a precise sense, dual to each other:

▶ **Definition 20** (Gale Duality)**.** *Two vector configurations* $V \in \mathbf{R}^{r \times n}$ *and* $W \in \mathbf{R}^{(n-r)\times n}$ *are called* Gale duals *of one another if the rows of* V *and the rows of* W *span subspaces of* $\mathbf{R}^n$ *that are orthogonal complements of one another.*

Since we always assume that V and W are in general position and of full rank, V and W are Gale dual to each other iff $VW^\top = 0$.

[3] Note that $f^*_{s,t}(V) = 0$ unless $r+1 \leqslant s \leqslant n$ and $0 \leqslant t \leqslant s$.

It is well-known that Gale dual configurations determine each other up to linear isomorphisms of their ambient spaces $\mathbf{R}^r$ and $\mathbf{R}^{n-r}$, respectively [30, Sec. 5.6]. Thus, we will speak of *the Gale dual* of V, which we denote by V^*. It follows from the definition that $(V^*)^* = V$.

As another consequence of the definition, $\mathcal{F}^*(V) = \mathcal{F}(V^*)$, hence $f^*_{s,t}(V) = f_{n-s,t}(V^*)$ for all s,t; equivalently, $f^*_V(x,y) = f_{V^*}(x,y)$.

By the well-known separation theorem for convex sets [30, Sec. 1.2], a vector configuration V in general position is either pointed (equivalently, $f_{0,0}(V) = 1$), or there is a linear dependence of the vectors all of whose coefficients are positive (i.e., $f^*_{n,0}(V) = 1$); the same holds for all subsets $W \subseteq V$. It follows from this that $\mathcal{F}(V)$ and $\mathcal{F}^*(V)$ determine each other (see, e.g., [40, Sec. 2] for more details). In particular, a vector configuration $V \subset \mathbf{R}^r$ is neighborly iff $f^*_{s,t}(V) = 0$ for $t \leqslant \lfloor \frac{r-1}{2} \rfloor$, i.e., V is neighborly iff V^* is coneighborly.

Moreover, one can show that the f-matrix and the f^*-matrix of a vector configuration, or equivalently the polynomials $f_V(x,y)$ and $f^*_V(x,y)$ determine each other as well [40, Sec. 2]:

▶ **Theorem 21.** *Let $V \in \mathbf{R}^{r\times n}$ be a vector configuration in general position. Then*

$$f^*_V(x,y) = (x+y+1)^n - (-1)^r x^n - (x+1)^n f_V(-\tfrac{x}{x+1}, \tfrac{x+y}{x+1}) \tag{12}$$

and

$$f_V(x,y) = (x+y+1)^n - (-1)^{n-r} x^n - (x+1)^n f^*_V(-\tfrac{x}{x+1}, \tfrac{x+y}{x+1}) \tag{13}$$

By Gale duality, Theorem 17 immediately gives a complete description of the affine space $\mathfrak{F}^*_{n,r}$ spanned by the f^*-matrices of vector configurations $V \in \mathbf{R}^{r\times n}$, and there is an analogous characterization for the subspace $(\mathfrak{F}^*_{n,r})^0$ spanned by the f^*-matrices of pointed vector configurations in $\mathbf{R}^r$ (which count the number of Radon partitions of given types for the corresponding point sets in $\mathbf{R}^d$), see [40, Sec. 1.2]

We say that two vector configurations $V, W \in \mathbf{R}^{r\times n}$ have the same *combinatorial type* if (up to a permutation of the vectors) $\mathcal{F}(V) = \mathcal{F}(W)$ (equivalently, $\mathcal{F}^*(V) = \mathcal{F}^*(W)$). Furthermore, we call V and W *weakly equivalent* if they have identical f-matrices (equivalently, identical f^*-matrices).

▶ Remark 22. For readers familiar with *oriented matroids* (see [48, Ch. 6] or [17]), $\mathcal{F}^*(V)$ and $\mathcal{F}(V)$ are precisely the sets of *vectors* and *covectors*, respectively, of the oriented matroid realized by V. However, speaking of "(co)vectors of a vector configuration" seems potentially confusing, and we hope that the terminology of dissection and dependency patterns is more descriptive. The Dehn–Sommerville relations hold for (uniform, not necessarily realizable) oriented matroids. The definition of the g-matrix via continuous motion does not carry over to the oriented matroid setting, but one can still define the g-matrix formally, by the properties described in Theorem 10. We plan to treat this in detail in a future paper.

3 Continuous Motion and the g-Matrix

3.1 The g-Matrix of a Pair

Any two configurations $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$ of n vectors in general position in $\mathbf{R}^r$ can be deformed into one another through a continuous family $V(t) = \{v_1(t), \dots, v_n(t)\}$ of vector configurations, where $v_i(t)$ describes a continuous path from $v_i(0) = v_i$ to $v_i(1) = w_i$ in $\mathbf{R}^r$. If we choose this continuous motion sufficiently generically, then there is only a finite set of events $0 < t_1 < \dots < t_N < 1$, called *mutations*, during which the combinatorial type of $V(t)$ (which is encoded by $\mathcal{F}(V(t))$), changes, in a controlled way (see Figures 1 and 2 for an illustration in the case $d = 2$). Specifically, during a mutation, a unique r-tuple of vectors in $V(t)$, indexed by some $R = \{i_1, \dots, i_r\} \subset [n]$, momentarily becomes linearly dependent and the orientation of this r-tuple (i.e., the sign of $\det[v_{i_1}| \dots |v_{i_r}]$) changes, while the orientations of all other r-tuples of vectors remain unchanged. Thus, any two vector configurations are connected by a finite sequence $V = V_0, V_1, \dots, V_N = W$ such that V_{s-1} and V_s differ by a mutation, $1 \leqslant s \leqslant N$.

We describe the change from $\mathcal{F}(V)$ to $\mathcal{F}(W)$ when V and W differ by a mutation. Let $R \in \binom{[n]}{r}$ index the unique r-tuple of vectors that become linearly dependent. In terms of the polar dual arrangements, this means that the r-tuple of great $(d-1)$-spheres H_i, $i \in R$, momentarily intersect in an antipodal pair $u, -u$ of points in S^d. Immediately before and immediately after the mutation, these r great $(d-1)$-spheres bound an antipodal pair of simplicial d-faces $\sigma, -\sigma$ in $\mathcal{A}(V)$ and a corresponding pair of simplicial d-faces $\tau, -\tau$ in $\mathcal{A}(W)$, respectively. We have $F \in \mathcal{F}(V) \setminus \mathcal{F}(W)$ iff the face of $\mathcal{A}(V)$ with signature F is contained in σ or $-\sigma$, and $F \in \mathcal{F}(W) \setminus \mathcal{F}(V)$ iff the face of $\mathcal{A}(W)$ with signature F is contained in τ or $-\tau$. All other faces are preserved, i.e., they belong to $\mathcal{F}(V) \cap \mathcal{F}(W)$.

Let $Y \in \mathcal{F}(W)$ be the signature of τ. We define a partition $[n] = I \sqcup J \sqcup A \sqcup B$ by

$$I := R \cap Y_+, \quad J := R \cap Y_-, \quad A := ([n] \setminus R) \cap Y_+, \qquad B := ([n] \setminus R) \cap Y_-$$

Define $j := |J|$ and $k := |B|$. We call the pair (j, k) the *type* of the simplicial face τ. The signature $X \in \mathcal{F}(V)$ of the corresponding simplicial face σ of $\mathcal{A}(V)$ satisfies $X_i = -Y_i$ for $i \in R$ and $X_i = Y_i$ for $i \in [n] \setminus R$. Thus, σ is of type $(r-j, k)$. Analogously, $-\tau$ and $-\sigma$ are of type $(r-j, n-r-k)$ and $(j, n-r-k)$, respectively, see Figures 1 and 2.

Let us define $f_\sigma(x,y) := \sum_{F \subseteq \sigma} x^{|F_0|} y^{|F_-|}$, where we use the notation $F \subseteq \sigma$ to indicate that the sum ranges over all $F \in \mathcal{F}(V)$ corresponding to faces of $\mathcal{A}(V)$ contained in σ. The polynomials $f_{-\sigma}(x,y)$, $f_\tau(x,y)$, and $f_{-\tau}(x,y)$ are defined analogously. These four polynomials have a simple form:

$$f_\sigma(x,y) = y^k \left[(x+1)^j (x+y)^{r-j} - x^r\right], \quad f_{-\sigma}(x,y) = y^{n-r-k} \left[(x+1)^{r-j}(x+y)^j - x^r\right]$$
$$f_\tau(x,y) = y^k \left[(x+1)^{r-j} (x+y)^{j} - x^r\right], \quad f_{-\tau}(x,y) = y^{n-r-k} \left[(x+1)^{j}(x+y)^{r-j} - x^r\right]$$

We say that the mutation $V \to W$ is of *Type* $(j,k) \equiv (r-j, n-r-k)$. The reverse mutation $W \to V$ is of Type $(r-j, k) \equiv (j, n-r-k)$. We can summarize the discussion as follows:

▶ **Lemma 23.** *Let $V \to W$ be a mutation of Type $(j,k) \equiv (r-j, n-r-k)$ between configurations of n vectors in $\mathbf{R}^r$. Then*

$$f_W(x,y) - f_V(x,y) = \left(y^k - y^{n-r-k}\right)\left[(x+1)^{r-j}(x+y)^j - (x+1)^j(x+y)^{r-j}\right] \tag{14}$$

Note that the right-hand side of (14) is zero if $2j = r$ or $2k = n - r$.

We are now ready to define the g-matrix $g(V \to W)$ of a pair of vector configurations.

▶ **Definition 24** (g-Matrix of a pair). *Let V, W be configurations of n vectors in $\mathbf{R}^r$.*

If $V \to W$ is a single mutation of Type $(i, \ell) \equiv (r-i, n-r-\ell)$ then we define the g-matrix $g(V \to W) = [g_{j,k}(V \to W)]$, $0 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant n-r$, as follows:

If $2i = r$ or $2\ell = n-r$, then $g_{j,k}(V \to W) = 0$ for all j, k. If $2i \neq r$ and $2\ell \neq n-r$, then

$$g_{j,k}(V \to W) := \begin{cases} +1 & \textit{if } (j,k) = (i,\ell) \textit{ or } (j,k) = (r-i, n-r-\ell) \\ -1 & \textit{if } (j,k) = (r-i,\ell) \textit{ or } (j,k) = (i, n-r-\ell) \\ 0 & \textit{else.} \end{cases}$$

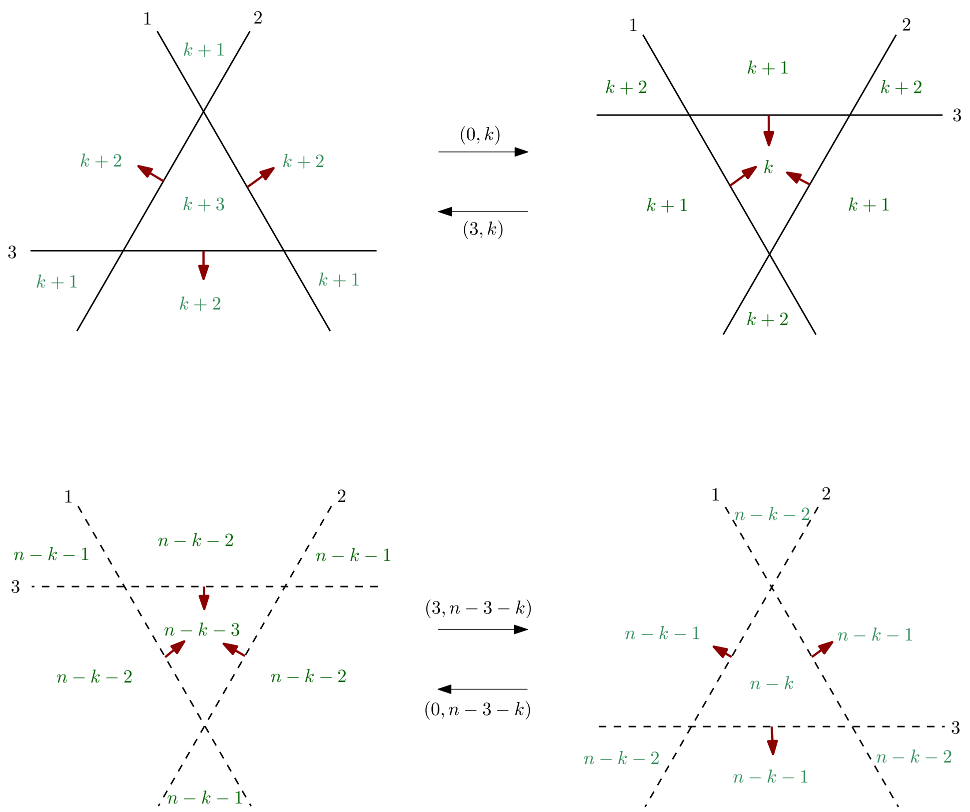

Figure 1 A mutation of Type $(0, k) \equiv (3, n-3-k)$ (from left to right), respectively $(3, k) \equiv (0, n-3-k)$ (from right to left) in S^2. The upper row shows the triangular faces σ and τ before and after the mutation, and the lower row shows the corresponding antipodal faces $-\sigma$ and $-\tau$. The little arrows indicate positive halfspaces, and the labels in full-dimensional faces indicate levels.

More generally, if V and W are connected by a sequence $V = V_0, V_1, \ldots, V_N = W$, where V_{s-1} and V_s differ by a single mutation, then we define

$$g_{j,k}(V \to W) := \sum_{s=1}^{N} g_{j,k}(V_{s-1} \to V_s)$$

Proof of Thm 10. All three properties follow directly from Definition 24 and Lemma 23. ◀

A priori, it may seem that the definition of the g-matrix depends on the choice of a particular sequence of mutations transforming V to W. However, this is not the case:

▶ **Lemma 25.** *Let $f(x, y) = \sum_{s,t} f_{s,t} x^s y^t$ and $g(x, y) = \sum_{j,k} g_{j,k} x^j y^k$ be polynomials (with real coefficients $f_{s,t}$ and $g_{j,k}$ that are zero unless $0 \leqslant s \leqslant d$ and $0 \leqslant t \leqslant n$, respectively $0 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant n - r$). Suppose that $f(x, y)$ and $g(x, y)$ satisfy the identity*

$$f(x, y) = (1 + x)^r g(\tfrac{x+y}{1+x}, y) = \textstyle\sum_{j=0}^{r} \sum_{k=0}^{n-r} g_{j,k} \cdot (x + y)^j (1 + x)^{r-j} y^k \tag{15}$$

Then, for every fixed t, the numbers $g_{j,t}$, $0 \leqslant j \leqslant r$, are linear combinations of the numbers $f_{s,\ell}$, $0 \leqslant s \leqslant d$ and $0 \leqslant \ell \leqslant t$, with coefficients given inductively by the polynomial equations

$$\sum_j g_{j,t} x^{r-j} = \sum_s f_{s,t} (x - 1)^s - \sum_j \sum_{k<t} g_{j,k} \binom{j}{t-k} x^{r-j}$$

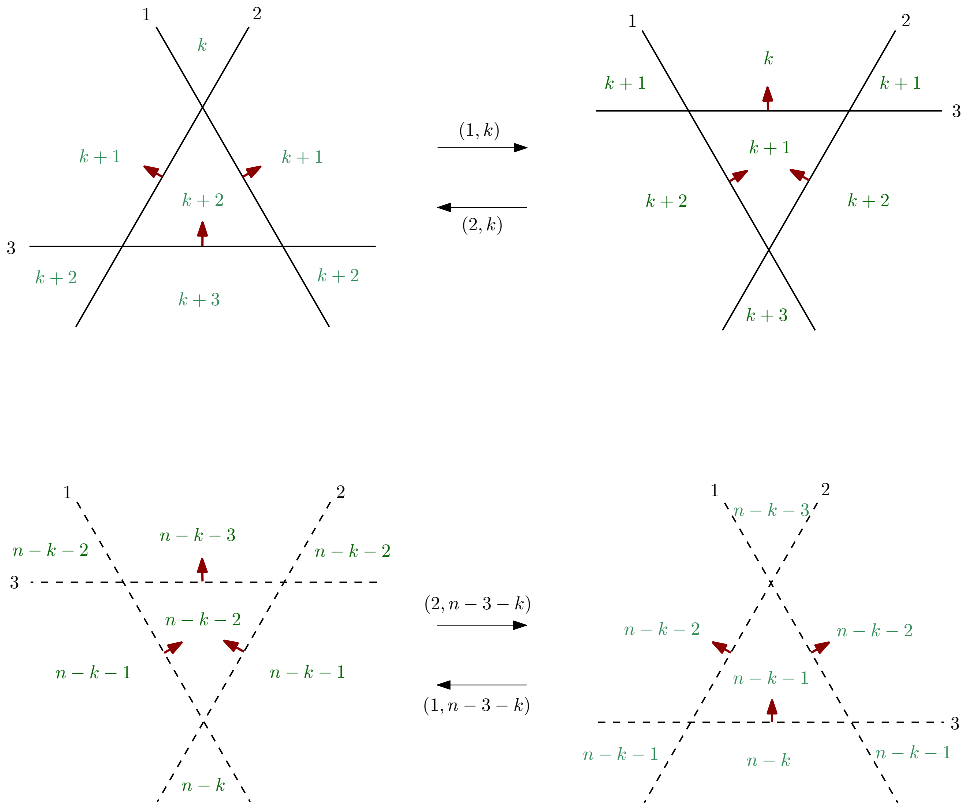

Figure 2 A mutation of Type $(1, k) \equiv (2, n-3-k)$ (from left to right), respectively $(2, k) \equiv (1, n-3-k)$ (from right to left) in S^2.

Proof. The coefficient of y^t in $(x+y)^j(x+1)^{r-j}y^k$ equals $\binom{j}{t-k}(x+1)^{r-j}$ (which is zero unless $0 \leqslant k \leqslant t$). Thus, fixing t and collecting terms in (15) according to y^t, we get

$$\sum_s f_{s,t}x^s = \sum_j \sum_{k \leqslant t} g_{j,k} \binom{j}{t-k} (1+x)^{r-j}$$

Moving the terms with $k < t$ to the other side yields

$$\sum_j g_{j,t}(1+x)^{r-j} = \sum_s f_{s,t}x^s - \sum_j \sum_{k<t} g_{j,k} \binom{j}{t-k} (1+x)^{r-j}$$

The result follows by a change of variable from x to $x-1$ (inductively, the numbers $g_{j,k}$, $k < t$, are determined by the numbers $f_{s,\ell}$, $\ell < t$.) ◀

By Theorem 21, the f-polynomial and the f^*-polynomial of a vector configuration determine each other. This yields the following analogue of Theorem 10 (which can also proved directly, by studying how $\mathcal{F}^*$ changes during mutations):

▶ **Theorem 26.** *Let V, W be configurations of n vectors in $\mathbf{R}^r$. Then*

$$f^*_W(x,y) - f^*_V(x,y) = \sum_{j,k} \underbrace{g_{j,k}(W \to V)}_{=-g_{j,k}(V \to W)} (x+y)^k (x+1)^{n-r-k} y^j \tag{16}$$

Theorems 10 and 26 imply the following:

▶ **Corollary 27.** *Let $V, W \in \mathbf{R}^{r\times n}$ be vector configurations, and let $V^*, W^* \in \mathbf{R}^{(n-r)\times n}$ be their Gale duals. Then $g_{j,k}(V \to W) = -g_{k,j}(V^* \to W^*)$.*

Using the above results, one can show [40, Sec. 3.1] that any pair of neighborly configurations of n vectors in $\mathbf{R}^r$ have the same f-matrix and the same f^*-matrix (equivalently, the g-matrix of the pair is identically zero); analogously for coneighborly configurations.

Moreover, an inductive argument based on *deletions* and *contractions* [40, Sec. 3.2] yields the following result. We say that a vector configuration $V \subset \mathbf{R}^r$ is *j-neighborly* if every subset of V of size j is extremal, and we call V is *k-coneighborly* if $f_{s,t} = 0$ for $t \leqslant k$, i.e., if every open linear halfspace contains at least $k+1$ vectors from V.

▶ **Lemma 28.** *Let $0 \leqslant j \leqslant \frac{r-1}{2}$, $0 \leqslant k \leqslant \frac{n-r-1}{2}$, and let $V, W \in \mathbf{R}^{r\times n}$ be vector configurations such that V is k-coneighborly and W is j-neighborly. Then*

$$g_{j,k}(V \to W) = \binom{n-k-r+j}{j}\binom{k+r-1-j}{k} - \binom{n-k-r+j-1}{j-1}\binom{k+r-j}{k} > 0 \tag{17}$$

3.2 The g-Matrix and the g^*-Matrix

We are now ready to define the g-matrix and the g^*-matrix of a vector configuration.

▶ **Definition 29** (g-matrix and g^*-matrix)**.** *Let V be a configuration of n vectors in $\mathbb{R}^r$. Set*

$$g_{j,k}(V) := g_{j,k}(V_{\text{cocyclic}}(n,r) \to V), \qquad \text{and} \qquad g^*_{j,k}(V) := g_{j,k}(V \to V_{\text{cyclic}}(n,r))$$

for $0 \leqslant j \leqslant r$ and $0 \leqslant k \leqslant n-r$. We call $g(V) = [g_{j,k}(V)]$ and $g^*(V) = [g^*_{j,k}(V)]$ the g-matrix *and the* g^*-matrix *of V, respectively. By the skew-symmetries* (2)*, both matrices are determined by their "upper left" quadrants indexed by $0 \leqslant j \leqslant \lfloor\frac{r-1}{2}\rfloor$ and $0 \leqslant k \leqslant \lfloor\frac{n-r-1}{2}\rfloor$, which we call the* small g-matrix *and the* small g^*-matrix, respectively.

By Gale duality, we get $g^*_{j,k}(V) = g_{k,j}(V^*)$, i.e., the g^*-matrix of V is the transpose of the g-matrix of the Gale dual V^*. Moreover, by Lemma 28,

$$g_{j,k}(V) + g^*_{j,k}(V) = \binom{n-k-r+j}{j}\binom{k+r-1-j}{k} - \binom{n-k-r+j-1}{j-1}\binom{k+r-j}{k} > 0 \tag{18}$$

for $0 \leqslant j \leqslant \frac{r-1}{2}$ and $0 \leqslant k \leqslant \frac{n-r-1}{2}$.

The 0-th column $[g_{j,0}(V) \mid 0 \leqslant j \leqslant r]$ of the g-matrix corresponds to the classical g-vector of a simple polytope. The *Generalized Lower Bound Theorem* (first proved by Stanley [39], as part of a full characterization of g-vectors, and hence f-vectors of simple polytopes, see [48, Sec. 8.6]), asserts that $g_{j,0}(V) \geqslant 0$ for $0 \leqslant j \leqslant \frac{r-1}{2}$.

The 0-th row $[g^*_{0,k} \mid 0 \leqslant k \leqslant n-r]$ corresponds to a Gale dual version of the g-vector of convex polytopes studied by Lee [25] and Welzl [46]. The following is implicit in [46, Sec. 4] (and implies McMullen's Upper Bound Theorem):

▶ **Theorem 30.** *Let $V \subset \mathbf{R}^r$ be a configuration of n vectors in general position. Then*

$$g^*_{0,k}(V) \leqslant \binom{k+r-1}{r-1} \qquad (0 \leqslant k \leqslant \lfloor\frac{n-r-1}{2}\rfloor)$$

Equivalently, by (18)*, $g_{0,k} \geqslant 0$ for $0 \leqslant k \leqslant \lfloor\frac{n-r-1}{2}\rfloor$. Equality holds if V is coneighborly.*

As a common generalization of the Upper Bound Theorem, the Generalized Lower Bound Theorem, and the Eckhoff–Linhart–Welzl Conjecture (Conj. 5), we propose the following:

▶ **Conjecture 31** (Nonnegativity of the Small g-Matrix)**.** *Let $V \subset \mathbf{R}^r$ be a configuration of n vectors in general position. Then $g_{j,k}(V) \geqslant 0$ for $0 \leqslant j \leqslant \frac{r-1}{2}$ and $0 \leqslant k \leqslant \frac{n-r-1}{2}$.*

Our main result (which implies Theorems 9 and 6) is the following:

▶ **Theorem 32.** *Let $V \subset \mathbf{R}^3$ be a configuration of n vectors in general position. Then*

$$g^*_{1,k}(V) \leqslant (k+1)n - 3\binom{k+2}{2} \qquad (0 \leqslant k \leqslant \lfloor \frac{n-4}{2} \rfloor)$$

Equivalently, $g_{1,k}(V) \geqslant 0$ for $0 \leqslant k \leqslant \lfloor \frac{n-4}{2} \rfloor$. Equality holds if V is coneighborly.

4 The Space $\mathfrak{G}_{n,r}$ Spanned by g-Matrices

Here, we prove Theorem 17. (The proof of the corresponding result for pointed configurations – Theorem 18 – uses similar ideas but is more involved, see [40]). By Theorem 10, the description of the space $\mathfrak{F}_{n,r}$ follows from the description of the space $\mathfrak{G}_{n,r}$, so it remains to prove the latter. Recall that $\mathcal{V}_{n,r}$ is the set of all vector configurations $V \in \mathbf{R}^{r\times n}$ in general position.

By Theorem 10, the g-matrix $g = g(V \to W)$ of any pair $V, W \in \mathcal{V}_{n,r}$ satisfies the skew-symmetries $g_{j,k} = -g_{r-j,k} = -g_{j,n-r-k} = g_{r-j,n-r-k}$ in (2). Thus, in order to prove Theorem 17, it remains to show that $\mathfrak{G}_{n,r} = \mathrm{lin}\{g(V \to W) \colon V, W \in \mathcal{V}_{n,r}\}$ has dimension $\lfloor \frac{r+1}{2} \rfloor \lfloor \frac{n-r+1}{2} \rfloor$. To see this, consider a generic continuous deformation from a coneighborly configuration V_0 to a neighborly configuration V_N, and let V_t, $0 \leqslant i \leqslant N$, be the intermediate vector configurations, i.e., V_t and V_{t-1} differ by a mutation. Thus, the g-matrices $g(V_0 \to V_t)$ and $g(V_0 \to V_{t-1})$ differ by the g-matrix of a mutation, i.e., their first quadrants (small g-matrices) differ in at most one coordinate, by $+1$ or -1. Moreover, $g(V_0 \to V_0)$ is identically zero, and every entry of the first quadrant of $g(V_0 \to V_N)$ is strictly positive by Lemma 28. Thus, the proof of Theorem 17 is completed by the following lemma:

▶ **Lemma 33.** *Let $X_0, X_1, \ldots, X_N$ be vectors in $\mathbf{R}^m$ such that*
1. *$X_0 = 0$;*
2. *X_t and X_{t-1} differ in one coordinate, by $+1$ or -1;*
3. *All coordinates of X_N are non-zero.*

Then there is a subset $X_{t_1}, \ldots, X_{t_m}$ of vectors that form a basis of $\mathbf{R}^m$.

Proof. For $1 \leqslant i \leqslant m$, let t_i be the smallest $t \in \{1, \ldots, N\}$ such that the i-th coordinate of X_{t_i} is non-zero; the index t_i exists by Properties 1 and 3. Moreover, by Property 2, no two coordinates can become non-zero at the same time, i.e., the indices t_i are pairwise distinct. Up to re-labeling the coordinates, we may assume $t_1 < t_2 < \ldots < t_m$. Then, for $1 \leqslant i \leqslant m$, the vector X_{t_i} is linearly independent from the vectors $X_{t_1}, \ldots X_{t_{i-1}}$, since all of the latter vectors have i-th coordinate zero. Thus, the X_{t_i} form a basis. ◀

5 Bounding the Spherical Arc Crossing Number

In this section, we outline the proof of Theorem 32 and explain how it implies Theorems 6 and 9. By Gale duality, Theorem 6 is equivalent to the following:

▶ **Theorem 34.** *Let $V = \{v_1, \ldots, v_n\} \subset \mathbf{R}^3$ be a vector configuration in general position. Then, for all $s \leqslant n$, the numbers $f^*_{s,0}(V)$ and $f^*_{s,\leqslant 1}(V) := f^*_{s,0}(V) + f^*_{s,1}(V)$ are maximized if V is coneighborly.*

Theorem 26 and the skew-symmetries of the g^*-matrix imply the following (see [41]):

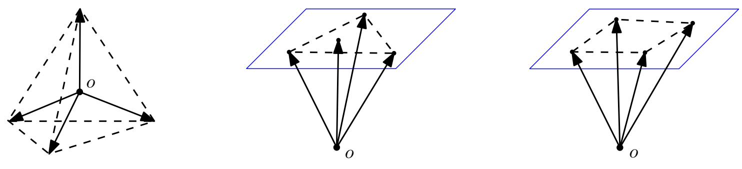

Figure 3 The three combinatorial types of four vectors in general position in $\mathbf{R}^3$.

▶ **Lemma 35.** *Let $V \in \mathbf{R}^{3\times n}$ be a vector configuration in general position and let $s \geqslant 4$. Then there are non-negative integers $\alpha(n,s,k)$ and $\beta(n,s,k)$, $0 \leqslant k \leqslant \lfloor\frac{n-4}{2}\rfloor$, such that*

$$f^*_{s,0}(V) = \sum_{k=0}^{\lfloor\frac{n-4}{2}\rfloor} \alpha(n,k,s)\cdot g^*_{0,k}(V)$$

and

$$f^*_{s,\leqslant 1}(V) = \sum_{k=0}^{\lfloor\frac{n-4}{2}\rfloor} \alpha(n,k,s)\cdot g^*_{1,k}(V) + \sum_{k=0}^{\lfloor\frac{n-4}{2}\rfloor} \beta(n,s,k)\cdot g^*_{0,k}(V)$$

Specifically, $\alpha(n,k,s) = \binom{n-3-k}{s-3} - \binom{k}{s-3}$ and $\beta(n,s,k) = \binom{n-3-k}{s-4}k - \binom{k}{s-4}(n-3-k)$.

Proof of Theorem 34. By Lemma 35, for $s \geqslant 4$, $f^*_{s,0}(V)$ and $f^*_{s,\leqslant 1}(V)$ are non-negative linear combinations of the numbers $g^*_{0,k}(V)$ and $g^*_{1,k}(V)$, $0 \leqslant k \leqslant \lfloor\frac{n-4}{2}\rfloor$. Hence, Theorem 34 follows from Theorems 30 and 32. ◀

Proof of Theorem 9. Let $V \subset \mathbf{R}^3$ be a configuration of n vectors in general position. There are three combinatorial types of quadruples $W \subset V$, which we call Type 0, Type 1, and Type 2, respectively, see Fig. 3. Each quadruple of Type i supports exactly two dependency patterns $F, -F \in \mathcal{F}^*(V)$ with four non-zero signs, one with $|F_-| = i$ negative signs, and the other one with $|F_+| = 4 - i$ negative signs. It follows that

$$f^*_{4,0}(V) + f^*_{4,1}(V) + \frac{1}{2}f^*_{4,2}(V) = \binom{n}{4}$$

Moreover, if all the vectors in V have unit length, then the number of crossings in the induced spherical arc drawing of K_n equals $\mathrm{cr}(V) = \frac{1}{2}f^*_{4,2}(V)$. Thus Theorem 9 is equivalent to the statement that $f^*_{4,2}(V) \geqslant 2X(n)$, equivalently, $f^*_{4,0}(V) + f^*_{4,1}(V) \leqslant \binom{n}{4} - X(n)$, where equality holds in both bounds if V is coneighborly. By the special case $s = 4$ of Theorem 34,

$$f^*_{4,\leqslant 1}(V) \leqslant \sum_{k=0}^{\lfloor\frac{n-4}{2}\rfloor} (n-3-2k)\left((k+1)n - 3\binom{k+2}{2}\right) =: Y(n), \tag{19}$$

with equality if V is coneighborly. By a direct calculation (see [41]), $Y(n) = \binom{n}{4} - X(n)$. ◀

▶ Remark 36. We mention a connection to geometric probability. Let μ be a probability distribution on $\mathbf{R}^3$; we assume that μ is non-degenerate in the sense that every plane through the origin has μ-measure zero. A beautiful argument due to Wendel [47] shows that if μ is *centrally symmetric* and $W = \{w_1, w_2, w_3, w_4\} \subset \mathbf{R}^3$ is a set of four independent μ-random vectors then the probability that W is of Type $i \in \{0,1,2\}$ (Fig. 3) equals q_i, where $q_0 = \frac{\binom{4}{0}+\binom{4}{4}}{2^4} = \frac{1}{8}$, $q_1 = \frac{\binom{4}{1}+\binom{4}{3}}{2^4} = \frac{1}{2}$, and $q_2 = \frac{\binom{4}{2}}{2^4} = \frac{3}{8}$; it follows that for any set $V \subset \mathbf{R}^3$ of

n independent μ-random vectors, the expected number of quadruples of type i equals $q_i\binom{n}{4}$. In particular, if the vectors in V are chosen independently and uniformly at random from S^2 then the expected number of crossings in the spherical arc drawing given by V is $\frac{3}{8}\binom{n}{4}$, which was also independently shown by Moon [32]. Theorem 9, together with a well-known limiting argument [38] implies that for an arbitrary (not necessarily centrally symmetric) non-degenerate probability distribution μ on S^2, the expected number of crossings in the spherical arc drawing given by n independent μ-random points is *at least* $\frac{3}{8}\binom{n}{4}$.

To conclude, we outline some of the main ideas for the proof of Theorem 32 (see [41] for the details). Let $V = \{v_1, \ldots, v_n\} \subset \mathbf{R}^3$ be a configuration of n vectors in general position. We will view V as the set $S - o$ of differences $v_i = p_i - o$ between a point set $S = \{p_1, \ldots, p_n\} \subset \mathbf{R}^3$ (the *tips* of the vectors) and another point $o \in \mathbf{R}^3$ (the *origin*).

Choose a line ℓ in $\mathbf{R}^3$ through o in general position. By positively rescaling the vectors v_i, we may assume that S is a subset of the cylinder Z consisting of the points in $\mathbf{R}^3$ at Euclidean distance 1 from ℓ; in particular, S is a point set in convex position.

The point set $S \subset \mathbf{R}^3$ corresponds to a neighborly configuration $U = \{u_1, \ldots, u_n\}$ of vectors in $\mathbf{R}^4$, and the origin $o \in \mathbf{R}^3$ corresponds to an additional vector $u_0 \in \mathbf{R}^4$. Let $\mathcal{A}(\{u_0\} \sqcup U) = \{H_0^+, H_1^+, \ldots, H_n^+\}$ be the polar dual arrangement of hemispheres in S^3. The intersection $\mathcal{A}(U) \cap H_0$ is a spherical arrangment in the equatorial 2-sphere $H_0 \cong S^2$, which is combinatorially isomorphic to the arrangement $\mathcal{A}(V)$.

Let $C = H_i \cap H_j$ be the great circle in S^3 formed by the intersection of two great 2-spheres of the arrangement $\mathcal{A}(U)$, $1 \leqslant i < j \leqslant n$. For $k \leqslant \lfloor \frac{n-4}{2} \rfloor$, consider the subgraph of $\mathcal{A}(U)$ consisting of the vertices and edges of $\mathcal{A}(U)$ contained in C and at sublevel $(\leqslant k)$. This subgraph cannot cover the entire circle C, since C contains at least one pair of antipodal vertices at levels $\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil$. Thus, the connected components of this subgraph form closed intervals, which we call *k-arcs* of $\mathcal{A}(U)$. We define $\lambda_k(U, u_0)$ as the number of k-arcs of $\mathcal{A}(U)$ that are completely contained in the negative open hemisphere H_0^-.

Suppose we move the origin o continuously along the line ℓ, while keeping the set S fixed. Consider the corresponding continuous motions of the vector configurations $V = S - o$ in $\mathbf{R}^3$ and $\{u_0\} \sqcup U$ in $\mathbf{R}^4$. A careful analysis of how $\lambda_k(U, u_0)$ and $g_{1,k}(V)$ change during this continuous motion yields

$$g_{1,k}(V) = \lambda_k(U, u_0) \geqslant 0$$

for $0 \leqslant k \leqslant \lfloor \frac{n-4}{2} \rfloor$, which proves Theorem 32.

References

1 Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Dan McQuillan, Bojan Mohar, Petra Mutzel, Pedro Ramos, R. Bruce Richter, and Birgit Vogtenhuber. Bishellable drawings of K_n. *SIAM J. Discrete Math.*, 32(4):2482–2492, 2018. doi:10.1137/17M1147974.

2 Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. The 2-page crossing number of K_n. *Discrete Comput. Geom.*, 49(4):747–777, 2013. doi:10.1007/S00454-013-9514-0.

3 Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. Shellable drawings and the cylindrical crossing number of K_n. *Discrete Comput. Geom.*, 52(4):743–753, 2014. doi:10.1007/S00454-014-9635-0.

4 Bernardo M. Ábrego, József Balogh, Silvia Fernández-Merchant, Jesús Leaños, and Gelasio Salazar. An extended lower bound on the number of $(\leqslant k)$-edges to generalized configurations of points and the pseudolinear crossing number of K_n. *J. Combin. Theory Ser. A*, 115(7):1257–1264, 2008. doi:10.1016/J.JCTA.2007.11.004.

5 Bernardo M. Ábrego, Mario Cetina, Silvia Fernández-Merchant, Jesús Leaños, and Gelasio Salazar. On $\leqslant k$-edges, crossings, and halving lines of geometric drawings of K_n. *Discrete Comput. Geom.*, 48(1):192–215, 2012. doi:10.1007/S00454-012-9403-Y.

6 Bernardo M. Ábrego and Silvia Fernández-Merchant. A lower bound for the rectilinear crossing number. *Graphs Combin.*, 21(3):293–300, 2005. doi:10.1007/S00373-005-0612-5.

7 Bernardo M. Ábrego, Silvia Fernández-Merchant, Jesús Leaños, and Gelasio Salazar. A central approach to bound the number of crossings in a generalized configuration. In *The IV Latin-American Algorithms, Graphs, and Optimization Symposium*, volume 30 of *Electron. Notes Discrete Math.*, pages 273–278. Elsevier Sci. B. V., Amsterdam, 2008. doi:10.1016/J.ENDM.2008.01.047.

8 Bernardo M. Ábrego, Silvia Fernández-Merchant, and Gelasio Salazar. The rectilinear crossing number of K_n: closing in (or are we?). In *Thirty essays on geometric graph theory*, pages 5–18. Springer, New York, 2013.

9 Oswin Aichholzer, Jesús García, David Orden, and Pedro Ramos. New lower bounds for the number of ($\leqslant k$)-edges and the rectilinear crossing number of K_n. *Discrete Comput. Geom.*, 38(1):1–14, 2007. doi:10.1007/S00454-007-1325-8.

10 Noga Alon and E. Győri. The number of small semispaces of a finite set of points in the plane. *J. Combin. Theory Ser. A*, 41(1):154–157, 1986. doi:10.1016/0097-3165(86)90122-6.

11 Artur Andrzejak, Boris Aronov, Sariel Har-Peled, Raimund Seidel, and Emo Welzl. Results on k-sets and j-facets via continuous motion. In *Proceedings of the Fourteenth Annual Symposium on Computational Geometry*, SCG '98, pages 192–199, New York, NY, USA, 1998. Association for Computing Machinery. doi:10.1145/276884.276906.

12 Artur Andrzejak and Emo Welzl. In between k-sets, j-facets, and i-faces: (i, j)-partitions. *Discrete Comput. Geom.*, 29(1):105–131, 2003.

13 Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete Comput. Geom.*, 53(1):107–143, 2015.

14 József Balogh, Bernard Lidický, and Gelasio Salazar. Closing in on Hill's conjecture. *SIAM J. Discrete Math.*, 33(3):1261–1276, 2019. doi:10.1137/17M1158859.

15 József Balogh and Gelasio Salazar. k-sets, convex quadrilaterals, and the rectilinear crossing number of K_n. *Discrete Comput. Geom.*, 35(4):671–690, 2006. doi:10.1007/S00454-005-1227-6.

16 Ranita Biswas, Sebastiano Cultrera di Montesano, Herbert Edelsbrunner, and Morteza Saghafian. Depth in arrangements: Dehn-Sommerville-Euler relations with applications. *J. Appl. Comput. Topol.*, 8(3):557–578, 2024. doi:10.1007/S41468-024-00173-W.

17 Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

18 Kenneth L. Clarkson and Peter W. Shor. Applications of random sampling in computational geometry. II. *Discrete Comput. Geom.*, 4(5):387–421, 1989. doi:10.1007/BF02187740.

19 Tamal K. Dey. Improved bounds for planar k-sets and related problems. *Discrete Comput. Geom.*, 19(3):373–382, 1998. doi:10.1007/PL00009354.

20 Jürgen Eckhoff. Helly, Radon, and Carathéodory type theorems. In *Handbook of convex geometry, Vol. A, B*, pages 389–448. North-Holland, Amsterdam, 1993.

21 Herbert Edelsbrunner. *Algorithms in combinatorial geometry*, volume 10 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1987. doi:10.1007/978-3-642-61568-9.

22 Paul Erdős, László. Lovász, A. Simmons, and Ernst G. Straus. Dissection graphs of planar point sets. In *A survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971)*, pages 139–149. North-Holland, Amsterdam-London, 1973.

23 Branko Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003.

24 T. Gulliksen and A. Hole. On the number of linearly separable subsets of finite sets in $\mathbf{R}^n$. *Discrete Comput. Geom.*, 18(4):463–472, 1997.

25 Carl W. Lee. Winding numbers and the generalized lower-bound conjecture. In *Discrete and computational geometry (New Brunswick, NJ, 1989/1990)*, volume 6 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 209–219. Amer. Math. Soc., Providence, RI, 1991. `doi:10.1090/DIMACS/006/13`.

26 Johann Linhart. The upper bound conjecture for arrangements of halfspaces. *Beiträge Algebra Geom.*, 35(1):29–35, 1994.

27 Johann Linhart, Yanling Yang, and Martin Philipp. Arrangements of hemispheres and halfspaces. *Discrete Math.*, 223(1-3):217–226, 2000. `doi:10.1016/S0012-365X(98)00360-4`.

28 László Lovász. On the number of halving lines. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 14:107–108, 1971.

29 László Lovász, Katalin Vesztergombi, Uli Wagner, and Emo Welzl. Convex quadrilaterals and k-sets. In *Towards a theory of geometric graphs*, volume 342 of *Contemp. Math.*, pages 139–148. Amer. Math. Soc., Providence, RI, 2004.

30 Jiří Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

31 Peter McMullen. The maximum numbers of faces of a convex polytope. *Mathematika*, 17:179–184, 1970.

32 John W. Moon. On the distribution of crossings in random complete graphs. *J. Soc. Indust. Appl. Math.*, 13:506–510, 1965.

33 Ketan Mulmuley. Dehn-sommerville relations, upper bound theorem, and levels in arrangements. In Chee Yap, editor, *Proceedings of the Ninth Annual Symposium on Computational Geometry (San Diego, CA, USA, May 19-21, 1993)*, pages 240–246, 1993. `doi:10.1145/160985.161141`.

34 Petra Mutzel and Lutz Oettershagen. The crossing number of seq-shellable drawings of complete graphs. In *Combinatorial algorithms*, volume 10979 of *Lecture Notes in Comput. Sci.*, pages 273–284. Springer, Cham, 2018. `doi:10.1007/978-3-319-94667-2_23`.

35 Arnau Padrol. Many neighborly polytopes and oriented matroids. *Discrete Comput. Geom.*, 50(4):865–902, 2013. `doi:10.1007/S00454-013-9544-7`.

36 G. W. Peck. On "k-sets" in the plane. *Discrete Math.*, 56(1):73–74, 1985.

37 Marcus Schaefer. *Crossing numbers of graphs*. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2018.

38 Edward R. Scheinerman and Herbert S. Wilf. The rectilinear crossing number of a complete graph and Sylvester's "four point problem" of geometric probability. *Amer. Math. Monthly*, 101(10):939–943, 1994.

39 Richard P. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.*, 35(3):236–238, 1980.

40 Elizaveta Streltsova and Uli Wagner. Linear relations between face numbers of levels in arrangements. Preprint, 2025. `arXiv:2504.07752`.

41 Elizaveta Streltsova and Uli Wagner. Sublevels in arrangements and the spherical arc crossing number of complete graphs. Preprint, 2025. `arXiv:2504.07770`.

42 László A. Székely. Turán's brick factory problem: the status of the conjectures of Zarankiewicz and Hill. In *Graph theory – Favorite conjectures and open problems. 1*, Probl. Books in Math., pages 211–230. Springer, 2016.

43 Géza Tóth. Point sets with many k-sets. *Discrete Comput. Geom.*, 26(2):187–194, 2001. `doi:10.1007/S004540010022`.

44 Uli Wagner. On a geometric generalization of the upper bound theorem. In *Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science*, pages 635–645, 2006. `doi:10.1109/FOCS.2006.53`.

45 Uli Wagner. k-Sets and k-facets. In *Surveys on Discrete and Computational Geometry*, volume 453 of *Contemp. Math.*, pages 443–513. Amer. Math. Soc., Providence, RI, 2008.

46 Emo Welzl. Entering and leaving j-facets. *Discrete Comput. Geom.*, 25(3):351–364, 2001. doi:10.1007/S004540010085.

47 James G. Wendel. A problem in geometric probability. *Math. Scand.*, 11:109–111, 1962.

48 Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.