# On Spheres with $k$ Points Inside

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**Abstract**

We generalize a classical result by Boris Delaunay that introduced Delaunay triangulations. In particular, we prove that for a locally finite and coarsely dense generic point set $A$ in $\mathbb{R}^d$, every generic point of $\mathbb{R}^d$ belongs to exactly $\binom{d+k}{d}$ simplices whose vertices belong to $A$ and whose circumspheres enclose exactly $k$ points of $A$. We extend this result to the cases in which the points are weighted, and when $A$ contains only finitely many points in $\mathbb{R}^d$ or in $\mathbb{S}^d$. Furthermore, we use the result to give a new geometric proof for the fact that volumes of hypersimplices are Eulerian numbers.

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## 1 Introduction

In the seminal paper [5], Boris Delaunay (also spelled Delone) introduced the Delaunay triangulation of a finite point sets using simplices with empty circumspheres. His construction can be reformulated as follows: for a (finite and generic) point set, $A \subseteq \mathbb{R}^d$, the simplices with vertices in $A$ that contain no points of $A$ inside their circumspheres cover the convex hull of $A$ in one layer. In this paper, we generalize Delaunay's construction and prove similar properties for simplices with circumspheres that enclose exactly $k$ points of $A$, for some fixed non-negative integer $k$. We call these simplices the $k$-hefty simplices of $A$.

We introduce the main concepts we will work with. A set $A \subseteq \mathbb{R}^d$ is *locally finite* if every closed ball contains at most a finite number of the points of $A$, and it is *coarsely dense* if every closed half-space contains at least one and therefore infinitely many of the points of $A$. If $A$ has both properties, we call it a *thin Delone set*; compare with the more restrictive class of *Delone sets*, which are *uniformly discrete* and *relatively dense*, meaning the smallest inter-point distance is bounded away from 0, and the radius of the largest empty ball is bounded away from $\infty$. We call $A$ *generic* if no $d+1$ of its points lie on a common hyperplane, and no $d+2$ of its points lie on a common $(d-1)$-sphere. Any $(d-1)$-sphere bounds a closed $d$-ball and thus partitions $\mathbb{R}^d$ into points *inside* the sphere (in the interior of the ball), points *on* the sphere, and points *outside* the sphere (in the complement of the ball). Assuming $A$ is generic, there is a unique $(d-1)$-sphere passing through any $d+1$ points of $A$, which we call the *circumscribed sphere* of the $d$-simplex spanned by the points.

▶ **Main Definition.** *Let $k$ be a non-negative integer and $A \subseteq \mathbb{R}^d$ a generic thin Delone set or a generic finite set. A $d$-simplex with vertices in $A$ is $k$-*hefty *if exactly $k$ points of $A$ lie inside the circumsphere of the $d$-simplex.*

For example, the 0-hefty simplices are the top-dimensional simplices in the Delaunay triangulation of $A$, and $k$-hefty simplices with $k > 0$ are related to the cells in higher-order Delaunay triangulations [2, 7]. Our main result is Theorem 2.2, which we restate here in less technical terms:

▶ **Main Theorem.** *Let $k$ be a non-negative integer and $A \subseteq \mathbb{R}^d$ a generic thin Delone set. Then the $k$-hefty simplices of $A$ cover $\mathbb{R}^d$ exactly $\binom{d+k}{d}$ times.*

More specifically, almost every point of $\mathbb{R}^d$ is covered by exactly $\binom{d+k}{d}$ $k$-hefty simplices, while boundary points of $k$-hefty simplices may be contained in more than this number of such simplices. We also prove versions of this theorem for finitely many points, points on the $d$-dimensional sphere, and weighted points. In addition, we apply the covering multiplicities to get a new proof that the volumes of hypersimplices are Eulerian numbers, and to get new proofs for some bounds on $k$-sets.

The paper is organized as follows. In Section 2, we introduce the main definitions, prove the main result for thin Delone sets (Theorem 2.2) and finite sets (Corollary 2.3), and formulate their local versions (Theorem 2.4). In Section 3, we apply the results to obtain a new proof for the fact that volumes of hypersimplices are Eulerian numbers and new proofs for old bounds on $k$-sets. In the concluding Section 4, we discuss extensions of the results to points in hyperbolic and spherical spaces and to points with real weights in Euclidean space.

# 2 Hefty Simplices in Euclidean Space

This section presents the main result of this paper, which is stated for infinite and finite point sets in Euclidean space. We begin with the main technical lemma before stating and proving the main theorem.

## 2.1 Main Technical Lemma

For technical reasons we first show that the $k$-hefty simplices of a thin Delone set $A$ are "locally uniform" in size. Specifically, we prove an upper bound for the radii of spheres that enclose a fixed point, $x \in \mathbb{R}^d$, as well as at most $k$ points of $A$. To this end, we write $B(x, R)$ for the closed ball with center $x$ and radius $R$, and note that the number of points of $A$ in this ball goes to infinity when $R$ goes to infinity.

▶ **Lemma 2.1.** *Let $A \subseteq \mathbb{R}^d$ be coarsely dense, $k$ a non-negative integer, and $x \in \mathbb{R}^d$. Then there exists $R = R(x, A, k)$ such that if $x$ is inside a sphere that is not fully contained in $B(x, R)$, then there are at least $k+1$ point of $A \cap B(x, R)$ inside this sphere.*

**Proof.** Without loss of generality, assume $x = 0$. For every unit vector, $u \in \mathbb{S}^{d-1}$, the open halfspace of points $y$ that satisfy $\langle y, u \rangle > 0$ contains infinitely many points of $A$. It follows that the function $f_u \colon (0, \infty) \to \mathbb{Z}$ that maps $r > 0$ to the number of points of $A$ inside the sphere with center $ru$ and radius $r$ is non-decreasing and unbounded.

We introduce $g\colon \mathbb{S}^{d-1} \to \mathbb{R}$ defined by $g(u) = \inf\{r > 0 \mid f_u(r) \geq k+1\}$ and claim that $g$ is bounded. To derive a contradiction, suppose $g$ is unbounded, and let $u_1, u_2, \ldots$ be a sequence of unit vectors with $g(u_n) \geq n$. Since $\mathbb{S}^{d-1}$ is compact, there is a subsequence that converges to a vector $u_0 \in \mathbb{S}^{d-1}$. Let $S$ be the sphere with radius $g(u_0)+1$ and center $(g(u_0)+1)u_0$. By construction, there are at least $k+1$ points of $A$ inside $S$. Since these points are (strictly) inside the sphere, there is a sufficiently small $\varepsilon > 0$ such that moving the center of the sphere by at most $\varepsilon$ while adjusting its radius so the origin remains on the sphere, retains at least $k+1$ point of $A$ inside the sphere. But this contradicts the unboundedness of $g$ as there are points $u_i$ in the subsequence that are within distance $\varepsilon$ from $u_0$ with $g(u_i)$ much larger than $g(u_0)+1$.

Since $g$ is bounded, $M = \sup\{g(u) \mid u \in \mathbb{S}^{d-1}\}$ is finite and, by construction of $g$, there are at least $k+1$ points of $A$ inside any sphere with center $y$ and radius $\|y\|$ as long as $\|y\| \geq M$. Setting $R = 2M$, every sphere with center $y$ that encloses the origin and is not contained in $B(0,R)$ has radius $r > M$. This sphere encloses the ball with center $M\frac{y}{\|y\|}$ and radius $M$, so there are at least $k+1$ points of $A$ inside this sphere that all belong to $B(0,R)$. ◀

As an immediate consequence of Lemma 2.1, the circumsphere of any $k$-hefty simplex of $A$ that encloses $x$ is completely contained in $B(x,R)$, in which $R$ depends on $x$, $A$, and $k$.

## 2.2 Global Covering

Our first goal is to generalize the classic result of Delaunay that the 0-hefty simplices of every thin Delone set cover $\mathbb{R}^d$ in one layer; that is: every point of $\mathbb{R}^d$ is contained in at least one 0-hefty simplex and almost every point of $\mathbb{R}^d$ is contained in exactly one 0-hefty simplex. Specifically, we show that for every generic thin Delone set, $A \subseteq \mathbb{R}^d$, the family of $k$-hefty simplices covers $\mathbb{R}^d$ $\binom{d+k}{d}$ times. We call $\binom{d+k}{d}$ the *$k$-th covering number* and note that it depends on the dimension, $d$, and the parameter, $k$, but not on the set $A$. We call $x \notin A$ *generic with respect to* $A$ if $A \cup \{x\}$ is generic. Almost every point $x \in \mathbb{R}^d$ is generic with respect to a generic thin Delone set, $A$. To see this, observe that by local finiteness of $A$ there are only countably many hyperplanes spanned by $d$ points each or spheres spanned by $d+1$ points each, so the union of these hyperplanes and spheres has Lebesgue measure zero.

▶ **Theorem 2.2.** *Let $k$ be a non-negative integer and $A \subseteq \mathbb{R}^d$ a generic thin Delone set. Then any point $x \in \mathbb{R}^d$ that is generic with respect to $A$ belongs to exactly $\binom{d+k}{d}$ $k$-hefty simplices of $A$.*

**Proof.** The case $d = 1$ is obvious since every $k$-hefty simplex of $A$ is an interval with endpoints in $A$ and exactly $k$ points between the two endpoints. Every point that is generic with respect to $A$, i.e. in $\mathbb{R} \setminus A$, is contained in exactly $k+1$ such intervals. For $d \geq 2$, the proof splits into three steps.

**Step 1.** Letting $k$ be a non-negative integer and $A \subseteq \mathbb{R}^d$ a generic thin Delone set, we prove that there is a constant $c = c(k, A)$ such that any point that is generic with respect to $A$ is contained in exactly $c$ $k$-hefty simplices of $A$. Write $cover_k(x, A)$ for the number of $k$-hefty simplices of $A$ that contain $x$. By Lemma 2.1, $cover_k(x, A)$ is finite. Indeed, every $k$-hefty simplex of $A$ that contains $x$ must select its vertices from the finitely many points inside the ball $B(x, 2M)$. To show that $cover_k(x, A)$ is the same for all generic points, we move $x$ continuously from one point to another. The only time $cover_k(x, A)$ can change is when $x$ passes through the boundary of a $k$-hefty simplex. It suffices to show that for every $(d-1)$-simplex, $\Delta$, with vertices in $A$, the number of $k$-hefty simplices with facet $\Delta$ is the same on both sides of $\Delta$.

Consider the line, $L$, that consists of all points equidistant to the vertices of $\Delta$ and mark each point $y \in L$ with the number of points of $A$ inside the sphere with center $y$ that passes through the vertices of $\Delta$. This partitions $L$ into labeled intervals, and since $A$ is generic, the labels of two consecutive intervals differ by exactly one. Fix a *left to right* direction on $L$, move $y$ in this direction, and observe that the portion of space inside the sphere centered at $y$ that lies to the left of the hyperplane spanned by $\Delta$ shrinks, while the portion to the right of this hyperplane grows. The transitions from an interval labeled $k+1$ to another labeled $k$ are in bijection with the $k$-hefty simplices with facet $\Delta$ to the left of $\Delta$. Indeed, as $y$ makes the transition, there is a point of $A$ that passes from inside to outside the sphere centered at $y$, so this point is to the left of $\Delta$. Similarly, the transitions from an interval labeled $k$ to another labeled $k+1$ are in bijection with the $k$-hefty simplices with facet $\Delta$ to the right of $\Delta$. There are equally many transitions of either kind because the labels go to infinity on both sides. This proves that $cover_k(x, A)$ does not depend on $x$.

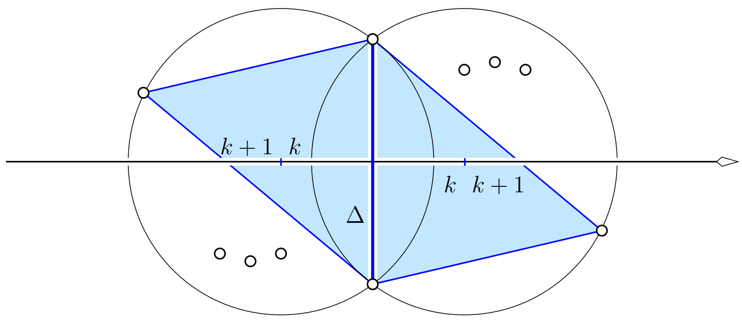


**Figure 1** Two circles in the 1-parameter family of circles that pass through the endpoint of the edge $\Delta$. Both are the circumcircles of $k$-hefty triangles, with $k = 3$ in the case displayed. As we move the center from left to right, every point that leaves the inside of the circle lies to the left of $\Delta$, and every point that enters the inside of the circle lies to the right of $\Delta$.

**Step 2.** We strengthen the claim proved in Step 1 by showing that the constant depends on $d$ and $k$ but not on $A$. Specifically, we prove that for every dimension $d$ and non-negative integer $k$, there exists a number $f(d, k)$ such that for any generic thin Delone set, $A \subseteq \mathbb{R}^d$, any point $x \in \mathbb{R}^d$ generic with respect to $A$ belongs to exactly $f(d, k)$ $k$-hefty simplices of $A$.

It suffices to show that for two thin Delone sets, $A$ and $A'$, and two points, $x, x' \in \mathbb{R}^d$, that are generic with repect to both sets, $cover_k(x, A) = cover_k(x', A')$. By Lemma 2.1, there exists $R > 0$ such that if a sphere encloses $x$ and a point outside $B(x, R)$, then there are at least $k+1$ points of $A \cap B(x, R)$ inside this sphere. It follows that the circumspheres of all $k$-hefty simplices that enclose $x$ are contained in $B(x, R)$. Similarly, let $R'$ be the constant from Lemma 2.1 for $x'$ and $A'$. We construct a new thin Delone set, $A''$: picking points $y$ and $y'$ at distance larger than $R + R'$ from each other, we let $A'' \cap B(y, R)$ be a translate of $A \cap B(x, R)$ and $A'' \cap B(y', R')$ a translate of $A' \cap B(x', R')$. We perturb the points if necessary to achieve genericity, and add more points outside $B(y, R)$ and $B(y', R')$ until $A''$ is thin Delone. By choice of $R$ and $R'$ we have $cover_k(x, A) = cover_k(y, A'')$, because every $k$-hefty simplex of $A$ whose circumsphere encloses $x$ translates into a $k$-hefty simplex of $A''$ whose circumsphere encloses $y$, and vice versa. Similarly, we have $cover_k(x', A') = cover_k(y', A'')$. Finally, $cover_k(x, A) = cover_k(x', A')$ because $cover_k(y, A'') = cover_k(y', A'')$ as proved in Step 1.

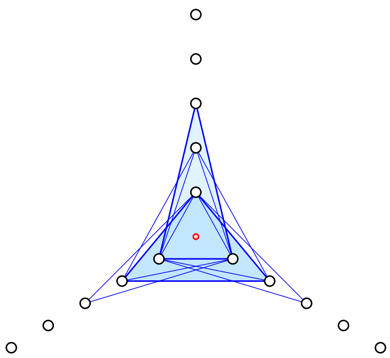

**Figure 2** Before perturbation, the points of $A$ lie on three half-lines emanating from the origin. We show six 2-hefty triangles, and emphasize two of them by shading.

**Step 3.** We provide an explicit example of a generic thin Delone set, $A \subseteq \mathbb{R}^d$, and a point, $x \in \mathbb{R}^d$, with $cover_k(x, A) = \binom{d+k}{d}$. Specifically, we prove that for every dimension $d$ and non-negative integer $k$, there exists a generic thin Delone set $A \subseteq \mathbb{R}^d$, and a point $x$ generic with respect to $A$, such that $cover_k(x, A) = \binom{d+k}{d}$.

Let $S$ be a regular $d$-simplex with vertices $v_0, v_1, \ldots, v_d$ and barycenter $0 = \frac{1}{d+1} \sum_i v_i$, and $A$ the set of points of the form $iv_j$, for integers $i \geq 1$ and $j = 0, 1, \ldots, d$. We call $A$ a *radial* set and the points with fixed $j$ a *direction* of $A$; see Figure 2. Set $x = 0$. Every $k$-hefty simplex of $A$ that contains 0 has exactly one vertex from each direction. Let $i_0 v_0, i_1 v_1, \ldots, i_d v_d$ be the vertices of such a simplex. The number of points of $A$ inside its circumsphere is $\sum_j (i_j - 1) = k$. Enumerating these simplices is the same as writing $k$ as an ordered sum of $d+1$ non-negative integers, and there are exactly $\binom{d+k}{d}$ ways to do that. To complete the proof, we perturb $A$ while making sure that $d+1$ points span a simplex that contains 0 before the perturbation iff they span such a simplex after the perturbation. This completes the proof of our main result. ◀

## 2.3 The Finite Case

For a finite set, $A \subseteq \mathbb{R}^d$, the covering multiplicity of Theorem 2.2 holds sufficiently deep inside the set but acts only as an upper bound near the fringes. To formalize these claims, we introduce a parametrized generalization of the convex hull: for each integer $k \geq 0$, the *$k$-hull* of $A$, denoted $H_k(A)$, is the common intersection of all closed half-spaces that miss at most $k$ of the points in $A$. Clearly, $H_0(A) = \mathrm{conv}\, A$, and $H_k(A) \supseteq H_{k+1}(A)$ for every $k$. It is also easy to see that $H_k(A)$ is contained in the convex hull of $A$ after removing at most $k$ of its points, and indeed $H_k(A)$ *is* the intersection of all such convex hulls. By the Centerpoint Theorem of discrete geometry [6, Section 4.1], $H_k(A) \neq \emptyset$ if $k < \frac{n}{d+1}$, in which $n = \#A$ is the number of points in $A$.

▶ **Theorem 2.3.** *Let $A \subseteq \mathbb{R}^d$ be finite and generic, $k \geq 0$ an integer, and $x \in \mathbb{R}^d$ generic with respect to $A$. Then $x$ is covered by at most $\binom{d+k}{d}$ $k$-hefty simplices of $A$, with equality iff $x \in H_k(A)$.*

**Proof.** To prove the upper bound, we add points outside all circumspheres of $d+1$ points in $A$ to construct a thin Delaunay set $A' \subseteq \mathbb{R}^d$. This is possible because the union of balls bounded by such spheres is bounded. Hence, $A \subseteq A'$, and any $k$-hefty simplex of $A$ is also a $k$-hefty simplex of $A'$. Assuming $x$ is generic with respect to $A'$, Theorem 2.2 implies that exactly $\binom{d+k}{d}$ $k$-hefty simplices of $A'$ cover $x$. If all of them are also $k$-hefty simplices of $A$, then $x$ is covered by exactly $\binom{d+k}{d}$ $k$-hefty simplices of $A$, but if not, then $x$ is covered by fewer than $\binom{d+k}{d}$ $k$-hefty simplices of $A$.

"$\Longleftarrow$". We prove $x \in H_k(A)$ implies that every $k$-hefty simplex of $A'$ covering $x$ is also a $k$-hefty simplex of $A$. Since $A'$ has exactly $\binom{d+k}{d}$ $k$-hefty simplices that cover $x$, this will imply that $A$ has the same number of such simplices. Consider such a $k$-hefty simplex of $A'$, and let $B \subseteq A'$ be the points inside its circumsphere. Since $x \in H_k(A)$, every closed half-space that misses at most $k$ points of $A$ contains $x$. But $\#B = k$, so $x$ belongs to the convex hull of $A \setminus B$. The top-dimensional simplices of $\mathrm{Del}(A \setminus B)$ cover the convex hull once, hence there is a unique 0-hefty simplex of $A \setminus B$ that covers $x$. All points of $A' \setminus A$ lie outside its circumsphere, which implies that the same simplex is the unique 0-hefty simplex of $A' \setminus B$ that covers $x$. Therefore $B \subseteq A$ because the vertices of the simplex are in $A$, and all vertices we removed must come from $A$ as well. So this is a $k$-hefty simplex of $A$.

"$\Longrightarrow$". Assuming $x \notin H_k(A)$, we modify the construction of $A'$ to show that there is a set $A'' \supseteq A$ such that all $k$-hefty simplices of $A$ are $k$-hefty simplices of $A''$ but at least one $k$-hefty simplex of $A''$ that covers $x$ is not a $k$-hefty simplex of $A$. In contrast to $A'$, the set $A''$ will be finite. Since $x \notin H_k(A)$, there are $d$ points in $A$ such that the open half-space bounded by the hyperplane passing through these points that contains $x$ contains at most $k$ points of $A$. Let $\Delta$ be the $(d-1)$-simplex spanned by the $d$ points, write $\Delta^+$ for the open half-space that contains $x$, and let $B \subseteq A$ be the points in $\Delta^+$. By assumption, $\#B \leq k$. We get $A''$ by adding $1 + k - \#B$ points to $A$ as follows. First, we add $k - \#B$ points in $\Delta^+$ but outside all circumspheres of $d+1$ points in $A$. After that, we add a point $y \in \mathbb{R}^d$ so that the $d$-simplex that is the pyramid with apex $y$ and base $\Delta$ covers $x$, and all other $k$ points in $\Delta^+$ are inside the circumsphere of this $d$-simplex. We also require that $y$ is outside all circumspheres of $d+1$ points in $A$. It is clear that such a point $y$ exists on a sufficiently large sphere that passes through all vertices of $\Delta$. As proved above, there are at most $\binom{d+k}{d}$ $k$-hefty simplices of $A''$ that cover $x$. The pyramid with apex $y$ and base $\Delta$ is such a simplex, but it is not a $k$-hefty simplex of $A$. Hence, the number of $k$-hefty simplices of $A$ that cover $x$ is strictly less than $\binom{d+k}{d}$, as claimed. ◀

▶ Remark. Observe that our arguments in part "$\Longleftarrow$" imply that once we fix a subset $B \subseteq A$ of size $k$, then the $k$-hefty simplices of $A$ that have exactly these points inside their circumspheres have disjoint interiors. This is because they are 0-hefty simplices of $A \setminus B$.

## 2.4 Local Covering

By Theorem 2.2, the $k$-hefty simplices of a thin Delone set cover $\mathbb{R}^d$ without gap an integer number of times. However, beyond one dimension, it is generally not possible to split such a cover into sub-covers (separate layers) that enjoy the same property. As an example, consider the thin Delone set of which Figure 3 shows one point surrounded by five others. Assuming the remaining points are sufficiently far from the six, the point at the center is incident to five 1-hefty triangles, as shown. Globally, the 1-hefty triangles cover any small neighborhood of this point three times, but it is not possible to decompose this cover into three; see [9] for indecomposable coverings in other contexts. It is however possible to split it locally into two. Such a local split exists for every $k \geq 1$, and we prove that the hefty simplices incident to the point cover any small neighborhood a specific number of times.

▶ **Theorem 2.4.** *Let $A \subseteq \mathbb{R}^d$ be a generic thin Delone set and $k$ a non-negative integer. Then the $k$-hefty simplices of $A$ incident to a vertex $a \in A$ cover any sufficiently small neighborhood of $a$ exactly $\binom{d+k-1}{d-1}$ times.*

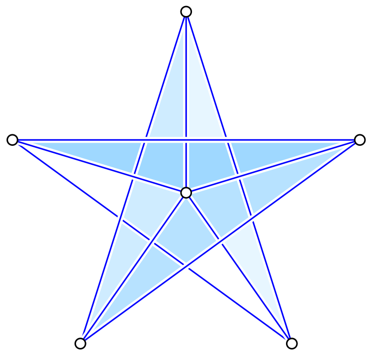

**Figure 3** Six points of a Delone set, in which one is surrounded by five others, and the remaining points lie outside the circumcircles defined by these points. The five 1-hefty triangles incident to the surrounded point cover any small neighborhood of that point twice, but they cannot be split into two disjoint covers of such a neighborhood.

**Proof.** The case $k = 0$ is that of the Delaunay triangulation of $A$. Its top-dimensional simplices cover $\mathbb{R}^d$ exactly once, and thus also every neighborhood of $a \in A$ exactly once. We therefore assume $k > 0$. Let $A' = A \setminus \{a\}$. Observe that $A'$ is also a generic thin Delone set and $a$ is generic with respect to $A'$. We can therefore apply Theorem 2.2 to the $(k-1)$-hefty simplices of $A'$ that contain $a \in \mathbb{R}^d$. There are only finitely many such simplices and each of them contains a small neighborhood of $a$.

Let $y$ be a point in a sufficiently small neighborhood of $a$ that is inside the intersection of all $(k-1)$-hefty simplices of $A'$ that contain $a$, and assume that $y$ is generic with respect to $A$. By Theorem 2.2, there are exactly $\binom{d+k}{d}$ $k$-hefty simplices of $A$ that cover $y$. They split into simplices incident and not incident to $a$. The latter are the $(k-1)$-hefty simplices of $A'$ that contain $a$. Indeed, since $y$ is sufficiently close to $a$, a $k$-hefty simplex of $A$ with vertices in $A'$ contains $a$ in its interior iff it contains $y$ in its interior. The circumsphere of each such simplex encloses exactly $k-1$ points of $A'$ as we removed $a$ from $A$ to get $A'$. This implies that the number of $k$-hefty simplices of $A$ that contain $y$ and are not incident to $a$ is $\binom{d+k-1}{d}$. Thus, the number of $k$-hefty simplices of $A$ that contain $y$ and are incident to $a$ is

$$\binom{d+k}{d} - \binom{d+k-1}{d} = \binom{d+k-1}{d-1}, \tag{1}$$

as claimed. ◀

Similar to Theorem 2.3, we establish an inequality for finite sets that follows from Theorem 2.4. We are satisfied with a less refined statement than formulated for the global case in Section 2.3.

▶ **Corollary 2.5.** *Let $A \subseteq \mathbb{R}^d$ be finite and generic and $k$ be a non-negative integer. Then every generic point in a small neighborhood of any point $a \in A$ belongs to at most $\binom{d+k-1}{d-1}$ $k$-hefty simplices of $A$ incident to $a$.*

# 3 Applications

We discuss two applications of Theorem 2.2: its relation to volumes of hypersimplices and Eulerian numbers in Section 3.1, and the connection to $k$-sets and $k$-facets in Section 3.2.

### 3.1 Worpitzky's Identity for Eulerian Numbers

As an application of our covering results, we show how the multiplicities from Theorem 2.2 are related to the volumes of hypersimplices and to Eulerian numbers. Specifically, we show that de Laplace's relation for hypersimplices [4] implies Worpitzky's identity for Eulerian numbers [14], and vice versa.

We begin by introducing the three main concepts we need in this subsection. Letting $d$ be a positive integer, the number of *descents* in a permutation $j_1, j_2, \ldots, j_d$ of $1, 2, \ldots, d$ is the number of indices $1 \leq i \leq d-1$ such that $j_i > j_{i+1}$. For $0 \leq k \leq d-1$, the *Eulerian number* for $d$ and $k$ is the number of permutations of $1, 2, \ldots, d$ with exactly $k$ descents. For example, $A(d, 0) = A(d, d-1) = 1$, for every $d$, and $\sum_{k=0}^{d-1} A(d, k) = d!$. Less obvious is Worpitzky's identity [14] between two polynomials from more than a century ago:

$$\sum_{k=0}^{d-1} A(d,k) \binom{x+k}{d} = x^d. \tag{2}$$

Next, let $x_0, x_1, \ldots, x_d$ be $d+1$ affinely independent points and write $\Delta = \operatorname{conv}\{x_0, x_1, \ldots, x_d\}$ for the $d$-simplex they span. A *$p$-fold sum* is obtained by selecting $p$ of the $d+1$ points, considering them as vectors, and returning the point that corresponds to the sum of these vectors. For each $1 \leq p \leq d$, the convex hull of the $p$-fold sums of the $d+1$ points is a $d$-dimensional convex polytope referred to as a *$d$-hypersimplex of order $p$*, denoted $\Delta_d^{(p)}$. Clearly, $\Delta_d^{(1)} = \Delta$, and more generally, $\Delta_d^{(p)}$ is a homothetic copy of the convex hull of the barycenters of all $(p-1)$-dimensional faces of $\Delta$. Since the barycenter of a $(p-1)$-simplex is $1/p$ times the sum of its vertices, the volume of that polytope is $1/p^d$ times the volume of the homothetic hypersimplex. Define the *relative volume* of $\Delta_d^{(p)}$ as $v(d,p) = \operatorname{vol}_d(\Delta_d^{(p)}) / \operatorname{vol}_d(\Delta)$, and observe that it does not depend on the choice of $\Delta$. Again more than a century ago, de Laplace [4] proved that these relative volumes are Eulerian numbers:

$$v(d, k+1) = A(d, k); \tag{3}$$

see also the combinatorial proof of the same equation by Stanley [11]. The third concept is the dual of the order-$n$ Voronoi tessellation of a finite set $A \subseteq \mathbb{R}^d$, introduced in 1990 by Aurenhammer [2]. Referring to this dual as the *order-$n$ Delaunay mosaic* of $A$, denoted $\operatorname{Del}_n(A)$, it is defined by its $d$-cells, each the convex hull of a collection of averages of $n$ points selected from $A$. Specifically, for each $1 \leq p \leq d$ and every $(n-p)$-hefty simplex, take all sets of cardinality $n$ that contain all $n-p$ points inside the circumsphere together with any $p$ points on the circumsphere. E.g. for $p = 1$, we get $d+1$ averages whose convex hull is a homothetic copy of the original $d$-simplex, and for $p = 2$, we get a homothetic copy of the convex hull of the midpoints of the edges of the $d$-simplex. Collecting these polytopes, we get $\operatorname{Del}_n(A)$. For $n = 1$, we have $\operatorname{Del}_1(A) = \operatorname{Del}(A)$, and more generally $\operatorname{Del}_n(A)$ has a $d$-cell for every $(n-p)$-hefty simplex in which $p$ varies from 1 to $d$. Since the vertices are averages of $n$ points, each $d$-cell in $\operatorname{Del}_n(A)$ has volume $1/n^d$ times the volume of the corresponding hypersimplex. It is now easy to prove the following relation for the relative volumes of the hypersimplices.

▶ **Theorem 3.1.** *For integers $d, n \geq 1$, the relative volumes of the hypersimplices satisfy*

$$\sum_{p=1}^{d} v(d,p) \binom{n+d-p}{n-p} = n^d, \tag{4}$$

*in which $\binom{n+d-p}{n-p} = 0$ whenever $n - p < 0$.*

**Proof.** Let $A$ be any Delone set – and not just a thin Delone set – in $\mathbb{R}^d$, so that every ball whose radius exceeds some given constant contains at least one point of $A$. Let $R > 0$ be sufficiently large and consider all $d$-cells in $\mathrm{Del}_n(A)$ that are contained in $[-R, R]^d$. Setting $n' = \max\{0, n-d\}$, there are $n - n' \leq d$ different types of $d$-cells to be considered, namely homothetic copies of hypersimplices of orders 1 to $n - n'$ defined by $(n-p)$-hefty simplices for $1 \leq p \leq n - n'$. The total volume of these $d$-cells is $(2R)^d + \mathcal{O}\left(R^{d-1}\right)$, since we miss only a constant width neighborhood of each facet of the hypercube.

Consider now an $(n-p)$-hefty simplex of $A$. Generically, its circumsphere passes through $d+1$ points and encloses $n-p$ of the points in $A$. By definition of Delaunay set, the radius of this circumsphere is bounded from above by a constant times $n-p$. Furthermore, the $(n-p)$-hefty simplex contains every $d$-cell in $\mathrm{Del}_n(A)$ it may determine since the vertices of the latter are averages of the vertices of the $(n-p)$-hefty simplex. By Theorem 2.2, the $(n-p)$-hefty simplices that define $d$-cells of $\mathrm{Del}_n(A)$ inside the hypercube therefore cover most of the hypercube exactly $\binom{n+d-p}{n-p}$ times. It follows that the total volume of these $p$-hefty simplices is $\binom{n+d-p}{n-p}(2R)^d + \mathcal{O}\left(R^{d-1}\right)$. By definition of relative volume, the total volume of the corresponding hypersimplices thus is $v(d,p)\binom{n+d-p}{n-p}(2R)^d/n^d + \mathcal{O}\left(R^{d-1}\right)$. Taking the sum for $1 \leq p \leq n - n'$, dividing by $(2R)^d$, and taking the limit as $R$ goes to infinity, we get the claimed relation. ◀

To see that Theorem 3.1 together with de Laplace's relation implies Worpitzky's identity and together with Worpitzky's identity implies de Laplace's relation, we change the summation index in (4) from $p$ to $k = d - p$ and apply $\binom{n+k}{n-d+k} = \binom{n+k}{d}$ to get

$$\sum\nolimits_{k=0}^{d-1} v(d, d-k)\binom{n+k}{d} = n^d. \tag{5}$$

Since this relation holds for every positive integer, $n$, it also holds if we treat $\binom{n+k}{d}$ and $n^d$ as polynomials of degree $d$ in $n$. Substituting $A(d,k) = A(d, d-k-1)$ for $v(d, d-k)$ using de Laplace's relation (3), we get Worpitzky's identity (2). To see the other direction, we observe that the polynomials given by the binomial coefficients are linearly independent, so there is only one way to write $n^d$ as their linear combination, and it is given by Worpitzky's identity. Comparing (5) with (2), we get $v(d, d-k) = A(d,k) = A(d, d-k-1)$, which is (3).

## 3.2 $k$-sets and $k$-facets

In this section, we briefly discuss connections between the covering numbers studied in Section 2 and the $k$-sets and $k$-facets studied in discrete geometry; see e.g. [8, Chapter 11]. Letting $A$ be a generic set of $n$ points in $\mathbb{R}^d$, a *$k$-set* is a set of $k$ points, $B \subseteq A$, such that $B$ and $A \setminus B$ can be separated by a hyperplane, and a *$k$-facet* is a set of $d$ points, $\Delta \subseteq A$, such that the hyperplane passing through these points partitions $A \setminus \Delta$ into $k$ and $n-d-k$ points on its two sides. We refer to [12, Section 2.2] for a discussion of the relation between these two concepts.

We present alternative proofs of two well-established results, which we state in terms of $k$-facets. Both proofs make use of the *inversion* of $A$ through the unit sphere centered at a point $x \in \mathbb{R}^d \setminus A$, which maps a point $a \in \mathbb{R}^d \setminus \{x\}$ to $\iota_x(a) = x + (a-x)/\|a-x\|^2$. It is not difficult to see that the image under this inversion of a hyperplane that avoids $x$ is a sphere that passes through $x$. Similarly, the image of the open half-space that does not contain $x$ is the open ball bounded by this sphere. If the hyperplane passes through the points of a $k$-facet, $\Delta \subseteq A$, and separates $x$ from the $k$-set on the other side, then the $d$-simplex spanned by $x$ and the points in $\iota_x(\Delta)$ is a $k$-hefty simplex of $A' = \iota_x(A) \cup \{x\}$ incident to $x$.

Write $F_k(A)$ for the number of $k$-facets of $A$. We first reprove the following 2-dimensional result by Alon and Györi [1] for $k$ smaller than a third of the number of points.

▶ **Proposition 3.2.** *Let $A$ be a generic set of $n$ points in $\mathbb{R}^2$ and $k < \frac{n}{3}$ a non-negative integer. Then $\sum_{j=0}^{k} F_j(A) \leq (k+1)n$.*

**Proof.** Recall that the $k$-hull of $A$ is the intersection of all closed half-spaces that miss at most $k$ points of $A$, denoted $H_k(A)$. By the Centerpoint Theorem of discrete geometry, $H_k(A)$ has a non-empty interior if $k < \frac{n}{3}$; see e.g. [6, Section 4.1]. Let $x$ be a point in the interior of $H_k(A)$ and set $A' = \iota_x(A) \cup \{x\}$. Let $j \leq k$. As explained above, the inversion through the unit circle centered at $x$ maps every $j$-facet of $A$ to a $j$-hefty triangle of $A'$ incident to $x$. By Corollary 2.5, the $j$-hefty triangles incident to $x$ cover a small neighborhood of $x$ at most $j+1$ times, so if we consider all $j$ between 0 and $k$, we get the neighborhood of $x$ covered at most $1 + 2 + \ldots + (k+1) = \binom{k+2}{2}$ times.

To continue, we draw a half-line emanating from $x$ through every point in $\iota_x(A)$, thus splitting the neighborhood of $x$ into $n$ angles. Every hefty triangle incident to $x$ covers a contiguous sequence of these angles, and for each $i \geq 1$, there are at most $n$ triangles that cover exactly $i$ of these angles. Each angle is covered some integer number of times, and we take the sum of these numbers over all angles. If $\sum_{j=0}^{k} F_j(A) > (k+1)n$, then this sum is strictly greater than $n(1 + 2 + \ldots + (k+1)) = n\binom{k+2}{2}$. This implies that one of the angles is covered more than $\binom{k+2}{2}$ times, which contradicts Corollary 2.5. ◀

We continue with Lovász Lemma – see [3] but also [8, Lemma 11.3.2] – which is a crucial ingredient in many arguments about $k$-sets; see e.g. [10]. This lemma gives an asymptotic upper bound on the number of $(d-1)$-simplices spanned by the points of $k$-facets a line can intersect. We give a short proof of a more general version by Welzl [13]. To state the lemma, we say a directed line, $L$, *enters* a $k$-facet, $\Delta \subseteq A$, if $L$ intersects the $(d-1)$-simplex spanned by $\Delta$ at an interior point and moves from the side with $k$ points to the side with $n-d-k$ points as it passes through the $(d-1)$-simplex.

▶ **Proposition 3.3.** *Let $A \subseteq \mathbb{R}^d$ be a generic finite set, $k$ a non-negative integer, and $L$ a directed line. Then $L$ enters at most $\binom{d+k-1}{d-1}$ $k$-facets of $A$.*

**Proof.** We may assume that $L$ passes through the convex hull of $A$ and let $x \in L$ be a point reached after passing through $\operatorname{conv} A$. The inversion through the unit sphere centered at $x$ maps every $k$-facet entered by $L$ to a $k$-hefty simplex incident to $x$ such that $L$ passes through this simplex before it reaches $x$. By Corollary 2.5, there are at most $\binom{d+k-1}{d-1}$ such $k$-hefty simplices. ◀

Since a line can be directed in two different ways, the number of $(d-1)$-simplices spanned by $k$-facets of the set $A$ one orientation of this line or the other can enter is at most $2\binom{d+k-1}{d-1}$.

## 4 Concluding Remarks

The results on covering numbers presented in Section 2 extend to other settings of which we mention three:

- locally finite sets in hyperbolic space;
- finite sets on the sphere;
- weighted points in Euclicean space.

A claim about global covering analogous to Theorem 2.2 holds in all three settings, while one about local covering analogous to Theorem 2.4 holds only for the first two. The first setting is most similar to the situation in Euclidean space studied in Section 2: the proofs are almost verbatim the same, with the main difference being a modified radial set since the design illustrated in Figure 2 is not thin Delone in hyperbolic space. For the second setting, there is an ambiguity in the definition of a $k$-hefty simplex since any $(d-1)$-sphere on the $d$-sphere bounds two complementary $d$-balls. We thus require that the finite set is *$k$-balanced*, by which we mean that every open hemisphere contains at least $k+1$ points. With this assumption, only the smaller of the two open $d$-balls has a chance to contain $k$ of the points, so we consider it the inside of the $(d-1)$-sphere. The proofs of the extensions of Theorems 2.2 and 2.4 to $k$-balanced sets on the sphere are similar to those for thin Delone sets in Euclidean space. Alternatively, we may use stereographic projection to reduce the spherical to the Euclidean setting.

The generalization to points with real weights is akin the generalization of Voronoi to power tessellations or diagrams; see e.g. [6, Chapter 13]. We interpret a point with weight $w \in \mathbb{R}$ as a sphere with squared radius $w$ (which may be negative), and we generalize the circumsphere of a $d$-simplex to the *orthosphere*, which in the generic case is the unique sphere orthogonal to the $d+1$ spheres that are the vertices of the $d$-simplex. The orthosphere *encloses* a weighted point if the two centers are strictly closer than required for the two spheres to be orthogonal to each other, and the $d$-simplex is *$k$-hefty* if it encloses exactly $k$ of the weighted points. With these adaptations of the definitions, a claim analogous to Theorem 2.2 holds also in the weighted case, provided the weights satisfy a mild condition, which is for example satisfied if the weights are bounded. On the other hand, Theorem 2.4 fails to extend. Indeed, the $k$-hefty simplices incident to a weighted point cover the neighborhood of the point some integer number of times, but depending on the point and its surrounding, this integer varies between 0 and $\binom{d+k}{d}$.

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