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Integral points on cubic surfaces: heuristics and numerics

Tim Browning [1] · Florian Wilsch [2]

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Abstract

We develop a heuristic for the density of integer points on affine cubic surfaces. Our heuristic applies to smooth surfaces defined by cubic polynomials that are log K3, but it can also be adjusted to handle singular cubic surfaces. We compare our heuristic to Heath-Brown's prediction for sums of three cubes, as well as to asymptotic formulae in the literature around Zagier's work on the Markoff cubic surface, and work of Baragar and Umeda on further surfaces of Markoff-type. We also test our heuristic against numerical data for several families of cubic surfaces.

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Contents

✉ Tim Browning
tdb@ist.ac.at

Florian Wilsch
florian.wilsch@mathematik.uni-goettingen.de

1 IST Austria, Am Campus 1, Klosterneuburg 3400, Austria

2 Göttingen University, Bunsenstrasse 3–5, Göttingen 37073, Germany

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1 Introduction

Let $U \subset \mathbb{A}^3$ be a cubic surface defined by an irreducible polynomial $f \in \mathbb{Z}[x, y, z]$ of degree 3, such that the surface is smooth over $\mathbb{Q}$. This paper develops heuristics for the expected asymptotic behaviour of the counting function

$$N_U(B) = \#\left\{(x, y, z) \in \mathbb{Z}^3 : \max\{|x|, |y|, |z|\} \leqslant B,\ f(x, y, z) = 0\right\},$$

as $B \to \infty$. A well-studied example is the cubic polynomial

$$f(x, y, z) = x^3 + y^3 + z^3 - k, \tag{1.1}$$

for non-zero $k \in \mathbb{Z}$. When k is cube-free, it has been conjectured by Heath-Brown [24] that $N_U(B) \sim c_k \log B$ for an appropriate constant c_k, which is positive if and only if $k \not\equiv \pm 4 \bmod 9$. Thus, in this example, $U(\mathbb{Z})$ is expected to be infinite if and only if $k \not\equiv \pm 4 \bmod 9$. For some values of k, this follows from the presence of parametric solutions. When $k = 1$, for example, the parameterisation

$$(9t^4)^3 + (3t - 9t^4)^3 + (1 - 9t^3)^3 = 1 \tag{1.2}$$

was discovered by Mahler [33] in 1936. In 1956, Lehmer [30] discovered an infinite family of parameterisations for the case $k = 1$. In general, however, even showing that $U(\mathbb{Z})$ is non-empty has proved very challenging. Indeed, for the cases $k = 33$ and $k = 42$, this has only recently been established by Booker [4] and Booker–Sutherland [5], respectively. In fact, Booker and Sutherland [5, Sec. 2A] also provide experimental evidence for Heath-Brown's conjecture by comparing $\sum_k N_U(B)$ with $\sum_k c_k \log B$, where the sum runs over cube-free integers $k \in [3, 1000]$ and B runs over the interval $[10^{7.5}, 10^{15}]$.

In some cases, the cubic surface admits a group action that renders an analysis of $N_U(B)$ more tractable. When $\mathrm{N}_{K/\mathbb{Q}}(x, y, z)$ is the norm form associated to a cubic extension $K/\mathbb{Q}$, the proof of Dirichlet's unit theorem allows one to study the counting function for the polynomial $f(x, y, z) = \mathrm{N}_{K/\mathbb{Q}}(x, y, z) - k$, for any non-zero $k \in \mathbb{Z}$. Assuming that $U(\mathbb{Z}) \neq \emptyset$, it follows from [47, Sec. 5] that

$$N_U(B) \sim c_k (\log B)^{r-1}, \tag{1.3}$$

for a suitable constant $c_k > 0$, where r is the number of infinite places in K.

The Markoff surface $U \subset \mathbb{A}^3$ is defined by the polynomial

$$f(x, y, z) = x^2 + y^2 + z^2 - 3xyz. \tag{1.4}$$

It follows from work of Zagier [49] that there exists a constant $c > 0$ such that $N_U(B) \sim c(\log B)^2$, as $B \to \infty$. For given $k \in \mathbb{Z}$, the arithmetic of the surfaces

$$x^2 + y^2 + z^2 - xyz = k \tag{1.5}$$

has been investigated deeply by Ghosh and Sarnak [21], who raise interesting questions about failures of the integral Hasse principle. In particular, it follows from [21, Thm. 1.2] that the integral Hasse principle fails for at least $\sqrt{K}(\log K)^{-\frac{1}{4}}$ integer coefficients $|k| \leqslant K$. These observations have been refined and put into the context of the Brauer–Manin obstruction by Loughran–Mitankin [31] and Colliot-Thélène–Wei–Xu [15]. In particular, the numerical evidence presented in [21, Conj. 10.2] for the density of Hasse failures is not wholly accounted for by the Brauer group. For the surfaces (1.5), an asymptotic formula of the shape $N_U(B) \sim c_{\mathrm{GS}}(\log B)^2$ can be deduced by taking $n = 4$ and $a = 1$ in recent work by Gamburd, Magee and Ronan [19, Thm. 3]. These surfaces are smooth when $k \notin \{0, 4\}$, providing many examples to compare with our heuristic.

The Markoff surface defined by (1.4) is singular and only fits into the scope of our heuristic after passing to a minimal desingularisation (as explained in Section 8). However, Baragar and Umeda [1] have shown how to adapt Zagier's argument to study $N_U(B)$ for surfaces $U \subset \mathbb{A}^3$ defined by the polynomial

$$f(x, y, z) = ax^2 + by^2 + cz^2 - dxyz - 1, \tag{1.6}$$

for $a, b, c, d \in \mathbb{N}$ such that $4abc - d^2 \neq 0$ and such that d is divisible by a, b and c. This surface is smooth over $\mathbb{Q}$. Moreover, U admits three non-commuting involutions defined over $\mathbb{Z}$, which are the so-called *Vieta involutions*. The induced action by the free product $(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})*(\mathbb{Z}/2\mathbb{Z})$ has finitely many orbits and so, as for (1.4), this can be used to study the set $U(\mathbb{Z})$ of integral points. The surfaces (1.6) generalise cubic surfaces considered by Mordell [35], and were first studied by Jin and Schmidt [29], who show $U(\mathbb{Z}) \neq \emptyset$ if and only if f is one of seven possibilities (up to permutation of the coefficients), with one of them being given by

$$f(x, y, z) = x^2 + by^2 + bz^2 - 2bxyz - 1, \tag{1.7}$$

for any $b \in \mathbb{N}$. This case is ignored, however, since the surface contains the line $x - 1 = y - z = 0$, which contributes at least $2B$ points to $N_U(B)$. Baragar and Umeda [1, Thm. 5.1] have shown that in each of the six remaining cases, there is a constant $c_{\mathrm{BU}} > 0$ such that

$$N_U(B) \sim c_{\mathrm{BU}}(\log B)^2, \tag{1.8}$$

as $B \to \infty$. The coefficient vectors for the six surfaces, together with a numerical value for c_{BU}, are presented in Table 1. In fact, the article [1, Sec. 4] contains a small oversight that affects the leading constant. The authors multiply their constant by 3 to account for negative coordinates, whereas it should be multiplied by 4: for each solution $(x, y, z) \in \mathbb{N}^3$, there are the three additional solutions $(x, -y, -z)$, $(-x, y, -z)$, and $(-x, -y, z)$. The same oversight applies to [1, Sec. 5] and the constants in our Table 1 are thus corrected by a factor of 4/3. It is worth highlighting that while Baragar and Umeda use the height $|x|+|y|+|z|$, rather than the sup-norm, this makes no difference to the leading term, since these norms are equivalent and the counting function grows logarithmically in B.

Table 1 Surfaces studied by Baragar and Umeda [1]

	a	b	c	d	c_{BU}
(i)	1	5	5	5	5.22750241554 ...
(ii)	1	3	6	6	2.96508393913 ...
(iii)	2	7	14	14	2.46790596426 ...
(iv)	2	2	3	6	4.05640933744 ...
(v)	6	10	15	30	2.49318310680 ...
(vi)	1	2	2	2	4.92081804684 ...

We are now ready to discuss our main heuristic, which comes from the circle method. Such heuristics are typically obtained by examining the major arc contribution, for a suitable set of major arcs, and ignoring the contribution from the minor arcs. This approach would suffice for surfaces with trivial Picard group, since then the associated singular series converges. However, for surfaces with non-trivial Picard group, such as the cubic surface $x^3 + ky^3 + kz^3 = 1$ considered in Section 6.2, the singular series diverges and the precise choice of major arcs would have a strong effect on the purported value of the leading constant. We shall avoid this difficulty by adopting a variant of the smooth δ-function version of the circle method originating in work of Duke, Friedlander and Iwaniec [18], and later developed by Heath-Brown [25, Thm. 1]. Once coupled with Poisson summation, the main idea is to ignore the contribution from the non-trivial characters, in order to obtain a heuristic for $N_U(B)$ for any cubic surface $U \subset \mathbb{A}^3$ that is smooth and log K3 over $\mathbb{Q}$. Here, we say that a smooth cubic surface $U_{\mathbb{Q}} \subset \mathbb{A}^3_{\mathbb{Q}}$ is *log K3* if the minimal desingularisation $\tilde{X}_{\mathbb{Q}}$ of the compactification $X_{\mathbb{Q}}$ of $U_{\mathbb{Q}}$ in $\mathbb{P}^3_{\mathbb{Q}}$ satisfies the property that the *boundary* $\tilde{D} = \tilde{X}_{\mathbb{Q}} \setminus U_{\mathbb{Q}}$ is a divisor with strict normal crossings whose class in Pic $\tilde{X}$ is $\omega^{\vee}_{\tilde{X}}$. In particular, it follows from the adjunction formula that U is log K3 if X itself is smooth over $\mathbb{Q}$ and $X \setminus U$ has strict normal crossings.

In general, it may happen that U contains $\mathbb{A}^1$-curves that are defined over $\mathbb{Z}$; for instance, this happens if any of the lines on $X_{\overline{\mathbb{Q}}} = X \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$ are defined over $\mathbb{Z}$, as in the example (1.7). It is therefore natural to try and classify those log K3 surfaces which admit infinitely many $\mathbb{A}^1$-curves, a programme that is already under way over $\overline{\mathbb{Q}}$, thanks to Chen and Zhu [12]. In the presence of $\mathbb{A}^1$-curves it is natural to study the subset $U(\mathbb{Z})^\circ$ obtained by removing those points in $U(\mathbb{Z})$ that are contained in any $\mathbb{A}^1$-curves defined over $\mathbb{Z}$, since we expect the contribution from integer points on these curves to dominate the counting function. As $B \to \infty$, this leads us to analyse the modified counting function

$$N_U^\circ(B) = \#\left\{(x, y, z) \in U(\mathbb{Z})^\circ : \max\{|x|, |y|, |z|\} \leqslant B\right\}. \tag{1.9}$$

We are now ready to reveal the main conjecture issuing from our investigation.

Conjecture 1.1 *Let $U \subset \mathbb{A}^3$ be a cubic surface that is smooth and log K3 over $\mathbb{Q}$ and that is defined by a cubic polynomial $f \in \mathbb{Z}[x, y, z]$. Denote by ϱ_U the Picard number of U over $\mathbb{Q}$ and by b the maximal number of components of $\tilde{D}(\mathbb{R})$ that share a real* *point. Then*

$$N_U^\circ(B) = O_U\left((\log B)^{\varrho_U+b}\right).$$

This resonates with a conjecture of Harpaz [23, Conj. 1.2], where an unspecified logarithmic growth is predicted for a certain class of log K3 surfaces. Conjecture 1.1 is the crudest conclusion that one can draw from our heuristic, which actually predicts an asymptotic formula for $N_U^\circ(B)$.

Heuristic 1.2 *Let $U \subset \mathbb{A}^3$ be a cubic surface satisfying the hypotheses of Conjecture 1.1, with Picard number ϱ_U, such that $U(\mathbb{Z})$ is not thin. Denote by b the maximal number of components of $\tilde{D}(\mathbb{R})$ that share a real point, and by $\mathcal{A}$ all such sets of cardinality b; for $A = \{\tilde{D}_1(\mathbb{R}), \dots, \tilde{D}_b(\mathbb{R})\} \in \mathcal{A}$, denote by $Z_A = \tilde{D}_1(\mathbb{R}) \cap \cdots \cap \tilde{D}_b(\mathbb{R})$ the corresponding intersection. Then*

$$N_U^\circ(B) \sim c_{\mathrm{h}}(\log B)^{\varrho_U+b},$$

as $B \to \infty$, with

$$c_{\mathrm{h}} = \gamma_U \cdot \tau_{U,H}(V), \tag{1.10}$$

where $\gamma_U \in \mathbb{Q}_{>0}$, $\tau_{U,H}$ is the Tamagawa measure induced by the standard height H, and V is the set of limit points of $U(\mathbb{Z})$ in $\bigcup_{A\in\mathcal{A}} Z_A \times U(\mathbf{A}_{\mathbb{Z}}^{\mathrm{fin}})$.

It is natural to impose the assumption that $U(\mathbb{Z})$ is Zariski dense in this heuristic. We have been led to further require that the set of integral points is not thin, since we do not expect the leading constant in (1.3) to agree with this heuristic and $U(\mathbb{Z})$ is thin in this case.

The Tamagawa measure $\tau_{U,H}$ is defined using residue measures at the archimedean place, as described in work of Chambert-Loir and Tschinkel [10, Secs. 2.1.9 and 2.1.12]. The set V of limit points can be studied via the Brauer–Manin obstruction, whose use to study integral points goes back to Colliot-Thélène and Xu [17]; Santens [41] has developed a variant that can also explain failures of accumulation phenomena at the infinite place. A further obstruction to approximation over $\mathbb{R}$ is analytic in nature, as expounded in work of Wilsch [48] and Santens [41]. For log Fano varieties whose Brauer group modulo constants is finite, it would follow from a conjecture of Santens [41, Conj. 6.6 and Thm. 6.11] and an equidistribution theorem of Chambert-Loir and Tschinkel [10, Prop. 2.10] that the algebraic Brauer–Manin obstruction is the only one. However, as observed in [15, 31] for the Markoff-type surfaces (1.5), the Brauer–Manin obstruction is not always sufficient for log K3 surfaces.

In order to illustrate our work, we state here a concrete conjecture for the polynomial $f(x, y, z) = x^3 + ky^3 + kz^3 - 1$, where $k > 1$ is square-free. We shall see in Section 6.2 that $b = 1$ and $\varrho_U = 2$ for the surface $U \subset \mathbb{A}^3$ defined by f.

Conjecture 1.3 *Let $U \subset \mathbb{A}^3$ be the cubic surface defined by $x^3 + ky^3 + kz^3 = 1$, for a square-free integer $k > 1$. Then Heuristic 1.2 holds with $\gamma_U = \frac{3}{8}$ and $V = D(\mathbb{R}) \times U(\mathbf{A}_{\mathbb{Z}}^{\mathrm{fin}})$.*

By adapting the parameterisation of Lehmer [30] to the setting of Conjecture 1.3, we are led to infinitely many $\mathbb{A}^1$-curves of increasing degree contained in U. We expect that a modification to work of Coccia [13] would yield analogues of his results for $k = 1$: the set of integer points on the Lehmer curves is thin (cf. [13, p. 371]), while its complement is not (cf. [13, Thm. 8]).

Summary of the article

Section 2

Formally speaking, our heuristic will involve the quantities

$$\int_{-\infty}^{\infty} \int_{[-B,B]^3} e(tf(x, y, z)) \mathrm{d}x \mathrm{d}y \mathrm{d}z \mathrm{d}t$$

and

$$\mu_\infty(B) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \operatorname{vol}\{\mathbf{x} \in [-B, B]^3 : |f(\mathbf{x})| < \varepsilon\},$$

both of which capture the real density of points on U. We shall introduce some hypotheses concerning the convergence properties of the oscillatory integral. Moreover, in Proposition 2.4 we shall apply work of Chambert-Loir and Tschinkel [10] to deduce that $\mu_\infty(B)$ grows like a power of $\log B$, as $B \to \infty$.

Section 3

This is the heart of our paper and concerns a circle method heuristic applied to $N_U(B)$. We shall derive an asymptotic expansion of the contribution from the trivial character, as $B \to \infty$, for a smoothly weighted variant of the counting function $N_U(B)$. This is achieved in Theorem 3.9, which will align with Conjecture 1.1. When $\varrho_U = 0$, we will arrive at a precise asymptotic prediction for $N_U(B)$ in Heuristic 3.11. Furthermore, we shall place Heuristic 1.2 in the context of the Manin conjecture for rational points on Fano varieties.

Section 4

We show that the exponent of $\log B$ in Heuristic 3.11 matches the asymptotic formula in (1.3).

Section 5

We demonstrate that Heuristic 3.11 matches the heuristic developed by Heath-Brown [24] for the sums of cubes example in (1.1), when k is cube-free.

Section 6

We shall adapt our work in Section 5 to develop a heuristic for the surface (1.1) when k is a cube, together with the cubic surface $x^3 + ky^3 + kz^3 = 1$, when $k > 1$ is a square-free integer. These surfaces will be seen to have Picard rank 3 and 2, respectively. In the former case we compare with Heuristic 1.2 when $k = 1$, using numerical data provided for us by Andrew Sutherland. For the second family of surfaces, we will gather numerical data for all square-free integer values of $2 \leqslant k \leqslant 1000$ and discuss Conjecture 1.3.

Section 7

We test Heuristic 3.11 against the asymptotic formula (1.8) of Baragar and Umeda. We will find that it correctly predicts the exponent of $\log B$, but that it fails to explain the leading constant. (Although we omit the details, similar arguments should go through for the surfaces (1.5).) In line with Heuristic 1.2, we shall modify the heuristic leading constant to take into account failures of strong approximation. All of the surfaces in Table 1 are equipped with a group action that makes it very efficient to test numerically for failures of strong approximation. In addition to uncovering failures coming from the Brauer–Manin obstruction, we will find failures of strong approximation that occur at infinitely many primes. In particular, we observe a failure of the *relative Hardy–Littlewood property*, as introduced by Borovoi and Rudnick [6, Def. 2.3]. On the other hand, we conduct a numerical investigation of equidistribution in Section 7.3, finding that the observed frequencies of reductions modulo m occur with the expected frequency, for various $m \in \mathbb{N}$. Nonetheless, it seems unlikely that Heuristic 1.2 is compatible with the numerical values occurring in (1.8).

Section 8

We extend our heuristic to the singular Markoff surface, as defined by the polynomial (1.4). We will find that the situation is similar to the examples of Baragar and Umeda in Section 7. However, while failures of strong approximation don't explain the leading constant, as in Section 7, such a modification does help to explain the power of $\log B$.

Section 9

We gather numerical evidence for two further cubic surfaces. The first is the cubic surface $U \subset \mathbb{A}^3$ defined by

$$(x^2 - ky^2)z = y - 1,$$

for a square-free integer $k > 1$. Under suitable assumptions, it has been shown by Harpaz [23] that $U(\mathbb{Z})$ is Zariski dense, prompting him to ask in [23, Qn. 4.4] about the exponent of $\log B$ in the associated counting function $N_U^\circ(B)$, after removing the $\mathbb{A}^1$-curve $z = y - 1 = 0$. We apply Heuristics 1.2 and 3.11 to deduce that the expected exponent is 2 and we gather numerical evidence for all square-free integers $2 \leqslant k \leqslant 1000$, which strongly supports this.

The second surface, which takes the shape

$$(ax + 1)(bx + 1) + (cy + 1)(dy + 1) = xyz,$$

for suitable $a, b, c, d \in \mathbb{Z}$, was also suggested to us by Harpaz (in private communication). This example will be seen to have non-trivial Picard group and contain several $\mathbb{A}^1$-curves that are defined over $\mathbb{Z}$. In this case, moreover, there are no involutions defined over $\mathbb{Q}$ and so we do not have a way to approach an asymptotic formula for the counting function, nor do we have an efficient way of enumerating integer points. By searching for points of height $\leqslant 10^{10}$, and identifying problematic $\mathbb{A}^1$-curves of degrees $1, \ldots, 4$, we find that the numerical data bears little resemblance to Heuristic 1.2.

Section 10

We offer some concluding remarks.

2 Archimedean densities

Let $f \in \mathbb{Z}[x, y, z]$ be a cubic polynomial and put

$$g(\mathbf{x}) = B^{-3} f(B\mathbf{x}), \tag{2.1}$$

where $\mathbf{x} = (x, y, z)$. Note that $g(\mathbf{x}) \ll 1$, where the implied constant is only allowed to depend on the coefficients of f. Given a compactly supported bounded function $w : \mathbb{R}^3 \to \mathbb{R}_{\geqslant 0}$, our work will feature the oscillatory integral

$$I(t) = \int_{\mathbb{R}^3} w(\mathbf{x}) e\left(t g(\mathbf{x})\right) \mathrm{d}\mathbf{x}, \tag{2.2}$$

for $t \in \mathbb{R}$. Note that $I(t)$ also depends on B, in view of the definition of g. In traditional applications of the circle method the real density often arises formally via the oscillatory integral

$$\sigma_\infty(B) = \int_{-\infty}^{\infty} I(t) \mathrm{d}t. \tag{2.3}$$

An alternative formulation is via the limit

$$\mu_\infty(B) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \operatorname{vol}\{\mathbf{x} \in [-B, B]^3 : |f(\mathbf{x})| < \varepsilon\}, \tag{2.4}$$

and it usually possible to prove that $\sigma_\infty(B)$ and $\mu_\infty(B)$ both converge to the same quantity, as $B \to \infty$. However, there are subtleties in the present setting and it will be convenient to build this into our assumptions.

Hypothesis 2.1 Assume that $\sigma_\infty(B)$ and $\mu_\infty(B)$ both converge, in the notation of (2.3) and (2.4). Then $\sigma_\infty(B) \sim \mu_\infty(B)$, as $B \to \infty$.

For the polynomial (1.1), we shall verify this hypothesis in Section 5.2.

2.1 Oscillatory integrals

Let us begin by discussing some properties and assumptions around the oscillatory integral $I(t)$ in (2.2) and the real density (2.3). To begin with, it is clear that $I(t)$ is infinitely differentiable and satisfies

$$I(t) \ll 1, \tag{2.5}$$

for any $t \in \mathbb{R}$, where the implied constant depends only on w. If w is a smooth function and $|\nabla f(\mathbf{x})| > 0$ throughout $\mathrm{supp}(w)$, then it is possible to establish exponential decay for $I(t)$, by using repeated applications of integration by parts. This would lead to a bound of the form

$$\int_{-\infty}^{\infty} |I(t)| \mathrm{d}t \ll 1,$$

which is the most favourable situation and underpins many applications of the circle method. When $\nabla f(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \in \mathrm{supp}(w)$, on the other hand, the situation is much more subtle, as indicated in works of Greenblatt [22] and Varchenko [46].

Example 2.2 It is instructive to consider the polynomial $f(x, y, z) = x^3 + y^3 + z^3 - k$, for non-zero $k \in \mathbb{Z}$. Taking $w(x, y, z) = \nu(x)\nu(y)\nu(z)$, where $\nu : \mathbb{R} \to \mathbb{R}$ is a smooth even bump function such that $\nu(x) = 1$ on $[-1, 1]$, it follows that

$$I(t) = e(-kt/B^3) R_\nu(t)^3,$$

where

$$R_\nu(t) = \int_{-\infty}^{\infty} \nu(x) e(tx^3) \mathrm{d}x.$$

The second derivative test [44, Lem. 4.4] yields $R_\nu(t) \ll \min\{1, |t|^{-1/3}\}$. Hence $I(t) \ll \min\{1, |t|^{-1}\}$ and we have

$$\int_{-\infty}^{\infty} |t|^{-\delta} |I(t)| \mathrm{d}t \ll_\delta 1,$$

for any $\delta \in (0, 1)$, where the implied constant depends on δ. We can get an asymptotic formula for $I(t)$, as $|t| \to \infty$, by noting that

$$R_\nu(t) = \int_{-\infty}^{\infty} e(tx^3) \mathrm{d}x - R_{1-\nu}(t).$$

The first term can be evaluated as

$$\int_{-\infty}^{\infty} e(tx^3) \mathrm{d}x = \frac{1}{|t|^{1/3}} \cdot \frac{\Gamma(\frac{1}{3})}{(2\pi)^{1/3}\sqrt{3}},$$

for any $t \in \mathbb{R}^*$. Since $1-\nu$ is smooth and supported on the region $\mathbb{R}^2 \setminus [-1,1]$, the second term is easily seen to be $O_N(|t|^{-N})$ on repeated integration by parts. Hence

$$I(t) = \frac{\Gamma(\frac{1}{3})^3}{6\pi\sqrt{3}} \cdot \frac{e(-kt/B^3)}{|t|} + O_N(|t|^{-N}). \tag{2.6}$$

This formula can be used to check that the integral

$$\int_{-\infty}^{\infty} \frac{I(t)}{(2+\pi^2 t^2)^{\frac{s+1}{2}}} \mathrm{d}t$$

is a holomorphic function of $s \in \mathbb{C}$ in the half-plane $\Re(s) \geqslant -1$.

Motivated by this example, our circle method heuristic will proceed under the following assumptions about $I(t)$.

Hypothesis 2.3 Let $I(t)$ be given by (2.2). Then the following hold:

(i) We have

$$\int_{-\infty}^{\infty} |t|^{-\delta} |I(t)| \mathrm{d}t \ll_\delta 1,$$

for any $\delta \in (0,1)$, where the implied constant depends only on w, f and δ.

(ii) The integral

$$\int_{-\infty}^{\infty} \frac{I(t)}{(2+\pi^2 t^2)^{\frac{s+1}{2}}} \mathrm{d}t$$

is a holomorphic function of $s \in \mathbb{C}$ in the half-plane $\Re(s) \geqslant -1$.

2.2 The real density

We now proceed to an analysis of $\mu_\infty(B)$, as defined in (2.4), using work of Chambert-Loir and Tschinkel [10, Thm. 4.7]. To begin with, we write

$$\mu_\infty(B) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \operatorname{vol} \left\{ P \in \mathbb{A}^3(\mathbb{R}) : H_\infty(P) \leqslant B,\ |f(x,y,z)| < \varepsilon \right\},$$

where $H_\infty(P) = \max\{|x|, |y|, |z|, 1\}$ for a real point $P = (x,y,z) \in \mathbb{A}^3(\mathbb{R})$. This is a volume on the surface U defined by f. Let $f_0(t_0, x_0, y_0, z_0)$ be the homogenisation of $f(x,y,z)$, with $x = x_0/t_0$, $y = y_0/t_0$, and $z = z_0/t_0$, so that $f = f_0/t_0^3$. Let $X = V(f_0)$ be the closure of U in $\mathbb{P}^3$, which we assume to be normal. Let $\varrho \colon \tilde{X} \to X$ be a minimal desingularisation. We shall assume that $\tilde{D} = \tilde{X} \setminus \tilde{U}$ has strict normal crossings and that U is log K3, so that the log canonical bundle $\omega_{\tilde{X}}(\tilde{D}) \cong \mathcal{O}_{\tilde{X}}$ is trivial. As a consequence of the adjunction formula, this condition is equivalent to $\varrho^* \mathcal{O}_X(D) \cong \mathcal{O}_{\tilde{X}}(\tilde{D})$, where $D = X \setminus U$; in particular, this is automatic if X is smooth.

The Leray form on U is a regular 2-form ω on U such that $\mathrm{d}f \wedge \omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. This allows us to write

$$\mu_\infty(B) = \int_{P \in U(\mathbb{R}),\ H_\infty(P) \leqslant B} |\omega| \,. \tag{2.7}$$

We shall endow certain line bundles with adelic metrics. On $\mathcal{O}_{\mathbb{P}^3}(d)$, for $d \in \mathbb{Z}$, consider the standard sup-norm

$$\|g(P)\|_v = \frac{|g(P)|_v}{\max\{|t_0|_v\,, |x_0|_v\,, |y_0|_v\,, |z_0|_v\}^d}, \tag{2.8}$$

where $P = (t_0 : x_0 : y_0 : z_0) \in \mathbb{P}^3(\mathbb{Q}_v)$ is a point over one of the local fields, and $g \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(d), U)$ is a local section. We have $\omega_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$ mapping $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ to t_0^{-4}. This induces a metric on $\omega_{\mathbb{P}^3}$ with

$$\|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\|_{\omega_{\mathbb{P}^3}} = \left\| t_0^{-4} \right\|_{\mathcal{O}_{\mathbb{P}^3}(-4)} = \max\{1, |x|\,, |y|\,, |z|\}^4,$$

after dividing numerator and denominator of (2.8) by t_0^{-4}. We have an isomorphism $\mathcal{O}_{\mathbb{P}^3}(X) \to \mathcal{O}_{\mathbb{P}^3}(3)$, mapping the canonical section 1_X to f_0, inducing an adelic metric on the former bundle. Now the adjunction isomorphism $\omega_X \to \omega_{\mathbb{P}^3}(X)|_X$ induces a metric on ω_X. We follow [10, Sec. 2.1.13] to get an explicit description. Consider the local equation $f \in \Gamma(\mathbb{A}^3, \mathcal{O}_{\mathbb{P}^3}(-X))$ of X. On this bundle, we have an adelic metric (induced by the one on $\mathcal{O}_{\mathbb{P}^3}(X)$), with

$$\|f\|_{\mathcal{O}_{\mathbb{P}^3}(-X)} \, \|1_X\|_{\mathcal{O}_{\mathbb{P}^3}(X)} = |f|\,,$$

since the product of the two sections on the left is f in $\mathcal{O}_{\mathbb{P}^3} \subset \mathcal{K}_{\mathbb{P}^3}$. As the adjunction isomorphism sends $\omega \mapsto \omega \wedge f^{-1}\,\mathrm{d}f$, we get

$$\begin{aligned} \|\omega\|_{\omega_X} &= \left\| \omega \wedge f^{-1}\,\mathrm{d}f \right\|_{\omega_{\mathbb{P}^3}(X)} \\ &= \|f\|^{-1}_{\mathcal{O}_{\mathbb{P}^3}(-X)} \, \|\omega \wedge \mathrm{d}f\|_{\omega_{\mathbb{P}^3}} \\ &= \frac{\|1_X\|_{\mathcal{O}_{\mathbb{P}^3}(X)}}{|f|} \, \|\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\|_{\omega_{\mathbb{P}^3}} \\ &= \frac{|f|}{\max\{1, |x|\,, |y|\,, |z|\}^3} \frac{1}{|f|} \max\{1, |x|\,, |y|\,, |z|\}^4 \\ &= \max\{1, |x|\,, |y|\,, |z|\}. \end{aligned} \tag{2.9}$$

Using all this, we can reformulate (2.7) as

$$\mu_\infty(B) = \int_{\{P \in X(\mathbb{R}) : \|t_0\|^{-1} \leqslant B\}} \|1_D\|^{-1} \, \frac{|\omega|}{\|\omega\|}.$$

As ϱ is crepant, the metric on ω_X can be pulled back to one on $\omega_{\tilde{X}}$, and the one on its dual bundle $\mathcal{O}_X(D)$ to $\mathcal{O}_{\tilde{X}}(\tilde{D})$ by the log K3 assumption. It follows that the above volume can be expressed as

$$\mu_\infty(B) = \int_{\{P \in \tilde{X}(\mathbb{R}) : \|\varrho^* t_0\|^{-1} \leqslant B\}} \left\|1_{\tilde{D}}\right\|^{-1} \frac{|\varrho^*\omega|}{\|\varrho^*\omega\|}$$

on the desingularisation: the exceptional set and its image are null sets and the integrands and height conditions coincide outside them. In the notation of [10, Sec. 4.2], we have

$$\left\|1_{\tilde{D}}\right\|^{-1} \frac{|\varrho^*\omega|}{\|\varrho^*\omega\|} = \mathrm{d}\tau_{(\tilde{X},\tilde{D})},$$

and an asymptotic expansion of this quantity is studied by means of its Mellin transform and a Tauberian theorem [10, Thm. 4.7]. In this analysis, Tamagawa measures on certain subsets of D arise naturally.

Let b be the maximal number of components of $\tilde{D}(\mathbb{R})$ that share a common real point, as in Conjecture 1.1. Note that if the set of integral points is Zariski dense, the set $U(\mathbb{R})$ of real points cannot be compact, so that $\tilde{D}(\mathbb{R}) \neq \emptyset$ and $b \geqslant 1$. Denote by $\mathcal{A}$ all sets of such components of cardinality b. For each $A \in \mathcal{A}$, let $Z_A = \bigcap_{D' \in A} D'$ be the intersection, which is a nonempty subset of $\tilde{D}(\mathbb{R})$ by assumption, and set $D_A = \sum_{D \in A} D$ and $\Delta_A = \sum_{D' \in \mathcal{A} \setminus A} D'$. In [10, Sec. 2.1.12], Chambert-Loir and Tschinkel define a *residue measure* τ_A on Z_A, which we normalise with a factor of 2^b as in [10, Sec. 4.1]. This measure depends on a metric on $\omega_{\tilde{X}}(D_A)$, but $\left\|1_{\Delta_A}\right\|^{-1} \tau_A$ does not, provided that the metrics on $\omega_{\tilde{X}}(D_A)$ and $\mathcal{O}_{\tilde{X}}(\Delta_A)$ are chosen so that their product coincides with the one on $\omega_{\tilde{X}}(D)$. Moreover, since the residue measure is finite and $\left\|1_{\Delta_A}\right\|$ is bounded from below on Z_A as a consequence of the maximality assumption and compactness, we get a finite volume

$$\mu_A = 2^b \int_{Z_A} \left\|1_{\Delta_A}\right\|^{-1} \mathrm{d}\tau_A. \tag{2.10}$$

We may now record the following asymptotic formula.

Proposition 2.4 *Under the above assumptions and notation, including that the set of integral points is Zariski dense, we have*

$$\mu_\infty(B) = c_\infty (\log B)^b + O\left((\log B)^{b-1}\right),$$

where

$$c_\infty = \frac{1}{b!} \sum_{A \in \mathcal{A}} \mu_A. \tag{2.11}$$

Proof This follows from [10, Thm. 4.7] with $d_\alpha = \lambda_\alpha = 0$ for all $\alpha \in \mathcal{A}$, whence $\sigma = 0$. Then [10, Eq. (4.3)] coincides with μ_A once evaluated at $s = 0$. □

In the next result we derive an explicit expression for c_∞ for certain polynomials featuring in our work.

Lemma 2.5 *Suppose that X is a smooth compactification of the affine surface defined by $f(x, y, z) = q(x, y, z) - dxyz$, where q is a quadratic polynomial. Then*

$$\sum_{A \in \mathcal{A}} \mu_A = \frac{12}{|d|},$$

so that $c_\infty = 6/|d|$ in (2.11).

Proof Write $q(x, y, z) = Q(x, y, z) + L(x, y, z) + e$ with Q and L homogeneous of degrees 2 and 1, respectively. Then X is defined by the cubic form

$$f_0 = dx_0y_0z_0 - Q(x_0, y_0, z_0)t_0 - L(x_0, y_0, z_0)t_0^2 - et_0^3.$$

In particular, the complement D of U in X is $V(dx_0y_0z_0)$, a union of three lines $D = L_1 + L_2 + L_3$. The Clemens complex associated with D is a triangle with three edges $\{L_i, L_j\}$, for $1 \leqslant i < j \leqslant 3$. Associated with each of these edges is a *residue measure* $\tau_{i,j}$ on $L_i \cap L_j$, this intersection consisting of only one rational point $P_{i,j}$. For the case $(i, j) = (1, 2)$ we have

$$L_1 = V(t_0, x_0), \quad L_2 = V(t_0, y_0), \quad P_{1,2} = (0 : 0 : 0 : 1).$$

We are interested in the norm $\left\| 1_{L_k} \right\|_{\omega_P}$ induced by the adjunction formula. Consider the affine chart around $P_{1,2}$ given by the coordinate functions $x' = x_0/z_0$, $y' = y_0/z_0$, and $t' = t_0/z_0$. In these coordinates, $P_{1,2} = (0, 0, 0)$ and X is cut out by

$$f' = f_0/z_0^3 = dx'y' - Q(x', y', 1)t' - L(x', y', 1)t'^2 - et'^3.$$

Note that the two partial derivatives $\mathrm{d}f'/\mathrm{d}x'$, $\mathrm{d}f'/\mathrm{d}y'$ vanish in $(0, 0, 0)$, so that

$$\frac{\mathrm{d}f'}{\mathrm{d}t'}(0, 0, 0) = Q(0, 0, 1) \neq 0,$$

by the smoothness assumption. Since f' is analytic, so is t' as a function of x' and y' by the implicit function theorem. Note that

$$f' = dx'y' - Q(0, 0, 1)t'(1 + O(x') + O(y')) + O(t'^2)$$

as a formal power series. Hence

$$t' = \frac{dx'y'}{Q(0, 0, 1)}(1 + O(x') + O(y')).$$

Now

$$\left\|1_{L_3}\right\| \left\|x'^{-1}\,\mathrm{d}x' \wedge y'^{-1}\,\mathrm{d}y\right\| = \left\|f'^{-1}\right\| \left\|\mathrm{d}x' \wedge \mathrm{d}y' \wedge \mathrm{d}f'\right\| \frac{\left\|1_{L_1}1_{L_2}1_{L_3}\right\|}{|x'y'|}, \quad (2.12)$$

by arguments similar to those appearing in the proof of Proposition 2.4. Analogously to there, $\left\|f'^{-1}\right\| = \max\{\left|t'\right|, \left|x'\right|, \left|y'\right|, 1\}^{-3} = 1 + O(x') + O(y')$. Note that

$$\mathrm{d}f' = (Q(0,0,1) + O(x') + O(y'))\,\mathrm{d}t' + f_1\,\mathrm{d}x + f_2\,\mathrm{d}y$$

for some f_1 and $f_2 \in \mathbb{Q}[x', y']$, so that

$$\begin{aligned}\left\|\mathrm{d}x' \wedge \mathrm{d}y' \wedge \mathrm{d}f'\right\| &= |Q(0,0,1)| \left\|\mathrm{d}x' \wedge \mathrm{d}y' \wedge \mathrm{d}t'(1 + O(x') + O(y'))\right\| \\ &= |Q(0,0,1)| \max\{\left|t'\right|, \left|x'\right|, \left|y'\right|, 1\}^4 (1 + O(x') + O(y')) \\ &= |Q(0,0,1)| + O(x') + O(y').\end{aligned}$$

Finally,

$$\begin{aligned}\left\|1_{L_1}1_{L_2}1_{L_3}\right\| = \|t_0\| &= \frac{|t_0|}{\max\{|t_0|, |x_0|, |y_0|, |z_0|\}} \\ &= \frac{\left|t'\right|}{\max\{|t'|, |x'|, |y'|, 1\}} \\ &= \left|\frac{dx'y'}{Q(0,0,1)}\right| (1 + O(x') + O(y')).\end{aligned}$$

Hence (2.12) becomes $|d| + O(x') + O(y')$. The integral (2.10) is over a single point and its value is simply the inverse of (2.12) evaluated at $P_{1,2} = (0,0)$ in the chosen chart. It still has to be renormalised by multiplying with $c_{\mathbb{R}}^2 = 4$, as in [10, Sec. 4.1]. The sum in (2.11) now runs over the three edges of the Clemens complex and each of the summands μ_A is equal to $4/\,|d|$, finishing the proof. □

3 A circle method heuristic

In this section we explore a heuristic based on the smooth δ-function version of the circle method due to Duke, Friedlander and Iwaniec [18]. This was developed and applied to quadratic forms by Heath-Brown [25] and put on an adelic footing by Getz [20] and Tran [45], in an effort to detect lower order terms. It is the latter approach that we shall adopt here. We begin, however, by analysing a certain Dirichlet series whose coefficients are complete exponential sums.

To fix notation, let $U = V(f) \subset \mathbb{A}^3_{\mathbb{Q}}$ and $\mathfrak{U} = V(f) \subset \mathbb{A}^3_{\mathbb{Z}}$ be the $\mathbb{Q}$-variety and $\mathbb{Z}$-scheme defined by our irreducible, cubic polynomial f. Throughout, we shall assume that U is smooth, and only briefly sketch a key difference of the singular case in Section 3.5. Denote by $\mathfrak{X}$ and X the closures of $\mathfrak{U}$ and U in $\mathbb{P}^3_{\mathbb{Z}}$ and $\mathbb{P}^3_{\mathbb{Q}}$, respectively, and assume that X is normal. If X is singular, some parts of our arguments will require us to pass to a minimal desingularisation $\tilde{X} \to X$, described by a sequence of blow-ups of $\mathbb{P}^3$. Let $\varrho\colon \tilde{\mathfrak{X}} \to \mathfrak{X}$ be the model described by the sequence of blow-ups in the closures of the centres. As U is smooth, these blow-ups keep it and its model $\mathfrak{U}$ invariant. Finally, denote by $\tilde{\mathfrak{D}} = \tilde{\mathfrak{X}} \setminus \mathfrak{U}$ and $\tilde{D} = \tilde{X} \setminus U$ the *boundary divisor*. If $\tilde{D}$ does not have strict normal crossings, we replace $\tilde{X}$ and $\tilde{\mathfrak{X}}$ by varieties arising as blow-ups with centres outside U that achieve this condition. Finally, let S be the set of primes of bad reduction of $\tilde{\mathfrak{X}}$.

3.1 Exponential sums and global L-functions

Let $e_q(\cdot) = \exp(\frac{2\pi i \cdot}{q})$, for any $q \in \mathbb{N}$. A key role in our work will be played by the Dirichlet series

$$F(s) = \sum_{q=1}^{\infty} q^{-s-3} \sum_{\substack{a \bmod q \\ \gcd(a,q)=1}} \sum_{\mathbf{b} \in (\mathbb{Z}/q\mathbb{Z})^3} e_q(af(\mathbf{b})), \tag{3.1}$$

for $s \in \mathbb{C}$ and a given cubic polynomial $f \in \mathbb{Z}[x, y, z]$. It is easy to see that $F(s)$ is absolutely convergent for $\Re(s) > 2$. In this section we shall relate $F(s)$ to an infinite Euler product involving the quantities

$$\nu(p^k) = \#\left\{\mathbf{x} \in (\mathbb{Z}/p^k\mathbb{Z})^3 : f(\mathbf{x}) \equiv 0 \bmod p^k\right\}, \tag{3.2}$$

for prime powers p^k. The following result is standard but we include its proof for the sake of completeness.

Lemma 3.1 *Assume that $\Re(s) > 2$. Then*

$$F(s) = \prod_p \sigma_p(s),$$

where

$$\sigma_p(s) = 1 + \sum_{k=1}^{\infty} \frac{1}{p^{ks}} \left(\frac{\nu(p^k)}{p^{2k}} - \frac{\nu(p^{k-1})}{p^{2(k-1)}} \right).$$

If $p \notin S$ then

$$\sigma_p(s) = 1 - \frac{1}{p^s} + \frac{\nu(p)}{p^{s+2}}. \tag{3.3}$$

Proof Since we are working with $s \in \mathbb{C}$ such that $\Re(s) > 2$, the infinite sum in $F(s)$ is absolutely convergent. Define the exponential sum

$$S_q = \sum_{\substack{a \bmod q \\ \gcd(a,q)=1}} \sum_{\mathbf{b} \in (\mathbb{Z}/q\mathbb{Z})^3} e_q(af(\mathbf{b})) = \sum_{\mathbf{b} \in (\mathbb{Z}/q\mathbb{Z})^3} c_q(f(\mathbf{b})),$$

for $q \in \mathbb{N}$, where $c_q(\cdot)$ is the Ramanujan sum. Then S_q is a multiplicative function of q and so we obtain an Euler product

$$F(s) = \prod_p \sigma_p(s),$$

where

$$\sigma_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^{k(s+3)}} \sum_{\mathbf{b} \bmod p^k} c_{p^k}(f(\mathbf{b})).$$

Let $a \in \mathbb{Z}$. At prime powers the Ramanujan sum takes the values

$$c_{p^k}(a) = \begin{cases} 0 & \text{if } p^{k-1} \nmid a, \\ -p^{k-1} & \text{if } p^{k-1} \mid a \text{ but } p^k \nmid a, \\ p^k - p^{k-1} & \text{if } p^k \mid a. \end{cases} \tag{3.4}$$

It follows that

$$\sigma_p(s) = 1 + \sum_{k=1}^{\infty} \frac{\nu(p^k) - p^2 \nu(p^{k-1})}{p^{ks+2k}},$$

as claimed in the first part of the lemma. Moreover, if U is smooth and $p \notin S$, then p is a prime of good reduction and Hensel's lemma yields $\nu(p^k) = p^{2(k-1)}\nu(p)$ for $k \geqslant 1$. The second part easily follows. □

Lemma 3.1 can be used to give a meromorphic continuation of $F(s)$, provided one has enough information about $\nu(p)$ for large primes p. Let $\tilde{X}_{\overline{\mathbb{Q}}} = \tilde{X} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ and let $\mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}})$ be the geometric Picard group of $\tilde{X}$. The global L-function that plays a role here is defined as an Euler product

$$\begin{aligned} L(s, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}})) &= \prod_{p<\infty} L_p(s, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}})), \\ L_p(s, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}})) &= \det\left(1 - p^{-s}\,\mathrm{Fr}_p \mid (\mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q})^{I_p}\right)^{-1}, \end{aligned} \tag{3.5}$$

where $\Re(s) > 1$, Fr_p is a geometric Frobenius element, and I_p is an inertia subgroup at p. Let $\varrho_{\tilde{X}}$ be the rank of the Picard group $\mathrm{Pic}(\tilde{X})$. Then, as described by Peyre [36, Sec. 2.1], $L(s, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))$ is an Artin L-function which has a meromorphic continuation to the whole complex plane, with a pole of order $\varrho_{\tilde{X}}$ at $s = 1$.

Bearing this notation in mind, we will need to examine $\nu(p)$ carefully. Note that

$$\nu(p) = \#\mathfrak{U}(\mathbb{F}_p) = \#\tilde{\mathfrak{X}}(\mathbb{F}_p) - \#\tilde{\mathfrak{D}}(\mathbb{F}_p).$$

As described by Manin [34, Thm. 23.1], a result of Weil yields

$$\frac{\#\tilde{\mathfrak{X}}(\mathbb{F}_p)}{p^2} = 1 + \frac{a_p(\tilde{X})}{p} + \frac{1}{p^2}, \tag{3.6}$$

where $a_p(\tilde{X})$ is the trace of the Frobenius element Fr_p acting on the Picard group $\mathrm{Pic}(\tilde{\mathfrak{X}}_{\overline{\mathbb{F}}_p})$, which is isomorphic to $\mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}})$ for almost all primes by [36, Lem. 2.2.1]. We note that $a_p(\tilde{X})$ is bounded independently of p, by Deligne's resolution of the Weil conjectures. Hence

$$\frac{\nu(p)}{p^2} = 1 + \frac{a_p(\tilde{X})}{p} - \frac{\#\tilde{\mathfrak{D}}(\mathbb{F}_p)}{p^2} + O(p^{-3/2}).$$

Returning to the Dirichlet series $F(s)$, it therefore follows from applying this in Lemma 3.1 that

$$\sigma_p(s) = 1 - \frac{1}{p^s} + \frac{\nu(p)}{p^{s+2}} = 1 + \frac{a_p(\tilde{X}) - \#\tilde{\mathfrak{D}}(\mathbb{F}_p)/p}{p^{s+1}} + O(p^{-\Re(s)-3/2}), \tag{3.7}$$

for any $p \notin S$. For sufficiently large primes, the Hasse–Weil bound implies that $\#\tilde{\mathfrak{D}}(\mathbb{F}_p) = O(p)$ and so $\sigma_p(s) = 1 + O(p^{-\Re(s)-1})$, for any $p \notin S$, and so $F(s)$ is an absolutely convergent Euler product for $\Re(s) > 0$.

We can relate the analytic properties of $F(s)$ to those of the global L-function introduced in (3.5). For $s \in \mathbb{C}$ with $\Re(s) > -1/2$ and sufficiently large primes, we find that

$$L_p(s, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))^{-1} = 1 - \frac{a_p(\tilde{X})}{p^s} + \frac{1}{p^{s+1}}.$$

Since $\Re(s) > -1/2$, we deduce from (3.7) that

$$\sigma_p(s) = L_p(s+1, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))\left(1 - \frac{\#\tilde{\mathfrak{D}}(\mathbb{F}_p)}{p^{s+2}}\right)\left(1 + O(p^{-\Re(s)-3/2})\right).$$

Let us define another Euler product

$$\zeta(s, \tilde{D}) = \prod_p \zeta_p(s, \tilde{D}), \quad \zeta_p(s, \tilde{D}) = \left(1 - \frac{\#\tilde{\mathfrak{D}}(\mathbb{F}_p)}{p^{s+1}}\right)^{-1},$$

for $\Re(s) > 1$. We will see in the proof of Proposition 3.2 that $\zeta(s, \tilde{D})$ has a meromorphic continuation to the region $\Re(s) > -1/2$ with a pole at $s = 1$. Thus, it will follow that there is a function $\tilde{F}(s)$ which is holomorphic in the half-plane $\Re(s) > -1/2$, such that

$$F(s) = L(s+1, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))\zeta(s+1, \tilde{D})^{-1}\tilde{F}(s). \tag{3.8}$$

An expression like this is essentially implied by work of Chambert-Loir and Tschinkel [10, Thm. 2.5] on convergence factors on adelic spaces, but we have chosen to include our own deduction for the sake of completeness and to deal explicitly with F as a function in s. (In particular, we have used factors associated with the easier zeta function $\zeta(s+1, \tilde{D})$, rather than $L(s+1, \mathrm{CH}^0(\tilde{D}_{\overline{\mathbb{Q}}}))$.)

Proposition 3.2 *Assume that $\mathfrak{U}(\mathbb{Z})$ is Zariski dense. Then the function $F(s)$ has a meromorphic continuation to the half-plane $\Re(s) > -1/2$ with a singularity at $s = 0$ of order ϱ_U. Moreover, letting $\sigma_p = \lim_{k\to\infty} p^{-2k}\nu(p^k)$, we have*

$$\lim_{s\to 0}\left(s^{\varrho_U}F(s)\right) = \lambda_0 \prod_p \lambda_p \sigma_p,$$

where

$$\lambda_0 = \lim_{s\to 0} s^{\varrho_U}\frac{L(s+1, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))}{\zeta(s+1, \tilde{D})} \quad \textit{and} \quad \lambda_p = \zeta_p(1, \tilde{D})L_p(1, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))^{-1}.$$

Proof Our starting point is the observation that $\sigma_p(0) = \sigma_p$ in Lemma 3.1. Recall that $L(s, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))$ has a pole of order $\varrho_{\tilde{X}}$ at $s = 1$. Moreover, we claim that $\zeta(s, \tilde{D})$ has a meromorphic continuation to the region $\Re(s) > -1/2$ with a pole of order $r_{\tilde{D}}$ at $s = 1$, where $r_{\tilde{D}}$ is the number of irreducible components of $\tilde{D}$ as a divisor over $\mathbb{Q}$. For this, for any $\sigma = \Re(s) > 1$, we take logarithms of both sides to obtain

$$\log\zeta(s, \tilde{D}) = \sum_p \frac{\#\tilde{\mathfrak{D}}(\mathbb{F}_p)}{p^{s+1}} + O_\sigma(1),$$

where the implied constant depends on σ. According to work of Serre [43, Cor. 7.13], we have

$$\sum_{p\leqslant x}\#\tilde{\mathfrak{D}}(\mathbb{F}_p) = \frac{r_{\tilde{D}}x}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

But then, for any $1 \leqslant y < x$, we may combine this with Abel summation to deduce that

$$\sum_{y<p\leqslant x}\frac{\#\tilde{\mathfrak{D}}(\mathbb{F}_p)}{p^{s+1}} = \frac{(s+1)r_{\tilde{D}}}{2}\int_y^x \frac{\mathrm{d}u}{u^s\log u} + O_\sigma\left(\frac{1}{\log y}\right).$$

Similarly, it follows from the prime number theorem that

$$\int_y^x \frac{\mathrm{d}u}{u^s\log u} = \frac{1}{s}\sum_{y<p\leqslant x}\frac{1}{p^s} + O_\sigma\left(\frac{1}{\log y}\right) = -\frac{1}{s}\sum_{y<p\leqslant x}\log\left(1 - \frac{1}{p^s}\right) + O_\sigma\left(\frac{1}{\log y}\right),$$

for $\sigma > 1$. Hence, we obtain $\zeta(s, \tilde{D}) = \zeta(s)^\alpha G(s)$ in the region $\sigma > 1$, where $G(s)$ is holomorphic in the region $\sigma > -1/2$ and $\alpha = (s+1)r_{\tilde{D}}/(2s)$. This therefore establishes the claim.

It follows from (3.8) that $F(s)$ has a meromorphic continuation to the region $\Re(s) > -1/2$ with a pole of order $\varrho_{\tilde{X}} - r_{\tilde{D}}$ at $s = 0$. If the set of integral points on $\mathfrak{U}$ is Zariski dense then, as explained in the proof of [48, Thm. 2.4.1(ii)], U cannot have invertible regular functions inducing a relation between the components of $\tilde{D}$ in $\mathrm{Pic}\,\tilde{X}$. Thus the left morphism in the localisation sequence $\mathrm{CH}_0(\tilde{D}) \to \mathrm{Pic}\,\tilde{X} \to \mathrm{Pic}\,U \to 0$ is injective and it follows that $\varrho_{\tilde{X}} - r_{\tilde{D}} = \varrho_U$. □

3.2 A smooth δ-function

We now come to record the version of the smooth δ-function that we shall use in our analysis. Let

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{Z} \text{ and } n \neq 0. \end{cases}$$

A smooth interpretation of this δ-function goes back to work of Duke, Friedlander and Iwaniec [18], but was developed for Diophantine equations by Heath-Brown [25, Thm. 1]. The version recorded below is essentially due to Tran [45], but we have elected to reprove it here, since Tran is missing a factor 2 in his statement.

Proposition 3.3 *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a Schwartz function satisfying the hypotheses*

(i) $\Phi(-x, -y) = \Phi(x, y)$ *for all* $x, y \in \mathbb{R}$,
(ii) $\Phi(x, 0) = 0$ *for all* $x \in \mathbb{R}$,
(iii) $\int_{-\infty}^{\infty} \Phi(0, y)\mathrm{d}y = 1$.

Then for any $n \in \mathbb{Z}$ and sufficiently large Q, there exists $c_Q > 0$ such that

$$\delta(n) = \frac{2c_Q}{Q} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \bmod q} e\left(\frac{an}{q}\right) h\left(\frac{n}{qQ}, \frac{q}{Q}\right),$$

where $h(x, y) = \Phi(x, y) - \Phi(y, x)$. Moreover, $c_Q = 1 + O_N(Q^{-N})$ for any $N \geqslant 1$.

Proof Since n/q runs over all divisors of n as q does, we are easily led to the expression

$$\sum_{\substack{q \in \mathbb{N} \\ q \mid n}} \left(\Phi\left(\frac{n}{qQ}, \frac{q}{Q}\right) - \Phi\left(\frac{q}{Q}, \frac{n}{qQ}\right) \right) = \delta(n) \sum_{q \in \mathbb{N}} \Phi\left(0, \frac{q}{Q}\right),$$

by (ii). We take Q large enough to ensure that the point $(0, 1/Q)$ is contained in the support of Φ, so that the right hand side doesn't vanish. It follows from (i) and Poisson summation that

$$\sum_{q \in \mathbb{N}} \Phi\left(0, \frac{q}{Q}\right) = \frac{1}{2} \sum_{q \in \mathbb{Z}} \Phi\left(0, \frac{q}{Q}\right) = \frac{1}{2} \sum_{c \in \mathbb{Z}} \int_{-\infty}^{\infty} \Phi\left(0, \frac{t}{Q}\right) e(-ct)\mathrm{d}t.$$

Part (iii) shows that the inner integral is Q when $c = 0$. On the other hand, repeated integration by parts shows that the integral is $O_N(Q(Q|c|)^{-N})$ when $c \neq 0$. Defining c_Q via

$$\sum_{q \in \mathbb{N}} \Phi\left(0, \frac{q}{Q}\right) = c_Q^{-1} \frac{Q}{2},$$

we deduce that $c_Q = 1 + O_N(Q^{-N})$ and

$$\delta(n) = \frac{2c_Q}{Q} \sum_{\substack{q \in \mathbb{N} \\ q \mid n}} h\left(\frac{n}{qQ}, \frac{q}{Q}\right).$$

The proposition follows on using additive characters to detect the condition $q \mid n$. □

The main difference between Proposition 3.3 and the version in Heath-Brown [25, Thm. 1] is that one has a sum over all additive characters, rather than just over primitive characters. We note that the function

$$\Phi(x, y) = \frac{e^{-x^2}(e^{-y^2} - e^{-2y^2})}{\sqrt{\pi}(1 - 2^{-1/2})} \tag{3.9}$$

is a Schwartz function that clearly satisfies the conditions (i)–(iii) in Proposition 3.3.

3.3 Application of the circle method

Let $w : \mathbb{R}^3 \to \mathbb{R}_{\geqslant 0}$ be a compactly supported smooth weight function. Rather than studying $N_U(B)$, we shall begin by considering the weighted counting function

$$N_U(B, w) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^3 \\ f(\mathbf{x}) = 0}} w(B^{-1}\mathbf{x}),$$

as $B \to \infty$. As is well-known, on assuming a reasonable dependence on w in all the error terms, it is possible to approximate the characteristic function of $[-B, B]^3$ by suitable weight functions to deduce the asymptotic behaviour of $N_U(B)$, as $B \to \infty$.

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be the function (3.9), which satisfies the hypotheses (i)–(iii) in Proposition 3.3. Let $h(x, y) = \Phi(x, y) - \Phi(y, x)$. Then it follows from this result that

$$N_U(B, w) = \frac{2c_Q}{Q} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \bmod q} \sum_{\mathbf{x} \in \mathbb{Z}^3} w(B^{-1}\mathbf{x}) e_q(af(\mathbf{x})) h\left(\frac{f(\mathbf{x})}{qQ}, \frac{q}{Q}\right),$$

where $c_Q = 1 + O_N(Q^{-N})$. Breaking the sum over $\mathbf{x}$ into residue classes modulo q and applying the 3-dimensional Poisson summation formula, one readily obtains

$$N_U(B, w) = \frac{2c_Q}{Q} \sum_{\mathbf{c} \in \mathbb{Z}^3} \sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{c}) \int_{\mathbb{R}^3} w(B^{-1}\mathbf{x}) h\left(\frac{f(\mathbf{x})}{qQ}, \frac{q}{Q}\right) e_q(-\mathbf{c}.\mathbf{x}) \mathrm{d}\mathbf{x},$$

where

$$S_q(\mathbf{c}) = \sum_{a \bmod q} \sum_{\mathbf{b} \bmod q} e_q(af(\mathbf{b}) + \mathbf{c}.\mathbf{b}).$$

Since $h(x, y)$ is a Schwartz function, we expect that only $q, \mathbf{x}$ with $q \ll Q$ and $f(\mathbf{x}) \ll Q^2$ make a dominant contribution. Moreover, the integrand is zero unless $w(B^{-1}\mathbf{x}) \neq 0$ and it can be shown that $|f(\mathbf{x})|$ has exact order of magnitude B^3 for typical such $\mathbf{x}$. In this way we are led to make the choice $Q = B^{3/2}$ in our analysis. (In fact, one can take $Q = cB^{3/2}$ for any constant $c > 0$ without affecting the heuristic main term, while taking $Q = B^\theta$ for $\theta > \frac{3}{2}$ would cause problems in the analysis of the oscillatory integral.)

Our circle method heuristic arises from asymptotically evaluating the contribution from the trivial character, corresponding to $\mathbf{c} = \mathbf{0}$. (In fact, there is evidence to suggest that the contribution from possible accumulating subvarieties is accounted for by the non-trivial characters, as discussed by Heath-Brown [26] for diagonal cubic surfaces in $\mathbb{P}^3$.) For our heuristic, we shall take $\mathbf{c} = \mathbf{0}$ and $c_Q = 1$, leaving us to estimate

$$M(B, w) = \frac{2}{Q} \sum_{q=1}^{\infty} q^{-4} S_q(\mathbf{0}) \int_{\mathbb{R}^3} w(B^{-1}\mathbf{x}) h\left(\frac{f(\mathbf{x})}{qQ}, \frac{q}{Q}\right) \mathrm{d}\mathbf{x}, \tag{3.10}$$

with $Q = B^{3/2}$. This will eventually be achieved in Theorem 3.9.

Let

$$D(s) = \sum_{q=1}^{\infty} q^{-s-4} S_q(\mathbf{0}),$$

for $s \in \mathbb{C}$. In view of the trivial bound $|S_q(\mathbf{0})| \leqslant q^4$, this is absolutely convergent for $\Re(s) > 1$. We proceed by proving the following result.

Lemma 3.4 *Let $\Re(s) > 1$. Then we have*

$$D(s) = F(s+1)\zeta(s+1),$$

where $F(s)$ is given by (3.1).

Proof Let $s \in \mathbb{C}$ such that $\Re(s) > 1$, so that $D(s)$ is absolutely convergent. Breaking the sum according to the greatest common divisor of a and q, we find that

$$D(s) = \sum_{q=1}^{\infty} q^{-s-4} \sum_{r \mid q} \sum_{\substack{a \bmod q \\ \gcd(a,q)=r}} \sum_{\mathbf{b} \bmod q} e_q(af(\mathbf{b})).$$

Making the change of variables $q = rq'$ and $a = ra'$, one concludes that

$$D(s) = \sum_{q'=1}^{\infty} \sum_{r=1}^{\infty} q'^{-s-4} r^{-s-4} \sum_{\substack{a' \bmod q' \\ \gcd(a',q')=1}} r^3 \sum_{\mathbf{b} \bmod q'} e_{q'}(a' f(\mathbf{b})).$$

The statement of the lemma is now obvious. □

An application of the Mellin inversion theorem yields

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} D(s)\int_0^\infty\int_{\mathbb{R}^3} w(B^{-1}\mathbf{x})h\left(\frac{f(\mathbf{x})}{yQ},\frac{y}{Q}\right)\mathrm{d}\mathbf{x}y^{s-1}\mathrm{d}y\mathrm{d}s$$
$$=\sum_{q=1}^\infty q^{-4}S_q(\mathbf{0})\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\int_0^\infty\int_{\mathbb{R}^3} w(B^{-1}\mathbf{x})h\left(\frac{f(\mathbf{x})}{yQ},\frac{y}{Q}\right)y^{s-1}\mathrm{d}\mathbf{x}\mathrm{d}y\right)\frac{\mathrm{d}s}{q^s}$$
$$=\sum_{q=1}^\infty q^{-4}S_q(\mathbf{0})\int_{\mathbb{R}^3} w(B^{-1}\mathbf{x})h\left(\frac{f(\mathbf{x})}{qQ},\frac{q}{Q}\right)\mathrm{d}\mathbf{x},$$

which we recognise as appearing in our expression for $M(B,w)$. Replacing y/Q by y and $\mathbf{x}$ by $B\mathbf{x}$, we obtain

$$\int_0^\infty\int_{\mathbb{R}^3} w(B^{-1}\mathbf{x})h\left(\frac{f(\mathbf{x})}{yQ},\frac{y}{Q}\right)y^{s-1}\mathrm{d}\mathbf{x}\mathrm{d}y$$
$$= B^3Q^s\int_0^\infty\int_{\mathbb{R}^3} w(\mathbf{x})h\left(\frac{g(\mathbf{x})}{y},y\right)y^{s-1}\mathrm{d}\mathbf{x}\mathrm{d}y,$$

where $g(\mathbf{x}) = Q^{-2}f(B\mathbf{x})$. (Note that this coincides with the definition (2.1), since $Q = B^{3/2}$.) Returning to (3.10), it follows from Lemma 3.4 that

$$M(B,w) = \frac{2Q^{-1}B^3}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s+1)\zeta(s+1)G(s)Q^s\mathrm{d}s, \tag{3.11}$$

where

$$G(s) = \int_0^\infty\int_{\mathbb{R}^3} w(\mathbf{x})h\left(\frac{g(\mathbf{x})}{y},y\right)y^{s-1}\mathrm{d}\mathbf{x}\mathrm{d}y.$$

We seek to obtain a meromorphic continuation of $G(s)$ sufficiently far to the left of the line $\Re(s) = 2$.

Let

$$k_t(y) = \int_{-\infty}^\infty h(x,y)e(-txy)\mathrm{d}x, \tag{3.12}$$

for any $t, y \in \mathbb{R}$. The following result evaluates this integral.

Lemma 3.5 *We have*

$$k_t(y) = \frac{1}{1-2^{-1/2}}\left(2^{-1/2}e^{-\frac{1}{2}y^2(2+\pi^2t^2)} - e^{-y^2(2+\pi^2t^2)}\right).$$

In particular $k_{\varepsilon_1 t}(\varepsilon_2 y) = k_t(y)$ *for* $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.

Proof On recalling that $h(x,y) = \Phi(x,y) - \Phi(y,x)$, in the notation of (3.9), it follows that $k_t(y) = M(t,y) - M^{\mathrm{sw}}(t,y)$ where

$$M(t,y) = \int_{-\infty}^\infty \Phi(x,y)e(-txy)\mathrm{d}x \quad \text{and} \quad M^{\mathrm{sw}}(t,y) = \int_{-\infty}^\infty \Phi(y,x)e(-txy)\mathrm{d}x.$$

It will be convenient to put $C = \sqrt{\pi}(1 - 2^{-1/2})$ in the proof, so that we may write $\Phi(x, y) = C^{-1}e^{-x^2}(e^{-y^2} - e^{-2y^2})$. Then

$$\begin{aligned} k_t(y) &= \frac{1}{C}\left(-e^{-2y^2}\int_{-\infty}^{\infty} e^{-x^2}e(-txy)\mathrm{d}x + e^{-y^2}\int_{-\infty}^{\infty} e^{-2x^2}e(-txy)\mathrm{d}x\right) \\ &= \frac{\sqrt{\pi}}{C}\left(-e^{-y^2(2+\pi^2t^2)} + 2^{-1/2}e^{-\frac{1}{2}y^2(2+\pi^2t^2)}\right), \end{aligned}$$

on completing the square and executing the integral over x. Substituting in the value of C completes the proof of the lemma. □

We may now assess the analytic properties of the function $G(s)$, in which it will be convenient to recall the definition (2.2) of $I(t)$.

Lemma 3.6 *Assume that Hypothesis 2.3 holds. Then $G(s)$ has a meromorphic continuation to the region $\Re(s) \geqslant -1$ with a simple pole at $s = -1$. Moreover, in this region we have*

$$G(s) = \frac{1}{2}\cdot\frac{2^{s/2}-1}{1-2^{-1/2}}\cdot\Gamma\left(\frac{s+1}{2}\right)R(s), \tag{3.13}$$

where

$$R(s) = \int_{-\infty}^{\infty} \frac{I(t)}{(2+\pi^2t^2)^{\frac{s+1}{2}}}\mathrm{d}t \tag{3.14}$$

is holomorphic.

Proof Let $\sigma = \Re(s) > 2$. In this region we have

$$G(s) = \int_0^{\infty}\int_{\mathbb{R}^3} w(\mathbf{x})h\left(\frac{g(\mathbf{x})}{y}, y\right)y^{s-1}\mathrm{d}\mathbf{x}\mathrm{d}y.$$

We denote by

$$H(t, y) = \int_{-\infty}^{\infty} h(x, y)e(-tx)\mathrm{d}x$$

the Fourier transform of $h(x, y)$ with respect to the first variable. Then the Fourier inversion theorem yields

$$\begin{aligned} \int_{\mathbb{R}^3} w(\mathbf{x})h\left(\frac{g(\mathbf{x})}{y}, y\right)\mathrm{d}\mathbf{x} &= \int_{-\infty}^{\infty} H(t, y)\left(\int_{\mathbb{R}^3} w(\mathbf{x})e\left(\frac{tg(\mathbf{x})}{y}\right)\mathrm{d}\mathbf{x}\right)\mathrm{d}t \\ &= y\int_{-\infty}^{\infty} H(yt, y)I(t)\mathrm{d}t, \end{aligned}$$

on replacing t/y by t and recalling the definition (2.2) of $I(t)$. But $H(yt, y) = k_t(y)$, in the notation of (3.12). Hence it follows from Lemma 3.5 that

$$\int_{\mathbb{R}^3} w(\mathbf{x})h\left(\frac{g(\mathbf{x})}{y}, y\right)\mathrm{d}\mathbf{x} = \frac{y}{1-2^{-1/2}}\sum_{i\in\{0,1\}}\frac{(-1)^{i+1}}{2^{i/2}}\int_{-\infty}^{\infty} I(t)e^{-\frac{1}{2^i}y^2(2+\pi^2t^2)}\mathrm{d}t.$$

Thus

$$G(s)=\frac{1}{1-2^{-1/2}}\sum_{i\in\{0,1\}}\frac{(-1)^{i+1}}{2^{i/2}}\int_0^\infty y^s\int_{-\infty}^\infty I(t)e^{-\frac{1}{2^i}y^2(2+\pi^2t^2)}\mathrm{d}t\mathrm{d}y.$$

Taking absolute values we see that

$$G(s)\ll\int_0^\infty y^\sigma e^{-y^2}J(y)\mathrm{d}y,$$

where

$$J(y)=\int_{-\infty}^\infty |I(t)|e^{-\frac{1}{2}\pi^2t^2y^2}\mathrm{d}t.$$

If $|ty|\leqslant 1$ then we take $e^{-\frac{1}{2}\pi^2t^2y^2}\ll 1$ and it follows from (2.5) that

$$\int_{|t|\leqslant 1/y}|I(t)|e^{-\frac{1}{2}\pi^2t^2y^2}\mathrm{d}t\ll y^{-1}.$$

When $|ty|\geqslant 1$ we may take $e^{-\frac{1}{2}\pi^2t^2y^2}\ll|ty|^{-1/2}$, and it follows from part (i) of Hypothesis 2.3 that

$$\int_{|t|>1/y}|I(t)|e^{-\frac{1}{2}\pi^2t^2y^2}\mathrm{d}t\ll y^{-1/2}\int_{-\infty}^\infty |t|^{-1/2}|I(t)|\mathrm{d}t\ll y^{-1/2}.$$

Hence

$$G(s)\ll\int_0^\infty y^{\sigma-1}e^{-y^2}\mathrm{d}y+\int_0^\infty y^{\sigma-1/2}e^{-y^2}\mathrm{d}y,$$

which is bounded for $\sigma>0$. Thus $G(s)$ is absolutely convergent in the region $\sigma>0$.

Working in the region $\sigma>0$, an application of Fubini's theorem allows us to interchange the order of integration. This leads to the expression

$$G(s)=\frac{1}{1-2^{-1/2}}\sum_{i\in\{0,1\}}\frac{(-1)^{i+1}}{2^{i/2}}\int_{-\infty}^\infty I(t)J_i(t,s)\mathrm{d}t,$$

where

$$\begin{aligned}J_i(t,s)&=\int_0^\infty y^s e^{-\frac{1}{2^i}y^2(2+\pi^2t^2)}\mathrm{d}y\\&=\frac{2^{i(s+1)/2}}{(2+\pi^2t^2)^{\frac{s+1}{2}}}\int_0^\infty y^s e^{-y^2}\mathrm{d}y\\&=\frac{2^{i(s+1)/2}}{(2+\pi^2t^2)^{\frac{s+1}{2}}}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{s+1}{2}\right).\end{aligned}$$

We may now execute the sum over $i \in \{0, 1\}$ and finally arrive at the expression for $G(s)$ recorded in the statement of the lemma. Since part (ii) of Hypothesis 2.3 ensures that $R(s)$ is holomorphic in the region $\sigma \geqslant -1$, this gives the desired meromorphic continuation of $G(s)$ to the region $\sigma \geqslant -1$. $\square$

We will need to understand the derivatives of the function (3.14) in the region $\Re(s) \geqslant -1$. Let $R^{(\ell)}(s)$ be the ℓth derivative with respect to s, for any integer $\ell \geqslant 0$. Then it follows that

$$R^{(\ell)}(s) = \frac{(-1)^\ell}{2^\ell} \int_{-\infty}^{\infty} \frac{I(t)(\log(2+\pi^2 t^2))^\ell}{(2+\pi^2 t^2)^{\frac{s+1}{2}}} \mathrm{d}t,$$

whence

$$R^{(\ell)}(-1) = (-1)^\ell \int_{-\infty}^{\infty} I(t)(\log \sqrt{2+\pi^2 t^2})^\ell \mathrm{d}t.$$

In the light of (2.5), the interval $[-2, 2]$ contributes $O(1)$ to the integral. On the other hand, when $|t| \geqslant 2$ we have

$$\log \sqrt{2+\pi^2 t^2} = \log |t| + \log \pi + O(|t|^{-2}),$$

since $\log(1 + \frac{2}{\pi^2 t^2}) = O(|t|^{-2})$. According to part (i) of Hypothesis 2.3, we conclude that

$$R^{(\ell)}(-1) = (-1)^\ell \int_{-\infty}^{\infty} I(t)(\log|t| + \log \pi)^\ell \mathrm{d}t + O_\ell(1).$$

The following result summarises our analysis of this function.

Lemma 3.7 *Assume Hypothesis 2.3 and let $\ell \geqslant 0$ be an integer. Then there is a monic degree ℓ polynomial $P \in \mathbb{R}[x]$ such that*

$$R^{(\ell)}(-1) = (-1)^\ell \int_{-\infty}^{\infty} I(t) P(\log|t|) \mathrm{d}t + O_\ell(1).$$

3.4 Conclusions and heuristics

It is now time to return to our expression (3.11) for $M(B, w)$, in order to record our main circle method heuristic. Assume Hypothesis 2.3 holds. If $Q = B^{3/2}$ then $Q^{-2} B^3 = 1$ and we find that

$$M(B, w) = 2 \cdot \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s+1)\zeta(s+1)G(s)Q^{s+1} \mathrm{d}s.$$

Proposition 3.2 implies that $F(s+1)$ has a meromorphic continuation to the region $\Re(s) > -\frac{3}{2}$ with a pole of order ϱ_U at $s = -1$. Moreover, Lemma 3.6 implies that

$G(s)$ has a meromorphic continuation to the region $\Re(s) \geqslant -1$ with a simple pole at $s = -1$. In addition to this, it follows from (3.13) that $G(0) = 0$, so that the integrand is holomorphic at $s = 0$. Overall, we conclude that in the region $\Re(s) \geqslant -1$ the function $F(s+1)\zeta(s+1)G(s)Q^s$ has a pole of order $\varrho_U + 1$ at $s = -1$ and is holomorphic everywhere else.

In the usual way the asymptotic behaviour of $M(B, w)$ is obtained by moving the line of integration to the left in order to capture the pole at $s = -1$. We shall not delve into details here, but content ourselves with recording the expected asymptotic formula

$$\begin{aligned} M(B, w) &\sim 2 \cdot \mathrm{Res}_{s=-1} \left(F(s+1)\zeta(s+1)G(s)Q^{s+1} \right) \\ &= \mathrm{Res}_{s=0} \left(\frac{2^{(s-1)/2} - 1}{1 - 2^{-1/2}} F(s)\zeta(s)\Gamma\left(\frac{s}{2}\right) R(s-1)Q^s \right), \end{aligned}$$

on making the substitution (3.13).

We recall that we have $F(s) = s^{-\varrho_U}\tilde{F}(s)$, for some function $\tilde{F}(s)$ which is holomorphic for $\Re(s) > -\frac{1}{2}$. Moreover, we have $\Gamma(\frac{s}{2}) = s^{-1}(2 + O(s))$. Let

$$U(s) = \frac{2^{(s-1)/2} - 1}{1 - 2^{-1/2}} \zeta(s) R(s-1) Q^s.$$

This is holomorphic for $\Re(s) \geqslant 0$. Taking the Taylor expansion about the point $s = 0$, we obtain

$$U(s) = U(0) + \frac{U'(0)}{1!}s + \frac{U''(0)}{2!}s^2 + \cdots + \frac{U^{(\varrho_U)}(0)}{\varrho_U!}s^{\varrho_U} + O(s^{\varrho_U+1}).$$

It therefore follows that

$$\mathrm{Res}_{s=0} \left(U(s)F(s)\zeta(s)\Gamma\left(\frac{s}{2}\right) \right) = \frac{2\tilde{F}(0)U^{(\varrho_U)}(0)}{\varrho_U!} + O\left(\max_{0 \leqslant \ell \leqslant \varrho_U - 1} |U^{(\ell)}(0)| \right).$$

At this point it is convenient to make another assumption about the asymptotic behaviour of the integral $I(t)$.

Hypothesis 3.8 Let $I(t)$ be given by (2.2) and define

$$J_\ell(B) = \int_{-\infty}^{\infty} I(t)(\log|t|)^\ell \mathrm{d}t, \tag{3.15}$$

for $\ell \geqslant 0$. Then $J_\ell(B) \ll_\ell (\log B)^{b+\ell}$, where b is defined in Conjecture 1.1 and the implied constant is allowed to depend on w, f and ℓ.

Under Hypotheses 2.3 and 3.8, it is clear from Lemma 3.7 and the Leibniz rule that

$$
\begin{aligned}
U^{(\ell)}(0) &= -\zeta(0) \sum_{j=0}^{\ell} \binom{\ell}{j} R^{(j)}(-1)(\log Q)^{\ell-j} + O\left((\log B)^{b+\ell-1}\right) \\
&= -\zeta(0) \cdot \ell! \sum_{j=0}^{\ell} \frac{(-1)^j J_j(B)(\log Q)^{\ell-j}}{j!(\ell-j)!} + O\left((\log B)^{b+\ell-1}\right),
\end{aligned}
$$

for any integer $\ell \geqslant 0$. Since $\zeta(0) = -\frac{1}{2}$ and $Q = B^{3/2}$, we therefore deduce the following result.

Theorem 3.9 *Under Hypotheses 2.3 and 3.8, the contribution from the trivial character is*

$$
M(B, w) = \lim_{s \to 0} \left(s^{\varrho_U} F(s)\right) \cdot r(B) + O((\log B)^{\varrho_U + b - 1}),
$$

where if $J_j(B)$ is given by (3.15) then

$$
r(B) = \sum_{j=0}^{\varrho_U} \frac{(-1)^j \left(\frac{3}{2}\right)^{\varrho_U - j} J_j(B)(\log B)^{\varrho_U - j}}{j!(\varrho_U - j)!}.
$$

According to Hypothesis 3.8, we have $r(B) = O((\log B)^{\varrho_U + b})$, which therefore accords with Conjecture 1.1. When $\varrho_U > 0$ the sum $r(B)$ features multiple terms, some of which have negative coefficients, but all with seemingly equal order of magnitude. This is very different to classical applications of the circle method. When $\varrho_U = 0$, however, the contribution from the trivial character is more straightforward. Thus, under Hypotheses 2.3 and 3.8, we obtain

$$
M(B, w) = F(0) \cdot J_0(B) + O((\log B)^{b-1}),
$$

where $J_0(B)$ is given by (3.15). In particular, we have $J_0(B) = \sigma_\infty(B)$, in the notation of (2.3). It follows from Proposition 3.2 that $F(0) = \prod_p \sigma_p$, where σ_p are the local densities. For our heuristic we shall suppose that the characteristic function of the region $[-B, B]^3$ is approximated by an appropriate compactly supported smooth weight function w. This leads to the following expectation.

Heuristic 3.10 *Let $U \subset \mathbb{A}^3$ be a smooth cubic surface that is log K3 over $\mathbb{Q}$ and that is defined by a cubic polynomial $f \in \mathbb{Z}[x, y, z]$. Assume that $\varrho_U = 0$. Then*

$$
N_U^\circ(B) \sim \prod_p \sigma_p \cdot \sigma_\infty(B),
$$

as $B \to \infty$, where $\sigma_\infty(B)$ is given by (2.3).

We may further refine this by supposing that Hypothesis 2.1 holds. Combining Heuristic 3.10 with Proposition 2.4, we are therefore led to the following expectation.

Heuristic 3.11 *Let $U \subset \mathbb{A}^3$ be a smooth cubic surface that is log K3 over $\mathbb{Q}$ and that is defined by a cubic polynomial $f \in \mathbb{Z}[x, y, z]$. Assume that $U(\mathbb{Z})$ is Zariski dense and $\varrho_U = 0$. Then*

$$N_U^\circ(B) \sim c_\infty \prod_p \sigma_p \cdot (\log B)^b,$$

as $B \to \infty$, where c_∞ is given by (2.11).

Later, in Section 6, we shall return to the heuristic suggested by Theorem 3.9 for some explicit cubic surfaces $U \subset \mathbb{A}^3$ with $\varrho_U > 0$. It remains to offer some justification for Heuristic 1.2, which is concerned with arbitrary $\varrho_U \geqslant 0$.

Analogy with Manin's conjecture

It is natural to draw comparisons with Manin's conjecture, which concerns the distribution of $\mathbb{Q}$-rational points on smooth, projective, Fano varieties V defined over $\mathbb{Q}$. A value for the leading constant in this conjecture has been suggested by Peyre [36, 37]. Let $H: V(\mathbb{Q}) \to \mathbb{R}_{\geqslant 0}$ be an anticanonical height function and define the counting function $N(Z; B) = \#\{P \in Z : H(P) \leqslant B\}$, for any subset $Z \subset V(\mathbb{Q})$. Then, as put forward in [37, Sec. 8], we expect there to exist a *thin set* $\Omega \subset V(\mathbb{Q})$ for which

$$N(V(\mathbb{Q}) \setminus \Omega; B) \sim c_V B(\log B)^{\varrho_V - 1},$$

as $B \to \infty$. (Note that rational points are much more prolific for Fano varieties than integral points are expected to be in the setting of log K3 surfaces.) The conjectured leading constant has the structure

$$c_V = \alpha_V \cdot \beta_V \cdot \tau_{V,H}\left(V(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}\, V}\right), \tag{3.16}$$

where $V(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}\, V}$ is the set of adelic points on V that are orthogonal to the Brauer–Manin pairing, and $\tau_{V,H}$ is the *Tamagawa measure* defined in [36, Sec. 2]. The constants α_V and β_V are rational numbers; the latter is the order of the Brauer group $\mathrm{Br}(V)/\mathrm{Br}(\mathbb{Q})$ and the former is the volume of a certain polytope in the dual of the effective cone of divisors, as defined in [36, Déf. 2.4]. In particular, if the Picard group is trivial and the height function is associated with q times a generator, then $\alpha_V = 1/q$.

Suppose that $V \subset \mathbb{P}^n$ is a smooth complete intersection of r degree d hypersurfaces, with $n \geqslant r + 2^{d-1}(d-1)r(r+1)$. Then it follows from work of Birch [3], which is proved using the circle method, that an asymptotic formula is available for $N(V(\mathbb{Q}); B)$, with $\alpha_V = 1/(n+1-rd)$ and $\beta_V = 1$, and where $\tau_{V,H}(V(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}\, V})$ is the usual product of local densities. However, there exist many Fano varieties V for which the full Manin–Peyre conjecture holds with $\beta_V \neq 1$ or with a more complicated expression for α_V. An example of the latter is provided by the smooth quartic del Pezzo surface (corresponding to $(d, n, r) = (2, 4, 2)$ in the above notation) studied by de la Bretèche and Browning [7], in which an asymptotic formula is obtained with $\alpha_V = 1/36$ and $\beta_V = 1$.

A refined heuristic

Returning to the setting of log K3 surfaces $U \subset \mathbb{A}^3$, as in Conjecture 1.1, we have seen that Theorem 3.9 suggests an asymptotic behaviour $N_U^\circ(B) \sim c_{\mathrm{h}}(\log B)^{\varrho_U+b}$, for a suitable constant c_{h}. Inspired by our discussion of Peyre's constant, we have been led to modify the product of local densities to account for failures of strong approximation, in addition to allowing for an unspecified positive rational factor. This has led us to the value for c_{h} proposed in (1.10), which concludes our discussion of Heuristic 1.2.

3.5 Singularities on *U*

We conclude by briefly remarking on the case of singular U. In this case, we still have $\sigma_p = \lim_{k\to\infty} p^{-2k}\nu(p^k)$ and we proceed by comparing this quantity to the analogous one defined on a minimal desingularisation. To this end, let $\varrho\colon \tilde{\mathfrak{X}} \to \mathfrak{X}$ and $\tilde{X} \to X$ be minimal desingularisations as before, but now without ϱ being an isomorphism above U and without the requirement that D have strict normal crossings. Define $\tilde{\nu}(p) = \#\tilde{\mathfrak{U}}(\mathbb{F}_p)$.

Lemma 3.12 *There is a finite set S of places (containing the archimedean one and those of bad reduction of $\tilde{\mathfrak{X}}$) such that*

$$\lim_{k\to\infty} \frac{\nu(p^k)}{p^{2k}} = \frac{\tilde{\nu}(p)}{p^2}$$

for all $p \notin S$.

Proof The minimal desingularisation is crepant, so that $\varrho^*\omega_X \cong \omega_{\tilde{X}}$. Moreover, this isomorphism spreads out to an isomorphism between $\varrho^*\omega_{\mathfrak{X}}$ and $\omega_{\tilde{\mathfrak{X}}}$, except possibly above a finite set of places. Let S be the union of these places, the places of bad reduction of $\tilde{\mathfrak{X}}$, and the archimedean place. Equip ω_X with a metric that is the model metric outside S, and $\omega_{\tilde{X}}$ with the pullback metric. After possibly enlarging S, this pullback is the model metric outside S. Let $p \notin S$ be a prime, and denote by τ_p and $\tilde{\tau}_p$ the resulting Tamagawa measures on $X(\mathbb{Q}_p)$ and $\tilde{X}(\mathbb{Q}_p)$, respectively, satisfying $\tau_p = \varrho_*\tilde{\tau}_p$. (Their definitions coincide on $X_{\mathrm{reg}} \cong \varrho^{-1}X_{\mathrm{reg}}$.)

We construct a countable disjoint covering $\mathcal{B}$ of $\mathfrak{U}(\mathbb{Z}_p) \cap X_{\mathrm{reg}}(\mathbb{Q}_p)$ as follows. Let $\mathbf{x} \in \mathfrak{U}(\mathbb{Z}_p) \cap X_{\mathrm{reg}}(\mathbb{Q}_p)$ be a $\mathbb{Z}_p$-point whose generic point is regular. Denote by $e_{\mathbf{x}}$ the minimal power of p annihilating the torsion of $H^0(\operatorname{Spec}\mathbb{Z}_p, \mathbf{x}^*\Omega_{\mathfrak{U}/\mathbb{Z}_p})$. Define

$$U_{\mathbf{x}} = \{\mathbf{x}' \in \mathfrak{U}(\mathbb{Z}_p) : \mathbf{x}' \equiv \mathbf{x} \bmod p^{e_{\mathbf{x}}}\}.$$

For fixed E, those $\mathbf{x}$ with $e_{\mathbf{x}} \leqslant E$ form a finite set of balls $\mathcal{B}_E = \{U_{\mathbf{x}_1}, \dots U_{\mathbf{x}_{s_E}}\}$; let $\mathcal{B} = \bigcup_{E=1}^{\infty} \mathcal{B}_E$ be their union.

For each of the $U_{\mathbf{x}}$, the arguments used by Salberger [40, Thm. 2.13] are applicable and show that

$$\tau_p(U_{\mathbf{x}}) = p^{-2e_{\mathbf{x}}} = \lim_{l\to\infty} \frac{\#\{\mathbf{x}' \in \mathfrak{U}(\mathbb{Z}/p^l\mathbb{Z}) : \mathbf{x}' \equiv \mathbf{x} \bmod p^{e_{\mathbf{x}}}\}}{p^{2l}}.$$

As l grows, apart from the at most finitely many singularities, eventually every point modulo p^l is counted this way; it follows that

$$\tau_p(\mathfrak{U}(\mathbb{Z}_p) \cap X_{\mathrm{reg}}(\mathbb{Q}_p)) = \sum_{U_{\mathbf{x}} \in \mathcal{B}} \tau_p(U_{\mathbf{x}}) = \lim_{l \to \infty} \frac{\nu(p^l)}{p^{2l}}.$$

According to [40, Cor. 2.15], we also get $\tilde{\tau}_p(\tilde{\mathfrak{U}}(\mathbb{Z}_p)) = \tilde{\nu}(p)/p^2$. Moreover, ϱ restricts to a measure-preserving homeomorphism $\varrho^{-1} X_{\mathrm{reg}}(\mathbb{Q}_p) \to X_{\mathrm{reg}}(\mathbb{Q}_p)$ by construction of the Tamagawa measures. As the complement $\varrho^{-1} X_{\mathrm{sing}}(\mathbb{Q}_p)$ of the former set is a null set, we obtain

$$\frac{\tilde{\nu}(p)}{p^2} = \tilde{\tau}_p(\tilde{\mathfrak{U}}(\mathbb{Z}_p) \cap \varrho^{-1} X_{\mathrm{reg}}(\mathbb{Q}_p)) = \tau_p(\mathfrak{U}(\mathbb{Z}_p) \cap X_{\mathrm{reg}}(\mathbb{Q}_p)) = \lim_{l \to \infty} \frac{\nu(p^l)}{p^{2l}},$$

as claimed. □

As a consequence of this result, we deduce that

$$\prod_{p \notin S} \frac{L_p(1, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))}{\zeta(1, \tilde{D})} \sigma_p = \prod_{p \notin S} \frac{L_p(1, \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}}))}{\zeta(1, \tilde{D})} \frac{\tilde{\nu}(p)}{p^2}$$

is absolutely convergent, suggesting an asymptotic behaviour

$$N_U^\circ(B) \sim c(\log B)^{\varrho_{\tilde{U}} + b},$$

for a suitable constant c.

4 Norm form equations

Let $K/\mathbb{Q}$ be a cubic number field and let $k \in \mathbb{Z}$ be non-zero. Let $\mathfrak{U} \subset \mathbb{A}^3_{\mathbb{Z}}$ be the smooth cubic surface defined by the polynomial $f(x, y, z) = \mathrm{N}_{K/\mathbb{Q}}(x, y, z) - k$, where $\mathrm{N}_{K/\mathbb{Q}}(x, y, z)$ is the *norm form* associated to the number field, and let U be its generic fibre. Since $U_{\overline{\mathbb{Q}}}$ is a torus over $\overline{\mathbb{Q}}$, it is an open subset of affine space over $\overline{\mathbb{Q}}$. The Picard group of affine space vanishes and so it follows that the geometric Picard group of U vanishes. Thus $\varrho_U = 0$, since the Picard group $\mathrm{Pic}(U)$ is a subgroup of the geometric Picard group.

We proceed by showing that the exponent $r - 1$ in (1.3) agrees with the exponent of $\log B$ in Conjecture 1.1 and Heuristic 3.11. Observe that the divisor D is given by $V(\mathrm{N}_{K/\mathbb{Q}}(x_0, y_0, z_0))$. If K is totally real, then $b = 2$. On the other hand, if K has a complex embedding, then $b = 1$. Thus $b = r - 1$, and so the exponents of $\log B$ do indeed match. We do not expect the leading constant in (1.3) to agree with the leading constant in Heuristic 3.11, since $U(\mathbb{Z})$ is a thin set.

5 Sums of cubes: rank zero

In this section, we specialise to the smooth cubic surface $U \subset \mathbb{A}^3$ defined by the polynomial $f(x, y, z) = x^3 + y^3 + z^3 - k$, for an integer $k \not\equiv \pm 4 \bmod 9$. Our main aim is to check that Heuristic 3.11 aligns with the prediction worked out by Heath-Brown [24, p. 622] when k is cube-free.

The compactification $X \subset \mathbb{P}^3$ is the smooth cubic surface defined by the polynomial $f_0 = x_0^3 + y_0^3 + z_0^3 - kt_0^3$. The divisor D is the smooth genus 1 curve $V(x_0^3 + y_0^3 + z_0^3)$. In particular, D is geometrically irreducible and we have $b = 1$ in the notation of Conjecture 1.1. We claim that

$$\varrho_U = \begin{cases} 0 & \text{if } k \text{ is not a cube,} \\ 3 & \text{if } k \text{ is a cube.} \end{cases} \tag{5.1}$$

Since $\varrho_U = \varrho_X - 1$, it will suffice to calculate ϱ_X. When k is a cube the surface X is $\mathbb{Q}$-isomorphic to the surface $x_0^3 + y_0^3 + z_0^3 + t_0^3 = 0$. But then it follows from [39, Prop. 6.1] that $\varrho_X = 4$. When k is not a cube it follows from work of Segre [42, Thm. IX] that $\varrho_X = 1$. This establishes the claim.

5.1 Non-archimedean densities

For any prime p the relevant p-adic density is $\sigma_p = \lim_{\ell \to \infty} p^{-2\ell} \nu(p^\ell)$, where

$$\nu(p^\ell) = \#\{(x, y, z) \in (\mathbb{Z}/p^\ell\mathbb{Z})^3 : x^3 + y^3 + z^3 \equiv k \bmod p^\ell\}.$$

Heath-Brown [24, p. 622] has calculated these explicitly when k is cube-free, beginning with

$$\sigma_3 = \frac{\nu(27)}{27^2}. \tag{5.2}$$

Moreover,

$$\sigma_p = \begin{cases} 1 & \text{if } p \equiv 2 \bmod 3 \text{ and } p \nmid k, \\ 1 - \frac{1}{p^2} & \text{if } p \equiv 2 \bmod 3 \text{ and } p \mid k. \end{cases} \tag{5.3}$$

On the other hand, if $p \equiv 1 \bmod 3$, let a_p, b_p the unique choice of integers such that $4p = a_p^2 + 27b_p^2$, with $a_p \equiv 1 \bmod 3$ and $b_p > 0$. Define

$$c_p(k) = \begin{cases} 2 & \text{if } k^{(p-1)/3} \equiv 1 \bmod p, \\ -1 & \text{if } k^{(p-1)/3} \not\equiv 1 \bmod p. \end{cases} \tag{5.4}$$

Then

$$\sigma_p = \begin{cases} 1 + \frac{3c_p(k)}{p} - \frac{a_p}{p^2} & \text{if } p \equiv 1 \bmod 3 \text{ and } p \nmid k, \\ 1 + \frac{(p-1)a_p - 1}{p^2} & \text{if } p \equiv 1 \bmod 3 \text{ and } p \mid k. \end{cases} \tag{5.5}$$

When k is not cube-free it is still possible to calculate explicit expressions for σ_p, but we have chosen not to do so. However, if $p \nmid k$ then (5.2), (5.3) and (5.5) remain true.

5.2 Archimedean density

We begin by discussing the integral $I(t)$ in (2.2) in the special case (1.1). We already saw in Example 2.2 that Hypothesis 2.3 holds in this case. In the development of Heuristic 3.11 we introduced Hypothesis 3.8, which concerns the asymptotic behaviour of the integral $J_\ell(B)$, as defined in (3.15). The following result confirms this hypothesis, since $b = 1$ in this setting.

Lemma 5.1 *Let $\ell \geqslant 0$ be an integer. Then*

$$J_\ell(B) = \kappa_\ell (\log B)^{\ell+1} + O((\log B)^\ell),$$

where

$$\kappa_\ell = \frac{3^\ell}{\ell+1} \cdot \frac{\Gamma(\frac{1}{3})^3}{\pi\sqrt{3}}.$$

Proof Combining (2.5) with (2.6) in (3.15), we readily obtain

$$J_\ell(B) = \frac{\Gamma(\frac{1}{3})^3}{3\pi\sqrt{3}} \int_2^\infty \frac{\cos(2\pi t/B^3)(\log t)^\ell}{t} \mathrm{d}t + O_\ell(1).$$

We have $\cos(\theta) = 1 + O(\theta^2)$ for $|\theta| \leqslant 1$. Hence

$$\begin{aligned}\int_2^{B^3/(2\pi)} \frac{\cos(2\pi t/B^3)(\log t)^\ell}{t} \mathrm{d}t &= \int_2^{B^3/(2\pi)} \frac{(\log t)^\ell}{t} \mathrm{d}t + O_\ell((\log B)^\ell) \\ &= \frac{3^{\ell+1}}{\ell+1} \cdot (\log B)^{\ell+1} + O_\ell((\log B)^\ell).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{B^3/(2\pi)}^\infty \frac{\cos(2\pi t/B^3)(\log t)^\ell}{t} \mathrm{d}t &= \int_1^\infty \frac{\cos(t)(\log(B^3 t/(2\pi)))^\ell}{t} \mathrm{d}t + O_\ell((\log B)^\ell) \\ &= O_\ell((\log B)^\ell),\end{aligned}$$

since

$$\int_1^\infty \frac{\cos(t)(\log t)^j}{t} \mathrm{d}t \ll_j 1,$$

for any $j \geqslant 0$. Putting these estimates together yields the lemma. □

Taking $\ell = 0$, it follows from this result that

$$\int_{-\infty}^\infty I(t)\mathrm{d}t \sim \frac{\Gamma(\frac{1}{3})^3}{\pi\sqrt{3}} \cdot \log B.$$

On the other hand, it follows from Proposition 2.4 that $\mu_\infty(B) \sim \mu_D \log B$, where μ_D is defined in (2.10). Although we omit details, one can adopt the argument in (2.9) to prove that

$$\mu_D = \frac{\Gamma(\frac{1}{3})^3}{\pi\sqrt{3}}. \tag{5.6}$$

Thus Hypothesis 2.1 is also true in this case.

5.3 Application of the heuristic

We have already seen in (1.2) that the surface U can contain $\mathbb{A}^1$-curves over $\mathbb{Z}$, depending on the choice of k. (In fact, the $\mathbb{A}^1$-curves of degree at most 4 have all been identified by Segre [42, Thm. XII].) Thus, we let $N_U^\circ(B)$ be the counting function defined in (1.9), where $U(\mathbb{Z})^\circ$ is obtained by removing those points in $U(\mathbb{Z})$ that are contained in any such curve. We are now ready to reveal what our heuristic says about $N_U^\circ(B)$ when k is not a cube, so that $\varrho_U = 0$ and $b = 1$ in Heuristic 3.10. On applying Lemma 5.1, we are therefore led to the following expectation, which fully recovers Heath-Brown's heuristic [24].

Heuristic 5.2 *Let $k \in \mathbb{Z}$ be a non-cube that is not congruent to $\pm 4 \bmod 9$. Let $U \subset \mathbb{A}^3$ be the cubic surface defined by (1.1). Then*

$$N_U^\circ(B) \sim \frac{\Gamma(\frac{1}{3})^3}{\pi\sqrt{3}} \cdot \prod_p \sigma_p \cdot \log B,$$

as $B \to \infty$. Explicit expressions for σ_p are given by (5.2)–(5.5) when k is cube-free.

When k is not a cube, the arithmetic of U has been studied by Colliot-Thélène and Wittenberg [16], with the aim of understanding the effect of the Brauer group on the integral Hasse principle. In this setting, it follows from [16, Props. 2.1 and 3.1] that $\operatorname{Br}(U)/\operatorname{Br}(\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$. Although it is found in [16, Thm. 4.1] that there is no obstruction to the Hasse principle, it can certainly happen that the Brauer group obstructs strong approximation. When $k = 3$, for example, it was discovered by Cassels [9] that any point $(x, y, z) \in U(\mathbb{Z})$ must satisfy $x \equiv y \equiv z \bmod 9$. (This is explained by the Brauer–Manin obstruction in [16, Remark 5.7].) In general, for any cube-free $k \in \mathbb{Z}$, let $\mathcal{A}$ be a non-trivial class of the Brauer group from [16, Prop. 2.1]. For any prime $p \neq 3$ such that $v_p(k) = 0$, the evaluation of $\mathcal{A}$ at any local point at p is equal to 0 since both the surface and the class $\mathcal{A}$ have good reduction at such primes. Thus there is no obstruction to strong approximation at these primes. On the other hand, it follows from [16, Prop. 4.6] that there is an obstruction to strong approximation at any prime $p \neq 3$ for which $v_p(k) \in \{1, 2\}$ and that these are the only obstructions. It is natural to expect that the the local factors σ_p in Heuristic 5.2 should be modified to take into account the possible failures of strong approximation that occur when $p \mid 3k$ and we conjecture that Heuristic 1.2 holds with $\gamma_U = 3$ and $V = (D(\mathbb{R}) \times U(\mathbf{A}_{\mathbb{Z}}^{\mathrm{fin}}))^{\operatorname{Br} U}$, where the pairing with $\operatorname{Br}(U) \cong \operatorname{Br}(X)$ is the restriction of the usual Brauer–Manin pairing on $X(\mathbf{A}_{\mathbb{Q}})$. In their work [5, Sec. 2A], Booker and Sutherland have provided numerical evidence that the constant in Heuristic 5.2 is correct on average, and so we expect that Brauer–Manin obstruction cuts out $\frac{1}{3}$ of the adelic points, as for the case $k = 3$.

6 Sums of cubes: higher rank

We proceed by investigating $N_U^\circ(B)$ for the polynomial (1.1) when k is a cube, and secondly, for the polynomial $f(x, y, z) = x^3 + ky^3 + kz^3 - 1$ when $k > 1$ is a square-free integer. In both cases we have $b = 1$. We shall find that $\varrho_U = 3$ in the former case and $\varrho_U = 2$ in the latter.

6.1 Representations of a cube as a sum of three cubes

We begin by studying the cubic surface $U \subset \mathbb{A}^3$ defined by (1.1) when k is a cube, having already seen in (5.1) that $\varrho_U = 3$. Thus it follows from Conjecture 1.1 that $N_U^\circ(B) = O((\log B)^4)$. It is natural to appeal to Theorem 3.9 in order to get an analogue of Heuristic 5.2 for the case that k is a cube. On returning to the setting of Proposition 3.2, we have $\zeta(s, D) = \zeta(s)$. Moreover, if $X \subset \mathbb{P}^3$ is the compactification of U, then it follows from Lemma 3.3 and Proposition 3.6 in [39] that $L(s, \mathrm{Pic}(X_{\overline{\mathbb{Q}}})) = \zeta(s)\zeta_K(s)^3$, where $\zeta_K(s)$ is the Dedekind zeta function associated to $K = \mathbb{Q}(\sqrt{-3})$. But $\zeta_K(s) = \zeta(s)L(s, \chi)$, where $L(s, \chi)$ is the Dirichlet L-function associated to the real Dirichlet character

$$\chi(n) = \begin{cases} (\frac{-3}{n}) & \text{if } 3 \nmid n, \\ 0 & \text{if } 3 \mid n. \end{cases} \tag{6.1}$$

It therefore follows that $L(s, \mathrm{Pic}(X_{\overline{\mathbb{Q}}})) = \zeta(s)^4 L(s, \chi)^3$, whence

$$\lambda_0 = \lim_{s \to 0} s^3 \zeta(s+1)^3 L(s+1, \chi)^3 = L(1, \chi)^3 = \frac{\pi^3}{3^4\sqrt{3}},$$

by Dirichlet's class number formula. Moreover,

$$\lambda_p = \zeta_p(1, D) L_p(1, \mathrm{Pic}(X_{\overline{\mathbb{Q}}}))^{-1} = \left(1 - \frac{1}{p}\right)^3 \left(1 - \frac{\chi(p)}{p}\right)^3.$$

Thus Theorem 3.9 suggests the heuristic

$$N_U^\circ(B) \sim \lambda_0 \prod_p \lambda_p \sigma_p \cdot r(B),$$

where

$$r(B) = \frac{\left(\frac{3}{2}\right)^3 J_0(B)(\log B)^3}{3!} - \frac{\left(\frac{3}{2}\right)^2 J_1(B)(\log B)^2}{2!} + \frac{\left(\frac{3}{2}\right) J_2(B) \log B}{2!} - \frac{J_3(B)}{3!}.$$

But it follows from Lemma 5.1 that $r(B) \sim C(\log B)^4$, with

$$C = \frac{\left(\frac{3}{2}\right)^3 \kappa_0}{6} - \frac{\left(\frac{3}{2}\right)^2 \kappa_1}{2} + \frac{\left(\frac{3}{2}\right) \kappa_2}{2} - \frac{\kappa_3}{6} = 0.$$

Thus we seem to run into trouble when applying our circle method heuristic to this particular case.

Instead, we appeal to Heuristic 1.2. When k is a cube it follows from Segre [42] that the compactification $X \subset \mathbb{P}^3$ is $\mathbb{Q}$-rational. In particular, the Brauer group $\mathrm{Br}(X)$ is trivial. Since we also have $\mathrm{Br}(U) \cong \mathrm{Br}(X)$, by [16, Prop. 3.1], it follows that Brauer group considerations don't require us to make any adjustment to the leading constant. In this way, on recalling Lemma 5.1, we are led to expect that

$$N_U^\circ(B) \sim \gamma_U \cdot \frac{\pi^2 \Gamma(\frac{1}{3})^3}{3^5} \cdot \prod_p \left(1 - \frac{1}{p}\right)^3 \left(1 - \frac{\chi(p)}{p}\right)^3 \sigma_p \cdot (\log B)^4 \qquad (6.2)$$

for some $\gamma_U \in \mathbb{Q}_{>0}$, where $\sigma_p = \lim_{\ell \to \infty} p^{-2\ell} \nu(p^\ell)$.

We proceed to study this numerically when $k = 1$. We shall need to remove the set of integral points lying on the infinite family of $\mathbb{A}^1$-curves found by Lehmer [30]. Coccia has shown that this set is thin [13, p. 371], while its complement is not [13, Thm. 8]. We can easily get explicit expressions for the local densities σ_p in this case. Thus it follows from (5.2) that $\sigma_3 = \nu(27)/27^2 = 2$. If $p \neq 3$ we can assess σ_p via (5.3) and (5.5), which leads to the expression

$$\sigma_p = 1 + \frac{3(1 + \chi(p))}{p} - \frac{a_p(1 + \chi(p))}{2p^2}.$$

The expectation $N_U^\circ(B) \sim \gamma_U \cdot c \cdot (\log B)^4$ now follows from (6.2), with

$$c = \frac{56\pi^2 \Gamma(\frac{1}{3})^3}{3^{10}} \cdot \prod_{p \equiv 2 \bmod 3} \left(1 - \frac{1}{p^2}\right)^3 \prod_{p \equiv 1 \bmod 3} \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} - \frac{a_p}{p^2}\right).$$

Evaluating the Euler product for $p \leqslant 10^8$ results in $c \approx 0.0958$.

Based on his work with Booker, Sutherland has determined all integer solutions of $x^3 + y^3 + z^3 = 1$ with $\max\{|x|, |y|, |z|\} \leqslant B_{\max} = \sqrt[3]{2} \cdot 10^{15}$, excluding those on lines. We filtered out solutions on the first three embedded $\mathbb{A}^1$-curves that were discovered by Lehmer [30, Thm. A]. The remaining curves have degree $\geqslant 22$ and contribute negligibly many points. Let $N(B)$ denote the contribution to $N_U(B)$ from the points not on one of the three curves of lowest degree. We determined a least squares linear regression of $\log N(B)$ with respect to $\log \log B$. In this, as in all regressions in this paper, the input is the unweighted set of vectors $(\log \log H(P), \log N(H(P)))$ such that P is an integral point with $\sqrt{B_{\max}} \leqslant H(P) \leqslant B_{\max}$. In this way, we obtain the estimate

$$\log N(B) \approx \sigma \log \log B + \delta,$$

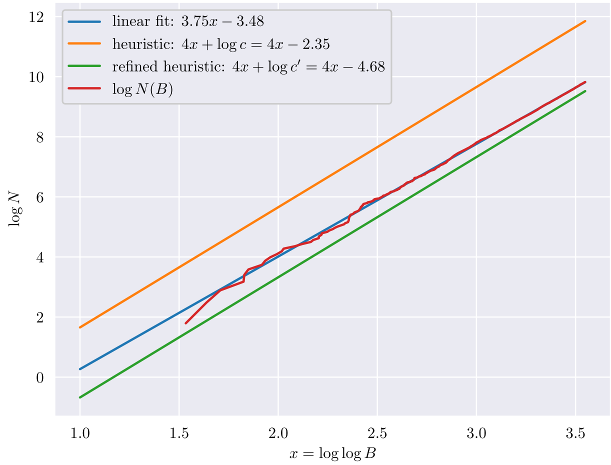

Fig. 1 A comparison of $N(B)$ and a linear fit with the heuristic

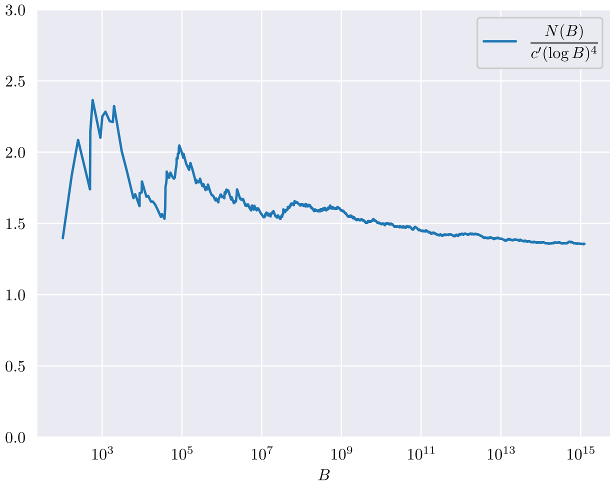

Fig. 2 A comparison of $N(B)$ with $c'(\log B)^4$

with $\sigma = 3.75$ and $\delta = -3.48$, as illustrated in Fig. 1. This seems to be compatible with (6.2), which predicts $\sigma = 4$. We will take $\gamma_U = \frac{7}{72}$ in (6.2), which yields the modified constant $c' = \frac{7}{72} \cdot c \approx 0.00931$. The estimate

$$c_{\mathrm{exp}} = \frac{N(B_{\max})}{(\log B_{\max})^4} \approx 0.013$$

for the leading constant is roughly four thirds the size of this prediction, though as reflected in Fig. 2, it most likely overestimates the true leading constant. In summary, the modified leading constant seems to bring the prediction closer to the actual data.

It remains to justify the numerical value $\gamma_U = \frac{7}{72}$ in (6.2). In the setting of rational points on Fano varieties, as in (3.16) (and further described in [37, p. 335]), Peyre's prediction for the leading constant involves a factor α_X that depends on the geometry of the effective cone $\mathrm{Eff}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}} \cong \mathbb{R}^4$. Denoting by $\mathrm{Eff}(X)^\vee \subset \mathrm{Pic}(X)^*_{\mathbb{R}}$ the dual of this cone, it can be described as an integral

$$\alpha_X = \frac{1}{3!} \int_{\mathrm{Eff}(X)^\vee} e^{-\langle t, \omega_X^\vee \rangle}, \tag{6.3}$$

or as a volume

$$\alpha_X = \mathrm{vol}_H \{ t \in \mathrm{Eff}(X)^\vee : \langle t, \omega_X^\vee \rangle = 1 \}, \tag{6.4}$$

for the hyperplane volume normalised by ω_X and the Picard lattice. More generally, as explained by Batyrev and Tschinkel [2, Def. 2.3.13], for arbitrary height functions $H_{\mathcal{L}}$ associated with a metrised line bundle $\mathcal{L}$ such that $\omega_X^\vee = \mathcal{L}^{\otimes a}$ is a multiple of it, the anticanonical bundle $\omega_X^\vee$ has to be replaced by $\mathcal{L}$ in both formulas.

In the context of integral points, formulae such as those described by Santens [41, Conj. 6.6 and Thm. 6.11] and Wilsch [48, Sec. 2.5] have the feature that the effective cone appearing in (6.3) and (6.4) needs to be replaced by that of a certain subvariety V that depends on intersection properties of the boundary divisor D. If all components of D share a real point, however, then this subvariety is simply $V = X$, by [48, Rem. 2.2.9 (i)]. Moreover, the log anticanonical bundle $\omega_X(D)^\vee$ assumes the role of the canonical bundle in this setting. For the Fermat cubic, the bundle associated with the height function $H(x, y, z) = \max\{|x|, |y|, |z|, 1\}$ is $\mathcal{O}(1) \cong \omega_X^\vee$. Since the log anticanonical bundle is its multiple $\omega_X(D)^\vee \cong \mathcal{O}_X \cong \omega_X^{\otimes 0}$, it would seem natural to include the factor $\alpha_X = \frac{7}{18}$, as determined by Peyre and Tschinkel [39, Prop. 6.1]. However, one further modification seems prudent. When α_X appears in its form (6.3), the factor $\frac{1}{3!}$ comes from an application of Cauchy's residue theorem to $s^{-1}Z(s)B^s$ for a suitable meromorphic function $Z(s)$ whose right-most pole is at $s = 1$ and is of order 4. This results in an expected main term of order $B(\log B)^3$ in the Manin conjecture for X. If such a pole is at $s = 0$ then $s^{-1}Z(s)B^s$ has a pole of order 5 at $s = 0$, resulting in a main term of order $(\log B)^4$ and the relevant factor becomes $\frac{1}{4!}$ instead of $\frac{1}{3!}$. It therefore seems natural to believe that

$$\gamma_U = \frac{3!}{4!} \cdot \alpha_X = \frac{7}{72}.$$

6.2 An example with Picard rank two

We now consider the smooth cubic surface $U_k \subset \mathbb{A}^3$ defined by the polynomial $f(x, y, z) = x^3 + ky^3 + kz^3 - 1$, for a square-free integer $k > 1$. This time, we shall see that Theorem 3.9 suggests a meaningful heuristic for $N_{U_k}^\circ(B)$. The compactification $X_k \subset \mathbb{P}^3$ is the smooth cubic surface $x_0^3 + ky_0^3 + kz_0^3 = t_0^3$. The geometry of X_k has been studied by Peyre and Tschinkel [39] and it follows from [39, Prop. 6.1] that $\varrho_{X_k} = 3$. The divisor D is the smooth genus 1 curve $V(x_0^3 + ky_0^3 + kz_0^3)$. In particular, we have $b = 1$ and $\varrho_{U_k} = \varrho_{X_k} - 1 = 2$. It follows from [14, Lemme 1] that the Brauer group $\mathrm{Br}(X_k)$ is trivial and from [32, Thm. 1.1] that $\mathrm{Br}(U_k) \cong \mathrm{Br}(X_k)$.

6.2.1 Local densities

Adapting Lemma 5.1, it is straightforward to prove that

$$J_\ell(B) = \kappa_\ell (\log B)^{\ell+1} + O((\log B)^\ell), \tag{6.5}$$

in the notation of (3.15), where

$$\kappa_\ell = \frac{3^\ell}{\ell+1} \cdot \frac{\Gamma(\frac{1}{3})^3}{\pi\sqrt{3}k^{2/3}}.$$

Turning to the non-archimedean densities, we have $\sigma_p = \lim_{\ell\to\infty} p^{-2\ell}\nu(p^\ell)$, with $\nu(p^\ell) = \#\left\{(x,y,z) \in (\mathbb{Z}/p^\ell\mathbb{Z})^3 : x^3 + ky^3 + kz^3 \equiv 1 \bmod p^\ell\right\}$. When $p \mid 3k$, the densities can be calculated using a computer, with the outcome that

$$\sigma_p = \begin{cases} 1 & \text{if } p \mid k \text{ and } p \equiv 2 \bmod 3, \\ 3 & \text{if } p \mid k \text{ and } p \equiv 1 \bmod 3, \end{cases} \tag{6.6}$$

and

$$\sigma_3 = \begin{cases} 3 & \text{if } k \equiv 0 \bmod 9, \\ 2 & \text{if } k \equiv \pm 1 \bmod 9, \\ \frac{5}{3} & \text{if } k \equiv \pm 2 \bmod 9, \\ 1 & \text{if } k \equiv \pm 3 \bmod 9, \\ \frac{4}{3} & \text{if } k \equiv \pm 4 \bmod 9. \end{cases} \tag{6.7}$$

Recall that any prime $p \equiv 1 \bmod 3$ admits a unique representation as $4p = a_p^2 + 27b_p^2$, for $a_p, b_p \in \mathbb{Z}$ such that $a_p \equiv 1 \bmod 3$ and $b_p > 0$. We can then write $p = \pi\overline{\pi}$ in $\mathbb{Q}(\sqrt{-3})$, with $\pi = \frac{1}{2}(a_p + 3b_p\sqrt{-3})$. Denote by $\omega = \frac{1}{2}(-1 + \sqrt{-3})$, a primitive cube root of unity.

Lemma 6.1 *Let $p \nmid 3k$. Then*

$$\sigma_p = \begin{cases} 1 & \text{if } p \equiv 2 \bmod 3, \\ 1 + \frac{6}{p} - \frac{a_p}{p^2} & \text{if } p \equiv 1 \bmod 3 \text{ and } (\frac{k}{\pi})_3 = 1, \\ 1 + \frac{3}{p} + \frac{\frac{1}{2}(a_p + 9b_p)}{p^2} & \text{if } p \equiv 1 \bmod 3 \text{ and } (\frac{k}{\pi})_3 = \omega, \\ 1 + \frac{3}{p} + \frac{\frac{1}{2}(a_p - 9b_p)}{p^2} & \text{if } p \equiv 1 \bmod 3 \text{ and } (\frac{k}{\pi})_3 = \omega^2, \end{cases}$$

where $(\frac{\cdot}{\pi})_3$ is the cubic residue symbol associated to π.

Proof It follows from Hensel's lemma that $\sigma_p = \nu(p)/p^2$. We can use cubic characters to evaluate $\nu(p)$, following the approach in [39, Rem. 4.2] and the various identities recorded in [27, Chapter 8]. We begin by writing

$$\nu(p) = \sum_{\substack{\chi_i^3=1\\ i=1,2,3}} \chi_1(k^{-1})\chi_2(k^{-1})J(\chi_1,\chi_2,\chi_3),$$

where the sum is over all characters $\chi_i : \mathbb{F}_p^* \to \mathbb{C}^*$ of order dividing 3, and

$$J(\chi_1,\chi_2,\chi_3) = \sum_{\substack{u,v,w\in\mathbb{F}_p\\ u+v+w=1}} \chi_1(u)\chi_2(v)\chi_3(w)$$

is the Jacobi sum. If $p \equiv 2 \bmod 3$ then there is only the trivial character and it follows that $\nu(p) = p^2$, which gives the result.

Suppose next that $p \equiv 1 \bmod 3$. Then $\chi_i^3 = 1$ if and only if $\chi_i \in \{1, \psi, \overline{\psi}\}$, where $\psi(\cdot) = (\frac{\cdot}{\pi})_3$ is the cubic residue symbol associated to π. $J(\chi_1,\chi_2,\chi_3) = 0$ whenever precisely one or two of the characters χ_1,χ_2,χ_3 is trivial. Hence

$$\begin{aligned}\nu(p) = \ & p^2 + J(\psi,\psi,\overline{\psi})(\psi(k^{-2})+2) + J(\overline{\psi},\overline{\psi},\psi)(\overline{\psi}(k^{-2})+2)\\ &+ J(\psi,\psi,\psi)\psi(k^{-2}) + J(\overline{\psi},\overline{\psi},\overline{\psi})\overline{\psi}(k^{-2}).\end{aligned}$$

We note that $\psi(-1) = 1$ since -1 is a cube, and moreover $\psi(k^{-2}) = \psi(k)$ and $\overline{\psi}(k^{-2}) = \overline{\psi}(k)$. On appealing to the standard formulae for Jacobi sums, we therefore find that $J(\psi,\psi,\overline{\psi}) = \tau(\psi)\tau(\overline{\psi}) = p\psi(-1) = p$, where

$$\tau(\psi) = \sum_{t\in\mathbb{F}_p} \psi(t)e_p(t)$$

is the Gauss sum. Similarly, $J(\overline{\psi},\overline{\psi},\psi) = p$. Moreover, we have

$$J(\psi,\psi,\psi) = -\tau(\psi)^3/p = -J(\psi,\psi), \quad J(\overline{\psi},\overline{\psi},\overline{\psi}) = -\tau(\overline{\psi})^3/p = -J(\overline{\psi},\overline{\psi}).$$

Hence it follows that

$$\frac{\nu(p)}{p^2} = 1 + \frac{4+c_p(k)}{p} - \frac{2\Re\left(J(\chi,\chi)\psi(k)\right)}{p^2},$$

in the notation of (5.4). Now it follows from [27, Prop. 8.3.4] and its corollary that $J(\psi,\psi) = \frac{1}{2}(a_p+3b_p)+3b_p\omega$. Let us write $A_k = 2\Re\left(J(\psi,\psi)\psi(k)\right)$ for simplicity. If $\psi(k) = 1$ then $A_k = 2\Re J(\psi,\psi) = a_p$, as claimed in the lemma. If $\psi(k) = \omega$ then $A_k = 2\Re(\omega J(\psi,\psi)) = -\frac{1}{2}(a_p+9b_p)$. Finally, if $\psi(k) = \omega^2$ we get $A_k = 2\Re(\omega^2 J(\psi,\psi)) = -\frac{1}{2}(a_p-9b_p)$. □

6.2.2 Application of the heuristic

We can adapt the parameterisation of Lehmer [30] to the present setting. On substituting $k^{\lfloor n/3\rfloor}t^n$ for t^n in the Lehmer parametrisation, we are led to infinitely many $\mathbb{A}^1$-curves of increasing degree. The curves of lowest degree are given parametrically by

$$x(t) = 9kt^3 + 1, \quad \{y(t), z(t)\} = \{-9kt^4 - 3t, 9kt^4\},$$

and

$$x(t) = 2^4 3^5 k^3 t^9 - 3^2 kt^3 + 2^3 3^4 k^2 t^6 + 1,$$
$$\{y(t), z(t)\} = \{-2^4 3^5 k^3 t^{10} - 2^4 3^4 k^2 t^7 - 3^4 kt^4 + 3t, 2^4 3^5 k^3 t^{10} - 135kt^4\}.$$

Let $N_k(B) = N^{\circ}_{U_k}(B)$ be the counting function defined in (1.9), where $U_k(\mathbb{Z})^{\circ}$ is obtained by removing those points in $U_k(\mathbb{Z})$ that are contained in any such curve. We are now ready to reveal what our heuristic says about $N_k(B)$.

We have already seen that we may take $\varrho_{U_k} = 2$ and $b = 1$ in Theorem 3.9. Returning to Proposition 3.2, we have $\zeta(s, D) = \zeta(s)$ and it follows from Lemma 3.3 and Proposition 3.6 in [39] that $L(s, \mathrm{Pic}(X_{k,\overline{\mathbb{Q}}})) = \zeta(s)^2\zeta_K(s)L(s,\chi)^2$, where $\zeta_K(s)$ is the Dedekind zeta function associated to $K = \mathbb{Q}(k^{1/3})$ and $L(s,\chi)$ is the Dirichlet L-function associated to the real Dirichlet character (6.1). Hence we have $F(s) = \zeta(s+1)\zeta_K(s+1)L(s+1,\chi)^2\tilde{F}(s)$ in (3.8). But then, on recalling Lemma 3.1, we see that $\tilde{F}(0) = \prod_p \lambda_p\sigma_p$, where

$$\lambda_p = \left(1 - \frac{1}{p}\right)\left(1 - \frac{\chi(p)}{p}\right)^2 \zeta_{K,p}(1)^{-1}.$$

Moroeover,

$$\lambda_0 = \lim_{s\to 0} s^2\zeta(s+1)\zeta_K(s+1)L(s+1,\chi)^2 = \frac{\pi^2}{27}\cdot \lim_{s\to 1}(s-1)\zeta_K(s),$$

since $L(1,\chi)^2 = \pi^2/27$.

Next, we clearly have

$$r(B) = \frac{\left(\frac{3}{2}\right)^2 J_0(B)(\log B)^2}{2!} - \frac{\left(\frac{3}{2}\right) J_1(B)\log B}{1!} + \frac{J_2(B)}{2!},$$

in Theorem 3.9. But then (6.5) yields $r(B) = C_\infty(\log B)^3$, with

$$C_\infty = \frac{\left(\frac{3}{2}\right)^2\kappa_0}{2!} - \frac{\left(\frac{3}{2}\right)\kappa_1}{1!} + \frac{\kappa_2}{2!} = \frac{\sqrt{3}\Gamma(\frac{1}{3})^3}{8\pi k^{2/3}}.$$

Similarly to (5.6), we note that $C_\infty = \frac{3}{8}\cdot\mu_D$, where μ_D is the constant appearing in Proposition 2.4, related to the real density. In summary, we are led to the following expectation.

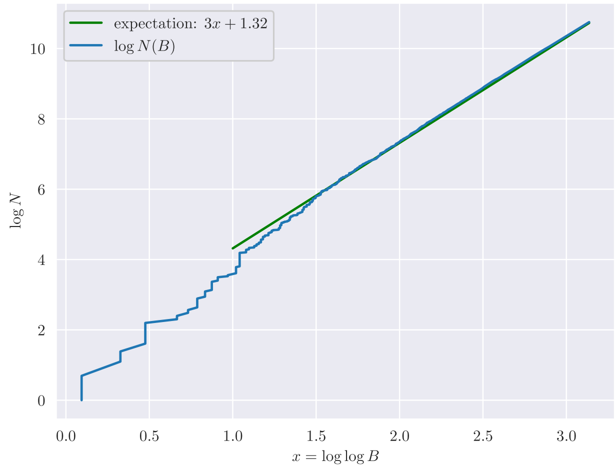

Fig. 3 A comparison of $N(B)$ with the circle method prediction

Heuristic 6.2 *Let $k > 1$ be a square-free integer and let $U_k \subset \mathbb{A}^3$ be the cubic surface defined by $x^3 + ky^3 + kz^3 = 1$. Then*

$$N_k(B) \sim c_{\text{circle}}^{(k)} (\log B)^3,$$

as $B \to \infty$, where

$$c_{\text{circle}}^{(k)} = \frac{\pi \Gamma(\frac{1}{3})^3}{72\sqrt{3}k^{2/3}} \cdot \lim_{s \to 1} (s-1)\zeta_K(s) \cdot \prod_p \left(1 - \frac{1}{p}\right)\left(1 - \frac{\chi(p)}{p}\right)^2 \zeta_{K,p}(1)^{-1} \sigma_p.$$

Explicit expressions for σ_p are given by Lemma 6.1 for $p \nmid 3k$, and by (6.6)–(6.7) for $p \mid 3k$.

6.2.3 Numerical data

We have determined all integer points $(x, y, z) \in U_k(\mathbb{Z})$ with $\max\{|x|, |y|, |z|\} \leqslant 10^{10}$, for all square-free integers $2 \leqslant k \leqslant 1000$. We removed all points contained in the $\mathbb{A}^1$-curves of degrees 4 and 10, that we identified above. The higher degree curves contribute negligibly and the numerics don't suggest the presence of any further $\mathbb{A}^1$-curves of low degree. Let

$$N(B) = \sum_{\substack{2 \leqslant k \leqslant 1000 \\ k \text{ square-free}}} N_k(B).$$

Fig. 4 A comparison of $N(B)$ with $(\log B)^3$

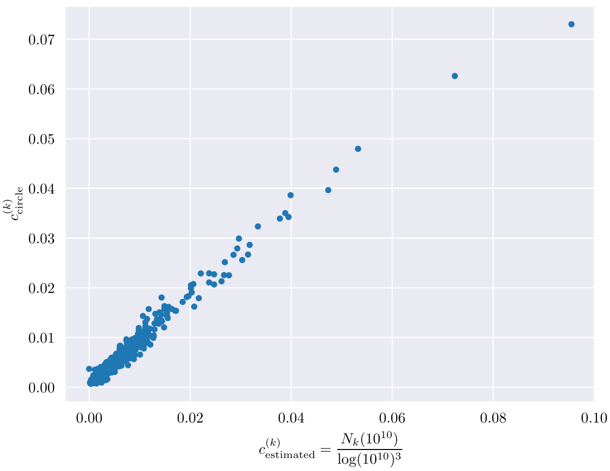

Fig. 5 A scatter plot comparing the predicted leading constants to the heuristic leading constants determined from the data

The sum of the predicted constant over all relevant k is

$$c_{\text{circle}} = \sum_{\substack{2 \leqslant k \leqslant 1000 \\ k \text{ square-free}}} c^{(k)}_{\text{circle}} \approx 3.73.$$

Figure 3 confirms that our prediction is very close to the numerical data. Moreover, a least squares linear regression of $\log N(B)$ against $\log\log B$ results in a fit

$$\log N(B) \approx 3.02 \log\log B + 1.31,$$

which suggests the experimental leading constant $c_{\text{exp}} = \exp(1.31) \approx 3.71$. This agrees with Heuristic 6.2, which predicts slope 3 and leading constant $c_{\text{circle}} \approx 3.73$.

In fact, as seen in Fig. 4, both the cumulative counting function $N(B)$ as well as individual counting functions (depicted for $k = 2$ and $k = 3$) align rather well with the circle method prediction for $B \leqslant 10^{10}$. Moreover, in Fig. 5 we have included a scatter plot, in which each blue dot represents a surface in the family; on the x-axis is the prediction for the constant coming from the circle method and on the y-axis is the ratio $N^{\circ}_{U_k}(B)/(\log B)^3$, for $B = 10^{10}$. The correlation is very good. Note that both C_∞ and the product of non-archimedean densities vary significantly with the parameter k, and the presence of both factors in $c^{(k)}_{\text{circle}}$ is necessary to achieve the correlation seen in Fig. 5. Indeed, estimating

$$\log(c^{(k)}_{\text{exp}}) \approx \log(c^{(k)}_{\text{circle}}) + \mu$$

results in an R^2-value of 0.90, while analogous estimates using only C_∞ or $\lambda_0 \prod_p \lambda_p \sigma_p$ instead of the full circle method constant result in R^2-values of 0.40 and 0.38, respectively.

Finally, we compare our findings with Heuristic 1.2, recalling that $\text{Br}(U_k)$ is trivial in this case. Thus Heuristic 1.2 predicts that $N^{\circ}_U(B) \sim \gamma_U \cdot \tau_{U,H}(V)(\log B)^3$ for $\gamma_U \in \mathbb{Q}_{>0}$ and $V = D(\mathbb{R}) \times U(\mathbf{A}^{\text{fin}}_{\mathbb{Z}})$. If we take $\gamma_U = \frac{3}{8}$, we will therefore have $c_{\text{h}} = c^{(k)}_{\text{circle}}$, in the notation of Heuristic 6.2, whence Conjecture 1.3.

7 The Baragar–Umeda examples

In this section we examine the surfaces appearing in Table 1 that were studied by Baragar and Umeda [1]. In Fig. 6 we have plotted the integer points of low height on the first surface in the table. Let $U \subset \mathbb{A}^3$ be any surface in Table 1 and let $X \subset \mathbb{P}^3$ be its compactification. In particular X is a clearly a smooth cubic surface. An analysis of the lines contained in X, similar to the calculations in [15], reveals that $\varrho_U = 0$. The divisor at infinity is $D = X \setminus U$, which is equal to $V(dx_0y_0z_0)$, a union of three lines

$$L_1 = V(t_0, x_0), \quad L_2 = V(t_0, y_0), \quad L_3 = V(t_0, z_0) \tag{7.1}$$

It follows that $\varrho_U = 0$ and $b = 2$ in Heuristic 3.11, and so the exponent of $\log B$ in the heuristic agrees with the asymptotic formula (1.8).

7.1 The leading constant

We proceed by studying the constant in Heuristic 3.11 and comparing it to the constant c_{BU} in Table 1, for the different choices of coefficient vectors. We shall find that they do not agree, even after making natural modifications along the lines suggested in Heuristic 1.2.

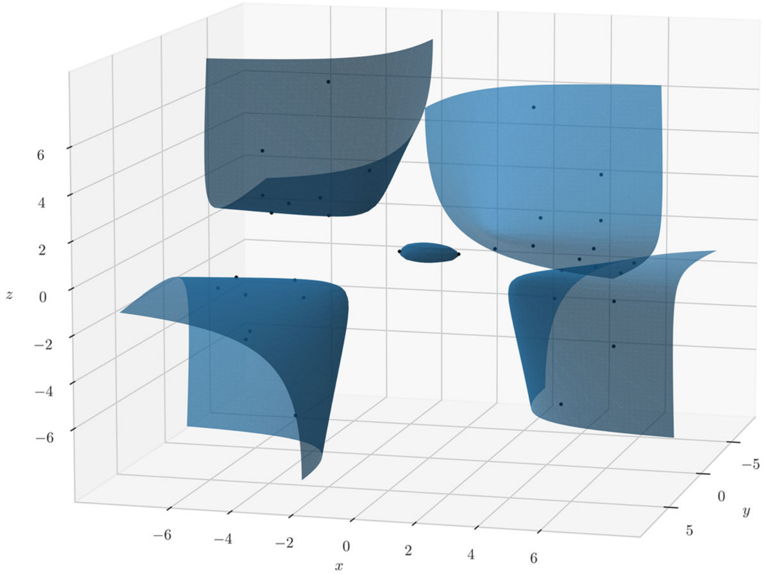

Fig. 6 Integer points on the surface $x^2 + 5y^2 + 5z^2 = 5xyz + 1$ of height $\leqslant 10$

7.1.1 The number of solutions modulo *p*

Henceforth we focus on the surfaces (1.6) for square-free $a, b, c, d \in \mathbb{N}$ such that $4abc - d^2 \neq 0$ and d is divisible by a, b and c. Moreover, we assume that none of $d^2 - 4abc, a(d^2 - 4abc), \ldots, c(d^2 - 4abc)$ is the square of an integer. These conditions are clearly satisfied by the six surfaces in Table 1. We let S be the set of prime divisors of $2abcd(d^2 - 4abc)$.

Let $p \notin S$ and recall the definition (3.2) of $\nu(p^k)$. We need to calculate this quantity when $k = 1$ While it is possible to evaluate $\nu(p)$ using (3.6), we shall give an elementary treatment using character sums, based on the expression

$$\nu(p) = p^2 + \frac{1}{p} \sum_{h \in \mathbb{F}_p^*} \sum_{x,y,z \in \mathbb{F}_p} e_p\left(hf(x, y, z)\right). \tag{7.2}$$

We will need to recollect some relevant facts about character sums. Let $A, B, C \in \mathbb{Z}$. The quadratic Gauss sum is

$$\sum_{x \in \mathbb{F}_p} e_p(Ax^2 + Bx) = \varepsilon_p \sqrt{p} \left(\frac{A}{p}\right) e_p(-\overline{4A}B^2), \quad \text{if } p \nmid 2A, \tag{7.3}$$

where $\overline{4A}$ is the multiplicative inverse of $4A$ modulo p and

$$\varepsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ i & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

When $B = 0$ and $p \nmid 2A$, we note that the sum on the left hand side of (7.3) can be written in the equivalent form

$$\sum_{x\in\mathbb{F}_p} e_p(Ax^2) = \sum_{x\in\mathbb{F}_p} \left(1+\left(\frac{x}{p}\right)\right) e_p(Ax) = \sum_{x\in\mathbb{F}_p} \left(\frac{x}{p}\right) e_p(Ax).$$

Next, the Legendre sum is

$$\sum_{x\in\mathbb{F}_p} \left(\frac{Ax^2+Bx+C}{p}\right) = \begin{cases} -(\frac{A}{p}) & \text{if } p\nmid A \text{ and } p \nmid B^2-4AC, \\ (p-1)(\frac{A}{p}) & \text{if } p\nmid A \text{ and } p \mid B^2-4AC. \end{cases} \tag{7.4}$$

We are now ready to reveal our calculation of $\nu(p)$.

Lemma 7.1 *For any $p \notin S$, we have*

$$\frac{\nu(p)}{p^2} = 1 + \frac{1}{p}\left(\frac{d^2-4abc}{p}\right)\left(1+\left(\frac{a}{p}\right)+\left(\frac{b}{p}\right)+\left(\frac{c}{p}\right)\right)+\frac{1}{p^2}.$$

Proof Recall from (1.6) that $f(x, y, z) = ax^2 + by^2 + cz^2 - dxyz - 1$. Applying the formula (7.3) for Gauss sums in (7.2), we deduce that

$$\begin{aligned}\nu(p) &= p^2 + \frac{\varepsilon_p}{\sqrt{p}} \sum_{h\in\mathbb{F}_p^*} \left(\frac{ha}{p}\right) \sum_{y,z\in\mathbb{F}_p} e_p\left(h(by^2+cz^2-1-\overline{4a}d^2y^2z^2)\right) \\ &= p^2 + \frac{\varepsilon_p}{\sqrt{p}} \sum_{h\in\mathbb{F}_p^*} \left(\frac{ha}{p}\right) \sum_{z\in\mathbb{F}_p} e_p\left(h(cz^2-1)\right) \sum_{y\in\mathbb{F}_p} e_p\left(hy^2(b-\overline{4a}d^2z^2)\right).\end{aligned}$$

Next we evaluate the sum over y. If $b - \overline{4a}d^2z^2 \not\equiv 0 \bmod p$ then the inner sum is $\varepsilon_p\sqrt{p}(\frac{h(b-\overline{4a}d^2z^2)}{p})$ by (7.3). Alternatively, it takes the value p. Thus

$$\nu(p) = p^2 + \Sigma_1 + \Sigma_2, \tag{7.5}$$

where

$$\Sigma_1 = \varepsilon_p^2 \left(\frac{a}{p}\right) \sum_{z\in\mathbb{F}_p} \left(\frac{b-\overline{4a}d^2z^2}{p}\right) c_p\left(cz^2-1\right)$$

and

$$\Sigma_2 = \varepsilon_p\sqrt{p} \sum_{h\in\mathbb{F}_p^*} \left(\frac{ha}{p}\right) \sum_{\substack{z\in\mathbb{F}_p \\ b-\overline{4a}d^2z^2\equiv 0 \bmod p}} e_p\left(h(cz^2-1)\right).$$

It follows from (3.4) that

$$\Sigma_1 = \varepsilon_p^2 \left(\frac{a}{p}\right) \left(- \sum_{z \in \mathbb{F}_p} \left(\frac{b - \overline{4a}d^2z^2}{p}\right) + p \left(\frac{b - \overline{4ac}d^2}{p}\right) \left(1 + \left(\frac{c}{p}\right)\right)\right).$$

We evaluate the sum over z by appealing to (7.4). This yields

$$-\sum_{z \in \mathbb{F}_p} \left(\frac{b - \overline{4a}d^2z^2}{p}\right) = \left(\frac{-\overline{4a}d^2}{p}\right) = \left(\frac{-a}{p}\right).$$

Thus

$$\begin{aligned} \Sigma_1 &= \varepsilon_p^2 \left(\frac{a}{p}\right)\left(\frac{-a}{p}\right) + p\varepsilon_p^2 \left(\frac{ab - \overline{4c}d^2}{p}\right)\left(1 + \left(\frac{c}{p}\right)\right) \\ &= 1 + p\left(\frac{d^2 - 4abc}{p}\right)\left(1 + \left(\frac{c}{p}\right)\right), \end{aligned}$$

since $\varepsilon_p^2 = (\frac{-1}{p})$.

Next, we see that

$$\begin{aligned} \Sigma_2 &= \varepsilon_p \sqrt{p} \sum_{h \in \mathbb{F}_p} \left(\frac{ha}{p}\right) e_p\left(h(4abc\overline{d}^2 - 1)\right)\left(1 + \left(\frac{ab}{p}\right)\right) \\ &= \varepsilon_p \sqrt{p} \left(\left(\frac{a}{p}\right) + \left(\frac{b}{p}\right)\right) \sum_{h \in \mathbb{F}_p} \left(\frac{h}{p}\right) e_p\left(h(4abc\overline{d}^2 - 1)\right). \end{aligned}$$

The inner sum is another Gauss sum and can be evaluated using (7.3). Thus

$$\Sigma_2 = p\left(\frac{d^2 - 4abc}{p}\right)\left(\left(\frac{a}{p}\right) + \left(\frac{b}{p}\right)\right).$$

Combining our expressions for Σ_1 and Σ_2 in (7.5) and dividing by p^2, we arrive at the statement of the lemma. □

7.1.2 Non-archimedean densities

Throughout this subsection, let $k = d^2 - 4abc$ and let S be the set of prime divisors of $2abcdk$. It is convenient to define Dirichlet characters $\chi_1, \ldots, \chi_4$ via the Kronecker symbols $\chi_i(n) = \left(\frac{D_i}{n}\right)$, where $D_i = \mathrm{disc}(K_i)$, for

$$K_1 = \mathbb{Q}(\sqrt{k}), \quad K_2 = \mathbb{Q}(\sqrt{ka}), \quad K_3 = \mathbb{Q}(\sqrt{kb}), \quad K_3 = \mathbb{Q}(\sqrt{kc}).$$

In particular, we have $\chi_1(p) = (\frac{d^2-4abc}{p})$ and

$$\chi_2(p) = \chi_1(p)\left(\frac{a}{p}\right), \quad \chi_3(p) = \chi_1(p)\left(\frac{b}{p}\right), \quad \chi_4(p) = \chi_1(p)\left(\frac{c}{p}\right),$$

for $p \notin S$. Thus Lemma 7.1 yields

$$\frac{\nu(p)}{p^2} = 1 + \frac{\chi_1(p)+\chi_2(p)+\chi_3(p)+\chi_4(p)}{p} + \frac{1}{p^2}, \tag{7.6}$$

for any such prime. It follows from (3.3) that

$$\sigma_p(s) = 1 + \frac{\chi_1(p)+\chi_2(p)+\chi_3(p)+\chi_4(p)}{p^{s+1}} + \frac{1}{p^{s+2}}.$$

Define $\lambda(s) = L(s,\chi_1)L(s,\chi_2)L(s,\chi_3)L(s,\chi_4)$ and

$$\lambda_p(s) = (L_p(s,\chi_1)L_p(s,\chi_2)L_p(s,\chi_3)L_p(s,\chi_4))^{-1}.$$

With this notation we have

$$F(s) = \prod_p \sigma_p(s) = \lambda(s+1)\prod_p \lambda_p(s+1)\sigma_p(s)$$

in Lemma 3.1, for $\Re(s) > 2$. Now

$$\begin{aligned}\lambda_p(s+1)\sigma_p(s) &= \left(1-\frac{\chi_1(p)}{p^{s+1}}\right)\cdots\left(1-\frac{\chi_4(p)}{p^{s+1}}\right)\\ &\quad\times\left(1+\frac{\chi_1(p)+\chi_2(p)+\chi_3(p)+\chi_4(p)}{p^{s+1}}+\frac{1}{p^{s+2}}\right),\end{aligned} \tag{7.7}$$

for $p \notin S$, and so the Euler product $\prod_p \lambda_p(s+1)\sigma_p(s)$ converges absolutely for $\Re(s) > -\frac{1}{2}$. In particular, we have

$$\lim_{s\to 0}\left(s^{\varrho_U}F(s)\right) = F(0) = \lambda(1)\prod_p \lambda_p(1)\sigma_p = \prod_p \sigma_p. \tag{7.8}$$

This expression could also have been deduced from Proposition 3.2, but we have chosen to present an explicit derivation using Dirichlet L-functions.

7.1.3 The expected leading constant

We are now ready to record an explicit expression for the expected leading constant c_{circle}, say, in Heuristic 3.11, with $\varrho_U = 0$ and $b = 2$. Combining Lemma 2.5 with (7.8), it follows that

$$c_{\mathrm{circle}} = \frac{6}{d}\cdot\lambda(1)\cdot c_S\cdot c_{S^c}, \tag{7.9}$$

Table 2 The circle method prediction and a comparison to the actual leading constant

	c_{circle}	$c_{\text{circle}}/c_{\text{BU}}$
(i)	2.997816	0.5734700
(ii)	2.997094	1.0107957
(iii)	1.484675	0.6015930
(iv)	2.397675	0.5910831
(v)	1.16853	0.4686900
(vi)	3.331807	0.6770839

with

$$c_S = \prod_{p \in S} \lambda_p(1)\sigma_p \quad \text{and} \quad c_{S^c} = \prod_{p \notin S} \lambda_p(1)\sigma_p.$$

We determine $\lambda(1)$ using Dirichlet's class number formula, c_S by a computer search for points modulo small powers of $p \in S$, and c_{S^c} by multiplying the factors $\lambda_p(1)\sigma_p$ for $p < 10^7$. (Note that the latter are obtain by taking $s = 0$ in (7.7).) The results of these computations are summarised in Table 2.

7.2 Modified expectations

For each of the surfaces in Table 1, we note that $U(\mathbb{R})$ has five connected components: one bounded component and four unbounded ones. This is illustrated in Fig. 6 for the first surface in the table. On the unbounded components, we have $xyz > 0$, and the four components can be distinguished by imposing conditions on the signs of the variables that are compatible with this observation. Denote by U_0 the unbounded component with $x, y, z > 0$. Due to the symmetry of the equation, it suffices to study this component.

7.2.1 Hensel's lemma and the place 2

While not a failure of strong approximation, we make the following observation.

Lemma 7.2 *Let $U \subset \mathbb{A}^3$ be one of the surfaces in Table 1. Then the map*

$$\mathfrak{U}(\mathbb{Z}_2) \to \mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z})$$

is not surjective for any k. Indeed, its image consists of half the points in $\mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z})$ if $k \geqslant 3$.

Proof Let $k \geqslant 2$, and let (x, y, z) be a solution modulo 2^k. Then all eight points of the form $(x + \delta_1 2^{k-1}, y + \delta_2 2^{k-1}, z + \delta_3 2^{k-1})$ with $\delta_i \in \{0, 1\}$ are solutions modulo 2^k. Indeed, changing x by 2^{k-1} results in

$$\begin{aligned} f(x + 2^{k-1}, y, z) &= a(x + 2^{k-1})^2 + by^2 + cz^2 - d(x + 2^{k-1})yz - 1 \\ &= f(x, y, z) + 2^k ax + 2^{2k-2}a + d2^{k-1}yz. \end{aligned} \tag{7.10}$$

Clearly, 2^k divides $2^k ax + 2^{2k-2}a$. In case (i), precisely two of x, y, and z are even so that $2 \mid yz$, while in the remaining cases, d is even, so that $2^k \mid d2^{k-1}yz$ in any case. Hence, $f(x + 2^{k-1}, y, z) \equiv f(x, y, z) \equiv 0 \bmod 2^k$, and modifications of y or z can be treated analogously.

Let the parameters a, b, c, d be as in case (i) for now. If (x, y, z) is a solution modulo 2^k, then precisely one of x, y, z is odd, say x (the other two cases are analogous). We note that $4 \mid yz$ by the assumption on the parities of the coordinates, while $2^{k+1} \mid 2^{2k-2}$ by the assumption on k, so that (7.10) implies that

$$f(x + 2^{k-1}, y, z) - f(x, y, z) \equiv 2^k ax \equiv 2^k \bmod 2^{k+1},$$

noting that both a and x are odd by assumption for the second equivalence. Using that $f(x, y, z) \equiv f(x + 2^{k-1}, y, z) \equiv 0 \bmod 2^k$, this implies that precisely one of $f(x, y, z)$ and $f(x + 2^{k-1}, y, z)$ vanishes modulo 2^{k+1}. (And then f also vanishes modulo 2^{k+1} on the other seven points coinciding with this one modulo 2^k, but on none of the points coinciding with the other one modulo 2^k.)

The remaining cases can be dealt with similarly, using that precisely one of x and y is odd in case (ii), that y is always odd in case (iii), that z is always odd in cases (iv) and (v), and that x is always odd in case (vi). □

Remark 7.3 As a consequence of this and by [28, Ch. II, Lem. 6.6], the Tamagawa volume of each residue disc in $\mathfrak{U}(\mathbb{Z}_2)$ modulo 2^k is 2^{1-2k}. (Note that this makes Lemma 7.2 compatible with [6, Lem. 1.8.1].) Thus, whenever we count points in the image $\mathfrak{U}(\mathbb{Z}) \to \mathfrak{U}(\mathbb{Z}/m\mathbb{Z})$ with $2 \mid m$, we shall multiply the result by 2 when using it as part of our modified leading constant.

On the other hand, for odd primes in S, with the help of a computer, we find that each point P modulo p lifts to a point modulo p^2 and the p-adic norm of a least one of the partial derivatives is at least p^{-1} at P. Hence, Hensel's lemma implies that all points modulo odd primes lift to p-adic points and (3.3) holds for all odd places.

7.2.2 Failures of strong approximation

As usual let $U \subset \mathbb{A}^3$ be one of the Baragar–Umeda surfaces (1.6) and let $\mathfrak{U}$ be its integral model over U. If a is a square modulo p, then there are obvious solutions $(\pm 1/\sqrt{a}, 0, 0) \in \mathfrak{U}(\mathbb{Z}_p)$ modulo p. However, the group Γ acts trivially on these, meaning that they only lift to the trivial solutions $(\pm 1, 0, 0) \in \mathfrak{U}(\mathbb{Z})$ if $a \in \{\pm 1\}$ is an integral square, or not at all if it is not. (For instance, in case (i), there is a solution $(1, 0, 0) \in \mathfrak{U}(\mathbb{F}_p)$ for all primes p which lifts only to $(1, 0, 0)$, while $(0, 3, 0) \in \mathfrak{U}(\mathbb{F}_{11})$ does not lift at all.) In the light of these observations, we are led to set

$$\mathfrak{U}(\mathbb{Z}/p^k\mathbb{Z})' = \left\{ P \in \mathfrak{U}(\mathbb{Z}/p^k\mathbb{Z}) : \begin{array}{l} P \not\equiv \{(\alpha, 0, 0),\ (0, \beta, 0), (0, 0, \gamma)\} \bmod p \\ \text{for any } \alpha, \beta, \gamma \in \mathbb{Z}/p\mathbb{Z} \end{array} \right\}$$

for odd primes p and $k \geqslant 1$, and

$$\mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z})' = \left\{ P \in \mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z}) : \begin{array}{l} P \in \operatorname{im}(\mathfrak{U}(\mathbb{Z}_2) \to \mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z})), \\ P \not\equiv \{(\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)\} \bmod 8 \\ \text{for any } \alpha,\ \beta, \gamma \in \mathbb{Z}/8\mathbb{Z} \end{array} \right\}$$

for $k \geqslant 3$.

For any integer $m > 0$, the description of $\mathfrak{U}(\mathbb{Z}) \cap U_0$ as the orbit of one or more primitive solutions under the group generated by the Vieta involutions allows us to efficiently compute the image of

$$\varphi_m : \mathfrak{U}(\mathbb{Z}) \cap U_0 \to \mathfrak{U}(\mathbb{Z}/m\mathbb{Z}). \tag{7.11}$$

Although we omit the details it is possible to extend the work of Colliot-Thélène–Wei–Xu [15] and Loughran–Mitankin [31], in order to study the Brauer–Manin obstruction for the Baragar–Umeda surfaces. In the case of the surface (i), for example, one can check that the Brauer–Manin obstruction precisely cuts out this image for $m = 2^3 \cdot 3 \cdot 5$; in other words,

$$\operatorname{im} \varphi_{2^3 \cdot 3 \cdot 5} = \left(\mathfrak{U}(\mathbb{Z}/8\mathbb{Z})' \times \mathfrak{U}(\mathbb{Z}/3\mathbb{Z})' \times \mathfrak{U}(\mathbb{Z}/5\mathbb{Z})'\right)^{\operatorname{Br} U}.$$

(This makes sense, since the pairing is constant over all places different from ∞, 2, 3 and 5, so that the set cut out does not depend on choices of points over the remaining primes.) Motivated by this, for any of the surfaces in Table 1, we define

$$m_S = \prod_{p \in S} p^{k_p}, \quad \text{with } k_p = \begin{cases} 3 & \text{if } p = 2, \\ 1 & \text{if } p \in S \setminus \{2\}. \end{cases} \tag{7.12}$$

We can then prove the following facts about $\operatorname{im} \varphi_m$, for various choices of $m \in \mathbb{N}$.

Proposition 7.4 *Let $I_m = \operatorname{im} \varphi_m$ for $m \in \mathbb{N}$. Then*

(1) $I_p = \mathfrak{U}(\mathbb{F}_p)'$ *if* $p \notin S$ *and* $p \leqslant 1000$, *provided* U *is not as in case (ii) or (iv);*
(2) $I_{pq} = I_p \times I_q$ *if* $p, q \leqslant 73$ *are distinct primes and* $p \notin S$;
(3) $I_{pql} = I_{pq} \times I_l$, *up to reordering of* p, q, l, *if* $p, q, l \leqslant 23$ *are distinct primes;*
(4) $I_m = I_{m/m_S} \times I_{m_S}$ *for* $m = 2^3 \cdot 3 \cdots 11$; *and*
(5) $\#I_{m \cdot m_S} = m^2 \# I_{m_S}$, *where* $m = \prod_{p \in S} p$.

Proof These equalities are established by determining the respective orbits using a computer. More precisely, for U as in case (ii), the first equality fails for $p \equiv \pm 1 \bmod 24$, and for U as in case (iv), the behaviour seems to depend on p modulo 120. The second computation reveals failures of strong approximation for precisely one pair (p, q) with $p, q \in S$ in all cases except (iv), similar to the one in case (i) that is explained by the Brauer–Manin obstruction. □

We expect that the failures of strong approximation encountered in the numerical analysis of part (2) of Proposition 7.4 are all explained by the Brauer–Manin obstruction.

7.2.3 The modified constant

Based on our observations in the previous section, we propose modifying our constant along the lines of Heuristic 1.2. Let P_1, P_2, P_3 be the three vertices of the triangle at infinity. Let m_S be defined by (7.12) and recall the definition (7.11) of the map φ_m, for any $m \in \mathbb{N}$. We apply Heuristic 1.2 with the set

$$V_0 = \{P_1, P_2, P_3\} \times \pi_{m_S}^{-1}\left(\operatorname{im}\varphi_{m_S}\right) \times \prod_{p \notin S} \mathfrak{U}(\mathbb{F}_p)',$$

where $\pi_k \colon \prod_{p|k} \mathfrak{U}(\mathbb{Z}_p) \to \mathfrak{U}(\mathbb{Z}/k\mathbb{Z})$ is the reduction modulo k, for any $k \in \mathbb{N}$. Note that taking a different unbounded component U_i to U_0 would give a different set V_i of equal volume. We do not take the union, however, since the set V_0 only approaches each vertex of the triangle at infinity from one of the four possible directions, so in fact the resulting volume would be $\frac{1}{4}\tau_{U,H}(V_0 \cup V_1 \cup V_2 \cup V_3) = \tau_{U,H}(V_0)$.

We proceed to calculate the value of $\tau_{U,H}(V_0)$. Let

$$c'_S = 2\frac{\#\operatorname{im}(\mathfrak{U}(\mathbb{Z}) \cap U_0 \to \mathfrak{U}(\mathbb{Z}/m_S\mathbb{Z}))}{m_S^2} \prod_{p \in S} \lambda_p(1),$$

noting that the leading 2 is a consequence of Remark 7.3. For $p \notin S$, we set

$$\sigma'_p = \frac{\#\mathfrak{U}(\mathbb{F}_p)'}{p^2}. \tag{7.13}$$

Explicitly, on modifying (7.6) to remove the solutions $(\pm\sqrt{a}, 0, 0)$, etc., if they exist, we find that

$$\sigma'_p = 1 + \frac{\chi_1(p) + \chi_2(p) + \chi_3(p) + \chi_4(p)}{p} - \left(2 + \left(\frac{a}{p}\right) + \left(\frac{b}{p}\right) + \left(\frac{c}{p}\right)\right)\frac{1}{p^2}.$$

We set

$$c'_{S^c} = \prod_{p \notin S} \lambda_p(1)\sigma'_p.$$

Then we are led to modify the circle method constant in (7.9) to

$$c'_{\text{circle}} = \frac{6}{d} \cdot \lambda(1) \cdot c'_S \cdot c'_{S^c}.$$

Numerical approximations of these new constants and a comparison to the constants in Table 1 are recorded in Table 3. (It is interesting to note that our modified circle method constant is always smaller than the actual constant.)

Recalling that not all points counted in (7.13) lift to $\mathbb{Z}$-points in cases (ii) and (iv), we further set

$$c''_{S^c} = \prod_{p \notin S} \lambda_p(1)\frac{\#\operatorname{im}(\mathfrak{U}(\mathbb{Z}) \cap U_0 \to \mathfrak{U}(\mathbb{F}_p))}{p^2}$$

Table 3 The modified expected constant and a comparison to the actual leading constant

	c'_{circle}	c''_{circle}	$c'_{\text{circle}}/c_{\text{BU}}$	$c''_{\text{circle}}/c_{\text{BU}}$
(i)	0.8127795		0.1554814	
(ii)	0.6682904	0.63	0.2253867	0.21
(iii)	0.5012050		0.2030892	
(iv)	1.038439	0.51	0.2559995	0.13
(v)	0.4472312		0.1793816	
(vi)	0.7655632		0.1555764	

and arrive at the modified constant

$$c''_{\text{circle}} = \frac{6}{d} \cdot \lambda(1) \cdot c'_S \cdot c''_{S^c},$$

by computing the images for primes $p < 10^3$. It follows from Proposition 7.4 that this modification does not make a difference in cases (i), (iii), (v) and (vi), except for a reduction in the bound for p that we can use to numerically calculate it.

It is interesting to speculate on the constant $\gamma_U \in \mathbb{Q}_{>0}$ in Heuristic 1.2, led by the situation (3.16) for rational points on Fano varieties. An integral variant of the α-constant has been described by Wilsch [48, Def. 2.2.8] for split log Fano varieties, but this rational number is the same for the six surfaces considered here, since the relevant cones are all isometric. Turning to the β-constant, it is possible to expand on the arguments in [15] to deduce that the algebraic part of the Brauer group up to constants has order 8 in cases (i) and (vi), and order 4 in the remaining cases. We note that the quotients $c'_{\text{circle}}/c_{\text{BU}}$ are not integers, nor are they rational numbers of small height. Thus Table 3 does not seem to be compatible with a version of Heuristic 1.2 with γ_U of small height.

7.3 Equidistribution

As a consequence of the failures of strong approximation, the equidistribution property also fails. However, we can still ask about equidistribution to the uniform probability measure on the image of the map φ_m in (7.11). In other words, we can ask whether a variant of the *relative Hardy–Littlewood property* holds, as defined by Borovoi and Rudnick [6, Def. 2.3], with respect to the density function $\delta_{\overline{\mathfrak{U}(\mathbb{Z})}}$, which is the indicator function of the closure of $\mathfrak{U}(\mathbb{Z})$ in the adelic points $\mathfrak{U}(\mathbf{A}_{\mathbb{Z},\text{fin}}) = \prod_p \mathfrak{U}(\mathbb{Z}_p)$. In fact, the relative Hardy–Littlewood property fails: there are infinitely many places at which strong approximation fails, and so $\delta_{\overline{\mathfrak{U}(\mathbb{Z})}}$ is not locally constant. However, it is still measurable, and it is natural to investigate this weaker property.

We numerically tested equidistribution of integral points modulo m, where $m \in \{8, 3, 5, 7\}$. This set always includes all primes in S and at least one place not in S. In cases (ii), (v) and (vi), there is a failure of strong approximation simultaneously involving the primes 2 and 3; in case (i), there is a failure involving 3 and 5; in case (iii), there is one involving 2 and 7. To test for joint equidistribution modulo these primes,

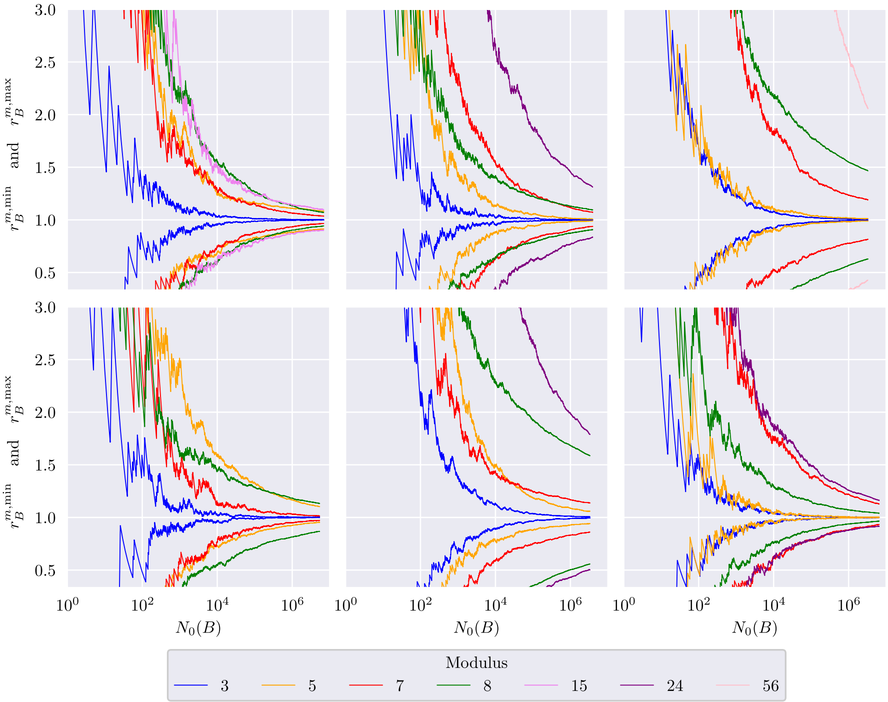

Fig. 7 A comparison of the maximal and minimal observed frequencies of reductions modulo m with the expected frequency

we have thus added $m = 24$, 15 and 56, respectively. In each case, we computed the set of integral points of height at most B for $B \leqslant 10^{1000}$, which can be done efficiently using the Vieta involutions, resulting in between $3 \cdot 10^6$ and $7 \cdot 10^6$ points. For m as before and $P_m \in \operatorname{im} \varphi_m$, we computed the frequencies

$$p_B^{(P_m)} = \frac{\#\{P \in \mathfrak{U}(\mathbb{Z}) \cap U_0 : H(P) \leqslant B,\ P \equiv P_m \bmod m\}}{\#\{P \in \mathfrak{U}(\mathbb{Z}) \cap U_0 : H(P) \leqslant B\}}.$$

Equidistribution modulo m means that $p_B^{(P_m)} \to 1/k$ as $B \to \infty$, where $k = \# \operatorname{im} \varphi_m$ is the number of points modulo m that lift to $\mathbb{Z}$. We thus determined

$$r_B^{m,\max} = k \max_{P_m \in \operatorname{im} \varphi_m} p_B^{(P_m)} \qquad \text{and} \qquad r_B^{m,\min} = k \min_{P_m \in \operatorname{im} \varphi_m} p_B^{(P_m)},$$

expecting that both quantities converge to 1. The results are recorded in Fig. 7. As $\# \operatorname{im} \varphi_m$ grows like m^2, we expect our order statistics to converge more slowly for larger values of m. With that in mind, our results seem compatible with equidistribution, even though we note that the distributions modulo 8 in cases (iii) and (v) are outliers.

8 The Markoff surface

The Markoff surface is defined by the cubic equation (1.4) and has an $\mathbf{A}_1$-singularity at $(0, 0, 0)$. Over the reals, this singularity is an isolated point, while the remaining four connected components are smooth. Again, let U_0 be the unbounded component on which $x, y, z > 0$.

Let $\tilde{X} \to X$ be a minimal desingularisation and E its exceptional divisor. Let $\varrho\colon \tilde{\mathfrak{X}} \to \mathfrak{X}$ be a model, let $\mathfrak{E}$ be the closure of E, let $\tilde{U} = \varrho^{-1}U$, and let $\tilde{\mathfrak{U}} = \varrho^{-1}\mathfrak{U}$. We note that the singular point $(0, 0, 0)$ is invariant under the Vieta involutions, both as an integral point and as an $\mathbb{F}_p$-point. It follows that any integral point on $\mathfrak{U}$ or $\tilde{\mathfrak{U}}$ that reduces to $(0, 0, 0) \in \mathfrak{U}(\mathbb{F}_p)$ must be $(0, 0, 0)$ or lie above $(0, 0, 0)$, respectively.

8.1 Non-archimedean local densities

The local densities, adjusted as in Section 7.2, coincide for the Markoff surface and its minimal desingularisation. More precisely, we note that for all primes, including 2 and 3, the point $(0, 0, 0)$ is the only singular point in $\mathfrak{U}(\mathbb{F}_p)$. In the light of this, we set

$$\mathfrak{U}(\mathbb{F}_p)' = \tilde{\mathfrak{U}}(\mathbb{F}_p)\backslash\mathfrak{E}(\mathbb{F}_p) \cong \{P \in \tilde{\mathfrak{U}}(\mathbb{F}_p) : \varrho(P) \neq (0, 0, 0)\};$$

this set contains the image of the reduction map $\mathfrak{U}(\mathbb{Z}) \cap U_0 \to \mathfrak{U}(\mathbb{F}_p)$ and only consists of smooth points. For $p = 2$, we computed the image of $\varphi_{2^k} : \mathfrak{U}(\mathbb{Z}) \cap U_0 \to \mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z})$ by the same method as in Section 7.2. For $2 \leqslant k \leqslant 10$, it consists of one fourth of the points in

$$\{P \in \mathfrak{U}(\mathbb{Z}/2^k\mathbb{Z}) : P \not\equiv (0, 0, 0) \bmod 2\}. \tag{8.1}$$

In contrast to the observation in Lemma 7.2, this is not a consequence of a failure of Hensel's lemma, as all points in the set (8.1) are smooth. Hence, we set

$$\sigma_2 = \frac{\#\operatorname{im}\varphi_4}{4^2},$$

without any of the normalisations described in Remark 7.3, and compute this to be $\sigma_2 = 1/4$. Computing $\operatorname{im}\varphi_m$ for m as in Proposition 7.4 does not reveal any further failures of strong approximation. In fact, it follows from recent work of Chen [11, Thm. 5.58] that the same is true when m is a product of primes, with each prime larger than some absolute constant. We are therefore led to set

$$\sigma_p = \frac{\#\mathfrak{U}(\mathbb{F}_p)'}{p^2},$$

for odd primes. It follows from [21, Lem. 6.4] (with $\alpha = 3$ and $\beta = 0$) that

$$\sigma_p = \begin{cases} \frac{8}{9} & \text{if } p = 3, \\ 1 + \frac{3\chi(p)}{p} & \text{if } p > 3, \end{cases}$$

Table 4 The circle method prediction for the Markoff surface and a comparison to the actual leading constant as determined by Zagier

c_{circle}	$c_{\text{circle}}/c_{\text{Zagier}}$
1.256791	0.2897693

where $\chi(p) = (\frac{-1}{p})$. We have $\varrho_U = 0$. Moreover, setting $\lambda_p = L_p(1,\chi)^{-3}$ clearly makes $\prod \lambda_p \sigma_p$ absolutely convergent. Letting $\lambda_0 = L(1,\chi)^3$, the analytic class number formula yields $\lambda_0 = \pi^3/2^6$.

Remark 8.1 This passage between points on $\mathfrak{U}$ and a desingularisation only works because of the exclusion of the singular point (0, 0, 0) modulo all primes. Its preimage on $\tilde{X}$ is a (-2)-curve E and geometrically isomorphic to $\mathbb{P}^1$. The ranks of Pic $\tilde{X}$ and Pic $\tilde{U}$ increase by one, so that $\operatorname{rk}\operatorname{Pic}\tilde{U} = 1$. As E splits over almost all primes p, the naïve local densities on $\tilde{U}$ would become $1 + \frac{1+3\chi(p)}{p}$ over these primes. In particular, $F_{\tilde{U}}(s)$ would have a pole of order 1 at $s = 0$. A similar heuristic for this desingularisation would thus predict a growth rate of $(\log B)^3$, which is larger than the $(\log B)^2$ obtained by Zagier [49]. Only by modifying the local densities to account for failures of strong approximation, can we remove this pole and return the expected order of growth to $(\log B)^2$.

8.2 Archimedean local densities

As ϱ is crepant and an isomorphism above the boundary, it follows that we have $\mathcal{O}_{\tilde{X}}(\varrho^{-1}(D_1 + D_2 + D_3)) \cong \varrho^*\omega_X \cong \omega_{\tilde{X}}$. Moreover, it is an isomorphism above the unbounded real components. Hence, arguing similarly to Lemma 2.5 (with $d = 3$), we have $c_\infty = 2$ in (2.11).

8.3 Conclusion

In summary, Proposition 3.2 and Heuristic 3.11 leave us with the prediction $N_U(B) \sim c_{\text{circle}}(\log B)^2$, as $B \to \infty$, where

$$c_{\text{circle}} = 2\lambda_0 \prod_p \lambda_p \sigma_p = \frac{4\pi^3}{3^5} \prod_{p>3} \left(1 - \frac{\chi(p)}{p}\right)^3 \left(1 + \frac{3\chi(p)}{p}\right).$$

We computed the Euler product for $p < 10^8$ and compared this constant with the constant 0.180717104712 obtained by Zagier [49]. (Note that, as pointed out in [1, p. 481], there is a typo in his paper.) Moreover, Zagier counts all ordered, positive Markoff triples and so his constant has to be multiplied by 24 to account for symmetries and signs before comparing it to our expectations. This is summarised in Table 4. We observe that the results are off by factors in a similar range to those present in Table 3.

9 Further examples

9.1 A question posed by Harpaz

In [23, Qn. 4.4], Harpaz asks about the number of integral points of bounded height on the surfaces $U_k \subset \mathbb{A}^3$ defined by the cubic polynomial $f(x, y, z) = (x^2 - ky^2)z - y + 1$, for a square-free integer $k > 1$. It will be useful to recall Harpaz' compactification, which is based on the map $U_k \to \mathbb{P}^2$ given by $(x, y, z) \mapsto (x : y : 1)$. This map factors through the blow-up X of $\mathbb{P}^2$ in the two points $(\pm\sqrt{k} : 1 : 1)$. Let $D_1 = V(z)$, $D_2 = V(x - \sqrt{k}y)$, and $D_3 = V(x + \sqrt{k}y)$. Then $D = D_1 + D_2 + D_3$ is defined over $\mathbb{Q}$, and U_k is isomorphic to $X \setminus D$.

Harpaz proves in [23, Prop. 4.3] that $U_k(\mathbb{Z})$ is Zariski dense whenever the real quadratic field $K = \mathbb{Q}(\sqrt{k})$ has class number one and is such that the reduction map $\mathfrak{o}_K^\times \to (\mathfrak{o}_K/\mathfrak{p})^\times$ is surjective for infinitely many prime ideals $\mathfrak{p}$ of degree 1 over $\mathbb{Q}$. Moreover, the surface U_k is smooth and admits a log K3 structure by [23, Ex. 2.13], and furthermore, its compactification is a del Pezzo surface of degree 7 having geometric Picard rank 3. Since the boundary is a triangle of three lines whose divisor classes are linearly independent, so it follows that the geometric Picard group of U_k is trivial. In particular, we have $\varrho_{U_k} = 0$ and $\mathrm{Br}_1(U_k)/\mathrm{Br}(\mathbb{Q}) = 0$. Moreover, note that the components of D intersect pairwise in a real point, so that $b = 2$. It now follows from Conjecture 1.1 that $N^\circ_{U_k}(B) = O\left((\log B)^2\right)$, where the implied constant depends on k.

We claim that the only $\mathbb{A}^1$-curve over $\mathbb{Z}$ is the line $z = y - 1 = 0$. Suppose for a contradiction that $z \neq 0$ and that U_k contains the $\mathbb{A}^1$-curve

$$x = a_0 t^k + \cdots + a_k, \quad y = b_0 t^k + \cdots + b_k, \quad z = c_0 t^l + \cdots + c_l,$$

with integer coefficients such that $\max\{|a_0|, |b_0|\} \neq 0$ and $c_0 \neq 0$. Comparing coefficients of t^{2k+l} yields $(a_0^2 - kb_0^2)c_0 = 0$, which implies that $a_0 = b_0 = 0$, since k is square-free. This is a contradiction and so U_k° is obtained by removing the line $z = y - 1 = 0$. Heuristic 3.11 then gives

$$N^\circ_{U_k}(B) \sim c_\infty \prod_p \sigma_p \cdot (\log B)^2, \tag{9.1}$$

where c_∞ is the leading constant in Proposition 2.4 and $\sigma_p = \lim_{k\to\infty} p^{-2k}\nu(p^k)$, in the notation of (3.2).

9.1.1 Real density

In this section we give a direct estimate for the real density $\mu_\infty(B)$, as defined in (2.4), as $B \to \infty$. However, it turns out that there is an analytic obstruction to the Zariski density of integral points near certain faces of the Clemens complex of a desingularisation of the compactification of U_k. The outcome of this is that we should redefine $\mu_\infty(B)$ to involve only $(x, y, z) \in U_k(\mathbb{R})$ for which

$$\min\{|x-\sqrt{k}y|, |x+\sqrt{k}y|\} < 1 < \max\{|x-\sqrt{k}y|, |x+\sqrt{k}y|\}, \tag{9.2}$$

and we redefine c_∞ to be the leading constant in the asymptotic formula for this modified real density. To check this it is convenient to make the change of variables $u = x+\sqrt{k}y$ and $v = x-\sqrt{k}y$. If $\max\{|u|, |v|\} < 1$, then $|y| = |u-v|/2\sqrt{k} \leqslant 1/\sqrt{k}$, leaving only the trivial solutions with $y = 0$. If $\min\{|u|, |v|\} > 1$, on the other hand, then $|z| = |(y-1)/uv| \ll \max(|u|, |v|)/|uv| < 1$, leaving only the non-dense set of solutions with small $|z|$.

Lemma 9.1 *We may take* $c_\infty = \frac{4}{\sqrt{k}}$.

Proof Using the Leray form to calculate the real density, it readily follows that

$$\mu_\infty(B) = \int_{\mathcal{R}} \frac{\mathrm{d}x\mathrm{d}y}{|x^2-ky^2|},$$

where $\mathcal{R} \subset \mathbb{R}^2$ is cut out by the inequalities $|x|, |y| \leqslant B$ and $|y-1| \leqslant B|x^2-ky^2|$, together with (9.2). Making the change of variables $u = x+\sqrt{k}y$ and $v = x-\sqrt{k}y$, we obtain

$$\mu_\infty(B) = \frac{1}{2\sqrt{k}} \int_{\mathcal{S}} \frac{\mathrm{d}x\mathrm{d}y}{|uv|},$$

where now $\mathcal{S} \subset \mathbb{R}^2$ is cut out by the inequalities

$$|u+v| \leqslant 2B, \quad |u-v| \leqslant 2\sqrt{k}B, \quad |u-v-2\sqrt{k}| \leqslant 2\sqrt{k}B|uv|,$$

together with $\min\{|u|, |v|\} < 1 < \max\{|u|, |v|\}$.

Summing over the possible signs of u and v, we deduce that

$$\mu_\infty(B) = \frac{1}{2\sqrt{k}} \sum_{\varepsilon_1,\varepsilon_2\in\{\pm1\}} \int_0^{2(1+\sqrt{k})B} \int_0^{2(1+\sqrt{k})B} \frac{\mathbf{1}_{\mathcal{S}}(\varepsilon_1 u, \varepsilon_2 v)}{uv} \mathrm{d}u\mathrm{d}v.$$

We isolate two subregions $\mathcal{S}_1 \sqcup \mathcal{S}_2 \subset \mathcal{S}$. Let $A > 0$ be a large parameter which doesn't depend on B and define $\mathcal{S}_1 = \left(\frac{A}{B}, \frac{1}{A}\right) \times \left(A, \frac{B}{A}\right)$ and $\mathcal{S}_2 = \left(A, \frac{B}{A}\right) \times \left(\frac{A}{B}, \frac{1}{A}\right)$. Taking $\mathbf{1}_{\mathcal{S}}(\varepsilon_1 u, \varepsilon_2 v) \leqslant 1$, the overall contribution to $\mu_\infty(B)$ from

$$(u, v) \in [0, 2(1+\sqrt{k})B]^2 \setminus \mathcal{S}_1 \sqcup \mathcal{S}_2$$

is readily found to be $O(\log B)$, where the implied constant is allowed to depend on A and k. Taking A sufficiently large, we clearly have $\mathbf{1}_{\mathcal{S}}(\varepsilon_1 u, \varepsilon_2 v) = 1$ whenever $(u, v) \in \mathcal{S}_1 \sqcup \mathcal{S}_2$. Hence

$$\mu_\infty(B) = \frac{2}{\sqrt{k}} \sum_{i\in\{1,2\}} \iint_{\mathcal{S}_i} \frac{\mathrm{d}u\mathrm{d}v}{uv} = \frac{4}{\sqrt{k}}(\log B)^2 + O(\log B),$$

with an implied constant that depends on A and k. □

9.1.2 Non-archimedean densities

Lemma 9.2 *Let p be a prime. Then*

$$\sigma_p = \begin{cases} 1 - \frac{1}{p^2} & \text{if } p > 2 \text{ and } (\frac{k}{p}) = -1, \\ 1 + \frac{1}{p^2} & \text{if } p > 2 \text{ and } (\frac{k}{p}) = +1, \\ 1 & \text{if } p \mid 2k. \end{cases}$$

Proof Let $\nu(p)$ be the number of zeros of f over $\mathbb{F}_p$. It follows from Hensel's lemma that $\sigma_p = \nu(p)/p$. Applying (7.2), we deduce that

$$\begin{aligned} \nu(p) &= p^2 + \frac{1}{p} \sum_{h \in \mathbb{F}_p^*} e_p(h) \sum_{x,y \in \mathbb{F}_p} e_p(-hy) \sum_{z \in \mathbb{F}_p} e_p\left(h(x^2 - ky^2)z\right) \\ &= p^2 + \sum_{h \in \mathbb{F}_p^*} e_p(h) U_p(h), \end{aligned}$$

by orthogonality of characters, where

$$U_p(h) = \sum_{\substack{x,y \in \mathbb{F}_p \\ x^2 = ky^2}} e_p(-hy).$$

Suppose first that $p \nmid 2k$. If $(\frac{k}{p}) = -1$ then $U_p(h) = 1$, since only $(x, y) = (0, 0)$ can occur. On the other hand, if $(\frac{k}{p}) = +1$, then

$$U_p(h) = 1 + \sum_{\substack{\eta \in \mathbb{F}_p \\ \eta^2 = k}} \sum_{\substack{x,y \in \mathbb{F}_p^* \\ x = \eta y}} e_p(-hy) = 1 + 2 \sum_{y \in \mathbb{F}_p^*} e_p(-hy) = -1,$$

since $h \in \mathbb{F}_p^*$. Suppose next that $p = 2$. Then

$$U_2(h) = \sum_{\substack{x,y \in \mathbb{F}_2 \\ x = ky}} e_2(y) = 0.$$

Finally, we suppose that $p > 2$ and $p \mid k$. In this case

$$U_p(h) = \sum_{y \in \mathbb{F}_p} e_p(-hy) = 0.$$

The lemma follows on putting these together and evaluating the sum over h. □

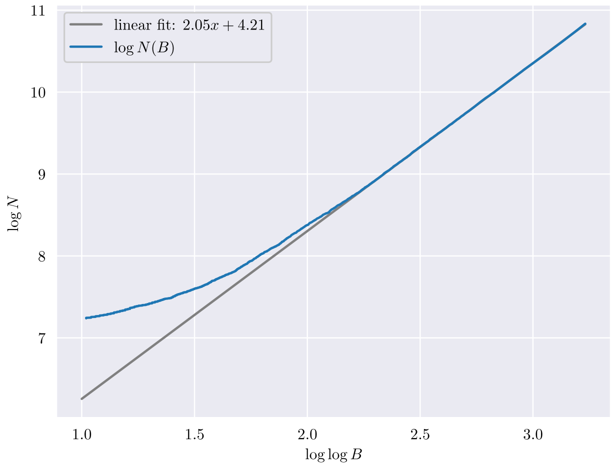

Fig. 8 The number of points on U_k for square-free $k \in [2, 1000]$ and a linear fit

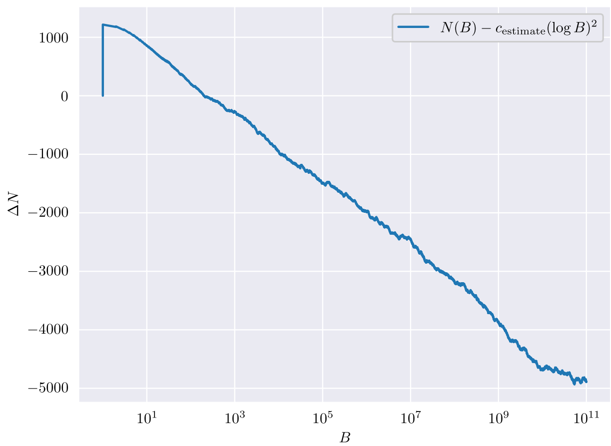

Fig. 9 Comparison of $N(B)$ with the prediction

9.1.3 Numerical data

Combining Lemmas 9.1 and 9.2 in (9.1), our heuristic leads us to expect that $N^{\circ}_{U_k}(B) \sim c^{(k)}_{\text{circle}}(\log B)$, with

$$c^{(k)}_{\text{circle}} = \frac{4}{\sqrt{k}} \prod_{p \nmid 2k} \left(1 + \frac{\left(\frac{k}{p}\right)}{p^2}\right).$$

We computed integral points of height at most 10^{11} on U_k for all square-free integers $k \in [2, 1000]$. Let

$$N(B) = \sum_{\substack{2 \leqslant k \leqslant 1000 \\ k \text{ square-free}}} N^{\circ}_{U_k}(B).$$

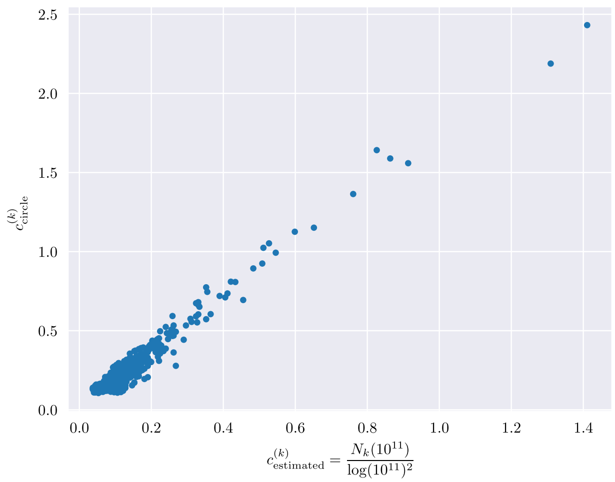

Fig. 10 A scatter plot comparing the predicted leading constants to the heuristic leading constants determined from the data

The sum of the predicted constant over all relevant k is

$$c_{\text{circle}} = \sum_{\substack{2 \leqslant k \leqslant 1000 \\ k \text{ square-free}}} c^{(k)}_{\text{circle}} \approx 148.8.$$

A linear regression of $\log N(B)$ against $\log \log B$, as in the previous sections, provides evidence for the exponent 2 of $\log B$ (Fig. 8). Based on this, a polynomial regression of degree 2 suggests a behaviour $N(B) = c_{\text{estimate}}(\log B)^2 + O(\log B)$, where $c_{\text{estimate}} = 87$. Note that $c_{\text{estimate}}/c_{\text{circle}} \approx \frac{3}{5}$, but we can offer no explanation for this disparity. This is consistent with taking $\gamma_U = \frac{3}{5}$ and $V = D(\mathbb{R}) \times U(\mathbf{A}^{\text{fin}}_{\mathbb{Z}})$ in Heuristic 1.2. In Fig. 9 we have plotted the difference $N(B) - c_{\text{estimate}}(\log B)^2$, for $B \leqslant 10^{11}$, which looks convincingly linear in $\log B$. Finally, in Fig. 10 we have included a scatter plot, in which each blue dot represents a surface in the family; on the x-axis is an estimated leading constant and on the y-axis is the circle method prediction for the leading constant associated to that particular surface. The correlation is rather good and a similar calculation to that recorded at the end of Section 6.2 results in $R^2 = 0.84$. This further illustrates that $\gamma_U \approx \frac{3}{5}$ is an appropriate value in Heuristic 1.2.

9.2 An example with higher Picard rank

Finally, we compare Conjecture 1.1 with numerical data for a smooth affine cubic surface of the shape

$$(ax + 1)(bx + 1) + (cy + 1)(dy + 1) = xyz,$$

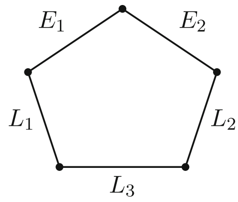

Fig. 11 Pentagon at infinity

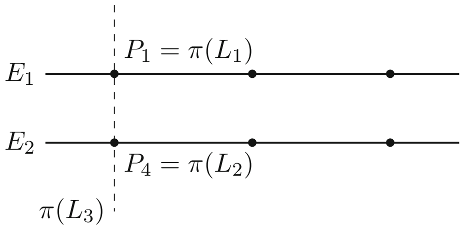

Fig. 12 Configuration of blown up points, with the images of the pentagon at infinity labeled

for $a, b, c, d \in \mathbb{Z}$. Such a surface $U = U_{a,b,c,d} \subset \mathbb{A}^3$ is smooth if $(a-b)(c-d) \neq 0$ and none of a, b, c or d are ± 1. Let X be the completion of U in $\mathbb{P}^3$, with homogeneous coordinates t_0, x_0, y_0, z_0, as in Section 7. The divisor at infinity is again a union of three lines L_1, L_2, and L_3 defined as in (7.1). In particular, $b = 2$ in Conjecture 1.1.

Next, we note that the point $Q = (0 : 0 : 0 : 1)$ is an $\mathbf{A}_2$-singularity. Let $\tilde{X}$ be a minimal desingularisation. This is a weak del Pezzo surface of degree 3 and so it has geometric Picard rank 7. As illustrated in Fig. 11, the triangle at infinity becomes a pentagon on $\tilde{X}$, formed by the strict transforms of L_1, L_2, and L_3 (still denoted by the same names) and two (-2)-curves E. The projection away from Q induces a morphism $\tilde{X} \to \mathbb{P}^2$. This morphism is a blow-up of six points, two sets of three on a line, as in Fig. 12. All negative curves are rational, and those making up $\tilde{D}$ are linearly independent, whence $\varrho_U = 2$. Moreover, this description of $\tilde{X}$ as a blow-up shows that $E_1 + E_2 + L_1 + L_2 + L_3$ has anticanonical class in the Picard group, and so U is log K3. Finally, since the five negative curves making up $\tilde{D}$ are linearly independent in $\mathrm{Pic}(\tilde{X}) = \mathrm{Pic}(\tilde{X}_{\overline{\mathbb{Q}}})$, the subvariety U does not have invertible regular functions, whence $\mathrm{Br}(U)/\,\mathrm{Br}(\mathbb{Q}) \cong H^1(\mathbb{Q}, \mathrm{Pic}(U_{\overline{\mathbb{Q}}})) = 0$.

It follows from Conjecture 1.1 that $N_U^\circ(B) = O\left((\log B)^4\right)$, and we proceed to investigate numerically the predicted power of $\log B$. While there is an obstruction as in [48, Thm. 2.4.1(i)], it only affects three of the minimal strata of $\tilde{D}$, namely those lying above Q. Thus this obstruction does not change the predicted order of growth, but merely the leading constant. Let $(a, b, c, d) = (-2, 3, -3, 5)$. Computing all integral $\mathbb{A}^1$-curves of degree at most 8, we found curves of degrees 1, 2, 3, and 4. We computed the set of integral points of height at most $2.5 \cdot 10^{10}$ on the surface and filtered out those on the $\mathbb{A}^1$-curves that we found. We ran a least squares estimate to compare $\log N(B)$ and $\log\log B$, where $N(B) = N_U^\circ(B)$, and we plotted the result in Fig. 13. This results in an empirical exponent of $\log B$ of 1.78, which is much less than the prediction of 4.

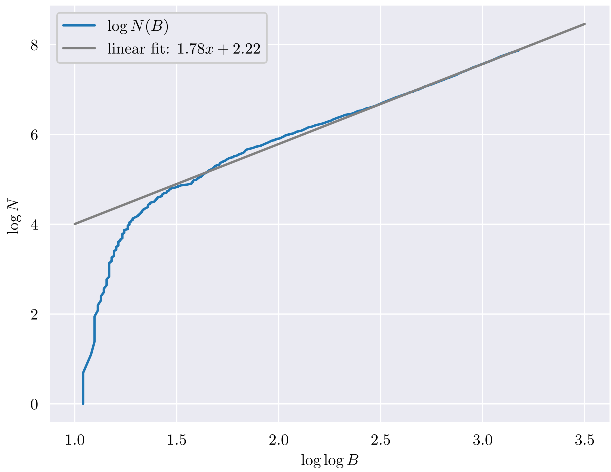

Fig. 13 The number of points on $U_{(-2,3,-3,5)}$ of height at most B and a linear fit

10 Conclusion

We end this article by summarising the numerical data that we have gathered. All surfaces that we studied are compatible with Conjecture 1.1. In fact, apart from the individual surface studied in Section 9.2, all of the examples seem to have an associated counting function $N_U^\circ(B)$ that behaves asymptotically like $c(\log B)^{\varrho_U+b}$, for suitable $c>0$, where ϱ_U and b are defined in Conjecture 1.1. In the case of the singular Markoff surface studied in Section 8, this was only true after modifying the heuristic to account for defects of strong approximation: these defects are big enough to not just make the product of p-adic densities smaller in size, but they also affect its convergence properties. It would be interesting to know whether a similar phenomenon, or perhaps the presence of large lower order terms, can explain the disparity observed in Section 9.2 regarding the exponent of $\log B$.

Turning to the leading constant c, the results of our investigation are more mixed. While our heuristic specialises to Heath-Brown's conjecture [24] for sums of three cubes, for which Booker and Sutherland [5] have provided evidence on average, in Section 6.1 we supplied a prediction for the surface $x^3+y^3+z^3=1$ with less compelling numerical data. On the other hand, the circle method heuristic aligned very well with numerical data for the the surfaces $x^3+ky^3+kz^3=1$ in Section 6.2. Moreover, in this case, we noted that the circle method heuristic is equivalent to allowing for an explicit low height rational number γ_U in Heuristic 1.2. For the surfaces $(x^2-ky^2)z=y-1$ in Section 9.1, we saw that the circle method heuristic only agrees with the numerical data when adjoining a suitable γ_U-factor, as in Heuristic 1.2. While in Section 9.1, this correlation is almost exclusively explained by the dependence of the archimedean volume on the parameter k, in Section 6.2 it is both the Euler product and the archimedean volume that depend highly on the parameter k.

Finally, for the Markoff surface and its variants studied in Sections 7 and 8, the circle method prediction became systematically too small after accounting for failures of strong approximation, and there is no obvious choice of γ_U-factor explaining the discrepancies. We suspect that the presence of a group action (generated by the Vieta involutions) makes these surfaces incompatible with the circle method.

In summary, we suspect that Heuristic 1.2 is true for most cubic surfaces and that there are only finitely possibilities for the γ_U-factor in the moduli space of all affine cubic surfaces over $\mathbb{Q}$. It would be very interesting to find a geometric interpretation for its value.

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Data availability The data used in Sections 6 and 9 is hosted on the *Göttingen Research Online Data* repository [8]. The code used to determine the data in Sections 6.2, 9.1 and 9.2 is found on the second author's github page.

References

1. Baragar, A., Umeda, K.: The asymptotic growth of integer solutions to the Rosenberger equations. Bull. Austral. Math. Soc. **69**, 481–497 (2004)
2. Batyrev, V.V., Tschinkel, Y.: Tamagawa numbers of polarized algebraic varieties. Astérisque **251**, 299–340 (1998)
3. Birch, B.J.: Forms in many variables. Proc. Roy. Soc. Ser. A **265**, 245–263 (1961/62)
4. Booker, A.: Cracking the problem with 33. Res. Number Theory **5**, 5–26 (2019)
5. Booker, A., Sutherland, A.V.: On a question of Mordell. Proc. Natl. Acad. Sci. U.S.A. **118**(11), e2022377118 (2021)
6. Borovoi, M., Rudnick, Z.: Hardy-Littlewood varieties and semisimple groups. Invent. Math. **119**, 37–66 (1995)
7. de la Bretèche, R., Browning, T.D.: Manin's conjecture for quartic del Pezzo surfaces with a conic fibration. Duke Math. J. **160**, 1–69 (2011)
8. Browning, T., Sutherland, A., Wilsch, F.: Data and code for Integral points on cubic surfaces: heuristics and numerics, (2024), https://doi.org/10.25625/4FLFH8
9. Cassels, J.W.S.: A note on the Diophantine equation $x^3 + y^3 + z^3 = 3$. Math. Comp. **44**, 265–266 (1985)
10. Chambert-Loir, A., Tschinkel, Y.: Igusa integrals and volume asymptotics in analytic and adelic geometry. Confluentes Math. **2**, 351–429 (2010)

11. Chen, W.: Nonabelian level structures, Nielsen equivalence, and Markoff triples. Annals of Math. **199**, 301–443 (2024)
12. Chen, X., Zhu, Y.: $\mathbb{A}^1$ curves on log K3 surfaces. Advances Math. **313**, 718–745 (2017)
13. Coccia, S.: The Hilbert Property for integral points of affine smooth cubic surfaces. J. Number Theory **200**, 353–379 (2019)
14. Colliot-Thélène, J.-L., Kanevsky, D., Sansuc, J.-J.: Arithmétique des surfaces cubiques diagonales. Diophantine approximation and transcendence theory (Bonn, 1985), 1–108, Lecture Notes in Math. **1290**, Springer, Berlin, (1987)
15. Colliot-Thélène, J.-L., Wei, D., Xu, F.: Brauer-Manin obstruction for Markoff surfaces. Ann. Sc. Norm. Super. Pisa Cl. Sci. **21**, 1257–1313 (2020)
16. Colliot-Thélène, J.-L., Wittenberg, O.: Groupe de Brauer et points entiers de deux familles de surfaces cubiques affines. Amer. J. Math. **134**, 1303–1327 (2012)
17. Colliot-Thélène, J.-L., Xu, F.: Brauer-Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms. Compositio Math. **145**, 309–363 (2009)
18. Duke, W., Friedlander, J.B., Iwaniec, H.: Bounds for automorphic L-functions. Invent. Math. **112**, 1–8 (1993)
19. Gamburd, A., Magee, M., Ronan, R.: An asymptotic formula for integer points on Markoff-Hurwitz varieties. Annals of Math. **190**, 751–809 (2019)
20. Getz, J.: Secondary terms in asymptotics for the number of zeros of quadratic forms over number fields. J. London Math. Soc. **98**, 275–305 (2018)
21. Ghosh, A., Sarnak, P.: Integral points on Markoff type cubic surfaces. Invent. Math. **229**, 689–749 (2022)
22. Greenblatt, M.: Resolution of singularities, asymptotic expansions of integrals and related phenomena. J. d'Analyse Math. **111**, 221–245 (2010)
23. Harpaz, Y.: Geometry and arithmetic of certain log K3 surfaces. Ann. Inst. Fourier **67**, 2167–2200 (2017)
24. Heath-Brown, D.R.: The density of zeros of forms for which weak approximation fails. Math. Comput. **59**, 613–623 (1992)
25. Heath-Brown, D.R.: A new form of the circle method and its application to quadratic forms. J. Reine Angew. Math. **481**, 149–206 (1996)
26. Heath-Brown, D. R.: The circle method and diagonal cubic forms. R. Soc. Lond. Philos. Trans. Ser. A **356**, 673–699 (1998)
27. Ireland, K., Rosen, M.: A classical introduction to modern number theory, 2nd edn. Springer-Verlag (1990)
28. Jahnel, J.: Brauer groups, Tamagawa measures, and rational points on algebraic varieties. American Math. Soc., (2014)
29. Jin, Y., Schmidt, A.: A Diophantine equation appearing in Diophantine approximation. Indag. Mathem. **12**, 477–482 (2001)
30. Lehmer, D.H.: On the Diophantine equation $x^3 + y^3 + z^3 = 1$. J. London Math. Soc. **31**, 275–280 (1956)
31. Loughran, D., Mitankin, V.: Integral Hasse principle and strong approximation for Markoff surfaces. Int. Math. Res. Not. IMRN **18**, 14086–14122 (2021)
32. Lyczak, J., Mitankin, V., Uppal, H.: Cubic surfaces failing the integral Hasse principle. Preprint, (2024). arXiv:2311.10008
33. Mahler, K.: Note on Hypothesis K of Hardy and Littlewood. J. London Math. Soc. **11**, 136–138 (1936)
34. Manin, Y.I.: Cubic forms, 2nd edn. North- Holland, Amsterdam, New York and Oxford (1986)
35. Mordell, L.J.: On the integer solutions of the equation $x^2 + y^2 + z^2 + 2xyz = n$. J. London Math. Soc. **28**, 500–510 (1953)
36. Peyre, E.: Hauteurs et mesures de Tamagawa sur les variétés de Fano. Duke Math. J. **79**, 101–218 (1995)
37. Peyre, E.: Points de hauteur bornée, topologie adélique et mesures de Tamagawa. J. Théor. Nombres Bordeaux **15**, 319–345 (2003)
38. Peyre, E.: Beyond heights: slopes and distribution of rational points. Arakelov geometry and diophantine applications, 1–67, Lecture Notes Math. **2276**, Springer, (2021)
39. Peyre, E., Tschinkel, Yu.: Tamagawa numbers of diagonal cubic surfaces of higher rank. Rational points on algebraic varieties, 275–305, Progr. Math., **199**, Birkhäuser, (2001)

40. Salberger, P.: Tamagawa measures on universal torsors and points of bounded height on Fano varieties. Astérisque **251**, 91–258 (1998)
41. Santens, T.: Manin's conjecture for integral points on toric varieties. Preprint, (2024). arXiv:2312.13914
42. Segre, B.: A note on arithmetical properties of cubic surfaces. J. London Math. Soc. **18**, 24–31 (1943)
43. Serre, J.-P.: Lectures on $N_X(p)$. Chapman & Hall/CRC Research Notes in Mathematics, vol. 11. CRC Press, Boca Raton, FL (2012)
44. Titchmarsh, E.C.: The theory of the Riemann zeta-function. 2nd ed., edited by D.R. Heath-Brown, Oxford University Press, (1986)
45. Tran, T.: Secondary terms in asymptotics for the number of zeros of quadratic forms. Ph.D. thesis, Duke Univeristy, (2020)
46. Varchenko, A.N.: Newton polyhedra and estimates of oscillatory integrals. Functional Anal. Appl. **18**, 175–196 (1976)
47. Wei, D., Xu, F.: Counting integral points in certain homogeneous spaces. J. Algebra **448**, 350–398 (2016)
48. Wilsch, F.: Integral points of bounded height on a certain toric variety. Trans. Amer. Math. Soc. **11**, 567–599 (2024)
49. Zagier, D.: On the number of Markoff numbers below a given bound. Math. Comp. **39**, 709–723 (1982)

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