Optimal Transport Methods for Kinetic Equations, Boundary Value Problems, and Discretization of Measures

by

Filippo Quattrocchi

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Abstract

The theory of optimal transport provides an elegant and powerful description of many evolution equations as gradient flows. The primary objective of this thesis is to adapt and extend the theory to deal with important equations that are not covered by the classical framework, specifically boundary value problems and kinetic equations. Additionally, we establish new results in periodic homogenization for discrete dynamical optimal transport and in quantization of measures.

Section 1.1 serves as an invitation to the classical theory of optimal transport, including the main definitions and a selection of well-established theorems. Sections 1.2-1.5 introduce the main results of this thesis, outline the motivations, and review the current state of the art.

In Chapter 2, we consider the Fokker–Planck equation on a bounded set with positive Dirichlet boundary conditions. We construct a time-discrete scheme involving a modification of the Wasserstein distance and, under weak assumptions, prove its convergence to a solution of this boundary value problem. In dimension 1, we show that this solution is a gradient flow in a suitable space of measures.

Chapter 3 presents joint work with Giovanni Brigati and Jan Maas. We introduce a new theory of optimal transport to describe and study particle systems at the *mesoscopic* scale. We prove adapted versions of some fundamental theorems, including the Benamou–Brenier formula and the identification of absolutely continuous curves of measures.

Chapter 4 presents joint work with Lorenzo Portinale. We prove convergence of dynamical transportation functionals on periodic graphs in the large-scale limit when the cost functional is asymptotically linear. Additionally, we show that discrete 1-Wasserstein distances converge to 1-Wasserstein distances constructed from *crystalline norms* on $\mathbb{R}^d$.

Chapter 5 concerns *optimal empirical quantization*: the problem of approximating a measure by the sum of n equally weighted Dirac deltas, so as to minimize the error in the p-Wasserstein distance. Our main result is an analog of Zador's theorem, providing asymptotic bounds for the minimal error as n tends to infinity.

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About the Author

From 2016 to 2021, Filippo Quattrocchi was a student in mathematics at the University of Pisa and at the Scuola Normale Superiore. He wrote the Bachelor's thesis *A variational approach to evolution problems* under the guidance of Prof. Massimo Gobbino in 2019, and the Master's thesis *Quantization of Probability Measures and the Import-Export Problem* under the supervision of Prof. Dario Trevisan in 2021. During the Master's program, he specialized in mathematical analysis, numerical analysis, and probability. He joined the Institute of Science and Technology Austria in September 2021 and subsequently became part of Prof. Jan Maas's group. His main research interests are gradient flows in spaces of measures, quantization of measures, and homogenization of optimal transport.

List of Collaborators and Publications

During my PhD at the Institute of Science and Technology Austria, I have had the opportunity and the honor to work with many researchers at ISTA, in Vienna, and abroad. To date, these collaborations have led to three publications and preprints, coauthored with Giovanni Brigati, Lorenzo Dello Schiavo, Jan Maas, and Lorenzo Portinale. I have written two other works as a sole author.

This thesis is based on the following works:[1]

- Filippo Quattrocchi. Variational structures for the Fokker–Planck equation with general Dirichlet boundary conditions. To appear in *Calculus of Variations and Partial Differential Equations*, 2025+. doi:10.1007/s00526-025-03193-1

  This work, referred to as [Qua25] throughout the thesis, is presented in Chapter 2.

- Giovanni Brigati, Jan Maas, and Filippo Quattrocchi. Kinetic Optimal Transport (OTIKIN) – Part 1: Second-Order Discrepancies Between Probability Measures. *arXiv preprint* arXiv:2502.15665, 2025.

  This work, referred to as [BMQ25] throughout the thesis, is presented in Chapter 3.

- Lorenzo Portinale and Filippo Quattrocchi. Discrete-to-continuum limits of optimal transport with linear growth on periodic graphs. *European Journal of Applied Mathematics*, 2024. doi:10.1017/S0956792524000810

  This work, referred to as [PQ24] throughout the thesis, is presented in Chapter 4.

- Filippo Quattrocchi. Asymptotics for Optimal Empirical Quantization of Measures. *arXiv preprint* arXiv:2408.12924, 2024.

  This work, referred to as [Qua24] throughout the thesis, is presented in Chapter 5. Some of the results of [Qua24] are contained in my Master's thesis [Qua21]. Details on the intersections and the novelties are given at the beginning of Chapter 5.

The following work [DSQ23]—not included in the thesis—also originated from a collaboration at ISTA:

- Lorenzo Dello Schiavo and Filippo Quattrocchi. Multivariate Dirichlet Moments and a Polychromatic Ewens Sampling Formula. *arXiv preprint* arXiv:2309.11292, 2023.

[1] Authors are listed in alphabetical order.

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CHAPTER 1

Introduction

Over the last four decades, the theory of optimal transport has consistently attracted considerable interest among both theoretical mathematicians and applied researchers. Its relevance is well-established not only in mathematical analysis (calculus of variations, PDE theory, functional analysis), but also in mathematical physics, probability, geometry, and statistics [Vil09b, San15, CNWR25, MRTV24, Mik21]. Furthermore, applications have been found in many other fields, ranging from economics [Gal16] to machine learning [MPHA25], from biology [BSK+24] to geophysics [MBM+16]. The originating mathematical problem, formulated by G. Monge almost 250 years ago [Mon81], stems from a remarkably simple and natural question: What is the *optimal* way to move a certain amount of mass to a different spatial configuration? The modern mathematical formulation of the problem, due to L. Kantorovich [Kan42] is as follows. Given are a cost function $c\colon X \times Y \to \mathbb{R}$, which quantifies the effort required to move a unit of mass from a location $x \in X$ to another one $y \in Y$, and measures μ, ν that represent the initial and final configurations on X, Y. The minimization problem reads

$$\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times Y} c(x,y) \, \mathrm{d}\pi(x,y) \,, \tag{1.0.1}$$

where $\Pi(\mu, \nu)$ is the set of *couplings*

$$\Pi(\mu,\nu) := \Big\{ \pi \in \mathcal{M}(X \times Y) \, : \, \pi(\cdot \times Y) = \mu(\cdot) \text{ and } \pi(X \times \cdot) = \nu(\cdot) \Big\} \,.$$

Note that $\Pi(\mu, \nu)$ is nonempty if and only if μ and ν have the same total mass m: in this case, $\frac{\mu \otimes \nu}{m} \in \Pi(\mu, \nu)$. For this reason, it is not restrictive to work with probability measures (i.e., to assume $m = 1$). Of particular interest is the case where $X = Y$, this space is endowed with a metric d, and the cost function c is set equal to a power d^p of the distance. For example, it is reasonable that on $X = Y = \mathbb{R}^d$, the cost of physically moving a unit of mass is proportional to the distance it covers. When $c = \mathrm{d}^p$, $p \geq 1$, the minimal total cost

$$W_p^p(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} \mathrm{d}^p(x,y) \, \mathrm{d}\pi(x,y) \tag{1.0.2}$$

turns out to be the p-th power of a metric on the space $\mathcal{P}_p(X)$ of the probability measures with finite p-th moment, called p-*Wasserstein* or p-*Kantorovich–Rubinstein distance*. We will refer to $\mathcal{P}_p(X)$ endowed with W_p as the p-*Wasserstein space*.

The interest in Wasserstein distances is not only due to the natural problem they derive from and their simple definition, but also to their many favorable properties, which give rise to an

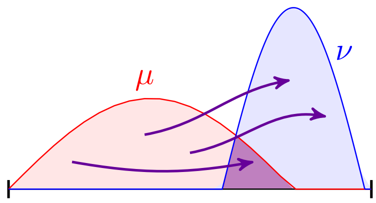

Figure 1.1: The mass displacement problem in dimension 1.

elegant and useful theory. The simplest example of this is that they generalize the underlying distance d: if δ_x, δ_y are Dirac deltas at $x, y \in X$, then $W_p(\delta_x, \delta_y) = \mathrm{d}(x, y)$. Many other results—more involved to prove, but generally with surprisingly neat statements—are, by now, classical and can be found, e.g., in the monographs [Vil03, Vil09b, San15]. In Section 1.1, we will present four of them: existence of optimal transport maps, the Benamou–Brenier formula, the Riemannian structure of the 2-Wasserstein space, and the gradient-flow representation of evolution equations. The latter, in particular, reveals a profound connection to PDEs and random processes, which still seems to hold great potential for further development. This leads to the first main topic of this thesis: the treatment of boundary value problems, kinetic equations, and—more indirectly—evolutions in a discrete (or discretized) setting by means of optimal transport techniques. To deal with these problems, it is often necessary to adapt the classical theory, e.g., by modifying the Wasserstein geometry. Determining the best modified framework and exploring the results that can be obtained within it are among our main objectives.

The second main topic is *discrete approximation* via optimal transport methods, which is ultimately motivated by computational problems such as, e.g., the design of numerical schemes and data compression. Indeed, first, assessing the quality of numerical approximations of certain PDEs can benefit from a theory of gradient flows of measures in a discrete setting. Second, Wasserstein distances are natural tools to quantify the error introduced by discretizing a measure. In Sections 1.2-1.5, we will contextualize and discuss the contributions of this thesis.

1.1 Classical Optimal Transport

Optimal transport maps

When X, Y are separable and completely metrizable topological spaces, and c is bounded from below and lower semicontinuous, it is not difficult to show that the problem (1.0.1) admits a minimizer $\pi \in \Pi(\mu, \nu)$ for every choice of $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$; see [Vil09b, Theorem 4.1]. More challenging is the question of existence of *deterministic* optimal couplings π, namely such that, additionally, there exists a map $T \colon X \to Y$ with

$$\int_{X \times Y} \varphi(x, y) \, \mathrm{d}\pi(x, y) = \int_X \varphi\big(x, T(x)\big) \, \mathrm{d}\mu(x) \qquad \text{for all } \varphi \in C_b(X \times Y) \,.$$

In general, the answer is negative. For example, when μ is a Dirac delta and ν is not, the mass must necessarily *split*. The pioneering work of M. Knott and C. S. Smith [KS84], Y. Brenier [Bre87], L. Rüschendorf and S. T. Rachev [RR90] provided the first positive result.

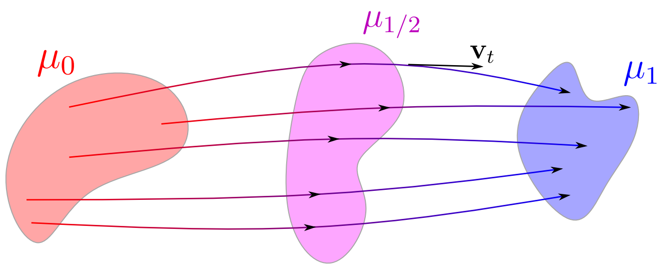

Figure 1.2: Displacement interpolation.

Theorem 1.1.1 (Knott–Smith, Brenier, Rüschendorf–Rachev)**.** *Assume that $X = Y = \mathbb{R}^d$, that μ, ν have finite second moment, that μ is absolutely continuous with respect to the Lebesgue measure, and that the cost c is the squared Euclidean distance. Then:*

1. *The problem* (1.0.1) *has a* unique *solution π. This coupling is induced by a map T of the form $T = \nabla\psi$ for a lower semicontinuous, convex function $\psi\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$.*
2. *Conversely, if $\psi\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and convex, and if the coupling π induced by $\nabla\psi$ belongs to $\Pi(\mu, \nu)$, then π is optimal for* (1.0.1)*.*

Since then, research on this topic has been intense, and many generalizations are now known. For example, analogous versions of Theorem 1.1.1 hold when we replace the squared Euclidean distance with any of its p-powers, $p \geq 1$ (without uniqueness for $p = 1$). For details, we refer to [Vil03, Chapter 4] and [Vil09b, Chapters 9 & 10], and the references therein.

The Benamou–Brenier formula

The Benamou–Brenier formula is a *dynamical* formulation of the Wasserstein distances due to J.-D. Benamou and Y. Brenier [BB00]. Let $X = Y = \mathbb{R}^d$, choose $c(x, y) = |x - y|^p$ with $p > 1$. Let us assume—for simplicity—that there exists an optimal map T between $\mu_0 = \mu$ and $\mu_1 = \nu$, and that the maps $T_t(x) := (1 - t)x + tT(x)$, with $t \in (0, 1)$, are invertible. We naturally find an *interpolating curve* of measures $(\mu_t)_{t\in[0,1]}$ by setting

$$\int_{\mathbb{R}^d} \varphi(z)\, \mathrm{d}\mu_t(z) = \int_{\mathbb{R}^d} \varphi\big(T_t(x)\big)\, \mathrm{d}\mu(x) \qquad \text{for all } \varphi \in C_b(\mathbb{R}^d)\,.$$

Interestingly, the curve $(\mu_t)_t$ solves a *continuity equation*: there exists a vector field $\mathbf{v}_t\colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\partial_t \mu_t + \operatorname{div}(\mathbf{v}_t \mu_t) = 0 \tag{1.1.1}$$

in the distributional sense. One such vector field can be found by setting $\mathbf{v}_t := (\partial_t T_t) \circ T_t^{-1}$ and, with this choice, one has

$$W_p^p(\mu_0, \mu_1) = \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^p\, \mathrm{d}\mu_t\, \mathrm{d}t\,. \tag{1.1.2}$$

Even more, the curves $(\mu_t, \mathbf{v}_t)_t$ we defined are exactly the *minimizers* of the action at the right-hand side of (1.1.2) among the solutions to the continuity equation. This means that the p-Wasserstein distance is characterized by a variational *dynamical* problem.

Theorem 1.1.2 (Benamou–Brenier [San15, Section 6.1])**.** *Assume that μ_0, μ_1 have finite p-th moments. Then*

$$W_p^p(\mu_0, \mu_1) = \min_{(\mu_t, \mathbf{v}_t)_{t\in[0,1]} \in \mathrm{CE}(\mu_0,\mu_1)} \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^p \, \mathrm{d}\mu_t \, \mathrm{d}t \,, \tag{1.1.3}$$

where $\mathrm{CE}(\mu_0, \mu_1)$ is the set of all narrowly continuous curves of probability measures $t \mapsto \mu_t$ between μ_0 and μ_1, and all vector fields $\mathbf{v}_t$ such that the continuity equation (1.1.1) is satisfied.

Equation (1.1.3) was derived for the first time (for $p = 2$) in the work of J.-D. Benamou and Y. Brenier [BB00]. Various generalizations are possible, for example when replacing $\mathbb{R}^d$ with a manifold; see the references in [Vil09b, Chapter 7]. A similar characterization, known as *Beckmann's problem* [Bec52], holds for $p = 1$.

Theorem 1.1.3 ([San15, Theorem 4.6])**.** *Assume that μ_0, μ_1 have finite 1-st moments. Then*

$$W_1(\mu_0, \mu_1) = \min \left\{ |\mathbf{w}| \left(\mathbb{R}^d\right) \, : \, \mu_1 - \mu_0 + \mathrm{div}(\mathbf{w}) = 0 \right\} \tag{1.1.4}$$

$$= \min_{(\mu_t, \mathbf{w}_t)_{t\in[0,1]}} \left\{ \int_0^1 |\mathbf{w}_t| \left(\mathbb{R}^d\right) \mathrm{d}t \, : \, \partial_t \mu_t + \mathrm{div}(\mathbf{w}_t) = 0 \right\} , \tag{1.1.5}$$

where $\mathbf{w}$ and $\mathbf{w}_t$, for $t \in [0, 1]$, are vector measures on $\mathbb{R}^d$. In (1.1.5), $t \mapsto \mu_t$ is taken among the curves of probability measures connecting μ_0 to μ_1.

Riemannian structure

The Benamou–Brenier formula (1.1.3) for $p = 2$ hints at a formal *Riemannian structure* on the space of probability measures $\mathcal{P}_2(\mathbb{R}^d)$ endowed with the 2-Wasserstein distance. The idea, first introduced by F. Otto [Ott01], is the following. The role of "smooth" curves is played by solutions $(\mu_t)_t$ to the continuity equation (1.1.1) (for some vector field), and the Hilbert norm on the tangent at μ_t is given by

$$\|\partial_t \mu_t\|_{\mu_t}^2 := \inf_{\mathbf{v}_t} \int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \, \mathrm{d}\mu_t \,, \tag{1.1.6}$$

where the infimum is taken among all $\mathbf{v}_t$'s such that (1.1.1) holds. It can be shown that the optimal $\mathbf{v}_t$ is the *only* solution to (1.1.1) in the $L^2(\mu_t)$-closure of the set of gradients $\left\{ \nabla\psi \, : \, \psi \in C_c^\infty(\mathbb{R}^d) \right\}$. In this way, (1.1.3) (for $p = 2$) becomes

$$W_2^2(\mu_0, \mu_1) = \min_{(\mu_t)_{t\in[0,1]}} \int_0^1 \|\partial_t \mu_t\|_{\mu_t}^2 \, \mathrm{d}t \,, \tag{1.1.7}$$

under the constraint that $(\mu_t)_t$ connects μ_0 to μ_1.

L. Ambrosio, N. Gigli, and G. Savaré [AGS08] established a similar compatibility between this (formal) Riemannian structure and the metric W_2. Precisely, they proved that solutions to the continuity equations coincide with *absolutely continuous curves* in the 2-Wasserstein space, and that the *metric derivative* equals the norm in (1.1.6).

Definition 1.1.4 ([AGS08, Definition 1.1.1])**.** Let (X, d) be a metric space. We say that an X-valued curve $(x_t)_{t\in[a,b]}$ is 2-absolutely continuous if there exists $\ell \in L^2(a, b)$ such that

$$\mathrm{d}(x_s, x_t) \leq \int_s^t \ell(r) \, \mathrm{d}r \qquad \text{for all } a \leq s \leq t \leq b \,. \tag{1.1.8}$$

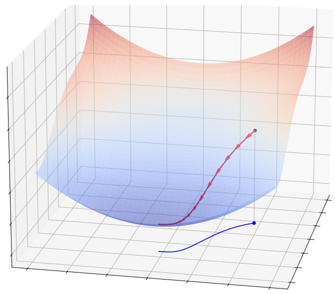

Figure 1.3: A gradient flow of an energy functional $E\colon \mathbb{R}^2 \to \mathbb{R}$.

Theorem 1.1.5 ([AGS08, Theorem 1.1.2]). *Let* (X, d) *be a metric space. If* $(x_t)_{t\in[a,b]}$ *is* 2*-absolutely continuous, then the* metric derivative

$$\left|x'\right|(t) \coloneqq \lim_{s\to t} \frac{\mathrm{d}(x_s, x_t)}{|t-s|} \tag{1.1.9}$$

exists for a.e. $t \in (a,b)$*. Moreover, the function* $t \mapsto |x'|(t)$ *is square-integrable, it is an admissible* ℓ *for* (1.1.8)*, and it is minimal, meaning that*

$$\left|x'\right|(t) \leq \ell(t) \qquad \text{for a.e. } t \in (a,b) \tag{1.1.10}$$

whenever ℓ *satisfies* (1.1.8)*.*

Theorem 1.1.6 (Ambrosio–Gigli–Savaré [AGS08, Theorems 8.3.1 & 8.4.5]). *Let* $(\mu_t)_{t\in[a,b]}$ *be a* 2*-absolutely continuous curve of measures in the metric space* $\big(\mathcal{P}_2(\mathbb{R}^d), W_2\big)$*. Then there exists* $(\mathbf{v}_t)_{t\in[a,b]}$ *such that the continuity equation* (1.1.1) *is satisfied, and*

$$\int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \,\mathrm{d}\mu_t \leq \left|\mu'\right|^2(t) \qquad \text{for a.e. } t \in [a,b]\,. \tag{1.1.11}$$

Conversely, if $(\mu_t)_{t\in[a,b]}$ *is a narrowly continuous curve that satisfies the continuity equation for some* $(\mathbf{v}_t)_{t\in[a,b]}$ *with* $\int_a^b \int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t < \infty$*, then* $(\mu_t)_{t\in[a,b]}$ *is* 2*-absolutely continuous with*

$$\left|\mu'\right|^2(t) \leq \int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \,\mathrm{d}\mu_t \qquad \text{for a.e. } t \in [a,b]\,. \tag{1.1.12}$$

In either case,

$$\left\|\partial_t \mu_t\right\|_{\mu_t} = \left|\mu'\right|(t) \qquad \text{for a.e. } t \in [a,b]\,. \tag{1.1.13}$$

Wasserstein gradient flows

Let $E\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth function. The gradient flow equation in $\mathbb{R}^d$ is the ODE

$$\dot{x}_t = -\nabla E(x_t)\,, \qquad x_t \in \mathbb{R}^d \text{ for all } t \geq 0\,. \tag{1.1.14}$$

Namely, the solution flows in the direction of steepest descent for E. This equation makes perfect sense in any Riemannian manifold M as well: given $E\colon M \to \mathbb{R}$, we define its gradient at $x \in M$ as the only vector $\nabla E(x) \in T_x M$ such that

$$(d_x E)(w) = \langle \nabla E, w \rangle_{T_x M} \qquad \text{for all } w \in T_x M\,.$$

As we have a Riemannian structure on $\mathcal{P}_2(\mathbb{R}^d)$, we can define gradient flows in this space. For example, if $\mathcal{E}\colon \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is an integral functional of the form

$$\mathcal{E}(\mu) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} F\big(\rho(x)\big)\, \mathrm{d}x & \text{if } \mu = \rho(x)\, \mathrm{d}x\,, \\ +\infty & \text{if } \mu \not\ll \mathrm{d}x\,, \end{cases}$$

for some $F\colon \mathbb{R} \to \mathbb{R}$, then the corresponding gradient flow equation reads

$$\partial_t \rho_t = \operatorname{div}\Big(\nabla F'(\rho_t)\rho_t\Big)\,, \qquad \mu_t = \rho_t(x)\, \mathrm{d}x\,.$$

This suggests that several important evolution equations can be interpreted as gradient flows in the 2-Wasserstein space. For instance, the heat equation $\partial_t \rho_t = \Delta \rho_t$ is found by choosing $F(\rho) \coloneqq \rho \log \rho$, but also several *nonlinear* equations naturally fit into this theory; see [San15, Section 8.4.2].

There are (at least) three other common notions of *gradient flow* in the metric setting:

1. Minimizing Movement approximation,
2. Curves of Maximal Slope,
3. Evolution Variational Inequalities.

In the 2-Wasserstein space, under certain regularity assumptions on the functional $\mathcal{E}$, the first two notions are substantially equivalent to the differential-geometric one discussed above. When $\mathcal{E}$ additonally enjoys a suitable convexity property, all notions coincide. Details are given in [AGS08, Chapter 11]. Let us present the Minimizing Movement approximation and the Curves of Maximal Slope, which will play a role later in the thesis, especially in Chapter 2.

In the Euclidean setting, (1.1.14) can be discretized in time using the Implicit Euler Scheme

$$x^\tau_{(k+1)\tau} - x^\tau_{k\tau} = -\tau \nabla E\Big(x^\tau_{(k+1)\tau}\Big)\,, \qquad k \in \mathbb{N}\,, \tag{1.1.15}$$

where $\tau > 0$ is the discretization parameter. Given $x^\tau_{k\tau}$, one can find $x^\tau_{(k+1)\tau}$ by solving

$$x^\tau_{(k+1)\tau} \in \operatorname*{arg\,min}_x \Big(2\tau E(x) + |x^\tau_{k\tau} - x|^2\Big)\,. \tag{1.1.16}$$

Interestingly, the differential structure of $\mathbb{R}^d$ is invisible in the last formula; only the metric structure needs to be defined. The following is due to E. De Giorgi [DG93].

Definition 1.1.7. Let (X, d) be a metric space, let $E\colon \mathrm{X} \to \mathbb{R}$, and let $x_\cdot\colon [0,\infty) \to \mathrm{X}$. We say that $(x_t)_{t\geq 0}$ is a *Minimizing Movements curve* if, for a sequence of discretization parameters $\tau_j \to 0$, the following holds. There exist curves $x^{\tau_j}_\cdot\colon [0,\infty) \to \mathrm{X}$ such that:

1. $t \mapsto x^{\tau_j}_t$ is constant on each interval $[k\tau_j, (k+1)\tau_j)$ for $k \in \mathbb{N}$,
2. for every $k \in \mathbb{N}$ and j, we have the the inclusion
$$x^{\tau_j}_{(k+1)\tau_j} \in \operatorname*{arg\,min}_{x\in \mathrm{X}} \Big(2\tau_j E(x) + \mathrm{d}^2(x^{\tau_j}_{k\tau_j}, x)\Big)\,, \tag{1.1.17}$$
3. we have the convergence $x^{\tau_j}_\cdot \to x_\cdot$ uniformly on compact sets as $j \to \infty$.

The application of this definition to the 2-Wassertstein space by R. Jordan, D. Kinderlehrer, and F. Otto [JKO98] allowed for the first identification of an evolution equation as a gradient flow in the 2-Wasserstein space. Indeed, the authors of [JKO98] showed that the solutions to the Fokker–Planck equation

$$\partial_t \rho_t = \Delta \rho_t + \operatorname{div}\Big(\nabla \Psi(x) \rho_t\Big) ,$$

for a smooth $\Psi\colon \mathbb{R}^d \to [0,\infty)$, are Minimizing Movement curves in $\big(\mathcal{P}_2(\mathbb{R}^d), W_2\big)$.

Finally, let us turn to Curves of Maximal Slope. Let $E\colon \mathbb{R}^d \to \mathbb{R}$ and $t \mapsto x_t \in \mathbb{R}^d$ be smooth functions. The chain rule, and Cauchy–Schwarz and Young's inequalities give

$$-\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) = -\nabla E(x_t) \cdot \dot{x}_t \leq |\nabla E(x_t)|\, |\dot{x}_t| \leq \frac{1}{2} |\nabla E(x_t)|^2 + \frac{1}{2} |\dot{x}_t|^2 .$$

The inequalities in the latter become equalities if and only if $\dot{x}$ is negatively proportional to $\nabla E(x_t)$ and $|\nabla E(x_t)| = |\dot{x}_t|$; hence, if and only if the gradient flow equation (1.1.14) is satisfied. In other words, (1.1.14) is equivalent to the opposite inequality

$$-\frac{\mathrm{d}}{\mathrm{d}t} E(x_t) \geq \frac{1}{2} |\nabla E(x_t)|^2 + \frac{1}{2} |\dot{x}_t|^2 .$$

Interestingly, the norms $|\nabla E(x_t)|$ and $|\dot{x}_t|$ can be written in purely metric terms. Indeed, we have already introduced the metric derivative (see Theorem 1.1.5), while the norm of the gradient can be seen—in the context of gradient flows—as the magnitude of the maximal *descending* slope, i.e.,

$$|\nabla E(x)| = \limsup_{y \to x} \frac{\big(E(x) - E(y)\big)_+}{|y - x|} .$$

These ideas are originally due to E. De Giorgi, A. Marino, and M. Tosques [DGMT80], and were later further developed by L. Ambrosio, N. Gigli, and G. Savaré [AGS08].

Definition 1.1.8 ([DGMT80, Definition 1.1], [AGS08, Definition 1.2.4])**.** Let (X, d) be a metric space, let $E\colon \mathrm{X} \to \mathbb{R} \cup \{+\infty\}$, and let $x \in \mathrm{X}$ be such that $E(x) < \infty$. We let the *descending slope* of E at x be

$$|\partial E|\,(x) \coloneqq \limsup_{y \to x} \frac{(E(x) - E(y))_+}{|y - x|} \tag{1.1.18}$$

if x is an accumulation point of X, and $|\partial E|\,(x) = 0$ otherwise.

Definition 1.1.9 ([AGS08, Definition 1.3.2])**.** Let (X, d) be a metric space, let $E\colon \mathrm{X} \to \mathbb{R} \cup \{+\infty\}$, and let $(x_t)_{t \in [a,b]}$ be an X-valued locally 2-absolutely continuous curve. We say that $(x_t)_{t \in [a,b]}$ is a *Curve of Maximal Slope* for E (with respect to its descending slope $|\partial E|$) if $t \mapsto E(x_t)$ is a.e. equal to a nonincreasing map φ such that

$$\varphi'(t) \leq -\frac{1}{2} \big|x'\big|\,(t) - \frac{1}{2} |\partial E|^2\,(x_t) \qquad \text{for a.e. } t \in [a,b] . \tag{1.1.19}$$

Wasserstein gradient flows have been deeply investigated. The connection they provide between optimal transport and PDEs has shed light on the geometric interpretation of many evolution equations, and, at the same time, has supplied new theoretical tools to prove existence, uniqueness, stability, speed of convergence, energy estimates, and functional inequalities. As an example, the Minimizing Movement scheme can be used to prove *existence*, even with irregular initial data (i.e., measures), and provides a *numerical method* to compute the solution. A comprehensive list of applications can be found in [AGS08, Section 11.1], together with many references to specific results in the literature. We refer to [San17] for a detailed overview on this topic.

1.2 Optimal Transport for Boundary Value Problems

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set. Any curve in $\big(\mathcal{P}(\Omega), W_2\big)$, by definition, conserves the total mass. Therefore, gradient flows in this space always satisfy appropriate *Neumann* boundary conditions; see [San17, Section 4.7]. For example, the gradient flows of the functional $\int_\Omega \rho \log \rho \, \mathrm{d}x$ are solutions to

$$\begin{cases} \partial_t \rho_t = \Delta \rho_t & \text{in } \Omega\,, \\ \partial_n \rho_t = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where ∂_n denotes the outer normal derivative.

At a first glance, therefore, the theory of optimal transport does not seem well-suited to describe and study equations with other types of boundary conditions. Nonetheless, in 2010, A. Figalli and N. Gigli [FG10] proposed a modified Wasserstein distance that provides a gradient-flow representation for equations with *Dirichlet* boundary conditions. The *boundary Wasserstein distance* $Wb_p(\mu, \nu)$ has almost the same definition of W_p:

$$Wb_p^p(\mu,\nu) := \inf_{\pi \in \Pi b(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} |y-x|^p \, \mathrm{d}\pi(x,y)\,, \qquad \mu,\nu \in \mathcal{M}(\Omega)\,, \quad p \geq 1\,, \tag{1.2.1}$$

but now the set of admissible transport plans is

$$\Pi b(\mu,\nu) := \Big\{\pi \in \mathcal{M}(\overline{\Omega}\times\overline{\Omega}) \,:\, \pi(A\times\Omega) = \mu(A) \text{ and } \pi(\Omega \times A) = \nu(A) \text{ for all } A \subseteq \Omega\Big\}\,. \tag{1.2.2}$$

The novelty is that transport plans are defined on the *closure* $\overline{\Omega}\times\overline{\Omega}$ of $\Omega\times\Omega$, although we prescribe only the restrictions to Ω of their marginals. Intuitively (see Figure 1.4), the allowed motion of mass is not only within Ω, but also from the interior to the boundary and *vice versa*. One can think of the amount of mass at each point of the boundary as infinite, in the sense that any amount can flow in and out of any region of $\partial\Omega$. Consider the functional

$$\mathcal{E}(\mu) := \begin{cases} \displaystyle\int_\Omega \Big(\rho(x)\log\rho(x) - \rho(x) + 1\Big)\,\mathrm{d}x & \text{if } \mu = \rho(x)\,\mathrm{d}x\,, \\ +\infty & \text{if } \mu \not\ll \mathrm{d}x\,, \end{cases}$$

and fix $\bar{\mu} = \bar{\rho}(x)\,\mathrm{d}x \in \mathcal{M}(\Omega)$. The main result of [FG10] is the convergence of the Minimizing Movement scheme

$$\begin{cases} \mu_0^\tau = \bar{\mu}\,, \\ \mu_{(k+1)\tau}^\tau \in \arg\min_{\mu\in\mathcal{M}(\Omega)}\Big(2\tau\mathcal{E}(\mu) + Wb_2^2(\mu_{k\tau}^\tau,\mu)\Big)\,, \end{cases} \qquad k \in \mathbb{N}\,, \quad \tau > 0\,,$$

to a solution to the heat equation with the constant Dirichlet boundary condition $\rho_t|_{\partial\Omega} \equiv 1$:

$$\begin{cases} \partial_t \rho_t = \Delta\rho_t & \text{in } \Omega\,, \\ \rho_t = 1 & \text{on } \partial\Omega\,, \\ \rho_0 = \bar{\rho} & \text{in } \Omega\,. \end{cases}$$

Remarkably, only one hypothesis is enforced: the finiteness of $\int_\Omega \bar{\rho}\log\bar{\rho}\,\mathrm{d}x$. No regularity is required on $\partial\Omega$, as long as the identity $\rho_t|_{\partial\Omega} \equiv 1$ is interpreted as $(\rho_t - 1) \in W_0^{1,1}(\Omega)$.

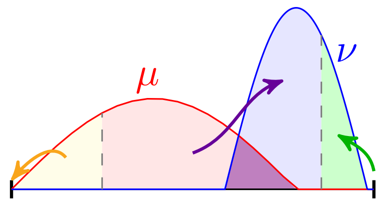

Figure 1.4: An admissible transport plan for Wb_p when Ω is an interval in dimension $d = 1$.

Recent research has shown a renewed interest in the optimal transport interpretation of equations with boundary conditions other than Neumann [Mor18, PS20, KKS25, CMS25, EM25, BMRv25]. With [Qua25] (Chapter 2), we aim at contributing to this line of research, by demonstrating its applicability to equations with *more general* Dirichlet boundary conditions, under *weak assumptions*.

The subject of [Qua25] is the Fokker–Planck equation with positive and temporally constant[1]—but otherwise arbitrary—boundary conditions:

$$\begin{cases} \partial_t \rho_t = \operatorname{div}(\nabla \rho_t + \rho_t \nabla V) & \text{in } \Omega\,, \\ \rho_t = g & \text{on } \partial\Omega\,, \\ \rho_0 = \bar{\rho} & \text{in } \Omega\,. \end{cases} \tag{1.2.3}$$

The domain Ω, the potential $V\colon \Omega \to \mathbb{R}$, the boundary value $g\colon \partial\Omega \to \mathbb{R}_{>0}$, and the initial datum $\bar{\rho}\colon \Omega \to \mathbb{R}_+$ are given. The main results are:

1. the construction of a scheme of Minimizing Movement type and its proof of convergence to a solution to (1.2.3),
2. a Curve of Maximal Slope formulation of (1.2.3) when Ω is an interval in $\mathbb{R}^1$.

This work builds upon the paper [FG10] discussed above and [Mor18] by J. Morales. In fact, a Minimizing Movement scheme for a problem similar to (1.2.3) is also described in [Mor18], but we significantly reduce the regularity hypotheses on $\partial\Omega$, on V, and on $\bar{\rho}$, thereby obtaining, in particular, the same assumptions as in [FG10] for Ω and $\bar{\rho}$. Another fundamental aspect of [Qua25] is the idea of *lifting* the problem to a larger space, namely a suitable subset $\mathscr{S}$ of the signed measures on the closure $\overline{\Omega}$. This idea is partially inspired by [Mon21, PS20], but this is its first use as a convenient way to handle *arbitrary* (positive) boundary conditions. More concretely, we define a transportation functional $\mathcal{T}$—not a distance—similar to the boundary Wasserstein distance Wb_2, but between signed measures in $\mathscr{S}$, and consider the driving functional[2]

$$\mathcal{H}(\mu) \coloneqq \begin{cases} \displaystyle\int_\Omega (\log \rho - 1 + V)\, \mathrm{d}\mu + \int_{\partial\Omega} (\log g + V)\, \mathrm{d}\mu & \text{if } \mu|_\Omega = \rho(x)\, \mathrm{d}x\,, \\ +\infty & \text{if } \mu|_\Omega \not\ll \mathrm{d}x\,. \end{cases}$$

Note that $\mathcal{H}$ depends on the measure μ on the full closure $\overline{\Omega}$, which justifies the necessity of a larger space of measures. Like in [FG10], we see the evolution (1.2.3) as a motion of mass that can be freely exchanged with the infinite reservoir at the boundary, but we additionally keep track of the balance of mass taken or deposited at each point of the boundary. For this reason we use *signed* measures.

[1] and not too irregular
[2] Assume here, for simplicity, that V continuously extends to the boundary.

Theorem 1.2.1 (Q., Theorem 2.1.1 (informal)). *Given $\bar{\mu} \in \mathscr{S}$, the scheme*

$$\begin{cases} \mu_0^\tau = \bar{\mu}\,, \\ \mu_{(k+1)\tau}^\tau \in \arg\min_{\mu \in \mathscr{S}} \Big(2\tau \mathcal{H}(\mu) + \mathcal{T}^2(\mu_{k\tau}^\tau, \mu) \Big)\,, \end{cases} \qquad k \in \mathbb{N}\,, \qquad \tau > 0 \tag{1.2.4}$$

converges to a curve $t \mapsto \mu_t$ such that its restriction to Ω satisfies (1.2.3).

In the case where $V \equiv 0$ and $g \equiv 1$, this result reduces to the theorem by A. Figalli and N. Gigli [FG10]. Indeed, Wb_2 can be seen as a projection[3] of $\mathcal{T}$ and, when $\log g + V \equiv 0$ on $\partial\Omega$, we have $\mathcal{E}(\mu|_\Omega) = \mathcal{H}(\mu)$.

Since $\mathcal{T}$ is not a distance, (1.2.4) is not exactly a Minimizing Movement scheme, and we cannot say that the limit $t \mapsto \mu_t$ is a gradient flow. Nonetheless, this theorem can be used as a prototype to prove existence in problems with general Dirichlet boundary conditions *under weak assumptions*, as well as, possibly, numerically construct a solution. Furthermore, in a similar way as with the classical Minimizing Movement scheme, this type of existence proof allows to establish properties of the solution that, even when formally derivable from the equation, may be difficult to directly prove under weak assumptions; see Remark 2.1.4.

We obtain a more refined result in dimension $d = 1$, that is, when Ω is an interval. In this case, we define a *true distance* $\widetilde{Wb}_2$ on $\mathscr{S}$—again, similar to Wb_2—and prove the following.

Theorem 1.2.2 (Q., Theorem 2.1.5 (informal)). *Assume that Ω is an interval in $\mathbb{R}^1$. Given $\bar{\mu} \in \mathscr{S}$, the limit curve $t \mapsto \mu_t$ found with Theorem 1.2.1 is a Curve of Maximal Slope for $\mathcal{H}$ in the space $(\mathscr{S}, \widetilde{Wb}_2)$.*

The main difficulty in the proof of this theorem is to ensure that the slope $|\partial\mathcal{H}|$ is lower semicontinuous. We overcome it by deriving an explicit formula for $|\partial\mathcal{H}|$. In [FG10], the identification of the slope $|\partial\mathcal{E}|$ was left as an open problem, which, as a byproduct of our proof, we resolve in the case $d = 1$.

1.3 Kinetic Optimal Transport

Kinetic equations describe time-evolving physical systems at a mesoscopic scale, when particles are not individually traceable, but we can write—for every time t—a statistical description of their positions and velocities (i.e., a distribution on the phase space). One example is the kinetic Fokker–Planck equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \operatorname{div}_v \Big(\nabla_v f(t,x,v) + f(t,x,v)\, v + f(t,x,v)\, \nabla_x U(x) \Big)\,, \qquad (t,x,v) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d\,,$$

where $U \colon \mathbb{R}^d \to \mathbb{R}$ is a potential. The study of these equations is an active research area, with many open questions, relative, e.g., to stability and convergence to equilibrium of their solutions, and to the precise mathematical links between micrsoscopic, mesoscopic, and macroscopic descriptions of a system, which is also part of Hilbert's sixth problem [Gor18].

One fruitful approach to the convergence-to-equilibrium problem—mainly for linear equations—is C. Villani's theory of *hypocoercivity* [Vil09a]. Optimal transport too has been employed in

[3] in the sense made precise in Lemma 2.4.1

kinetic theory, e.g., in the celebrated works [Dob79] by R. L. Dobrushin and [Loe06] by G. Loeper on the stability of Vlasov's equations. More recent stability results have been obtained by introducing *twisted* Wasserstein distances, inspired by hypocoercivity theory [BGM10, Iac16]. Furthermore, it has been shown that some kinetic equations can be approximated by time-discrete schemes that, step-by-step, solve an optimal transport minimization problem [Hua00, DPZ14, Par25], in a similar fashion to the Jordan–Kinderlehrer–Otto scheme [JKO98].

Although known contraction estimates and the aforementioned discretization schemes suggest the possibility of a gradient flow description, an analog of Wasserstein gradient flows has not yet been developed for kinetic equations. With [BMQ25] (Chapter 3), written in collaboration with G. Brigati and J. Maas, we put forward the foundations of one such theory. One of the main future goals is to obtain new convergence estimates for a large class of kinetic equations, including those nonlinear ones that are not covered by the theory of hypocoercivity.

The main object we introduce is a discrepancy d between probability measures on the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_v$. This discrepancy is based on the minimization of the *acceleration* of curves between coupled points. Its construction is as follows. First, for fixed $T > 0$ and $(x, v), (y, w) \in \mathbb{R}^d_x \times \mathbb{R}^d_v$, we define

$$\tilde{d}^2_T\Big((x,v),(y,w)\Big) := \inf_{\alpha \in H^2(0,T;\mathbb{R}^d_x)} \left\{ T \int_0^T \left|\alpha''(t)\right|^2 \,\mathrm{d}t \,:\, (\alpha,\alpha')(0) = (x,v)\,, \quad (\alpha,\alpha')(T) = (y,w) \right\} . \qquad (1.3.1)$$

Secondly, we consider the optimal transport problem

$$\tilde{\mathrm{d}}^2_T(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v \times \mathbb{R}^d_x \times \mathbb{R}^d_v} \tilde{d}^2_T\Big((x,v),(y,w)\Big) \,\mathrm{d}\pi(x,v,y,w)\,, \qquad \mu,\nu \in \mathcal{P}_2(\mathbb{R}^d_x \times \mathbb{R}^d_v)\,.$$

Thirdly, we set d equal to the W_2-lower semicontinuous envelope of the infimum over $T > 0$ of $\tilde{\mathrm{d}}_T$. In can be checked that replacing the second derivative $\alpha''(t)$ with $\alpha'(t)$ would give the squared Euclidean distance between x and y in (1.3.1), regardless of the choice of T, and the subsequent constructions would yield the classical 2-Wasserstein distance. Therefore, d can be thought of as a "second-order" version of W_2, although we emphasize that d is *not* a distance.

Our main results are a kinetic Benamou–Brenier formula and the identification of 2-absolutely continuous curves with time-reparametrized solutions to Vlasov's equations, which reminds [AGS08, Theorem 8.3.1] (Theorem 1.1.6) from the classical theory. The Benamou–Brenier formula has been independently obtained also in a recent work by K. Elamvazhuthi [Ela25].

Theorem 1.3.1 (Elamvazhuthi, Brigati–Maas–Q., Theorem 3.1.2 (simplified))**.** *For every $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and $T > 0$, we have*

$$\tilde{\mathrm{d}}^2_T(\mu,\nu) = \inf_{(\mu_t, F_t)_{t\in[0,T]}} T \int_0^T \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} |F_t|^2 \,\mathrm{d}\mu_t(x,v)\,\mathrm{d}t\,, \qquad (1.3.2)$$

where the infimum is taken among solutions to Vlasov's equation

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \nabla_v \cdot (F_t \mu_t) = 0 \qquad (1.3.3)$$

starting at $\mu_0 = \mu$ and ending at $\mu_T = \nu$.

Theorem 1.3.2 (Brigati–Maas–Q., Theorem 3.1.7 (simplified)). *Under appropriate regularity assumptions, the following hold.*

1. *Let $(\tilde{\mu}_s)_{s\in(\tilde{a},\tilde{b})}$ be* d-2*-absolutely continuous, i.e., there exists $\tilde{\ell} \in L^2(\tilde{a},\tilde{b})$ such that*

$$\mathsf{d}(\tilde{\mu}_s, \tilde{\mu}_t) \le \int_s^t \tilde{\ell}(r)\, \mathrm{d}r \qquad \text{for } s < t\,. \tag{1.3.4}$$

Then, there exist $(F_t)_{t\in(a,b)}$ and a bi-Lipschitz time-reparametrization $(\mu_t)_{t\in(a,b)}$ of the curve $(\tilde{\mu}_s)_{s\in(\tilde{a},\tilde{b})}$ such that $(\mu_t, F_t)_{t\in(a,b)}$ satisfies Vlasov's equation (1.3.3), and the right d*-derivative of $(\mu_t)_{t\in(a,b)}$ is $\|F_t\|_{L^2(\mu_t)}$, namely,*

$$\lim_{h\downarrow 0} \frac{\mathsf{d}(\mu_t, \mu_{t+h})}{h} = \|F_t\|_{L^2(\mu_t)} \qquad \text{for a.e. } t \in (a,b)\,. \tag{1.3.5}$$

2. *Let $(\mu_t, F_t)_{t\in(a,b)}$ be a solution to Vlasov's equation (1.3.3). Then,*

$$\mathsf{d}(\mu_s, \mu_t) \le 2\int_s^t \|F_r\|_{L^2(\mu_r)}\, \mathrm{d}r \qquad \text{for } s < t\,, \tag{1.3.6}$$

and

$$\limsup_{h\downarrow 0} \frac{\mathsf{d}(\mu_t, \mu_{t+h})}{h} \le \|F_t\|_{L^2(\mu_t)} \qquad \text{for a.e. } t \in (a,b)\,. \tag{1.3.7}$$

These theorems give the rigorous mathematical justification for a new formal degenerate Riemannian-like structure on $\mathcal{P}_2(\mathbb{R}^d_x \times \mathbb{R}^d_v)$: given a curve $(\mu_t)_t$ that satisfies Vlasov's equation (1.3.3) for some vector field $(F_t)_t$, the norm of its tangent vector is given by

$$\|\partial_t \mu_t\|^2_{\mu_t} := \inf_{F_t} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} |F_t|^2\, \mathrm{d}\mu_t\,,$$

where the infimum is taken among all F_t's such that (1.3.3) holds. See Remarks 3.1.9, 3.1.12, and 3.1.13 for further details. In future works, we plan to show that, with these definitions or suitable variants thereof (depending on the specific problem), the solutions to some significant kinetic equations are gradient flows.

Finally, let us point out that a "second-order" optimal transport theory is useful also in many applications requiring to construct *smooth interpolations* of measures:

1. optimal steering of a fleet of agents;
2. trajectory inference for particle motion or cell development;
3. image interpolation for computer graphics.

In fact, in [BMQ25], we also propose a variation on the smooth time interpolation of [CCLG+21]. With our theory, we prove that this new interpolation enjoys an injectivity property, which may be desirable in practical applications since it prevents sharp shrinkage at intermediate times.

1.4 Discrete Dynamical Optimal Transport

The geometry induced by W_2 is well-suited to describe many PDEs on the Euclidean space and on manifolds, but not the evolution of continuous-time Markov chains on *discrete* spaces. Indeed, as observed by J. Maas (see [Maa11, Remark 2.1]) the heat flow induced by an irreducible and reversible Markov kernel on a finite set X cannot[4] be 2-absolutely continuous in $\big(\mathcal{P}_2(X), W_2\big)$. The missing ingredient here is an analog of the Benamou–Brenier formula. The gradient-flow description in this setting was recovered with the introduction of an alternative distance $\mathcal{W}$ by J. Maas [Maa11], A. Mielke [Mie11], and S.-N. Chow, W. Huang, Y. Li, and H. Zhou [CHLZ12]. Its definition resembles the Benamou–Brenier formula. Let $G = (X, E)$ be a finite undirected graph (i.e, $E \subseteq X \times X$ is symmetric), let $\pi \in \mathcal{P}(X)$ be a reference measure, let $\omega\colon E \to \mathbb{R}_+$ be a symmetric weight function, let $\theta(a, b) := \int_0^1 a^s b^{1-s}\, \mathrm{d}s$ denote the logarithmic mean. Given $m_0, m_1 \in \mathcal{P}(X)$, we write $\mathrm{CE}_G(m_0, m_1)$ for the set of all $(m_t, J_t)_{t\in[0,1]}$ such that $t \mapsto m_t \in \mathcal{P}(X)$ connects m_0 to m_1, each $J_t \in \mathbb{R}^E$ is antisymmetric, i.e., $J_t(x, y) = -J_t(y, x)$, and the following *discrete continuity equation* is satisfied:

$$\partial_t m_t(x) + \sum_{y\in X\,:\,(x,y)\in E} J_t(x, y) = 0 \qquad \text{for all } x \in X\,. \tag{1.4.1}$$

We set

$$\mathcal{W}^2(m_0, m_1) := \min_{(m_t,J_t)_t\in\mathrm{CE}_G(m_0,m_1)} \int_0^1 \frac{1}{2} \sum_{(x,y)\in E} \frac{|J_t(x, y)|^2}{\theta\left(\frac{m_t(x)}{\pi(x)}, \frac{m_t(y)}{\pi(y)}\right)}\omega(x, y)\, \mathrm{d}t\,. \tag{1.4.2}$$

Let us remark that it is also possible to construct analogs of W_p for every $p \geq 1$; see [GKMP23, Remark 2.6]. For example, the counterpart of W_1 is

$$\min_{(m_t,J_t)_t\in\mathrm{CE}_G(m_0,m_1)} \int_0^1 \frac{1}{2} \sum_{(x,y)\in E} |J_t(x, y)|\, \omega(x, y)\, \mathrm{d}t\,. \tag{1.4.3}$$

Beyond gradient flows in the metric space induced by $\mathcal{W}$ (see also [Mie13, EM14]), research around this topic has mainly developed in two directions: functional inequalities for Markov chains [EM12, EMT15, FM16, EHMT17, EF18], and discrete-to-continuum limits [GM13, GT20, GKMP23, GMP25, GKM20, GKMP20, GK26, Lav21]. The latter—which is the subject of [PQ24], Chapter 4—deals with the following problem: If $G_n = (X_n, E_n)$ is a sequence of graphs *embedded* in $\mathbb{R}^d$ (or a manifold) that, in the limit, tend to fill the space, can we say that the corresponding (suitably rescaled) distances $\mathcal{W}_n$ (or variants thereof) are closer and closer to a Wasserstein distance? The graph G_n can represent, for instance, a numerical discretization of the space, or a model of atoms or neurons. In practice, G_n is typically periodic [GM13, GKMP23, PQ24] or randomly sampled [GT20, GMP25, GK26]. The question posed above is natural, given the similarity between the definition of $\mathcal{W}$ and the Benamou–Brenier formula for W_2. A positive answer may allow to infer properties of the space $\big(\mathcal{P}(X_n), \mathcal{W}_n\big)$ from those of the well-studied $\big(\mathcal{P}_2(\mathbb{R}^d), W_2\big)$, and to ensure consistency of numerical schemes to compute Wasserstein distances or Wasserstein gradient flows in $\mathcal{P}_2(\mathbb{R}^d)$ [GT20, Lav21].

The work [PQ24] (Chapter 4), coauthored with L. Portinale, builds upon [GKMP23] by P. Gladbach, E. Kopfer, J. Maas, and L. Portinale, and answers a question that remained

[4] unless one starts at the equilibrium

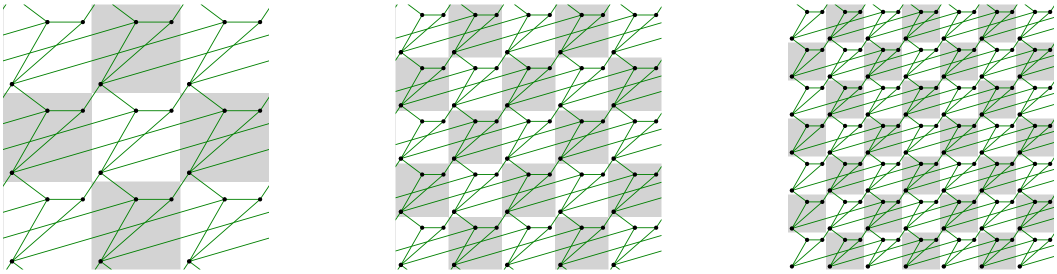

Figure 1.5: Successive scalings of a periodic graph.

open in this previous work. The problem is the Γ-convergence of dynamical transportation functionals—i.e., generalizations of $\mathcal{W}$—for periodic graphs in the large-scale limit. Given is a $\mathbb{Z}^d$-periodic graph $\bar{G} = (\bar{X}, \bar{E})$ in $\mathbb{R}^d$. For $\epsilon > 0$, we set $X_\epsilon := \epsilon\bar{X}/\mathbb{Z}^d$ and $E_\epsilon := \epsilon\bar{E}/\mathbb{Z}^d$, which define a graph G_ϵ in the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We fix a convex, local, and lower semicontinuous *cost function* $\bar{F}\colon \mathbb{R}_+^{\bar{X}} \times \mathbb{R}^{\bar{E}} \to \mathbb{R} \cup \{+\infty\}$, and suitably define a rescaled version $F_\epsilon\colon \mathbb{R}_+^{X_\epsilon} \times \mathbb{R}^{E_\epsilon} \to \mathbb{R} \cup \{+\infty\}$. Hence, we define the rescaled *action functional*

$$\mathcal{A}_\epsilon\Big((m_t, J_t)_t\Big) := \int_0^1 F_\epsilon(m_t, J_t)\,\mathrm{d}t\,, \tag{1.4.4}$$

and the *minimal action functional* (or *dynamical transportation functional*)

$$\mathcal{MA}_\epsilon(m_0, m_1) := \inf_{(m_t,J_t)_t \in \mathrm{CE}_{G_\epsilon}(m_0,m_1)} \mathcal{A}_\epsilon\Big((m_t, J_t)_t\Big)\,, \qquad m_0, m_1 \in \mathcal{P}(X_\epsilon)\,. \tag{1.4.5}$$

For example, the choice

$$\bar{F}(m, J) = \frac{1}{2} \sum_{(x,y)\in\bar{E}\,:\,x\in[0,1)^d} \frac{|J(x,y)|^2}{\theta\left(\frac{m_t(x)}{\bar{\pi}(x)}, \frac{m_t(y)}{\bar{\pi}(y)}\right)} \bar{\omega}(x,y)\,, \qquad m\colon \bar{X} \to \mathbb{R}_+\,, \qquad J\colon \bar{E} \to \mathbb{R} \tag{1.4.6}$$

yields[5] $\mathcal{MA}_\epsilon = \mathcal{W}$ for G_ϵ. The main result of [GKMP23] is the Γ-convergence of both $\mathcal{A}_\epsilon$ and $\mathcal{MA}_\epsilon$. More precisely:

1. [GKMP23, Theorem 5.1]: Under the assumption that $\bar{F}$ grows at least linearly (in the sense of Assumption 4.2.8), the functionals $\mathcal{A}_\epsilon$ Γ-converge, as $\epsilon \to 0$, to a certain functional $\mathbb{A}_{\mathrm{hom}}$, which can be characterized by a *cell formula* (see [GKMP23, Definition 4.6]). The assumption on $\bar{F}$ is satisfied by the discrete analogs of all p-Wasserstein distances with $p \geq 1$.

2. [GKMP23, Theorem 5.10]: Under an assumption of *superlinear* growth at infinity on $\bar{F}$ (see Remark 4.2.6), the functionals $\mathcal{MA}_\epsilon$ Γ-converge, as $\epsilon \to 0$, to

$$\mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) = \inf_{(\boldsymbol{\mu},\boldsymbol{\nu})\in\mathsf{CE}(\mu_0,\mu_1)} \mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}, \boldsymbol{\nu})\,, \qquad \mu_0, \mu_1 \in \mathcal{P}(\mathbb{T}^d)\,, \tag{1.4.7}$$

where $\mathsf{CE}(\mu_0, \mu_1)$ is a set of generalized solutions to the continuity equation with $\boldsymbol{\mu} = \mu_t \otimes \mathrm{d}t \in \mathcal{P}\Big((0,1)\times\mathbb{T}^d\Big)$ connecting μ_0 to μ_1, and $\boldsymbol{\nu}$ being a vector measure on $(0,1)\times\mathbb{T}^d$; see Definition 4.2.1.[6] The assumption of superlinearity on $\bar{F}$ is satisfied by the discrete analogs of the p-Wasserstein distances for $p > 1$, but not for $p = 1$.

[5] by suitably choosing ω and π in (1.4.2) in terms of $\bar{\omega}$ and $\bar{\pi}$

[6] When $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE}(\mu_0, \mu_1)$, the measure $\boldsymbol{\mu}$ disintegrates as $\boldsymbol{\mu} = \mu_t \otimes \mathrm{d}t$, but $t \mapsto \mu_t \in \mathcal{P}(\mathbb{T}^d)$ is not necessarily continuous; see [GKMP23, Lemma 3.13].

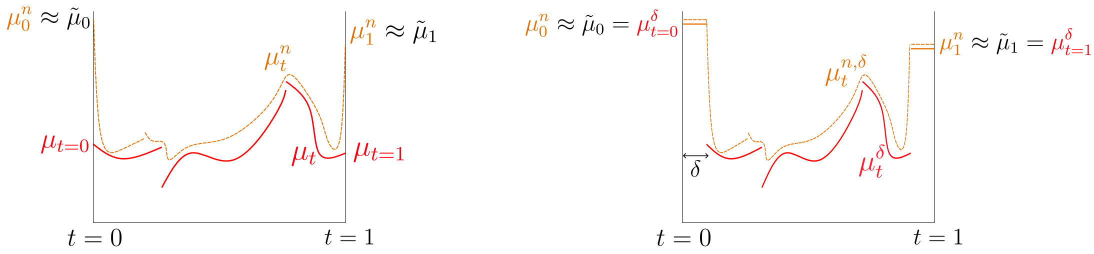

Figure 1.6: Schematic proof of the semicontinuity of $\mathbb{MA}_{\text{hom}}$. On the left: The boundary condition may not pass to the limit. On the right: The modified curves of measures $t \mapsto \mu_t^{n,\delta}$.

In the case where $\bar{F}$ is given by (1.4.6), one deduces ([GKMP23, Corollary 5.3]) the convergence of $\mathcal{MA}_\epsilon$ to the 2-Wasserstein distance on $\mathcal{P}(\mathbb{T}^d)$ with respect to some underlying metric d on the torus. Whether or not d is the Euclidean metric depends on an isotropy condition on the graph $\bar{G}$; see [GKM20].

The main contribution in [PQ24] is the following.

Theorem 1.4.1 (Portinale–Q., Theorem 4.3.9 (informal))**.** *The convergence*

$$\mathcal{MA}_\epsilon \xrightarrow{\Gamma} \mathbb{MA}_{\text{hom}} \quad \textit{as } \epsilon \to 0\,, \qquad \mathbb{MA}_{\text{hom}} \textit{ as in } (1.4.7), \tag{1.4.8}$$

holds when one of the following two is satisfied:

1. *the function $\bar{F}$ has* linear *growth at infinity, or*
2. *$\bar{F}$ does not depend on the variable m.*

In this way, we obtain convergence, e.g., in the previously excluded case of the discrete "1-Wasserstein" minimal action; see (1.4.3). In this specific case, we additionally prove that the functional $\mathbb{MA}_{\text{hom}}$ is, in fact, a 1-Wasserstein distance, but never with respect to the Euclidean metric if $d \geq 2$, which is a significant difference compared to the case $p = 2$ described above.

Theorem 1.4.2 (Portinale–Q., Proposition 4.4.4)**.** *If*

$$\bar{F}(m, J) = \frac{1}{2} \sum_{(x,y) \in \bar{E}\,:\, x \in [0,1)^d} |J(x,y)|\,\bar{\omega}(x,y)\,, \qquad m \colon \bar{X} \to \mathbb{R}_+\,, \qquad J \colon \bar{E} \to \mathbb{R}\,, \tag{1.4.9}$$

then $\mathbb{MA}_{\text{hom}}$ is the 1-Wasserstein distance on $\mathcal{P}(\mathbb{T}^d)$ with respect to an underlying distance $\mathrm{d} \colon \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}_+$ that depends on $\bar{G}$ and $\bar{\omega}$. This distance is induced by a norm $\|\cdot\|$ on $\mathbb{R}^d$ via the formula

$$\mathrm{d}(x, y) = \inf_{z \in \mathbb{Z}^d} \|x - y + z\|\,,$$

and $\|\cdot\|$ is a crystalline *norm, i.e., the unit ball for $\|\cdot\|$ is a* polytope*. Consequently, $\|\cdot\|$ can be equal to the Euclidean norm only in dimension $d = 1$.*

To conclude, let us briefly discuss how to prove that $\mathbb{MA}_{\text{hom}}$ defined by (1.4.7) is lower semicontinuous in the case of asymptotically linear $\bar{F}$. *A posteriori*, this property follows from the Γ-convergence, but its direct proof reveals the main difficulty in proving $\mathcal{MA}_\epsilon \xrightarrow{\Gamma} \mathbb{MA}_{\text{hom}}$ and how to solve it. Let $(\mu_0^n, \mu_1^n)_n$ be a sequence of pairs of measures weakly converging to $(\tilde{\mu}_0, \tilde{\mu}_1)$. For every n, let us choose an approximate minimizer $(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) \in \mathrm{CE}(\mu_0^n, \mu_1^n)$, i.e., such that $\mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) \approx \mathbb{MA}_{\mathrm{hom}}(\mu_0^n, \mu_1^n)$. Up to extracting a subsequence, we can assume convergence $(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) \rightharpoonup (\boldsymbol{\mu}, \boldsymbol{\nu})$. If $\bar{F}$ is not superlinear, the obtainable integrability bounds on $(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n)$ are insufficient to ensure that $\tilde{\mu}_0, \tilde{\mu}_1$ are equal to the boundary values $\mu_{t=0}, \mu_{t=1}$ of the limit curve $t \mapsto \mu_t$ (where $\mu_t \otimes \mathrm{d}t = \boldsymbol{\mu}$); see Figure 1.6. Therefore, $\boldsymbol{\mu}$ might not be an admissible competitor for the problem defining $\mathbb{MA}_{\mathrm{hom}}(\tilde{\mu}_0, \tilde{\mu}_1)$. Our solution is to perturb each $(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n)$ into $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta})$ by squeezing it into a smaller time interval $(\delta, 1-\delta)$, and defining $\boldsymbol{\mu}^{n,\delta}$ constantly equal to μ_0^n (resp. μ_1^n) in the interval $(0, \delta)$ (resp. $(1-\delta, 1)$). This procedure does not significantly change the value of the action $\mathbb{A}_{\mathrm{hom}}$, and the new sequence $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta})_n$ converges to some $(\boldsymbol{\mu}^\delta, \boldsymbol{\nu}^\delta)$ that satisfies the continuity equation and such that $t \mapsto \mu_t^\delta$ is constantly equal to $\tilde{\mu}_0$ (resp. $\tilde{\mu}_1$) for $t \in (0, \delta)$ (resp. $t \in (1-\delta, 1)$); hence, it has the desired boundary conditions. At this point, we use the lower semicontinuity of $\mathbb{A}_{\mathrm{hom}}$ on the sequence $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta})_n$—which, by [GKMP23], holds also when $\bar{F}$ is not superlinear—and conclude with the chain of inequalities

$$\begin{aligned}\mathbb{MA}_{\mathrm{hom}}(\tilde{\mu}_0, \tilde{\mu}_1) \le \mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}^\delta, \boldsymbol{\nu}^\delta) &\le \liminf_{n\to\infty} \mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \\ &\approx \liminf_{n\to\infty} \mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) \approx \liminf_{n\to\infty} \mathbb{MA}_{\mathrm{hom}}(\mu_0^n, \mu_1^n)\,.\end{aligned}$$

1.5 Quantization of measures

Discretizing measures is a problem that frequently arises in applications to economics (urban planning), numerics (numerical integration), data science (clustering and data compression), and many other fields; see Section 5.1.5. As Wasserstein distances generalize the Euclidean metric, they provide a natural way to quantify the error to be minimized in the discretization process.

Fix $p \ge 1$, $n \in \mathbb{N}_1$, and $\mu \in \mathcal{P}_p(\mathbb{R}^d)$. The *$n$-th optimal quantization error of order p for μ* is

$$e_{p,n}(\mu) \coloneqq \min_{\mu_n \in \mathcal{P}_p(\mathbb{R}^d)} \left\{ W_p(\mu, \mu_n) \,:\, \#\mathrm{supp}(\mu_n) \le n \right\}. \tag{1.5.1}$$

In other words, for a chosen number n, the problem (1.5.1) seeks the "best" compressed description of μ on n points. One equivalent formulation[7] is the following. Every set of n points $x_1, \ldots, x_n \in \mathbb{R}^d$ determines a *Voronoi tessellation* of $\mathbb{R}^d$ (see Figure 1.7), i.e., the sets

$$V_i \coloneqq \left\{ x \in \mathbb{R}^d \,:\, |x - x_i| \le |x - x_j| \text{ for all } j \in \{1, \ldots, n\} \right\}, \qquad i = 1, \ldots, n\,,$$

and the error $e_{p,n}(\mu)$ is given by

$$e_{p,n}^p(\mu) = \inf_{x_1,\ldots,x_n} \sum_{i=1}^n \int_{V_i} |x - x_i|^p \,\mathrm{d}\mu(x)\,.$$

One of the most fundamental results in *quantization theory* is the asymptotic behavior of $e_{p,n}(\mu)$ as $n \to \infty$, found by P. L. Zador [Zad64, Zad82]; see also [GL00].

Theorem 1.5.1 (Zador [GL00, Theorem 6.2])**.** *Let $\mu \in \mathcal{P}_\theta(\mathbb{R}^d)$ for some $\theta > p$ and let ρ be the density of the absolutely continuous part of μ. Then:*

$$\lim_{n\to\infty} n^{1/d} e_{p,n}(\mu) = q_{p,d} \left(\int_{\mathbb{R}^d} \rho(x)^{\frac{d}{d+p}} \,\mathrm{d}x \right)^{\frac{d+p}{dp}}, \tag{1.5.2}$$

[7] We assume here, for simplicity, that μ is absolutely continuous.

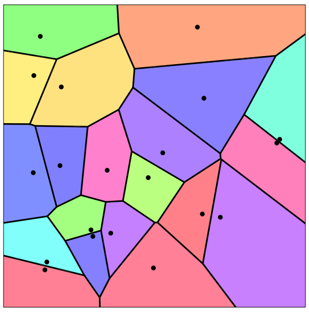

Figure 1.7: A Voronoi tessellation of $\mathbb{R}^2$.

where the optimal quantization coefficient $q_{p,d}$ *is a positive constant defined by*

$$q_{p,d} := \inf_{n \in \mathbb{N}_1} n^{1/d} e_{p,n}\left(\mathrm{d}x|_{[0,1]^d}\right) . \tag{1.5.3}$$

The subject of [Qua24] (Chapter 5) is a natural—albeit less extensively studied—variant of the quantization problem (1.5.1): *optimal empirical quantization*. Given $\mu \in \mathcal{P}_p(\mathbb{R}^d)$, we are interested in the problem

$$\tilde{e}_{p,n}(\mu) := \min_{\mu_n \in \mathcal{P}_p(\mathbb{R}^d)} \left\{ W_p(\mu, \mu_n) \; : \; \mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \text{ for some } x_1, \ldots, x_n \in \mathbb{R}^d \right\} , \tag{1.5.4}$$

which defines the n*-th optimal empirical quantization error of order* p *for* μ. Namely, we consider the same minimization as in (1.5.1), with the difference that admissible competitors are restricted to sums of n equally weighted Dirac deltas. More precisely, we investigate the asymptotics for $\tilde{e}_{p,n}(\mu)$ and find, as a main result, an adapted version of Zador's theorem.

Theorem 1.5.2 (Q., Theorem 5.1.1). *Assume that* $1 \le p < d$*, let* $p^* := \frac{dp}{d-p}$ *be the Sobolev conjugate of* p*, let* $\mu \in \mathcal{P}_\theta(\mathbb{R}^d)$ *for some* $\theta > p^*$*, let* ρ *be the density of the absolutely continuous part of* μ*, and let* $\operatorname{supp} \mu^s$ *be the support of the singular part of* μ*. Then:*

$$q_{p,d} \left(\int_{\mathbb{R}^d \setminus \operatorname{supp}(\mu^s)} \rho(x)^{\frac{d-p}{d}} \, \mathrm{d}x \right)^{1/p} \le \liminf_{n \to \infty} n^{1/d} \tilde{e}_{p,n}(\mu) , \tag{1.5.5}$$

$$\limsup_{n \to \infty} n^{1/d} \tilde{e}_{p,n}(\mu) \le \tilde{q}_{p,d} \left(\int_{\mathbb{R}^d} \rho(x)^{\frac{d-p}{d}} \, \mathrm{d}x \right)^{1/p} , \tag{1.5.6}$$

where

$$q_{p,d} := \inf_{n \in \mathbb{N}_1} n^{1/d} e_{p,n}\left(\mathrm{d}x|_{[0,1]^d}\right) > 0 \quad \textit{and} \quad \tilde{q}_{p,d} := \inf_{n \in \mathbb{N}_1} n^{1/d} \tilde{e}_{p,n}\left(\mathrm{d}x|_{[0,1]^d}\right) > 0 . \tag{1.5.7}$$

The main novelty of this theorem is the identification—in both the lower and the upper bound—of the prefactor $\left(\int \rho^{\frac{d-p}{d}} \, \mathrm{d}x \right)^{1/p}$ (when $\operatorname{supp}(\mu^s)$ is μ-negligible). Note that this is different from the one in Zador's theorem, but it appears in other related discretization problems, e.g. [DSS13]. We provide an heuristic derivation in Section 5.2.

The existence of the limit of $n^{1/d}\tilde{e}_{p,n}(\mu)$ remains, in general, an open problem, but we establish it in dimension $d = 2$ (with $p < 2$) for general measures, and in arbitrary dimension for certain classes of measures. Note that the identity $q_{p,d} = \tilde{q}_{p,d}$—combined with Theorem 1.5.2—would essentially imply the existence of the limit. Whether this equality holds is a problem closely related to an open conjecture by A. Gersho [Ger79].

The case $p \geq d$ is more complex. In this regime, "many" measures exhibit the error asymptotic

$$\limsup_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\mu) = \infty\,,$$

for example all those having compact and *disconnected* support; see Example 5.4.4. Nonetheless, we prove that there is a class of regular measures for which $\tilde{e}_{p,n}(\mu) \asymp n^{-1/d}$.

Theorem 1.5.3 (Q., Corollary 5.1.4 (simplified))**.** *Let $\Omega \subseteq \mathbb{R}^d$ be an open, convex, bounded set with $C^{1,1}$-regular boundary. Let $\rho\colon \Omega \to \mathbb{R}_+$ be a uniformly positive and globally Hölder continuous probability density. Then, for every $p \geq 1$:*

$$0 < \liminf_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\rho\,\mathrm{d}x) \leq \limsup_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\rho\,\mathrm{d}x) < \infty\,. \tag{1.5.8}$$

One of the fascinating aspects of quantization lies in the combination of elementary combinatorial, geometric, and measure-theoretic arguments with powerful tools from optimal transport theory. As a first example, consider the inequalities (1.5.5) and (1.5.6). A key property of the p-Wasserstein distance that we employ in proving the upper bound is its convexity:

$$W_p^p\Big(\lambda\mu^1 + (1-\lambda)\mu^2, \lambda\nu^1 + (1-\lambda)\nu^2\Big) \leq \lambda W_p^p(\mu^1,\nu^1) + (1-\lambda)W_p^p(\mu^2,\nu^2)\,, \qquad \mu^1,\mu^2,\nu^1,\nu^2 \in \mathcal{P}_p(\mathbb{R}^d)\,,\quad \lambda \in [0,1]\,.$$

Conversely, to get the lower bound, we need some sort of *concavity*. Surprisingly, the boundary Wasserstein distance by A. Figalli and N. Gigli [FG10], which we described in Section 1.2, enjoys such a property (and trivially bounds W_p from below). More precisely, if $\Omega_1,\dots,\Omega_n$ are open and *pairwise disjoint* subsets of a set $\Omega \subseteq \mathbb{R}^d$, then[8]

$$Wb^p_{\Omega,p}(\mu,\nu) \geq \sum_{i=1}^n Wb^p_{\Omega_i,p}(\mu|_{\Omega_i},\nu|_{\Omega_i})\,, \qquad \mu,\nu \in \mathcal{M}(\Omega)\,;$$

see [AGT22, Section 2.2]. A second example is contained in the proof of Theorem 1.5.3, where we use a result by S. Chen, J. Liu, and X.-J. Wang [CLW21] on the (global) regularity of optimal transport maps. Additionally, let us mention that the study of *random matching*, a combinatorial problem similar to quantization, has also benefited from advanced tools from PDE theory and Fourier analysis; see [AST19, BL21].

[8]We specify with a subscript the set on which the boundary Wasserstein distance is constructed.

CHAPTER 2

Variational structures for the Fokker–Planck equation with general Dirichlet boundary conditions

This chapter contains the following publication [Qua25]:

F. Quattrocchi. Variational structures for the Fokker–Planck equation with general Dirichlet boundary conditions. To appear in *Calculus of Variations and Partial Differential Equations*, 2025+, CC BY 4.0. doi:10.1007/s00526-025-03193-1

This version of the article has been accepted for publication, after peer review but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: http://dx.doi.org/10.1007/s00526-025-03193-1.

Abstract

We prove the convergence of a modified Jordan–Kinderlehrer–Otto scheme to a solution to the Fokker–Planck equation in $\Omega \Subset \mathbb{R}^d$ with general—strictly positive and temporally constant—Dirichlet boundary conditions. We work under mild assumptions on the domain, the drift, and the initial datum.

In the special case where Ω is an interval in $\mathbb{R}^1$, we prove that such a solution is a gradient flow—curve of maximal slope—within a suitable space of measures, endowed with a modified Wasserstein distance.

Our discrete scheme and modified distance draw inspiration from contributions by A. Figalli and N. Gigli [J. Math. Pures Appl. 94, (2010), pp. 107–130], and J. Morales [J. Math. Pures Appl. 112, (2018), pp. 41–88] on an optimal-transport approach to evolution equations with Dirichlet boundary conditions. Similarly to these works, we allow the mass to flow from/to the boundary $\partial\Omega$ throughout the evolution. However, our leading idea is to also keep track of the mass at the boundary by working with measures defined on the whole closure $\overline{\Omega}$.

The driving functional is a modification of the classical relative entropy that also makes use of the information at the boundary. As an intermediate result, when Ω is an interval in $\mathbb{R}^1$, we find a formula for the descending slope of this geodesically nonconvex functional.

2.1 Introduction

The subject of this paper is the linear Fokker–Planck equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \operatorname{div}\left(\nabla \rho_t + \rho_t \nabla V\right) \tag{2.1.1}$$

on a bounded Euclidean domain $\Omega \subseteq \mathbb{R}^d$ combined with general—strictly positive and constant in time—Dirichlet boundary conditions, and with nonnegative initial data. We want to approach this problem by applying the theory of *optimal transport*, which, since the seminal works of R. Jordan, D. Kinderlehrer, and F. Otto [JKO98, Ott99, Ott01], has proven effective in the study of a number of evolution equations.

Existence, uniqueness, and appropriate estimates are often consequence of a peculiar structure of the problem. Important instances are those PDEs which can be seen as *gradient flows*. In fact, it has been proven that several equations, including Fokker–Planck on $\mathbb{R}^d$, are gradient flows in a space of probability measures endowed with the 2-Wasserstein distance

$$W_2(\mu,\nu) := \inf_\gamma \sqrt{\int |x-y|^2 \,\mathrm{d}\gamma(x,y)}\,,$$

where the infimum is taken among all couplings γ between μ and ν, i.e., measures with marginals $\pi^1_\# \gamma = \mu$ and $\pi^2_\# \gamma = \nu$. For such PDEs, existence can be deduced from the convergence of the discrete-time approximations given by the Jordan–Kinderlehrer–Otto variational scheme (also known, in a more general metric setting, as De Giorgi's minimizing movement scheme [DG93])

$$\rho^\tau_{(n+1)\tau}\,\mathrm{d}x \in \arg\min\nolimits_\mu \left(\mathcal{F}(\mu) + \frac{W_2^2(\mu, \rho^\tau_{n\tau}\,\mathrm{d}x)}{2\tau}\right)\,, \qquad n \in \mathbb{N}_0\,, \tag{2.1.2}$$

where $\mathcal{F}$ is a functional that depends on the equation, and $\tau > 0$ is the time step.

When applied on a bounded Euclidean domain, this approach produces solutions *with Neumann boundary conditions*. This fact is inherent in the choice of the metric space (probability measures with the distance W_2) in which the flow evolves. Intuitively, Neumann boundary conditions are natural because a curve of probability measures, by definition, conserves the total mass; see also the discussion in [San17].

In order to deal with Dirichlet boundary conditions, A. Figalli and N. Gigli defined in [FG10] a modified Wasserstein distance Wb_2 that gives a special role to the boundary $\partial\Omega$. Despite measuring a distance between nonnegative measures on Ω, the metric Wb_2 is defined as an infimum over measures γ on the product of the topological closures $\overline{\Omega} \times \overline{\Omega}$, and only the restrictions of the marginals $\pi^1_\#\gamma$ and $\pi^2_\#\gamma$ to Ω are prescribed (see the original paper [FG10] or Section 2.3.6). In this sense, the boundary $\partial\Omega$ can be interpreted as an infinite reservoir, where mass can be taken and deposited freely. The main result in [FG10] is the convergence of the scheme

$$\rho^\tau_{(n+1)\tau} \in \arg\min\nolimits_\rho \left(\int_\Omega \Big(\rho\log\rho - \rho + 1\Big)\,\mathrm{d}x + \frac{Wb_2^2(\rho\,\mathrm{d}x, \rho^\tau_{n\tau}\,\mathrm{d}x)}{2\tau}\right)\,, \qquad n \in \mathbb{N}_0\,,$$

as $\tau \downarrow 0$, to a solution to the heat equation with the *constant* Dirichlet boundary condition $\rho|_{\partial\Omega} = 1$. More generally, it was observed in [FG10, Section 4] that the same scheme with a suitably modified entropy functional converges to solutions to the linear Fokker–Planck equation (2.1.1) with the boundary condition $\rho|_{\partial\Omega} = e^{-V}$. In particular, this theory covers the heat equation with *any constant and strictly positive* Dirichlet boundary condition.

In a more recent contribution, J. Morales [Mor18] proved convergence of a similar discrete scheme for a family of reaction-diffusion equations with drift, subject to rather *general* Dirichlet boundary conditions. In this scheme, the distance between measures is replaced by τ-dependent transportation costs. Morales' work, together with [FG10], is the starting point of the present paper.

Related literature

The case of the heat flow with *vanishing* Dirichlet boundary conditions was studied by A. Profeta and K.-T. Sturm in [PS20]. They defined 'charged probabilities' and a suitable distance on them. This metric is built upon the idea that mass can touch the boundary and be reflected, as with the classical Wasserstein distance, but possibly changing the charge (positive to negative or vice versa). One of their results is the *Evolution Variational Inequality* (see [AGS08]) for such a heat flow.

D. Kim, D. Koo and G. Seo [KKS25] adapted the setting of [FG10] to porous medium equations $\partial_t \rho_t = \Delta \rho^\alpha$ $(\alpha > 1)$ with *constant* boundary conditions.

M. Erbar and G. Meglioli [EM25] generalized the result of [KKS25] to a larger class of diffusion equations with constant boundary conditions. They also established a dynamical characterization of Wb_2, in the spirit of the Benamou–Brenier formula for W_2 [BB00].

J.-B. Casteras, L. Monsaingeon, and F. Santambrogio [CMS25] found the Wasserstein gradient flow structure for the equation arising from the so-called Sticky Brownian Motion, i.e., the Fokker–Planck equation together with boundary conditions of Dirichlet type that also evolve in time subject to diffusion and drift on the boundary. Namely, denoting by $\partial_{\boldsymbol{n}}$ the outer normal derivative,

$$\begin{cases} \partial_t \rho = \Delta \rho & \text{in } \Omega\,, \\ \rho = \gamma & \text{on } \partial\Omega\,, \\ \partial_t \gamma = \Delta_{\partial\Omega}\gamma - \partial_{\boldsymbol{n}}\rho & \text{in } \partial\Omega\,. \end{cases} \tag{2.1.3}$$

M. Bormann, L. Monsaingeon, D. R. M. Renger, and M. von Renesse [BMRv25] recently proved a negative result. If we modify (2.1.3) by weakening the diffusion on the boundary (i.e., we multiply $\Delta_{\partial\Omega}\gamma$ by a factor $a \in (0,1)$) the resulting problem is *not* a gradient flow of the entropy in the 2-Wasserstein space built from any reasonably regular metric on $\overline{\Omega}$.

Our contribution

In this work, we present two novel results:

1. We prove convergence of a modified Jordan–Kinderlehrer–Otto scheme to a solution to the Fokker–Planck equation with general Dirichlet boundary conditions under mild regularity assumptions. To do this, we adopt a *different point of view* compared to [FG10, Mor18, KKS25]: our scheme is defined on a subset $\mathscr{S}$ of the signed measures *on the closure* $\overline{\Omega}$, rather than on measures on Ω.

2. In dimension $d = 1$, we determine that this solution is also a *curve of maximal slope* for a functional $\mathcal{H}$ in an appropriate metric space $(\mathscr{S}, \widetilde{W}b_2)$.

Let us now explain in detail the extent of these contributions and provide precise statements.

Convergence of a modified JKO scheme

We look at the boundary-value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \operatorname{div}(\nabla \rho_t + \rho_t \nabla V) & \text{in } \Omega\,, \\ \rho_t|_{\partial\Omega} = e^{\Psi - V} & \text{on } \partial\Omega\,, \\ \rho_{t=0} = \rho_0\,. \end{cases} \tag{2.1.4}$$

Here, $\Omega \subseteq \mathbb{R}^d$ is a *bounded* open set and ρ_0, Ψ, V are given functions, with $\rho_0 \geq 0$. The function Ψ can be tuned to obtain the desired boundary condition.

We introduce the set $\mathscr{S}$ of all signed measures on $\overline{\Omega}$ with

$$\mu|_\Omega \geq 0 \quad \text{and} \quad \mu(\overline{\Omega}) = 0\,. \tag{2.1.5}$$

We also define

$$\mathcal{E}(\rho) := \int_\Omega \Big(\rho \log \rho + (V-1)\rho + 1\Big)\,\mathrm{d}x\,, \qquad \rho\colon \Omega \to \mathbb{R}_+\,, \tag{2.1.6}$$

and, for $\mu \in \mathscr{S}$,

$$\mathcal{H}(\mu) := \begin{cases} \mathcal{E}(\rho) + \displaystyle\int \Psi \,\mathrm{d}\mu|_{\partial\Omega} & \text{if } \mu|_\Omega = \rho\,\mathrm{d}x\,, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1.7}$$

In Section 2.3.7, we will define a transportation-cost functional $\mathcal{T}$ on $\mathscr{S}$. With it, we can consider the scheme

$$\mu^\tau_{(n+1)\tau} \in \operatorname*{arg\,min}_{\mu \in \mathscr{S}} \left(\mathcal{H}(\mu) + \frac{\mathcal{T}^2(\mu, \mu^\tau_{n\tau})}{2\tau}\right)\,, \qquad n \in \mathbb{N}_0\,,\ \tau > 0\,, \tag{2.1.8}$$

starting from some $\mu^\tau_0 = \mu_0 \in \mathscr{S}$, independent of τ, such that the restriction $\mu_0|_\Omega$ is absolutely continuous with density ρ_0. These sequences are extended to maps $t \mapsto \mu^\tau_t$, constant on the intervals $\big[n\tau, (n+1)\tau\big)$ for every $n \in \mathbb{N}_0$, namely:

$$\mu^\tau_t := \mu^\tau_{\lfloor t/\tau \rfloor \tau}\,, \qquad t \in [0, \infty)\,. \tag{2.1.9}$$

Theorem 2.1.1. *Assume that $\int_\Omega \rho_0 \log \rho_0 \,\mathrm{d}x < \infty$, that $\Psi\colon \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous, and that*[1] *$V \in W^{1,d+}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$. Then:*

1. Well-posedness: *The maps $(t \mapsto \mu^\tau_t)_\tau$ resulting from the scheme (2.1.8) are well-defined and uniquely defined: for every n and τ, there exists a minimizer in (2.1.8) and it is unique.*

[1] By $V \in W^{1,d+}_{\mathrm{loc}}(\Omega)$ we mean that for every $\omega \Subset \Omega$ open there exists $p = p(\omega) > d$ such that $V \in W^{1,p}(\omega)$; see also Definition 2.3.1.

2. Convergence: *When $\tau \to 0$, up to subsequences, the maps $\left(t \mapsto \mu_t^\tau|_\Omega\right)_\tau$ converge pointwise w.r.t. the Figalli–Gigli distance Wb_2 to a curve of absolutely continuous measures $t \mapsto \rho_t \, \mathrm{d}x$. For every $q \in [1, \frac{d}{d-1})$, convergence holds also in $L^1_{\mathrm{loc}}\big((0,\infty); L^q(\Omega)\big)$.*

3. Equation: *This limit curve is a weak solution to the Fokker–Planck equation* (2.1.1)*; see Section 2.3.4.*

4. Boundary condition: *The function $t \mapsto \left(\sqrt{\rho_t e^V} - e^{\Psi/2}\right)$ belongs to the space $L^2_{\mathrm{loc}}\big([0,\infty); W_0^{1,2}(\Omega)\big)$.*

Remark 2.1.2. We assume that Ψ is defined on the whole set $\overline{\Omega}$ in order to make sense of the inclusion $\sqrt{\rho_t e^V} - e^{\Psi/2} \in W_0^{1,2}(\Omega)$ also when $\partial\Omega$ is not smooth enough to have a trace operator. Note that, if we are given a Lipschitz continuous function $\Psi_0 \colon \partial\Omega \to \mathbb{R}$, we can extend it to a Lipschitz function on $\overline{\Omega}$ via

$$\Psi(x) \coloneqq \inf_{y \in \partial\Omega} \left(\Psi_0(y) + (\operatorname{Lip}\Psi_0)|x-y|\right) .$$

Remark 2.1.3. If V is Lipschitz continuous *only in a neighborhood of* $\partial\Omega$, then it is possible to find Ψ, Lipschitz as well, in order for $e^{\Psi - V}$ to match *any* uniformly positive and Lipschitz boundary condition.

Remark 2.1.4. Throughout the proof of Theorem 2.1.1, we also show:

- time contractivity of suitably truncated and weighted L^q norms of $\mu_t^\tau|_\Omega$ (see Proposition 2.5.15),
- upper bounds on the L^q norms of $\mu_t^\tau|_\Omega$, for every $t > 0$ (see Lemma 2.5.23),
- upper bounds on time averages of the $W^{1,2}$ norm of $\sqrt{\rho_t^\tau e^V}$, where ρ_t^τ is the density of $\mu_t^\tau|_\Omega$ (see Lemma 2.5.22).

Furthermore, these estimates (assuming $q \in [1, \frac{d}{d-1})$ in the first two) pass to the limit as $\tau \to 0$, i.e., analogous properties hold for the curve $t \mapsto \rho_t$.

As mentioned, the conceptual difference between the present work and [FG10, Mor18, KKS25] is that we make use of signed measures on the full closure $\overline{\Omega}$. In this regard, our approach is similar to those of [CMS25, Mon21]. The idea is that, due to the boundary condition we have to match, it is convenient to keep track of the mass at the boundary and to consider a functional that makes use of this information (namely, $\mathcal{H}$).

On a more technical note, although Theorem 2.1.1 is similar to [Mor18, Theorem 4.1], the latter is not applicable to the Fokker–Planck equation (2.1.1) without reaction term due to [Mor18, Assumptions (C1)-(C9)] (see in particular (C7)). Furthermore, we achieve significant improvements in the hypotheses:

- The boundary $\partial\Omega$ does not need to have *any* regularity, as opposed to Lipschitz and with the interior ball condition.
- There is no uniform bound on ρ_0 from above or below by positive constants. Only nonnegativity and the integrability of $\rho_0 \log \rho_0$ are assumed.

- The function V is not necessarily Lipschitz continuous. Rather, it is required to be bounded and to have suitable local Sobolev regularity.

These weak assumptions make it more involved to prove Lebesgue and Sobolev bounds for μ_t^τ, as well as the strong convergence of the scheme, which in turn allows us to characterize the limit. Indeed:

- When ρ_0 is bounded, or lies in some L^q, it is possible to propagate these properties along $t \mapsto \mu_t^\tau|_\Omega$; see [Mor18, Proposition 5.3] and Proposition 2.5.15. With our weak assumptions on ρ_0, we are still able to propagate the L^1 bound, but also need to establish suitable Sobolev estimates (see Proposition 2.5.9 and Lemma 2.5.22) and make use of the Sobolev embedding theorem in order to get stronger integrability (see Lemma 2.5.23) and convergence in $L^1_{\text{loc}}\big((0,\infty); L^q(\Omega)\big)$ (see Lemma 2.5.26).
- If $\partial\Omega$ is not regular enough, we cannot directly apply the Sobolev embedding theorem for $W^{1,2}$ functions. Since the Sobolev continuous embedding holds for $W_0^{1,2}$ functions regardless of the domain regularity, we are still able to apply it after establishing suitable boundary conditions for $\mu_t^\tau|_\Omega$; see Proposition 2.5.9.
- When V is not Lipschitz, we need an extra approximation procedure to prove that $\mu_t^\tau|_\Omega$ is Sobolev regular and satisfies a precursor of the Fokker–Planck equation; see Proposition 2.5.9 and Lemma 2.5.10.
- Another issue with $\partial\Omega$ not being regular is in applying (a variant of) the Aubin–Lions lemma to prove convergence of the scheme. One of its assumptions is a compact embedding of functional spaces, which would follow from the Rellich–Kondrachov theorem if Ω were regular enough. To overcome it, we use the Rellich–Kondrachov theorem on *smooth subdomains* and take advantage of the integrability estimates to promote local L^q convergence to convergence in $L^q(\Omega)$; see Lemma 2.5.26.

Curve of maximal slope

Our second main result is a strengthened version of Theorem 2.1.1 in the case where Ω is an interval in $\mathbb{R}^1$ and $V \in W^{1,2}(\Omega)$. In this setting, we are able to define a *true* metric $\widetilde{Wb}_2$ on $\mathscr{S}$, construct piecewise constant maps with the scheme

$$\begin{aligned} &\mu^\tau_{(n+1)\tau} \in \operatorname*{arg\,min}_{\mu\in\mathscr{S}} \left(\mathcal{H}(\mu) + \frac{\widetilde{Wb}_2^2(\mu,\mu^\tau_{n\tau})}{2\tau}\right), \qquad n\in\mathbb{N}_0\,,\ \tau>0\,, \\ &\mu_0^\tau = \mu_0\,, \end{aligned} \tag{2.1.10}$$

for a fixed μ_0 with $\mu_0|_\Omega = \rho_0\,\mathrm{d}x$, show that they *coincide* with those of Theorem 2.1.1, and prove that their limit is a *curve of maximal slope* in $(\mathscr{S}, \widetilde{Wb}_2)$.

Theorem 2.1.5. *Assume that $\Omega = (0,1)$, that $\int_0^1 \rho_0 \log\rho_0 \,\mathrm{d}x < \infty$, and that $V \in W^{1,2}(0,1)$. Then:*

1. *If τ is sufficiently small, the maps $(t \mapsto \mu_t^\tau)_\tau$ resulting from the scheme (2.1.10) are well-defined, uniquely defined, and* coincide with those of Theorem 2.1.1.

2. *When $\tau \to 0$, up to subsequences, the maps $(t \mapsto \mu_t^\tau)_\tau$ converge pointwise w.r.t. $\widetilde{Wb}_2$ to a curve $t \mapsto \mu_t$.*

3. *The convergence $\mu^\tau|_\Omega \to_\tau \mu|_\Omega$ also holds in $L^1_{\mathrm{loc}}\big((0,\infty); L^q(0,1)\big)$ for every $q \in [1,\infty)$. The curve $t \mapsto \mu_t|_\Omega$ is a weak solution to the Fokker–Planck equation. Denoting by ρ_t the density of $\mu_t|_\Omega$, the map $t \mapsto \left(\sqrt{\rho_t e^V} - e^{\Psi/2}\right)$ belongs to $L^2_{\mathrm{loc}}\big([0,\infty); W_0^{1,2}(0,1)\big)$.*

4. *The map $t \mapsto \mu_t$ is a curve of maximal slope for the functional $\mathcal{H}$ in the metric space $(\mathscr{S}, \widetilde{Wb}_2)$, with respect to the descending slope $\left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|$; see Section 2.3.5.*

Within the general theory of gradient flows in metric spaces developed by L. Ambrosio, N. Gigli, and G. Savaré in [AGS08] (see [San17] for an overview), the 'curve of maximal slope' is one of the metric counterparts of the gradient flow in the Euclidean space. In the context of PDEs with Dirichlet boundary conditions, other proofs of this metric characterization in a (Wasserstein-like) space of measures are given in [PS20, KKS25, EM25]. To be precise, the result of [PS20, Proposition 1.20] is an 'Evolution Variational Inequality' (EVI) characterization, which *implies* a formulation as curve of maximal slope by [AG13, Proposition 4.6]. By Proposition 2.8.5, our functional $\mathcal{H}$ is not semiconvex and, therefore, we do not expect an EVI characterization in our setting; see [DS08, Theorem 3.2]. Let us also point out that the 'curve of maximal slope' characterizations in [KKS25, EM25] use the *relaxed* descending slope (see [AGS08, Equation (2.3.1)]), which yields a weaker notion of gradient flow compared to ours. In fact, establishing that the descending slope is lower semicontinuous is the main difficulty in proving Theorem 2.1.5. Indeed, the lower semicontinuity of the slope is usually derived from the geodesic (semi)convexity of the functional via [AGS08, Corollary 2.4.10], but $\mathcal{H}$ is not geodesically semiconvex by Proposition 2.8.5.

Nonetheless, in dimension $d = 1$, we are able to find an *explicit formula* for the descending slope of $\mathcal{H}$ in $(\mathscr{S}, \widetilde{Wb}_2)$ without resorting to geodesic convexity. As a corollary, we also give an answer, again in dimension $d = 1$, to the problem left open in [FG10] of identifying the descending slope $|\partial_{Wb_2}\mathcal{E}|$ of $\mathcal{E}$ with respect to the Figalli–Gigli distance Wb_2.

Theorem 2.1.6 (see Corollary 2.6.5)**.** *Assume that $V \in W^{1,2}(0,1)$. For every $\rho \in L^1_+(0,1)$, we have the formula*

$$|\partial_{Wb_2}\mathcal{E}|^2(\rho) = \begin{cases} 4\displaystyle\int_0^1 \left(\partial_x \sqrt{\rho e^V}\right)^2 e^{-V}\,\mathrm{d}x & \text{if } \sqrt{\rho e^V} - 1 \in W_0^{1,2}(0,1)\,, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1.11}$$

Additionally, $|\partial_{Wb_2}\mathcal{E}|$ is lower semicontinuous with respect to Wb_2.

We believe that the same formula should hold true also in higher dimension. A similar open problem is [CMS25, Conjecture 2].

Plan of the work

In Section 2.2, we formally derive the objects (entropy and transportation functionals) that appear in the schemes (2.1.8) and (2.1.10).

In Section 2.3, we introduce notation, terminology, and assumptions that are in place throughout the paper, we recall some definitions from the theory of gradient flows in metric spaces, as well as the Figalli–Gigli distance of [FG10], and we define rigorously the transportation functionals $\mathcal{T}$ and $\widetilde{Wb}_2$.

In Section 2.4, we gather the main properties of these functionals and of the corresponding admissible transport plans. In particular, we show that $\widetilde{Wb}_2$ is a true metric when Ω is a finite union of one-dimensional intervals.

In Section 2.5, we prove Theorem 2.1.1.

In Sections 2.6-2.7, we focus on the case where $\Omega = (0,1) \subseteq \mathbb{R}^1$. In Section 2.6, we find a formula for the slope of $\mathcal{H}$ in the metric space $(\mathscr{S}, \widetilde{Wb}_2)$ and prove, as a corollary, Theorem 2.1.6. In Section 2.7, making use of Theorem 2.1.1 and of the slope formula, we prove Theorem 2.1.5.

Section 2.8 contains some additional results on $\widetilde{Wb}_2$. Particularly, we prove the lack of geodesic λ-convexity for $\mathcal{H}$ when $\Omega = (0,1)$.

2.2 Formal derivation

Let us work at a completely formal level and postulate that a solution to the Fokker–Planck equation (2.1.4) is the "Wasserstein-like" gradient flow of some functional $\mathcal{F}$. By this we mean the following:

1. the motion of ρ_t in Ω is governed by the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\operatorname{div}(\rho_t \boldsymbol{v}_t)\,, \tag{2.2.1}$$

for some velocity field $\boldsymbol{v}_t$,

2. the time-derivative of ρ_t equals the inverse of the Wasserstein gradient of $\mathcal{F}$ at ρ_t for every t, in the sense that for every sufficiently nice curve $s \mapsto f_s$ of functions on Ω starting at $f_0 = \rho_t$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}(f_s\,\mathrm{d}x)\Big|_{s=0} = -\int_\Omega \langle \boldsymbol{v}_t, \nabla\psi\rangle \rho_t\,\mathrm{d}x\,, \quad \text{where } \frac{\mathrm{d}}{\mathrm{d}s}f_s\Big|_{s=0} = -\operatorname{div}(\rho_t\nabla\psi)\,. \tag{2.2.2}$$

As we want to retrieve the Fokker–Planck equation, a reasonable choice for $\mathcal{F}$ seems to be

$$\mathcal{F}_0(\rho\,\mathrm{d}x) := \int_\Omega \Big(\rho\log\rho + (V-1)\rho + 1\Big)\,\mathrm{d}x\,. \tag{2.2.3}$$

For a fixed $t \geq 0$ and a curve $s \mapsto f_s$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}_0(f_s\,\mathrm{d}x) = \int_\Omega (V + \log f_s)\frac{\mathrm{d}}{\mathrm{d}s}f_s\,\mathrm{d}x\,,$$

and, therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}_0(f_s\,\mathrm{d}x)\Big|_{s=0} &= -\int_\Omega (V + \log\rho_t)\operatorname{div}(\rho_t\nabla\psi)\,\mathrm{d}x\\ &= \int_\Omega \langle(\nabla V + \nabla\log\rho_t), \nabla\psi\rangle\rho_t\,\mathrm{d}x - \int_{\partial\Omega}\Psi\rho_t\langle\nabla\psi, \boldsymbol{n}\rangle\,\mathrm{d}\mathscr{H}^{d-1}\,,\end{aligned}$$

where, in the last identity, we used the boundary conditions in (2.1.4). Let us choose

$$\boldsymbol{v}_t := -\nabla V - \nabla \log \rho_t \,,$$

which makes the continuity equation (2.2.1) true, since ρ_t solves (2.1.4). Then,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}_0(f_s)\Big|_{s=0} = -\int_\Omega \langle \boldsymbol{v}_t, \nabla\psi\rangle \rho_t \,\mathrm{d}x - \int_{\partial\Omega} \Psi \rho_t \langle \nabla\psi, \boldsymbol{n}\rangle \,\mathrm{d}\mathscr{H}^{d-1} \,,$$

and we see that $\mathcal{F}_0$ is not the right functional because of the integral on the boundary. The measure $\langle \nabla\psi, \boldsymbol{n}\rangle \rho_t \mathscr{H}^{d-1}$ on $\partial\Omega$ can be seen as the flux of mass (coming from $f_0 = \rho_t$) that is moving away from Ω along the flow $s \mapsto f_s$ at $s = 0$. Thus, if we let this mass settle on the boundary, $\langle \nabla\psi, \boldsymbol{n}\rangle \rho_t \mathscr{H}^{d-1}$ is the time-derivative of the mass on $\partial\Omega$. For this reason, it makes sense to consider not just measures on Ω, but rather on the closure $\overline{\Omega}$, and to define

$$\mathcal{F}(\mu) := \mathcal{F}_0(\mu|_\Omega) + \int \Psi \,\mathrm{d}\mu|_{\partial\Omega} \,.$$

Our entropy functional $\mathcal{H}$ is defined precisely like this, and, as we will see in Section 2.3, the transportation functionals $\mathcal{T}$ and $\widetilde{Wb}_2$ are extensions of Wb_2 to the subset $\mathscr{S}$ of the signed measures on $\overline{\Omega}$, constructed so as to encode the idea that mass can leave Ω to settle on $\partial\Omega$ (and vice versa).

This argument is simple, but let us also emphasize the hidden difficulties:

- we assume low regularity on $\partial\Omega$ and on the functions ρ_0 and V;
- the transportation-cost functionals $\widetilde{Wb}_2$ and $\mathcal{T}$ will not be, in general, distances;
- the functional $\mathcal{H}$ is not bounded from below on $\mathscr{S}$ (if Ψ is nonconstant), nor it is strictly convex. Indeed, it is linear along lines of the form $\mathbb{R} \ni l \mapsto \mu + l\eta$ with $\mu, \eta \in \mathscr{S}$ and η concentrated on $\partial\Omega$;
- when $(\mathscr{S}, \widetilde{Wb}_2)$ is a geodesic metric space, the functional $\mathcal{H}$ is *not* geodesically semi-convex; see [FG10, Remark 3.4] and Section 2.8.3.

2.3 Preliminaries

2.3.1 Setting

Throughout the paper, Ω is an open, bounded, and nonempty subset of $\mathbb{R}^d$. Without loss of generality, we assume that $0 \in \Omega$. No assumption is made on the regularity of its boundary.

Three functions are given: the initial datum $\rho_0 \colon \Omega \to \mathbb{R}_+$, the potential $V \colon \Omega \to \mathbb{R}$, and the function $\Psi : \overline{\Omega} \to \mathbb{R}$ that determines the boundary condition. We assume that Ψ is Lipschitz continuous and that the integral $\int_\Omega \rho_0 \log \rho_0 \,\mathrm{d}x$ is finite. In addition, we suppose that V is bounded (i.e., in $L^\infty(\Omega)$) and in the set of locally Sobolev functions $W^{1,d+}_{\mathrm{loc}}(\Omega)$.[2]

Definition 2.3.1. We say that $V \in W^{1,d+}_{\mathrm{loc}}(\Omega)$ if, for every $\omega \Subset \Omega$ open, there exists $p = p(\omega) > d$ such that $V \in W^{1,p}(\omega)$.

[2] In particular, $V \in C(\Omega)$.

The set $\mathscr{S}$ is the convex cone of all finite and signed Borel measures μ on $\overline{\Omega}$ such that (2.1.5) holds.

Proposition 2.3.2. *The set $\mathscr{S}$ is closed w.r.t. the weak convergence, i.e., in duality with continuous and bounded functions on $\overline{\Omega}$.*

Proof. If $\mathscr{S} \ni \mu^n \to_n \mu$, then $\mu(\overline{\Omega}) = \lim_{n\to\infty} \mu^n(\overline{\Omega}) = 0$ and, for every $f\colon \overline{\Omega} \to \mathbb{R}_+$ continuous and compactly supported in Ω,

$$\int f \,\mathrm{d}\mu_\Omega = \int f \,\mathrm{d}\mu = \lim_{n\to\infty} \int f \,\mathrm{d}\mu^n = \lim_{n\to\infty} \int f \,\mathrm{d}\mu^n_\Omega \geq 0\,.$$

The conclusion follows from the Riesz–Markov–Kakutani theorem. □

The entropy functionals $\mathcal{E}\colon L^1_+(\Omega) \to \mathbb{R} \cup \{\infty\}$ and $\mathcal{H}\colon \mathscr{S} \to \mathbb{R} \cup \{\infty\}$ are defined in (2.1.6) and (2.1.7), respectively.

2.3.2 Convention on constants

The symbol $\mathfrak{c}$ is reserved for strictly positive real constants. The number it represents *may change from formula to formula* and possibly depends on the dimension d, the set Ω, the functions V and Ψ, and the initial datum ρ_0. We also allow $\mathfrak{c}$ to depend on other quantities, which are, in case, explicitly displayed as a subscript.

2.3.3 Measures

For every signed Borel measure μ and Borel set A, we write $\mu_A = \mu|_A$ for the restriction of μ to A. Similarly, and following the notation of [FG10, Mor18], if γ is a measure on a product space and A, B are Borel, we write $\gamma_A^B = \gamma_{A\times B}$ for the restriction of γ to $A \times B$. We use the notation μ_+, μ_- for the positive and negative parts of a given measure μ, and $\|\mu\|$ for the total-variation norm of μ, i.e., the total mass of $\mu_+ + \mu_-$.

For every Borel function f and signed Borel measure μ, we denote by $\mu(f)$ the integral $\int f \,\mathrm{d}\mu$.

On the set of the finite signed Borel measures on $\overline{\Omega}$, we also consider the (modified) Kantorovich–Rubinstein norm (see [Bog07, Section 8.10(viii)])

$$\|\mu\|_{\widetilde{\mathrm{KR}}} := \left|\mu(\overline{\Omega})\right| + \sup\left\{\mu(f) \,:\, f\colon \overline{\Omega} \to \mathbb{R}\,,\ \operatorname{Lip}(f) \leq 1 \text{ and } f(0) = 0\right\}\,. \tag{2.3.1}$$

We write $F_\# \mu$ for the push-forward of a (signed) Borel measure μ via a Borel map F. Often, we use as F the projection onto some coordinate: we write π^i for the projection on the i^{th} coordinate (or π^{ij} for the projection on the two coordinates i and j).

We denote by $\mathscr{L}^d$ the d-dimensional Lebesgue measure on $\mathbb{R}^d$. We also use the notation $|A| := \mathscr{L}^d(A)$ when $A \subseteq \mathbb{R}^d$ is a Borel set. We write δ_x for the Dirac delta measure at x.

2.3.4 Weak solution to the Fokker–Planck equation

We say that a family of nonnegative measures $(\mu_t)_{t\geq 0}$ on Ω is a weak solution to the Fokker–Planck equation if:

1. it is continuous in duality with the space of continuous and compactly supported functions $C_c(\Omega)$;
2. for every open set $\omega \Subset \Omega$, both $t \mapsto \mu_t(\omega)$ and $t \mapsto \int |\nabla V| \,\mathrm{d}\mu_t|_\omega$ belong to $L^1_{\mathrm{loc}}\big([0,\infty)\big)$, i.e., their restrictions to $(0,\bar{t}\,)$ are integrable for every $\bar{t} > 0$;
3. for every $\varphi \in C^2_c(\Omega)$ and $0 \leq s \leq t$, the following identity holds:

$$\int \varphi \,\mathrm{d}\mu_t - \int \varphi \,\mathrm{d}\mu_s = \int_s^t \int \Big(\Delta\varphi - \langle \nabla\varphi, \nabla V\rangle\Big) \,\mathrm{d}\mu_r \,\mathrm{d}r\,. \tag{2.3.2}$$

2.3.5 Metric gradient flows

The general theory of gradient flows in metric spaces was developed in [AGS08]; we refer to this book and to the survey [San17] for a comprehensive exposition of the topic. We collect here only the definitions we need from this theory.

Let (X, d) be a metric space, let $[0,\infty) \ni t \mapsto x_t$ be an X-valued map, and let $f\colon X \to \mathbb{R} \cup \{\infty\}$ be a function.

Definition 2.3.3 (Metric derivative [AGS08, Theorem 1.1.2])**.** We say that $(x_t)_{t\in[0,\infty)}$ is *locally absolutely continuous* if there exists a function $m \in L^1_{\mathrm{loc}}\big([0,\infty)\big)$ such that

$$\mathsf{d}(x_s, x_t) \leq \int_s^t m(r)\,\mathrm{d}r \tag{2.3.3}$$

for every $0 \leq s < t$. If $(x_t)_{t\in[0,\infty)}$ is locally absolutely continuous, for $\mathscr{L}^1_{[0,\infty)}$-a.e. t there exists the limit

$$|\dot{x}_t| := \lim_{s\to t} \frac{\mathsf{d}(x_s, x_t)}{|s-t|}\,, \tag{2.3.4}$$

and this function, called *metric derivative*, is the $\mathscr{L}^1_{[0,\infty)}$-a.e. minimal function m that satisfies (2.3.3); see [AGS08, Theorem 1.1.2].[3]

Definition 2.3.4 (Descending slope [AGS08, Definition 1.2.4])**.** The *descending slope* of f at $x \in X$ is the number

$$\big|\partial f\big|(x) = \big|\partial_{\mathsf{d}} f\big|(x) := \limsup_{y \xrightarrow{\mathsf{d}} x} \frac{\big(f(x) - f(y)\big)_+}{\mathsf{d}(x,y)}\,, \tag{2.3.5}$$

where $a_+ := \max\{0, a\}$ is the positive part of $a \in \mathbb{R} \cup \{\pm\infty\}$. The slope is conventionally set equal to ∞ if $f(x) = \infty$, and to 0 if x is isolated and $f(x) < \infty$.

[3] In [AGS08, Theorem 1.1.2], the completeness of the space is assumed but not necessary, as can be easily checked.

Definition 2.3.5 (Curve of maximal slope [AGS08, Definition 1.3.2])**.** We say that a locally absolutely continuous X-valued map $(x_t)_{t\in[0,\infty)}$ is a *curve of maximal slope* (with respect to $|\partial_{\mathsf{d}} f|$) if $t \mapsto f(x_t)$ is a.e. equal to a nonincreasing map $\phi\colon [0,\infty) \to \mathbb{R}$ such that

$$\dot{\phi}(t) \leq -\frac{1}{2}|\dot{x}_t|^2 - \frac{1}{2}\left|\partial_{\mathsf{d}} f\right|^2(x_t) \quad \text{for } \mathscr{L}^1_{[0,\infty)}\text{-a.e. } t\,. \tag{2.3.6}$$

Definition 2.3.5 is motivated by the observation that, when (X, d) is a Euclidean space and f is smooth, the inequality (2.3.6) is equivalent to the gradient-flow equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t = -\nabla f(x_t)\,, \qquad t \geq 0\,,$$

see for instance [San17, Section 2.2]. As noted in [AGS08, Remark 1.3.3],[4] even in the general metric setting, (2.3.6) actually implies the identities

$$-\dot{\phi}(t) = |\dot{x}_t|^2 = \left|\partial_{\mathsf{d}} f\right|^2(x_t) \quad \text{for a.e. } t \geq 0\,.$$

2.3.6 The Figalli–Gigli distance

We briefly recall the definition and some properties of the distance Wb_2 introduced in [FG10]. We denote by $\mathcal{M}_2(\Omega)$ the set of nonnegative Borel measures μ on Ω such that

$$\int \inf_{y\in\partial\Omega} |x-y|^2 \,\mathrm{d}\mu(x) < \infty\,, \tag{2.3.7}$$

and, for every nonnegative Borel measure γ on $\overline{\Omega} \times \overline{\Omega}$, define the cost functional

$$\mathcal{C}(\gamma) := \int |x-y|^2 \,\mathrm{d}\gamma(x,y)\,. \tag{2.3.8}$$

Definition 2.3.6 ([FG10, Problem 1.1])**.** Let $\mu, \nu \in \mathcal{M}_2(\Omega)$. We say that a nonnegative Borel measure γ on $\overline{\Omega} \times \overline{\Omega}$ is a Wb_2*-admissible transport plan* between μ and ν, and write $\gamma \in \mathrm{Adm}_{Wb_2}(\mu,\nu)$, if

$$\left(\pi^1_{\#}\gamma\right)_{\Omega} = \mu \quad \text{and} \quad \left(\pi^2_{\#}\gamma\right)_{\Omega} = \nu\,. \tag{2.3.9}$$

The distance $Wb_2(\mu,\nu)$ is then defined as

$$Wb_2(\mu,\nu) := \inf\left\{\sqrt{\mathcal{C}(\gamma)} \;:\; \gamma \in \mathrm{Adm}_{Wb_2}(\mu,\nu)\right\}\,. \tag{2.3.10}$$

In [FG10, Section 2], it was observed that for every $\mu, \nu \in \mathcal{M}_2(\Omega)$ there exists at least one Wb_2-optimal transport plan, that is, a measure $\gamma \in \mathrm{Adm}_{Wb_2}(\mu,\nu)$ that attains the infimum in (2.3.10).

Later, we will make use of the following consequences of [FG10, Proposition 2.7]: the convergence w.r.t. the metric Wb_2 implies the convergence in duality with $C_c(\Omega)$, and it is implied by the convergence in duality with $C_b(\Omega)$.

[4] Once again, completeness is not necessary.

2.3.7 Transportation functionals

We now define the transportation functionals $\mathcal{T}$ and $\widetilde{Wb}_2$ that appear in the schemes (2.1.8) and (2.1.10).

Definition 2.3.7. For every $\mu, \nu \in \mathscr{S}$, let $\mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$ be the set of all finite nonnegative Borel measures γ on $\overline{\Omega} \times \overline{\Omega}$ such that

(1) $\left(\pi^1_{\#}\gamma\right)_{\Omega} = \mu_{\Omega}$,

(2) $\left(\pi^2_{\#}\gamma\right)_{\Omega} = \nu_{\Omega}$,

(3) $\pi^1_{\#}\gamma - \pi^2_{\#}\gamma = \mu - \nu$.

We call such measures $\widetilde{Wb}_2$-*admissible transport plans* between μ and ν. We set

$$\widetilde{Wb}_2(\mu, \nu) := \inf\left\{\sqrt{\mathcal{C}(\gamma)} \,:\, \gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)\right\}, \tag{2.3.11}$$

and write

$$\mathrm{Opt}_{\widetilde{Wb}_2}(\mu, \nu) := \operatorname*{arg\,min}_{\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu,\nu)} \mathcal{C}(\gamma) \tag{2.3.12}$$

for the set of all $\widetilde{Wb}_2$-*optimal tranport plans* between μ and ν.

Remark 2.3.8. There is some redundancy in the properties (1)-(3), indeed,

$$(1) + (3) \Rightarrow (2) \quad \text{and} \quad (2) + (3) \Rightarrow (1).$$

Definition 2.3.9. For every $\mu, \nu \in \mathscr{S}$, let $\mathrm{Adm}_{\mathcal{T}}(\mu, \nu)$ be the set of all measures $\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$ such that, additionally,

(4) $\gamma^{\partial\Omega}_{\partial\Omega} = 0$.

We define the $\mathcal{T}$-admissible/optimal tranport plans as in (2.3.11) and (2.3.12), by replacing $\widetilde{Wb}_2$ with $\mathcal{T}$.

Remark 2.3.10. If $\gamma \in \mathrm{Adm}_{\mathcal{T}}(\mu, \nu)$ for some $\mu, \nu \in \mathscr{S}$, then

$$\|\gamma\| \le \left\|\gamma^{\overline{\Omega}}_{\Omega}\right\| + \left\|\gamma^{\Omega}_{\overline{\Omega}}\right\| = \|\mu_{\Omega}\| + \|\nu_{\Omega}\| . \tag{2.3.13}$$

Remark 2.3.11. Fix $\mu, \nu \in \mathscr{S}$. For every $\eta \in \mathscr{S}$ concentrated on $\partial\Omega$, it is easy to check that

$$\mathrm{Adm}_{\widetilde{Wb}_2}(\mu + \eta, \nu + \eta) = \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu) \quad \text{and} \quad \mathrm{Adm}_{\mathcal{T}}(\mu + \eta, \nu + \eta) = \mathrm{Adm}_{\mathcal{T}}(\mu, \nu).$$

Hence,

$$\widetilde{Wb}_2(\mu + \eta, \nu + \eta) = \widetilde{Wb}_2(\mu, \nu) \quad \text{and} \quad \mathcal{T}(\mu + \eta, \nu + \eta) = \mathcal{T}(\mu, \nu). \tag{2.3.14}$$

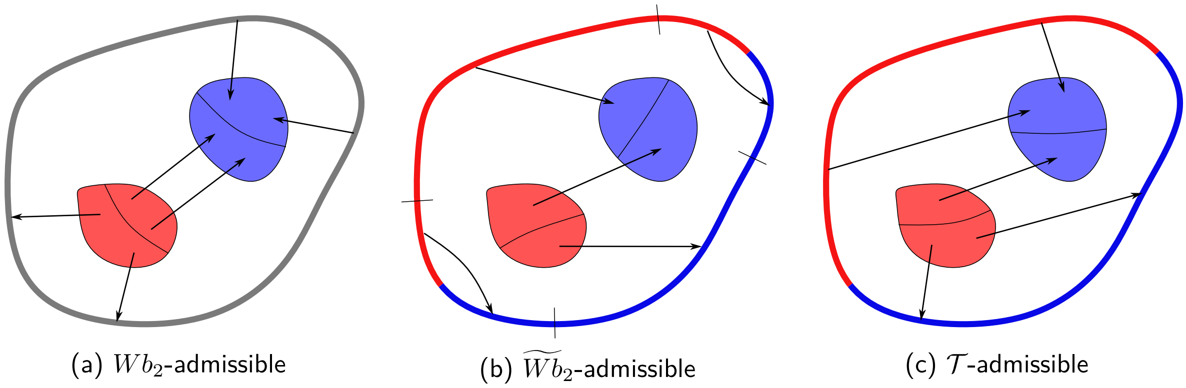

Figure 2.1: Examples of admisssible plans. Red (resp. blue) regions are those with an abundance of initial (resp. final) mass μ (resp. ν). Admissible plans for Wb_2 do not have any restriction on the mass departing from and arriving to $\partial\Omega$. Admissible plans for $\widetilde{Wb}_2$ must agree—in the sense of Condition (3)—with the configurations μ, ν also on $\partial\Omega$. Admissible plans for $\mathcal{T}$ are $\widetilde{Wb}_2$-admissible and, additionally, do not move mass from $\partial\Omega$ to $\partial\Omega$.

Let us briefly comment on these definitions. Conditions (1) and (2) are precisely the same as (2.3.9). They are needed to ensure that the mass that departs from (resp. arrives in) Ω is precisely μ_Ω (resp. ν_Ω). Condition (3) is needed to also keep track of the mass that is exchanged with the boundary. Namely, it ensures that, on each subregion of $A \subseteq \overline{\Omega}$ (possibly including part of the boundary, which was neglected by Conditions (2)-(3)), the mass $\nu(A)$ after the transportation equals the initial mass $\mu(A)$, plus the imported mass $\gamma(\overline{\Omega} \times A)$, minus the exported mass $\gamma(A \times \overline{\Omega})$. Observe that, since μ and ν normally have a negative mass on some subregions of $\partial\Omega$, it does not make sense to naively impose $\pi^1_\# \gamma = \mu$ and $\pi^2_\# \gamma = \nu$.

The difference between $\widetilde{Wb}_2$ and $\mathcal{T}$ is Condition (4): $\mathcal{T}$-admissible transport plans cannot move mass from $\partial\Omega$ to $\partial\Omega$. This results in the loss of the triangle inequality.

Example 2.3.12. Consider, for the domain $\Omega \coloneqq (0, 1)$, the measures

$$\mu_1 \coloneqq \delta_0 - \delta_1 \in \mathscr{S}\,, \qquad \mu_2 \coloneqq \delta_{1/2} - \delta_1 \in \mathscr{S}\,, \qquad \mu_3 \coloneqq 0 \in \mathscr{S}\,.$$

The transport plans $\gamma_{12} \coloneqq \delta_{(0,1/2)}$ and $\gamma_{23} \coloneqq \delta_{(1/2,1)}$ are $\mathcal{T}$-admissible, between μ_1 and μ_2, and between μ_2 and μ_3, respectively. Thus, both $\mathcal{T}(\mu_1, \mu_2)$ and $\mathcal{T}(\mu_2, \mu_3)$ are bounded above by $1/2$. However, there is no $\gamma_{13} \in \mathrm{Adm}_{\mathcal{T}}(\mu_1, \mu_3)$, whence $\mathcal{T}(\mu_1, \mu_3) = \infty$. Indeed, Conditions (1) and (2) in Definition 2.3.7 would imply $(\gamma_{13})^{\overline{\Omega}}_{\Omega} = (\gamma_{13})^{\Omega}_{\overline{\Omega}} = 0$. Together with (4) in Definition 2.3.9, this means that γ_{13} equals the zero measure, which contradicts (3) in Definition 2.3.7.

Nonetheless, it is shown in Proposition 2.8.1 that Condition (4) is needed in dimension $d \geq 2$, because the information about $\mu_{\partial\Omega}$ and $\nu_{\partial\Omega}$ may otherwise be lost. This does not happen when Ω is just a finite union of intervals in $\mathbb{R}^1$, because points in $\partial\Omega$ are distant from each other. We will see that, in this case, Definition 2.3.7 defines a distance.

These remarks reveal part of the difficulties in building cost functionals for signed measures that behave like W_2. See [Mai11] for further details. However, it seems at least convenient to use signed measures, given that a modified JKO scheme that mimics [FG10] should allow for a virtually unlimited amount of mass to be taken from points of $\partial\Omega$, step after step.

2.4 Properties of the transportation functionals

We gather some useful properties of $\mathcal{T}$ and $\widetilde{Wb}_2$.

2.4.1 Relation with the Figalli–Gigli distance

For every $\mu, \nu \in \mathscr{S}$, we have the inclusions

$$\mathrm{Adm}_{\mathcal{T}}(\mu,\nu) \subseteq \mathrm{Adm}_{\widetilde{Wb}_2}(\mu,\nu) \subseteq \mathrm{Adm}_{Wb_2}(\mu_\Omega,\nu_\Omega)\,.$$

As a consequence,

$$Wb_2(\mu_\Omega,\nu_\Omega) \le \widetilde{Wb}_2(\mu,\nu) \le \mathcal{T}(\mu,\nu)\,, \qquad \mu,\nu \in \mathscr{S}\,. \tag{2.4.1}$$

In fact, $\widetilde{Wb}_2$ and $\mathcal{T}$ can be seen as extensions of Wb_2 in the following sense.

Lemma 2.4.1. *Let μ, ν be finite nonnegative Borel measures on Ω. For every $\tilde\mu \in \mathscr{S}$ with $\tilde\mu_\Omega = \mu$, we have the identities*

$$Wb_2(\mu,\nu) = \inf_{\tilde\nu\in\mathscr{S}} \left\{\widetilde{Wb}_2(\tilde\mu,\tilde\nu) \,:\, \tilde\nu_\Omega = \nu\right\} = \inf_{\tilde\nu\in\mathscr{S}} \left\{\mathcal{T}(\tilde\mu,\tilde\nu) \,:\, \tilde\nu_\Omega = \nu\right\}\,. \tag{2.4.2}$$

Proof. In light of (2.4.1), it suffices to prove that

$$\inf_{\tilde\nu\in\mathscr{S}} \left\{\mathcal{T}(\tilde\mu,\tilde\nu) \,:\, \tilde\nu_\Omega = \nu\right\} \le Wb_2(\mu,\nu)\,.$$

Let $\gamma \in \mathrm{Adm}_{Wb_2}(\mu,\nu)$. Define $\tilde\gamma := \gamma - \gamma_{\partial\Omega}^{\partial\Omega}$ and

$$\tilde\nu := \tilde\mu + \pi^2_\#\tilde\gamma - \pi^1_\#\tilde\gamma\,.$$

It is easy to check that $\tilde\nu_\Omega = \nu$, that $\tilde\gamma \in \mathrm{Adm}_{\mathcal{T}}(\tilde\mu,\tilde\nu)$, and that $\mathcal{C}(\tilde\gamma) \le \mathcal{C}(\gamma)$. As a consequence,

$$\inf_{\tilde\nu\in\mathscr{S}} \left\{\mathcal{T}(\tilde\mu,\tilde\nu) \,:\, \tilde\nu_\Omega = \nu\right\} \le \sqrt{\mathcal{C}(\gamma)}\,,$$

and we conclude by arbitrariness of γ. □

2.4.2 Relation with the Kantorovich–Rubinstein norm

Interestingly, an inequality relates $\widetilde{Wb}_2$ and $\|\cdot\|_{\widetilde{\mathrm{KR}}}$.

Lemma 2.4.2. *For every $\mu, \nu \in \mathscr{S}$, we have*

$$\widetilde{Wb}_2^2(\mu,\nu) \le \mathrm{diam}(\Omega)\|\mu-\nu\|_{\widetilde{\mathrm{KR}}}\,. \tag{2.4.3}$$

Proof. Define the nonnegative measures

$$\tilde\mu := \mu_\Omega + (\mu_{\partial\Omega} - \nu_{\partial\Omega})_+\,, \qquad \tilde\nu := \nu_\Omega + (\mu_{\partial\Omega} - \nu_{\partial\Omega})_-\,,$$

and note that $\tilde\mu - \tilde\nu = \mu - \nu$. In particular, $\tilde\mu(\overline\Omega) = \tilde\nu(\overline\Omega)$.

Let γ be a coupling between $\tilde\mu$ and $\tilde\nu$, i.e., γ is a nonnegative Borel measure on $\overline\Omega \times \overline\Omega$ such that $\pi^1_\#\gamma = \tilde\mu$ and $\pi^2_\#\gamma = \tilde\nu$. Notice that γ is $\widetilde{Wb}_2$-admissible between μ and ν. Therefore,

$$\widetilde{Wb}_2^2(\mu,\nu) \le \mathcal{C}(\gamma) = \int |x-y|^2 \,\mathrm{d}\gamma \le \mathrm{diam}(\Omega)\int |x-y|\,\mathrm{d}\gamma\,.$$

After taking the infimum over γ, the Kantorovich–Rubinstein duality [Bog07, Theorem 8.10.45] implies

$$\widetilde{Wb}_2^2(\mu,\nu) \le \mathrm{diam}(\Omega)\|\tilde\mu-\tilde\nu\|_{\widetilde{\mathrm{KR}}} = \mathrm{diam}(\Omega)\|\mu-\nu\|_{\widetilde{\mathrm{KR}}}\,.$$ □

2.4.3 $\mathcal{T}$ is an extended semimetric

The functional $\mathcal{T}$ may take the value infinity and does not satisfy the triangle inequality; see Example 2.3.12. Nonetheless, we have the following proposition, which we prove together with two useful lemmas.

Proposition 2.4.3. *The functional $\mathcal{T}$ is an extended semimetric, i.e., it is nonnegative, symmetric, and we have*

$$\mathcal{T}(\mu,\nu)=0 \quad\Longleftrightarrow\quad \mu=\nu\,. \tag{2.4.4}$$

Lemma 2.4.4. *Let $(\mu^n)_{n\in\mathbb{N}_0}$ and $(\nu^n)_{n\in\mathbb{N}_0}$ be two sequences in $\mathscr{S}$, and let $\gamma^n\in\mathrm{Adm}_{\mathcal{T}}(\mu^n,\nu^n)$ for every $n\in\mathbb{N}_0$. Assume that*

(a) $\mu^n\to_n\mu$ and $\nu^n\to_n\nu$ weakly for some μ,ν,

(b) $\mu^n_\Omega\to_n\mu_\Omega$ and $\nu^n_\Omega\to_n\nu_\Omega$ setwise, *i.e., on all Borel sets,*

(c) $\gamma^n\to_n\gamma$ weakly.

Then $\mu,\nu\in\mathscr{S}$ and $\gamma\in\mathrm{Adm}_{\mathcal{T}}(\mu,\nu)$.

In particular, for any $\mu,\nu\in\mathscr{S}$, the set $\mathrm{Adm}_{\mathcal{T}}(\mu,\nu)$ is sequentially closed with respect to the weak convergence.

The proof of this lemma is inspired by part of that of [Mor18, Lemma 3.1].

Proof. The total mass of γ^n is bounded and, therefore, the same can be said for the total mass of $(\gamma^n)^\Omega_\Omega,(\gamma^n)^{\partial\Omega}_\Omega,(\gamma^n)^\Omega_{\partial\Omega}$. Hence, up to taking a subsequence, we may assume that

$$\begin{aligned}
(\gamma^n)^\Omega_\Omega &\to_n \sigma_1 \quad \text{in duality with } C(\overline{\Omega}\times\overline{\Omega})\,,\\
(\gamma^n)^{\partial\Omega}_\Omega &\to_n \sigma_2 \quad \text{in duality with } C(\overline{\Omega}\times\partial\Omega)\,,\\
(\gamma^n)^\Omega_{\partial\Omega} &\to_n \sigma_3 \quad \text{in duality with } C(\partial\Omega\times\overline{\Omega})
\end{aligned}$$

for some $\sigma_1,\sigma_2,\sigma_3$. In particular, $\gamma^n\to_n\gamma:=\sigma_1+\sigma_2+\sigma_3$.

We *claim* that $\sigma_1,\sigma_2,\sigma_3$ are concentrated on $\Omega\times\Omega,\Omega\times\partial\Omega,\partial\Omega\times\Omega$ respectively. If this is true, then Condition (4) in Definition 2.3.9 for γ is obvious, and those in Definition 2.3.7 follow by testing them with a function $f\in C_b(\overline{\Omega})$ for every n and passing to the limit. For instance, to prove Condition (1) in Definition 2.3.7, we write the chain of equalities

$$\begin{aligned}
\mu_\Omega(f) &= \lim_{n\to\infty}\mu^n_\Omega(f)=\lim_{n\to\infty}\int f(x)\,\mathrm{d}(\gamma^n)^{\overline{\Omega}}_\Omega(x,y)\\
&=\int f(x)\,\mathrm{d}(\sigma_1+\sigma_2)(x,y)=\int f(x)\,\mathrm{d}\gamma^{\overline{\Omega}}_\Omega(x,y)=\left(\pi^1_\#\gamma^{\overline{\Omega}}_\Omega\right)(f)\,.
\end{aligned}$$

Let us prove the claim. Let $A\subseteq\overline{\Omega}$ be an open set, in the relative topology of $\overline{\Omega}$, that contains $\partial\Omega$. We have

$$\begin{aligned}
\sigma_1(\partial\Omega\times\overline{\Omega}) &\le \sigma_1(A\times\overline{\Omega})\le\liminf_{n\to\infty}(\gamma^n)^\Omega_\Omega(A\times\overline{\Omega})\\
&\le\liminf_{n\to\infty}(\gamma^n)^{\overline{\Omega}}_\Omega(A\times\overline{\Omega})=\liminf_{n\to\infty}\mu^n_\Omega(A)=\mu_\Omega(A)\,,
\end{aligned}$$

where the second inequality follows from the semicontinuity of the mass on open sets (in the topology of $\overline{\Omega} \times \overline{\Omega}$) and the last equality from the setwise convergence. Since μ_Ω has finite total mass and $\mu_\Omega(\partial\Omega) = 0$, we have $\sigma_1(\partial\Omega \times \overline{\Omega}) = 0$. Analogously, using Condition (2) in place of Condition (1), we obtain $\sigma_1(\overline{\Omega} \times \partial\Omega) = 0$. For σ_2 and σ_3, the proof is similar. □

Lemma 2.4.5. *If $\mathcal{T}(\mu, \nu) < \infty$, then $\mathrm{Opt}_{\mathcal{T}}(\mu, \nu) \neq \emptyset$.*

Proof. It suffices to prove that $\mathrm{Adm}_{\mathcal{T}}(\mu, \nu)$ is nonempty and weakly sequentially compact. It is nonempty if $\mathcal{T}(\mu, \nu) < \infty$. It is sequentially compact because

$$\gamma \in \mathrm{Adm}_{\mathcal{T}}(\mu, \nu) \quad \overset{(2.3.13)}{\Longrightarrow} \quad \|\gamma\| \leq \|\mu_\Omega\| + \|\nu_\Omega\| ,$$

and thanks to Lemma 2.4.4. □

Proof of Proposition 2.4.3. Only the implication $\Rightarrow$ in (2.4.4) is not immediate. Let us assume that $\mathcal{T}(\mu, \nu) = 0$ and let $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu, \nu)$. Since $\mathcal{C}(\gamma) = 0$, the measure γ is concentrated on the diagonal of $\overline{\Omega} \times \overline{\Omega}$. Thus, the equality $\mu = \nu$ follows from Condition (3) in Definition 2.3.7. □

We conclude with a corollary of Lemma 2.4.4: a semicontinuity property of $\mathcal{T}$.

Corollary 2.4.6. *Let $(\mu^n)_{n \in \mathbb{N}_0}$ and $(\nu^n)_{n \in \mathbb{N}_0}$ be two sequences in $\mathscr{S}$. Assume that*

(a) $\mu^n \to_n \mu$ and $\nu^n \to_n \nu$ weakly for some μ, ν,

(b) $\mu^n_\Omega \to_n \mu_\Omega$ and $\nu^n_\Omega \to_n \nu_\Omega$ setwise, *i.e., on all Borel sets.*

Then

$$\mathcal{T}(\mu, \nu) \leq \liminf_{n \to \infty} \mathcal{T}(\mu^n, \nu^n) . \tag{2.4.5}$$

Proof. We may assume that the right-hand side in (2.4.5) exists as a finite limit and that, for every $n \in \mathbb{N}_0$, there exists $\gamma^n \in \mathrm{Adm}_{\mathcal{T}}(\mu, \nu)$ such that

$$\mathcal{C}(\gamma^n) \leq \mathcal{T}^2(\mu^n, \nu^n) + \frac{1}{n} .$$

The total variation of each measure γ^n is bounded by $\|\mu^n_\Omega\| + \|\nu^n_\Omega\|$, which is in turn bounded thanks to the assumption. Therefore, we can extract a subsequence $(\gamma^{n_k})_{k \in \mathbb{N}_0}$ that converges weakly to a measure γ. We know from Lemma 2.4.4 that $\gamma \in \mathrm{Adm}_{\mathcal{T}}(\mu, \nu)$; thus,

$$\mathcal{T}^2(\mu, \nu) \leq \mathcal{C}(\gamma) = \lim_{k \to \infty} \mathcal{C}(\gamma^{n_k}) = \lim_{k \to \infty} \mathcal{T}^2(\mu^{n_k}, \nu^{n_k}) = \lim_{n \to \infty} \mathcal{T}^2(\mu^n, \nu^n) . \qquad \square$$

2.4.4 $\mathcal{H}$ is "semicontinuous w.r.t $\mathcal{T}$"

Albeit not being a distance, the transportation functional $\mathcal{T}$ makes $\mathcal{H}$ lower semicontinuous, in the following sense.

Proposition 2.4.7. *Let $(\mu^n)_{n \in \mathbb{N}_0}$ be a sequence in $\mathscr{S}$ and suppose that*

$$\lim_{n \to \infty} \mathcal{T}(\mu^n, \mu) = 0 \tag{2.4.6}$$

for some $\mu \in \mathscr{S}$. Then

$$\mathcal{H}(\mu) \leq \liminf_{n \to \infty} \mathcal{H}(\mu^n) . \tag{2.4.7}$$

For the proof we need a lemma, to which we will also often refer later. This lemma, inspired by [Mor18, Lemma 5.8] allows to control $(\mu-\nu)_{\partial\Omega}$ in terms of $\mathcal{T}(\mu,\nu)$ and of the restrictions μ_Ω and ν_Ω. This fact is convenient for two reasons:

- the part of the functional $\mathcal{H}$ that depends on μ_Ω is superlinear,
- we will see (Remark 2.5.17) that *the restrictions to* Ω of the measures produced by the scheme (2.1.8) have bounded (in time) mass.

Lemma 2.4.8. *Let $\tau > 0$, let $\mu,\nu \in \mathscr{S}$, and let $\Phi\colon \overline{\Omega} \to \mathbb{R}$ be Lipschitz continuous. Then,*

$$|\mu(\Phi) - \nu(\Phi)| \le \tau(\operatorname{Lip}\Phi)^2\Big(\|\mu_\Omega\| + \|\nu_\Omega\|\Big) + \frac{\mathcal{T}^2(\mu,\nu)}{4\tau}\,. \tag{2.4.8}$$

In particular,

$$\mu_{\partial\Omega}(\Phi) - \nu_{\partial\Omega}(\Phi) \le \nu_\Omega(\Phi) - \mu_\Omega(\Phi) + \tau(\operatorname{Lip}\Phi)^2\Big(\|\mu_\Omega\| + \|\nu_\Omega\|\Big) + \frac{\mathcal{T}^2(\mu,\nu)}{4\tau}\,. \tag{2.4.9}$$

Proof. Let $\gamma \in \operatorname{Opt}_{\mathcal{T}}(\mu,\nu)$. By Definition 2.3.7 and Definition 2.3.9, we have

$$\begin{aligned}
|\mu(\Phi) - \nu(\Phi)| &= \Big|(\pi^1_\#\gamma - \pi^2_\#\gamma)(\Phi)\Big| = \left|\int \Big(\Phi(x) - \Phi(y)\Big)\,\mathrm{d}\gamma(x,y)\right| \\
&\le \int \sqrt{2\tau}(\operatorname{Lip}\Phi)\cdot\frac{|x-y|}{\sqrt{2\tau}}\,\mathrm{d}\gamma(x,y) \\
&\le \tau(\operatorname{Lip}\Phi)^2\|\gamma\| + \frac{1}{4\tau}\int |x-y|^2\,\mathrm{d}\gamma(x,y) \\
&\le \tau(\operatorname{Lip}\Phi)^2\Big(\|\mu_\Omega\| + \|\nu_\Omega\|\Big) + \frac{\mathcal{T}^2(\mu,\nu)}{4\tau}\,.
\end{aligned}$$

□

Proof of Proposition 2.4.7. We may assume that the right-hand side in (2.4.7) exists as a finite limit and that $\mathcal{H}(\mu^n)$ is finite for every n. In particular, μ^n_Ω is absolutely continuous w.r.t. $\mathscr{L}^d_\Omega$. Denote by ρ^n its density. Owing to Lemma 2.4.8, for every $\tau > 0$ and n, we have

$$\begin{aligned}
\mathcal{H}(\mu^n) &= \mathcal{E}(\rho^n) + \mu^n_{\partial\Omega}(\Psi) \\
&\ge \int_\Omega (\log\rho^n + V - 1 - \mathfrak{c}\tau - \Psi)\rho^n\,\mathrm{d}x + |\Omega| + \mu(\Psi) - \mathfrak{c}\tau\|\mu_\Omega\| - \frac{\mathcal{T}^2(\mu^n,\mu)}{4\tau}\,.
\end{aligned}$$

It follows that the sequence $(\rho^n)_n$ is uniformly integrable. By the Dunford–Pettis theorem, it admits a (not relabeled) subsequence that converges, weakly in $L^1(\Omega)$, to some function ρ. From (2.4.1) and [FG10, Proposition 2.7], we infer that $\mu^n_\Omega \to \mu_\Omega$ in duality with $C_c(\Omega)$ and, therefore, ρ is precisely the density of μ_Ω. The functional $\mathcal{E}$ is convex and lower semicontinuous on $L^1(\Omega)$ (by Fatou's lemma), hence weakly lower semicontinuous. Thus, we are only left with proving that

$$\mu_{\partial\Omega}(\Psi) \le \liminf_{n\to\infty} \mu^n_{\partial\Omega}(\Psi)\,.$$

Once again, we make use of Lemma 2.4.8 and of the weak convergence in $L^1(\Omega)$ to write, for every $\tau > 0$,

$$\limsup_{n\to\infty}(\mu - \mu^n)_{\partial\Omega}(\Psi) \le \limsup_{n\to\infty}\mathfrak{c}\tau\Big(\|\mu^n_\Omega\| + \|\mu_\Omega\|\Big) + \limsup_{n\to\infty}\frac{\mathcal{T}^2(\mu^n,\mu)}{4\tau} \le \mathfrak{c}\tau\|\mu_\Omega\|\,.$$

We conclude by arbitrariness of τ. □

2.4.5 $\widetilde{Wb}_2$ is a pseudodistance

The functional $\widetilde{Wb}_2$ is a pseudodistance on $\mathscr{S}$, meaning that it fulfills the properties of a distance, except, possibly, $\mu = \nu$ when $\widetilde{Wb}_2(\mu,\nu) = 0$. As before, nonnegativity, symmetry, and the implication

$$\mu = \nu \quad \Longrightarrow \quad \widetilde{Wb}_2(\mu,\nu) = 0$$

are obvious. To prove finiteness, it suffices to produce a single $\gamma \in \operatorname{Adm}_{\widetilde{Wb}_2}(\mu,\nu)$ for every $\mu,\nu \in \mathscr{S}$. Let us arbitrarily fix a probability measure ζ on $\partial\Omega$ and set

$$\eta \coloneqq \mu_{\partial\Omega} - \nu_{\partial\Omega} + \Big(\|\mu_\Omega\| - \|\nu_\Omega\|\Big)\zeta\,.$$

The following is $\widetilde{Wb}_2$-admissible:

$$\gamma \coloneqq \begin{cases} \mu_\Omega \otimes \zeta + \zeta \otimes \nu_\Omega + \frac{\eta_+ \otimes \eta_-}{\|\eta_+\|} & \text{if } \eta \neq 0\,, \\ \mu_\Omega \otimes \zeta + \zeta \otimes \nu_\Omega & \text{if } \eta = 0\,. \end{cases}$$

Only the triangle inequality is still missing.

Proposition 2.4.9. *The functional $\widetilde{Wb}_2$ satisfies the triangle inequality. Hence, it is a pseudodistance.*

Proof. Let $\mu_1,\mu_2,\mu_3 \in \mathscr{S}$, and let us view them as measures on three different copies of $\overline{\Omega}$, that we denote by $\overline{\Omega}_1,\overline{\Omega}_2,\overline{\Omega}_3$, respectively. We write π^2 for both the projections from $\overline{\Omega}_1 \times \overline{\Omega}_2$ and $\overline{\Omega}_2 \times \overline{\Omega}_3$ onto $\overline{\Omega}_2$.

Choose two transport plans $\gamma_{12} \in \operatorname{Adm}_{\widetilde{Wb}_2}(\mu_1,\mu_2)$ and $\gamma_{23} \in \operatorname{Adm}_{\widetilde{Wb}_2}(\mu_2,\mu_3)$. Let $\eta \coloneqq (\pi^2_\#\gamma_{23} - \pi^2_\#\gamma_{12})_{\partial\Omega}$ and consider

$$\tilde\gamma_{12} \coloneqq \gamma_{12} + (\mathrm{Id},\mathrm{Id})_\#\eta_+, \quad \tilde\gamma_{23} \coloneqq \gamma_{23} + (\mathrm{Id},\mathrm{Id})_\#\eta_-\,.$$

It is easy to check that these are admissible too, i.e., $\tilde\gamma_{12} \in \operatorname{Adm}_{\widetilde{Wb}_2}(\mu_1,\mu_2)$ and $\tilde\gamma_{23} \in \operatorname{Adm}_{\widetilde{Wb}_2}(\mu_2,\mu_3)$, as well as that $\mathcal{C}(\gamma_{12}) = \mathcal{C}(\tilde\gamma_{12})$ and $\mathcal{C}(\gamma_{23}) = \mathcal{C}(\tilde\gamma_{23})$. Furthermore, $\pi^2_\#\tilde\gamma_{12}$ equals $\pi^2_\#\tilde\gamma_{23}$. The gluing lemma [AGS08, Lemma 5.3.2] supplies a nonnegative Borel measure $\tilde\gamma_{123}$ such that

$$\pi^{12}_\#\tilde\gamma_{123} = \tilde\gamma_{12} \quad \text{and} \quad \pi^{23}_\#\tilde\gamma_{123} = \tilde\gamma_{23}\,.$$

The measure $\gamma \coloneqq \pi^{13}_\#\tilde\gamma_{123}$ is $\widetilde{Wb}_2$-admissible between μ_1 and μ_2. By the Minkowski inequality,

$$\widetilde{Wb}_2(\mu_1,\mu_2) \le \sqrt{\mathcal{C}(\gamma)} \le \sqrt{\mathcal{C}(\tilde\gamma_{12})} + \sqrt{\mathcal{C}(\tilde\gamma_{23})} = \sqrt{\mathcal{C}(\gamma_{12})} + \sqrt{\mathcal{C}(\gamma_{23})}\,,$$

from which, by arbitrariness of γ_{12} and γ_{23}, the triangle inequality follows. □

In general, $\widetilde{Wb}_2$ is *not* a true metric on $\mathscr{S}$. This is proven in Proposition 2.8.1. However, an analogue of Lemma 2.4.4 holds (proof omitted).

Lemma 2.4.10. *Let $(\mu^n)_{n\in\mathbb{N}_0}$ and $(\nu^n)_{n\in\mathbb{N}_0}$ be two sequences in $\mathscr{S}$, and let $\gamma^n \in \operatorname{Adm}_{\widetilde{Wb}_2}(\mu^n,\nu^n)$ for every $n \in \mathbb{N}_0$. Assume that*

(a) $\mu^n \to_n \mu$ and $\nu^n \to_n \nu$ weakly for some μ,ν,

(b) $\mu_\Omega^n \to_n \mu_\Omega$ *and* $\nu_\Omega^n \to_n \nu_\Omega$ setwise, *i.e., on all Borel sets,*

(c) $\gamma^n \to_n \gamma$ *weakly.*

Then $\mu, \nu \in \mathscr{S}$ *and* $\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$.

In particular, for any $\mu, \nu \in \mathscr{S}$*, the set* $\mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$ *is sequentially closed with respect to the weak convergence.*

2.4.6 When Ω is a finite union of intervals, $\widetilde{Wb}_2$ is a distance

When Ω is a finite union of 1-dimensional intervals (equivalently, when $\partial\Omega$ is a finite set) we also have

$$\widetilde{Wb}_2(\mu, \nu) = 0 \quad \Longleftrightarrow \quad \mu = \nu\,.$$

Proposition 2.4.11. *If* $d = 1$ *and* Ω *is a finite union of intervals, then* $(\mathscr{S}, \widetilde{Wb}_2)$ *is a metric space.*

This proposition is an easy consequence of the following remark and lemma, analogous to Remark 2.3.10 and Lemma 2.4.5, respectively.

Remark 2.4.12. Fix $\mu, \nu \in \mathscr{S}$ and pick $\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$. If $\partial\Omega$ is finite and the diagonal of $\partial\Omega \times \partial\Omega$ is γ-negligible, then

$$\begin{aligned} \|\gamma\| &\le \left\|\gamma_\Omega^{\overline{\Omega}}\right\| + \left\|\gamma_{\overline{\Omega}}^{\Omega}\right\| + \left\|\gamma_{\partial\Omega}^{\partial\Omega}\right\| \le \|\mu_\Omega\| + \|\nu_\Omega\| + \frac{1}{\min_{\substack{x,y\in\partial\Omega \\ x\neq y}} |x-y|^2} \int |x-y|^2 \,\mathrm{d}\gamma(x,y) \\ &\le \|\mu_\Omega\| + \|\nu_\Omega\| + \mathfrak{c}\,\mathcal{C}(\gamma)\,. \end{aligned} \tag{2.4.10}$$

Lemma 2.4.13. *Assume that* $d = 1$ *and that* Ω *is a finite union of intervals. Then the set* $\mathrm{Opt}_{\widetilde{Wb}_2}(\mu, \nu)$ *is nonempty for every* $\mu, \nu \in \mathscr{S}$.

Proof. We already know that $\mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu) \neq \emptyset$. Let us take a minimizing sequence $(\gamma^n)_{n\in\mathbb{N}_0} \subseteq \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$ for the cost functional $\mathcal{C}$. Let Δ be the diagonal of $\partial\Omega \times \partial\Omega$. It is easy to see that $(\gamma^n - \gamma^n|_\Delta)_n$ is still an admissible and minimizing sequence. Therefore, we can assume that $\gamma^n|_\Delta = 0$. By Remark 2.4.12, the total variation of γ^n is bounded. Therefore, there exists a subsequence of $(\gamma^n)_n$ that converges weakly to a limit γ and, by Lemma 2.4.10, $\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \nu)$. Since the sequence is minimizing, γ is also $\widetilde{Wb}_2$-optimal. □

Two further useful facts about $\widetilde{Wb}_2$ are the counterparts of Lemma 2.4.8 and Proposition 2.4.7 in the case where Ω is a finite union of intervals.

Lemma 2.4.14. *Assume that* $d = 1$ *and that* Ω *is a finite union of intervals. Let* $\mu, \nu \in \mathscr{S}$ *and let* $\Phi \colon \overline{\Omega} \to \mathbb{R}$ *be Lipschitz continuous. Then,*

$$|\mu(\Phi) - \nu(\Phi)| \le \mathfrak{c}_\Phi \widetilde{Wb}_2(\mu, \nu) \sqrt{\|\mu_\Omega\| + \|\nu_\Omega\| + \widetilde{Wb}_2^2(\mu, \nu)}\,. \tag{2.4.11}$$

Proof. By Condition (3) in Definition 2.3.7, for every $\mu, \nu \in \mathscr{S}$ and every $\gamma \in \mathrm{Opt}_{\widetilde{Wb}_2}(\mu, \nu)$, we have

$$|\mu(\Phi) - \nu(\Phi)| = \left| \int \big(\Phi(x) - \Phi(y)\big) \, \mathrm{d}\gamma(x, y) \right| \le (\mathrm{Lip}\,\Phi) \int |x - y| \, \mathrm{d}\gamma(x, y)$$
$$\le (\mathrm{Lip}\,\Phi) \sqrt{\mathcal{C}(\gamma) \, \|\gamma\|} = (\mathrm{Lip}\,\Phi) \widetilde{Wb}_2(\mu, \nu) \sqrt{\|\gamma\|} \,.$$

We can assume that the diagonal of $\partial\Omega \times \partial\Omega$ is γ-negligible; hence, we conclude by Remark 2.4.12. □

Proposition 2.4.15. *Assume that $d = 1$ and that Ω is a finite union of intervals. Then $\mathcal{H}$ is lower semicontinuous w.r.t. $\widetilde{Wb}_2$.*

Proof. Similar to the proof of Proposition 2.4.7, making use of Lemma 2.4.14 in place of Lemma 2.4.8. □

When $\widetilde{Wb}_2$ defines a metric, a natural question is whether or not this metric is complete. In general, the answer is *no*; this is proven in Proposition 2.8.2. Nonetheless, we prove in Proposition 2.8.3 that the *sublevels* of $\mathcal{H}$ are complete for $\widetilde{Wb}_2$.

Another interesting problem is to find a convergence criterion for $\widetilde{Wb}_2$. Exploiting Lemma 2.4.2, we find a simple sufficient condition for convergence in the 1-dimensional setting.

Lemma 2.4.16. *Assume that $d = 1$ and that Ω is a finite union of intervals. If $(\mu^n)_{n \in \mathbb{N}_0} \subseteq \mathscr{S}$ converges weakly to $\mu \in \mathscr{S}$, then $\mu^n \overset{\widetilde{Wb}_2}{\to}_n \mu$.*

Proof. The idea is to use Lemma 2.4.2 together with the measure-theoretic result [Bog07, Theorem 8.3.2]: the metric induced by $\|\cdot\|_{\widetilde{\mathrm{KR}}}$ metrizes the weak convergence[5] of *nonnegative* Borel measures on $\overline{\Omega}$. For every $x \in \partial\Omega$, let $a_x := -\inf_n \mu_n(x)$. Every number a_x is finite because, by the uniform boundedness principle, the total variation of μ^n is bounded. By the considerations above, we have

$$\mu^n \to_n \mu \text{ weakly} \quad \implies \quad \mu^n + \sum_{x \in \partial\Omega} a_x \delta_x \to_n \mu + \sum_{x \in \partial\Omega} a_x \delta_x \text{ weakly}$$
$$\implies \quad \|\mu^n - \mu\|_{\widetilde{\mathrm{KR}}} \to_n 0 \quad \overset{(2.4.3)}{\implies} \quad \widetilde{Wb}_2(\mu^n, \mu) \to_n 0 \,. \quad \square$$

Remark 2.4.17. The converse of Lemma 2.4.16 is not true: in the case $\Omega := (0, 1)$, consider the sequence

$$\mu^n := n(\delta_{1/n} - \delta_0) \,, \qquad n \in \mathbb{N}_1 \,,$$

which converges to $\mu := 0$ w.r.t. $\widetilde{Wb}_2$.

2.4.7 Estimate on the directional derivative

The following lemma will be used in Proposition 2.5.9 to characterize the solutions of the variational problem (2.1.8). We omit its simple proof, almost identical to that of [FG10, Proposition 2.11].

[5] In [Bog07], two Kantorovich–Rubinstein norms are defined. Here, we implicitly use that they are equivalent on measures on a bounded metric space; see [Bog07, Section 8.10(viii)].

Lemma 2.4.18. *Let $\mu, \nu \in \mathscr{S}$ and $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu,\nu)$. Let $\boldsymbol{w}\colon \Omega \to \mathbb{R}^d$ be a bounded and Borel vector field with compact support. For $t > 0$ sufficiently small, define $\mu_t := (\mathrm{Id} + t\boldsymbol{w})_{\#}\mu$. Then*

$$\limsup_{t \to 0^+} \frac{\mathcal{T}^2(\mu_t,\nu) - \mathcal{T}^2(\mu,\nu)}{t} \le -2 \int \langle \boldsymbol{w}(x), y - x \rangle \, \mathrm{d}\gamma(x,y)\,. \tag{2.4.12}$$

2.4.8 Existence of transport maps

Proposition 2.4.19. *Let $\mu, \nu \in \mathscr{S}$, let $A, B \subseteq \overline{\Omega} \times \overline{\Omega}$ be Borel sets, and let γ be a nonnegative Borel measure on $\overline{\Omega} \times \overline{\Omega}$. If*

(a) $\gamma \in \mathrm{Opt}_{\widetilde{Wb}_2}(\mu,\nu)$,

(b) or: $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu,\nu)$ *and* $(A \times B) \cap (\partial\Omega \times \partial\Omega) = \emptyset$,

then γ_A^B is optimal for the classical 2-Wasserstein distance *between its marginals.*

Consequently: under the assumptions of this proposition, if one of the two marginals of γ_A^B is absolutely continuous, we can apply Brenier's theorem [Bre87] and deduce the existence of an optimal transport map. For instance, whenever μ_Ω is absolutely continuous, there exists a Borel map $T\colon \Omega \to \overline{\Omega}$ such that $\gamma_\Omega^{\overline{\Omega}} = (\mathrm{Id}, T)_{\#}\mu_\Omega$.

Proof of Proposition 2.4.19. Let $\tilde{\gamma}$ be any nonnegative Borel coupling between $\pi^1_{\#}\gamma_A^B$ and $\pi^2_{\#}\gamma_A^B$. In particular, $\tilde{\gamma}$ is concentrated on $A \times B$. Define the nonnegative measure

$$\gamma' := \gamma - \gamma_A^B + \tilde{\gamma}\,.$$

Note that

$$\pi^1_{\#}\gamma' = \pi^1_{\#}\gamma \quad \text{and} \quad \pi^2_{\#}\gamma' = \pi^2_{\#}\gamma\,,$$

which yields

$$\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu,\nu) \quad \Longrightarrow \quad \gamma' \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu,\nu)\,.$$

Furthermore, if $\gamma_{\partial\Omega}^{\partial\Omega} = 0$, then $(\gamma')_{\partial\Omega}^{\partial\Omega} = \tilde{\gamma}_{\partial\Omega}^{\partial\Omega}$. Thus,

$$\Big[\gamma \in \mathrm{Adm}_{\mathcal{T}}(\mu,\nu) \text{ and } (A \times B) \cap (\partial\Omega \times \partial\Omega) = \emptyset\Big] \quad \Longrightarrow \quad \gamma' \in \mathrm{Adm}_{\mathcal{T}}(\mu,\nu)\,.$$

Hence, if $\gamma \in \mathrm{Opt}_{\widetilde{Wb}_2}(\mu,\nu)$, or $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu,\nu)$ and $(A \times B) \cap (\partial\Omega \times \partial\Omega) = \emptyset$, then, by optimality, $\mathcal{C}(\gamma) \le \mathcal{C}(\gamma')$, and we infer that $\mathcal{C}(\gamma_A^B) \le \mathcal{C}(\tilde{\gamma})$. We conclude by arbitrariness of $\tilde{\gamma}$. ☐

In [FG10, Proposition 2.3] and [Mor18, Proposition 3.2], the authors give more precise characterizations of the optimal plans for their respective transportation functionals in terms of suitable c-cyclical monotonicity of the support, as in the classical optimal transport theory; see, e.g., [ABS21, Lecture 3]. Existence of transport plans is then derived as a consequence. We believe that a similar analysis can be carried out for the transport plans in $\mathrm{Opt}_{\mathcal{T}}$ and $\mathrm{Opt}_{\widetilde{Wb}_2}$, but it is not necessary for the purpose of this work.

2.5 Proof of Theorem 2.1.1

Recall the scheme (2.1.8): we first fix a measure $\mu_0 \in \mathscr{S}$ such that its restriction to Ω is absolutely continuous (w.r.t. the Lebesgue measure) with density equal to ρ_0. Then, for every $\tau > 0$ and $n \in \mathbb{N}_0$, we iteratively choose

$$\mu^\tau_{(n+1)\tau} \in \operatorname*{arg\,min}_{\mu \in \mathscr{S}} \left(\mathcal{H}(\mu) + \frac{\mathcal{T}^2(\mu, \mu^\tau_{n\tau})}{2\tau} \right) .$$

For all $\tau > 0$, these sequences are extended to maps $t \mapsto \mu^\tau_t$, constant on the intervals $\big[n\tau, (n+1)\tau\big)$ for every $n \in \mathbb{N}_0$.

Remark 2.5.1. The choice of $(\mu_0)_{\partial\Omega}$ is inconsequential, in the sense that, for every t and τ the restriction $(\mu^\tau_t)_\Omega$ does not depend on it. In fact, from Remark 2.3.11 and the uniqueness of the minimizer in (2.1.8) (i.e., Proposition 2.5.11), it is possible to infer the following proposition (proof omitted).

Proposition 2.5.2. *Fix $\tau > 0$, and let $\mu_0, \tilde{\mu}_0 \in \mathscr{S}$ be such that $(\mu_0)_\Omega = (\tilde{\mu}_0)_\Omega$. Let $t \mapsto \mu^\tau_t$ and $t \mapsto \tilde{\mu}^\tau_t$ be the maps constructed with the scheme* (2.1.8)*, starting from μ_0 and $\tilde{\mu}_0$, respectively. Then, for every $t \geq 0$,*

$$\mu^\tau_t - \tilde{\mu}^\tau_t = \mu_0 - \tilde{\mu}_0 = (\mu_0)_{\partial\Omega} - (\tilde{\mu}_0)_{\partial\Omega} \,. \tag{2.5.1}$$

We are going to prove Theorem 2.1.1 in seven steps, corresponding to as many (sub)sections:

1. Existence: The scheme is well-posed, in the sense that there exists a minimizer for the variational problem (2.1.8).

2. Boundary condition: The minimizers of (2.1.8) approximately satisfy the boundary condition $\rho|_{\partial\Omega} = e^{\Psi - V}$.

3. Sobolev regularity: There are minimizers such that their restriction to Ω enjoy some Sobolev regularity, with quantitative estimates, and satisfy a "precursor" of the Fokker–Planck equation.

4. Uniqueness: There is only one minimizer for (2.1.8) (given $\mu^\tau_{n\tau}$).

5. Contractivity: Suitably truncated L^q norms decrease in time along $t \mapsto \mu^\tau_t$. This result is useful in proving convergence of the scheme, both w.r.t. Wb_2 and in $L^1_{\text{loc}}\big((0,\infty); L^q(\Omega)\big)$.

6. Convergence w.r.t. Wb_2.

7. Fokker–Planck with Dirichlet boundary conditions: The limit solves the Fokker–Planck equation with the desired Dirichlet boundary conditions. Moreover, the convergence holds in $L^1_{\text{loc}}\big((0,\infty); L^q(\Omega)\big)$ for $q \in [1, \frac{d}{d-1})$.

Each (sub)section starts with the precise statement of the corresponding main proposition and ends with its proof. When needed, some preparatory lemmas precede the proof.

2.5.1 One step of the scheme

In this section, we gather together the subsections corresponding to the first five steps of our plan for Theorem 2.1.1. The reason is that they all involve only one step of the discrete scheme.

Throughout this section, $\bar{\mu}$ is any measure in $\mathscr{S}$ whose restriction to Ω is absolutely continuous and such that, denoting by $\bar{\rho}$ the density of $\bar{\mu}_\Omega$, the quantity $\mathcal{E}(\bar{\rho})$ is finite. We also fix $\tau > 0$. We aim to find one/all minimizer(s) of

$$\mathcal{H}(\cdot) + \frac{\mathcal{T}^2(\cdot, \bar{\mu})}{2\tau} \, : \, \mathscr{S} \to \mathbb{R} \tag{2.5.2}$$

and determine some of its/their properties.

Existence

Proposition 2.5.3. *There exists at least one minimizer of the function in* (2.5.2)*. Every minimizer μ satisfies the following:*

1. *Both $\mathcal{H}(\mu)$ and $\mathcal{T}(\mu, \bar{\mu})$ are finite. In particular, μ_Ω admits a density ρ.*
2. *The total variation of μ and the integral $\int_\Omega \rho \log \rho \, \mathrm{d}x$ can be bounded by a constant $\mathfrak{c}_{\tau,\bar{\mu}}$ that depends on V only through $\|V\|_{L^\infty}$.*
3. *The following inequality holds:*

$$\frac{\mathcal{T}^2(\mu, \bar{\mu})}{4\tau} \le \mathcal{E}(\bar{\rho}) - \mathcal{E}(\rho) + \mu_\Omega(\Psi) - \bar{\mu}_\Omega(\Psi) + \mathfrak{c}\tau\Big(\|\mu_\Omega\| + \|\bar{\mu}_\Omega\|\Big) \, . \tag{2.5.3}$$

The proof of this proposition, partially inspired by [Mor18, Propositions 4.3 & 5.9], is essentially an application of the *direct method in the calculus of variations*, although some care is needed due to the unboundedness of $\mathcal{H}$ from below.

Proof of Proposition 2.5.3. Let $(\mu^n)_{n \in \mathbb{N}_1} \subseteq \mathscr{S}$ be a minimizing sequence for (2.5.2). We may assume that

$$\mathcal{H}(\mu^n) + \frac{\mathcal{T}^2(\mu^n, \bar{\mu})}{2\tau} \le \mathcal{H}(\bar{\mu}) + \frac{\mathcal{T}^2(\bar{\mu}, \bar{\mu})}{2\tau} + \frac{1}{n} = \mathcal{H}(\bar{\mu}) + \frac{1}{n} < \infty \, , \qquad n \in \mathbb{N}_1 \, , \tag{2.5.4}$$

where the finiteness of $\mathcal{H}(\bar{\mu})$ is consequence of $\mathcal{E}(\bar{\rho}) < \infty$. For every n, let ρ^n be the density of μ^n_Ω and let $\gamma^n \in \mathrm{Opt}_{\mathcal{T}}(\mu^n, \bar{\mu})$.

Step 1 (preliminary bounds). Firstly, we shall do some work towards the proof of (2.5.3) and establish uniform integrability for $\{\rho^n\}_n$. By (2.5.4) and Lemma 2.4.8,

$$\begin{aligned}\frac{\mathcal{T}^2(\mu^n, \bar{\mu})}{2\tau} &\le \mathcal{H}(\bar{\mu}) - \mathcal{H}(\mu^n) + \frac{1}{n} = \mathcal{E}(\bar{\rho}) - \mathcal{E}(\rho^n) + \bar{\mu}_{\partial\Omega}(\Psi) - \mu^n_{\partial\Omega}(\Psi) + \frac{1}{n} \\ &\le \mathcal{E}(\bar{\rho}) - \mathcal{E}(\rho^n) + \mu^n_\Omega(\Psi) - \bar{\mu}_\Omega(\Psi) + \tau(\mathrm{Lip}\, \Psi)^2\Big(\|\mu^n_\Omega\| + \|\bar{\mu}_\Omega\|\Big) + \frac{\mathcal{T}^2(\mu^n, \bar{\mu})}{4\tau} + \frac{1}{n} \, ,\end{aligned} \tag{2.5.5}$$

from which,

$$\int_\Omega \rho^n \log \rho^n \le \int_\Omega \left(\bar\rho \log \bar\rho + (\|V\|_{L^\infty} + \|\Psi\|_{L^\infty} + 1 + \tau(\operatorname{Lip}\Psi)^2)(\bar\rho + \rho^n)\right) \mathrm{d}x + \frac{1}{n} . \tag{2.5.6}$$

Since $l \mapsto l \log l$ is superlinear, we have uniform integrability of $\{\rho^n\}_n$. In particular, $\|\mu^n_\Omega\|$ is bounded.

Also the total variation $\|\mu^n\|$ is bounded. Indeed,

$$\|\mu^n\| \le 2\|\gamma^n\| + \|\bar\mu\| \le 2\|\mu^n_\Omega\| + 3\|\bar\mu\| , \tag{2.5.7}$$

where the first inequality follows from Condition (3) in Definition 2.3.7, and the second one from Remark 2.3.10.

Step 2 (existence). We can extract a (not relabeled) subsequence such that:

1. $\mu^n_{\partial\Omega} \to_n \eta$ for some η weakly in duality with $C(\partial\Omega)$,
2. $\rho^n \rightharpoonup_n \rho$ for some ρ weakly in $L^1(\Omega)$,
3. $\mu^n \to_n \mu := \rho\,\mathrm{d}x + \eta$ weakly in duality with $C(\overline{\Omega})$, and $\mu \in \mathscr{S}$.

Since the functional $\mathcal{E}$ is sequentially lower semicontinuous w.r.t. the weak convergence in $L^1(\Omega)$, and sum of lower semicontinuous functions is lower semicontinuous, Corollary 2.4.6 yields

$$\mathcal{H}(\mu) + \frac{\mathcal{T}^2(\mu,\bar\mu)}{2\tau} \le \liminf_{n\to\infty} \left(\mathcal{H}(\mu^n) + \frac{\mathcal{T}^2(\mu^n,\bar\mu)}{2\tau}\right) = \inf\left(\mathcal{H}(\cdot) + \frac{\mathcal{T}^2(\cdot,\bar\mu)}{2\tau}\right) .$$

Step 3 (inequalities). If μ is *any* minimizer for (2.5.2), the inequality (2.5.3), and the bounds on $\|\mu\|$ and $\int_\Omega \rho \log\rho \,\mathrm{d}x$ directly follow from (2.5.5), (2.5.6), and (2.5.7) by taking the constant sequence equal to μ in place of $(\mu^n)_n$. □

Boundary condition

Pick any minimizer μ for (2.5.2) and denote by ρ the density of μ_Ω. Let $\gamma \in \operatorname{Opt}_{\mathcal{T}}(\mu,\bar\mu)$ and let $S\colon \Omega \to \overline{\Omega}$ be such that $\gamma^{\overline{\Omega}}_\Omega = (\mathrm{Id}, S)_\# \mu_\Omega$.

Proposition 2.5.4. *There exists a $\mathscr{L}^d$-negligible set $N \subseteq \Omega$ such that:*

1. *For all $x \in \Omega \setminus N$ and $y \in \partial\Omega$, the inequalities*

$$-\frac{|x-y|^2}{2\tau} \le \log\rho(x) - \Psi(y) + V(x) \le \mathfrak{c}\frac{|x-y|}{\tau} + \mathfrak{c}\tau \tag{2.5.8}$$

   hold. The constant $\mathfrak{c}$ can be chosen independent of V.

2. *For all $x \in \Omega \setminus N$ such that $S(x) \in \partial\Omega$, we have the identity*

$$\log\rho(x) = \Psi(S(x)) - V(x) - \frac{|x - S(x)|^2}{2\tau} . \tag{2.5.9}$$

Remark 2.5.5. Proposition 2.5.4 implies in particular that $\rho \in L^\infty(\Omega)$ and that ρ is bounded from below by a positive constant (depending on τ). In particular, the measure μ_Ω is equivalent to the Lebesgue measure on Ω.

Remark 2.5.6. Define

$$g := \sqrt{\rho e^V} - e^{\Psi/2}\,, \qquad g^{(\kappa)} := (g-\kappa)_+ - (g+\kappa)_-\,, \qquad \kappa > 0\,.$$

It follows from (2.5.8) that, when $\kappa \geq c(e^{c\tau}-1)$, for a suitable constant c independent of V and τ, the function $g^{(\kappa)}$ is compactly supported in Ω (up to changing its value on a Lebesgue-negligible set).

Remark 2.5.7. The term $\mathfrak{c}\tau$ at the right-hand side of (2.5.8) can be removed when Ψ is constant. This fact can be easily checked in the proof of Proposition 2.5.4 and is consistent with [AGS08, Proposition 3.7 (27)]. However, the following example proves that, in general, this extra term is necessary, i.e., the boundary condition need not be satisfied *exactly* by the map $t \mapsto \mu_t^\tau$ (even for $t \geq \tau$).

Example 2.5.8. Let $\Omega := (0,1)$ and $V \equiv 0$, and choose $\bar\mu = 0$. Since $\bar\mu = 0$, we necessarily have $S(x) \in \partial\Omega = \{0,1\}$ for μ_Ω-a.e. x, hence for $\mathscr{L}^1$-a.e. $x \in \Omega$ by Remark 2.5.5. Additionally, by Proposition 2.5.4, for $\mathscr{L}^1$-a.e. $x \in S^{-1}(0)$ we have

$$\Psi(1) - \frac{|1-x|^2}{2\tau} \overset{(2.5.8)}{\leq} \log\rho(x) \overset{(2.5.9)}{=} \Psi(0) - \frac{|x|^2}{2\tau}$$

and, after rearranging,

$$x \leq \frac{1}{2} + \tau\Big(\Psi(0) - \Psi(1)\Big)\,.$$

Therefore, when Ψ and τ are such that $\tau\Big(\Psi(0)-\Psi(1)\Big) < -\frac{1}{2}$, the set $S^{-1}(0)$ is negligible, i.e., $S(x) = 1$ for $\mathscr{L}^1$-a.e. $x \in \Omega$. Then, (2.5.9) gives

$$\log\rho(x) = \Psi(1) - \frac{|1-x|^2}{2\tau} \qquad \text{for } \mathscr{L}^1\text{-a.e. } x \in \Omega\,,$$

and, therefore, the trace of ρ at 0 is $\exp\Big(\Psi(1) - \frac{1}{2\tau}\Big) > \exp\Big(\Psi(0)\Big)$.

Proposition 2.5.4 is analogous to [FG10, Proposition 3.7 (27) & (28)] and [Mor18, Proposition 5.2 (5.39) & (5.40)]. Like those, ours is proven by taking suitable variations of the minimizer μ.

Proof of Proposition 2.5.4. We shall prove the inequalities in the statement for x out of negligible sets N_y *that depend on* y. This is sufficient because the set $\partial\Omega$ is separable and all the functions in the statement are continuous in the variable y. Fix $y \in \partial\Omega$.

Step 1 (first inequality in (2.5.8)). Let $\epsilon > 0$, take a Borel set $A \subseteq \Omega$, and define

$$\tilde\mu_1 := \mu + \epsilon\mathscr{L}^d_A - \epsilon|A|\,\delta_y \in \mathscr{S}\,, \qquad \tilde\gamma_1 := \gamma + \epsilon\mathscr{L}^d_A \otimes \delta_y \in \mathrm{Adm}_\mathcal{T}(\tilde\mu_1, \bar\mu)\,.$$

By the minimality property of μ and the optimality of γ,

$$0 \leq \int_A \left(\frac{(\rho+\epsilon)\log(\rho+\epsilon) - \rho\log\rho}{\epsilon} + V - 1 - \Psi(y) + \frac{|x-y|^2}{2\tau}\right) \mathrm{d}x\,.$$

Since the function $l \mapsto l \log l$ is convex, we can use the monotone convergence theorem ("downwards") to find

$$0 \le \int_A \left(\log\rho + V - \Psi(y) + \frac{|x-y|^2}{2\tau} \right) \mathrm{d}x \,.$$

By arbitrariness of A, we have the first inequality in (2.5.8) for x out of a $\mathscr{L}^d$-negligible set (possibly dependent on y). In particular, $\rho > 0$.

Step 2 (second indequality in (2.5.8) on $S^{-1}(\Omega)$). Let $\epsilon \in (0,1)$, take a Borel set $A \subseteq S^{-1}(\Omega)$, define

$$\begin{aligned} \tilde\mu_2 &\coloneqq \mu + \epsilon\mu(A)\delta_y - \epsilon\mu_A \in \mathscr{S} \,, \\ \tilde\gamma_2 &\coloneqq \gamma - \epsilon(\mathrm{Id}, S)_\#\mu_A + \epsilon\delta_y \otimes S_\#\mu_A \in \mathrm{Adm}_\mathcal{T}(\tilde\mu_2, \bar\mu) \,. \end{aligned}$$

Note that $A \subseteq S^{-1}(\Omega)$ is needed to ensure that $(\tilde\gamma_2)^{\partial\Omega}_{\partial\Omega} = 0$. This time, the minimality property gives

$$0 \le \int \left(\frac{(1-\epsilon)\log(1-\epsilon)}{\epsilon} - \log\rho - V + 1 + \Psi(y) + \frac{\langle y - \mathrm{Id}, y + \mathrm{Id} - 2S\rangle}{2\tau} \right) \mathrm{d}\mu_A \,.$$

We conclude by arbitrariness of A, after letting $\epsilon \to 0$, that

$$\log\rho(x) + V(x) - \Psi(y) \le \frac{\langle y - x, y + x - 2S(x)\rangle}{2\tau} \le \mathrm{diam}(\Omega)\frac{|x-y|}{\tau}$$

for μ-a.e. $x \in S^{-1}(\Omega)$. Since $\rho > 0$, the same is true $\mathscr{L}^d_{S^{-1}(\Omega)}$-a.e.

Step 3 (identity (2.5.9)). Let $\epsilon \in (0,1)$, take a Borel set $A \subseteq S^{-1}(\partial\Omega)$, define

$$\begin{aligned} \tilde\mu_3 &\coloneqq \mu + \epsilon S_\#\mu_A - \epsilon\mu_A \in \mathscr{S} \,, \\ \tilde\gamma_3 &\coloneqq \gamma - \epsilon(\mathrm{Id}, S)_\#\mu_A \in \mathrm{Adm}_\mathcal{T}(\tilde\mu_3, \bar\mu) \,. \end{aligned}$$

By the minimality property,

$$0 \le \int \left(\frac{(1-\epsilon)\log(1-\epsilon)}{\epsilon} - \log\rho - V + 1 + \Psi \circ S - \frac{|\mathrm{Id} - S|^2}{2\tau} \right) \mathrm{d}\mu_A \,,$$

from which, by arbitrariness of ϵ and A, we infer the inequality $\le$ in (2.5.9) $\mathscr{L}^d_{S^{-1}(\partial\Omega)}$-a.e. The inequality $\ge$ follows from the first inequality in (2.5.8).

Step 4 (second inequality in (2.5.8) on $S^{-1}(\partial\Omega)$). We make use of (2.5.9), the Lipschitz continuity of Ψ, the triangle inequality, and the inequality $2ab - b^2 \le a^2$:

$$\begin{aligned} \log\rho(x) - \Psi(y) + V(x) &\overset{(2.5.9)}{=} \Psi(S(x)) - \Psi(y) - \frac{|x - S(x)|^2}{2\tau} \\ &\le (\mathrm{Lip}\,\Psi)|S(x) - y| - \frac{|x - S(x)|^2}{2\tau} \\ &\le (\mathrm{Lip}\,\Psi)|x - S(x)| - \frac{|x - S(x)|^2}{2\tau} + (\mathrm{Lip}\,\Psi)|x - y| \\ &\le \frac{\tau(\mathrm{Lip}\,\Psi)^2}{2} + (\mathrm{Lip}\,\Psi)|x - y| \,. \end{aligned}$$

Eventually, we conclude with the estimate

$$|x - y| \le \frac{|x-y|}{2\tau} + \frac{\tau|x-y|}{2} \le \frac{|x-y|}{2\tau} + \frac{\tau\,\mathrm{diam}(\Omega)}{2} \,. \qquad \square$$

Sobolev regularity

Proposition 2.5.9. *Let μ be a minimizer of (2.5.2) and denote by ρ the density of μ_Ω.*

1. *The function ρ belongs to $W^{1,(2\wedge d)}_{\mathrm{loc}}(\Omega)$, and $\sqrt{\rho e^V}$ belongs to $W^{1,2}(\Omega)$. We have the estimates*
$$\left\| \nabla \sqrt{\rho e^V} \right\|_{L^2} \le \mathfrak{c} \frac{\mathcal{T}(\mu, \bar{\mu})}{\tau}, \tag{2.5.10}$$
and, for every $q \in [1, \infty)$ such that $q(d-2) \le d$,
$$\|\rho\|_{L^q} \le \mathfrak{c}_q \left(e^{\mathfrak{c}\tau} + \left\| \nabla \sqrt{\rho e^V} \right\|_{L^2}^2 + \|\rho\|_{L^1} \right). \tag{2.5.11}$$
If $d = 1$, the same is true with $q = \infty$ too.

2. *For every $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu, \bar{\mu})$, writing $\gamma_\Omega^{\overline{\Omega}} = (\mathrm{Id}, S)_{\#}\mu_\Omega$, we have*
$$\frac{S - \mathrm{Id}}{\tau} \rho = \nabla \rho + \rho \nabla V = e^{-V} \nabla (\rho e^V) \qquad \mathscr{L}^d\text{-a.e. on } \Omega. \tag{2.5.12}$$

The core idea to prove Proposition 2.5.9 is to compute the first variation of the functional (2.5.2) at a minimizer and exploit Lemma 2.4.18, like in [FG10, Proposition 3.6]. However, the proof is complicated by the weak assumptions on V and the lack of regularity of the boundary $\partial\Omega$. To manage V, we rely on an approximation argument (in the next lemma). The issue with $\partial\Omega$ is that the the Sobolev embedding theorem is not available for functions in $W^{1,2}(\Omega)$. Nonetheless, we can still apply it to functions in $W^{1,2}_0(\Omega)$. To do this, we leverage the approximate boundary conditions of Proposition 2.5.4.

Lemma 2.5.10. *Let μ be a minimizer of (2.5.2) and denote by ρ the density of μ_Ω. Let $\boldsymbol{w} \colon \Omega \to \mathbb{R}^d$ be a C^∞-regular vector field with compact support. For $\epsilon > 0$ sufficiently small, define $\mu^\epsilon := (\mathrm{Id} + \epsilon \boldsymbol{w})_{\#}\mu$. Then*
$$\lim_{\epsilon \to 0^+} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu^\epsilon)}{\epsilon} = \int_\Omega \Big(\operatorname{div} \boldsymbol{w} - \langle \nabla V, \boldsymbol{w} \rangle \Big) \rho \, \mathrm{d}x. \tag{2.5.13}$$

Proof. Let $R_\epsilon(x) := x + \epsilon \boldsymbol{w}(x)$. Fix ϵ sufficiently small and an open set $\omega \Subset \Omega$ so that $R_{s\epsilon}$ is a diffeomorphism from ω to itself and equals the identity on $\Omega \setminus \omega$ for every $s \in (0,1)$, and $\inf_{s \in (0,1), x \in \Omega} |\det \nabla R_{s\epsilon}(x)| > 0$. It can be easily checked that the density ρ^ϵ of μ^ϵ_Ω satisfies
$$\rho^\epsilon \circ R_\epsilon = \frac{\rho}{\det \nabla R_\epsilon} \qquad \mathscr{L}^d\text{-a.e. on } \Omega;$$
therefore,
$$\begin{aligned} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu^\epsilon)}{\epsilon} &= \int_\Omega \frac{\log \rho - \log(\rho^\epsilon \circ R_\epsilon) + V - V \circ R_\epsilon}{\epsilon} \, \mathrm{d}\mu_\Omega \\ &= \int_\Omega \frac{\log \det \nabla R_\epsilon}{\epsilon} \, \mathrm{d}\mu_\Omega + \int_\Omega \frac{V - V \circ R_\epsilon}{\epsilon} \, \mathrm{d}\mu_\Omega. \end{aligned} \tag{2.5.14}$$

By the dominated convergence theorem,
$$\lim_{\epsilon \to 0^+} \int_\Omega \frac{\log \det \nabla R_\epsilon}{\epsilon} \, \mathrm{d}\mu_\Omega = \int_\Omega (\operatorname{div} \boldsymbol{w}) \rho \, \mathrm{d}x.$$

To deal with the last term in (2.5.14), we choose an open set $\tilde{\omega}$ such that $\omega \Subset \tilde{\omega} \Subset \Omega$. By Definition 2.3.1, we have $V \in W^{1,p}(\tilde{\omega})$ for some $p > d$ and, by Friedrichs' theorem [Bre11, Theorem 9.2], the function $V|_\omega$ is the limit in $W^{1,p}(\omega)$ and a.e. of (the restriction to ω of) a sequence of equibounded functions $(V_k)_{k\in\mathbb{N}_0} \subseteq C_c^\infty(\mathbb{R}^d)$. For every k, we have

$$\int \frac{V - V\circ R_\epsilon}{\epsilon}\,\mathrm{d}\mu_\Omega = \int_\omega \frac{V - V_k}{\epsilon}\rho\,\mathrm{d}x + \int_\omega \frac{V_k\circ R_\epsilon - V\circ R_\epsilon}{\epsilon}\rho\,\mathrm{d}x - \int_\omega \langle \nabla V_k, \boldsymbol{w}\rangle \rho\,\mathrm{d}x \\ - \int_0^1 \int_\omega \langle (\nabla V_k)\circ R_{s\epsilon} - \nabla V_k, \boldsymbol{w}\rangle \rho\,\mathrm{d}x\,\mathrm{d}s\,.$$

With a change of variables, we rewrite the last integral as

$$\int_0^1 \int_\omega \langle (\nabla V_k)\circ R_{s\epsilon} - \nabla V_k, \boldsymbol{w}\rangle \rho\,\mathrm{d}x\,\mathrm{d}s = \int_\omega \left\langle \nabla V_k, \int_0^1 \frac{(\boldsymbol{w}\rho)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}}\,\mathrm{d}s - \boldsymbol{w}\rho \right\rangle \mathrm{d}x\,.$$

Recall that $\rho \in L^\infty(\Omega)$ by Remark 2.5.5. Passing to the limit in k, we find that

$$\int \frac{V - V\circ R_\epsilon}{\epsilon}\,\mathrm{d}\mu_\Omega + \int_\Omega \langle \nabla V, \boldsymbol{w}\rangle \rho\,\mathrm{d}x = \int_\omega \left\langle \nabla V, \int_0^1 \frac{(\boldsymbol{w}\rho)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}}\,\mathrm{d}s - \boldsymbol{w}\rho \right\rangle \mathrm{d}x\,.$$

It only remains to prove that the right-hand side in the latter is negligible as $\epsilon \to 0$. Let $(\rho_l)_{l\in\mathbb{N}_0}$ be a sequence of continuous and equibounded functions that converge to ρ almost everywhere (hence in $L^{p'}$). Using the triangle inequality and Minkowski's integral inequality, for $l \in \mathbb{N}_0$, we write

$$\left\| \int_0^1 \frac{(\boldsymbol{w}\rho)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}}\,\mathrm{d}s - \boldsymbol{w}\rho \right\|_{L^{p'}} \le \int_0^1 \left\| \frac{(\boldsymbol{w}\rho - \boldsymbol{w}\rho_l)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}} \right\|_{L^{p'}} \mathrm{d}s + \|\boldsymbol{w}\rho_l - \boldsymbol{w}\rho\|_{L^{p'}} \\ + \int_0^1 \left\| \frac{(\boldsymbol{w}\rho_l)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}} - \boldsymbol{w}\rho_l \right\|_{L^{p'}} \mathrm{d}s\,.$$

A change of variables yields

$$\left\| \frac{(\boldsymbol{w}\rho - \boldsymbol{w}\rho_l)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}} \right\|_{L^{p'}} = \left\| \frac{\boldsymbol{w}\rho - \boldsymbol{w}\rho_l}{|\det \nabla R_{s\epsilon}|^{1/p}} \right\|_{L^{p'}}\,.$$

Hence, when we let $\epsilon \to 0$, using that ρ_l is continuous, we find

$$\limsup_{\epsilon\to 0} \left\| \int_0^1 \frac{(\boldsymbol{w}\rho)\circ R_{s\epsilon}^{-1}}{\det \nabla R_{s\epsilon}\circ R_{s\epsilon}^{-1}}\,\mathrm{d}s - \boldsymbol{w}\rho \right\|_{L^{p'}} \le 2\|\boldsymbol{w}\rho - \boldsymbol{w}\rho_l\|_{L^{p'}}\,,$$

and we conclude by arbitrariness of l. □

Proof of Proposition 2.5.9. Step 1 (inequality (2.5.10)). Let $\boldsymbol{w}\colon \Omega \to \mathbb{R}^d$ be a C^∞-regular vector field with compact support. For $\epsilon > 0$ sufficiently small, define $\mu^\epsilon := (\mathrm{Id} + \epsilon\boldsymbol{w})_\#\mu \in \mathscr{S}$. Since μ is optimal for (2.5.2),

$$\frac{\mathcal{H}(\mu) - \mathcal{H}(\mu^\epsilon)}{\epsilon} \le \frac{\mathcal{T}^2(\mu^\epsilon, \bar{\mu}) - \mathcal{T}^2(\mu, \bar{\mu})}{2\epsilon\tau}\,.$$

We can pass to the limit $\epsilon \to 0$ using Lemma 2.4.18 and Lemma 2.5.10 to find that

$$\int_\Omega \Big(\operatorname{div}\boldsymbol{w} - \langle \nabla V, \boldsymbol{w}\rangle\Big)\rho\,\mathrm{d}x \le -\frac{1}{\tau}\int \langle \boldsymbol{w}(x), y - x\rangle\,\mathrm{d}\gamma(x,y) \le \|\boldsymbol{w}\|_{L^2(\rho)} \frac{\mathcal{T}(\mu,\bar{\mu})}{\tau}\,, \tag{2.5.15}$$

for any $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu, \bar{\mu})$. By the Riesz representation theorem, this means that there exists a vector field $\boldsymbol{u} \in L^2(\rho; \mathbb{R}^d)$ such that

$$\|\boldsymbol{u}\|_{L^2(\rho)} \le \frac{\mathcal{T}(\mu, \bar{\mu})}{\tau}\,, \tag{2.5.16}$$

and

$$\int_\Omega \big(\operatorname{div} \boldsymbol{w} - \langle \nabla V, \boldsymbol{w} \rangle\big) \rho \,\mathrm{d}x = \int_\Omega \langle \boldsymbol{u}, \boldsymbol{w} \rangle \rho \,\mathrm{d}x\,,$$

for all smooth and compactly supported vector fields $\boldsymbol{w}$. In other words, $-\rho(\boldsymbol{u} + \nabla V)$ is the distributional gradient of ρ. Since $\rho \in L^\infty(\Omega)$ (see Remark 2.5.5) and $V \in W^{1,d+}_{\mathrm{loc}}(\Omega)$, we now know that $\rho \in W^{1,(2\wedge d)}_{\mathrm{loc}}(\Omega)$. Hence, for every smooth $\boldsymbol{w}$ that is compactly supported,

$$\begin{aligned}\int_\Omega \sqrt{\rho e^V} \operatorname{div} \boldsymbol{w} \,\mathrm{d}x &= \lim_{\epsilon \downarrow 0} \int_\Omega \sqrt{\rho e^V + \epsilon} \operatorname{div} \boldsymbol{w} \,\mathrm{d}x = \lim_{\epsilon \downarrow 0} \int_\Omega \frac{\rho e^V}{2\sqrt{\rho e^V + \epsilon}} \langle \boldsymbol{u}, \boldsymbol{w} \rangle \,\mathrm{d}x \\ &\le \frac{\|\boldsymbol{u}\|_{L^2(\rho)}}{2} \liminf_{\epsilon \downarrow 0} \sqrt{\int_\Omega \frac{\rho e^{2V} |\boldsymbol{w}|^2}{\rho e^V + \epsilon} \,\mathrm{d}x} = \frac{\|\boldsymbol{u}\|_{L^2(\rho)} \|\boldsymbol{w}\|_{L^2(e^V)}}{2}\,,\end{aligned}$$

where, for the second equality, we used a standard property of the composition of Sobolev functions (cf. [Bre11, Proposition 9.5]) and, in the last one, the monotone convergence theorem. It follows that that $\sqrt{\rho e^V} \in W^{1,2}(\Omega)$ with

$$\int_\Omega \left|\nabla \sqrt{\rho e^V}\right|^2 e^{-V} \,\mathrm{d}x \le \left(\frac{\|\boldsymbol{u}\|_{L^2(\rho)}}{2}\right)^2 \overset{(2.5.16)}{\le} \frac{\mathcal{T}^2(\mu, \bar{\mu})}{4\tau^2}\,, \tag{2.5.17}$$

which, since V is bounded, yields (2.5.10).

Step 2 (inequality (2.5.11)). Pick q as in the statement, i.e., $1 \le q < \infty$ with $q(d-2) \le d$ or, if $d = 1$, $q \in [1, \infty]$. Inequality (2.5.11) would follow from the Sobolev embedding theorem [Bre11, Corollary 9.14] if $\partial\Omega$ were regular enough. Nonetheless, by [Bre11, Remark 20, Chapter 9], even with no regularity on $\partial\Omega$, we still have that the inclusion $W^{1,2}_0(\Omega) \hookrightarrow L^q(\Omega)$ is continuous. Consider the functions g and $g^{(\kappa)}$ of Remark 2.5.6 and fix $\kappa = c(e^{c\tau} - 1)$ for a suitable constant c independent of τ (and q), so that $g^{(\kappa)}$ is compactly supported, hence in $W^{1,2}_0(\Omega)$. From the Sobolev embedding theorem we obtain $\left\|g^{(\kappa)}\right\|_{L^{2q}} \le \mathfrak{c}_q \left\|g^{(\kappa)}\right\|_{W^{1,2}}$ and, therefore,

$$\begin{aligned}\left\|\sqrt{\rho e^V}\right\|_{L^{2q}} &\le \mathfrak{c}_q + \|g\|_{L^{2q}} \le \mathfrak{c}_q(1+\kappa) + \left\|g^{(\kappa)}\right\|_{L^{2q}} \le \mathfrak{c}_q \left(1 + \kappa + \left\|g^{(\kappa)}\right\|_{W^{1,2}}\right) \\ &\le \mathfrak{c}_q \left(1 + \kappa + \|g\|_{W^{1,2}}\right) \le \mathfrak{c}_q \left(1 + \kappa + \left\|\sqrt{\rho e^V}\right\|_{W^{1,2}}\right) \\ &\le \mathfrak{c}_q \left(1 + \kappa + \left\|\nabla \sqrt{\rho e^V}\right\|_{L^2} + \sqrt{\|\rho\|_{L^1}}\right)\,,\end{aligned}$$

which can be easily transformed into (2.5.11).

Step 3 (identity (2.5.12)). Let $\gamma \in \mathrm{Opt}_{\mathcal{T}}(\mu, \bar{\mu})$ and let S be such that $\gamma^{\Omega}_{\Omega} = (\mathrm{Id}, S)_{\#}\mu_\Omega$. From (2.5.15) we infer that

$$-2\int_\Omega \sqrt{\rho e^{-V}} \left\langle \nabla \sqrt{\rho e^V}, \boldsymbol{w} \right\rangle \mathrm{d}x \le -\frac{1}{\tau} \int \langle \boldsymbol{w}(x), y - x \rangle \,\mathrm{d}\gamma(x,y) = -\frac{1}{\tau} \int \langle \boldsymbol{w}, S - \mathrm{Id} \rangle \rho \,\mathrm{d}x\,.$$

By arbitrariness of $\boldsymbol{w}$, (2.5.12) follows. $\square$

Uniqueness

Let us assume that μ and μ' are two minimizers for (2.5.2) such that their restrictions to Ω are absolutely continuous; let ρ and ρ' be their respective densities. Let $\gamma \in \mathrm{Opt}_\mathcal{T}(\mu, \bar\mu)$ and $\gamma' \in \mathrm{Opt}_\mathcal{T}(\mu', \bar\mu)$. By Proposition 2.4.19, we can write

$$\begin{aligned} \gamma_\Omega^{\overline{\Omega}} &= (\mathrm{Id}, S)_\# \mu_\Omega\,, \qquad (\gamma')_\Omega^{\overline{\Omega}} = (\mathrm{Id}, S')_\# \mu_\Omega\,, \\ \gamma_{\overline{\Omega}}^{\Omega} &= (T, \mathrm{Id})_\# \bar\mu_\Omega\,, \qquad (\gamma')_{\overline{\Omega}}^{\Omega} = (T', \mathrm{Id})_\# \bar\mu_\Omega\,, \end{aligned}$$

for some appropriate Borel maps.

Proposition 2.5.11. *The two measures μ and μ' are equal.*

Note that uniqueness is not immediate, given that the functional $\mathcal{H}$ is not strictly convex. This setting is different from that of [Mor18] and [FG10]: therein, measures are defined *only on* Ω. Instead, we claim here that the measure μ, on the *whole* $\overline{\Omega}$, is uniquely determined.

The proof of Proposition 2.5.11 is preceded by three lemmas: the first one concerns the identification of S and S'; the second one, similar to [Mor18, Proposition A.3 (A.5)], shows that $T|_{T^{-1}(\partial\Omega)}$ and $T'|_{(T')^{-1}(\partial\Omega)}$ enjoy one same property, inferred from the minimality of μ and μ'; the third one ensures that this property identifies uniquely T (i.e., $T = T'$) on $T^{-1}(\partial\Omega) \cap (T')^{-1}(\partial\Omega)$.

Lemma 2.5.12. *If $\mu_\Omega = \mu'_\Omega$, then $S(x) = S'(x)$ for $\mathscr{L}^d_\Omega$-a.e x.*

Proof. This statement immediately follows from (2.5.12) in Proposition 2.5.9. □

Lemma 2.5.13. *For $\bar\mu$-a.e. point $x \in \Omega$ such that $T(x) \in \partial\Omega$, we have*

$$T(x) \in \operatorname*{arg\,min}_{y \in \partial\Omega} \left(\Psi(y) + \frac{|x-y|^2}{2\tau} \right) . \tag{2.5.18}$$

An analogous statement holds for T'.

Proof. Set

$$f(x, y) := \Psi(y) + \frac{|x-y|^2}{2\tau}\,, \qquad x \in \Omega\,,\ y \in \partial\Omega\,. \tag{2.5.19}$$

By [AB06, Theorem 18.19] there exists a Borel function $R \colon \Omega \to \partial\Omega$ such that

$$R(x) \in \operatorname*{arg\,min}_{y \in \partial\Omega} f(x, y)$$

for all $x \in \Omega$. Let $A \subseteq T^{-1}(\partial\Omega)$ be a Borel set and consider the measure

$$\tilde\mu := \mu - T_\# \bar\mu_A + R_\# \bar\mu_A\,,$$

which lies in $\mathscr{S}$. Additionally define

$$\tilde\gamma := \gamma - (T, \mathrm{Id})_\# \bar\mu_A + (R, \mathrm{Id})_\# \bar\mu_A$$

and notice that $\tilde\gamma \in \mathrm{Adm}_\mathcal{T}(\tilde\mu, \bar\mu)$. By the minimality property of μ and the optimality of γ, we must have

$$\mathcal{H}(\mu) + \frac{1}{2\tau}\mathcal{C}(\gamma) \le \mathcal{H}(\tilde\mu) + \frac{1}{2\tau}\mathcal{C}(\tilde\gamma)\,,$$

which, after rearranging the terms, gives

$$\int f(x, T(x))\,\mathrm{d}\bar{\mu}_A(x) \le \int f(x, R(x))\,\mathrm{d}\bar{\mu}_A(x) = \int \min_{y\in\partial\Omega} f(x,y)\,\mathrm{d}\bar{\mu}_A(x)\,.$$

We conclude the proof by arbitrariness of A. □

Lemma 2.5.14. *For $\bar{\mu}$-a.e. point $x \in \Omega$ such that $T(x) \in \partial\Omega$ and $T'(x) \in \partial\Omega$, we have*

$$T(x) = T'(x)\,.$$

Proof. We can resort to [Cox20, Lemma 1] by G. Cox. Adopting the notation of this lemma, we set

$$Q(t,z) := \Psi(t) + \frac{|z-t|^2}{2\tau}\,, \qquad P := c\,\bar{\mu}|_{T^{-1}(\partial\Omega)\cap(T')^{-1}(\partial\Omega)}\,,$$

for some constant c that makes P a probability distribution. Four assumptions are made therein and need to be checked:

- Absolute Continuity: It follows from $\mathcal{E}(\bar{\mu}) < \infty$ that $\bar{\mu}_\Omega$ is absolutely continuous. Hence, so is the probability P.
- Continuous Differentiability: Conditions (a) and (b) are easy to check. Condition (c) is vacuously true by setting $A(t) := \emptyset$ for every t.
- Generic: Condition (d) is true and easy to check.
- Manifold: This condition is not true if $\partial\Omega$ does not enjoy any kind of regularity. However, one can check that that $\partial\Omega$ does not need to be a union of manifolds if the condition Generic holds with $A(t) := \emptyset$ for every t. The other topological properties, namely second-countability and Hausdorff, are trivially true, since $\partial\Omega \subseteq \mathbb{R}^d$. □

Proof of Proposition 2.5.11. Step 1 (uniqueness of ρ and S). The identity $\rho = \rho'$ follows from the strict convexity of the function $l \mapsto l \log l$. To see why, notice that $\frac{\gamma+\gamma'}{2} \in \mathrm{Adm}_{\mathcal{T}}(\frac{\mu+\mu'}{2}, \bar{\mu})$; therefore, by minimality,

$$\frac{\mathcal{H}(\mu) + \frac{1}{2\tau}\mathcal{C}(\gamma) + \mathcal{H}(\mu') + \frac{1}{2\tau}\mathcal{C}(\gamma')}{2} \le \mathcal{H}\left(\frac{\mu+\mu'}{2}\right) + \frac{1}{2\tau}\mathcal{C}\left(\frac{\gamma+\gamma'}{2}\right)\,.$$

Most of the terms simplify by linearity. What remains is

$$\int_\Omega \frac{\rho\log\rho + \rho'\log\rho'}{2}\,\mathrm{d}x \le \int_\Omega \left(\frac{\rho+\rho'}{2}\right)\log\left(\frac{\rho+\rho'}{2}\right)\mathrm{d}x\,,$$

which implies $\rho(x) = \rho'(x)$ for $\mathscr{L}^d$-a.e. $x \in \Omega$. The identity $S = S'$ out of a $\mathscr{L}^d_\Omega$-negligible set follows from Lemma 2.5.12.

Step 2 (uniqueness of $\gamma^\Omega_{\partial\Omega}$). We can write

$$\gamma = \gamma^{\overline{\Omega}}_{\Omega} + \gamma^{\Omega}_{\partial\Omega} \quad\text{and}\quad \gamma' = (\gamma')^{\overline{\Omega}}_{\Omega} + (\gamma')^{\Omega}_{\partial\Omega}\,.$$

Because of the uniqueness of μ_Ω and S, we have the equality $\gamma_\Omega^{\overline{\Omega}} = (\gamma')_\Omega^{\overline{\Omega}}$. If we combine this fact with Condition (2) in Definition 2.3.7, we find

$$\begin{aligned} 0 &= \left(\pi^2_\#(\gamma-\gamma')\right)_\Omega = \pi^2_\# \left(\gamma^\Omega_{\partial\Omega} - (\gamma')^\Omega_{\partial\Omega}\right) \\ &= \pi^2_\# \left((T,\mathrm{Id})_\#\bar\mu_{T^{-1}(\partial\Omega)} - (T',\mathrm{Id})_\#\bar\mu_{(T')^{-1}(\partial\Omega)}\right) = \bar\mu_{T^{-1}(\partial\Omega)} - \bar\mu_{(T')^{-1}(\partial\Omega)}\,. \end{aligned}$$

This proves that $T^{-1}(\partial\Omega)$ and $(T')^{-1}(\partial\Omega)$ are $\bar\mu$-essentially equal. Together with Lemma 2.5.14, this gives

$$\gamma^\Omega_{\partial\Omega} = (T,\mathrm{Id})_\#\bar\mu_{T^{-1}(\partial\Omega)} = (T',\mathrm{Id})_\#\bar\mu_{(T')^{-1}(\partial\Omega)} = (\gamma')^\Omega_{\partial\Omega}\,.$$

Step 3 (conclusion). We have determined that $\gamma=\gamma'$. Condition (3) in Definition 2.3.9 gives

$$\mu = \pi^1_\#\gamma - \pi^2_\#\gamma + \bar\mu = \pi^1_\#\gamma' - \pi^2_\#\gamma' + \bar\mu = \mu'\,,$$

which is what we wanted to prove. □

Contractivity

In this section, we establish time monotonicity for some truncated and weighted L^q norm ($q\geq 1$) of the densities ρ^τ_t.

Here, too, only one step of the scheme is involved. We let μ be the unique minimum point of (2.5.2) and ρ be the density of its restriction to Ω.

Proposition 2.5.15. *Let $q\geq 1$. For every $\vartheta \geq \vartheta_0 := \max_{\partial\Omega} e^\Psi$, the following inequality holds (possibly, with one or both sides being infinite):*

$$\int_\Omega \max\left\{\rho, \vartheta e^{-V}\right\}^q e^{(q-1)V}\,\mathrm{d}x \leq \int_\Omega \max\left\{\bar\rho, \vartheta e^{-V}\right\}^q e^{(q-1)V}\,\mathrm{d}x\,. \tag{2.5.20}$$

Remark 2.5.16. For a solution to the Fokker–Planck equation (2.1.4), a monotonicity property like (2.5.20) is expected. Indeed, *formally*:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \max\left\{\rho_t,\vartheta e^{-V}\right\}^q e^{(q-1)V}\,\mathrm{d}x &= q\int_{\left\{\rho_t>\vartheta e^{-V}\right\}} (\rho_t e^V)^{q-1}\operatorname{div}(\nabla\rho_t+\rho_t\nabla V)\,\mathrm{d}x \\ &= q\int_{\partial\left\{\rho_t>\vartheta e^{-V}\right\}} (\rho_t e^V)^{q-1} e^{-V}\langle\nabla(\rho_t e^V),\boldsymbol{n}\rangle\,\mathrm{d}\mathscr{H}^{d-1} \\ &\quad \underbrace{-q(q-1)\int_{\left\{\rho_t>\vartheta e^{-V}\right\}} (\rho_t e^V)^{q-2} e^V |\nabla\rho_t+\rho_t\nabla V|^2\,\mathrm{d}x}_{\leq 0}\,. \end{aligned}$$

If $\vartheta \gneq \vartheta_0$, the boundary condition forces the set $\partial\left\{\rho_t>\vartheta e^{-V}\right\}\cap\partial\Omega$ to be negligible. Moreover, on $\partial\left\{\rho_t>\vartheta e^{-V}\right\}\cap\Omega$, the scalar product $\langle\nabla(\rho_t e^V),\boldsymbol{n}\rangle$ is nonpositive. The case $\vartheta=\vartheta_0$ can be deduced by approximation.

Remark 2.5.17 (Mass bound). Note that Proposition 2.5.15 implies that the mass of $(\mu^\tau_t)_\Omega$ is bounded by a constant $\mathfrak{c}$ indepentent of t and τ. Indeed,

$$\begin{aligned} \int_\Omega \rho^\tau_t\,\mathrm{d}x &\leq \int_\Omega \max\left\{\rho^\tau_t,\vartheta_0 e^{-V}\right\}\mathrm{d}x \leq \cdots \leq \int_\Omega \max\left\{\rho_0,\vartheta_0 e^{-V}\right\}\mathrm{d}x \\ &\leq \int_\Omega \rho_0\,\mathrm{d}x + \vartheta_0\int_\Omega e^{-V}\,\mathrm{d}x\,. \end{aligned}$$

The proof of the first Step in Proposition 2.5.15, i.e., the case $q = 1$, and of the preliminary lemma Lemma 2.5.18 follow the lines of [FG10, Proposition 3.7 (24)] and [Mor18, Proposition 5.3]. In all these proofs, the key is to leverage the optimality of μ by constructing small variations. In the proof of Step 2, i.e., the case $q > 1$, instead, our idea is to take the inequality for $q = 1$, multiply it by a suitable power of ϑ, and integrate it w.r.t. the variable ϑ itself. This is the reason why, while Proposition 2.5.15 will later be used only with $\vartheta = \vartheta_0$—or in the form of Remark 2.5.17—it is convenient to have it stated and proven (at least for $q = 1$) for a continuum of values of ϑ.

Lemma 2.5.18. *For μ-a.e. $x \in \Omega$ such that $S(x) \in \Omega$, we have*

$$\log \rho(x) + V(x) \le \log \rho(S(x)) + V(S(x)) - \frac{|x - S(x)|^2}{2\tau}\,. \tag{2.5.21}$$

Proof. Let $\epsilon \in (0,1)$ and let $A \subseteq S^{-1}(\Omega)$ be a Borel set. We define

$$\begin{aligned}
\tilde\mu &:= \mu + \epsilon S_\# \mu_A - \epsilon \mu_A \in \mathscr{S}\,,\\
\tilde\gamma &:= \gamma - \epsilon(\mathrm{Id}, S)_\# \mu_A + \epsilon (S,S)_\# \mu_A \in \mathrm{Adm}_{\mathcal{T}}(\tilde\mu, \bar\mu)\,.
\end{aligned}$$

Let $\hat\rho$ be the density of $S_\#\mu_A$ and note that $\hat\rho \le \bar\rho$. By the minimality of μ, we have

$$0 \le \underbrace{\int_\Omega \frac{\big(\rho + \epsilon(\hat\rho - \mathbb{1}_A\rho)\big)\log\big(\rho + \epsilon(\hat\rho - \mathbb{1}_A \rho)\big) - \rho\log\rho}{\epsilon}\,\mathrm{d}x}_{:=I_1} + \int \left(V\circ S - V - \frac{|\mathrm{Id} - S|^2}{2\tau}\right)\mathrm{d}\mu_A\,.$$

We use the convexity of $l \mapsto l \log l$ to write

$$\begin{aligned}
I_1 &\le \int_\Omega (\hat\rho - \mathbb{1}_A\rho)\left(1 + \log\big(\rho + \epsilon(\hat\rho - \mathbb{1}_A\rho)\big)\right)\mathrm{d}x\\
&= \int_\Omega (\hat\rho - \mathbb{1}_A\rho)\log\big(\rho + \epsilon(\hat\rho - \mathbb{1}_A\rho)\big)\,\mathrm{d}x\\
&= \int_\Omega \hat\rho\log\big(\rho + \epsilon(\hat\rho - \mathbb{1}_A\rho)\big)\,\mathrm{d}x - \int_A \rho\log\big((1-\epsilon)\rho + \epsilon\hat\rho\big)\,\mathrm{d}x\\
&\le \int_\Omega \hat\rho \log(\rho + \epsilon\hat\rho\big)\,\mathrm{d}x - \int_A \rho\big(\log\rho + \log(1-\epsilon)\big)\,\mathrm{d}x\,.
\end{aligned}$$

On the first integral on the last line, we use the monotone convergence theorem ("downwards"): its hypotheses are satisfied because $\hat\rho \le \bar\rho$. By passing to the limit $\epsilon \to 0$, we obtain

$$\begin{aligned}
0 &\le \int_\Omega \hat\rho\log\rho\,\mathrm{d}x + \int\left(-\log\rho + V\circ S - V - \frac{|\mathrm{Id} - S|^2}{2\tau}\right)\mathrm{d}\mu_A\\
&= \int\left(\log\rho\circ S - \log\rho + V\circ S - V - \frac{|\mathrm{Id} - S|^2}{2\tau}\right)\mathrm{d}\mu_A\,,
\end{aligned}$$

and we conclude by arbitrariness of A. □

Proof of Proposition 2.5.15. Step 1 ($q = 1$). Consider the case $q = 1$. Let

$$A := \left\{x \in \Omega : \ \rho e^V > \vartheta\right\} . \tag{2.5.22}$$

Thanks to (2.5.9), we know that $A \cap S^{-1}(\partial\Omega)$ is $\mathscr{L}^d$-negligible. Therefore, we can extract a $\mathscr{L}^d_A$-full-measure Borel subset $\tilde{A}$ of $A \cap S^{-1}(\Omega)$ where (2.5.21) holds (recall that $\mathscr{L}^d_\Omega \ll \mu_\Omega$). It is easy to check that $S(\tilde{A}) \subseteq A$. Therefore, we have

$$\begin{aligned}\int_A \max\left\{\rho, \vartheta e^{-V}\right\} \mathrm{d}x &\overset{(2.5.22)}{=} \int_A \rho \,\mathrm{d}x = \int_{\tilde{A}} \rho \,\mathrm{d}x \le \int_{S^{-1}(A)} \rho \,\mathrm{d}x = S_\# \mu_\Omega(A) \\ &= \pi^2_\# \gamma^{\overline{\Omega}}_\Omega(A) \overset{(A \subseteq \Omega)}{=} \pi^2_\# \gamma^{\Omega}_\Omega(A) \le \pi^2_\# \gamma^{\Omega}_{\overline{\Omega}}(A) = \bar{\mu}_\Omega(A) \le \int_A \max\left\{\bar{\rho}, \vartheta e^{-V}\right\} \mathrm{d}x \,. \end{aligned} \tag{2.5.23}$$

On the other hand,

$$\int_{\Omega \setminus A} \max\left\{\rho, \vartheta e^{-V}\right\} \mathrm{d}x \overset{(2.5.22)}{=} \int_{\Omega\setminus A} \vartheta e^{-V} \,\mathrm{d}x \le \int_{\Omega\setminus A} \max\left\{\bar{\rho}, \vartheta e^{-V}\right\} \mathrm{d}x \,, \tag{2.5.24}$$

and we conclude by taking the sum of (2.5.23) and (2.5.24).

Step 2 ($q > 1$) Assume now that $q > 1$. Define

$$f := \max\left\{\rho, \vartheta e^{-V}\right\} , \qquad g := \max\left\{\bar{\rho}, \vartheta e^{-V}\right\} .$$

Note that the case $q = 1$ implies

$$\int_\Omega \max\left\{f, \tilde{\vartheta} e^{-V}\right\} \mathrm{d}x \le \int_\Omega \max\left\{g, \tilde{\vartheta} e^{-V}\right\} \mathrm{d}x \tag{2.5.25}$$

for every $\tilde{\vartheta} > 0$. After multiplying (2.5.25) by $\tilde{\vartheta}^{q-2}$, integrating w.r.t. $\tilde{\vartheta}$ from 0 to some $\Theta > 0$, and changing the order of integration with Tonelli's theorem, we find

$$\begin{aligned}&\int_\Omega \left(\int_0^{\min\left\{fe^V, \Theta\right\}} \tilde{\vartheta}^{q-2} \,\mathrm{d}\tilde{\vartheta}\right) f \,\mathrm{d}x + \int_\Omega \left(\int_{\min\left\{fe^V, \Theta\right\}}^{\Theta} \tilde{\vartheta}^{q-1} \,\mathrm{d}\tilde{\vartheta}\right) e^{-V} \,\mathrm{d}x \\ &\qquad \le \int_\Omega \left(\int_0^{\min\left\{ge^V, \Theta\right\}} \tilde{\vartheta}^{q-2} \,\mathrm{d}\tilde{\vartheta}\right) g \,\mathrm{d}x + \int_\Omega \left(\int_{\min\left\{ge^V, \Theta\right\}}^{\Theta} \tilde{\vartheta}^{q-1} \,\mathrm{d}\tilde{\vartheta}\right) e^{-V} \,\mathrm{d}x \,,\end{aligned}$$

whence

$$\begin{aligned}&\frac{1}{q-1}\int_\Omega \min\left\{fe^V, \Theta\right\}^{q-1} f \,\mathrm{d}x - \frac{1}{q}\int_\Omega \min\left\{fe^V, \Theta\right\}^{q} e^{-V} \,\mathrm{d}x \\ &\qquad \le \frac{1}{q-1}\int_\Omega \min\left\{ge^V, \Theta\right\}^{q-1} g \,\mathrm{d}x - \frac{1}{q}\int_\Omega \min\left\{ge^V, \Theta\right\}^{q} e^{-V} \,\mathrm{d}x \,.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\frac{1}{q-1} - \frac{1}{q}\right)\int_\Omega \min\left\{fe^V, \Theta\right\}^{q} e^{-V} \,\mathrm{d}x + \frac{1}{q}\int_\Omega \min\left\{ge^V, \Theta\right\}^{q} e^{-V} \,\mathrm{d}x \\ &\qquad \le \frac{1}{q-1}\int_\Omega \min\left\{ge^V, \Theta\right\}^{q-1} g \,\mathrm{d}x \,.\end{aligned}$$

We now let $\Theta \to \infty$ and deduce from the monotone convergence theorem that

$$\left(\frac{1}{q-1} - \frac{1}{q}\right)\int_\Omega f^q e^{(q-1)V} \,\mathrm{d}x + \frac{1}{q}\int_\Omega g^q e^{(q-1)V} \,\mathrm{d}x \le \frac{1}{q-1}\int_\Omega g^q e^{(q-1)V} \,\mathrm{d}x \,.$$

Eventually, we can rearrange, and, noted that $\left(\frac{1}{q-1} - \frac{1}{q}\right) > 0$, simplify to finally obtain (2.5.20). □

2.5.2 Convergence w.r.t Wb_2

In this section, we prove convergence w.r.t. Wb_2 of the measures built with the scheme (2.1.8). The argument is standard. In fact, we shall give a short proof that relies on the 'refined version of Ascoli-Arzelà theorem' [AGS08, Proposition 3.3.1].

Proposition 2.5.19. *As $\tau \to 0$, up to subsequences, the maps $\big(t \mapsto (\mu_t^\tau)_\Omega\big)_\tau$ converge pointwise w.r.t. Wb_2 to a curve $t \mapsto \rho_t \, \mathrm{d}x$ of absolutely continuous measures, continuous w.r.t. Wb_2.*

Once again, we first need a lemma.

Lemma 2.5.20. *Let $t \geq 0$ and $\tau > 0$. Then*

$$\tau \int_\Omega \rho_t^\tau \log \rho_t^\tau \, \mathrm{d}x + \sum_{i=0}^{\lfloor t/\tau \rfloor - 1} \mathcal{T}^2\Big(\mu_{i\tau}^\tau, \mu_{(i+1)\tau}^\tau\Big) \leq \mathfrak{c}\, \tau(1+t+\tau)\,. \tag{2.5.26}$$

As a consequence,

$$Wb_2\Big((\mu_s^\tau)_\Omega, (\mu_t^\tau)_\Omega\Big) \leq \widetilde{Wb}_2\Big(\mu_s^\tau, \mu_t^\tau\Big) \leq \mathfrak{c}\sqrt{(t-s+\tau)(1+t+\tau)}\,, \qquad s \in [0,t]\,. \tag{2.5.27}$$

Proof. We use (2.5.3) to write

$$\sum_{i=0}^{\lfloor t/\tau \rfloor - 1} \frac{\mathcal{T}^2\Big(\mu_{i\tau}^\tau, \mu_{(i+1)\tau}^\tau\Big)}{4\tau} \leq \mathcal{E}(\rho_0) - \mathcal{E}(\rho_t^\tau) + (\mu_t^\tau)_\Omega(\Psi) - (\mu_0)_\Omega(\Psi) + \mathfrak{c}\tau \sum_{i=0}^{\lfloor t/\tau \rfloor} \|(\mu_{i\tau}^\tau)_\Omega\|\,,$$

and conclude (2.5.26) by using Remark 2.5.17.

The first inequality in (2.5.27) follows from (2.4.1). As for the second one, since $\widetilde{Wb}_2$ is a pseudometric, and by the Cauchy–Schwarz inequality and (2.4.1), we have the chain of inequalities

$$\begin{aligned}\widetilde{Wb}_2(\mu_s^\tau, \mu_t^\tau) &\leq \sum_{i=\lfloor s/\tau \rfloor}^{\lfloor t/\tau \rfloor - 1} \widetilde{Wb}_2(\mu_{i\tau}^\tau, \mu_{(i+1)\tau}^\tau) \leq \sum_{i=\lfloor s/\tau \rfloor}^{\lfloor t/\tau \rfloor - 1} \mathcal{T}\Big(\mu_{i\tau}^\tau, \mu_{(i+1)\tau}^\tau\Big) \\ &\leq \sqrt{\frac{t-s+\tau}{\tau}} \sqrt{\sum_{i=\lfloor s/\tau \rfloor}^{\lfloor t/\tau \rfloor - 1} \mathcal{T}^2\Big(\mu_{i\tau}^\tau, \mu_{(i+1)\tau}^\tau\Big)}\,.\end{aligned}$$

We combine the latter with (2.5.26) to infer (2.5.27). □

Proof of Proposition 2.5.19. Fix $t > 0$. We know from Lemma 2.5.20 that, for every $s \in [0,t]$ and $\tau \in (0,1)$, we have

$$(\mu_s^\tau)_\Omega \in K_t := \left\{ \rho \, \mathrm{d}x \;:\; \int_\Omega \rho \log \rho \, \mathrm{d}x \leq c\,(2+t) \right\},$$

where c is the constant in (2.5.26). We claim that K_t is *compact* in $(\mathcal{M}_2(\Omega), Wb_2)$. By identifying an absolutely continuous measure with its density, K_t can be seen as a subset of $L^1(\Omega)$. This set is closed and convex, as well as weakly sequentially compact by the Dunford–Pettis theorem. From [FG10, Proposition 2.7] we know that weak convergence in $L^1(\Omega)$ implies convergence w.r.t. Wb_2; hence the claim is true.

Furthermore, for every $r, s \in [0,t]$, we have

$$\limsup_{\tau \to 0} Wb_2\big((\mu_r^\tau)_\Omega, (\mu_s^\tau)_\Omega\big) \overset{(2.5.27)}{\leq} \mathfrak{c}\sqrt{|s-r|\,(1+t)}\,.$$

All the hypotheses of [AGS08, Proposition 3.3.1] are satisfied; thus, we conclude the existence of a subsequence of $\big(s \mapsto (\mu_s^\tau)_\Omega\big)_\tau$ that converges, pointwise in $[0,t]$ w.r.t. Wb_2, to a continuous curve of measures. Each limit measure lies in K_t; hence it is absolutely continuous. With a diagonal argument, we find a single subsequence that converges pointwise on the whole half-line $[0,\infty)$. □

2.5.3 Solution to the Fokker–Planck equation with Dirichlet boundary conditions

We are now going to conclude the proof of Theorem 2.1.1 by showing that the limit curve is, in fact, a solution to the linear Fokker–Planck equation with the desired boundary conditions.

Proposition 2.5.21. *If the sequence $\big(t \mapsto (\mu_t^\tau)_\Omega\big)_\tau$ converges, pointwise w.r.t. Wb_2 as $\tau \to 0$, to $t \mapsto \rho_t \,\mathrm{d}x$, then $\rho^\tau \to_\tau \rho$ also in $L^1_{\mathrm{loc}}\big((0,\infty); L^q(\Omega)\big)$ for every $q \in [1, \frac{d}{d-1})$. The curve $t \mapsto \rho_t\,\mathrm{d}x$ solves the linear Fokker–Planck equation in the sense of Section 2.3.4, and the map $t \mapsto \Big(\sqrt{\rho_t e^V} - e^{\Psi/2}\Big)$ belongs to $L^2_{\mathrm{loc}}\big([0,\infty); W_0^{1,2}(\Omega)\big)$.*

Like in the proofs of [FG10, Theorem 3.5] and [Mor18, Theorem 4.1], the key to Proposition 2.5.21 is to first determine (see Lemma 2.5.24) that the measures constructed with (2.1.8) already solve approximately the Fokker–Planck equation. In order to prove that the limit curve has the desired properties and that convergence holds in $L^1_{\mathrm{loc}}\big((0,\infty); L^q(\Omega)\big)$ (Lemma 2.5.26), two further preliminary lemmas turn out to be particularly useful. Both provide quantitative bounds at the discrete level: one (Lemma 2.5.22) for $\sqrt{\rho^\tau e^V}$ in $L^2_{\mathrm{loc}}\big((0,\infty); W^{1,2}(\Omega)\big)$; the other (Lemma 2.5.23) for ρ^τ in $L^\infty_{\mathrm{loc}}\big((0,\infty); L^q(\Omega)\big)$, for suitable values of q. In turn, these bounds are deduced from Proposition 2.5.9 and Proposition 2.5.15.

Lemma 2.5.22 (Sobolev bound)**.** *If $\tau \leq t$, then,*

$$\int_\tau^t \left\| \sqrt{\rho_r^\tau e^V} \right\|_{W^{1,2}}^2 \mathrm{d}r \leq \mathfrak{c}(1+t)\,. \tag{2.5.28}$$

Proof. Let $r \geq \tau$. By (2.5.10), we have

$$\left\| \nabla \sqrt{\rho_r^\tau e^V} \right\|_{L^2}^2 \leq \mathfrak{c}\frac{\mathcal{T}^2\left(\mu^\tau_{\lfloor r/\tau \rfloor \tau}, \mu^\tau_{\lfloor r/\tau \rfloor \tau - \tau}\right)}{\tau^2}\,.$$

Thus,

$$\int_\tau^t \left\| \nabla \sqrt{\rho_r^\tau e^V} \right\|_{L^2}^2 \mathrm{d}r \leq \mathfrak{c} \sum_{i=0}^{\lfloor t/\tau \rfloor - 1} \frac{\mathcal{T}^2\left(\mu^\tau_{(i+1)\tau}, \mu^\tau_{i\tau}\right)}{\tau}\,,$$

which, using Lemma 2.5.20, can be easily reduced to the desired inequality. □

Lemma 2.5.23 (Lebesgue bound)**.** *Let $q \in [1,\infty)$ be such that $q(d-2) \leq d$. If $\tau < t$, then*

$$\|\rho_t^\tau\|_{L^q} \leq \mathfrak{c}_q e^{\mathfrak{c}\tau} \frac{1+t}{t-\tau}\,. \tag{2.5.29}$$

Proof. For every $r \in [0,t]$, Proposition 2.5.15 gives

$$\begin{aligned}\|\rho_t^\tau\|_{L^q} &\leq \mathfrak{c}_q \left(\int_\Omega \max\left\{\rho_t^\tau e^V, \vartheta_0\right\}^q e^{-V}\,\mathrm{d}x\right)^{1/q} \\ &\leq \mathfrak{c}_q \left(\int_\Omega \max\left\{\rho_r^\tau e^V, \vartheta_0\right\}^q e^{-V}\,\mathrm{d}x\right)^{1/q} \leq \mathfrak{c}_q\left(1 + \|\rho_r^\tau\|_{L^q}\right),\end{aligned}$$

and if, additionally, $r \geq \tau$, then (2.5.11) yields

$$\|\rho_t^\tau\|_{L^q} \leq \mathfrak{c}_q \left(e^{\mathfrak{c}\tau} + \left\|\nabla\sqrt{\rho_r^\tau e^V}\right\|_{L^2}^2 + \|\rho_r^\tau\|_{L^1}\right).$$

After integrating w.r.t. r from τ to t, Lemma 2.5.22 and Remark 2.5.17 imply (2.5.29). □

Lemma 2.5.24 (Approximate Fokker–Planck)**.** *Let $\omega \Subset \Omega$ be open, let $\varphi \in C_0^2(\omega)$, and let s,t be such that $0 \leq s \leq t$. Then, $\rho^\tau, \rho^\tau\nabla V \in L^1_{\mathrm{loc}}\big((\tau,\infty); L^1(\omega)\big)$, and*

$$\left|\int_\Omega (\rho_t^\tau - \rho_s^\tau)\varphi\,\mathrm{d}x - \int_{\lfloor\frac{s}{\tau}\rfloor\tau+\tau}^{\lfloor\frac{t}{\tau}\rfloor\tau+\tau}\int_\Omega (\Delta\varphi - \langle\nabla\varphi,\nabla V\rangle)\rho_r^\tau\,\mathrm{d}x\,\mathrm{d}r\right| \leq \mathfrak{c}_\omega\,\tau(1+t+\tau)\|\varphi\|_{C_0^2(\omega)}. \tag{2.5.30}$$

Moreover, for $\epsilon > 0$, the inequality

$$\|\rho_t^\tau - \rho_s^\tau\|_{(C_0^2(\omega))^*} \leq \mathfrak{c}_{\omega,\epsilon}(t - s + \tau) \tag{2.5.31}$$

holds whenever $0 < 2\tau \leq \epsilon \leq s \leq t \leq 1/\epsilon$.

Remark 2.5.25. In (2.5.31), we identify $\rho_t^\tau - \rho_s^\tau$ with the continuous linear functional

$$C_0^2(\omega) \ni \varphi \longrightarrow \int_\omega (\rho_t^\tau - \rho_s^\tau)\varphi\,\mathrm{d}x\,.$$

Proof of Lemma 2.5.24. Step 1 (integrability). From Remark 2.5.17, it follows trivially that $\rho^\tau \in L^1_{\mathrm{loc}}\big([0,\infty); L^1(\Omega)\big)$.

We shall prove that the function $\rho^\tau\nabla V$ belongs to $L^1_{\mathrm{loc}}\big((\tau,\infty); L^1(\omega)\big)$ for every $\omega \Subset \Omega$ open. Fix $a,b > 0$ with $\tau < a \leq b$. Let p be as in Definition 2.3.1. Its conjugate exponent p' satisfies $p' \in [1,\infty)$ and $p'(d-2) \leq d$. By Hölder's inequality and Lemma 2.5.23, we have

$$\begin{aligned}\int_a^b \|\rho_r^\tau\nabla V\|_{L^1}\,\mathrm{d}r \leq \|\nabla V\|_{L^p(\omega)}\int_a^b \|\rho_r^\tau\|_{L^{p'}}\,\mathrm{d}r &\overset{(2.5.29)}{\leq} \mathfrak{c}_p\|\nabla V\|_{L^p(\omega)}\,e^{\mathfrak{c}\tau}\int_a^b \frac{1+r}{r-\tau}\,\mathrm{d}r \\ &\leq \mathfrak{c}_p\|\nabla V\|_{L^p(\omega)}\,e^{\mathfrak{c}\tau}\frac{1+b}{a-\tau}(b-a) \leq \mathfrak{c}_\omega e^{\mathfrak{c}\tau}\frac{1+b}{a-\tau}(b-a)\,.\end{aligned} \tag{2.5.32}$$

The last passage is due to the fact that both p and $\|\nabla V\|_{L^p(\omega)}$ can be seen as functions of V and ω.

Step 2 (inequality (2.5.30)). Let $i \in \mathbb{N}_0$, and choose $\gamma^i \in \mathrm{Opt}_{\mathcal{T}}\big(\mu^\tau_{(i+1)\tau}, \mu^\tau_{i\tau}\big)$ and $S_i \colon \Omega \to \overline{\Omega}$ as in (2.5.12). By the triangle inequality and the fact that $\rho^\tau_r = \rho^\tau_{(i+1)\tau}$ when $r \in \big[(i+1)\tau, (i+2)\tau\big)$, we have

$$\left| \int_\Omega (\rho^\tau_{(i+1)\tau} - \rho^\tau_{i\tau})\varphi \,\mathrm{d}x - \int_{(i+1)\tau}^{(i+2)\tau} \int_\Omega (\Delta\varphi - \langle \nabla\varphi, \nabla V\rangle)\rho^\tau_r \,\mathrm{d}x \,\mathrm{d}r \right|$$
$$\le \underbrace{\left| \int_\Omega \left(\varphi - \varphi \circ S_i - \tau\Delta\varphi + \tau\langle\nabla\varphi, \nabla V\rangle\right) \rho^\tau_{(i+1)\tau} \,\mathrm{d}x \right|}_{=: I^i_1}$$
$$+ \underbrace{\left| \int_\Omega \Big((\varphi \circ S_i)\rho^\tau_{(i+1)\tau} - \varphi\rho^\tau_{i\tau}\Big) \,\mathrm{d}x \right|}_{=: I^i_2} .$$

Using (2.5.12), we rewrite I^i_1 as

$$I^i_1 = \left| \int_\Omega \left(\varphi - \varphi \circ S_i + \langle \nabla\varphi, S_i - \mathrm{Id}\rangle\right) \rho^\tau_{(i+1)\tau} \,\mathrm{d}x \right| ,$$

and then, by means of Taylor's theorem with remainder in Lagrange form, we establish the upper bound

$$I^i_1 \le \mathfrak{c} \|\varphi\|_{C^2_0(\omega)} \int_\Omega |S_i - \mathrm{Id}|^2 \, \rho^\tau_{(i+1)\tau} \,\mathrm{d}x \le \mathfrak{c}\|\varphi\|_{C^2_0(\omega)} \, \mathcal{T}^2\left(\mu^\tau_{(i+1)\tau}, \mu^\tau_{i\tau}\right) .$$

By Condition (2) in Definition 2.3.7 and the fact that φ is supported in the closure of ω, we have

$$I^i_2 = \left| \int_{\overline{\Omega}} \varphi(y) \,\mathrm{d}\pi^2_\#(\gamma^{\overline{\Omega}}_{\Omega} - \gamma^{\Omega}_{\overline{\Omega}}) \right| = \left| \int_{\overline{\Omega}} \varphi(y) \,\mathrm{d}\pi^2_\#(\gamma^{\omega}_{\Omega} - \gamma^{\omega}_{\overline{\Omega}}) \right| \le \|\varphi\|_{L^\infty(\omega)} \|\gamma^\omega_{\partial\Omega}\|$$
$$\le \mathfrak{c}_\omega \|\varphi\|_{L^\infty(\omega)} \int_{\partial\Omega\times\omega} |x-y|^2 \,\mathrm{d}\gamma(x,y) \le \mathfrak{c}_\omega \|\varphi\|_{L^\infty(\omega)} \, \mathcal{T}^2\left(\mu^\tau_{(i+1)\tau}, \mu^\tau_{i\tau}\right) ,$$

where $\mathfrak{c}_\omega$ actually only depends on the (strictly positive) distance of ω from $\partial\Omega$. Taking the sum over i, we obtain

$$\left| \int_\Omega (\rho^\tau_t - \rho^\tau_s)\varphi \,\mathrm{d}x - \int_{\lfloor \frac{s}{\tau} \rfloor\tau+\tau}^{\lfloor \frac{t}{\tau} \rfloor\tau+\tau} \int_\Omega \rho^\tau_r(\Delta\varphi - \langle\nabla\varphi, \nabla V\rangle) \,\mathrm{d}x \,\mathrm{d}r \right| \le \sum_{i=\lfloor s/\tau \rfloor}^{\lfloor t/\tau \rfloor - 1} (I^i_1 + I^i_2)$$
$$\le \mathfrak{c}_\omega \|\varphi\|_{C^2_0(\omega)} \sum_{i=0}^{\lfloor t/\tau \rfloor - 1} \mathcal{T}^2\left(\mu^\tau_{(i+1)\tau}, \mu^\tau_{i\tau}\right) .$$

At this point, (2.5.30) follows from the last estimate and Lemma 2.5.20.

Step 3 (inequality (2.5.31)). Assume that $2\tau \le \epsilon \le s \le t \le 1/\epsilon$. From (2.5.30), we obtain

$$\left| \int_\Omega (\rho^\tau_t - \rho^\tau_s)\varphi \,\mathrm{d}x \right| \le \mathfrak{c}_{\omega,\epsilon} \, \tau \|\varphi\|_{C^2_0(\omega)} + \underbrace{\int_{\lfloor \frac{s}{\tau} \rfloor\tau+\tau}^{\lfloor \frac{t}{\tau} \rfloor\tau+\tau} \|\rho^\tau_r(\Delta\varphi - \langle\nabla\varphi, \nabla V\rangle)\|_{L^1} \,\mathrm{d}r}_{=: I_3} .$$

Taking into account Remark 2.5.17 and the estimate (2.5.32) of Step 1,

$$\begin{aligned} I_3 &\leq \|\varphi\|_{C_0^2(\omega)} \int_{\lfloor \frac{s}{\tau} \rfloor \tau + \tau}^{\lfloor \frac{t}{\tau} \rfloor \tau + \tau} \left(\|\rho_r^\tau\|_{L^1} + \|\rho_r^\tau \nabla V\|_{L^1} \right) \mathrm{d}r \\ &\leq \mathfrak{c}_\omega e^{\mathfrak{c}\tau} \|\varphi\|_{C_0^2(\omega)} (t - s + \tau) \left(1 + \frac{1 + t + \tau}{\lfloor s/\tau \rfloor \tau} \right) \\ &\leq \mathfrak{c}_{\omega,\epsilon} \|\varphi\|_{C_0^2(\omega)} (t - s + \tau) . \end{aligned}$$

The inequality (2.5.31) easily follows. □

Lemma 2.5.26 (Improved convergence)**.** *Assume that the sequence* $\left(t \mapsto (\mu_t^\tau)_\Omega\right)_\tau$ *converges pointwise w.r.t.* Wb_2 *as* $\tau \to 0$ *to a limit* $t \mapsto \rho_t \, \mathrm{d}x$*. Then, for every* $q \in [1, \frac{d}{d-1})$*, the sequence* $(\rho^\tau)_\tau$ *converges to* ρ *in* $L^1_{\mathrm{loc}}\big((0,\infty); L^q(\Omega)\big)$*.*

Proof. Step 1. Fix $\epsilon \in (0,1)$ and an open set $\omega \Subset \Omega$ with C^1-regular boundary. As a first step, we shall prove strong convergence of $(\rho^\tau)_\tau$ in $L^1\big(\epsilon, \epsilon^{-1}; L^q(\omega)\big)$. The idea is to use a variant of the Aubin–Lions lemma by M. Dreher and A. Jüngel [DJ12]. Consider the Banach spaces

$$X := W^{1,1}(\omega) , \qquad B := L^q(\omega) , \qquad Y := \left(C_0^2(\omega)\right)^* ,$$

and note that the embeddings $X \hookrightarrow B$ and $B \hookrightarrow Y$ are respectively compact (by the Rellich–Kondrachov theorem [Bre11, Theorem 9.16]) and continuous. Inequality (2.5.31) in Lemma 2.5.24 provides one of the two bounds needed to apply [DJ12, Theorem 1]. The other one, namely

$$\limsup_{\tau \to 0} \|\rho^\tau\|_{L^1\left((\epsilon,\epsilon^{-1}); W^{1,1}(\omega)\right)} < \infty ,$$

can be derived from our previous lemmas. Indeed, Remark 2.5.17 provides the bound on the $L^1\big(\epsilon, \epsilon^{-1}; L^1(\omega)\big)$ norm, and we have

$$\begin{aligned} \|\nabla \rho_t^\tau\|_{L^1(\omega)} &\leq \mathfrak{c} \left\| \sqrt{\rho_t^\tau} \, \nabla \sqrt{\rho_t^\tau e^V} \right\|_{L^1(\omega)} + \|\rho_t^\tau \nabla V\|_{L^1(\omega)} \\ &\leq \mathfrak{c} \sqrt{\|\rho_t^\tau\|_{L^1}} \left\| \nabla \sqrt{\rho_t^\tau e^V} \right\|_{L^2} + \|\rho_t^\tau\|_{L^{p'}(\omega)} \|\nabla V\|_{L^p(\omega)} , \end{aligned}$$

where $p = p(\omega)$ is given by Definition 2.3.1. When $\tau \leq \epsilon$, Remark 2.5.17 and Lemma 2.5.22 yield

$$\int_\epsilon^{\frac{1}{\epsilon}} \sqrt{\|\rho_t^\tau\|_{L^1}} \left\| \nabla \sqrt{\rho_t^\tau e^V} \right\|_{L^2} \mathrm{d}t \leq \sqrt{\int_\epsilon^{\frac{1}{\epsilon}} \|\rho_t^\tau\|_{L^1} \, \mathrm{d}t} \sqrt{\int_\epsilon^{\frac{1}{\epsilon}} \left\| \nabla \sqrt{\rho_t^\tau e^V} \right\|_{L^2}^2 \mathrm{d}t} \leq \mathfrak{c}_\epsilon .$$

Moreover, since $p' \in [1,\infty)$ and $p'(d-2) \leq d$, we can apply Lemma 2.5.23 to bound $\|\rho_t^\tau\|_{L^{p'}(\omega)}$. To be precise, there is still a small obstruction to applying Dreher and Jüngel's theorem: it requires ρ^τ to be constant on equally sized subintervals of the time domain, i.e., $(\epsilon, \epsilon^{-1})$; instead, here, τ and $(\epsilon^{-1} - \epsilon)$ may even be incommensurable. Nonetheless, it is not difficult to check that the proof in [DJ12] can be adapted.[6] In the end, we obtain the convergence of $\left(\rho^\tau\right)_\tau$, *along*

[6] The adaptation is the following. In place of [DJ12, Inequality (7)], we write, in our notation:

$$\sum_{i \,:\, \epsilon < i\tau < \epsilon^{-1}} \left\| \rho_{i\tau}^\tau - \rho_{(i-1)\tau}^\tau \right\|_Y \overset{(2.5.31)}{\leq} \mathfrak{c}_{\omega,\epsilon} \tau \left(\lceil 1/(\epsilon\tau) - 1 \rceil - \lfloor \epsilon/\tau \rfloor \right) \leq \mathfrak{c}_{\omega,\epsilon} (\epsilon^{-1} - \epsilon + \tau) .$$

a subsequence $(\tau_k)_{k\in\mathbb{N}_0}$, to some function $f\colon (\epsilon,\epsilon^{-1})\times\omega\to\mathbb{R}_+$ in $L^1\big(\epsilon,\epsilon^{-1};L^q(\omega)\big)$. Up to extracting a further subsequence, we can also require that convergence holds in $L^q(\omega)$ for $\mathscr{L}^1_{(\epsilon,\epsilon^{-1})}$-a.e. t. For any such t, and for any $\varphi\in C_c(\omega)$, we thus have

$$\int_\omega \varphi f_t \,\mathrm{d}x = \lim_{k\to\infty}\int_\omega \varphi\rho_t^{\tau_k}\,\mathrm{d}x = \int_\omega \varphi\rho_t\,\mathrm{d}x\,,$$

where the last identity follows from the convergence w.r.t. Wb_2 and [FG10, Proposition 2.7]. Therefore, $f_t(x)=\rho_t(x)$ for $\mathscr{L}^{d+1}_{(\epsilon,\epsilon^{-1})\times\omega}$-a.e. (t,x), and, *a posteriori*, there was no need to extract subsequences.

Step 2. Secondly, we prove that, for every $\epsilon\in(0,1)$, the sequence $(\rho^\tau)_\tau$ is Cauchy in the complete space $L^1\big(\epsilon,\epsilon^{-1};L^q(\Omega)\big)$. Pick an open subset $\omega\Subset\Omega$ and cover it with a *finite* number of open balls $\{A_i\}_i$, all compactly contained in Ω. Additionally choose $\beta\in(q,\infty)$ with $\beta(d-2)\le d$. We have

$$\|\cdot\|_{L^1\left(\epsilon,\epsilon^{-1};L^q(\Omega)\right)} \le \sum_i \|\cdot\|_{L^1\left(\epsilon,\epsilon^{-1};L^q(A_i)\right)} + \|\cdot\|_{L^1\left(\epsilon,\epsilon^{-1};L^q(\Omega\setminus\omega)\right)}\,,$$

and, by Hölder's inequality,

$$\|\cdot\|_{L^1\left(\epsilon,\epsilon^{-1};L^q(\Omega\setminus\omega)\right)} \le |\Omega\setminus\omega|^{\frac1q-\frac1\beta}\|\cdot\|_{L^1\left(\epsilon,\epsilon^{-1};L^\beta(\Omega)\right)}\,.$$

Hence, by Step 1,

$$\limsup_{\tau_1,\tau_2\to0}\|\rho^{\tau_1}-\rho^{\tau_2}\|_{L^1\left(\epsilon,\epsilon^{-1};L^q(\Omega)\right)} \le 2|\Omega\setminus\omega|^{\frac1q-\frac1\beta}\limsup_{\tau\to0}\|\rho^\tau\|_{L^1\left(\epsilon,\epsilon^{-1};L^\beta(\Omega)\right)}\,.$$

Recall Lemma 2.5.23: we have

$$\limsup_{\tau\to0}\|\rho^\tau\|_{L^1\left(\epsilon,\epsilon^{-1};L^\beta(\Omega)\right)} \le \mathfrak{c}_\beta\int_\epsilon^{\epsilon^{-1}}\left(1+\frac1t\right)\mathrm{d}t\le \mathfrak{c}_{\beta,\epsilon}\,.$$

We conclude, by arbitrariness of ω, the desired Cauchy property.

By Step 1, the limit of $(\rho^\tau)_\tau$ in $L^1\big(\epsilon,\epsilon^{-1};L^q(\Omega)\big)$ must coincide $\mathscr{L}^{d+1}_{(\epsilon,\epsilon^{-1})\times\omega}$-a.e. with ρ for every $\omega\Subset\Omega$ open; hence, this limit is precisely ρ on Ω. □

Proof of Proposition 2.5.21. Convergence in $L^1_{\mathrm{loc}}\big((0,\infty);L^q(\Omega)\big)$ was proven in the previous lemma. Thus, we shall only prove the properties of the limit curve.

Step 1 (continuity). Continuity in duality with $C_c(\Omega)$ follows from Proposition 2.5.19 and [FG10, Proposition 2.7].

Step 2 (identity (2.3.2) for $s>0$). Let $0<s\le t$ and let $\varphi\in C^2_c(\Omega)$. Thanks to the convergences

$$\rho_s^\tau\,\mathrm{d}x \xrightarrow{Wb_2}_\tau \rho_s\,\mathrm{d}x \quad\text{and}\quad \rho_t^\tau\,\mathrm{d}x \xrightarrow{Wb_2}_\tau \rho_t\,\mathrm{d}x\,,$$

we have (see [FG10, Proposition 2.7])

$$\int_\Omega(\rho_t^\tau-\rho_s^\tau)\varphi\,\mathrm{d}x \to_\tau \int_\Omega(\rho_t-\rho_s)\varphi\,\mathrm{d}x\,.$$

Moreover, since every p as in Definition 2.3.1 has a conjugate exponent p' that satisfies $p'(d-1) < d$, Lemma 2.5.26 yields

$$\int_{\lfloor \frac{s}{\tau} \rfloor \tau + \tau}^{\lfloor \frac{t}{\tau} \rfloor \tau + \tau} \int_\Omega \rho_r^\tau (\Delta\varphi - \langle \nabla\varphi, \nabla V \rangle) \, \mathrm{d}x \, \mathrm{d}r \to_\tau \int_s^t \int_\Omega \rho_r (\Delta\varphi - \langle \nabla\varphi, \nabla V \rangle) \, \mathrm{d}x \, \mathrm{d}r \, .$$

Thus, (2.3.2) is true by Lemma 2.5.24.

Step 3 (Sobolev regularity and boundary condition). In analogy with Remark 2.5.6, we define

$$g_r^\tau \coloneqq \sqrt{\rho_r^\tau e^V} - e^{\Psi/2} \, , \qquad g_r^{\tau,(\kappa)} \coloneqq (g_r^\tau - \kappa)_+ - (g_r^\tau + \kappa)_- \, , \qquad \tau, \kappa > 0 \, , \; r \geq 0 \, ,$$

and

$$g_r \coloneqq \sqrt{\rho_r e^V} - e^{\Psi/2} \, , \qquad g_r^{(\kappa)} \coloneqq (g_r - \kappa)_+ - (g_r + \kappa)_- \, , \qquad \kappa > 0 \, , \; r \geq 0 \, .$$

Recall that, if $\kappa \geq c(e^{c\tau} - 1)$ for an appropriate constant c, and if $r \geq \tau$, then the function $g_r^{\tau,(\kappa)}$ is compactly supported in Ω. Let us fix one such κ and $0 < s < t$. Lemma 2.5.22 implies that the sequence $\left(g^{\tau,(\kappa)}\right)_\tau$ is eventually norm-bounded in the space $L^2\left(s,t; W_0^{1,2}(\Omega)\right)$. As a consequence, it admits a subsequence $\left(g^{\tau_k,(\kappa)}\right)_k$ (possibly dependent on s,t,κ) that converges weakly in $L^2\left(s,t; W_0^{1,2}(\Omega)\right)$. Using Lemma 2.5.26 and Mazur's lemma [Bre11, Corollary 3.8 & Exercise 3.4(.1)], one can easily show that this limit indeed coincides with $g^{(\kappa)}$. By means of the weak semicontinuity of the norm, the definition of $g^{\tau,(\kappa)}$, and Lemma 2.5.22, we find

$$\int_s^t \left\| g_r^{(\kappa)} \right\|_{W^{1,2}}^2 \mathrm{d}r \leq \liminf_{k\to\infty} \int_s^t \left\| g_r^{\tau_k,(\kappa)} \right\|_{W^{1,2}}^2 \mathrm{d}r \leq \liminf_{k\to\infty} \int_s^t \| g_r^{\tau_k} \|_{W^{1,2}}^2 \, \mathrm{d}r \leq \mathfrak{c}(1+t) \, ,$$

and, by arbitrariness of s,

$$\int_0^t \left\| g_r^{(\kappa)} \right\|_{W^{1,2}}^2 \mathrm{d}r \leq \mathfrak{c}(1+t)$$

for every $\kappa, t > 0$. We can thus extract a subsequence $\left(g^{(\kappa_l)}\right)_l$ (possibly dependent on t) that converges weakly in $L^2\left(0,t; W_0^{1,2}(\Omega)\right)$. As before, one can check that this limit is g; hence $g \in L^2\left(0,t; W_0^{1,2}(\Omega)\right)$ with

$$\int_0^t \| g_r \|_{W^{1,2}}^2 \, \mathrm{d}r \leq \mathfrak{c}(1+t) \tag{2.5.33}$$

Step 4 (integrability, and (2.3.2) for $s = 0$). Fix an open set $\omega \Subset \Omega$. Let $p = p(\omega) > d$ be as in Definition 2.3.1 and let p' be its conjugate exponent. Since $g \in L^2_{\mathrm{loc}}\left([0,\infty); W_0^{1,2}(\Omega)\right)$, the Sobolev embedding theorem implies $g \in L^2_{\mathrm{loc}}\left([0,\infty); L^{2p'}(\Omega)\right)$. Given that $V \in L^\infty(\Omega)$, we obtain $\rho \in L^1_{\mathrm{loc}}\left([0,\infty); L^{p'}(\Omega)\right)$. In particular, $t \mapsto \int_\omega \rho_t \, \mathrm{d}x$ and $t \mapsto \int_\omega |\nabla V| \, \rho_t \, \mathrm{d}x$ are both locally integrable on $[0,\infty)$. Given $\varphi \in C_c^2(\omega)$, the identity (2.3.2) for $s = 0$ thus follows from the one with $s > 0$ by taking the limit $s \downarrow 0$: on the one side,

$$\lim_{s\downarrow 0} \int_\Omega \rho_s \varphi \, \mathrm{d}x = \int_\Omega \rho_0 \varphi \, \mathrm{d}x$$

by continuity in duality with $C_c(\Omega)$; on the other,

$$\lim_{s\downarrow 0} \int_s^t \int_\Omega \rho_r (\Delta\varphi - \langle \nabla\varphi, \nabla V \rangle) \, \mathrm{d}x \, \mathrm{d}r = \int_0^t \int_\Omega \rho_r (\Delta\varphi - \langle \nabla\varphi, \nabla V \rangle) \, \mathrm{d}x \, \mathrm{d}r$$

by the dominated convergence theorem. □

2.6 Slope formula in dimension $d = 1$

In this section, we only work in dimension $d = 1$ and we take $\Omega = (0,1)$. Recall (Proposition 2.4.11) that, in this setting, $\widetilde{Wb}_2$ is a metric on $\mathscr{S}$. Our purpose is to find an explicit formula for the descending slope $\left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|$ and to derive Theorem 2.1.6 as a corollary. Specifically, the main result of this section is the following.

Proposition 2.6.1. *Assume that $V \in W^{1,2}(\Omega)$. Take $\mu \in \mathscr{S}$ such that $\mathcal{H}(\mu) < \infty$ and let ρ be the density of μ_Ω. Then,*

$$\left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|^2(\mu) = \begin{cases} 4\displaystyle\int_\Omega \left(\partial_x\sqrt{\rho e^V}\right)^2 e^{-V}\,\mathrm{d}x & \text{if } \sqrt{\rho e^V} - e^{\Psi/2} \in W_0^{1,2}(\Omega)\,, \\ \infty & \text{otherwise.}\end{cases} \tag{2.6.1}$$

Remark 2.6.2. In the current setting, i.e., $\Omega = (0,1)$ and $V \in W^{1,2}(\Omega)$, the function V is Hölder continuous; thus it extends to the boundary $\partial\Omega = \{0,1\}$. When $\sqrt{\rho e^V} \in W^{1,2}(\Omega)$, the function ρ belongs to $W^{1,2}(\Omega)$, is continuous, and extends to the boundary as well.

Remark 2.6.3. The functional

$$W^{1,2}(\Omega) \ni f \longmapsto \begin{cases} 4\displaystyle\int_\Omega (\partial_x f)^2 e^{-V}\,\mathrm{d}x & \text{if } f - e^{\Psi/2} \in W_0^{1,2}(\Omega)\,, \\ \infty & \text{if } f - e^{\Psi/2} \in W^{1,2}(\Omega)\setminus W_0^{1,2}(\Omega)\,.\end{cases} \tag{2.6.2}$$

is particularly well-behaved: it is convex, strongly continuous, weakly lower semicontinuous, and has weakly compact sublevels. As a consequence, $\left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|$ turns out to be lower semicontinuous w.r.t. $\widetilde{Wb}_2$. Indeed, assume that $\mu^n \overset{\widetilde{Wb}_2}{\to} \mu$ and $\sup_n \left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|(\mu^n) < \infty$. Let ρ^n be the density of μ^n_Ω. Then the functions $f_n := \sqrt{\rho^n e^V}$ converge, up to subsequences, weakly in $W^{1,2}(\Omega)$ and—by the Rellich–Kondrachov theorem [Bre11, Theorem 8.8]—strongly in $C(\overline{\Omega})$ to a function f such that $f - e^{\Psi/2} \in W_0^{1,2}(\Omega)$ and

$$4\int_\Omega (\partial_x f)^2 e^{-V}\,\mathrm{d}x \le \liminf_{n\to\infty}\left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|^2(\mu^n)\,.$$

Additionally, $\rho^n = f_n^2 e^{-V} \to f^2 e^{-V}$ in $C(\overline{\Omega})$, hence $\mu_\Omega = f^2 e^{-V}\,\mathrm{d}x$ (we use (2.4.1) and [FG10, Proposition 2.7]).

While (2.6.1) reminds the classical slope of the relative entropy (i.e., the relative Fisher information), the crucial difference is in the role of the boundary condition: if ρ does not satisfy the correct one, the slope is infinite.

We are going to prove the two opposite inequalities in (2.6.1) separately. Proving $\ge$ is easier: for the case where $\sqrt{\rho e^V} - e^{\Psi/2} \in W_0^{1,2}$, it amounts to taking small variations of μ in an arbitrary direction; for the other case, it suffices to find appropriate sequences that make the difference quotient diverge. To handle the opposite inequality, we have to bound $\big(\mathcal{H}(\mu) - \mathcal{H}(\tilde\mu)\big)_+$ from above for every sufficiently close measure $\tilde\mu \in \mathscr{S}$. Classical proofs (e.g., [ABS21, Theorem 15.25] or [AGS08, Theorem 10.4.6]) take advantage of geodesic convexity of the functional, which we do not to have; see Section 2.8.3. One of the perks of geodesic convexity is that it automatically ensures lower semicontinuity of the descending slope, which in turn allows to assume stronger regularity on μ and then argue by approximation. To overcome this problem, we combine different ideas on different parts of μ and $\tilde{\mu}$. Away from the boundary $\partial\Omega = \{0,1\}$, the transport plans move absolutely continuous measures to absolutely continuous measures. The Jacobian equation (change of variables formula) relates the two densities and makes the computations rather easy. Estimating the contribution of the parts of $\mu, \tilde{\mu}$ closest to the boundary is more technical: we need to exploit the boundary condition and the Sobolev regularity of the functions ρ, $\log\rho$, and V. Note, indeed, that since the boundary condition is positive, also $\log\rho$ has a square-integrable derivative in a neighborhood of $\partial\Omega$.

To be in dimension $d = 1$ is necessary for $\widetilde{Wb}_2$ to be a distance, but is also extremely convenient because optimal transport maps are monotone and $W^{1,2}$-regular functions are Hölder continuous. For these reasons, it seems difficult (but maybe still possible) to adapt our proof of Proposition 2.6.1 for an analogue of Theorem 2.1.6 in higher dimension.

We first prove a variant of the Lebesgue differentiation theorem that is needed for the subsequent proof of Proposition 2.6.1. We prove Theorem 2.1.6 at the end of the section.

Lemma 2.6.4. *Let $(\gamma^n)_{n\in\mathbb{N}_0}$ be a sequence of nonnegative Borel measures on $\Omega\times\overline{\Omega}$ such that $\lim_{n\to\infty}\mathcal{C}(\gamma^n) = 0$. Additionally assume that $\pi^1_\#\gamma^n$ is absolutely continuous for every $n \in \mathbb{N}_0$, with a density that is uniformly bounded in $L^\infty(\Omega)$. Then, for every $f \in L^2(\Omega)$,*

$$\lim_{n\to\infty}\int\left(⨍_x^y\big(f(z)-f(x)\big)\,\mathrm{d}z\right)^2\mathrm{d}\,\gamma^n(x,y) = 0\,. \tag{2.6.3}$$

Proof. Denote by ρ^n the density of $\pi^1_\#\gamma^n$. Let $g\colon\Omega\to\mathbb{R}$ be Lipschitz continuous. For every $n\in\mathbb{N}_0$, we have

$$\begin{aligned}I_n &:= \int\left(⨍_x^y\big(f(z)-f(x)\big)\,\mathrm{d}z\right)^2\mathrm{d}\gamma^n\\ &\le 3\int\left(⨍_x^y(f-g)\,\mathrm{d}z\right)^2\mathrm{d}\gamma^n + 3\int\left(⨍_x^y g\,\mathrm{d}z - g(x)\right)^2\mathrm{d}\gamma^n\\ &\quad + 3\int_\Omega (g-f)^2\rho^n\,\mathrm{d}x\,.\end{aligned}$$

Consider the Hardy–Littlewood maximal function of (the extension to $\mathbb{R}$ of) $f-g$, that is,

$$(f-g)^*(x) := \sup_{r>0}\frac{1}{2r}\int_{\max\{x-r,0\}}^{\min\{x+r,1\}}|f-g|\,\mathrm{d}z\,,\qquad x\in\mathbb{R}\,.$$

By the (strong) Hardy–Littlewood maximal inequality,

$$\begin{aligned}\int\left(⨍_x^y(f-g)\,\mathrm{d}z\right)^2\mathrm{d}\,\gamma^n &\le 4\int\big((f-g)^*(x)\big)^2\,\mathrm{d}\,\gamma^n = 4\int_\Omega\big((f-g)^*\big)^2\rho^n\,\mathrm{d}x\\ &\le 4\sup_n\|\rho^n\|_{L^\infty}\|(f-g)^*\|^2_{L^2(\mathbb{R})} \le \mathfrak{c}\sup_n\|\rho^n\|_{L^\infty}\|f-g\|^2_{L^2}\,.\end{aligned}$$

The Lipschitz-continuity of g gives

$$\int\left(⨍_x^y g\,\mathrm{d}z - g(x)\right)^2\mathrm{d}\,\gamma^n \le (\operatorname{Lip} g)^2\int(x-y)^2\,\mathrm{d}\,\gamma^n \le (\operatorname{Lip} g)^2\mathcal{C}(\gamma^n)\,,$$

and, moreover, we have

$$\int_\Omega (g-f)^2 \rho^n \, \mathrm{d}x \leq \|\rho^n\|_{L^\infty} \|f-g\|_{L^2}^2 \, .$$

In conclusion,

$$I_n \leq \mathfrak{c} \sup_n \|\rho^n\|_{L^\infty} \|f-g\|_{L^2}^2 + 3(\mathrm{Lip}\, g)^2 \mathcal{C}(\gamma^n) \, .$$

After passing to the limit superior in n, we conclude by arbitrariness of g. □

Proof of Proposition 2.6.1. We omit the subscript $_{\widetilde{Wb}_2}$ in $\left|\partial_{\widetilde{Wb}_2}\mathcal{H}\right|$ throughout the proof.

Step 1 (inequality $\geq$, finite case). Assume that $\sqrt{\rho e^V} - e^{\Psi/2} \in W_0^{1,2}$; hence, in particular, $\rho \in L^\infty(\Omega)$. Let $w \colon \Omega \to \mathbb{R}$ be C^∞-regular with compact support (and not identically equal to 0), and, for $\epsilon > 0$, define $R_\epsilon(x) := x + \epsilon w(x)$. Set $\mu^\epsilon := (R_\epsilon)_\# \mu$ and $\gamma^\epsilon := (\mathrm{Id}, R_\epsilon)_\# \mu$. When ϵ is sufficiently small, $\mu^\epsilon \in \mathscr{S}$ and $\gamma^\epsilon \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \mu^\epsilon)$. Therefore, arguing as in the proof of Lemma 2.5.10,

$$\lim_{\epsilon \to 0^+} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu^\epsilon)}{\epsilon} = \int_\Omega (\partial_x w - w\, \partial_x V) \rho \, \mathrm{d}x \, .$$

Thus,

$$\int_\Omega (\partial_x w - w\, \partial_x V)\rho \, \mathrm{d}x \leq \left|\partial \mathcal{H}\right|(\mu) \liminf_{\epsilon \downarrow 0} \frac{\sqrt{\mathcal{C}(\gamma^\epsilon)}}{\epsilon} \leq \left|\partial \mathcal{H}\right|(\mu) \|w\|_{L^2(\rho)} \, ,$$

and we conclude that

$$\int_\Omega \left|\partial_x \sqrt{\rho e^V}\right|^2 e^{-V} \, \mathrm{d}x \leq \frac{1}{4} \left|\partial \mathcal{H}\right|^2(\mu) \, .$$

Step 2 (inequality $\geq$, infinite case). The case $\sqrt{\rho e^V} \notin W^{1,2}(\Omega)$ is trivial. Thus, let us assume now that $\sqrt{\rho e^V} \in W^{1,2}(\Omega)$ with $\mathrm{Tr}\, \rho \neq \mathrm{Tr}\, e^{\Psi - V}$. Without loss of generality, we may consider the case where $\rho(0) \neq e^{\Psi(0) - V(0)}$. If $\rho(0) > e^{\Psi(0) - V(0)}$, for $\epsilon > 0$ define

$$\mu^\epsilon := \mu - \epsilon \mu_{(0, \epsilon^2)} + \left(\epsilon \int_0^{\epsilon^2} \rho \, \mathrm{d}x \right) \delta_0 \in \mathscr{S} \, ,$$

$$\gamma^\epsilon := \epsilon \mu_{(0,\epsilon^2)} \otimes \delta_0 + (\mathrm{Id}, \mathrm{Id})_\# (\mu_\Omega - \epsilon \mu_{(0,\epsilon^2)}) \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \mu^\epsilon) \, .$$

Since all the functions involved are continuous up to the boundary, we get

$$\begin{aligned} \mathcal{H}(\mu) - \mathcal{H}(\mu^\epsilon) &= \int_0^{\epsilon^2} \Big(\rho \log \rho - (1-\epsilon)\rho \log\big((1-\epsilon)\rho\big) + \epsilon \big(V - 1 - \Psi(0)\big)\rho \Big) \, \mathrm{d}x \\ &\sim_{\epsilon \downarrow 0} \epsilon^3 \big(\log \rho(0) + V(0) - \Psi(0) \big) \rho(0) \, . \end{aligned}$$

On the other hand,

$$\widetilde{Wb}_2(\mu, \mu^\epsilon) \leq \sqrt{\mathcal{C}(\gamma^\epsilon)} = \sqrt{\epsilon \int_0^{\epsilon^2} x^2 \rho \, \mathrm{d}x} \leq \sqrt{\epsilon^5 \int_0^{\epsilon^2} \rho \, \mathrm{d}x} \sim_{\epsilon \downarrow 0} \epsilon^{\frac{7}{2}} \sqrt{\rho(0)} \, ,$$

from which we find

$$\begin{aligned} \left|\partial \mathcal{H}\right|(\mu) &\geq \limsup_{\epsilon \downarrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu^\epsilon)}{\widetilde{Wb}_2(\mu, \mu^\epsilon)} \\ &\geq \underbrace{\sqrt{\rho(0)} \big(\log \rho(0) + V(0) - \Psi(0) \big)}_{>0} \limsup_{\epsilon \downarrow 0} \epsilon^{-\frac{1}{2}} = \infty \, . \end{aligned}$$

If, instead, $\rho(0) < e^{\Psi(0)-V(0)}$, we consider, for $\epsilon > 0$,

$$\mu^\epsilon := \mu + \epsilon \mathscr{L}^1_{(0,\epsilon^2)} - \epsilon^3 \delta_0 \in \mathscr{S}\,, \qquad \gamma^\epsilon := \epsilon\delta_0 \otimes \mathscr{L}^1_{(0,\epsilon^2)} + (\mathrm{Id},\mathrm{Id})_{\#}\mu_\Omega \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu,\mu^\epsilon)\,.$$

and conclude with similar computations as before.

Step 3 (preliminaries for $\leq$). We suppose again that $\sqrt{\rho e^V} - e^{\Psi/2} \in W^{1,2}_0(\Omega)$. In particular, there exist $\bar\lambda, \bar\epsilon > 0$ such that

$$\rho|_{[0,\bar\epsilon]\cup[1-\bar\epsilon,1]} > \bar\lambda\,.$$

Let us take a sequence $(\mu^n)_{n\in\mathbb{N}_0}$ that converges to μ w.r.t. $\widetilde{Wb}_2$, with $\mathcal{H}(\mu^n) < \mathcal{H}(\mu)$ for every n. We aim to prove that

$$\limsup_{n\to\infty} \frac{\mathcal{H}(\mu)-\mathcal{H}(\mu^n)}{\widetilde{Wb}_2(\mu,\mu^n)} \leq 2\sqrt{\int_\Omega \left(\partial_x\sqrt{\rho e^V}\right)^2 e^{-V}\,\mathrm{d}x}\,.$$

For every $n\in\mathbb{N}_0$, we write:

- ρ^n for the density of μ^n_Ω;
- γ^n for some (arbitrarily chosen) $\widetilde{Wb}_2$-optimal transport plan between μ and μ^n such that the diagonal Δ of $\partial\Omega\times\partial\Omega$ (i.e., the set with the two points $(0,0)$ and $(1,1)$) is γ^n-negligible;
- T_n, S_n for maps such that $(\gamma^n)^{\overline{\Omega}}_{\Omega} = (\mathrm{Id}, T_n)_{\#}\mu_\Omega$ and $(\gamma^n)^{\Omega}_{\overline{\Omega}} = (S_n,\mathrm{Id})_{\#}\mu^n_\Omega$. We can and will assume that these two maps are nondecreasing, hence $\mathscr{L}^1_\Omega$-a.e. differentiable;
- $a_n, b_n \in \overline{\Omega} = [0,1]$ for the infimum and supremum of the set $T_n^{-1}(\Omega)$, respectively. Note that, since T_n is monotone, $T_n^{-1}(\Omega)$ is an interval. Conventionally, we set $a_n = 1$ and $b_n = 0$ if $T_n^{-1}(\Omega) = \emptyset$.

Observe that, since $(0,a_n)\subseteq T_n^{-1}(\{0,1\})$, we have

$$\widetilde{Wb}_2^2(\mu,\mu^n) \geq \int_0^{a_n} \min\{x,1-x\}^2 \rho\,\mathrm{d}x \geq \bar\lambda \int_0^{\min\{a_n,\bar\epsilon\}} x^2\,\mathrm{d}x = \frac{\bar\lambda}{3}\min\{a_n,\bar\epsilon\}^3\,.$$

In particular,

$$\limsup_{n\to\infty}\frac{a_n^3}{\widetilde{Wb}_2^2(\mu,\mu^n)} < \infty \text{ and, similarly, } \limsup_{n\to\infty}\frac{(1-b_n)^3}{\widetilde{Wb}_2^2(\mu,\mu^n)} < \infty\,; \tag{2.6.4}$$

thus, up to taking subsequences, we may and will assume that $a_n < \bar\epsilon < 1-\bar\epsilon < b_n$ for every n. In particular, $(\gamma^n)^\Omega_\Omega \neq 0$ and $\mathscr{L}^1_{(0,a_n)\cup(b_n,1)} \ll \mu_{(0,a_n)\cup(b_n,1)}$. Furthermore, since γ^n is W_2-optimal between its marginals (cf. Proposition 2.4.19), it is concentrated on a monotone set Γ_n. This implies that $\gamma(0,1)$ and $\gamma(1,0)$ equal 0 as soon as $\gamma^\Omega_\Omega \neq 0$. Combining this observation with the fact that Δ is γ-negligible, we infer that $\gamma^{\partial\Omega}_{\partial\Omega} = 0$. By the same argument, $T|_{(b_n,1)} \equiv 1$ and $T|_{(0,a_n)} \equiv 0$.

Another assumption that we can and will make is

$$\rho^n|_{S_n^{-1}(\partial\Omega)} \leq \Lambda := \left(\sup_{\partial\Omega} e^\Psi\right)\cdot\left(\sup_\Omega e^{-V}\right)\,. \tag{2.6.5}$$

Indeed, if this is not the case, we can consider the new measures

$$\tilde{\gamma}^n := \gamma^n - (S_n, \mathrm{Id})_\# \left(\rho^n|_{S_n^{-1}(\partial\Omega)} - \Lambda\right)_+ \mathscr{L}^1_\Omega\,,$$
$$\tilde{\mu}^n := \mu - \pi^1_\#(\tilde{\gamma}^n) + \pi^2_\#(\tilde{\gamma}^n) \in \mathscr{S}\,,$$

and notice that $\tilde{\gamma}^n \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu, \tilde{\mu}^n)$. We have

$$\mathcal{H}(\tilde{\mu}^n) - \mathcal{H}(\mu^n) = \int_{S_n^{-1}(\partial\Omega)\cap\{\rho^n>\Lambda\}} \Lambda(\log\Lambda + V - 1 - \Psi\circ S_n)\,\mathrm{d}x$$
$$- \int_{S_n^{-1}(\partial\Omega)\cap\{\rho^n>\Lambda\}} \rho^n(\log\rho^n + V - 1 - \Psi\circ S_n)\,\mathrm{d}x\,,$$

and, because of the definition of Λ, we obtain $\mathcal{H}(\tilde{\mu}^n) \leq \mathcal{H}(\mu^n)$. At the same time, $\widetilde{Wb}_2(\mu, \tilde{\mu}^n) \leq \widetilde{Wb}_2(\mu, \mu^n)$ because $\tilde{\gamma}^n \leq \gamma^n$. This concludes the proof of the claim that we can assume (2.6.5).

Step 4 (inequality $\leq$). By Proposition 2.4.19, $(\gamma^n)^\Omega_\Omega$ is a W_2-optimal transport plan between its marginals $\rho\mathscr{L}^1_{T_n^{-1}(\Omega)}$ and $\rho^n\mathscr{L}^1_{S_n^{-1}(\Omega)}$, and it is induced by the map T_n. Hence, by [ABS21, Theorem 7.3], the Jacobian equation

$$\left(\rho^n|_{S_n^{-1}(\Omega)} \circ T_n\right) \cdot \partial_x T_n = \rho \tag{2.6.6}$$

holds $\rho\mathscr{L}^1_{T_n^{-1}(\Omega)}$-a.e. Consequently, we have the chain of identities

$$\begin{aligned}\int_{S_n^{-1}(\Omega)} (\log\rho^n + V - 1)\rho^n\,\mathrm{d}x &= \int (\log\rho^n + V - 1)\,\mathrm{d}\pi^2_\#(\gamma^n)^\Omega_\Omega \\ &= \int_{T_n^{-1}(\Omega)} \left((\log\rho^n + V - 1)\circ T_n\right)\rho\,\mathrm{d}x \\ &\overset{(2.6.6)}{=} \int_{T_n^{-1}(\Omega)} (\log\rho - \log(\partial_x T_n) + V\circ T_n - 1)\,\rho\,\mathrm{d}x\,.\end{aligned} \tag{2.6.7}$$

Thus, we can decompose the difference $\mathcal{H}(\mu) - \mathcal{H}(\mu^n)$ as

$$\begin{aligned}\mathcal{H}(\mu) - \mathcal{H}(\mu^n) &\overset{(2.6.7)}{=} \int_{T_n^{-1}(\Omega)} \left(\log(\partial_x T_n) + V - V\circ T_n\right)\rho\,\mathrm{d}x + (\mu-\mu^n)_{\partial\Omega}(\Psi) \\ &\quad + \int_{T_n^{-1}(\partial\Omega)} (\log\rho + V - 1)\rho\,\mathrm{d}x - \int_{S_n^{-1}(\partial\Omega)} (\log\rho^n + V - 1)\rho^n\,\mathrm{d}x\,.\end{aligned} \tag{2.6.8}$$

Let us focus on the integral on $T_n^{-1}(\Omega)$. By making the estimate $\log(\partial_x T_n) \leq \partial_x T_n - 1$ and using the properties of the Riemann–Stieltjes integral, we obtain

$$\begin{aligned}\int_{T_n^{-1}(\Omega)} \log(\partial_x T_n)\rho\,\mathrm{d}x &\leq \int_{T_n^{-1}(\Omega)} (\partial_x T_n - 1)\rho\,\mathrm{d}x = \int_{a_n}^{b_n} (\partial_x T_n)\rho\,\mathrm{d}x - \int_{a_n}^{b_n} \rho\,\mathrm{d}x \\ &\leq \lim_{\epsilon\downarrow 0} \int_{a_n+\epsilon}^{b_n-\epsilon} \rho\,\mathrm{d}T_n - b_n\rho(b_n) + a_n\rho(a_n) + \int_{a_n}^{b_n} x\partial_x\rho\,\mathrm{d}x \\ &= (T(b_n^-) - b_n)\rho(b_n) - (T(a_n^+) - a_n)\rho(a_n) - \int_{a_n}^{b_n} (T_n - \mathrm{Id})\partial_x\rho\,\mathrm{d}x\,,\end{aligned} \tag{2.6.9}$$

where we employ the notation $T(a_n^+) := \lim_{\epsilon\downarrow 0} T(a_n+\epsilon)$, and similarly with $T(b_n^-)$.

Let $f := \partial_x V$. By the fundamental theorem of calculus,

$$\int_{T_n^{-1}(\Omega)} (V - V\circ T_n)\rho \,\mathrm{d}x = \int_{a_n}^{b_n} \left(\int_{T_n(x)}^{x} f(z)\,\mathrm{d}z \right) \rho\,\mathrm{d}x\,.$$

By adding and subtracting $f(x)$, we get

$$\begin{aligned}
&\int_{T_n^{-1}(\Omega)} (V - V\circ T_n)\rho\,\mathrm{d}x \\
&\quad= \int_{a_n}^{b_n} f(x) \left(\int_{T_n(x)}^{x} \mathrm{d}z \right) \rho\,\mathrm{d}x + \int_{a_n}^{b_n} \left(\int_{T_n(x)}^{x} \big(f(z) - f(x)\big)\,\mathrm{d}z \right) \rho\,\mathrm{d}x \\
&\quad= -\int_{a_n}^{b_n} (T_n - \mathrm{Id})\rho\, f\,\mathrm{d}x + \int_{a_n}^{b_n} \left(\int_{T_n(x)}^{x} \big(f(z) - f(x)\big)\,\mathrm{d}z \right) \rho\,\mathrm{d}x\,.
\end{aligned} \tag{2.6.10}$$

At this point, we observe that, by Hölder's inequality and Lemma 2.6.4 (applied to the restriction $(\gamma^n)_\Omega^\Omega$), the last double integral is negligible, i.e., it is of the order $o_n\big(\widetilde{W}b_2(\mu,\mu^n)\big)$.

To handle the rest of (2.6.8), we exploit the convexity of $l \mapsto l\log l$ and write

$$-\int_{S_n^{-1}(\partial\Omega)} (\log\rho^n + V - 1)\rho^n\,\mathrm{d}x \le -\int_{S_n^{-1}(\partial\Omega)} (\log\rho + V)\rho^n\,\mathrm{d}x + \int_{S_n^{-1}(\partial\Omega)\cap\{\rho^n>0\}} \rho\,\mathrm{d}x\,. \tag{2.6.11}$$

By Condition (3) in Definition 2.3.7 and the boundary condition of ρ,

$$(\mu - \mu^n)_{\partial\Omega}(\Psi) = \int (\log\rho + V)\,\mathrm{d}\left(\pi^1_\#(\gamma^n)^{\overline{\Omega}}_{\partial\Omega} - \pi^2_\#(\gamma^n)^{\partial\Omega}_{\overline{\Omega}} \right)\,. \tag{2.6.12}$$

In summary, recalling that $(\gamma^n)^{\partial\Omega}_{\partial\Omega} = 0$, from (2.6.8), (2.6.9), (2.6.10), (2.6.11), and (2.6.12) follows the inequality

$$\begin{aligned}
\mathcal{H}(\mu) - \mathcal{H}(\mu^n) \le o_n\left(\widetilde{W}b_2(\mu,\mu^n)\right) &\underbrace{- \int_{a_n}^{b_n} (T_n - \mathrm{Id})(\partial_x\rho + \rho\partial_x V)\,\mathrm{d}x}_{=:L_1^n} \\
&\underbrace{+ \int (\log\rho + V)\,\mathrm{d}\Big(\pi^1_\#\big(\gamma^n - (\gamma^n)^\Omega_\Omega\big) - \pi^2_\#\big(\gamma^n - (\gamma^n)^\Omega_\Omega\big) \Big)}_{=:L_2^n} \\
&\underbrace{+ \big(T(b_n^-) - b_n\big)\rho(b_n) + \int_{S_n^{-1}(1)\cap\{\rho^n>0\}} \rho\,\mathrm{d}x - \int_{T_n^{-1}(1)} \rho\,\mathrm{d}x}_{=:L_3^n} \\
&\underbrace{- \big(T(a_n^+) - a_n\big)\rho(a_n) + \int_{S_n^{-1}(0)\cap\{\rho^n>0\}} \rho\,\mathrm{d}x - \int_{T_n^{-1}(0)} \rho\,\mathrm{d}x}_{=:L_4^n}\,.
\end{aligned} \tag{2.6.13}$$

We claim that the last three lines in (2.6.13), i.e., L_2^n, L_3^n and L_4^n, are bounded from above by negligible quantities, of the order $o_n\left(\widetilde{W}b_2(\mu,\mu^n)\right)$. Let us start with L_3^n. Since every left-neighborhood of 1 is *not* μ_Ω-negligible,

$$\sup\{x \in \Omega :\ (x, T_n(x)) \in \Gamma_n\} = 1\,,$$

which, together with the monotonicity of Γ_n, implies

$$T_n(1^-) \le \mu^n_\Omega\text{-ess}\inf S^{-1}(1)\,. \tag{2.6.14}$$

We now distinguish two cases: either $b_n < 1$ or $b_n = 1$. If $b_n < 1$, given that $T_n|_{(b_n,1)} \equiv 1$, the set $S^{-1}(1)$ is μ^n_Ω-negligible by (2.6.14). Thus

$$\begin{aligned} L_3^n &\le \int_{b_n}^1 \Big(\rho(b_n) - \rho(x)\Big)\,\mathrm{d}x = -\int_{b_n}^1 \left(\int_{b_n}^x \partial_x\rho\,\mathrm{d}z\right)\mathrm{d}x \\ &\le \sqrt{\int_{b_n}^1 |x-b_n|^2\,\mathrm{d}x}\sqrt{\int_{b_n}^1 \left(⨍_{b_n}^x \partial_x\rho\,\mathrm{d}z\right)^2\mathrm{d}x} \\ &\overset{(2.6.4)}{=} O_n\Big(\widetilde{W}b_2(\mu,\mu^n)\Big)\sqrt{\int_{b_n}^1 \left(⨍_{b_n}^x \partial_x\rho\,\mathrm{d}z\right)^2\mathrm{d}x}\,. \end{aligned}$$

Knowing that $\rho \in W^{1,2}(\Omega)$ and that $b_n \to_n 1$, it can be easily proven with Hardy's inequality that the last square root tends to 0 as $n \to \infty$.

Assume now that $b_n = 1$. This time, Inequality (2.6.14) yields

$$L_3^n \le (T_n(1^-) - 1)\rho(1) + \int_{T_n(1^-)}^1 \rho\,\mathrm{d}x = \int_{T_n(1^-)}^1 \Big(\rho(x) - \rho(1)\Big)\,\mathrm{d}x\,.$$

We conclude as in the case $b_n < 1$, because the computations that led to (2.6.4) can be easily adapted to show that $(1 - T_n(1^-))^3 = O_n\Big(\widetilde{W}b_2^2(\mu,\mu^n)\Big)$. Indeed, the monotonicity of T_n gives

$$\widetilde{W}b_2^2(\mu,\mu^n) \ge \int_{T_n(1^-)}^1 \Big(x - T_n(x)\Big)^2 \rho(x)\,\mathrm{d}x \ge \bar\lambda \int_{\max\{1-\bar\epsilon, T_n(1^-)\}}^1 \Big(x - T_n(1^-)\Big)^2\,\mathrm{d}x\,.$$

The proof for L_4^n is similar to that for L_3^n.

Let us now deal with the term L_2^n:

$$L_2^n = \int \Big(\log\rho(x) + V(x) - \log\rho(y) - V(y)\Big)\,\mathrm{d}\Big((\gamma^n)_\Omega^{\partial\Omega} + (\gamma^n)_{\partial\Omega}^\Omega\Big)\,.$$

Define the square-integrable function

$$g := \begin{cases} \frac{\partial_x\rho}{\rho} + \partial_x V & \text{on } (0,\bar\epsilon) \cup (1-\bar\epsilon, 1)\,, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\gamma_\Omega^{\{1\}}$ is concentrated on $(b_n, 1) \times \{1\}$, and $\gamma_{\{1\}}^\Omega$ is concentraded on $\{1\} \times (T_n(1^-), 1)$, as soon as n is large enough for b_n and $T_n(1^-)$ to be greater than $1 - \bar\epsilon$, we have the equality

$$\Big(\log\rho(x) + V(x) - \log\rho(y) - V(y)\Big) = \int_y^x g\,\mathrm{d}z \quad \text{for } \Big((\gamma^n)_\Omega^{\{1\}} + (\gamma^n)_{\{1\}}^\Omega\Big)\text{-a.e. } (x,y)\,.$$

Moreover,

$$\int \left(\int_y^x g\,\mathrm{d}z\right)\mathrm{d}(\gamma^n)_\Omega^{\{1\}} \le \widetilde{W}b_2(\mu,\mu^n)\sqrt{\int_{b_n}^1 \left(⨍_x^1 g\,\mathrm{d}z\right)^2 \underbrace{\rho}_{\le\|\rho\|_{L^\infty}}\,\mathrm{d}x}\,,$$

and

$$\int \left(\int_y^x g \, \mathrm{d}z \right) \mathrm{d}(\gamma^n)_{\{1\}}^{\Omega} \leq \widetilde{W}b_2(\mu, \mu^n) \sqrt{\int_{T_n(1^-)}^1 \left(\fint_x^1 g \, \mathrm{d}z \right)^2 \underbrace{\rho^n|_{S_n^{-1}(1)}}_{\leq \Lambda} \mathrm{d}x} \,.$$

In both cases, since b_n and $T_n(1^-)$ tend to 1 as $n \to \infty$, and $g \in L^2(\Omega)$, the square roots are infinitesimal. The same argument can be easily applied at 0 (i.e. for the integrals w.r.t. $(\gamma^n)_\Omega^{\{0\}}$ and $(\gamma^n)_{\{0\}}^\Omega$), and this brings us to the conclusion that L_2^n is negligible.

In the end, (2.6.13) reduces to

$$\begin{aligned} \mathcal{H}(\mu) - \mathcal{H}(\mu^n) &\leq - \int_{a_n}^{b_n} (T_n - \mathrm{Id})(\partial_x \rho + \rho \, \partial_x V) \, \mathrm{d}x + o_n \left(\widetilde{W}b_2(\mu, \mu^n) \right) \\ &\leq \widetilde{W}b_2(\mu, \mu^n) \sqrt{\int_\Omega \left(\frac{\partial_x \rho}{\sqrt{\rho}} + \sqrt{\rho} \, \partial_x V \right)^2 \mathrm{d}x + o_n(1)} \,, \end{aligned}$$

which is precisely the statement that we wanted to prove. □

Corollary 2.6.5 (Theorem 2.1.6)**.** *Assume that $V \in W^{1,2}(\Omega)$. Let $\mu \in \mathcal{M}_2(\Omega)$. Then,*

$$\left| \partial_{Wb_2} \hat{\mathcal{E}} \right|^2 (\mu) = \begin{cases} 4 \displaystyle\int_0^1 \left(\partial_x \sqrt{\rho e^V} \right)^2 e^{-V} \, \mathrm{d}x & \text{if } \mu = \rho \, \mathrm{d}x \\ & \text{and } \sqrt{\rho e^V} - 1 \in W_0^{1,2}(\Omega) \,, \\ \infty & \text{otherwise,} \end{cases} \tag{2.6.15}$$

where $\hat{\mathcal{E}}$ is defined as

$$\mathcal{M}_2(\Omega) \ni \mu \overset{\hat{\mathcal{E}}}{\longmapsto} \begin{cases} \mathcal{E}(\rho) & \text{if } \mu = \rho \, \mathrm{d}x \,, \\ \infty & \text{otherwise.} \end{cases} \tag{2.6.16}$$

Additionally, $\left| \partial_{Wb_2} \hat{\mathcal{E}} \right|$ is lower semicontinuous w.r.t. Wb_2.

Proof. We may assume that $\mu = \rho \, \mathrm{d}x$ for some $\rho \in L_+^1(\Omega)$, and that $\mathcal{E}(\rho) < \infty$. In particular, μ is finite and we can fix some $\tilde{\mu} \in \mathscr{S}$ such that $\tilde{\mu}_\Omega = \mu$

Step 1 (inequality $\leq$). Let $(\mu^n)_{n \in \mathbb{N}_0} \subseteq \mathcal{M}_2(\Omega)$ be such that $Wb_2(\mu^n, \mu) \to_n 0$ (and $\mu^n \neq \mu$). We want to prove that the limit superior

$$\limsup_{n \to \infty} \frac{\left(\hat{\mathcal{E}}(\mu) - \hat{\mathcal{E}}(\mu^n) \right)_+}{Wb_2(\mu, \mu^n)}$$

is bounded from above by the right-hand side of (2.6.15). To this aim, we may assume that the limit superior is actually a limit and that $\hat{\mathcal{E}}(\mu^n) \leq \hat{\mathcal{E}}(\mu) = \mathcal{E}(\rho)$ for every $n \in \mathbb{N}_0$. In particular, each measure μ^n is finite and has a density ρ^n. By Lemma 2.4.1, for every $n \in \mathbb{N}_0$,

$$\inf_{\tilde{\nu} \in \mathscr{S}} \left\{ \widetilde{W}b_2(\tilde{\mu}, \tilde{\nu}) \, : \, \tilde{\nu}_\Omega = \mu^n \right\} = Wb_2(\mu, \mu^n) \,,$$

which ensures the existence of $\tilde{\mu}^n \in \mathscr{S}$ such that $\tilde{\mu}_\Omega^n = \mu^n$ and

$$\lim_{n \to \infty} \frac{\widetilde{W}b_2(\tilde{\mu}, \tilde{\mu}^n)}{Wb_2(\mu, \mu^n)} = 1 \,, \quad \text{as well as, consequently,} \quad \lim_{n \to \infty} \widetilde{W}b_2(\tilde{\mu}, \tilde{\mu}^n) = 0 \,. \tag{2.6.17}$$

By (2.6.17) and Proposition 2.6.1 (with $\Psi \equiv 0$), we conclude that

$$\lim_{n\to\infty} \frac{\left(\hat{\mathcal{E}}(\mu) - \hat{\mathcal{E}}(\mu^n)\right)_+}{Wb_2(\mu, \mu^n)} \le \limsup_{n\to\infty} \frac{\left(\mathcal{E}(\rho) - \mathcal{E}(\rho^n)\right)_+}{\widetilde{Wb}_2(\tilde{\mu}, \tilde{\mu}^n)} \le \text{RHS of (2.6.15)}.$$

Step 2 (inequality $\geq$). By Proposition 2.6.1 (with $\Psi \equiv 0$), we know that there exists a sequence $(\tilde{\mu}^n)_{n\in\mathbb{N}_0} \subseteq \mathscr{S}$ such that $\widetilde{Wb}_2(\tilde{\mu}^n, \tilde{\mu}) \to_n 0$ (with $\tilde{\mu}^n \neq \tilde{\mu}$) and

$$\lim_{n\to\infty} \frac{\left(\hat{\mathcal{E}}(\mu) - \hat{\mathcal{E}}(\tilde{\mu}^n_\Omega)\right)_+}{\widetilde{Wb}_2(\tilde{\mu}, \tilde{\mu}^n)} = \text{RHS of (2.6.15)}.$$

If this number is 0, then there is nothing to prove. Otherwise, we may assume that $\mu \neq \tilde{\mu}^n_\Omega$ for every n, and we conclude by using (2.4.1).

Step 3 (semicontinuity). The lower semicontinuity is proven as in Remark 2.6.3: if $\mu^n \overset{Wb_2}{\to} \mu$ and $\sup_n \left|\partial_{Wb_2}\hat{\mathcal{E}}\right|(\mu^n) < \infty$, then, up to subsequences, $\left(\sqrt{\rho^n e^V}\right)_n$ converges weakly in $W^{1,2}(\Omega)$ and (strongly) in $C(\overline{\Omega})$, the limit is $\sqrt{\rho e^V}$ by [FG10, Proposition 2.7], and $\sqrt{\rho e^V} - 1 \in W^{1,2}_0(\Omega)$. We conclude by the weak semicontinuity of the functional in (2.6.2). □

2.7 Proof of Theorem 2.1.5

As in Section 2.6, throughout this section we restrict to the case where $\Omega = (0,1) \subseteq \mathbb{R}^1$. Fix $\mu_0 \in \mathscr{S}$ such that its restriction to $(0,1)$ is absolutely continuous with density equal to ρ_0. Recall the scheme (2.1.10): for every $\tau > 0$ and $n \in \mathbb{N}_0$, we iteratively choose

$$\mu^\tau_{(n+1)\tau} \in \operatorname*{arg\,min}_{\mu\in\mathscr{S}} \left(\mathcal{H}(\mu) + \frac{\widetilde{Wb}_2^2(\mu, \mu_{n\tau})}{2\tau}\right). \tag{2.7.1}$$

These sequences of measures are extended to maps $t \mapsto \mu^\tau_t$, constant on the intervals $\left[n\tau, (n+1)\tau\right)$ for every $n \in \mathbb{N}_0$.

The purpose of this section is to prove Theorem 2.1.5. Observe the following fact: Statement 3 follows directly from Statements 1-2. Indeed, given the sequence of maps $(t \mapsto \mu^\tau_t)_\tau$ that converges to $t \mapsto \mu_t$ pointwise w.r.t. $\widetilde{Wb}_2$, we infer from (2.4.1) that $\left(t \mapsto (\mu^\tau_t)_\Omega\right)_\tau$ converges to $t \mapsto (\mu_t)_\Omega$ pointwise w.r.t. Wb_2. Since the approximating maps are precisely the same as those built with (2.1.8), we can apply Proposition 2.5.21 to conclude Statement 3. The proof of Theorem 2.1.6 is thus split into only three parts.

2.7.1 Equivalence of the schemes

Let us fix a measure $\bar{\mu} \in \mathscr{S}$ such that its restriction to $\Omega = (0,1)$ is absolutely continuous. Denote by $\bar{\rho}$ the density of this restriction and assume that $\mathcal{E}(\bar{\rho}) < \infty$.

Proposition 2.7.1. *If $2\tau|\Psi(1) - \Psi(0)| < 1$, then $\mu \in \mathscr{S}$ is a minimizer of*

$$\mathcal{H}(\cdot) + \frac{\widetilde{Wb}_2^2(\cdot, \bar{\mu})}{2\tau} : \mathscr{S} \to \mathbb{R} \cup \{\infty\} \tag{2.7.2}$$

if and only if it is a minimizer of

$$\mathcal{H}(\cdot) + \frac{\mathcal{T}^2(\cdot, \bar{\mu})}{2\tau} : \mathscr{S} \to \mathbb{R} \cup \{\infty\} . \tag{2.7.3}$$

In particular, there exists one single such μ; see Proposition 2.5.3 and Proposition 2.5.11.

Proof. Let $\mathscr{F}$ be the function in (2.7.2) and $\mathscr{G}$ be that in (2.7.3). Recall that $\widetilde{Wb}_2 \leq \mathcal{T}$, which implies that $\mathscr{F} \leq \mathscr{G}$. Let $\mu \in \mathscr{S}$, let $\gamma \in \mathrm{Opt}_{\widetilde{Wb}_2}(\mu, \bar{\mu})$ be such that the diagonal Δ of $\partial\Omega \times \partial\Omega$ is γ-negligible, and define

$$\tilde{\mu} := \mu - \pi^1_\# \gamma^{\partial\Omega}_{\partial\Omega} + \pi^2_\# \gamma^{\partial\Omega}_{\partial\Omega} \in \mathscr{S} , \qquad \tilde{\gamma} := \gamma - \gamma^{\partial\Omega}_{\partial\Omega} \in \mathrm{Adm}_{\mathcal{T}}(\tilde{\mu}, \bar{\mu}) .$$

We have

$$\begin{aligned} \mathscr{G}(\tilde{\mu}) &\leq \mathcal{H}(\tilde{\mu}) + \frac{\mathcal{C}(\tilde{\gamma})}{2\tau} = \mathscr{F}(\mu) + \Big(\pi^2_\# \gamma^{\partial\Omega}_{\partial\Omega} - \pi^1_\# \gamma^{\partial\Omega}_{\partial\Omega}\Big)(\Psi) - \frac{\mathcal{C}(\gamma^{\partial\Omega}_{\partial\Omega})}{2\tau} \\ &= \mathscr{F}(\mu) + \Big(\Psi(1) - \Psi(0)\Big)\Big(\gamma(0,1) - \gamma(1,0)\Big) - \frac{\gamma(0,1) + \gamma(1,0)}{2\tau} \leq \mathscr{F}(\mu) , \end{aligned} \tag{2.7.4}$$

where, in the last inequality, we used the assumption on τ.

Step 1. It follows from (2.7.4) that $\inf \mathscr{G} \leq \mathscr{F} \leq \mathscr{G}$. This is enough to conclude that every minimizer of $\mathscr{G}$ is a minimizer of $\mathscr{F}$ too.

Step 2. Assume now that μ is a minimizer of $\mathscr{F}$. Again by (2.7.4),

$$\mathscr{F}(\mu) \leq \mathscr{F}(\tilde{\mu}) \leq \mathscr{G}(\tilde{\mu}) \leq \mathscr{F}(\mu) .$$

Therefore, it must be true that $\mathscr{F}(\mu) = \mathscr{G}(\tilde{\mu})$ and that all inequalities in (2.7.4) are equalities. This can only happen if $\gamma_{(\partial\Omega\times\partial\Omega)\setminus\Delta} = \gamma^{\partial\Omega}_{\partial\Omega}$ has zero mass, which implies $\mu = \tilde{\mu}$. It is now easy to conclude from $\mathscr{F} \leq \mathscr{G}$ and $\mathscr{F}(\mu) = \mathscr{G}(\mu)$ that μ is a minimizer of $\mathscr{G}$. □

2.7.2 Convergence

Proposition 2.7.2. *As $\tau \to 0$, up to subsequences, the maps $(t \mapsto \mu^\tau_t)_\tau$ converge pointwise w.r.t. $\widetilde{Wb}_2$ to a curve $t \mapsto \mu_t$, continuous w.r.t $\widetilde{Wb}_2$. The restrictions $(\mu_t)_\Omega$ are absolutely continuous.*

Lemma 2.7.3. *For every $t \geq 0$ and $\tau > 0$ such that $2\tau|\Psi(1) - \Psi(0)| < 1$, we have the upper bound*

$$\|\mu^\tau_t\| \leq \mathfrak{c}(1 + t + \tau) . \tag{2.7.5}$$

Proof. Let $t \geq 0$ be fixed. We already know from Remark 2.5.17 that $\|(\mu^\tau_t)_\Omega\| \leq \mathfrak{c}$. By applying Lemma 2.4.8 with $\Phi(x) := 1 - x$, we find

$$\mu^\tau_{(i+1)\tau}(0) - \mu^\tau_{i\tau}(0) \leq \int (1 - x)\, \mathrm{d}\Big(\mu^\tau_{i\tau} - \mu^\tau_{(i+1)\tau}\Big)_\Omega + \mathfrak{c}\,\tau + \frac{\mathcal{T}^2\Big(\mu^\tau_{(i+1)\tau}, \mu^\tau_{i\tau}\Big)}{4\tau} ,$$

for every $i \in \mathbb{N}_0$. By summing over $i \in \{0, 1, \ldots, \lfloor t/\tau \rfloor - 1\}$ and using Lemma 2.5.20,

$$\mu^\tau_t(0) - \mu_0(0) \leq \int (1 - x)\, \mathrm{d}(\mu_0 - \mu^\tau_t)_\Omega + \mathfrak{c}(1 + t + \tau) \leq \mathfrak{c}(1 + t + \tau) .$$

Thus, the sequence $\Big(\mu^\tau_t(0)\Big)_\tau$ is bounded from above as $\tau \to 0$. By suitably choosing Φ, we can find a similar bound from below and bounds for $\mu^\tau_t(1)$. □

Proof of Proposition 2.7.2. We can assume that $\tau < 1$ and that $2\tau|\Psi(1) - \Psi(0)| < 1$. The proof goes as in Proposition 2.5.19: for a fixed $t \geq 0$, we need to prove that

$$\limsup_{\tau \to 0} \widetilde{Wb}_2(\mu_s^\tau, \mu_t^\tau) \leq \mathfrak{c}\sqrt{|r-s|\,(1+t)}\,, \qquad r, s \in [0,t]\,, \tag{2.7.6}$$

and that

$$\tilde{K}_t := \left\{ \mu \in \mathscr{S} \, : \|\mu\| \leq c_1(2+t)\,, \text{ and } \mu_\Omega = \rho \, \mathrm{d}x \text{ with } \int_\Omega \rho \log \rho \, \mathrm{d}x \leq c_2(2+t) \right\}$$

is compact in $(\mathscr{S}, \widetilde{Wb}_2)$, where the constants c_1 and c_2 are given by Lemma 2.7.3 and Lemma 2.5.20, respectively.

The inequality (2.7.6) follows from (2.5.27). If $(\mu^n)_{n \in \mathbb{N}_0}$ is a sequence in $\tilde{K}_t$, thanks to the bound on the total mass, we can extract a (not relabeled) subsequence that converges weakly to some $\mu \in \mathscr{S}$. Let ρ^n be the density of μ_Ω^n for every $n \in \mathbb{N}_0$. We exploit the bound on the integral $\int_\Omega \rho^n \log \rho^n$ to extract a further subsequence such that $(\rho^n)_{n \in \mathbb{N}_0}$ converges weakly in $L^1(\Omega)$ to some ρ. We have $\mu_\Omega = \rho \, \mathrm{d}x$, as well as $\|\mu\| \leq c_1(2+t)$ and $\int_\Omega \rho \log \rho \, \mathrm{d}x \leq c_2(2+t)$; hence $\mu \in \tilde{K}_t$. The convergence $\mu^n \to_n \mu$ holds also w.r.t. $\widetilde{Wb}_2$ thanks to Lemma 2.4.16. □

2.7.3 Curve of maximal slope

Proposition 2.7.4. *Assume that $V \in W^{1,2}(\Omega)$. If the sequence $(t \mapsto \mu_t^\tau)_\tau$ converges pointwise w.r.t. $\widetilde{Wb}_2$ to a curve $t \mapsto \mu_t$, then the latter is a curve of maximal slope for the functional $\mathcal{H}$ in the metric space $(\mathscr{S}, \widetilde{Wb}_2)$.*

To prove this proposition, we employ the classical [AGS08, Theorem 2.3.1], but we also crucially need the results of Section 2.6. In particular, we rely on the explicit formula for the slope of Proposition 2.6.1 and on the consequent semicontinuity observed in Remark 2.6.3.

Proof. Consider the subspace $\widetilde{\mathscr{S}} := \{\mu \in \mathscr{S} \, : \, \mathcal{H}(\mu) \leq \mathcal{H}(\mu_0)\}$. Note that, since $\mathcal{H}$ is $\widetilde{Wb}_2$-lower semicontinuous (Proposition 2.4.15), $t \mapsto \mu_t$ entirely lies in $\widetilde{\mathscr{S}}$. Moreover, $\left|\partial_{\widetilde{Wb}_2} \mathcal{H}\right|$ coincides with $\left|\partial_{\widetilde{Wb}_2} (\mathcal{H}|_{\widetilde{\mathscr{S}}})\right|$ on $\widetilde{\mathscr{S}}$. Therefore, it suffices to prove that $t \mapsto \mu_t$ is a curve of maximal slope in $\widetilde{\mathscr{S}}$.

We invoke [AGS08, Theorem 2.3.1]. Let us check the assumptions. Firstly, the space $(\widetilde{\mathscr{S}}, \widetilde{Wb}_2)$ is complete by Proposition 2.8.3. Secondly, [AGS08, (2.3.2)] is satisfied because the slope $\left|\partial_{\widetilde{Wb}_2} \mathcal{H}\right|$ is $\widetilde{Wb}_2$-lower semicontinuous; see Remark 2.6.3 and [AGS08, Remark 2.3.2]. Thirdly, [AGS08, Assumptions 2.1a,b] follow from Proposition 2.4.15 and Proposition 2.7.1. Finally, to prove [AGS08, (2.3.3)], let us pick a sequence $(\mu^n)_{n \in \mathbb{N}_0} \subseteq \widetilde{\mathscr{S}}$ that converges to some μ w.r.t. $\widetilde{Wb}_2$ and such that $\sup_n \left|\partial_{\widetilde{Wb}_2} \mathcal{H}\right|(\mu^n) < \infty$. We will show that $\mathcal{H}(\mu^n) \to \mathcal{H}(\mu)$. Note that it is enough to prove this convergence *up to subsequences*. Let ρ^n, ρ be the densities of μ_Ω^n, μ_Ω, respectively. Since $\sup_n \left|\partial_{\widetilde{Wb}_2} \mathcal{H}\right|(\mu^n) < \infty$, up to subsequences, the functions $\left(\sqrt{\rho^n e^V}\right)_n$ converge in $C(\overline{\Omega})$ to $\sqrt{\rho e^V}$. Since V is bounded, we also have the convergence $\rho^n \to \rho$ in $C(\overline{\Omega})$. We write

$$\begin{aligned} |\mathcal{H}(\mu^n) - \mathcal{H}(\mu)| &= |\mathcal{E}(\mu^n) - \mathcal{E}(\mu) + (\mu^n - \mu)_{\partial\Omega}(\Psi)| \\ &\leq |\mathcal{E}(\mu^n) - \mathcal{E}(\mu) - (\mu^n - \mu)_\Omega(\Psi)| + |\mu^n(\Psi) - \mu(\Psi)| \end{aligned}$$

Thanks to the uniform convergence $\rho^n \to \rho$, we have $|\mathcal{E}(\mu^n) - \mathcal{E}(\mu) - (\mu_n - \mu)_\Omega(\Psi)| \to 0$. Additionally, by Lemma 2.4.14,

$$|\mu^n(\Psi) - \mu(\Psi)| \le \mathfrak{c}\widetilde{Wb}_2(\mu^n, \mu)\sqrt{\|\mu^n_\Omega\| + \|\mu_\Omega\| + \widetilde{Wb}_2^2(\mu^n, \mu)}\,,$$

from which we conclude, because $\sup_n\|\mu^n_\Omega\| \le \sup_n\|\rho^n\|_{L^\infty} < \infty$. □

Remark 2.7.5. To be precise, [AGS08, Theorem 2.3.1] applies to the limit of the maps $t \mapsto \tilde{\mu}^\tau_t := \mu_{\lceil t/\tau \rceil \tau}$ (as opposed to $\mu^\tau_t = \mu_{\lfloor t/\tau \rfloor \tau}$). It can be easily checked that the distance $\widetilde{Wb}_2(\mu^\tau_t, \tilde{\mu}^\tau_t)$ converges to 0 locally uniformly in time; see (2.5.27).

2.8 Appendix: Additional properties of $\widetilde{Wb}_2$

2.8.1 $\widetilde{Wb}_2$ is not a distance when $d \ge 2$

We are going to prove that, when $d \ge 2$, the property

$$\widetilde{Wb}_2(\mu, \nu) = 0 \quad \implies \quad \mu = \nu$$

in general breaks down. In fact, when applying $\widetilde{Wb}_2$ to two measures $\mu, \nu \in \mathscr{S}$ the information about $\mu_{\partial\Omega}$ and $\nu_{\partial\Omega}$ is completely lost, as soon as $\partial\Omega$ is connected and "not too irregular". A similar result is [Mai11, Theorem 2.2] by E. Mainini.

Proposition 2.8.1. *If $\alpha\colon [0,1] \to \partial\Omega$ is $\left(\frac{1}{2} + \epsilon\right)$-Hölder continuous for some $\epsilon > 0$, then*

$$\widetilde{Wb}_2\Big(\delta_{\alpha(0)} - \delta_{\alpha(1)}, 0\Big) = 0\,. \tag{2.8.1}$$

Consequently: Assume that $\partial\Omega$ is $C^{0,\frac{1}{2}+}$-path-connected, meaning that for every pair of points $x, y \in \partial\Omega$ there exist $\epsilon > 0$ and a $\left(\frac{1}{2} + \epsilon\right)$-Hölder curve $\alpha\colon [0,1] \to \partial\Omega$ with $\alpha(0) = x$ and $\alpha(1) = y$; then, for every $\mu, \nu \in \mathscr{S}$, we have

$$\widetilde{Wb}_2(\mu, \nu) = Wb_2(\mu_\Omega, \nu_\Omega)\,. \tag{2.8.2}$$

Proof. Step 1. Let $\alpha\colon [0,1] \to \partial\Omega$ be $\left(\frac{1}{2} + \epsilon\right)$-Hölder continuous for some $\epsilon > 0$. For $n \in \mathbb{N}_1$, consider the points

$$x_i := \alpha(i/n), \qquad i \in \{0, 1, \dots, n\}\,,$$

and the measure

$$\gamma^n := \sum_{i=0}^{n-1} \delta_{(x_i, x_{i+1})}\,.$$

It is easy to check that $\gamma^n \in \mathrm{Adm}_{\widetilde{Wb}_2}\Big(\delta_{\alpha(0)} - \delta_{\alpha(1)}, 0\Big)$; moreover,

$$\mathcal{C}(\gamma^n) = \sum_{i=0}^{n-1}|x_i - x_{i+1}|^2 \le \mathfrak{c}_\alpha \sum_{i=0}^{n-1} n^{-1-2\epsilon} = \mathfrak{c}_\alpha n^{-2\epsilon}\,,$$

where the inequality follows from the Hölder continuity of α. We conclude (2.8.1) by letting $n \to \infty$.

Step 2. Assume now that $\partial\Omega$ is $C^{0,\frac{1}{2}+}$-path-connected. Fix a finite signed Borel measure η on $\partial\Omega$ with $\eta(\partial\Omega) = 0$, that is, $\|\eta_+\| = \|\eta_-\| =: \lambda$. We shall prove that $\widetilde{Wb}_2(\eta, 0) = 0$.

Fix $\epsilon_1, \epsilon_2 > 0$ and let $X = \{x_1, x_2, \dots, x_N\} \subseteq \partial\Omega$ be a ϵ_1-covering for $\partial\Omega$, meaning that there exists a function $P\colon \partial\Omega \to X$ such that $|x - P(x)| \le \epsilon_1$ for every $x \in \partial\Omega$. We pick one such P that is also Borel measurable (we can by [AB06, Theorem 18.19]). From the previous Step, for every $i, j \in \{1, 2, \dots, N\}$, we get $\gamma_{i,j}$ (nonnegative and concentrated on $\partial\Omega \times \partial\Omega$) such that

$$\pi^1_\# \gamma_{i,j} - \pi^2_\# \gamma_{i,j} = \delta_{x_i} - \delta_{x_j} \quad \text{and} \quad \mathcal{C}(\gamma_{i,j}) \le \epsilon_2 .$$

We define

$$\gamma := (\mathrm{Id}, P)_\# \eta_+ + (P, \mathrm{Id})_\# \eta_- + \frac{1}{\lambda} \sum_{i,j=1}^N \eta_+\big(P^{-1}(x_i)\big)\eta_-\big(P^{-1}(x_j)\big)\gamma_{i,j} .$$

The $\widetilde{Wb}_2$-admissibility of γ, i.e., $\gamma \in \mathrm{Adm}_{\widetilde{Wb}_2}(\eta, 0)$, is straightforward. Furthermore,

$$\begin{aligned} \mathcal{C}(\gamma) &= \int |\mathrm{Id} - P|^2 \,\mathrm{d}(\eta_+ + \eta_-) + \frac{1}{\lambda} \sum_{i,j=1}^N \eta_+(P^{-1}(x_i))\eta_-(P^{-1}(x_j))\mathcal{C}(\gamma_{i,j}) \\ &\le 2\lambda\epsilon_1^2 + \lambda\epsilon_2 , \end{aligned}$$

which brings us to the conclusion that $\widetilde{Wb}_2(\eta, 0) = 0$ by arbitrariness of ϵ_1, ϵ_2.

Step 3. Let us assume again that $\partial\Omega$ is $C^{0,\frac{1}{2}+}$-path-connected, and fix $\mu, \nu \in \mathscr{S}$ and $\epsilon_3 > 0$. Let γ be a Wb_2-optimal transport plan between μ_Ω and ν_Ω, and set $\tilde{\mu} := \pi^1_\# \gamma + (\nu - \pi^2_\# \gamma)_{\partial\Omega}$. It is easy to check that $\tilde{\mu} \in \mathscr{S}$ and that $\mu_\Omega = \tilde{\mu}_\Omega$. Therefore, the previous Step is applicable to $\eta := \mu_{\partial\Omega} - \tilde{\mu}_{\partial\Omega}$, and produces γ_η on $\partial\Omega \times \partial\Omega$ such that

$$\pi^1_\# \gamma_\eta - \pi^2_\# \gamma_\eta = \eta \quad \text{and} \quad \mathcal{C}(\gamma_\eta) \le \epsilon_3 .$$

The measure $\gamma' := \gamma + \gamma_\eta$ is $\widetilde{Wb}_2$-admissible between μ and ν. Therefore,

$$\widetilde{Wb}_2^2(\mu, \nu) \le \mathcal{C}(\gamma') \le \mathcal{C}(\gamma) + \epsilon_3 = Wb_2^2(\mu_\Omega, \nu_\Omega) + \epsilon_3 ,$$

which yields one of the two inequalities in (2.8.2) by arbitrariness of ϵ_3. The other inequality is (2.4.1). □

2.8.2 (Lack of) completeness

We prove here two claims from Section 2.4.6: in the setting where Ω is a finite union of intervals, the metric space $(\mathscr{S}, \widetilde{Wb}_2)$ is *not* complete, but the sublevels of $\mathcal{H}$ are.

Proposition 2.8.2. *Assume that $d = 1$ and that Ω is a finite union of intervals. Then the metric space $(\mathscr{S}, \widetilde{Wb}_2)$ is not complete.*

Proof. Without loss of generality, we may assume that $(0, 1)$ is a connected component of Ω, i.e., $(0, 1) \subseteq \Omega$ and $\{0, 1\} \subseteq \partial\Omega$.

Consider the sequence

$$\mu^n := \frac{1}{x}\mathscr{L}^1_{(2^{-n},1)} - \delta_0 \int_{2^{-n}}^1 \frac{1}{x} \,\mathrm{d}x \in \mathscr{S} , \qquad n \in \mathbb{N}_1 .$$

For every n, there exists the admissible transport plan

$$\gamma^n := \delta_0 \otimes \left(\frac{1}{x}\mathscr{L}^1_{(2^{-n-1},2^{-n})}\right) + (\mathrm{Id}, \mathrm{Id})_\# \left(\frac{1}{x}\mathscr{L}^1_{(2^{-n},1)}\right) \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu^n, \mu^{n+1}) ,$$

which yields

$$\sum_{n=1}^{\infty} \widetilde{Wb}_2(\mu^n, \mu^{n+1}) \leq \sum_{n=1}^{\infty} \sqrt{\int_{2^{-n-1}}^{2^{-n}} \frac{x^2}{x} \, \mathrm{d}x} = \sum_{n=1}^{\infty} \sqrt{\frac{3}{8}} \, 2^{-n} = \sqrt{\frac{3}{8}}\,;$$

hence $(\mu^n)_n$ is Cauchy.

Assume now that $\mu^n \xrightarrow{\widetilde{Wb}_2}_n \mu$ for some $\mu \in \mathscr{S}$ and, for every $n \in \mathbb{N}_1$, fix $\tilde{\gamma}^n \in \mathrm{Opt}_{\widetilde{Wb}_2}(\mu^n, \mu)$. Also fix $\epsilon > 0$. We have

$$\widetilde{Wb}_2^2(\mu^n, \mu) = \int |x - y|^2 \, \mathrm{d}\tilde{\gamma}^n(x, y) \geq \epsilon^2 \tilde{\gamma}^n\Big([\epsilon, 1-\epsilon] \times \partial\Omega\Big)\,,$$

and, using the conditions in Definition 2.3.7,

$$\begin{aligned} \|\mu_\Omega\| &\geq \tilde{\gamma}^n\Big([\epsilon, 1-\epsilon] \times \Omega\Big) = \mu^n\Big([\epsilon, 1-\epsilon]\Big) - \tilde{\gamma}^n\Big([\epsilon, 1-\epsilon] \times \partial\Omega\Big) \\ &\geq \mu^n\Big([\epsilon, 1-\epsilon]\Big) - \frac{\widetilde{Wb}_2^2(\mu^n, \mu)}{\epsilon^2}\,. \end{aligned}$$

Passing to the limit $n \to \infty$, we find

$$\|\mu_\Omega\| \geq \int_\epsilon^{1-\epsilon} \frac{1}{x} \, \mathrm{d}x$$

from which, by arbitrariness of ϵ, it follows that the total mass of μ_Ω is infinite, contradicting the finiteness required in Definition 2.3.7. □

Proposition 2.8.3. *Assume that $d = 1$ and that Ω is a finite union of intervals. Then the sublevels of $\mathcal{H}$ in $\mathscr{S}$ are complete w.r.t. $\widetilde{Wb}_2$.*

Proof. Take a Cauchy sequence $(\mu^n)_{n\in\mathbb{N}_0} \subseteq \mathscr{S}$ for $\widetilde{Wb}_2$ in a sublevel of $\mathcal{H}$, that is, $\mathcal{H}(\mu^n) \leq M$ for some $M \in \mathbb{R}$, for every $n \in \mathbb{N}_0$. Thanks to Lemma 2.4.14, for every $n \in \mathbb{N}_0$ we have

$$\begin{aligned} M \geq \mathcal{H}(\mu^n) &\geq \int_\Omega \rho^n \log \rho^n \, \mathrm{d}x - \Big(\|V\|_{L^\infty} + 1\Big)\|\mu^n_\Omega\| + \mu^n_{\partial\Omega}(\Psi) \\ &\geq \int_\Omega \rho^n \log \rho^n \, \mathrm{d}x - \Big(\|V\|_{L^\infty} + 1\Big)\|\mu^n_\Omega\| + \mu^0(\Psi) - \mu^n_\Omega(\Psi) \\ &\quad - \mathfrak{c}\widetilde{Wb}_2(\mu^n, \mu^0)\sqrt{\|\mu^n_\Omega\| + \left\|\mu^0_\Omega\right\| + \widetilde{Wb}_2^2(\mu^n, \mu^0)}\,, \end{aligned}$$

and, since $\widetilde{Wb}_2(\mu^n, \mu^0)$ is bounded, the family $(\rho^n)_{n\in\mathbb{N}_0}$ is uniformly integrable. Let $(\rho^{n_k})_{k\in\mathbb{N}_0}$ be a subsequence that converges to some ρ weakly in $L^1(\Omega)$. For each of the finitely many $\bar{x} \in \partial\Omega$, let $\Phi_{\bar{x}}$ be a Lipschitz continuous function such that

$$\Phi_{\bar{x}}(\bar{x}) = 1 \quad \text{and} \quad \Phi_{\bar{x}}(x) = 0 \text{ if } x \in \partial\Omega \setminus \{\bar{x}\}\,.$$

Again by Lemma 2.4.14, for every $\bar{x} \in \partial\Omega$ and $n, m \in \mathbb{N}_0$, we have

$$\begin{aligned} |\mu^n(\bar{x}) - \mu^m(\bar{x})| \leq& \left|\mu^n_\Omega(\Phi_{\bar{x}}) - \mu^m_\Omega(\Phi_{\bar{x}})\right| \\ &+ \mathfrak{c}_{\Phi_{\bar{x}}}\widetilde{Wb}_2(\mu^n, \mu^m)\sqrt{\|\mu^n_\Omega\| + \|\mu^m_\Omega\| + \widetilde{Wb}_2^2(\mu^n, \mu^m)} \\ =& \left|\int_\Omega \Phi_{\bar{x}} \cdot (\rho^n - \rho^m) \, \mathrm{d}x\right| \\ &+ \mathfrak{c}_{\Phi_{\bar{x}}}\widetilde{Wb}_2(\mu^n, \mu^m)\sqrt{\|\rho^n\|_{L^1} + \|\rho^m\|_{L^1} + \widetilde{Wb}_2^2(\mu^n, \mu^m)}\,, \end{aligned}$$

which implies that $(\mu^{n_k}(\bar{x}))_{k\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathbb{R}$, thus convergent to some number $l_{\bar{x}}$. Define

$$\mu := \rho\,\mathrm{d}x + \sum_{\bar{x}\in\partial\Omega} l_{\bar{x}}\delta_{\bar{x}}\,.$$

It is easy to check that $\mu^{n_k} \to_k \mu$ weakly; therefore, by Lemma 2.4.16, also w.r.t. $\widetilde{W}b_2$. The limit μ also lies in the sublevel, i.e., $\mathcal{H}(\mu) \le M$, by Proposition 2.4.15. □

2.8.3 If Ω is an interval, $\widetilde{W}b_2$ is geodesic, but $\mathcal{H}$ is not geodesically convex

We prove that $(\mathscr{S}, \widetilde{W}b_2)$ is geodesic when $\Omega = (0,1)$, by using the analogous well-known property of the classical 2-Wasserstein distance. However, as we expect in light of [FG10, Remark 3.4], $\mathcal{H}$ is *not* geodesically λ-convex for any λ. We provide a short proof by adapting the aforementioned remark.

Proposition 2.8.4. *If $\Omega = (0,1)$, then $(\mathscr{S}, \widetilde{W}b_2)$ it is a geodesic metric space.*

Proof. We already know from Proposition 2.4.11 that $(\mathscr{S}, \widetilde{W}b_2)$ is a metric space.

For any two measures $\mu_0, \mu_1 \in \mathscr{S}$, we need to find a curve $t \mapsto \mu_t$ such that

$$\widetilde{W}b_2(\mu_s, \mu_t) \le (t-s)\widetilde{W}b_2(\mu_0, \mu_1)\,, \qquad 0 \le s \le t \le 1\,. \tag{2.8.3}$$

The opposite inequality follows from the triangle inequality and (2.8.3) itself. Indeed,

$$\begin{aligned}\widetilde{W}b_2(\mu_0, \mu_1) &\le \widetilde{W}b_2(\mu_0, \mu_s) + \widetilde{W}b_2(\mu_s, \mu_t) + \widetilde{W}b_2(\mu_t, \mu_1)\\ &\overset{(2.8.3)}{\le} (s + t - s + 1 - t)\widetilde{W}b_2(\mu_0, \mu_1) = \widetilde{W}b_2(\mu_0, \mu_1)\,,\end{aligned}$$

and, in order for the inequalities to be equalities, the identity $\widetilde{W}b_2(\mu_s, \mu_t) = (t-s)\widetilde{W}b_2(\mu_0, \mu_1)$ must be true.

Take $\gamma \in \mathrm{Opt}_{\widetilde{W}b_2}(\mu_0, \mu_1)$. By Proposition 2.4.19, γ is optimal, between its marginals, for the classical 2-Wasserstein distance. Since the set $\overline{\Omega} = [0,1]$, endowed with the Euclidean metric, is geodesic, the classical theory of optimal transport (see, e.g., [ABS21, Theorem 10.6]) ensures the existence of a curve (geodesic) $t \mapsto \nu_t$ of nonnegative measures on $\overline{\Omega}$ with constant total mass, such that

$$W_2(\nu_s, \nu_t) \le (t-s)W_2(\pi^1_{\#}\gamma, \pi^2_{\#}\gamma) = (t-s)\sqrt{\mathcal{C}(\gamma)} = (t-s)\widetilde{W}b_2(\mu_0, \mu_1) \tag{2.8.4}$$

for $0 \le s \le t \le 1$. After noticing that $\nu_1 - \nu_0 = \mu_1 - \mu_0$ by Condition (3) in Definition 2.3.7, we define

$$\mu_t := \mu_0 + \nu_t - \nu_0\,, \qquad t \in (0,1)\,.$$

We claim that this is the sought curve. Firstly, since

$$(\mu_t)_\Omega = (\mu_0)_\Omega + (\nu_t)_\Omega - (\nu_0)_\Omega = (\nu_t)_\Omega \ge 0$$

by Condition (1) in Definition 2.3.7, and since $\nu_0(\overline{\Omega}) = \nu_t(\overline{\Omega})$, we can be sure that $\mu_t \in \mathscr{S}$ for every t. Secondly, every W_2-optimal transport plan γ_{st} between ν_s and ν_t is $\widetilde{W}b_2$-admissible between μ_s and μ_t. Hence,

$$\widetilde{W}b_2(\mu_s, \mu_t) \le \sqrt{\mathcal{C}(\gamma_{st})} = W_2(\nu_s, \nu_t) \overset{(2.8.4)}{\le} (t-s)\widetilde{W}b_2(\mu_0, \mu_1)\,.$$ □

Proposition 2.8.5. *Let $\Omega = (0,1)$. The functional $\mathcal{H}$ is* not *geodesically λ-convex on the metric space $(\mathscr{S}, \widetilde{Wb}_2)$ for any $\lambda \in \mathbb{R}$.*

Proof. Consider the curve

$$t \longmapsto \mu_t := \begin{cases} \frac{1}{t}\mathscr{L}^1_{(0,t)} - \delta_0 & \text{if } t \in (0,1]\,, \\ 0 & \text{if } t = 0\,. \end{cases}$$

Clearly, $\mu_t \in \mathscr{S}$ for every $t \in [0,1]$. We claim that this curve is a geodesic, that $\mathcal{H}(\mu_0) < \infty$, and that $\lim_{t\to 0} \mathcal{H}(\mu_t) = \infty$, which would conclude the proof. The second claim, namely $\mathcal{H}(\mu_0) < \infty$, is obvious. The third claim is true because

$$\mathcal{H}(\mu_t) = -\log t + \fint_0^t V \,\mathrm{d}x - \Psi(0)\,, \qquad t \in (0,1]\,,$$

and, since $V \in L^\infty(0,1)$, the right-hand side tends to ∞ as $t \to 0$. To prove the first claim, fix $0 \le s < t \le 1$ and define

$$\gamma_{st} := \left(\mathrm{Id}, \frac{s}{t}\,\mathrm{Id}\right)_{\#} \mu_t \in \mathrm{Adm}_{\widetilde{Wb}_2}(\mu_t, \mu_s)\,,$$

which gives

$$\widetilde{Wb}_2^2(\mu_s, \mu_t) \le \mathcal{C}(\gamma_{st}) = \int \left| x - \frac{s}{t}x \right|^2 \mathrm{d}\mu_t = \frac{(t-s)^2}{3}\,. \tag{2.8.5}$$

Conversely, for every $\gamma \in \mathrm{Opt}_{\widetilde{Wb}_2}(\mu_1, \mu_0)$, Condition (3) in Definition 2.3.7 implies

$$\gamma(1,1) + \gamma(1,0) + \gamma(\{1\} \times \Omega) = \gamma(1,1) + \gamma(0,1) + \gamma(\Omega \times \{1\})\,,$$

and, since $\gamma(\{1\} \times \Omega) = 0$ by Condition (2) in Definition 2.3.7, we have $\gamma(1,0) \ge \gamma(\Omega \times \{1\})$. Therefore,

$$\begin{aligned} \widetilde{Wb}_2^2(\mu_1, \mu_0) = \mathcal{C}(\gamma) &\ge \mathcal{C}\left(\gamma_\Omega^{\{0\}}\right) + \int |x-1|^2 \,\mathrm{d}\pi^1_\# \gamma_\Omega^{\{1\}} + \gamma(1,0) \\ &\ge \mathcal{C}\left(\gamma_\Omega^{\{0\}}\right) + \int \left(|x-1|^2 + 1\right) \mathrm{d}\pi^1_\# \gamma_\Omega^{\{1\}} \ge \int x^2 \,\mathrm{d}\pi^1_\# \gamma_\Omega^{\partial\Omega}\,. \end{aligned}$$

By Conditions (1) and (2) in Definition 2.3.7,

$$\int x^2 \,\mathrm{d}\pi^1_\# \gamma_\Omega^{\partial\Omega} = \int x^2 \,\mathrm{d}\pi^1_\# \gamma_\Omega^{\overline{\Omega}} = \int_0^1 x^2 \,\mathrm{d}x = \frac{1}{3}\,;$$

hence

$$\widetilde{Wb}_2^2(\mu_s, \mu_t) \overset{(2.8.5)}{\le} \frac{(t-s)^2}{3} \le (t-s)^2 \widetilde{Wb}_2^2(\mu_1, \mu_0)\,,$$

and this concludes the proof. □

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CHAPTER 3

Kinetic Optimal Transport (OTIKIN) – Part 1: Second-Order Discrepancies Between Probability Measures

This chapter contains (with minimal modifications) the following preprint [BMQ25]:

G. Brigati, J. Maas, and F. Quattrocchi. Kinetic Optimal Transport (OTIKIN) – Part 1: Second-Order Discrepancies Between Probability Measures. *arXiv preprint* arXiv:2502.15665v2, 2025.

Abstract

This is the first part of a general description in terms of mass transport for time-evolving interacting particles systems, at a mesoscopic level. Beyond kinetic theory, our framework naturally applies in biology, computer vision, and engineering.

The central object of our study is a new discrepancy d between two probability distributions in position and velocity states, which is reminiscent of the 2-Wasserstein distance, but of second-order nature. We construct d in two steps. First, we optimise over transport plans. The cost function is given by the minimal *acceleration* between two coupled states on a fixed time horizon T. Second, we further optimise over the time horizon $T > 0$.

We prove the existence of optimal transport plans and maps, and study two time-continuous characterisations of d. One is given in terms of dynamical transport plans. The other one —in the spirit of the Benamou–Brenier formula— is formulated as the minimisation of an action of the acceleration field, constrained by Vlasov's equations. Equivalence of static and dynamical formulations of d holds true. While part of this result can be derived from recent, parallel developments in optimal control between measures, we give an original proof relying on two new ingredients: Galilean regularisation of Vlasov's equations and a kinetic Monge–Mather shortening principle.

Finally, we establish a first-order differential calculus in the geometry induced by d, and identify solutions to Vlasov's equations with curves of measures satisfying a certain d-absolute continuity condition. One consequence is an explicit formula for the d-derivative of such curves.

3.1 Introduction

Scientific background Kinetic equations describe systems of *many* interacting particles, at an intermediate level between the microscopic scale, where each particle is tracked individually, and the macroscopic scale of observable quantities, corresponding to, e.g., fluid dynamics or diffusion models. Particles are characterised via their position $x \in \mathcal{X}$ and velocity $v \in \mathcal{V}$. At the kinetic scale—which is our point of view thorough this paper—we do not track the evolution of each single particle. Rather, particles are indistinguishable, and the only available information is their distribution in x, v. The evolution of the system over time $t \in [0, \infty)$ is modelled in a statistical mechanics fashion, as a time-dependent probability distribution on the phase space $\Gamma := \mathcal{X} \times \mathcal{V}$.

The hierarchy between scales was already considered by J.-C. Maxwell and L. Boltzmann [Max67, Bol72], and later included in D. Hilbert's problems for the XX$^{\text{th}}$ century (Problem VI) [Gor18]. Kinetic equations, their derivation from microscopic dynamics, and their macroscopic limit regimes—fluid dynamics or diffusion—have been a vast research field ever since, with important open questions still under active investigation.

On the other side, the classical optimal transport (OT) theory [San15, Vil09b], see §3.1.2, is naturally connected to the macroscopic description of particle systems. Indeed, OT can be reformulated in terms of fluid mechanics [BB00]. In addition, OT provides a deep interpretation of diffusion equations as gradient flows in the space of probability measures [AGS08], as well as variational (JKO [JKO98]) discrete approximation schemes.

In this paper, we take a step towards a new *kinetic optimal transport* (OTIKIN) theory, specifically tailored to the kinetic description of particle systems. Indeed, our main object, a new *second-order discrepancy* d between measures on Γ, preserves the distinct nature of the variables x and v. We consider the case where particles are subject to Newton's laws of mechanics.

Structure of the paper

Section 3.1. The main definitions and results are formulated in §3.1.1. In §3.1.2, we draw connections with related works and collect some motivations, applications, and perspectives.

Section 3.2. We consider the case of Dirac masses. In §3.2.1, we study the minimal acceleration problem between states in Γ. In §3.2.2, we introduce a non-parametric minimal-acceleration discrepancy.

Section 3.3. In §3.3.1, we generalise the construction to a minimal-acceleration discrepancy d between *probability measures*. The definition is given as a static mass transportation problem. Optimisers (transport plans and maps) are shown to exist in §3.3.2. Additional results are given in §3.3.3.

Section 3.4. We analyse two equivalent dynamical formulations of the minimal-acceleration discrepancy d. These are defined, respectively, by means of dynamical transport plans (§3.4.1) and minimal action of solutions to Vlasov's equations (§3.4.3). Further results on dynamical plans and Vlasov's equations are collected in §3.4.2 and §3.4.4, respectively.

Section 3.5. We study a differential calculus induced by the structure of d. In §3.5.1-3.5.2, we prove the equivalence between solutions to Vlasov's equations and a class of *physical* d-absolutely continuous curves. In §3.5.3, we compute the d-derivative of solutions to Vlasov's equations. Moreover, we show that, along such curves, the optimal transport plans are *tangent* to the curve itself. Finally, in §3.5.4, we extend the result to reparametrisations of solutions to Vlasov's equations.

3.1.1 Definitions and main results

Static formulation Set $\mathcal{X} := \mathbb{R}^n$ and $\mathcal{V} := \mathbb{R}^n$, and let the phase space be $\Gamma := \mathcal{X} \times \mathcal{V}$. Let $\mathcal{P}_2(\Gamma)$ be the set of probability measures $\mu \in \mathcal{P}(\Gamma)$, such that the second-order moments of μ are finite, i.e.,

$$\int_\Gamma \left(|x|^2 + |v|^2\right) \mathrm{d}\mu(x,v) < \infty\,.$$

We aim at defining a minimal acceleration discrepancy between measures $\mu, \nu \in \mathcal{P}_2(\Gamma)$. Let us start with the case of Dirac masses $\mu = \delta_{(x,v)}$ and $\nu = \delta_{(y,w)}$. We can see the squared Euclidean distance between x and y as a variational problem where we minimise the integral of the squared velocity for all paths α joining x and y in one unit of time:

$$|y-x|^2 = \inf_{\alpha \in \mathrm{H}^1(0,1;\mathcal{X})} \left\{ \int_0^1 \left|\alpha'(t)\right|^2 \mathrm{d}t \quad \text{subject to } \alpha(0) = x \text{ and } \alpha(1) = y \right\}. \tag{3.1.1}$$

Therefore, one reasonable definition for an acceleration-based discrepancy would be

$$\inf_{\alpha \in \mathrm{H}^2(0,1;\mathcal{X})} \left\{ \int_0^1 \left|\alpha''(t)\right|^2 \mathrm{d}t \quad \text{subject to } (\alpha,\alpha')(0) = (x,v) \text{ and } (\alpha,\alpha')(1) = (y,w) \right\}, \tag{3.1.2}$$

namely, we compute the minimal squared L^2-norm of a force F_t that moves (x,v) to (y,w) in one unit of time, under Newton's law

$$\dot{x}_t = v_t\,, \qquad \dot{v}_t = F_t\,.$$

However, unlike in the first-order case, the choice of the time interval $[0,1]$ is now arbitrary. Indeed, while we can write

$$|y-x|^2 = \inf_{\alpha \in \mathrm{H}^1(0,T;\mathcal{X})} \left\{ T \int_0^T \left|\alpha'(t)\right|^2 \mathrm{d}t \quad \text{subject to } \alpha(0) = x \text{ and } \alpha(1) = y \right\} \tag{3.1.3}$$

for every $T > 0$, a direct calculation (see §3.2) shows that

$$\begin{aligned}\inf_{\alpha \in \mathrm{H}^2(0,T;\mathcal{X})} &\left\{ T \int_0^T \left|\alpha''(t)\right|^2 \mathrm{d}t \quad \text{s.t. } (\alpha,\alpha')(0) = (x,v) \text{ and } (\alpha,\alpha')(T) = (y,w) \right\} \\ &= 12\left|\frac{y-x}{T} - \frac{v+w}{2}\right|^2 + |w-v|^2 =: \tilde{d}_T^2\big((x,v),(y,w)\big)\,,\end{aligned} \tag{3.1.4}$$

which is *not* independent of T.

Thus, we introduce a relaxed version of (3.1.2), where the time parameter is an additional resource to optimise:

$$\tilde{d}\big((x,v),(y,w)\big) := \inf_{T>0} \tilde{d}_T\big((x,v),(y,w)\big)\,. \tag{3.1.5}$$

Problem (3.1.5) admits a solution (see §3.2). Namely, for all $(x,v),(y,w)\in\Gamma$,

$$\tilde{d}^2\Big((x,v),(y,w)\Big)=\begin{cases}3|v+w|^2-3\left(\frac{y-x}{|y-x|}\cdot(v+w)\right)_+^2+|w-v|^2 & \text{if } x\neq y\,,\\ 3|v+w|^2+|w-v|^2 & \text{if } x=y\,.\end{cases}\tag{3.1.6}$$

This quantity is not lower-semicontinuous, but we can write its lower-semicontinuous envelope explicitly: for $(x,v),(y,w)\in\Gamma$,

$$d^2\Big((x,v),(y,w)\Big)=\begin{cases}3|v+w|^2-3\left(\frac{y-x}{|y-x|}\cdot(v+w)\right)_+^2+|w-v|^2 & \text{if } x\neq y\,,\\ |w-v|^2 & \text{if } x=y\,.\end{cases}\tag{3.1.7}$$

This function d is our second-order discrepancy in the case of Dirac deltas. Notice that d and $\tilde{d}$ are not distances. A collection of their properties is given in §3.2.

For general probability measures $\mu,\nu\in\mathcal{P}_2(\Gamma)$, we define $\tilde{\mathsf{d}}_T(\mu,\nu)$, $\tilde{\mathsf{d}}(\mu,\nu)$, and $\mathsf{d}(\mu,\nu)$, by optimising over couplings (or transport plans) π as follows:

$$\tilde{\mathsf{d}}_T^2(\mu,\nu):=\inf_{\pi\in\Pi(\mu,\nu)}\left(12\left\|\frac{y-x}{T}-\frac{v+w}{2}\right\|_{\mathrm{L}^2(\pi)}^2+\|w-v\|_{\mathrm{L}^2(\pi)}^2\right),\tag{3.1.8}$$

$$\tilde{\mathsf{d}}^2(\mu,\nu):=\inf_{\pi\in\Pi(\mu,\nu)}\begin{cases}3\|v+w\|_{\mathrm{L}^2(\pi)}^2-3\frac{\left((y-x,v+w)_\pi\right)_+^2}{\|y-x\|_{\mathrm{L}^2(\pi)}^2}+\|w-v\|_{\mathrm{L}^2(\pi)}^2 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}>0,\\ 3\|v+w\|_{\mathrm{L}^2(\pi)}^2+\|w-v\|_{\mathrm{L}^2(\pi)}^2 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}=0,\end{cases}\tag{3.1.9}$$

$$\mathsf{d}^2(\mu,\nu):=\inf_{\pi\in\Pi(\mu,\nu)}\begin{cases}3\|v+w\|_{\mathrm{L}^2(\pi)}^2-3\frac{\left((y-x,v+w)_\pi\right)_+^2}{\|y-x\|_{\mathrm{L}^2(\pi)}^2}+\|w-v\|_{\mathrm{L}^2(\pi)}^2 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}>0,\\ \|w-v\|_{\mathrm{L}^2(\pi)}^2 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}=0,\end{cases}\tag{3.1.10}$$

where $(\cdot,\cdot)_\pi$ is the scalar product in $\mathrm{L}^2(\pi)$, and

$$\Pi(\mu,\nu)=\Big\{\pi\in\mathcal{P}(\Gamma\times\Gamma)\;:\;(\mathrm{pr}_{x,v})_\#\pi=\mu\,,\quad(\mathrm{pr}_{y,w})_\#\pi=\nu\Big\}\,.$$

Observe that (3.1.8)-(3.1.10) define finite non-negative quantities for every choice of $\mu,\nu\in\mathcal{P}_2(\Gamma)$. This is proved as in the classical OT theory, by testing with $\pi:=\mu\otimes\nu\in\Pi(\mu,\nu)$. Another observation from OT theory is that (3.1.8) admits a minimiser for all μ,ν.

Our first result establishes the existence of optimisers for (3.1.10) and, under the assumption of absolute continuity for μ, of an optimal transport map, in analogy with the classical OT theory [San15, AGS08, Vil09b].

Theorem 3.1.1 (Optimal plans and maps)**.** *The following statements hold.*

1. *(Proposition 3.3.2) We have*

$$\tilde{\mathsf{d}}(\mu,\nu)=\inf_{T>0}\tilde{\mathsf{d}}_T(\mu,\nu)\,,\qquad\mu,\nu\in\mathcal{P}_2(\Gamma)\,.\tag{3.1.11}$$

2. *(Proposition 3.3.4) The second-order discrepancy* d *is the lower-semicontinuous envelope of* $\tilde{\mathrm{d}}$ *with respect to the 2-Wasserstein distance on* $\mathcal{P}_2(\Gamma)$.

3. *(Proposition 3.3.5) For all* $\mu, \nu \in \mathcal{P}_2(\Gamma)$, *there exists a minimiser for* (3.1.10).

4. *(Proposition 3.3.10) If* μ *is absolutely continuous with respect to the Lebesgue measure, then, there exists a* d*-optimal transport map between* μ *and* ν, *i.e., a measurable function* $M\colon \Gamma \to \Gamma$ *such that* $M_{\#}\mu = \nu$ *and* $\pi := (\mathrm{id}, M)_{\#}\mu$ *is a minimiser for* (3.1.10).

While Proposition 3.3.5 shows that d-optimal transport plan exist, we will see that this is not always the case for $\tilde{\mathrm{d}}$, i.e., minimisers for (3.1.9) may not exist, see Example 3.3.6. In §3.3.3, we show that *uniqueness* of d-optimal kinetic transport plans (i.e., minimisers for (3.1.10)) and maps is not to be expected in general.

Dynamical formulations Even though (3.1.8) generalises (3.1.5)—which is derived from the dynamical optimal control problem (3.1.4)—it is not immediate to recognise a minimal acceleration in the cost of (3.1.8). However, there are at least two natural ways to generalise $\tilde{d}_T$ to a discrepancy between probability measures via *dynamical formulations*. In what follows, we discuss them and state our second theorem: these formulations are indeed equivalent to the static one.

Fix $\mu, \nu \in \mathcal{P}_2(\Gamma)$ and $T > 0$. To build our first dynamical formulation, the idea is to take a mixture of curves $(\alpha, \alpha')\colon [0, T] \to \Gamma$ connecting points of $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\nu)$. Precisely, we consider measures $\mathbf{m} \in \mathcal{P}\big(\mathrm{H}^2(0, T; \mathcal{X})\big)$ such that

$$\left(\mathrm{pr}_{\alpha(0),\alpha'(0)}\right)_{\#} \mathbf{m} = \mu\,, \qquad \left(\mathrm{pr}_{\alpha(T),\alpha'(T)}\right)_{\#} \mathbf{m} = \nu\,, \tag{3.1.12}$$

where $\mathrm{pr}_{\alpha(t),\alpha'(t)}$ denotes the evaluation map $\alpha \mapsto \big(\alpha(t), \alpha'(t)\big)$, and we define

$$\tilde{\mathsf{n}}_T^2(\mu, \nu) := \inf_{\mathbf{m} \in \mathcal{P}\left(\mathrm{H}^2(0,T;\mathcal{X})\right)} \left\{ T \int_0^T \int \left|\alpha''(t)\right|^2 \, \mathrm{d}\mathbf{m}(\alpha)\, \mathrm{d}t \quad \text{subject to (3.1.12)} \right\}. \tag{3.1.13}$$

The function $\tilde{\mathsf{n}}_T$ is a natural generalisation of $\tilde{d}_T$, i.e.,

$$\tilde{d}_T\big((x, v), (y, w)\big) = \tilde{\mathsf{n}}_T\left(\delta_{(x,v)}, \delta_{(y,w)}\right), \qquad (x, v)\,,\ (y, w) \in \Gamma\,, \quad T > 0\,.$$

To write our second dynamical formulation, we observe that, for any $\alpha \in \mathrm{H}^2(0, T; \mathcal{X})$ and $\varphi \in \mathrm{C}_\mathrm{c}^\infty\big((0, T) \times \mathcal{X} \times \mathcal{V}\big)$, we have

$$\begin{aligned} 0 &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\varphi\big(t, \alpha, \alpha'\big)\, \mathrm{d}t = \int_0^T \Big(\partial_t\varphi(t, \alpha, \alpha') + \alpha' \cdot \nabla_x\varphi(t, \alpha, \alpha') + \alpha'' \cdot \nabla_v\varphi(t, \alpha, \alpha'')\Big)\, \mathrm{d}t \\ &= \int_0^T \int_\Gamma \Big(\partial_t\varphi(t, x, v) + v \cdot \nabla_x\varphi(t, x, v) + \alpha''(t) \cdot \nabla_v\varphi(t, x, v)\Big)\, \mathrm{d}\delta_{\left(\alpha(t),\alpha'(t)\right)}(x, v)\, \mathrm{d}t\,, \end{aligned}$$

meaning that $\mu_t := \delta_{\left(\alpha(t),\alpha'(t)\right)}$ and $F_t(x, v) := \alpha''(t)$ satisfy Vlasov's equation

$$\partial_t\mu_t + v \cdot \nabla_x\mu_t + \nabla_v \cdot (F_t\, \mu_t) = 0 \tag{3.1.14}$$

weakly in $(0,T)\times\Gamma$. For given $\mu,\nu\in\mathcal{P}_2(\Gamma)$ we define the T-*minimal action* as

$$\widetilde{\mathcal{MA}}_T^2(\mu,\nu) := \inf_{(\mu_t,F_t)_{t\in[0,T]}} \left\{ T\int_0^T \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\,\mathrm{d}t \quad \text{s.t. (3.1.14) and } \mu_0=\mu\,,\ \mu_T=\nu \right\}\,, \tag{3.1.15}$$

which is reminiscent of the Benamou–Brenier formulation of the 2-Wasserstein distance [BB00], see (3.1.35) below. As before,

$$\tilde{d}_T\Big((x,v),(y,w)\Big) = \widetilde{\mathcal{MA}}_T\left(\delta_{(x,v)},\delta_{(y,w)}\right)\,, \qquad (x,v)\,,\ (y,w)\in\Gamma\,, \quad T>0\,. \tag{3.1.16}$$

Indeed, the inequality $\geq$ follows from the discussion above. To justify the converse: when a given curve $(\mu_t,F_t)_{t\in(0,T)}$ solves (3.1.14), then $t\mapsto\alpha(t):=\int_\Gamma x\,\mathrm{d}\mu_t$ (formally) satisfies

$$\alpha_i'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma x_i\,\mathrm{d}\mu_t \overset{(3.1.14)}{=} \int_\Gamma \Big(v\cdot\nabla_x x_i + F_t\cdot\nabla_v x_i\Big)\,\mathrm{d}\mu_t = \int_\Gamma v_i\,\mathrm{d}\mu_t\,, \quad i\in\{1,\dots,n\}\,,$$
$$\alpha_i''(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma v_i\,\mathrm{d}\mu_t \overset{(3.1.14)}{=} \int_\Gamma \Big(v\cdot\nabla_x v_i + F_t\cdot\nabla_v v_i\Big)\,\mathrm{d}\mu_t = \int_\Gamma (F_t)_i\,\mathrm{d}\mu_t\,, \quad i\in\{1,\dots,n\}\,,$$

and, by Jensen's inequality, $|\alpha''(t)|\leq\|F_t\|_{\mathrm{L}^2(\mu_t)}$ for all $t\in(0,T)$, which yields $\leq$ in (3.1.16). The following result extends (3.1.16) from Dirac measures to all of $\mathcal{P}_2(\Gamma)$.

Theorem 3.1.2 (Equivalence of static and dynamic formulations)**.** *For every $\mu,\nu\in\mathcal{P}_2(\Gamma)$ and $T>0$, the problems* (3.1.13) *and* (3.1.15) *admit a minimiser. Moreover, we have the identities*

$$\tilde{\mathsf{n}}_T(\mu,\nu) = \widetilde{\mathcal{MA}}_T(\mu,\nu) = \tilde{\mathsf{d}}_T(\mu,\nu)\,. \tag{3.1.17}$$

This result is proved in two steps, corresponding to Theorem 3.4.1 and Theorem 3.4.10. After posting a first version of this manuscript on arXiv, we were informed of the preprint [Ela25] by K. Elamvazhuthi—building on a previous work [ELLO23]—which contains a generalised version of the second equality in Theorem 3.1.2, in the context of optimal control systems. In Theorem 3.1.2, we prove further equivalence with the formulation $\tilde{\mathsf{n}}_T$, and existence of minimisers for all three problems. Distinctive features of our approach are an original kinetic Monge–Mather principle (cf. Proposition 3.2.6 and Lemma 3.4.4) and the regularisation of solutions to Vlasov's equation via Galilean convolution (cf. Lemma 3.4.9), which may be of independent interest.

A variational characterisation of Vlasov's equations In the classical optimal transport theory, the Benamou–Brenier formula is constrained by the continuity equation $\partial_t\rho_t+\nabla\cdot(V_t\rho_t)=0$, for a velocity field V_t. Solutions to the continuity equation on a bounded open interval (a,b) turn out to coincide with absolutely continuous curves in the Wasserstein space, under appropriate integrability conditions [AGS08, San15]. Although d is not a distance, we will give a similar characterisation for solutions to Vlasov's equations.

Definition 3.1.3 (Physical curves)**.** Let $(\mu_t)_t:(a,b)\to\Gamma$ be a 2-Wasserstein absolutely continuous curve (see §3.1.2). We say that $(\mu_t)_t$ is physical if, in addition, for all $s<t\in(a,b)$, and for a function $\ell\in\mathrm{L}^2_{\geq 0}(a,b)$, it holds true that

$$\tilde{\mathsf{d}}_{t-s}(\mu_s,\mu_t) \leq \int_s^t \ell(r)\,\mathrm{d}r\,. \tag{3.1.18}$$

Heuristically, the physicality condition for a curve $(\mu_t)_t$ yields some differentiable control in the velocity marginal $(\mathrm{pr}_v)_\# \mu_t$, together with the fact that the variation of the spatial marginal $(\mathrm{pr}_x)_\# \mu_t$ is given by v (hence, μ_t would solve (3.1.14)). This idea is made rigorous in the next result.

Theorem 3.1.4 (Identification of the tangent I: physical curves and Vlasov's equations)**.** *The following hold true.*

1. *If $(\mu_t, F_t)_t$ is a weak solution to (3.1.14) on (a,b) for some force field $(F_t)_t$ such that*

$$\int_a^b \left(\|v\|^2_{\mathrm{L}^2(\mu_t)} + \|F_t\|^2_{\mathrm{L}^2(\mu_t)} \right) \, \mathrm{d}t < \infty \,, \tag{3.1.19}$$

then the curve $(\mu_t)_t$ is physical with

$$\ell(t) = 2 \, \|F_t\|_{\mathrm{L}^2(\mu_t)} \,. \tag{3.1.20}$$

2. *Assume that $(\mu_t)_{t \in (a,b)}$ is a physical curve. Then, there exists a vector field $(F_t)_t$ with $\|F_t\|_{\mathrm{L}^2(\mu_t)} \leq \ell(t)$ for a.e. $t \in (a,b)$, such that $(\mu_t, F_t)_t$ is a weak solution to Vlasov's equation (3.1.14) and we have the limit*

$$\lim_{h \downarrow 0} \frac{\tilde{\mathsf{d}}_h(\mu_t, \mu_{t+h})}{h} = \|F_t\|_{\mathrm{L}^2(\mu_t)} \tag{3.1.21}$$

for a.e. $t \in (a,b)$.

The proof of this result can be found in §3.5, as a combination of Proposition 3.5.4, Corollary 3.5.13, and Proposition 3.5.23.

Hypoelliptic Riemannian structure The class of solutions to Vlasov's equations (3.1.14) is rather rigid, as it is not closed under Lipschitz time-reparametrisation. The latter is a desirable property for "absolutely continuous" curves, which we define below.

Definition 3.1.5 (d-absolutely continuous curves)**.** Let $(\tilde{\mu}_s)_{s \in (\tilde{a}, \tilde{b})}$ be a 2-Wasserstein absolutely continuous curve (see §3.1.2). We say that $(\tilde{\mu}_s)_{s \in (\tilde{a}, \tilde{b})}$ is d-absolutely continuous if there exists a function $\tilde{\ell} \in \mathrm{L}^2_{\geq 0}(\tilde{a}, \tilde{b})$ such that for every $s, t \in (\tilde{a}, \tilde{b})$ with $s < t$, we have

$$\mathsf{d}(\tilde{\mu}_s, \tilde{\mu}_t) \leq \int_s^t \tilde{\ell}(r) \, \mathrm{d}r \,. \tag{3.1.22}$$

All physical curves are d-absolutely continuous. The converse is not true, e.g., a time-reparametrisation of a physical curve is still absolutely continuous (but not physical). We may wonder how general this example is and, consequently, how large the class of absolutely continuous curves is compared to that of physical curves.

We find that, under a suitable regularity condition (Assumption 3.1.10), d-absolutely continuous curves coincide with the closure of physical curves under regular reparametrisations in time. Heuristically, d-absolute continuity is enough to have a differentiable control on the velocity marginal for a curve $(\tilde{\mu}_s)_s$, together with the fact that the variation of the space marginal of $\tilde{\mu}_s$ is positively proportional to v, i.e, it amounts to $\tilde{\lambda}(s)v$, for some $\tilde{\lambda}(s) \geq 0$ (independent of the space-velocity variables). The proportionality factor $\tilde{\lambda}(s)$ can be renormalised to one via a time reparametrisation.

The factor $\tilde{\lambda}(s)$ is related to the infinitesimal ratio between the optimal time horizon in the definition of $\mathsf{d}(\tilde{\mu}_s, \tilde{\mu}_{s+\tilde{h}})$ and the physical time $\tilde{h}$. More precisely, given $s, \tilde{h}$, we define $T(s, s+\tilde{h}) := \arg\min_T \mathsf{d}_T(\tilde{\mu}_s, \tilde{\mu}_{s+\tilde{h}})$. Then, whenever $\tilde{h} \mapsto T(s, s+\tilde{h})$ is right-differentiable at 0, we have that $\tilde{\lambda}(s)$ is finite, and

$$\tilde{\lambda}(s) = \lim_{\tilde{h}\to 0^+} \frac{T(s, s+\tilde{h})}{\tilde{h}}.$$

We state these ideas precisely in Theorem 3.1.7 below.

Remark 3.1.6. If $(\mu_t, F_t)_{t\in(a,b)}$ solves Vlasov's equation (3.1.14), and $\tau\colon (\tilde{a}, \tilde{b}) \to (a, b)$ is a bi-Lipschitz reparametrisation, then the curve $s \mapsto \tilde{\mu}_s := \mu_{\tau(s)}$ solves the reparametrised Vlasov equation

$$\partial_s \tilde{\mu}_s + \tilde{\lambda}(s)\, v \cdot \nabla_x \tilde{\mu}_s + \nabla_v \cdot (\tilde{F}_s \tilde{\mu}_s) = 0\,, \qquad s \in (\tilde{a}, \tilde{b})\,, \tag{3.1.23}$$

with $\tilde{\lambda}(s) := \tau'(s)$ and $\tilde{F}_s := \tilde{\lambda}(s) F_{\tau(s)}$.

Theorem 3.1.7 (Identification of the tangent II: d-absolutely continuous curves, d-derivative)**.** *The following hold true.*

1. *Assume that $(\tilde{\mu}_s, \tilde{F}_s)_s$ is a weak solution to (3.1.23) on $(\tilde{a}, \tilde{b})$ for some force field $(\tilde{F}_s)_s$ such that*
$$\int_{\tilde{a}}^{\tilde{b}} \left(\|v\|^2_{\mathrm{L}^2(\tilde{\mu}_s)} + \left\|\tilde{F}_s\right\|^2_{\mathrm{L}^2(\tilde{\mu}_s)} \right) \mathrm{d}s < \infty\,, \tag{3.1.24}$$
and for a function $\tilde{\lambda}$ bounded from above and below by positive constants.
If the Wasserstein metric derivative of the spatial marginal $\tilde{\rho}_s(\cdot) := \tilde{\mu}_s(\cdot \times \mathcal{V})$ satisfies
$$\left|\tilde{\rho}'_s\right|_{\mathrm{W}_2} > 0 \quad \textit{for a.e. } s \in (\tilde{a}, \tilde{b})\,, \tag{3.1.25}$$
then, the curve $(\tilde{\mu}_s)_s$ is d*-absolutely continuous and satisfies Assumption 3.1.10, with*
$$\tilde{\ell}(s) = 2\,\|\tilde{F}_s\|_{\mathrm{L}^2(\tilde{\mu}_s)} \quad \textit{and} \quad \tilde{\lambda}_{\mathrm{ac}}(s) = \tilde{\lambda}(s)\,, \qquad s \in (\tilde{a}, \tilde{b})\,. \tag{3.1.26}$$

2. *Assume that $(\tilde{\mu}_s)_{s\in(\tilde{a},\tilde{b})}$ is a* d*-absolutely continuous curve satisfying Assumption 3.1.10. If $\tilde{\rho}_s$ satisfies (3.1.25), then there exists a vector field $(\tilde{F}_s)_s$ with $\|\tilde{F}_s\|_{\mathrm{L}^2(\tilde{\mu}_s)} \le \tilde{\ell}(s)$ for a.e. $s \in (\tilde{a}, \tilde{b})$, such that $(\tilde{\mu}_s, \tilde{F}_s)_s$ is a solution to (3.1.23) with $\tilde{\lambda} = \tilde{\lambda}_{\mathrm{ac}}$, and we have the limit*
$$\lim_{\tilde{h}\downarrow 0} \frac{\mathsf{d}(\tilde{\mu}_s, \tilde{\mu}_{s+\tilde{h}})}{\tilde{h}} = \left\|\tilde{F}_s\right\|_{\mathrm{L}^2(\tilde{\mu}_s)} \tag{3.1.27}$$
for a.e. $s \in (\tilde{a}, \tilde{b})$.

The proof of this result can be found in §3.5, see in particular Theorem 3.5.24.

Remark 3.1.8 (The flow velocity)**.** In both statements in Theorem 3.1.7, we assume positivity a.e. of the quantity $|\tilde{\rho}'_s|_{\mathrm{W}_2}$, i.e., of the Wasserstein metric derivative of the spatial density $(\tilde{\rho}_s)_s$. This can be interpreted as macroscopic non-steadiness of the system. Using the theory of optimal transport (cf. [AGS08] and Lemma 3.5.3 below) it is possible to prove the following.

When $(\tilde\mu_s)_s$ solves the reparametrised Vlasov equation (3.1.23) the metric derivative $|\tilde\rho_s'|_{W_2}$ is equal to the $L^2(\tilde\rho_s)$-norm of the irrotational part of the vector $\tilde\lambda(s)\,\tilde{j}_s(x)$, where $\tilde{j}_s(x)$ is the *flow velocity*

$$\tilde{j}_s(x) := \int_{\mathcal{V}} v \,\mathrm{d}\tilde\mu_s(x,v)\,.$$

Notice that $(\tilde\rho_s, \tilde{j}_s)_s$ solves Euler's equation

$$\partial_s\tilde\rho_s + \tilde\lambda(s)\nabla_x\cdot\tilde{j}_s = 0\,.$$

Remark 3.1.9 (The tangent cone I: admissible directions)**.** Theorem 3.1.7 asserts that d-absolutely continuous curves are identified with solutions to (3.1.23), which in turn are induced by vector fields $(\tilde\lambda(s)v, \tilde{F}_s)_{s\in(\tilde a,\tilde b)}$. A narrowly continuous curve of measures $(\tilde\mu_s)_s$ solves (3.1.23) for two different vector fields $(\tilde F_s)_s$ and $(\tilde G_s)_s$ if and only if, by linearity of (3.1.23), we have $\nabla_v\cdot\Big((\tilde F_s - \tilde G_s)\,\tilde\mu_s\Big) = 0$ in the sense of distributions, that is, if and only if $\tilde F_s$ and $\tilde G_s$ have the same projection onto

$$\mathrm{cl}_{L^2(\tilde\mu_s)}\left\{\nabla_v\varphi \,:\, \varphi\in C^1_c(\Gamma)\right\}.$$

As in the classical case, see §3.1.2, we can define the hypoelliptic tangent cone at μ as

$$\mathcal{T}_{\mu,\mathsf{d}}\mathcal{P}_2(\Gamma) := \mathrm{cl}_{L^2(\mu)}\Big\{(\lambda\, v, \nabla_v\varphi) \,:\, \lambda\in\mathbb{R}_{\geq 0},\ \varphi\in C^1_c(\Gamma)\Big\}, \tag{3.1.28}$$

and equip $\mathcal{T}_{\mu,\mathsf{d}}\mathcal{P}_2(\Gamma)$ with the degenerate Riemannian form

$$\Big\langle(\lambda^{(1)}v, F^{(1)}), (\lambda^{(2)}v, F^{(2)})\Big\rangle_{\mathcal{T}_{\mu,\mathsf{d}}\mathcal{P}_2(\Gamma)} := \int F^{(1)}\cdot F^{(2)}\,\mathrm{d}\mu\,. \tag{3.1.29}$$

Finally, recall that Equation (3.1.23) is the closure under time-reparametrisation of (3.1.14). At the geometric level, we formally interpret this fact as follows. The hypoelliptic tangent $\mathcal{T}_{\mu,\mathsf{d}}\mathcal{P}_2(\Gamma)$ is the conical envelope of the vectors $\Big\{(v,\nabla_v\varphi),\quad \varphi\in C^1_c(\Gamma)\Big\}$, which correspond exactly to the vector fields inducing (3.1.14). See also Remark 3.1.13 below.

Assumption 3.1.10 (Regularity)**.** Let $(\tilde\mu_s)_{s\in(\tilde a,\tilde b)}$ be a d-absolutely continuous (hence 2-Wasserstein a.c.) curve, and define the open set of times

$$\tilde\Omega := \Big\{s\in(\tilde a,\tilde b) \,:\, \|v\|_{L^2(\tilde\mu_s)} > 0\Big\}\,. \tag{3.1.30}$$

We assume that there exist

1. a measurable selection $(s,t)\mapsto\tilde\pi_{s,t}\in\Pi(\tilde\mu_s,\tilde\mu_t)$ of d-optimal transport plans, i.e., minimisers in (3.1.10),
2. a measurable function $\tilde\lambda_{\mathrm{ac}}\colon(\tilde a,\tilde b)\to\mathbb{R}_{>0}$ bounded from above and below by positive constants,

such that, defining the *optimal time*[1]

$$\tilde T_{s,t} = \begin{cases} 2\dfrac{\|y-x\|^2_{L^2(\tilde\pi_{s,t})}}{(y-x,v+w)_{\tilde\pi_{s,t}}} & \text{if } (y-x,v+w)_{\tilde\pi_{s,t}} > 0\,,\\ 0 & \text{if } \|y-x\|_{L^2(\tilde\pi_{s,t})} = 0\,,\\ \infty & \text{otherwise,}\end{cases} \qquad \tilde a < s < t < \tilde b\,, \tag{3.1.31}$$

[1]This is a minimiser for (3.1.11) between $\tilde\mu_s$ and $\tilde\mu_t$, see Proposition 3.3.2

then, the convergence

$$\frac{\tilde{T}_{s,s+\tilde{h}}}{\tilde{h}} \to \tilde{\lambda}_{\mathrm{ac}}(s) \quad \text{as } \tilde{h} \downarrow 0\,, \tag{3.1.32}$$

holds for a.e. $s \in (\tilde{a}, \tilde{b})$, with L^1-domination on every compact subset of $\tilde{\Omega}$.

We conclude by stating an immediate consequence of Theorem 3.1.4 and Theorem 3.1.7.

Corollary 3.1.11. *Let $(\mu_t)_t$ be a 2-Wasserstein absolutely continuous curve with $|\rho_t'|_{\mathrm{W}_2} > 0$ for a.e. t. The curve $(\mu_t)_t$ is physical if and only if:*

1. *it is* d*-absolutely continuous, and*
2. *it satisfies Assumption 3.1.10 with $\tilde{\lambda}_{\mathrm{ac}} \equiv 1$.*

3.1.2 Motivation, related contributions, and perspectives

In this section, we review the recent literature that motivated or inspired our construction. We also give a perspective on future developments after this work in §3.1.2.

Comparison with standard optimal transport

Optimal transport (OT) Let $\rho_0, \rho_1 \in \mathcal{P}_2(\mathcal{X})$. One way to define the standard 2-Wasserstein distance between ρ_0 and ρ_1 is given by

$$\mathrm{W}_2(\rho_0, \rho_1) := \inf_{\pi \in \Pi(\rho_0,\rho_1)} \sqrt{\int |y - x|^2 \, \mathrm{d}\pi(x, y)}\,, \tag{3.1.33}$$

where $\Pi(\rho_0, \rho_1)$ is the set of all couplings $\pi \in \mathcal{P}_2(\mathcal{X} \times \mathcal{X})$ of ρ_0 and ρ_1. This variational problem, in which the average distance between coupled points $x \in \mathrm{supp}(\rho_0)$ and $y \in \mathrm{supp}(\rho_1)$ is minimised, is known as *Kantorovich formulation*. The existence of minimisers is classical [San15] and they are referred to as *optimal transport plans*. Under mild conditions on ρ_0, it is also possible to establish existence of an *optimal transport map* between ρ_0 and ρ_1, i.e., a function $M : \mathcal{X} \to \mathcal{X}$ for which $\pi := (\mathrm{id}, M)_{\#}\rho_0$ is an optimal plan [Bre91]. The optimal transport map is ρ_0-a.e. uniquely determined and can be found by solving a Monge–Ampère equation. Its regularity is a major research topic [San15, Fig17].

Recalling (3.1.1), $|y - x|^2$ is the squared length of the line joining x with y that minimises $\int_0^1 |\alpha'(t)|^2 \, \mathrm{d}t$, among all H^1-regular curves $\alpha : (0, 1) \to \mathcal{X}$ with x and y as endpoints. Thus, the 2-Wasserstein distance can also be written in its *dynamical formulation*

$$\mathrm{W}_2^2(\rho_0, \rho_1) = \inf\left\{ \int_0^1 \int \left|\alpha'(t)\right|^2 \mathrm{d}\mathbf{m}\, \mathrm{d}t\,, \quad \text{s.t. } \mathbf{m} \in \mathcal{P}\Big(\mathrm{H}^1(0, 1; \mathcal{X})\Big)\,, \right.$$
$$\left. \Big(\mathrm{pr}_{\alpha(0)}\Big)_{\#} \mathbf{m} = \rho_0\,, \; \Big(\mathrm{pr}_{\alpha(1)}\Big)_{\#} \mathbf{m} = \rho_1 \right\}. \tag{3.1.34}$$

Moreover, J.-D. Benamou and Y. Brenier [BB00] provided the following fluid-mechanical characterisation:

$$\mathrm{W}_2^2(\rho_0,\rho_1) = \inf\left\{ \int_0^1 \|V_t\|^2_{\mathrm{L}^2(\rho_t)}\,\mathrm{d}t\,, \quad \text{s.t. } \partial_t\rho_t + \nabla_x\cdot(V_t\rho_t) = 0 \text{ in } \mathcal{D}^\star\big((0,1)\times\mathcal{X}\big)\,, \right.$$
$$\left. \rho_{t=i} = \rho_i \text{ for } i=0,1 \right\}. \tag{3.1.35}$$

The idea is that the characteristic ODE of the continuity equation $\partial_t\rho_t + \nabla_x\cdot(V_t\rho_t) = 0$ in (3.1.35) is $\dot{x}_t = V_t$. Thus, the squared norm of the velocity field $(V_t)_t$ is equal to the average squared path-wise speed, cf. [AGS08, Chapter 8]. As it turns out, the curves $(\rho_t)_t$ solving the continuity equation for some vector field $(V_t)_t$ such that $\int_0^1 \|V_t\|^2_{\mathrm{L}^2(\rho_t)}\,\mathrm{d}t < \infty$ are exactly the W_2-2-absolutely continuous curves [AGS08], i.e., those satisfying

$$\mathrm{W}_2(\rho_s,\rho_t) \le \int_s^t \ell(r)\,\mathrm{d}r\,, \qquad 0<s<t<1\,. \tag{3.1.36}$$

for some function $\ell \in \mathrm{L}^2_{\ge 0}(0,1)$. For such curves, the *metric derivative* is

$$\left|\rho_t'\right|_{\mathrm{W}_2} \coloneqq \lim_{h\to 0}\frac{\mathrm{W}_2(\rho_t,\rho_{t+h})}{h} = \|V_t\|_{\mathrm{L}^2(\rho_t)}\,, \qquad \text{for a.e. } t\in(0,1)\,, \tag{3.1.37}$$

if $(\rho_t, V_t)_t$ solves the continuity equation and V_t is chosen in the $\mathrm{L}^2(\rho_t)$-closure of the set $\left\{\nabla_x\phi : \phi\in \mathrm{C}^1_c(\mathcal{X})\right\}$. Formally, the distance W_2 induces a Riemannian structure [Ott01]:

$$\mathcal{T}_\rho\mathcal{P}_2(\mathcal{X}) \coloneqq \mathrm{cl}_{\mathrm{L}^2(\rho)}\left\{\nabla_x\phi : \phi\in \mathrm{C}^1_c(\mathcal{X})\right\}\,, \qquad \langle F,G\rangle_{\mathcal{T}_\rho\mathcal{P}_2(\mathcal{X})} \coloneqq \int F\cdot G\,\mathrm{d}\rho\,. \tag{3.1.38}$$

Comparison of OTIKIN and OT

- At the level of static problems (i.e., optimal transport plans), existence of minimisers is true for both OT (i.e., in the problem (3.1.33) defining W_2) and OTIKIN (i.e., in (3.1.10), defining d). For both, also existence of an optimal transport map holds under the assumption of absolute continuity of the starting measure w.r.t. Lebesgue (see Proposition 3.3.10), but uniqueness in OTIKIN does not hold, see §3.3.3.
- By minimising the acceleration as in (3.1.13), we find a discrepancy $\tilde{\mathrm{n}}_T = \tilde{\mathrm{d}}_T$ that depends on the time parameter T. The non-parametric discrepancy d is then found by optimising in T and taking the W_2-relaxation, see Theorem 3.1.1. Note that this relaxation is necessary in order to ensure existence of optimal transport plans, see Proposition 3.3.5 and Example 3.3.6. In OT, the Wasserstein distance W_2 is, instead, naturally non-parametric, in the sense that the choice of the time interval in (3.1.34) is inconsequential. This follows from our discussion in §3.1.1.
- In (3.1.34), an optimiser exists and is supported on constant-speed straight lines connecting points $x\in\mathrm{supp}(\rho_0)$ to points $y\in\mathrm{supp}(\rho_1)$, according to the optimal coupling in (3.1.33). In this case, there is no loss of generality in considering only curves on $[0,1]$, see (3.1.3). In (3.1.13), we have the existence of a minimiser $\mathbf{m}_T$, and this measure is supported on cubic T-splines [BGV19] (see §3.2), for every T. However, time-reparametrisations of $\mathbf{m}_T$ on another interval $[0,T']$ might no longer satisfy the

desired boundary conditions. This happens even for one single *spline* between two Dirac masses. Furthermore, even if a reparametrisation satisfies the boundary conditions, it may not be optimal for the problem with time T'.

- While the class of continuity equations $\partial_t\rho_t + \nabla_x \cdot (V_t\rho_t) = 0$ is invariant under time-reparametrisation, the class of Vlasov's equations $\partial_t\mu_t + v \cdot \nabla_x\mu_t + \nabla_v \cdot (F_t\mu_t) = 0$ is not. When working with Dirac deltas, the same observation arises by comparing the first-order ODE $\dot{x}_t = V_t$ with the second-order ODE $\dot{x}_t = v_t\,,\ \dot{v}_t = F_t$.

Remark 3.1.12 (The tangent cone II: tangency of optimal plans)**.** In [AGS08, Chapter 8], the Wasserstein tangent space at $\rho \in \mathcal{P}_2(\mathcal{X})$—denoted by $\mathcal{T}_\rho\mathcal{P}_2(\mathcal{X})$—is identified with the closure in $\mathrm{L}^2(\rho;\mathbb{R}^d)$ of $\left\{\nabla_x\varphi \,:\, \varphi \in \mathrm{C}^1_\mathrm{c}(\mathcal{X})\right\}$. A time-dependent vector field $V_t \in \mathcal{T}_{\rho_t}\mathcal{P}_2(\mathcal{X})$ can be interpreted as the velocity field of a curve solving $\partial_t\rho_t + \nabla_x \cdot (\rho_t\, V_t) = 0$ (which is a necessary and sufficient condition for a curve to be W_2-2-a.c.). In [AGS08, Proposition 8.4.6], it is shown that, taken two measures ρ_t, ρ_{t+h} along such a curve, one has $y - x - h\,V_t = o(h)$, as $h \to 0$, on the support of any W_2-optimal plan between ρ_t and ρ_{t+h}.

In our setting, we establish a similar structure. Let $(\mu_t)_t$ be a solution to the Vlasov equation (3.1.14), i.e. $\partial_t\mu_t + v\cdot\nabla_x\mu_t + \nabla_v\cdot(F_t\mu_t) = 0$, with $(v, F_t) \in \mathcal{T}_{\mu_t,\mathsf{d}}\mathcal{P}_2(\Gamma)$ (see (3.1.28)). Note that, by Theorem 3.1.4, this is equivalent to physicality. In §3.5.3, we will prove that, if $\pi_{t,t+h}$ is an optimal transport plan for $\mathsf{d}(\mu_t, \mu_{t+h})$, then, $\pi_{t,t+h}$-almost everywhere,

$$
\begin{aligned}
&y - x - h\,v = o(h)\,,\\
&w - v - h\,F_t(x, v) = o(h)\,.
\end{aligned}
$$

Whenever the *total momentum* of μ_t is non-zero, we gain a further order of precision in our Taylor expansions on the support of $\pi_{t,t+h}$:

$$
y = x + hv + \frac{1}{2}h^2F_t(x, v) + o(h^2)\,.
$$

Remark 3.1.13 (The tangent cone III: geometry of the tangent bundle)**.** We continue the formal geometric considerations of Remark 3.1.9. Formally, the 2-Wasserstein distance induces a Riemannian structure on $\mathcal{P}_2(\mathcal{X})$, with a clear identification of the tangent bundle [Ott01, AGS08]. The discrepancy d of OTIKIN yields a sort of *hypoelliptic* Riemannian structure [Hör67]. Vlasov's equation (3.1.14) and its time-reparametrisations (3.1.23), can be rewritten as $\partial_s\tilde{\mu}_s + \nabla_{x,v}\cdot\left((\tilde{\lambda}(t)v, \tilde{F}_t)\,\tilde{\mu}_t\right) = 0$, which is a special case of the continuity equation $\partial_s\tilde{\mu}_s + \nabla_{x,v}\cdot(X_s\tilde{\mu}_s) = 0$ associated with the W_2-distance over $\mathcal{P}_2(\Gamma)$, for a $2n$-component vector field $X_s : \Gamma \to \mathbb{R}^{2n}$. Thus, we can formally see the geometry of d—i.e., the hypoelliptic tangent cone $\mathcal{T}_{\tilde{\mu}_s,\mathsf{d}}\mathcal{P}_2(\Gamma)$ with the form (3.1.29)—as a distribution of vectors in $\mathcal{T}_{\tilde{\mu}_s,\mathrm{W}_2}\mathcal{P}_2(\Gamma)$, equipped with a degenerate version of (3.1.38) that measures only the acceleration component $\tilde{F}_t$.

Remark 3.1.14 (Comparison with sub-Riemannian optimal transport)**.** A. Figalli and L. Rifford developed a theory for optimal transport on sub-Riemannian manifolds [FR10]. They consider a m-dimensional Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$, equipped with a distribution of vector fields $\Delta = \operatorname{span}\{X_1, \cdots, X_k\}$, with $k < m$, such that $\operatorname{Lie}(\Delta) = \mathcal{TM}$, i.e., the *Hörmander condition* [Hör67] is satisfied. The Wasserstein distance is replaced by

$$
\mathrm{W}^2_{\mathrm{SR},2}(\mu, \nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int \mathrm{d}^2_{\mathrm{SR}}(x, y)\ \mathrm{d}\pi(x, y)\,,
$$

where $\mathrm{d_{SR}}$ denotes the sub-Riemannian distance on $(\mathcal{M}, \langle\cdot,\cdot\rangle, \Delta)$. Notice that $\mathrm{d_{SR}}$ is obtained via minimal length of curves tangent to Δ,

$$\mathrm{d}^2_{\mathrm{SR}}(x,y) = \inf_{\substack{\alpha:[0,1]\to\mathcal{M} \\ \alpha'\in\Delta}} \int_0^1 \langle \alpha'(t), \alpha'(t)\rangle_{\alpha(t)} \,\mathrm{d}t \,.$$

Our case is different, since d-absolutely continuous curves are induced by vector fields tangent to[2]

$$\Delta_{\mathrm{kin}} := \mathrm{span}\,\{Z, Y_1, \cdots, Y_n\}\,, \qquad \text{with } Z = v\cdot\nabla_x\,, \qquad Y_i = \frac{\partial}{\partial_{v_i}}\,,$$

but we only measure the speed of curves in the directions $\{Y_i\}_{i=1}^n$, while the vector field Z acts solely as a constraint. The resulting hypoelliptic geometry combines degenerate Riemannian with symplectic effects. Heuristically, the difference between sub-Riemannian geometry and our hypoelliptic geometry is analogous to the distinction between Hörmander operators of the *first kind*, like the sub-elliptic Laplacian $\sum_{i=1}^k (X_i)^2$, and Hörmander operators of the *second kind*, like the Kolmogorov operator $Z + \sum_{i=1}^n (Y_i)^2$.

Minimal acceleration costs, kinetic Wasserstein, and related distances

Optimal transport with minimal acceleration cost has appeared in the context of variational schemes for fluid dynamics [GW09, CSW19], see also §3.1.2. There, a discrete time-step $T>0$ is fixed, and the authors consider both $\tilde{\mathrm{d}}_T$ and

$$\mathbf{W}^2_T(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int \left(12\left|\frac{y-x}{T} - \frac{w-v}{2}\right|^2 + |w-v|^2\right) \mathrm{d}\pi\,, \qquad \mu,\nu\in\mathcal{P}_2(\Gamma)\,, \tag{3.1.39}$$

which differ in the sign of $\frac{w-v}{2}$.

For our purposes, the functionals $\tilde{\mathrm{d}}_T$ and $\mathbf{W}_T$ cannot be used as such, indeed:

1. smooth curves $t\mapsto\mu_t$ are not, in general, $\tilde{\mathrm{d}}_T$-continuous, in the sense that

   $$\liminf_{h\downarrow 0} \tilde{\mathrm{d}}_T(\mu_t,\mu_{t+h}) > 0\,.$$

   Therefore, absolute continuity is not meaningful for $\tilde{\mathrm{d}}_T$;

2. on the one hand, for all $T>0$, the functional $\mathbf{W}_T$ is a distance on $\mathcal{P}_2(\Gamma)$, which is equivalent—with equivalence constants depending on T—to W_2, as the cost function satisfies $\mathrm{w}_T\big((x,v),(y,w)\big) := 12\left|\frac{y-x}{T} - \frac{w-v}{2}\right|^2 + |w-v|^2 \asymp |(x,v)-(y,w)|^2$. However, the derivative of w_T along Newton's ODE $\dot{x}_t = v_t$, $\dot{v}_t = F_t$ is given by

   $$\lim_{h\to 0} \frac{\mathrm{w}^2_T\big((x_t,v_t),(x_{t+h},v_{t+h})\big)}{h^2} = 12\left|\frac{v_t}{T} - \frac{F_t}{2}\right|^2 + |F_t|^2\,.$$

   This quantity is not *natural* in our setting, which demands the squared force $|F_t|^2$ instead, as we build a purely acceleration-based theory.

[2] Note that $\mathrm{Lie}(\Delta_{\mathrm{kin}}) = \mathcal{T}\mathbb{R}^{2n}$, so that Hörmander's condition holds.

In a recent paper [Iac22], M. Iacobelli explored two other families of 'kinetic Wasserstein distances', yielding to new results for Vlasov's PDEs. The first idea is to build perturbations of the standard Wasserstein distance W_2 on $\mathcal{P}_2(\Gamma)$, using transportation costs of the form $a\,|y-x|^2 + b\,(y-x)\cdot(w-v) + c\,|w-v|^2$, with $a,b,c\in\mathbb{R}$ to be tuned. Then, time-dependent and non-linear generalisations are considered. Twisting the reference distance (usually H^2 or L^2) to better capture the interaction between space and velocity variables has been a fruitful technique in kinetics, to prove both regularity (*hypoellipticity* [Kol34, Hör67]) and long-time convergence to equilibrium (*hypocoercivity* [Vil09a, DMS15]). The interplay between hypocoercivity and optimal transport has been analysed in a few papers [Bau17, Sal21]. A second class of distances is constructed by adding a time-shift as follows [Iac22]:

$$\inf_{\pi\in\Pi(\mu,\nu)} \int \left(|(x-tv)-(y-tw)|^2 + |w-v|^2\right) \mathrm{d}\pi\,, \qquad t>0\,, \qquad \mu,\nu\in\mathcal{P}_2(\Gamma)\,.$$

Our discrepancy d differs from previous constructions, as it involves an optimisation over $T>0$. As a result, d is not a distance, but it has the physical dimension of a speed, so that its time derivative along curves of measures is naturally an acceleration.

Variational approximation schemes for kinetic equations

Minimising the squared acceleration in optimal transport originated from a series of papers about variational approximation schemes for dissipative kinetic PDEs [HJ00, Hua00, DPZ13, DPZ14, Par25], before being readapted to fluid dynamics [GW09, CSW19]. The goal there is to approximate, by means of De Giorgi minimising movement schemes [AGS08], the solution to Kramer's equation

$$\partial_t f + v\cdot\nabla_x f = \Delta_v f\,, \qquad f\colon (a,b)\times\Gamma\to\mathbb{R}\,, \tag{3.1.40}$$

and various generalisations thereof. One prototypical result is the following.

Theorem 3.1.15 ([DPZ14, Hua00]). *Let $\mathcal{E}\colon\mathcal{P}_2(\Gamma)\to[0,\infty]$ be the Boltzmann–Gibbs entropy*

$$\mathcal{E}(\mu) := \begin{cases} \int_\Gamma f\log f\,\mathrm{d}x\,\mathrm{d}v & \text{if } \mu = f(x,v)\,\mathrm{d}x\,\mathrm{d}v\,,\\ +\infty & \text{otherwise.}\end{cases} \tag{3.1.41}$$

Given an initial datum $f_0\in\mathrm{L}^1(\Gamma)$, define the sequence

$$\begin{aligned} \mu_0^h &:= f_0\,\mathrm{d}x\,\mathrm{d}v\,,\\ \mu_{(k+1)h}^h &\in \operatorname*{arg\,min}_{\nu\in\mathcal{P}_2(\Gamma)} \left(\mathcal{E}(\nu) + \frac{1}{2h}\tilde{\mathrm{d}}_h^2\left(\mu_{kh}^h,\nu\right)\right)\,, \qquad k\in\mathbb{N}\,. \end{aligned} \tag{3.1.42}$$

Then, as $h\to 0$, the piece-wise constant interpolation $(\mu_t^h)_{t\geq 0}$ converges to the solution to (3.1.40) with initial datum f_0.

As observed in [DPZ14], the choice of the entropy $\mathcal{E}$ and the penalisation $\tilde{\mathrm{d}}_h$ can be motivated by a *large deviation principle*. However, the penalisation $\tilde{\mathrm{d}}_h$ depends on the timestep h, so that (3.1.42) does not exactly define a De Giorgi scheme [AGS08]. Instead, the discrepancy d we introduce does not depend on the timestep (in fact, we optimise over the time parameter). This way, the intrinsic time parameter of d will depend on the evolution itself, without being conditioned by the time discretisation. The analysis of a scheme akin to (3.1.42), with d in place of $\tilde{\mathrm{d}}_h$, will be the subject of a forthcoming work.

Wasserstein splines and interpolation

Splines between probability measures Splines arise in numerics as an interpolation method for a set of data $(t_i, x_i)_{i=1,\dots,k} \subseteq [0,1]\times\mathcal{X}$. An interpolating curve is a function $\alpha\colon [0,1]\to\mathcal{X}$ with $\alpha(t_i)=x_i$, subject to certain constraints (e.g., regularity). One interesting and common example is the solution to

$$\bar\alpha \in \operatorname*{arg\,min}_{\alpha\in \mathrm{H}^2(0,1;\mathcal{X})} \left\{ \int_0^1 \left|\alpha''(t)\right|^2 \,\mathrm{d}t\,,\quad \text{s.t. } \alpha(t_i)=x_i \text{ for every } i \right\} .$$

Spline interpolation between probability measures is the problem of finding a *minimal acceleration curve* $(\rho_t)_{t\in[0,1]}$ such that $\rho_{t_i}=\rho_i$, for a given dataset $(t_i,\rho_i)_{i=1,\dots,k}\subseteq[0,1]\times\mathcal{P}_2(\mathcal{X})$. This matter has recently attracted increasing interest and has been the subject of both theoretical and numerical investigations [BGV19, CCG18, Cla21, CCLG+21]. One possible approach is to interpret $(\rho_t)_t$ as a curve taking values in the Riemannian-like space $\big(\mathcal{P}_2(\mathcal{X}),\mathrm{W}_2\big)$ (see §3.1.2) and to measure the *covariant acceleration* of ρ via the Levi–Civita connection [Lot08, Gig12]. A more tractable strategy is to consider measure-valued path splines:

$$\bar{\mathbf{m}} \in \operatorname*{arg\,min}_{\mathbf{m}\in\mathcal{P}\left(\mathrm{H}^2(0,1;\mathcal{X})\right)} \left\{ \int_0^1 \int |\alpha''(t)|^2 \,\mathrm{d}\mathbf{m}\,\mathrm{d}t\,,\quad \text{s.t. } (\mathrm{pr}_{t_i})_\#\mathbf{m}=\rho_i \text{ for every } i \right\} . \tag{3.1.43}$$

Notice that this problem differs from (3.1.13) with $T=1$, where we interpolate only between two measures and, most notably, we also fix the velocity marginals. However, (3.1.43) writes as a relaxation of (3.1.13), by further minimising over all possible velocity marginals [CCG18].

Kinetic Optimal Transport for measure interpolation We address two issues from the recent literature related to spline interpolation:

1. Theorem 3.1.2 provides a fully rigorous proof of the kinetic Benamou–Brenier formula with fixed space-velocity marginals. As a corollary, we prove a version thereof where only the space marginals are assigned, as conjectured by Y. Chen, G. Conforti, and T. T. Georgiou [CCG18, Claim 4.1]. Details are given in Remark 3.4.12.

2. We outline a variation on the algorithm by S. Chewi, J. Clancy, T. Le Gouic, P. Rigollet, G. Stepaniants, and A. Stromme [CCLG+21, Section 3] for the construction of splines in $\mathcal{P}_2(\mathcal{X})$. As above, the problem is to construct a curve interpolating a given dataset $(t_i,\rho_i)_{i=1,\dots,k}\subseteq[0,1]\times\mathcal{P}_2(\mathcal{X})$, with $t_1<t_2<\dots<t_k$. The procedure in [CCLG+21] is briefly described as follows. Firstly, one computes the optimal multimarginal 2-Wasserstein coupling $\pi\in\mathcal{P}_2(\mathcal{X}^k)$. Secondly, one connects each tuple $(x_1,\dots,x_k)\in\mathcal{X}^k$ by means of an acceleration-minimising spline $\bar\alpha_{x_1,\dots,x_k}$, and defines the interpolating spline of measures $t\mapsto\rho_t$ at time t as the push-forward of π through the map $\bar\alpha_\cdot(t)\colon(x_1,\dots,x_k)\mapsto\bar\alpha_{x_1,\dots,x_k}(t)$. The modification we propose is to use the construction above only to assign velocity marginals to each ρ_i, i.e., we set $\mu_i:=(\bar\alpha_\cdot(t),\bar\alpha'_\cdot(t))_\#\pi\in\mathcal{P}_2(\Gamma)$. Given $t\in(t_i,t_{i+1})$, we take the optimal $\tilde{\mathbf{n}}_{(t_{i+1}-t_i)}$-optimal dynamical plan $\mathbf{m}$ between μ_i and μ_{i+1}, set

$$\mu_t := \left(\mathrm{pr}_{\alpha(t),\alpha'(t)}\right)_\# \mathbf{m}\,, \tag{3.1.44}$$

and define ρ_t as the space marginal of μ_t. Remarkably, our interpolation is deterministic and injective, in the sense that

$$\left(\mathrm{pr}_{\left(\alpha(t_i),\alpha'(t_i)\right),\left(\alpha(t),\alpha'(t)\right)}\right)_{\#}\mathbf{m}$$

is induced by an *injective map* $\Gamma \to \Gamma$ if μ_i is absolutely continuous, see §3.4.2.

Applications to biology and engineering

The problem of finding a *minimal acceleration path* between two measures μ, ν appears naturally in applications.

- *Trajectory inference* is aimed at reconstructing a time-continuous evolution from a few time-separated observations. This technique has recently gained relevance in mathematical biology to study the development of cells [LZKS24, Sch21, BSK+24, CZHS22], with potential applications in regenerative medicine. Wasserstein splines (see §3.1.2) and our Proposition 3.4.3 provide a smooth interpolation scheme to this purpose [RBDB+24, CGP21]. In addition, the action functional in (3.1.13), which encodes minimal acceleration/consumption along paths, can be easily adapted to the specific model under investigation (taking into account, e.g., potential energy, drift).

- Images in computer vision can be cast into probability measures. Various applications involve continuous interpolations between images, which are often performed using classical OT [RPDB12, San15]. More recently, an alternative has been formulated using Wasserstein splines [BGV19, Cla21, FP23, JRR23, ZSS22], with applications to texture generation models [JRE24]. Our construction, see §3.4, proposes a twofold variation. Firstly, we remove the dependence on the timespan T, which is usually a datum of the problem in the literature. Secondly, we also assign the velocity marginals.

- The minimal acceleration problem (3.1.4)-(3.1.5) arises naturally also in optimal control. Indeed, we look for the optimal time-dependent force $F_t = \alpha''(t)$ and timespan T required to connect two states (x, v) and (y, w) in Γ. The quantity we minimise is the squared norm of F_t over time, which is reminiscent of resource consumption in steering of robots and space vehicles [MR24, LGP14]. In particular, (3.1.4)-(3.1.5) is used for rockets powered by Variable Specific Impulse engines [Mar12, Kec95a]. Our work yields a natural generalisation, i.e., a mathematical framework for the optimal steering of a fleet of agents between two configurations, described by $\mu, \nu \in \mathcal{P}_2(\Gamma)$.

Conclusions and perspectives

- In the current work, we build an optimal-transportation discrepancy d between probability measures, which is based on the minimal acceleration. Also the timespan of the minimal-acceleration path is optimised. In addition, we give a characterisation of d-absolutely continuous curves, see Theorem 3.1.7. Such a result allows us to recast kinetic equations of Vlasov type, driven by the transport operator $v \cdot \nabla_x$ and various collision terms, as paths in the space of probability measures, with the metric derivative depending only on the collisional effects. This is the starting point of the forthcoming *Part II - Kinetic Gradient Flows*. There, we are going to recast dissipative kinetic PDEs as steepest descent curves, among all d-absolutely continuous curves, for some given free-energy functionals. The steepest descent will be characterised via optimality in an energy-dissipation functional inequality. As a further justification, we aim at finding a large-deviation principle behind our kinetic gradient flow structure [DPZ14].

- In *Part II - Kinetic Gradient Flows*, we aim to define JKO discrete variational schemes [JKO98] with the discrepancy d and prove their convergence to dissipative kinetic PDEs. To this end, we will treat a kinetic equation as a whole, exploiting the interplay between space and velocity variables, which is also the *leitmotiv* of the current work. This strategy is also at the core of the *hypocoercivity* theory [Vil09a, DMS15], where the reference norm is twisted precisely in order to capture the interaction of x and v via $v \cdot \nabla_x$. This is in contrast with the *splitting* numerical schemes [Par25], where the transport and the collision terms are treated separately at each iteration.

- In this work, we specialise to a model case: particles are subject to Newton's laws $\dot{x}_t = v_t\,,\ \dot{v}_t = F_t$, without external confinement. However, this suggests a general scheme to build adapted versions of the discrepancy d for systems of interacting particles. The collision/irreversible effects in the phenomenon are measured via an action functional to be minimised under a constraint, given by a suitable continuity equation, see (3.1.15). Such a continuity equation (Vlasov's equation (3.1.14) in our setting) is determined by the conservative/reversible dynamics of the system. The resulting discrepancy captures the interaction between reversible and irreversible effects, while clearly distinguishing their roles. The induced geometry on $\mathcal{P}_2(\Gamma)$ formally reads as a degenerate Riemann-like structure (where the d-derivative of curves corresponds to the instantaneous action), constrained on a symplectic form, which allows only the *physically* admissible directions. We will explore generalisations of our theory in forthcoming papers. In particular, we aim at giving an optimal-transport interpretation of systems belonging to the GENERIC (General Equation for Non-Equilibrium Reversible-Irreversible Coupling) framework [GÖ97, ÖG97].

3.2 The particle model

In this section, we describe the kinetic optimal transport model for Dirac measures.

3.2.1 The fixed-time discrepancy

Let $T > 0$ be a time parameter, and fix two points (x, v) and (y, w) in the phase space $\Gamma := \mathcal{X} \times \mathcal{V} = \mathbb{R}^n \times \mathbb{R}^n$. Recall the minimisation problem

$$\inf_{\alpha \in \mathrm{H}^2(0,T;\mathcal{X})} \left\{ T \int_0^T \left|\alpha''(t)\right|^2 \mathrm{d}t \quad \text{s.t. } (\alpha, \alpha')(0) = (x, v) \text{ and } (\alpha, \alpha')(T) = (y, w) \right\} . \tag{3.2.1}$$

This problem is strictly convex and coercive, hence, it admits a unique minimiser $\alpha^T_{x,v,y,w}$. This curve satisfies the Euler–Lagrange equation $\alpha'''' \equiv 0$ (i.e., it is a degree-3 polynomial in t) with the prescribed boundary conditions. Straightforward computations yield

$$\alpha^T_{x,v,y,w}(t) = \left(\frac{v + w}{T^2} - 2\frac{y - x}{T^3} \right) t^3 + \left(3\frac{y - x}{T^2} - \frac{2v + w}{T} \right) t^2 + vt + x\,, \qquad t \in (0, T)\,, \tag{3.2.2}$$

or, equivalently,

$$\alpha^T_{x,v,y,w}(\xi T) = x + \xi^2(3 - 2\xi)(y - x) + T\xi(1 - \xi)\Big((1 - \xi)v - \xi w\Big)\,, \qquad \xi \in (0, 1)\,.$$

We thus find the identity (3.1.4), i.e., the minimal value of (3.2.1) coincides with

$$\tilde{d}_T^2\Big((x,v),(y,w)\Big) \coloneqq 12\left|\frac{y-x}{T}-\frac{v+w}{2}\right|^2 + |w-v|^2 \,. \tag{3.2.3}$$

Remark 3.2.1. It was shown by Kolmogorov [Kol34] that the function

$$\Psi\Big((x,v),(y,w),t\Big) \coloneqq \left(\frac{3}{2\pi t^2}\right)^d \exp\left(-\frac{\tilde{d}_t^2\Big((x,v),(y,w)\Big)}{4t}\right)$$

is the fundamental solution to the Kramers equation

$$\partial_t\Psi + v\cdot\nabla_x\Psi = \Delta_v\Psi \,.$$

Remark 3.2.2. Contrary to the minimal L^2-norm of the velocity (see (3.1.3)), the function $\tilde{d}_T$ is not a distance on Γ. Indeed, it is not symmetric, does not vanish on the diagonal of $\Gamma\times\Gamma$ (i.e., points $(x,v)=(y,w)$), and does not satisfy the triangle inequality. The latter can be easily checked on

$$(x_1,v_1)\coloneqq(0,\bar{v})\,,\qquad (x_2,v_2)\coloneqq(T\bar{v},\bar{v})\,,\qquad (x_3,v_3)\coloneqq(2T\bar{v},\bar{v})\,,$$

which satisfy $\tilde{d}_T\Big((x_1,v_1),(x_2,v_2)\Big)=\tilde{d}_T\Big((x_2,v_2),(x_3,v_3)\Big)=0$, while $\tilde{d}_T\Big((x_1,v_1),(x_3,v_3)\Big)=\sqrt{12}\,|\bar{v}|\neq 0$ for any $\bar{v}\in\mathcal{V}\setminus\{0\}$. However, as observed in [GW09], we have the equivalence

$$\tilde{d}_T^2\Big((x,v),(y,w)\Big)=0 \quad\iff\quad y=x+Tv \text{ and } v=w\,. \tag{3.2.4}$$

In this case, following [GW09], we say that (y,w) is the T-*free transport* of (x,v) and write $(y,w)=\mathcal{G}_T(x,v)$. Note that $(\mathcal{G}_T)_{T\geq 0}$ enjoys the semigroup property

$$\mathcal{G}_{T_1}\circ\mathcal{G}_{T_2}=\mathcal{G}_{T_1+T_2}\,,\qquad T_1,T_2\geq 0\,. \tag{3.2.5}$$

Remark 3.2.3. The fact that $\tilde{d}_T\Big((x,v),(y,w)\Big)$ is finite for every $(x,v),(y,w)\in\Gamma$ can be seen as an elementary version of *hypoellipticity* (see [Hör67]). In fact, solutions to Newton's equation $\dot{x}_t=v_t\,,\ \dot{v}_t=F_t$ are generated by vector fields in the space $\{(v,F)\,:\,F\in\mathbb{R}^n\}$. This vector space has dimension n, but it generates a Lie algebra of full rank $2n$:

$$\Big[(v,0),(v,e_i)\Big]=(e_i,0)\,,\qquad i\in\{1,\dots,n\}\,.$$

Remark 3.2.4. Fix $(x,v),(y,w)\in\Gamma$. Let $(v_t,F_t)_{t\in(0,T)}$ be the solution to Newton's equations, connecting (x,v) to (y,w) with the minimal action (i.e., $F_t=\Big(\alpha^T_{x,v,y,w}\Big)''$). The first norm in (3.2.3) (i.e., $\left|\frac{y-x}{T}-\frac{v+w}{2}\right|$) is the distance between the average velocity $⨍_0^T v_t\,\mathrm{d}t$ and the arithmetic mean of the velocities at the endpoints. The second norm (i.e., $|w-v|$) can be written as $\left|\int_0^T F_t\,\mathrm{d}t\right|$.

Remark 3.2.5. It is interesting to analyse the behaviour of the optimal curves $\alpha^T_{x,v,y,w}$ for large T. When $t,T\to\infty$ with $t\geq a\,T$ for some $a>0$, the formula (3.2.2) gives

$$\begin{aligned}\Big(\alpha^T_{x,v,y,w}\Big)'(t)&=\frac{t^2}{T^2}(3v+3w)-\frac{t}{T}(4v+2w)+v+\left(-\frac{t^2}{T^2}+\frac{t}{T}\right)\frac{6(y-x)}{T}\\&=\xi^2(3v+3w)-\xi\,(4v+2w)+\xi\,(1-\xi)\frac{6(y-x)}{T}\,,\qquad \xi\coloneqq t/T\,.\end{aligned}$$

Therefore, two cases may occur.

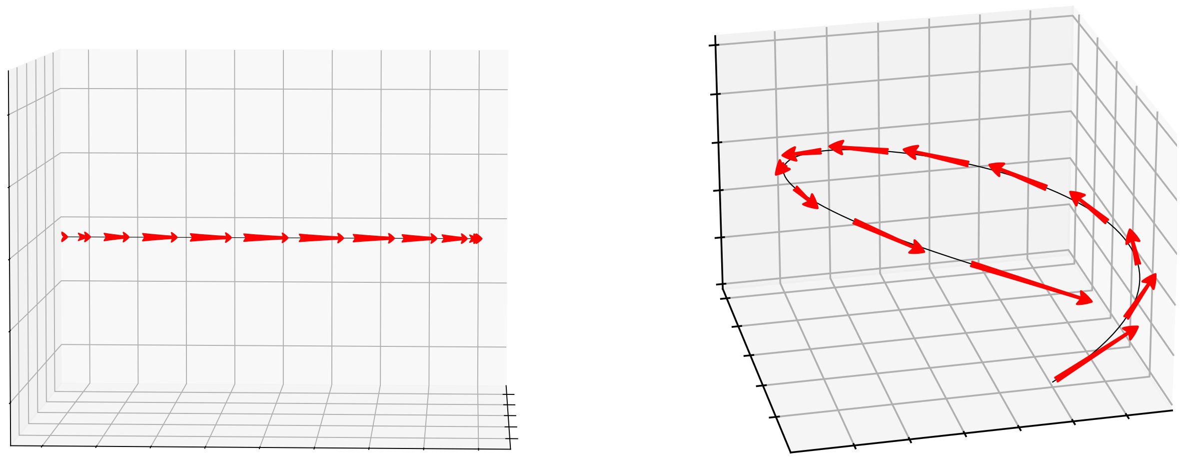

Figure 3.1: Examples of trajectories. In the left figure, $v = w = 0$.

- **Case 1: slow line.** If the vector-valued polynomial
$$p(\xi) := \xi^2(3v + 3w) - \xi\,(4v + 2w) + v$$
is identically equal to 0, then $v = w = 0$, the curve $t \mapsto \alpha^T_{x,0,y,0}(t)$ is a parametrisation of the segment connecting $(x, 0)$ to $(y, 0)$, and $\left(\alpha_{x,0,y,0}\right)'(t) = O(T^{-1})$ uniformly in $t \in (0, T)$ as $T \to \infty$.

- **Case 2: long curve.** If $\xi \mapsto p(\xi)$ is not identically equal to 0, then there must exist an interval $[a, b] \subseteq [0, 1]$ (with $0 \le a < b \le 1$) where it never vanishes. On such an interval, we have
$$\left|\left(\alpha^T_{x,v,y,w}\right)'(t)\right| \ge \min_{\xi\in[a,b]} |p(\xi)| - O(T^{-1})\,, \qquad \text{as } T \to \infty\,, \quad \text{uniformly in } t \in [a, b]\,,$$
which shows that *the total length of* $\alpha^T_{x,v,y,w}$ *is of order* $\Theta(T)$. On the other hand, the curvature is bounded as
$$\kappa(t) \le \frac{\left|\left(\alpha^T_{x,v,y,w}\right)''(t)\right|}{\left|\left(\alpha^T_{x,v,y,w}\right)'(t)\right|^2} = \frac{T^{-1}|p'(t/T)| + O(T^{-2})}{|p(t/T)|^2 + O(T^{-1})} = O(T^{-1}) \qquad \text{as } T \to \infty\,,$$
uniformly in $t \in [a, b]$. Moreover, in dimension $d \ge 2$, we can often choose $[a, b] = [0, 1]$. This is because the set
$$\{(v, w) \in \mathcal{V} \times \mathcal{V} \,:\, \exists \xi \in [0, 1] \text{ with } p(\xi) = 0\}$$
has dimension $d+1$ (hence, it is negligible). All these observations indicate that, typically, this second case corresponds to $\alpha^T_{x,v,y,w}$ resembling a large loop in the limit $T \to \infty$.

We conclude this section with a version of the Monge–Mather's shortening principle, cf. [Vil09b, Chapter 8]. Namely, we show that, given the initial and final configurations of two indistinguishable particles[3] in different locations, their optimal trajectories for the minimal acceleration problem cannot meet *at the same time, at the same point, with the same velocity.*

[3] We also optimise—with respect to the average squared acceleration—how particles in the initial configuration are coupled with those in the final configuration. Namely, a coupling is a matching of each particle in the initial configuration to one in the final configuration. Then, for every pair, one can compute the minimal acceleration as in (3.2.1), and average such contribution over all pairs.

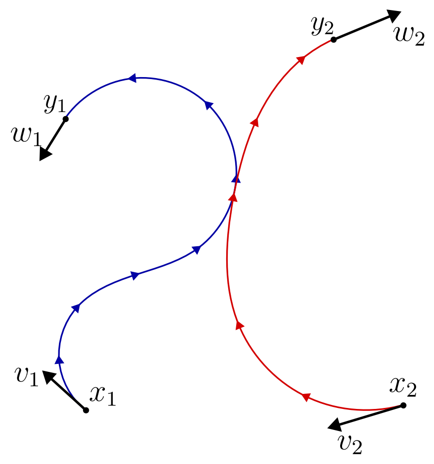

Figure 3.2: Trajectories that meet at the same time with the same velocity are not optimal.

Proposition 3.2.6. *Fix $T > 0$ and let $(x_1, v_1), (y_1, w_1), (x_2, v_2), (y_2, w_2) \in \Gamma$. Let α_1, α_2 be the optimal (polynomial) curves for the problem (3.1.4) between (x_1, v_1) and (y_1, w_1), and between (x_2, v_2) and (y_2, w_2), respectively, i.e.,*

$$\tilde{d}_T^2\Big((x_1, v_1), (y_1, w_1)\Big) = T \int_0^T \Big|\alpha_1''(t)\Big|^2 \,\mathrm{d}t \quad \textit{and} \quad \tilde{d}_T^2\Big((x_2, v_2), (y_2, w_2)\Big) = T \int_0^T \Big|\alpha_2''(t)\Big|^2 \,\mathrm{d}t\,. \tag{3.2.6}$$

If there exists $\bar{t} \in (0, T)$ such that $(\alpha_1(\bar{t}), \alpha_1'(\bar{t})) = (\alpha_2(\bar{t}), \alpha_2'(\bar{t}))$, and if

$$\begin{aligned}\tilde{d}_T^2\Big((x_1, v_1), (y_1, w_1)\Big) &+ \tilde{d}_T^2\Big((x_2, v_2), (y_2, w_2)\Big) \\ &\leq \tilde{d}_T^2\Big((x_1, v_1), (y_2, w_2)\Big) + \tilde{d}_T^2\Big((x_2, v_2), (y_1, w_1)\Big)\,,\end{aligned} \tag{3.2.7}$$

then, $(x_1, v_1) = (x_2, v_2)$ and $(y_1, w_1) = (y_2, w_2)$.

Proof. Define the curves

$$\tilde{\alpha}_1(t) := \begin{cases} \alpha_1(t) & \text{if } t \in [0, \bar{t}]\,, \\ \alpha_2(t) & \text{if } t \in [\bar{t}, T]\,, \end{cases} \qquad \tilde{\alpha}_2(t) := \begin{cases} \alpha_2(t) & \text{if } t \in [0, \bar{t}]\,, \\ \alpha_1(t) & \text{if } t \in [\bar{t}, T]\,, \end{cases}$$

which, by our assumptions, are of class H^2. They are competitors for the problem (3.2.1) between $X_1 := (x_1, v_1)$ and $Y_2 := (y_2, w_2)$, and between $X_2 := (x_2, v_2)$ and $Y_1 := (y_1, w_1)$, respectively. We also notice that, by additivity of the integral in the domain of integration,

$$\int_0^T \Big|\alpha_1''(t)\Big|^2 \,\mathrm{d}t + \int_0^T \Big|\alpha_2''(t)\Big|^2 \,\mathrm{d}t = \int_0^T \Big|\tilde{\alpha}_1''(t)\Big|^2 \,\mathrm{d}t + \int_0^T \Big|\tilde{\alpha}_2''(t)\Big|^2 \,\mathrm{d}t. \tag{3.2.8}$$

Exploiting the assumption (3.2.7), we obtain

$$\begin{aligned}\tilde{d}_T^2(X_1, Y_1) + \tilde{d}_T^2(X_2, Y_2) &\overset{(3.2.7)}{\leq} \tilde{d}_T^2(X_1, Y_2) + \tilde{d}_T^2(X_2, Y_1) \\ &\leq T \int_0^T \Big|\tilde{\alpha}_1''(t)\Big|^2 \,\mathrm{d}t + T \int_0^T \Big|\tilde{\alpha}_2''(t)\Big|^2 \,\mathrm{d}t \\ &\overset{(3.2.8)}{=} T \int_0^T \Big|\alpha_1''(t)\Big|^2 \,\mathrm{d}t + T \int_0^T \Big|\alpha_2''(t)\Big|^2 \,\mathrm{d}t \\ &= \tilde{d}_T^2(X_1, Y_1) + \tilde{d}_T^2(X_2, Y_2)\,.\end{aligned}$$

We infer that the two inequalities in the latter formula are in fact equalities and, therefore, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are *optimal* for the problem (3.2.1). Consequently, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are polynomials. Since $\alpha_1(t) = \tilde{\alpha}_1(t)$ for $t \in [0, \bar{t}]$ and $\bar{t} > 0$, the two polynomials α_1 and $\tilde{\alpha}_1$ coincide. Similarly, $\tilde{\alpha}_1 \equiv \alpha_2$; therefore, $\alpha_1 \equiv \alpha_2$. We conclude that

$$(x_1, v_1) = \alpha_1(0) = \alpha_2(0) = (x_2, v_2) \quad \text{and} \quad (y_1, w_1) = \alpha_1(T) = \alpha_2(T) = (y_2, w_2)\,. \qquad \square$$

3.2.2 The non-parametric discrepancy

Using $\tilde{d}_T$, we shall now define a discrepancy d which is not parametric in time.

Definition 3.2.7. For all $(x, v), (y, w) \in \Gamma$, set

$$\tilde{d}\big((x, v), (y, w)\big) := \inf_{T>0} \tilde{d}_T\big((x, v), (y, w)\big)\,. \tag{3.2.9}$$

We denote by $d\colon \Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$ the lower-semicontinuous envelope of $\tilde{d}$. We give d the name *second-order discrepancy* between particles.

Proposition 3.2.8. *The following hold.*

1. *The function $\tilde{d}\colon \Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$ is upper-semicontinuous. For every $(x, v), (y, w) \in \Gamma$, we have*

$$\tilde{d}\big((x, v), (y, w)\big) = \begin{cases} \lim_{T\to\infty} \tilde{d}_T\big((x, v), (y, w)\big) & \text{if } (y - x) \cdot (v + w) \leq 0\,, \\ \tilde{d}_{T^*}\big((x, v), (y, w)\big) & \text{if } (y - x) \cdot (v + w) > 0\,, \end{cases} \tag{3.2.10}$$

   where

$$T^* := 2\frac{|y - x|^2}{(y - x) \cdot (v + w)}\,. \tag{3.2.11}$$

   Hence, the following formula

$$\tilde{d}^2\big((x, v), (y, w)\big) = \begin{cases} 3|v + w|^2 - 3\left(\frac{y-x}{|y-x|} \cdot (v + w)\right)_+^2 + |w - v|^2 & \text{if } x \neq y\,, \\ 3|v + w|^2 + |w - v|^2 & \text{if } x = y \end{cases}$$
$$(x, v), (y, w) \in \Gamma\,. \tag{3.2.12}$$

2. *The second-order discrepancy $d\colon \Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$ is given by the formula*

$$d^2\big((x, v), (y, w)\big) = \begin{cases} 3|v + w|^2 - 3\left(\frac{y-x}{|y-x|} \cdot (v + w)\right)_+^2 + |w - v|^2 & \text{if } x \neq y\,, \\ |w - v|^2 & \text{if } x = y\,, \end{cases}$$
$$(x, v), (y, w) \in \Gamma\,. \tag{3.2.13}$$

3. *We have $d\big((x, v), (y, w)\big) = 0$ if and only if either $(y, w) = \mathcal{G}_T(x, v)$ for some $T \geq 0$, or $x \neq y$ and $v = w = 0$.*

Proof. **Proof of 1.** Upper-semicontinuity is trivial, because $\tilde{d}$ is defined as an infimum of continuous functions.

Fix $(x,v),(y,w) \in \Gamma$. If $y = x$, then $T \mapsto \tilde{d}_T^2\big((x,v),(y,w)\big)$ is constant, hence always equal to its limit as $T \to \infty$. Otherwise, let us rewrite (3.2.3) as a convex quadratic polynomial in T^{-1}, namely:

$$\tilde{d}_T^2\big((x,v),(y,w)\big) = 12T^{-2}|y-x|^2 - 24T^{-1}(y-x)\cdot(v+w) + 3|v+w|^2 + |w-v|^2 .$$

The vertex of this parabola is found at

$$T^{-1} = \frac{(y-x)\cdot(v+w)}{2|y-x|^2} .$$

Therefore, when $(y-x)\cdot(v+w) \le 0$, the minimum of $\tilde{d}_T^2$, constrained to $T > 0$, is approached as $T^{-1} \to 0$. In formulae:

$$\tilde{d}^2\big((x,v),(y,w)\big) = \lim_{T\to\infty} \tilde{d}_T^2\big((x,v),(y,w)\big) = 3|v+w|^2 + |w-v| .$$

Instead, when $(y-x)\cdot(v+w) > 0$, we have

$$\tilde{d}^2\big((x,v),(y,w)\big) = \tilde{d}_{T^*}^2\big((x,v),(y,w)\big) = 3|v+w|^2 - 3\left(\frac{y-x}{|y-x|}\cdot(v+w)\right)^2 + |w-v|^2 .$$

Proof of 2. The right-hand side in (3.2.13) is lower-semicontinuous. Since it coincides with $\tilde{d}\big((x,v),(y,w)\big)$ when $x \ne y$ or $v + w = 0$, we are only left with showing that, for every x, v, w with $v + w \ne 0$, there exists a sequence $y_k \to x$ such that

$$|w-v|^2 \ge \limsup_{k\to\infty} \tilde{d}^2\big((x,v),(y_k,w)\big) .$$

We simply choose

$$y_k \coloneqq x + \frac{1}{k}(v+w) , \qquad k \in \mathbb{N}_1 .$$

Proof of 3. Assume that the right-hand side of (3.2.13) equals 0. We infer that $v = w$. If $x = y$, then $(y,w) = \mathcal{G}_0(x,v)$. If $x \ne y$, then

$$2|v| = 2\frac{y-x}{|y-x|}\cdot v$$

and, therefore, either $v = 0$, or $y = x + Tv$ for some $T > 0$, that is, $(y,w) = \mathcal{G}_T(x,v)$. The converse implication is a direct computation. □

Remark 3.2.9. It follows from (3.2.12) and (3.2.13) that neither $\tilde{d}$ nor d is symmetric. Neither of the two satisfies the triangle inequality: consider

$$(x_1, v_2) \coloneqq (0, \bar{v}) , \qquad (x_2, v_2) \coloneqq (\bar{v}, 0) , \qquad (x_3, v_3) \coloneqq (-\bar{v}, 0)$$

for any $\bar{v} \in \mathcal{V} \setminus \{0\}$. Moreover, we have the characterisation

$$d^2\big((x,v),(y,w)\big) = 0 \qquad \Longleftrightarrow \qquad [\, v = w = 0 \text{ or } (y,w) = \mathcal{G}_T(x,v) \text{ for some } T \ge 0\,] . \tag{3.2.14}$$

In analogy with the metric setting of [AGS08, Theorem 1.1.2]), we give the following.

Definition 3.2.10. We say that a curve $\gamma = (x_\cdot, v_\cdot)\colon (a,b) \to \Gamma$ is *d-differentiable* at $t \in (a,b)$ when the one-sided limit

$$\lim_{h\downarrow 0} \frac{d\big(\gamma(t), \gamma(t+h)\big)}{h} \,. \tag{3.2.15}$$

exists. In this case, we denote it by $|\gamma'|_d\,(t)$ and call it the *d-derivative of* γ at t.

Remark 3.2.11. As the discrepancy d is not symmetric, taking left or right limits to define d-differentiability is not the same, even for smooth curves. We argue that (3.2.15) is the natural definition. Indeed, if, for example, we consider the straight constant-speed line $\gamma(t) \coloneqq (t\bar v, \bar v)$, $t \in \mathbb{R}$, for some $\bar v \in \mathcal{V} \setminus \{0\}$, we have

$$d\big(\gamma(t), \gamma(t+h)\big) = 0 \quad \text{for every } h > 0\,,$$

and

$$d\big(\gamma(t), \gamma(t-h)\big) = |2v| > 0 \quad \text{for every } h > 0\,.$$

In particular, γ is d-differentiable in the sense of Definition 3.2.10, but

$$\lim_{h\downarrow 0} \frac{d\big(\gamma(t), \gamma(t-h)\big)}{|h|} = \infty\,.$$

Our next aim is to formulate necessary and sufficient conditions for d-differentiability.

Proposition 3.2.12. *Let $\gamma = (x_\cdot, v_\cdot)\colon (a,b) \to \Gamma$ be a curve such that $x_\cdot$ is of class C^1 and $v_\cdot$ is of class C^0. If γ is d-differentiable at $t \in (a,b)$ and $v_t \neq 0$, then there exists $\lambda(t) \geq 0$ such that $\dot x_t = \lambda(t) v_t$.*

Proof. If γ is d-differentiable at $t \in (a,b)$, then $d\big(\gamma(t), \gamma(t+h)\big) \leq Ch$ for suitable $C > 0$, whenever $h > 0$ is small enough. If $\dot x_t = 0$, then it suffices to choose $\lambda(t) = 0$. Otherwise, we have $x_{t+h} \neq x_t$ for small enough $h > 0$, hence $d^2\big(\gamma(t), \gamma(t+h)\big) = \tilde d^2\big(\gamma(t), \gamma(t+h)\big)$. Pick $T(h) > 0$ such that $\tilde d^2_{T(h)}\big(\gamma(t), \gamma(t+h)\big) \leq \tilde d^2\big(\gamma(t), \gamma(t+h)\big) + h^2$. In particular, for $h > 0$ small enough,

$$\begin{aligned} 12\left|\frac{x_{t+h} - x_t}{T(h)} - \frac{v_t + v_{t+h}}{2}\right|^2 &\leq \tilde d^2_{T(h)}\big(\gamma(t), \gamma(t+h)\big) \leq \tilde d^2\big(\gamma(t), \gamma(t+h)\big) + h^2 \\ &= d^2\big(\gamma(t), \gamma(t+h)\big) + h^2 \leq (C^2+1)h^2\,. \end{aligned}$$

Since $\frac{x_{t+h}-x_t}{h} \to \dot x_t \neq 0$ and $\frac{v_t + v_{t+h}}{2} \to v_t \neq 0$, we infer that $T(h)/h \to \lambda(t)$ as $h \to 0$ for some $\lambda(t) \in (0,\infty)$ satisfying $\dot x_t = \lambda(t) v_t$. □

Remark 3.2.13 (Reparametrisation)**.** Let $\gamma = (x_\cdot, v_\cdot)\colon (a,b) \to \Gamma$ be a curve such that $x_\cdot$ is of class C^{k+1} and $v_\cdot$ is of class C^k for some $k \in \mathbb{N}_0$. Assume that $\dot x_t = \lambda(t) v_t$ for every $t \in (a,b)$, for a function $\lambda\colon (a,b) \to (0,\infty)$ of class C^k. Let $\tau\colon (\tilde a, \tilde b) \to (a,b)$ be a function of class C^{k+1} with $\tau' > 0$. Set

$$\tilde x_s \coloneqq x_{\tau(s)}\,, \qquad \tilde v_s \coloneqq v_{\tau(s)}\,, \qquad \tilde\gamma(s) \coloneqq (\tilde x_s, \tilde v_s)\,, \qquad s \in (\tilde a, \tilde b)\,.$$

Then, we have

$$\tilde{v}_s = v_{\tau(s)} = \frac{1}{\lambda\big(\tau(s)\big)}\dot{x}_{\tau(s)} = \frac{1}{\lambda\big(\tau(s)\big)\tau'(s)}\,\dot{\tilde{x}}_s\,, \qquad s \in (\tilde{a},\tilde{b})\,,$$

that is, also $\tilde{\gamma}$ has the property $\dot{\tilde{x}}_s = \tilde{\lambda}(s)\tilde{v}_s$ for every $s \in (\tilde{a},\tilde{b})$, for a function $\tilde{\lambda}\colon (\tilde{a},\tilde{b}) \to (0,\infty)$ of class C^k.

Let $\bar{t} \in (a,b)$ and assume that $\lambda > 0$ on (a,b). By solving the Cauchy problem

$$\tau' = \frac{1}{\lambda(\tau)}\,, \qquad \tau(0) = \bar{t}\,, \tag{3.2.16}$$

we can find a reparametrisation $\tilde{\gamma}$ with $\dot{\tilde{x}}_s = \tilde{v}_s$. Indeed, by classical results, the ODE (3.2.16) admits a maximal solution on a neighbourhood of $\bar{t}$. Moreover, since λ is bounded on every compact $K \Subset (a,b)$, the values of τ exit K in finite time. Therefore, the maximal solution has the full set (a,b) as its image.

Proposition 3.2.14. *Let $\gamma = (x_\cdot, v_\cdot)\colon (a,b) \to \Gamma$ be a curve such that $x_\cdot$ is of class C^2 and $v_\cdot$ is of class C^1. Assume that there exists $\lambda\colon (a,b) \to (0,\infty)$ continuous, such that $\dot{x}_t = \lambda(t)v_t$ for every $t \in (a,b)$. Then, γ is d-differentiable on (a,b) with $|\gamma'|_d(t) = |\dot{v}_t|$.*

Proof. On the one hand, we have

$$\liminf_{h\downarrow 0} \frac{d^2\big(\gamma(t),\gamma(t+h)\big)}{h^2} \geq \liminf_{h\downarrow 0} \frac{|v_t - v_{t+h}|^2}{h^2} = |\dot{v}_t|^2\,, \qquad t \in (a,b)\,.$$

To prove the opposite inequality, momentarily assume that $\lambda \equiv 1$, i.e., $\dot{x}_t = v_t$ for every $t \in (a,b)$. We obtain

$$\begin{aligned}\limsup_{h\downarrow 0} \frac{d^2\big(\gamma(t),\gamma(t+h)\big)}{h^2} &\leq \limsup_{h\downarrow 0} \frac{\tilde{d}_h^2\big(\gamma(t),\gamma(t+h)\big)}{h^2}\\ &= \limsup_{h\downarrow 0} \frac{1}{h^2}\left(12\left|\frac{x_{t+h}-x_t}{h} - \frac{v_{t+h}+v_t}{2}\right|^2 + |v_{t+h}-v_t|^2\right)\\ &= |\dot{v}_t|^2 + \limsup_{h\downarrow 0} \frac{12}{h^2}\left|\dot{x}_t + \frac{h}{2}\ddot{x}_t - v_t - \frac{h}{2}\dot{v}_t + o(h)\right|^2 = |\dot{v}_t|^2\end{aligned} \tag{3.2.17}$$

for every $t \in (a,b)$. In the general case, we apply the reparametrisation of Remark 3.2.13 to find a diffeomorphism $\tau\colon (\tilde{a},\tilde{b}) \to (a,b)$ such that $\tau'(s) = \frac{1}{\lambda(\tau(s))}$ for every $s \in (\tilde{a},\tilde{b})$, so that the computation (3.2.17) can be performed on $\tilde{\gamma}\colon s \mapsto \gamma\big(\tau(s)\big)$. Given $t = \tau(s) \in (a,b)$, we thus find

$$\begin{aligned}\limsup_{h\downarrow 0}\frac{d^2\big(\gamma(t),\gamma(t+h)\big)}{h^2}&=\limsup_{h\downarrow 0}\frac{d^2\Big(\gamma\big(\tau(s)\big),\gamma\big(\tau(s)+h\big)\Big)}{h^2}\\&=\limsup_{\tilde h\downarrow 0}\frac{d^2\Big(\gamma\big(\tau(s)\big),\gamma\big(\tau(s+\tilde h)\big)\Big)}{\big(\tau(s+\tilde h)-\tau(s)\big)^2}\\&=\limsup_{\tilde h\downarrow 0}\frac{d^2\big(\tilde\gamma(s),\tilde\gamma(s+\tilde h)\big)}{\tilde h^2}\frac{\tilde h^2}{\big(\tau(s+\tilde h)-\tau(s)\big)^2}\\&\overset{(3.2.17)}{=}\lambda\big(\tau(s)\big)^2\left|\frac{\mathrm{d}}{\mathrm{d}s}v_{\tau(s)}\right|^2=|\dot v_t|^2\ ,\end{aligned}$$

and this concludes the proof. □

Remark 3.2.15. The last result, specialised to curves satisfying $\dot x_t = v_t$ (i.e., with $\lambda\equiv 1$), yields $|\gamma'|_d\,(t) = |\dot v_t| = |\ddot x_t|$, for all $t\in(a,b)$. In view of Newton's second law of motion, we can say that the d-derivative equals the magnitude of the force driving the motion.

Remark 3.2.16. A curve $\gamma=(x_\cdot,v_\cdot)\to\Gamma$ is everywhere d-differentiable also when $v_\cdot\equiv 0$, regardless of $x_\cdot$. In this case, $|\gamma'|_d\equiv 0$.

Corollary 3.2.17. *Let $\gamma=(x_\cdot,v_\cdot)\colon(a,b)\to\Gamma$ be a curve such that $x_\cdot$ is of class C^2 and $v_\cdot$ is of class C^1. Assume that $\dot x_t\neq 0$ and $v_t\neq 0$ for every $t\in(a,b)$. Then, γ is everywhere d-differentiable if and only if there exists $\lambda\colon(a,b)\to(0,\infty)$ of class C^1 such that $\dot x_t=\lambda(t)v_t$ for every $t\in(a,b)$.*

Proof. If γ is everywhere d-differentiable, by Proposition 3.2.12, there exists $\lambda\colon(a,b)\to\mathbb{R}_{\geq 0}$ such that $\dot x_t=\lambda(t)v_t$. This function is of class C^1 because $v_t\neq 0$ for every $t\in(a,b)$ and because both $\dot x_\cdot$ and $v_\cdot$ are of class C^1. Moreover $\lambda(t)\neq 0$ for every t, because this property holds for $\dot x_t$. The converse follows from Proposition 3.2.14. In this case, $|\gamma'|_d\,(t)=|\dot v_t|$. □

3.3 Kinetic optimal plans and maps

This section is divided into three parts:

1. In §3.3.1, we prove Statements 1-3 in Theorem 3.1.1, including the semicontinuity of d and the existence of optimal transport *plans*.
2. In §3.3.2, we prove Statement 4 in Theorem 3.1.1, i.e., the existence of optimal *maps*.
3. In §3.3.3, we discuss additional results, including *non-uniqueness* of optimal plans and maps, and the characterisation of the pairs (μ,ν) for which $\mathsf{d}(\mu,\nu)=0$.

3.3.1 Semicontinuity and existence of optimal plans

Let $\mathcal{P}_2(\Gamma\times\Gamma)$ be the set of probability measures on $\Gamma\times\Gamma$ with finite second moment. For every $\pi\in\mathcal{P}_2(\Gamma\times\Gamma)$, set

$$\tilde{\mathsf{c}}_T(\pi) := 12\left\|\frac{y-x}{T}-\frac{v+w}{2}\right\|^2_{\mathrm{L}^2(\pi)} + \|w-v\|^2_{\mathrm{L}^2(\pi)}\,, \qquad T>0 \tag{3.3.1}$$

$$\tilde{\mathsf{c}}(\pi) := \begin{cases} 3\|v+w\|^2_{\mathrm{L}^2(\pi)} - 3\frac{\left((y-x,v+w)_\pi\right)^2_+}{\|y-x\|^2_{\mathrm{L}^2(\pi)}} + \|w-v\|^2_{\mathrm{L}^2(\pi)} & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}>0\,,\\ 3\|v+w\|^2_{\mathrm{L}^2(\pi)} + \|w-v\|^2_{\mathrm{L}^2(\pi)}\,, & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}=0\,,\end{cases} \tag{3.3.2}$$

$$\mathsf{c}(\pi) := \begin{cases} 3\|v+w\|^2_{\mathrm{L}^2(\pi)} - 3\frac{\left((y-x,v+w)_\pi\right)^2_+}{\|y-x\|^2_{\mathrm{L}^2(\pi)}} + \|w-v\|^2_{\mathrm{L}^2(\pi)} & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}>0\,,\\ \|w-v\|^2_{\mathrm{L}^2(\pi)}\,, & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}=0\,.\end{cases} \tag{3.3.3}$$

Note that $\tilde{\mathsf{c}}(\pi)=\mathsf{c}(\pi)$ whenever $\|y-x\|_{\mathrm{L}^2(\pi)}>0$.

Remark 3.3.1. It may appear tempting to consider instead of $\tilde{\mathsf{c}}$ a different object, namely $\hat{\mathsf{c}}(\pi) := \int_{\Gamma\times\Gamma}\tilde{d}^2\big((x,v),(y,w)\big)\,\mathrm{d}\pi$. Let us start by noticing that $\hat{\mathsf{c}}(\pi)$ involves first a pointwise optimisation of $\tilde{d}^2_T((x,v),(y,w))$ in T, for each pair of states (x,v) and (y,w), and then an integration over all pairs $((x,v),(y,w))\in\operatorname{supp}(\pi)$. By contrast, as the next result shows, $\tilde{\mathsf{c}}(\pi)$ is given by one *synchronous* minimisation over $T>0$ for the cost $\tilde{\mathsf{c}}_T(\pi)$. We justify why $\tilde{\mathsf{c}}$ is more natural for our purposes. Let us consider $\mu\in\mathcal{P}_2(\Gamma)$, such that $(\mathrm{pr}_v)_\#\mu\neq\delta_0$. Let $\sigma:\Gamma\to[0,\infty)$ be a measurable map, and let $G_\sigma:\Gamma\to\Gamma$ be defined by the formula $G_\sigma(x,v)=(x+\sigma(x,v),v)$. Then, by calling $\nu_\sigma=(G_\sigma)_\#\mu$, we have that

$$\inf_{\pi\in\Pi(\mu,\nu_\sigma)}\hat{\mathsf{c}}(\pi)=0\,.$$

By contrast,

$$\inf_{\pi\in\Pi(\mu,\nu_\sigma)}\tilde{\mathsf{c}}(\pi)=0$$

if and only if $\sigma\equiv T$, for some $T\in[0,\infty)$. This shows that the optimal-transport problem associated with $\hat{\mathsf{c}}$ is much more degenerate than the one associated with $\tilde{\mathsf{c}}$. Finally, by taking the curve $t\mapsto\mu_t:=\nu_{t\sigma}$, we have that

$$\forall\, 0<t<s\,, \qquad \inf_{\pi\in\Pi(\mu_t,\mu_s)}\hat{\mathsf{c}}(\pi)=0\,,$$

which means that the curve $(\mu_t)_t$ is everywhere differentiable in the optimal-transport discrepancy induced by $\hat{\mathsf{c}}$. On the other hand, it is easy to see that this curve does not solve any Vlasov's equation (3.1.14) in general. Thus, we would not be able to recover the PDE representation of Theorem 3.1.7 in case we used $\hat{\mathsf{c}}$ instead of $\tilde{\mathsf{c}}$.

Proposition 3.3.2 (Theorem 3.1.1, Statement 1)**.** *For every $\pi\in\mathcal{P}_2(\Gamma\times\Gamma)$, we have*

$$\tilde{\mathsf{c}}(\pi)=\inf_{T>0}\tilde{\mathsf{c}}_T(\pi)\,. \tag{3.3.4}$$

The infimum is obtained for

$$\begin{cases} T=2\frac{\|y-x\|^2_{\mathrm{L}^2(\pi)}}{(y-x,v+w)_\pi} & \text{if } (y-x,v+w)_\pi>0\,,\\ \text{any } T>0 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi)}=0\,,\\ T\to\infty & \text{otherwise.}\end{cases} \tag{3.3.5}$$

In particular, the function $\pi \mapsto \tilde{\mathsf{c}}(\pi)$ is concave, and (3.1.11) holds for every $\mu, \nu \in \mathcal{P}_2(\Gamma)$.

Proof. The proof is identical to that of Proposition 3.2.8, 1. The function $\tilde{\mathsf{c}}$ is concave because it is an infimum of linear functions. □

Lemma 3.3.3. *Let $\mu_k \to \mu$ and $\nu_k \to \nu$ be two converging sequences in the 2-Wasserstein distance. Let $\pi_k \in \Pi(\mu_k, \nu_k)$ for every k, and assume that $(\pi_k)_k$ narrowly converges to a measure $\pi \in \mathcal{P}(\Gamma \times \Gamma)$. Then, convergence holds in W_2, we have $\pi \in \Pi(\mu, \nu)$, and*

$$\mathsf{c}(\pi) \leq \liminf_{k \to \infty} \mathsf{c}(\pi_k) \,. \tag{3.3.6}$$

Proof. Convergence holds in W_2 by [AGS08, Remark 7.1.11]. The measure π lies in $\Pi(\mu, \nu)$ by narrow continuity of the projection maps. We claim that the four functions

$$F_1(x,v,y,w) := |v+w|^2 \,, \qquad F_2(x,v,y,w) := |(y-x)\cdot(v+w)| \,,$$
$$F_3(x,v,y,w) := |y-x|^2 \,, \qquad F_4(x,v,y,w) := |w-v|^2$$

are uniformly integrable with respect to $(\pi_k)_k$. Indeed, as

$$F_i(x,v,y,w) \leq 4 \max\left\{|x|^2 + |v|^2, |y|^2 + |w|^2\right\} \,, \qquad i \in \{1,2,3,4\} \,,$$

for $a > 0$, we find that

$$\int_{\{F_i \geq a\}} F_i \,\mathrm{d}\pi_k \leq 4 \int_{\left\{|x|^2 + |v|^2 \geq \frac{a}{4}\right\}} \left(|x|^2 + |v|^2\right) \mathrm{d}\mu_k + 4 \int_{\left\{|y|^2 + |w|^2 \geq \frac{a}{4}\right\}} \left(|y|^2 + |w|^2\right) \mathrm{d}\nu_k \,.$$

Then, the claim follows from the uniform integrability of the second moments of $(\mu_k)_k$ and $(\nu_k)_k$, given by [AGS08, Proposition 7.1.5].

If $\|y - x\|_{\mathrm{L}^2(\pi)} > 0$, then $\|y - x\|_{\mathrm{L}^2(\pi_k)} > 0$ eventually; hence $\mathsf{c}(\pi) = \tilde{\mathsf{c}}(\pi)$ and $\mathsf{c}(\pi_k) = \tilde{\mathsf{c}}(\pi_k)$ for every k sufficiently large. Since the functions F_i are uniformly integrable, through [AGS08, Lemma 5.1.7], we find that $\tilde{\mathsf{c}}(\pi_k) \to \tilde{\mathsf{c}}(\pi)$. Therefore,

$$\mathsf{c}(\pi) = \tilde{\mathsf{c}}(\pi) = \lim_{k \to \infty} \tilde{\mathsf{c}}(\pi_k) = \lim_{k \to \infty} \mathsf{c}(\pi_k) \,.$$

If, instead, $\|y - x\|_{\mathrm{L}^2(\pi)} = 0$, then

$$\mathsf{c}(\pi) = \|w - v\|^2_{\mathrm{L}^2(\pi)} \leq \liminf_{k \to \infty} \|w - v\|^2_{\mathrm{L}^2(\pi_k)} \leq \liminf_{k \to \infty} \mathsf{c}(\pi_k) \,.$$

□

Proposition 3.3.4 (Theorem 3.1.1, Statement 2)**.** *The lower-semicontinuous envelope of $\tilde{\mathsf{d}}$ w.r.t. the 2-Wasserstein distance over $\mathcal{P}_2(\Gamma)$ is the discrepancy d.*

Proof. Firstly, let us show that d is lower-semicontinuous. Let $\mu_k \to \mu$ and $\nu_k \to \nu$ be two W_2-convergent sequences. For every $k \in \mathbb{N}$, choose $\pi_k \in \Pi(\mu_k, \nu_k)$ so that we have $\left|\mathsf{c}(\pi_k) - \mathsf{d}^2(\mu_k, \nu_k)\right| \to 0$. By Prokhorov's theorem, see [Bog07, Theorem 8.6.2], up to subsequences, $(\pi_k)_k$ is narrowly convergent to a certain measure π. By Lemma 3.3.3 we deduce that $\pi \in \Pi(\mu, \nu)$, and therefore

$$\mathsf{d}^2(\mu, \nu) \overset{(3.1.10)}{\leq} \mathsf{c}(\pi) \overset{(3.3.6)}{\leq} \liminf_{k \to \infty} \mathsf{c}(\pi_k) = \liminf_{k \to \infty} \mathsf{d}^2(\mu_k, \nu_k) \,.$$

Secondly, we shall find sequences $\bar{\mu}_k \to \mu$ and $\bar{\nu}_k \to \nu$ (w.r.t. W_2) such that

$$\mathsf{d}^2(\mu,\nu) \geq \limsup_{k\to\infty} \tilde{\mathsf{d}}^2(\bar{\mu}_k,\bar{\nu}_k)\,.$$

Let $(\bar{\pi}_k)_{k\in\mathbb{N}} \subseteq \Pi(\mu,\nu)$ be such that $\mathsf{c}(\bar{\pi}_k) \to \mathsf{d}^2(\mu,\nu)$ as $k\to\infty$. If $\mathsf{c}(\bar{\pi}_k) = \tilde{\mathsf{c}}(\bar{\pi}_k)$ for infinitely many k's, then, up to subsequences,

$$\mathsf{d}^2(\mu,\nu) = \lim_{k\to\infty} \mathsf{c}(\bar{\pi}_k) = \lim_{k\to\infty} \tilde{\mathsf{c}}(\bar{\pi}_k) \geq \tilde{\mathsf{d}}^2(\mu,\nu)\,,$$

i.e., it suffices to take the constant sequences $\bar{\mu}_k := \mu$ and $\bar{\nu}_k := \nu$. Otherwise, up to subsequences, we have $\mathsf{c}(\bar{\pi}_k) < \tilde{\mathsf{c}}(\bar{\pi}_k)$ for every k, which implies that

$$\|y-x\|_{\mathrm{L}^2(\bar{\pi}_k)} = 0 \quad \text{and} \quad \|v+w\|_{\mathrm{L}^2(\bar{\pi}_k)} > 0\,, \qquad k\in\mathbb{N}\,. \tag{3.3.7}$$

In this case, we set

$$R_k(x,v,y,w) := \left(x, v, y+\frac{v+w}{k+1}, w\right), \qquad \tilde{\pi}_k := (R_k)_\#\bar{\pi}_k\,, \qquad k\in\mathbb{N}\,, \tag{3.3.8}$$

as well as

$$\bar{\mu}_k := (\mathrm{pr}_{x,v})_\#\tilde{\pi}_k = \mu\,, \qquad \bar{\nu}_k := (\mathrm{pr}_{y,w})_\#\tilde{\pi}_k \in \mathcal{P}_2(\Gamma)\,, \qquad k\in\mathbb{N}\,.$$

From (3.3.7) and (3.3.8), it follows that $\|y-x\|_{\mathrm{L}^2(\tilde{\pi}_k)} = \frac{1}{k+1}\|v+w\|_{\mathrm{L}^2(\bar{\pi}_k)} > 0$, and since $x=y$ holds $\bar{\pi}_k$-a.e., we infer that $y-x = \frac{v+w}{k+1}$ holds $\tilde{\pi}_k$-a.e. Consequently,

$$\tilde{\mathsf{c}}(\tilde{\pi}_k) = \|w-v\|^2_{\mathrm{L}^2(\tilde{\pi}_k)} = \|w-v\|^2_{\mathrm{L}^2(\bar{\pi}_k)}\,.$$

Thus,

$$\tilde{\mathsf{d}}^2(\bar{\mu}_k,\bar{\nu}_k) \leq \tilde{\mathsf{c}}(\tilde{\pi}_k) = \|w-v\|^2_{\mathrm{L}^2(\bar{\pi}_k)} \leq \mathsf{c}(\bar{\pi}_k) \to \mathsf{d}^2(\mu,\nu) \quad \text{as } k\to\infty\,.$$

Using that $(\mathrm{pr}_{y,w}, \mathrm{pr}_{y,w}\circ R_k)_\#\bar{\pi}_k \in \Pi(\nu,\bar{\nu}_k)$, we find

$$\mathrm{W}_2(\nu,\bar{\nu}_k) \leq \frac{\|v+w\|_{\mathrm{L}^2(\bar{\pi}_k)}}{k+1} \leq \frac{\|v\|_{\mathrm{L}^2(\mu)} + \|v\|_{\mathrm{L}^2(\nu)}}{k+1} \to 0 \quad \text{as } k\to\infty\,,$$

which shows that $\bar{\nu}_k \to \nu$, as desired. □

Proposition 3.3.5 (Theorem 3.1.1, Statement 3)**.** *Problem* (3.1.10) *admits a minimiser.*

Proof. By Lemma 3.3.3, the function c is narrowly lower-semicontinuous on $\Pi(\mu,\nu)$. The set $\Pi(\mu,\nu)$ is narrowly compact by Prokhorov's theorem, hence a minimiser of c exists. □

We will denote by $\Pi_{\mathrm{o,d}}(\mu,\nu)$ the set of minimisers. An analogue of Proposition 3.3.5 does not hold for $\tilde{\mathsf{d}}$, namely, it is possible that no minimiser in (3.1.9) exists.

Example 3.3.6. Let $n=2$. For every $\epsilon \geq 0$ and $t\in\mathbb{R}$, set

$$\begin{aligned} M^x_\epsilon(t) &:= (\sin 2\pi(t+\epsilon), \cos 2\pi(t+\epsilon)) \in \mathcal{X}\,,\\ M^v_\epsilon(t) &:= \frac{\mathrm{d}}{\mathrm{d}t} M^x_\epsilon(t) \in \mathcal{V}\\ M_\epsilon(t) &:= \left(M^x_\epsilon(t), M^v_\epsilon(t)\right) \in \Gamma\,, \end{aligned}$$

and observe that these functions are of class C^∞ with bounded derivatives, uniformly in t and ϵ. Define the measure $\mu := (M_\epsilon)_\#\big(\mathrm{d}t|_{(0,1)}\big)$, which is independent of ϵ, and choose

$$\pi_\epsilon := (M_0, M_\epsilon)_\#\big(\mathrm{d}t|_{(0,1)}\big) \in \Pi(\mu,\mu)\,, \qquad \epsilon \geq 0\,.$$

If $0 < \epsilon \ll 1$, then $\|y-x\|_{\mathrm{L}^2(\pi_\epsilon)} > 0$, and we can write

$$\begin{aligned}\tilde{\mathsf{c}}(\pi_\epsilon) &\overset{(3.3.2)}{=} 12\int_0^1 \left|M_0^v(t)\right|^2\,\mathrm{d}t - 12\frac{\left(\int_0^1 M_0^v(t)\cdot\left(M_\epsilon^x(t)-M_0^x(t)\right)\,\mathrm{d}t + o(\epsilon)\right)_+^2}{\int_0^1\left|M_\epsilon^x(t)-M_0^x(t)\right|^2\,\mathrm{d}t} + o(1)\\ &= 12\int_0^1 \left|M_0^v(t)\right|^2\,\mathrm{d}t - 12\frac{\left(\int_0^1 \epsilon\left|M_0^v(t)\right|^2\,\mathrm{d}t + o(\epsilon)\right)_+^2}{\epsilon^2\int_0^1\left|M_0^v(t)\right|^2\,\mathrm{d}t + o(\epsilon^2)} + o(1) = o(1)\,,\end{aligned}$$

where, in the last identity, we used that $\int_0^1\left|M_0^v(t)\right|^2\,\mathrm{d}t = \|v\|^2_{\mathrm{L}^2(\mu)} > 0$. This proves that $\tilde{\mathsf{d}}(\mu,\mu) = 0$. However, assume that there exists $\pi \in \Pi(\mu,\mu)$ such that $\tilde{\mathsf{c}}(\pi) = 0$. If $\|y-x\|_{\mathrm{L}^2(\pi)} = 0$, then $\|v+w\|_{\mathrm{L}^2(\pi)} = \|v-w\|_{\mathrm{L}^2(\pi)} = 0$, which implies that $\|v\|_{\mathrm{L}^2(\mu)} = 0$, which is absurd. If, instead, $\|y-x\|_{\mathrm{L}^2(\pi)} > 0$, then $v = w$ for π-a.e. (v,w), and we have equality in the Cauchy–Schwarz inequality

$$(y-x, v+w)_\pi \leq \|y-x\|_{\mathrm{L}^2(\pi)}\|v+w\|_{\mathrm{L}^2(\pi)}\,,$$

which means that either $v = w = 0$ for π-a.e. (v,w) (hence $\|v\|_{\mathrm{L}^2(\mu)} = 0$), or $y = x + Tv$ for some $T > 0$, for π-a.e. (x,y,v). The latter case is excluded by observing that $(\mathcal{G}_T)_\#\mu \neq \mu$ for every $T > 0$, as its space marginal $(\mathrm{pr}_x \circ \mathcal{G}_T)_\#\mu$ lies on a circle with radius strictly larger than 1.

Corollary 3.3.7. *There exists a* measurable *selection* $(\mu,\nu) \mapsto \pi_{\mu,\nu} \in \Pi_{\mathsf{o,d}}(\mu,\nu)$.

Remark 3.3.8. We prove measurability w.r.t. the Borel σ-algebra of the 2-Wasserstein topology, which is the same as that of the narrow topology, e.g., by the Lusin–Suslin theorem [Kec95b, Theorem 15.1].

Proof of Corollary 3.3.7. We shall invoke [BP73, Corollary 1]. By [Vil09b, Theorem 6.18], the metric spaces $X := \big(\mathcal{P}_2(\Gamma)\times\mathcal{P}_2(\Gamma), \mathrm{W}_2 \oplus \mathrm{W}_2\big)$ and $Y := \big(\mathcal{P}_2(\Gamma\times\Gamma), \mathrm{W}_2\big)$ are complete and separable. The set

$$D := \left\{\big((\mu,\nu),\pi\big) \in \big(\mathcal{P}_2(\Gamma)\times\mathcal{P}_2(\Gamma)\big)\times\mathcal{P}_2(\Gamma\times\Gamma) \,:\, \pi \in \Pi(\mu,\nu)\right\}$$

is Borel, as it is the preimage of 0 through the continuous map

$$\big((\mu,\nu),\pi\big) \longmapsto \mathrm{W}_2\big((\mathrm{pr}_{x,v})_\#\pi, \mu\big) + \mathrm{W}_2\big((\mathrm{pr}_{y,w})_\#\pi, \nu\big)\,.$$

Each section

$$D_{\mu,\nu} = \Pi(\mu,\nu)\,, \qquad \mu,\nu \in \mathcal{P}_2(\Gamma)$$

is compact by Prokhorov's theorem and Lemma 3.3.3. Again by Lemma 3.3.3, the real-valued function c is lower-semicontinuous on $D_{\mu,\nu}$, for every μ,ν. Furthermore, by Proposition 3.3.5, for every μ,ν, there exists $\pi \in D_{\mu,\nu}$ such that $\mathsf{c}(\pi) = \inf_{\tilde{\pi}\in D_{\mu,\nu}} \mathsf{c}(\tilde{\pi})$. Therefore, the hypotheses of [BP73, Corollary 1] are satisfied, and there exists a measurable function $(\mu,\nu) \mapsto \pi_{\mu,\nu} \in D_{\mu,\nu}$ such that $\mathsf{c}(\pi_{\mu,\nu}) = \inf_{\tilde{\pi}\in D_{\mu,\nu}} \mathsf{c}(\tilde{\pi})$ for every $\mu,\nu \in \mathcal{P}_2(\Gamma)$. □

Remark 3.3.9. With a similar proof, one can show the existence of a measurable selection $(T,\mu,\nu) \mapsto \pi_{T,\mu,\nu}$, where $\pi_{T,\mu,\nu}$ is a $\tilde{\mathsf{d}}_T$-optimal plan between μ and ν.

3.3.2 Existence of kinetic optimal maps

Proposition 3.3.10 (Theorem 3.1.1, Statement 4)**.** *Let $\mu, \nu \in \mathcal{P}_2(\Gamma)$. Assume that μ is absolutely continuous with respect to the Lebesgue measure. Then, for every $T > 0$, there exists a* unique *transport plan $\pi_T \in \Pi(\mu, \nu)$ optimal for $\tilde{\mathsf{d}}_T(\mu, \nu)$. Moreover, π_T is induced by a measurable function $M_T := \Gamma \to \Gamma$, i.e., $\pi_T = (\mathrm{id}, M_T)_\# \mu$.*

Furthermore, there exists a transport map M such that $(\mathrm{id}, M)_\# \mu$ is optimal for the time-independent discrepancy $\mathsf{d}(\mu, \nu)$, i.e., $(\mathrm{id}, M)_\# \mu \in \Pi_{\mathrm{o}, \mathsf{d}}(\mu, \nu)$.

Note that we state uniqueness of the map for $\tilde{\mathsf{d}}_T$, but not for d, see also §3.3.3 below.

Proof. The first part of the statement, namely the uniqueness of π_T and its representation $\pi_T = (\mathrm{id}, M_T)_\# \mu$, follows from the classical theory of optimal transport, see in particular [Vil09b, Theorems 10.26 & 10.38]. To apply these theorems, we observe that

1. the function $\tilde{d}_T^2$ is smooth,
2. the *twist condition* is satisfied, i.e.,
$$(y, w) \longmapsto \nabla_{x,v} \tilde{d}_T^2\big((x, v), (y, w)\big)$$
   is injective for every $(x, v) \in \Gamma$.

Let us now move to the proof of the second part of the statement. Let $\pi \in \Pi_{\mathrm{o},\mathsf{d}}(\mu, \nu)$ be an optimal transport plan. We will distinguish three cases.

Case 1. Assume that $\|y - x\|_{\mathrm{L}^2(\pi)} = 0$, i.e., $y = x$ for π-a.e. (x, y). By disintegration, there exists a measure-valued measurable map $x \mapsto \pi_x \in \mathcal{P}(\mathcal{V} \times \mathcal{V})$ with

$$\int \varphi(x, v, y, w)\, \mathrm{d}\pi = \iint \varphi(x, v, x, w)\, \mathrm{d}\pi_x(v, w)\, \mathrm{d}\eta(x)\,, \qquad \varphi \in \mathrm{C_b}(\Gamma \times \Gamma)\,, \tag{3.3.9}$$

where $\eta := (\mathrm{pr}_x)_\# \pi = (\mathrm{pr}_x)_\# \mu$. Note that we can also write $\eta = (\mathrm{pr}_y)_\# \pi = (\mathrm{pr}_x)_\# \nu$. Set

$$\mu_x := (\mathrm{pr}_v)_\# \pi_x\,, \qquad \nu_x := (\mathrm{pr}_w)_\# \pi_x\,, \qquad x \in \mathcal{X}\,. \tag{3.3.10}$$

Since μ admits a density, so does μ_x for η-a.e. $x \in \mathcal{X}$. In particular, there exists a unique W_2-optimal transport map from μ_x to ν_x for η-a.e. x, see [AGS08, Theorem 6.2.4]. Therefore, we can apply [FGM10, Theorem 1.1] and get a Borel map $M_2 \colon \Gamma \to \mathcal{V}$ such that, for η-a.e. $x \in \mathcal{V}$, the transport plan $\big(\mathrm{id}, M_2(x, \cdot)\big)_\# \mu_x$ is optimal for the 2-Wasserstein distance between μ_x and ν_x. In particular,

$$\begin{aligned}
\int |v - M_2(x, v)|^2\, \mathrm{d}\mu &= \int |v - M_2(x, v)|^2\, \mathrm{d}\pi \\
&\overset{(3.3.9)}{=} \iint |v - M_2(x, v)|^2\, \mathrm{d}(\mathrm{pr}_v)_\# \pi_x(v)\, \mathrm{d}\eta(x) \\
&\overset{(3.3.10)}{=} \iint |v - M_2(x, v)|^2\, \mathrm{d}\mu_x(v)\, \mathrm{d}\eta(x) \\
&\leq \iint |w - v|^2\, \mathrm{d}\pi_x(v, w)\, \mathrm{d}\eta(x) \\
&\overset{(3.3.9)}{=} \int |w - v|^2\, \mathrm{d}\pi = \mathsf{d}^2(\mu, \nu)\,,
\end{aligned}$$

where the inequality follows from the optimality of M_2. Moreover, $M\colon (x,v) \mapsto \big(x, M_2(x,v)\big)$ defines a transport map from μ to ν. We conclude that $(\mathrm{id}, M)_\#\mu \in \Pi_{\mathsf{o},\mathsf{d}}(\mu,\nu)$.

Case 2. Assume that $\|y-x\|_{L^2(\pi)} > 0$ and $(y-x, v+w)_\pi > 0$. Define T as in (3.3.5). In this case, π is optimal for $\tilde{\mathsf{d}}_T(\mu,\nu)$, and it is induced by a map.

Case 3. Assume that $\|y-x\|_{L^2(\pi)} > 0$ and $(y-x, v+w)_\pi \leq 0$. We apply disintegration to μ and ν to find maps $v \mapsto \mu_v$ and $v \mapsto \nu_v$ such that

$$\mu(\mathrm{d}x, \mathrm{d}v) = \mu_v(\mathrm{d}x)(\mathrm{pr}_v)_\#\mu(\mathrm{d}v) \quad \text{and} \quad \nu(\mathrm{d}x, \mathrm{d}v) = \nu_v(\mathrm{d}x)(\mathrm{pr}_v)_\#\nu(\mathrm{d}v)\,.$$

Note that $(\mathrm{pr}_v)_\#\mu$ and $(\mathrm{pr}_v)_\#\mu$-almost every measure μ_v are absolutely continuous.

Consider the cost $(v,w) \mapsto 3|v+w|^2 + |w-v|^2$. Since this function is smooth and satisfies the twist condition, once again we infer the existence of a Borel map $B \coloneqq \mathcal{V} \to \mathcal{V}$ optimal for such a cost from $(\mathrm{pr}_v)_\#\mu$ to $(\mathrm{pr}_v)_\#\nu$. Let $A\colon \Gamma \to \mathcal{X}$ be any Borel function such that $A(\cdot, v)_\#\mu_v = \nu_{B(v)}$ for $(\mathrm{pr}_v)_\#\mu$-a.e. $v \in \mathcal{V}$. The existence of A can be deduced, e.g., from [FGM10, Theorem 1.1]. We claim that the map $M\colon (x,v) \mapsto \big(A(x,v), B(v)\big)$ defines an optimal transport map between μ and ν. By construction, $M_\#\mu = \nu$ and, by optimality of B, we conclude the proof of our claim:

$$\begin{aligned}\mathsf{c}\big((\mathrm{id}, M)_\#\mu\big) &\overset{(3.3.3)}{\leq} 3\|v + B(v)\|^2_{L^2(\mu)} + \|v - B(v)\|^2_{L^2(\mu)} \\ &\leq 3\|v+w\|^2_{L^2(\pi)} + \|w-v\|^2_{L^2(\pi)} = \mathsf{d}^2(\mu,\nu)\,. \quad \square\end{aligned}$$

3.3.3 Additional results

Non-uniqueness

Fix $\mu, \nu \in \mathcal{P}_2(\Gamma)$. If μ is absolutely continuous, then for every $T > 0$, there exists a unique minimiser for $\tilde{\mathsf{c}}_T$ in $\Pi(\mu,\nu)$, and this plan is induced by a map. This fact follows from the classical theory of optimal transport, cf. [Vil09b, Theorems 10.26 & 10.38]. Nonetheless, non-uniqueness in $\Pi_{\mathsf{o},\mathsf{d}}(\mu,\nu)$ may arise in several ways, for example:

- there may be two different times $T_1, T_2 > 0$ for which

$$\mathsf{d}^2(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \tilde{\mathsf{c}}_{T_1}(\pi) = \inf_{\pi \in \Pi(\mu,\nu)} \tilde{\mathsf{c}}_{T_2}(\pi)$$

- in the proof of Case 3 in Proposition 3.3.10 there is freedom in the choice of the map $A\colon \Gamma \to \mathcal{X}$ for which $M\colon (x,v) \mapsto \big(A(x,v), B(v)\big)$ is optimal.

Let us provide an example of non-uniqueness.

Example 3.3.11. Fix

$$X_1 = (x_1, v_1) \in \Gamma\,, \qquad X_2 = (x_2, v_2) \in \Gamma\,, \qquad S > 0\,,$$

and set

$$\mu \coloneqq \frac{1}{2}\delta_{X_1} + \frac{1}{2}\delta_{X_2}\,, \qquad \nu \coloneqq \frac{1}{2}\delta_{\mathcal{G}_S(X_1)} + \frac{1}{2}\delta_{X_2}\,.$$

Note that $\Pi(\mu,\nu)$ coincides with the set of all the convex combinations of

$$\pi_1 \coloneqq \frac{1}{2}\delta_{(X_1, \mathcal{G}_S(X_1))} + \frac{1}{2}\delta_{(X_2,X_2)}\,, \qquad \pi_2 \coloneqq \frac{1}{2}\delta_{(X_1,X_2)} + \frac{1}{2}\delta_{(X_2, \mathcal{G}_S(X_1))}\,.$$

Let us assume that $\{x_1, x_2\} \neq \{x_1 + Sv_1, x_2\}$. Then, $\mathsf{c} = \tilde{\mathsf{c}}$ on $\Pi(\mu, \nu)$. Taking also into account the concavity of $\tilde{\mathsf{c}}$ (see Proposition 3.3.2), we deduce that

$$\mathsf{d}^2(\mu,\nu) = \inf_{\pi \in \Pi(\mu,\nu)} \mathsf{c}(\pi) = \inf_{\pi \in \Pi(\mu,\nu)} \tilde{\mathsf{c}}(\pi) = \min\{\tilde{\mathsf{c}}(\pi_1), \tilde{\mathsf{c}}(\pi_2)\} .$$

Straightforward computations yield

$$\tilde{\mathsf{c}}(\pi_1) = 6|v_2|^2 , \qquad \tilde{\mathsf{c}}(\pi_2) = 3|v_1 + v_2|^2 - \frac{3S^2}{2} \frac{\left(|v_1|^2 + v_1 \cdot v_2\right)_+^2}{|x_2 - x_1|^2 + |x_2 - x_1 + Sv_1|^2} + |v_2 - v_1|^2 .$$

If, for example, we choose $x_1 = x_2$ and $v_1 \perp v_2$, we find

$$\tilde{\mathsf{c}}(\pi_1) = \tilde{\mathsf{c}}_S(\pi_1) = 6|v_2|^2 , \qquad \tilde{\mathsf{c}}(\pi_2) = \tilde{\mathsf{c}}_{2S}(\pi_2) = \frac{5}{2}|v_1|^2 + 4|v_2|^2 .$$

Therefore, when, additionally, $5|v_1|^2 = 4|v_2|^2$, both plans π_1 and π_2 are optimal. Note that they are induced by maps, and that their corresponding optimal times are different: S for π_1 and $2S$ for π_2. We also observe that S is exactly the optimal time for (3.1.5) between (x_1, v_1) and $(x_1 + Sv_1, v_1)$, while $2S$ is the optimal time between (x_1, v_2) and $(x_1 + Sv_1, v_1)$. In this case, the structure of c disadvantages intermediate times between S and $2S$.

Characterisation of $\mathsf{d} = 0$

We provide a characterisation of the measures μ, ν such that $\mathsf{d}(\mu, \nu) = 0$, analogous to the particle case of Remark 3.2.9.

Proposition 3.3.12. *Let $\mu, \nu \in \mathcal{P}_2(\Gamma)$. We have $\mathsf{d}(\mu, \nu) = 0$ if and only if one of the following holds:*

1. *$\nu = (\mathcal{G}_T)_\# \mu$ for some $T \geq 0$,*
2. *or $(\mathrm{pr}_v)_\# \mu = (\mathrm{pr}_v)_\# \nu = \delta_0$.*

If $(\mathrm{pr}_v)_\# \mu \neq \delta_0$ and $\nu = (\mathcal{G}_T)_\# \mu$ for a $T \geq 0$, then such a T is unique, and $\Pi_{\mathrm{o},\mathsf{d}}(\mu, \nu) = \left\{(\mathrm{id}, \mathcal{G}_T)_\# \mu\right\}$.

Proof. If $\nu = (\mathcal{G}_T)_\# \mu$ for some $T \geq 0$, we have $\mathsf{c}\big((\mathrm{id}, \mathcal{G}_T)_\# \mu\big) = 0$. If $(\mathrm{pr}_v)_\# \mu = (\mathrm{pr}_v)_\# \nu = \delta_0$, then every $\pi \in \Pi(\mu, \nu)$ has zero cost.

Conversely, assume that $\mathsf{d}(\mu, \nu) = 0$. By Proposition 3.3.5, there exists $\pi \in \Pi(\mu, \nu)$ with $\mathsf{c}(\pi) = 0$. By the definition of c, we must have $v = w$ for π-a.e. (v, w). If $x = y$ for π-a.e. (x, y), then $\pi = (\mathrm{id}, \mathrm{id})_\# \mu$. Otherwise, we have equality in the inequality

$$(y - x, v + w)_\pi \leq \|y - x\|_{\mathrm{L}^2(\pi)} \|v + w\|_{\mathrm{L}^2(\pi)} .$$

This can happen only if $v = w = 0$ for π-a.e. (v, w), or if there exists $T \geq 0$ such that $y = x + Tv$ for π-a.e. (x, y, v).

Assume that $(\mathrm{pr}_v)_\# \mu \neq \delta_0$. We have already proved that every $\pi \in \Pi_{\mathrm{o},\mathsf{d}}(\mu, \nu)$ is of the form $\pi = (\mathrm{id}, \mathcal{G}_T)_\# \mu$ for some T. Let us assume, by contradiction, that $\nu = (\mathcal{G}_{T_1})_\# \mu = (\mathcal{G}_{T_2})_\# \mu$ for some $T_1, T_2 \geq 0$ with $T_1 < T_2$. By the semigroup property:

$$\nu = (\mathcal{G}_{T_2})_\# \mu = (\mathcal{G}_{T_2 - T_1})_\# (\mathcal{G}_{T_1})_\# \mu = (\mathcal{G}_{T_2 - T_1})_\# \nu = \cdots = (\mathcal{G}_{(T_2 - T_1)k})_\# \nu$$

for every $k \in \mathbb{N}_{>0}$. For every $\varphi \in \mathrm{C_c}(\Gamma)$ and $k \in \mathbb{N}_{>0}$, we thus find

$$\int \varphi(x,v)\, \mathrm{d}\nu = \int \varphi\big(x + k(T_2 - T_1)v, v\big)\, \mathrm{d}\nu\,.$$

Note that $\lim_{k\to\infty}|x + k(T_2 - T_1)v| = \infty$ whenever $v \neq 0$; hence, since φ is compactly supported, the dominated convergence theorem yields

$$\int \varphi(x,v)\, \mathrm{d}\nu = \int_{\{v=0\}} \varphi(x,0)\, \mathrm{d}\nu\,.$$

Hence,

$$\int \varphi(x + T_1 v, v)\, \mathrm{d}\mu = \int_{\{v=0\}} \varphi(x,0)\, \mathrm{d}\mu\,,$$

which can hold for every $\varphi \in \mathrm{C_c}(\Gamma)$ only if $(\mathrm{pr}_v)_\# \mu = \delta_0$. □

Corollary 3.3.13. *Let $\mu_k \rightharpoonup \mu$ and $\nu_k \rightharpoonup \nu$ be two* narrowly *convergent sequences in $\mathcal{P}_2(\Gamma)$. For every $k \in \mathbb{N}$, pick one $\pi_k \in \Pi_{\mathrm{o,d}}(\mu_k, \nu_k)$. Assume:*

(a) $\lim_{k\to\infty} \mathsf{d}(\mu_k, \nu_k) = 0$,

(b) $(\mathrm{pr}_v)_\# \mu \neq \delta_0$,

(c) $\sup_k \min\left\{\|v\|_{\mathrm{L}^2(\mu_k)}, \|v\|_{\mathrm{L}^2(\nu_k)}\right\} < \infty$.

Then, $\mathsf{d}(\mu, \nu) = 0$, so that $\nu = (\mathcal{G}_T)_\# \mu$ for a $T \geq 0$. Finally, $\pi_k \rightharpoonup (\mathrm{id}, \mathcal{G}_T)_\# \mu \in \Pi_{\mathrm{o,d}}(\mu, \nu)$.

Remark 3.3.14. This corollary would easily follow from Proposition 3.3.12, Lemma 3.3.3, and the lower semicontinuity of d if we assumed convergence of $(\mu_k)_k$ and $(\nu_k)_k$ w.r.t. W_2. Instead, we assume here only narrow convergence.

Proof of Corollary 3.3.13. Since $(\mu_k)_k$ and $(\nu_k)_k$ are convergent, they are tight. Therefore, the same is true for $(\pi_k)_k$. By Prokhorov's theorem, the sequence $(\pi_k)_k$ admits at least one narrow limit. If one such limit π satisfies $\mathsf{c}(\pi) = 0$, then $\pi \in \Pi_{\mathrm{o,d}}(\mu, \nu)$ and $\mathsf{d}(\mu, \nu) = 0$, which yields, by Proposition 3.3.12, $\nu = (\mathcal{G}_T)_\# \mu$ and $\pi = (\mathrm{id}, \mathcal{G}_T)_\# \mu$ for some unique $T \geq 0$ (independent of the limit π). If, *every* limit π satisfies $\mathsf{c}(\pi) = 0$, then there exists only one limit of the sequence $(\pi_k)_k$, namely $(\mathrm{id}, \mathcal{G}_T)_\# \mu$.

Let us thus prove that every limit π satisfies $\mathsf{c}(\pi) = 0$. Up to extracting a subsequence, $\pi_k \rightharpoonup \pi$. To begin with, let us note that, by lower semicontinuity of the norm w.r.t. narrow convergence,

$$\|w - v\|_{\mathrm{L}^2(\pi)} \leq \liminf_{k\to\infty} \|w - v\|_{\mathrm{L}^2(\pi_k)} \overset{(3.1.10)}{\leq} \liminf_{k\to\infty} \mathsf{d}(\mu_k, \nu_k) = 0\,.$$

Moreover, by the triangle inequality,

$$\begin{aligned}\limsup_{k\to\infty} \|v + w\|_{\mathrm{L}^2(\pi_k)} &\leq \limsup_{k\to\infty} \left(\|w - v\|_{\mathrm{L}^2(\pi_k)} + 2\min\left\{\|v\|_{\mathrm{L}^2(\mu_k)}, \|v\|_{\mathrm{L}^2(\nu_k)}\right\}\right) \\ &\leq 2 \sup_k \min\left\{\|v\|_{\mathrm{L}^2(\mu_k)}, \|v\|_{\mathrm{L}^2(\nu_k)}\right\}\,,\end{aligned}$$

and the last term is bounded by Assumption (c). Thus, up to subsequences, we may assume that $\|v + w\|_{\mathrm{L}^2(\pi_k)}$ converges to a number $a \in \mathbb{R}_{\geq 0}$. Up to subsequences, we can also assume that $\|y-x\|_{\mathrm{L}^2(\pi_k)}$ converges, to an either real or infinite quantity $b \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. If $a=0$, then, by lower semicontinuity of the norm w.r.t. narrow convergence,

$$\mathsf{c}(\pi) \overset{(3.3.3)}{\leq} 3\|v+w\|^2_{\mathrm{L}^2(\pi)} + \|w-v\|^2_{\mathrm{L}^2(\pi)} \leq 3 \liminf_{k\to\infty} \|v+w\|^2_{\mathrm{L}^2(\pi_k)} = 3a^2 = 0\,.$$

If $b=0$, again by lower semicontinuity, $\|y-x\|_{\mathrm{L}^2(\pi)}=0$, hence $\mathsf{c}(\pi) = \|w-v\|^2_{\mathrm{L}^2(\pi)} = 0$.

From now on, let us assume $a, b > 0$ and, possibly up to subsequences, that $\|v+w\|_{\mathrm{L}^2(\pi_k)}$ and $\|y-x\|_{\mathrm{L}^2(\pi_k)}$ are strictly positive for every k. Define

$$c_k := \int \left| \frac{v+w}{\|v+w\|_{\mathrm{L}^2(\pi_k)}} - \frac{y-x}{\|y-x\|_{\mathrm{L}^2(\pi_k)}} \right|^2 \mathrm{d}\pi_k\,, \qquad k \in \mathbb{N}\,.$$

We find that

$$\|v+w\|^2_{\mathrm{L}^2(\pi)} - \frac{\left((y-x, v+w)_\pi\right)_+^2}{\|y-x\|^2_{\mathrm{L}^2(\pi)}} = \|v+w\|^2_{\mathrm{L}^2(\pi)} \cdot \left(1 - \left(1 - \frac{c_k}{2}\right)_+^2\right)$$

and, consequently,

$$\limsup_{k\to\infty} \min\left\{\frac{c_k}{2}, 1\right\} \leq \limsup_{k\to\infty} \left(1 - \left(1-\frac{c_k}{2}\right)_+^2\right) \leq \limsup_{k\to\infty} \frac{\mathsf{d}^2(\mu_k,\nu_k)/3}{\|v+w\|^2_{\mathrm{L}^2(\pi)}} = \frac{0}{a^2} = 0\,.$$

This proves that $c_k \to 0$. Let $\varphi \in \mathrm{C_c}(\Gamma\times\Gamma)$ be non-negative. The convergence

$$\left| \frac{v+w}{\|v+w\|_{\mathrm{L}^2(\pi_k)}} - \frac{y-x}{\|y-x\|_{\mathrm{L}^2(\pi_k)}} \right|^2 \varphi \to \left| \frac{v+w}{a} - \frac{y-x}{b}\right|^2 \varphi$$

is *uniform*. Thus, the narrow convergence $\pi_k \rightharpoonup \pi$ yields

$$\begin{aligned} \int \left| \frac{v+w}{a} - \frac{y-x}{b}\right|^2 \varphi \,\mathrm{d}\pi &= \lim_{k\to\infty} \int \left| \frac{v+w}{\|v+w\|_{\mathrm{L}^2(\pi_k)}} - \frac{y-x}{\|y-x\|_{\mathrm{L}^2(\pi_k)}} \right|^2 \varphi \,\mathrm{d}\pi_k \\ &\leq \|\varphi\|_\infty \liminf_{k\to\infty} c_k = 0\,, \end{aligned}$$

and, by arbitrariness of φ,

$$\frac{v+w}{a} = \frac{y-x}{b} \quad \text{for } \pi\text{-a.e. } (x,v,y,w)\,.$$

Using the definition (3.3.3) of c, we infer that $\mathsf{c}(\pi) = \|w-v\|^2_{\mathrm{L}^2(\pi)} = 0$. □

Corollary 3.3.15. *In the setting of Corollary 3.3.13, additionally set*

$$T_k := \begin{cases} 2\dfrac{\|y-x\|_{\mathrm{L}^2(\pi_k)}}{(y-x, v+w)_{\pi_k}} & \text{if } (y-x, v+w)_{\pi_k} > 0\,, \\ 0 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi_k)} = 0\,, \\ \infty & \text{otherwise.} \end{cases} \qquad k \in \mathbb{N}\,. \tag{3.3.11}$$

Then, $T := \lim_{k\to\infty} T_k$ exists, is finite, and $\nu = (\mathcal{G}_T)_{\#}\mu$.

Proof. We know from Corollary 3.3.13 that $\pi_k \rightharpoonup (\mathrm{id}, \mathcal{G}_{\tilde{T}})_\# \mu$ for some $\tilde{T} \geq 0$. Up to extracting a subsequence, we may assume that $T := \lim_{k\to\infty} T_k$ exists in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. We shall prove that $T = \tilde{T}$.

If $\|y-x\|_{\mathrm{L}^2(\pi_k)} = 0$ frequently, then $T = 0$ and, by semicontinuity, we obtain

$$0 = \|y-x\|_{\mathrm{L}^2\left((\mathrm{id},\mathcal{G}_{\tilde{T}})_\#\mu\right)} = \tilde{T}\|v\|_{\mathrm{L}^2(\mu)}\ ;$$

hence, $\tilde{T} = 0$. Up to subsequences, we can from now on assume that $\|y-x\|_{\mathrm{L}^2(\pi_k)} > 0$ for every k. By Proposition 3.3.2, we have

$$\mathsf{d}(\mu_k,\nu_k) = \mathsf{c}(\pi_k) = \tilde{\mathsf{c}}(\pi_k) \geq \int \left|\frac{y-x}{T_k} - \frac{v+w}{2}\right|^2 \mathrm{d}\pi_k\,, \qquad k \in \mathbb{N}\,.$$

Let $\varphi \in \mathrm{C_c}(\Gamma\times\Gamma)$ be non-negative and assume that $T > 0$. Then, the convergence

$$\left|\frac{y-x}{T_k} - \frac{v+w}{2}\right|^2 \varphi \to \left|\frac{y-x}{T} - \frac{v+w}{2}\right|^2 \varphi$$

is uniform; hence,

$$\int \left|\frac{\tilde{T}}{T}v - v\right|^2 \varphi\ \mathrm{d}\mu = \lim_{k\to\infty} \int \left|\frac{y-x}{T_k} - \frac{v+w}{2}\right|^2 \varphi\ \mathrm{d}\pi_k \leq \|\varphi\|_\infty \liminf_{k\to\infty} \mathsf{d}(\mu_k,\nu_k) = 0\,.$$

This proves, by arbitrariness of φ, that $\frac{\tilde{T}}{T}v = v$ for $(\mathrm{pr}_v)_\#\mu$-a.e. v. Since, by assumption, $(\mathrm{pr}_v)_\#\mu \neq \delta_0$, we conclude that $\tilde{T} = T$.

Let again $\varphi \in \mathrm{C_c}(\Gamma\times\Gamma)$ be non-negative and assume that $T = 0$. Now the convergence

$$\left|y - x - T_k\frac{v+w}{2}\right|^2 \varphi \to |y-x|^2\,\varphi$$

is uniform; hence,

$$\int \left|\tilde{T}\,v\right|^2 \varphi\ \mathrm{d}\mu = \lim_{k\to\infty} \int \left|y-x-T_k\frac{v+w}{2}\right|^2 \varphi\ \mathrm{d}\pi_k \leq \|\varphi\|_\infty \liminf_{k\to\infty} T_k \mathsf{d}(\mu_k,\nu_k) = 0\,.$$

We conclude, as before, that $\tilde{T} = 0 = T$. □

3.4 Dynamical formulations of kinetic optimal transport

This section, devoted to the dynamical formulations of kinetic optimal transport that we introduced in §3.1 (see (3.1.13) and (3.1.15)), is organised as follows.

- In §3.4.1, we explore dynamical transport plans in the kinetic setting and prove the equality of $\tilde{\mathsf{d}}_T$ and $\tilde{\mathsf{n}}_T$, together with the existence of a minimiser for (3.1.13).
- In §3.4.2, we better characterise the *optimal spline interpolations* stemming from Theorem 3.4.1 and we discuss injectivity of optimal-spline flows.
- In §3.4.3, we study Vlasov's equation (3.1.14), and we conclude the proof of Theorem 3.1.2 with the kinetic Benamou–Brenier formula of Theorem 3.4.10.
- In §3.4.4, we show propagation of second-order moments along solutions to (3.1.14).

3.4.1 Dynamical plans

For a fixed $T > 0$, a *T-dynamical (transport) plan* between $\mu, \nu \in \mathcal{P}_2(\Gamma)$ is a probability measure $\mathbf{m} \in \mathcal{P}\big(\mathrm{H}^2(0,T;\mathcal{X})\big)$ subject to the endpoint conditions (3.1.12).

The theorem below shows that minimising the acceleration functional

$$\alpha \longmapsto T \int_0^T \int_{\mathrm{H}^2(0,T;\mathcal{X})} \left|\alpha''(t)\right|^2 \,\mathrm{d}\mathbf{m}(\alpha)\,\mathrm{d}t$$

along T-dynamical plans between μ and ν is equivalent to computing $\tilde{\mathsf{d}}_T^2(\mu,\nu)$ via (3.1.8).

The leading idea is that optimal T-dynamical plans between μ and ν are supported on *T-splines* between points $(x,v) \in \operatorname{supp}(\mu)$ and $(y,w) \in \operatorname{supp}(\nu)$. Splines are uniquely determined by their endpoints, and their total squared acceleration equals $\tilde{d}_T^2\big((x,v),(y,w)\big)$. Endpoints chosen according to an optimal coupling π for $\tilde{\mathsf{d}}_T^2(\mu,\nu)$ determine an optimal $\mathbf{m}$.

Theorem 3.4.1. *For every $\mu, \nu \in \mathcal{P}_2(\Gamma)$ and $T > 0$, the problem (3.1.13) admits a minimiser. Moreover, we have the identity*

$$\tilde{\mathsf{n}}_T(\mu,\nu) = \tilde{\mathsf{d}}_T(\mu,\nu)\,. \tag{3.4.1}$$

Proof. Fix $T > 0$. We build a correspondence between admissible dynamical transport plans for (3.1.13) and plans in $\Pi(\mu,\nu)$. If $\mathbf{m} \in \mathcal{P}\big(\mathrm{H}^2(0,T;\mathcal{X})\big)$ is admissible (i.e., it satisfies (3.1.12)), we have that

$$\pi_{\mathbf{m}} := \Big(\mathrm{pr}_{\big(\alpha(0),\alpha'(0)\big)},\, \mathrm{pr}_{\big(\alpha(T),\alpha'(T)\big)}\Big)_{\#} \mathbf{m} \in \Pi(\mu,\nu)\,. \tag{3.4.2}$$

Conversely, given $\pi \in \Pi(\mu,\nu)$, we construct a T-dynamical plan as follows: $\mathbf{m}_\pi$ is the push-forward of π through the map $\big((x,v),(y,w)\big) \mapsto \alpha^T_{x,v,y,w}(\cdot)$, see (3.2.2). Note $\pi_{\mathbf{m}_\pi} = \pi$ for $\pi \in \Pi(\mu,\nu)$, and $\mathbf{m}_{\pi_{\mathbf{m}}} = \mathbf{m}$ for all T-dynamical plans $\mathbf{m}$ *concentrated on T-splines*.

Let $\mathbf{m}$ be any admissible dynamical transport plan in (3.1.13). For every α, it is clear from (3.1.4) that $T\int_0^T |\alpha''(t)|^2 \,\mathrm{d}t \geq 12\left|\frac{y-x}{T} - \frac{v+w}{2}\right|^2 + |v-w|^2$, where $(x,v),(y,w)$ are the endpoints of α. Equality holds if and only if α is minimal in (3.1.4). Thus,

$$\begin{aligned}
T\int_0^T \int_{\mathrm{H}^2(0,T;\mathcal{X})} |\alpha''(t)|^2 \,\mathrm{d}\mathbf{m}(\alpha)\,\mathrm{d}t &= \int_{\mathrm{H}^2(0,T;\mathcal{X})} T\int_0^T |\alpha''(t)|^2 \,\mathrm{d}t\,\mathrm{d}\mathbf{m}(\alpha) \\
&\geq \int_{\mathrm{H}^2(0,T;\mathcal{X})} \left(12\left|\frac{\alpha(T)-\alpha(0)}{T} - \frac{\alpha'(0)+\alpha'(T)}{2}\right|^2 + |\alpha'(0)-\alpha'(T)|^2\right) \mathrm{d}\mathbf{m}(\alpha) \\
&= \int_{\Gamma\times\Gamma} \left(12\left|\frac{y-x}{T} - \frac{v+w}{2}\right|^2 + |v-w|^2\right) \mathrm{d}\pi_{\mathbf{m}}\big((x,v),(y,w)\big) \geq \tilde{\mathsf{d}}_T^2(\mu,\nu)\,.
\end{aligned}$$

Optimising in $\mathbf{m}$, we find $\tilde{\mathsf{n}}_T(\mu,\nu) \geq \tilde{\mathsf{d}}_T(\mu,\nu)$.

Turning to the converse inequality, let π be optimal in the definition (3.1.8) of $\tilde{\mathsf{d}}_T(\mu,\nu)$. By definition of $\mathbf{m}_\pi$, we have

$$\begin{aligned}\tilde{\mathsf{n}}_T^2(\mu,\nu) &\le T\int_0^T\int_{\mathrm{H}^2(0,T;\mathcal{X})}|\alpha''(t)|^2 d\mathbf{m}_\pi(\alpha)\ \mathrm{d}t\\ &=\int_{\Gamma\times\Gamma}T\int_0^T\left|(\alpha^T_{x,v,y,w})''(t)\right|^2\mathrm{d}t\ \mathrm{d}\pi\big((x,v),(y,w)\big)\\ &=\int_{\Gamma\times\Gamma}\left(\left|\frac{y-x}{T}-\frac{v+w}{2}\right|^2+|v-w|^2\right)\mathrm{d}\pi=\tilde{\mathsf{d}}_T^2(\mu,\nu)\,,\end{aligned}$$

thanks to the fact that $\mathbf{m}_\pi$ is supported on T-splines $\alpha^T_{x,v,y,w}$. As a by-product, we have that $\mathbf{m}_\pi$ is optimal in the minimisation problem for $\tilde{\mathsf{n}}_T(\mu,\nu)$. □

Remark 3.4.2. By optimising in T, and then taking the lower semi-continuous relaxation $\mathrm{sc}^-_{\mathrm{W}_2}$ in $\tilde{\mathsf{d}}_T(\mu,\nu)=\tilde{\mathsf{n}}_T(\mu,\nu)$, we also have that

$$\tilde{\mathsf{d}}^2(\mu,\nu)=\inf_{T>0}\inf_{\mathbf{m}\in\mathcal{P}\left(\mathrm{H}^2(0,T;\mathcal{X})\right)}\left\{T\int_0^T\int\left|\alpha''(t)\right|^2\mathrm{d}\mathbf{m}(\alpha)\ \mathrm{d}t\quad\text{subject to (3.1.12)}\right\},$$

$$\mathsf{d}^2(\mu,\nu)=\mathrm{sc}^-_{\mathrm{W}_2}\inf_{T>0}\inf_{\mathbf{m}\in\mathcal{P}\left(\mathrm{H}^2(0,T;\mathcal{X})\right)}\left\{T\int_0^T\int\left|\alpha''(t)\right|^2\mathrm{d}\mathbf{m}(\alpha)\ \mathrm{d}t\quad\text{subject to (3.1.12)}\right\}.$$

3.4.2 Spline interpolation and injectivity

As pointed out in §3.1.2, interpolation of measures based on splines is relevant for various applications. For all $\mu,\nu\in\mathcal{P}_2(\Gamma)$, all $T>0$, and $\pi\in\Pi(\mu,\nu)$ such that π is optimal for $\tilde{\mathsf{d}}_T(\mu,\nu)$, the proof of Theorem 3.4.1 provides us with an optimal dynamical transport plan $\mathbf{m}_\pi$. The plan $\mathbf{m}_\pi$ is supported on splines (parametrised on $[0,T]$) joining points of $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\nu)$. Hence, we can interpret the curve

$$[0,T]\ni t\longmapsto\bar{\mu}_t:=\left(\mathrm{pr}_{\left(\alpha(t),\alpha'(t)\right)}\right)_{\#}\mathbf{m}_\pi \tag{3.4.3}$$

as an optimal *spline interpolation* between μ and ν. In §3.4.3, we will show that the curve $(\bar{\mu}_t)_t$ satisfies an optimality criterion and it is a solution to Vlasov's equation (3.1.14), for a suitable force field $(F_t)_t$.

We start with proving injectivity of the interpolation $\bar{\mu}_t$, whenever $\mu\ll\mathrm{d}x\ \mathrm{d}v$.

Proposition 3.4.3 (Injective optimal spline interpolation)**.** *Fix $T>0$ and $\mu,\nu\in\mathcal{P}_2(\Gamma)$ such that μ is absolutely continuous with respect to the Lebesgue measure on Γ. Then, there exists a unique optimal T-dynamical transport plan $\bar{\mathbf{m}}$ for $\tilde{\mathsf{n}}_T(\mu,\nu)$ and, therefore, a unique spline interpolation $\bar{\mu}_\cdot$. Moreover, for every $t\in[0,T]$, there exists a μ-a.e. injective (and measurable) map $M_t\colon\Gamma\to\Gamma$ such that*

$$\bar{\mu}_t=(M_t)_{\#}\mu\,. \tag{3.4.4}$$

Proof. Existence of $\bar{\mathbf{m}}$ is ensured by Theorem 3.4.1. The plan $\pi_{\bar{\mathbf{m}}}$ is a minimiser for $\tilde{c}_T$ in $\Pi(\mu,\nu)$. Since μ is absolutely continuous, Proposition 3.3.10 shows that $\pi_{\bar{\mathbf{m}}}$ equals the *unique* $\bar{\pi}$ that is optimal for $\tilde{\mathsf{d}}_T(\mu,\nu)$. Therefore,

$$\bar{\mathbf{m}}=\bar{\mathbf{m}}_{\pi_{\bar{\mathbf{m}}}}=\bar{\mathbf{m}}_{\bar{\pi}}\,.$$

Once $(\bar{\mu}_t)_{t\in[0,T]}$ has been defined as in (3.4.3), we use Proposition 3.3.10 to find, for every $t \in [0,T]$, an optimal map M_t for $\tilde{\mathsf{d}}_t(\mu,\bar{\mu}_t)$. Injectivity of M_t follows from Lemma 3.4.4. □

Lemma 3.4.4. *Fix $T > 0$, and let $\mu,\nu \in \mathcal{P}_2(\Gamma)$. Assume that, for a Borel map $M_T : \Gamma \to \Gamma$, the transport plan $\pi = (\mathrm{id}, M_T)_{\#}\mu$ is optimal for $\tilde{\mathsf{d}}_T(\mu,\nu)$. Then there exists a Borel set $A \subseteq \Gamma$ of full μ-measure such that, for every $t \in (0,T)$, the map*

$$M_t(x,v) := \Big(\alpha^T_{x,v,M_T(x,v)}(t), \big(\alpha^T_{x,v,M_T(x,v)}\big)'(t)\Big), \qquad (x,v) \in A \tag{3.4.5}$$

is injective, where $\alpha^T_{x,y,v,w}$ is the solution of (3.1.4). Moreover, the set $B := \bigcup_{t\in(0,T)}\{t\} \times M_t(A)$ and the map $B \ni (t,y,w) \mapsto M_t^{-1}(y,w)$ are Borel measurable.

Proof. Define

$$A := \Big\{(x,v) \in \Gamma \,:\, \big((x,v), M_T(x,v)\big) \in \operatorname{supp}\pi\Big\} = (\mathrm{id}, M_T)^{-1}(\operatorname{supp}\pi)\,,$$

and notice that

$$\mu(A) = \mu\big((\mathrm{id}, M_T)^{-1}(\operatorname{supp}\pi)\big) = \pi(\operatorname{supp}\pi) = 1\,.$$

By cyclical monotonicity [San15, Theorem 1.38], we know that

$$\begin{aligned}\tilde{\mathsf{d}}_T^2\big((x_1,v_1), M_T(x_1,v_1)\big) &+ \tilde{\mathsf{d}}_T^2\big((x_2,v_2), M_T(x_2,v_2)\big)\\ &\le \tilde{\mathsf{d}}_T^2\big((x_1,v_1), M_T(x_2,v_2)\big) + \tilde{\mathsf{d}}_T^2\big((x_2,v_2), M_T(x_1,v_1)\big)\,,\end{aligned}$$

for every $(x_1,v_1),(x_2,v_2) \in A$. Hence, injectivity for every $t \in (0,T)$ comes from Proposition 3.2.6. Consequently, the map

$$(0,T) \times A \ni (t,x,v) \longmapsto \big(t, M_t(x,v)\big) \in B$$

is (Borel and) bijective. By the Lusin–Suslin theorem [Kec95b, Corollary 15.2], images of Borel sets through injective maps are Borel, from which the second assertion follows. □

3.4.3 Vlasov's equations and the kinetic Benamou–Brenier formula

The class of Vlasov's equations

Definition 3.4.5. Let $a,b \in \mathbb{R}\cup\{\pm\infty\}$ with $a<b$. Let $(\mu_t)_{t\in(a,b)} \subseteq \mathcal{P}_2(\Gamma)$ be a Borel family of probability measures, and let $F = (F_t)_t \colon (a,b)\times\Gamma \to \mathbb{R}^n$ be a time-dependent measurable vector field. Assume that

$$\int_a^b \int_\Gamma \big(|v| + |F_t|\big)\,\mathrm{d}\mu_t\,\mathrm{d}t < \infty\,. \tag{3.4.6}$$

We say that $(\mu_t, F_t)_{t\in(a,b)}$ is a solution to *Vlasov's equation*

$$\partial_t\mu_t + v\cdot\nabla_x\mu_t + \nabla_v\cdot(F_t\,\mu_t) = 0 \tag{3.4.7}$$

if (3.4.7) is solved in the weak sense, namely

$$\forall\varphi \in \mathrm{C}_c^\infty\big((a,b)\times\Gamma\big) \qquad \int_a^b\int_\Gamma \big(\partial_t\varphi + v\cdot\nabla_x\varphi + F_t\cdot\nabla_v\varphi\big)\,\mathrm{d}\mu_t\,\mathrm{d}t = 0 \tag{3.4.8}$$

(or, equivalently, for every $\varphi \in \mathrm{C}_c^1\big((a,b)\times\Gamma\big)$, cf. [AGS08, Remark 8.1.1]).

Proposition 3.4.6 ([San15, Section 4.1.2], [AGS08, Section 8.1])**.** *Let $(\mu_t, F_t)_t$ be a solution to (3.4.7). Then, up to changing the representative of $(\mu_t)_t$ (i.e., changing μ_t for a negligible set of times t), the following hold.*

- *The curve $(\mu_t)_t$ is continuous w.r.t. the narrow convergence of measures, and extends continuously to the closure $[a, b]$.*
- *For all functions $\psi \in \mathrm{C}_c^\infty(\Gamma)$, the mapping $t \mapsto \int_\Gamma \psi \, \mathrm{d}\mu_t(x, v)$ is absolutely continuous and it holds true that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \psi \, \mathrm{d}\mu_t(x, v) = \int_\Gamma \nabla_{x,v}\psi \cdot (v, F_t) \, \mathrm{d}\mu_t(x, v) \,, \qquad \textit{for a.e. } t \in (a, b) \,. \tag{3.4.9}$$

- *If $(\mu_t)_t$ has a Lipschitz continuous density (in t, x, v) and F_t is Lipschitz continuous in x, v, then (3.4.7) is also solved in the a.e. sense.*

Let $(F_t)_t$ be a vector field $(a, b) \times \Gamma \to \mathbb{R}^d$, such that

$$\int_a^b \left(\sup_B |F_t| + \mathrm{Lip}_B(F_t) \right) \mathrm{d}t < \infty \,, \tag{3.4.10}$$

for every compact set $B \Subset \Gamma$. Then, for every $(x, v) \in \Gamma$, the associated flow $t \mapsto M_t = (x_t, v_t)$ given by

$$\begin{cases} M_a(x, v) = (x, v) \,, \\ \partial_t M_t = \Big(v_t, F_t(x_t, v_t) \Big) \,, \end{cases} \tag{3.4.11}$$

is well-posed in an interval $[a, a + \epsilon)$ with $\epsilon > 0$, see [AGS08, Lemma 8.1.4]. In case

$$\int_a^b \left(\sup_\Gamma |F_t| + \mathrm{Lip}_\Gamma(F_t) \right) \mathrm{d}t < \infty \,, \tag{3.4.12}$$

we have global existence of the flow M_t, i.e., (3.4.11) is well-posed in $[a, b]$.

Proposition 3.4.7 ([AGS08, Lemma 8.1.6 & Proposition 8.1.8])**.** *Let $\mu \in \mathcal{P}_2(\Gamma)$ and let $(F_t)_{t \in (a,b)}$ be a vector field satisfying (3.4.6) and (3.4.10).*

- *Assume that, for μ-a.e. $(x, v) \in \Gamma$, the flow $t \mapsto M_t(x, v)$ defined by (3.4.11) is well-posed in the interval $[a, b)$. Then, $t \mapsto \mu_t := (M_t)_\# \mu$ is narrowly continuous and $(\mu_t, F_t)_t$ is a weak solution to (3.4.7) in (a, b).*
- *Conversely, given a narrowly continuous curve $(\mu_t)_{t \in [a,b)}$ such that $(\mu_t, F_t)_t$ solves (3.4.7) on (a, b), and $\mu_a = \mu$, then the flow $t \mapsto M_t(x, v)$ associated with $(F_t)_t$ is well-defined on (a, b) for μ-a.e. (x, v), and*

$$\mu_t = (M_t)_\# \mu \,, \qquad t \in (a, b) \,. \tag{3.4.13}$$

We conclude the section by adapting the results of [AGS08, Section 8.2] to our framework.

Proposition 3.4.8 ([AGS08, Theorem 8.2.1])**.** *Let $(\mu_t)_{t \in [a,b)} \subseteq \mathcal{P}_2(\Gamma)$ be a narrowly continuous curve such that $(\mu_t, F_t)_t$ is a solution to (3.4.7) on (a, b), and*

$$\int_a^b \int_\Gamma \left(|v|^2 + |F_t|^2 \right) \mathrm{d}\mu_t \, \mathrm{d}t < \infty \,. \tag{3.4.14}$$

Then, there exists $\boldsymbol{\eta} \in \mathcal{P}\Big(\mathcal{X} \times \mathcal{V} \times \mathrm{H}^2(a, b; \mathcal{X}) \Big)$ such that

1. *the measure $\boldsymbol{\eta}$ is supported on triples (x, v, α) such that (α, α') is an absolutely continuous curve solving (3.4.11), with initial conditions $\big(\alpha(a), \alpha'(a)\big) = (x, v) \in \mathrm{supp}(\mu_a)$;*

2. *we have*

$$\left(\mathrm{pr}_{\left(\alpha(t), \alpha'(t)\right)}\right)_{\#} \boldsymbol{\eta} = \mu_t\,, \qquad \text{for all } t \in [a, b]\,. \tag{3.4.15}$$

Conversely, any $\boldsymbol{\eta} \in \mathcal{P}\big(\mathcal{X} \times \mathcal{V} \times \mathrm{H}^2(a, b; \mathcal{X})\big)$ satisfying Condition 1 and

$$\int_a^b \int_{\mathrm{H}^2(a,b;\mathcal{X})} \left(\left|\alpha'(t)\right|^2 + \left|F_t(\alpha(t), \alpha'(t))\right|^2\right) \mathrm{d}\boldsymbol{\eta}\ \mathrm{d}t < \infty \tag{3.4.16}$$

induces a solution to (3.4.7) via

$$\mu_t \coloneqq \left(\mathrm{pr}_{\left(\alpha(t), \alpha'(t)\right)}\right)_{\#} \boldsymbol{\eta}\,, \qquad t \in (a, b)\,. \tag{3.4.17}$$

The measure $\boldsymbol{\eta}$ is usually referred to as the *lift* of the curve $(\mu_t, F_t)_t$.

Regularising Vlasov's equations

In various technical passages of the next sections, a suitable *regularisation* of Vlasov's equation (3.4.7) will be necessary. Namely, given a solution $(\mu_t, F_t)_t$ to (3.4.7), we aim at finding a family $\big((\mu_t^\epsilon, F_t^\epsilon)_t\big)_\epsilon$, for $\epsilon > 0$, such that each curve $(\mu_t^\epsilon, F_t^\epsilon)_t$ is a classical solution to (3.4.7) and $\lim_{\epsilon\to 0}(\mu_t^\epsilon, F_t^\epsilon)_t = (\mu_t, F_t)_t$ in a suitable sense. In particular, a desirable feature is that the approximation is tight enough to ensure that $\lim_{\epsilon\to 0} \int_a^b \|F_t^\epsilon\|^2_{\mathrm{L}^2(\mu_t^\epsilon)}\ \mathrm{d}t = \int_a^b \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\ \mathrm{d}t$, where the non-trivial inequality is $\leq$.

Classically, such arguments are obtained by convolution with *some* regularising kernel (e.g., Gaussian mollifiers). A statement like [AGS08, Lemma 8.1.10]—where $\int_0^T \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\ \mathrm{d}t$ is proved to decrease under any convolution operation—holds true also in our setting, with natural adaptations, see also the proof of the lemma below.

By contrast, we need a novel argument to get a counterpart of [AGS08, Lemma 8.1.9]. There, distributional solutions to the continuity equation $\partial_t\mu_t + \nabla\cdot(X_t\mu_t) = 0$ are approximated with regular solutions to the same equation, via standard convolution. Starting from a solution to Vlasov's equation $\partial_t\mu_t + \nabla_{x,v}\cdot\big((v, F_t)\mu_t\big) = 0$, the standard convolution simply yields a solution to $\partial_t\mu_t^\epsilon + \nabla_{x,v}\cdot(X_t^\epsilon\mu_t^\epsilon) = 0$, without ensuring the structure $X_t^\epsilon = (v, F_t^\epsilon)$. Indeed, the operator $v\cdot\nabla_x$ is not preserved under this regularisation.

To overcome this difficulty, we use a natural convolution product for kinetic equations, taken from [Sil22]. Consider the Lie group of Galilean translations of $\mathbb{R}^{1+2n}$:

$$(t, x, v) \diamond (s, y, w) = (t + s, x + sv + y, v + w)\,, \qquad s, t \in \mathbb{R}\,, \quad x, y \in \mathcal{X} = \mathbb{R}^n\,, \quad v, w \in \mathcal{V} = \mathbb{R}^n\,. \tag{3.4.18}$$

The Galilean inverse is given by $(t, x, v)^{-1} = (-t, -(x - tv), -v)$. The Lebesgue measure on $\mathbb{R}^{1+2n}$ is invariant under left and right translations, i.e., the Galilean group is unimodular.

For finite Borel measures μ, ν on $\mathbb{R}^{1+2n}$, their Galilean convolution $\mu \star \nu$ is the Borel measure defined by

$$\int_{\mathbb{R}^{1+2n}} \varphi \, \mathrm{d}(\mu \star \nu) = \int_{\mathbb{R}^{1+2n}} \int_{\mathbb{R}^{1+2n}} \varphi(\mathfrak{a} \diamond \mathfrak{b}) \, \mathrm{d}\nu(\mathfrak{b}) \, \mathrm{d}\mu(\mathfrak{a}) \,, \qquad \varphi \in \mathrm{C_b}(\mathbb{R}^{1+2n}) \,. \tag{3.4.19}$$

Measures that are absolutely continuous with respect to the Lebesgue measure will be identified with their density. In particular, for $f, g \in \mathrm{L}^1(\mathbb{R}^{1+2n})$, we have

$$(f \star \nu)(\mathfrak{a}) = \int f(\mathfrak{a} \diamond \mathfrak{b}^{-1}) \, \mathrm{d}\nu(\mathfrak{b}) \,, \qquad (\mu \star g)(\mathfrak{b}) = \int g(\mathfrak{a}^{-1} \diamond \mathfrak{b}) \, \mathrm{d}\mu(\mathfrak{a}) \,. \tag{3.4.20}$$

For vector-valued measures we apply this definition component-wise. For $\mathfrak{b} \in \mathbb{R}^{1+2n}$, we consider the left shift $L^{\mathfrak{b}} : \mathfrak{a} \mapsto \mathfrak{b} \diamond \mathfrak{a}$, and the right shift $R^{\mathfrak{b}} : \mathfrak{a} \mapsto \mathfrak{a} \diamond \mathfrak{b}$. Then,

$$L^{\mathfrak{b}}_{\#}(\mu \star \nu) = (L^{\mathfrak{b}}_{\#}\mu) \star \nu \,, \qquad R^{\mathfrak{b}}_{\#}(\mu \star \nu) = \mu \star (R^{\mathfrak{b}}_{\#}\nu) \,. \tag{3.4.21}$$

The relevance of the Galilean group for Vlasov's equation becomes apparent when considering infinitesimal Galilean translations. Indeed, for fixed $\mathfrak{b} = (t, x, v) \in \mathbb{R}^{1+2n}$ with $tv = 0$, the left translation operators $\left(T^{\mathfrak{b}}_s\right)_{s \in \mathbb{R}}$ act on functions $f \in \mathrm{L}^1(\mathbb{R}^{1+2n})$ via

$$\left(T^{\mathfrak{b}}_s f\right)(\mathfrak{a}) = f(\mathfrak{a} \diamond s\mathfrak{b}) \,, \qquad \mathfrak{a} \in \mathbb{R}^{1+2n} \,.$$

These operators satisfy the group property $T^{\mathfrak{b}}_r \circ T^{\mathfrak{b}}_s = T^{\mathfrak{b}}_{s+r}$ for $r, s \in \mathbb{R}$, since $tv = 0$. For smooth functions we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} T^{\mathfrak{b}}_s f = \begin{cases} \partial_t f + v \cdot \nabla_x f & \text{if } \mathfrak{b} = (1, 0, 0) \,, \\ \partial_{x_i} f & \text{if } \mathfrak{b} = (0, e_i, 0) \,, \\ \partial_{v_i} f & \text{if } \mathfrak{b} = (0, 0, e_i) \,. \end{cases}$$

Hence, in view of the commutation relation $T^{\mathfrak{b}}_s (f \star g) = f \star T^{\mathfrak{b}}_s g$, we infer that

$$\begin{aligned} (\partial_t + v \cdot \nabla_x)(f \star g) = f \star (\partial_t + v \cdot \nabla_x) g \,, \qquad & \partial_{x_i}(f \star g) = f \star (\partial_{x_i} g) \,, \\ & \partial_{v_i}(f \star g) = f \star (\partial_{v_i} g) \,. \end{aligned} \tag{3.4.22}$$

Lemma 3.4.9. *Let $(\mu_t, F_t)_t$ be a solution on (a, b) to the Vlasov equation (3.4.7), with*

$$\int_a^b \int_\Gamma \left(|v|^2 + |F_t|^2\right) \mathrm{d}\mu_t \, \mathrm{d}t < \infty \,. \tag{3.4.23}$$

Then, there exists an approximating sequence $\left((\mu^\epsilon_t, F^\epsilon_t)_t\right)_{\epsilon > 0}$, such that

1. *For all $\epsilon > 0$, the function $(\mu^\epsilon_t, F^\epsilon_t)_t$ is smooth in (t, x, v), satisfies (3.4.6)-(3.4.10), and solves the Vlasov equation (3.4.7) on (a, b) in the classical sense.*

2. *The following bounds hold true:*

$$\int_a^b \int_\Gamma |F^\epsilon_t|^2 \, \mathrm{d}\mu^\epsilon_t \, \mathrm{d}t \le \int_a^b \int_\Gamma |F_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \,, \tag{3.4.24}$$

$$\int_a^b \int_\Gamma |v|^2 \, \mathrm{d}\mu^\epsilon_t \, \mathrm{d}t \le \int_a^b \int_\Gamma |v|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t + C\,\epsilon^2 \,, \tag{3.4.25}$$

   for some constant $C > 0$, and for all $\epsilon > 0$.

3. *The sequence* $\left((\mu_t^\epsilon, F_t^\epsilon)_t\right)_\epsilon$ *converges to* $(\mu_t, F_t)_t$ *as* $\epsilon \downarrow 0$ *in the following sense:*

$$\forall t \in (a,b) \qquad \mu_t^\epsilon \rightharpoonup \mu_t \quad \text{and} \quad F_t^\epsilon \mu_t^\epsilon \rightharpoonup F_t \mu_t \,, \tag{3.4.26}$$

narrowly, and

$$\|(v, F_t^\epsilon)\|^2_{\mathrm{L}^2(\mu_t^\epsilon\,\mathrm{d}t)} \to \|(v, F_t)\|^2_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \,. \tag{3.4.27}$$

Proof. Let us take a smooth function $\eta = \eta(t,x,v) : \mathbb{R}^{1+2n} \to \mathbb{R}_{\geq 0}$, with globally bounded derivatives, unitary integral, and the moment bound

$$\int_{-1}^{1} \int_\Gamma |v|^2 \eta \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t < \infty \,,$$

that is also symmetric w.r.t. the variable v (i.e., $\eta(\cdot,\cdot,-v) = \eta(\cdot,\cdot,v)$, for all $v \in \mathcal{V}$), strictly positive when the variable t lies in $(-1,1)$, and equal to 0 otherwise. We introduce the mollifiers

$$\eta_\epsilon(t,x,v) := \epsilon^{-2-4n} \eta(\epsilon^{-2} t, \epsilon^{-3} x, \epsilon^{-1} v) \,, \qquad \epsilon > 0 \,.$$

Given $(\mu_t, F_t)_{t \in (a,b)}$, we consider their trivial extension to curves defined on $\mathbb{R}$:

$$F_t = 0 \quad \text{and} \quad \partial_t \mu_t + v \cdot \nabla_x \mu_t = 0 \quad \text{for } t \notin [a,b] \,.$$

We define $E := F\,\mu$ and consider the regularised measures

$$\mu^\epsilon := \eta_\epsilon \star \mu \,, \qquad E^\epsilon := \eta_\epsilon \star E \,, \qquad \epsilon > 0 \,,$$

Smoothness of μ^ϵ and E^ϵ are indeed a consequence of the last display in [Sil22, Page 6]:

$$(\nabla_v + t\nabla_x, \nabla_{t,x})\,(\mu^\epsilon, E^\epsilon) = (\nabla_v + t\nabla_x, \nabla_{t,x}) \eta_\epsilon \star (\mu, E) \,.$$

Let now $F^\epsilon := \frac{E^\epsilon}{\mu^\epsilon}$, where we identify regular measures with their densities. Following the proof of [Sil22, Lemma 4.2], we will show that $(\mu^\epsilon, F^\epsilon)$ solves the Vlasov equation

$$\partial_t \mu_t^\epsilon + v \cdot \nabla_x \mu_t^\epsilon + \nabla_v \cdot (F_t^\epsilon \mu_t^\epsilon) = 0 \,.$$

Indeed, write $\tilde{\eta}^\epsilon(t,x,v) = \eta^\epsilon((t,x,v)^{-1})$ where $(t,x,v)^{-1}$ denotes the Galilean inverse. Using (3.4.22) and the fact that (μ, F) solves the Vlasov equation, we obtain for any test function $\varphi \in \mathrm{C}_\mathrm{c}^\infty\big((a,b) \times \Gamma\big)$,

$$\begin{aligned} &\int_\mathbb{R} \int_\Gamma (\partial_t + v \cdot \nabla_x)\varphi \,\mathrm{d}\mu^\epsilon = \int_\mathbb{R} \int_\Gamma \tilde{\eta}_\epsilon \star (\partial_t + v \cdot \nabla_x)\varphi \,\mathrm{d}\mu = \int_\mathbb{R} \int_\Gamma (\partial_t + v \cdot \nabla_x)(\tilde{\eta}_\epsilon \star \varphi) \,\mathrm{d}\mu \\ &= -\int_\mathbb{R} \int_\Gamma \nabla_v (\tilde{\eta}_\epsilon \star \varphi) \,\mathrm{d}(F\mu) = -\int_\mathbb{R} \int_\Gamma \tilde{\eta}_\epsilon \star \nabla_v \varphi \,\mathrm{d}(F\mu) = -\int_\mathbb{R} \int_\Gamma \nabla_v \varphi \cdot F^\epsilon \,\mathrm{d}\mu^\epsilon \,, \end{aligned}$$

which proves the claim. Since η^ϵ is an approximation of the identity, it holds true that

$$\mu^\epsilon \rightharpoonup \mu_t \,\mathrm{d}t \,, \qquad E^\epsilon \rightharpoonup F_t\,\mu_t \,, \qquad \text{as } \epsilon \to 0 \,.$$

Using Jensen's inequality for the jointly convex function $(E, \mu) \mapsto \frac{|E|^2}{\mu}$ as in [AGS08, Lemma 8.1.10], we obtain the pointwise inequality

$$|F^\epsilon|^2\,\mu^\epsilon = \frac{|\eta^\epsilon \star E|^2}{\eta^\epsilon \star \mu} \leq \eta^\epsilon \star (|F|^2\,\mu) \,.$$

Integration over $\mathbb{R}\times\Gamma$ yields

$$\|F_t^\epsilon\|^2_{\mathrm{L}^2(\mu_t^\epsilon\,\mathrm{d}t)} \le \|F_t\|^2_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)}\,, \qquad \epsilon>0\,.$$

By [San15, Proposition 5.18], we have

$$\|(v,F_t)\|^2_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \le \liminf_{\epsilon\downarrow 0}\ \|(v,F_t^\epsilon)\|^2_{\mathrm{L}^2(\mu_t^\epsilon\,\mathrm{d}t)}\,.$$

Finally, using that $\int v\,\eta(\cdot,\cdot,v)\,\mathrm{d}v=0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}\times\Gamma}|v|^2\,\mu_t^\epsilon(x,v)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = \int_{\mathbb{R}\times\Gamma}|v|^2\,\mathrm{d}(\eta^\epsilon\star\mu) = \int_{\mathbb{R}\times\Gamma}\eta^\epsilon\star|v|^2\,\mathrm{d}\mu\\
&= \int_{\mathbb{R}\times\Gamma}\int_{\mathbb{R}\times\Gamma}\eta_\epsilon(s,y,w)\,|v-w|^2\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\mu_t(x,v)\,\mathrm{d}t\\
&= \int_{\mathbb{R}\times\Gamma}\int_{\mathbb{R}\times\Gamma}\eta_\epsilon(s,y,w)\,\left(|v|^2+|w|^2\right)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}\mu_t(x,v)\,\mathrm{d}t\\
&= \|v\|^2_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} + \int_{\mathbb{R}\times\Gamma}\eta_\epsilon(s,y,w)\,|w|^2\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}w\\
&\le \|v\|^2_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} + C\epsilon^2,
\end{aligned}$$

with an explicit constant $C>0$ independent of ϵ. □

Proof of the kinetic Benamou–Brenier formula

In classical optimal transport, the Kantorovich problem admits an equivalent fluid-dynamics formulation, as was shown by J.-D. Benamou and Y. Brenier [BB00]. The idea is that optimally transporting ρ_0 to ρ_1 is equivalent to finding the minimal velocity field $(V_t)_t$ one should apply to make particles *flow* from one measure to the other. This velocity field induces an evolution of measures $t\mapsto\rho_t$ that satisfies the continuity equation

$$\partial_t\rho_t+\nabla\cdot(V_t\rho_t)=0\,.$$

Here, we recover a similar interpretation for the second-order discrepancy d. The *kinetic optimal transport* between μ,ν is given by the minimal force field $(F_t)_t$ required to *push* particles from μ to ν. In this case, $t\mapsto\mu_t$ evolves according to the Vlasov equation (3.4.7).

Theorem 3.4.10 (Kinetic Benamou–Brenier formula)**.** *For every $\mu,\nu\in\mathcal{P}_2(\Gamma)$ and $T>0$, the problem* (3.1.15) *admits a minimiser. Moreover, we have the identities*

$$\tilde{\mathsf{n}}_T(\mu,\nu)=\tilde{\mathsf{d}}_T(\mu,\nu)=\widetilde{\mathcal{MA}}_T(\mu,\nu)\,. \tag{3.4.28}$$

Proof. Fix $T>0$. We say that a curve $(\mu_t)_t:[0,T]\to\mathcal{P}_2(\Gamma)$ is admissible and belongs to the class $\mathcal{N}_T(\mu,\nu)$ if $(\mu_t,F_t)_t$ solves (3.4.7) for a vector field $(F_t)_t$ satisfying

$$\int_0^T\int_\Gamma\left(|v|^2+|F_t|^2\right)\,\mathrm{d}\mu_t\,\mathrm{d}t<\infty \tag{3.4.29}$$

and

$$\mu_0=\mu\,,\qquad \mu_T=\nu\,. \tag{3.4.30}$$

We shall prove that $\tilde{\mathsf{d}}_T \geq \widetilde{\mathcal{MA}}_T \geq \tilde{\mathsf{n}}_T$, which is sufficient, since $\tilde{\mathsf{n}}_T = \tilde{\mathsf{d}}_T$, in view of Theorem 3.4.1. To this end, fix $\mu, \nu \in \mathcal{P}_2(\Gamma)$ and, for now, assume that μ is absolutely continuous with respect to the Lebesgue measure. We shall prove that

$$\inf_{\pi \in \Pi(\mu,\nu)} \tilde{\mathsf{c}}_T(\pi) \geq \inf_{(\mu_t)_t \in \mathcal{N}_T(\mu,\nu)} T \int_0^T \int_\Gamma |F_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t \,.$$

Notice first that the value $\inf_{\pi \in \Pi(\mu,\nu)} \tilde{\mathsf{c}}_T(\pi)$ is attained at a transport plan of the form $\pi = (\mathrm{id}, M_T)_{\#}\mu$, for a map $M_T = (Y_T, W_T) \colon \Gamma \to \Gamma$, see Theorem 3.1.1. Now define the flow $(M_t)_t$, for $t \in [0,T]$, via

$$M_t(x,v) = (x_t, v_t), \qquad (x_t, v_t) = \left(\alpha^T_{x,v,M_T(x,v)}(t), (\alpha^T_{x,v,M_T(x,v)})'(t)\right) , \qquad t \in [0,T] ,$$

using the same notation as Lemma 3.4.4. For every $t \in (0,T)$, the map M_t is injective on a full μ-measure set, as shown in Lemma 3.4.4. Let now

$$F_t(y,w) := \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=t} \alpha^T_{M_t^{-1}(y,w), M_T \circ M_t^{-1}(y,w)}(s) \,, \qquad t \in (0,T) \,. \tag{3.4.31}$$

It is clear that $(M_T)_{\#}\mu = \nu$, and, setting $\bar{\mu}_t = (M_t)_{\#}\mu$, we claim that $(\bar{\mu}_t)_t$ is a narrowly-continuous curve such that (3.4.7) holds, with the vector field $(F_t)_t$ given above. To prove this, we fix a smooth test function $\varphi \in \mathrm{C}^\infty_\mathrm{c}\big([0,T] \times \Gamma\big)$ and compute

$$\begin{aligned} &\int_0^T \int_\Gamma \partial_t \varphi(t,x,v) \, \mathrm{d}\bar{\mu}_t(x,v) \, \mathrm{d}t = \int_0^T \int_\Gamma \partial_t \varphi(t, x_t, v_t) \, \mathrm{d}\mu_0(x,v) \, \mathrm{d}t \\ &= \int_0^T \int_\Gamma \left(\frac{\mathrm{d}}{\mathrm{d}t} \varphi(t,x_t,v_t) - v_t \cdot \nabla_x \varphi(t,x_t,v_t) - F_t(x_t,v_t) \cdot \nabla_v \varphi(t,x_t,v_t) \right) \mathrm{d}\mu_0(x,v) \, \mathrm{d}t \\ &= \int_\Gamma \varphi(0,x,v) \, \mathrm{d}\mu_0 - \int_\Gamma \varphi(T, x_T, v_T) \, \mathrm{d}\mu_0 - \int_0^T \int_\Gamma (v \cdot \nabla_x \varphi + \nabla_v \varphi \cdot \nabla_v F_t) \, \mathrm{d}\bar{\mu}_t \, \mathrm{d}t \,, \end{aligned}$$

which is the weak formulation of (3.4.7) with fixed endpoints, as

$$\int_\Gamma \varphi(T, x_T, v_T) \, \mathrm{d}\mu_0(x,v) = \int_\Gamma \varphi(T,\cdot,\cdot) \, \mathrm{d}\mu_T = \int_\Gamma \varphi(T,\cdot,\cdot) \, \mathrm{d}\nu \,.$$

It it easy to check—using (3.2.2)—that

$$\int_0^T \int_\Gamma |v_t|^2 \, d\bar{\mu}_t \, \mathrm{d}t \lesssim_T \iint \left(|x|^2 + |v|^2 + |y|^2 + |w|^2 \right) \mathrm{d}\mu(x,v) \, \mathrm{d}\nu(y,w) < \infty \,.$$

In addition,

$$\begin{aligned} T \int_0^T \int_\Gamma |F_t|^2 d\bar{\mu}_t \, \mathrm{d}t &= T \int_0^T \int_\Gamma |F_t(x_t,v_t)|^2 \, \mathrm{d}\mu \, \mathrm{d}t = \int T \int_0^T |F_t(x_t,v_t)|^2 \, \mathrm{d}t \, \mathrm{d}\mu \\ &= \int_\Gamma \left(12 \left| \frac{Y_T(x,v) - x}{T} - \frac{W_T(x,v) + v}{2} \right|^2 + |v - W_T(x,v)|^2 \right) \mathrm{d}\mu(x,v) \\ &= \tilde{\mathsf{c}}_T(\pi) = \tilde{\mathsf{d}}^2_T(\mu,\nu) \,, \end{aligned}$$

since π is optimal. Then, $(\bar{\mu}_t)$ solves (3.4.7)—in particular it belongs to the class $\mathcal{N}_T(\mu,\nu)$—and $\widetilde{\mathcal{MA}}_T^2(\mu,\nu) \leq \tilde{\mathsf{d}}^2_T(\mu,\nu)$, every time μ is absolutely continuous.

We get rid of this additional assumption. Let $(\mu^k)_k$ be an approximation of μ in the Wasserstein metric, such that μ^k is absolutely continuous for all $k \in \mathbb{N}$, and let M_T^k be the optimal transport map for $\tilde{\mathrm{d}}_T(\mu^k, \nu)$. Define the splines flow $(M_t^k)_t$ associated with M_T^k via (3.4.5), let $\mu_t^k := (M_t^k)_\# \mu^k$, and let $(F_t^k)_t$ be given by (3.4.31). Note that $(\mu_t^k, F_t^k)_t$ solves (3.4.7), for all $k \in \mathbb{N}$. Using the explicit expressions of (3.2.2), and indicating with $(x_t, v_t)_t = M_t^k$ the solution of (3.4.11), we find

$$
\begin{aligned}
&\int_0^T \int \left(|v|^2 + |F_t^k|^2\right) \mathrm{d}\mu_t^k \,\mathrm{d}t = \int_0^T \int \left(|v_t|^2 + |F^k(x_t, v_t)|^2\right) \mathrm{d}\mu_0^k \,\mathrm{d}t \\
&= \int \int_0^T \left(|v_t|^2 + |F^k(x_t, v_t)|^2\right) \mathrm{d}t \,\mathrm{d}\mu_0^k \lesssim_T \int \left(|x|^2 + |v|^2 + |M_T^k(x,v)|^2\right) \mathrm{d}\mu_0^k \\
&\leq \int \left(|x|^2 + |v|^2 + |y|^2 + |w|^2\right) \mathrm{d}\mu_0^k(x,v)\, d\nu(y,w) \\
&\lesssim 1 + \int \left(|x|^2 + |v|^2 + |y|^2 + |w|^2\right) \mathrm{d}\mu_0(x,v)\, \mathrm{d}\nu(y,w) \leq C < \infty\,.
\end{aligned}
$$

In addition,

$$
\int (|x|^2 + |v|^2)\, \mathrm{d}\mu_t^k(x,v) = \int (|x_t|^2 + |v_t|^2)\, \mathrm{d}\mu_0^k \leq C' < \infty, \ \text{uniformly in}\, t \in [0,T],\ k \in \mathbb{N}\,.
$$

Then, following [DNS09, Lemma 4.5] we have that, up to a subsequence, $\mu_t^k \rightharpoonup \bar{\mu}_t$ for all $t \in [0,T]$, and $(v, F_t^k)\mu_t^k \,\mathrm{d}t \rightharpoonup Z$ narrowly, for some measures $\bar{\mu}_t \in \mathcal{P}_2(\Gamma)$ and $Z \in \mathcal{M}([0,T] \times \Gamma; \mathbb{R}^{2n})$. By uniform integrability of $t \mapsto \int |(v, F_t^k)|\, \mathrm{d}\mu_t$ with respect to k, we have that $Z = \Xi_t\, \mathrm{d}t$, for a vector-valued measure Ξ_t satisfying

$$
\int \int_\Gamma \frac{|\Xi_t|^2}{\bar{\mu}_t}\, \mathrm{d}t \leq \liminf_{k\to\infty} \int_0^T \int_\Gamma \left(|v|^2 + |F_t^k|^2\right) \mathrm{d}\mu_t^k \,\mathrm{d}t\,.
$$

Finally, by [San15, Proposition 5.18], we have that $\Xi_t = X_t \bar{\mu}_t$ for a vector field $X_t = (X_t^{(1)}, X_t^{(2)}) \in \mathrm{L}^2(\bar{\mu}_t; \mathbb{R}^{2n})$ and a.e. $t \in [0,T]$. By weak convergence, $\nabla_x \cdot (X_t^{(1)} \mu_t) = v \cdot \nabla_x \bar{\mu}_t$. Let $F_t := X_t^{(2)}$. Passing to the limit in the weak formulation of (3.4.7), we have that $(\bar{\mu}_t, F_t)_t$ is a solution to (3.4.7), such that $(\bar{\mu}_t)_t$ is admissible for $\widetilde{\mathcal{MA}}_T(\mu,\nu)$.

Using lower semi-continuity (see again [San15, Proposition 5.18]), we achieve

$$
\begin{aligned}
\widetilde{\mathcal{MA}}_T^2(\mu,\nu) \leq T \int_0^T \int |F_t|^2 \,\mathrm{d}\bar{\mu}_t \,\mathrm{d}t \leq \liminf_{k\to\infty} T \int_0^T \int |F_t^k|^2 \,\mathrm{d}\mu_t^k \,\mathrm{d}t = \liminf_{k\to\infty} \tilde{\mathrm{d}}_T^2(\mu^k,\nu) \\
= \tilde{\mathrm{d}}_T^2(\mu,\nu)\,,
\end{aligned}
$$

where the second to last equality holds because $(\mu_t^k, F_t^k)_t$ are optimal spline interpolations, and the last equality is a consequence of the Wasserstein convergence $\mu^k \to \mu$ and of the sequential Wasserstein continuity of $\tilde{\mathrm{d}}_T$.

For the inequality $\tilde{\mathrm{n}}_T \leq \widetilde{\mathcal{MA}}_T$, let $(\mu_t, F_t)_t$ be any admissible curve in (3.1.15). By the smoothing procedure of Lemma 3.4.9, we can find a sequence $\left((\mu_t^\epsilon, F_t^\epsilon)_t\right)_\epsilon$ of classical solutions to (3.4.7) such that

$$
\int_0^T \int |F_t^\epsilon|^2 \,\mathrm{d}\mu_t^\epsilon \,\mathrm{d}t \leq \int_0^T \int |F_t|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t\,, \qquad \int_0^T \int |v|^2 \,\mathrm{d}\mu_t^\epsilon \,\mathrm{d}t \leq 1 + \int_0^T \int |v|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t\,.
$$

By Proposition 3.4.7—more precisely following [AGS08, Proposition 8.1.8]—for all $\epsilon > 0$, we have $\mu_t^\epsilon = (M_t^\epsilon)_\# \mu_0^\epsilon$ for all $t \in [0,T]$, where M^ϵ is the flow generated by the vector field $(v, F_t^\epsilon)_t$. Let $\mathbf{m}^\epsilon \in \mathcal{P}\big(\mathrm{H}^2(0,T;\mathcal{X})\big)$ be defined via

$$\mathbf{m}^\epsilon := \int \delta_{M^\epsilon(x,v)} \, \mathrm{d}\mu_0^\epsilon(x,v) \, .$$

For all $\epsilon > 0$ and $t \in [0,T]$, it holds true that

$$(\mathrm{pr}_{(\alpha(t),\alpha'(t))})_\# \mathbf{m}^\epsilon = \mu_t^\epsilon \quad \text{and} \quad (\mathrm{pr}_{(\alpha(t),\alpha'(t))})_\# \Big(\big|\alpha''(t)\big|^2 \mathbf{m}^\epsilon \Big) = |F_t^\epsilon|^2 \, \mu_t^\epsilon \, .$$

Then, as in [AGS08], we have that the sequence $(\mathbf{m}^\epsilon)_\epsilon$ is tight, and we call $\mathbf{m}$ any narrow limit point of $(\mathbf{m}^\epsilon)_\epsilon$. Narrow convergence, together with Lemma 3.4.9, ensures that $(\mathrm{pr}_{(\alpha(t),\alpha'(t))})_\# \mathbf{m} = \mu_t$, for all $t \in [0,T]$, and, in particular, $(\mathrm{pr}_{(\alpha(0),\alpha'(0))})_\# \mathbf{m} = \mu$, and $(\mathrm{pr}_{(\alpha(T),\alpha'(T))})_\# \mathbf{m} = \nu$. By semicontinuity,

$$\begin{aligned} \tilde{\mathrm{n}}_T^2(\mu,\nu) &\leq T \int_0^T \int_{\mathrm{H}^2(0,T;\mathcal{X})} |\alpha''(t)|^2 \, \mathrm{d}\mathbf{m}(\alpha) \, \mathrm{d}t \leq \liminf_{\epsilon \downarrow 0} T \int_0^T \int_{\mathrm{H}^2(0,T;\mathcal{X})} |\alpha''(t)|^2 \, \mathrm{d}\mathbf{m}^\epsilon \, \mathrm{d}t \\ &= \liminf_{\epsilon \downarrow 0} T \int_0^T \int |F_t^\epsilon|^2 \, \mathrm{d}\mu_t^\epsilon \, \mathrm{d}t = T \int_0^T \int |F_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t, \end{aligned}$$

where we used the strong convergence induced by Lemma 3.4.9. This concludes the equivalence, by taking the infimum over $(\mu_t, F_t)_t$. As a by-product, the curve $(\bar{\mu}_t)_t$ built above is a minimiser in (3.1.15). □

Remark 3.4.11. *A posteriori*, the proof shows that optimal curves in (3.1.15) are given by injective interpolation along splines, when μ is absolutely continuous, see also Proposition 3.4.3. Indeed, in this case, when $\mu \ll \mathrm{d}x \, \mathrm{d}v$, the curve $\mu_t = (M_t)_\# \mu$ is optimal in (3.1.15), where M_t is the flow of (3.4.5). The general case is a mixture of spline interpolations.

Remark 3.4.12. Our result proves[4] a conjecture of [CCG18], i.e., the equivalence of [CCG18, Formula (14)] and [CCG18, Formula (3)]. Indeed, in our language [CCG18, Formula (14)] reads

$$\inf_{\mu_0,\mu_1} \left\{ \tilde{\mathrm{n}}_1^2(\mu_0,\mu_1) \, : \, (\mathrm{pr}_x)_\# \mu_i = \rho_i, \, i = 0,1 \right\},$$

while [CCG18, Formula (3)] corresponds to

$$\inf_{\mu_0,\mu_1} \left\{ \widetilde{\mathcal{MA}}_1^2(\mu_0,\mu_1) \, : \, (\mathrm{pr}_x)_\# \mu_i = \rho_i, \, i = 0,1 \right\},$$

and equality between the two is a straightforward consequence of Theorem 3.4.10.

Y. Chen, G. Conforti, and T. T. Georgiou conjecture such an equivalence in [CCG18, Claim 4.1], and provide a formal argument in favour of it. At the same time, the authors remark the lack of a rigorous proof. Our Theorem 3.4.10 fills the gap and completes the proof, by building on the argument of [CCG18] with the crucial addition of two new ingredients: the injectivity of the map M_t (allowing for the definition of F_t) and the Galilean approximation of solutions to (3.4.7) via Lemma 3.4.9.

[4] in case only two measures are considered

3.4.4 Moment estimates for Vlasov's equations

In this section we prove propagation estimates for moments along solutions to (3.4.7). In particular, the following results show that a solution $(\mu_t)_{t\in[a,b]}$ of (3.4.7) stays in $\mathcal{P}_2(\Gamma)$, provided the initial datum $\mu_a \in \mathcal{P}_2(\Gamma)$.

Before turning to the rigorous estimates, let us give a heuristic argument. Let $(\mu_t, F_t)_t$ be a solution to (3.4.7). Then, formally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\mathrm{L}^2(\mu_t)} = -\int |v|^2 \nabla_{x,v}\cdot((v,F_t)\mu_t) = 2\int v\cdot F_t \,\mathrm{d}\mu_t \le 2\|v\|_{\mathrm{L}^2(\mu_t)}\|F_t\|_{\mathrm{L}^2(\mu_t)} ,$$

from which we obtain $\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{\mathrm{L}^2(\mu_t)} \le \|F_t\|_{\mathrm{L}^2(\mu_t)}$. Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2_{\mathrm{L}^2(\mu_t)} = -\int |x|^2 \nabla_x\cdot(v\,\mu_t) = 2\|x\|_{\mathrm{L}^2(\mu_t)}\|v\|_{\mathrm{L}^2(\mu_t)} ,$$

and, therefore, $\frac{\mathrm{d}}{\mathrm{d}t}\|x\|_{\mathrm{L}^2(\mu_t)} \le \|v\|_{\mathrm{L}^2(\mu_t)}$.

Lemma 3.4.13 (Moment estimate)**.** *Let $(\mu_t, F_t)_t$ be a narrowly continuous solution on $[a,b]$ to the Vlasov equation (3.4.7) with $\mu_a \in \mathcal{P}_2(\Gamma)$ and $\int_a^b\int |F_t|^2 \,\mathrm{d}\mu_t\,\mathrm{d}t < \infty$. Then, for every $t\in(a,b)$:*

$$\|v\|_{\mathrm{L}^2(\mu_t)} \le \|v\|_{\mathrm{L}^2(\mu_a)} + \int_a^t \|F_s\|_{\mathrm{L}^2(\mu_s)}\,\mathrm{d}s \tag{3.4.32}$$

and

$$\begin{aligned}\|x\|_{\mathrm{L}^2(\mu_t)} &\le \|x\|_{\mathrm{L}^2(\mu_a)} + \int_a^t \|v\|_{\mathrm{L}^2(\mu_s)}\,\mathrm{d}s & (3.4.33)\\ &\le \|x\|_{\mathrm{L}^2(\mu_a)} + (t-a)\|v\|_{\mathrm{L}^2(\mu_a)} + \int_a^t (t-s)\|F_s\|_{\mathrm{L}^2(\mu_s)}\,\mathrm{d}s\,. & (3.4.34)\end{aligned}$$

Proof. Let $\psi\in \mathrm{C}^\infty_{\mathrm{c}}((a,b)\times\mathcal{V})$ and $\zeta\in\mathrm{C}^\infty_{\mathrm{c}}(\mathcal{X})$ with $\zeta(0)=1$. For every $\epsilon>0$, the definition of solution to the Vlasov equation implies

$$\int_a^b\int (\zeta(\epsilon x)\partial_t\psi + \epsilon\psi\, v\cdot(\nabla_x\zeta)(\epsilon x) + \zeta(\epsilon x)F_t\cdot\nabla_v\psi)\,\mathrm{d}\mu_t\,\mathrm{d}t = 0\,.$$

Note that $v\psi$ is compactly supported, hence bounded. The dominated convergence theorem (for $\epsilon\to 0$) yields

$$\int_a^b\int (\partial_t\psi + F_t\cdot\nabla_v\psi)\,\mathrm{d}\mu_t\,\mathrm{d}t = 0\,.$$

For every t, consider the disintegration $\mu_t = \int \mu_t^v\,\mathrm{d}(\mathrm{pr}_v)_\#\mu_t$, which gives

$$\int_a^b\int\left(\partial_t\psi + \left(\int F_t\,\mathrm{d}\mu_t^v\right)\cdot\nabla_v\psi\right)\mathrm{d}(\mathrm{pr}_v)_\#\mu_t\,\mathrm{d}t = 0\,. \tag{3.4.35}$$

By arbitrariness of ψ, this argument shows that $\left((\mathrm{pr}_v)_\#\mu_t, \int F_t\,d\mu_t^v\right)$ satisfies the classical continuity equation. Note that the vector field satisfies [AGS08, (8.1.21)]: by assumption

$$\int_a^b\int\left|\int F_t\,\mathrm{d}\mu_t^v\right|^2\mathrm{d}(e_v)_\#\mu_t\,\mathrm{d}t \le \int_a^b\int\int |F_t|^2\,\mathrm{d}\mu_t^v\,\mathrm{d}(\mathrm{pr}_v)_\#\mu_t\,\mathrm{d}t < \infty\,.$$

Therefore, by [AGS08, Theorem 8.2.1], there exists a probability measure $\boldsymbol{\eta}$ such that:

(i) $\boldsymbol{\eta}$ is concentrated on the set of pairs $(v, \beta) \in \mathcal{V} \times \mathrm{H}^1(a, b; \mathcal{V})$ such that $\dot{\beta}(t) = \int F_t(x, \beta(t)) \, \mathrm{d}\mu_t^{\beta(t)}$ for a.e. $t \in (a, b)$, with $\beta(0) = v$;

(ii) for every $t \in [a, b]$, $(\mathrm{pr}_v)_{\#}\mu_t$ equals the push-forward of $\boldsymbol{\eta}$ via the map $(v, \beta) \mapsto \beta(t)$.

For $t \in (a, b)$, using Property (ii), and the Minkowski and Cauchy–Schwarz inequalities,

$$\begin{aligned}
\|v\|_{\mathrm{L}^2(\mu_t)} &= \sqrt{\int |v|^2 \, \mathrm{d}(\mathrm{pr}_v)_{\#}\mu_t} = \sqrt{\int |\beta(t)|^2 \, \mathrm{d}\boldsymbol{\eta}} \\
&\leq \sqrt{\int |\beta(a)|^2 \, \mathrm{d}\boldsymbol{\eta}} + \sqrt{\int \left| \int_a^t \left|\dot{\beta}(s)\right| \, \mathrm{d}s \right|^2 \, \mathrm{d}\boldsymbol{\eta}} \\
&= \|v\|_{\mathrm{L}^2(\mu_a)} + \sqrt{\int \left| \int_a^t \int F_s(x, \beta(s)) \, \mathrm{d}\mu_s^{\beta(s)} \, \mathrm{d}s \right|^2 \, \mathrm{d}\boldsymbol{\eta}} \\
&\leq \|v\|_{\mathrm{L}^2(\mu_a)} + \int_a^t \sqrt{\int \left| \int F_s(x, \beta(s)) \, \mathrm{d}\mu_s^{\beta(s)} \right|^2 \, \mathrm{d}\boldsymbol{\eta}} \, \mathrm{d}s \\
&\leq \|v\|_{\mathrm{L}^2(\mu_a)} + \int_a^t \sqrt{\int \int |F_s(x, \beta(s))|^2 \, \mathrm{d}\mu_s^{\beta(s)} \, \mathrm{d}\boldsymbol{\eta}} \, \mathrm{d}s \\
&= \|v\|_{\mathrm{L}^2(\mu_a)} + \int_a^t \sqrt{\int |F_s(x, v)|^2 \, \mathrm{d}\mu_s} \, \mathrm{d}s \, .
\end{aligned}$$

Let us focus on the other inequality we need to prove. The Vlasov equation can be seen as a classical continuity equation with vector field (v, F_t). It follows from the previous estimates that this vector field satisfies [AGS08, (8.1.21)] and, by [AGS08, Theorem 8.2.1], there exists a probability measure $\boldsymbol{\xi}$ such that:

(i) $\boldsymbol{\xi}$ is concentrated on the set of triples $(x, v, \gamma) \in \mathcal{X} \times \mathcal{V} \times \mathrm{H}^1(a, b; \mathcal{X} \times \mathcal{V})$ such that $\dot{\gamma}_x(t) = \gamma_v(t)$ and $\dot{\gamma}_v(t) = F_t\big(\gamma_x(t), \gamma_v(t)\big)$ for a.e. $t \in (a, b)$, with $\gamma(0) = (x, v)$;

(ii) for every $t \in [a, b]$, μ_t equals the push-forward of $\boldsymbol{\xi}$ via the map $(x, v, \gamma) \mapsto \gamma(t)$.

Hence, for every $t \in (a, b)$, we have:

$$\begin{aligned}
\|x\|_{\mathrm{L}^2(\mu_t)} &= \sqrt{\int |x|^2 \, \mathrm{d}\mu_t} = \sqrt{\int |\gamma_x(t)|^2 \, \mathrm{d}\boldsymbol{\xi}} \\
&\leq \sqrt{\int |\gamma_x(a)|^2 \, \mathrm{d}\boldsymbol{\xi}} + \sqrt{\int \left| \int_a^t \dot{\gamma}_x(s) \, ds \right|^2 \, \mathrm{d}\boldsymbol{\xi}} \\
&= \|x\|_{\mathrm{L}^2(\mu_a)} + \sqrt{\int \left| \int_a^t \gamma_v(s) \, \mathrm{d}s \right|^2 \, \mathrm{d}\boldsymbol{\xi}} \\
&\leq \|x\|_{\mathrm{L}^2(\mu_a)} + \int_a^b \sqrt{\int |\gamma_v(s)|^2 \, \mathrm{d}\boldsymbol{\xi}} \, \mathrm{d}s \\
&= \|x\|_{\mathrm{L}^2(\mu_a)} + \int_a^b \sqrt{\int |v|^2 \, \mathrm{d}\mu_s} \, \mathrm{d}s \, .
\end{aligned}$$

□

3.5 Hypoelliptic Riemannian structure

In this section, we develop a differential calculus induced by d, with the main contributions organised as follows.

- In §3.5.1, we show that solutions $(\mu_t, F_t)_t$ to (3.4.7) are physical and d-absolutely continuous, with the optimal time for $\mathsf{d}(\mu_t, \mu_{t+h})$ being asymptotically h, for $h \downarrow 0$.
- In §3.5.2, we prove the converse: d-absolutely continuous curves of measures $(\mu_t)_t$ can be represented as solutions to (3.4.7), provided the optimal time for $\mathsf{d}(\mu_t, \mu_{t+h})$ is asymptotically h as $h \downarrow 0$. Similarly, we show that physical curves solve (3.4.7).
- In §3.5.3, we show that the minimal $\mathrm{L}^2(\mu_t)$-norm of a force field $(F_t)_t$ such that $(\mu_t, F_t)_t$ solves (3.4.7) can be interpreted as a *metric derivative*, namely, it is, for a.e. t, the limit of $\frac{\mathsf{d}(\mu_t,\mu_{t+h})}{h}$ and $\frac{\tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h})}{h}$ as $h \downarrow 0$.
- In §3.5.4, we extend these results to reparametrisations of (3.4.7) and complete the proof of Theorem 3.1.7.

Henceforth, we assume that $(\mu_t)_{t\in(a,b)} \subseteq \mathcal{P}_2(\Gamma)$ is a narrowly continuous curve. We set

$$\Omega := \left\{ t \in (a,b) \, : \, \|v\|_{\mathrm{L}^2(\mu_t)} > 0 \right\} \tag{3.5.1}$$

and define the *spatial density*

$$\rho_t := (\mathrm{pr}_x)_\# \mu_t \in \mathcal{P}(\mathcal{X}) \,, \qquad t \in (a,b) \,. \tag{3.5.2}$$

Using the disintegration theorem we write $\mathrm{d}\mu_t(x,v) = \mathrm{d}\mu_{t,x}(v)\, \mathrm{d}\rho_t(x)$, where $\mu_{t,x} \in \mathcal{P}(\mathcal{V})$ denotes the distribution of velocities at $x \in \mathcal{X}$, defined ρ_t-a.e.

For every $t \in (a,b)$, let $\overline{V}_t$ be the closure of the space $V := \{\nabla\phi \, : \, \phi \in \mathrm{C}_c^\infty(\mathcal{X})\}$ in $\mathrm{L}^2(\rho_t; \mathbb{R}^d)$. Additionally let $\mathrm{pr}_{\overline{V}_t} \colon \mathrm{L}^2(\rho_t; \mathbb{R}^d) \to \overline{V}_t$ be the corresponding projection operator, and define the *flow velocity*

$$j_t(x) := \int_{\mathcal{V}} v \, \mathrm{d}\mu_{t,x} \,, \qquad (t,x) \in (a,b) \times \mathcal{X} \,, \tag{3.5.3}$$

and the *total momentum*

$$\langle v \rangle_t := \int_\Gamma v \, \mathrm{d}\mu_t = \int_{\mathcal{X}} j_t \, \mathrm{d}\rho_t \,, \qquad t \in (a,b) \,. \tag{3.5.4}$$

Remark 3.5.1. For any $\rho \in \mathcal{P}(\mathcal{X})$, the closure of V in $\mathrm{L}^2(\rho; \mathbb{R}^d)$ contains all constant vector fields. Indeed, fix $u_0 \in \mathbb{R}^d$ and a $\mathrm{C}_c^\infty(\mathbb{R}^d)$ function ζ with support contained in the unit ball, and such that $\zeta \equiv 1$ in a neighbourhood of 0. For every $\epsilon > 0$, set

$$\psi_\epsilon := \zeta(\epsilon x)\, x \cdot u_0 \,, \qquad x \in \mathcal{X} \,.$$

We have

$$\nabla\psi_\epsilon(x) = \zeta(\epsilon x) u_0 + \epsilon(x \cdot u_0) \nabla\zeta(\epsilon x) \in V \,, \qquad x \in \mathcal{X} \,.$$

As $\epsilon \to 0$, the dominated convergence theorem gives

$$\zeta(\epsilon x) u_0 \xrightarrow{\mathrm{L}^2(\rho;\mathbb{R}^d)} \zeta(0) u_0 = u_0 \,,$$

as well as

$$\int_{\mathcal{X}} \left|\epsilon(x\cdot u_0)\nabla\zeta(\epsilon x)\right|^2 \,\mathrm{d}\rho \leq \left|u_0\right|^2 \int_{\left\{|x|<\frac{1}{\epsilon}\right\}} \epsilon^2 |x|^2 |\nabla\zeta(\epsilon x)| \,\mathrm{d}\rho \leq \left|u_0\right|^2 \int_{\mathcal{X}} |\nabla\zeta(\epsilon x)| \,\mathrm{d}\rho$$
$$\to \left|u_0\right|^2 |\nabla\zeta(0)| = 0\,.$$

Let $(s,t)\mapsto \pi_{s,t}\in\Pi_{\mathrm{o,d}}(\mu_s,\mu_t)$ be a measurable selection of d-optimal transport plans, and $T_{s,t}$ the corresponding optimal times,[5] i.e.,

$$T_{s,t} = \begin{cases} 2\dfrac{\|y-x\|^2_{\mathrm{L}^2(\pi_{s,t})}}{(y-x,v+w)_{\pi_{s,t}}} & \text{if } (y-x,v+w)_{\pi_{s,t}}>0\,,\\ 0 & \text{if } \|y-x\|_{\mathrm{L}^2(\pi_{s,t})}=0\,,\\ \infty & \text{otherwise.}\end{cases} \tag{3.5.5}$$

3.5.1 d-regularity of solutions to Vlasov's equations

The results of this subsection are given under the following.

Assumption 3.5.2 (Solution to Vlasov's equation)**.** The curve $(\mu_t)_{t\in(a,b)}$ in $\mathcal{P}_2(\Gamma)$ is a distributional solution to Vlasov's equation (3.4.7) for a field $(F_t)_{t\in(a,b)}$ such that

$$\int_a^b \left(\|v\|^2_{\mathrm{L}^2(\mu_t)} + \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\right) \mathrm{d}t < \infty\,. \tag{3.5.6}$$

Under this assumption, the curve $t\mapsto\mu_t$ is W_2-2-absolutely continuous by [AGS08, Theorem 8.3.1]. It is readily shown that the maps $t\mapsto\|x\|_{\mathrm{L}^2(\mu_t)}$ and $t\mapsto\|v\|_{\mathrm{L}^2(\mu_t)}$ are continuous. Indeed, for any W_2-optimal plan $\pi_{s,t}\in\Pi(\mu_s,\mu_t)$ we have

$$\left|\|v\|^2_{\mathrm{L}^2(\mu_t)} - \|v\|^2_{\mathrm{L}^2(\mu_s)}\right| = \left|\int_{\Gamma\times\Gamma}(v+w)\cdot(v-w)\,\mathrm{d}\pi_{s,t}\right| \leq \mathrm{W}_2(\mu_s,\mu_t)\Big(\|v\|_{\mathrm{L}^2(\mu_t)}+\|v\|_{\mathrm{L}^2(\mu_s)}\Big).$$

Since $t\mapsto\|x\|^2_{\mathrm{L}^2(\mu_t)}+\|v\|^2_{\mathrm{L}^2(\mu_t)}=\mathrm{W}_2^2(\mu_t,\delta_{(0,0)})$ is continuous and thus locally bounded, the continuity of $t\mapsto\|v\|_{\mathrm{L}^2(\mu_t)}$ follows. In particular, the set Ω is open in (a,b). The continuity of $t\mapsto\|x\|_{\mathrm{L}^2(\mu_t)}$ is proved analogously.

Lemma 3.5.3. *Under Assumption 3.5.2, the space-marginal curve $t\mapsto\rho_t$ is W_2-2-a.c. with*

$$|\langle v\rangle_t| \leq \left|\rho_t'\right|_{\mathrm{W}_2} = \left\|\mathrm{pr}_{\overline{V}_t}(j_t)\right\|_{\mathrm{L}^2(\rho_t)} \leq \|v\|_{\mathrm{L}^2(\mu_t)} \quad \textit{for a.e. } t\in(a,b)\,. \tag{3.5.7}$$

Proof. Fix $\psi\in\mathrm{C}_c^\infty\big((a,b)\times\mathcal{X}\big)$. With the same argument as in the proof of Lemma 3.4.13:

$$\int_a^b\int_\Gamma(\partial_t\psi+v\cdot\nabla_x\psi)\,\mathrm{d}\mu_t\,\mathrm{d}t=0\,,$$

from which we get

$$0=\int_a^b\int_{\mathcal{X}}(\partial_t\psi+j_t(x)\cdot\nabla_x\psi)\,\mathrm{d}\rho_t\,\mathrm{d}t=\int_a^b\int_{\mathcal{X}}\Big(\partial_t\psi+\mathrm{pr}_{\overline{V}_t}(j_t)\cdot\nabla_x\psi\Big)\,\mathrm{d}\rho_t\,\mathrm{d}t\,, \tag{3.5.8}$$

[5] All times $T>0$ are optimal when $\|y-x\|_{\mathrm{L}^2(\pi_{s,t})}=0$. In this case, we conventionally choose $T_{s,t}=0$.

where we used that $\nabla_x\psi(t,\cdot) \in \overline{V}_t$ in the last equality. Since ψ is arbitrary, we deduce that $t \mapsto \rho_t$ is a solution to the continuity equation with vector field $\left(\mathrm{pr}_{\overline{V}_t}(j_t)\right)_{t\in(a,b)}$. The identity $|\rho_t'|_{\mathrm{W}_2} = \left\|\mathrm{pr}_{\overline{V}_t}(j_t)\right\|_{\mathrm{L}^2(\rho_t)}$ thus follows from [AGS08, Proposition 8.4.5].

The inequality $\left\|\mathrm{pr}_{\overline{V}_t}(j_t)\right\|_{\mathrm{L}^2(\rho_t)} \le \|v\|_{\mathrm{L}^2(\mu_t)}$ follows from the definition of j_t using Jensen's inequality. Finally, by definition of $\langle v\rangle_t$ and Remark 3.5.1, we write

$$\langle v\rangle_t^2 = \int_{\mathcal{X}} j_t \cdot \langle v\rangle_t \,\mathrm{d}\rho_t = \int_{\mathcal{X}} \mathrm{pr}_{\overline{V}_t}(j_t)\cdot\langle v\rangle_t \,\mathrm{d}\rho_t \le |\langle v\rangle_t| \left\|\mathrm{pr}_{\overline{V}_t}(j_t)\right\|_{\mathrm{L}^2(\rho_t)},$$

which yields $|\langle v\rangle_t| \le \left\|\mathrm{pr}_{\overline{V}_t}(j_t)\right\|_{\mathrm{L}^2(\rho_t)}$. □

Our goal is to prove the following three propositions.

Proposition 3.5.4. *Under Assumption 3.5.2, for every s,t with $a<s<t<b$, we have*

$$\mathsf{d}(\mu_s,\mu_t) \le \tilde{\mathsf{d}}_{t-s}(\mu_s,\mu_t) \le 2\int_s^t \|F_r\|_{\mathrm{L}^2(\mu_r)}\,\mathrm{d}r\,. \tag{3.5.9}$$

Proposition 3.5.5. *Under Assumption 3.5.2, for almost every $t\in(a,b)$, we have*

$$\limsup_{h\downarrow 0}\frac{\mathsf{d}(\mu_t,\mu_{t+h})}{h} \le \limsup_{h\downarrow 0}\frac{\tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h})}{h} \le \|F_t\|_{\mathrm{L}^2(\mu_t)}\,. \tag{3.5.10}$$

Proposition 3.5.6. *Under Assumption 3.5.2, the following assertions hold.*

1. *For a.e. $t\in(a,b)$ such that $|\rho_t'|_{\mathrm{W}_2}>0$, we have*

$$\lim_{h\downarrow 0}\frac{T_{t,t+h}}{h} = 1 \quad \textit{for a.e. } t\in(a,b)\,. \tag{3.5.11}$$

2. *For every $[a',b']\subseteq\Omega$, there exist $\bar{h}>0$ and $g\in \mathrm{L}^2(a',b')$ such that*

$$\sup_{h\in(0,\bar{h})}\frac{T_{t,t+h}}{h} \le g(t) \quad \textit{for all } t\in[a',b']\,. \tag{3.5.12}$$

Consequently, if $|\rho_t'|_{\mathrm{W}_2}>0$ for a.e. $t\in\Omega$, we have $\frac{T_{\cdot,\cdot+h}}{h}\to 1$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ as $h\downarrow 0$.

Proposition 3.5.4 and Proposition 3.5.5 provide upper bounds for the kinetic discrepancies between successive states along $(\mu_t)_t$. The first one applies to any two times s,t with $s<t$, while the second one concerns the infinitesimal change, i.e., it provides an upper bound on the d-derivative. Proposition 3.5.6 shows that the optimal time for d between two successive nearby states along a solution to Vlasov's equation is comparable to the *physical* time. In Proposition 3.5.20 below, the convergence $\frac{T_{t,t+h}}{h}\to 1$ will be improved to $\frac{T_{t,t+h}-h}{h^2}\to 0$ at a.e. times $t\in(a,b)$ where $\langle v\rangle_t\neq 0$.

Remark 3.5.7. Comparing Proposition 3.5.4 and Proposition 3.5.5, we see the presence of an extra factor 2 in the former. Note that a version of Proposition 3.5.5 with the extra factor 2 follows immediately from Proposition 3.5.4. Also notice that, if d were a distance, these two propositions, together, would allow dropping the constant 2 in (3.5.9), see [AGS08, Theorem 1.1.2]. However, this factor is *sharp*, as demonstrated by the following example.

Example 3.5.8. Let $\mathcal{X} = \mathcal{V} = \mathbb{R}$. For $\epsilon \in (0,1)$ we define

$$\alpha(t) := \begin{cases} \epsilon t^2 , & \text{for } t \in \Big[0,1\Big] , \\ -\epsilon + 2\epsilon t - (t-1)^2 & \text{for } t \in \Big[1, 1+\sqrt{\epsilon}\Big] , \end{cases}$$

and, by means of α,

$$\mu_t := \delta_{\big(\alpha(t), \alpha'(t)\big)} , \qquad t \in \Big[0, 1+\sqrt{\epsilon}\Big] .$$

This curve solves Vlasov's equation with $F_t(x,v) := \alpha''(t)$. In particular,

$$\int_0^{1+\sqrt{\epsilon}} \|F_t\|_{\mathrm{L}^2(\mu_t)} \, \mathrm{d}t = 2\epsilon + 2\sqrt{\epsilon} .$$

On the other hand, recalling the definition (3.1.7) of d,

$$\begin{aligned} d^2\Big((\alpha(0), \alpha'(0)), (\alpha(1+\sqrt{\epsilon}), \alpha'(1+\sqrt{\epsilon})\Big) &= d^2\Big((0,0), (2\epsilon\sqrt{\epsilon}, 2\epsilon - 2\sqrt{\epsilon})\Big) \\ &= 3\Big|2\epsilon - 2\sqrt{\epsilon}\Big|^2 - 3\left(\frac{2\epsilon\sqrt{\epsilon}}{\Big|2\epsilon\sqrt{\epsilon}\Big|} \cdot (2\epsilon - 2\sqrt{\epsilon})\right)_+^2 + \Big|2\epsilon - 2\sqrt{\epsilon}\Big|^2 \\ &= 4\Big|2\epsilon - 2\sqrt{\epsilon}\Big|^2 - 3(2\epsilon - 2\sqrt{\epsilon})_+^2 = 4\Big|2\epsilon - 2\sqrt{\epsilon}\Big|^2 , \end{aligned}$$

where the last equality is true because $\epsilon < 1$. Hence,

$$\frac{d\Big((\alpha(0), \alpha'(0)), (\alpha(1+\sqrt{\epsilon}), \alpha'(1+\sqrt{\epsilon})\Big)}{\int_0^{1+\sqrt{\epsilon}} \|F_t\|_{\mathrm{L}^2(\mu_t)} \, \mathrm{d}t} = 2\frac{\Big|2\epsilon - 2\sqrt{\epsilon}\Big|}{2\epsilon + 2\sqrt{\epsilon}} ,$$

and the latter tends to 2 as $\epsilon \to 0$.

Proof of Proposition 3.5.4. Let us fix $s, t \in (a,b)$ with $s < t$. By [AGS08, Theorem 8.2.1], there exists a measure $\boldsymbol{\eta} \in \mathcal{P}\Big(\Gamma \times \mathrm{H}^1(s,t;\Gamma)\Big)$ supported on tuples $(x, v, \gamma_x, \gamma_v)$ such that:

1. $\gamma_x(s) = x$ and $\gamma_v(s) = v$;
2. $\dot{\gamma}_x(r) = \gamma_v(r)$ and $\dot{\gamma}_v(r) = F_r\Big(\gamma_x(r), \gamma_v(r)\Big)$ for a.e. $r \in (s,t)$;
3. $\Big(\mathrm{pr}_{\gamma(r)}\Big)_\# \boldsymbol{\eta} = \mu_r$ for every $r \in (s,t)$.

By definition of $\tilde{\mathsf{d}}_{t-s}$ and by the properties of $\boldsymbol{\eta}$, we write

$$\begin{aligned} \tilde{\mathsf{d}}^2_{t-s}(\mu_s, \mu_t) &\le \tilde{\mathsf{c}}_{t-s}\left(\Big(\mathrm{pr}_{\gamma(s),\gamma(t)}\Big)_\# \boldsymbol{\eta}\right) \\ &= \int \left(12\left|\frac{\gamma_x(t) - \gamma_x(s)}{t-s} - \frac{\gamma_v(t) + \gamma_v(s)}{2}\right|^2 + |\gamma_v(t) - \gamma_v(s)|^2\right) \mathrm{d}\boldsymbol{\eta} \\ &= 3\int \left|\int_s^t \frac{t+s-2r}{t-s} F_r\Big(\gamma_x(r), \gamma_v(r)\Big) \, \mathrm{d}r\right|^2 \mathrm{d}\boldsymbol{\eta} + \int \left|\int_s^t F_r\Big(\gamma_x(r), \gamma_v(r)\Big) \, \mathrm{d}r\right|^2 \mathrm{d}\boldsymbol{\eta} , \end{aligned}$$

which yields, by Minkowski's integral inequality,

$$\begin{aligned}\tilde{\mathsf{d}}^2_{t-s}(\mu_s,\mu_t) &\le 3\left(\int_s^t \frac{|t+s-2r|}{t-s}\sqrt{\int\left|F_r\Big(\gamma_x(r),\gamma_v(r)\Big)\right|^2 \mathrm{d}\boldsymbol{\eta}}\;\mathrm{d}r\right)^2 \\ &\quad + \left(\int_s^t \sqrt{\int\left|F_r\Big(\gamma_x(r),\gamma_v(r)\Big)\right|^2 \mathrm{d}\boldsymbol{\eta}}\;\mathrm{d}r\right)^2 \\ &= 3\left(\int_s^t \frac{|t+s-2r|}{t-s}\|F_r\|_{\mathrm{L}^2(\mu_r)}\,\mathrm{d}r\right)^2 + \left(\int_s^t \|F_r\|_{\mathrm{L}^2(\mu_r)}\,\mathrm{d}r\right)^2 . \end{aligned} \tag{3.5.13}$$

The conclusion follows by estimating $\frac{|t+s-2r|}{t-s} \le 1$. □

Proof of Proposition 3.5.5. Let $t \in (a,b)$ be a Lebesgue point for $\tilde{t} \mapsto \|F_{\tilde{t}}\|_{\mathrm{L}^2(\mu_{\tilde{t}})}$. By the kinetic Benamou–Brenier formula of Theorem 3.4.10 we have, for every $h > 0$,

$$\frac{\tilde{\mathsf{d}}^2_h(\mu_t,\mu_{t+h})}{h^2} \le \frac{\widetilde{\mathcal{MA}}^2_h(\mu_t,\mu_{t+h})}{h^2} \le \fint_t^{t+h} \|F_s\|^2_{\mathrm{L}^2(\mu_s)}\,\mathrm{d}s\,.$$

We conclude by letting $h \downarrow 0$. □

Proof of Proposition 3.5.6

The core idea in the proof of Proposition 3.5.6 is to equate two interpretations of μ_t and μ_{t+h}, as marginals of two different plans in $\Pi(\mu_t,\mu_{t+h})$. One is the d-optimal plan $\pi_{t,t+h}$, i.e., an evolution along a $T_{t,t+h}$-long curve; while the other is the dynamical transport plan induced by Vlasov's equation, hence an evolution taking time h. One of the lemmas we prove after this idea—namely, Lemma 3.5.10—will also be used to compute the d-derivative, see Proposition 3.5.22.

Another key passage in the proof below is the derivation of the local L^2-domination (3.5.12) by means of the upper bound (3.5.9).

Lemma 3.5.9. *Assume Assumption 3.5.2. Fix $[a',b'] \subseteq \Omega$. Then, there exist $\bar{h} > 0$ and a function $g \in \mathrm{L}^2(a',b')$ such that*

$$\frac{T_{t,t+h}}{h} \le g(t) \quad \textit{for all } t \in [a',b'] \textit{ and every } h \in (0,\bar{h})\,. \tag{3.5.14}$$

In particular, for a.e. $t \in \Omega$ (hence, for a.e. t such that $|\rho'_t|_{\mathrm{W}_2} > 0$), we have

$$\limsup_{h\downarrow 0} \frac{T_{t,t+h}}{h} < \infty\,. \tag{3.5.15}$$

Proof. Recall that the functions $\tilde{t} \mapsto \|x\|_{\mathrm{L}^2(\mu_{\tilde{t}})}$ and $\tilde{t} \mapsto \|v\|_{\mathrm{L}^2(\mu_{\tilde{t}})}$ are continuous on (a,b). Let $c > 0$ be an upper bound on the restriction of these functions to $\left[a', \frac{b'+b}{2}\right]$, and $\epsilon > 0$ be the minimum of $\tilde{t} \mapsto \|v\|_{\mathrm{L}^2(\mu_{\tilde{t}})}$ on $[a',b']$. By Assumption 3.5.2 and Proposition 3.5.4, we can find $\bar{h} \in \left(0, \frac{b-b'}{2}\right)$ depending on ϵ, c and $\int_a^b \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\,\mathrm{d}t$, such that

$$h \in (0,\bar{h}) \quad\Longrightarrow\quad \|w-v\|_{\mathrm{L}^2(\pi_{t,t+h})} \le \mathsf{d}(\mu_t,\mu_{t+h}) \le \tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h}) \le \frac{\epsilon^2}{6c}\,. \tag{3.5.16}$$

Fix $t \in [a', b']$ and $h \in (0, \bar{h})$. Let $\boldsymbol{\eta} \in \mathcal{P}\Big(\Gamma \times \mathrm{H}^1(t, t+h; \Gamma)\Big)$ be as in the proof of Proposition 3.5.4 (after replacing (s,t) with $(t, t+h)$) and let $\mathbf{P}$ be a probability measure on $\Gamma \times \Gamma \times \mathrm{H}^1(t, t+h; \Gamma)$ constructed by gluing $\pi_{t,t+h}$ and $\boldsymbol{\eta}$ at μ_t, i.e., such that

$$\Big(\mathrm{pr}_{x,v,y,w}\Big)_{\#}\mathbf{P} = \pi_{t,t+h} \quad \text{and} \quad \Big(\mathrm{pr}_{x,v,\gamma}\Big)_{\#}\mathbf{P} = \boldsymbol{\eta}\,.$$

Using the properties of $\boldsymbol{\eta}$ and $\mathbf{P}$, we write

$$\begin{aligned}
&\int y \cdot w \,\mathrm{d}\mathbf{P} - \int x \cdot v \,\mathrm{d}\mathbf{P} \\
&\quad = \int \gamma_x(t+h) \cdot \gamma_v(t+h) \,\mathrm{d}\mathbf{P} - \int \gamma_x(t) \cdot \gamma_v(t) \,\mathrm{d}\mathbf{P} \\
&\quad = \int \int_t^{t+h} (\dot{\gamma}_x(s) \cdot \gamma_v(t+h) + \dot{\gamma}_v(s) \cdot \gamma_x(t)) \,\mathrm{d}s \,\mathrm{d}\mathbf{P} \\
&\quad = \int \int_t^{t+h} \Big(\gamma_v(s) \cdot \gamma_v(t+h) + F_s\big(\gamma_x(s), \gamma_v(s)\big) \cdot \gamma_x(t)\Big) \,\mathrm{d}s \,\mathrm{d}\mathbf{P}\,.
\end{aligned}$$

Rearranging terms, this identity writes, for every $\theta > 0$, as

$$\begin{aligned}
&\int \Big(\theta |v|^2 + \theta(w - v) \cdot v + (y - x - \theta v) \cdot w\Big) \,\mathrm{d}\mathbf{P} \\
&\quad = \int x \cdot (v - w) \,\mathrm{d}\mathbf{P} + \int \int_t^{t+h} \Big(\gamma_v(s) \cdot \gamma_v(t+h) + F_s\big(\gamma_x(s), \gamma_v(s)\big) \cdot \gamma_x(t)\Big) \,\mathrm{d}s \,\mathrm{d}\mathbf{P}\,.
\end{aligned}$$

By the triangle and Cauchy–Schwarz inequalities, we obtain

$$\begin{aligned}
&\theta \cdot \left(\|v\|^2_{\mathrm{L}^2(\mu_t)} - \|v\|_{\mathrm{L}^2(\mu_t)} \|w - v\|_{\mathrm{L}^2(\pi_{t,t+h})} - \|v\|_{\mathrm{L}^2(\mu_{t+h})} \left\| \frac{y - x}{\theta} - v \right\|_{\mathrm{L}^2(\pi_{t,t+h})} \right) \\
&\leq \|x\|_{\mathrm{L}^2(\mu_t)} \|w - v\|_{\mathrm{L}^2(\pi_{t,t+h})} + \|v\|_{\mathrm{L}^2(\mu_{t+h})} \int_t^{t+h} \|v\|_{\mathrm{L}^2(\mu_s)} \,\mathrm{d}s + \|x\|_{\mathrm{L}^2(\mu_t)} \int_t^{t+h} \|F_s\|_{\mathrm{L}^2(\mu_s)} \,\mathrm{d}s\,;
\end{aligned}$$

hence, since $\|v\|_{\mathrm{L}^2(\mu_t)} > \epsilon$ and $\max\left\{\sup_{\tilde{t}} \|x\|_{\mathrm{L}^2(\mu_{\tilde{t}})}, \sup_{\tilde{t}} \|v\|_{\mathrm{L}^2(\mu_{\tilde{t}})}\right\} \leq c$, we have

$$\begin{aligned}
&\theta \cdot \left(\epsilon^2 - c\, \|w - v\|_{\mathrm{L}^2(\pi_{t,t+h})} - c \left\| \frac{y - x}{\theta} - v \right\|_{\mathrm{L}^2(\pi_{t,t+h})} \right) \\
&\qquad\qquad \leq c\, \|w - v\|_{\mathrm{L}^2(\pi_{t,t+h})} + c^2 h + c \int_t^{t+h} \|F_s\|_{\mathrm{L}^2(\mu_s)} \,\mathrm{d}s\,. \qquad (3.5.17)
\end{aligned}$$

It remains to bound the term $\left\| \frac{y-x}{\theta} - v \right\|_{\mathrm{L}^2(\pi_{t,t+h})}$ for a suitable choice of θ.

If $T_{t,t+h} \in (0, \infty)$, we choose $\theta := T_{t,t+h}$. Using the triangle inequality, the definition of $\tilde{\mathsf{c}}$, the fact that $\|y - x\|_{\mathrm{L}^2(\pi_{t,t+h})} > 0$, and the optimality of $\pi_{t,t+h}$, we obtain

$$\begin{aligned}
&\left\| \frac{y - x}{T_{t,t+h}} - v \right\|_{\mathrm{L}^2(\pi_{t,t+h})} + \|w - v\|_{\mathrm{L}^2(\pi_{t,t+h})} \leq \left\| \frac{y - x}{T_{t,t+h}} - \frac{v + w}{2} \right\|_{\mathrm{L}^2(\pi_{t,t+h})} + \frac{3}{2} \|w - v\|_{\mathrm{L}^2(\pi_{t,t+h})} \\
&\qquad \leq 3\sqrt{\tilde{\mathsf{c}}(\pi_{t,t+h})} = 3\sqrt{\mathsf{c}(\pi_{t,t+h})} = 3\,\mathsf{d}(\mu_t, \mu_{t+h}) \overset{(3.5.16)}{\leq} \frac{\epsilon^2}{2c}\,. \qquad (3.5.18)
\end{aligned}$$

We arrive to the same conclusion in the case where $T_{t,t+h} = \infty$ by letting $\theta \to \infty$. In all cases (trivially when $T_{t,t+h} = 0$), we get the inequality

$$\frac{\epsilon^2 T_{t,t+h}}{2} \le c\,\mathsf{d}(\mu_t, \mu_{t+h}) + c^2 h + c \int_t^{t+h} \|F_s\|_{\mathrm{L}^2(\mu_s)} \,\mathrm{d}s\,,$$

proving in particular that $T_{t,t+h} < \infty$. Bounding $\mathsf{d}(\mu_t, \mu_{t+h})$ with Proposition 3.5.4, we find

$$\frac{T_{t,t+h}}{h} \le \frac{1}{\epsilon^2} \fint_t^{t+h} \Big(6c\|F_s\|_{\mathrm{L}^2(\mu_s)} + 2c^2\Big) \,\mathrm{d}s \le \underbrace{\frac{1}{\epsilon^2} \sup_{\tilde h \in (0,\bar h)} \fint_t^{t+\tilde h} \Big(6c\|F_s\|_{\mathrm{L}^2(\mu_s)} + 2c^2\Big) \,\mathrm{d}s}_{=:g(t)}\,.$$

Since $\tilde t \mapsto \|F_{\tilde t}\|_{\mathrm{L}^2(\tilde t)}$ is L^2, so is g by the strong Hardy–Littlewood maximal inequality. □

Lemma 3.5.10. *Fix $\varphi \in \mathrm{C}^{1,1}_{\mathrm{b}}(\Gamma)$, i.e., φ is bounded and continuously differentiable, with bounded and Lipschitz gradient. Under Assumption 3.5.2, for a.e.*[6] *$t \in \Omega$, we have*

$$\lim_{h \downarrow 0} \bigg(\int_{\Gamma\times\Gamma} \frac{w-v}{h} \cdot \nabla_v \varphi \,\mathrm{d}\pi_{t,t+h} + \frac{T_{t,t+h}}{h} \int_\Gamma v \cdot \nabla_x \varphi \,\mathrm{d}\mu_t \bigg) = \int_\Gamma (v, F_t) \cdot \nabla_{x,v} \varphi \,\mathrm{d}\mu_t\,. \tag{3.5.19}$$

Proof. Fix $t \in \Omega$ satisfying (3.5.15). We also assume that t is a Lebesgue point of

$$s \longmapsto \int_\Gamma \Big(\tilde v, F_s\Big) \cdot \nabla_{x,v}\varphi \,\mathrm{d}\mu_s \quad \text{and} \quad s \longmapsto \|F_s\|_{\mathrm{L}^2(\mu_s)}\,. \tag{3.5.20}$$

Fix $h \in (0, b-t)$ such that $T_{t,t+h} < \infty$, and let $\boldsymbol{\eta}, \mathbf{P}$ be as in Lemma 3.5.9. In particular

$$\int \varphi(y,w) \,\mathrm{d}\mathbf{P} - \int \varphi(x,v) \,\mathrm{d}\mathbf{P} = \int \varphi\Big(\gamma(t+h)\Big) \,\mathrm{d}\mathbf{P} - \int \varphi\Big(\gamma(t)\Big) \,\mathrm{d}\mathbf{P}\,.$$

By the fundamental theorem of calculus and the properties of $\boldsymbol{\eta}$ and $\mathbf{P}$, we deduce that

$$\begin{aligned} &\frac{1}{h} \int_{\Gamma\times\Gamma} \int_0^1 (y-x, w-v) \cdot \nabla_{x,v}\varphi\Big(x + r(y-x), v + r(w-v)\Big) \,\mathrm{d}r \,\mathrm{d}\pi_{t,t+h} \\ &\quad = \fint_t^{t+h} \int \dot\gamma(s) \nabla_{x,v}\varphi\Big(\gamma(s)\Big) \,\mathrm{d}\boldsymbol{\eta} \,\mathrm{d}s = \fint_t^{t+h} \int_\Gamma \Big(\tilde v, F_s(\tilde x, \tilde v)\Big) \cdot \nabla_{x,v}\varphi(\tilde x, \tilde v) \,\mathrm{d}\mu_s \,\mathrm{d}s\,. \end{aligned} \tag{3.5.21}$$

The right-hand side in the latter equality converges to the right-hand side of (3.5.19) as $h \downarrow 0$, since t is a Lebesgue point for (3.5.20).

Focusing on the left-hand side of (3.5.21), we observe that

$$\begin{aligned} &\left| \frac{1}{h} \int_{\Gamma^2} \int_0^1 (y-x, w-v) \cdot \Big(\nabla_{x,v}\varphi\Big(x + r(y-x), v + r(w-v)\Big) - \nabla_{x,v}\varphi(x,v) \Big) \,\mathrm{d}r \,\mathrm{d}\pi_{t,t+h} \right| \\ &\quad \lesssim \frac{\|y-x\|^2_{\mathrm{L}^2(\pi_{t,t+h})} + \|w-v\|^2_{\mathrm{L}^2(\pi_{t,t+h})}}{h} \\ &\quad \lesssim \frac{1}{h} \left(\left\| y - x - T_{t,t+h} \frac{w+v}{2} \right\|^2_{\mathrm{L}^2(\pi_{t,t+h})} + \Big(1 + T^2_{t,t+h}\Big) \|w-v\|^2_{\mathrm{L}^2(\pi_{t,t+h})} + T^2_{t,t+h} \|v\|^2_{\mathrm{L}^2(\mu_t)} \right) \\ &\quad \lesssim \frac{\Big(1 + T^2_{t,t+h}\Big) \tilde{\mathsf{d}}(\mu_t, \mu_{t+h})^2 + T^2_{t,t+h} \|v\|^2_{\mathrm{L}^2(\mu_t)}}{h}\,, \end{aligned} \tag{3.5.22}$$

[6] We allow the negligible set of times where (3.5.19) does not hold to possibly depend on φ.

where the constants hidden in $\lesssim$ do not depend on h. Combining the latter with Proposition 3.5.4, (3.5.15), and the fact that t is a Lebesgue point for $s \mapsto \|F_s\|_{L^2(\mu_s)}$, we infer that (3.5.22) is $o(1)$ for $h \downarrow 0$. Finally, we notice that

$$\left| \int_\Gamma \frac{y - x - T_{t,t+h} v}{h} \cdot \nabla_x \varphi \, d\mu_t \right| \lesssim \frac{\left\| y - x - T_{t,t+h} v \right\|_{L^2(\pi_{t,t+h})}}{h} \overset{(*)}{\lesssim} \frac{T_{t,t+h}}{h} \tilde{d}(\mu_t, \mu_{t+h}) = o(1) \,,$$

where $(*)$ can be proved as in (3.5.18) if $T_{t,t+h} > 0$, and is trivial otherwise. From (3.5.21) and these observations, the conclusion follows. □

Corollary 3.5.11. *Under Assumption 3.5.2, for a.e. $t \in (a, b)$ such that $|\rho'_t|_{W_2} > 0$, we have*

$$\lim_{h \downarrow 0} \frac{T_{t,t+h}}{h} = 1 \,. \tag{3.5.23}$$

Proof. Let $\{\phi_k\}_{k \in \mathbb{N}}$ be a C^1-dense set of $C_c^\infty(\mathcal{X})$. We apply Lemma 3.5.10 with $\varphi \colon (x, v) \mapsto \phi_k(x)$ for every $k \in \mathbb{N}$ to deduce that, for a.e. $t \in (a, b)$ such that $|\rho'_t|_{W_2} > 0$, we have

$$\lim_{h \downarrow 0} \frac{T_{t,t+h}}{h} \int v \cdot \nabla_x \phi_k \, d\mu_t = \int v \cdot \nabla_x \phi_k \, d\mu_t \,, \qquad k \in \mathbb{N} \,. \tag{3.5.24}$$

Let us take any such t for which, additionally, (3.5.7) holds. By Lemma 3.5.3, there exists $\bar{\phi} \in C_c^\infty(\mathcal{X})$ such that $\nabla \bar{\phi}$ is sufficiently close to $\mathrm{pr}_{\overline{V}_t}(j_t)$ in $L^2(\rho_t; \mathbb{R}^d)$, in the sense that

$$\left| \int \nabla \bar{\phi}(x) \cdot v \, d\mu_t \right| = \left| \int \nabla \bar{\phi} \cdot \mathrm{pr}_{\overline{V}_t}(j_t) \, d\rho_t \right| > \frac{|\rho'_t|_{W_2}}{2} > 0 \,.$$

To conclude, it suffices to choose k in (3.5.24) such that $\left\| \bar{\phi} - \phi_k \right\|_{C^1}$ is sufficiently small. □

Proof of Proposition 3.5.6. The result is immediate after Lemma 3.5.9 and Corollary 3.5.11. □

3.5.2 Physical curves solve Vlasov's equations

Let $(\mu_t)_{t \in (a,b)}$ be a narrowly continuous curve in $\mathcal{P}_2(\Gamma)$, let $(s, t) \mapsto \pi_{s,t}$ be a measurable selection of optimal transport plans for d, and let $T_{s,t}$ be the corresponding optimal times, see (3.5.5). Recall $\rho_t := (\mathrm{pr}_x)_\# \mu_t$ and $\Omega := \left\{ t \in (a, b) \, : \, \|v\|_{L^2(\mu_t)} > 0 \right\}$.

Choose measurable functions $(s, t) \mapsto \theta_{s,t} \in [0, \infty]$ and $(s, t) \mapsto \pi^\theta_{s,t} \in \Pi(\mu_s, \mu_t)$. Let $\widehat{\Omega} \subseteq (a, b)$ be an *open* set of times.

Proposition 3.5.12. *Assume the following:*

(a) The curve $(\mu_t)_{t \in (a,b)}$ is W_2-2-absolutely continuous. (Consequently, $t \mapsto \rho_t$ is W_2-2-a.c., and $t \mapsto \|v\|_{L^2(\mu_t)}$ is 2-a.c.)

(b) For every $s < t$ such that $\theta_{s,t} = 0$, we have $y = x$ $\pi^\theta_{s,t}$-a.e.

(c) There exists a function $\ell \in L^2(a, b)$ such that

$$\|w - v\|_{L^2(\pi^\theta_{s,t})} \leq \int_s^t \ell(r) \, dr \,, \qquad a < s < t < b \,. \tag{3.5.25}$$

(d) The set $\widehat{\Omega}$ *has full measure in* (a,b)*. For every* $[a',b'] \subseteq \widehat{\Omega}$*, we have the limits*

$$\lim_{h\downarrow 0} \int_{a'}^{b'} \left| \frac{\theta_{t,t+h}}{h} - 1 \right| \,\mathrm{d}t = 0 \tag{3.5.26}$$

and

$$\lim_{h\downarrow 0} \int_{\left\{t\in(a',b')\,:\,\theta_{t,t+h}\in(0,\infty)\right\}} \frac{\theta_{t,t+h}}{h} \sqrt{\tilde{\mathsf{c}}_{\theta_{t,t+h}}(\pi^{\theta}_{t,t+h})} \,\mathrm{d}t = 0\,. \tag{3.5.27}$$

Then, there exists a force field $(F_t)_t$ *such that* $(\mu_t, F_t)_t$ *solves Vlasov's equation* (3.4.7)*, and* $(F_t)_t$ *belongs to the* $\mathrm{L}^2(\mu_t\,\mathrm{d}t)$*-closure of* $\left\{\nabla_v\varphi \,:\, \varphi\in \mathrm{C}^\infty_\mathrm{c}\big((a,b)\times\Gamma\big)\right\}$.

We discuss the assumptions of Proposition 3.5.12 below, and give a proof at the end of this section. Choosing $\theta_{s,t} := t-s$ and $\theta_{s,t} := T_{s,t}$, we immediately obtain two corollaries.

Corollary 3.5.13. *Assume (a) in Proposition 3.5.12 holds. If there exists* $\ell\in \mathrm{L}^2(a,b)$ *with*

$$\tilde{\mathsf{d}}_{t-s}(\mu_s,\mu_t) \le \int_s^t \ell(r)\,\mathrm{d}r\,, \qquad a<s<t<b\,, \tag{3.5.28}$$

then the conclusion of Proposition 3.5.12 holds.

Proof. Set $\theta_{s,t} := t-s$, and let $(s,t)\mapsto \pi^{\theta}_{s,t}\in\Pi(\mu_s,\mu_t)$ be a measurable selection of $\tilde{\mathsf{d}}_{t-s}$-optimal plans. We set $\widehat{\Omega} := (a,b)$. Then, Assumption (b) in Proposition 3.5.12 is vacuously true, (3.5.25) follows from (3.5.28) because $\tilde{\mathsf{d}}_{t-s}(\mu_s,\mu_t) \ge \|w-v\|_{\mathrm{L}^2(\pi^{\theta}_{s,t})}$, (3.5.26) is obvious, and (3.5.27) follows from (3.5.28). □

Corollary 3.5.14. *Assume (a) in Proposition 3.5.12 and, in addition, the following.*

(c') There exists a function $\ell\in \mathrm{L}^2(a,b)$ *such that*

$$\mathsf{d}(\mu_s,\mu_t) \le \int_s^t \ell(r)\,\mathrm{d}r\,, \qquad a<s<t<b\,. \tag{3.5.29}$$

(d') The set $\widehat{\Omega}$ *has full measure in* (a,b)*. For every* $[a',b']\subseteq\widehat{\Omega}$*, we have the limit*

$$\lim_{h\downarrow 0} \int_{a'}^{b'} \left| \frac{T_{t,t+h}}{h} - 1 \right| \,\mathrm{d}t = 0\,. \tag{3.5.30}$$

Then the conclusion of Proposition 3.5.12 holds.

Proof. Set $\theta_{s,t} := T_{s,t}$ and choose $\pi^{\theta}_{s,t} := \pi_{s,t}$, which has the following property:

$$T_{s,t}\in(0,\infty) \quad\Longrightarrow\quad \mathsf{d}^2(\mu_s,\mu_t) = \mathsf{c}(\pi_{s,t}) = \tilde{\mathsf{c}}_{T_{s,t}}(\pi_{s,t})\,. \tag{3.5.31}$$

Then, Assumption (b) in Proposition 3.5.12 follows from (3.5.5), and (3.5.25) follows from (3.5.29) because $\mathsf{d}(\mu_s,\mu_t) \ge \|w-v\|_{\mathrm{L}^2(\pi_{s,t})}$. Finally, (3.5.27) follows from (3.5.31), (3.5.29), and (3.5.30); indeed,

$$\begin{aligned}\int_{\{t\in(a',b')\,:\,T_{t,t+h}\in(0,\infty)\}} \frac{T_{t,t+h}}{h}\sqrt{\tilde{\mathsf{c}}_{T_{s,t}}(\pi_{s,t})}\ \mathrm{d}t &\le \int_{a'}^{b'} \frac{T_{t,t+h}}{h}\int_t^{t+h} \ell(r)\ \mathrm{d}r\ \mathrm{d}t\,,\\ &\le \|\ell\|_{\mathrm{L}^1}\int_{a'}^{b'}\left|\frac{T_{t,t+h}}{h}-1\right|\ \mathrm{d}t\\ &\quad + h\int_{a'}^{b'+h}\ell(r)\ \mathrm{d}r\,,\end{aligned}$$

which tends to 0 as $h\downarrow 0$ by (3.5.30). □

Remark 3.5.15. The combination of Corollary 3.5.14 and Proposition 3.5.6 yields a self-improvement result for the convergence $\frac{T_{t,t+h}}{h}\to 1$. Indeed, let us make the assumptions of Corollary 3.5.14 with $\widehat{\Omega}=\Omega$. In particular, we assume the $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ convergence of $\frac{T_{t,t+h}}{h}$. From Corollary 3.5.14, we get Assumption 3.5.2 and therefore, by Proposition 3.5.6, the $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$-and almost everywhere convergence of $\frac{T_{t,t+h}}{h}$.

Proposition 3.5.12 and its corollaries reproduce [AGS08, Theorem 8.3.1] from the classical OT theory. We will also adopt a similar proof strategy, namely we prove that a certain linear functional is bounded, so as to apply the Riesz representation theorem. Naively, one could try to work with the same functional as in [AGS08, Theorem 8.3.1], i.e.

$$\varphi\longmapsto\int_a^b\int_\Gamma \partial_t\varphi\ \mathrm{d}\mu_t\ \mathrm{d}t\,,$$

and prove that the function representing it is of the form (v,F). However, it turns out being more natural to treat $\partial_t + v\cdot\nabla_x$ as a single differential operator, in the spirit of hypoellipticity [Hör67]. Then, we work with the linear functional $L=L(\varphi)$ defined via

$$\varphi\overset{L}{\longmapsto}\int_a^b\int_\Gamma(\partial_t\varphi+v\cdot\nabla_x\varphi)\ \mathrm{d}\mu_t\ \mathrm{d}t=\int_a^b\int_\Gamma\lim_{h\downarrow 0}\frac{\varphi(t,x,v)-\varphi(t-h,x-hv,v)}{h}\ \mathrm{d}\mu_t\ \mathrm{d}t\,.$$

We shall prove that, in fact, $L=L(\nabla_v\varphi)$, i.e., L is a linear and bounded functional of $\nabla_v\varphi$, with operator norm $\|L\|\le\|\ell\|_{\mathrm{L}^2}$. One key ingredient in the proof is that x and $y-hw$ coincide to the first order on the support of $\pi^\theta_{t,t+h}$. More precisely, by means of Assumption (d), we show that

$$\lim_{h\downarrow 0}\int_{a'}^{b'}\frac{|y-x-hw|}{h}\ \mathrm{d}\pi^\theta_{t,t+h}\ \mathrm{d}t=0\,,\qquad [a',b']\subseteq\widehat{\Omega}\,.$$

Let us briefly comment on the assumptions of Proposition 3.5.12. Assumption (a) is certainly true for any solution to Vlasov's equation with moment bounds by [AGS08, Theorem 8.3.1]. Furthermore, it is independent of the other assumptions, and not even replaceable by Wasserstein BV-continuity of $(\mu_t)_{t\in(a,b)}$. The following example produces a bounded variation curve satisfying Assumptions (b) to (d) which does not solve (3.4.7).

Example 3.5.16. Let $x_0\colon(0,1)\to(0,1)$ be the Cantor function, namely a continuous, surjective, nondecreasing function of bounded variation, having a Cantor measure—concentrated on the Cantor set C—as derivative. In dimension $n=1$, set

$$\begin{aligned} v(t) &:= \inf\{|t-c|\ :\ c\in C\}\,, && t\in(0,1)\,,\\ x(t) &:= x_0(t)+\int_0^t v(s)\ \mathrm{d}s\,, && t\in(0,1)\,,\end{aligned}$$

and define the curve $\mu_{\cdot} := \delta_{\left(x(\cdot),v(\cdot)\right)}$. Choose $\widehat{\Omega} := (0,1) \setminus C$ and

$$\theta_{s,t} := T_{s,t} \overset{(3.3.5)}{=} 2\frac{|x(t)-x(s)|^2}{\big(x(t)-x(s)\big)\big(v(t)+v(s)\big)} = \frac{x_0(t)-x_0(s)+\int_s^t v(r)\,\mathrm{d}r}{v(t)+v(s)}\,. \tag{3.5.32}$$

In this case, there is only one admissible plan for every s,t, namely $\pi^\theta_{s,t} = \mu_s \otimes \mu_t$.

Assumption (b) is vacuously true, because x is strictly increasing. Assumption (c) holds because

$$|v(t)-v(s)| \le t-s\,, \qquad 0 < s < t < 1\,.$$

The function v is uniformly bounded away from 0 on any compact subset of $\widehat{\Omega}$, and x_0 is constant on any interval in $\widehat{\Omega}$. Therefore, the convergence $\frac{\theta_{t,t+h}}{h} \to 1$ holds locally uniformly on $\widehat{\Omega}$. When $\theta_{s,t} \in (0,\infty)$, we have

$$\sqrt{\tilde{\mathsf{c}}_{\theta_{s,t}}(\pi^\theta_{s,t})} = |v(t)-v(s)| \le t-s\,,$$

and with this we verify Assumption (d).

Nevertheless, this curve does not solve Vlasov's equation for any force field $(F_t)_t$ such that $\int_0^1 \left|F_t\big(x(t),v(t)\big)\right|^2 \,\mathrm{d}t < \infty$. If it did, then, by Lemma 3.4.13, we would have

$$|x(t)| \overset{(3.4.33)}{\le} |x(0)| + \int_0^t |v(s)|\,\mathrm{d}s\,, \qquad t \in (0,1)\,,$$

which would imply $x_0 \equiv 0$.

Assumption (b), Assumption (c), and (3.5.27), together, are a weakened version of the natural absolute continuity condition

$$\sqrt{\tilde{\mathsf{c}}_{\theta_{t,t+h}}(\pi^\theta_{t,t+h})} \le \int_s^t \ell(r)\,\mathrm{d}r$$

(calling $\tilde{\mathsf{c}}_0(\pi^\theta_{t,t+h})$ the limit of $\tilde{\mathsf{c}}_\epsilon(\pi^\theta_{t,t+h})$ for $\epsilon \to 0$), as can be easily checked (using (3.5.26)).

Assuming $\frac{\theta_{t,t+h}}{h} \to 1$ is needed to select a solution to Vlasov's equation (3.4.7) among all its possible reparametrisations. Recall also the hypothesis that $\widehat{\Omega}$ has full measure in (a,b). This ensures that $(\mu_t)_t$ solves Vlasov's equation *on the whole* (a,b). In the next lemma, we show that there are conditions under which Ω has full measure and, therefore, it may be a viable choice for $\widehat{\Omega}$. More precisely, we establish a connection between $|\rho'|_{\mathrm{W}_2}$ and $\|v\|_{\mathrm{L}^2(\mu_t)}$ in terms of $\liminf_{h\downarrow 0} \frac{\theta_{t,t+h}}{h}$ *a priori*, i.e., without knowing that $(\mu_t)_t$ solves Vlasov's equation. For solutions to Vlasov's equation, the analogous statement is Lemma 3.5.3.

Lemma 3.5.17. *Let $t \in (a,b)$ be a W_2-differentiability point for $\tilde{t} \to \rho_{\tilde{t}}$, and assume that there exists a sequence $h_k \downarrow 0$ such that*

$$l := \lim_{k\to\infty} \frac{\theta_{t,t+h_k}}{h_k} < \infty\,, \qquad \theta_{t,t+h_k} \in (0,\infty)\,\forall\, k\,, \qquad \textit{and} \qquad \lim_{k\to\infty} \tilde{\mathsf{c}}_{\theta_{t,t+h_k}}(\pi^\theta_{t,t+h_k}) = 0\,. \tag{3.5.33}$$

Then,

$$\left|\rho'_t\right|_{\mathrm{W}_2} \le l\,\|v\|_{\mathrm{L}^2(\mu_t)}\,. \tag{3.5.34}$$

As a consequence, if

$$\liminf_{h\downarrow 0} \frac{T_{t,t+h}}{h} < \infty\,, \qquad \lim_{h\downarrow 0} \mathsf{d}(\mu_t,\mu_{t+h}) = 0\,, \qquad \textit{and} \quad \left|\rho_t'\right|_{\mathrm{W}_2} > 0 \tag{3.5.35}$$

for a.e. $t \in (a,b)$, then Ω has full measure in (a,b).

Remark 3.5.18. If $|\rho_t'|_{\mathrm{W}_2} > 0$ and Assumption (b) holds, then $\theta_{t,t+h} > 0$ for small h.

Proof of Lemma 3.5.17. We write

$$\begin{aligned}\frac{\|y-x\|_{\mathrm{L}^2(\pi^\theta_{t,t+h_k})}}{\theta_{t,t+h_k}} &\le \left\|\frac{y-x}{\theta_{t,t+h_k}} - \frac{v+w}{2}\right\|_{\mathrm{L}^2(\pi^\theta_{t,t+h_k})} + \left\|\frac{w-v}{2}\right\|_{\mathrm{L}^2(\pi^\theta_{t,t+h_k})} + \|v\|_{\mathrm{L}^2(\mu_t)} \\ &\le \sqrt{\tilde{\mathsf{c}}_{\theta_{t,t+h_k}}(\pi^\theta_{t,t+h_k})} + \|v\|_{\mathrm{L}^2(\mu_t)}\,.\end{aligned}$$

Therefore, by hypothesis,

$$\left|\rho_t'\right|_{\mathrm{W}_2} \le l\, \liminf_{k\to\infty} \frac{\|y-x\|_{\mathrm{L}^2(\pi^\theta_{t,t+h_k})}}{\theta_{t,t+h_k}} = l\,\|v\|_{\mathrm{L}^2(\mu_t)}\,.$$

The proof that Ω has full measure under (3.5.35) is consequence of Remark 3.5.18 and the fact that $\mathsf{d}(\mu_t,\mu_{t+h}) = \tilde{\mathsf{c}}_{T_{t,t+h}}(\pi_{t,t+h})$ if $T_{t,t+h} \in (0,\infty)$. □

Proof of Proposition 3.5.12. Let us fix $[a',b'] \subseteq \widehat{\Omega}$ and $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}\big((a',b')\times\mathcal{X}\times\mathcal{V}\big)$. We write

$$\begin{aligned}L(\varphi) &:= \int_a^b \int_\Gamma (\partial_t\varphi + v\cdot\nabla_x\varphi)\;\mathrm{d}\mu_t\;\mathrm{d}t \\ &\le \left|\int_{a'}^{b'}\int_\Gamma \lim_{h\downarrow 0}\frac{\varphi(t,x,v)-\varphi(t-h,x-hv,v)}{h}\;\mathrm{d}\mu_t\;\mathrm{d}t\right|\end{aligned}$$

and, by the dominated convergence theorem and a change of time variable,

$$\begin{aligned}L(\varphi) &\le \lim_{h\downarrow 0}\left|\int_{a'}^{b'}\int \frac{\varphi(t,x,v)-\varphi(t-h,x-hv,v)}{h}\;\mathrm{d}\mu_t\;\mathrm{d}t\right| \\ &= \lim_{h\downarrow 0}\left|\frac{\int_{a'}^{b'}\int\varphi(t,x,v)\;\mathrm{d}\mu_t\;\mathrm{d}t - \int_{a'}^{b'}\int\varphi(t,x-hv,v)\;\mathrm{d}\mu_{t+h}\;\mathrm{d}t}{h}\right|\,.\end{aligned}$$

Hence, we have

$$\begin{aligned}L(\varphi) &\le \lim_{h\downarrow 0}\left|\int_{a'}^{b'}\int_\Gamma \frac{\varphi(t,x,v)-\varphi(t,y-hw,w)}{h}\;\mathrm{d}\pi^\theta_{t,t+h}\;\mathrm{d}t\right| \\ &\le \underbrace{\liminf_{h\downarrow 0}\int_{a'}^{b'}\int_{\Gamma\times\Gamma}\frac{|\varphi(t,x,v)-\varphi(t,x,w)|}{h}\;\mathrm{d}\pi^\theta_{t,t+h}\;\mathrm{d}t}_{=:I_{\varphi,1}} \\ &\quad + \underbrace{\limsup_{h\downarrow 0}\int_{a'}^{b'}\int_{\Gamma\times\Gamma}\frac{|\varphi(t,x,w)-\varphi(t,y-hw,w)|}{h}\;\mathrm{d}\pi^\theta_{t,t+h}\;\mathrm{d}t}_{=:I_{\varphi,2}}\,.\end{aligned}$$

We start by estimating $I_{\varphi,1}$. By the Cauchy–Schwarz inequality:

$$I_{\varphi,1} \leq \liminf_{h\downarrow 0} \sqrt{\int_{a'}^{b'} \int_{\Gamma\times\Gamma} \frac{|\varphi(t,x,v)-\varphi(t,x,w)|^2}{|v-w|^2} \,\mathrm{d}\pi^\theta_{t,t+h}\,\mathrm{d}t} \cdot \sqrt{\int_{a'}^{b'} \int_{\Gamma\times\Gamma} \frac{|v-w|^2}{h^2} \,\mathrm{d}\pi^\theta_{t,t+h}\,\mathrm{d}t}\,. \qquad (3.5.36)$$

By Assumption (c), we have $\|w-v\|_{\mathrm{L}^2(\pi^\theta_{t,t+h})} \to 0$ for a.e. t. Therefore, the first square root in (3.5.36) converges to $\|\nabla_v\varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)}$ by the dominated convergence theorem. As for the second one, again by Assumption (c), we write

$$\int_{a'}^{b'} \int_{\Gamma\times\Gamma} \frac{|v-w|^2}{h^2} \,\mathrm{d}\pi^\theta_{t,t+h}\,\mathrm{d}t \leq \int_{a'}^{b'} \left(⨍_t^{t+h} \ell(r)\,\mathrm{d}r \right)^2 \mathrm{d}t \leq \int_{a'}^{b'+h} \ell^2(r)\,\mathrm{d}r \qquad (3.5.37)$$

from which we conclude that $I_{\varphi,1} \leq \|\nabla_v\varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \|\ell\|_{\mathrm{L}^2}$.

We claim that $I_{\varphi,2} = 0$. To prove it, we estimate

$$I_{\varphi,2} \leq \|\varphi\|_{C^1} \limsup_{h\downarrow 0} \int_{a'}^{b'} \int_{\Gamma\times\Gamma} \frac{|y-x-h\,w|}{h} \,\mathrm{d}\pi^\theta_{t,t+h}\,\mathrm{d}t\,.$$

Momentarily fix t and h. If $\theta_{t,t+h} \in (0,\infty)$, then the triangle inequality gives

$$\frac{|y-x-h\,w|}{h} \leq \left| \frac{y-x}{h} - \frac{\theta_{t,t+h}}{h}\frac{v+w}{2} \right| + \frac{|w-v|}{2} + \left| \frac{\theta_{t,t+h}}{h} - 1 \right| \frac{|v+w|}{2}\,,$$

and, therefore,

$$\int_{\Gamma\times\Gamma} \frac{|y-x-h\,w|}{h} \,\mathrm{d}\pi^\theta_{t,t+h} \leq \frac{\theta_{t,t+h}}{h} \sqrt{\tilde{\mathsf{c}}_{\theta_{t,t+h}}(\pi^\theta_{t,t+h})} + ⨍_t^{t+h} \ell(r)\,\mathrm{d}r + \left| \frac{\theta_{t,t+h}}{h} - 1 \right| \frac{\|v\|_{\mathrm{L}^2(\mu_t)} + \|v\|_{\mathrm{L}^2(\mu_{t+h})}}{2}$$

If $\theta_{t,t+h} = 0$, then by Assumption (b),

$$\int_{\Gamma\times\Gamma} \frac{|y-x-h\,w|}{h} \,\mathrm{d}\pi^\theta_{t,t+h} = \|v\|_{\mathrm{L}^2(\mu_{t+h})} = \left| \frac{\theta_{t,t+h}}{h} - 1 \right| \|v\|_{\mathrm{L}^2(\mu_{t+h})}\,.$$

If $\theta_{t,t+h} = \infty$, then, trivially,

$$\int_{\Gamma\times\Gamma} \frac{|y-x-h\,w|}{h} \,\mathrm{d}\pi^\theta_{t,t+h} \leq \left| \frac{\theta_{t,t+h}}{h} - 1 \right|.$$

Hence, we find

$$\begin{aligned} \int_{a'}^{b'} \int_{\Gamma\times\Gamma} \frac{|y-x-h\,w|}{h} \,\mathrm{d}\pi^\theta_{t,t+h}\,\mathrm{d}t \leq &\int_{\{t\in(a',b')\,:\,\theta_{t,t+h}\in(0,\infty)\}} \frac{\theta_{t,t+h}}{h} \sqrt{\tilde{\mathsf{c}}_{\theta_{t,t+h}}(\pi^\theta_{t,t+h})}\,\mathrm{d}t \\ &+ h \int_{a'}^{b'+h} \ell(r)\,\mathrm{d}r \\ &+ \left(\sup_{\frac{a+a'}{2}\leq t\leq\frac{b+b'}{2}} \|v\|_{\mathrm{L}^2(\mu_t)} + 1 \right) \int_{a'}^{b'} \left| \frac{\theta_{t,t+h}}{h} - 1 \right| \mathrm{d}t\,, \end{aligned}$$

and Assumption (d) allows to conclude that $I_{\varphi,2} = 0$.

We established that $L(\varphi) \leq \|\nabla_v \varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \|\ell\|_{\mathrm{L}^2}$ for every $\varphi \in \mathrm{C}_\mathrm{c}^\infty\big((a',b') \times \mathcal{X} \times \mathcal{V}\big)$ with $[a',b'] \subseteq \Omega$. By linearity of L and arbitrariness of a', b', we have, in fact, that

$$L(\varphi) \leq \|\nabla_v \varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \|\ell\|_{\mathrm{L}^2} \quad \text{for every } \varphi \in \mathrm{C}_\mathrm{c}^\infty(\widehat{\Omega} \times \mathcal{X} \times \mathcal{V})\,. \tag{3.5.38}$$

We claim that the same inequality holds for every $\varphi \in \mathrm{C}_\mathrm{c}^\infty\big((a,b) \times \mathcal{X} \times \mathcal{V}\big)$. Given one such φ, and a function $\eta \in \mathrm{C}_\mathrm{c}^\infty(\widehat{\Omega})$, we write

$$\begin{aligned} L(\varphi) &= \int_a^b \int_\Gamma (\partial_t \varphi + v \cdot \nabla_x \varphi)\,\mathrm{d}\mu_t\,\mathrm{d}t \\ &= \int_a^b \big(1 - \eta(t)\big) \int_\Gamma (\partial_t \varphi + v \cdot \nabla_x \varphi)\,\mathrm{d}\mu_t\,\mathrm{d}t \\ &\quad + \int_a^b \left(\eta(t) \int_\Gamma (\partial_t \varphi + v \cdot \nabla_x \varphi)\,\mathrm{d}\mu_t + \partial_t \eta(t) \int_\Gamma \varphi\,\mathrm{d}\mu_t\right) \mathrm{d}t - \int_a^b \partial_t \eta(t) \int_\Gamma \varphi\,\mathrm{d}\mu_t\,\mathrm{d}t\,. \end{aligned}$$

Since $(\mu_t)_{t\in(a,b)}$ is W_2-2-a.c., and φ is smooth and compactly supported, the function $t \mapsto \int \varphi(t,\cdot)\,\mathrm{d}\mu_t$ is 2-a.c. Therefore, an integration by parts yields

$$\begin{aligned} L(\varphi) = &\int_a^b \big(1 - \eta(t)\big) \int_\Gamma (\partial_t \varphi + v \cdot \nabla_x \varphi)\,\mathrm{d}\mu_t\,\mathrm{d}t \\ &+ \int_a^b \int_\Gamma \big(\partial_t(\eta\varphi) + v \cdot \nabla_x(\eta\varphi)\big)\,\mathrm{d}\mu_t\,\mathrm{d}t + \int_a^b \eta(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \varphi\,\mathrm{d}\mu_t\,\mathrm{d}t\,. \end{aligned}$$

Since $\eta\varphi \in \mathrm{C}_\mathrm{c}^\infty(\widehat{\Omega} \times \mathcal{X} \times \mathcal{V})$, we apply (3.5.38) to write

$$\begin{aligned} L(\varphi) \leq \int_a^b \big(1 - \eta(t)\big) \int (\partial_t \varphi + v \cdot \nabla_x \varphi)\,\mathrm{d}\mu_t\,\mathrm{d}t + \|\eta \nabla_v \varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \|\ell\|_{\mathrm{L}^2(a,b)} \\ + \int_a^b \eta(t) \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \varphi\,\mathrm{d}\mu_t\,\mathrm{d}t\,. \end{aligned}$$

By Assumption (d), the complement of $\widehat{\Omega}$ is Lebesgue negligible. Thus, η can approximate the constant function 1 in $\mathrm{L}^2(\widehat{\Omega}) = \mathrm{L}^2(a,b)$. This gives

$$L(\varphi) \leq \|\nabla_v \varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \|\ell\|_{\mathrm{L}^2(a,b)} + \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \varphi\,\mathrm{d}\mu_t\,\mathrm{d}t = \|\nabla_v \varphi\|_{\mathrm{L}^2(\mu_t\,\mathrm{d}t)} \|\ell\|_{\mathrm{L}^2}\,,$$

which was our claim.

Finally, we apply the Riesz representation theorem on the closure in $\mathrm{L}^2(\mu_t\,\mathrm{d}t)$ of the set $\left\{\nabla_v \varphi \,:\, \varphi \in \mathrm{C}_\mathrm{c}^\infty\big((a,b) \times \mathcal{X} \times \mathcal{V}\big)\right\}$ to find $(F_t)_t$ such that $(\mu_t, F_t)_{t\in(a,b)}$ solves (3.4.7). □

3.5.3 First-order differential calculus

Let $(\mu_t)_{t\in(a,b)}$ be a narrowly continuous curve in $\mathcal{P}_2(\Gamma)$, let $(s,t) \mapsto \pi_{s,t}$ be a measurable selection of optimal transport plans for d, and let $T_{s,t}$ be the corresponding optimal times, see (3.5.5). Recall that $\rho_t := (\mathrm{pr}_x)_\# \mu_t$ and $\langle v \rangle_t = \int v\,\mathrm{d}\mu_t$.

The tangent

In the light of the previous sections, we can now give a more rigorous description of the geometric intuitions of Remark 3.1.9, Remark 3.1.12, and Remark 3.1.13. Given a solution $(\mu_t, F_t)_{t\in(a,b)}$ to Vlasov's equation (3.4.7), we can see v as the infinitesimal x-variation (i.e. the *velocity*), and the field $(F_t)_t$ driving $(\mu_t)_t$ as the infinitesimal v-variation (i.e., the *acceleration* or *force*). In the case of the particle model—Section 3.2—we have that, along a *regular* solution to Newton's equations $\dot{x}_t = v_t, \quad \dot{v}_t = F_t(x_t, v_t)$, it holds true:

$$\begin{aligned}
x_t &= x_0 + t\, v_0 + o(t), \quad t \to 0\,,\\
v_t &= v_0 + t\, F_0(x_0, v_0) + o(t), \quad t \to 0\,,\\
x_t &= x_0 + tv_0 + \frac{1}{2}\, t^2\, F_0(x_0, v_0) + o(t^2)\,, \quad t \to 0\,.
\end{aligned}$$

In the next propositions, we recover analogous formulae in the case of evolutions of measures along Vlasov's equations. The heuristic argument—given in Remark 3.1.12—is the following. Along a solution $(\mu_t)_t$ to (3.4.7), the optimal plan $\pi_{t,t+h}$ for $\mathsf{d}(\mu_t, \mu_{t+h})$ is *close* to the projection

$$\Big(\mathrm{pr}_{(\alpha(t),\alpha'(t)),(\alpha(t+h),\alpha'(t+h))}\Big)_\#\, \mathbf{m}$$

of the dynamical transport plan $\mathbf{m}$ induced by Vlasov's equation itself (cf. [AGS08, Theorem 8.2.1]). Quantitative statements are given below.

Proposition 3.5.19. *Suppose that Assumption 3.5.2 holds (i.e., $(\mu_t, F_t)_t$ is a solution to Vlasov's equation for a field $(F_t)_t$), with $(F_t)_t$ belonging to the $\mathrm{L}^2(\mu_t\,\mathrm{d}t)$-closure of the set $\Big\{\nabla_v\varphi \,:\, \varphi \in \mathrm{C}^\infty_c\big((a,b)\times\Gamma\big)\Big\}$. Then, for a.e. $t \in (a,b)$ such that $|\rho'_t|_{\mathrm{W}_2} > 0$, we have*

$$\lim_{h\downarrow 0} \frac{1}{h}\|w - v - hF_t(x,v)\|_{\mathrm{L}^2(\pi_{t,t+h})} = 0\,, \tag{3.5.39}$$

$$\lim_{h\downarrow 0} \frac{1}{h^2}\left\|y - x - T_{t,t+h}\frac{v+w}{2}\right\|_{\mathrm{L}^2(\pi_{t,t+h})} = 0\,. \tag{3.5.40}$$

Proposition 3.5.20. *In the setting of Proposition 3.5.19, for a.e. t such that $\langle v\rangle_t \neq 0$, we have*

$$\lim_{h\downarrow 0} \frac{T_{t,t+h} - h}{h^2} = 0 \tag{3.5.41}$$

and

$$\lim_{h\downarrow 0} \frac{1}{h^2}\left\|y - x - hv - \frac{h^2}{2}F_t(x,v)\right\|_{\mathrm{L}^2(\pi_{t,t+h})} = 0\,. \tag{3.5.42}$$

Proof of Proposition 3.5.19. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a C^1-dense set of $\mathrm{C}^\infty_c(\Gamma)$ functions. For almost every $t \in (a,b)$, we have

1. $|\rho'_t|_{\mathrm{W}_2} > 0$,
2. $\frac{T_{t,t+h}}{h} \to 1$, cf. Corollary 3.5.11,
3. Equation (3.5.19) holds with $\varphi := \varphi_k$ for every $k \in \mathbb{N}$,
4. $\limsup_{h\downarrow 0} \frac{\mathsf{d}(\mu_t,\mu_{t+h})}{h} \leq \|F_t\|_{\mathrm{L}^2(\mu_t)} < \infty$, cf. Proposition 3.5.5,

5. F_t belongs to the $\mathrm{L}^2(\mu_t)$-closure of $\{\nabla_v\varphi : \varphi \in \mathrm{C}^\infty_\mathrm{c}(\Gamma)\}$.

Let t be one such time. To prove (3.5.39), we compute

$$\begin{aligned}\|w-v-hF_t(x,v)\|^2_{\mathrm{L}^2(\pi_{t,t+h})} &= \|w-v\|^2_{\mathrm{L}^2(\pi_{t,t+h})} + h^2\|F_t\|^2_{\mathrm{L}^2(\mu_t)} \\ &\qquad - 2h\int (w-v)\cdot F_t(x,v)\,\mathrm{d}\pi_{t,t+h}\,,\end{aligned}$$

hence, resorting to Proposition 3.5.5,

$$\limsup_{h\downarrow 0}\frac{\|w-v-hF_t\|^2_{\mathrm{L}^2(\pi_{t,t+h})}}{h^2} \overset{(3.5.10)}{\le} 2\|F_t\|^2_{\mathrm{L}^2(\mu_t)} - 2\liminf_{h\downarrow 0}\int\frac{w-v}{h}\cdot F_t(x,v)\,\mathrm{d}\pi_{t,t+h}\,. \tag{3.5.43}$$

We estimate the last integral. For every k, we have

$$\begin{aligned}&\int F_t\cdot\nabla_v\varphi_k\,\mathrm{d}\mu_t \\ &\overset{(3.5.19)}{=} \lim_{h\downarrow 0}\left(\int\frac{w-v}{h}\cdot\nabla_v\varphi_k(x,v)\,\mathrm{d}\pi_{t,t+h} + \left(\frac{T_{t,t+h}}{h}-1\right)\int v\cdot\nabla_x\varphi_k\,\mathrm{d}\mu_t\right) \\ &\overset{(3.5.23)}{=} \lim_{h\downarrow 0}\int\frac{w-v}{h}\cdot\nabla_v\varphi_k(x,v)\,\mathrm{d}\pi_{t,t+h} \\ &\le \liminf_{h\downarrow 0}\int\frac{w-v}{h}\cdot F_t\,\mathrm{d}\pi_{t,t+h} + \|\nabla_v\varphi_k - F_t\|_{\mathrm{L}^2(\mu_t)}\limsup_{h\downarrow 0}\frac{\|w-v\|_{\mathrm{L}^2(\pi_{t,t+h})}}{h} \\ &\overset{(3.5.10)}{\le} \liminf_{h\downarrow 0}\int\frac{w-v}{h}\cdot F_t\,\mathrm{d}\pi_{t,t+h} + \|\nabla_v\varphi_k - F_t\|_{\mathrm{L}^2(\mu_t)}\|F_t\|_{\mathrm{L}^2(\mu_t)}\,.\end{aligned}$$

By arbitrariness of k,

$$\int F_t\cdot\nabla_v\varphi\,\mathrm{d}\mu_t \le \liminf_{h\downarrow 0}\int\frac{w-v}{h}\cdot F_t\,\mathrm{d}\pi_{t,t+h} + \|\nabla_v\varphi - F_t\|_{\mathrm{L}^2(\mu_t)}\|F_t\|_{\mathrm{L}^2(\mu_t)}$$

for every $\varphi \in \mathrm{C}^\infty_\mathrm{c}(\Gamma)$. Since F_t belongs to the $\mathrm{L}^2(\mu_t)$-closure of $\{\nabla_v\varphi : \varphi \in \mathrm{C}^\infty_\mathrm{c}(\Gamma)\}$,

$$\|F_t\|^2_{\mathrm{L}^2(\mu_t)} \le \liminf_{h\downarrow 0}\int\frac{w-v}{h}\cdot F_t\,\mathrm{d}\pi_{t,t+h}\,,$$

which, together with (3.5.43), yields (3.5.39).

Let us now prove (3.5.40). By definition of $\tilde{\mathrm{d}}$, we write

$$\frac{12}{h^4}\left\|y-x-T_{t,t+h}\frac{v+w}{2}\right\|^2_{\mathrm{L}^2(\pi_{t,t+h})} = \frac{T^2_{t,t+h}}{h^2}\,\frac{\tilde{\mathrm{d}}^2(\mu_t,\mu_{t+h}) - \|w-v\|^2_{\mathrm{L}^2(\pi_{t,t+h})}}{h^2}\,,$$

therefore, it suffices that

$$\lim_{h\downarrow 0}\frac{\tilde{\mathrm{d}}^2(\mu_t,\mu_{t+h}) - \|w-v\|^2_{\mathrm{L}^2(\pi_{t,t+h})}}{h^2} = 0\,,$$

which follows from (3.5.10) and (3.5.39). □

Proof of Proposition 3.5.20. Let $t \in (a,b)$ be such that $\langle v\rangle_t \neq 0$. In the light of Lemma 3.5.3, Corollary 3.5.11, Proposition 3.5.19, we may assume that (3.5.23), (3.5.39), (3.5.40) hold, and, additionally, t is a Lebesgue point for

$$s \longmapsto \int F_s \,\mathrm{d}\mu_s \,.$$

Fix $h \in (0, b-t)$ such that $T_{t,t+h} < \infty$, and let $\boldsymbol{\eta}, \mathbf{P}$ be as in Lemma 3.5.9. In particular,

$$\int y \cdot \langle v\rangle_t \,\mathrm{d}\mathbf{P} - \int x \cdot \langle v\rangle_t \,\mathrm{d}\mathbf{P} = \int \gamma_x(t+h) \cdot \langle v\rangle_t \,\mathrm{d}\mathbf{P} - \int \gamma_x(t) \cdot \langle v\rangle_t \,\mathrm{d}\mathbf{P}\,,$$

thus,

$$\begin{aligned}
\int (y-x)\cdot\langle v\rangle_t \,\mathrm{d}\pi_{t,t+h} &= \int\int_t^{t+h} \dot\gamma_x(s)\cdot\langle v\rangle_t \,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta} = \int\int_t^{t+h} \gamma_v(s)\cdot\langle v\rangle_t \,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta} \\
&= \int\int_t^{t+h} \gamma_v(s)\cdot\langle v\rangle_t \,\mathrm{d}s\,\mathrm{d}\boldsymbol{\eta} = \int \left(h\gamma_v(t) + \int_t^{t+h} (t+h-s)\dot\gamma_v(s)\,\mathrm{d}s\right)\cdot\langle v\rangle_t \,\mathrm{d}\boldsymbol{\eta} \\
&= \int \left(h\gamma_v(t) + \int_t^{t+h} (t+h-s)F_s\big(\gamma_x(r),\gamma_v(r)\big)\,\mathrm{d}s\right)\cdot\langle v\rangle_t \,\mathrm{d}\boldsymbol{\eta} \\
&= h\int v\cdot\langle v\rangle_t \,\mathrm{d}\mu_t + \int_t^{t+h} (t+h-s)\int F_s\cdot\langle v\rangle_t \,\mathrm{d}\mu_s\,\mathrm{d}s\,.
\end{aligned}$$

We infer that

$$\begin{aligned}
\frac{T_{t,t+h}-h}{h^2}\langle v\rangle_t^2 &= \fint_t^{t+h} \frac{t+h-s}{h}\left(\int F_s\cdot\langle v\rangle_t \,\mathrm{d}\mu_s - \int F_t\cdot\langle v\rangle_t \,\mathrm{d}\mu_t\right)\mathrm{d}s \\
&+ \int F_t\cdot\langle v\rangle_t \,\mathrm{d}\mu_t \fint_t^{t+h}\frac{t+h-s}{h}\,\mathrm{d}s \\
&+ \frac{T_{t,t+h}}{h}\int \frac{v-w}{2h}\cdot\langle v\rangle_t \,\mathrm{d}\pi_{t,t+h} - \frac{1}{h^2}\int\left(y-x-T_{t,t+h}\frac{v+w}{2}\right)\cdot\langle v\rangle_t \,\mathrm{d}\pi_{t,t+h}\,.
\end{aligned}$$

Let us analyse the four terms at the right-hand side one by one. The first one is negligible, because we can bound $\frac{t+h-s}{h} \leq 1$ and use the Lebesgue differentiation theorem. The second one is equal, for every $h > 0$, to $\frac{1}{2}\int F_t\cdot\langle v\rangle_t \,\mathrm{d}\mu_t$. The third one converges to this same quantity with inverse sign (i.e., $-\frac{1}{2}\int F_t\cdot\langle v\rangle_t \,\mathrm{d}\mu_t$) by (3.5.23) and (3.5.39). The fourth one is negligible by (3.5.40).

Since $\langle v\rangle_t \neq 0$, the proof of (3.5.41) is complete, and (3.5.42) follows from (3.5.39), (3.5.40), (3.5.41). □

d-**derivative**

For a solution $(\mu_t)_t$ to Vlasov's equation, we are going to prove that the limits of the incremental ratios $\frac{\mathsf{d}(\mu_t,\mu_{t+h})}{h}$ and $\frac{\tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h})}{h}$ as $h \downarrow 0$ exist and are equal to the smallest norm of a force field $(F_t)_t$ driving $(\mu_t)_t$. This is similar to a consequence of [AGS08, Theorem 1.1.2 & Theorem 8.3.1] in classical OT. A major obstacle in replicating these results is that d is not a distance.

The classical way to show that $\frac{\mathsf{d}(\mu_t,\mu_{t+h})}{h}$ has a limit for almost every t—without extracting a subsequence—is [AGS08, Thereom 1.1.2], which relies on the triangle inequality. The proof we give here is, instead, based on Lemma 3.5.10, which, for a.e. $t \in (a,b)$ with $|\rho'_t|_{W_2} > 0$, identifies F_t with the direction of infinitesimal change of the velocities, which in turn bounds the incremental ratio of d from below.

Remark 3.5.21. Proposition 3.5.22 below is a generalised version of Proposition 3.2.14, which provided the d-derivative in the particle-model case.

Proposition 3.5.22. *Under Assumption 3.5.2 with $(F_t)_t$ belonging to the $L^2(\mu_t\,dt)$-closure of the set $\Big\{\nabla_v\varphi \,:\, \varphi \in C_c^\infty\big((a,b)\times\Gamma\big)\Big\}$, for a.e. t such that $|\rho'_t|_{W_2} > 0$, we have the limits*

$$\lim_{h\downarrow 0} \frac{\mathsf{d}(\mu_t, \mu_{t+h})}{h} = \lim_{h\downarrow 0} \frac{\tilde{\mathsf{d}}_h(\mu_t, \mu_{t+h})}{h} = \|F_t\|_{L^2(\mu_t)} . \tag{3.5.44}$$

Proof. The inequality $\le$ is given by Proposition 3.5.5. The inequality $\ge$ follows from Proposition 3.5.19:

$$\liminf_{h\downarrow 0} \frac{\tilde{\mathsf{d}}_h(\mu_t, \mu_{t+h})}{h} \ge \liminf_{h\downarrow 0} \frac{\mathsf{d}(\mu_t, \mu_{t+h})}{h} \ge \liminf_{h\downarrow 0} \frac{\|w-v\|_{L^2(\pi_{t,t+h})}}{h} \overset{(3.5.39)}{=} \|F_t\|_{L^2(\mu_t)}$$

for a.e. $t \in (a,b)$ such that $|\rho'_t|_{W_2} > 0$. □

The limit

$$\lim_{h\downarrow 0} \frac{\tilde{\mathsf{d}}_h(\mu_t, \mu_{t+h})}{h} = \|F_t\|_{L^2(\mu_t)} , \tag{3.5.45}$$

can be obtained without the assumption $|\rho'_t|_{W_2} > 0$.

Proposition 3.5.23. *Under Assumption 3.5.2 with $(F_t)_t$ belonging to the $L^2(\mu_t\,dt)$-closure of the set $\Big\{\nabla_v\varphi \,:\, \varphi \in C_c^\infty\big((a,b)\times\Gamma\big)\Big\}$, for a.e. t, we have (3.5.45).*

Proof. The inequality $\le$ is given by Proposition 3.5.5. To prove $\ge$, we adopt a similar strategy as in the proof of Lemma 3.5.10 and Proposition 3.5.20. Let us fix a C^1-dense set of functions $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq C_c^\infty(\Gamma)$, and t such that:

1. t is a Lebesgue point of the functions

$$s \longmapsto \int_\Gamma (\tilde{v}, F_s)\cdot\nabla_{x,v}\varphi_k \,d\mu_s \quad \text{for every } k\in\mathbb{N}\,, \quad \text{and} \quad s \longmapsto \|F_s\|_{L^2(\mu_s)} .$$

2. F_t belongs to the $L^2(\mu_t)$-closure of $\{\nabla_v\varphi \,:\, \varphi\in C_c^\infty(\Gamma)\}$.

The points t satisfying the previous conditions form a full-measure set in (a,b).

Fix $h \in (0, b-t)$, let $\bar\pi \in \Pi(\mu_t, \mu_{t+h})$ be $\tilde{\mathsf{d}}_h$-optimal, let $\boldsymbol{\eta} \in \mathcal{P}\big(\Gamma\times H^1(t,t+h;\Gamma)\big)$ be as in the proof of Proposition 3.5.4 (after replacing (s,t) with $(t,t+h)$). By gluing $\bar\pi$ and $\boldsymbol{\eta}$ at μ_t, construct $\mathbf{P} \in \mathcal{P}\big(\Gamma\times\Gamma\times H^1(t,t+h;\Gamma)\big)$. For every k, we have

$$\int \varphi_k(y,w)\,d\mathbf{P} - \int \varphi_k(x,v)\,d\mathbf{P} = \int \varphi_k\big(\gamma(t+h)\big)\,d\mathbf{P} - \int \varphi_k\big(\gamma(t)\big)\,d\mathbf{P}\,,$$

which yields, by the fundamental theorem of calculus and the properties of $\boldsymbol{\eta}$ and $\mathbf{P}$,

$$\frac{1}{h}\int_{\Gamma\times\Gamma}\int_0^1 (y-x,w-v)\cdot\nabla_{x,v}\varphi_k\Big(x+r(y-x),v+r(w-v)\Big)\,\mathrm{d}r\,\mathrm{d}\bar{\pi} \\ = \fint_t^{t+h}\int_\Gamma \Big(\tilde{v},F_s(\tilde{x},\tilde{v})\Big)\cdot\nabla_{x,v}\varphi_k(\tilde{x},\tilde{v})\,\mathrm{d}\mu_s\,\mathrm{d}s\,.$$

Observe that

$$\left|\frac{1}{h}\int_{\Gamma^2}\int_0^1 (y-x,w-v)\cdot\Big(\nabla_{x,v}\varphi_k\Big(x+r(y-x),v+r(w-v)\Big)-\nabla_{x,v}\varphi_k(x,v)\Big)\,\mathrm{d}r\,\mathrm{d}\bar{\pi}\right| \\ \lesssim \frac{\|y-x\|^2_{\mathrm{L}^2(\bar{\pi})}+\|w-v\|^2_{\mathrm{L}^2(\bar{\pi})}}{h} \\ \lesssim \frac{1}{h}\left(\left\|y-x-h\frac{w+v}{2}\right\|^2_{\mathrm{L}^2(\bar{\pi})}+\Big(1+h^2\Big)\|w-v\|^2_{\mathrm{L}^2(\bar{\pi})}+h^2\|v\|^2_{\mathrm{L}^2(\mu_t)}\right) \\ \lesssim \frac{\Big(1+h^2\Big)\tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h})^2+h^2\|v\|^2_{\mathrm{L}^2(\mu_t)}}{h}\,,$$

and the last contribution is negligible by Proposition 3.5.4. Furthermore,

$$\left|\int_\Gamma \frac{y-x-hv}{h}\cdot\nabla_x\varphi_k\,\mathrm{d}\bar{\pi}\right| \lesssim \frac{\|y-x-hv\|_{\mathrm{L}^2(\bar{\pi})}}{h}\lesssim \tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h})\to 0\,,$$

as $h\downarrow 0$. We deduce that

$$\lim_{h\downarrow 0}\int_\Gamma \frac{w-v}{h}\cdot\nabla_v\varphi_k\,\mathrm{d}\bar{\pi} = \int_\Gamma F_t\cdot\nabla_v\varphi_k\,\mathrm{d}\mu_t\,.$$

Consequently,

$$\int_\Gamma F_t\cdot\nabla_v\varphi_k\,\mathrm{d}\mu_t \le \|\nabla_v\varphi_k\|_{\mathrm{L}^2(\mu_t)}\liminf_{h\downarrow 0}\frac{\|w-v\|_{\mathrm{L}^2(\bar{\pi})}}{h}\le \|\nabla_v\varphi_k\|_{\mathrm{L}^2(\mu_t)}\liminf_{h\downarrow 0}\frac{\tilde{\mathsf{d}}_h(\mu_t,\mu_{t+h})}{h}\,.$$

The conclusion follows, as F_t can be approximated by $\nabla_v\varphi_k$. □

3.5.4 Reparametrisations

Let $(\tilde{\mu}_s)_{s\in(\tilde{a},\tilde{b})}\subseteq\mathcal{P}_2(\Gamma)$ be a W_2-2-absolutely continuous curve, and $(s,t)\mapsto\tilde{\pi}_{s,t}$ a measurable selection of d-optimal transport plans. Define $\tilde{\Omega}$, $\tilde{\rho}_s$, $\tilde{T}_{s,t}$ in the same way as in the introduction to §3.5.

Theorem 3.5.24. *Let $\tilde{\lambda}\colon(\tilde{a},\tilde{b})\to\mathbb{R}_{>0}$ be measurable, bounded, and bounded away from zero. Assume that $|\tilde{\rho}'_s|_{\mathrm{W}_2}>0$ for a.e. $s\in(\tilde{a},\tilde{b})$. Then, the following are equivalent:*

1. *The curve $(\tilde{\mu}_s)_{s\in(\tilde{a},\tilde{b})}$ is a distributional solution to*

$$\partial_s\tilde{\mu}_s+\tilde{\lambda}(s)\,v\cdot\nabla_x\tilde{\mu}_s+\nabla_v\cdot(\tilde{F}_s\tilde{\mu}_s)=0 \tag{3.5.46}$$

for some force field $(\tilde{F}_s)_{s\in(\tilde{a},\tilde{b})}$ with

$$\int_{\tilde{a}}^{\tilde{b}}\left(\|v\|^2_{\mathrm{L}^2(\tilde{\mu}_s)}+\left\|\tilde{F}_s\right\|^2_{\mathrm{L}^2(\tilde{\mu}_s)}\right)\,\mathrm{d}s<\infty\,. \tag{3.5.47}$$

2. *There exists* $\tilde{\ell} \in \mathrm{L}^2(\tilde{a}, \tilde{b})$ *such that*

$$\mathsf{d}(\tilde{\mu}_s, \tilde{\mu}_t) \le \int_s^t \tilde{\ell}(r)\, \mathrm{d}r\,, \qquad \tilde{a} < s < t < \tilde{b}\,. \tag{3.5.48}$$

Moreover,

$$\lim_{\tilde{h}\downarrow 0} \frac{\tilde{T}_{s,s+\tilde{h}}}{\tilde{h}} = \tilde{\lambda}(s) \quad \text{for a.e. } s \in (\tilde{a}, \tilde{b})\,. \tag{3.5.49}$$

and, for every $[\tilde{a}', \tilde{b}'] \subseteq \tilde{\Omega}$, *there exist* $\bar{h} > 0$ *and a function* $\tilde{g} \in \mathrm{L}^1(\tilde{a}', \tilde{b}')$ *such that*

$$\sup_{\tilde{h}\in(0,\bar{h})} \frac{\tilde{T}_{s,s+\tilde{h}}}{\tilde{h}} \le \tilde{g}(s) \quad \text{for all } s \in [\tilde{a}', \tilde{b}']\,. \tag{3.5.50}$$

When the first statement holds for some $(\tilde{F}_s)_{s\in(\tilde{a},\tilde{b})}$, *we can choose* $\tilde{l} \coloneqq 2\big\|\tilde{F}_\cdot\big\|_{\mathrm{L}^2(\mu_\cdot)}$ *in* (3.5.48).

When either of the two statements is true, a force field $(\tilde{F}_s)_{s\in(\tilde{a},\tilde{b})}$ *for which* (3.5.46) *and* (3.5.47) *hold exists in the* $\mathrm{L}^2(\tilde{\mu}_s\, \mathrm{d}s)$*-closure of* $\left\{\nabla_v \tilde{\varphi} \,:\, \tilde{\varphi} \in \mathrm{C}_c^\infty\big((\tilde{a}, \tilde{b}) \times \Gamma\big)\right\}$. *Given such a force field, for a.e.* $s \in (\tilde{a}, \tilde{b})$ *we have*

$$\lim_{\tilde{h}\downarrow 0} \frac{\mathsf{d}(\tilde{\mu}_s, \tilde{\mu}_{s+\tilde{h}})}{\tilde{h}} = \big\|\tilde{F}_s\big\|_{\mathrm{L}^2(\mu_s)}\,. \tag{3.5.51}$$

Proof. Set $a \coloneqq 0$, $b \coloneqq \int_{\tilde{a}}^{\tilde{b}} \tilde{\lambda}(s)\, \mathrm{d}s$, and define the bi-Lipschitz continuous function

$$\tau(s) \coloneqq \int_0^s \tilde{\lambda}(r)\, \mathrm{d}r\,, \qquad s \in (\tilde{a}, \tilde{b})\,.$$

Define

$$\mu_t \coloneqq \tilde{\mu}_{\tau^{-1}(t)}\,, \qquad t \in (a, b)\,,$$

as well as

$$\pi_{s,t} \coloneqq \tilde{\pi}_{\tau^{-1}(s),\tau^{-1}(t)} \in \Pi_{\mathrm{o},\mathsf{d}}(\mu_s, \mu_t)\,, \qquad a < s < t < b\,,$$

so that

$$T_{s,t} = \tilde{T}_{\tau^{-1}(s),\tau^{-1}(t)}\,, \qquad a < s < t < b\,.$$

Note that $(\mu_t)_{t\in(a,b)}$ is W_2-2-absolutely continuous. Indeed, for all $a < s < t < b$, we have

$$\mathrm{W}_2(\mu_s, \mu_t) = \mathrm{W}_2\Big(\tilde{\mu}_{\tau^{-1}(s)}, \tilde{\mu}_{\tau^{-1}(t)}\Big) \le \int_{\tau^{-1}(s)}^{\tau^{-1}(t)} \big|\tilde{\mu}'_{\tilde{r}}\big|_{\mathrm{W}_2}\, \mathrm{d}\tilde{r} = \int_s^t \frac{\big|\tilde{\mu}'_{\tau^{-1}(r)}\big|_{\mathrm{W}_2}}{\tilde{\lambda}\big(\tau^{-1}(r)\big)}\, \mathrm{d}r$$

and

$$\int_a^b \frac{\big|\tilde{\mu}'_{\tau^{-1}(r)}\big|_{\mathrm{W}_2}^2}{\tilde{\lambda}\big(\tau^{-1}(r)\big)^2}\, \mathrm{d}r = \int_{\tilde{a}}^{\tilde{b}} \frac{|\tilde{\mu}'_{\tilde{r}}|_{\mathrm{W}_2}^2}{\tilde{\lambda}(\tilde{r})}\, \mathrm{d}\tilde{r} \le \left\|\frac{1}{\tilde{\lambda}}\right\|_{\mathrm{L}^\infty} \int_{\tilde{a}}^{\tilde{b}} \big|\tilde{\mu}'_{\tilde{r}}\big|_{\mathrm{W}_2}^2\, \mathrm{d}\tilde{r} < \infty\,.$$

Moreover, at every differentiability point s for $\tilde{s} \mapsto \tilde{\rho}_{\tilde{s}}$ and τ, for which $\tau(s)$ is a W_2-differentiability point for $\tilde{t} \mapsto \rho_{\tilde{t}}$ and $\tau'(s) = \tilde{\lambda}(s) > 0$, we have

$$\begin{aligned} 0 < \big|\tilde{\rho}'_s\big|_{\mathrm{W}_2} &= \lim_{h\to 0} \frac{\mathrm{W}_2(\tilde{\rho}_s, \tilde{\rho}_{s+h})}{|h|} \\ &\le \lim_{h\to 0} \frac{\mathrm{W}_2\Big(\rho_{\tau(s)}, \rho_{\tau(s+h)}\Big)}{|\tau(s+h) - \tau(s)|} \lim_{h\to 0} \frac{|\tau(s+h) - \tau(s)|}{|h|} = \big|\rho'_{\tau(s)}\big|\, \tau'(s)\,. \end{aligned} \tag{3.5.52}$$

This proves that $|\rho_t'|_{\mathrm{W}_2} > 0$ for a.e. $t \in (a,b)$. Observe that

$$\Omega = \Big\{t \in (a,b) \,:\, \|v\|_{\mathrm{L}^2(\mu_t)} > 0\Big\} = \Big\{\tau(s) \,:\, s \in (\tilde a, \tilde b)\,,\ \|v\|_{\mathrm{L}^2(\tilde\mu_s)} > 0\Big\} = \tau(\tilde\Omega)\,.$$

Proof of 1 ⇒ 2. Define

$$F_t := \frac{\tilde F_{\tau^{-1}(t)}}{\tilde\lambda\big(\tau^{-1}(t)\big)}\,, \qquad t \in (a,b)\,.$$

We have

$$\int_a^b \Big(\|v\|^2_{\mathrm{L}^2(\mu_t)} + \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\Big)\,\mathrm{d}t = \int_{\tilde a}^{\tilde b}\left(\tilde\lambda(s)\|v\|^2_{\mathrm{L}^2(\tilde\mu_s)} + \frac{\big\|\tilde F_s\big\|^2_{\mathrm{L}^2(\tilde\mu_s)}}{\tilde\lambda(s)}\right)\mathrm{d}s < \infty\,. \tag{3.5.53}$$

Fix $\varphi \in \mathrm{C}^\infty_\mathrm{c}\big((a,b)\times\Gamma\big)$ and define $\tilde\varphi(s,\cdot) := \varphi\big(\tau(s),\cdot\big)$. Let $(\tau_k)_{k\in\mathbb N}$ be a sequence of C^∞ functions converging to τ in $\mathrm{H}^1(\tilde a,\tilde b)$ (hence uniformly), and let $\tilde\varphi_k(s,\cdot) := \varphi\big(\tau_k(s),\cdot\big)$ for $s \in (\tilde a,\tilde b)$ and $k \in \mathbb N$. At least when k is large, we have $\tilde\varphi_k \in \mathrm{C}^\infty_\mathrm{c}\big((\tilde a,\tilde b)\times\Gamma\big)$, so that

$$\begin{aligned}0 &\overset{(3.5.46)}{=} \lim_{k\to\infty}\int_{\tilde a}^{\tilde b}\int \big(\partial_s\tilde\varphi_k + \tilde\lambda\, v\cdot\nabla_x\tilde\varphi_k + \tilde F_s\nabla_v\tilde\varphi_k\big)\,\mathrm{d}\tilde\mu_s\,\mathrm{d}s\\ &= \int_{\tilde a}^{\tilde b}\int\big(\partial_s\tilde\varphi + \tilde\lambda\, v\cdot\nabla_x\tilde\varphi + \tilde F_s\nabla_v\tilde\varphi\big)\,\mathrm{d}\tilde\mu_s\,\mathrm{d}s = \int_a^b\int\big(\partial_t\varphi + v\cdot\nabla_x\varphi + F_t\cdot\nabla_v\varphi\big)\,\mathrm{d}\mu_t\,\mathrm{d}t\,.\end{aligned} \tag{3.5.54}$$

This proves that $(\mu_t, F_t)_{t\in(a,b)}$ satisfies Assumption 3.5.2. We apply Proposition 3.5.4 to write

$$\mathsf{d}(\tilde\mu_s,\tilde\mu_t) = \mathsf{d}\big(\mu_{\tau(s)},\mu_{\tau(t)}\big) \overset{(3.5.9)}{\leq} 2\int_{\tau(s)}^{\tau(t)}\|F_r\|_{\mathrm{L}^2(\mu_r)}\,\mathrm{d}r = 2\int_s^t\big\|\tilde F_r\big\|_{\mathrm{L}^2(\tilde\mu_{\tilde r})}\,\mathrm{d}\tilde r \tag{3.5.55}$$

and deduce (3.5.48).

Let $[\tilde a',\tilde b'] \subseteq \tilde\Omega$. By Lemma 3.5.9 there exist $\bar h > 0$ and a function g in L^2 such that

$$\frac{T_{t,t+h}}{h} \leq g(t)\,, \qquad \text{for all } t \in \big[\tau(\tilde a'),\tau(\tilde b')\big]\,, \quad \text{and every } h \in (0,\bar h)\,.$$

Then, there exists a constant $C_{\tilde\lambda} > 0$ such that

$$\frac{\tilde T_{s,s+\tilde h}}{\tilde h} = \frac{\tau(s+\tilde h)-\tau(s)}{\tilde h}\,\frac{T_{\tau(s),\tau(s+\tilde h)}}{\tau(s+\tilde h)-\tau(s)} \leq C_{\tilde\lambda}\,g\big(\tau(s)\big) \tag{3.5.56}$$

for $s \in [\tilde a',\tilde b']$ and $h \in \big(0,\bar h/C_{\tilde\lambda}\big)$. Observe that $s \mapsto g\big(\tau(s)\big)$ is square-integrable (hence integrable), thus (3.5.50) follows. By Corollary 3.5.11, we have

$$\lim_{h\downarrow 0}\frac{T_{t,t+h}}{h} = 1 \quad \text{for a.e. } t \in (a,b)\,,$$

hence (3.5.49) thanks to

$$\frac{\tilde T_{s,s+\tilde h}}{\tilde h} = \frac{\tau(s+\tilde h)-\tau(s)}{\tilde h}\,\frac{T_{\tau(s),\tau(s+\tilde h)}}{\tau(s+\tilde h)-\tau(s)} \to \tilde\lambda(s) \quad \text{as } h\downarrow 0 \text{ for a.e. } s \in (\tilde a,\tilde b)\,. \tag{3.5.57}$$

Proof of 2 ⇒ 1. By (3.5.49), for a.e. $t \in (a,b)$, we have

$$\frac{T_{t,t+h}}{h} = \frac{\tau^{-1}(t+h)-\tau^{-1}(t)}{h}\,\frac{\tilde{T}_{\tau^{-1}(t),\tau^{-1}(t+h)}}{\tau^{-1}(t+h)-\tau^{-1}(t)} \to \frac{1}{\tilde{\lambda}\big(\tau^{-1}(t)\big)}\tilde{\lambda}\big(\tau^{-1}(t)\big) = 1\,. \quad (3.5.58)$$

For every s,t with $a<s<t<b$, we have

$$\mathsf{d}(\mu_s,\mu_t) = \mathsf{d}(\tilde{\mu}_{\tau^{-1}(s)},\tilde{\mu}_{\tau^{-1}(t)}) \overset{(3.5.48)}{\leq} \int_{\tau^{-1}(s)}^{\tau^{-1}(t)} \tilde{\ell}(\tilde{r})\,\mathrm{d}\tilde{r} = \int_s^t \frac{\tilde{\ell}\big(\tau^{-1}(r)\big)}{\tilde{\lambda}\big(\tau^{-1}(r)\big)}\,\mathrm{d}r\,,$$

and we notice that

$$\int_a^b \frac{\tilde{\ell}\big(\tau^{-1}(r)\big)^2}{\tilde{\lambda}\big(\tau^{-1}(r)\big)^2}\,\mathrm{d}r = \int_{\tilde{a}}^{\tilde{b}} \frac{\tilde{\ell}(\tilde{r})^2}{\tilde{\lambda}(\tilde{r})}\,\mathrm{d}\tilde{r} < \infty\,.$$

This proves Assumption (c') in Corollary 3.5.14, which together with (3.5.58), fulfil the hypotheses of Lemma 3.5.17. Then, Ω has full measure in (a,b).

To prove Assumption (d'), let us fix $[a',b'] \subseteq \Omega$. For every $t \in [a',b']$, we have

$$\frac{T_{t,t+h}}{h} = \frac{\tau^{-1}(t+h)-\tau^{-1}(t)}{h}\,\frac{\tilde{T}_{\tau^{-1}(t),\tau^{-1}(t+h)}}{\tau^{-1}(t+h)-\tau^{-1}(t)} \leq \left\|\frac{1}{\tilde{\lambda}}\right\|_{\mathrm{L}^\infty}\frac{\tilde{T}_{\tau^{-1}(t),\tau^{-1}(t+h)}}{\tau^{-1}(t+h)-\tau^{-1}(t)}\,,$$

and, by (3.5.50),

$$\frac{T_{t,t+h}}{h} \leq C_{\tilde{\lambda}}\tilde{g}\big(\tau^{-1}(t)\big)$$

for $h \in (0,\bar{h}/C_{\tilde{\lambda}})$, for some constant $C_{\tilde{\lambda}}$. The function $t \mapsto \tilde{g}\big(\tau^{-1}(t)\big)$ is integrable on $[a',b']$, therefore (3.5.30) follows from (3.5.58) and the dominated convergence theorem. Corollary 3.5.14 provides a force field $(F_t)_t$ such that $(\mu_t,F_t)_t$ solves (3.4.7) on $(a,b)\times\Gamma$ and $(F_t)_t$ belongs to the $\mathrm{L}^2(\mu_t\,\mathrm{d}t)$-closure of $\Big\{\nabla_v\varphi\,:\,\varphi\in\mathrm{C}_\mathrm{c}^\infty\big((a,b)\times\Gamma\big)\Big\}$. As we did in (3.5.54), it is possible to prove that the curve $(\tilde{\mu}_s)_s$ and the field

$$\tilde{F}_s := \tilde{\lambda}(s)F_{\tau(s)}\,, \qquad s\in(\tilde{a},\tilde{b})\,,$$

solve (3.5.46). From (3.5.53), we infer (3.5.47). By Proposition 3.5.22, we find (3.5.51): for a.e. s,

$$\frac{\mathsf{d}(\tilde{\mu}_s,\tilde{\mu}_{s+\tilde{h}})}{\tilde{h}} = \frac{\tau(s+\tilde{h})-\tau(s)}{\tilde{h}}\,\frac{\mathsf{d}(\mu_{\tau(s)},\mu_{\tau(s+\tilde{h})})}{\tau(s+\tilde{h})-\tau(s)} \overset{(3.5.44)}{\to} \tilde{\lambda}(s)\big\|F_{\tau(s)}\big\|_{\mathrm{L}^2(\mu_{\tau(s)})} = \big\|\tilde{F}_s\big\|_{\mathrm{L}^2(\tilde{\mu}_s)}\,.$$

We show that $(\tilde{F}_s)_s$ lies in the $\mathrm{L}^2(\tilde{\mu}_s\,\mathrm{d}s)$-closure of $\Big\{\nabla_v\tilde{\varphi}\,:\,\tilde{\varphi}\in\mathrm{C}_\mathrm{c}^\infty\big((\tilde{a},\tilde{b})\times\Gamma\big)\Big\}$. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of $\mathrm{C}_\mathrm{c}^\infty\big((a,b)\times\Gamma\big)$ functions such that $\nabla_v\varphi_k\to F$ in $\mathrm{L}^2(\mu_t\,\mathrm{d}t)$, and let $(\tau_l)_{l\in\mathbb{N}}$ be a sequence of C^∞ functions converging to τ in $\mathrm{H}^1(\tilde{a},\tilde{b})$. For every k,

$$\tilde{\varphi}_{k,l}\colon (s,x,v)\longmapsto \tau_l'(s)\varphi_k\big(\tau_l(s),x,v\big)$$

belongs to $\mathrm{C}_\mathrm{c}^\infty\big((\tilde{a},\tilde{b})\times\Gamma\big)$, at least for large l. As $l\to\infty$, the sequence $(\nabla_v\varphi_{k,l})_l$ converges to the v-gradient of the function

$$\tilde{\varphi}_k\colon (s,x,v)\longmapsto \tilde{\lambda}(s)\varphi_k\big(\tau(s),x,v\big)$$

in $\mathrm{L}^2(\mu_s\,\mathrm{d}s)$, since $\tau_l \to \tau$ uniformly, $\tau_l' \to \tilde{\lambda}$ in $\mathrm{L}^2(\mathrm{d}s)$, and $\nabla_v \varphi_k\big(\tau_l(s), x, v\big)$ is uniformly bounded in l for fixed k. Moreover, by a change of variables

$$\int_{\tilde{a}}^{\tilde{b}} \int \Big|\tilde{\lambda}(s)\nabla\varphi_k\big(\tau(s),\cdot\big) - \tilde{F}_s\Big|^2 \,\mathrm{d}\tilde{\mu}_s\,\mathrm{d}s = \int_{\tilde{a}}^{\tilde{b}} \big|\tilde{\lambda}(s)\big|^2 \int \Big|\nabla\varphi_k\big(\tau(s),\cdot\big) - F_{\tau(s)}\Big|^2 \,\mathrm{d}\mu_{\tau(s)}\,\mathrm{d}s$$
$$= \int_a^b \big|\tilde{\lambda}(\tau^{-1}(t))\big| \int |\nabla\varphi_k - F_t|^2 \,\mathrm{d}\mu_t\,\mathrm{d}t \le \big\|\tilde{\lambda}\big\|_{\mathrm{L}^\infty} \int_a^b \int |\nabla\varphi_k - F_t|^2 \,\mathrm{d}\mu_t\,\mathrm{d}t\,,$$

and the latter tends to 0 as $k \to \infty$. □

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CHAPTER 4

Discrete-to-continuum limits of optimal transport with linear growth on periodic graphs

This chapter contains (with minimal modifications) the following publication [PQ24]:

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Abstract

We prove discrete-to-continuum convergence for dynamical optimal transport on $\mathbb{Z}^d$-periodic graphs with cost functional having linear growth at infinity. This result provides an answer to a problem left open by Gladbach, Kopfer, Maas, and Portinale (Calc Var Partial Differential Equations 62(5), 2023), where the convergence behaviour of discrete boundary-value dynamical transport problems is proved under the stronger assumption of superlinear growth. Our result extends the known literature to some important classes of examples, such as scaling limits of 1-Wasserstein transport problems. Similarly to what happens in the quadratic case, the geometry of the graph plays a crucial role in the structure of the limit cost function, as we discuss in the final part of this work, which includes some visual representations.

4.1 Introduction

In the Euclidean setting, the Benamou–Brenier [BB00] formulation of the distance on the space $\mathscr{P}_2(\mathbb{R}^d)$ known as *2-Wasserstein* or *Kantorovich–Rubinstein* distance is given by the minimisation problem

$$\mathbb{W}_2(\mu_0,\mu_1)^2 = \inf\left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\nu_t|^2}{\mu_t} \,\mathrm{d}x\,\mathrm{d}t \;:\; \partial_t\mu_t + \nabla\cdot\nu_t = 0, \quad \mu_{t=0}=\mu_0, \quad \mu_{t=1}=\mu_1 \right\}, \tag{4.1.1}$$

for every $\mu_0,\mu_1 \in \mathscr{P}_2(\mathbb{R}^d)$. The PDE constraint is called *continuity equation* (we write $(\boldsymbol{\mu},\boldsymbol{\nu}) \in \mathsf{CE}$ when $(\boldsymbol{\mu},\boldsymbol{\nu})$ is a solution). Over the years, the Benamou–Brenier formula (4.1.1) has revealed significant connections between the theory of optimal transport and different fields of mathematics, including partial differential equations [JKO98], functional inequalities [OV00], and the novel notion of Lott–Sturm–Villani's synthetic Ricci curvature bounds for metric measure spaces [LV07, LV09, Stu06a, Stu06b]. Inspired by the dynamical formulation (4.1.1), in independent works, Maas [Maa11] (in the setting of Markov chains) and Mielke [Mie11] (in the context of reaction-diffusion systems) introduced a notion of optimal transport in discrete settings structured as a dynamical formulation *à la* Benamou–Brenier as in (4.1.1). One of the features of this discretisation procedure is the replacement of the continuity equation with a discrete counterpart: when working on a (finite) graph $(\mathcal{X}, \mathcal{E})$ (resp. vertices and edges), the discrete continuity equation reads

$$\partial_t m_t(x) + \sum_{y \sim x} J_t(x,y) = 0\,, \quad \forall x \in \mathcal{X}\,, \quad \Big(\text{we write } (\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_{\mathcal{X}}\Big)$$

where (m_t, J_t) corresponds to discrete masses and fluxes (s.t. $J_t(x,y) = -J_t(y,x)$). Maas' proposed distance $\mathcal{W}$ [Maa11] is obtained by minimising, under the above constraint, a discrete analogue of the Benamou–Brenier action functional with reference measure $\pi \in \mathscr{P}(\mathcal{X})$ and weight function $\omega \in \mathbb{R}_+^{\mathcal{E}}$, of the form

$$\int_0^1 \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \frac{|J_t(x,y)|^2}{\hat{r}_t(x,y)} \omega(x,y) \,\mathrm{d}t\,, \quad \text{where} \quad \hat{r}_t(x,y) := \theta_{\log}(r_t(x), r_t(y))\,, \quad r_t(x) := \frac{m_t(x)}{\pi(x)}\,,$$

and where $\theta_{\log}(a,b) := \int_0^1 a^s b^{1-s} \,\mathrm{d}s$ denotes the 1-homogeneous, positive mean called *logarithmic mean*. With this particular choice of the mean, it was proved [Maa11, Mie11] (see also [CHLZ12]) that the discrete heat flow coincides with the gradient flow of the relative entropy with respect to the discrete distance $\mathcal{W}$. In discrete settings, the equivalence between static and dynamical optimal transport breaks down, and the latter stands out in applications to evolution equations, discrete Ricci curvature, and functional inequalities [EM12, Mie13, EM14, EMT15, FM16, EHMT17, EF18]. Subsequently, several contributions have been devoted to the study of the scaling behaviour of discrete transport problems, in the setting of discrete-to-continuum approximation problems. The first convergence results were obtained in [GM13] for symmetric grids on a d-dimensional torus, and by [GT20] in a stochastic setting. In both cases, the authors obtained convergence of the discrete distances towards $\mathbb{W}_2$ in the limit of the discretisation getting finer and finer.

Nonetheless, it turned out that the geometry of the graph plays a crucial role in the game. A general result was obtained in [GKM20], where it is proved that the convergence of discrete distances associated with finite-volume partitions with vanishing size to the 2-Wasserstein space is substantially equivalent to an *asymptotic isotropy condition* on the mesh. The first complete characterisation of limits of transport costs on periodic graphs in arbitrary dimension for general action functionals (not necessarily quadratic) was established in [GKMP20, GKMP23]: in this setting, the limit action functional (more precisely, the energy density) can be explicitly characterised in terms of a cell formula, which is a finite-dimensional constrained minimisation problem depending on the initial graph and the cost function at the discrete level. The *action functionals* considered in [GKMP23] are of the form

$$(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE} \quad \mapsto \quad \mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) := \int_{(0,1)\times\mathbb{T}^d} f\Big(\rho, j\Big) \,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\Big(\rho^\perp, j^\perp\Big) \,\mathrm{d}\boldsymbol{\sigma}\,, \tag{4.1.2}$$

where we used the Lebesgue decomposition

$$\boldsymbol{\mu} = \rho \mathscr{L}^{d+1} + \boldsymbol{\mu}^\perp, \quad \boldsymbol{\nu} = j \mathscr{L}^{d+1} + \boldsymbol{\nu}^\perp, \quad \text{and} \quad \boldsymbol{\mu}^\perp = \rho^\perp \boldsymbol{\sigma}, \quad \boldsymbol{\nu}^\perp = j^\perp \boldsymbol{\sigma} \quad \left(\boldsymbol{\sigma} \perp \mathscr{L}^{d+1}\right)$$

and where the *energy density* $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is some given convex, lower semicontinuous function with *at least linear growth*, i.e. satisfying

$$f(\rho, j) \geq c|j| - C(\rho + 1), \qquad \forall \rho \in \mathbb{R}_+ \quad \text{and} \quad j \in \mathbb{R}^d, \tag{4.1.3}$$

whereas f^∞ denotes its *recession function* (see (4.2.3) for the precise definition). The choice $f(\rho, j) := |j|^2/\rho$ corresponds to the $\mathbb{W}_2$ distance. At the discrete level, on a locally finite connected graph $(\mathcal{X}, \mathcal{E})$ embedded in $\mathbb{R}^d$, the natural counterpart is represented by action functionals of the form

$$(\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_{\mathcal{X}} \quad \mapsto \quad \mathcal{A}(\boldsymbol{m}, \boldsymbol{J}) := \int_0^1 F(m, J) \, \mathrm{d}t, \tag{4.1.4}$$

for a given lower semicontinuous, convex, and local *cost function* F which also has at least linear growth with respect to the second variable (see (4.2.16) for the precise definition).

The main result in [GKMP23] is the Γ-convergence for constrained functionals as in (4.1.4), after a suitable rescaling of the graph $\mathcal{X}_\varepsilon := \varepsilon \mathcal{X}$, $\mathcal{E}_\varepsilon := \varepsilon \mathcal{E}$, and of the cost F_ε (and associated action $\mathcal{A}_\varepsilon$), in the framework of $\mathbb{Z}^d$-periodic graphs. In particular, the limit action is of the form (4.1.2), where the energy density $f = f_{\mathrm{hom}}$ is given in terms of a *cell formula*, explicitly reading

$$f_{\mathrm{hom}}(\rho, j) := \inf\left\{F(m, J) \ : \ (m, J) \in \mathsf{Rep}(\rho, j)\right\}, \qquad \rho \in \mathbb{R}_+, \quad j \in \mathbb{R}^d, \tag{4.1.5}$$

where $\mathsf{Rep}(\rho, j)$ denotes the set of discrete *representatives* of ρ and j, given by all $\mathbb{Z}^d$-periodic functions $m : \mathcal{X} \to \mathbb{R}_+$ with

$$\sum_{x \in \mathcal{X} \cap [0,1)^d} m(x) = \rho \tag{4.1.6}$$

and all $\mathbb{Z}^d$-periodic anti-symmetric discrete vector fields $J : \mathcal{E} \to \mathbb{R}$ with zero *discrete divergence* and with *effective flux* equal to j, i.e.,

$$\operatorname{div} J(x) := \sum_{y \sim x} J(x, y) = 0 \quad \forall x \in \mathcal{X} \quad \text{and} \quad \mathsf{Eff}(J) := \frac{1}{2} \sum_{\substack{(x,y) \in \mathcal{E} \\ x \in [0,1)^d}} J(x, y)(y - x) = j. \tag{4.1.7}$$

The result covers several examples, both for what concerns the geometric properties of the graph (such as isotropic meshes of $\mathbb{T}^d$, or the simple nearest-neighbors interaction on the symmetric grid) as well as the choice of the cost functionals (including discretisation of p-Wasserstein distances in arbitrary dimension and flow-based models, i.e. when F – or f – does not depend on the first variable).

As a consequence of this Γ-convergence (in time-space) and a compactness result for curves of measures with bounded action [GKMP23, Theorem 5.9], one obtains as a corollary [GKMP23, Theorem 5.10] that, under the stronger assumption of *superlinear growth* on F, also the corresponding discrete boundary-value problems (i.e. the associated squared distances, in the

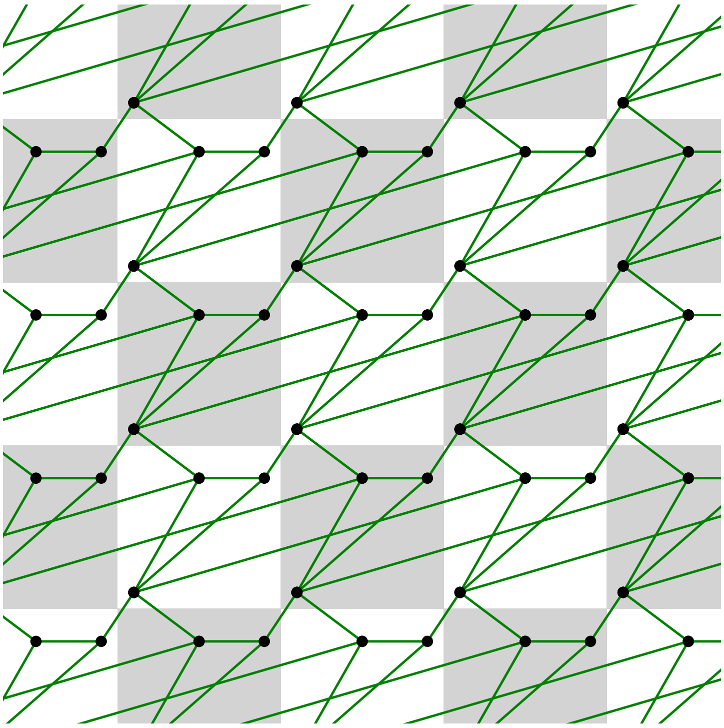

Figure 4.1: Example of $\mathbb{Z}^d$-periodic graph embedded in $\mathbb{R}^d$

case of the quadratic Wasserstein problems) Γ-converge to the corresponding continuous one, namely $\mathcal{MA}_\varepsilon \xrightarrow{\Gamma} \mathbb{MA}_{\hom}$ (with respect to the weak topology), where

$$\mathcal{MA}_\varepsilon(m_0, m_1) := \inf \{\mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J}) \ : \ (\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_{\mathcal{X}_\varepsilon} \quad \text{and} \quad \boldsymbol{m}_{t=0} = m_0, \ \boldsymbol{m}_{t=1} = m_1\} \ ,$$
$$\mathbb{MA}_{\hom}(\mu_0, \mu_1) := \inf \{\mathbb{A}_{\hom}(\boldsymbol{\mu}, \boldsymbol{\nu}) \ : \ (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE} \quad \text{and} \quad \boldsymbol{\mu}_{t=0} = \mu_0, \ \boldsymbol{\mu}_{t=1} = \mu_1\}$$

are the *minimal* discrete and homogenised action functionals, respectively. The superlinear growth condition, at the continuous level, is a reinforcement of the condition (4.1.3) and assumes the existence of a function $\theta : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty$ and a constant $C \in \mathbb{R}$ such that

$$f(\rho, j) \geq (\rho + 1)\theta\left(\frac{|j|}{\rho + 1}\right) - C(\rho + 1)\,, \qquad \forall \rho \in \mathbb{R}_+\,, \quad j \in \mathbb{R}^d\,. \tag{4.1.8}$$

In particular, this forces every $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE}$ with finite action to satisfy $\boldsymbol{\nu} \ll \boldsymbol{\mu} + \mathscr{L}^{d+1}$ [GKMP23, Remark 6.1], and it ensures compactness in $\mathcal{C}_{\mathrm{KR}}\big([0, 1]; \mathcal{M}_+(\mathbb{T}^d)\big)$ [GKMP23, Theorem 5.9], i.e., with respect to the time-uniform convergence in the Kantorovich–Rubinstein norm (recall that the KR norm metrises weak convergence on $\mathcal{M}_+(\mathbb{T}^d)$, see [GKMP23, Appendix A]). This compactness property makes the proof of the convergence $\mathcal{MA}_\varepsilon \xrightarrow{\Gamma} \mathbb{MA}_{\hom}$ an easy corollary of the convergence of the time-space energies.

Without the assumption of superlinear growth the situation is much more subtle: in particular, the lower semicontinuity of $\mathbb{MA}$ obtained minimising the functional $\mathbb{A}$ associated to a function f satisfying only (4.1.3) is not trivial. This is due to the fact that, in this framework, being a solution to CE with bounded action only ensures bounds for $\boldsymbol{\mu} \in \mathrm{BV}_{\mathrm{KR}}\big((0, 1); \mathcal{M}_+(\mathbb{T}^d)\big)$, which does not suffice to pass to the limit in the constraint given by the boundary conditions: *jumps* may occur at $t \in \{0, 1\}$ in the limit. Therefore, when the cost F grows linearly (linear bounds from both below and above), the scaling behaviour of the discrete boundary-value problems $\mathcal{MA}_\varepsilon$, as well as the lower semicontinuity of $\mathbb{MA}$, cannot be understood with the techniques utilised in [GKMP23]. The main goal of this work is, thus, to provide discrete-to-continuum results for $\mathcal{MA}_\varepsilon$ for cost functionals with linear growth, as well as for every flow-based type of cost, i.e. $F(m, J) = F(J)$. With similar arguments, we can also show the lower semicontinuity of $\mathbb{MA}$ for a general energy density f under the same assumptions, see Section 4.3.3.

Theorem 4.1.1 (Main result)**.** *Assume that either F satisfies the linear growth condition, i.e.*

$$F(m,J) \le C\left(1 + \sum_{\substack{(x,y)\in\mathcal{E}\\ |x|\le R}} |J(x,y)| + \sum_{\substack{x\in\mathcal{X}\\ |x|\le R}} m(x)\right)$$

for some constant $C < \infty$ and some $R > 0$, or that F does not depend on the m-variable (flow-based type). Then, as $\varepsilon \to 0$, the discrete functionals $\mathcal{MA}_\varepsilon$ Γ-converge to the continuous functional $\mathbb{MA}_{\mathrm{hom}}$ with respect to weak convergence.

The contribution of this paper is twofold. On one side, thanks to our main result, we can now include important examples, such as the $\mathbb{W}_1$ distance and related approximations, see in particular Section 4.4 for some explicit computations of the cell formula, including the equivalence between static and dynamical formulations (4.4.6), as well as some simulations. As typical in this discrete-to-continuum framework, also for $\mathbb{W}_1$-type problems, the geometry of the graph plays an important role in the homogenised norm obtained in the limit, giving rise to a whole class of *crystalline norms*, see Proposition 4.4.4 as well as Figure 4.2. On the other hand, this work provides ideas and techniques on how to handle the presence of singularities/jumps in the framework of curves of measures which are only of bounded variation, which is of independent interest.

Related literature. In their seminal work [JKO98], Jordan, Kinderlehrer, and Otto showed that the heat flow in $\mathbb{R}^d$ can be seen as the gradient flow of the relative entropy with respect to the 2-Wasserstein distance. In the same spirit, a discrete counterpart was proved in [Maa11] and [Mie11], independently, for the discrete heat flow and discrete relative entropy on Markov chains. In [FMP22], the authors proved the evolutionary Γ-convergence of the discrete gradient-flow structures associated with finite-volume partitions and discrete Fokker–Planck equations, generalising a previous result obtained in [DL15] in the setting of isotropic, one-dimensional meshes. Similar results were later obtained in [HT23, HST24] for the study of the limiting behaviour of random walks on tessellations in the diffusive limit. Generalised gradient-flow structures associated to jump processes and approximation results of nonlocal and local-interaction equations have been studied in a series of works [EPSS21, EPS23, EHPS23]. Recently, [EM24] considered the more general setting where the graph also depends on time.

4.2 General framework: continuous and discrete transport problems

In this section, we first introduce the general class of problems at the continuous level we are interested in, discussing main properties and known results. We then move to the discrete, periodic framework in the spirit of [GKMP23], summarise the known convergence results, and discuss the open problems we want to treat in this work. In contrast with [GKMP23], for the sake of the exposition we restrict our analysis to the time interval $\mathcal{I} := (0,1)$. Nonetheless, our main results easily extend to a general bounded, open interval $\mathcal{I} \subset \mathbb{R}$.

4.2.1 The continuous setting: transport problems on the torus

We start by recalling the definition of solutions to the continuity equation on $\mathbb{T}^d$.

Definition 4.2.1 (Continuity equation)**.** A pair of measures $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathcal{M}_+\big((0,1) \times \mathbb{T}^d\big) \times \mathcal{M}^d\big((0,1) \times \mathbb{T}^d\big)$ is said to be a solution to the continuity equation

$$\partial_t \boldsymbol{\mu} + \nabla \cdot \boldsymbol{\nu} = 0 \tag{4.2.1}$$

if, for all functions $\varphi \in \mathcal{C}_c^1\big((0,1) \times \mathbb{T}^d\big)$, the identity

$$\int_{(0,1)\times\mathbb{T}^d} \partial_t \varphi \, \mathrm{d}\boldsymbol{\mu} + \int_{(0,1)\times\mathbb{T}^d} \nabla\varphi \cdot \mathrm{d}\boldsymbol{\nu} = 0$$

holds. We use the notation $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE}$.

Throughout the whole paper, we consider energy densities f with the following properties.

Assumption 4.2.2. *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function, whose domain $\mathsf{D}(f)$ has nonempty interior. We assume that there exist constants $c > 0$ and $C < \infty$ such that the (at least) linear growth condition*

$$f(\rho, j) \geq c|j| - C(\rho + 1) \tag{4.2.2}$$

holds for all $\rho \in \mathbb{R}_+$ and $j \in \mathbb{R}^d$.

The corresponding *recession function* $f^\infty : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$f^\infty(\rho, j) := \lim_{t \to +\infty} \frac{f(\rho_0 + t\rho, j_0 + tj)}{t}\,, \tag{4.2.3}$$

for every $(\rho_0, j_0) \in \mathsf{D}(f)$. It is well established that the function f^∞ is lower semicontinuous, convex, and it satisfies the inequality

$$f^\infty(\rho, j) \geq c|j| - C\rho, \qquad \rho \in \mathbb{R}_+\,,\ j \in \mathbb{R}^d\,, \tag{4.2.4}$$

see [AFP00, Section 2.6].

Let $\mathscr{L}^{d+1}$ denote the Lebesgue measure on $(0,1) \times \mathbb{T}^d$. For $\boldsymbol{\mu} \in \mathcal{M}_+\big((0,1) \times \mathbb{T}^d\big)$ and $\boldsymbol{\nu} \in \mathcal{M}^d\big((0,1) \times \mathbb{T}^d\big)$, we write their Lebesgue decompositions as

$$\boldsymbol{\mu} = \rho \mathscr{L}^{d+1} + \boldsymbol{\mu}^\perp\,, \qquad \boldsymbol{\nu} = j \mathscr{L}^{d+1} + \boldsymbol{\nu}^\perp\,,$$

for some $\rho \in L^1_+\big((0,1) \times \mathbb{T}^d\big)$ and $j \in L^1\big((0,1) \times \mathbb{T}^d; \mathbb{R}^d\big)$. Given these decompositions, there always exists a measure $\boldsymbol{\sigma} \in \mathcal{M}_+\big((0,1) \times \mathbb{T}^d\big)$ such that

$$\boldsymbol{\mu}^\perp = \rho^\perp \boldsymbol{\sigma}\,, \qquad \boldsymbol{\nu}^\perp = j^\perp \boldsymbol{\sigma}\,, \tag{4.2.5}$$

for some $\rho^\perp \in L^1_+(\boldsymbol{\sigma})$ and $j^\perp \in L^1(\boldsymbol{\sigma}; \mathbb{R}^d)$ (take for example $\boldsymbol{\sigma} := |\boldsymbol{\mu}^\perp| + |\boldsymbol{\nu}^\perp|$).

Definition 4.2.3 (Action functionals)**.** We define the action functionals by

$$\begin{aligned}
&\mathbb{A} : \mathcal{M}_+\Big((0,1)\times\mathbb{T}^d\Big)\times\mathcal{M}^d\Big((0,1)\times\mathbb{T}^d\Big)\to\mathbb{R}\cup\{+\infty\}\,,\\
&\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) := \int_{(0,1)\times\mathbb{T}^d} f\Big(\rho,j\Big)\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\Big(\rho^\perp,j^\perp\Big)\,\mathrm{d}\boldsymbol{\sigma}\,,\\
&\mathbb{A}(\boldsymbol{\mu}) := \inf_{\boldsymbol{\nu}}\left\{\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) \ : \ (\boldsymbol{\mu},\boldsymbol{\nu})\in\mathsf{CE}\right\}\,.
\end{aligned}$$

Remark 4.2.4. This definition does not depend on the choice of $\boldsymbol{\sigma}$, due to the 1-homogeneity of f^∞. As $f(\rho,j)\geq -C(1+\rho)$ and $f^\infty(\rho,j)\geq -C\rho$ from (4.2.2) and (4.2.4), the fact that $\boldsymbol{\mu}\Big((0,1)\times\mathbb{T}^d\Big)<\infty$ ensures that $\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu})$ is well-defined in $\mathbb{R}\cup\{+\infty\}$.

The natural setting to work in is the space $\mathrm{BV}_{\mathrm{KR}}\Big((0,1);\mathcal{M}_+(\mathbb{T}^d)\Big)$ of the curves of measures $\boldsymbol{\mu}:(0,1)\to\mathcal{M}_+(\mathbb{T}^d)$ such that the BV-seminorm $\|\boldsymbol{\mu}\| = \|\boldsymbol{\mu}\|_{\mathrm{BV}_{\mathrm{KR}}\left((0,1);\mathcal{M}_+(\mathbb{T}^d)\right)}$ defined by

$$\|\boldsymbol{\mu}\| := \sup\left\{\int_{(0,1)}\int_{\mathbb{T}^d}\partial_t\varphi_t\,\mathrm{d}\mu_t\,\mathrm{d}t \ : \ \varphi\in\mathcal{C}^1_c\Big((0,1);\mathcal{C}^1(\mathbb{T}^d)\Big),\ \max_{t\in(0,1)}\|\varphi_t\|_{\mathcal{C}^1(\mathbb{T}^d)}\leq 1\right\}$$

is finite. Note that, by the trace theorem in BV, curves of measures in $\mathrm{BV}_{\mathrm{KR}}\Big((0,1);\mathcal{M}_+(\mathbb{T}^d)\Big)$ have a well defined trace at $t=0$ and $t=1$. As shown in [GKMP23, Lemma 3.13], any solution $(\boldsymbol{\mu},\boldsymbol{\nu})\in\mathsf{CE}$ can be disintegrated as $\mathrm{d}\boldsymbol{\mu}(t,x) = \mathrm{d}\mu_t(x)\,\mathrm{d}t$ for some measurable curve $t\mapsto\mu_t\in\mathcal{M}_+(\mathbb{T}^d)$ with finite constant mass. If $\mathbb{A}(\boldsymbol{\mu})<\infty$, then this curve belongs to $\mathrm{BV}_{\mathrm{KR}}\Big((0,1);\mathcal{M}_+(\mathbb{T}^d)\Big)$ and

$$\|\boldsymbol{\mu}\|_{\mathrm{BV}_{\mathrm{KR}}\left((0,1);\mathcal{M}_+(\mathbb{T}^d)\right)}\leq|\boldsymbol{\nu}|\Big((0,1)\times\mathbb{T}^d\Big)\,. \tag{4.2.6}$$

Boundary conditions and lower semicontinuity

Define the minimal homogenised action for $\mu_0,\mu_1\in\mathcal{M}_+(\mathbb{T}^d)$ with $\mu_0(\mathbb{T}^d)=\mu_1(\mathbb{T}^d)$ as

$$\mathbb{MA}(\mu_0,\mu_1) := \inf_{\boldsymbol{\mu}\in\mathrm{BV}_{\mathrm{KR}}\left((0,1);\mathcal{M}_+(\mathbb{T}^d)\right)}\left\{\mathbb{A}(\boldsymbol{\mu}) \ : \ \boldsymbol{\mu}_{t=0}=\mu_0,\boldsymbol{\mu}_{t=1}=\mu_1\right\}\,. \tag{4.2.7}$$

Note that, in general, $\mathbb{MA}$ may be infinite (although the measures have equal masses). Despite the lower semicontinuity property of $\mathbb{A}$ (cfr. [GKMP23, Lemma 3.14]), the lower semicontinuity of $\mathbb{MA}$ with respect to the natural weak topology of $\mathcal{M}_+(\mathbb{T}^d)\times\mathcal{M}_+(\mathbb{T}^d)$ is, in general, nontrivial. More precisely, it is a challenging question to prove (or disprove) that for any two sequences μ_0^n, $\mu_1^n\in\mathcal{M}_+(\mathbb{T}^d)$, such that $\mu_i^n\to\mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ as $n\to\infty$ for $i=0,1$, the inequality

$$\liminf_{n\to\infty}\mathbb{MA}(\mu_0^n,\mu_1^n)\geq\mathbb{MA}(\mu_0,\mu_1) \tag{4.2.8}$$

holds. In this work, we provide a positive answer in the case when f has linear growth or it is flow-based (i.e. it does not depend on the first variable), see Remark 4.3.14 and Proposition 4.3.15 below. First, we discuss the main challenges and the setup where the lower semicontinuity is already known to hold.

Remark 4.2.5 (Lack of compatible compactness). We know from [GKMP23, Lemma 3.14] that $(\boldsymbol{\mu}, \boldsymbol{\nu}) \mapsto \mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $\boldsymbol{\mu} \mapsto \mathbb{A}(\boldsymbol{\mu})$ are lower semicontinuous w.r.t. the weak topology. Moreover, if $\boldsymbol{\mu}^n$ is a sequence with

$$\sup_n \mathbb{A}(\boldsymbol{\mu}^n) < \infty \quad \text{and} \quad \sup_n \boldsymbol{\mu}^n\Big((0,1) \times \mathbb{T}^d\Big) < \infty \tag{4.2.9}$$

then $\boldsymbol{\mu}^n$ is weakly compact and any limit $\boldsymbol{\mu}$ belongs to $\mathrm{BV}_{\mathrm{KR}}\Big((0,1); \mathcal{M}_+(\mathbb{T}^d)\Big)$. This can be proved as in [GKMP23, Theorem 5.4]. Nonetheless, this property does not ensure the lower semicontinuity of $\mathbb{MA}$, because weak convergence does not preserve the boundary conditions (at time $t = 0$ and $t = 1$). For similar issues in the setting of functionals of $\mathbb{R}^d$-valued curves with bounded variations and their minimisation, see e.g. [AV98].

Remark 4.2.6 (Superlinear growth). Under the strengthened assumption of superlinear growth on f (with respect to the momentum variable), it is possible to prove the lower semicontinuity property (4.2.8), in the same way as in the proof of the discrete-to-continuum Γ-convergence of boundary-value problems of [GKMP23, Theorem 5.10]. More precisely, we say that f is of *superlinear growth* if there exists a function $\theta : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty$ and a constant $C \in \mathbb{R}$ such that

$$f(\rho, j) \geq (\rho + 1)\theta\left(\frac{|j|}{\rho + 1}\right) - C(\rho + 1)\,, \qquad \forall \rho \in \mathbb{R}_+\,, \quad j \in \mathbb{R}^d\,. \tag{4.2.10}$$

Arguing as in [GKMP23, Remark 5.6], one shows that any function of superlinear growth must satisfy the growth condition given by Assumption 4.2.2. Moreover, in this case, the recession function satisfies $f^\infty(0, j) = +\infty$, for every $j \neq 0$. See [GKMP23, Examples 5.7 & 5.8] for some examples belonging to this class. By arguing similarly as in the proof of [GKMP23, Theorem 5.9], assuming superlinear growth one can show that if $\boldsymbol{\mu}^n$ is a sequence with bounded action $\mathbb{A}(\boldsymbol{\mu}^n)$ and bounded total mass $\boldsymbol{\mu}^n\Big((0,1) \times \mathbb{T}^d\Big)$, then, up to a (nonrelabeled) subsequence, we have $\boldsymbol{\mu}^n \to \boldsymbol{\mu}$ in $\mathcal{M}_+((0,1) \times \mathbb{T}^d)$ and $\mu_t^n \to \mu_t$ in KR norm *uniformly* in $t \in (0,1)$, with limit curve $\boldsymbol{\mu} \in W^{1,1}_{\mathrm{KR}}((0,1); \mathcal{M}_+(\mathbb{T}^d))$. Using this fact, it is clear that the problem of "jumps" in the limit explained in Remark 4.2.5 does not occur, and the lower semicontinuity (4.2.8) directly follows from the lower semicontinuity of $\mathbb{A}$.

Remark 4.2.7. (Nonnegativity) Without loss of generality, we can assume that $f \geq 0$. Indeed, thanks to the linear growth assumption 4.2.2, we can define a new function

$$\widetilde{f}(\rho, j) := f(\rho, j) + C(\rho + 1) \geq c|j| \geq 0 \tag{4.2.11}$$

which is now nonnegative and with (at least) linear growth. Furthermore, we can compute the recess $\widetilde{f}^\infty$ and from the definition we see that

$$\widetilde{f}^\infty(\rho, j) = f^\infty(\rho, j) + C\rho\,. \tag{4.2.12}$$

Denote by $\widetilde{\mathbb{A}}$ the action functional obtained by replacing f with $\widetilde{f}$. Thanks to (4.2.11), (4.2.12), we have that

$$\widetilde{\mathbb{A}}(\boldsymbol{\mu}) := \inf_{\boldsymbol{\nu}} \left\{ \widetilde{\mathbb{A}}(\boldsymbol{\mu}, \boldsymbol{\nu}) \ : \ (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE} \right\} \tag{4.2.13}$$

$$= \inf_{\boldsymbol{\nu}} \left\{ \mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) \ : \ (\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE} \right\} + C(\boldsymbol{\mu}\Big((0,1) \times \mathbb{T}^d\Big) + 1)\,. \tag{4.2.14}$$

It follows that the corresponding boundary value problems are given by

$$\widetilde{\mathbb{MA}}(\mu_0, \mu_1) = \mathbb{MA}(\mu_0, \mu_1) + C(\mu_0(\mathbb{T}^d) + 1)\,, \quad \text{if } \mu_0(\mathbb{T}^d) = \mu_1(\mathbb{T}^d)\,. \tag{4.2.15}$$

Therefore, the (weak) lower semicontinuity for $\widetilde{\mathbb{MA}}$ is equivalent to that of $\mathbb{MA}$.

4.2.2 The discrete framework: transport problems on periodic graphs

We recall the framework of [GKMP23]: let $(\mathcal{X}, \mathcal{E})$ be a locally finite and $\mathbb{Z}^d$-periodic connected graph of bounded degree. We encode the set of vertices as $\mathcal{X} = \mathbb{Z}^d \times V$, where V is a finite set, and we use coordinates $x = (x_\mathsf{z}, x_\mathsf{v}) \in \mathcal{X}$. The set of edges $\mathcal{E} \subseteq \mathcal{X} \times \mathcal{X}$ is symmetric and $\mathbb{Z}^d$-periodic, and we use the notation $x \sim y$ whenever $(x, y) \in \mathcal{E}$. Let $R_0 := \max_{(x,y)\in\mathcal{E}} |x_\mathsf{z} - y_\mathsf{z}|_\infty$ be the maximal edge length in the supremum norm $|\cdot|_\infty$ on $\mathbb{R}^d$. We use the notation $\mathcal{X}^Q := \{x \in \mathcal{X} \, : \, x_\mathsf{z} = 0\}$ and $\mathcal{E}^Q := \Big\{(x, y) \in \mathcal{E} \, : \, x_\mathsf{z} = 0\Big\}$. For a discussion concerning abstract and embedded graphs, see [GKMP23, Remark 2.2].

In what follows, we denote by $\mathbb{R}_+^{\mathcal{X}}$ the set of functions $m \colon \mathcal{X} \to \mathbb{R}_+$, and by $\mathbb{R}_a^{\mathcal{E}}$ the set of *anti-symmetric* functions $J \colon \mathcal{E} \to \mathbb{R}$, that is, such that $J(x, y) = -J(y, x)$. The elements of $\mathbb{R}_a^{\mathcal{E}}$ will often be called *(discrete) vector fields*.

Assumption 4.2.8 (Admissible cost function)**.** *The function $F : \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ is assumed to have the following properties:*

(1) F is convex and lower semicontinuous.

(2) F is local, *meaning that, for some number $R_1 < \infty$, we have $F(m, J) = F(m', J')$ whenever $m, m' \in \mathbb{R}_+^{\mathcal{X}}$ and $J, J' \in \mathbb{R}_a^{\mathcal{E}}$ agree within a ball of radius R_1, i.e.*

$$\begin{aligned} m(x) &= m'(x) && \text{for all } x \in \mathcal{X} \text{ with } |x_\mathsf{z}|_\infty \le R_1\,, \quad \text{and} \\ J(x,y) &= J'(x,y) && \text{for all } (x,y) \in \mathcal{E} \text{ with } |x_\mathsf{z}|_\infty, |y_\mathsf{z}|_\infty \le R_1\,. \end{aligned}$$

(3) F is of at least linear growth, *i.e. there exist $c > 0$ and $C < \infty$ such that*

$$F(m, J) \ge c \sum_{(x,y)\in\mathcal{E}^Q} |J(x,y)| - C\left(1 + \sum_{\substack{x\in\mathcal{X} \\ |x_\mathsf{z}|_\infty \le R_{\max}}} m(x)\right) \tag{4.2.16}$$

for any $m \in \mathbb{R}_+^{\mathcal{X}}$ and $J \in \mathbb{R}_a^{\mathcal{E}}$. Here, $R_{\max} := \max\{R_0, R_1\}$.

(4) There exist a $\mathbb{Z}^d$-periodic function $m^\circ \in \mathbb{R}_+^{\mathcal{X}}$ and a $\mathbb{Z}^d$-periodic and divergence-free vector field $J^\circ \in \mathbb{R}_a^{\mathcal{E}}$ such that

$$(m^\circ, J^\circ) \in \mathsf{D}(F)^\circ\,. \tag{4.2.17}$$

Remark 4.2.9. Important examples that satisfy the growth condition (4.2.16) are of the form

$$F(m, J) = \frac{1}{2} \sum_{(x,y)\in\mathcal{E}^Q} \frac{|J(x,y)|^p}{\Lambda\Big(q_{xy} m(x), q_{yx} m(y)\Big)^{p-1}}\,, \tag{4.2.18}$$

where $1 \le p < \infty$, the constants $q_{xy}, q_{yx} > 0$ are fixed parameters defined for $(x, y) \in \mathcal{E}^Q$, and Λ is a suitable mean. Functions of this type naturally appear in discretisations of Wasserstein gradient-flow structures [Maa11, Mie11, CHLZ12], see also [GKMP23, Remark 2.6].

The rescaled graph. Let $\mathbb{T}_\varepsilon^d = (\varepsilon\mathbb{Z}/\mathbb{Z})^d$ be the discrete torus of mesh size $\varepsilon \in 1/\mathbb{N}$. We denote by $[\varepsilon z]$ for $z \in \mathbb{Z}^d$ the corresponding equivalence classes. Equivalently, $\mathbb{T}_\varepsilon^d = \varepsilon\mathbb{Z}_\varepsilon^d$ where $\mathbb{Z}_\varepsilon^d = \left(\mathbb{Z}/\frac{1}{\varepsilon}\mathbb{Z}\right)^d$. The rescaled graph $(\mathcal{X}_\varepsilon, \mathcal{E}_\varepsilon)$ is defined as

$$\mathcal{X}_\varepsilon := \mathbb{T}_\varepsilon^d \times V \quad \text{and} \quad \mathcal{E}_\varepsilon := \Big\{\Big(T_\varepsilon^0(x), T_\varepsilon^0(y)\Big) \, : \, (x,y) \in \mathcal{E}\Big\}$$

where, for $\bar{z} \in \mathbb{Z}^d_\varepsilon$,

$$T^{\bar{z}}_\varepsilon : \mathcal{X} \to \mathcal{X}_\varepsilon \,, \qquad (z,v) \mapsto \Big([\varepsilon(\bar{z}+z)], v \Big) \,. \tag{4.2.19}$$

For $x = \Big([\varepsilon z], v\Big) \in \mathcal{X}_\varepsilon$ we write

$$x_{\mathsf{z}} := z \in \mathbb{Z}^d_\varepsilon \,, \qquad x_{\mathsf{v}} := v \in V \,.$$

The equivalence relation $\sim$ on $\mathcal{X}$ is equivalently defined on $\mathcal{X}_\varepsilon$ by means of $\mathcal{E}_\varepsilon$. Hereafter, we always assume that $\varepsilon R_0 < \frac{1}{2}$.

The rescaled energies. Let $F : \mathbb{R}^{\mathcal{X}}_+ \times \mathbb{R}^{\mathcal{E}}_a \to \mathbb{R} \cup \{+\infty\}$ be a cost function satisfying Assumption 4.2.8. For $\varepsilon > 0$ satisfying the conditions above, we can define a corresponding energy functional $\mathcal{F}_\varepsilon$ in the rescaled periodic setting: following [GKMP23], for $\bar{z} \in \mathbb{Z}^d_\varepsilon$, each function $\psi : \mathcal{X}_\varepsilon \to \mathbb{R}$ induces a $\frac{1}{\varepsilon}\mathbb{Z}^d$-periodic function

$$\tau^{\bar{z}}_\varepsilon \psi : \mathcal{X} \to \mathbb{R} \,, \qquad \Big(\tau^{\bar{z}}_\varepsilon \psi\Big)(x) := \psi\Big(T^{\bar{z}}_\varepsilon(x)\Big) \quad \text{for } x \in \mathcal{X} \,.$$

Similarly, each function $J : \mathcal{E}_\varepsilon \to \mathbb{R}$ induces a $\frac{1}{\varepsilon}\mathbb{Z}^d$-periodic function

$$\tau^{\bar{z}}_\varepsilon J : \mathcal{E} \to \mathbb{R} \,, \qquad \Big(\tau^{\bar{z}}_\varepsilon J\Big)(x,y) := J\Big(T^{\bar{z}}_\varepsilon(x), T^{\bar{z}}_\varepsilon(y)\Big) \quad \text{for } (x,y) \in \mathcal{E} \,.$$

Definition 4.2.10 (Discrete energy functional)**.** The rescaled energy is defined by

$$\mathcal{F}_\varepsilon : \mathbb{R}^{\mathcal{X}_\varepsilon}_+ \times \mathbb{R}^{\mathcal{E}_\varepsilon}_a \to \mathbb{R} \cup \{+\infty\} \,, \qquad (m,J) \overset{\mathcal{F}_\varepsilon}{\longmapsto} \sum_{z \in \mathbb{Z}^d_\varepsilon} \varepsilon^d F\left(\frac{\tau^z_\varepsilon m}{\varepsilon^d}, \frac{\tau^z_\varepsilon J}{\varepsilon^{d-1}} \right) .$$

Remark 4.2.11. As observed in [GKMP23, Remark 2.8], the functional $\mathcal{F}_\varepsilon(m,J)$ is well-defined as an element in $\mathbb{R} \cup \{+\infty\}$. Indeed, the condition (4.2.16) yields

$$\begin{aligned} \mathcal{F}_\varepsilon(m,J) = \sum_{z \in \mathbb{Z}^d_\varepsilon} \varepsilon^d F\left(\frac{\tau^z_\varepsilon m}{\varepsilon^d}, \frac{\tau^z_\varepsilon J}{\varepsilon^{d-1}} \right) &\geq -C \sum_{z \in \mathbb{Z}^d_\varepsilon} \varepsilon^d \left(1 + \sum_{\substack{x \in \mathcal{X} \\ |x_{\mathsf{z}}|_\infty \leq R_{\max}}} \frac{\tau^z_\varepsilon m(x)}{\varepsilon^d} \right) \\ &\geq -C \left(1 + (2R_{\max}+1)^d \sum_{x \in \mathcal{X}_\varepsilon} m(x) \right) > -\infty \,. \end{aligned}$$

Definition 4.2.12 (Discrete continuity equation)**.** A pair $(\boldsymbol{m}, \boldsymbol{J})$ is said to be a solution to the discrete continuity equation if $\boldsymbol{m} : (0,1) \to \mathbb{R}^{\mathcal{X}_\varepsilon}_+$ is continuous, $\boldsymbol{J} : (0,1) \to \mathbb{R}^{\mathcal{E}_\varepsilon}_a$ is Borel measurable, and

$$\partial_t m_t(x) + \sum_{y \sim x} J_t(x,y) = 0 \tag{4.2.20}$$

holds for all $x \in \mathcal{X}_\varepsilon$ in the sense of distributions. We use the notation

$$(\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_\varepsilon \,.$$

Remark 4.2.13. We may write (4.2.20) as $\partial_t m_t + \operatorname{div} J_t = 0$ using the discrete divergence operator, given by

$$\operatorname{div} J \in \mathbb{R}^{\mathcal{X}_\varepsilon} \,, \qquad \operatorname{div} J(x) := \sum_{y \sim x} J(x,y) \,, \qquad \forall J \in \mathbb{R}^{\mathcal{E}_\varepsilon}_a \,.$$

The proof of the following lemma can be found in [GKMP23].

Lemma 4.2.14 (Mass preservation). *Let* $(\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_\varepsilon$. *Then we have* $m_s(\mathcal{X}_\varepsilon) = m_t(\mathcal{X}_\varepsilon)$ *for all* $s, t \in (0,1)$.

We are now ready to define one of the main objects in this paper.

Definition 4.2.15 (Discrete action functional). For any continuous function $\boldsymbol{m} : (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ such that $t \mapsto \sum_{x \in \mathcal{X}_\varepsilon} m_t(x) \in L^1\big((0,1)\big)$ and any Borel measurable function $\boldsymbol{J} : (0,1) \to \mathbb{R}_a^{\mathcal{E}_\varepsilon}$, we define

$$\mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J}) := \int_0^1 \mathcal{F}_\varepsilon(m_t, J_t)\,\mathrm{d}t \in \mathbb{R} \cup \{+\infty\}\,.$$

Furthermore, we set

$$\mathcal{A}_\varepsilon(\boldsymbol{m}) := \inf_{\boldsymbol{J}} \Big\{ \mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J}) \ : \ (\boldsymbol{m}, \boldsymbol{J}) \in \mathcal{CE}_\varepsilon \Big\}\,.$$

Arguing as in Remark 4.2.11, one can show [GKMP23, Remark 2.13] that $\mathcal{A}_\varepsilon(\boldsymbol{m}, \boldsymbol{J})$ is well-defined as an element in $\mathbb{R} \cup \{+\infty\}$, as a consequence of the growth condition (4.2.16).

Definition 4.2.16 (Minimal discrete action functional). For any pair of boundary data m_0, $m_1 \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$, we define the associated discrete boundary value problem as

$$\mathcal{MA}_\varepsilon(m_0, m_1) := \inf \Big\{ \mathcal{A}_\varepsilon(\boldsymbol{m}) \ : \ \boldsymbol{m} : (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}\,, \quad \boldsymbol{m}_{t=0} = m_0 \quad \text{and} \quad \boldsymbol{m}_{t=1} = m_1 \Big\}\,.$$

The aim of this work is to study the asymptotic behaviour of the energies $\mathcal{MA}_\varepsilon$ as $\varepsilon \to 0$ under the Assumption 4.2.8.

4.3 Statement and proof of the main result

In this paper we extend the Γ-convergence result for the functionals $\mathcal{MA}_\varepsilon$ towards $\mathbb{MA}_{\mathrm{hom}}$, proved in [GKMP23] for superlinear cost functionals to two cases: under the assumption of linear growth (bound both from below and above) and when the function F does not depend on ρ.

Assumption 4.3.1 (Linear growth). *We say that a function* $F : \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ *has* linear growth *if it satisfies*

$$F(m, J) \le C \left(1 + \sum_{\substack{(x,y) \in \mathcal{E} \\ |x_{\mathrm{z}}|_\infty \le R}} |J(x,y)| + \sum_{\substack{x \in \mathcal{X} \\ |x_{\mathrm{z}}|_\infty \le R}} m(x) \right)$$

for some constant $C < \infty$ *and some* $R > 0$.

Assumption 4.3.2 (Flow-based). *We say that a function* $F : \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ *is of* flow-based type *if it depends only on the the second variable, i.e. (with a slight abuse of notation)* $F(m, J) = F(J)$, *for some* $F : \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$.

Similarly, we say that $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is of *flow-based type* if it does not depend on the ρ variable, i.e., $f(\rho, j) = f(j)$. In this case, the problem simplifies significantly, and the dynamical variational problem described in (4.2.7) admits an equivalent, static formulation (see (4.3.35)).

Remark 4.3.3 (Linear growth vs Lipschitz). While working with convex functions, to assume a linear growth condition (from above) is essentially equivalent to require Lipschitz continuity with respect to the second variable.

Lemma 4.3.4 (Lipschitz continuity)**.** *Let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a function, convex in the second variable. Let $C > 0$. Then the following are equivalent:*

1. *for every $\rho \in \mathbb{R}_+$ and $j \in \mathbb{R}^d$ the inequality $f(\rho, j) \leq C(1 + \rho + |j|)$ holds.*
2. *for every $\rho \in \mathbb{R}_+$, the function $f(\rho, \cdot) : \mathbb{R}^d \to \mathbb{R}_+$ is Lipschitz continuous (uniformly in ρ) with constant C, and the inequality $f(\rho, 0) \leq C(1 + \rho)$ holds.*

In the very same spirit, one can show the analogous result at the discrete level.

Lemma 4.3.5 (Lipschitz continuity II)**.** *Let $F : \mathbb{R}_+^{\mathcal{X}} \times \mathbb{R}_a^{\mathcal{E}} \to \mathbb{R} \cup \{+\infty\}$ be convex in the second variable. Let $C, R > 0$. Then the following are equivalent:*

1. *F is of linear growth, in the sense of Assumption 4.3.1, with the same constants C and R.*
2. *For every $m \in \mathbb{R}_+^{\mathcal{X}}$, we have that*

$$F(m, 0) \leq C\Bigg(1 + \sum_{\substack{x \in \mathcal{X} \\ |x_z|_\infty \leq R}} m(x)\Bigg),$$

as well as that F is Lipschitz continuous with constant C in the second variable, in the sense that

$$|F(m, J_1) - F(m, J_2)| \leq C \sum_{\substack{(x,y) \in \mathcal{E} \\ |x_z|_\infty \leq R}} |J_1(x, y) - J_2(x, y)|, \tag{4.3.1}$$

for every $J_1, J_2 \in \mathbb{R}_a^{\mathcal{E}}$.

Proof of Lemma 4.3.4. Let us assume the first condition and fix $\rho \in \mathbb{R}_+$ as well as $j_1, j_2 \in \mathbb{R}^d$. It follows from the convexity in the second variable that the function

$$\mathbb{R} \ni t \mapsto f(\rho, j_1 + t(j_2 - j_1))$$

is convex. In particular, the inequalities

$$\begin{aligned} f(\rho, j_2) - f(\rho, j_1) &\leq \frac{f(\rho, j_1 + t(j_2 - j_1)) - f(\rho, j_1)}{t} \\ &\leq \frac{C\big(1 + \rho + \big|j_1 + t(j_2 - j_1)\big|\big) - f(\rho, j_1)}{t} \end{aligned}$$

hold for every $t \geq 1$. Letting $t \to \infty$, we thus find

$$f(\rho, j_2) - f(\rho, j_1) \leq C|j_2 - j_1|$$

and, by arbitrariness of the arguments, the claimed Lipschitz continuity. The fact that $f(\rho, 0) \leq C(1+\rho)$ trivially follows from the first condition.

Conversely, if the second condition holds, we necessarily have

$$f(\rho, j) \leq C|j| + f(\rho, 0) \leq C(1 + \rho + |j|)\,,$$

for every $\rho \in \mathbb{R}_+$ and $j \in \mathbb{R}^d$, which is precisely the first condition in the statement. □

Let us recall the homogenised energy density $f_{\hom}$, which describes the limit energy and is given by a cell formula. For given $\rho \geq 0$ and $j \in \mathbb{R}^d$, $f_{\hom}(\rho, j)$ is obtained by minimising over the unit cube the cost among functions m and vector fields J *representing* ρ and j. More precisely, the function $f_{\hom} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is given by

$$f_{\hom}(\rho, j) := \inf_{m,J}\Big\{F(m, J) \;:\; (m, J) \in \mathsf{Rep}(\rho, j)\Big\}\,, \tag{4.3.2}$$

where the set of *representatives* $\mathsf{Rep}(\rho, j)$ consists of all $\mathbb{Z}^d$-periodic functions $m : \mathcal{X} \to \mathbb{R}_+$ and all $\mathbb{Z}^d$-periodic anti-symmetric discrete vector fields $J : \mathcal{E} \to \mathbb{R}$ satisfying

$$\sum_{x \in \mathcal{X}^Q} m(x) = \rho\,, \quad \operatorname{div} J = 0\,, \quad \text{and} \quad \mathsf{Eff}(J) := \frac{1}{2}\sum_{(x,y)\in\mathcal{E}^Q} J(x,y)(y_{\mathsf{z}} - x_{\mathsf{z}}) = j\,. \tag{4.3.3}$$

The set of representatives is nonempty for every choice of ρ and j by [GKMP23, Lemma 4.5 (iv)]. In the case of embedded graphs, the definition of effective flux coincide with the one provided in the introduction (cfr. (4.1.7)), see [GKMP23, Proposition 9.1].

Remark 4.3.6. It is not hard to show that if F is of linear growth, then $f_{\hom}$ is also of linear growth (and therefore, in view of Lemma 4.3.4, it is Lipschitz in the second variable uniformly w.r.t. the first one), see e.g. [GMP25].

We denote by $\mathbb{A}_{\hom}$ and $\mathbb{MA}_{\hom}$ the action functionals corresponding to the choice $f = f_{\hom}$. In order to talk about Γ-convergence, we need to specificy which type of discrete-to-continuum topology/convergence we adopt (in the same spirit of [GKMP23]).

Definition 4.3.7 (Embedding)**.** For $\varepsilon > 0$ and $z \in \mathbb{R}^d$, let $Q_\varepsilon^z := \varepsilon z + [0, \varepsilon)^d \subseteq \mathbb{T}^d$ denote the projection of the cube with side length ε based at εz to the quotient $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For $m \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ and $J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon}$, we define $\iota_\varepsilon m \in \mathcal{M}_+(\mathbb{T}^d)$ and $\iota_\varepsilon J \in \mathcal{M}^d(\mathbb{T}^d)$ by

$$\iota_\varepsilon m := \varepsilon^{-d} \sum_{x \in \mathcal{X}_\varepsilon} m(x) \mathscr{L}^d|_{Q_\varepsilon^{x_{\mathsf{z}}}}\,, \tag{4.3.4}$$

$$\iota_\varepsilon J := \varepsilon^{-d+1} \sum_{(x,y)\in\mathcal{E}_\varepsilon} \frac{J(x,y)}{2}\bigg(\int_0^1 \mathscr{L}^d|_{Q_\varepsilon^{(1-s)x_{\mathsf{z}}+sy_{\mathsf{z}}}}\,\mathrm{d}s\bigg)(y_{\mathsf{z}} - x_{\mathsf{z}})\,. \tag{4.3.5}$$

With a slight abuse of notation, given $\boldsymbol{m} : (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ we also write $\iota_\varepsilon \boldsymbol{m} \in \mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$ for the continuous space-time measure with time disintegration given by $t \mapsto \iota_\varepsilon m_t$. Moreover, for a given sequence of nonnegative discrete measures $m^\varepsilon \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$, we write

$$m_\varepsilon \to \mu \in \mathcal{M}_+(\mathbb{T}^d) \quad \text{weakly} \qquad \text{iff} \qquad \iota_\varepsilon m^\varepsilon \to \mu \quad \text{weakly in } \mathcal{M}_+(\mathbb{T}^d)\,.$$

Similarly, for $\boldsymbol{m}^\varepsilon : (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ we write $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu} \in \mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$ with an analogous meaning. Similar notation is used for (Borel, possibly discontinuous) curves of fluxes $\boldsymbol{J} : (0,1) \to \mathbb{R}_a^{\mathcal{E}_\varepsilon}$ and convergent sequences of (curves of) fluxes.

Remark 4.3.8 (Preservation of the continuity equation). The definition of this embedding for masses and fluxes ensures that solutions to the discrete continuity equation are mapped to solutions of CE, cfr. [GKMP23, Lemma 4.9].

We are ready to state our main result.

Theorem 4.3.9 (Main result)**.** *Let $(\mathcal{X}, \mathcal{E}, F)$ be as described in Section 4.2.2 and Assumption 4.2.8. Assume that F is either of flow-based type (Assumption 4.3.2) or with linear growth (Assumption 4.3.1). Then, in either case, the functionals $\mathcal{MA}_\varepsilon$ Γ-convergence to $\mathbb{MA}_{\rm hom}$ as $\varepsilon \to 0$ with respect to the weak topology of $\mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d)$. More precisely, we have:*

(1) ***Liminf inequality**: For any sequences $m_0^\varepsilon, m_1^\varepsilon \in \mathcal{M}_+(\mathcal{X}_\varepsilon)$ such that $m_i^\varepsilon \to \mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ for $i = 0,1$, we have that*

$$\liminf_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \geq \mathbb{MA}_{\rm hom}(\mu_0, \mu_1)\,. \tag{4.3.6}$$

(2) ***Limsup inequality**: For any $\mu_0, \mu_1 \in \mathcal{M}_+(\mathbb{T}^d)$, there exist sequences $m_0^\varepsilon, m_1^\varepsilon \in \mathcal{M}_+(\mathcal{X}_\varepsilon)$ such that $m_i^\varepsilon \to \mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ for $i = 0,1$, and*

$$\limsup_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \leq \mathbb{MA}_{\rm hom}(\mu_0, \mu_1)\,. \tag{4.3.7}$$

Remark 4.3.10 (Convergence of the actions and superlinear regime). The Γ-convergence of the energies $\mathcal{A}_\varepsilon$ towards $\mathbb{A}_{\rm hom}$ under Assumption 4.2.8 is the main result of [GKMP23, Theorem 5.1]. Related to it, similarly as discussed in Remark 4.2.6, the superlinear case [GKMP23, Assumption 5.5], not included in the statement, has already been proved in [GKMP23], and it follows directly from the aforementioned convergence $\mathcal{A}_\varepsilon \xrightarrow{\Gamma} \mathbb{A}_{\rm hom}$ and a strong compactness result which holds in such a framework, see in particular [GKMP23, Theorems 5.9 & 5.10]. Without the superlinear growth assumption, the proof is much more involved and requires extra work and new ideas, which are the main contribution of this paper.

Remark 4.3.11 (Compactness under linear growth from below). Just assuming Assumption 4.2.8, the following compactness result for sequences of bounded action was proved in [GKMP23, Theorem 5.4]: if $\boldsymbol{m}^\varepsilon : (0,1) \to \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ is such that

$$\sup_{\varepsilon>0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon) < \infty \quad \text{and} \quad \sup_{\varepsilon>0} \boldsymbol{m}^\varepsilon\Big((0,1) \times \mathcal{X}_\varepsilon\Big) < \infty\,,$$

then there exists a curve $\boldsymbol{\mu} = \mu_t(\mathrm{d}x)\,\mathrm{d}t \in \mathrm{BV}_{\rm KR}\Big((0,1); \mathcal{M}_+(\mathbb{T}^d)\Big)$ such that, up to a (nonrelabeled) subsequence, we have

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu} \quad \text{weakly in } \mathcal{M}_+\Big((0,1) \times \mathbb{T}^d\Big) \quad \text{and} \quad m_t^\varepsilon \to \mu_t \quad \text{weakly in } \mathcal{M}_+(\mathbb{T}^d)\,,$$

for a.e. $t \in (0,1)$. This is going to be an important tool in the proof of our main result.

4.3.1 Proof of the limsup inequality

In this section, we prove the limsup inequality in Theorem 4.3.9. This proof does not require Assumption 4.3.1 or Assumption 4.3.2, but rather a weaker hypothesis, which is satisfied under either of the two assumptions.

Proposition 4.3.12 (Γ-limsup). *Let μ_0, μ_1 be nonnegative measures on $\mathbb{T}^d$. Assume that there exists a $\mathbb{Z}^d$-periodic and divergence-free vector field $\bar{J} \in \mathbb{R}^{\mathcal{E}}_a$ such that*

$$F(m, \bar{J}) \le C\left(1 + \sum_{\substack{x\in\mathcal{X} \\ |x|_\infty \le R}} m(x)\right), \qquad m \in \mathbb{R}^{\mathcal{X}}_+ , \tag{4.3.8}$$

for some finite constants C and R. Then there exist two sequences $(m_0^\varepsilon)_{\varepsilon>0}$ and $(m_1^\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}^{\mathcal{X}_\varepsilon}_+$ such that $m_i^\varepsilon \to \mu_i$ weakly in $\mathcal{M}_+(\mathbb{T}^d)$ for $i = 0, 1$, and

$$\limsup_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) . \tag{4.3.9}$$

Proof. We may and will assume that $\mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) < \infty$. We also claim that it suffices to prove the statement with $\mathbb{MA}(\mu_0, \mu_1) + 1/k$ in place of the right-hand side of (4.3.9) for every $k \in \mathbb{N}_1$. Indeed, assume that we know of the existence of sequences $(m_i^{\varepsilon,k})_\varepsilon$ such that $m_i^{\varepsilon,k} \to \mu_i$, and

$$\limsup_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^{\varepsilon,k}, m_1^{\varepsilon,k}) \le \mathbb{MA}(\mu_0, \mu_1) + 1/k ,$$

for every $k \in \mathbb{N}_1$. Since $\mathbb{T}^d$ is compact, the weak convergence is equivalent to convergence in the Kantorovich–Rubinstein norm. Hence, for every k we can find ε_k such that, when $\varepsilon \le \varepsilon_k$,

$$\mathcal{MA}_\varepsilon(m_0^{\varepsilon,k}, m_1^{\varepsilon,k}) \le \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) + 2/k \quad \text{and} \quad \max_{i=0,1} \left\| \iota_\varepsilon m_i^{\varepsilon,k} - \mu_i \right\|_{\mathrm{KR}} \le 1/k .$$

We can also assume that $\varepsilon_{k+1} \le \frac{\varepsilon_k}{2}$, for every k. It now suffices to set

$$k_\varepsilon := \max\{k \in \mathbb{N}_1 \,:\, \varepsilon_k \ge \varepsilon\} \quad \text{and} \quad m_i^\varepsilon := m_i^{\varepsilon,k_\varepsilon} ,$$

for every ε and $i = 0, 1$ to get

$$\limsup_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) + \limsup_{\varepsilon\to 0} \frac{2}{k_\varepsilon}$$

as well as

$$\limsup_{\varepsilon\to 0} \max_{i=0,1} \|\iota_\varepsilon m_i^\varepsilon - \mu_i\|_{\mathrm{KR}} \le \limsup_{\varepsilon\to 0} \frac{1}{k_\varepsilon} .$$

The claim is proved, since $k_\varepsilon \to_\varepsilon \infty$, as can be readily verified.

Thus, let us now choose k and keep it fixed. By definition of $\mathbb{MA}_{\mathrm{hom}}$, there exists $\boldsymbol{\mu} = \mu_t(\mathrm{d}x)\,\mathrm{d}t \in \mathrm{BV}_{\mathrm{KR}}\big((0,1); \mathcal{M}_+(\mathbb{T}^d)\big)$ with $\boldsymbol{\mu}_{t=0} = \mu_0$, $\boldsymbol{\mu}_{t=1} = \mu_1$ and such that

$$\mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}) \le \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) + 1/k .$$

Recall from Remark 4.3.10 that $\mathcal{A}_\varepsilon \xrightarrow{\Gamma} \mathbb{A}_{\mathrm{hom}}$; in particular, there exists a recovery sequence $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$ such that $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu}$ weakly and

$$\limsup_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \le \mathbb{A}_{\mathrm{hom}}(\boldsymbol{\mu}) .$$

We shall prove that $\|\iota_\varepsilon m_t^\varepsilon - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)} \to 0$ in ($\mathscr{L}^1$-)measure or, equivalently, that

$$\lim_{\varepsilon\to 0} \int_0^1 \min\left\{\|\iota_\varepsilon m_t^\varepsilon - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)} , 1\right\} \mathrm{d}t = 0 . \tag{4.3.10}$$

In order to do this, assume by contradiction that there exists a subsequence such that

$$\int_0^1 \min\left\{\|\iota_{\varepsilon_n} m_t^{\varepsilon_n} - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)}, 1\right\} \,\mathrm{d}t > \delta\,, \qquad n \in \mathbb{N}\,,$$

for some $\delta > 0$. Up to possibly extracting a further subsequence, it can be easily checked that the hypotheses of [GKMP23, Theorem 5.4] are satisfied (cfr. Remark 4.3.11); hence, there exists a further (not relabeled) subsequence such that, for almost every $t \in (0,1)$, $m_t^{\varepsilon_n} \to \mu_t$ weakly and thus $\|\iota_{\varepsilon_n} m_t^{\varepsilon_n} - \mu_t\|_{\mathrm{KR}(\mathbb{T}^d)} \to 0$. The dominated convergence theorem yields an absurd.

From (4.3.10) we deduce that for every $T \in (0,1/2)$ there exists a sequence of times $(a_\varepsilon^T)_\varepsilon \subseteq (0,T)$ such that

$$\lim_{\varepsilon\to0}\left\|\iota_\varepsilon m^\varepsilon_{a_\varepsilon^T} - \mu_{a_\varepsilon^T}\right\|_{\mathrm{KR}(\mathbb{T}^d)} = 0\,.$$

With a diagonal argument, we find a sequence $(a_\varepsilon)_\varepsilon \subseteq (0,1/2)$ such that

$$\lim_{\varepsilon\to0} a_\varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon\to0}\left\|\iota_\varepsilon m^\varepsilon_{a_\varepsilon} - \mu_{a_\varepsilon}\right\|_{\mathrm{KR}(\mathbb{T}^d)} = \lim_{\varepsilon\to0}\left\|\iota_\varepsilon m^\varepsilon_{a_\varepsilon} - \mu_0\right\|_{\mathrm{KR}(\mathbb{T}^d)} = 0\,.$$

Similarly, we can find another sequence $(b_\varepsilon)_\varepsilon \subseteq (1/2,1)$ such that

$$\lim_{\varepsilon\to0} b_\varepsilon = 1 \quad \text{and} \quad \lim_{\varepsilon\to0}\left\|\iota_\varepsilon m^\varepsilon_{b_\varepsilon} - \mu_1\right\|_{\mathrm{KR}(\mathbb{T}^d)} = 0\,.$$

We claim the sought recovery sequences is provided by $m_0^\varepsilon := m^\varepsilon_{a_\varepsilon}$ and $m_1^\varepsilon := m^\varepsilon_{b_\varepsilon}$. In order to show this, let us define $\hat{J}^\varepsilon : \mathcal{E}_\varepsilon \to \mathbb{R}$ via the formula[1] (recall the assumption (4.3.8))

$$\frac{\tau_\varepsilon^z \hat{J}^\varepsilon}{\varepsilon^{d-1}} := \bar{J}\,, \qquad z \in \mathbb{Z}_\varepsilon^d\,,$$

so that $\hat{J}^\varepsilon$ is divergence-free. Now define

$$\widetilde{m}_t^\varepsilon := \begin{cases} m^\varepsilon_{a_\varepsilon} & \text{if } t \in [0,a_\varepsilon) \\ m_t^\varepsilon & \text{if } t \in [a_\varepsilon, b_\varepsilon] \\ m^\varepsilon_{b_\varepsilon} & \text{if } t \in (b_\varepsilon, 1] \end{cases} \quad \text{and} \quad \widetilde{J}_t^\varepsilon := \begin{cases} \hat{J}^\varepsilon & \text{if } t \in [0,a_\varepsilon) \\ J_t^\varepsilon & \text{if } t \in [a_\varepsilon, b_\varepsilon] \\ \hat{J}^\varepsilon & \text{if } t \in (b_\varepsilon, 1] \end{cases}.$$

It is readily verified that $(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon)$ solves the continuity equation for every ε. Therefore

$$\mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \mathcal{A}_\varepsilon(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon) = \int_0^1 \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z \widetilde{m}_t^\varepsilon}{\varepsilon^d}, \frac{\tau_\varepsilon^z \widetilde{J}_t^\varepsilon}{\varepsilon^{d-1}}\right) \mathrm{d}t \tag{4.3.11}$$

$$= \int_0^{a_\varepsilon} \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m^\varepsilon_{a_\varepsilon}}{\varepsilon^d}, \bar{J}\right) \mathrm{d}t + \int_{a_\varepsilon}^{b_\varepsilon} \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m_t^\varepsilon}{\varepsilon^d}, \frac{\tau_\varepsilon^z J_t^\varepsilon}{\varepsilon^{d-1}}\right) \mathrm{d}t \tag{4.3.12}$$

$$+ \int_{b_\varepsilon}^1 \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m^\varepsilon_{b_\varepsilon}}{\varepsilon^d}, \bar{J}\right) \mathrm{d}t \tag{4.3.13}$$

$$=: I_1 + I_2 + I_3\,. \tag{4.3.14}$$

[1] The definition is well-posed because εR_0 is assumed to be smaller than $1/2$.

The first and last integral can be estimated using the assumption (4.3.8). Indeed,

$$
\begin{aligned}
I_1 + I_3 &\le C \sum_{z \in \mathbb{Z}_\varepsilon^d} \left((a_\varepsilon + 1 - b_\varepsilon)\varepsilon^d + \sum_{\substack{x \in \mathcal{X} \\ |x_z|_\infty \le R}} \left(a_\varepsilon (\tau_\varepsilon^z m_{a_\varepsilon}^\varepsilon)(x) + (1 - b_\varepsilon)(\tau_\varepsilon^z m_{b_\varepsilon}^\varepsilon)(x) \right) \right) \\
&\le C \left((a_\varepsilon + 1 - b_\varepsilon) + (2R+1)^d \sum_{x \in \mathcal{X}_\varepsilon} \left(a_\varepsilon m_{a_\varepsilon}^\varepsilon(x) + (1-b_\varepsilon) m_{b_\varepsilon}^\varepsilon(x) \right) \right) \\
&= C \left((a_\varepsilon + 1 - b_\varepsilon) + (2R+1)^d \left(a_\varepsilon \iota_\varepsilon m_{a_\varepsilon}^\varepsilon(\mathbb{T}^d) + (1-b_\varepsilon)\iota_\varepsilon m_{b_\varepsilon}^\varepsilon(\mathbb{T}^d) \right) \right) ,
\end{aligned}
$$

and in the limit we find

$$
\limsup_{\varepsilon \to 0} I_1 + I_3 \le C \left(0 + (2R+1)^d (0 \cdot \mu_0(\mathbb{T}^d) + 0 \cdot \mu_1(\mathbb{T}^d)) \right) = 0 . \tag{4.3.15}
$$

As for the second integral, thanks to Assumption 4.2.8(c) we have that

$$
\begin{aligned}
I_2 - \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) &= - \int_{(0,a_\varepsilon)\cup(b_\varepsilon,1)} \sum_{z \in \mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z m_t^\varepsilon}{\varepsilon^d}, \frac{\tau_\varepsilon^z J_t^\varepsilon}{\varepsilon^{d-1}} \right) \, \mathrm{d}t & (4.3.16) \\
&\le C' \left((a_\varepsilon + 1 - b_\varepsilon) + (2R_{\max}+1)^d \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(((0,a_\varepsilon) \cup (b_\varepsilon,1)) \times \mathbb{T}^d \right) \right) . & (4.3.17)
\end{aligned}
$$

Since $(\iota_\varepsilon \boldsymbol{m}^\varepsilon)_\varepsilon$ converges weakly, for every $a, b \in (0,1)$, we have that

$$
\begin{aligned}
\limsup_{\varepsilon \to 0} \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(((0,a_\varepsilon) \cup (b_\varepsilon,1)) \times \mathbb{T}^d \right) &\le \limsup_{\varepsilon \to 0} \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(\Big((0,a] \cup [b,1) \Big) \times \mathbb{T}^d \right) \\
&\le \boldsymbol{\mu} \left(\Big((0,a] \cup [b,1) \Big) \times \mathbb{T}^d \right) .
\end{aligned}
$$

Using the fact that the previous estimate holds for every $a, b \in (0,1)$, we obtain that

$$
\limsup_{\varepsilon \to 0} \iota_\varepsilon \boldsymbol{m}^\varepsilon \left(((0,a_\varepsilon) \cup (b_\varepsilon,1)) \times \mathbb{T}^d \right) = 0 .
$$

This, together with the estimate obtained in (4.3.16), gives us the inequality

$$
\limsup_{\varepsilon \to 0} I_2 \le \limsup_{\varepsilon \to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) . \tag{4.3.18}
$$

In conclusion, from (4.3.11), (4.3.15), and (4.3.18) we find

$$
\limsup_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \le \limsup_{\varepsilon \to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \le \mathbb{A}(\boldsymbol{\mu}) \le \mathbb{MA}(\mu_0, \mu_1) + 1/k ,
$$

which is sought upper bound. □

4.3.2 Proof of the liminf inequality

In this section, we provide the proof of the liminf inequality in Theorem 4.3.9. Let m_0^ε, m_1^ε be sequences of measures weakly converging to μ_0, μ_1, respectively. We want to show that

$$
\liminf_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \ge \mathbb{MA}_{\mathrm{hom}}(\mu_0, \mu_1) . \tag{4.3.19}
$$

Without loss of generality, we will assume that the limit inferior in the latter is a true finite limit, and that $m_0^\varepsilon(\mathcal{X}_\varepsilon) = m_1^\varepsilon(\mathcal{X}_\varepsilon)$ for every $\varepsilon > 0$.

We split the proof into two parts: first for F with linear growth and then for F of flow-based type, respectively Assumption 4.3.1 and Assumption 4.3.2.

Case 1: F with linear growth

Assume that F satisfies Assumption 4.3.1. Recall that, as a consequence of Lemma 4.3.5, F is Lipschitz continuous as well, in the sense of (4.3.1).

Proof of the liminf inequality (linear growth). With a very similar argument as the one provided by Remark 4.2.7 in the continuous setting, we can with no loss of generality assume that F is nonnegative. Let $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$ be approximate optimal solutions associated to $\mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon)$, i.e. such that

$$\lim_{\varepsilon \to 0} \Big(\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) - \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \Big) = 0\,. \tag{4.3.20}$$

As usual, we write $\mathrm{d}\boldsymbol{m}^\varepsilon(t,x) = m_t^\varepsilon(\mathrm{d}x)\,\mathrm{d}t$ for some measurable curve $t \mapsto m_t^\varepsilon \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ of constant, finite mass. By compactness (Remark 4.3.11), we know that up to a further non-relabeled subsequence, $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu}$ weakly in $\mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$ with $\boldsymbol{\mu} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$. Due to the lack of continuity of the trace operators in BV, a priori we cannot conclude that $\boldsymbol{\mu}_{t=0} = \mu_0$ and $\boldsymbol{\mu}_{t=1} = \mu_1$. In other words, there might be a "jump" in the limit as $\varepsilon \to 0$ at the boundary of $(0,1)$. In order to take care of this problem, we rescale our measures $\boldsymbol{m}^\varepsilon$ in time, so as to be able to "see" the jump in the interior of $(0,1)$.

To this purpose, for $\delta \in (0,1/2)$, we define $\mathcal{I}_\delta := (\delta, 1-\delta)$ and $\boldsymbol{m}^{\varepsilon,\delta} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ as

$$m_t^{\varepsilon,\delta} := \begin{cases} m_0^\varepsilon & \text{if } t \in (0,\delta] \\ m^\varepsilon_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ m_1^\varepsilon & \text{if } t \in [1-\delta, 1) \end{cases}, \quad \mathrm{d}\boldsymbol{m}^{\varepsilon,\delta}(t,x) := m_t^{\varepsilon,\delta}(\mathrm{d}x)\,\mathrm{d}t\,. \tag{4.3.21}$$

By construction, the convergence of the boundary data, and the fact that, by assumption, $\boldsymbol{m}^\varepsilon \to \boldsymbol{\mu}$ weakly, it is straightforward to see that $\boldsymbol{m}^{\varepsilon,\delta} \to \hat{\boldsymbol{\mu}}^\delta$ weakly, where

$$\hat{\mu}_t^\delta := \begin{cases} \mu_0 & \text{if } t \in (0,\delta] \\ \mu_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ \mu_1 & \text{if } t \in [1-\delta, 1) \end{cases}, \quad \mathrm{d}\hat{\boldsymbol{\mu}}^\delta(t,x) := \hat{\mu}_t^\delta(\mathrm{d}x)\,\mathrm{d}t\,. \tag{4.3.22}$$

Note that the rescaled curve $t \mapsto \hat{\mu}_t^\delta$ might have discontinuities at $t = \delta$ and $t = 1-\delta$, which correspond to the possible jumps in the limit as $\varepsilon \to 0$ for $\boldsymbol{m}^\varepsilon$ at $\{0,1\}$. Nevertheless, $\hat{\boldsymbol{\mu}}^\delta$ is a competitor for $\mathbb{MA}(\mu_0,\mu_1)$, which, by the Γ-convergence of $\mathcal{A}_\varepsilon$ towards $\mathbb{A}_{\mathrm{hom}}$ (Remark 4.3.10), ensures that

$$\liminf_{\varepsilon \to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}) \geq \mathbb{A}_{\mathrm{hom}}(\hat{\boldsymbol{\mu}}^\delta) \geq \mathbb{MA}_{\mathrm{hom}}(\mu_0,\mu_1)\,. \tag{4.3.23}$$

We are left with estimating from above the left-hand side of the latter displayed equation. To do so, we seek a suitable curve of discrete vector fields $\boldsymbol{J}^{\varepsilon,\delta}$ with $(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) \in \mathcal{CE}_\varepsilon$ and having an action $\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta})$ comparable with $\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon)$ for small $\delta > 0$. We set

$$J_t^{\varepsilon,\delta} := \begin{cases} 0 & \text{if } t \in (0,\delta] \\ \dfrac{1}{1-2\delta} J^\varepsilon_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ 0 & \text{if } t \in [1-\delta, 1) \end{cases}, \quad \mathrm{d}\boldsymbol{J}^{\varepsilon,\delta}(t,x) := J_t^{\varepsilon,\delta}(\mathrm{d}x)\,\mathrm{d}t\,.$$

We claim that $(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) \in \mathcal{CE}_\varepsilon$ and

$$\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) \leq \Big(1 + C(F)\delta\Big)\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) + C(F)\delta\Big(1 + \iota_\varepsilon \boldsymbol{m}^\varepsilon((0,1)\times\mathbb{T}^d)\Big), \tag{4.3.24}$$

where $C(F) \in \mathbb{R}_+$ only depends on F (specifically on the constants in Assumption 4.2.8 and Assumption 4.3.1). This would suffice to conclude the proof of the sought liminf inequality. Indeed, from (4.3.20) and (4.3.24) we infer

$$\begin{aligned}\liminf_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) &= \liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \\ &\geq \frac{1}{1+C(F)\delta} \liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) - \frac{C(F)\delta}{1+C(F)\delta}\Big(1 + \iota_\varepsilon \boldsymbol{m}^\varepsilon((0,1)\times\mathbb{T}^d)\Big)\end{aligned}$$

which, combined with (4.3.23), yields

$$\liminf_{\varepsilon\to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \geq \frac{\mathbb{MA}_{\mathrm{hom}}(\mu_0,\mu_1)}{1+C(F)\delta} - \frac{C(F)\delta}{1+C(F)\delta}\Big(1 + \mu_0(\mathbb{T}^d)\Big)$$

for any $\delta \in (0, 1/2)$. We conclude by letting $\delta \to 0$.

We are left with the proof of $(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) \in \mathcal{CE}_\varepsilon$ and of the claim (4.3.24).

Proof of $(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) \in \mathcal{CE}_\varepsilon$. Let us fix $x \in \mathcal{X}_\varepsilon$ and $\varphi \in C_c^1\big((0,1)\big)$. Set $\widetilde{\varphi} := \varphi \circ r_\delta$, with $r_\delta(s) := (1-2\delta)s + \delta$. We have

$$\begin{aligned}\int_0^1 \partial_t \varphi m_t^{\varepsilon,\delta}(x)\,\mathrm{d}t &= \int_0^\delta \partial_t\varphi\, m_0^\varepsilon(x)\,\mathrm{d}t + \int_{1-\delta}^1 \partial_t\varphi\, m_1^\varepsilon(x)\,\mathrm{d}t + \int_{\mathcal{I}_\delta} \partial_t\varphi\, m^\varepsilon_{r_\delta^{-1}(t)}(x)\,\mathrm{d}t \\ &= \varphi(\delta)\, m_0^\varepsilon(x) - \varphi(1-\delta)\, m_1^\varepsilon(x) + (1-2\delta)\int_0^1 (\partial_t\varphi)\circ r_\delta\, m_s^\varepsilon(x)\,\mathrm{d}s \\ &= \widetilde{\varphi}(0)\, m_0^\varepsilon(x) - \widetilde{\varphi}(1)\, m_1^\varepsilon(x) + \int_0^1 \partial_s\widetilde{\varphi}\, m_s^\varepsilon(x)\,\mathrm{d}s \\ &= \int_0^1 \widetilde{\varphi} \sum_{y\sim x} J_s^\varepsilon(x,y)\,\mathrm{d}s = \frac{1}{1-2\delta}\int_{\mathcal{I}_\delta} \varphi \sum_{y\sim x} J^\varepsilon_{r_\delta^{-1}(t)}(x,y)\,\mathrm{d}t \\ &= \int_0^1 \varphi \sum_{y\sim x} J_t^{\varepsilon,\delta}(x,y)\,\mathrm{d}t\,,\end{aligned} \tag{4.3.25}$$

where, in the fourth equality, we used that $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$.

Proof of the action estimate. Define $r_\delta(s) := (1-2\delta)s + \delta$. Note that, by construction, for $(t,(x,y)) \in \mathcal{I}_\delta \times \mathcal{E}_\varepsilon$,

$$m_t^{\varepsilon,\delta}(x) = m^\varepsilon_{r_\delta^{-1}(t)}(x)\,, \qquad J_t^{\varepsilon,\delta}(x,y) = \frac{1}{1-2\delta} J^\varepsilon_{r_\delta^{-1}(t)}(x,y)\,. \tag{4.3.26}$$

On the other hand, for $(t,(x,y)) \in \Big((0,\delta] \cup [1-\delta, 1)\Big) \times \mathcal{E}_\varepsilon$, we have that

$$m_t^{\varepsilon,\delta}(x) = \begin{cases} m_0^\varepsilon(x) & \text{if } t \in (0,\delta] \\ m_1^\varepsilon(x) & \text{if } t \in [1-\delta, 1) \end{cases} \qquad \text{and} \qquad J_t^{\varepsilon,\delta}(x,y) = 0\,.$$

It follows that the action of $(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta})$ is given by

$$\mathcal{A}_\varepsilon(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) = \int_0^1 \mathcal{F}_\varepsilon(m_t^{\varepsilon,\delta}, J_t^{\varepsilon,\delta}) \, \mathrm{d}t = \mathcal{A}_\varepsilon^{\mathcal{I}_\delta}(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) + \delta \sum_{i=0,1} \mathcal{F}_\varepsilon(m_i^\varepsilon, 0) \,, \tag{4.3.27}$$

where we used the notation

$$\mathcal{A}_\varepsilon^{\mathcal{I}_\delta}(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) := \int_{\mathcal{I}_\delta} \mathcal{F}_\varepsilon(m_t^{\varepsilon,\delta}, J_t^{\varepsilon,\delta}) \, \mathrm{d}t = (1-2\delta) \int_0^1 \mathcal{F}_\varepsilon\Big(m_t^\varepsilon, \frac{1}{1-2\delta} J_t^\varepsilon\Big) \, \mathrm{d}t \,. \tag{4.3.28}$$

Using Assumption 4.3.1, we see that, for $i = 0, 1$,

$$\mathcal{F}_\varepsilon(m_i^\varepsilon, 0) \le C(m_i^\varepsilon(\mathcal{X}_\varepsilon) + 1) = C\Big(\iota_\varepsilon \boldsymbol{m}^\varepsilon\Big((0,1) \times \mathbb{T}^d\Big) + 1\Big)$$

and, by the Lipschitz continuity exhibited in Lemma 4.3.5, we also infer that

$$\begin{aligned} \mathcal{A}_\varepsilon^{\mathcal{I}_\delta}(\boldsymbol{m}^{\varepsilon,\delta}, \boldsymbol{J}^{\varepsilon,\delta}) &\le (1-2\delta) \left(\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) + C\Big(\frac{1}{1-2\delta} - 1\Big) \sum_{z \in \mathbb{Z}_\varepsilon^d} \varepsilon^d \sum_{\substack{(x,y) \in \mathcal{E} \\ |x_z|_\infty \le R}} \int_0^1 \frac{|\tau_\varepsilon^z J_t^\varepsilon(x,y)|}{\varepsilon^{d-1}} \, \mathrm{d}t \right) \\ &\le (1-2\delta) \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) + 2\delta C (2R+1)^d \sum_{z \in \mathbb{Z}_\varepsilon^d} \varepsilon^d \sum_{(x,y) \in \mathcal{E}^Q} \int_0^1 \frac{|\tau_\varepsilon^z J_t^\varepsilon(x,y)|}{\varepsilon^{d-1}} \, \mathrm{d}t \,. \end{aligned}$$

Since we assumed F to be nonnegative, we can estimate

$$(1-2\delta)\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \le \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon)$$

and, using (4.2.16),

$$\sum_{z \in \mathbb{Z}_\varepsilon^d} \varepsilon^d \sum_{(x,y) \in \mathcal{E}^Q} \int_0^1 \frac{|\tau_\varepsilon^z J_t^\varepsilon(x,y)|}{\varepsilon^{d-1}} \, \mathrm{d}t \le \frac{1}{c} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) + \frac{C}{c} \Big(1 + (1 + 2R_{\max})^d \|\iota_\varepsilon \boldsymbol{m}^\varepsilon\|_{\mathrm{TV}}\Big) \,.$$

Combining these estimates with (4.3.27), we find (4.3.24). □

Case 2: F is flow-based

In this section we show (4.3.19) in the case F (and hence $f_{\hom}$) is of flow-based type, i.e. it satisfies Assumption 4.3.2. We start by observing that, in this particular setting, both the discrete and the continuous formulations of the boundary-value problems admit an equivalent, static formulation.

Let $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathsf{CE}$, and consider the Lebesgue decomposition

$$\boldsymbol{\mu} = \rho \mathscr{L}^{d+1} + \rho^\perp \boldsymbol{\sigma} \,, \qquad \boldsymbol{\nu} = j \mathscr{L}^{d+1} + j^\perp \boldsymbol{\sigma} \,.$$

We know that every solution to the continuity equation can be disintegrated in the form $\boldsymbol{\mu}(\mathrm{d}t, \mathrm{d}x) = \mu_t(\mathrm{d}x)\,\mathrm{d}t$ for some measurable curve $t \mapsto \mu_t \in \mathcal{M}_+(\mathbb{T}^d)$ of constant, finite mass. If f is a function as in Assumption 4.2.2 that further does *not* depend on ρ, then Jensen's inequality yields

$$\int_0^1 \int_{\mathbb{T}^d} f(j_t) \, \mathrm{d}x \, \mathrm{d}t \ge \int_{\mathbb{T}^d} f\left(\int_0^1 j_t \, \mathrm{d}t \right) \mathrm{d}x \,. \tag{4.3.29}$$

In order to take care of the singular part, consider the disintegration of $\boldsymbol{\sigma}$ with respect to the projection map $\pi : (t,x) \mapsto x$, in the form

$$\boldsymbol{\sigma}(\mathrm{d}t, \mathrm{d}x) = \sigma^x(\mathrm{d}t)(\pi_\# \boldsymbol{\sigma})(\mathrm{d}x)\,, \tag{4.3.30}$$

for some measurable $x \mapsto \sigma^x \in \mathscr{P}\big((0,1)\big)$. Due to the convexity of f^∞, by Jensen's inequality we also obtain

$$\int_{(0,1)\times\mathbb{T}^d} f^\infty(j^\perp)\,\mathrm{d}\boldsymbol{\sigma} \geq \int_{\mathbb{T}^d} f^\infty\left(\int j^\perp\,\mathrm{d}\sigma^x\right)\mathrm{d}\pi_\#\boldsymbol{\sigma}(x)\,. \tag{4.3.31}$$

Now, we define the new space-time measures

$$\begin{gathered}\widetilde{\boldsymbol{\mu}} := \widetilde{\mu}_t(\mathrm{d}x)\,\mathrm{d}t \quad \text{and} \quad \widetilde{\boldsymbol{\nu}} := \hat{j}\mathscr{L}^{d+1} + \hat{j}^\perp\,\mathrm{d}t \otimes \pi_\#\boldsymbol{\sigma}\,, \quad \text{where} \\ \widetilde{\mu}_t := \mu_0 + t(\mu_1 - \mu_0)\,, \quad \hat{j}(x) := \int_0^1 j_t(x)\,\mathrm{d}t\,, \quad \text{and} \quad \hat{j}^\perp(x) := \int j^\perp\,\mathrm{d}\sigma^x\,,\end{gathered} \tag{4.3.32}$$

and note that $(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\nu}}) \in \mathsf{CE}$. By (4.3.29) and (4.3.31), we therefore have

$$\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) \geq \int_{\mathbb{T}^d} f(\hat{j})\,\mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\hat{j}^\perp)\,\mathrm{d}\pi_\#\boldsymbol{\sigma}(x)\,. \tag{4.3.33}$$

We need to be careful here: the decomposition of $\widetilde{\boldsymbol{\nu}}$ in (4.3.32) may not be a Lebesgue decomposition, in the sense that $\mathrm{d}t \otimes \pi_\#\boldsymbol{\sigma}$ can have a nonzero absolutely continuous part. Let $\widetilde{\sigma} \in \mathcal{M}_+(\mathbb{T}^d)$ be singular w.r.t. $\mathscr{L}^d$ and such that $\mu_0, \mu_1, \pi_\#\boldsymbol{\sigma} \ll \mathscr{L}^d + \widetilde{\sigma}$. We can write the Lebesgue decompositions

$$\widetilde{\boldsymbol{\mu}} = \widetilde{\rho}\mathscr{L}^{d+1} + \widetilde{\rho}^\perp\,\mathrm{d}t \otimes \widetilde{\sigma}\,, \qquad \widetilde{\boldsymbol{\nu}} = \widetilde{j}\mathscr{L}^{d+1} + \widetilde{j}^\perp\,\mathrm{d}t \otimes \widetilde{\sigma}\,.$$

If we write

$$\pi_\#\boldsymbol{\sigma} = \alpha\mathscr{L}^d + \beta\widetilde{\sigma}$$

for some functions $\alpha, \beta : \mathbb{T}^d \to \mathbb{R}_+$, then

$$\widetilde{j} = \hat{j} + \alpha\hat{j}^\perp \quad \text{and} \quad \widetilde{j}^\perp = \beta\hat{j}^\perp\,.$$

The inequality (4.3.33) becomes, recalling that f^∞ is 1-homogeneous,

$$\mathbb{A}(\boldsymbol{\mu},\boldsymbol{\nu}) \geq \int_{\mathbb{T}^d} \left(f(\hat{j}) + f^\infty(\alpha\hat{j}^\perp)\right)\mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\beta\hat{j}^\perp)\,\mathrm{d}\widetilde{\sigma}\,. \tag{4.3.34}$$

At this point, we need a lemma.

Lemma 4.3.13. *For every $j_1, j_2 \in \mathbb{R}^d$, we have that $f(j_1 + j_2) \leq f(j_1) + f^\infty(j_2)$.*

Proof. Let $g \leq f$ be a convex and Lipschitz continuous function. By convexity, for every $\epsilon \in (0,1)$, we have

$$g(j_1 + j_2) = g\left((1-\epsilon)\frac{j_1}{1-\epsilon} + \epsilon\frac{j_2}{\epsilon}\right) \leq (1-\epsilon)g\left(\frac{j_1}{1-\epsilon}\right) + \epsilon g\left(\frac{j_2}{\epsilon}\right)\,.$$

Let $j_0 \in \mathsf{D}(f)$. By the Lipschitz continuity of g,

$$g(j_1 + j_2) \leq (1-\epsilon)\left(g(j_1) + (\mathrm{Lip} g)\left(\frac{1}{1-\epsilon} - 1\right)|j_1|\right) + \epsilon g\left(\frac{j_2}{\epsilon} + j_0\right) + \epsilon(\mathrm{Lip} g)|j_0|$$

and, since $g \leq f$,

$$g(j_1 + j_2) \leq (1-\epsilon) f(j_1) + \epsilon f\left(\frac{j_2}{\epsilon} + j_0\right) + \epsilon(\mathrm{Lip} g)\big(|j_0| + |j_1|\big) .$$

As we let $\epsilon \to 0$, we find

$$g(j_1 + j_2) \leq f(j_1) + f^\infty(j_2) .$$

Since f is convex and lower semicontinuous, we conclude by an approximation argument. □

Applying this lemma with $j_1 = \hat{j}(x)$ and $j_2 = \alpha \hat{j}^\perp(x)$ for every $x \in \mathbb{T}^d$, (4.3.34) finally becomes

$$\mathbb{A}(\boldsymbol{\mu}, \boldsymbol{\nu}) \geq \int_{\mathbb{T}^d} f(\widetilde{j}) \, \mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\widetilde{j}^\perp) \, \mathrm{d}\widetilde{\sigma} = \mathbb{A}(\widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\nu}}) .$$

In other words, we have shown that an optimal curve $\boldsymbol{\mu}$ between two given boundary data is always given by the affine interpolation (and a constant-in-time flux). We conclude that

$$\mathbb{MA}(\mu_0, \mu_1) = \mathbb{A}(\widetilde{\boldsymbol{\mu}}) \tag{4.3.35}$$

$$= \inf_\nu \left\{ \int_{\mathbb{T}^d} f(\widetilde{j}) \, \mathrm{d}x + \int_{\mathbb{T}^d} f^\infty(\widetilde{j}^\perp) \, \mathrm{d}\widetilde{\sigma} \ : \ \nu = \widetilde{j} \mathscr{L}^d + \widetilde{j}^\perp \widetilde{\sigma} \,, \ \mathscr{L}^d \perp \widetilde{\sigma} \ \text{ and } \ \nabla \cdot \nu = \mu_0 - \mu_1 \right\} .$$

We refer to the latter expression as the *static formulation* of the boundary value problem described by $\mathbb{MA}(\mu_0, \mu_1)$ (in the case when f is of flow-based type).

Remark 4.3.14. Using this equivalence, the lower semicontinuity of $\mathbb{MA}$ directly follows from the semicontinuity of $\mathbb{A}$ given by [GKMP23, Lemma 3.14].

Arguing in a similar way (in fact, via an even simpler argument, due to the lack of singularities), we obtain a static formulation of the discrete transport problem in terms of a discrete divergence equation, when $F(m, J) = F(J)$. Precisely, in this case we obtain

$$\mathcal{MA}_\varepsilon(m_0, m_1) = \inf \left\{ \mathcal{F}_\varepsilon(J) \ : \ J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon} \,, \quad \operatorname{div} J = m_0 - m_1 \right\} . \tag{4.3.36}$$

The sought Γ-liminf inequality easily follows from such static formulations.

Proof of the liminf inequality (flow-based type). Let m_0^ε, $m_1^\varepsilon \in \mathbb{R}_+^{\mathcal{X}_\varepsilon}$ be a sequence of discrete nonnegative measures which converge weakly (via ι_ε in the usual sense) to μ_0, μ_1, and such that $m_0^\varepsilon(\mathcal{X}_\varepsilon) = m_1^\varepsilon(\mathcal{X}_\varepsilon)$ for every $\varepsilon > 0$. Let $(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \in \mathcal{CE}_\varepsilon$ be (almost-)optimal solutions associated with $\mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon)$, namely

$$\liminf_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) = \liminf_{\varepsilon \to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon) = \liminf_{\varepsilon \to 0} \mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) . \tag{4.3.37}$$

Consider the discrete equivalent of the measure constructed in (4.3.32), namely

$$\widetilde{m}_t^\varepsilon := m_0^\varepsilon + t(m_1^\varepsilon - m_0^\varepsilon) \quad \text{ and } \quad \widetilde{J}_t^\varepsilon \equiv \widetilde{J}^\varepsilon := \int_0^1 J_s^\varepsilon \, \mathrm{d}s \,,$$

which still solves the continuity equation. By applying Jensen's inequality, the convexity of F ensures that

$$\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \geq \mathcal{A}_\varepsilon(\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon) = \mathcal{F}_\varepsilon(\widetilde{J}^\varepsilon) \quad \text{ and } \quad (\widetilde{\boldsymbol{m}}^\varepsilon, \widetilde{\boldsymbol{J}}^\varepsilon) \in \mathcal{CE}_\varepsilon . \tag{4.3.38}$$

Thus $\mathcal{A}_\varepsilon(\boldsymbol{m}^\varepsilon, \boldsymbol{J}^\varepsilon) \geq \mathcal{A}_\varepsilon(\widetilde{\boldsymbol{m}}^\varepsilon)$. Note that, by construction, $\widetilde{\boldsymbol{m}}^\varepsilon \to \widetilde{\boldsymbol{\mu}}$ weakly, where

$$\widetilde{\boldsymbol{\mu}} := \widetilde{\mu}_t(\,\mathrm{d}x)\,\mathrm{d}t \quad \text{with} \quad \widetilde{\mu}_t := \mu_0 + t(\mu_1 - \mu_0)\,.$$

Hence, from (4.3.37), (4.3.38), and the Γ-convergence of $\mathcal{A}_\varepsilon$ to $\mathbb{A}_{\text{hom}}$ (cfr. [GKMP23, Theorem 5.1]), we infer that

$$\liminf_{\varepsilon \to 0} \mathcal{MA}_\varepsilon(m_0^\varepsilon, m_1^\varepsilon) \geq \mathbb{A}_{\text{hom}}(\widetilde{\boldsymbol{\mu}}) \geq \mathbb{MA}_{\text{hom}}(\mu_0, \mu_1)\,,$$

which concludes the proof of the liminf inequality. □

4.3.3 About the lower semicontinuity of $\mathbb{MA}$

In view of our main result, whenever F satisfies either Assumption 4.3.1 or Assumption 4.3.2, the limit boundary-value problem $\mathbb{MA}_{\text{hom}}(\cdot,\cdot)$ is necessarily jointly lower semicontinuous with respect to the weak topology on $\mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d)$. This indeed follows from the general fact that any Γ-limit with respect to a given topology is always lower semicontinuous with respect to that same topology. Using a very similar proof to that of the Γ-liminf inequality, we can actually show that, if f is with linear growth or it is of flow-based type, then the associated $\mathbb{MA}$ is always lower semicontinuous (even if, a priori, f is not of the form $f = f_{\text{hom}}$), thus providing a positive answer in this framework to the validity of (4.2.8). In the flow-based setting, this fact has been observed in Remark 4.3.14.

Proposition 4.3.15. *Assume that f is with linear growth, namely it satisfies one of the two equivalent conditions appearing in Lemma 4.3.4, and assume that $(\mu_0^n,\ \mu_1^n) \to (\mu_0,\ \mu_1) \in \mathcal{M}_+(\mathbb{T}^d) \times \mathcal{M}_+(\mathbb{T}^d)$ weakly. Then:*

$$\liminf_{n \to \infty} \mathbb{MA}(\mu_0^n, \mu_1^n) \geq \mathbb{MA}(\mu_0, \mu_1)\,.$$

The proof goes along the same line of the proof of the Γ-liminf inequality for discrete energies F with linear growth. We sketch it here and add details whenever we encounter nontrivial differences between the two proofs.

Proof. Let $(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) \in \mathsf{CE}$ be (almost-)optimal solutions associated to $\mathbb{MA}(\mu_0^n, \mu_1^n)$, i.e.,

$$\liminf_{n \to \infty} \mathbb{MA}(\mu_0^n, \mu_1^n) = \liminf_{n \to \infty} \mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n)\,. \tag{4.3.39}$$

With no loss of generality we can assume $\sup_n \mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) < \infty$ and that the limits inferior are true limits. By the compactness of Remark 4.2.5, we know that, up to a non-relabeled subsequence, $\boldsymbol{\mu}^n \to \boldsymbol{\mu}$ weakly in $\mathcal{M}_+\big((0,1) \times \mathbb{T}^d\big)$. Moreover, we also have $\mathrm{d}\boldsymbol{\mu}(t,x) = \mu_t(\,\mathrm{d}x)\,\mathrm{d}t \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ for some measurable curve $t \mapsto \mu_t \in \mathcal{M}_+(\mathbb{T}^d)$ of constant, finite mass. Once again, due to the lack of continuity of the trace operators in BV, we cannot ensure that $\boldsymbol{\mu}_{t=0} = \mu_0$ and $\boldsymbol{\mu}_{t=1} = \mu_1$. To solve this issue, we rescale our measures $\boldsymbol{\mu}^n$ in time in the same spirit as in (4.3.21). For a given $\delta > 0$, we define $\mathcal{I}_\delta := (\delta, 1-\delta)$ and $\boldsymbol{\mu}^{n,\delta} \in \mathrm{BV}_{\mathrm{KR}}\big((0,1);\mathcal{M}_+(\mathbb{T}^d)\big)$ as

$$\mu_t^{n,\delta} := \begin{cases} \mu_0^n & \text{if } t \in (0,\delta] \\ \mu^n_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ \mu_1^n & \text{if } t \in [1-\delta, 1) \end{cases}, \qquad \mathrm{d}\boldsymbol{\mu}^{n,\delta}(t,x) := \mu_t^{n,\delta}(\,\mathrm{d}x)\,\mathrm{d}t\,. \tag{4.3.40}$$

By construction, it is not hard to see that $\boldsymbol{\mu}^{n,\delta} \to \hat{\boldsymbol{\mu}}^\delta$ weakly, where

$$\hat{\mu}_t^\delta := \begin{cases} \mu_0 & \text{if } t \in (0,\delta] \\ \mu_{\frac{t-\delta}{1-2\delta}} & \text{if } t \in \mathcal{I}_\delta \\ \mu_1 & \text{if } t \in [1-\delta, 1) \end{cases}, \qquad \mathrm{d}\hat{\boldsymbol{\mu}}^\delta(t,x) := \hat{\mu}_t^\delta(\,\mathrm{d}x)\,\mathrm{d}t\,. \tag{4.3.41}$$

We stress that, as in (4.3.22), the rescaled curve $t \mapsto \hat{\mu}_t^\delta$ could have discontinuities at $t=\delta$ and $t = 1-\delta$, corresponding to the possible jumps in the limit as $n \to \infty$ for $\boldsymbol{\mu}^n$ at $\{0,1\}$. Nevertheless, $\hat{\boldsymbol{\mu}}^\delta$ is a competitor for $\mathbb{MA}(\mu_0,\mu_1)$, which, by lower semicontinuity of $\mathbb{A}$, ensures that

$$\liminf_{n\to\infty} \mathbb{A}(\boldsymbol{\mu}^{n,\delta}) \geq \mathbb{A}(\hat{\boldsymbol{\mu}}^\delta) \geq \mathbb{MA}(\mu_0,\mu_1)\,. \tag{4.3.42}$$

In order to estimate the left-hand side of the latter displayed equation, we seek a suitable vector meausure $\boldsymbol{\nu}^{n,\delta}$ so that $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \in \mathsf{CE}$ and whose action $\mathbb{A}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta})$ is comparable with $\mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n)$ for small $\delta > 0$.

It is useful to introduce the following notation: for $\delta \in (0,1/2)$,

$$r_\delta : (0,1) \to \mathcal{I}_\delta\,, \quad r_\delta(s) := (1-2\delta)s + \delta\,, \tag{4.3.43}$$
$$R_\delta : (0,1)\times\mathbb{T}^d \to \mathcal{I}_\delta \times \mathbb{T}^d\,, \quad R_\delta(s,x) := (r_\delta(s), x)\,. \tag{4.3.44}$$

Define $\hat{\iota}_\delta : \mathcal{M}^d(\mathcal{I}_\delta \times \mathbb{T}^d) \to \mathcal{M}^d\big((0,1)\times\mathbb{T}^d\big)$ as the natural embedding obtained by extending to 0 any measure outside $\mathcal{I}_\delta$, and set

$$\boldsymbol{\nu}^{n,\delta} := \hat{\iota}_\delta\Big[(R_\delta)_\# \boldsymbol{\nu}^n\Big] \in \mathcal{M}^d((0,1)\times\mathbb{T}^d)\,. \tag{4.3.45}$$

The proof that $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \in \mathsf{CE}$ works as in (4.3.25). In the same spirit as in (4.3.24), we claim that

$$\mathbb{A}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) \leq \Big(1 + C(f)\delta\Big)\mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) + C(f)\delta\Big(1 + \boldsymbol{\mu}^n((0,1)\times\mathbb{T}^d)\Big)\,, \tag{4.3.46}$$

where $C(f) \in \mathbb{R}_+$ only depends on f. The combination of (4.3.39), (4.3.42) and (4.3.46), and the arbitrariness of δ would then suffice to conclude the proof.

We are left with the proof of the claim (4.3.46), which is a bit more involved, compared to that of (4.3.24), due to the presence of the singular part at the continuous level. We need the following.

Lemma 4.3.16. *Let $\boldsymbol{\sigma} \in \mathcal{M}_+\big((0,1)\times\mathbb{T}^d\big)$ be a singular measure with respect to $\mathscr{L}^{d+1}$. Then, the measure $(R_\delta)_\#\boldsymbol{\sigma} \in \mathcal{M}_+(\mathcal{I}_\delta \times \mathbb{T}^d)$ is also singular with respect to $\mathscr{L}^{d+1}$. Moreover, for every measure $\boldsymbol{\xi} = f\mathscr{L}^{d+1} + f^\perp\boldsymbol{\sigma} \in \mathcal{M}^n((0,1)\times\mathbb{T}^d)$, we have the decomposition*

$$(R_\delta)_\#\boldsymbol{\xi} = f^\delta\mathscr{L}^{d+1} + f^{\delta,\perp}(R_\delta)_\#\boldsymbol{\sigma}\,, \tag{4.3.47}$$

where the respective densities are given by the formulas

$$f^\delta(t,x) = \frac{1}{1-2\delta} f\Big(r_\delta^{-1}(t), x\Big) \quad \textit{and} \quad f^{\delta,\perp}(t,x) = f^\perp\Big(r_\delta^{-1}(t),x\Big)\,. \tag{4.3.48}$$

Proof. By assumption, $\boldsymbol{\sigma}$ is singular with respect to $\mathscr{L}^{d+1}$, which means there exists a set $A \subset (0,1) \times \mathbb{T}^d$ such that $\mathscr{L}^{d+1}(A) = 0 = \boldsymbol{\sigma}(A^c)$. By the very definition of push-forward and the bijectivity of R_δ, we then have that

$$(R_\delta)_\# \boldsymbol{\sigma}\Big((R_\delta(A))^c\Big) = \boldsymbol{\sigma}\Big(R_\delta^{-1}\Big(R_\delta(A^c)\Big)\Big) = \boldsymbol{\sigma}(A^c) = 0\,, \tag{4.3.49}$$

whereas, by the scaling properties of the Lebesgue measure, we have that $\mathscr{L}^{d+1}(R_\delta(A)) = (1-2\delta)\mathscr{L}^{d+1}(A) = 0$, which shows the claimed singularity. The second part of the lemma follows from the fact that $(R_\delta)_\# \mathscr{L}^{d+1} = (1-2\delta)^{-1}\mathscr{L}^{d+1}$ and the following statement: for every $\boldsymbol{\xi}' = f'\boldsymbol{\sigma}'$ with $\boldsymbol{\sigma}' \in \mathcal{M}_+((0,1)\times\mathbb{T}^d)$, we claim that

$$\frac{\mathrm{d}(R_\delta)_\#\boldsymbol{\xi}'}{\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}'}(t,x) = f'(R_\delta^{-1}(t,x))\,, \quad \forall (t,x) \in \mathcal{I}_\delta \times \mathbb{T}^d\,. \tag{4.3.50}$$

Indeed, by definition of push-forward, we have for every test function $\varphi \in C_b$

$$\int \varphi \,\mathrm{d}(R_\delta)_\#\boldsymbol{\xi}' = \int (\varphi\circ R_\delta)\,\mathrm{d}\boldsymbol{\xi}' = \int (\varphi \circ R_\delta) f' \,\mathrm{d}\boldsymbol{\sigma}' = \int \varphi\cdot(f'\circ R_\delta^{-1})\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}'\,,$$

which indeed shows (4.3.50). □

Let

$$\boldsymbol{\mu}^n = \rho^n \,\mathrm{d}\mathscr{L}^{d+1} + \rho^{n,\perp}\,\mathrm{d}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\nu}^n = j^n \,\mathrm{d}\mathscr{L}^{d+1} + j^{n,\perp}\,\mathrm{d}\boldsymbol{\sigma} \tag{4.3.51}$$

be Lebesgue decompositions. We apply Lemma 4.3.16 to both $\boldsymbol{\mu}^n$ and $\boldsymbol{\nu}^n$ and find that, on $\mathcal{I}_\delta \times \mathbb{T}^d$, we have

$$\boldsymbol{\mu}^{n,\delta} = \rho^{n,\delta}\,\mathrm{d}\mathscr{L}^{d+1} + \rho^{n,\delta,\perp}\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{\nu}^{n,\delta} = j^{n,\delta}\,\mathrm{d}\mathscr{L}^{d+1} + j^{n,\delta,\perp}\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}\,,$$

with $(R_\delta)_\#\boldsymbol{\sigma}$ singular with respect to $\mathscr{L}^{d+1}$ and

$$\begin{aligned} \rho^{n,\delta}(t,x) &= \Big(\rho^n \circ R_\delta^{-1}\Big)(t,x)\,, & \rho^{n,\delta,\perp}(t,x) &= (1-2\delta)\Big(\rho^{n,\perp}\circ R_\delta^{-1}\Big)(t,x)\,, \\ j^{n,\delta}(t,x) &= \frac{1}{1-2\delta}\Big(j^n \circ R_\delta^{-1}\Big)(t,x)\,, & j^{n,\delta,\perp}(t,x) &= \Big(j^{n,\perp}\circ R_\delta^{-1}\Big)\,. \end{aligned} \tag{4.3.52}$$

Further consider the Lebesgue decompositions

$$\mu_i^n = \rho_i^n \,\mathrm{d}\mathscr{L}^d + \rho_i^{n,\perp}\,\mathrm{d}\sigma_i\,, \qquad i \in \{0,1\}$$

for some $\sigma_1, \sigma_2 \in \mathcal{M}_+(\mathbb{T}^d)$ singular w.r.t. $\mathscr{L}^d$. The action of $(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta})$ is given by[2]

$$\mathbb{A}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) = \mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) + \sum_{i=0,1}\delta\Big(\int_{\mathbb{T}^d} f(\rho_i^n,0)\,\mathrm{d}\mathscr{L}^d + \int_{\mathbb{T}^d} f^\infty(\rho_i^{n,\perp},0)\,\mathrm{d}\sigma_i\Big)\,,$$

where we used the notation

$$\mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta},\boldsymbol{\nu}^{n,\delta}) := \int_{\mathcal{I}_\delta\times\mathbb{T}^d} f(\rho^{n,\delta}, j^{n,\delta})\,\mathrm{d}\mathscr{L}^{d+1} + \int_{\mathcal{I}_\delta\times\mathbb{T}^d} f^\infty(\rho^{n,\delta,\perp}, j^{n,\delta,\perp})\,\mathrm{d}(R_\delta)_\#\boldsymbol{\sigma}\,.$$

[2] Note that the definition of the action does not depend on the choice of the measure which is singular with respect to $\mathscr{L}^{d+1}$, therefore we can use $(R_\delta)_\#\boldsymbol{\sigma}$ instead of $\boldsymbol{\sigma}$.

Making use of the formulas (4.3.52) and the homogeneity of f^∞, we find

$$\begin{aligned}
&\mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) &(4.3.53)\\
&\quad = (1-2\delta)\int_{(0,1)\times\mathbb{T}^d} f\Big(\rho^n, \frac{j^n}{1-2\delta}\Big)\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\Big((1-2\delta)\rho^{n,\perp}, j^{n,\perp}\Big)\,\mathrm{d}\boldsymbol{\sigma}\\
&\quad = (1-2\delta)\Big(\int_{(0,1)\times\mathbb{T}^d} f\Big(\rho^n, \frac{j^n}{1-2\delta}\Big)\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} f^\infty\Big(\rho^{n,\perp}, \frac{j^{n,\perp}}{1-2\delta}\Big)\,\mathrm{d}\boldsymbol{\sigma}\Big)\,. &(4.3.54)
\end{aligned}$$

Furthermore, it follows from the linear growth assumption that, for $i = 0, 1$,

$$\int_{\mathbb{T}^d} f(\rho_i^n, 0)\,\mathrm{d}\mathscr{L}^d + \int_{\mathbb{T}^d} f^\infty(\rho_i^{n,\perp}, 0)\,\mathrm{d}\sigma_i \le C(\mu_i^n(\mathbb{T}^d)+1) = C\Big(\boldsymbol{\mu}^n((0,1)\times\mathbb{T}^d)+1\Big)$$

as well as, by (4.3.53), the nonnegativity of f, and Assumption 4.2.2,

$$\begin{aligned}
\mathbb{A}^{\mathcal{I}_\delta}(\boldsymbol{\mu}^{n,\delta}, \boldsymbol{\nu}^{n,\delta}) &\le \mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) + 2\delta(\mathrm{Lip} f)\Big(\int_{(0,1)\times\mathbb{T}^d} |j^n|\,\mathrm{d}\mathscr{L}^{d+1} + \int_{(0,1)\times\mathbb{T}^d} |j^{n,\perp}|\,\mathrm{d}\boldsymbol{\sigma}\Big)\\
&\le \mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) + \frac{2\delta(\mathrm{Lip} f)}{c}\Big(\mathbb{A}(\boldsymbol{\mu}^n, \boldsymbol{\nu}^n) + C(1+\|\boldsymbol{\mu}^n\|_{\mathrm{TV}})\Big)\,.
\end{aligned}$$

We thus conclude (4.3.46). □

4.4 Analysis of the cell problem with examples

This section is devoted to the characterisation and illustration of f_{hom} in the case where the function F is of the form

$$F(m,J) = F(J) = \sum_{(x,y)\in\mathcal{E}^Q} \alpha_{xy}|J(x,y)| \tag{4.4.1}$$

for some strictly positive function $\alpha : \mathcal{E}^Q \ni (x,y) \mapsto \alpha_{xy} > 0$. A natural problem of interest is to determine whether/when the Γ-limit $\mathbb{MA}_{\mathrm{hom}}$ can be the W_1-distance. The analogous problem for the W_2-distance has been extensively studied in [GKM20] and [GKMP23] in the case where the graph stucture is associated with finite-volume partitions.

4.4.1 Discrete 1-Wasserstein distance

We start the analysis of this special setting by observing that, in this case, the discrete functional $\mathcal{MA}_\varepsilon$ actually coincides with the $\mathbb{W}_1$ distance associated to a natural induced metric structure. In order to prove this fact, we first define $\widetilde{\alpha}^\varepsilon : \mathcal{E}_\varepsilon \to \mathbb{R}_+$ as the unique function such that

$$\frac{\tau_\varepsilon^z \widetilde{\alpha}^\varepsilon}{\varepsilon}\Big|_{\mathcal{E}^Q} := \alpha \qquad z \in \mathbb{Z}_\varepsilon^d\,.$$

It is easy to check that this definition is well-posed and determines the value of $\widetilde{\alpha}_{xy}$ for every $(x,y) \in \mathcal{E}_\varepsilon$. Further let

$$\alpha_{xy}^\varepsilon = \frac{\widetilde{\alpha}_{xy}^\varepsilon + \widetilde{\alpha}_{yx}^\varepsilon}{2}\,, \qquad (x,y) \in \mathcal{E}_\varepsilon$$

be the symmetrisation of $\widetilde{\alpha}^\varepsilon$. Given $J \in \mathbb{R}_a^{\mathcal{E}_\varepsilon}$, we can write $\mathcal{F}_\varepsilon(J)$ in terms of α^ε. Precisely,

$$\begin{aligned}\mathcal{F}_\varepsilon(J) &= \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d F\left(\frac{\tau_\varepsilon^z J}{\varepsilon^{d-1}}\right) = \sum_{z\in\mathbb{Z}_\varepsilon^d} \varepsilon^d \sum_{(\hat{x},\hat{y})\in\mathcal{E}^Q} \alpha_{\hat{x}\hat{y}} \frac{|\tau_\varepsilon^z J(\hat{x},\hat{y})|}{\varepsilon^{d-1}} \\ &= \sum_{z\in\mathbb{Z}_\varepsilon^d} \sum_{(\hat{x},\hat{y})\in\mathcal{E}^Q} \tau_\varepsilon^z \widetilde{\alpha}^\varepsilon(\hat{x},\hat{y}) |\tau_\varepsilon^z J(\hat{x},\hat{y})| = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \widetilde{\alpha}_{xy}^\varepsilon |J(x,y)| \\ &= \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha_{xy}^\varepsilon |J(x,y)| \ ,\end{aligned}$$

where in the last passage we used that $|J|$ is symmetric. We define a distance on $\mathcal{X}_\varepsilon$ given by

$$d_\varepsilon(x,y) := \mathcal{MA}_\varepsilon(\delta_x,\delta_y)\,, \qquad \forall x,y\in\mathcal{X}_\varepsilon\,. \tag{4.4.2}$$

One can easily show that d_ε indeed defines a metric on $\mathcal{X}_\varepsilon$. In fact, d_ε can be seen as a weighted graph distance.

Proposition 4.4.1. *For every $x,y\in\mathcal{X}_\varepsilon$, we have*

$$d_\varepsilon(x,y) = \inf\left\{\sum_{i=0}^{k-1} 2\alpha_{x_i x_{i+1}}^\varepsilon \ : \ x_0 = x\,,\ x_k = y\,,\ (x_i,x_{i+1})\in\mathcal{E}_\varepsilon \ \ \forall i\,,\ k\in\mathbb{N}\right\}.$$

Proof. The inequality $\le$ directly follows by choosing unit fluxes along admissible paths: let $x_0 = x, x_1, \ldots, x_{k-1}, x_k = y$ be a path, i.e., $(x_i,x_{i+1})\in\mathcal{E}_\varepsilon$ for every $i = 0,1,\ldots,k$, and consider

$$J^P := \sum_{i=0}^{k-1}\Big(\delta_{(x_i,x_{i+1})} - \delta_{(x_{i+1},x_i)}\Big)\,, \tag{4.4.3}$$

which has divergence equal to $\delta_x - \delta_y$. Then,

$$\begin{aligned} d_\varepsilon(x,y) = \mathcal{MA}_\varepsilon(\delta_x,\delta_y) &= \inf\Big\{\mathcal{F}_\varepsilon(J) \ : \ \operatorname{div} J = \delta_x - \delta_y\Big\} \\ &= \inf\left\{\sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha_{xy}^\varepsilon |J(x,y)| \ : \ \operatorname{div} J = \delta_x - \delta_y\right\} \\ &\le \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha_{xy}^\varepsilon \Big|J^P(x,y)\Big| \le \sum_{i=0}^{k-1} 2\alpha_{x_i x_{i+1}}^\varepsilon\,,\end{aligned}$$

where in the last inequality we used that α^ε is symmetric.

To prove the converse, let $\bar{J}\in\mathbb{R}_a^{\mathcal{E}_\varepsilon}$ be an optimal flux for $\mathcal{MA}_\varepsilon(\delta_x,\delta_y)$, that is,

$$\operatorname{div}\bar{J} = \delta_x - \delta_y \quad \text{and} \quad \mathcal{MA}_\varepsilon(\delta_x,\delta_y) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha_{xy}^\varepsilon \Big|\bar{J}(x,y)\Big|\,.$$

Since the graph $\mathcal{E}_\varepsilon$ is finite, in order for $\bar{J}$ to satisfy the divergence condition, there must exist a simple path $x_0 = x, x_1, \ldots, x_k = y$ such that $(x_i,x_{i+1})\in\mathcal{E}_\varepsilon$ and $\bar{J}(x_i,x_{i+1}) > 0$ for every i. Let J^P be the associated vector field as in (4.4.3). Note that, for every $\lambda\in\mathbb{R}$, we have $\operatorname{div}\Big((1-\lambda)\bar{J} + \lambda J^P\Big) = \delta_x - \delta_y$. Furthermore, the function

$$\lambda \mapsto \mathcal{F}_\varepsilon((1-\lambda)\bar{J} + \lambda J^P\Big) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha_{xy}^\varepsilon \Big|(1-\lambda)\bar{J}(x,y) + \lambda J^P(x,y)\Big|$$

is differentiable at $\lambda = 0$, since $(\bar{J}(x,y) = 0) \Rightarrow (J^P(x,y) = 0)$. By optimality, the derivative at $\lambda = 0$ must equal 0, i.e.,

$$0 = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}\Big(J^P(x,y) - \bar{J}(x,y)\Big) \operatorname{sgn} \bar{J}(x,y) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy} J^P(x,y) \operatorname{sgn} \bar{J}(x,y) - d_\varepsilon(x,y)\,,$$

and, since $(J^P(x,y) \neq 0) \Rightarrow (\operatorname{sgn} J^P(x,y) = \operatorname{sgn} \bar{J}(x,y))$, we have

$$d_\varepsilon(x,y) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}\Big|J^P(x,y)\Big| = 2\sum_{i=1}^k \alpha^\varepsilon_{x_i x_{i+1}}\,,$$

where, in the last equality, we used that the path is simple (and the symmetry of α^ε). This shows the inequality $\geq$ and concludes the proof. □

Consider the 1-Wasserstein distance associted to d_ε, that is,

$$\mathbb{W}_{1,\varepsilon}(m_0,m_1) = \inf\left\{\int_{\mathcal{X}_\varepsilon\times\mathcal{X}_\varepsilon} d_\varepsilon(x,y)\,\mathrm{d}\pi(x,y) \ : \ (e_0)_\#\pi = m_0\,, \quad (e_1)_\#\pi = m_1\right\}\,, \tag{4.4.4}$$

as well as, by Kantorovich duality,

$$\mathbb{W}_{1,\varepsilon}(m_0,m_1) = \sup\left\{\int_{\mathcal{X}_\varepsilon} \varphi\,\mathrm{d}(m_0 - m_1) \ : \ \operatorname{Lip}_{d_\varepsilon}(\varphi) \leq 1\right\}\,, \tag{4.4.5}$$

for every $m_0, m_1 \in \mathscr{P}(\mathcal{X}_\varepsilon)$.

Proposition 4.4.2. *For every $m_0, m_1 \in \mathscr{P}(\mathcal{X}_\varepsilon)$, we have*

$$\mathcal{MA}_\varepsilon(m_0,m_1) = \mathbb{W}_{1,\varepsilon}(m_0,m_1)\,. \tag{4.4.6}$$

Proof of $\geq$. Fix $m_0, m_1 \in \mathscr{P}(\mathcal{X}_\varepsilon)$ and set $m := m_0 - m_1$. Let $\bar{J} \in \mathbb{R}^{\mathcal{E}_\varepsilon}_a$ be an optimal flux for $\mathcal{MA}_\varepsilon(m_0,m_1)$, that is,

$$\operatorname{div} \bar{J} = m \quad \text{and} \quad \mathcal{MA}_\varepsilon(m_0,m_1) = \sum_{(x,y)\in\mathcal{E}_\varepsilon} \alpha^\varepsilon_{xy}\Big|\bar{J}(x,y)\Big|\,. \tag{4.4.7}$$

Let $\varphi : \mathcal{X}_\varepsilon \to \mathbb{R}$ be such that $\operatorname{Lip}_{d_\varepsilon}\varphi \leq 1$, i.e., $|\varphi(y) - \varphi(x)| \leq d_\varepsilon(x,y)$ for $x, y \in \mathcal{X}_\varepsilon$. Then,

$$\int_{\mathcal{X}_\varepsilon} \varphi\,\mathrm{d}m = \int_{\mathcal{X}_\varepsilon} \varphi\,\mathrm{d}\operatorname{div}\bar{J} = \sum_{x\in\mathcal{X}_\varepsilon} \varphi(x) \sum_{y\sim x} \bar{J}(x,y) = \frac{1}{2}\sum_{(x,y)\in\mathcal{E}_\varepsilon} \varphi(x)\Big(\bar{J}(x,y) - \bar{J}(y,x)\Big) \tag{4.4.8}$$

$$= \frac{1}{2}\sum_{(x,y)\in\mathcal{E}_\varepsilon} \Big(\varphi(y) - \varphi(x)\Big)\bar{J}(x,y) \leq \frac{1}{2}\sum_{(x,y)\in\mathcal{E}_\varepsilon} d_\varepsilon(x,y)\Big|\bar{J}(x,y)\Big|\,. \tag{4.4.9}$$

In order to conclude, we make the following crucial observation: as a consequence of the optimality of $\bar{J}$, we claim that

$$\bar{J}(x,y) \neq 0 \quad \Longrightarrow \quad d_\varepsilon(x,y) = 2\alpha^\varepsilon_{xy}\,. \tag{4.4.10}$$

To this end, assume that $\bar{J}(x,y) \neq 0$ and consider an optimal $J^{(x,y)}$ for $d_\varepsilon(x,y) = \mathcal{MA}_\varepsilon(\delta_x,\delta_y)$. Note that, by construction,

$$\operatorname{div}\Big(J^{(x,y)}\Big) = \delta_x - \delta_y = \operatorname{div}\widetilde{J}\,, \qquad \text{where } \widetilde{J} := \delta_{(x,y)} - \delta_{(y,x)}\,, \tag{4.4.11}$$

which in turns also implies that

$$\operatorname{div}\left(\bar{J}+\bar{J}(x,y)\left(J^{(x,y)}-\widetilde{J}\right)\right)=\operatorname{div}\bar{J}\,. \tag{4.4.12}$$

By optimality of $J^{(x,y)}$, we have

$$\mathcal{F}_\varepsilon(\widetilde{J})=2\alpha^\varepsilon_{xy}\geq\mathcal{F}_\varepsilon\left(J^{(x,y)}\right)=\sum_{(\widetilde{x},\widetilde{y})\in\mathcal{E}_\varepsilon}\alpha^\varepsilon_{\widetilde{x},\widetilde{y}}\left|J^{(x,y)}(\widetilde{x},\widetilde{y})\right|\,, \tag{4.4.13}$$

whereas the optimality of $\bar{J}$ yields

$$\begin{aligned}\mathcal{F}_\varepsilon\left(\bar{J}+\bar{J}(x,y)\left(J^{(x,y)}-\widetilde{J}\right)\right)&=\sum_{(\widetilde{x},\widetilde{y})\in\mathcal{E}_\varepsilon\setminus\{(x,y),(y,x)\}}\alpha^\varepsilon_{\widetilde{x}\widetilde{y}}\left|\bar{J}(\widetilde{x},\widetilde{y})+\bar{J}(x,y)J^{(x,y)}(\widetilde{x},\widetilde{y})\right|\\&\quad+\alpha^\varepsilon_{xy}\left|\bar{J}(x,y)J^{(x,y)}(x,y)\right|+\alpha^\varepsilon_{yx}\left|\bar{J}(x,y)J^{(x,y)}(y,x)\right|\\&\geq\mathcal{F}_\varepsilon(\bar{J})=\sum_{(\widetilde{x},\widetilde{y})\in\mathcal{E}_\varepsilon}\alpha^\varepsilon_{\widetilde{x}\widetilde{y}}\left|\bar{J}(\widetilde{x},\widetilde{y})\right|\,.\end{aligned}$$

By applying the triangle inequality and simplifying the latter formula, we find

$$\sum_{(\widetilde{x},\widetilde{y})\in\mathcal{E}_\varepsilon}\alpha^\varepsilon_{\widetilde{x}\widetilde{y}}\left|\bar{J}(x,y)J^{(x,y)}(\widetilde{x},\widetilde{y})\right|\geq2\alpha^\varepsilon_{xy}\left|\bar{J}(x,y)\right|\,. \tag{4.4.14}$$

The combination of (4.4.13) and (4.4.14) implies $d_\varepsilon(x,y)=2\alpha^\varepsilon_{xy}$. With (4.4.10) at hand, we can write

$$\int_{\mathcal{X}_\varepsilon}\varphi\,\mathrm{d}m\leq\sum_{(x,y)\in\mathcal{E}_\varepsilon}\alpha^\varepsilon_{xy}\left|\bar{J}(x,y)\right|=\mathcal{MA}_\varepsilon(m_0,m_1)\,,$$

and we conclude by arbitrariness of φ. □

Proof of $\leq$. Let π be such that $(e_i)_\#\pi=m_i$ for $i=0,1$. Further, for every $x,y\in\mathcal{X}_\varepsilon$, let $J^{(x,y)}\in\mathbb{R}^{\mathcal{E}_\varepsilon}_a$ be optimal for $\mathcal{MA}_\varepsilon(\delta_x,\delta_y)$. It follows from a direct computation that the divergence of the asymmetric flux

$$J:=\sum_{x,y\in\mathcal{X}_\varepsilon}\pi(x,y)J^{(x,y)}$$

is equal to m_0-m_1. Thus,

$$\begin{aligned}\mathcal{MA}_\varepsilon(m_0,m_1)\leq\sum_{(\widetilde{x},\widetilde{y})\in\mathcal{E}_\varepsilon}\alpha^\varepsilon_{\widetilde{x}\widetilde{y}}|J(\widetilde{x},\widetilde{y})|\leq\sum_{x,y\in\mathcal{X}_\varepsilon}\pi(x,y)\sum_{(\widetilde{x},\widetilde{y})\in\mathcal{E}_\varepsilon}\alpha^\varepsilon_{\widetilde{x}\widetilde{y}}\left|J^{(x,y)}(\widetilde{x},\widetilde{y})\right|\\=\int_{\mathcal{X}_\varepsilon\times\mathcal{X}_\varepsilon}d_\varepsilon\,\mathrm{d}\pi\,,\end{aligned}$$

and we conclude by arbitrariness of π. □

In view of the equality $\mathcal{MA}_\varepsilon=\mathbb{W}_{1,\varepsilon}$, it is worth noting that for cost functions of the form (4.4.1) there are (at least) two different possible methods to show discrete-to-continuum limits for $\mathcal{MA}_\varepsilon$. One such method is provided by the current work and makes use of the Γ-convergence of $\mathcal{A}_\varepsilon$ to $\mathbb{A}_{\mathrm{hom}}$ proved in [GKMP23, Theorem 5.4]. The convergence of the "weighted graph distance" d_ε follows *a posteriori*. Another approach is to study directly the scaling limits of the distance d_ε as $\varepsilon\to0$ and, from that, infer the convergence of the associated 1-Wasserstein distances, in a similar spirit as in [BDPF+01].

4.4.2 General properties of f_{hom}

For $j \in \mathbb{R}^d$, recall that

$$f_{\mathrm{hom}}(j) := \inf\left\{ F(J) \ : \ J \in \mathsf{Rep}(j) \right\}, \tag{4.4.15}$$

where $\mathsf{Rep}(j)$ is the set of all $\mathbb{Z}^d$-periodic functions $J \in \mathbb{R}^{\mathcal{E}}_a$ such that

$$\mathsf{Eff}(J) := \frac{1}{2} \sum_{(x,y) \in \mathcal{E}^Q} J(x,y)(y_{\mathsf{z}} - x_{\mathsf{z}}) = j \quad \text{and} \quad \operatorname{div} J \equiv 0 \, .$$

As noted in [GKMP23, Lemma 4.7], we may as well write $\min$ in place of $\inf$ in (4.4.15).

Our first observation is that, indeed, the homogenised density is a norm. This has already been proved in [GKMP23, Corollary 5.3]; for the sake of completeness we provide here a simple proof in our setting.

Proposition 4.4.3. *The function f_{hom} is a norm.*

Proof. Finiteness follows from the nonemptiness of the set of representatives proved in [GKMP23, Lemma 4.5]. To prove positiveness, take any $j \in \mathbb{R}^d$ and $J \in \mathsf{Rep}(j)$. For every norm $\|\cdot\|$, we have

$$\|j\| = \|\mathsf{Eff}(J)\| \leq \frac{1}{2} \sum_{(x,y) \in \mathcal{E}^Q} |J(x,y)| \|y_{\mathsf{z}} - x_{\mathsf{z}}\| \leq \frac{F(J)}{2} \max_{(x,y) \in \mathcal{E}^Q} \frac{\|y_{\mathsf{z}} - x_{\mathsf{z}}\|}{\alpha_{xy}} \, . \tag{4.4.16}$$

The constant that multiplies $F(J)$ at the right-hand side is finite because every α_{xy} is strictly positive and the graph $(\mathcal{X}, \mathcal{E})$ is locally finite. Absolute homogeneity and the triangle inequality follow from the absolute homogeneity and subadditivity of F, and the affinity of the constraints. □

Hence, $\mathbb{MA}_{\mathrm{hom}}$ is *always* (i.e. for any choice of $(\alpha_{xy})_{x,y}$ and of the graph $(\mathcal{X}, \mathcal{E})$) the W_1-distance w.r.t. some norm. However, the norm f_{hom} can equal the 2-norm $|\cdot|_2$ only in dimension $d = 1$. In fact, the unit ball for f_{hom} has to be a polytope, namely the associated sphere is contained in the union of finitely many hyperplanes. These types of norms are also known as *crystalline norms*.

Proposition 4.4.4. *The unit ball associated to the norm f_{hom}, namely*

$$B := \left\{ j \in \mathbb{R}^d \ : \ f_{\mathrm{hom}}(j) \leq 1 \right\} ,$$

is the convex hull of finitely many points. In particular, the associated unit sphere is contained in the union of finitely many hyperplanes, i.e., f_{hom} is a crystalline norm.

Proof. Let X be the vector space of all $\mathbb{Z}^d$-periodic functions $J \in \mathbb{R}^{\mathcal{E}}_a$ such that $\operatorname{div} J \equiv 0$. The sublevel set

$$X_1 := \{ J \in X \ : \ F(J) \leq 1 \}$$

is clearly compact (due to the strict positivity of $(\alpha_{xy})_{x,y}$) and can be written as finite intersection of half-spaces, namely

$$X_1 = \bigcap_{r \in \{-1,1\}^{\mathcal{E}^Q}} \left\{ J \in X \ : \ \sum_{(x,y) \in \mathcal{E}^Q} \alpha_{xy} r_{xy} J(x,y) \leq 1 \right\} .$$

Thus, X_1 is the convex hull of some finite set of points A, that is, $X_1 = \mathrm{conv}(A)$. Since f_{hom} is defined as a minimum, we have

$$B = \Big\{ j \in \mathbb{R}^d \ : \ \exists J \in \mathsf{Rep}(j),\ F(J) \le 1 \Big\} = \mathsf{Eff}(X_1) = \mathsf{Eff}(\mathrm{conv}(A)) = \mathrm{conv}(\mathsf{Eff}(A))\,,$$

where the last equality is due to the linearity of Eff. □

4.4.3 Embedded graphs

To visualise some examples, we shall now focus on the case where $(\mathcal{X}, \mathcal{E})$ is embedded, in the sense that V is a subset of $[0,1)^d$ and we use the identification $(z, v) \equiv z + v$ (see also [GKMP23, Remark 2.2]). It has been proved in [GKMP23, Proposition 9.1] that, for embedded graphs, the identity

$$\mathsf{Eff}(J) = \frac{1}{2} \sum_{(x,y)\in\mathcal{E}^Q} J(x,y)(y-x) \tag{4.4.17}$$

holds for every $\mathbb{Z}^d$-periodic and divergence-free vector field $J \in \mathbb{R}^{\mathcal{E}}_a$. In what follows, we also make the choice

$$\alpha_{xy} := \frac{1}{2}|x-y|_2\,, \qquad (x,y) \in \mathcal{E}^Q\,.$$

One-dimensional case with nearest-neighbor interaction

Assume $d = 1$, let $x_1 < x_2 < \cdots < x_k$ be an enumeration of V, and set

$$\mathcal{E} := \{(x,y) \in \mathcal{X} \times \mathcal{X} \text{ s.t. there is no } z \in \mathcal{X} \text{ strictly between } x \text{ and } y\}.$$

In other words, denoting $x_0 = x_k - 1$ and $x_{k+1} = x_1 + 1$,

$$\mathcal{E} = \bigcup_{z\in\mathbb{Z}} \bigcup_{i=1}^{k} \{(x_i, x_{i+1})\} \cup \{(x_i, x_{i-1})\}\,.$$

By rewriting (4.4.16) using (4.4.17), and by the definition of f_{hom}, we find

$$|j| \le f_{\mathrm{hom}}(j)\,, \qquad j \in \mathbb{R}^d\,.$$

On the other hand, given $j \in \mathbb{R}^d$, choose

$$J(x,y) := j\,\mathrm{sgn}(y-x)\,, \qquad (x,y) \in \mathcal{E}\,.$$

This vector field is in $\mathsf{Rep}(j)$, because

$$\mathrm{div}\, J(x_i) = J(x_i, x_{i+1}) + J(x_i, x_{i-1}) = j - j = 0\,,$$

for every i, and

$$\begin{aligned}
\mathsf{Eff}(J) &= \frac{1}{2}\sum_{i=1}^{k}\Big(J(x_i,x_{i+1})(x_{i+1}-x_i) + J(x_i,x_{i-1})(x_{i-1}-x_i)\Big) \\
&= \frac{j}{2}\sum_{i=1}^{k}\Big(|x_{i+1}-x_i| + |x_i - x_{i-1}|\Big) \\
&= \frac{j}{2}\Big(x_{k+1} - x_1 + x_k - x_0\Big) \\
&= j.
\end{aligned}$$

A similar computation shows that $F(J) = |j|$, from which $f_{\mathrm{hom}}(j) = |j|$.

Cubic partition

Consider the case where $\mathcal{X} = \mathbb{Z}^d$ and

$$\mathcal{E} := \left\{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \ : \ |x - y|_\infty = 1\right\}.$$

It is a result of [GKMP23, Section 9.2] that

$$f_{\text{hom}}(j) = |j|_1 , \qquad j \in \mathbb{R}^d .$$

Notice that, in this case, the 2-norm is evaluated only at vectors on the coordinate axes. Therefore, the same result holds when $\alpha_{xy} = \frac{1}{2}|x - y|_p$, for any p.

Graphs in $\mathbb{R}^2$

A few other examples in dimension $d = 2$ are shown in Figure 4.2: for each one, we display the graph and the unit ball in the corresponding norm f_{hom}. To algorithmically construct the unit balls, we solve the variational problem (4.4.15) for every j on a discretisation of the circle $\mathbb{S}^1$. In turn, this is achieved with the help of the Python library CVXPY [DB16, AVDB18]. For visualisation, we make use of the library matplotlib [Hun07].

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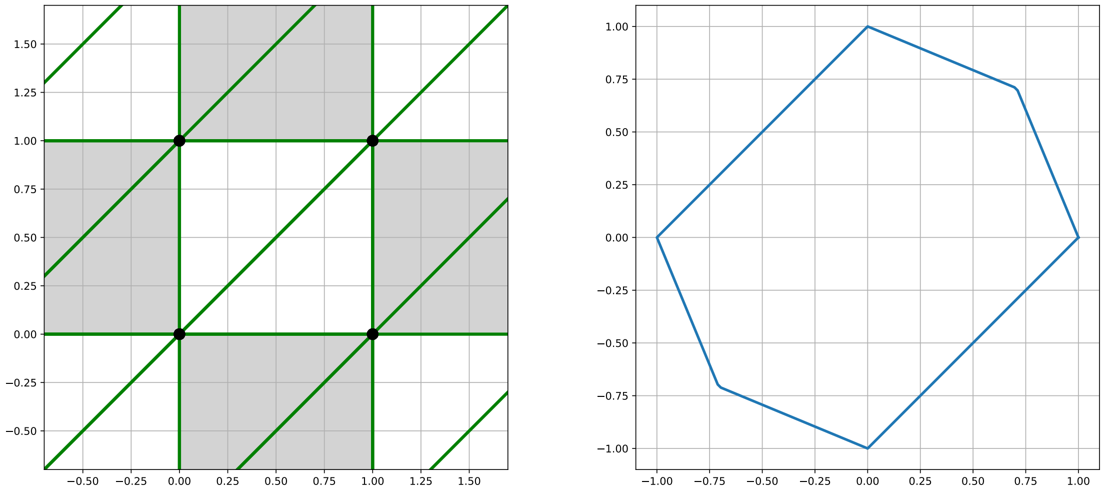
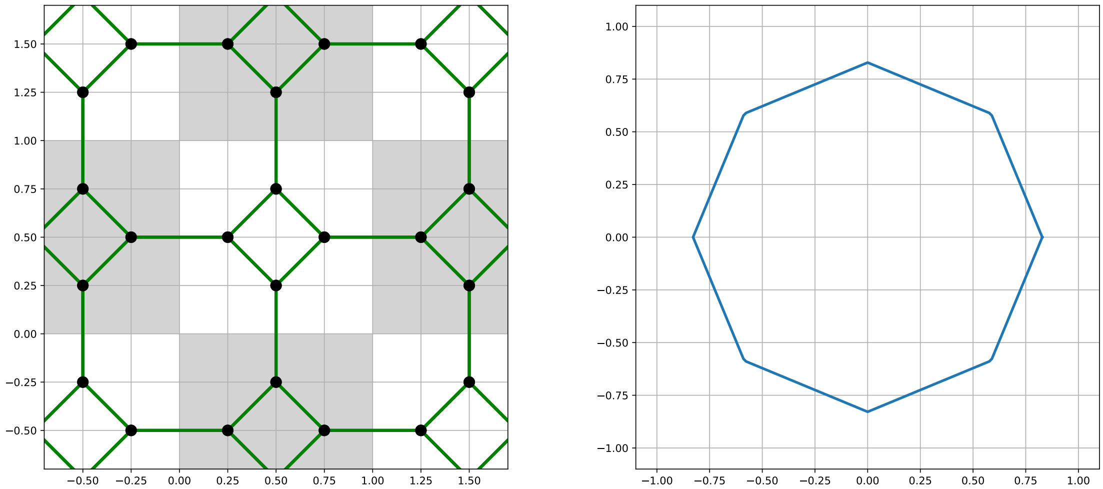
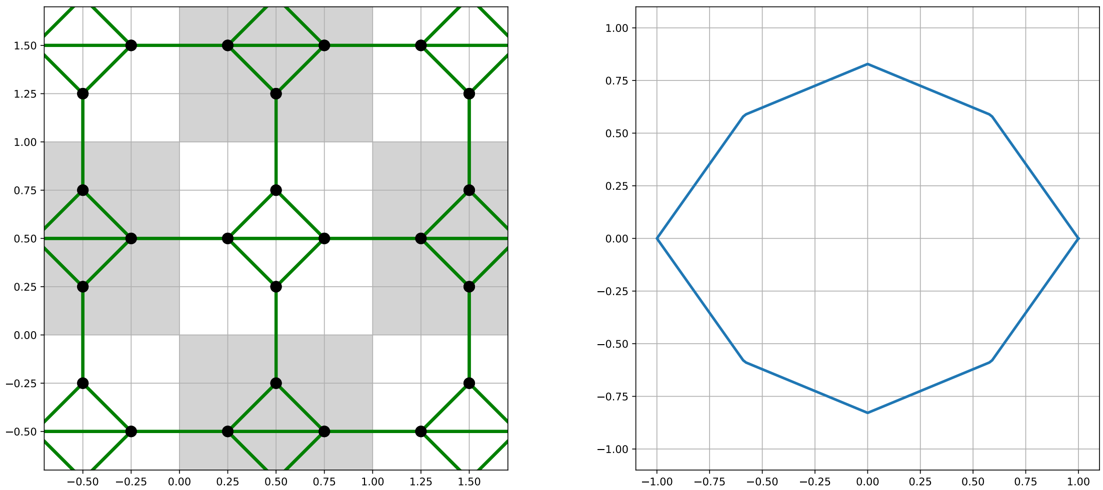


(a)



(b)



(c)

Figure 4.2: Examples of graphs in $\mathbb{R}^2$ and corresponding unit balls for f_{hom}

CHAPTER 5

Asymptotics for Optimal Empirical Quantization of Measures

This chapter contains (with minimal modifications) the following preprint [Qua24]:

F. Quattrocchi. Asymptotics for Optimal Empirical Quantization of Measures. *arXiv preprint* arXiv:2408.12924v1, 2024.

I wrote this preprint during my PhD at ISTA, but some partial results were already included in my Master's thesis [Qua21], written under the supervision of Prof. Dario Trevisan at the University of Pisa. More precisely:

- the upper bound (U) in Theorem 5.1.1, in the special case of absolutely continuous and compactly supported measures, is the statement of [Qua21, Theorem 3.2.3]. The generalization to arbitrary measures with a moment condition, and the lower bound (L) are new;
- Theorem 5.1.3 was proven in [Qua21, Theorem 3.2.1] in the special case where A is a bounded 2-dimensional convex set. The generalisation to sets that are bi-Lipschitz equivalent to a disk is new. The other special cases in which existence of the renormalized limit is proven are Proposition 5.6.1 and Corollary 5.9.1, which correspond to [Qua21, Theorem 3.2.2] and [Qua21, Corollary 3.5.7], respectively;
- Corollary 5.1.4 is new;
- Theorem 5.1.7 is new.

Abstract

We investigate the *minimal* error in approximating a general probability measure μ on $\mathbb{R}^d$ by the *uniform* measure on a finite set with prescribed cardinality n. The error is measured in the p-Wasserstein distance. In particular, when $1 \le p < d$, we establish *asymptotic* upper and lower bounds as $n \to \infty$ on the rescaled minimal error that have the *same, explicit* dependency on μ.

In some instances, we prove that the rescaled minimal error has a limit. These include general measures in dimension $d = 2$ with $1 \le p < 2$, and uniform measures in arbitrary dimension with $1 \le p < d$. For some uniform measures, we prove the limit existence for $p \ge d$ as well.

For a class of compactly supported measures with Hölder densities, we determine the convergence speed of the minimal error for *every* $p \geq 1$.

Furthermore, we establish a new Pierce-type (i.e., nonasymptotic) upper estimate of the minimal error when $1 \leq p < d$.

In the initial sections, we survey the state of the art and draw connections with similar problems, such as classical and random quantization.

5.1 Introduction

Quantization is the problem of optimally approximating a probability measure μ on $\mathbb{R}^d$ by another one, say μ_n, supported on a finite number n of points. For instance, we can think of μ as the description of a picture and of μ_n as its digital *compression*. Another typical example comes from *urban planning*: if μ represents the population distribution in a city, then the support of the approximating measure μ_n determines good locations to build schools, supermarkets, parks, etc.

The mathematical formulation is as follows: for a given number of points n and a fixed parameter $p \in [1, \infty)$, find a solution to the minimization problem

$$e_{p,n}(\mu) \coloneqq \min_{\mu_n \in \mathcal{P}(\mathbb{R}^d)} \left\{ W_p(\mu, \mu_n) \, : \, \# \operatorname{supp}(\mu_n) \leq n \right\} , \tag{5.1.1}$$

where W_p is the p-Wasserstein–Kantorovich–Rubinstein distance [Vil09b, San15]. We will call this minimal number the n^{th} *optimal quantization error* of order p for μ. Equivalently (see [GL00]), $e_{p,n}(\mu)$ can be written as the following minimum over maps $T \colon \mathbb{R}^d \to \mathbb{R}^d$ *onto at most n points*:

$$e_{p,n}(\mu) = \min_T \left(\int \|x - T(x)\|^p \, \mathrm{d}\mu \right)^{1/p} = \min_T \mathbb{E}_{X \sim \mu} \left[\|X - T(X)\|^p \right]^{1/p} ,$$

or, also,

$$e_{p,n}(\mu) = \min_{x_1, \ldots, x_n \in \mathbb{R}^d} \left(\int \min_i \|x - x_i\|^p \, \mathrm{d}\mu \right)^{1/p} . \tag{5.1.2}$$

One of the questions that have attracted considerable attention over the years is the *asymptotic* behavior of $e_{p,n}(\mu)$ as $n \to \infty$, see [GL00]. The most fundamental result in this direction is Zador's Theorem: given a probability measure μ on $\mathbb{R}^d$ enjoying a suitable moment condition, and denoting by ρ the density of its absolutely continuous part, we have

$$\lim_{n \to \infty} n^{1/d} e_{p,n}(\mu) = q_{p,d} \left(\int \rho^{\frac{d}{d+p}} \, \mathrm{d}\mathscr{L}^d \right)^{\frac{d+p}{dp}} , \tag{5.1.3}$$

with

$$q_{p,d} \coloneqq \inf_{n \in \mathbb{N}_1} n^{1/d} e_{p,n} \left(\mathscr{L}^d|_{[0,1]^d} \right) > 0 \, , \tag{5.1.4}$$

see [Zad64, Zad82, BW82, GL00] and Theorem 5.4.1 below. That is, when μ is not purely singular (otherwise the limit (5.1.3) equals zero), Zador's Theorem determines the speed of convergence $n^{-1/d}$ of $e_{p,n}(\mu)$ and the explicit dependency of the prefactor on μ. For a heuristic derivation, see [Der09] or Section 5.2 below.

We will focus on a variation of the classical quantization problem: *optimal* (or *deterministic*) *empirical quantization* (also known as *optimal/deterministic uniform quantization*). The subject of our study is the *optimal empirical quantization error* $\tilde{e}_{p,n}(\mu)$, defined by

$$\tilde{e}_{p,n}(\mu) := \min_{\mu_n \in \mathcal{P}(\mathbb{R}^d)} \left\{ W_p(\mu, \mu_n) \, : \, \mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \text{ for some } x_1, \dots, x_n \in \mathbb{R}^d \right\} , \tag{5.1.5}$$

or, equivalently,

$$\tilde{e}_{p,n}(\mu) = \min_{x_1,\dots,x_n \in \mathbb{R}^d} \min_{\mu^1,\dots,\mu^n} \left(\sum_{i=1}^{n} \int \|x - x_i\|^p \, \mathrm{d}\mu^i \right)^{1/p} ,$$

where $\mu^1, \dots, \mu^n$ are subprobabilities, *each having total mass equal to* $1/n$, that sum up to μ. The two numbers $e_{p,n}(\mu)$ and $\tilde{e}_{p,n}(\mu)$ are similarly defined, but the second one is a minimum over a *smaller* set of measures, hence $e_{p,n}(\mu) \le \tilde{e}_{p,n}(\mu)$. Our aim is to find formulas analogous to (5.1.3) for $\tilde{e}_{p,n}(\mu)$.

Several results are available, both for the case $d = 1$ [JR16, XB19, GHMR19b, BJ22a, BJ22b, GSM23], and in arbitrary dimension [GHMR19b, MM16, Che18, GHMR19a, BS20], but a general statement like Zador's Theorem is still missing. As we will see in greater detail in Section 5.4, the works [MM16, Che18] contain the proof of the following. For "sufficiently nice" probability measures μ (and assuming, for simplicity, $p \neq d$), we have

$$0 < \liminf_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) \quad \text{and} \quad \limsup_{n\to\infty} n^{\frac{1}{\max(p,d)}} \tilde{e}_{p,n}(\mu) < \infty \, .$$

In particular, the speed of convergence to 0 of $\tilde{e}_{p,n}(\mu)$ is $n^{-1/d}$ in the regime $p < d$. However, the known bounds from below and above of the limits inferior and superior are rather loose, in that they do not depend in the same way on μ (see (5.4.5) below), and the existence of this limit is unknown.

5.1.1 Main theorem

Our main theorem addresses the first of the two matters above by providing a *high-resolution formula* for $p < d$ (in the same spirit of [DSS13] for random empirical quantization).

Theorem 5.1.1. *Assume that* $1 \le p < d$ *and let* $p^* := \frac{dp}{d-p}$ *be the Sobolev conjugate of* p. *Let* μ *be a probability measure on* $\mathbb{R}^d$ *and assume that, for some* $\theta > p^*$, *the* θ^{th} *moment of* μ *is finite. Let* ρ *be the density of the absolutely continuous part of* μ *and let* $\operatorname{supp} \mu^s$ *be the support of the singular part of* μ *(w.r.t. the Lebesgue measure* $\mathscr{L}^d$*). Then:*

$$q_{p,d} \left(\int_{\mathbb{R}^d \setminus \operatorname{supp}(\mu^s)} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d \right)^{1/p} \le \liminf_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) \, , \tag{L}$$

$$\limsup_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) \le \tilde{q}_{p,d} \left(\int_{\mathbb{R}^d} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d \right)^{1/p} , \tag{U}$$

where

$$q_{p,d} := \inf_{n \in \mathbb{N}_1} n^{1/d} e_{p,n}\Big(\mathscr{L}^d|_{[0,1]^d}\Big) > 0 \quad \textit{and} \quad \tilde{q}_{p,d} := \inf_{n \in \mathbb{N}_1} n^{1/d} \tilde{e}_{p,n}\Big(\mathscr{L}^d|_{[0,1]^d}\Big) > 0 \, . \tag{5.1.6}$$

Note that the dependence on the measure in (5.1.3) and in Theorem 5.1.1 is different; we will give a heuristic explanation of this phenomenon in Section 5.2. It is also worth noting that the integral $\int \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d$ has already appeared in the asymptotic study of other (related) problems in *combinatorial optimization* [BHH59, Ste97, Yuk98, BB13, GT24] and *random (empirical) quantization* [GL00, DSS13].

In general, the two constants $q_{p,d}$ and $\tilde{q}_{p,d}$ in (5.1.6) are not known explicitly, but it is possible to establish upper and lower bounds, see [GL00, Chapters 8 & 9]. We pose the following.

Conjecture 5.1.2. *The identity $q_{p,d} = \tilde{q}_{p,d}$ holds (for every $p \geq 1$ and $d \in \mathbb{N}_1$).*

This is tightly linked with a famous conjecture by A. Gersho [Ger79], which, in essence, states the following: if $A \subseteq \mathbb{R}^d$ is convex and we denote by U_A its uniform measure, then the optimal quantizers μ_n for $e_{p,n}(U_A)$ are asymptotically *uniform*, and "most" of the Voronoi regions generated by $\operatorname{supp}(\mu_n)$ are congruent to one another. Weak versions of Gersho's Conjecture have been proven in [GLP12, Zhu11, Zhu20], but they seem to be insufficient to settle Conjecture 5.1.2. As noted in [DSS13, Remark 2], proving the equality of the constants appearing in the upper and lower bounds «*seems to be a general open problem in transport problems*» (see also [BB13, GT24]).

Nonetheless, with Remark 5.6.3 and Theorem 5.1.3 (see below), we show that Conjecture 5.1.2 is true for $d = 1$ and $d = 2$.

5.1.2 Existence of the limit

The second matter, namely the convergence of the renormalized error (i.e., $n^{1/d}\tilde{e}_{p,n}(\mu)$ if the speed of convergence of $\tilde{e}_{p,n}(\mu)$ is $n^{-1/d}$), remains, in general, an open question. For $p < d$, however, Theorem 5.1.1 reduces this problem to Conjecture 5.1.2. Indeed, assuming Conjecture 5.1.2, that $1 \leq p < d$, that the θ^{th} moment of μ is finite for some $\theta > p^*$, and $\int_{\operatorname{supp}(\mu^s)} \rho \, \mathrm{d}\mathscr{L}^d = 0$, the limit of $n^{1/d}\tilde{e}_{p,n}(\mu)$ exists by the combination of (L) and (U).

Moreover, with the results of this work, we are able to prove the limit existence in some cases:

1. for every $p \geq 1$ and $d \in \mathbb{N}_1$, when μ is the uniform measure on a cube, see Proposition 5.6.1;
2. when $1 \leq p < d$ and μ is the uniform measure on a bounded nonnegligible Borel set, see Corollary 5.9.1;
3. for every $p \geq 1$, when $d = 2$ and μ is the uniform measure on a set which is bi-Lipschitz equivalent to a closed disk;
4. when $1 \leq p < d = 2$, the θ^{th} moment of μ is finite for some $\theta > p^*$, and $\operatorname{supp}(\mu^s)$ is μ^a-negligible, where μ^a and μ^s are the absolutely continuous and singular parts of μ, respectively.

In all these cases, the upper bound (U) is attained in the limit:

$$\lim_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\mu) = \tilde{q}_{p,d} \left(\int_{\{\rho>0\}} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d \right)^{1/p}. \tag{5.1.7}$$

The points (3),(4) descend directly from Theorem 5.1.1 and the following.

Theorem 5.1.3. *Let $A \subseteq \mathbb{R}^2$ be bi-Lipschitz equivalent to a closed disk*[1] *and let U_A be its uniform measure. Then, for every $p \geq 1$, the limit of $\sqrt{n}\tilde{e}_{p,n}(U_A)$ exists and coincides with $\lim_{n\to\infty} \sqrt{n} e_{p,n}(U_A)$, that is (by (5.1.3)),*

$$\lim_{n\to\infty} \sqrt{n}\tilde{e}_{p,n}(U_A) = q_{p,2}\sqrt{|A|}\,. \tag{5.1.8}$$

In particular, we can choose $A := [0,1]^2$ and obtain $q_{p,2} = \tilde{q}_{p,2}$.

By [XB19, Theorem 5.15] (restated as Theorem 5.4.2 below), the limit exists also when $d = 1$ and the upper quantile function of μ is absolutely continuous.

5.1.3 Asymptotic behavior for $p \in [1, \infty)$

As a first step towards the proof of Theorem 5.1.1, we will prove (5.1.7) for the uniform measure $\mathscr{L}^d|_{[0,1]^d}$ *for every* $p \geq 1$ (Proposition 5.6.1). In particular, we have[2]

$$\limsup_{n\to\infty} n^{1/d}\tilde{e}_{p,n}\Big(\mathscr{L}^d|_{[0,1]^d}\Big) < \infty\,, \tag{5.1.9}$$

from which we derive one corollary which may be of independent interest. Note that, while Theorem 5.1.1 assumes $p < d$, this corollary applies when $p \geq d$ as well.

Corollary 5.1.4. *Let $\tilde{\Omega}, \Omega$ be open bounded sets in $\mathbb{R}^d$ and let $\mu = \rho\mathscr{L}^d$ be an absolutely continuous probability measure concentrated on Ω. Assume that:*

1. *the set $\tilde{\Omega}$ is convex with $C^{1,1}$ boundary;*
2. *there exists a diffeomorphism $M : \tilde{\Omega} \to \Omega$ of class C^1 with (globally) Hölder continuous and uniformly nonsingular Jacobian;*
3. *the restriction $\rho|_\Omega$ is uniformly positive and Hölder continuous (globally on Ω).*

Then, for every $p \geq 1$,

$$0 < \liminf_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\mu) \leq \limsup_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\mu) < \infty\,. \tag{5.1.10}$$

For general measures and $p \geq d$, it is possible, and often expected, that

$$\limsup_{n\to\infty} n^{1/d}\tilde{e}_{p,n}(\mu) = \infty\,,$$

see [XB19, Example 5.8 & Remark 5.22] and Example 5.4.4 below. Corollary 5.1.4 states that the error convergence is still fast (of order $n^{-1/d}$) if the measure is "smooth and well-concentrated".

[1] Note that every convex body A is bi-Lipschitz equivalent to a closed disk: further assuming, without loss of generality, that 0 lies in the interior part of A, the map

$$A \ni x \longmapsto \frac{\inf\{r > 0 \,:\, x \in rA\}}{\|x\|} x$$

(deformation by the Minkowski functional) is bi-Lipschitz onto the unit disk.

[2] *For* $d \geq 3$, the bound (5.1.9) can also be easily derived from the theory of random empirical quantization, see [Led23, Formula (8)].

Remark 5.1.5. Corollary 5.1.4 applies in particular when Ω itself is convex with $C^{1,1}$ boundary and $\rho|_\Omega$ is uniformly positive and Hölder: the identity is admissible as a diffeomorphism M onto Ω.

Remark 5.1.6. The proof of Corollary 5.1.4 relies on a theorem by S. Chen, J. Liu, and X.-J. Wang [CLW21] on the regularity of optimal transport maps. Using other results from this field, e.g. [CLW19, CLW18], it is possible to adapt Corollary 5.1.4 to other sets of assumptions.

5.1.4 Nonasymptotic upper bound

Along the way, we also prove a nonasymptotic upper bound on the optimal empirical quantization error. This is analogous to what is known as *Pierce's Lemma* [Pie70] in classical quantization.

Theorem 5.1.7. *Under the assumptions of Theorem 5.1.1, there exists a constant $c_{p,d,\theta}$ (independent of μ and n) such that*

$$n^{1/d}\tilde{e}_{p,n}(\mu) \leq c_{p,d,\theta} \left(\int \|x\|^\theta \, \mathrm{d}\mu \right)^{1/\theta}, \qquad n \in \mathbb{N}_1 . \tag{5.1.11}$$

5.1.5 Related literature

The theory of quantization has been studied since the 1940s by electrical engineers interested in the compression of analog signals. Early works include [Sha48] by C. E. Shannon, [OPS48] by B. M. Oliver, J. R. Pierce and C. E. Shannon, [Ben48] by W. R. Bennett, and [PD51] by P. F. Panter and W. Dite. We refer the reader to [GN98] for a survey of the related literature in the fields of *signal processing* and *information theory* until the late 1990s.

Algorithms to solve the quantization problem in $\mathbb{R}^d$ are known since the works of H. Steinhaus [Ste56] and S. P. Lloyd [Llo82]. Arguably, the most popular ones are *Lloyd's method* (also known as *k-means algorithm*) and the *Competitive Learning Vector Quantization*, see [Pag15, Section 3].

Over the years, quantization theory has found applications to *data science* (*clustering*, *recommender systems*, etc.) [LP20, LDX⁺24], mathematical models in *economics* [BS72, Bol73, BS09], *computer graphics* [BD23], *geometry* (approximation of convex bodies and *Alexandrov's Problem*) [Gru04, MO16]. The survey [Pag15] describes its applications to numerics, particularly to *numerical integration* [Pag98], *numerical probability* [PP05], and numerical solving of (stochastic) (partial) *differential equations*, relevant, e.g., in *mathematical finance* [PP09]. Quantization has been studied also beyond the finite-dimensional Euclidean setting, particularly in *Riemannian manifolds* [Gru01, Gru04, Klo12, Iac16, AI25, LBP19b, LBP19a] and *infinite-dimensional Banach spaces* (*functional quantization*), see [LP23b] and references therein.

For a more comprehensive and detailed picture of this extensive mathematical subject, we refer to the following monographs. In chronological order: [GG92] by A. Gersho and R. M. Gray, [GL00] by S. Graf and H. Luschgy, and [LP23b] by H. Luschgy and G. Pagès.

As previously noted, asymptotics for $\tilde{e}_{p,n}(\mu)$ have been investigated in [JR16, MM16, Che18, XB19, GHMR19b, BJ22a, BJ22b, GSM23, GHMR19a, BS20], see also Section 5.4. Algorithms and other theoretical properties of optimal empirical quantization (and of the slightly more general *capacity-constrained quantization*[3]) have been proposed and studied, e.g., in [AHA98, BHM08, dGBOD12, Bak15, XLC⁺16, MSS24, BSD09, Cor10, PFB14]. Furthermore, this theory has been used as a tool, e.g., for the approximation of *variational problems* and *(stochastic) differential equations* [MM16, Sar22, BJ22b, KX22, LP23a, GKX25], to prove convergence rates for *regularized optimal transport* [EN24], to analyze *restricted Monte Carlo methods* for quadrature [GHMR19b], to optimally *place robots* in an environment [Cor10, PFB14, CE17], in *computer graphics* (e.g., to generate *blue-noise* distributions) [BSD09, dGBOD12, XLC⁺16], in *neuronal evolution* modeling [CDLO19], and in *material* modeling [DZW⁺23, ZYS⁺24].

Several versions of optimal empirical quantization with respect to different metrics/divergences/discrepancies (in place of W_p) have also been studied. We mention [MJ18, BX20, XKS22], as well as the series of works [Bec84, AD14, ABN18, FGW21, Wei23] on the generalized *star-discrepancy*, which is used to bound numerical integration error by means of the generalized *Koksma–Hlawka inequality* [AD15].

Closely related to optimal empirical quantization is *random* empirical quantization, i.e., the problem of approximating a measure μ using *random* empirical measures $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}$, where $(X_i)_{i\in\mathbb{N}}$ is a sequence of random variables (typically independent and identically distributed), see [WB19, Led23] and references therein. In recent times, some asymptotic results for this problem have been proven using the theory of partial differential equations [AST19] and Fourier analysis [BL21].

5.1.6 Open questions

It may be interesting to further investigate the following problems.

1. In (L), the domain of the integral is $\mathbb{R}^d \setminus \operatorname{supp}(\mu^s)$. Is this just an artifact of our proof? That is: Can we replace this domain with the whole space $\mathbb{R}^d$?

2. We already stated Conjecture 5.1.2 on the equality $q_{p,d} = \tilde{q}_{p,d}$. Unclear is also the relation between $q_{p,d}, \tilde{q}_{p,d}$ and the constants that appear in [DSS13, Theorem 2] and [CGPT24, Theorem 1.6] in the context of random empirical quantization. Numerical estimates of the constants may also help understand this relation.

3. Depending on μ, several asymptotic behaviors are possible for the error $\tilde{e}_{p,n}(\mu)$ when $p \geq d$, see [BJ22a, Table 1] as well as Corollary 5.1.4, Example 5.4.4, and Proposition 5.6.1. It may be worth determining precise characterizations of the measures that exhibit a certain error decay. For example, given $p \geq d$, for which absolutely continuous and compactly supported measures μ is the limit superior of $n^{1/d}\tilde{e}_{p,n}(\mu)$ finite?

4. What can we say about the error asymptotics for *singular* measures?

5. It would be natural to also study the problem on manifolds (as in [Klo12, Iac16, AI25] for classical quantization) and infinite-dimensional spaces.

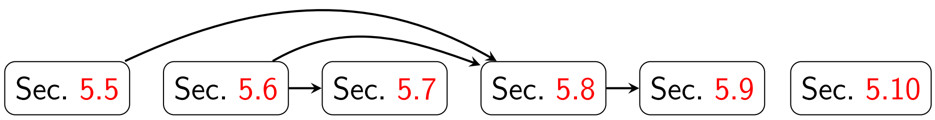
[3]In this version, the competitors μ_n are of the form $\mu_n = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$, where n and $\lambda_1, \ldots, \lambda_n \in [0,1]$ are prescribed, and $x_1, \ldots, x_n$ are free.

5.1.7 Plan of the work

The first four sections are preparatory. In Section 5.2, we give a simple heuristic argument that justifies the integral $\int \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d$ in Theorem 5.1.1. In Section 5.3, we fix the notation and give all necessary definitions. In Section 5.4, we present some of the existing results in the literature, both to provide context and because we will use some of them.

The subsequent sections contain proofs. The major ones will be preceded by comments on the core ideas and techniques. In Section 5.5, we prove the nonasymptotic upper bound of Theorem 5.1.7. In Section 5.6, we begin the proof of Theorem 5.1.1 by proving the limit (5.1.7) for the uniform measure on the unit cube. In Section 5.7, we prove Corollary 5.1.4. In Section 5.8, we complete the proof of Theorem 5.1.1. In Section 5.9, we prove the limit (5.1.7) for uniform measures in the regime $p < d$. In Section 5.10, we prove Theorem 5.1.3.

Not all sections are necessary for the later arguments in this manuscript. The following scheme outlines the logical dependencies among Sections 5.5-5.10.



5.2 Heuristics

Firstly, let us formally derive Zador's Theorem. A similar heuristic argument is given in [Der09]. Fix a "nice" probability measure μ, say absolutely continuous, compactly supported, and with continuous density ρ. Let $S_n = \{x_1, \dots, x_n\} \subseteq \mathbb{R}^d$ be the support of an optimal classical quantizer (i.e., a minimizer in (5.1.2)) and let $\sigma_n \mathscr{L}^d$ be a "nice" approximation of the measure $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$. For n large, the number of points in S_n that fall within a small ball $B_\epsilon(\bar{x})$ of radius ϵ centered at $\bar{x} \in \mathbb{R}^d$ is, approximately and up to a dimensional constant, $\epsilon^d n \sigma_n(\bar{x})$. Since ρ is continuous and ϵ is small, we can expect the points of $S_n \cap B_\epsilon(\bar{x})$ to be evenly spread on $B_\epsilon(\bar{x})$; therefore, the distance r of a generic point in such a ball from S_n is roughly equal to the d^{th} root of the ratio between the volume of the ball and the cardinality $\#(S_n \cap B_\epsilon(\bar{x}))$, i.e.,[4] $r \approx \sqrt[d]{\frac{\epsilon^d}{\epsilon^d n \sigma_n(\bar{x})}} = \big(n\sigma_n(\bar{x})\big)^{-1/d}$. Hence,

$$\int \min_i \|x - x_i\|^p \, \mathrm{d}\mu \approx n^{-p/d} \int \sigma_n^{-p/d} \rho \, \mathrm{d}\mathscr{L}^d \,.$$

Thus, we can rephrase the problem in (5.1.2) as a minimization over functions:

$$e^p_{p,n}(\mu) \approx n^{-p/d} \inf_\sigma \int \sigma^{-p/d} \rho \, \mathrm{d}\mathscr{L}^d \,,$$

[4] In this work, the symbols $\approx, \lessapprox, \gtrapprox$ do not have a rigorous meaning. They are used in heuristic arguments as shorthands for 'is approximately equal to' and 'is approximately smaller/greater than'.

under the constraint $\int \sigma \, \mathrm{d}\mathscr{L}^d = 1$. By Hölder's inequality,

$$\int \rho^{\frac{d}{d+p}} \, \mathrm{d}\mathscr{L}^d \leq \left(\int \left(\rho^{\frac{d}{d+p}} \sigma^{-\frac{p}{d+p}} \right)^{\frac{d+p}{d}} \, \mathrm{d}\mathscr{L}^d \right)^{\frac{d}{d+p}} \left(\int \left(\sigma^{\frac{p}{d+p}} \right)^{\frac{d+p}{p}} \, \mathrm{d}\mathscr{L}^d \right)^{\frac{p}{d+p}}$$
$$= \left(\int \sigma^{-p/d} \rho \, \mathrm{d}\mathscr{L}^d \right)^{\frac{d}{d+p}} \underbrace{\left(\int \sigma \, \mathrm{d}\mathscr{L}^d \right)^{\frac{p}{d+p}}}_{=1},$$

and the inequality is an equality for $\sigma := c\, \rho^{\frac{d}{d+p}}$, where c is a normalizing constant.

In the case of optimal empirical quantization, we expect that the optimal locations $S_n = \{x_1, \dots, x_n\}$ are, instead, approximately distributed according to ρ: to keep the Wasserstein distance minimal, we should approximately match the mass in every small ball $B_\epsilon(\bar{x})$ to the points in (or closest to) such a ball, which means, in particular,

$$\epsilon^d \rho(\bar{x}) \approx \mu(B_\epsilon(\bar{x})) \approx n^{-1} \#(S_n \cap B_\epsilon(\bar{x})) \approx \epsilon^d \sigma_n(\bar{x}),$$

where, as before, σ_n is an approximation of the uniform measure on S_n.[5] Since, once again, the points $S_n \cap B_\epsilon(\bar{x})$ are evenly spread on $B_\epsilon(\bar{x})$, a generic point $x \in B_\epsilon(\bar{x})$ should be matched by an optimal transport plan to the *closest* $x_i \in S_n$. Recall that the typical distance from S_n is of order $\big(n\sigma_n(\bar{x})\big)^{-1/d}$, which, combined with the considerations above, yields

$$\tilde{e}^p_{p,n}(\mu) \approx n^{-p/d} \int \rho^{-p/d} \rho \, \mathrm{d}\mathscr{L}^d .$$

We conclude this section with another simple observation. *Postulate* that

$$\tilde{e}_{p,n}(\mu) \approx n^{-a} \left(\int \rho^b \, \mathrm{d}\mathscr{L}^d \right)^c$$

for some $a, b, c \in \mathbb{R}$ and for every (sufficiently "nice") probability measure $\mu = \rho \mathscr{L}^d$. It is easy to check that $\tilde{e}_{p,n}(\lambda^{-d} \rho\big(\lambda^{-1} \cdot\big) \mathscr{L}^d) = \lambda \tilde{e}_{p,n}(\mu)$ for every $\lambda > 0$. Therefore,

$$\lambda \left(\int \rho^b \, \mathrm{d}\mathscr{L}^d \right)^c = \left(\int_{\mathbb{R}^d} \big(\lambda^{-d} \rho(\lambda^{-1} \cdot) \big)^b \, \mathrm{d}\mathscr{L}^d \right)^c = \lambda^{dc(1-b)} \left(\int \rho^b \, \mathrm{d}\mathscr{L}^d \right)^c ,$$

from which we obtain the identity $1 = dc(1-b)$. Note that this is coherent with the statement of Theorem 5.1.1.

5.3 Preliminaries

5.3.1 Notation

We regard $\mathbb{R}^d$ as a measure space endowed with the σ-algebra of *Borel sets* $\mathscr{B}(\mathbb{R}^d)$, on which the *Lebesgue measure* $\mathscr{L}^d$ is defined, and as a normed space with the *Euclidean*

[5] This explains why the same formula (up to constant) appears in random empirical quantization, see [DSS13, Theorem 2].

norm $\|\cdot\| = \|\cdot\|_2$. For every $x \in \mathbb{R}^d$, we let δ_x be the Dirac delta measure at x. Given a Borel set $A \in \mathscr{B}(\mathbb{R}^d)$, we sometimes write $|A|$ in place of $\mathscr{L}^d(A)$. If $|A| \neq 0, \infty$, it is well defined the *uniform measure*

$$U_A \coloneqq \frac{\mathscr{L}^d|_A}{|A|}$$

of A. For convenience, we further define

$$U_d \coloneqq U_{[0,1]^d}\,.$$

For every set $A \subseteq \mathbb{R}^d$, we denote by $\operatorname{diam}(A)$ its diameter, i.e.,

$$\operatorname{diam}(A) \coloneqq \begin{cases} 0 & \text{if } A = \emptyset\,, \\ \sup_{x,y\in A}\|x-y\| & \text{otherwise,} \end{cases}$$

and by $\#A \in \mathbb{N}_0 \cup \{\infty\}$ its cardinality. We write $\operatorname{int}(A)$ and $\overline{A}$ for its interior part and topological closure, respectively.

For every pair of sets $A, B \subseteq \mathbb{R}^d$, we denote by $\operatorname{dist}(A, B)$ their minimal distance

$$\operatorname{dist}(A,B) \coloneqq \begin{cases} \inf\{\|x-y\| \,:\, x\in A\,,\ y\in B\} & \text{if } A, B \neq \emptyset\,, \\ \infty & \text{otherwise,} \end{cases}$$

and, similarly, we write $\operatorname{dist}(x, A) \coloneqq \operatorname{dist}(\{x\}, A)$.

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of Borel *probability measures* on $\mathbb{R}^d$ and by $\mathcal{M}(\mathbb{R}^d)$ the space of Borel *nonnegative finite measures* on $\mathbb{R}^d$, i.e., $\mathcal{M}(\mathbb{R}^d) \coloneqq \mathbb{R}_{\geq 0} \cdot \mathcal{P}(\mathbb{R}^d)$. For every $p \geq 1$, it is also convenient to introduce the space $\mathcal{P}_p(\mathbb{R}^d)$ of probability measures with *finite* p^{th} *moment*

$$\mathcal{P}_p(\mathbb{R}^d) \coloneqq \left\{\mu \in \mathcal{P}(\mathbb{R}^d) :\ \int \|x\|^p \,\mathrm{d}\mu(x) < \infty\right\}$$

and the space $\mathcal{P}_\mathrm{c}(\mathbb{R}^d)$ of *compactly supported probability measures*

$$\mathcal{P}_\mathrm{c}(\mathbb{R}^d) \coloneqq \left\{\mu \in \mathcal{P}(\mathbb{R}^d) :\ \exists K \subseteq \mathbb{R}^d \text{ compact such that } \mu(K) = 1\right\}.$$

For $n \in \mathbb{N}_1$, we further define the set

$$\mathcal{P}_{(n)}(\mathbb{R}^d) \coloneqq \left\{\mu_n \in \mathcal{P}(\mathbb{R}^d) \,:\, \exists x_1, x_2, \ldots, x_n \in \mathbb{R}^d\,,\ \mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right\}.$$

Analogously, we set

$$\mathcal{M}_p(\mathbb{R}^d) \coloneqq \mathbb{R}_{\geq 0} \cdot \mathcal{P}_p(\mathbb{R}^d)\,,\ \ \mathcal{M}_\mathrm{c}(\mathbb{R}^d) \coloneqq \mathbb{R}_{\geq 0} \cdot \mathcal{P}_\mathrm{c}(\mathbb{R}^d)\,,\ \ \mathcal{M}_{(n)}(\mathbb{R}^d) \coloneqq \mathbb{R}_{\geq 0} \cdot \mathcal{P}_{(n)}(\mathbb{R}^d)\,,$$

and $\mathcal{M}_{(0)}(\mathbb{R}^d) \coloneqq \{0\}$.

The (total variation) *norm of a measure* $\mu \in \mathcal{M}(\mathbb{R}^d)$ is $\|\mu\| \coloneqq \mu(\mathbb{R}^d)$. For every measurable function $T \colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$, we denote by $T_\# \colon \mathcal{M}(\mathbb{R}^{d_1}) \to \mathcal{M}(\mathbb{R}^{d_2})$ the *pushforward* operator, defined by

$$T_\#\mu(A) \coloneqq \mu\big(T^{-1}(A)\big)\,, \qquad A \in \mathscr{B}(\mathbb{R}^{d_2})\,.$$

Note that the norm is invariant under pushforward, i.e., $\left\|T_\#\mu\right\| = \|\mu\|$. For $\mu \in \mathcal{M}(\mathbb{R}^d)$, we write $\operatorname{supp}(\mu)$ for the support of μ, i.e., the smallest closed set on which μ is concentrated.

For ease of notation, when $\rho \in L^1_{\geq 0}(\mathbb{R}^d)$, we sometimes write ρ to denote the measure $\rho\mathscr{L}^d \in \mathcal{M}(\mathbb{R}^d)$.

We use the notation $a \lesssim b$ when there exists a constant $c > 0$ for which $a \leq cb$. Given two sequences $(a_n)_n$ and $(b_n)_n$ of positive real numbers (defined for an unbounded set of natural indices), we write $a_n \asymp b_n$ if

$$\frac{1}{c} \leq \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} \leq c$$

for some constant $c > 0$. Possible dependencies of the constant c are explicitly displayed as subscripts of the symbols $\lesssim$ and $\asymp$.

5.3.2 Wasserstein distance

Let $p \geq 1$, and take two measures $\mu, \nu \in \mathcal{M}_p(\mathbb{R}^d)$ such that $\|\mu\| = \|\nu\|$. We denote by $\Gamma(\mu, \nu)$ the set of couplings between μ and ν, i.e., the nonnegative Borel measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ that have μ and ν as marginals. The *Wasserstein distance* of order p between μ and ν is given by the formula

$$W_p(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \left(\int \|x - y\|^p \, \mathrm{d}\gamma(x, y) \right)^{1/p} . \tag{5.3.1}$$

The function W_p is really a distance on $\lambda\mathcal{P}_p(\mathbb{R}^d)$ for every $\lambda \geq 0$ (the case $\lambda = 0$ is trivial), as shown, for instance, in [San15, Proposition 5.1], and we have $W_p(\lambda\mu, \lambda\nu) = \lambda^{1/p} W_p(\mu, \nu)$ for every admissible choice of μ, ν, λ. Moreover, by a simple compactness argument (see [San15, Theorem 1.7]), the infimum in (5.3.1) is actually a minimum.

The following two nice features of W_p will be used in this work. The first one is a subadditivity property.

Lemma 5.3.1 (Subadditivity). *Let $\mu^1, \mu^2, \nu^1, \nu^2 \in \mathcal{M}_p(\mathbb{R}^d)$ be such that $\left\|\mu^1\right\| = \left\|\nu^1\right\|$ and $\left\|\mu^2\right\| = \left\|\nu^2\right\|$. Then we have*

$$W_p^p\left(\mu^1 + \mu^2, \nu^1 + \nu^2\right) \leq W_p^p(\mu^1, \nu^1) + W_p^p(\mu^2, \nu^2) . \tag{5.3.2}$$

Proof. This result follows from the implication

$$\gamma^i \in \Gamma(\mu^i, \nu^i) , \; i \in \{1, 2\} \quad \Longrightarrow \quad \gamma^1 + \gamma^2 \in \Gamma\left(\mu^1 + \mu^2, \nu^1 + \nu^2\right)$$

and the linearity of

$$\gamma \mapsto \int \|x - y\|^p \, \mathrm{d}\gamma(x, y) .$$

□

The second one is: on a fixed compact set, the Wasserstein distance of two a.c. measures can be controlled by the L^1-distance of their densities.

Lemma 5.3.2 (Comparison with $\|\cdot\|_{L^1}$). *Let $\mu = \rho\mathscr{L}^d, \nu = \sigma\mathscr{L}^d$ be compactly supported and absolutely continuous measures, with $\|\mu\| = \|\nu\|$. Let $A \subseteq \mathbb{R}^d$ be a bounded set on which both μ and ν are concentrated. Then:*

$$W_p(\mu, \nu) \leq \operatorname{diam}(A) \|\rho - \sigma\|_{L^1}^{1/p} . \tag{5.3.3}$$

Proof. We can and will assume that $\mu \neq \nu$. Set

$$\mu^1 = \nu^1 := \min(\rho, \sigma)\mathscr{L}^d\,, \qquad \mu^2 = \max(\rho - \sigma, 0)\mathscr{L}^d\,, \qquad \nu^2 = \max(\sigma - \rho, 0)\mathscr{L}^d\,,$$

and notice that neither μ^2 nor ν^2 is equal to the zero measure. The hypotheses of Lemma 5.3.1 are satisfied. Hence,

$$W_p^p(\mu,\nu) \le \underbrace{W_p^p(\mu^1,\nu^1)}_{=0} + W_p^p(\mu^2,\nu^2)\,.$$

Therefore, it suffices to find a suitable coupling between μ^2 and ν^2. We choose

$$\gamma := \frac{\mu^2 \otimes \nu^2}{\|\mu^2\|} = \frac{\mu^2 \otimes \nu^2}{\|\nu^2\|}\,,$$

which yields

$$\begin{aligned}\int \|x-y\|^p \,\mathrm{d}\gamma(x,y) &\le \int \|x-y\|^p \,\frac{\mu^2 \otimes \nu^2}{\|\nu^2\|}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \operatorname{diam}(A)^p \frac{\left\|\mu^2\right\|\left\|\nu^2\right\|}{\|\nu^2\|} = \operatorname{diam}(A)^p \left\|\mu^2\right\|\,.\end{aligned}$$

We conclude by the inequality $\left\|\mu^2\right\| \le \|\rho - \sigma\|_{L^1}$. □

5.3.3 Boundary Wasserstein pseudodistance

A. Figalli and N. Gigli introduced in [FG10] a modified Wasserstein distance Wb for measures defined on a bounded Euclidean domain, giving a special role to the boundary of such a domain: it can be interpreted as an infinite reservoir, where mass can be deposited and taken freely. We give here a slightly modified definition of a *pseudodistance* between measures *defined on the whole* $\mathbb{R}^d$.

Let $p \ge 1$ and fix an open bounded nonempty set $\Omega \subseteq \mathbb{R}^d$. Take two measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$, possibly having different total mass. Let $\Gamma b_\Omega(\mu,\nu)$ be the set of the nonnegative Borel measures γ on the closure $\overline{\Omega} \times \overline{\Omega}$ such that $\gamma|_{\Omega \times \overline{\Omega}}$ has $\mu|_\Omega$ as first marginal, and $\gamma|_{\overline{\Omega} \times \Omega}$ has $\nu|_\Omega$ as second marginal.

Definition 5.3.3. The *boundary Wasserstein pseudodistance* of order p for Ω between μ and ν is given by the formula

$$Wb_{\Omega,p}(\mu,\nu) := \inf_{\gamma \in \Gamma b_\Omega(\mu,\nu)} \left(\int \|x-y\|^p \,\mathrm{d}\gamma(x,y) \right)^{1/p}\,. \tag{5.3.4}$$

It is easy to check that $Wb_{\Omega,p}(\mu,\nu)$ is nonnegative and finite for every μ,ν, that the symmetry property $Wb_{\Omega,p}(\mu,\nu) = Wb_{\Omega,p}(\nu,\mu)$ holds, and that $Wb_{\Omega,p}(\mu,\mu) = 0$. The triangle inequality can be proven as in [FG10, Theorem 2.2] (or directly deduced from this theorem). Clearly, with our definition, $Wb_{\Omega,p}$ cannot be a true distance, as it does not distinguish measures that differ out of Ω: for every $\mu,\nu \in \mathcal{M}(\mathbb{R}^d)$, we have $Wb_{\Omega,p}(\mu,\nu) = Wb_{\Omega,p}(\mu|_\Omega, \nu|_\Omega)$. As with W_p, we have the identity $Wb_{\Omega,p}(\lambda\mu,\lambda\nu) = \lambda^{1/p} W_{\Omega,p}(\mu,\nu)$ for $\lambda \ge 0$. Further notice that $Wb_{\Omega,p}(\mu,\nu) \le W_p(\mu,\nu)$ when $\mu,\nu \in \mathcal{M}_p(\mathbb{R}^d)$ and $\|\mu\| = \|\nu\|$, for any Ω.

A crucial property of $Wb_{\Omega,p}$ is its geometric superadditivity, which will be used in the proof of the lower bound (L).

Lemma 5.3.4 (Superadditivity). *If $\{\Omega_i\}_i$ is a (finite or countably infinite) family of open, bounded, nonempty, and pairwise* disjoint *subsets of Ω, then*

$$Wb^p_{\Omega,p}(\mu,\nu) \geq \sum_i Wb^p_{\Omega_i,p}(\mu,\nu)\,, \qquad \mu,\nu \in \mathcal{M}(\mathbb{R}^d)\,. \tag{5.3.5}$$

Proof. The proof of this lemma can be found in [AGT22, Section 2.2]. □

5.3.4 Quantization errors and coefficients

Definition 5.3.5. The n^{th} *optimal quantization error* of order p is

$$e_{p,n}(\mu) \coloneqq \inf\left\{W_p(\mu,\mu_n) \,:\, \#\operatorname{supp}(\mu_n) \leq n \text{ and } \|\mu\| = \|\mu_n\|\right\}\,, \qquad \mu \in \mathcal{M}_p(\mathbb{R}^d)\,, \tag{5.3.6}$$

and the *optimal quantization coefficient* of order p is

$$q_{p,d} \coloneqq \inf_{n\in\mathbb{N}_1} n^{1/d} e_{p,n}(U_d)\,. \tag{5.3.7}$$

Definition 5.3.6. The n^{th} *optimal empirical quantization error* of order p is

$$\tilde{e}_{p,n}(\mu) \coloneqq \inf\left\{W_p(\mu,\mu_n) \,:\, \mu_n \in \mathcal{M}_{(n)}(\mathbb{R}^d)\,,\ \|\mu_n\| = \|\mu\|\right\}\,, \qquad \mu \in \mathcal{M}_p(\mathbb{R}^d)\,, \tag{5.3.8}$$

and the *optimal empirical quantization coefficient* of order p is

$$\tilde{q}_{p,d} \coloneqq \inf_{n\in\mathbb{N}_1} n^{1/d} \tilde{e}_{p,n}(U_d)\,. \tag{5.3.9}$$

We leave $e_{p,0}(\mu)$ and $\tilde{e}_{p,0}(\mu)$ undefined when $\mu \neq 0$.

In words, the optimal quantization error measures the minimal distance to atomic measures supported on at most n points (with the same total mass); the optimal empirical quantization error measures the minimal distance to (appropriately rescaled) sums of n Dirac deltas.

Remark 5.3.7. For every $\mu \in \mathcal{M}_p(\mathbb{R}^d)$, the following inequality holds:

$$e_{p,n}(\mu) \leq \tilde{e}_{p,n}(\mu)\,. \tag{5.3.10}$$

Both errors are $\frac{1}{p}$-homogeneous and $e_{p,n}(0) = \tilde{e}_{p,n}(0) = 0$ for every n, including $n = 0$. Moreover, if $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is an affine transformation of the form $T(x) = v + \lambda x$, with $v \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ then

$$e_{p,n}(T_\#\mu) = |\lambda|\, e_{p,n}(\mu)\,, \quad \tilde{e}_{p,n}(T_\#\mu) = |\lambda|\, \tilde{e}_{p,n}(\mu)\,. \tag{5.3.11}$$

Remark 5.3.8. From (5.3.10), we deduce also $q_{p,d} \leq \tilde{q}_{p,d}$. Moreover, the quantization coefficients are strictly positive, see Theorem 5.4.1.

Remark 5.3.9. Let $\mu^1, \mu^2 \in \mathcal{M}_p(\mathbb{R}^d)$ and $n_1, n_2 \in \mathbb{N}_0$ be such that

$$\left[n_i = 0 \Rightarrow \mu^i = 0\right]\,, \qquad i \in \{1,2\}\,.$$

Then it follows from Lemma 5.3.1 that

$$e^p_{p,n_1+n_2}(\mu^1+\mu^2) \leq e^p_{p,n_1}(\mu^1) + e^p_{p,n_2}(\mu^2)\,. \tag{5.3.12}$$

If, moreover, $\left\|\mu^1\right\| n_2 = \left\|\mu^2\right\| n_1$, then

$$\tilde{e}^p_{p,n_1+n_2}(\mu^1+\mu^2) \leq \tilde{e}^p_{p,n_1}(\mu^1) + \tilde{e}^p_{p,n_2}(\mu^2)\,. \tag{5.3.13}$$

The infima in (5.3.6) and (5.3.8) are, in fact, minima. For $e_{p,n}(\mu)$, the proof can be found in [GL00, Theorem 4.12] (which, in turn, follows the lines of [Pol82, Lemma 8]) or in [Qua21, Appendix A.4]. Let us prove the existence of the minimum in (5.3.8).

Lemma 5.3.10. *Let $\mu \in \mathcal{M}_p(\mathbb{R}^d)$. For every $n \in \mathbb{N}_1$ there exists a measure $\mu_n \in \mathcal{M}_{(n)}(\mathbb{R}^d)$ with $\|\mu_n\| = \|\mu\|$ and such that $\tilde{e}_{p,n}(\mu) = W_p(\mu, \mu_n)$.*

Proof. If $\mu = 0$, then $\mu_n := 0 \in \mathcal{M}_{(n)}(\mathbb{R}^d)$ is the sought measure. Otherwise, we may renormalize and assume that $\|\mu\| = 1$, we have to prove that the function

$$\psi \colon \mathbb{R}^{nd} \ni (x_1, \dots, x_n) \longmapsto W_p\left(\mu, \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right)$$

admits a minimizer. This function is continuous: by the triangle inequality and Lemma 5.3.1,

$$\begin{aligned}|\psi(x_1, \dots, x_n) - \psi(y_1, \dots, y_n)|^p &\le W_p^p\left(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}, \frac{1}{n}\sum_{i=1}^n \delta_{y_i}\right)\\ &\le \frac{1}{n}\sum_{i=1}^n W_p^p(\delta_{x_i}, \delta_{y_i}) = \frac{1}{n}\sum_{i=1}^n \|x_i - y_i\|^p\end{aligned}$$

for every $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^d$. Again by the triangle inequality,

$$\psi(x_1, \dots, x_n) \ge W_p\left(\delta_0, \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right) - W_p(\mu, \delta_0) = \frac{1}{n}\sum_{i=1}^n \|x_i\|^p - \underbrace{\int \|x\|^p \,\mathrm{d}\mu}_{<\infty},$$

which implies that the sublevels of ψ are bounded. We conclude by applying the extreme value theorem on a sufficiently large compact set. □

Let us show that the sequence $\left(\tilde{e}_{p,n}(\mu)\right)_n$ is infinitesimal as $n \to \infty$ *for every* μ with finite p^{th} moment. In dimension $d = 1$, this was established in [XB19, Corollary 5.12]. The analogous result for $e_{p,n}(\mu)$ follows as a corollary, but was also proven, e.g., in [GL00, Lemma 6.1].

Proposition 5.3.11. *For every $\mu \in \mathcal{M}_p(\mathbb{R}^d)$, we have*

$$\lim_{n \to \infty} \tilde{e}_{p,n}(\mu) = 0\,. \tag{5.3.14}$$

Proof. We may and will assume that μ is a probability measure, not concentrated on a single point. Set

$$M := \int \|x\|^p \,\mathrm{d}\mu\,,$$

and fix $r > 0$ large enough for the ball $B_r := \{x : \|x\| < r\}$ to have nonzero μ-measure.

If μ is concentrated on B_r, then the conclusion follows from Theorem 5.4.3. Otherwise, let us define

$$R := \left(\frac{2M}{1 - \mu(B_r)}\right)^{1/p}$$

and notice that, by Markov's inequality,

$$\mu\{x : \|x\| \ge R\} \le \frac{M}{R^p} = \frac{1 - \mu(B_r)}{2}\,. \tag{5.3.15}$$

For every natural number $n > \frac{2}{1-\mu(B_r)}$, define $n_1 := \lceil n\mu(B_r) \rceil$ and $n_2 := n - n_1$. Since the μ-measure of the ball $B_R := \{x : \|x\| < R\}$ can be estimated with

$$\mu(B_R) \overset{(5.3.15)}{\geq} 1 - \frac{1-\mu(B_r)}{2} \geq \mu(B_r) + \frac{1-\mu(B_r)}{2} > \mu(B_r) + \frac{1}{n},$$

there exists a measure μ^1 (dependent on n) such that $\mu|_{B_r} \leq \mu^1 \leq \mu|_{B_R}$ and $\|\mu^1\| = n_1/n$. Let $\mu^2 := \mu - \mu^1$. By Remark 5.3.9,

$$\tilde{e}^p_{p,n}(\mu) \leq \tilde{e}^p_{p,n_1}(\mu^1) + \tilde{e}^p_{p,n_2}(\mu^2).$$

By Theorem 5.4.3, there exists an infinitesimal function $f_{p,d}$ such that

$$\tilde{e}^p_{p,n_1}(\mu^1) \leq \frac{n_1}{n} R^p f_{p,d}(n_1),$$

and, since μ^2 is concentrated on $\mathbb{R}^d \setminus B_r$,

$$\tilde{e}^p_{p,n_2}(\mu^2) \leq W^p_p\left(\mu^2, \|\mu^2\| \delta_0\right) = \int \|x\|^p \, \mathrm{d}\mu^2 \leq \int_{\mathbb{R}^d \setminus B_r} \|x\|^p \, \mathrm{d}\mu.$$

Note that

$$\limsup_{n\to\infty} \frac{n_1}{n} R^p f_{p,d}(n_1) = \mu(B_r) R^p \limsup_{n_1\to\infty} f_{p,d}(n_1) = 0;$$

therefore, we infer that

$$\limsup_{n\to\infty} \tilde{e}^p_{p,n}(\mu) \leq \int_{\mathbb{R}^d \setminus B_r} \|x\|^p \, \mathrm{d}\mu,$$

and we conclude by arbitrariness of r. □

Remark 5.3.12. The minimizers of (5.3.6) and (5.3.8) are *not*, in general, unique. For example, let μ be invariant under orthogonal transformations and not concentrated at the origin. If n is large enough, by Proposition 5.3.11, no minimizer can be concentrated at the origin; hence, infinitely many orthogonal transformations map any minimizer to *other* minimizers (via pushforward).

Let us conclude this section with a lemma that relates the classical quantization error and the boundary Wasserstein pseudodistance. This result will be used in the proof of the lower bound (L).

Lemma 5.3.13. *Let Ω be an open bounded nonempty subset of $\mathbb{R}^d$. Choose $\epsilon > 0$ and define the "tightened" open set*

$$\Omega^-_\epsilon := \left\{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega) > \epsilon\right\}. \tag{5.3.16}$$

Fix $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then, for every $n \in \mathbb{N}_0$ and $\mu_n \in \mathcal{M}(\mathbb{R}^d)$ with $\#\operatorname{supp}(\mu_n|_\Omega) \leq n$, we have

$$Wb_{\Omega,p}(\mu, \mu_n) \geq e_{p,n+N}(\mu|_{\Omega^-_\epsilon}), \quad \textit{where} \quad N := \left\lceil \frac{\sqrt{d}\,\operatorname{diam}(\Omega)}{\epsilon} \right\rceil^d. \tag{5.3.17}$$

Proof. By considering the vertices of a suitable regular grid, it is easy to check that there exist a set $\mathcal{Y} \subseteq \mathbb{R}^d$ with $\#\mathcal{Y} \le N$ and a Borel function $T \colon \Omega_\epsilon^- \to \mathcal{Y}$ such that $\|x - T(x)\| \le \epsilon$ for every $x \in \Omega_\epsilon^-$. Given μ, μ_n as in the statement, let γ be a nonnegative Borel measure on $\overline{\Omega} \times \overline{\Omega}$ such that $\gamma|_{\Omega \times \overline{\Omega}}$ has $\mu|_\Omega$ as first marginal, and $\gamma|_{\overline{\Omega} \times \Omega}$ has $\mu_n|_\Omega$ as second marginal. Let $\pi^1 \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the projection onto the first d coordinates, and define

$$\gamma' \coloneqq \gamma|_{\Omega_\epsilon^- \times \Omega} + (\pi^1, T \circ \pi^1)_\#(\gamma|_{\Omega_\epsilon^- \times \partial\Omega}) \,.$$

Let ν be the second marginal of γ'. Notice that $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(\mu_n|_\Omega) \cup \mathcal{Y}$, which implies $\#\operatorname{supp}(\nu) \le n + N$. Moreover, since the norm is invariant under pushforward,

$$\|\nu\| = \left\|\gamma|_{\Omega_\epsilon^- \times \Omega}\right\| + \left\|\gamma|_{\Omega_\epsilon^- \times \partial\Omega}\right\| = \left\|\gamma|_{\Omega_\epsilon^- \times \overline{\Omega}}\right\| = \left\|\mu|_{\Omega_\epsilon^-}\right\| \,.$$

Consequently, $e_{p,n+N}(\mu|_{\Omega_\epsilon^-}) \le W_p(\mu|_{\Omega_\epsilon^-}, \nu)$. By noticing that $\gamma' \in \Gamma(\mu|_{\Omega_\epsilon^-}, \nu)$, we deduce that

$$e_{p,n+N}^p(\mu|_{\Omega_\epsilon^-}) \le \int \|x - y\|^p \, \mathrm{d}\gamma|_{\Omega_\epsilon^- \times \Omega} + \int \|x - T(x)\|^p \, \mathrm{d}\gamma|_{\Omega_\epsilon^- \times \partial\Omega} \,.$$

Moreover, by definition of T and Ω_ϵ^-,

$$\int \|x - T(x)\|^p \, \mathrm{d}\gamma|_{\Omega_\epsilon^- \times \partial\Omega} \le \int \epsilon^p \, \mathrm{d}\gamma|_{\Omega_\epsilon^- \times \partial\Omega} \le \int \|x - y\|^p \, \mathrm{d}\gamma|_{\Omega_\epsilon \times \partial\Omega} \,;$$

therefore,

$$e_{p,n+N}^p(\mu|_{\Omega_\epsilon^-}) \le \int \|x - y\|^p \, \mathrm{d}\gamma|_{\Omega_\epsilon^- \times \overline{\Omega}} \le \int \|x - y\|^p \, \mathrm{d}\gamma \,.$$

We conclude by arbitrariness of γ. □

5.4 Previous results

There is a rich literature studying asymptotics for classical quantization, see, e.g., [GL00, LP23b]. The following is a fundamental result by P. Zador [Zad64, Zad82].

Theorem 5.4.1 (Zador's Theorem, [GL00, Theorem 6.2])**.** *Let $\mu \in \mathcal{P}_\theta(\mathbb{R}^d)$ for some $\theta > p \ge 1$ and let ρ be the density of the absolutely continuous part of μ. Then:*

$$\lim_{n\to\infty} n^{1/d} e_{p,n}(\mu) = q_{p,d} \left(\int \rho^{\frac{d}{d+p}} \, \mathrm{d}\mathscr{L}^d \right)^{\frac{d+p}{dp}} \,, \tag{5.4.1}$$

and the optimal quantization coefficient $q_{p,d}$ is strictly positive.

This theorem establishes the exact asymptotic of $e_{p,n}(\mu)$ as $n \to 0$ for every μ which is not purely singular, under a moment condition (which is not dispensable, see [GL00, Example 6.4]).

Less is known about the rate of convergence of $\tilde{e}_{p,n}(\mu)$. The case $d = 1$ has been studied in [JR16, XB19, GHMR19b, BJ22a, BJ22b, GSM23]. In particular, the following theorem determines the exact convergence rate under a suitable assumption.

Theorem 5.4.2 (C. Xu and A. Berger, [XB19, Theorem 5.15])**.** *Let $\mu \in \mathcal{P}_p(\mathbb{R})$. If the (upper) quantile function*

$$F_\mu^{-1}(t) := \sup\left\{x \in \mathbb{R} \,:\, \mu\big((-\infty, x]\big) \le t\right\}, \qquad t \in (0,1) \tag{5.4.2}$$

is absolutely continuous, then

$$\lim_{n\to\infty} n\tilde{e}_{p,n}(\mu) = q_{p,1} \left\| \frac{\mathrm{d}F_\mu^{-1}}{\mathrm{d}t} \right\|_{L^p} . \tag{5.4.3}$$

For general measures in arbitrary dimension, we have the following theorem, independently proven in [MM16] (only for $p = 2$) and [Che18].

Theorem 5.4.3 (Q. Mérigot and J.-M. Mirebeau, [MM16, Proposition 12]; J. Chevallier, [Che18, Theorem 3])**.** *If $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ is supported in $[-r, r]^d$ for some $r > 0$, then:*

$$\tilde{e}_{p,n}(\mu) \lesssim_{p,d} r \cdot \begin{cases} n^{-1/d} & \text{if } p < d\,, \\ (1 + \log n)^{1/d} n^{-1/d} & \text{if } p = d\,, \\ n^{-1/p} & \text{if } p > d\,, \end{cases} \qquad n \in \mathbb{N}_1 . \tag{5.4.4}$$

Combined with Theorem 5.4.1 and Remark 5.3.7, this theorem determines the speed of convergence $\tilde{e}_{p,n}(\mu) \asymp_{p,d,\mu} n^{-1/d}$ in the regime $p < d$ for every μ which is compactly supported and not purely singular:

$$q_{p,d} \left(\int \rho^{\frac{d}{d+p}} \, \mathrm{d}\mathscr{L}^d \right)^{\frac{d+p}{dp}} \le \liminf_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) \le \limsup_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) \lesssim_{p,d} r\,, \tag{5.4.5}$$

where ρ is as in Theorem 5.4.1 and r is as in Theorem 5.4.3.

We note that also for $p > d$ the upper bound (5.4.4) is tight, in the sense that there exist compactly supported measures—even absolutely continuous and with smooth densities—for which

$$\limsup_{n\to\infty} n^{1/p} \tilde{e}_{p,n}(\mu) > 0\,, \tag{5.4.6}$$

as demonstrated by the following example; see also the 1-dimensional case in [XB19, Remark 5.22].

Example 5.4.4. Assume that $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ is concentrated on the union of two distant sets $A, B \subseteq \mathbb{R}^d$ with $\mu(A) > 0$ and $\mu(B) > 0$. Then (5.4.6) holds.

Proof. Le us write $r := \mathrm{dist}(A, B)$. Define

$$\tilde{A} := \left\{x \in \mathbb{R}^d \,:\, \mathrm{dist}(x, A) \le \mathrm{dist}(x, B)\right\}, \qquad \tilde{B} := \mathbb{R}^d \setminus \tilde{A} .$$

Given $n \in \mathbb{N}_1$, take any $\mu_n \in \mathcal{P}_{(n)}(\mathbb{R}^d)$ and $\gamma \in \Gamma(\mu, \mu_n)$. We have

$$\begin{aligned} \int \|x - y\|^p \, \mathrm{d}\gamma &\ge \left(\frac{r}{2}\right)^p \Big(\gamma(A \times \tilde{B}) + \gamma(B \times \tilde{A})\Big) \ge \left(\frac{r}{2}\right)^p \left|\gamma(A \times \tilde{B}) - \gamma(B \times \tilde{A})\right| \\ &= \left(\frac{r}{2}\right)^p \left|\mu_n(\tilde{B}) - \gamma(B \times \tilde{B}) - \gamma(B \times \tilde{A})\right| = \left(\frac{r}{2}\right)^p \left|\mu_n(\tilde{B}) - \mu(B)\right| . \end{aligned}$$

Let us denote by $\tau(n)$ the fractional part of $n\mu(B)$. Since $n\mu_n(\tilde{B}) \in \mathbb{N}_0$, we get

$$\int \|x-y\|^p \, d\gamma \geq \frac{1}{n}\left(\frac{r}{2}\right)^p \min\bigl(\tau(n), 1-\tau(n)\bigr)\,,$$

and, by arbitrariness of γ and μ_n, we find

$$n^{1/p}\tilde{e}_{p,n}(\mu) \geq \frac{r}{2}\min\bigl(\tau(n), 1-\tau(n)\bigr)^{1/p}\,, \qquad n \in \mathbb{N}_1\,.$$

To conclude (5.4.6), it suffices to prove that, for infinitely many numbers $n \in \mathbb{N}_1$, we have $\tau(n) \in [1/3, 2/3]$. Firstly, we note that

$$\tau(n) = 0 \quad \Rightarrow \quad \tau(n+1) = \mu(B) \in (0,1)\,;$$

hence $\tau(n) \in (0,1)$ frequently. Finally, it is easy to check that

$$\tau(n) \in (0,1) \setminus [1/3, 2/3] \quad \Rightarrow \quad \tau\left(\left\lceil \frac{1}{3\min\bigl(\tau(n), 1-\tau(n)\bigr)} \right\rceil n\right) \in [1/3, 2/3]\,. \qquad \square$$

For $p = d$, it is still unknown whether the logarithmic term in (5.4.4) is necessary (for compactly supported measures), see [Che18, Remark 1].

In addition to Theorem 5.4.3, we mention the results in [GHMR19b, GHMR19a], applicable to certain measures in infinite-dimensional Banach spaces, [BS20] for the volume measure on a compact manifold, and the upper bounds that can be deduced from the theory of random empirical quantization [Led23] using the trivial inequality

$$\tilde{e}_{p,n}(\mu) \leq \mathbb{E}\left[W_p\left(\mu, \frac{1}{n}\sum_{i=1}^n \delta_{X_i}\right)\right]\,,$$

valid for every family of random variables $\{X_1, \dots, X_n\}$. In particular, the following theorems already provide an upper estimate of the form (U) and a nonasymptotic upper bound like Theorem 5.1.7 *in the regime* $p < d/2$.

Theorem 5.4.5 (S. Dereich, M. Scheutzow, and R. Schottstedt, [DSS13, Theorem 2]). *Under the assumptions of Theorem 5.1.1, further suppose that $p < d/2$, and that ρ is Riemann integrable or $p = 1$. If $X_1, X_2, \dots$ is a sequence of μ-distributed i.i.d. random variables, then*

$$\mathbb{E}\left[W_p^p\left(\mu, \frac{1}{n}\sum_{i=1}^n \delta_{X_i}\right)\right]^{1/p} \asymp_{p,d} n^{-1/d}\left(\int_{\mathbb{R}^d} \rho^{\frac{d-p}{d}} \, d\mathscr{L}^d\right)^{1/p} \quad \text{as } n \to \infty\,. \tag{5.4.7}$$

Theorem 5.4.6 (S. Dereich, M. Scheutzow, and R. Schottstedt, [DSS13, Theorem 1]). *Under the assumptions of Theorem 5.1.1, further suppose that $p < d/2$. If $X_1, X_2, \dots$ is a sequence of μ-distributed i.i.d. random variables, then*

$$\mathbb{E}\left[W_p^p\left(\mu, \frac{1}{n}\sum_{i=1}^n \delta_{X_i}\right)\right]^{1/p} \lesssim_{p,d,\theta} n^{-1/d}\left(\int \|x\|^\theta \, d\mu\right)^{1/\theta}\,, \qquad n \in \mathbb{N}_1\,. \tag{5.4.8}$$

In a recent preprint, E. Caglioti, M. Goldman, F. Pieroni, and D. Trevisan [CGPT24] extended [DSS13, Theorem 2] to $p \le d$ (with some modifications when $p = d$) for μ in a certain class of radially symmetric and rapidly decaying probability laws, including, e.g., the normal distribution.

As noted in the introduction, it remains unknown whether any of the hidden constants appearing in (5.4.7) and in [CGPT24, Theorem 1.6] coincides with $q_{p,d}$ or $\tilde{q}_{p,d}$.

With our Theorem 5.1.1 and Theorem 5.1.7, we obtain several improvements over what was previously known:

- We establish the speed of convergence $\tilde{e}_{p,n}(\mu) \asymp_{p,d,\mu} n^{-1/d}$ for general (not purely singular) measures in the whole range $p \in [1, d)$, under a moment condition, but without assuming compactness of the support or Riemann integrability of the density.
- We prove a nonasymptotic upper bound also for $d/2 \le p < d$ without assuming compactness of the support.
- We find the *same explicit* dependence on the measure in the asymptotic upper and lower bounds (L) and (U) (assuming $\mu^a(\operatorname{supp}(\mu^s)) = 0$, where μ^a and μ^s are the absolutely continuous and singular parts of μ, respectively).
- We establish the asymptotic upper bound with the constant $\tilde{q}_{p,d}$, which is optimal, since (U) is an equality for $\mu = U_d$.

Furthermore, we determine the existence of the limit in some instances (Section 5.1.2) and we find the speed of convergence $\tilde{e}_{p,n}(\mu) \asymp_{p,d,\mu} n^{-1/d}$ for every $p \ge 1$ for a certain class of measures (Corollary 5.1.4).

5.5 Nonasymptotic upper bound (Theorem 5.1.7)

The proof of Theorem 5.1.7 is similar to those of its counterpart for compactly supported measures in [MM16, Che18] (Theorem 5.4.3). Iteratively n times, we extract from the given measure μ a sufficiently concentrated subprobability with mass equal to $1/n$. In [MM16, Che18], where μ is compactly supported, the subprobabilities are found by splitting the support into a finite number of pieces (of small, comparable size) and applying a pigeonhole-like principle: the measure of at least one of these pieces is sufficiently large. Since our measure is not compactly supported, we use the moment condition to first identify, at each iteration, a compact region (small relative to the moment of μ) where enough mass is concentrated; we then proceed as before.

Proof of Theorem 5.1.7. Fix $n \in \mathbb{N}_1$, let $M := \int \|x\|^\theta \, \mathrm{d}\mu$, and define

$$r_k := \left(\frac{2nM}{n-k}\right)^{\frac{1}{\theta}}, \qquad k \in \{1, 2, \dots, n-1\} .$$

With this choice, for every k we have

$$\begin{aligned} \mu\Big([-r_k, r_k]^d\Big) &= 1 - \int_{\mathbb{R}^d \setminus [-r_k, r_k]^d} \mathrm{d}\mu \ge 1 - r_k^{-\theta} \int_{\mathbb{R}^d \setminus [-r_k, r_k]^d} \|x\|^\theta \, \mathrm{d}\mu \\ &\ge 1 - \frac{n-k}{2nM} M = \frac{1}{2} + \frac{k}{2n} . \end{aligned} \tag{5.5.1}$$

Using [Che18, Lemma 1], we now argue as in the proof of [Che18, Theorem 2]. By applying this lemma to the measure $\nu_1 := \mu|_{[-r_1,r_1]^d}$, we find a measure $\eta_1 \le \nu_1$ with total mass $\|\eta_1\| = 1/n$ and supported on a set with diameter bounded by $4\sqrt{d}\, r_1(n\|\nu_1\|)^{-1/d}$. We repeat this procedure with $\nu_2 := \mu|_{[-r_2,r_2]^d} - \eta_1$ to find η_2, then with $\nu_3 := \mu|_{[-r_3,r_3]^d} - \eta_1 - \eta_2$, and so on. At the end, we have a family of measures $\eta_1, \eta_2, \dots, \eta_{n-1}$ with

$$\eta_1 + \eta_2 + \cdots + \eta_{n-1} \le \mu \quad \text{and} \quad \|\eta_1\| = \|\eta_2\| = \cdots = \|\eta_{n-1}\| = \frac{1}{n},$$

and

$$\begin{aligned} \operatorname{diam}(\operatorname{supp}(\eta_k)) \lesssim_d r_k (n\|\nu_k\|)^{-1/d} &= r_k \Big(n\mu\big([-r_k, r_k]^d\big) - (k-1) \Big)^{-1/d} \\ &\overset{(5.5.1)}{\le} r_k \left(\frac{n-k}{2} \right)^{-1/d} \end{aligned} \tag{5.5.2}$$

for every $k \in \{1, 2, \dots, n-1\}$.

Let us pick a point x_k from each $\operatorname{supp}(\eta_k)$. After defining $\eta_n := \mu - \eta_1 - \cdots \eta_{n-1}$, Lemma 5.3.1 gives

$$\begin{aligned} \tilde{e}^p_{p,n}(\mu) &\le W_p^p\Big(\eta_1 + \cdots + \eta_{n-1} + \eta_n, n^{-1}\big(\delta_{x_1} + \cdots + \delta_{x_{n-1}} + \delta_0\big)\Big) \\ &\le W_p^p(\eta_n, n^{-1}\delta_0) + \sum_{k=1}^{n-1} W_p^p(\eta_k, n^{-1}\delta_{x_k}) \\ &\le \int \|x\|^p \,\mathrm{d}\eta_n + \sum_{k=1}^{n-1} \frac{\big(\operatorname{diam}(\operatorname{supp}(\eta_k))\big)^p}{n}. \end{aligned} \tag{5.5.3}$$

Hölder's inequality and the fact that $\theta > p^*$ yield

$$\int \|x\|^p \,\mathrm{d}\eta_n \le M^{p/\theta} \|\eta_n\|^{1-\frac{p}{\theta}} \le M^{p/\theta} n^{\frac{p}{\theta}-1} \le M^{p/\theta} n^{-p/d}. \tag{5.5.4}$$

Moreover,

$$\sum_{k=1}^{n-1} \frac{\big(\operatorname{diam}(\operatorname{supp}(\eta_k))\big)^p}{n} \overset{(5.5.2)}{\lesssim_{p,d,\theta}} M^{p/\theta} n^{\frac{p}{\theta}-1} \sum_{k=1}^{n-1} (n-k)^{-\frac{p}{\theta}-\frac{p}{d}} \lesssim_{p,d,\theta} M^{p/\theta} n^{-p/d}, \tag{5.5.5}$$

where, in the last inequality, we used that $\frac{p}{\theta} + \frac{p}{d} < 1$. We conclude by combining (5.5.3), (5.5.4), and (5.5.5). □

5.6 Uniform measure on a cube

In this section, we establish the limiting behavior of the optimal empirical quantization error for the uniform measure U_d on $[0,1]^d$.

Proposition 5.6.1. *For every $p \ge 1$, we have the identity*

$$\lim_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(U_d) = \inf_{n \in \mathbb{N}_1} n^{1/d} \tilde{e}_{p,n}(U_d) =: \tilde{q}_{p,d}. \tag{5.6.1}$$

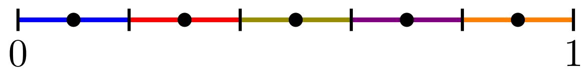


Figure 5.1: The points on which the optimal measure μ_n for U_1 is concentrated are evenly separated on $[0,1]$.

Note that this proposition applies also when $p \geq d$.

Proposition 5.6.1 is easy to prove in dimension $d = 1$, see Remark 5.6.3 below. Moreover, exploiting the self-similarity of the cube, we can build a simple "scale-and-copy" argument (Lemma 5.6.4, inspired by [GL00, Step 1 in Theorem 6.2], see also Figure 5.2) to write

$$\tilde{q}_{p,d} = \inf_{m \in \mathbb{N}_1} \limsup_{k \to \infty} k m^{1/d} \tilde{e}_{p,k^d m}(U_d)\,.$$

In order to prove that

$$\limsup_{n \to \infty} n^{1/d} \tilde{e}_{p,n}(U_d) \leq \inf_{m \in \mathbb{N}_1} \limsup_{k \to \infty} k m^{1/d} \tilde{e}_{p,k^d m}(U_d)\,,$$

we estimate the increase rate of the function $n \mapsto \tilde{e}_{p,n}(U_d)$: given n and m, we want to know how far $n^{1/d}\tilde{e}_{p,n}(U_d)$ is from the sequence $k \mapsto k m^{1/d} \tilde{e}_{p,k^d m}(U_d)$. The bound on the increase rate is proven in Lemma 5.6.5 by constructing a suitable competitor for the minimization problem that defines $\tilde{e}_{p,n}(U_d)$. This competitor is built by combining two optimal empirical quantizers: one for U_d and one for the uniform measure U_{d-1} on the $(d-1)$-dimensional (!) cube. In the end, this procedure shifts the problem to estimating the optimal empirical quantization error for the uniform measure on a *lower dimensional* cube. In fact, (5.6.1) is proven *by induction* on the dimension.

Remark 5.6.2. While it is obvious that $n \mapsto e_{p,n}(\mu)$ is nonincreasing, the same cannot in general be said for the optimal empirical quantization error. For example, if $\mu = \frac{\delta_x + \delta_y}{2}$ for two distinct points $x, y \in \mathbb{R}^d$, then $\tilde{e}_{p,2}(\mu) = 0$ but $\tilde{e}_{p,3}(\mu) > 0$.

Remark 5.6.3 (1-dimensional case). The values of $e_{p,n}(U_1)$ and $\tilde{e}_{p,n}(U_1)$ are easy to compute and coincide. For both the problems, the optimal measure $\mu_n \in \mathcal{P}_{(n)}(\mathbb{R}^d)$ and the optimal transport plan $\gamma \in \Gamma(U_1, \mu_n)$ are simply:

$$\mu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{\frac{2i-1}{2n}}\,, \qquad \gamma := \sum_{i=1}^{n} \mathscr{L}^1|_{\left(\frac{i-1}{n}, \frac{i}{n}\right)} \otimes \delta_{\frac{2i-1}{2n}} \in \Gamma(U_1, \mu_n)\,,$$

see [GL00, Theorem 4.16], [XB19, Theorem 5.5], and Figure 5.1. Hence,

$$e^p_{p,n}(U_1) = \tilde{e}^p_{p,n}(U_1) = \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left| x - \frac{2i-1}{n} \right|^p \mathrm{d}x = \frac{1}{(p+1)(2n)^p}\,.$$

Proof of Proposition 5.6.1. For simplicity, we write $\tilde{e}_{p,n,d}$ in place of $\tilde{e}_{p,n}(U_d)$. The proof is by induction on the dimension d. Base step: By Remark 5.6.3, $n\tilde{e}_{p,n,1}$ is constantly equal to $\frac{1}{2\sqrt[p]{p+1}}$.

For the inductive step, we make use of two lemmas.

Lemma 5.6.4. *For every $m, k \in \mathbb{N}_1$, we have the inequality*

$$\tilde{e}_{p,k^d m,d} \leq \frac{1}{k} \tilde{e}_{p,m,d}\,. \tag{5.6.2}$$

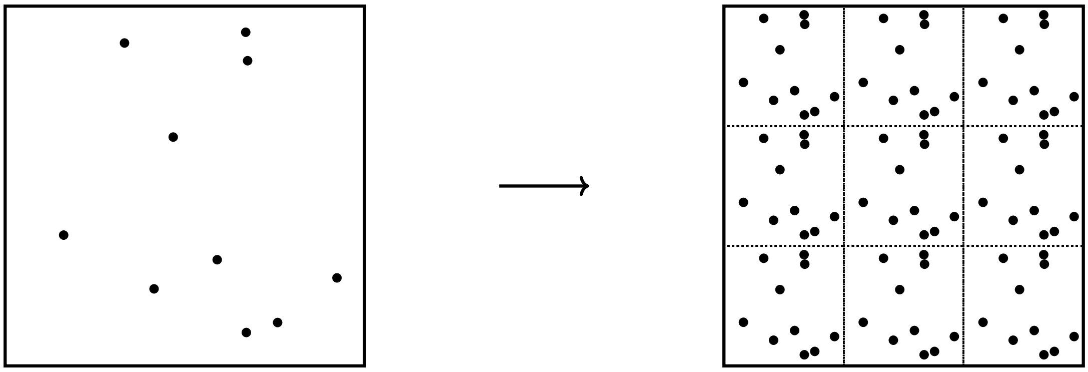


Figure 5.2: Idea for Lemma 5.6.4. Given the measure μ_n, concentrated on the black dots in the left square, the competitor $\mu'_{k^d n}$ is built by making k^d rescaled copies of μ_n.

Proof. Let $\mu_m \in \mathcal{P}_{(m)}(\mathbb{R}^d)$ and $\gamma \in \Gamma(U_d, \mu_m)$. For every $i \in \{0, 1, \dots, k-1\}^d$, we define

$$T_i(x) := \frac{1}{k}(i + x), \qquad x \in [0,1]^d.$$

Notice that T_i maps $[0,1]^d$ to $i/k + [0, 1/k]^d$. The idea is to use these transformations to make smaller copies of μ_n, which, together, constitute an appropriate competitor for the infimum that defines $e_{p,k^d m,d}$. Precisely, we set

$$\mu'_{k^d m} := \frac{1}{k^d} \sum_i (T_i)_\# \mu_m \in \mathcal{P}_{(k^d m)}(\mathbb{R}^d),$$
$$\gamma' := \frac{1}{k^d} \sum_i (T_i, T_i)_\# \gamma \in \Gamma(U_d, \mu'_{k^d m}).$$

With these choices, we obtain

$$\begin{aligned} \tilde{e}^p_{p,k^d m,d} &\le \int \|x - y\|^p \, \mathrm{d}\gamma' = \frac{1}{k^d} \sum_i \int |T_i(x) - T_i(y)|^p \, \mathrm{d}\gamma \\ &= \frac{1}{k^{p+d}} \sum_i \int \|x - y\|^p \, \mathrm{d}\gamma = \frac{1}{k^p} \int \|x - y\|^p \, \mathrm{d}\gamma. \end{aligned}$$

We conclude by arbitrariness of γ and μ_m. □

Lemma 5.6.5. *There exists a constant $c_p > 0$ such that, for every $n, l \in \mathbb{N}_1$, we have*

$$\tilde{e}^p_{p,n+l,d+1} \le \frac{n}{n+l} \tilde{e}^p_{p,n,d+1} + c_p \frac{l}{n+l} \tilde{e}^p_{p,l,d} + c_p \left(\frac{l}{n+l} \right)^{p+1}. \tag{5.6.3}$$

Proof. Let μ_n, ν_l be probability measures of the form

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, t_i)} \in \mathcal{P}_{(n)}(\mathbb{R}^{d+1}), \qquad \nu_l = \frac{1}{l} \sum_{i=n+1}^{n+l} \delta_{x_i} \in \mathcal{P}_{(l)}(\mathbb{R}^d),$$

for some $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+l} \in \mathbb{R}^d$ and $t_1, \dots, t_n \in \mathbb{R}$. Pick $\gamma \in \Gamma(U_{d+1}, \mu_n)$ and $\eta \in \Gamma(U_d, \nu_l)$. Consider the linear 1-Lipschitz function $T \colon \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ given by the formula

$$T(x, t) := \left(x, \frac{n}{n+l} t \right), \qquad x \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

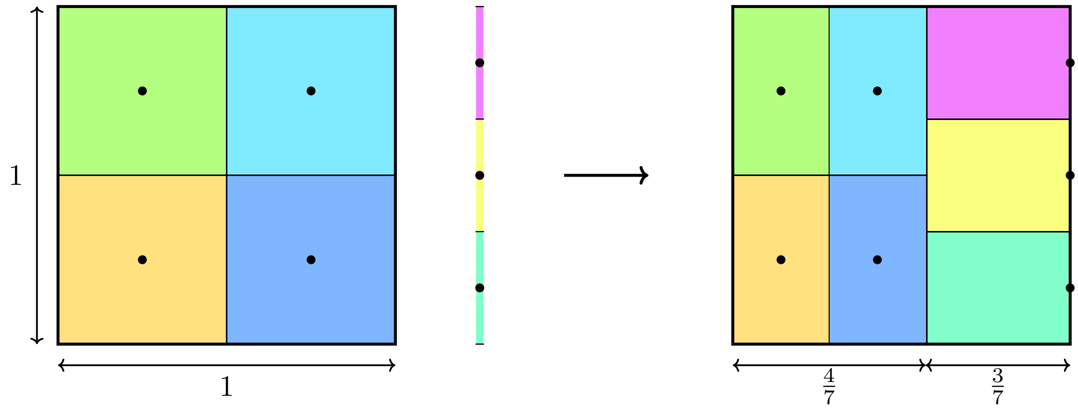


Figure 5.3: Idea for Lemma 5.6.5 with $d = 1, n = 4, l = 3$. From a transport plan for U_{d+1} with n points and one for U_d with l points, we construct a new plan for U_{d+1} with $n+l$ points by "shrinking" the first one and "expanding" the second one.

and define $\gamma' \in \mathcal{P}(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ via

$$\int \varphi \, \mathrm{d}\gamma' := \frac{n}{n+l} \int \varphi\Big(T(x,t), T(x',t')\Big) \, \mathrm{d}\gamma + \int_{\frac{n}{n+l}}^{1} \int \varphi(x,t,x',1) \, \mathrm{d}\eta(x,x') \, \mathrm{d}t \,,$$

for every continuous and bounded test function $\varphi \colon \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}$. It is not difficult to check that the first marginal of γ' is U_{d+1}. Indeed, given a test function $\psi \colon \mathbb{R}^{d+1} \to \mathbb{R}$ and denoting by $\pi^1 \colon \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ the projection onto the first $d+1$ coordinates, we have

$$\begin{aligned} \int \psi \, \mathrm{d}\pi^1_{\#}\gamma' &= \frac{n}{n+l} \int \psi\Big(T(x,t)\Big) \, \mathrm{d}\pi^1_{\#}\gamma + \int_{\frac{n}{n+l}}^{1} \int \psi(x,t) \, \mathrm{d}\pi^1_{\#}\eta(x) \, \mathrm{d}t \\ &= \frac{n}{n+l} \int_0^1 \int \psi\left(x, \frac{nt}{n+l}\right) \mathrm{d}U_d(x) \, \mathrm{d}t + \int_{\frac{n}{n+l}}^{1} \int \psi(x,t) \, \mathrm{d}U_d(x) \, \mathrm{d}t \\ &= \int \psi \, \mathrm{d}U_{d+1} \,. \end{aligned}$$

The second marginal is

$$\pi^2_{\#}\gamma' = \frac{1}{n+l} \sum_{i=1}^{n} \delta_{T(x_i,t_i)} + \frac{1}{n+l} \sum_{i=n+1}^{n+l} \delta_{(x_i,1)} \in \mathcal{P}_{(n+l)}(\mathbb{R}^d) \,,$$

because

$$\begin{aligned} \int \psi \, \mathrm{d}\pi^2_{\#}\gamma' &= \frac{n}{n+l} \int \psi\Big(T(x,t)\Big) \, \mathrm{d}\pi^2_{\#}\gamma + \int_{\frac{n}{n+l}}^{1} \int \psi(x',1) \, \mathrm{d}\pi^2_{\#}\eta(x') \, \mathrm{d}t \\ &= \frac{\cancel{n}}{n+l} \frac{1}{\cancel{n}} \sum_{i=1}^{n} \psi\Big(T(x_i,t_i)\Big) + \frac{\cancel{l}}{n+l} \frac{1}{\cancel{l}} \sum_{i=n+1}^{n+l} \psi(x_i,1) \,. \end{aligned}$$

We infer the inequality

$$\begin{aligned} \tilde{e}^p_{p,n+l,d+1} \le \frac{n}{n+l} \int \Big\| T(x,t) - T(x',t') \Big\|^p \, \mathrm{d}\gamma & \\ + \int_{\frac{n}{n+l}}^{1} \int \Big\| (x,t) - (x',1) \Big\|^p \, \mathrm{d}\eta(x,x') \, \mathrm{d}t \,. \end{aligned}$$

Now we make the following two observations:

1. since T is 1-Lipschitz, $\|T(x,t)-T(x',t')\| \le \|(x,t)-(x',t')\|$,
2. there exists a constant c_p such that

$$\left\|(x,t)-(x',t')\right\|^p \le c_p\left(\left\|x-x'\right\|^p + \left|t-t'\right|^p\right)$$

for every $x, x' \in \mathbb{R}^d$ and $t, t' \in \mathbb{R}$. Precisely, $c_p = \max\left(1, 2^{\frac{p}{2}-1}\right)$.

Therefore, we obtain

$$\tilde{e}^p_{p,n+l,d+1} \le \frac{n}{n+l}\int \left\|(x,t)-(x',t')\right\|^p \,\mathrm{d}\gamma + c_p\frac{l}{n+l}\int \left\|x-x'\right\|^p \,\mathrm{d}\eta + c_p\int_{\frac{n}{n+l}}^1 (1-t)^p \,\mathrm{d}t\,.$$

By arbitrariness of $\mu_n, \nu_l, \gamma, \eta$,

$$\tilde{e}^p_{p,n+l,d+1} \le \frac{n}{n+l}\tilde{e}^p_{p,n,d+1} + c_p\frac{l}{n+l}\tilde{e}^p_{p,l,d} + c_p\int_{\frac{n}{n+l}}^1 (1-t)^p \,\mathrm{d}t\,,$$

and the conclusion follows. □

Assume that (5.6.1) is true for a certain dimension d, and fix $m \in \mathbb{N}_1$. For every $n \ge 2^{d+1}m$, set

$$k_n := \left\lfloor \left(\frac{n}{m}\right)^{\frac{1}{d+1}} \right\rfloor - 1\,, \qquad l_n := n - k_n^{d+1}m\,.$$

Observe that $k_n, l_n \ge 1$ for every n (they are integer and strictly positive), and that $l_n \asymp_{d,m} n^{\frac{d}{d+1}}$. Indeed, on the one hand,

$$\frac{n}{m} \le \left(\left\lfloor \left(\frac{n}{m}\right)^{\frac{1}{d+1}} \right\rfloor + 1\right)^{d+1} = (k_n+2)^{d+1}\,,$$

from which we get

$$l_n \le \left((k_n+2)^{d+1} - k_n^{d+1}\right)m \lesssim_d k_n^d m \le n^{\frac{d}{d+1}} m^{\frac{1}{d+1}}\,.$$

On the other hand,

$$l_n \ge \left((k_n+1)^{d+1} - k_n^{d+1}\right)m \gtrsim_d (k_n+2)^d m \ge n^{\frac{d}{d+1}} m^{\frac{1}{d+1}}\,.$$

Lemma 5.6.5 gives the estimate

$$\tilde{e}^p_{p,n,d+1} - \tilde{e}^p_{p,k_n^{d+1}m,d+1} \lesssim_p \frac{l_n}{n}\tilde{e}^p_{p,l_n,d} + \left(\frac{l_n}{n}\right)^{p+1}\,,$$

and, by inductive hypothesis, $\tilde{e}^p_{p,l_n,d} \lesssim_{p,d} l_n^{-p/d}$. Thus,

$$\tilde{e}^p_{p,n,d+1} - \tilde{e}^p_{p,k_n^{d+1}m,d+1} \lesssim_{p,d} \frac{l_n^{1-\frac{p}{d}}}{n} + \left(\frac{l_n}{n}\right)^{p+1}\,.$$

The combination of the latter with $l_n \asymp_{d,m} n^{\frac{d}{d+1}}$ gives

$$\limsup_{n\to\infty} n^{\frac{p}{d+1}} \tilde{e}^p_{p,n,d+1} \le \limsup_{n\to\infty} n^{\frac{p}{d+1}} \tilde{e}^p_{p,k_n^{d+1}m,d+1}\,.$$

Now we use Lemma 5.6.4 to write

$$\limsup_{n\to\infty} n^{\frac{1}{d+1}} \tilde{e}_{p,n,d+1} \le \tilde{e}_{p,m,d+1} \limsup_{n\to\infty} \frac{n^{\frac{1}{d+1}}}{k_n} = m^{\frac{1}{d+1}} \tilde{e}_{p,m,d+1}\,.$$

We conclude the inductive step (and therefore the proof) by arbitrariness of m. □

5.7 Asymptotic behavior for $p \in [1,\infty)$ (Corollary 5.1.4)

This section is devoted to Corollary 5.1.4. Note that this result will *not* be used later in this work. The following simple observation is at the core of the proof.

Remark 5.7.1. The property

$$\limsup_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) < \infty \tag{5.7.1}$$

is invariant under pushforward via Lipschitz maps. In particular, if $T\colon [0,1]^d \to \mathbb{R}^d$ is Lipschitz, then (5.7.1) holds with $\mu \coloneqq T_\# U_d$ for every $p \ge 1$.

Proof of Corollary 5.1.4. Step 1 ($\Omega = \tilde{\Omega}$). Assume at first that $\Omega = \tilde{\Omega}$, i.e., Ω itself is convex and with $C^{1,1}$ boundary. The idea is to use the regularity theory for optimal transport to find a Lipschitz map T such that $\mu = T_\# U_d$ in order to apply Remark 5.7.1. Precisely, we use [CLW21, Theorem 1.1] (see also [CLW19, Theorem 1.1 (i)]): given a measure $\mu_0 = \rho_0 \mathscr{L}^d$ concentrated on an open set Ω_0, with the same assumptions as ρ and Ω, there exists a Lipschitz transport map[6] T_1 pushing μ_0 to μ. If we manage to find one such μ_0 of the form $\mu_0 = (T_0)_\# U_d$ for some Lipschitz map T_0, then we can set $T \coloneqq T_1 \circ T_0$ and conclude. The obstruction to simply taking $\mu_0 = U_d$ is that the boundary of $[0,1]^d$ is not of class $C^{1,1}$. Let us also note that it makes no difference if we find μ_0 as Lipschitz pushforward of the uniform measure on *another* d-dimensional cube, such as the unit ball w.r.t. 1-norm $\|\cdot\|_1$.

In light of the previous discussion, proving the following lemma suffices to complete this Step.

Lemma 5.7.2. *The map*

$$T_0(x) \coloneqq \left(1 - \|x\|_1 + \frac{\|x\|_1^2}{\|x\|_2}\right) x\,, \qquad \|x\|_1 < 1\,,$$

is Lipschitz continuous. Moreover, the measure $\mu_0 \coloneqq (T_0)_\# U\Big(\big\{\|\cdot\|_1 < 1\big\}\Big)$ is concentrated on the Euclidean ball $\Omega_0 \coloneqq \big\{\|\cdot\|_2 < 1\big\}$ and, therein, it has Lipschitz continuous and uniformly positive density.

Proof. We omit the simple proofs that T_0 is Lipschitz and that μ_0 is concentrated on Ω_0, and focus on the computation of the density of μ_0. Let $\varphi\colon \Omega_0 \to \mathbb{R}$ be a Borel measurable and bounded test function. We have

$$\int \varphi \,\mathrm{d}\mu_0 = \frac{1}{c_d} \int_{\mathbb{S}^{d-1}} \int_0^{\|v\|_1^{-1}} \varphi\big(T_0(rv)\big) r^{d-1} \,\mathrm{d}r \,\mathrm{d}\mathscr{H}^{d-1}(v)\,,$$

[6] In fact, the map T_1 is of class C^1 with Hölder Jacobian. Since Ω_0 is convex, T_1 is Lipschitz.

where $c_d := \left|\left\{\|\cdot\|_1 < 1\right\}\right| = \frac{2^d}{d!}$, the set $\mathbb{S}^{d-1}$ is the $(d-1)$-dimensional sphere (w.r.t. the 2-norm), and $\mathscr{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on it. Let us write

$$T_0(rv) = \underbrace{r\left(1 - r\|v\|_1 + r\|v\|_1^2\right)}_{=:\xi_v(r)} v\,, \qquad r \in \left(0, \|v\|_1^{-1}\right)\,, \qquad v \in \mathbb{S}^{d-1}\,,$$

and notice that

$$(\partial_r \xi_v)(r) = 1 + 2r\|v\|_1\left(\|v\|_1 - 1\right) \geq 1\,,$$

where, in the last inequality, we used that $\|\cdot\|_2 \leq \|\cdot\|_1$. In particular, ξ_v is invertible. Thus, by changing variables, we find

$$\int \varphi \,\mathrm{d}\mu_0 = \frac{1}{c_d}\int_{\mathbb{S}^{d-1}}\int_0^1 \varphi(\tilde{r}v)\frac{\xi_v^{-1}(\tilde{r})^{d-1}}{(\partial_r\xi_v)\left(\xi_v^{-1}(\tilde{r})\right)}\,\mathrm{d}\tilde{r}\,\mathrm{d}\mathscr{H}^{d-1}(v)\,,$$

and, therefore, the density of μ_0 on Ω_0 is

$$\rho_0(x) := \frac{\xi_{v_x}^{-1}\left(\|x\|_2\right)^{d-1}}{c_d\|x\|_2^{d-1}\,(\partial_r\xi_{v_x})\left(\xi_{v_x}^{-1}\left(\|x\|_2\right)\right)}\,, \qquad \text{where } v_x := \frac{x}{\|x\|_2}\,, \qquad \|x\|_2 < 1\,.$$

If we set

$$\alpha(t) := \begin{cases} \frac{1}{c_d} & \text{if } t = 0\,,\\ \frac{1}{c_d\sqrt{1+4t}}\left(\frac{\sqrt{1+4t}-1}{2t}\right)^{d-1} & \text{if } t > 0\,,\end{cases} \qquad \beta(x) := \|x\|_1\left(\frac{\|x\|_1}{\|x\|_2} - 1\right)\,,$$

tedious but simple computations (passing through the explicit formula for ξ_v^{-1}) reveal that $\rho_0|_{\Omega_0} = \alpha\circ\beta$. When $\|x\|_2 < 1$, the values of $\beta(x)$ range between 0 and $d - \sqrt{d}$. On this interval, the function α is Lipschitz continuous and positive. Since $\beta|_{\Omega_0}$ is Lipschitz too, the proof is complete. □

Step 2 ($\Omega \neq \tilde{\Omega}$). Let us now generalize to the case where, possibly, $\Omega \neq \tilde{\Omega}$, but there exists $M\colon \tilde{\Omega} \to \Omega$ as in the assumptions. Consider the probability measure $\tilde{\mu}$ defined by

$$\tilde{\rho} := \begin{cases} (\rho\circ M)|\det\nabla M| & \text{on } \tilde{\Omega}\,,\\ 0 & \text{on } \mathbb{R}^d\setminus\tilde{\Omega}\,,\end{cases} \qquad \tilde{\mu} := \tilde{\rho}\mathscr{L}^d\,.$$

Thanks to the assumptions on M and ρ, to this new measure we can apply Step 1; thus (5.7.1) holds for $\tilde{\mu}$. Moreover, by the change of variables formula, $\mu = M_\#\tilde{\mu}$, and the map M is Lipschitz because its Jacobian is bounded and $\tilde{\Omega}$ is convex. Hence, we conclude by Remark 5.7.1. □

5.8 Main theorem (Theorem 5.1.1)

This section is subdivided into four parts: we first establish two preliminary lemmas, then we prove Theorem 5.1.1 for *singular* measures, the upper bound (U) (in general), and eventually the lower bound (L).

5.8.1 Preliminary lemmas

Lemma 5.8.1. *Let $(b_k)_{k\in\mathbb{N}_1}$ be a sequence of nonnegative numbers, infinitesimal as $k\to\infty$. Then there exists a sequence $(k_n)_{n\in\mathbb{N}_1}\subseteq\mathbb{N}_1$ such that $k_n\to\infty$ as $n\to\infty$ and*

$$\lim_{n\to\infty} n^{-1/d}2^{k_n} = \lim_{n\to\infty} n^{1/d}2^{-k_n}b_{k_n} = 0\,. \tag{5.8.1}$$

Proof. The existence of such a sequence is established in the proof of [BB13, Theorem 5] by F. Barthe and C. Bordenave. □

Lemma 5.8.2. *Let $C\subseteq\mathbb{R}^d$ be a closed set and let $\rho\in L^1_{\geq 0}(\mathbb{R}^d)$. For every $k\in\mathbb{N}_1$ and $s\geq 0$ define the open sets*

$$\Omega_i := (0,2^{-k})^d + 2^{-k}i\,, \qquad i\in\mathbb{Z}^d\,, \tag{5.8.2}$$

and

$$\Omega^{(k)} := \operatorname{int}\left(\bigcup_{i\in\mathbb{Z}^d\,:\,\overline{\Omega_i}\cap C=\emptyset}\overline{\Omega_i}\right)\,, \tag{5.8.3}$$

(see Figure 5.4), the set of indices

$$I_{k,s} := \left\{i\in\mathbb{Z}^d\ :\ \|x-y\|>s\ \forall x\in\Omega_i\ \forall y\in\mathbb{R}^d\setminus\Omega^{(k)}\right\}\,, \tag{5.8.4}$$

and the function

$$\rho_{k,s} := \sum_{i\in I_{k,s}}\left(\fint_{\Omega_i}\rho\,\mathrm{d}\mathscr{L}^d\right)\mathbb{1}_{\Omega_i}\,. \tag{5.8.5}$$

Then $\rho_{k,s}\to\rho|_{\mathbb{R}^d\setminus C}$ almost everywhere and in $L^1(\mathbb{R}^d)$ as $k\to\infty$ and $s\to 0$.

Proof. Almost every point $x\in\mathbb{R}^d\setminus C$ (for example, the points out of C for which all coordinates are irrational) is contained in some Ω_i with $i\in I_{k,s}$ as soon as its distance from C is larger than $\sqrt{d}2^{1-k}+s$. Therefore, by [Coh13, Theorem 6.2.3], we have $\rho_{k,s}\to\rho|_{\mathbb{R}^d\setminus C}$ almost everywhere and, by Scheffé's Lemma [Wil91, Theorem 5.10], in $L^1(\mathbb{R}^d)$. □

Remark 5.8.3. With the notation of Lemma 5.8.2, note the following:

$$\bigcup_{i\in I_{k,s}}\Omega_i\subseteq\bigcup_{i\in I_{k,0}}\Omega_i\subseteq\Omega^{(k)}\subseteq\bigcup_{i\in I_{k,0}}\overline{\Omega_i}\subseteq\mathbb{R}^d\setminus C\,,\qquad k\in\mathbb{N}_1\,,\qquad s\geq 0\,.$$

5.8.2 Singular measures

The proof of Theorem 5.1.1 for singular measures is inspired by [DSS13, Proposition 3]. We will combine the following three observations:

- we can split μ into measures μ^i supported on small *cubes* (plus a remainder that we control with Theorem 5.1.7);
- the error $\tilde{e}^p_{p,n}(\mu)$ is subadditive in the sense of Remark 5.3.9 and, by Theorem 5.4.3, for every $\mu^i\in\mathcal{P}_c(\mathbb{R}^d)$, we can bound $n^{1/d}\tilde{e}_{p,n}(\mu^i)$ in terms of the diameter of the support of μ^i;
- since μ is singular, it is concentrated on open sets with arbitrarily small Lebesgue measure.

Proof of Theorem 5.1.1 for $\mu \perp \mathscr{L}^d$. Choose any open set $\Omega \subseteq \mathbb{R}^d$ such that $\mu(\Omega) = 1$, and write it as a countable *disjoint* union of (half-open) *cubes* $\{Q_i\}_{i\in\mathbb{N}_1}$, see, e.g., [WZ15, Theorem 1.11]. Note, in particular, that $\operatorname{diam}(Q_i) \lesssim_d |Q_i|^{1/d}$.

Pick two numbers $n, i_{\max} \in \mathbb{N}_1$ and define

$$n_i := \lfloor n\mu(Q_i) \rfloor \,, \qquad \mu^i := \begin{cases} \frac{n_i}{n\mu(Q_i)} \mu|_{Q_i} & \text{if } n_i \geq 1\,, \\ 0 & \text{otherwise,} \end{cases} \qquad i \in \{1, \dots, i_{\max}\}\ .$$

Notice that $\mu^i \leq \mu$ for every i and, since the cubes are all disjoint, also the sum $\sum_{i=1}^{i_{\max}} \mu^i$ is not larger than μ. Define

$$n_0 := n - \sum_{i=1}^{i_{\max}} n_i\,, \qquad \mu^0 := \mu - \sum_{i=1}^{i_{\max}} \mu^i\,,$$

and notice that $\left\|\mu^0\right\| = n_0/n$.

Owing to Remark 5.3.9, we have

$$\tilde{e}^p_{p,n}(\mu) \leq \sum_{i=0}^{i_{\max}} \tilde{e}^p_{p,n_i}(\mu^i)\,.$$

Theorem 5.1.7 (or Theorem 5.4.3 for $i \geq 1$) yields

$$\begin{aligned} \tilde{e}^p_{p,n}(\mu) &\lesssim_{p,d,\theta} n^{\frac{p}{\theta}-1} n_0^{1-\frac{p}{d}-\frac{p}{\theta}} \left(\int \|x\|^\theta \, \mathrm{d}\mu_0 \right)^{\frac{p}{\theta}} + \sum_{i=1}^{i_{\max}} n^{-1} n_i^{1-\frac{p}{d}} \operatorname{diam}(Q_i)^p \\ &\lesssim_{p,d,\theta,\mu} n^{\frac{p}{\theta}-1} n_0^{1-\frac{p}{d}-\frac{p}{\theta}} + n^{-1} \sum_{i=1}^{i_{\max}} n_i^{1-\frac{p}{d}} |Q_i|^{p/d}\,, \end{aligned} \tag{5.8.6}$$

where $\theta > p^*$ is such that $\mu \in \mathcal{P}_\theta(\mathbb{R}^d)$. Note that $a := 1 - \frac{p}{d} - \frac{p}{\theta} > 0$. Since $p < d$, we can apply Hölder's inequality to the last sum and obtain

$$\sum_{i=1}^{i_{\max}} n_i^{1-\frac{p}{d}} |Q_i|^{p/d} \leq \left(\sum_{i=1}^{i_{\max}} n_i \right)^{1-\frac{p}{d}} \left(\sum_{i=1}^{i_{\max}} |Q_i| \right)^{p/d} \leq n^{1-\frac{p}{d}} |\Omega|^{p/d}\,. \tag{5.8.7}$$

Furthermore, we notice that

$$n_0 \leq n - \sum_{i=1}^{i_{\max}} \big(n\mu(Q_i) - 1 \big) = n \left(1 - \sum_{i=1}^{i_{\max}} \mu(Q_i) \right) + i_{\max}\,. \tag{5.8.8}$$

We now combine (5.8.6), (5.8.7), and (5.8.8) to infer

$$n^{p/d} \tilde{e}^p_{p,n}(\mu) \lesssim_{p,d,\theta,\mu} \left(1 - \sum_{i=1}^{i_{\max}} \mu(Q_i) + \frac{i_{\max}}{n} \right)^a + |\Omega|^{p/d}\,;$$

hence,

$$\limsup_{n\to\infty} n^{p/d} \tilde{e}^p_{p,n}(\mu) \lesssim_{p,d,\theta,\mu} \left(1 - \sum_{i=1}^{i_{\max}} \mu(Q_i) \right)^a + |\Omega|^{p/d}\,. \tag{5.8.9}$$

Since μ is concentrated on $\bigcup_{i\in\mathbb{N}_1} Q_i$, the first term at the right-hand side of (5.8.9) tends to 0 as $i_{\max} \to \infty$. Moreover, when μ is singular, $|\Omega|$ can be made arbitrarily small. $\square$

5.8.3 Upper bound

To prove the upper bound, we first assume that the measure μ is compactly supported and absolutely continuous. We split the domain into cubes $\{\Omega_i\}_i$ with edge length 2^{-k} and consider an approximating density ρ_k that is constant on each of these cubes. We then construct a further approximation $\rho_k^{(n)}$ having mass on each cube equal to an integer multiple of $1/n$, i.e., of the form

$$\rho_k^{(n)} := \sum_i \frac{n_i}{n} \frac{\mathbb{1}_{\Omega_i}}{|\Omega_i|}$$

with $n_i \approx n\mu(\Omega_i)$. Using Remark 5.3.7, Remark 5.3.9, and Proposition 5.6.1, it is possible to show that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} n^{1/d} \tilde{e}_{p,n}\left(\rho_k^{(n)}\right) \le \tilde{q}_{p,d} \left(\int_{\mathbb{R}^d} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d \right)^{1/p} .$$

Indeed, heuristically:

$$\begin{aligned}
\tilde{e}_{p,n}^p\left(\rho_k^{(n)}\right) &\le \sum_i \frac{n_i}{n} |\Omega_i|^{p/d} \, \tilde{e}_{p,n_i}^p(U_d) && \text{(Rmk. 5.3.7, Rmk. 5.3.9)} \\
&\approx \tilde{q}_{p,d}^p \sum_i \frac{n_i}{n} |\Omega_i|^{p/d} \, n_i^{-p/d} && \text{(Prop. 5.6.1)} \\
&\approx n^{-p/d} \tilde{q}_{p,d}^p \sum_i \mu(\Omega_i)^{\frac{d-p}{d}} |\Omega_i|^{p/d} && (n_i \approx n\mu(\Omega_i)) \\
&\approx n^{-p/d} \tilde{q}_{p,d}^p \int_{\mathbb{R}^d} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d .
\end{aligned}$$

Our argument is similar to the proof of Zador's Theorem (see [GL00, Steps 2 & 3 in Theorem 6.2]), but we have an additional obstacle: for fixed k, the approximating error explodes as $n \to \infty$. Even worse: the two errors made by replacing μ with ρ_k, and ρ_k with $\rho_k^{(n)}$ compete with each other, in the sense that (up to constant), each one is *almost* equal to a negative power of the other. However, in the upper bound for the error $W_p^p(\mu, \rho_k)$, thanks to Lemma 5.3.2, there is also the additional term $\|\rho - \rho_k\|_{L^1}$. This term is infinitesimal as $k \to \infty$ (here we use that μ is absolutely continuous and Lemma 5.8.2). Taking advantage of Lemma 5.8.1, we can let k tend to infinity with n in such a way that both approximating errors become negligible. This solution is partly inspired by the proof of [BB13, Theorem 5].

To deal with a general measure, we split it into its singular part, a compactly supported and absolutely continuous part, and a remainder. To the latter, we apply Theorem 5.1.7.

Proof of the upper bound in Theorem 5.1.1. Step 1 ($\mu \in \mathcal{P}_c(\mathbb{R}^d)$ and $\mu \ll \mathscr{L}^d$). We start by proving the upper bound (U) under the additional assumption that μ is absolutely continuous, i.e., $\mu = \rho\mathscr{L}^d$, and compactly supported. It is easy to check that, if $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is a homothety, then (U) for μ and for $T_\#\mu$ are equivalent. Thus, without loss of generality, we assume that μ is concentrated on $(0,1)^d$.

Fix $k, n \in \mathbb{N}_1$. Let us define $\{\Omega_i\}_{i\in\mathbb{Z}^d}$, $I_k = I_{k,0}$, and $\rho_k = \rho_{k,0}$ as in Lemma 5.8.2 with $C := \mathbb{R}^d \setminus (-1,2)^d$ and $s = 0$. Notice that $\|\rho_k\|_{L^1} = \|\mu\| = 1$. For every $i \in I_k$, we define $n_i = n_i(n,k) := \lfloor n\mu(\Omega_i) \rfloor$, and we let $n_0 := n - \sum_{i\in I_k} n_i$. We then set

$$\rho_k^{(n)} := \sum_{i\in I_k} \frac{n_i}{n} 2^{kd} \mathbb{1}_{\Omega_i} \le \rho_k .$$

By using the triangle inequality, it is immediate to check that

$$\tilde{e}_{p,n}(\mu) \le \tilde{e}_{p,n}(\rho_k) + W_p(\mu, \rho_k)\,. \tag{5.8.10}$$

Remark 5.3.9 yields

$$\tilde{e}^p_{p,n}(\rho_k) \le \tilde{e}^p_{p,n_0}\left(\rho_k - \rho_k^{(n)}\right) + \sum_{i\in I_k} \tilde{e}^p_{p,n_i}\left(\frac{n_i}{n} U(\Omega_i)\right)\,, \tag{5.8.11}$$

and can use Remark 5.3.7 to write

$$\tilde{e}^p_{p,n_i}\left(\frac{n_i}{n}U(\Omega_i)\right) = \frac{n_i}{n}\tilde{e}^p_{p,n_i}\left(U(\Omega_i)\right) = \frac{n_i}{n2^{kp}}\tilde{e}^p_{p,n_i}(U_d)\,, \qquad i \in I_k \text{ s.t. } n_i \ge 1\,. \tag{5.8.12}$$

The $\frac{1}{p}$-homogeneity of $\tilde{e}_{p,n_0}$, combined with Theorem 5.4.3 (recall that, currently, all measures are concentrated on $(0,1)^d$) gives

$$\tilde{e}^p_{p,n_0}\left(\rho_k - \rho_k^{(n)}\right) \lesssim_{p,d} \left\|\rho_k - \rho_k^{(n)}\right\|_{L^1} n_0^{-p/d} = \frac{n_0^{\frac{d-p}{d}}}{n}\,.$$

Thus, since

$$n_0 = n - \sum_{i\in I_k} \lfloor n\mu(\Omega_i)\rfloor \le n - \sum_{i\in I_k} n\mu(\Omega_i) + \#I_k = \#I_k \lesssim_d 2^{kd}\,,$$

we have

$$\tilde{e}^p_{p,n_0}\left(\rho_k - \rho_k^{(n)}\right) \lesssim_{p,d} \frac{2^{k(d-p)}}{n} \tag{5.8.13}$$

(here we use $p < d$). Moreover, by applying Lemma 5.3.1 and Lemma 5.3.2, we get

$$\begin{aligned} W_p^p(\mu, \rho_k) &\le \sum_{i\in I_k} W_p^p\left(\mu|_{\Omega_i}, \mu(\Omega_i)U(\Omega_i)\right) \\ &\le \sum_{i\in I_k} \operatorname{diam}(\Omega_i)^p \left\|(\rho - \rho_k)|_{\Omega_i}\right\|_{L^1} \lesssim_{p,d} 2^{-kp}\|\rho - \rho_k\|_{L^1}\,. \end{aligned} \tag{5.8.14}$$

By Lemma 5.8.1 and since $\rho_k \xrightarrow{L^1} \rho$ as $k \to \infty$ (Lemma 5.8.2), we can choose $k = k_n$ as a function of n in such a way that

$$\lim_{n\to\infty} n^{-1/d}2^{k_n} = \lim_{n\to\infty} n^{1/d}2^{-k_n}\|\rho - \rho_{k_n}\|_{L^1}^{1/p} = 0\,, \tag{5.8.15}$$

By (5.8.10), (5.8.11), (5.8.12), (5.8.13), (5.8.14), and (5.8.15) we thus have

$$\begin{aligned} \limsup_{n\to\infty} n^{p/d}\tilde{e}^p_{p,n}(\mu) &\le \limsup_{n\to\infty} 2^{-k_n p} \sum_{i\in I_{k_n}:\, n_i\ge 1} \left(\frac{n_i}{n}\right)^{\frac{d-p}{d}} n_i^{p/d}\tilde{e}^p_{p,n_i}(U_d) \\ &\le \limsup_{n\to\infty} \int \rho_{k_n}^{\frac{d-p}{d}} h(\lfloor n2^{-k_n d}\rho_{k_n}\rfloor)\, \mathrm{d}\mathscr{L}^d\,, \end{aligned}$$

where

$$h(m) := \begin{cases} m^{p/d}\tilde{e}^p_{p,m}(U_d) & \text{if } m \in \mathbb{N}_1\,, \\ 0 & \text{if } m = 0\,. \end{cases}$$

Note that h is nonnegative, bounded, and converges to $\tilde{q}^p_{p,d}$ as $m \to \infty$ by Proposition 5.6.1. In particular, since $n2^{-k_n d} \to \infty$ and $\rho_{k_n} \to \rho$ a.e., we have

$$\lim_{n\to\infty} h(\lfloor n2^{-k_n d}\rho_{k_n}\rfloor)\,\mathrm{d}\mathscr{L}^d = \tilde{q}^p_{p,d}\,, \qquad \text{a.e. on } \{\rho > 0\}\,. \tag{5.8.16}$$

Since $\frac{d-p}{d} \in (0,1)$, the function $t \mapsto t^{\frac{d-p}{d}}$ is subadditive. Therefore,

$$\int \rho_{k_n}^{\frac{d-p}{d}} h(\lfloor n2^{-k_n d}\rho_{k_n}\rfloor)\,\mathrm{d}\mathscr{L}^d \le \int \rho^{\frac{d-p}{d}} h(\lfloor n2^{-k_n d}\rho_{k_n}\rfloor)\,\mathrm{d}\mathscr{L}^d + \sup_{m\in\mathbb{N}_0} h(m) \int |\rho - \rho_{k_n}|^{\frac{d-p}{d}}\,\mathrm{d}\mathscr{L}^d\,.$$

Using that ρ and ρ_{k_n} are supported on $[0,1]^d$, Jensen's inequality gives

$$\int |\rho - \rho_{k_n}|^{\frac{d-p}{d}}\,\mathrm{d}\mathscr{L}^d \le \|\rho - \rho_{k_n}\|_{L^1}^{\frac{d-p}{d}} \to 0\,, \qquad \text{as } n \to \infty\,.$$

In the end, we obtain

$$\limsup_{n\to\infty} n^{p/d}\tilde{e}^p_{p,n}(\mu) \le \limsup_{n\to\infty} \int \rho^{\frac{d-p}{d}} h(\lfloor n2^{-k_n} d\rho_{k_n}\rfloor)\,\mathrm{d}\mathscr{L}^d = \tilde{q}^p_{p,d} \int \rho^{\frac{d-p}{d}}\,\mathrm{d}\mathscr{L}^d\,,$$

where the last identity follows from (5.8.16) and the dominated convergence theorem.

Step 2 (conclusion). Let $\mu \in \mathcal{P}_\theta(\mathbb{R}^d)$ and fix $r > 0$. We let:

- μ^1 be the *absolutely continuous part* of $\mu|_{[-r,r]^d}$,
- μ^2 be the *singular part* of $\mu|_{[-r,r]^d}$,
- $\mu^3 := \mu|_{\mathbb{R}^d \setminus [-r,r]^d}$.

Furthermore, for $n \in \mathbb{N}_1$, we define

$$n_i := \left\lfloor n\left\|\mu^i\right\|\right\rfloor\,, \qquad \mu^{i,n} := \begin{cases} \frac{n_i}{n\|\mu^i\|}\mu^i & \text{if } n_i \ge 1\,, \\ 0 & \text{otherwise,} \end{cases} \qquad i \in \{1,2,3\}\,,$$

as well as $n_0 := n - n_1 - n_2 - n_3$ and $\mu^{0,n} := \mu - \mu^{1,n} - \mu^{2,n} - \mu^{3,n}$. Note that $n_0 \le 3$.

By Remark 5.3.9, we can make the estimate

$$\limsup_{n\to\infty} n^{p/d}\tilde{e}^p_{p,n}(\mu) \le \sum_{i=0}^{3} \limsup_{n\to\infty} n^{p/d}\tilde{e}^p_{p,n_i}(\mu^{i,n})\,. \tag{5.8.17}$$

We shall bound the four terms in the sum separately.

When $n_0 \ge 1$, Remark 5.3.7 and Theorem 5.1.7 yield

$$\begin{aligned} n^{p/d}\tilde{e}^p_{p,n_0}(\mu^{0,n}) &= \left(\frac{n_0}{n}\right)^{1-\frac{p}{d}} n_0^{p/d}\tilde{e}^p_{p,n_0}\left(\frac{\mu^{0,n}}{\|\mu^{0,n}\|}\right) \\ &\overset{(5.1.11)}{\lesssim_{p,d,\theta}} \left(\frac{n_0}{n}\right)^{1-\frac{p}{d}} \left(\int \|x\|^\theta\,\mathrm{d}\frac{\mu^{0,n}}{\|\mu^{0,n}\|}\right)^{p/\theta} \\ &\le \left(\frac{n_0}{n}\right)^{1-\frac{p}{d}-\frac{p}{\theta}} \left(\int \|x\|^\theta\,\mathrm{d}\mu\right)^{p/\theta}\,. \end{aligned}$$

The exponent $a := 1 - \frac{p}{d} - \frac{p}{\theta}$ is positive; hence (the case $n_0 = 0$ is trivial),

$$n^{p/d} \tilde{e}^p_{p,n_0}(\mu^{0,n}) \lesssim_{p,d,\theta,\mu} n^{-a} ,$$

which means that the 0^{th} term of the sum in (5.8.17) is zero.

When $n_3 \geq 1$, similar computations give

$$n^{p/d} \tilde{e}^p_{p,n_3}(\mu^{3,n}) \lesssim_{p,d,\theta} \left(\frac{n_3}{n}\right)^a \left(\int \|x\|^\theta \, \mathrm{d}\mu^3\right)^{p/\theta} .$$

Using that $n_3 \leq n\left\|\mu^3\right\| \leq n\|\mu\|$, we thus obtain (trivially if $n_3 = 0$)

$$n^{p/d} \tilde{e}^p_{p,n_3}(\mu^{3,n}) \lesssim_{p,d,\theta,\mu} \left(\int \|x\|^\theta \, \mathrm{d}\mu|_{\mathbb{R}^d \setminus [-r,r]^d}\right)^{p/\theta} .$$

If $\left\|\mu^2\right\| > 0$, then $n_2 \to \infty$ as $n \to \infty$; therefore Theorem 5.1.1 for singular measures yields

$$\limsup_{n\to\infty} n^{p/d} \tilde{e}^p_{p,n_2}(\mu^{2,n}) = \limsup_{n\to\infty} \left(\frac{n_2}{n}\right)^{\frac{d-p}{d}} n_2^{p/d} \tilde{e}^p_{p,n_2}\left(\frac{\mu^2}{\|\mu^2\|}\right) = 0 ,$$

and the same conclusion holds trivially if $\left\|\mu^2\right\| = 0$.

If $\left\|\mu^1\right\| > 0$, then $n_1 \to \infty$ as $n \to \infty$; therefore the previous Step gives

$$\begin{aligned}
\limsup_{n\to\infty} n^{p/d} \tilde{e}^p_{p,n_1}(\mu^{1,n}) &= \limsup_{n\to\infty} \left(\frac{n_1}{n}\right)^{\frac{d-p}{d}} n_1^{p/d} \tilde{e}^p_{p,n_1}\left(\frac{\mu^1}{\|\mu^1\|}\right) \\
&\leq \tilde{q}^p_{p,d} \left\|\mu^1\right\|^{\frac{d-p}{d}} \int_{[-r,r]^d} \left(\frac{\rho}{\|\mu^1\|}\right)^{\frac{d-p}{d}} \mathrm{d}\mathscr{L}^d \\
&= \tilde{q}^p_{p,d} \int_{[-r,r]^d} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d ,
\end{aligned}$$

and the same conclusion holds trivially if $\left\|\mu^1\right\| = 0$.

In the end, (5.8.17) and the subsequent estimates prove that

$$\limsup_{n\to\infty} n^{p/d} \tilde{e}^p_{p,n}(\mu) \leq \tilde{q}^p_{p,d} \int_{[-r,r]^d} \rho^{\frac{d-p}{d}} \, \mathrm{d}\mathscr{L}^d + c_{p,d,\theta,\mu} \left(\int \|x\|^\theta \, \mathrm{d}\mu|_{\mathbb{R}^d \setminus [-r,r]^d}\right)^{p/\theta}$$

for some constant $c_{p,d,\theta,\mu}$ independent of r. We conclude by letting $r \to \infty$. □

5.8.4 Lower bound

To prove the lower bound, we again split the domain into cubes $\{\Omega_i\}_i$ with edge length 2^{-k} and approximate the density ρ of the given measure μ by the piecewise constant function

$$\rho_{k,0} = \sum_i \mu(\Omega_i) \frac{\mathbb{1}_{\Omega_i}}{|\Omega_i|} .$$

Given an optimal empirical quantizer μ_n for μ, we aim to bound from below the boundary Wasserstein pseudodistance between $\rho_{k,0}$ and μ_n. We make use of this pseudodistance—smaller than the Wasserstein distance—because its geometric superadditivity (Lemma 5.3.4) is well-suited to reduce the lower bound problem to the single cubes. On each cube, we use Lemma 5.3.13 and the definition of $q_{p,d}$ to obtain the integral of $\rho^{\frac{d-p}{d}}$. The argument can be sketched as follows:

$$\begin{aligned} W_p^p(\rho_{k,0},\mu_n) &\geq \sum_i Wb^p_{\Omega_i,p}\Big(\mu(\Omega_i)U(\Omega_i),\mu_n|_{\Omega_i}\Big) && \text{(Lem. 5.3.4)} \\ &\gtrapprox \sum_i e^p_{p,n_i}\Big(\mu(\Omega_i)U(\Omega_i)\Big) \text{ with } n_i \approx n\mu_n(\Omega_i) && \text{(Lem. 5.3.13)} \\ &\geq q^p_{p,d}\sum_i n_i^{-p/d}\mu(\Omega_i)|\Omega_i|^{p/d} && \text{(Rmk. 5.3.7, Def. 5.3.5)} \end{aligned}$$

and, since it is reasonable to expect that $\mu_n(\Omega_i) \approx \mu(\Omega_i)$, we get

$$n^{p/d}W_p^p(\rho_{k,0},\mu_n) \gtrapprox q^p_{p,d}\int_{\mathbb{R}^d}\rho^{\frac{d-p}{d}}\,\mathrm{d}\mathscr{L}^d\,.$$

The idea of using the boundary Wasserstein pseudodistance (or a similar object) to exploit its geometric superadditivity is not new. It has been used to prove lower bounds in similar problems, see, e.g., [BB13, DSS13, AGT22]. There is, however, a technical difference between these works, which estimate the expected value of a functional of i.i.d. random variables, and the current one. Given a set of μ-distributed i.i.d. random variables, the random number of those that fall within a certain region (cube) is a binomial r.v. whose law can be explicitly determined in terms of μ. Instead, given an optimal empirical quantizer μ_n, it does not seem immediate to rigorously justify the heuristic $\mu_n(\Omega_i) \approx \mu(\Omega_i)$. A considerable part of the proof is indeed devoted to this problem.

Proof of the lower bound in Theorem 5.1.1. Fix $k, n \in \mathbb{N}_1$ and $s \in (0, 2^{-k})$, choose two numbers $\epsilon_1, \epsilon_2 \in (0,1)$, and define $\{\Omega_i\}_{i\in\mathbb{Z}^d}, \Omega^{(k)}, I_{k,0}, I_{k,s}, \rho_{k,0}, \rho_{k,s}$ as in Lemma 5.8.2 with $C := \operatorname{supp}(\mu^s)$. Set

$$\Omega_i^- := \Big\{x \in \Omega_i \,:\, \operatorname{dist}(x, \mathbb{R}^d \setminus \Omega_i) > \epsilon_1 2^{-k-1}\Big\}\,, \qquad i \in \mathbb{Z}^d\,.$$

Note that each Ω_i^- is an open cube with edge length equal to $(1-\epsilon_1)2^{-k}$. It is also convenient to define the "enlarged" sets

$$\Omega_i^+ := \Big\{x \in \mathbb{R}^d \,:\, \operatorname{dist}(x, \Omega_i) < s\Big\}\,, \qquad i \in \mathbb{Z}^d\,.$$

An important observation that we are going to use later is:

$$\Big|\Omega_i^+ \cap \Omega_j\Big| \leq s2^{-k(d-1)} \quad \text{if } i \neq j\,. \tag{5.8.18}$$

We say that two cubes Ω_i and Ω_j are *adjacent*, and we write $i \sim j$, if $\overline{\Omega_i} \cap \overline{\Omega_j} \neq \emptyset$ (it suffices that their closures share a single vertex). Notice that each cube has 3^d adjacent cubes, including itself, and that, since $s < 2^{-k}$, the intersection $\Omega_i^+ \cap \Omega_j$ is nonempty iff $i \sim j$.

Using Lemma 5.3.10, pick $\mu_n \in \mathcal{P}_{(n)}(\mathbb{R}^d)$ such that $\tilde{e}_{p,n}(\mu) = W_p(\mu,\mu_n)$. We have

$$\begin{aligned} \tilde{e}_{p,n}(\mu) &\geq Wb_{\Omega^{(k)},p}(\mu,\mu_n) \geq Wb_{\Omega^{(k)},p}(\rho_{k,0},\mu_n) - Wb_{\Omega^{(k)},p}(\rho_{k,0},\mu) \\ &\geq Wb_{\Omega^{(k)},p}(\rho_{k,0},\mu_n) - W_p(\rho_{k,0},\mu|_{\Omega^{(k)}})\,, \end{aligned} \tag{5.8.19}$$

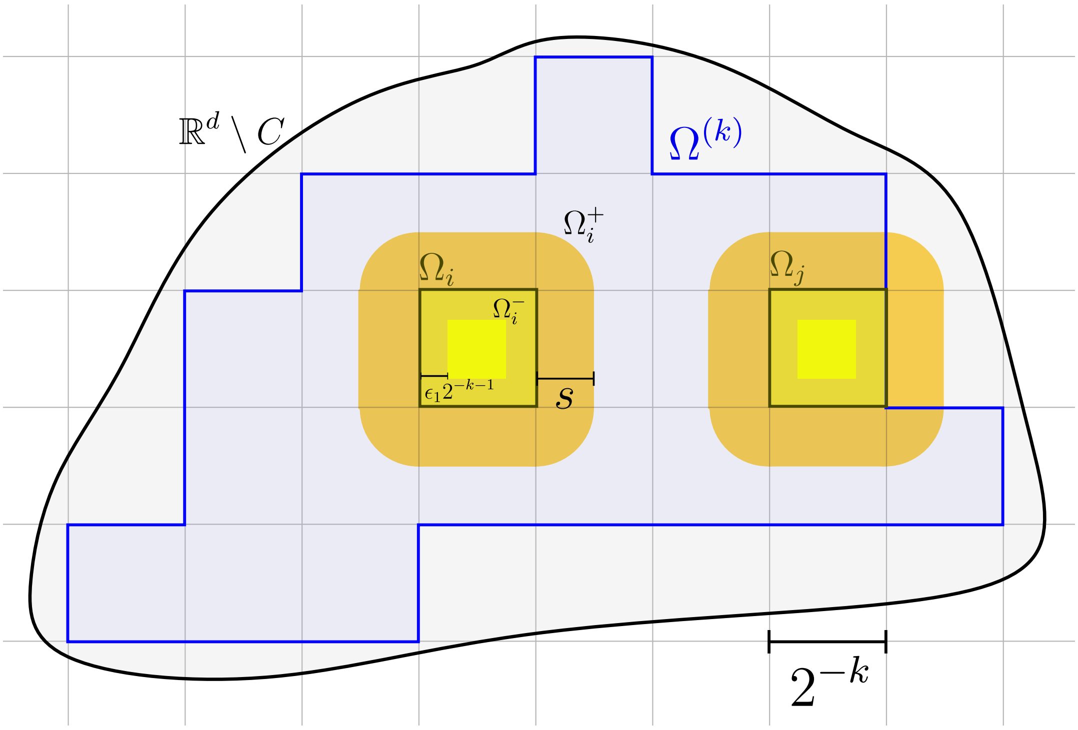


Figure 5.4: Geometric setup in the proof of the lower bound. In this example, $i \in I_{k,s}$ and $j \in I_{k,0} \setminus I_{k,s}$.

where, in the last inequality, we used that $\mu\left(\Omega^{(k)}\right) = \left\|\rho_{k,0}\right\|_{L^1}$ (recall Remark 5.8.3). Note that, in the same way we derived (5.8.14), we can deduce

$$W_p(\rho_{k,0}, \mu|_{\Omega^{(k)}}) \lesssim_{p,d} 2^{-k} \left\|\rho|_{\Omega^{(k)}} - \rho_{k,0}\right\|_{L^1}^{1/p} . \tag{5.8.20}$$

Let us focus on $Wb_{\Omega^{(k)},p}(\rho_{k,0}, \mu_n)$. Set $n_i := n\mu_n(\Omega_i) \in \mathbb{N}_0$ for every $i \in I_{k,s}$. By the superadditivity property of Lemma 5.3.4,

$$Wb^p_{\Omega^{(k)},p}\left(\rho_{k,0}, \mu_n\right) \geq \sum_{i \in I_{k,s} \,:\, \mu(\Omega_i) \geq \epsilon_2 2^{-kd}} Wb^p_{\Omega_i,p}\left(\rho_{k,0}, \mu_n\right) ,$$

and, by Lemma 5.3.13,

$$Wb^p_{\Omega_i,p}\left(\rho_{k,0}, \mu_n\right) \geq e^p_{p,n_i+N}\left(\rho_{k,0}|_{\Omega_i^-}\right) = \frac{(1-\epsilon_1)^{p+d}}{2^{kp}} \mu(\Omega_i) e^p_{p,n_i+N}(U_d) ,$$

where $N := \lceil 2d/\epsilon_1 \rceil^d$. Hence, by the definition of $q_{p,d}$, we have

$$Wb^p_{\Omega^{(k)},p}(\rho_{k,0}, \mu_n) \geq \frac{q^p_{p,d}\,(1-\epsilon_1)^{p+d}}{2^{kp}} \sum_{i \in I_{k,s} \,:\, \mu(\Omega_i) \geq \epsilon_2 2^{-kd}} \mu(\Omega_i)\left(n_i + N\right)^{-p/d} . \tag{5.8.21}$$

At this point, we need to estimate n_i from above. To this aim, pick $\gamma \in \Gamma b_{\Omega^{(k)}}(\rho_{k,0}, \mu_n)$, which gives:

$$\begin{aligned}\frac{n_i}{n} &= \mu_n(\Omega_i) = \gamma\Big(\overline{\Omega^{(k)}} \times \Omega_i\Big) \\ &= \gamma\Big(\Big(\overline{\Omega^{(k)}} \setminus \Omega_i^+\Big) \times \Omega_i\Big) + \gamma\Big(\Big(\Omega^{(k)} \cap \Omega_i^+\Big) \times \Omega_i\Big) + \gamma\Big(\Big(\partial\Omega^{(k)} \cap \Omega_i^+\Big) \times \Omega_i\Big) \\ &\le \frac{1}{s^p} \underbrace{\int \|x-y\|^p \,\mathrm{d}\gamma|_{\overline{\Omega^{(k)}}\times\Omega_i}}_{=:\alpha_i} + \int_{\Omega_i^+} \rho_{k,0}\,\mathrm{d}\mathscr{L}^d + \gamma\Big(\Big(\partial\Omega^{(k)} \cap \Omega_i^+\Big) \times \Omega_i\Big).\end{aligned}$$

If $i \in I_{k,s}$, then the last term is zero since $\partial\Omega^{(k)} \cap \Omega_i^+ = \emptyset$ by the definitions of $I_{k,s}$ and Ω_i^+. Moreover, since $s < 2^{-k}$, we have

$$\int_{\Omega_i^+} \rho_{k,0}\,\mathrm{d}\mathscr{L}^d = \sum_{j\in I_{k,0}\,:\,j\sim i} 2^{kd}\mu(\Omega_j)\Big|\Omega_i^+ \cap \Omega_j\Big| \overset{(5.8.18)}{\le} \mu(\Omega_i) + \sum_{j\in I_{k,0}\,:\,j\sim i} 2^k s\mu(\Omega_j)\,.$$

We thus obtain

$$n_i + N \le n\mu(\Omega_i) + N + n\left(\frac{\alpha_i}{s^p} + \sum_{j\in I_{k,0}\,:\,j\sim i} 2^k s\mu(\Omega_j)\right).$$

The elementary inequality $(a+b)^{-\zeta} \ge a^{-\zeta} - \zeta\frac{b}{a^{\zeta+1}}$, which holds for every $a,\zeta > 0$ and $b \ge 0$, yields

$$\frac{\mu(\Omega_i)}{\big(n_i+N\big)^{p/d}} \ge \frac{\mu(\Omega_i)}{\big(n\mu(\Omega_i)+N\big)^{p/d}} - \underbrace{\frac{p}{d}}_{<1} \cdot \underbrace{\frac{n\mu(\Omega_i)\left(\frac{\alpha_i}{s^p} + \sum_{j\in I_{k,0}\,:\,j\sim i} 2^k s\mu(\Omega_j)\right)}{\big(n\mu(\Omega_i)+N\big)^{\frac{p+d}{d}}}}_{=:\beta_i}, \qquad i \in I_{k,s}\,. \tag{5.8.22}$$

Note that

$$\frac{\mu(\Omega_i)}{\big(n\mu(\Omega_i)+N\big)^{p/d}} \ge \frac{n^{-p/d}\mu(\Omega_i)^{\frac{d-p}{d}}}{\Big(1+\frac{N2^{kd}}{\epsilon_2 n}\Big)^{p/d}} \quad \text{if } \mu(\Omega_i) \ge \epsilon_2 2^{-kd}\,.$$

Let us focus on the sum of the last terms in (5.8.22):

$$\begin{aligned}\sum_{i\in I_{k,s}\,:\,\mu(\Omega_i)\ge\epsilon_2 2^{-kd}} \beta_i &\le \epsilon_2^{-p/d} n^{-p/d} 2^{kp} \sum_{i\in I_{k,s}} \left(\frac{\alpha_i}{s^p} + \sum_{j\in I_{k,0}\,:\,j\sim i} 2^k s\mu(\Omega_j)\right) \\ &\le \epsilon_2^{-p/d} n^{-p/d} 2^{kp} \left(\frac{\sum_{i\in I_{k,s}} \alpha_i}{s^p} + \sum_{i,j\in I_{k,0}\,:\,i\sim j} 2^k s\mu(\Omega_j)\right) \\ &\le \epsilon_2^{-p/d} n^{-p/d} 2^{kp} \left(\frac{1}{s^p}\int \|x-y\|^p \,\mathrm{d}\gamma + 3^d 2^k s\right).\end{aligned} \tag{5.8.23}$$

We plug these estimates into (5.8.21), take the infimum over $\gamma \in \Gamma b_{\Omega^{(k)}}(\rho_{k,0},\mu_n)$, and find

$$\begin{aligned}n^{p/d} Wb^p_{\Omega^{(k)},p}(\rho_{k,0},\mu_n) \ge \frac{q^p_{p,d}\,(1-\epsilon_1)^{p+d}}{2^{kp}} &\sum_{i\in I_{k,s}\,:\,\mu(\Omega_i)\ge\epsilon_2 2^{-kd}} \frac{\mu(\Omega_i)^{\frac{d-p}{d}}}{\Big(1+\frac{N2^{kd}}{\epsilon_2 n}\Big)^{p/d}} \\ &- q^p_{p,d}\underbrace{(1-\epsilon_1)^{p+d}}_{<1}\epsilon_2^{-p/d}\left(\frac{1}{s^p} Wb^p_{\Omega^{(k)},p}(\rho_{k,0},\mu_n) + 3^d 2^k s\right).\end{aligned}$$

Now we make a choice for the values of s and k. Thanks to Lemma 5.8.1, Lemma 5.8.2, and the observation that $\Omega^{(k)} \nearrow \mathbb{R}^d \setminus C$ as $k \to \infty$, we can find $k = k_n$ such that

$$\lim_{n\to\infty} n^{-1/d} 2^{k_n} = \lim_{n\to\infty} n^{1/d} 2^{-k_n} \left\| \rho|_{\Omega^{(k_n)}} - \rho_{k_n,0} \right\|_{L^1}^{1/p} = 0\,. \tag{5.8.24}$$

We set $s_n := \sqrt{2^{-k_n} n^{-1/d}}$, which is smaller than 2^{-k_n}, at least for large values of n, and obtain

$$\begin{aligned}\left(n^{p/d} + \frac{q_{p,d}^p\, 2^{\frac{k_n p}{2}} n^{\frac{p}{2d}}}{\epsilon_2^{p/d}} \right) & Wb^p_{\Omega^{(k_n)},p}(\rho_{k_n,0}, \mu_n) \\ & \geq q_{p,d}^p \frac{(1-\epsilon_1)^{p+d}}{\left(1 + \frac{N 2^{k_n d}}{\epsilon_2 n}\right)^{p/d}} \int_{\{\rho_{k_n,s_n} \geq \epsilon_2\}} \rho_{k_n,s_n}^{\frac{d-p}{d}} \,\mathrm{d}\mathscr{L}^d - 3^d q_{p,d}^p\, \epsilon_2^{-p/d} \sqrt{2^{k_n} n^{-1/d}}\,.\end{aligned}$$

If we pass to the limit, keeping (5.8.24) in mind, we get

$$\begin{aligned}\liminf_{n\to\infty} n^{p/d} Wb^p_{\Omega^{(k_n)},p}(\rho_{k_n,0}, \mu_n) &\geq q_{p,d}^p (1-\epsilon_1)^{p+d} \liminf_{n\to\infty} \int_{\{\rho_{k_n,s_n} \geq \epsilon_2\}} \rho_{k_n,s_n}^{\frac{d-p}{d}} \,\mathrm{d}\mathscr{L}^d \\ &\geq q_{p,d}^p (1-\epsilon_1)^{p+d} \int_{\{\rho > \epsilon_2\}\setminus C} \rho^{\frac{d-p}{d}} \,\mathrm{d}\mathscr{L}^d\,,\end{aligned} \tag{5.8.25}$$

where the last inequality follows from Lemma 5.8.2 and Fatou's Lemma. By combining the formulas (5.8.19), (5.8.20), and (5.8.25), and by arbitrariness of ϵ_2, ϵ_1, we conclude:

$$\liminf_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(\mu) \geq q_{p,d} \left(\int_{\mathbb{R}^d \setminus C} \rho^{\frac{d-p}{d}} \,\mathrm{d}x \right)^{1/p} - c_{p,d} \underbrace{\limsup_{n\to\infty} \frac{\left\| \rho|_{\Omega^{(k_n)}} - \rho_{k_n,0} \right\|_{L^1}^{1/p}}{2^{k_n} n^{-1/d}}}_{=0}\,. \qquad \square$$

5.9 Limit existence for uniform measures

Combining the upper bound (U) and the existence of the limit for the uniform measure on a cube, it is possible to prove (for $p < d$) the existence of the limit for *any* uniform measure on a bounded set. The proof is inspired by [BB13, Theorem 24].

Corollary 5.9.1. *If $p < d$ and $A \subseteq \mathbb{R}^d$ is a bounded Borel set with $|A| \neq 0$, then*

$$\lim_{n\to\infty} n^{1/d} \tilde{e}_{p,n}(U_A) = \tilde{q}_{p,d} |A|^{1/d}\,. \tag{5.9.1}$$

Proof. Note that this result easily follows from Proposition 5.6.1 if A is a cube. Moreover, in general, one inequality is already given by (U) in Theorem 5.1.1.

We may and will assume that A is contained in (and not essentially equal to) $[0,1]^d$. Consider the measures:

- $\mu^1 := U_d|_A = |A|\, U_A$,
- $\mu^2 := U_d - \mu^1 = (1 - |A|)\, U_{[0,1]^d \setminus A}$.

For $n \in \mathbb{N}_1$, define

$$\hat{n} := \left\lfloor \frac{n}{|A|} \right\rfloor + 1\,, \qquad n_1 := n\,, \qquad n_2 := \hat{n} - n - 1\,, \qquad n_0 := 1\,.$$

Observe that that $0 \leq n_i \leq \hat{n}\left\|\mu^i\right\|$ for $i \in \{1,2\}$ and define

$$\mu^{i,n} := \frac{n_i}{\hat{n}\left\|\mu^i\right\|}\mu^i \quad \text{for } i \in \{1,2\}\,, \qquad \mu^{0,n} := U_d - \mu^{1,n} - \mu^{2,n}\,.$$

By definition of $\tilde{q}_{p,d}$ and Remark 5.3.9, we have

$$\tilde{q}^p_{p,d}\hat{n}^{-p/d} \leq \tilde{e}^p_{p,\hat{n}}(U_d) \leq \sum_{i=0}^{2} \tilde{e}^p_{p,n_i}(\mu^{i,n})\,.$$

The 0^{th} term at the right-hand side can be easily bounded:

$$\tilde{e}^p_{p,1}(\mu^{0,n}) \leq W^p_p\left(\mu^{0,n}, \left\|\mu^{0,n}\right\|\delta_0\right) = \int \|x\|^p \,\mathrm{d}\mu^{0,n} \leq \frac{d^{p/2}}{\hat{n}}\,.$$

Hence,

$$\tilde{q}^p_{p,d} \leq d^{p/2}\left(\frac{1}{\hat{n}}\right)^{\frac{d-p}{d}} + \left(\frac{n}{\hat{n}}\right)^{\frac{d-p}{d}} n^{p/d}\tilde{e}^p_{p,n}\left(U_A\right) + \left(\frac{n_2}{\hat{n}}\right)^{\frac{d-p}{d}} n_2^{p/d}\tilde{e}_{p,n_2}\left(U_{[0,1]^d\setminus A}\right),$$

which yields

$$\begin{aligned}\tilde{q}^p_{p,d} &\leq |A|^{\frac{d-p}{d}} \liminf_{n\to\infty} n^{p/d}\tilde{e}^p_{p,n}(U_A) + (1-|A|)^{\frac{d-p}{d}} \limsup_{n\to\infty} n^{p/d}\tilde{e}_{p,n}\left(U_{[0,1]^d\setminus A}\right)\\ &\overset{(\mathsf{U})}{\leq} |A|^{\frac{d-p}{d}} \liminf_{n\to\infty} n^{p/d}\tilde{e}^p_{p,n}(U_A) + \tilde{q}^p_{p,d}(1-|A|)\,.\end{aligned}$$

We conclude by rearranging the terms. □

5.10 Proof of Theorem 5.1.3

The proof of Theorem 5.1.3 is based on a fundamental result by L. Fejes Tóth [FT53, p. 81], see also [Gru99].

Theorem 5.10.1 (L. Fejes Tóth [FT53])**.** *Let $f\colon [0,\infty) \to \mathbb{R}$ be a nondecreasing function, let $H \subseteq \mathbb{R}^2$ be a convex hexagon centered at the origin, let $n \in \mathbb{N}_1$, and let $x_1, \dots, x_n \in \mathbb{R}^2$. Then*

$$\int_H f\left(\|x\|\right)\,\mathrm{d}x \leq \frac{1}{n}\int_{\sqrt{n}H} \min\left\{f\left(\|x - x_i\|\right) \;:\; i \in \{1,\dots,n\}\right\}\,\mathrm{d}x\,. \tag{5.10.1}$$

Let us fix $n \in \mathbb{N}_1$. To prove Theorem 5.1.3, we consider a hexagonal tiling $\left\{H_{i,n}\right\}_i$ of the plane, with the area of each $H_{i,n}$ being equal to $|A|/n$. The idea is to define an empirical quantizer by taking the centers of the hexagons contained in A. Theorem 5.10.1 (together with Theorem 5.4.1) is used to show that this quantizer is asymptotically optimal for the *classical* quantization problem; hence, $e_{p,n}(U_A) \leq \tilde{e}_{p,n}(U_A) \lessapprox e_{p,n}(U_A)$. The issue is that, in general, we cannot tile A perfectly with hexagons. Therefore, we carry out this construction only for the hexagons that are "well-contained" in A and leave out a strip of approximate thickness $n^{-1/2}$. We complete the quantizer by splitting the strip into approximately (and up to constant) $\sqrt{n}$ square-looking pieces $\{B_j\}_j$ of approximate size $n^{-1/2} \times n^{-1/2}$ and taking one point x_j from each piece. In this way, the contribution of the strip to the p^{th} power of the quantization error is bounded by

$$W_p^p\left(\frac{1}{n}\sum_j U(B_j), \frac{1}{n}\sum_j \delta_{x_j}\right) \overset{(5.3.2)}{\le} \frac{1}{n}\sum_j W_p^p\big(U(B_j), \delta_{x_j}\big) \le \frac{1}{n}\sum_j \operatorname{diam}(B_j)^p$$
$$\lessapprox \frac{1}{n}\sqrt{n}n^{-p/2},$$

which is negligible, i.e., much smaller than $n^{-p/2}$.

We use the bi-Lipschitz map to make the argument rigorous, by transforming the strip into a more explicit approximate annulus.

Lemma 5.10.2. *Let $D \subseteq \mathbb{R}^2$ be the open unit disk and $\overline{D}$ be its closure. Let $T\colon \overline{D} \to \mathbb{R}^2$ be a homeomorphism onto its image. Then $\partial\big(T(D)\big) = T(\partial D)$.*

Proof. By the Jordan–Schönflies Theorem (cf. [Tho92, Theorem 3.1]), there exists a homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi|_{\partial D} = T|_{\partial D}$. In particular, the connected set $\Phi^{-1}\big(T(D)\big)$ is contained in $\mathbb{R}^2 \setminus \partial D$, implying that, in fact, it is entirely contained in either D or $\mathbb{R}^2 \setminus \overline{D}$. The latter case is impossible: any retraction $r_1\colon \mathbb{R}^2 \setminus D \to \partial D$ would induce a retraction $r_1 \circ \Phi^{-1} \circ T\colon \overline{D} \to \partial D$, which is absurd by the No-Retraction Theorem. Hence, $\Phi^{-1}\big(T(D)\big) \subseteq D$. We claim that equality holds. Indeed, if there exists $z \in D \setminus \Phi^{-1}\big(T(D)\big)$, then we can find a retraction $r_2\colon \overline{D} \setminus \{z\} \to \partial D$, which induces a retraction $r_2 \circ \Phi^{-1} \circ T\colon \overline{D} \to \partial D$. Once again, this is absurd. Thus, $T(D) = \Phi(D)$ and, using that Φ is a homeomorphism and that it coincides with T on ∂D, we conclude:

$$\partial\big(T(D)\big) = \partial\big(\Phi(D)\big) = \Phi(\partial D) = T(\partial D). \qquad \square$$

Proof of Theorem 5.1.3. Fix a regular hexagon $H \subseteq \mathbb{R}^2$ with unit area, centered at the origin. For $n \in \mathbb{N}_1$, choose $x_1, \dots, x_n \in \mathbb{R}^2$. By Theorem 5.10.1 with $f(t) := t^p$, we have

$$\int_H \|x\|^p \, \mathrm{d}x \le \frac{1}{n}\int_{\sqrt{n}H} \min_i \big\|x - \sqrt{n}x_i\big\|^p \, \mathrm{d}x = n^{p/2}\int_H \min_i \|x - x_i\|^p \, \mathrm{d}x.$$

Hence, by arbitrariness of the points $x_1, \dots, x_n$ and by Theorem 5.4.1,

$$\int_H \|x\|^p \, \mathrm{d}x \le \lim_{n\to\infty} n^{p/2} e_{p,n}^p(U_H) = q_{p,2}^p.$$

Therefore, again thanks to Theorem 5.4.1, it will suffice to prove that

$$\limsup_{n\to\infty} n^{p/2} \tilde{e}_{p,n}^p(U_A) \le |A|^{p/2}\int_H \|x\|^p \, \mathrm{d}x. \tag{5.10.2}$$

We can and will assume that $|A| = 1$. Let $T\colon \overline{D} \to A$ be a (bijective) bi-Lipschitz map and let $c_T > 1$ be a Lipschitz constant for both T and T^{-1}. Let $\{H_i\}_{i\in\mathbb{N}_0}$ be a family of regular, unit-area, pairwise disjoint hexagons that cover $\mathbb{R}^2$. For $n \in \mathbb{N}_1$, define $H_{i,n} := H_i/\sqrt{n}$ and

$$I_n := \left\{ i \in \mathbb{N}_0 \,:\, H_{i,n} \subseteq T(D) \text{ and } \operatorname{dist}\Big(H_{i,n}, T(\partial D)\Big) > \frac{c_T}{\sqrt{n}} \right\}, \qquad A_n := \bigcup_{i \in I_n} H_{i,n}\,. \tag{5.10.3}$$

Note that we have $A_n \subseteq T(D)$ and

$$D \setminus T^{-1}(A_n) \subseteq \Bigg\{ x \in D \,:\, \operatorname{dist}(x, \partial D) \leq n^{-1/2} \underbrace{c_T \Big(c_T + \operatorname{diam}(H)\Big)}_{=:\bar{c}} \Bigg\}. \tag{5.10.4}$$

Indeed, for every $x \in D \setminus T^{-1}(A_n)$ there exists $i \notin I_n$ such that $T(x) \in H_{i,n}$. There are two cases. If $\operatorname{dist}\Big(H_{i,n}, T(\partial D)\Big) \leq \frac{c_T}{\sqrt{n}}$, then

$$\inf_{y \in \partial D} \|x - y\| \leq c_T \inf_{y \in \partial D} \|T(x) - T(y)\| \leq c_T \underbrace{\operatorname{diam}\Big(H_{i,n}\Big)}_{= n^{-1/2} \operatorname{diam}(H)} + c_T \underbrace{\operatorname{dist}\Big(H_{i,n}, T(\partial D)\Big)}_{\leq n^{-1/2} c_T}.$$

If $H_{i,n} \not\subseteq T(D)$, then, since $H_{i,n}$ is connected, we have $H_{i,n} \cap \partial\Big(T(D)\Big) \neq \emptyset$. By Lemma 5.10.2, we know that $\partial\Big(T(D)\Big) = T(\partial D)$; hence $\operatorname{dist}(x, \partial D) \leq c_T \operatorname{diam}\Big(H_{i,n}\Big)$.

By the measure-theoretic properties of Lipschitz maps (cf. [Mat95, Theorem 7.5]), we have $|T(\partial D)| = 0$ and

$$|A \setminus A_n| = |T(D) \setminus A_n| \leq c_T^2 \Big| D \setminus T^{-1}(A_n) \Big| \to 0 \quad \text{as } n \to \infty \text{ by (5.10.4)}. \tag{5.10.5}$$

Let $k_n := \# I_n$. By Remark 5.3.9, we have the inequality

$$\tilde{e}^p_{p,n}(U_A) \leq \sum_{i \in I_n} \tilde{e}^p_{p,1}\Big(U_A|_{H_{i,n}}\Big) + \tilde{e}^p_{p,n-k_n}\Big(U_A|_{A \setminus A_n}\Big),$$

and the sum over $i \in I_n$ is easy to bound:

$$\sum_{i \in I_n} \tilde{e}^p_{p,1}\Big(U_A|_{H_{i,n}}\Big) \leq k_n \int_{H/\sqrt{n}} \|x\|^p \,\mathrm{d}x \leq n^{-p/2} \int_H \|x\|^p \,\mathrm{d}x\,.$$

Therefore, (5.10.2) is verified once we prove that

$$\limsup_{n \to \infty} n^{p/2} \tilde{e}^p_{p,n-k_n}\Big(U_A|_{A \setminus A_n}\Big) = 0\,. \tag{5.10.6}$$

From now on, we use polar coordinates on $\overline{D}$. Consider the function

$$g(\theta) := \Big| T\left\{ (r, \phi) \in D \setminus T^{-1}(A_n) \,:\, \phi \in [0, \theta] \right\} \Big|, \qquad \theta \in [0, 2\pi]\,, \tag{5.10.7}$$

and note that g is continuous, $g(0) = 0$, and $g(2\pi) = |T(D) \setminus A_n| = |A \setminus A_n| = 1 - \frac{k_n}{n}$. Furthermore, g is strictly increasing: for $0 \leq \theta_1 < \theta_2 \leq 2\pi$, we have

$$\begin{aligned} g(\theta_2) - g(\theta_1) &= \Big| T\left\{ (r, \phi) \in D \setminus T^{-1}(A_n) \,:\, \phi \in (\theta_1, \theta_2] \right\} \Big| \\ &\geq c_T^{-2} \Big| \left\{ (r, \phi) \in D \setminus T^{-1}(A_n) \,:\, \phi \in (\theta_1, \theta_2] \right\} \Big| \\ &\geq c_T^{-2} \Big| \left\{ (r, \phi) \in D \setminus T^{-1}(A_n) \,:\, r \in [1 - n^{-1/2}, 1]\,,\ \phi \in (\theta_1, \theta_2] \right\} \Big| \\ &\geq c_T^{-2} \Big| \left\{ (r, \phi) \in D \,:\, r \in [1 - n^{-1/2}, 1]\,,\ \phi \in (\theta_1, \theta_2] \right\} \Big|\,, \end{aligned}$$

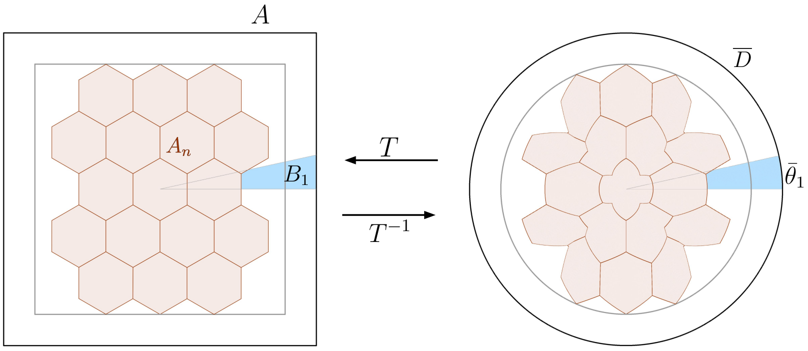


Figure 5.5: Idea for Theorem 5.1.3. We select the hexagons inside $T(D)$ that are sufficiently far from $T(\partial D)$. We define the sets B_i as T-images of intersections of $D \setminus T^{-1}(A_n)$ with angles.

where the last inequality follows from the definition of A_n in (5.10.3). Indeed, if $1 - r \leq n^{-1/2}$ and $T(r, \phi) \in H_{i,n}$, then

$$\operatorname{dist}\Big(H_{i,n}, T(\partial D)\Big) \leq \|T(r,\phi) - T(1,\phi)\| \leq c_T \|(r,\phi) - (1,\phi)\| \leq \frac{c_T}{\sqrt{n}} .$$

Therefore,

$$g(\theta_2) - g(\theta_1) \geq c_T^{-2} \frac{\theta_2 - \theta_1}{\sqrt{n}} \left(1 - \frac{1}{2\sqrt{n}}\right) \geq c_T^{-2} \frac{\theta_2 - \theta_1}{2\sqrt{n}} . \tag{5.10.8}$$

Let us define

$$\begin{aligned} \bar{\theta}_j &\coloneqq g^{-1}(j/n) , && j \in \{0, \dots, n - k_n\} \\ B_j &\coloneqq T\Big\{(r,\phi) \in D \setminus T^{-1}(A_n) \,:\, \phi \in (\bar{\theta}_{j-1}, \bar{\theta}_j]\Big\} , && j \in \{1, \dots, n - k_n\} . \end{aligned} \tag{5.10.9}$$

These sets enjoy two important properties: firstly, by (5.10.7) and (5.10.9),

$$\Big|B_j\Big| = g(\bar{\theta}_j) - g(\bar{\theta}_{j-1}) = \frac{1}{n} ; \tag{5.10.10}$$

secondly, by (5.10.4),

$$B_j \subseteq T\Big\{(r,\phi) \in D \,:\, r \geq 1 - n^{-1/2}\bar{c} ,\ \phi \in (\bar{\theta}_{j-1}, \bar{\theta}_j]\Big\} ,$$

which implies[7]

$$\begin{aligned} \operatorname{diam}(B_j) &\leq c_T(\bar{\theta}_j - \bar{\theta}_{j-1}) + 2c_T \frac{\bar{c}}{\sqrt{n}} \overset{(5.10.8)}{\leq} 2c_T^3 \sqrt{n}\Big(g(\bar{\theta}_j) - g(\bar{\theta}_{j-1})\Big) + 2c_T \frac{\bar{c}}{\sqrt{n}} \\ &\overset{(5.10.9)}{=} 2c_T^3 \frac{1}{\sqrt{n}} + 2c_T \frac{\bar{c}}{\sqrt{n}} \lesssim_T n^{-1/2} . \end{aligned} \tag{5.10.11}$$

We can conclude: by Remark 5.3.9,

$$\begin{aligned} \tilde{e}^p_{p,n-k_n}\Big(U_A|_{A\setminus A_n}\Big) &\overset{(5.10.10)}{\leq} \sum_{j=1}^{n-k_n} \tilde{e}^p_{p,1}\Big(U_A|_{B_j}\Big) \leq \sum_{i=1}^{n-k_n} \frac{\operatorname{diam}(B_j)^p}{n} \\ &\overset{(5.10.11)}{\lesssim_{T,p}} \left(1 - \frac{k_n}{n}\right) n^{-p/2} = |A \setminus A_n|\, n^{-p/2} , \end{aligned}$$

which, together with (5.10.5), implies (5.10.6). □

[7] To move from one point of D to another, one can (inefficiently) walk radially up to the circle ∂D, then along ∂D, and radially again.

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