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Heat flow, log-concavity, and Lipschitz transport maps*

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Abstract

In this paper we derive estimates for the Hessian of the logarithm (log-Hessian) for solutions to the heat equation. For initial data in the form of log-Lipschitz perturbation of strongly log-concave measures, the log-Hessian admits an explicit, uniform (in space) lower bound. This yields a new estimate for the Lipschitz constant of a transport map pushing forward the standard Gaussian to a measure in this class. On the other hand, we show that assuming only fast decay of the tails of the initial datum does not suffice to guarantee uniform log-Hessian upper bounds.

Keywords: heat semigroup; log-Hessian estimates; Lipschitz transport maps; log-concavity; logarithmic Sobolev inequality; score-based diffusion models.
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1 Introduction

Let $d \geqslant 1$. We say that a function $V \colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is α-convex, and that a probability density $\mu \in L^1_+(\mathbb{R}^d)$ is α-log-concave, if, respectively, $x \to V(x) - \frac{\alpha}{2}|x|^2$ is convex, and $\mu(x) = e^{-V(x)}$ for some α-convex function such that $\int_{\mathbb{R}^d} e^{-V(x)} dx = 1$. In case $\alpha = 0$, μ is a log-concave probability density; if $\alpha > 0$, μ is strongly log-concave. We also consider the *heat flow* over $\mathbb{R}^d$:

$$\begin{cases} \partial_t f = \frac{1}{2}\,\Delta f, \\ \lim_{t\to 0}\, f(t,\cdot) = \mu. \end{cases} \tag{1.1}$$

Taking $\mu = \delta_0$, the Dirac delta centered in zero, then the *fundamental solution* to (1.1) is

$$f(t,x) = \gamma_t(x) := (2\pi t)^{-d/2}\, \mathrm{e}^{-|x|^2/2t},$$

where γ_t is the isotropic Gaussian density with zero mean and covariance matrix equal to tI_d. Any other solution to (1.1) is then given by $\mu * \gamma_t$, where $*$ is the symbol of convolution: $(g_1 * g_2)(x) = \int_{\mathbb{R}^d} g_1(x-y)\, g_2(y)\, dy$. Denote by $(P_t)_t$ the corresponding heat semigroup

$$P_t\mu := \mu * \gamma_t, \quad t > 0, \tag{1.2}$$

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which is induced by the flow of (1.1). As solutions to (1.1) are Gaussian convolutions of the initial datum μ, it is expected that those would inherit some features from the Gaussian. There is a vast literature on the subject, which can be roughly classified into
(1). Properties holding as soon as $t > 0$. For example, for all $t > 0$, $f(t, \cdot)$ is smooth [25].
(2). Asymptotic behaviour, in the limit $t \to \infty$, for which we refer to [2, 22, 48].
(3). Properties which are satisfied by $f(t, \cdot)$ for $t \geqslant T$, after a finite time $T > 0$.

1.1 Log-concavity in finite time

As the fundamental solution to (1.1) is log-concave for all $t > 0$, we pose the following.

Question. Given a probability measure μ on $\mathbb{R}^d$, does there exist a time $T > 0$, such that the solution $f(t, x)$ to (1.1) is log-concave for $t \geqslant T$?

In general, we cannot expect *instantaneous* creation of log-concavity, as suggested by the example $\mu = \frac{1}{2}(\delta(1) + \delta(-1)) \in \mathcal{P}(\mathbb{R})$, see [8]. In addition, some hypotheses on the behaviour at infinity of μ shall be required, as suggested by [28]. On the other hand, our question has a positive answer in two known cases. First, if μ is already log-concave, the solution to (1.1) is log-concave at all times, as a consequence of the *Prékopa-Leindler inequality*, see e.g. [45, 44, 39, 6]. Then, by the semigroup property, if a solution to (1.1) is log-concave at a time $T > 0$, this property will be propagated to all $t \geqslant T$. Second, when μ is supported in $B(0, R)$, then $f(t, \cdot)$ is log-concave for all $t \geqslant R^2$, as pointed out first in [38]. More precisely, in [4] it is shown that for all $t > 0$

$$-\nabla^2 \log(\mu * \gamma_t) \succcurlyeq \frac{1}{t}\left(1 - \frac{R^2}{t}\right) I_d, \tag{1.3}$$

as an elementary consequence of (2.2). One aim of ours is to extend the class of measures for which creation of log-concavity in finite time holds, beyond the compactly supported case, motivated also by the series of papers [31, 32, 30, 33], concerning various concavity property of solutions for the heat flow. An analogous question can be posed in the context of functional inequalities satisfied by the Gaussian distribution. Starting from the case of compactly supported measures, previously analysed in [51, 50, 4], Chen, Chewi, and Niles-Weed prove in [12] that if μ is subgaussian, i.e. for some $\epsilon, \mathcal{K} > 0$

$$\int_{\mathbb{R}^d} \mathrm{e}^{\epsilon|x|^2}\,\mu(dx) \leqslant \mathcal{K}, \tag{1.4}$$

then the solution $\mu_t := f(t, \cdot)\,dx$ to (1.1) satisfies a log-Sobolev inequality, for $t \geqslant T(\epsilon, \mathcal{K})$. Moreover, the subgaussianity assumption is also necessary. Indeed, if μ_T satisfies a log-Sobolev inequality for some $T > 0$, then μ_T is also subgaussian [3, Prop. 5.4.1], which implies that μ is subgaussian in the first place. On the other hand, strongly log-concave measures do also satisfy a logarithmic Sobolev inequality, see [2]. Then, one might wonder if (1.4) would be sufficient for a measure to become log-concave along the heat flow. The following theorem implies that this is not the case.

Theorem 1.1. *For all non-decreasing function $\Psi\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, there exists an explicit probability measure on $\mathbb{R}$ such that*

- $\int_{\mathbb{R}} \mathrm{e}^{\Psi(x)}\mu(dx) < \infty$*;*
- *for all* $t > 0$, $\inf_{x\in\mathbb{R}}\left\{-\frac{d^2}{dx^2}\log \mu * \gamma_t\right\} = -\infty$.

Remark 1.2. Similar conclusions hold in arbitrary dimension, as one can see by considering the product probability measure $\mu \times \delta_0 \times \ldots \times \delta_0$, with μ given by Theorem 1.1.

Our result shows that the creation of log-concavity cannot be guaranteed by assuming only some control on the tails of the distributions μ. Therefore, we restrict our analysis to a perturbation regime, i.e. we take measures μ which are close to being strongly log-concave, and we show that they become log-concave after a finite time along (1.1).

Theorem 1.3. *Suppose that $\mu = \mathrm{e}^{-(V+H)} \in L^1_+(\mathbb{R}^d)$, where $V\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is α-convex and $H\colon \mathbb{R}^d \to \mathbb{R}$ is L-Lipschitz for some $\alpha \in \mathbb{R}$, $L \geqslant 0$. Then for every $t > 0$ such that $\alpha t + 1 > 0$ we have*

$$\frac{1}{t}\left[1 - \frac{1}{t}\left(\frac{L}{\alpha + \frac{1}{t}} + \sqrt{\frac{1}{\alpha + \frac{1}{t}}}\right)^2\right] I_d \preccurlyeq -\nabla^2 \log(\mu * \gamma_t) \preccurlyeq \frac{1}{t} I_d. \tag{1.5}$$

In particular, for $\alpha > 0$ and $t \geqslant \left(\frac{L}{\alpha} + \sqrt{\frac{1}{\alpha}}\right)^2$, we have that $\mu * \gamma_t$ is strongly log-concave.

Note that the upper bound in (1.5) is classical, while the lower bound is new. Moreover, for $L = \alpha = 0$, we read that the heat flow preserves log-concavity. Equation (1.5) goes beyond the problem of log-concavity, yielding interesting consequences.

1.2 Application to Lipschitz transport maps

In a seminal paper [9], Caffarelli showed that the Brenier map [7] from optimal transport between the standard Gaussian γ and an α-log-concave probability measure μ is $(1/\sqrt{\alpha})$-Lipschitz. This result is useful because Lipschitz transport maps transfer functional inequalities (including isoperimetric, log-Sobolev and Poincaré inequalities) from a probability measure to another one, and it is typically much easier to prove these inequalities for the Gaussian measure in the first place. For example, suppose that a probability measure μ satisfies the log-Sobolev inequality $\mathrm{LSI}(C)$ for some $C > 0$, i.e. for all regular enough probability measures $\rho \ll \mu$

$$\int \frac{d\rho}{d\mu} \log \frac{d\rho}{d\mu}\, d\mu \leqslant 2C \int \left|\nabla \sqrt{d\rho/d\mu}\right|^2 d\mu, \tag{LSI(C)}$$

where the two sides of the inequalities go under the name of *relative entropy* and *relative Fisher information*, respectively. Suppose, furthermore, that $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is L-Lipschitz and consider the pushforward probability measure $\nu \coloneqq T\#\mu$. Then, ν satisfies $\mathrm{LSI}(L^2 \cdot C)$. Therefore, Caffarelli's result (together with the Gaussian LSI [27]) immediately implies that strongly α-log-concave probability densities satisfy $\mathrm{LSI}(1/\alpha)$, recovering the celebrated result by Bakry and Émery [2]. Further details and many more applications of Lipschitz transport maps are discussed in [41, 20] and the references therein. More recently, Kim and Milman [35] generalized Caffarelli's result by constructing another transport map, which is obtained by reverting an appropriate heat flow, and is referred to as the *heat-flow map* (notation: T^{flow}). Other Lipschitz estimates for this transport map were then provided in [41], where the authors considered different types of assumptions on the target measure ν (namely, measures that satisfy a combination of boundedness and (semi-)log-concavity and some Gaussian convolutions). Several works dealt with the study of Lipschitz transport maps [36, 17, 21, 40, 46, 18, 10, 16]; the recent paper [26] in particular considers an analogous class of target measure as in the present contribution. For comparison, we recall below its main result in the Euclidean setting.

Theorem ([26], Theorem 1)**.** Let $\mu = e^{-(V+H)}$, $\nu = e^{-V}$ be probability densities on $\mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$ we have

$$|\nabla H| \leqslant L, \qquad \nabla^2 V(x) \geqslant \alpha I_d, \qquad \left|\nabla^3 V(x)(w,w)\right| \leqslant K \qquad \text{for all } w \in \mathbb{S}^{d-1},$$

for some $\alpha > 0$, L, $K \geqslant 0$. Then, there exists a transport map $T\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $T\#\nu = \mu$ and T is $\exp\left(\frac{5L^2}{\alpha} + \frac{5\sqrt{\pi}L}{\sqrt{\alpha}} + \frac{LK}{2\alpha^2}\right)$-Lipschitz.

Since Lipschitz transport maps can be composed, this result (combined with Caffarelli's theorem [9]) implies in particular the existence of transport map $\tilde{T}$ such that $\tilde{T}\#\gamma = \mu$ and $\tilde{T}$ is Lipschitz with constant

$$\frac{1}{\sqrt{\alpha}} \exp\left(\frac{5L^2}{\alpha} + \frac{5\sqrt{\pi}L}{\sqrt{\alpha}} + \frac{LK}{2\alpha^2}\right). \tag{1.6}$$

On the other hand, we will prove in Section 3 that our Theorem 1.3 implies new upper bounds on the Lipschitz norm for the heat-flow map from γ to μ.

Theorem 1.4. *Let $\mu = e^{-(V+H)} \in L^1_+(\mathbb{R}^d)$ be a probability density on $\mathbb{R}^d$ such that V is α-convex for $\alpha > 0$ and H is L-Lipschitz for $L \geqslant 0$. Then, there exists a map $T^{\text{flow}} \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $T^{\text{flow}}\#\gamma = \mu$ and T^{flow} is $\frac{1}{\sqrt{\alpha}} \exp\left(\frac{L^2}{2\alpha} + 2\frac{L}{\sqrt{\alpha}}\right)$-Lipschitz.*

Remark 1.5. Consider the case where $d = 1$, $V(x) = \frac{1}{2}x^2$ and $H(x) = L|x| + \log(Z)$ for a normalizing constant Z, so that the assumptions of Theorem 1.4 are satisfied with $\alpha = 1$. Then, it was observed in [26] that the Lipschitz norm of any map T such that $T\#\gamma = \mu$ is at least $\mathrm{e}^{\frac{L^2}{2}}$. Hence, the dependence on L^2 in Theorem 1.4 is sharp.

The estimate for the Lipschitz constant of T^{flow} in Theorem 1.4 improves in particular on the value in (1.6), yielding the best available bound in this setting. Moreover, Theorem 1.4 does not need any assumption on $\nabla^3 V$. On the technical side, in Theorem 1.4 we transport directly γ to μ via the heat-flow map, and our proof only exploits elementary log-Hessian estimates for the heat semigroup, as in Theorem 1.3. On the other hand, [26] employs a construction based on reverting the overdamped Langevin dynamics targeting the measure $\nu = \mathrm{e}^{-V}$: this requires estimates for the corresponding semigroup (cf. [26, Proposition 2]), which is less explicit and needs more sophisticated arguments. We remark that the results of [26] are of independent interest, due to the construction of a Lipschitz map transporting ν to μ therein, and the extension to non-Euclidean spaces.

1.3 Score-based diffusions models

A similar construction as in Section 1.2, based on reverting an ergodic diffusion process, has also recently found application in the machine learning community, within the framework of score-based diffusion models [47, 29]. Let μ be a probability measure, from which we want to generate random samples. Consider the Ornstein–Uhlenbeck process (initialized at μ)

$$X_0 \sim \mu, \qquad dX_t = -X_t dt + \sqrt{2} dB_t,$$

and denote by Q_t the associate semigroup, i.e.

$$Q_t f(x) = \int f\left(e^{-t}x + \sqrt{1 - e^{-2t}}\right)\gamma(x)\, dx. \tag{1.7}$$

The key observation is that this process can be reverted, i.e. for $T_1 > 0$ the *reverse SDE*

$$Y_0 \sim \mathrm{law}(X_{T_1}), \qquad dY_t = -Y_t\, dt + 2\nabla \log Q_{T_1 - t}\left(\frac{d\mu}{d\gamma}\right)(Y_t) dt + \sqrt{2} dB_t \tag{1.8}$$

is such that $Y_{T_1} \sim \mu$, see [1, 11, 47]. Therefore, one wishes simulate the process $(Y_t)_t$ until time T_1 to sample from μ. In practice, this requires to approximate the unknown "score functions" $\log Q_{T_1 - t}\left(\frac{d\mu}{d\gamma}\right)$ with appropriate "score matching" techniques and, since $\mathrm{law}(X_{T_1})$ is not known too, the reverse process is initialized according to γ, which is a good approximation provided that T_1 is big enough.

A common assumption in theoretical works aimed at analysing this method is some control on the Lipschitz constant of $\nabla \log Q_t\left(\frac{d\mu}{d\gamma}\right)$ [14, 15, 13] or on the one-sided one

[37, 43]. These assumptions are indeed useful to control the discretization errors when employing a numerical scheme to simulate the process or some sort of “contractivity” along the reverse dynamics. On the one hand, Theorem 1.3 enlarges the class of distributions μ for which these assumptions can be justified, by implying bounds on the Hessian $\nabla^2 \log Q_t\left(\frac{d\mu}{d\gamma}\right)$ (cf. Corollary 3.2), beyond the setting where the initial distribution μ has bounded support. On the other hand, Theorem 1.1 shows that, for some distributions μ, such assumptions can be too restrictive. Thus, complementary analysis is needed, as done in [19, 5, 13].

2 Log-Lipschitz perturbations of log-concave measures: Proof of Theorem 1.3

Let μ be a probability measure on $\mathbb{R}^d$. For $t > 0$ and $z \in \mathbb{R}^d$, define the probability measure $\mu_{z,t}$ by

$$\mu_{z,t} \propto \exp\left(\frac{z \cdot x}{t} - \frac{|x|^2}{2t}\right)\mu(x) \propto \gamma_{z,t}(x)\mu(x), \tag{2.1}$$

where $\gamma_{z,t}$ is the Gaussian density with mean z and covariance matrix tI_d. We will make frequent use of the following well-known probabilistic characterization of the Hessian of $\log(\mu * \gamma_t)$, cf. [4, 36]:

$$-\nabla^2 \log(\mu * \gamma_t)(z) = \frac{1}{t}\left(I_d - \frac{\mathrm{Cov}_{\mu_{z,t}}}{t}\right). \tag{2.2}$$

Consequently, bounds on $\nabla^2 \log(\mu * \gamma_t)$ are given by bounds on covariance matrices. For this purpose, we provide the following lemma, which gives an upper bound for the covariance matrix of a probability measure μ in terms of the covariance of another probability measure ν and of the Wasserstein distance between the two.

Lemma 2.1. *Let μ, ν be probability measures on $\mathbb{R}^d$. For any unit vector $w \in \mathbb{S}^{d-1}$*

$$\langle w, \mathrm{Cov}_\mu\, w\rangle \leqslant \left(W_2(\mu,\nu) + \sqrt{\langle w, \mathrm{Cov}_\nu\, w\rangle}\right)^2. \tag{2.3}$$

Proof. Let (X, Y) be an optimal coupling for $W_2(\mu,\nu)$. Fix a unit vector $w \in \mathbb{R}^d$ and let $X_w := \langle w, X\rangle$ and $Y_w := \langle w, Y\rangle$. We have that

$$\begin{aligned}
\langle w, \mathrm{Cov}_\mu\, w\rangle &= \mathbb{E}\Big[(X_w - \mathbb{E}[X_w])^2\Big] \leqslant \mathbb{E}\Big[(X_w - \mathbb{E}[Y_w])^2\Big] = \mathbb{E}\Big[(X_w - Y_w + Y_w - \mathbb{E}[Y_w])^2\Big] \\
&\leqslant \left(\sqrt{\mathbb{E}\Big[(X_w - Y_w)^2\Big]} + \sqrt{\mathbb{E}\Big[(Y_w - \mathbb{E}[Y_w])^2\Big]}\right)^2 \qquad \text{(by Cauchy–Schwarz)} \\
&\leqslant \left(W_2(\mu,\nu) + \sqrt{\mathbb{E}\Big[(Y_w - \mathbb{E}[Y_w])^2\Big]}\right)^2 = \left(W_2(\mu,\nu) + \sqrt{\langle w, \mathrm{Cov}_\mu\, w\rangle}\right)^2. \qquad \square
\end{aligned}$$

Proof of Theorem 1.3. The upper bound in (1.5) is well known, and holds for arbitrary probability measures μ, as it follows from (2.2) and the fact that covariance matrices are positive semidefinite (cf., [23, Lemma 1.3]). Let us then turn to the first inequality. Fix $t > 0$ and $z \in \mathbb{R}^d$. Define the probability density $\nu_{z,t} \in L^1_+(\mathbb{R}^d)$ by $\nu_{z,t} \propto e^{-V}\gamma_{z,t}$. Notice that $\nu_{z,t}$ is $(\alpha + \frac{1}{t})$-log-concave: therefore, $\mathrm{Cov}_{\nu_{z,t}} \preccurlyeq \frac{1}{\alpha + \frac{1}{t}} I_d$ by the Brascamp–Lieb inequality [6] (cf. also [24, Lemma 5]). To turn this into an upper bound for $\mathrm{Cov}_{\mu_{z,t}}$ with Lemma 2.1, the key step is now to control $W_2(\mu_{z,t}, \nu_{z,t})$. Observe that $\mu_{z,t} \propto e^{-H}\nu_{z,t}$: hence, $\mu_{z,t}$ is a log-Lipschitz perturbation of the strongly log-concave measure $\nu_{z,t}$. This is precisely the setting of [34, Corollary 2.4], which provides an upper bound for even the stronger L^∞-Wasserstein distance between the two:

$$W_2(\mu_{z,t},\nu_{z,t}) \leqslant W_\infty(\mu_{z,t},\nu_{z,t}) \leqslant \frac{L}{\alpha+\frac{1}{t}}.$$

We are now in position to apply Lemma 2.1: for any unit vector $v \in \mathbb{R}^d$ we have

$$\langle v, \mathrm{Cov}_{\mu_{z,t}} v\rangle \leqslant \left(W_2(\mu_{z,t},\nu_{z,t}) + \sqrt{\langle v, \mathrm{Cov}_{\nu_{z,t}} v\rangle}\right)^2 \leqslant \left(\frac{L}{\alpha+\frac{1}{t}} + \sqrt{\frac{1}{\alpha+\frac{1}{t}}}\right)^2.$$

This shows that $\mathrm{Cov}_{\mu_{z,t}} \preccurlyeq \left(\frac{L}{\alpha+\frac{1}{t}} + \sqrt{\frac{1}{\alpha+\frac{1}{t}}}\right)^2 I_d$, and the conclusion follows from (2.2). □

Remark 2.2. In the proof of Theorem 1.3, we estimated from above $W_2(\mu_{z,t},\nu_{z,t})$ with the L^∞-Wasserstein distance $W_\infty(\mu_{z,t},\nu_{z,t})$. Alternatively, we could have achieved the same conclusion as follows, using that $\nu_{z,t}$ satisfies $\mathrm{LSI}\left(\frac{t}{\alpha t+1}\right)$. First, a transport-entropy inequality [42] allows to estimate $W_2(\mu_{z,t},\nu_{z,t})$ in terms of the relative entropy of $\mu_{z,t}$ with respect to $\nu_{z,t}$; then, the relative entropy is bounded from above by the relative Fisher information using the logarithmic Sobolev inequality of $\nu_{z,t}$; finally, the relative Fisher information is easily estimated using that $\mu_{z,t} \propto e^{-H}\nu_{z,t}$ and H is L-Lipschitz.

2.1 Sufficient conditions

By Theorem 1.3, log-Lipschitz perturbations of strongly log-concave measures become log-concave in finite time along (1.1); by Theorem 1.4, they are the pushforward of the Gaussian measure via a Lipschitz transport map. The purpose of this subsection is to give sufficient conditions for a measure μ to be a log-Lipschitz perturbation of a strongly log-concave measure. Consider hence a probability density $\mu = \mathrm{e}^{-U} \in L^1_+(\mathbb{R}^d)$ for some $U \in C^2(\mathbb{R}^d)$. The following result asserts that, if we have a uniform positive lower bound for the Hessian of U outside some Euclidean ball, then we can rewrite μ as a log-Lipschitz perturbation of a strongly log-concave measure.

Lemma 2.3. *Let $U \in C^2(\mathbb{R}^d)$ be such that for some $\alpha,\beta,R \geqslant 0$ it holds that*

$$\begin{cases} \nabla^2 U(x) \succcurlyeq \alpha I_d & \text{if } |x| \geqslant R, \\ \nabla^2 U(x) \succcurlyeq -\beta I_d & \text{if } |x| < R. \end{cases}$$

Then, there exist $V, H \in C^1(\mathbb{R}^d)$ such that $U = V + H$, V is α-convex and H is $2(\alpha+\beta)R$-Lipschitz.

Proof. Let $H\colon \mathbb{R}^d \to \mathbb{R}$ be defined by

$$-H(x) = \begin{cases} (\alpha+\beta)|x|^2 & \text{if } |x| \leqslant R, \\ 2(\alpha+\beta)R|x| - 2(\alpha+\beta)R^2 & \text{if } |x| \geqslant R, \end{cases}$$

and set $V(x) = U(x) - H(x)$. Then we have that $U = V + H$, $V \in C^1(\mathbb{R}^d)$ is α-convex and $|\nabla H| \leqslant 2(\alpha+\beta)R$, as desired. □

Lemma 2.3 can be useful to study linear combinations of strongly log-concave densities, via the following

Proposition 2.4. *Consider a measure $\mu = \sum_{i=1}^N \alpha_i \, \mathrm{e}^{-U_i}$ for some $N > 0$, weights $\alpha_i > 0$ and potentials $U_i \in C^2(\mathbb{R}^d)$ such that $\mathrm{e}^{-U_i} \in L^1_+(\mathbb{R}^d)$. Assume $\nabla^2 U_i \succcurlyeq K I_d$ for all i and some $K > 0$. Then*

$$-\nabla^2 \log \, \mu \succcurlyeq K I_d - \frac{\sum_{i>j} \alpha_i \alpha_j \mathrm{e}^{-U_i - U_j} (\nabla U_i - \nabla U_j)^{\otimes 2}}{\mu^2} \tag{2.4}$$

$$\succcurlyeq K I_d - \sum_{i>j} \frac{(\nabla U_i - \nabla U_j)^{\otimes 2}}{\left(2 + \frac{\alpha_i}{\alpha_j} \mathrm{e}^{U_j - U_i} + \frac{\alpha_j}{\alpha_i} \mathrm{e}^{U_i - U_j}\right)}. \tag{2.5}$$

Proof. Notice that $-\nabla^2 \log \, \mu = \mu^{-2}(\nabla \mu \otimes \nabla \mu - \mu \nabla^2 \mu)$. Set $\mu_i := \alpha_i \mathrm{e}^{-U_i}$ so that $\mu = \sum_{i=1}^N \mu_i$. By construction $\nabla \mu_i = -\nabla U_i \, \mu_i$, and $\nabla^2 \mu_i = (-\nabla^2 U_i + \nabla U_i \otimes \nabla U_i)\mu_i$, for all $i = 1, \ldots, N$. Then,

$$\begin{aligned}
-\nabla^2 \log \, \mu &= \frac{\left(\sum_{i=1}^N \nabla U_i \, \mu_i\right)^{\otimes 2} - \left(\sum_{i=1}^N \mu_i\right) \left(\sum_{i=1}^N (-\nabla^2 U_i + \nabla U_i \otimes \nabla U_i)\mu_i\right)}{\mu^2} \\
&= \frac{\mu \sum_{i=1}^N \nabla^2 U_i \, \mu_i - \sum_{i,j=1}^N \mu_i \, \mu_j (\nabla U_i \otimes \nabla U_j - \nabla U_j \otimes \nabla U_j)}{\mu^2} \\
&\succcurlyeq K \, I_d - \frac{\sum_{i>j} \mu_i \mu_j (\nabla U_i - \nabla U_j)^{\otimes 2}}{\mu^2},
\end{aligned}$$

which shows (2.4). Since $\mu^2 = \sum_{l,m=1}^N \mu_l \, \mu_m \geqslant 2\mu_i \, \mu_j + \mu_i^2 + \mu_j^2$ for $i \neq j$, (2.5) follows. $\square$

From the above proposition, it is clear that when the right-hand-side of (2.4) is uniformly positive definite outside a Euclidean ball, then by Lemma 2.3 we can recast μ as a log-Lipschitz perturbation of a strongly log-concave measure. Therefore, the assumptions of Theorem 1.3 are satisfied, and $\mu * \gamma_t$ becomes strongly log-concave in finite time along the heat flow (1.1). We illustrate this in the following example, where μ is a finite mixture of Gaussians in dimension 1.

Example 2.5. Let μ be a linear combination of one-dimensional Gaussians, i.e. $\mu = \sum_{i=1}^N \alpha_i \, \mathrm{e}^{-U_i}$ for some $N \geqslant 2$, weights $\alpha_i > 0$ and potentials U_i of the form $U_i(x) = (x - m_i)^2/\sigma_i^2$ for some $m_i \in \mathbb{R}, \sigma_i^2 > 0$. Without loss of generality we can assume that $U_i \neq U_j$ for $i \neq j$. By Proposition 2.4, we have that

$$-\frac{d^2}{dx^2} \log \mu \succcurlyeq \frac{1}{\max_i \sigma_i^2} - \sum_{i>j} \frac{(U_i' - U_j')^2}{\left(2 + \frac{\alpha_i}{\alpha_j} \mathrm{e}^{U_j - U_i} + \frac{\alpha_j}{\alpha_i} \mathrm{e}^{U_i - U_j}\right)}.$$

It is then not difficult to see that the argument of the sum in the right-hand-side converges to 0 as $|x| \to \infty$. By the previous discussion, it follows that the assumptions of Theorem 1.3 are satisfied for some $L, \alpha > 0$: hence, a finite linear combination of Gaussian densities on $\mathbb{R}$ becomes strongly log-concave in finite time along the heat flow.

3 Lipschitz transport maps: Proof of Theorem 1.4

Construction of the heat-flow map by Kim and Milman Let $\mu \in L^1_+(\mathbb{R}^d)$ be a probability density on $\mathbb{R}^d$. Assume, furthermore, that μ has finite second-order moment. We begin by sketching the construction of the heat-flow map, and refer the reader to [35, 41] for details. The idea is to interpolate between μ and γ along the Ornstein–Uhlenbeck flow

$$X_0 \sim \mu, \qquad dX_t = -X_t dt + \sqrt{2} dB_t. \tag{3.1}$$

Let us denote by Q_t the associated transition semigroup (1.7) and by μ_t the law of X_t. Then, μ_t satisfies the Fokker–Planck equation

$$\partial\mu_t - \boldsymbol{\nabla}\cdot\left[\mu_t \nabla \log Q_t\left(\frac{d\mu}{d\gamma}\right)\right] = 0.$$

Correspondingly, we can consider the flow maps $(S_t)_{t\geqslant 0}$ obtained by solving

$$S_0(x) = x, \quad \frac{\mathrm{d}}{\mathrm{d}t} S_t(x) = -\nabla \log Q_t\left(\frac{d\mu}{d\gamma}\right)$$

for all $x \in \mathbb{R}^d$. Under some regularity assumptions (cf. [35, 41, 42, 49]), this defines a flow of diffeomorphisms such that $S_t\#\mu = \mu_t$; conversely, $T_t := S_t^{-1}$ is such that $T_t\#\mu_t = \mu$. The heat-flow map is then heuristically defined by $T^{\text{flow}} = \lim_{t\to\infty} T_t$ and is such that $T^{\text{flow}}\#\gamma = \mu$. To make things rigorous, we recall the following result from [41].

Lemma 3.1. *Suppose that $\mu \in L^1_+(\mathbb{R}^d)$ is a probability density with finite second-order moment. Suppose, furthermore, that for all $t > 0$ there exist $\theta_t^{\max}, \theta_t^{\min} \in \mathbb{R}$ such that*

$$\theta_t^{\min} I_d \preccurlyeq \nabla^2 \log Q_t\left(\frac{d\mu}{d\gamma}\right) \preccurlyeq \theta_t^{\max} I_d \tag{3.2}$$

and for all $s > 1$, $\sup_{\frac{1}{s}<t<s} \max\left\{|\theta_t^{\min}|, |\theta_t^{\max}|\right\} < \infty$.

Then, provided that $L := \limsup_{t\to\infty} \int_{\frac{1}{t}}^{t} \theta_t^{\max} dt < \infty$, there exists a map $T\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $T\#\gamma = \mu$ and T is e^L-Lipschitz.

Proof. Notice first of all that μ_t is a smooth density for every $t > 0$. Fix $s > 0$: by the assumptions in the Lemma and by [41, Lemma 10 and 11] there exists a map T_s which is $\exp\left(\int_{\frac{1}{s}}^{s} \theta_t^{\max}\, dt\right)$-Lipschitz and such that $T_s\#\mu_s = \mu_{\frac{1}{s}}$. Since $\mu_s \to \gamma$ and $\mu_{\frac{1}{s}} \to \mu$ in W_2-distance (hence weakly) as $s \to \infty$, the conclusion follows from [41, Lemma 9]. ☐

New estimates In view of Lemma 3.1, the goal is to provide estimates on $\nabla^2 \log Q_t\left(\frac{d\mu}{d\gamma}\right)$, for some probability measures μ on $\mathbb{R}^d$. The following is a consequence of Theorem 1.3.

Corollary 3.2 (Corollary of Thm. 1.3)**.** *Let $\mu = e^{-V-H} \in L^1_+(\mathbb{R}^d)$ be a probability density on $\mathbb{R}^d$ such that V is α-convex and H is L-Lipschitz, for some $\alpha \in \mathbb{R}, L \geqslant 0$. Then for every $t > 0$ such that $\alpha(e^{2t} - 1) + 1 > 0$ we have*

$$\begin{aligned} -\frac{1}{e^{2t}-1}\, I_d \preccurlyeq& \nabla^2 \log Q_t\left(\frac{d\mu}{d\gamma}\right) \\ &\preccurlyeq \left(\frac{1-\alpha}{\alpha(e^{2t}-1)+1} + \frac{e^{2t}L^2}{(\alpha(e^{2t}-1)+1)^2} + \frac{2Le^{2t}}{\sqrt{(e^{2t}-1)}\,(\alpha(e^{2t}-1)+1)^{3/2}}\right) I_d. \end{aligned} \tag{3.3}$$

Proof. Let us consider X_t as in (3.1) and denote by $\mu_t = \mathrm{law}(X_t)$ the probability density of X_t. It is well known that $Q_t\left(\frac{d\mu}{d\gamma}\right) = \frac{d\mu_t}{d\gamma}$ and that $\mu_t = \mathrm{law}\left(e^{-t}\left[X_0 + \sqrt{e^{2t}-1}Z\right]\right)$, where $Z \sim \gamma$ is independent of X_0. The conclusion then follows easily from Theorem 1.3 by rescaling and noticing that $\nabla^2 \log \frac{d\mu_t}{d\gamma} = \nabla^2 \log \mu_t + I_d$. ☐

Proof of Theorem 1.4. We integrate the upper bound in (3.3). An elementary computation using the change of variable $\tau = e^{2t} - 1$ shows that

$$\int_0^\infty \left(\frac{1-\alpha}{\alpha(e^{2t}-1)+1} + \frac{e^{2t}L^2}{(\alpha(e^{2t}-1)+1)^2} + \frac{2Le^{2t}}{\sqrt{(e^{2t}-1)}\,(\alpha(e^{2t}-1)+1)^{3/2}}\right) dt$$

$$
\begin{aligned}
&= \int_0^\infty \left(\frac{1-\alpha}{\tau\alpha+1} + L^2 \frac{\tau+1}{(\tau\alpha+1)^2} + 2L \frac{\tau+1}{\sqrt{\tau}\,(\tau\alpha+1)^{3/2}} \right) \frac{1}{2(\tau+1)} d\tau \\
&= -\frac{1}{2}\log(\alpha) + \frac{L^2}{2\alpha} + 2\frac{L}{\sqrt{\alpha}}.
\end{aligned}
$$

The desired conclusion then follows from Lemma 3.1. □

4 The negative result: Proof of Theorem 1.1

Before proving the actual theorem, we give some heuristics behind the proof. The leading idea is the following. If one considers (1.1) with $\mu = \delta_0$, then the solution is immediately log-concave for $t > 0$. However, this behaviour is not stable.

Proposition 4.1. *Fix $x_0 \in \mathbb{R}$. Let $\mu = \frac{\alpha}{\alpha+\beta}\delta_0 + \frac{\beta}{\alpha+\beta}\delta_{x_0}$, for some $\alpha, \beta > 0$. Then, $\mu * \gamma_t$ is log-concave (if and) only if $t \geqslant \frac{1}{4}x_0^2$.*

Proof. We prove only the *only if* part, since the other implication follows directly from (1.3). It is not difficult to see that with $x_0, t, \alpha, \beta > 0$ fixed, there exists $\bar{z} \in \mathbb{R}$ for which

$$
\alpha e^{-\frac{\bar{z}^2}{2t}} = \beta e^{-\frac{(\bar{z}-x_0)^2}{2t}}.
$$

Then, using (2.2), we have $\frac{d^2}{dx^2}(-\log \mu * \gamma_t)(\bar{z}) = \frac{1}{t}\left(1 - \frac{x_0^2}{4t}\right) < 0$ when $t < x_0^2/4$. □

From equation (1.3) we see that a compactly-supported distribution becomes log-concave along (1.1) after a time $T = O(R^2)$. Proposition 4.1 gives a simple account of this time scale being correct. In addition, we see that the time needed for the measure μ of Proposition 4.1 to become log-concave along (1.1) does not depend on the mass of the perturbation δ_{x_0}. Exploiting these observations allows us to create mixtures of Dirac deltas with arbitrarily thin tails, which never become log-concave along (1.1).

Proof of Theorem 1.1. For $i \geqslant 0$, set $x_i = \frac{i(i+1)}{2} \geqslant 0$. Define the probability measure μ on $\mathbb{R}$ by

$$
\mu \propto \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} e^{-\Psi(x_i)} \delta_{x_i}
$$

and let $X \sim \mu$. It is immediate that $\mathbb{E}\left[e^{\Psi(X)}\right] < \infty$. Fix $t \geqslant 0$ and recall from (2.2) that

$$
\begin{aligned}
-\frac{d^2}{dx^2}\log \mu * \gamma_t(z) &= \frac{1}{t}\left(1 - \frac{\mathrm{Var}_{\mu_{z,t}}}{t}\right), \\
\mu_{z,t}(x) \propto e^{\frac{zx}{t} - \frac{x^2}{2t}}\mu(x) &\propto \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} e^{-\Psi(x_i) + \frac{zx_i}{t} - \frac{x_i^2}{2t}} \delta_{x_i}.
\end{aligned}
$$

Therefore, it suffices to prove that, for every $M > 0$, there exists z such that $\mathrm{Var}_{\mu_{z,t}} \geqslant M^2$. To this end, fix M and choose $j \geqslant \sqrt{2}M$ so that $|x_j - x_{j-1}|^2 = j^2 \geqslant 2M^2$. To conclude, it suffices to show that there exists $z \in \mathbb{R}$ such that

$$
\mu_{z,t}([0, x_{j-1}]) = \frac{1}{2} = \mu_{z,t}([x_j, +\infty]). \tag{4.1}
$$

Indeed, the above implies that $\mathrm{Var}_{\mu_{z,t}} \geqslant M^2$. Notice now that (4.1) is equivalent to finding a solution to the equation $F(z) = 0$, where

$$
F(z) = \sum_{i=0}^{j-1} \frac{1}{(i+1)^2} e^{-\Psi(x_i) + \frac{zx_i}{t} - \frac{x_i^2}{2t}} - \sum_{i=j}^{\infty} \frac{1}{(i+1)^2} e^{-\Psi(x_i) + \frac{zx_i}{t} - \frac{x_i^2}{2t}}. \tag{4.2}
$$

It is straightforward to check that $F(0) \geqslant 0$, e.g. using that $1 > \sum_{i=1}^{\infty} \frac{1}{(i+1)^2}$ and that Ψ is non-decreasing. Moreover, F is continuous, since for any compact interval $[a, b] \subset \mathbb{R}$, the series in (4.2) converges uniformly in $C([a, b])$. To conclude, we show now that $\lim_{z\to\infty} F(z) = -\infty$. To this end, notice that

$$F(z) \leqslant je^{-\Psi(0)+\frac{zx_{j-1}}{t}} - \frac{1}{(j+1)^2} e^{-\Psi(x_j)-\frac{x_j^2}{2t}+\frac{zx_j}{t}} = e^{\frac{zx_{j-1}}{t}} \left(je^{-\Psi(0)} - \frac{e^{-\Psi(x_j)-\frac{x_j^2}{2t}} e^{\frac{zj}{t}}}{(j+1)^2} \right),$$

which yields the desired conclusion since $j > 0$. □

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