La Matematica (2026) 5:5
https://doi.org/10.1007/s44007-025-00180-y

REVIEW ARTICLE

Logarithmic Sobolev Inequalities: A Review on Stability and Instability Results

Giovanni Brigati[1] · Jean Dolbeault[2] · Nikita Simonov[3]

Received: 11 April 2025 / Revised: 7 November 2025 / Accepted: 11 November 2025
© The Author(s) 2025

Abstract
In this paper, we review recent results on stability and instability in logarithmic Sobolev inequalities, with a particular emphasis on strong norms. We consider several versions of these inequalities on the Euclidean space, for the Lebesgue and the Gaussian measures, and discuss their differences in terms of moments and stability. We give new and direct proofs, as well as examples and discuss the stability of a logarithmic uncertainty principle. Although we do not cover all aspects of the topic, we hope to contribute to establishing the state of the art.

Keywords Logarithmic Sobolev inequality · Stability · Logarithmic uncertainty principle

Mathematics Subject Classification Primary: 39B62 · Secondary: 47J20 · 49J40 · 35A23

Authors supported by their respective institutionsxm.

✉ Giovanni Brigati
giovanni.brigati@ist.ac.at

Jean Dolbeault
dolbeaul@ceremade.dauphine.fr

Nikita Simonov
nikita.simonov@sorbonne-universite.fr

[1] ISTA, Institute of Science and Technology Austria, Am Campus 1, Klosterneuburg 3400, Austria

[2] CEREMADE, Centre de Recherche en Mathématiques de la Décision (CNRS UMR n° 7534), PSL University, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

[3] Laboratoire Jacques-Louis Lions, LJLL, CNRS UMR 7598, Sorbonne Université, Université Paris Cité, F-75005 Paris, France

Published online: 08 January 2026

1 Introduction

Let $d \geq 1$, and let $d\gamma = \gamma(x)\,dx$, with $\gamma(x) = (2\,\pi)^{-d/2}\,\mathrm{e}^{-|x|^2/2}$, be the normalized Gaussian probability measure. The *Gaussian logarithmic Sobolev inequality* on $\mathbb{R}^d$ reads as

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma \geq \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \tag{1}$$

for any function $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$. Moreover, by Jensen's inequality, we know that the right-hand side of (1) is non-negative. Throughout the paper, we will consider only real-valued functions.

If *v* is a smooth and compactly supported function such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$, then an elementary computation shows that (1) written for *v* is equivalent for $u = v\,\sqrt{\gamma}$ to the *Euclidean logarithmic Sobolev inequality* on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \geq \frac{1}{2}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx + \frac{d}{4}\,\log\left(2\,\pi\,\mathrm{e}^2\right) \tag{2}$$

which, by density, holds for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. However, even if *u* is smooth and compactly supported, it does not mean that $\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx$ is uniformly bounded from below, whatever $\|u\|_{\mathrm{H}^1(\mathbb{R}^d,dx)}$ is.

On $(\mathbb{R}^d, dx)$, one can take advantage of scalings. For any $\lambda > 0$, let us consider

$$u_\lambda(x) := \lambda^{d/4}\,u\big(\sqrt{\lambda}\,x\big) \quad \forall\,x \in \mathbb{R}^d.$$

Inequality (2) applied to u_λ becomes

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \geq \frac{1}{2\,\lambda}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx + \frac{d}{4\,\lambda}\,\log\left(2\,\pi\,\mathrm{e}^2\,\lambda\right) \tag{3}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. After optimizing on $\lambda > 0$, we obtain the *Euclidean logarithmic Sobolev inequality in scale invariant form*;

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \geq \frac{\pi\,d\,\mathsf{e}}{2}\,\exp\left(\frac{2}{d}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx\right) \tag{4}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$.

Logarithmic Sobolev inequalities have a long history. The *Gaussian logarithmic Sobolev inequality* (1) is due to L. Gross in [1] and its equivalence with (2) is well-known (see for instance [2, Identity (3)]), while its scale invariant form (4) appeared in [3, Inequality (2.3)] in dimension $d = 1$ and in [4, Theorem 2] for any $d \geq 1$. Among earlier related results, one has to quote [5]. We refer to [6, Section 1.3.2] and also to [7–9] for further background references in information theory and to [2] for the equality case, as well as an early stability result. See [10–14] and references therein for more recent results and [15–18] for related books.

A function achieving equality in a given functional inequality, e.g., (1), is called an *optimal function* or an *optimiser*. We indicate with $\mathcal{M}$ the manifold of all optimisers. Optimality in (1), which can also be deduced from [19], is characterised as follows in [2]: v is optimal if and only if

$$v \in \mathcal{M} := \big\{ v_{a,b} := a\, \mathrm{e}^{b \cdot x} \;:\; a \in \mathbb{R},\, b \in \mathbb{R}^d \big\}.$$

The goal of this paper is to review some *stability properties* of Inequalities (1), (2), (3) and (4), mostly in strong norms. In the case of (1), the Gaussian *deficit* is defined by

$$\delta[v] := \int_{\mathbb{R}^d} |\nabla v|^2\, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \,\log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}}\right) d\gamma \tag{5}$$

and we aim either at an *improved inequality* showing that $\delta[v]$ is bounded from below by a functional evaluated on v under a constraint (otherwise (1) would not be optimal), or by a distance to the manifold of optimal functions (see Theorem 10).

For Sobolev's inequality, the issue was raised by H. Brezis and E. Lieb in [20]. Slightly earlier, in [21], P.-L. Lions proved a *sequential stability* property: a normalized sequence of optimizing functions $(u_n)_{n\in\mathbb{N}}$ converges in $\dot{\mathrm{H}}^1(\mathbb{R}^d, dx)$ to an Aubin-Talenti function via the concentration-compactness method, up to the extraction of a subsequence (relative compactness) and after taking advantage of the invariances. Soon after [20], G. Bianchi and H. Egnell proved in [22] that for some constant $\kappa_d > 0$, the deficit associated with Sobolev's inequality is bounded from below by $\kappa_d\, \mathrm{d}(v, \mathcal{M})^2$ where d is the distance induced by $\dot{\mathrm{H}}^1(\mathbb{R}^d, dx)$ and $\mathcal{M}$ is the manifold of the *Aubin-Talenti functions*. A lower bound on κ_d is known from the recent work [23].

For the logarithmic Sobolev inequality, it is thus natural to ask whether there is a *quantitative stability property* for (1), that is, whether there is some $\kappa > 0$ such that

$$\delta[v] \geq \kappa\, \mathrm{d}(v, \mathcal{M})^2 \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma),$$

where $\mathcal{M}$ is now the manifold of optimal functions for (1) and investigate for which distance d this stability inequality holds true. Going back to [10, 24], results are know when d is a Wasserstein distance. A conditional stability result in L^2 was obtained in [25, Proposition 4.7]. The stability inequality is true if d is induced by $\mathrm{L}^2(\mathbb{R}^d, d\gamma)$ according to [23] but it is not true without additional assumptions if d is based on $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ as shown for instance in [14, 26, 27]. We shall give details on known stability results in Sect. 3 and elaborate on examples of instabilities based on [26, 27] in Sect. 4, while in Sect. 2 we collect relevant facts on H^1 spaces, second-order moments, and entropy functionals. We also emphasize a few differences between (1), (2), (3), and (4). Second moment estimates play a crucial role in many partial results. We would like to draw the attention of the reader to a *logarithmic uncertainty principle* and its stability (see Sect. 2.3) which seems remarkable.

2 H^1 Spaces and Logarithmic Sobolev Inequalities

Let us start by collecting some observations on the differences between the H^1 spaces with respect to Lebesgue and Gaussian measures and the consequences for the corresponding forms of the logarithmic Sobolev inequalities on $\mathbb{R}^d$. The space $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ is obtained by the completion of smooth and compactly supported functions with respect to the norm $v \mapsto \|v\|_{\mathrm{H}^1(\mathbb{R}^d,d\gamma)}$ with $\|v\|^2_{\mathrm{H}^1(\mathbb{R}^d,d\gamma)} = \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}$, and it is classical in the study of the logarithmic Sobolev inequality, see, e.g., [17, Def. 3.1.11].

2.1 Integrability and Averages in the Euclidean Case

The *Euclidean logarithmic Sobolev inequality* (2) on $\mathbb{R}^d$ can be written for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. This is not enough to prove that $\int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx$ is uniformly bounded from below as shown by the following examples.

Example 1 Assume that $d = 1$ and let u be a smooth function on $\mathbb{R}$ with compact support in (0, 1). Let

$$u_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} u(x+k) \tag{6}$$

so that $\|u_n\|_{\mathrm{L}^2(\mathbb{R})} = \|u\|_{\mathrm{L}^2(\mathbb{R})}$ and $\|\nabla u_n\|_{\mathrm{L}^2(\mathbb{R})} = \|\nabla u\|_{\mathrm{L}^2(\mathbb{R})}$ for any $n \geq 1$, while

$$\int_{\mathbb{R}} |u_n|^2 \log |u_n|^2 \, dx = \int_{\mathbb{R}} |u|^2 \log |u|^2 \, dx - \|u\|^2_{\mathrm{L}^2(\mathbb{R})} \log n \to -\infty \quad \text{as} \quad n \to +\infty.$$

Example 2 On $\mathbb{R}^d$, let us consider the function

$$u(x) = \left(1 + |x|^2\right)^{-\frac{d}{4}} \left(\log\left(2 + |x|^2\right)\right)^{-\frac{a}{2}} \quad \forall\, x \in \mathbb{R}^d$$

for some $a \in (1, 2)$. This function is smooth and such that as $|x| \to +\infty$

$$|x|^2 \, |\nabla u(x)|^2 \sim \frac{d^2}{4} \, |u(x)|^2 = O\left(|x|^{-d} \, (\log |x|)^{-a}\right) ,$$
$$|u(x)|^2 \log |u(x)|^2 = O\left(|x|^{-d} \, (\log |x|)^{1-a}\right) .$$

One can check that $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ is such that

$$\lim_{R\to+\infty}\int_{|x|<R}|u|^2\,\log|u|^2\,dx=-\infty.$$

It is a natural question to ask under which additional condition on $u\in\mathrm{H}^1(\mathbb{R}^d,dx)$ one can guarantee that $|u|^2\,\log|u|^2\in\mathrm{L}^1(\mathbb{R}^d)$. If this is the case, let us observe that we can choose $\lambda>0$ such that $\int_{\mathbb{R}^d}|u_\lambda|^2\,\log|u_\lambda|^2\,dx=0$ where $u_\lambda:=\lambda^{d/2}\,u(\lambda\cdot)$ as

$$\int_{\mathbb{R}^d}|u_\lambda|^2\,\log|u_\lambda|^2\,dx=\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx+d\,\log\lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

uniquely determines λ. Interestingly, we have a reciprocal result that goes as follows. Let us consider the Gagliardo-Nirenberg-Sobolev inequality

$$\|\nabla u\|^\theta_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\ge\mathcal{C}_{\mathrm{GNS}}(d,p)\,\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d,dx)\tag{7}$$

where $\theta=d\,(p-2)/(2\,p)$ and $\mathcal{C}_{\mathrm{GNS}}(d,p)>0$ is the optimal constant. The exponent p is larger than 2, with the additional restriction that $p\le2\,d/(d-2)$ if $d\ge3$. If $d\ge3$ and $p=2\,d/(d-2)$, then $\theta=1$ and (7) is the classical Sobolev inequality.

Proposition 1 *With this notation and p as above, if u is a smooth and compactly supported function such that* $\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx=0$*, then*

$$\int_{\mathbb{R}^d}\big|\,|u|^2\,\log|u|^2\big|\,dx\le4\,\frac{\Big(\|\nabla u\|^\theta_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\Big)^p}{(p-2)\,\mathrm{e}\,\mathcal{C}_{\mathrm{GNS}}(d,p)^p}.$$

By density, the result of Proposition 1 also holds in $\mathrm{H}^1(\mathbb{R}^d,dx)$.

Proof A simple optimization shows that

$$\sup_{t>1}\frac{t^2\,\log t^2}{t^p}=\frac{2}{(p-2)\,\mathrm{e}}$$

for any $p>2$. As a consequence with $t=|u|$, we have

$$-\int_{|u|\le1}|u|^2\,\log|u|^2\,dx=\int_{|u|\ge1}|u|^2\,\log|u|^2\,dx\le2\,\frac{\|u\|^p_{\mathrm{L}^p(\mathbb{R}^d)}}{(p-2)\,\mathrm{e}},$$

which completes the proof by using $\int_{\mathbb{R}^d}\big|\,|u|^2\,\log|u|^2\big|\,dx=2\int_{|u|\ge1}|u|^2\,\log|u|^2\,dx$ and (7). □

We can deduce a criterion of integrability from Proposition 1.

Corollary 2 *If* $u\in\mathrm{H}^1(\mathbb{R}^d,dx)\setminus\{0\}$*, then*

(i) *either for any sequence* $(u_n)_{n\in\mathbb{N}}$ *of smooth and compactly supported functions* *on* $\mathbb{R}^d$ *such that* $\lim_{n\to+\infty}\big(\|\nabla u-\nabla u_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\|u-u_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\big)=0$*, we have*

$$\lim_{n\to+\infty}\int_{\mathbb{R}}|u_n|^2\,\log|u_n|^2\,dx=-\infty,$$

(ii) *or the function u is such that* $|u|^2\,\log|u|^2\in\mathrm{L}^1(\mathbb{R}^d)$.

Proof If (i) does not hold, then one can find a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\lambda_n=\exp\left(-\frac{1}{d}\,\frac{\int_{\mathbb{R}^d}|u_n|^2\,\log|u_n|^2\,dx}{\|u_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)$$

converges to some $\lambda\ge 0$. It is then clear that $\widetilde{u}_n=\lambda_n^{d/2}\,u_n\big(\sqrt{\lambda_n}\,\cdot\big)$ satisfies the conditions of Proposition 1: $\int_{\mathbb{R}^d}|\widetilde{u}_n|^2\,\log|\widetilde{u}_n|^2\,dx=0$, while we notice that $\int_{\mathbb{R}^d}|\nabla\widetilde{u}_n|^2\,dx\sim\lambda_n\int_{\mathbb{R}^d}|\nabla u_n|^2\,dx\to 0$ as $n\to+\infty$ if $\lambda=0$. This contradicts (2) applied to $\widetilde{u}_n$. As a consequence, we have that λ is a positive real number such that $(\widetilde{u}_n)_{n\in\mathbb{N}}$ converges to $\widetilde{u}=\lambda^{d/2}\,u\big(\sqrt{\lambda}\,\cdot\big)$ in $\mathrm{H}^1(\mathbb{R}^d,dx)$. By Proposition 1 and Fatou's lemma, $|\widetilde{u}|^2\,\log|\widetilde{u}|^2\in\mathrm{L}^1(\mathbb{R}^d)$ and, as a consequence, $|u|^2\,\log|u|^2\in\mathrm{L}^1(\mathbb{R}^d)$. □

2.2 Integrability and Moments. Gaussian and Euclidean Cases

It turns out that a second moment condition is a sufficient condition to guarantee that $|u|^2\,\log|u|^2\in\mathrm{L}^1(\mathbb{R}^d)$. Here is a statement and a proof of this classical result that apparently goes back to [28, Sec. 7] and [29, 30]. Similar estimates can be found, e.g., in [31] or [32]. Notice that such bounds are often (see for instance [33, 34], and earlier references therein: [35–37]) referred to as *Shannon's inequality, i.e.,*

$$-\int_{\mathbb{R}^d}f\,\log f\,dx\le\frac{d}{2}\,\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}\,\log\left(\frac{2\,\pi\,\mathrm{e}}{d}\,\frac{\int_{\mathbb{R}^d}|x-\bar{x}|^2\,f(x)\,dx}{\|f\|^{1+2/d}_{\mathrm{L}^1(\mathbb{R}^d)}}\right)\tag{8}$$

as stated in [33, Lemma 2.3], where $\bar{x}:=\int_{\mathbb{R}^d}x\,f(x)\,dx/\,\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}$ is the center of mass. A classical reference is [38], which is (for the concerned part) a reprint of [29, 30]. However, this work is written in the language of the early days of information theory and it is not easy to rephrase it as above. Notice that the (convex) mathematical entropy has to be understood as a *neg-entropy* (i.e, the physical entropy up to a sign). Ineq. (8) follows from Jensen's inequality $\int_{\mathbb{R}^d}f\,\log(f/g)\,dx\ge\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}\,\log\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}$ applied with

$$g(x)=(\sqrt{2\pi})^{-d}\lambda^{-d}\,\mathrm{e}^{-\frac{|x-\bar{x}|^2}{2\lambda^2}},\quad\lambda^2=\frac{1}{d}\,\frac{\int_{\mathbb{R}^d}|x-\bar{x}|^2\,f(x)\,dx}{\|f\|_{\mathrm{L}^1(\mathbb{R}^d)}}.$$

Proposition 3 *If* $u \in \mathrm{L}^2(\mathbb{R}^d)$ *is smooth, compactly supported with* $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ *such that* $\int_{\mathbb{R}^d} |u|^2 \log|u|^2\,dx$ *and* $\int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx$ *are finite, then* $|u|^2 \log|u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$ *and*

$$\int_{\mathbb{R}^d} \big|\,|u|^2 \log|u|^2\big|\,dx \le \int_{\mathbb{R}^d} |u|^2 \log|u|^2\,dx + \frac{d}{2}\log\left(\frac{2\,\pi}{d}\int_{\mathbb{R}^d} |x|^2\,|u(x)|^2\,dx\right) + \frac{d+1}{\mathrm{e}}.$$

Proof Let $f = |u|^2$ if $|u| \le 1$, $f \equiv 0$ otherwise. Since $\|f\|_{\mathrm{L}^1(\mathbb{R}^d)} \le \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = 1$, we have $-\,\|f\|_{\mathrm{L}^1(\mathbb{R}^d)} \log\|f\|^{1+2/d}_{\mathrm{L}^1(\mathbb{R}^d)} \le (d+2)/(d\,\mathrm{e})$. Applying (8) to

$$\int_{\mathbb{R}^d} \big|\,|u|^2 \log|u|^2\big|\,dx = \int_{\mathbb{R}^d} |u|^2 \log|u|^2\,dx - \int_{\mathbb{R}^d} f\,\log f\,dx$$

completes the proof using $\int_{\mathbb{R}^d} |x-\bar{x}|^2\,f(x)\,dx \le \int_{\mathbb{R}^d} |x|^2\,f(x)\,dx$. □

If v is a smooth and compactly supported function, we already observed in Sect. 1 that (1) written for v is equivalent to (2) written for $u = \sqrt{\gamma}\,v$. However, using an integration by parts, we can notice that

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = \int_{\mathbb{R}^d} \left|\nabla u + \frac{x}{2}\,u\right|^2 dx = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{1}{4}\int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx - \frac{d}{2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \tag{9}$$

involves a second moment in x of $|u|^2$. It follows that, for a function v in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$, the second moment of $|v|^2$ is always finite:

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} + \frac{d}{2}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} \ge \frac{1}{4}\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma. \tag{10}$$

This fact was already observed in [39, Ineq. (4)]: combined with the Gaussian Poincaré inequality, it shows for instance that

$$\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \le 2\,(d+1)\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma \quad \forall\,v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \quad \text{such that} \quad \int_{\mathbb{R}^d} v\,d\gamma = 0.$$

Inequality (10) cannot hold for an arbitrary function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$, as the example of u_n given by (6) shows: in that case, we have

$$\int_{\mathbb{R}^d} |x|^2\,|u_n|^2\,dx \ge \frac{1}{n}\sum_{k=0}^{n-1} k^2\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \to +\infty \quad \text{as} \quad n \to +\infty.$$

By taking (2) into account, we deduce that

$$\int_{\mathbb{R}^d} \big|\,|u|^2 \log|u|^2\big|\,dx \le 2\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{d}{2}\log\left(\frac{2\,\pi}{d}\int_{\mathbb{R}^d} |x|^2\,|u(x)|^2\,dx\right) + \frac{d+1}{\mathrm{e}}$$

for any $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx$ is finite and $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$.

In the Gaussian framework, we have a similar result as in Proposition 3 using f as in the proof and $\int_{\mathbb{R}^d} f\,\log f\,d\gamma \geq \|f\|_{\mathrm{L}^1(\mathbb{R}^d,d\gamma)}\,\log\|f\|_{\mathrm{L}^1(\mathbb{R}^d,d\gamma)} \geq -1/\mathrm{e}$.

Corollary 4 *If $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ is such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$, then $|v|^2\,\log|v|^2$ is in $\mathrm{L}^1(\mathbb{R}^d, d\gamma)$ and we have*

$$\int_{\mathbb{R}^d}\Big|\,|v|^2\,\log|v|^2\Big|\,d\gamma \leq 2\,\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} + \frac{2}{\mathrm{e}}.$$

2.3 Improved Inequalities Under Second Moment Conditions

Here we aim at a *logarithmic uncertainty principle* introduced in [13, Lemma 2] and recently considered in [40, Proposition 1.1], and related stability results with explicit constants. Based on ideas of [10, Th. 1.1], [12, 41] and [13, Proposition 1], the following result holds.

Lemma 5 *Let $d \geq 1$. With φ defined by*

$$\varphi(t) := \frac{d}{4}\left(\exp\left(\frac{2\,t}{d}\right) - 1 - \frac{2\,t}{d}\right) \quad \forall\,t \in \mathbb{R}, \tag{11}$$

we have

$$\int_{\mathbb{R}^d}|\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d}|v|^2\,\log|v|^2\,d\gamma \geq \varphi\left(\int_{\mathbb{R}^d}|v|^2\,\log|v|^2\,d\gamma + \frac{d}{2} - \frac{1}{2}\int_{\mathbb{R}^d}|x|^2\,|v|^2\,d\gamma\right) \tag{12}$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$.

Proof of Lemma 5 Let us give a short proof based on [13]. By (10), we know that $\int_{\mathbb{R}^d}|x|^2\,|v|^2\,d\gamma < \infty$. If $u = v\,\sqrt{\gamma}$ is such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$, the deficit associated to (1) is such that

$$\begin{aligned}
&\int_{\mathbb{R}^d}|\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d}|v|^2\,\log|v|^2\,d\gamma\\
&= \int_{\mathbb{R}^d}|\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^d}|x|^2\,|u|^2\,dx - \frac{d}{2}\\
&\qquad\qquad - \frac{1}{2}\left(\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx + \frac{d}{2}\,\log\left(2\,\pi\right) + \frac{1}{2}\int_{\mathbb{R}^d}|x|^2\,|u|^2\,dx\right)\\
&\geq \frac{\pi\,d\,\mathsf{e}}{2}\,\exp\left(\frac{2}{d}\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx\right) - \frac{1}{2}\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx - \frac{d}{4}\,\log\left(2\,\pi\,\mathrm{e}^2\right)
\end{aligned}$$

using (9),

$$\int_{\mathbb{R}^d} |v|^2 \log |v|^2 \, d\gamma = \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx + \frac{d}{2} \log \left(2\,\pi\right) + \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx,$$

and (4). With

$$t := \int_{\mathbb{R}^d} |v|^2 \log |v|^2 \, d\gamma + \frac{d}{2} - \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma,$$

we have

$$\int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx = t - \frac{d}{2} \log \left(2\,\pi\,\mathrm{e}\right)$$

which concludes the proof of (12). □

Lemma 5 can be rewritten in the Euclidean space via the change of variables

$$v(x) = \frac{\lambda^{d/4}}{\sqrt{\gamma(x)}} \, u\left(\sqrt{\lambda}\, x\right) \quad \forall\, x \in \mathbb{R}^d$$

as

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{1}{2\,\lambda} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx - \frac{d}{4\,\lambda} \log \left(2\,\pi\,\mathrm{e}^2\,\lambda\right) \\ \geq \frac{1}{\lambda}\, \varphi \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx + \frac{d}{2} \log \left(2\,\pi\,\mathrm{e}\,\lambda\right) \right). \end{aligned}$$

With the choice

$$\lambda = \frac{1}{d} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx,$$

if $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$, we obtain a stability result for the *logarithmic uncertainty principle*

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx - \frac{d}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx - \frac{d^2}{4} \log \left(\frac{2\,\pi\,\mathrm{e}^2}{d} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx \right) \\ \geq d\, \varphi \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2 \, dx + \frac{d}{2} \log \left(\frac{2\,\pi\,\mathrm{e}}{d} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx \right) \right) \end{aligned}$$

which can be found in [40, Proposition 1.1]. This inequality is invariant under scaling. The uncertainty principle, i.e., the fact that the left-hand side is non-negative, is remarkable as it can be seen as an improvement of the standard Heisenberg uncertainty principle, whose stability has been studied in [42]. The right-hand side deserves further attention.

For any function $u \in \mathrm{L}^2(\mathbb{R}^d, dx)$, let us define the *relative entropy* with respect to the positive function $g \in \mathrm{L}^1(\mathbb{R}^d)$ by

$$\mathcal{E}[u|g] := \int_{\mathbb{R}^d} |u|^2 \log\left(\frac{|u|^2}{g}\right) dx - \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|g\|_{\mathrm{L}^1(\mathbb{R}^d)}$$

and consider the set of all positive Gaussian functions

$$\mathcal{M}_+ := \left\{ g \in \mathrm{L}^1(\mathbb{R}^d) : g(x) := \frac{M}{(2\,\pi\,\lambda)^{d/2}}\,\mathrm{e}^{-\frac{|x-y|^2}{2\,\lambda}}\ \forall\, x \in \mathbb{R}^d \right\}$$

parametrized by $(M, y, \lambda) \in (0, +\infty) \times \mathbb{R}^d \times (0, +\infty)$. The *best matching* Gaussian function in $\mathcal{M}_+$ is the function that minimizes $g \mapsto \mathcal{E}[u|g]$ for a given function *u* and it is determined by the following elementary observation.

Lemma 6 *For any function* $u \in \mathrm{L}^2\big(\mathbb{R}^d, (1 + |x|^2)\, dx\big)$ *such that* $|u|^2 \log |u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$, *we have*

$$\min_{g \in \mathcal{M}_+} \mathcal{E}[u|g] = \mathcal{E}[u|g_u]$$

where g_u *is the Gaussian function with parameters* (M, y, λ) *such that*

$$M := \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \quad y = \frac{1}{M}\int_{\mathbb{R}^d} x\,|u(x)|^2\, dx \quad \textit{and} \quad \lambda = \frac{1}{d\,M}\int_{\mathbb{R}^d} |x-y|^2\,|u(x)|^2\, dx \tag{13}$$

and, if $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$,

$$\mathcal{E}[u|g_u] = \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, dx + \frac{d}{2} \log\left(\frac{2\,\pi\,\mathrm{e}}{d}\int_{\mathbb{R}^d} |x-y|^2\,|u|^2\, dx\right).$$

Proof By convexity, $\mathcal{E}(u|g)$ is non-negative and $\mathcal{E}[u|g] = 0$ if and only if $|u|^2 = g$. If *g* is a Gaussian function with parameters (M, y, λ), then

$$\mathcal{E}[u|g] = \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, dx - \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + M - \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \log M$$
$$+ \frac{d}{2} \log(2\,\pi\,\lambda)\ \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{1}{2\,\lambda}\int_{\mathbb{R}^d} |x-y|^2\,|u(x)|^2\, dx$$

and an optimization with respect to (M, y, λ) shows the result because the above right-hand side diverges if $M \to 0_+$, $M \to +\infty$, $\lambda \to 0_+$, $\lambda \to +\infty$, or $|y| \to +\infty$. □

From Lemma 5 and Lemma 6, we recover the result of [40, Proposition 1.1], with the additional property that the choice (13) of the parameters of the Gaussian g_u minimizes the distance to $\mathcal{M}_+$, measured in terms of the relative entropy, as follows

$$\int_{\mathbb{R}^d} |\nabla v|^2\, d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 \log |v|^2\, d\gamma \ge \min_{g \in \mathcal{M}_+} \varphi\left(\mathcal{E}[u|g]\right) = \mathcal{E}[u|g_u]$$

for any $u = v\,\sqrt{\gamma} \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\int_{\mathbb{R}^d} x\,|u|^2\,dx = 0$.

Coming back to Inequality (12), we have the following result on the deficit (5), which was already known from [14, Theorem 1, 1.] with a different proof.

Corollary 7 *Let $d \geq 1$. Let us consider a sequence $(v_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v_n\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$ for any $n \in \mathbb{N}$. If $\limsup_{n\to+\infty} \int_{\mathbb{R}^d} |x|^2\,|v_n|^2\,d\gamma \leq d$, then $\lim_{n\to+\infty} \delta[v_n] = 0$ is equivalent to the convergence of $(v_n)_{n\in\mathbb{N}}$ to 1 in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$, and then we have $\lim_{n\to+\infty} \int_{\mathbb{R}^d} |x|^2\,|v_n|^2\,d\gamma = d$.*

With minimal effort, as a consequence of (12), we can also recover the full statement of [14, Theorem 1] which asserts that, for any sequence $(v_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to+\infty} \delta[v_n] = 0$, the two following properties are equivalent:
(i) $v_n \to 1$ in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ as $n \to +\infty$,
(ii) $\lim_{n\to+\infty} \int_{\mathbb{R}^d} |x|^2\,|v_n|^2\,d\gamma = d$.
With φ defined by (11), we may notice that $\varphi''(t) = (1/d)\,\exp(2\,t/d)$ for any $t \in \mathbb{R}$ and, as a consequence, $\varphi''(t) \geq 1/d$ if $t \geq 0$. Thus,

$$\delta[v] \geq \frac{1}{2\,d}\left(\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma\right)^2 + \frac{1}{8\,d}\left(d - \int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma\right)^2 \tag{14}$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$ and $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \leq d$. Corollary 7 is a consequence of (14) under the condition that $\limsup_{n\to+\infty} \int_{\mathbb{R}^d} |x|^2\,|v_n|^2\,d\gamma \leq d$. Indeed, by using the Csiszár-Kullback-Pinsker inequality

$$\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \geq \frac{1}{4}\left(\int_{\mathbb{R}^d} \big|\,|v|^2 - 1\big|\,d\gamma\right)^2$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$, see [43–45], and the Brezis-Lieb lemma, see [46, Theorem 2], one can then prove that the above sequence $(v_n)_{n\in\mathbb{N}}$ converges to 1 in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$. See [14] for further details. In fact, one can directly obtain an explicit stability estimate from (14), which goes as follows.

Corollary 8 *Let $d \geq 1$. For any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$ and $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \leq d$, we have*

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \geq \frac{d}{4}\,\chi\left(\frac{4}{d}\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma\right)$$

with $\chi(s) := 1 + s - \sqrt{1 + 2\,s}$.

Proof With $\mathrm{e} := \int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma$ and $\mathrm{i} := \int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$, from (14) we obtain

$$\mathrm{e}^2 + d\,\mathrm{e} - 2\,d\,\mathrm{i} \leq 0$$

which can be inverted as $\mathrm{e} \le \left(\sqrt{d\,(d+8\,\mathsf{i})} - d\right)/2$ and shows that

$$\mathrm{i} - \frac{1}{2}\,\mathrm{e} \ge \frac{1}{4}\left(4\,\mathsf{i} + d - \sqrt{d\,(d+8\,\mathrm{i})}\right).$$

This completes the proof of Corollary 8. □

In fact, under the assumptions of Corollary 8, instead of using (14), we can rewrite (12) as

$$\mathrm{i} \ge \frac{\mathrm{e}}{2} + \varphi(\mathsf{e}) = \psi^{-1}\left(\frac{\mathrm{e}}{2}\right)$$

where the last equality defines the concave increasing function ψ such that

$$\psi(s) := \frac{d}{2}\,\log\left(1 + \frac{4\,s}{d}\right) \quad \forall\, s \ge 0$$

and obtain

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \ge \xi\left(\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma\right)$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$ and $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \le d$. This is an improvement of Corollary 8 because

$$\xi(s) := s - \frac{1}{2}\,\psi(s) \ge \frac{d}{4}\,\chi\left(\frac{4\,s}{d}\right) \quad \forall\, s > 0, \tag{15}$$

so that $\xi\left(\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma\right) = 0$ if and only if v is constant. Notice that ξ is convex and positive.

As a consequence, in Euclidean variables, we also obtain a stability result for the *logarithmic uncertainty principle* in the strongest possible norm, that goes as follows. If we define the *relative Fisher information* with respect to the positive function $g \in \mathrm{L}^1(\mathbb{R}^d)$ by

$$\mathcal{I}[u|g] := \int_{\mathbb{R}^d} \left|\nabla\left(\frac{u}{\sqrt{g}}\right)\right|^2 g\,dx$$

and, with y and λ given by (13), rewrite Lemma 5 in the Euclidean space via the change of variables

$$v(x) = \frac{\lambda^{d/4}}{\sqrt{\gamma(x)}}\,u\left(\sqrt{\lambda}\,(x+y)\right)$$

then we have

$$\lambda\,\mathcal{I}[u|g_u]-\frac{1}{2}\,\mathcal{E}[u|g_u]\geq\varphi\big(\mathcal{E}[u|g_u]\big)$$

Corollary 9 *Let $d\geq 1$. For any $u\in \mathrm{H}^1(\mathbb{R}^d)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}=1$ and $\int_{\mathbb{R}^d}|x|^2\,|u|^2\,dx$ is finite, with y and λ given by (13), we have*

$$\int_{\mathbb{R}^d}|\nabla u|^2\,dx\int_{\mathbb{R}^d}|x-y|^2\,|u|^2\,dx-\frac{d}{2}\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx-\frac{d^2}{4}\,\log\big(2\,\pi\,\mathrm{e}^2\lambda\big)=:\delta[u]$$

where $\delta[u]\geq d\,\varphi\big(\mathcal{E}[u|g_u]\big)$ and $\delta[u]\geq d\,\xi\big(\mathcal{I}[u|g_u]\big)$ with φ and ξ given respectively by (11) and (15).

The proof is a simple rewriting of the previous computations. In Corollary 9, the inequalities provide upper estimates of the distances to $\mathcal{M}_+$ using, for instance, the Csiszár-Kullback-Pinsker inequality.

3 Stability

3.1 Optimal Constants and Optimal Functions

Inequalities (1), (2), (3) and (4) can be rewritten for functions $u\in\mathrm{H}^1(\mathbb{R}^d,dx)$ and $v\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)$ respectively as

$$\int_{\mathbb{R}^d}|\nabla v|^2\,d\gamma\geq\tfrac{1}{2}\int_{\mathbb{R}^d}|v|^2\,\log\left(\tfrac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}}\right)d\gamma\,,\tag{16a}$$

$$\int_{\mathbb{R}^d}|\nabla u|^2\,dx\geq\tfrac{1}{2}\int_{\mathbb{R}^d}|u|^2\,\log\left(\tfrac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx+\tfrac{d}{4}\,\log\big(2\,\pi\,\mathrm{e}^2\big)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,,\tag{16b}$$

$$\int_{\mathbb{R}^d}|\nabla u|^2\,dx\geq\tfrac{1}{2\,\lambda}\int_{\mathbb{R}^d}|u|^2\,\log\left(\tfrac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx+\tfrac{d}{4\,\lambda}\,\log\big(2\,\pi\,\mathrm{e}^2\,\lambda\big)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,,\tag{16c}$$

$$\int_{\mathbb{R}^d}|\nabla u|^2\,dx\geq\tfrac{\pi\,d\,\mathrm{e}}{2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,\exp\left(\tfrac{2}{d}\int_{\mathbb{R}^d}\tfrac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,\log\left(\tfrac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx\right)\,,\tag{16d}$$

without any normalization in either $\mathrm{L}^2(\mathbb{R}^d,dx)$ nor $\mathrm{L}^2(\mathbb{R}^d,d\gamma)$. These inequalities are written with optimal constants as can be checked using $v_\varepsilon(x)=1+\varepsilon\,x\cdot\nu$ in the limit as $\varepsilon\to 0$ for any given $\nu\in\mathbb{S}^{d-1}$ in case of (1), $u=\sqrt{\gamma}$ in case of (2) and (4), and $u=\lambda^{d/4}\,\gamma^{1/2}(\cdot/\sqrt{\lambda})$ in case of (3). The next issue is to identify all optimal functions. The first explicit result for (1) is due to E. Carlen in [2], although the *carré du champ* method of D. Bakry and M. Emery in [19] applies: we refer to [47] for more detailed explanations. Since (16a), (16b), (16c) and (16d) are equivalent for smooth and sufficiently decreasing functions as explained in Sect. 1, cases of equality can be reduced to optimality for any of these inequalities.

Theorem 10

(1) *A function v is optimal in (16a) if and only if* $v(x) = v_{a,b}(x) := a\,\mathrm{e}^{b\cdot x}$*, for any* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^d$*.*
(2) *A function u is optimal in (16b) if and only if* $u(x) = u_{a,b}(x) := a\,\mathrm{e}^{-|x-b|^2/2}$*, for any* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^d$*.*
(3) *For any fixed* $\lambda > 0$*, a function u is optimal in (16c) if and only if* $u(x) = u_{a,b,\lambda}(x) := a\,\mathrm{e}^{-|x-b|^2/(2\,\lambda)}$*, for any* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^d$*.*
(4) *A function u is optimal in (16d) if and only if* $u(x) = u_{a,b,\lambda}(x) = a\,\mathrm{e}^{-|x-b|^2/(2\,\lambda)}$*, for any* $a \in \mathbb{R}$*,* $b \in \mathbb{R}^d$*, and* $\lambda > 0$*.*

Cases 1) and 2) were explicitly established by E. Carlen in [2, Theorem 4]. Alternatively, we give a proof based on the *carré du champ* method of [19], which directly shows Case 4) and has been used in this context only in [7]. Here we use the pressure variable in the computations, as for instance in [47].

Proof Let us give a sketch of a proof based on the *Rényi entropy power* computation. Here we work at formal level and refer to [7] for the origin of this method. Assume that $\rho = |u|^2 = \mathrm{e}^{\mathrm{P}}$ solves the heat equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho = \nabla \cdot (\rho\,\nabla \mathrm{P}) \tag{17}$$

so that the *pressure* variable $\mathrm{P} = \log \rho$ and $u > 0$ respectively solve

$$\frac{\partial \mathrm{P}}{\partial t} = \Delta \mathrm{P} + |\nabla \mathrm{P}|^2 \quad \text{and} \quad \frac{\partial u}{\partial t} = \Delta u + \frac{|\nabla u|^2}{u}.$$

Further assuming that the function *u* is smooth and rapidly decaying as $|x| \to +\infty$, a straightforward computation shows that the entropy decays according to

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho\,\log \rho\,dx = -\int_{\mathbb{R}^d} \rho\,|\nabla \mathrm{P}|^2\,dx = -\,4\int_{\mathbb{R}^d} |\nabla u|^2\,dx$$

while the Fisher information obeys to

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2\,dx &= -\;2\int_{\mathbb{R}^d} \Delta u\left(\Delta u + \frac{|\nabla u|^2}{u}\right)dx\\ &= -\;2\int_{\mathbb{R}^d}\left(\|\mathrm{Hess}\,u\|^2 - 2\,\mathrm{Hess}\,u : \frac{\nabla u \otimes \nabla u}{u} + \frac{\|\nabla u \otimes \nabla u\|^2}{u^2}\right)dx\end{aligned}$$

where $A : B = \sum_{i,j=1}^d a_{ij}\, b_{ij}$ denotes the standard contraction of matrices A and B and $\|A\|^2 = A : A$. Using $\mathrm{P} = 2\,\log u$, $u\,\nabla\mathrm{P} = 2\,\nabla u$,

$$\frac{\nabla u \otimes \nabla u}{u^2} = \frac{1}{4}\,\nabla\mathrm{P} \otimes \nabla\mathrm{P} \quad \text{and} \quad \mathrm{Hess}\, u = \frac{u}{2}\left(\mathrm{Hess}\,\mathrm{P} + \frac{1}{2}\,\nabla\mathrm{P} \otimes \nabla\mathrm{P}\right),$$

we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u|^2\,dx = -\,\frac{1}{2}\int_{\mathbb{R}^d} \rho\,\|\mathrm{Hess}\,\mathrm{P}\|^2\,dx.$$

By conservation of mass, we can assume that $\|\rho(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ for any $t \ge 0$ if ρ solves (17), so that

$$\begin{aligned}
\exp\left(\frac{2}{d}\int_{\mathbb{R}^d} \rho\,\log\rho\,dx\right) \frac{d}{dt}\left(\int_{\mathbb{R}^d} |\nabla u|^2\,dx\,\exp\left(-\frac{2}{d}\int_{\mathbb{R}^d} \rho\,\log\rho\,dx\right)\right) \\
= -\frac{1}{2}\int_{\mathbb{R}^d} \rho\,\|\mathrm{Hess}\,\mathrm{P}\|^2\,dx + \frac{1}{2\,d}\left(\int_{\mathbb{R}^d} \rho\,|\nabla\mathrm{P}|^2\,dx\right)^2 \\
= -\frac{1}{2}\int_{\mathbb{R}^d} \rho\,\left\|\mathrm{Hess}\,\mathrm{P} - \frac{1}{d}\int_{\mathbb{R}^d} \rho\,|\nabla\mathrm{P}|^2\,dx\,\mathrm{Id}\right\|^2\,dx.
\end{aligned}$$

These computations can be justified using approximations based on smooth and compactly supported initial data $\rho(0,\cdot)$, for which $\rho(t,\cdot)$ is uniformly-log-concave, thus fast-decaying at infinity as well as its derivatives (as they are also solving the heat equation with smooth and compactly-supported initial data). Now let us consider an optimizer *u*, which can be taken positive without loss of generality, and take it as an initial datum in the above computations. We obtain that

$$\int_{\mathbb{R}^d} \rho\,\left\|\mathrm{Hess}\,\mathrm{P} - \frac{1}{d}\int_{\mathbb{R}^d} \rho\,|\nabla\mathrm{P}|^2\,dx\,\mathrm{Id}\right\|^2\,dx = 0,$$

i.e., $\mathrm{P} = 2\,\log u = \alpha\,|x - x_0|^2 + \beta$ for some constants α and β and for some $x_0 \in \mathbb{R}^d$. Since (16a), (16b), (16c) and (16d) share the same optimizers up to obvious transformations, this completes the sketch of the proof of Theorem 10. □

3.2 Stability Results in the Gaussian Setting

3.2.1 Improved Inequalities

A first improvement of (1) has been formulated in [2] by E. Carlen, in terms of the Wiener transform $\mathcal{W}$ and the Beckner-Hirschman inequality (see [48, 49, Section IV.3], with sharp constant due to I. Białynicki-Birula and J. Mycielski in [50], and Beckner in [49]), which is also known as the *entropic uncertainty principle*,

$$\delta[v] \geq \frac{1}{2}\int_{\mathbb{R}^d} |\mathcal{W}v|^2 \, \log |\mathcal{W}v|^2 \, d\gamma, \quad \|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1.$$

Under a non-negativity assumption, E. Carlen proved that the right-hand side in the above inequality is non-negative and vanishes if and only if $v \in \mathcal{M}$. According to Theorem 10, the set $\mathcal{M}$ of optimisers is made of the functions $v_{a,b}(x) := a\,\mathrm{e}^{b\cdot x}$, for any $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$ (recall that we are considering real valued functions only). The method relies first on Cramér's convolution result in [51], proving the result for non-negative functions. Then, in [2, p. 198], the technique is extended to signed functions. It points in the direction of the *entropic uncertainty principle*, complex-valued functions and the Fourier transform. See for instance the results of [52] and the L^2 conditional stability result of [25, Corollary 4.7].

Another direct improvement of (1) can be obtained using the *carré du champ* method of [19], which we sketch briefly. With respect to γ, let us define the *relative Fisher information* and the *relative entropy* functionals by $\mathcal{I}[v] = \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}$ and $\mathcal{E}[v] = \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma$, for $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$. Next, assume that $|w|^2$ solves the Ornstein–Uhlenbeck equation so that $w = w(t,x)$ is the solution of

$$\frac{\partial w}{\partial t} = \Delta w + \frac{|\nabla w|^2}{w} - x\cdot\nabla w, \quad w(t=0,\cdot) = v. \tag{18}$$

According, for instance, to [47, Section 2.2], it holds true that

$$\frac{d}{dt}\mathcal{E}[w(t,\cdot)] = -\,4\,\mathcal{I}[w(t,\cdot)], \quad \frac{d}{dt}\mathcal{I}[w(t,\cdot)] + 2\,\mathcal{I}[w(t,\cdot)] = -\,2\int_{\mathbb{R}^d} \|\mathrm{Hess}\,\mathrm{P}\|^2 \, |w|^2 \, d\gamma, \tag{19}$$

where $\mathrm{P} = 2\,\log w$ is the pressure variable as in Sect. 3.1. Integrating on $(0,\infty)$, (19) implies

$$\delta[v] \geq \int_0^\infty \mathcal{R}[w(t,\cdot)]\,dt \quad \text{where} \quad \mathcal{R}[w] := 2\int_{\mathbb{R}^d} \|\mathrm{Hess}\,\mathrm{P}\|^2 \, |w|^2 \, d\gamma, \tag{20}$$

where $\mathcal{R}$ vanishes if and only if v is an optimiser of (1). Additional information can be extracted from $\mathcal{R}$, for some classes of functions v as we shall see next.

3.2.2 Functions with asymptotic exponential or Gaussian behaviour

If the measure $|v|^2\,d\gamma$ satisfies the *Poincaré inequality*

$$\int_{\mathbb{R}^d} |\nabla\phi|^2 \, |v|^2 \, d\gamma \geq C_P \int_{\mathbb{R}^d} \left|\phi - \int_{\mathbb{R}^d} \phi\,|v|^2\,d\gamma\right|^2 |v|^2\,d\gamma \quad \forall\,\phi \in \mathrm{C}^\infty_c(\mathbb{R}^d) \tag{21}$$

for some positive constant C_P and if w solves (18) with initial datum $w(t=0,\cdot) = v$, the same holds true for the measure $|w(t,\cdot)|^2\,d\gamma$ for all $t \geq 0$, with $C_P(0) = C_P$ and

$$C_P(t) = \frac{C_P}{C_P + \big(1 - C_P\big)\,\mathrm{e}^{-2t}}.$$

In addition, if v is centered, i.e., $\int_{\mathbb{R}^d} x\,|v|^2\,d\gamma = 0$, then $\mathrm{P}(t,\cdot) = 2\,\log w(t,\cdot)$ is such that $\int_{\mathbb{R}^d} \nabla\mathrm{P}(t,\cdot)\,|w(t,\cdot)|^2\,d\gamma = 0$, and by the above Poincaré inequality with constant $C_P(t)$ applied to $\partial\mathrm{P}/\partial x_i$ for each $i = 1$, 2,...d, we obtain

$$\mathcal{R}\big[w(t,\cdot)\big] \ge C_P(t)\int_{\mathbb{R}^d} |\nabla\mathrm{P}(t,\cdot)|^2\,|w(t,\cdot)|^2\,d\gamma = C_P(t)\,\mathcal{I}[w].$$

In [11], this argument allows M. Fathi, E. Indrei, and M. Ledoux to prove that

$$\delta[v] \ge \frac{C_P^2 - C_P - C_P\,\log C_P}{(1 - C_P)^2}\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$$

for all centered functions v satisfying (21).

The result of [11] can be generalised as follows. Let us call $\mathcal{V}$ the space of centered functions v such that v admits (21) for some positive constant C_P. The flow (18) preserves $\mathcal{V}$. In addition, assume that for some $T \in (0,\infty)$, the solution $w(t,\cdot)$ to (18) with initial datum v belongs to $\mathcal{V}$ at $t = T$, hence, for any $t \ge T$. Then we obtain

$$\delta[v] \ge \mathrm{e}^{-2\,T}\,\frac{C_P^2 - C_P - C_P\,\log C_P}{(1 - C_P)^2}\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$$

using the *backwards-in-time* estimate of [47] and the result of [11]. The existence of such a finite T is granted if v is a compactly supported function. In [53], Chen, Chewi, and Niles-Weed provide a more general sufficient condition: if for some $\varepsilon > 0$ and $\mathcal{C} > 0$,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v(x)|^2\,|v(y)|^2\,\mathrm{e}^{\varepsilon\,|x-y|^2}\,\gamma(x)\,dx\,\gamma(y)\,dy \le \mathcal{C}, \tag{22}$$

then the solution $w(t,\cdot)$ to (18) has the property for an explicit $T > 0$ depending on ε and $\mathcal{C}$ but not on the dimension d. Note that the Gaussian-tail condition (22) cannot be created along the flow (18), see [54], without additional assumptions. As a result, proved in [47], there is an explicit constant $\mathrm{c} = \mathrm{c}(\varepsilon,\mathcal{C})$ such that

$$\delta[v] \ge \mathrm{c}\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$$

under Condition (22).

It is currently an open question to decide whether T is finite for a function $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ under the more natural assumption $\int_{\mathbb{R}^d} |v|^2\,\mathrm{e}^{\theta\,|x|}\,d\gamma < \infty$ for some $\theta > 0$, which is weaker than Condition (22). For functions with a finite exponential moment, there are stability results based on a weaker notion of distance. See [10, 24] and [55, ineq. (33)]. If $|v|^2$ can be written in the form $|v|^2 = \mathrm{e}^{-h}\,d\gamma$ for h such that $-1+\varepsilon \le \mathrm{Hess}\,h \le M$ for some $\varepsilon > 0$, then

$$\delta[v] \ge \beta(\varepsilon, M)\,\mathrm{W}_2^2(|v|^2\,dx, \gamma)$$

where W_2 is the 2-Wasserstein distance, see [24, Theorem 1.1]. For a more recent insight upon the relation between log-Hessian bounds, the Ornstein–Uhlenbeck flow, and the stability of (1), we refer to [56].

Finally, we notice that all results in this section are optimal with respect to the exponent of the distance, which is sometimes referred in the literature as *sharp qualitative stability*.

3.2.3 Functions with Finite Second Moment

Another possible way to exploit the improvement (20) is described below, for functions v such that $\int_{\mathbb{R}^d} x\,|v|^2\,d\gamma = 0$ and $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \le d$. The resulting estimate has been written in [13] using the comparison of (1) and (4), when the second moment is exactly $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma = d$. Otherwise, we attribute the result and the corresponding proof to [10], even though the key-estimate appears in [7] as well.

Going back to (20), using the Cauchy–Schwarz and the arithmetic–geometric inequalities as in [47, proof of Lemma 3], we can write

$$\mathcal{R}[w(t,\cdot)] \ge \frac{1}{d}\left(\int_{\mathbb{R}^d} (\Delta \mathrm{P})\,|w|^2\,d\gamma\right)^2 \ge \frac{4}{d}\left(\int_{\mathbb{R}^d} |\nabla w|^2\,d\gamma\right)^2,$$

where the last estimate is achieved using the condition on the second moment. By solving the differential inequality obtained from (19) for $t \in \mathbb{R}^+$, we find

$$\delta[v] \ \ge \xi\left(\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma\right), \quad \text{where} \quad \xi(s) := s - \frac{d}{4}\log\left(1 + \frac{4}{d}\,s\right) \quad \forall\, s > 0$$

is defined as in (15). This provides an alternative proof to the results of Sect. 2.3.

For $s \to 0$, we notice that $\xi(s) = 2\,s^2/d + o(s^2)$, which means that the extra term we found is of the order of $\|\nabla v\|^4_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}$ for $\|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}$ small, as in Corollary 8. In Sect. 3.2.2, we found a remainder term of order 2. Identifying the minimal conditions for the existence of a positive constant β such that $\delta[v] \ge \beta \int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$, for centered functions v with $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \le d$, is however still an open question.

As discussed in Sect. 2.3, the condition $\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma \le d$ is sufficient for local stability results for (1) around constant functions. This is also true in weaker distances such as as W_2. On the other hand, an improvement of (1) for functions $|v|^2$ with arbitrarily large but finite second order moment holds in two known cases. As found out by E. Indrei:

- In [14, Theorem 1(2)], it is shown that, for all $b > 0$, there exists a constant $\beta > 0$, such that, for all centered functions $v \in \mathrm{L}^2(\mathbb{R}^d, d\gamma)$ such that $\int_{\mathbb{R}^d} |x|^4 \, |v|^2 \, d\gamma \le b$,

$$\big\| |v| - 1 \big\|^2_{\mathrm{H}^1(\mathbb{R}^d, d\gamma)} \le \beta \, \Big(\delta[v] + \delta[v]^{1/2} \Big) \,.$$

- Stability in $\mathrm{W}^{1,1}(\mathbb{R}, d\gamma)$ is proved in [57, Theorem 1.1], in dimension $d = 1$. For all $a > 0$, there exist $\alpha > 0$ such that for all non-negative, normalized and centered functions $v \in \mathrm{H}^1(\mathbb{R}, d\gamma)$ with $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \le a$, it holds

$$\Big\| |v|^2 - 1 \Big\|_{\mathrm{W}^{1,1}(\mathbb{R}, d\gamma)} \le \alpha \, \Big(\delta^{1/4}[v] + \delta^{3/4}[v] \Big) \,.$$

Whether the exponents in these last two results are optimal and how they can be extended to $d > 1$ in the second case, according to [57], are open questions.

3.2.4 Stability in L^2 Without Moment Bounds

We refer to [27, 58] for a review on stability results in L^p-norms, which still leaves some open questions like, for instance, the question of optimal exponents in the stability estimates. Stability in L^2-norm was an open problem until recently. In [23, 59], J. Dolbeault, M. Esteban, A. Figalli, R. Frank, and M. Loss construct an explicit, positive, dimension-free constant β such that

$$\forall \, v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma), \quad \delta[v] \ge \beta \inf_{v_{a,b} \in \mathcal{M}} \| v - v_{a,b} \|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}, \tag{23}$$

where $\mathcal{M}$ and $v_{a,b}$ are defined in Sect. 3.2.1. The exponent in the right-hand side of (23) is optimal (see for instance [14, Theorem 2]) for homogeneity reasons. In [14, Theorem 1], the author also studies the stability in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ along sequences of functions, depending on their moments.

Even though (23) can be proved directly (see [59]), an interesting feature of this estimate is that is can be recovered as a large-dimensional limit of the constructive stability estimate of Sobolev's inequality on the sphere, according to [23]. The striking optimality of the constant 1/2 in (1), regardless of the topological dimension d of the space, means that (1) can be interpreted as an *infinite-dimensional* inequality in terms of the modern theory of metric measure spaces and synthetic curvature-dimension conditions: we refer to the work of L. Ambrosio, N. Gigli, and G. Savaré [60] for further details. However, the heuristics that *the Gaussian measure behaves similarly to the unitary measure on a very large-dimensional sphere* is present in mathematics since the XIXth century, at least, and we refer to [61] for a complete historical account. For completeness, let us review next a few recent results of stability of functional inequalities on the sphere, that are related with (1).

3.2.5 Interpolation Inequalities on the Sphere

One feature of (1) is the *criticality*, a concept related to maximal embeddings of Orlicz spaces studied for instance by A. Cianchi and L. Pick in [62]. We specialise this notion to the particular case, of Beckner's *Gaussian interpolation inequalities* introduced in [63]. For all $p \in [1,2)$ and all $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$, the following inequality holds

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right) \geq 0. \tag{24}$$

Inequality (1) represents the *critical* upper endpoint as $p \uparrow 2$. Note that for $p = 1$, we recover the *Gaussian Poincaré inequality*.

On the *n*-dimensional unit sphere $\mathbb{S}^n$, we have a similar family of interpolation inequalities, due to [64, 65], and obtained independently later in [66]. Those are a family of *Gagliardo–Nirenberg–Sobolev inequalities*, defined by a parameter $p \in [1,2) \cup (2, 2^\star]$, where $2^\star = 2\,n/(n-2)$, for $n \geq 3$, and for any $p \in [1,2) \cup (2, +\infty)$ if $n = 1$ or 2, which interpolates between the Poincaré inequality ($p = 1$), and the critical Sobolev inequality ($p = 2^\star$) if $n \geq 3$. Under these conditions, for all $F \in \mathrm{H}^1(\mathbb{S}^n, d\mu_n)$, where $d\mu_n$ denotes the uniform probability measure on $\mathbb{S}^n$, we have

$$\int_{\mathbb{S}^n} |\nabla F|^2 \, d\mu_n - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^n,d\mu_n)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}\right) \geq 0 \quad \text{if} \quad p \neq 2 \tag{25}$$

and for the limit case $p = 2$, the (subcritical) *logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^n} |\nabla F|^2 \, d\mu_n - \frac{2}{d}\int_{\mathbb{S}^n} |F|^2 \, \log\left(\frac{|F|^2}{\|F\|_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}}\right) d\mu_n \geq 0. \tag{26}$$

Inequality (25) can be proved via the entropy method, using nonlinear diffusion flows. The interested reader may refer to [67–70], and [71–73], where further computations for the heat equation and the Fisher information on Riemannian manifolds are also carried out.

It turns out that for all $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$, the sequence of functions $(F_n)_{n\in\mathbb{N}}$ of functions of $\mathrm{H}^1(\mathbb{S}^n, d\mu_n)$ such that

$$F_n(\omega_1, \omega_2, \ldots \omega_d, \omega_{d+1} \ldots \omega_{n+1}) = v\left(\omega_1/\sqrt{n}, \omega_2/\sqrt{n}, \ldots, \omega_d/\sqrt{n}\right)$$

satisfies

$$\lim_{n\to\infty}\left(\int_{\mathbb{S}^n} |\nabla F_n|^2 \, d\mu_n - \frac{d}{p_n-2}\left(\|F_n\|^2_{\mathrm{L}^{p_n}(\mathbb{S}^n,d\mu_n)} - \|F_n\|^2_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{1}{2-p_n}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \|v\|^2_{\mathrm{L}^{p_n}(\mathbb{R}^d,d\gamma)}\right)$$

if $(p_n)_{n\in\mathbb{N}}$ is a sequence of exponents in $[1,2) \cup (2, 2^\star)$ such that $\lim_{n\to\infty} p_n = p \in [1,2)$, see [74]. The case $p_n = 2^\star = 2\,n/(n-2) \to p = 2$ as $n \to +\infty$ is covered in [23, 59]. Heuristically, the function *v* has to be seen as the stereographic projection of a d-marginal of F_n for any $n > d$, large enough, if we assume for instance that v is compactly supported. See [74] for a detailed statement.

- For $p = 2^\star$, (25) is the critical Sobolev inequality on $\mathbb{S}^n$ and the optimisers are given by the Aubin–Talenti manifold $\mathcal{M}$ made of the functions $G(x) = c\,(1 + b \cdot x)^{-(n-2)/2}$ such that $c \in \mathbb{R}$ and $b \in \mathbb{R}^{n+1}$ with $|b| < 1$. There is a well known stability result which follows from [22] using an inverse stereographic projection and shows that the deficit in (25) if $p = 2^\star$ is bounded from below, up to a constant, by $\mathrm{d}^2(F, G) := \inf_{G\in\mathcal{M}} \big(\|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^n, d\mu_n)} + \frac{d}{p-2}\,\|F - G\|^2_{\mathrm{L}^2(\mathbb{S}^n, d\mu_n)}\big)$. The main result of [59] is the fact that the stability constant is bounded from below by β/n, with β as in (23), and that the dimensional dependence is sharp. In fact (23) is obtained in [59] by taking the limit as $n \to +\infty$, after a rescaling by $\sqrt{n}$.
- For $p \in (1, 2^\star)$ the stability issue for the subcritical family of inequalities (25) and (26) has been completely solved in [75, 76], with the caveat that the stability term degenerates on a n-dimensional subspace. Analogous stability estimates have been established for the subcritical family (24) in [74].

3.2.6 The Euclidean Case

Let us briefly observe that (1) and (2) are equivalent, up to the issue that the two inequalities are formulated in two different spaces (and there is a cancellation of the second moments in proving the Euclidean form from the Gaussian form of the inequality, as already remarked in [2]). However, by density, the stability result (23) translates into an analogous estimate for (2): see for instance [23, Corollary 4.4].

4 Examples of Instability

In this last section we collect some observations on counter-examples in various norms.

4.1 Known Counter-Examples

The first observation of instability of $\delta[v]$ with respect to the Wasserstein distance W_2 appears in [24]. The authors note that if such a stability estimate held for all functions, it would imply an improvement of the optimal constant in the logarithmic Sobolev inequality in the form (1), a contradiction. The first explicit counter-example was later constructed in [26] (and later in [58]): there is a sequence $(v_n)_{n\in\mathbb{N}}$ for which

$$\lim_{n\to\infty} \delta[v_n] = 0, \quad \liminf_{n\to\infty} \mathrm{W}_2^2\big(|v_n|^2\,dx, d\gamma\big) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \|v_n - 1\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > 0.$$

The results presented in [58] and the simplified version in [77] are primarily based on the observation that one can construct minimizing sequences for (1), for which the second moment becomes arbitrarily large. Crucially, the deficit $\delta[v]$ is insensitive to the second moment, an insight made precise through a computation by E. Carlen in [2], whereas the W_2 distance is highly sensitive to it.

The H^1 instability of (1) was pointed out by E. Indrei in [14]. The author also clarified the role of moments (see Corollary 7) by constructing a sequence $(v_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \delta[v_n] = 0 \quad \text{and} \quad \liminf_{n\to\infty} \|\nabla v_n\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = +\infty.$$

4.2 A Counter-Example to $\dot{\mathrm{H}}^1$ Stability

Here we prove that the examples constructed in [27, 58] also provide an example of instability in the $\dot{\mathrm{H}}^1(\mathbb{R}^d, dx)$-distance. We recall that $\mathcal{M}$ denotes the set of optimisers of (1) defined in Sect. 3.2.1.

Proposition 11 *Let $d \geq 1$. For all $a > 0$, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of functions in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v_n\|_{\mathrm{L}^2(\mathbb{R},d\gamma)} = 1$ and*

$$\int_{\mathbb{R}^d} x\,|v_n|^2\,d\gamma = 0\,, \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} |x|^2\,|v_n|^2\,d\gamma = d + a\,, \tag{27a}$$

$$\lim_{n\to\infty} \delta[v_n] = \lim_{n\to\infty} \left(\int_{\mathbb{R}^d} |\nabla v_n|^2\,d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v_n|^2\,\log\left(|v_n|^2\right) d\gamma \right) = 0\,, \tag{27b}$$

$$\liminf_{n\to\infty} \inf_{w\in\mathcal{M}} \|\nabla w - \nabla v_n\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} \geq \frac{a}{4} > 0\,. \tag{27c}$$

Proof of Proposition 11 We notice that it is sufficient to find such a sequence in dimension $d = 1$, as in higher dimensions one can consider a sum of functions depending only on one coordinate. We use a construction inspired by [58, Lemma 1.7]. Let us consider $(g_n)_{n\in\mathbb{N}}$ defined for any $x \in \mathbb{R}$ by

$$g_n(x) := \begin{cases} 1 & \text{if } |x| \leq \frac{n}{2} - \frac{1}{2\,n}, \\ \psi_n(|x|) & \text{if } \frac{n}{2} - \frac{1}{2\,n} \leq |x| \leq \frac{n}{2}, \\ \sqrt{\varepsilon_n}\,e^{\frac{n\,|x|}{2} - \frac{|n|^2}{4}} & \text{if } |x| \geq \frac{n}{2}, \end{cases} \tag{28}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} 2\,\varepsilon_n\,n^2 = a$, and ψ_n is a cut-off function such that $\psi_n(\frac{n}{2} - \frac{1}{2\,n}) = 1$ and $\psi_n(\frac{n}{2}) = \sqrt{\varepsilon_n}$. We finally set $v_{n,a} = g_n/\|g_n\|_{\mathrm{L}^2(\mathbb{R},d\gamma)}$. By construction, we have that $\int_{\mathbb{R}^d} |v_n|^2\,d\gamma = 1$, and $\int_{\mathbb{R}^d} x\,|v_n|^2\,d\gamma = 0$ since $v_n(x) = v_n(-x)$. By symmetry we also have that

$$\frac{1}{2}\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} = \int_0^{\frac{n}{2}-\frac{1}{2\,n}} \gamma(x)\,dx + \int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}} |\psi_n(x)|^2\,\gamma(x)\,dx + \varepsilon_n \int_{\frac{n}{2}}^{\infty} e^{n\,x-\frac{|n|^2}{2}}\,\gamma(x)\,dx \tag{29}$$

and

$$\int_0^{\frac{n}{2}-\frac{1}{2\,n}} |g_n|^2\,\gamma(x)\,dx = \int_{-\frac{n}{2}+\frac{1}{2\,n}}^0 |g_n|^2\,\gamma(x)\,dx = \frac{1}{2} - \Phi\left(-\frac{n}{2}+\frac{1}{2\,n}\right)$$

where Φ is the *normal cumulative function* $\Phi(x) := \int_{-\infty}^x \gamma(x)\,dx$. By completing the square, we find that

$$\int_{\frac{n}{2}}^\infty \mathrm{e}^{n\,x-\frac{|n|^2}{2}}\,d\gamma = \int_{\frac{n}{2}}^\infty \mathrm{e}^{-\frac{|x-n|^2}{2}}\frac{dx}{\sqrt{2\,\pi}} = \int_{-\frac{n}{2}}^\infty \mathrm{e}^{-\frac{s^2}{2}}\frac{ds}{\sqrt{2\,\pi}} = 1-\Phi\left(-\frac{n}{2}\right)\cdot \tag{30}$$

By combining (29) and (30), we find

$$\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} = 1 + 2\,\varepsilon_n + o(\varepsilon_n^2).$$

This relies on

$$\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}} |\psi_n(x)|^2\,d\gamma \le \frac{1}{2\,n}\,\gamma\left(\frac{n}{2}-\frac{1}{2\,n}\right) = o(\varepsilon_n^2)$$

using $|\psi_n| \le 1$, and

$$\int_0^{\frac{n}{2}-\frac{1}{2\,n}} \gamma(x)\,dx = \int_{-\frac{n}{2}+\frac{1}{2\,n}}^0 \gamma(x)\,dx = \frac{1}{2} - \Phi\left(-\frac{n}{2}+\frac{1}{2\,n}\right) = \frac{1}{2} + o(\varepsilon_n^2).$$

A similar computation also shows that

$$\int_{\mathbb{R}} |x|^2\,|v_n|^2\,d\gamma = 1 + 2\,\varepsilon_n\,n^2 + 2\,\varepsilon_n + o\left(\varepsilon_n\right) \to 1 + a \quad \text{as} \quad n\to\infty,$$

which completes the proof of (27a). From the definition (28) we find that

$$\|v_n'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} = \frac{2}{\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\left(\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}} \left|\psi_n'(x)\right|^2\,d\gamma + \frac{1}{4}\,\varepsilon_n\,n^2\left(1-\Phi\left(-\frac{n}{2}\right)\right)\right)$$

and $\mathcal{E}[v_n] = \int_{\mathbb{R}^d} |v_n|^2\,\log|v_n|^2\,d\gamma$ is estimated by

$$\begin{aligned}\mathcal{E}[v_n] = \frac{2}{\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\Bigg(&\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}} |\psi_n(x)|^2\log|\psi_n(x)|^2\,d\gamma \\ &+ \varepsilon_n\left(\log\varepsilon_n + \frac{1}{2}\,n^2\right)\left(1-\Phi\left(-\frac{n}{2}\right)\right) - n\,\varepsilon_n\,\gamma\left(-\frac{n}{2}\right)\Bigg) \\ &- 2\,\varepsilon_n + o(\varepsilon_n)\,,\end{aligned}$$

so that

$$\delta[v_n] = \frac{1}{\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \big(\varepsilon_n \,|\log \varepsilon_n| + o(\varepsilon_n)\big) + \varepsilon_n + o(\varepsilon_n),$$

which yields (27b). To prove (27c), let us establish that

$$\inf_{w\in\mathcal{M}} \|v_n' - w'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} \ge \frac{\varepsilon_n\, n^2}{2\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \left(1 - \Phi\left(-\frac{n}{2}\right)\right) \to \frac{a}{4} > 0 \quad \text{as} \quad n\to\infty.$$

Let $w \in \mathcal{M}$: there exists b and $c \in \mathbb{R}$ such that $w = c\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}$. Then $w'(x) = c\,\frac{b}{2}\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}$, and we distinguish three cases:

- If $b\,c = 0$, then $w' = 0$, so

$$\|v_n' - w'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} \ge \int_{\frac{n}{2}}^{\infty} |v_n'(x)|^2\, d\gamma = \frac{\varepsilon_n\, n^2}{2\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \left(1 - \Phi\left(-\frac{n}{2}\right)\right).$$

- Assume now that $b\,c < 0$. For $x > n/2$ we have that

$$v_n'(x) - w'(x) = \frac{n\,\varepsilon_n}{2\,\|g_n\|_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\,\mathrm{e}^{\frac{n\,x}{2}-\frac{n^2}{4}} - b\,c\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}} = v_n'(x) + |b\,c|\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}},$$

that is, for $x > n/2$, the functions v_n' and $|b\,c|\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}$ have the same sign and are both positive. As a consequence, we have

$$\|v_n' - w'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} \ge \int_{\frac{n}{2}}^{\infty} |v_n'(x)|^2\, d\gamma = \frac{\varepsilon_n\, n^2}{2\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \left(1 - \Phi\left(-\frac{n}{2}\right)\right).$$

This lower bound is uniform in b and c. We take the infimum in $\|v_n' - w'\|_{\mathrm{L}^2(\mathbb{R},d\gamma)}$ and obtain the sought inequality.

- The case $b\,c > 0$ can be dealt similarly to the case $b\,c < 0$ but in the interval $(-\infty, -n/2)$. □

Acknowledgements This work has been supported by the Project *Conviviality* (ANR-23-CE40–0003) of the French National Research Agency. G.B. has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 101034413. The authors thank a referee for a careful reading and suggestions which result in a significant improvement of the manuscript.

Author contributions The authors equally contributed to this paper (content-wise and in all phases of the redaction).

Funding Open access funding provided by Institute of Science and Technology (IST Austria). The work of GB has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 101034413. This work has been supported by the Project Conviviality (ANR-23-CE40–0003) of the French National Research Agency.

Data availability This study produced no data set

Declarations

Conflict of interest The authors declare no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Gross, L.: Logarithmic Sobolev inequalities. Amer. J. Math. **97**(4), 1061–1083 (1975). https://doi.org/10.2307/2373688
2. Carlen, E.A.: Superadditivity of fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. **101**(1), 194–211 (1991). https://doi.org/10.1016/0022-1236(91)90155-X
3. Stam, A.J.: Some inequalities satisfied by the quantities of information of Fisher and Shannon. Inf. Control **2**, 101–112 (1959). https://doi.org/10.1016/S0019-9958(59)90348-1
4. Weissler, F.B.: Logarithmic Sobolev inequalities for the heat-diffusion semigroup. Trans. Amer. Math. Soc. **237**, 255–269 (1978). https://doi.org/10.2307/1997621
5. Federbush, P.: Partially alternate derivation of a result of Nelson. J. Math. Phys. **10**, 50–52 (1969). https://doi.org/10.1063/1.1664760
6. Villani, C.: Entropy Production and Convergence to Equilibrium, pp. 1–70. Springer, Berlin, Heidelberg (2008). https://doi.org/10.1007/978-3-540-73705-6_1
7. Villani, C.: A short proof of the "concavity of entropy power". IEEE Trans. Inf. Theory **46**(4), 1695–1696 (2000). https://doi.org/10.1109/18.850718
8. Toscani, G.: An information-theoretic proof of Nash's inequality. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **24**(1), 83–93 (2013). https://doi.org/10.4171/RLM/645
9. Toscani, G.: A concavity property for the reciprocal of Fisher information and its consequences on Costa's EPI. Phys. A **432**, 35–42 (2015). https://doi.org/10.1016/j.physa.2015.03.018
10. Bobkov, S.G., Gozlan, N., Roberto, C., Samson, P.-M.: Bounds on the deficit in the logarithmic Sobolev inequality. J. Funct. Anal. **267**(11), 4110–4138 (2014). https://doi.org/10.1016/j.jfa.2014.09.016
11. Fathi, M., Indrei, E., Ledoux, M.: Quantitative logarithmic Sobolev inequalities and stability estimates. Discrete Contin. Dyn. Syst. **36**(12), 6835–6853 (2016). https://doi.org/10.3934/dcds.2016097
12. Toscani, G.: A strengthened entropy power inequality for log-concave densities. IEEE Trans. Inf. Theory **61**(12), 6550–6559 (2015). https://doi.org/10.1109/TIT.2015.2495302
13. Dolbeault, J., Toscani, G.: Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities. Int. Math. Res. Not. IMRN **2**, 473–498 (2016). https://doi.org/10.1093/imrn/rnv131
14. Indrei, E.: Sharp stability for LSI. Mathematics **11**(12), 2670 (2023). https://doi.org/10.3390/math11122670
15. Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C., Scheffer, G.: Sur les Inégalités de Sobolev Logarithmiques. Panoramas et Synthèses [Panoramas and Syntheses], vol. 10, p. 217. Société Mathématique de France, Paris (2000). With a preface by Dominique Bakry and Michel Ledoux
16. Guionnet, A., Zegarlinski, B.: Lectures on logarithmic Sobolev inequalities. Séminaire de probabilités de Strasbourg **36**, 1–134 (2002). https://doi.org/10.1007/978-3-540-36107-7_1

17. Royer, G.: An Initiation to Logarithmic Sobolev Inequalities. SMF/AMS Texts and Monographs, vol. 14, p. 119. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris (2007). Translated from the 1999 French original by Donald Babbitt
18. Bakry, D., Gentil, I., Ledoux, M.: Analysis and Geometry of Markov Diffusion Operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, p. 552. Springer, Cham (2014).https://doi.org/10.1007/978-3-319-00227-9
19. Bakry, D., Émery, M.: Diffusions hypercontractives. In: Séminaire de Probabilités, XIX, 1983/84. Lecture Notes in Math., vol. 1123, pp. 177–206. Springer, Berlin (1985). https://doi.org/10.1007/BFb0075847
20. Brézis, H., Lieb, E.H.: Sobolev inequalities with remainder terms. J. Funct. Anal. **62**, 73–86 (1985). https://doi.org/10.1016/0022-1236(85)90020-5
21. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(2), 109–145 (1984)
22. Bianchi, G., Egnell, H.: A note on the Sobolev inequality. J. Funct. Anal. **100**(1), 18–24 (1991). https://doi.org/10.1016/0022-1236(91)90099-Q
23. Dolbeault, J., Esteban, M.J., Figalli, A., Frank, R.L., Loss, M.: Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence. Camb. J. Math. **13**(2), 359–430 (2025). https://doi.org/10.4310/cjm.250325022725
24. Indrei, E., Marcon, D.: A quantitative log-Sobolev inequality for a two parameter family of functions. Int. Math. Res. Not. IMRN **2014**(20), 5563–5580 (2014). https://doi.org/10.1093/imrn/rnt138
25. Feo, F., Indrei, E., Posteraro, M.R., Roberto, C.: Some remarks on the stability of the log-Sobolev inequality for the Gaussian measure. Potential Anal. **47**(1), 37–52 (2016). https://doi.org/10.1007/s11118-016-9607-5
26. Kim, D.: Instability results for the logarithmic Sobolev inequality and its application to the Beckner–Hirschman Inequality (2018). https://arxiv.org/abs/1805.06272
27. Indrei, E., Kim, D. 2021 Deficit estimates for the logarithmic Sobolev inequality. Differential Integral Equations 34(7–8), 437–466 https://doi.org/10.57262/die034-0708-437
28. Carleman, T.: Sur la théorie de l'équation intégrodifférentielle de Boltzmann. Acta Math. **60**(1), 91–146 (1933). https://doi.org/10.1007/BF02398270
29. Shannon, C.E.: A mathematical theory of communication, Parts I and II. Bell Syst. Tech. J. **27**, 379–423 (1948)
30. Shannon, C.E.: A mathematical theory of communication. Part III. Bell System Tech. J. **27**, 623–656 (1948)
31. Kubo, H., Ogawa, T., Suguro, T.: Beckner type of the logarithmic Sobolev and a new type of Shannon's inequalities and an application to the uncertainty principle. Proc. Am. Math. Soc. **147**(4), 1511–1518 (2019). https://doi.org/10.1090/proc/14350
32. Suguro, T.: Shannon's inequality for the Rényi entropy and an application to the uncertainty principle. J. Diff. Equ. **283**(6), 109566 (2022). https://doi.org/10.1016/j.jfa.2022.109566
33. Kurokiba, M., Ogawa, T.: Finite time blow up for a solution to system of the drift-diffusion equations in higher dimensions. Math. Z. **284**(1–2), 231–253 (2016). https://doi.org/10.1007/s00209-016-1654-5
34. Biler, P.: Existence and nonexistence of solutions for a model of gravitational interaction of particles. III. Colloq. Math. **68**(2), 229–239 (1995). https://doi.org/10.4064/cm-68-2-229-239
35. McEliece, R.J.: The Theory of Information and Coding, Student edn. Encyclopedia of Mathematics and its Applications, vol. 86, p. 397. Cambridge University Press, Cambridge (2004). With a foreword by Mark Kac. https://doi.org/10.1017/CBO9780511819896
36. Derriennic, Y.: Entropie, théorèmes limite et marches aléatoires, pp. 241–284. Springer, Berlin, Heidelberg (1986). https://doi.org/10.1007/BFb0077188
37. Lasota, A., Mackey, M.C.: Chaos, Fractals, and Noise. Springer, New York (1994). https://doi.org/10.1007/978-1-4612-4286-4
38. Shannon, C.E., Weaver, W.: The Mathematical Theory of Communication, p. 117. University of Illinois Press, Urbana, IL, USA (1949)
39. Dolbeault, J., Volzone, B.: Improved Poincaré inequalities. Nonlinear Anal. Theory Methods Appl. **75**(16), 5985–6001 (2012). https://doi.org/10.1016/j.na.2012.05.008
40. Suguro, T.: Stability of the logarithmic Sobolev inequality and uncertainty principle for the Tsallis entropy. Nonlinear Anal. **250**, 113644 (2025). https://doi.org/10.1016/j.na.2024.113644
41. Bakry, D., Ledoux, M.: A logarithmic Sobolev form of the Li-Yau parabolic inequality. Rev. Mat. Iberoam. **22**(2), 683–702 (2006). https://doi.org/10.4171/RMI/470

42. Fathi, M.: A short proof of quantitative stability for the Heisenberg-Pauli-Weyl inequality. Nonlinear Anal. **210**, 112403–3 (2021). https://doi.org/10.1016/j.na.2021.112403
43. Csiszár, I.: Information-type measures of difference of probability distributions and indirect observations. Studia Scientiarum Mathematicarum Hungarica. A Quarterly of the Hungarian Academy of Sciences 2, 299–318 (1967)
44. Kullback, S.: A lower bound for discrimination information in terms of variation (corresp.). IEEE Trans. Inf. Theory **13**(1), 126–127 (1967). https://doi.org/10.1109/tit.1967.1053968
45. Pinsker, M.S.: Information and Information Stability of Random Variables and Processes. Translated and edited by Amiel Feinstein, p. 243. Holden-Day Inc., San Francisco, Calif. (1964)
46. Brézis, H., Lieb, E.H.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486–490 (1983). https://doi.org/10.2307/2044999
47. Brigati, G., Dolbeault, J., Simonov, N.: Stability for the logarithmic Sobolev inequality. J. Funct. Anal. **287**(8), 11056110562 (2024). https://doi.org/10.1016/j.jfa.2024.110562
48. Hirschman, I.I. Jr.: A note on entropy. Amer. J. Math. 79, 152–156 (1957) https://doi.org/10.2307/2372390
49. Beckner, W.: Inequalities in Fourier analysis. Ann. of Math. **102**(1), 159–182 (1975). https://doi.org/10.2307/1970980
50. Białynicki-Birula, I., Mycielski, J.: Uncertainty relations for information entropy in wave mechanics. Comm. Math. Phys. **44**(2), 129–132 (1975)
51. Cramér, H.: Über eine Eigenschaft der normalen Verteilungsfunktion. Math. Z. **41**(1), 405–414 (1936). https://doi.org/10.1007/BF01180430
52. Beckner, W.: Pitt's inequality and the uncertainty principle. Proc. Amer. Math. Soc. **123**(6), 1897–1905 (1995). https://doi.org/10.2307/2161009
53. Chen, H.-B., Chewi, S., Niles-Weed, J.: Dimension-free log-Sobolev inequalities for mixture distributions. J. Funct. Anal. **281**(11), 10923 (2021). https://doi.org/10.1016/j.jfa.2021.109236
54. Herraiz, L.: Asymptotic behaviour of solutions of some semilinear parabolic problems. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **16**(1), 49–105 (1999). https://doi.org/10.1016/S0294-1449(99)80008-0
55. Bolley, F., Gentil, I., Guillin, A.: Dimensional improvements of the logarithmic Sobolev. Talagrand and Brascamp-Lieb inequalities. Ann. Probability (2018). https://doi.org/10.1214/17-aop1184
56. Brigati, G., Pedrotti, F.: Heat flow, log-concavity, and Lipschitz transport maps (2024). arxiv:2404.15205
57. Indrei, E.: $W^{1,1}$ stability for the LSI. J. Differential Equations 421, 196–207 (2025) https://doi.org/10.1016/j.jde.2024.11.054
58. Kim, D.: Instability results for the logarithmic Sobolev inequality and its application to related inequalities. Discrete Contin. Dyn. Syst. **42**(9), 4297–4320 (2022). https://doi.org/10.3934/dcds.2022053
59. Dolbeault, J., Esteban, M.J., Figalli, A., Frank, R.L., Loss, M.: A short review on improvements and stability for some interpolation inequalities, Proceedings ICIAM 2023 (2024) arxiv:2404.15205
60. Ambrosio, L., Gigli, N., Savaré, G.: Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. Ann. Probab. **43**(1), 339–404 (2015). https://doi.org/10.1214/14-AOP907
61. Vershik, A.M.: Does there exist a Lebesgue measure in the infinite-dimensional space? Proc. Steklov Inst. Math. **259**(1), 248–272 (2007). https://doi.org/10.1134/S0081543807040153
62. Cianchi, A., Pick, L.: Optimal Gaussian Sobolev embeddings. J. Funct. Anal. **256**(11), 3588–3642 (2009). https://doi.org/10.1016/j.jfa.2009.03.001
63. Beckner, W.: A generalized Poincaré inequality for Gaussian measures. Proc. Amer. Math. Soc. **105**(2), 397–400 (1989). https://doi.org/10.2307/2046956
64. Bidaut-Véron, M.-F., Véron, L.: Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. Invent. Math. **106**(3), 489–539 (1991). https://doi.org/10.1007/BF01243922
65. Bidaut-Véron, M.-F., Véron, L.: Erratum: "Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations" [Invent. Math. **106** (1991), no. 3, 489–539; MR1134481 (93a:35045)]. Invent. Math. 112(2), 445 (1993) https://doi.org/10.1007/BF01232442
66. Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. **138**(1), 213–242 (1993). https://doi.org/10.2307/2946638
67. Rothaus, O.S.: Diffusion on compact Riemannian manifolds and logarithmic Sobolev inequalities. J. Funct. Analysis **42**(1), 102–109 (1981). https://doi.org/10.1016/0022-1236(81)90049-5

68. Dolbeault, J., Esteban, M.J., Loss, M.: Nonlinear flows and rigidity results on compact manifolds. J. Funct. Anal. **267**(5), 1338–1363 (2014). https://doi.org/10.1016/j.jfa.2014.05.021
69. Licois, J.R., Véron, L.: Un théorème d'annulation pour des équations elliptiques non linéaires sur des variétés riemanniennes compactes. C. R. Acad. Sci. Paris Sér. I Math. **320**(11), 1337–1342 (1995)
70. Dolbeault, J., Esteban, M.J., Loss, M.: Interpolation inequalities on the sphere: linear vs. nonlinear flows (inégalités d'interpolation sur la sphère : flots non-linéaires vs. flots linéaires). Ann. Fac. Sci. Toulouse Math. **26**(2), 351–379 (2017). https://doi.org/10.5802/afst.1536
71. Ji, S.: Bounds for the optimal constant of the Bakry-Émery Γ_2 criterion inequality on RP^{d-1} (2024). arxiv:2408.13954
72. Ji, S.: Dissipation estimates of the Fisher information for the Landau equation (2024). arxiv:2410.09035
73. Ji, S.: Entropy dissipation estimates for the Landau equation with Coulomb potentials. Kinet. Relat. Models **18**(3), 368–388 (2025). https://doi.org/10.3934/krm.2024020
74. Brigati, G., Dolbeault, J., Simonov, N.: On Gaussian interpolation inequalities. C. R. Math. Acad. Sci. Paris **362**, 21–44 (2024). https://doi.org/10.5802/crmath.488
75. Frank, R.L.: Degenerate stability of some Sobolev inequalities. Ann. Inst. H. Poincaré C Anal. Non Linéaire (2022). https://doi.org/10.4171/aihpc/35
76. Brigati, G., Dolbeault, J., Simonov, N.: Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results. Ann. Inst. H. Poincaré C Anal. Non Linéaire **41**(5), 1289–1321 (2024). https://doi.org/10.4171/aihpc/106
77. Eldan, R., Lehec, J., Shenfeld, Y.: Stability of the logarithmic Sobolev inequality via the Föllmer process. Ann. Inst. Henri Poincaré, Probab. Stat. **56**(3), 2253–2269 (2020). https://doi.org/10.1214/19-AIHP1038

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.