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ORIGINAL PAPER

Counting Perfect Matchings in Dirac Hypergraphs

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Abstract

One of the foundational theorems of extremal graph theory is *Dirac's theorem*, which says that if an n-vertex graph G has minimum degree at least $n/2$, then G has a Hamilton cycle, and therefore a perfect matching (if n is even). Later work by Sárközy, Selkow and Szemerédi showed that in fact Dirac graphs have *many* Hamilton cycles and perfect matchings, culminating in a result of Cuckler and Kahn that gives a precise description of the numbers of Hamilton cycles and perfect matchings in a Dirac graph G (in terms of an entropy-like parameter of G). In this paper we extend Cuckler and Kahn's result to perfect matchings in hypergraphs. For positive integers $d < k$, and for n divisible by k, let $m_d(k, n)$ be the minimum d-degree that ensures the existence of a perfect matching in an n-vertex k-uniform hypergraph. In general, it is an open question to determine (even asymptotically) the values of $m_d(k, n)$, but we are nonetheless able to prove an analogue of the Cuckler–Kahn theorem, showing that if an n-vertex k-uniform hypergraph G has minimum d-degree at least $(1+\gamma)m_d(k, n)$ (for any constant $\gamma > 0$), then the number of perfect matchings in G is controlled by an entropy-like parameter of G. This strengthens cruder estimates arising from work of Kang–Kelly–Kühn–Osthus–Pfenninger and Pham–Sah–Sawhney–Simkin.

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1 Introduction

1.1 Counting Perfect Matching in Dirac Graphs

Dirac's theorem is one of the foundational theorems in extremal graph theory. It says that if an n-vertex graph G has minimum degree at least $n/2$, then G has a Hamilton cycle, and therefore has a perfect matching if n is even (taking every second edge of the Hamilton cycle). A huge number of variants and extensions of Dirac's theorem have been considered over the years; in particular, one important direction of research (seemingly first proposed by Bondy [1, p. 79]) is to *count* Hamilton cycles and perfect matchings, given the same minimum degree condition. As one of the first applications of the so-called *regularity method*, Sárközy, Selkow and Szemerédi [2] resolved this problem up to an $\exp(O(n))$ error term: every graph on n vertices with minimum degree at least $n/2$ has at least $n^n \exp(-O(n))$ Hamilton cycles and at least $n^{n/2} \exp(-O(n))$ perfect matchings.

A much sharper estimate was later proved in influential work of Cuckler and Kahn [3, 4]: they managed to resolve this problem up to an $\exp(o(n))$ error term, characterising the number of Hamilton cycles and perfect matchings in terms of a parameter they called *graph entropy*. We state the perfect matching case of their result, after introducing some notation and definitions.

Definition 1.1 (Fractional perfect matchings) A *fractional perfect matching* $\boldsymbol{x}$ of G is an assignment of a weight $\boldsymbol{x}[e] \in [0, 1]$ to each edge of G, such that $\sum_{e \ni v} \boldsymbol{x}[e] = 1$ for each vertex v in G.

Definition 1.2 (Graph entropy) For a fractional perfect matching $\boldsymbol{x}$ of a graph G, the *entropy* of $\boldsymbol{x}$ is defined as

$$h(\boldsymbol{x}) := \sum_{e \in E(G)} \boldsymbol{x}[e] \ln(1/\boldsymbol{x}[e]) \,,$$

where we use the convention $0 \ln(1/0) = 0$. The entropy of G is defined as

$$h(G) := \sup_{\boldsymbol{x}} h(\boldsymbol{x}) \,,$$

where the supremum is taken over all fractional perfect matchings in G.

Theorem 1.3 *([3, 4]) Let G be an n-vertex graph with minimum degree at least $n/2$. Then the number of perfect matchings in G is*

$$\mathrm{e}^{h(G)}(1/\mathrm{e} + o(1))^{n/2},$$

where asymptotics are as $n \to \infty$ (along a sequence of even integers).

Note that $h(G)$, as the solution to a convex optimisation problem, can be efficiently computed.

In addition to proving Theorem 1.3, Cuckler and Kahn also studied the minimum possible value of $h(G)$ among all graphs with specified minimum degree, and as a result they managed to deduce the following rather appealing theorem: for a given minimum degree (exceeding $n/2$), a *random* graph of the appropriate density has essentially the minimum possible number of perfect matchings.

Theorem 1.4 *([3, 4]) Let G be an n-vertex graph with minimum degree at least pn, for some* $p \in [1/2, 1]$. *Then the number of perfect matchings in G is at least*

$$\Phi(K_n) \cdot (p + o(1))^{n/2},$$

where $\Phi(K_n) = n!/((n/2)!2^{n/2})$ *is the number of perfect matchings in the complete graph on n vertices. (Here asymptotics are as* $n \to \infty$*, along a sequence of even integers).*

(Note that $\Phi(K_n) \cdot p^{n/2}$ is the expected number of perfect matchings in a random graph on n vertices in which every edge is present with probability p. It is not hard to show that such random graphs certify tightness of the bound in Theorem 1.4).

1.2 Extensions to Hypergraphs

A *k-uniform hypergraph* (which we sometimes abbreviate as a *k-graph*) is a generalisation of a graph where each edge is a set of exactly k vertices (so, graphs are 2-uniform hypergraphs). A central program in combinatorics is to understand the extent to which classical theorems about graphs can be generalised to hypergraphs, and there has been particular interest in hypergraph generalisations of Dirac's theorem. However, despite intensive effort over the last 20 years, even the most basic questions largely remain unanswered (see the surveys [5, 6] for much more than we will be able to discuss here).

The notion of a perfect matching generalises in the obvious way to hypergraphs (there are various ways to generalise the notion of a Hamilton cycle, too, though we will not consider these in the present paper). The study of Dirac-type theorems for hypergraphs generally concerns the following parameters.

Definition 1.5 (Hypergraph degrees) For nonnegative integers $d < k$, the *minimum d-degree* $\delta_d(G)$ of an n-vertex k-uniform hypergraph G is the minimum, over all sets of d vertices S, of the degree of S (i.e., the number of edges $e \supset S$).

Definition 1.6 (Hypergraph Dirac threshold) For n divisible by k, let $m_d(k, n)$ be the minimum d-degree that ensures the existence of a perfect matching in a k-uniform hypergraph on n vertices. Noting that a set of d vertices in an n-vertex k-uniform hypergraph can have degree at most $\binom{n-d}{k-d}$, let

$$\alpha_d(k) := \lim_{n \to \infty} \frac{m_d(k, n)}{\binom{n-d}{k-d}}$$

be the "asymptotic Dirac threshold" for d-degree in k-uniform hypergraphs (it was proved by Ferber and Kwan [7, Theorem 1.2] that this limit exists).

In general, the values of $\alpha_d(k)$ are unknown: determining them is one of the most important open problems in extremal hypergraph theory (with surprising connections to certain inequalities in probability theory [8]). Apart from sporadic values of (d, k) (discussed in the survey [6]), the most general result is due to Frankl and Kupavskii [9], who showed that $\alpha_d(k) = 1/2$ for $d \geq 3k/8$.

Despite the fact that the values of $\alpha_d(k)$ are in general not known, recently Kang, Kelly, Kühn, Osthus and Pfenninger [10] and Pham, Sah, Sawhney and Simkin [11] (see also [12]) were able to prove estimates on the number of perfect matchings above the Dirac threshold, as a consequence of their work on so-called *spread measures*[1]. Specifically, for an n-vertex k-graph G (with n divisible by k) which has minimum d-degree at least $(\alpha_d(k) + \gamma)\binom{n-d}{k-d}$ (for some positive $\gamma > 0$), the number of perfect matchings in G is

$$n^{(1-1/k)n} \exp(-O(n))$$

(where the implicit constant in the big-Oh notation is allowed to depend on d, k and γ). This can be viewed as an analogue of the Sárközy–Selkow–Szemerédi bound in the graph case (though the fact that the implicit constant depends on γ is not very desirable).

In the particular case where $d = k - 1$ (where the Dirac threshold is known exactly, thanks to influential work of Rödl, Ruciński and Szemerédi [13]), a hypergraph analogue of Cuckler and Kahn's Theorem 1.4 is available. Indeed, it follows from results of Ferber, Hardiman and Mond [14] (improving earlier bounds of Ferber, Krivelevich and Sudakov [15]) that if a k-uniform hypergraph G satisfies $\delta_{k-1}(G) \geq (\alpha_{k-1}(k) + \gamma)(n - (k - 1)) = (1/2 + \gamma)(n - (k - 1))$ for any constant $\gamma > 0$, then the number of perfect matchings in G is at least

$$\Phi(K_n^{(k)}) \cdot (p + o(1))^{n/k}\,, \tag{1.1}$$

where $p = \delta_{k-1}(G)/n$, and $\Phi(K_n^{(k)}) = n!/((n/k)!(k!)^{n/k})$ is the number of perfect matchings in the complete k-uniform hypergraph on n vertices.

Ferber, Hardiman and Mond also raised the problem of proving similar results for all $d < k$, but it was observed by Sauermann (see [14, Section 5]) that the particular bound in (1.1) does *not* generalise to all d. Specifically, there is a 3-uniform hypergraph G satisfying $\delta_1(G) \geq (\alpha_1(3) + 0.01)(n - 1)$ which has exponentially fewer than $\Phi(K_n^{(k)})(\alpha_1(3) + 0.01)^{n/k}$ perfect matchings.

1.3 Main Results

In this paper, we extend Theorem 1.3 (i.e., the estimate in terms of graph entropy) to hypergraphs, for all $d < k$, and we deduce a hypergraph analogue of Theorem 1.3 for $d \geq k/2$ (due to the above observation of Sauermann, some assumption on d is necessary for this result).

[1] Actually, similar estimates are also available using the so-called *weak regularity lemma* for hypergraphs, and the so-called *absorption method* (see [7] for more on both), though the results in [10–12] have much better quantitative aspects.

Note that the definition of graph entropy (Definition 1.2) extends naturally to hypergraphs. First, our hypergraph analogue of Theorem 1.3 is as follows.

Theorem 1.7 *Fix constants $1 \le d < k$ and $\gamma > 0$. Let G be an n-vertex k-uniform hypergraph with minimum d-degree at least $(\alpha_d(k) + \gamma)\binom{n-d}{k-d}$ (recalling Definition 1.6). Then the number of perfect matchings in G is*

$$\mathrm{e}^{h(G)}(1/\mathrm{e} + o(1))^{(1-1/k)n},$$

where asymptotics are as $n \to \infty$ (along a sequence of integers divisible by k).

We remark that Glock, Gould, Joos, Kühn and Osthus [16] explicitly asked if the entropy approach of Cuckler and Kahn can be generalized to hypergraphs; Theorem 1.7 answers this question in the setting of perfect matchings.

For $d \ge k/2$, we are able to determine the minimum possible value of $h(G)$ over all k-graphs with a given minimum d-degree, and we are therefore able to deduce an analogue of Theorem 1.4, as follows.

Theorem 1.8 *Fix constants $1 \le d < k$ satisfying $d \ge k/2$, and fix a constant $\gamma > 0$. Let G be an n-vertex k-uniform hypergraph with minimum d-degree $\delta_d(G) \ge (\alpha_d(k) + \gamma)\binom{n-d}{k-d}$ (recalling Definition 1.6), and let $p = \delta_d(G)/\binom{n-d}{k-d}$. Then the number of perfect matchings in G is at least*

$$\Phi(K_n^{(k)}) \cdot (p + o(1))^{n/k},$$

where $\Phi(K_n^{(k)}) = n!/((n/k)!(k!)^{n/k})$ is the number of perfect matchings in the complete k-uniform hypergraph on n vertices, and asymptotics are as $n \to \infty$ (along a sequence of integers divisible by k).

As mentioned in the last subsection, the $d = k-1$ case of Theorem 1.8 was already known, thanks to work of Ferber, Hardiman and Mond [14].

Our proof approach follows the same approximate strategy as Cuckler and Kahn [3, 4]. Indeed, Theorem 1.7 really consists of a lower bound and an upper bound, proved separately. The lower bound is proved via analysis of a so-called *random greedy process* (as in [4]), and the upper bound is proved using the so-called *entropy method* (as in [3]). Actually, the desired upper bound can be quite easily deduced from a general estimate proved by Kahn [17] in his work on *Shamir's conjecture*; the main content of this paper is the lower bound.

The most significant point of departure from [4] is that in the hypergraph setting it seems to be harder to study properties of the maximum-entropy fractional perfect matching. To give some context: the proof of Theorem 1.7 considers a random process guided by a fractional perfect matching $\boldsymbol{x}$, and to maintain good control over the trajectory of this process it is important that $\boldsymbol{x}$ is quite evenly distributed over the edges of G.

In [3], a simple weight-shifting argument is used to show that the maximum-entropy fractional perfect matching is always quite evenly distributed. Indeed, if $\boldsymbol{x}$ has "unevenly distributed weights", one can shift weights around a 4-cycle to "even out the weights" and increase the entropy. There does not seem to be a direct generalisation of this fact to hypergraphs: in the hypergraph setting we cannot rule out the possibility that the entropy-maximising fractional perfect matching $\boldsymbol{x}^\star$ has a few very high-weight edges (these can be viewed as "pockets of trapped energy"). So, we introduce an additional "annealing" technique: we slightly perturb $\boldsymbol{x}^\star$ using a *spread measure* on perfect matchings (obtained from [10, 12]) to "release the trapped energy": namely, such a perturbation enables entropy-increasing weight-shifting operations. The resulting fractional perfect matching might not be entropy-maximising, but it is *nearly* entropy-maximising, which is good enough. Similar ideas also appeared in a recent work of Joos and Schrodt [18] on counting oriented trees in digraphs with large semidegree.

1.4 Further Directions

Perhaps the most important problem left open by our work is to prove a result similar to Theorem 1.8 for $d < k/2$. As mentioned in the introduction, the specific bound in Theorem 1.8 cannot hold for all $d < k/2$, due to a construction of Sauermann (see [14, Section 5]) but we can still ask for the minimum possible number of perfect matchings given a d-degree condition. This seems to have been first explicitly asked by Ferber, Hardiman and Mond [14]. Given Theorem 1.7, this problem now comes down to an entropy-maximisation problem over fractional perfect matchings.

Second, there is the question of improving the quantitative aspects of Theorem 1.7 and 1.8. We have presented both theorems with a multiplicative error of the form $(1+o(1))^n$, but the proof really gives a multiplicative error of the form $(1+O(n^{-c_k}))^n$ for some $c_k \geq 0$. It is unclear what exactly is the sharpest bound one can hope for, without using more refined information about our hypergraph G than its entropy.

Third, the results in this paper only apply when the Dirac threshold is "asymptotically exceeded" (i.e., when the minimum d-degree is at least $(1+\gamma)m_d(k,n)$ for some constant $\gamma > 0$). It would be interesting to prove similar results under only the assumption $d \geq m_d(k,n)$. In particular, this may be within reach in the case $d \geq k/2$, as in this case the exact value of $m_d(k,n)$ is known (thanks to Treglown and Zhao [19, 20]).

Finally, we would like to ask if Theorem 1.7 and 1.8 can be extended to other types of spanning substructures other than perfect matchings. In particular, there are several different notions of Hamilton cycles in hypergraphs ("tight cycles", "loose cycles", "Berge cycles"...), which would be a good starting point. There has actually already been quite some work in this direction, but only in the case $d = k-1$ (i.e., where we impose a condition on $\delta_{k-1}(G)$). In particular, the work of Ferber, Hardiman and Mond [14], mentioned in the introduction, proves an analogue of the $d = k-1$ case of Theorem 1.8 for certain types of hypergraph Hamilton cycles.

1.5 Notation

As convention, we abbreviate a k-uniform hypergraph as a k-graph. We also call a subset of vertices of size d a d-set.

Our graph-theoretic notation is for the most part standard. For a graph/hypergraph G, we write $V(G)$ and $E(G)$ for its sets of vertices and edges, respectively. For $S \subseteq V(G)$, we write $\deg_G(S)$ to denote its degree. If G is a graph, we write $\delta(G)$ to denote the minimum degree of G. If G is a k-graph, for $0 \le d \le k-1$, we write $\delta_d(G)$ to denote its minimum d-degree. We implicitly treat an edge as a set of vertices. We say a bipartite graph on $2n$ vertices is *balanced* if each part has size n.

Our use of asymptotic notation is standard as well. For functions $f = f(n)$ and $g = g(n)$, we write $f = O(g)$ to mean that there is a constant C such that $|f(n)| \le C|g(n)|$ for all n, $f = \Omega(g)$ to mean that there is a constant $c > 0$ such that $f(n) \ge c|g(n)|$ for sufficiently large n, and $f = o(g)$ or $g = \omega(f)$ to mean that $f/g \to 0$ as $n \to \infty$. Slightly less standardly, for $a, b \in \mathbb{R}$ we write $a = \pm b$ to mean $a \in [-b, b]$.

Finally, we also use standard probabilistic notation: $\mathbb{P}[\mathcal{E}]$ denotes the probability of an event $\mathcal{E}$ and $\mathbb{E}[X]$ denotes the expected value of a random variable X. We say a sequence of events $(\mathcal{E}_n)_{n\in\mathbb{N}}$ happens with high probability (abbreviated as whp) if $\lim_{n\to\infty} \mathbb{P}(\mathcal{E}_n) = 1$.

1.6 Structure of the Paper

In Section 2 we start with some general preliminaries, before moving on to the proofs of Theorem 1.7 and 1.8.

Theorem 1.7 really consists of a lower bound and an upper bound on the number of perfect matchings, which are proved separately. Most of the paper is devoted to the lower bound. Indeed, in Section 3, we will show that in the setting of Theorem 1.7 we can always find a fractional perfect matching which simultaneously has almost maximum entropy, and which satisfies an “approximate uniformity” condition. In Section 4, we will show how to use such a fractional perfect matching to prove the lower bound in Theorem 1.7, via analysis of a random matching process.

Then, in Section 5, we deduce the upper bound in Theorem 1.7 from a result of Kahn [17] (proved using the so-called *entropy method*).

Finally, in Section 6, we deduce Theorem 1.8 by constructing an explicit fractional perfect matching and lower bounding its entropy. The primary ingredient here is an algebraic construction of fractional perfect matchings in bipartite graphs, due to Cuckler and Kahn [4].

2 Preliminaries

First, it is convenient to introduce some notation that will be used throughout the paper.

Definition 2.1 (Dirac hypergraphs) Fix $1 \le d \le k-1$ and $\gamma > 0$, and let G be a k-uniform hypergraph on n vertices. We say G is a (d,γ)-*Dirac k-graph* if it satisfies $\delta_d(G) \ge (\alpha_d(k)+\gamma)\binom{n-d}{k-d}$ (recalling Definition 1.6) and $k \mid n$.

Definition 2.2 For a hypergraph G, let $\Phi(G)$ be the number of perfect matchings in G.

Definition 2.3 (Well-distributed fractional perfect matchings) For a k-graph G an edge-weighting $\boldsymbol{x}$ of G, and $D \geq 1$, we say that $\boldsymbol{x}$ is *D-well-distributed* if

$$\frac{1}{Dn^{k-1}} \leq \boldsymbol{x}[e] \leq \frac{D}{n^{k-1}}$$

for all edges e.

2.1 Inequalities on (fractional) Perfect Matchings

We now collect some basic inequalities that will be used throughout the proof. First, the following inequality can be proved with an easy double-counting argument.

Fact 2.4 (Monotonicity of minimum degrees) Let G be an arbitrary k-uniform hypergraph. Then,

$$\frac{\delta_0(G)}{\binom{n}{k}} \geq \cdots \geq \frac{\delta_{k-1}(G)}{\binom{n-k+1}{1}}.$$

Proof Let $0 \leq i \leq j \leq k-1$. Fix an arbitrary $S \subseteq V$ of size i. Then, $\deg(S) \geq \binom{n-i}{j-i}\delta_j(G)/\binom{k-i}{j-i}$ because we may first extend S to an arbitrary set of size j, observe that this extended set is contained in at least $\delta_j(G)$ edges, and then divide by the times an edge is overcounted, which is at most $\binom{k-i}{j-i}$. Since $\binom{n-i}{j-i}/\binom{k-i}{j-i} = \binom{n-i}{k-i}/\binom{n-j}{k-j}$, this implies the desired conclusion. $\square$

Fact 2.4 implies that $\alpha_0(k) \geq \alpha_1(k) \geq \cdots \geq \alpha_{k-1}(k)$. Since $\alpha_{k-1}(k)$ is known to be equal to $1/2$ (see [13]), we have the following corollary.

Fact 2.5 For any $d < k$ we have $\alpha_d(k) \geq 1/2$.

Recall Jensen's inequality:

Fact 2.6 (Jensen's inequality) Let $\alpha_1, \ldots, \alpha_n \in [0,1]$ with $\sum_{i=1}^n \alpha_i = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, for any $x_1, \ldots, x_n \in \mathbb{R}$, we have $f(\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i f(x_i)$. If f is a concave function, then the reverse inequality holds.

Next, the following inequalities are easy corollaries of Jensen's inequality.

Lemma 2.7 *Let G be a k-graph and let $\boldsymbol{x}$ be a fractional perfect matching in G. Then,*

$$h(\boldsymbol{x}) \leq (1 - 1/k)n \ln n\,.$$

In addition, if $\boldsymbol{x}[e] \leq L$ for every edge e, then

$$h(\boldsymbol{x}) \geq \frac{n}{k} \ln\left(\frac{n}{L^2 k |E(G)|}\right).$$

Proof Because $\boldsymbol{x}$ is a fractional perfect matching, we know that $\sum_{e \ni v} \boldsymbol{x}[e] = 1$. Applying Fact 2.6 to each $v \in V$ with the concave function $t \mapsto \ln t$, $n = |\{e : e \ni v\}|$, $\alpha_i = \boldsymbol{x}[e]$ and $x_i = 1/\boldsymbol{x}[e]$, we conclude

$$\begin{aligned} h(\boldsymbol{x}) = \frac{1}{k} \sum_{v \in V} \sum_{e \ni v} \boldsymbol{x}[e] \ln(1/\boldsymbol{x}[e]) &\leq \frac{1}{k} \sum_{v \in V} \ln\big(|\{e \in E(G) : e \ni v\}|\big) \\ &\leq \frac{n}{k} \ln\left(\binom{n-1}{k-1}\right) \leq \frac{n}{k} \ln(n^{k-1}) = (1 - 1/k) n \ln n \,. \end{aligned}$$

For the lower bound, applying Fact 2.6 with the convex function $t \mapsto -\ln(t)$,$n = |E(G)|$, $\alpha_i = k\boldsymbol{x}[e]/n$ and $x_i = \boldsymbol{x}[e]$, we get

$$h(\boldsymbol{x}) = -\sum_{e \in E(G)} \boldsymbol{x}[e] \ln(\boldsymbol{x}[e]) = -\frac{n}{k} \sum_{e \in E(G)} \frac{k\boldsymbol{x}[e]}{n} \ln(\boldsymbol{x}[e]) \geq -\frac{n}{k} \ln\left(\frac{k}{n} \sum_{e \in E(G)} \boldsymbol{x}[e]^2\right).$$

Using the assumption that each $\boldsymbol{x}[e] \leq L$, and rearranging finishes the proof. □

2.2 Entropy

We will need the notion of entropy (of a random variable), and some of its properties. These can be found in any source on information theory, e.g. [21]. Recall that ln is the natural logarithm (base-e), and write $\operatorname{supp}(X)$ for the support of a random variable X

Definition 2.8 Let X be a random variable with finite support. Then the *entropy* of X is defined as:

$$H(X) := \sum_{x \in \operatorname{supp}(X)} \mathbb{P}[X = x] \cdot \ln(1/\mathbb{P}[X = x]) \,.$$

For an event $\mathcal{E}$ on the same probability space as X, we write $H(X \mid \mathcal{E})$ for the entropy of X in the conditional probability space given $\mathcal{E}$. Also, for a second random variable Y with finite support, also on the same probability space, the *conditional entropy of X given Y* is defined as

$$H(X \mid Y) := \sum_{y \in \operatorname{supp}(Y)} \mathbb{P}[Y = y] H(X \mid \{Y = y\}) \,.$$

(Here, for an *event* $\mathcal{E}$, we write $H(X \mid \mathcal{E})$ for the entropy of X in the conditional probability space given $\mathcal{E}$).

Dropping conditioning cannot decrease the entropy:

Fact 2.9 For any random variables X, Y with finite support, we have $H(X \mid Y) \leq H(X)$.

The entropy is maximised by a uniform distribution on its support:

Fact 2.10 Let X be a random variable with finite support. Then $H(X) \leq \ln(|\operatorname{supp}(X)|)$, with equality when X is a uniform distribution.

We will also need the chain rule for entropy, as follows.

Fact 2.11 Let $X_1, \ldots, X_n$ be random variables on the same probability space. Then

$$H(X_1, \ldots, X_n) = \sum_{i=1}^{n} H(X_i \mid X_1, \ldots, X_{i-1}).$$

3 *D*-Well-Distributed Fractional Perfect Matchings with Near-Maximal Entropy

In this section, for any (d, γ)-Dirac k-graph G we show how to construct an well-distributed fractional perfect matching with almost maximum entropy. Specifically, if D is sufficiently large in terms of $\varepsilon > 0$ (and k, and γ), we show that there is a D-well-distributed fractional perfect matching $\boldsymbol{x}$ satisfying $h(G) - h(\boldsymbol{x}) = \varepsilon n$. (Recall the definition of a Dirac hypergraph from Definition 2.1, and the definition of well-distributedness from Definition 2.3).

Lemma 3.1 *Fix constants* $1 \leq d \leq k-1$ *and* $\gamma > 0$*, and consider a* (d, γ)*-Dirac* k*-graph* G*. For any* $\varepsilon > 0$ *which is sufficiently small in terms of* k, γ*, there is a* $O(\varepsilon^{-3k})$*-well-distributed fractional perfect matching* $\boldsymbol{x}$ *satisfying* $h(G) - h(\boldsymbol{x}) \leq \varepsilon n$*.*

We emphasise that in Lemma 3.1 (and in fact throughout the paper), d, k, γ are treated as constants (that is to say, the implicit constant in big-Oh notation is allowed to depend on d, k, γ).

The construction to prove Lemma 3.1 consists of two steps. In the first step, we construct a fractional perfect matching $\hat{\boldsymbol{x}}$ that is $O(1)$-well-distributed (but whose entropy may be quite far from the maximum). In the second step, we use $\hat{\boldsymbol{x}}$ to "anneal" the entropy-maximising perfect matching $\boldsymbol{x}^\star$, which enables the use of a weight-shifting argument to obtain the desired fractional perfect matching.

3.1 *D*-Well-Distributed Fractional Perfect Matchings

As stated above, the first step is to find a C-well-distributed fractional perfect matching $\hat{\boldsymbol{x}}$, as follows.

Lemma 3.2 *Fix constants* $1 \leq d \leq k-1$ *and* $\gamma > 0$*, let* $k \mid n$*, and let* G *be an* n*-vertex* (d, γ)*-Dirac* k*-graph. Then,* G *has a* $O(1)$*-well-distributed fractional perfect matching* $\hat{\boldsymbol{x}}$*.*

We remark that the $d = k-1$ case of Lemma 3.2 was proved by Glock, Gould, Joos, Kühn and Osthus [16, Lemma 4.1], using a switching argument. It seems that this proof approach is also suitable for Lemma 3.2, but given recent developments, it is more convenient to deduce Lemma 3.2 from a so-called *spread measure* independently constructed by Kang, Kelly, Kühn, Osthus and Pfenninger [10] and Pham, Sah, Sawhney and Simkin [11]. Specifically, the following theorem is a corollary of [10, Theorem 1.5][2].

Theorem 3.3 *Fix constants $1 \le d \le k-1$ and $\gamma > 0$, let $k \mid n$, and let G be an n-vertex (d,γ)-Dirac k-graph. Then, there is a probability distribution on the set of perfect matchings of G, such that if a random perfect matching M is sampled from this distribution, we have $\mathbb{P}[e \in M] \le O(1/n^{k-1})$ for all edges e of G.*

Proof of Lemma 3.2 Throughout this proof, we may assume that n is large in terms of k, γ.

Note that any random perfect matching M naturally gives rise to a fractional perfect matching $\boldsymbol{x}$, taking $\boldsymbol{x}[e]{:=}\mathbb{P}[e \in M]$. Theorem 3.3 already gives us a random perfect matching M with the desired upper bound on $\mathbb{P}[e \in M]$; we just need to make an adjustment to M to guarantee the desired lower bound.

Specifically, we define our random perfect matching M by combining Theorem 3.3 with a few rounds of a random greedy matching process, as follows.

- Set $G_0 = G$ and $M_0 = \emptyset$, let $\beta = \gamma/(10k^2)$ and let $T = \lfloor \beta n \rfloor$.
- For $t \in \{1, \dots, T\}$, choose a uniformly random edge e in G_{t-1}, and add it to M_{t-1} to form M_t. Then delete all vertices of e from G_{t-1} to form G_t.
- Note that

  $$\delta_d(G_T) \ge (\alpha_d(k)+\gamma)\binom{n-d}{k-d} - k\cdot\beta n\cdot\binom{n-d-1}{k-d-1} \ge (\alpha_d(k)+\gamma/2)\binom{n-d}{k-d},$$

  because for a set S of d vertices, deleting any other vertex can reduce the degree of S by at most $\binom{n-d-1}{k-d-1}$. So, G_T is $(d,\gamma/2)$-Dirac, and we can apply Theorem 3.3 to obtain a random perfect matching of G_T. Add this to M_T to obtain a random perfect matching M of G.

The number of edges in G_t, for each $t \le T$, is at least

$$(\alpha_d(k)+\gamma)\binom{n}{k} - k\cdot\beta n\cdot\binom{n-1}{k-1} \ge \frac{1}{2}\binom{n}{k}, \tag{3.1}$$

recalling Fact 2.5. So, for each edge e, we have

$$\mathbb{P}[e \in M] = \mathbb{P}[e \in M_T] + \mathbb{P}[e \in M \setminus M_T] \le \frac{T}{\binom{n}{k}/2} + O(1/n^{k-1}) = O(1/n^{k-1}).$$

It remains to prove a corresponding lower bound $\mathbb{P}[e \in M] = \Omega_{k,\gamma}(1)$.

For $1 \le t \le T$, let $\mathcal{E}_t$ be the event that e is selected in the t-th round (i.e., $e \in M_t \setminus M_{t-1}$), and let $\mathcal{F}_t$ the event that $e \in G_{t-1}$ (i.e., none of the edges intersecting e have been selected by round t). By definition, the events $\mathcal{E}_1 \cap \mathcal{F}_1, \dots, \mathcal{E}_T \cap \mathcal{F}_T$ are disjoint, so

$$\mathbb{P}[e \in M] \ge \mathbb{P}[e \in M_T] = \mathbb{P}[\mathcal{F}_1 \cap \mathcal{E}_1] + \cdots + \mathbb{P}[\mathcal{F}_T \cap \mathcal{E}_T].$$

[2] In the language of [10, Theorem 1.5], we take $s = 0$, and use the definition of spreadness for a single edge.

So, recalling that $T = \Omega_{k,\gamma}(n)$, it suffices to prove that $\mathbb{P}[\mathcal{F}_t \cap \mathcal{E}_t] = \mathbb{P}[\mathcal{E}_t \mid \mathcal{F}_t]\mathbb{P}[\mathcal{F}_t]$ is at least $\Omega_k(n^{-k})$, for each $t \le T$. To see this, note that G has at most $k \cdot \binom{n-1}{k-1}$ edges intersecting e, and recall (3.1), so

$$\mathbb{P}[\mathcal{F}_t] \ge 1 - t \cdot \frac{k \cdot \binom{n-1}{k-1}}{\binom{n}{k}/2} \ge 1/2\,, \qquad \mathbb{P}[\mathcal{E}_t|\mathcal{F}_t] \ge \frac{1}{\binom{n}{k}} = \Omega_k(n^{-k})\,.$$

□

Any $O(1)$-well-distributed fractional matching already has entropy which differs by at most $O(n)$ from the maximum entropy, as follows.

Proposition 3.4 *Fix $k \ge 2$ and let G be a k-graph on n vertices. Suppose $\boldsymbol{x}$ is a C-well-distributed fractional perfect matching in G for some constant $C > 0$. Then, $h(G) - h(\boldsymbol{x}) \le (2 \ln C/k)n$.*

Proof Since $\boldsymbol{x}$ is C-well-distributed, we have $\boldsymbol{x}[e] \le C/n^{k-1}$ for all edges e. Thus, plugging in $L = C/n^{k-1}$ and using the trivial bound $|E(G)| \le \binom{n}{k}$ in Lemma 2.7, we have

$$h(\boldsymbol{x}) \ge (1 - 1/k)n \ln n - \frac{2 \ln C}{k} n\,.$$

On the other hand, the upper bound in Lemma 2.7 implies $h(G) \le (1 - 1/k)n \ln n$. The desired result follows. □

3.2 Shifting Structures

Next, we define the notion of a *shifting structure*, which we will use to “move weight around” in a fractional perfect matching. We remark that this same structure was considered by Hàn, Person and Schacht [22] in their proof of the so-called *strong absorbing lemma*.

Definition 3.5 (Shifting structure) Let $k \ge 2$ and consider a k-graph G. Let $e = \{v_1, \ldots, v_k\}$ and $f = \{u_1, \ldots, u_k\}$ be two edges in E such that $e \cap f = \{v_1\} = \{u_1\}$. A *shifting structure* on (e, f) is a sequence $\mathcal{U} = (U_2, \ldots, U_k)$ of sets of $k-1$ vertices, satisfying the following conditions:

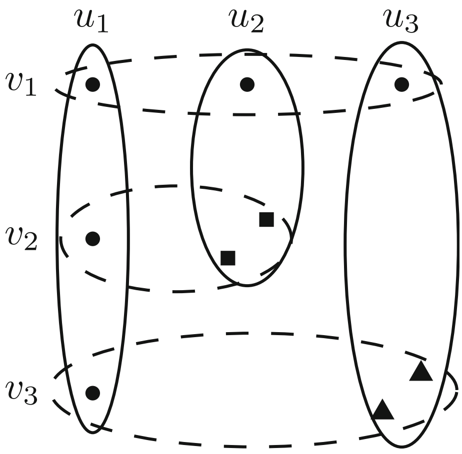
- $U_2, \ldots, U_k$ do not share any vertices with e or f.
- $U_2, \ldots, U_k$ are pairwise disjoint.
- For each $2 \le i \le k$, both $e_i := U_i \cup \{u_i\}$ and $f_i := U_i \cup \{v_i\}$ form edges of G.

For convenience, we let $e_1 = e$ and $f_1 = f$. See Figure 1 for an illustration.

We can use a shifting structure to move weight from $e_1, \ldots, e_k$ to $f_1, \ldots, f_k$, as follows.

Definition 3.6 (shift operation) Given a k-graph G and two edges $e, f \in E(G)$, consider a shifting structure $\mathcal{U} = (U_2, \ldots, U_k)$ on (e, f), and consider a fractional perfect matching $\boldsymbol{x}$. For $\Delta \ge 0$ satisfying $\Delta \le \min_i \boldsymbol{x}[e_i]$ and $\Delta \le \min_i (1 - \boldsymbol{x}[f_i])$, let $\mathrm{shift}_\Delta(\boldsymbol{x}, \mathcal{U})$ be the fractional perfect matching obtained from $\boldsymbol{x}$ by changing $\boldsymbol{x}[e_i]$ to $\boldsymbol{x}[e_i] - \Delta$ and changing $\boldsymbol{x}[f_i]$ to $\boldsymbol{x}[f_i] + \Delta$ for $1 \le i \le k$.

Fig. 1 We illustrate a shifting structure for $k = 3$. The three dotted edges, from top to bottom, are $f_1 (= f)$, f_2 and f_3, and the three solid edges, from left to right, are $e_1 (= e)$, e_2 and e_3. The set U_2 contains the two square-shaped vertices and the set U_3 contains the two triangle-shaped vertices



If a fractional matching is "imbalanced" on a shifting structure, in the sense that the product $\boldsymbol{x}[e_1] \cdots \boldsymbol{x}[e_k]$ is larger than the product $\boldsymbol{x}[f_1] \cdots \boldsymbol{x}[f_k]$, we can use that imbalance to perform a shifting operation which increases the entropy. In particular, we can do this if $\boldsymbol{x}[e_1]$ is much larger than $\boldsymbol{x}[f_1], \ldots, \boldsymbol{x}[f_k]$, and $\boldsymbol{x}[e_2], \ldots, \boldsymbol{x}[e_k]$ are not too small, as follows.

Lemma 3.7 *Consider a k-graph G and a fractional perfect matching* $\boldsymbol{x}$. *Let* $\mathcal{U}$ *be a shifting structure on* (e, f). *For some* $\Delta, \eta > 0$, *suppose that* $\boldsymbol{x}[e_i] \geq 2\Delta$ *and* $\boldsymbol{x}[f_i] \leq \eta - \Delta$ *for all* $i \in \{1, \ldots, k\}$. *Then*

$$h(\text{shift}_\Delta(\boldsymbol{x}, \mathcal{U})) - h(\boldsymbol{x}) \geq \Delta \ln \left(\frac{\boldsymbol{x}[e_1]\Delta^{k-1}}{2\eta^k} \right).$$

Note that if $\boldsymbol{x}[e_1]\Delta^{k-1} > 2\eta^k$, the right-hand side is positive, meaning that the shift operation increases the entropy.

Proof For $0 \leq q \leq \Delta$, let $f(q) = h(\text{shift}_q(\boldsymbol{x}, \mathcal{U}))$. Computing the derivative, we see that

$$f'(q) = \ln \left(\frac{(\boldsymbol{x}[e_1] - q) \cdots (\boldsymbol{x}[e_k] - q)}{(\boldsymbol{x}[f_1] + q) \cdots (\boldsymbol{x}[f_k] + q)} \right) \geq \ln \left(\frac{(\boldsymbol{x}[e_1]/2)\Delta^{k-1}}{\eta^k} \right),$$

for $q \leq \Delta$, and the desired result follows. □

If every vertex in $e \cup f$ has sufficiently large degree, then it is easy to find shifting structures, as follows.

Lemma 3.8 *Let G be an n-vertex k-graph, and fix two edges* $e = \{v_1, \ldots, v_k\}$ *and* $f = \{u_1, \ldots, u_k\}$ *intersecting in a single vertex* $u_1 = v_1$. *Suppose each of* $u_2, \ldots, u_k, v_2, \ldots, v_k$ *has degree at least* $(1/2 + \beta)\binom{n-1}{k-1}$. *Then, if n is sufficiently large (in terms of* k, β*), there is a shifting structure on* (e, f).

Proof A shifting structure is a sequence of sets $U_2, \ldots, U_k$; we choose these sets one-by-one in a greedy fashion. Indeed, suppose that we have made a choice for

$U_2, \ldots, U_{i-1}$. To choose U_i, first note that by inclusion-exclusion, there are at least $2 \cdot (1/2 + \beta)\binom{n-1}{k-1} - \binom{n}{k-1} \geq \beta\binom{n-1}{k-1}$ sets of $k-1$ vertices forming an edge with both v_i and u_i. At most $O_k(n^{k-2})$ of these sets involve other vertices of $e \cup f$ or other vertices of $U_2 \cup \cdots \cup U_{i-1}$, so there is at least one such set which is a valid choice for U_i. $\square$

3.3 Uniformising the Maximum-Entropy Fractional Perfect Matching

Now we use the preparations in Sections 3.1 and 3.2 to prove Lemma 3.1. As mentioned in the introduction, in the case $k = 2$ (i.e., the graph case), Cuckler and Kahn [4] showed that the maximum-entropy fractional matching $\boldsymbol{x}^\star$ is already well-distributed [3]. Specifically, they proved that if the entropy-maximising fractional perfect matching $\boldsymbol{x}^\star$ were not well-distributed, there would be a way to shift weight between high and low weight edges to increase the entropy, contradicting the entropy-maximality of $\boldsymbol{x}^\star$. (This shifting is done with a 4-cycle, which can actually be interpreted as the $k = 2$ case of the shifting structure defined in Definition 3.5).

One might hope to do the same for higher-uniformity hypergraphs, using the shifting operation defined in the previous section. Unfortunately, the Cuckler–Kahn argument does not seem to generalise to higher-uniformity hypergraphs: we were not able to rule out the possibility that the maximum-entropy fractional matching $\boldsymbol{x}^\star$ has a very high-weight edge e (and is therefore not well-distributed), but that there is a "conspiratorial" arrangement of zero-weight edges in all the shifting structures which involve e, which prevent us from using shifting operations to increase the entropy of $\boldsymbol{x}^\star$ by redistributing the weight of e.

To overcome this, we introduce an "annealing" technique, using the well-distributed fractional perfect matching $\hat{\boldsymbol{x}}$ from Lemma 3.2. Indeed, for the maximum-entropy fractional perfect matching $\boldsymbol{x}^\star$, we can consider the "annealed" fractional perfect matching $\boldsymbol{x} = (1-\delta)\boldsymbol{x}^\star + \delta\hat{\boldsymbol{x}}$, for some small $\delta > 0$. The contribution from $\hat{\boldsymbol{x}}$ means that all edges have weight at least $\Omega_{k,\gamma}(\delta/n^{k-1})$, meaning that we can use Lemma 3.7 and 3.8 to perform shifting operations to redistribute some of the weight on high-weight edges, increasing the entropy. Either we can do so many shifting operations that the entropy rises above $h(\boldsymbol{x}^\star)$ (i.e., the entropy gain from shifting overcomes the entropy loss from annealing), or we end up at a well-distributed fractional perfect matching (whose entropy is not much less than $h(\boldsymbol{x})$, if the annealing parameter δ is sufficiently small). The former case is impossible, by the entropy-maximality of $\boldsymbol{x}^\star$, so we obtain the desired well-distributed fractional perfect matching.

Proof of Lemma 3.1 Throughout this proof, we can (and do) assume that n is large, and ε is small, with respect to k, γ.

By Lemma 3.2 and 3.4, there is $C = O(1)$ and a fractional perfect matching $\hat{\boldsymbol{x}}$ such that $h(\boldsymbol{x}^\star) - h(\hat{\boldsymbol{x}}) \leq Cn$, and $\hat{\boldsymbol{x}}[e] \geq 1/(Cn^{k-1})$ for each edge e.

Let $\boldsymbol{x}_0 = (1-\varepsilon/C)\boldsymbol{x}^\star + (\varepsilon/C)\hat{\boldsymbol{x}}$. We think of $\boldsymbol{x}_0$ as an "annealed" version of $\boldsymbol{x}^\star$, which has slightly lower entropy but has a guaranteed lower bound on the weight of

[3] For the curious reader, this argument appears in [4, Lemma 3.2]. (It is stated with an assumption of "ξ-normality", which is automatically satisfied in the setting of this paper)

each edge. Indeed, by concavity of the entropy function $x \mapsto x \ln(1/x)$, we have

$$h(\boldsymbol{x}_0) \geq (1-\varepsilon/C)h(\boldsymbol{x}^\star) - (\varepsilon/C)h(\hat{\boldsymbol{x}}) = h(\boldsymbol{x}^\star) - (\varepsilon/C)(h(\boldsymbol{x}^\star) \\ -h(\hat{\boldsymbol{x}})) \geq h(\boldsymbol{x}^\star) - \varepsilon n\,, \tag{3.2}$$

i.e., $\boldsymbol{x}_0$ still has near-maximum entropy, and we have

$$\boldsymbol{x}_0[e] \geq \frac{\varepsilon}{C^2 n^{k-1}} = \frac{\Omega_{k,\gamma}(\varepsilon)}{n^{k-1}} \text{ for each edge } e\,. \tag{3.3}$$

Now, recall that our objective is to find a fractional perfect matching $\boldsymbol{x}$ satisfying $h(\boldsymbol{x}) \geq h(\boldsymbol{x}^\star) - \varepsilon n$ and $\Omega_{k,\gamma}(\varepsilon^{3k})/n^{k-1} \leq \boldsymbol{x}[e] \leq O(\varepsilon^{-3k})/n^{k-1}$ for every edge e. Note that $\boldsymbol{x}_0$ already satisfies the desired entropy lower bound, and the desired lower bound on all the edge-weights; the only problem is that it might not satisfy the desired upper bound on the edge-weights. We will use a sequence of shift operations, starting from $\boldsymbol{x}_0$, to obtain a fractional perfect matching which satisfies all the desired properties.

An iterated shifting algorithm. Define

$$\eta = \frac{4/\gamma}{\binom{n-1}{k-1}} = \frac{\Omega_{k,\gamma}(1)}{n^{k-1}}, \qquad \Delta = \frac{\varepsilon}{2C^2 n^{k-1}} = \frac{O(\varepsilon)}{n^{k-1}}, \qquad D = \varepsilon^{-3k}.$$

We greedily shift to obtain a sequence of fractional perfect matchings $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T$, according to the following algorithm. For each $t \geq 1$, a *good configuration* at step t is a pair of edges (e, f), intersecting in a single vertex, together with a shifting structure for (e, f), such that (with respect to this shifting structure) we have $\boldsymbol{x}_t[e_1] \geq D/n^{k-1} \geq 2\Delta$ (for small enough ε) and $\boldsymbol{x}_t[e_1], \ldots, \boldsymbol{x}_t[e_k] \geq 2\Delta$ and $\boldsymbol{x}_t[f_1], \ldots, \boldsymbol{x}_t[f_k] \leq \eta - \Delta$. If there is at least one good configuration at step i, then we choose one arbitrarily and set $\boldsymbol{x}_{i+1} = \mathrm{shift}_\Delta(\boldsymbol{x}_i, \mathcal{U})$ (where $\mathcal{U}$ is the shifting structure associated with the good configuration). Otherwise, if no good configuration exists, set $T = i$ and terminate the algorithm.

Assuming ε is sufficiently small, by Lemma 3.7 each step of our algorithm significantly increases the entropy. Namely, for each $0 \leq i < T$ we have

$$h(\boldsymbol{x}_{i+1}) \geq h(\boldsymbol{x}_i) + \Delta \ln\left(\frac{D\Delta^{k-1}}{2n^{k-1}\eta^k}\right) \geq h(\boldsymbol{x}_i) + \Delta \ln(D)/2$$

(here we are using that $\Delta^{k-1}/(2n^{k-1}\eta^k) \geq 1/\sqrt{D}$, assuming that ε is sufficiently small with respect to k, γ). So, recalling (3.2) and the definitions of the parameters Δ, D, we have

$$T \leq \frac{\varepsilon n}{\Delta \ln(D)/2} = O\left(\frac{n^k}{\ln(1/\varepsilon)}\right). \tag{3.4}$$

Reducing to an edge-weight upper bound. Recalling (3.2), and noting that each shifting operation can only increase the entropy, we have

$$h(\boldsymbol{x}_T) \geq h(\boldsymbol{x}_0) \geq h(\boldsymbol{x}^\star) - \varepsilon n\,.$$

Also, recall from (3.3) that in $\boldsymbol{x}_0$ every edge has weight at least 2Δ. We only ever decrease the weight of an edge (by Δ) when the weight of that edge is at least 2Δ, so in $\boldsymbol{x}_T$ every edge has weight at least $\Delta \geq O(\varepsilon)/n^{k-1}$. That is to say, to prove that $\boldsymbol{x} = \boldsymbol{x}_T$ is $O_{k,\gamma}(\varepsilon^{-3k})$-well-distributed (completing the proof of the lemma), it suffices to show that in $\boldsymbol{x}_T$ every edge has weight less than $D/n^{k-1} = O(\varepsilon^{-3k})/n^{k-1}$.
High-weight edges imply good configurations. Suppose for the purpose of contradiction that there is an edge e with $\boldsymbol{x}_T[e] \geq D/n^{k-1}$. We will show that there is a good configuration at step T (contradicting the definition of T as the final step of our iterated shifting algorithm). This will be accomplished by deleting a few edges of G (whose weights are not compatible with a good configuration) and then applying Lemma 3.8.

First, let $G_{>\eta-\Delta}$ be the k-graph consisting of all edges g of G with $\boldsymbol{x}_T[g] > \eta - \Delta$. By the definition of η and Δ, assuming ε (therefore Δ) is sufficiently small, we have $\eta - \Delta > 0$ and $1/(\eta - \Delta) \leq (\gamma/2)\binom{n-1}{k-1}$. For every vertex v, since $\boldsymbol{x}_T$ is a fractional perfect matching, the sum of weights of edges including v is exactly 1, so there are at most $1/(\eta - \Delta) \leq (\gamma/2)\binom{n-1}{k-1}$ edges containing v which have weight greater than $\eta - \Delta$. That is to say, $G_{>\eta-\Delta}$ has maximum degree at most $(\gamma/2)\binom{n-1}{k-1}$.

Then, let $G_{<2\Delta}$ be the k-graph consisting of the edges g of G which satisfy $\boldsymbol{x}_T[g] < 2\Delta$. Recalling (3.3), each edge of $G_{<2\Delta}$ must have been involved in one of our T shifting operations (each of which decreases the weight of exactly k edges), so by (3.4), $G_{<2\Delta}$ has at most $kT \leq O(n^k/\ln(1/\varepsilon)) \leq (\gamma/(8k))\binom{n}{k}$ edges (assuming ε is sufficiently small). That is to say, $G_{<2\Delta}$ has average degree at most $(\gamma/(8k))\binom{n-1}{k-1}$, meaning that it has at most $n/(4k)$ vertices which have degree greater than $(\gamma/2)\binom{n-1}{k-1}$. Call these vertices *bad*.

Fix a vertex $v_1 \in e$. By Fact 2.4, v_1 is contained in at least $(1/2 + \gamma)\binom{n-1}{k-1}$ edges of G. At most $(n/(4k) + k - 1)\binom{n-2}{k-2} \leq (1/2)\binom{n-1}{k-1}$ of these edges also contain a bad vertex or a vertex in $e \setminus \{v_1\}$. So, there is an edge f, containing no bad vertices, such that $e \cap f = \{v_1\}$.

Let G' be the hypergraph obtained from G by deleting all edges of $G_{>\eta-\Delta}$ which do not intersect f, and deleting all edges of $G_{<2\Delta}$ which do not intersect e. By construction, we have deleted at most $(\gamma/2)\binom{n-1}{k-1}$ edges incident to every vertex in $(e \cup f) \setminus \{v_1\}$. By Facts 2.4 and 2.5, we have $\delta_1(G) \geq (1/2 + \gamma)\binom{n-1}{k-1}$, so in G', every vertex in $(e \cup f) \setminus \{v_1\}$ has degree at least $(1/2 + \gamma/2)\binom{n-1}{k-1}$. Using Lemma 3.8, we deduce that there is a shifting structure $\mathcal{U}$ for (e, f) in G'. By the definition of G' via $G_{>\eta-\Delta}$ and $G_{<2\Delta}$, with respect to $\mathcal{U}$ we have $\boldsymbol{x}_2[e_1], \ldots, \boldsymbol{x}_2[e_k] \geq 2\Delta$ and $\boldsymbol{x}_2[f_1], \ldots, \boldsymbol{x}_2[f_k] \leq \eta - \Delta$. Recalling our assumption $\boldsymbol{x}_T[e_1] = \boldsymbol{x}_T[e] \geq D/n^{k-1}$, we see that $\mathcal{U}$, together with (e, f), forms a good configuration, yielding the desired contradiction. $\square$

4 A Random Greedy Matching Process

The goal of this section is prove the lower bound in Theorem 1.7. We do this by analysing a random greedy matching process, guided by a fractional perfect matching.

Definition 4.1 (Random greedy matching process) Consider a k-graph G, and let $\boldsymbol{x}$ be a fractional perfect matching in G. Then, the *greedy matching process* in G with respect to $\boldsymbol{x}$ is defined as follows. Let $G(0) = G$ and for each $i \geq 1$: choose a random edge $e(i)$ in $G(i-1)$, with probability proportional to $\boldsymbol{x}[e(i)]$, and delete all the vertices in $e(i)$ from $G(i-1)$ to obtain a k-graph $G(i)$. This process cannot continue forever: let $G(M)$ be the first graph in the sequence with no positive-weight edges.

First, in Section 4.1, we consider certain statistics that evolve with the process, and show that these statistics concentrate around certain explicit values. This analysis requires that $\boldsymbol{x}$ is well-distributed (recall the definition from Definition 2.3). Then, in Section 4.2, we use this analysis to show that the greedy matching process gives rise to a high-entropy probability distribution on perfect matchings. By Fact 2.10, this implies a lower bound on the number of perfect matchings.

4.1 Analysis of the Greedy Matching Process

Our objective in this section is to show that if $\boldsymbol{x}$ is n^c-well-distributed for sufficiently small c, then our greedy matching process is likely to continue until nearly all vertices are gone. We are also able to track the evolution of various important statistics of the process. These statistics are chosen to evince that the almost-perfect matching produced by the process can typically be completed to a perfect matching, and that the resulting distribution on perfect matchings has high entropy.

Theorem 4.2 *For any constant $k \geq 2$, there is a constant $c > 0$ such that the following holds. Let G be a k-graph, and let $\boldsymbol{x}$ be an n^c-well-distributed fractional perfect matching of G. Then, consider the greedy matching process defined in Definition 4.1 (in G, with respect to $\boldsymbol{x}$), producing a sequence of k-graphs $G(0), \ldots, G(M)$. With probability at least $1 - \exp(-n^c)$: we have*

$$M \geq (1 - n^{-c})n/k\,, \tag{4.1}$$

and for each $i \leq (1 - n^{-c})n/k$ and each set S of at most $k - 1$ vertices of $G(i)$,

$$\sum_{e \in E(G(i))} \boldsymbol{x}[e] = (1 \pm n^{-c}) \left(\frac{n/k - i}{n/k}\right)^k \frac{n}{k}\,, \tag{4.2}$$

$$\sum_{e \in E(G(i))} \boldsymbol{x}[e] \ln\left(\frac{1}{\boldsymbol{x}[e]}\right) = (1 \pm n^{-c}) \left(\frac{n/k - i}{n/k}\right)^k h(\boldsymbol{x})\,, \tag{4.3}$$

$$\deg_{G(i)}(S) = \left(\frac{n/k - i}{n/k}\right)^{k-|S|} \deg_G(S) \pm n^{k-|S|-c}\,. \tag{4.4}$$

To prove Lemma 4.2, we will use an approach popularised by Bohman, Frieze and Lubetzky [23, 24]: we anticipate the trajectories of the relevant statistics, and use the difference between the actual and anticipated trajectories, to define supermartingales to which we can apply an exponential tail bound for martingales, namely *Freedman's inequality*[4] [25, Theorem 1.6]. The statement of Freedman's inequality is as follows, writing $\Delta X(i)$ for the one-step change $X(i+1) - X(i)$.

Lemma 4.3 *Let $X(0), X(1), \dots$ be a supermartingale*[5] *with respect to a filtration $(\mathcal{F}_i)_i$. Suppose that $|\Delta X(i)| \leq K$ for all i, and let $V(i) = \sum_{j=0}^{i-1} \mathbb{E}\left[(\Delta X(j))^2 \,\middle|\, \mathcal{F}_j\right]$. Then for any $t, v > 0$,*

$$\mathbb{P}\left[X(i) \geq X(0) + t \text{ and } V(i) \leq v \text{ for some } i\right] \leq \exp\left(-\frac{t^2}{2(v + Kt)}\right).$$

We remark that, for the graph case ($k = 2$), Cuckler and Kahn [4] also analysed a random greedy process, though their analysis was much more complicated. Partially this is due to their consideration of Hamilton cycles as well as perfect matchings, but it is also due to the fact that the field of random (hyper)graph processes has become much more mature (with more streamlined methods available) since [4] first appeared.

Proof of Lemma 4.2 Instead of specifying the value of c at the outset, we will assume that c is sufficiently small to satisfy certain inequalities that arise in few places throughout the proof.

It is convenient to "freeze" the process instead of aborting it at the point where no positive-weight edges remain: recalling that $G(M)$ is the first graph in the sequence with no positive-weight edges, we artificially define $G(i) = G(M)$ for $i \geq M$.

Tracked statistics. First, we define an ensemble of statistics that we will keep track of as the process evolves.

For a set S of at most $k-1$ vertices, we write $\Gamma_S(i)$ for the collection of sets U of $k - |S|$ vertices such that $S \cup U$ is an edge of $G(i)$. For each vertex v we define a process f_v by

$$f_v(i) = \sum_{U \in \Gamma_v(i)} \boldsymbol{x}[\{v\} \cup U].$$

Also, for each set S of at most $k-1$ vertices of G, such that $\deg_G(S) \geq n^{k-|S|-c}$, define a process

$$d_S(i) = |\Gamma_S(i)|.$$

Finally, define a process a by

$$a(i) = \sum_{e \in E(G(i))} \boldsymbol{x}[e] \ln\left(\frac{1}{\boldsymbol{x}[e]}\right).$$

[4] Freedman's inequality was originally stated for martingales, but it also holds for supermartingales with the same proof.

[5] Recall that a process $(X(i))_i$ is a *supermartingale* with respect to a filtration $(\mathcal{F}_i)_i$ if $\mathbb{E}[\Delta X(i) \mid \mathcal{F}_i] \leq 0$ for all i.

Note that $E(G(i)) = \Gamma_\emptyset(G(i))$ (so we can interpret $a(i)$ as a sum over elements of $\Gamma_\emptyset(G(i))$).

Note that

$$f_v(0) = 1\,, \qquad d_S(0) = \deg_G(S) \geq n^{k-|S|-c}\,, \qquad a(0) = h(\boldsymbol{x}) = \Theta(n \ln n) \tag{4.5}$$

for each appropriate v, S (for the final estimate we used Lemma 2.7).

We organise all these processes f_v, d_S, a into collections $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$, as follows. Let $\mathcal{Q}_r$ contain the processes of the form d_S with $|S| = k - r$, and in addition put all n processes of the form f_v in $\mathcal{Q}_{k-1}$ and put a in $\mathcal{Q}_k$. The idea is that every process in $\mathcal{Q}_r$ can be interpreted as a sum over r-sets in $\Gamma_S(i)$, for some S of size $k - r$.

Then, let $\mathcal{Q} = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_k$.

Anticipated trajectories. Let

$$p(i) = \frac{n - ki}{n} = \frac{n/k - i}{n/k} = 1 - \frac{ki}{n}\,.$$

This is the proportion of vertices that remain in $G(i)$ (unless the process freezes before then). For $q \in \mathcal{Q}_r$, we anticipate that $q(i)$ will evolve like $p(i)^r q(0)$, because it is a sum over r-sets of vertices.

Error tolerances. Let $C = 1/(8c)$ and let $\varepsilon(i) = p(i)^{-C} n^{-1/4}$. We wish to show that, with high probability, for all $1 \leq r \leq k$ and $q \in \mathcal{Q}_r$ and all $i \leq (1 - n^{-c})n/k$,

$$(1 - \varepsilon(i))p(i)^r q(0) \leq q(i) \leq (1 + \varepsilon(i))p(i)^r q(0)\,.$$

Let T_0 be the smallest index i such that one of the these inequalities is violated (let $T_0 = \infty$ if this never happens). Writing $x \wedge y = \min(x, y)$, let $T = T_0 \wedge (1 - n^{-c})n/k$. For $r \leq k$ and $q \in \mathcal{Q}_r$, define processes $F^- q$, $F^+ q$ by

$$\begin{aligned} F^+ q(i) &= q(i \wedge T)/q(0) - (1 + \varepsilon(i \wedge T))p(i)^r, \\ F^- q(i) &= -q(i \wedge T)/q(0) + (1 - \varepsilon(i \wedge T))p(i)^r. \end{aligned}$$

Note that

$$F^+ q(0) = F^- q(0) = -n^{-1/4}. \tag{4.6}$$

The plan. We want to show that for each $q \in \mathcal{Q}$ and each $s \in \{+, -\}$, the process $F^s q$ is a supermartingale, and then we want to use Lemma 4.3 and the union bound to show that whp each $F^s q$ only takes negative values, i.e., $T_0 > (1 - n^{-c})n/k$.

Before doing this, we show that the statement of Lemma 4.2 follows. Indeed, assume that $T_0 > (1 - n^{-c})n/k$; we will deduce (4.1) to (4.4).

So, consider $i \leq (1 - n^{-c})n/k < T_0$. Note that $p(i) > n^{-c}$, so $\varepsilon(i) < n^{-1/8} < n^{-c}$. For each vertex v and $s \in \{-, +\}$, since $F^s f_v(i) < 0$ we have $f_v(i) = (1 \pm \varepsilon(i))p(i)^{k-1}$, and

$$\sum_{e\in E(G(i))} \boldsymbol{x}[e] = \frac{1}{k}\sum_{v\in V(G(i))} f_v(i) = \frac{1}{k}\cdot p(i)n\cdot(1\pm\varepsilon(i))p(i)^{k-1}\,, \tag{4.7}$$

which implies (4.2).

In particular, since the above expression is positive, whenever $i \le (1-n^{-c})n/k$ there is at least one positive-weight edge in $G(i)$. That is to say, the process cannot freeze until after step $(1-n^{-c})n/k$, i.e., $(1-n^{-c})n/k = T \le M$, which is (4.1). For (4.3), again consider $i \le T = (1-n^{-c})n/k < T_0$. Since $F^s a < 0$ for each $s \in \{-,+\}$, recalling (4.5) we have

$$\sum_{e\in E(G(i))} \boldsymbol{x}[e]\ln\left(\frac{1}{\boldsymbol{x}[e]}\right) = a(i) = (1\pm n^{-1/8})\cdot p(i)^k h(\boldsymbol{x})$$

which implies (4.2). Finally, note that (4.4) trivially holds when $\deg_G(S) \le n^{k-|S|-c}$. For any other $S \subseteq V(G(i))$, using that $F^s d_S(i) < 0$, we see that

$$\deg_{G(i)}(S) = |\Gamma_S(i)| = (1\pm n^{-1/8})p(i)^{k-|S|}\deg_{G(i)}(S)\,,$$

which implies (4.4).

Expected and worst-case differences. We now estimate the expected and worst-case per-step differences $\Delta f_v(i)$, $\Delta a(i)$, $\Delta d_S(i)$. (Recall that $\Delta X(i)$ means $X(i+1) - X(i)$).

Fix a set S of at most $k-1$ vertices, and consider $U \in \Gamma_S(i)$, for $i < T$. The only way we can have $U \notin \Gamma_S(i+1)$ is if $e(i)$ intersects U. For each vertex $u \in U$, the total weight of edges in $G(i)$ which intersect u is $f_u(i) = (1\pm\varepsilon(i))p(i)^{k-1}$ (since we are assuming $i < T \le T_0$). The total weight of the edges which intersect more than one vertex in U is at most $\binom{k}{2}\binom{n}{k-2}(n^c/n^{k-1}) = O(n^{-1+c})$, recalling that $\boldsymbol{x}$ is n^c-well-distributed.

So, the total weight of edges intersecting U in $G(i)$ is

$$(1\pm\varepsilon(i))(k-|S|)p(i)^{k-1} - O(n^{c-1}) = (1\pm O(\varepsilon(i)))(k-|S|)p(i)^{k-1}\,.$$

On the other hand, (4.7) gives an expression for the total weight of all edges in $G(i)$. Dividing these two quantities by each other, we see that

$$\mathbb{P}\left[U \notin \Gamma_S(i+1)\,\middle|\, G(1),\ldots,G(i)\right] = (1\pm O(\varepsilon(i)))\frac{k-|S|}{p(i)\cdot n/k}\,.$$

Now, for $1 \le r \le k$ and $q \in \mathcal{Q}_r$, we have $q(i) = (1\pm\varepsilon(i))p(i)^r q(0)$ (still assuming that $i < T$). Recall that $q(i)$ can be interpreted as a sum over elements of $\Gamma_S(i)$, for some S of size $k-r$, so by linearity of expectation,

$$
\begin{aligned}
\mathbb{E}\left[\Delta q(i) \,\middle|\, G(1), \ldots, G(i)\right] &= -(1 \pm O(\varepsilon(i))) \frac{rp(i)^{r-1}}{n/k} q(0) \\
&= -(1 \pm O(\varepsilon(i))) p(i)^r q(0) \cdot \frac{r}{p(i) \cdot n/k} \\
&= -\frac{rp(i)^{r-1}}{n/k} q(0) \pm O\left(\frac{\varepsilon(i) p(i)^{r-1}}{n} q(0)\right). \qquad (4.8)
\end{aligned}
$$

Note also that we have the deterministic bound

$$
|\Delta f_v(i)| \le k \binom{n}{k-2} (n^c / n^{k-1}) = O(n^{c-1}),
$$

using the n^c-well-distributedness of $\boldsymbol{x}$, and the fact that deleting the k vertices of an edge results in at most $k\binom{n}{k-2}$ edges of $\Gamma_v(i)$ being deleted. Similarly, we have

$$
|\Delta a(i)| \le k \binom{n}{k-1} \left((n^c / n^{k-1}) \ln \left(\frac{1}{n^{-c} / n^{k-1}} \right) \right) = O(n^c \ln n).
$$

and

$$
|\Delta d_S(i)| \le k \binom{n}{k - |S| - 1} = O(n^{k-|S|-1}).
$$

Recalling (4.5), for all $q \in \mathcal{Q}$ we deduce

$$
|\Delta q(i)| \le O(n^{c-1}) \cdot q(0). \qquad (4.9)
$$

Computations on the trajectories and tolerances. In order to estimate expressions of the form $\Delta F^s q$ (for $q \in \mathcal{Q}$ and $s \in \{-, +\}$), we also need some control over expressions of the form Δp^r and $\Delta(\varepsilon p^r)$ (where we write p^r for the function $i \mapsto p(i)^r$ and we write εp^r for the function $i \mapsto \varepsilon(i) p(i)^r$). The required computations are basically a calculus exercise.

For any $r = O(1)$, we have

$$
\begin{aligned}
\Delta p^r(i) &= \left(1 - \frac{i+1}{n/k}\right)^r - \left(1 - \frac{i}{n/k}\right)^r \\
&= \left(1 - \frac{i}{n/k}\right)^r \left(\left(\frac{n/k - i - 1}{n/k - i} \right)^r - 1 \right) \\
&= p^r(i) \left(\left(1 - \frac{1}{n/k - i}\right)^r - 1 \right) \\
&= p^r(i) \left(-\frac{r}{n/k - i} + O\left(\frac{1}{(n/k - i)^2} \right) \right) \\
&= -\frac{rp^{r-1}(i)}{n/k} \left(1 + O\left(\frac{p(i)}{n} \right) \right)
\end{aligned}
$$

$$= -\frac{rp^{r-1}(i)}{n/k} + o\left(\frac{\varepsilon(i)p^{r-1}(i)}{n}\right), \tag{4.10}$$

where we used the fact that $p(i)/\varepsilon(i) = o(n)$ in the last inequality (with lots of room to spare) and we deduce

$$\begin{aligned}
\Delta(\varepsilon p^r)(i) &= n^{-1/4} \Delta p^{r-C}(i) \\
&\geq n^{-1/4} \frac{(C - r - o(1))p^{r-C-1}(i)}{n/k} \\
&= \frac{(C - r - o(1))\varepsilon(i)\, p^{r-1}(i)}{n/k}.
\end{aligned} \tag{4.11}$$

Verifying the supermartingale property. We are assuming c is small (so $C = 1/(8c)$ is large). So, for any $r \leq k$, $q \in \mathcal{Q}_r$ and $i < T$, combining (4.8),(4.10) and (4.11) we see that

$$\mathbb{E}\left[\Delta F^+ q(i) \,\middle|\, G(1), \ldots, G(i)\right] = \mathbb{E}\left[\Delta q(i) \,\middle|\, G(i)\right]/q(0) - \Delta p^r(i) - \Delta(\varepsilon p^r)(i) \leq 0.$$

and similarly

$$\mathbb{E}\left[\Delta F^- q(i) \,\middle|\, G(1), \ldots, G(i)\right] = -\mathbb{E}\left[\Delta q(i) \,\middle|\, G(i)\right]/q(0) + \Delta p^r(i) - \Delta(\varepsilon p^r)(i) \leq 0.$$

We have proved that each $F^s q$ is a supermartingale.

With a similar calculation, we see

$$\mathbb{E}\left[|\Delta F^s q(i)| \,\middle|\, G(1), \ldots, G(i)\right] \leq O(1/n). \tag{4.12}$$

for all $q \in \mathcal{Q}$ and $s \in \{-, +\}$.

Applying Freedman's inequality. Combining (4.9), (4.10) and (4.11), we see that (with probability 1), for each $q \in \mathcal{Q}$ and $s \in \{-, +\}$,

$$|\Delta F^s q| \leq O(n^{c-1}).$$

So, using (4.12),

$$\begin{aligned}
\mathbb{E}\left[(\Delta F^s q(i))^2 \,\middle|\, G(1), \ldots, G(i)\right] &\leq O(n^{c-1}) \cdot \mathbb{E}\left[|\Delta F^s q(i)| \,\middle|\, G(1), \ldots, G(i)\right] \\
&\leq O(n^{c-2}).
\end{aligned}$$

Since $T \leq n/k$ (and $\Delta F^s q(i) = 0$ for $i \geq T$) this yields

$$\sum_{i=0}^{\infty} \mathbb{E}\left[(\Delta F^s q(i))^2 \,\middle|\, G(1), \ldots, G(i)\right] = O(n^{c-1}).$$

Applying Lemma 4.3 with $t = -F^s q(0) = n^{-1/4}$ (recalling (4.6)) and $v = O(n^{c-1})$ then gives

$$\mathbb{P}\left[F^s q(i) \geq 0 \text{ for some } i\right] \leq \exp\left(-\Omega\left(\frac{(n^{-1/4})^2}{n^{c-1} + n^{c-1} \cdot n^{-1/4}}\right)\right) = \exp(-n^{1/4}) .$$

The desired result then follows from a union bound over $q \in \mathcal{Q}$ -and $s \in \{-, +\}$. □

4.2 Deducing the Lower Bound in Theorem 1.7

Now we prove the lower bound in Theorem 1.7, using Lemma 4.2 and 3.1.

Proof of the lower bound in Theorem 1.7 Let $\mathcal{S}(G)$ be the set of *sequences* of disjoint edges $(e(1), \ldots, e(n/k))$ (i.e, perfect matchings with an ordering on their edges). Our objective is to prove that

$$\ln\left(|\mathcal{S}(G)|\right) = h(G) + \frac{n}{k} \ln\left(\frac{n}{k}\right) - n + o(n) , \tag{4.13}$$

Recalling that $\Phi(G)$ denotes the number of perfect matchings in G, we have $|\mathcal{S}(G)| = (n/k)!\Phi(G)$. So, given (4.13), the desired result on $\Phi(G)$ will follow, using Stirling's approximation $\ln((n/k)!) = (n/k)\ln(n/(ek)) + o(n)$.

The plan. Let c be as in Lemma 4.2. By Lemma 3.1, there is an n^c-well-distributed fractional perfect matching $\boldsymbol{x}$ with

$$h(\boldsymbol{x}) \geq h(G) - O(n^{1-c/(3k)}) = h(G) - o(n) .$$

We will consider the random greedy matching process in Definition 4.1, guided by the fractional perfect matching $\boldsymbol{x}$. By Lemma 4.2, this almost always produces a sequence of disjoint edges $e(1), \ldots, e(I)$, for some $I = (1 - o(1))n/k$, which can be completed to an ordered perfect matching $(e(1), \ldots, e(n/k))$ (by (4.4), and the fact that G is a (d, γ)-Dirac k-graph). That is to say, the random greedy matching process gives rise to a probability distribution on $\mathcal{S}(G)$, and the entropy of this probability distribution gives a lower bound on $\ln\left(|\mathcal{S}(G)|\right)$ (recalling Fact 2.10). We will estimate the desired entropy using the chain rule (Fact 2.11), and (4.3) and (4.2). (The details of this are a bit fiddly, as we need to handle the small probability that the random greedy matching process fails to satisfy the conclusion of Lemma 4.2).

Random greedy matching. Let $I = (1 - n^{-c/(2k)})n/k$. We run the random greedy matching process defined in Definition 4.1 for I steps to produce a sequence of disjoint edges $e(1), \ldots, e(M \wedge I)$, where M is the first step for which $G(M)$ has no positive-weight edges (here we write $x \wedge y = \min(x, y)$). If $M < I$, we artificially set $e(i) = *$ for $M \wedge I < i \leq I$.

Let $p(i) = (n/k - i)/(n/k)$ and let $\mathcal{E}$ be the event that the estimates in (4.1) to (4.4) (in Theorem 4.2) hold for the first I steps. In particular, this implies that $M \geq I$, and it implies that for $i \leq I$,

$$\sum_{e \in E(G(i))} \boldsymbol{x}[e] = (1 \pm n^{-c}) p(i)^k \frac{n}{k}, \tag{4.14}$$

$$\sum_{e \in E(G(i))} \boldsymbol{x}[e] \ln\left(\frac{1}{\boldsymbol{x}[e]}\right) \geq (1 \pm n^{-c}) p(i)^k h(\boldsymbol{x}) \geq p(i)^k (h(G) - o(n)), \tag{4.15}$$

$$\delta_d(G(i)) \geq p(i)^{k-d} \delta_d(G) - n^{k-|S|-c} \geq (\alpha_d(k) + \gamma/2) \binom{p(i)n - d}{k - d}.$$

When $\mathcal{E}$ occurs, the definition of $\alpha_d(k)$ implies that $G(i)$ has a perfect matching, meaning that we can extend $e(1), \ldots, e(I)$ to a sequence of n/k disjoint edges $e(1), \ldots, e(n/k)$. Using Fact 2.10, this implies that

$$\ln(|\mathcal{S}(G)|) \geq H\left(e(1), \ldots, e(I) \,\middle|\, \mathcal{E}\right).$$

Removing the conditioning. Now, let $I_{\mathcal{E}}$ be the indicator random variable for the event $\mathcal{E}$, and let $\mathcal{E}^c$ be the complement of $\mathcal{E}$. Observe that $H(I_{\mathcal{E}} \mid e(1), \ldots, e(I)) = 0$ since the outcome of $e(1), \ldots, e(I)$ completely determines $I_{\mathcal{E}}$. Thus, on the one hand, by Fact 2.11, we have

$$\begin{aligned} H(e(1), \ldots, e(I), I_{\mathcal{E}}) &= H(e(1), \ldots, e(I)) \\ &\quad + H(I_{\mathcal{E}} \mid e(1), \ldots, e(I)) = H(e(1), \ldots, e(I)). \end{aligned}$$

On the other hand, using Fact 2.11 again, we have

$$\begin{aligned} &H(e(1), \ldots, e(I), I_{\mathcal{E}}) \\ &= H(I_{\mathcal{E}}) + H\left(e(1), \ldots, e(I) \,\middle|\, I_{\mathcal{E}}\right) \\ &= H(I_{\mathcal{E}}) + \mathbb{P}[\mathcal{E}] \cdot H\left(e(1), \ldots, e(I) \,\middle|\, \mathcal{E}\right) + \mathbb{P}[\mathcal{E}^c] \cdot H\left(e(1), \ldots, e(I) \,\middle|\, \mathcal{E}^c\right). \end{aligned}$$

By Lemma 4.2, we have $\mathbb{P}[\mathcal{E}] = 1 - n^{-\omega(1)}$. Also note that $H\left(e(1), \ldots, e(I) \,\middle|\, \mathcal{E}^c\right) \leq \ln\left(\binom{n}{k}!\right) = O(n^k \ln n)$. So, we deduce

$$H\left(e(1), \ldots, e(I) \,\middle|\, \mathcal{E}\right) \geq H(e(1), \ldots, e(I)) - o(n). \tag{4.16}$$

So, it suffices to study $H(e(1), \ldots, e(I))$.

Computing the entropy. By Fact 2.11 we have

$$H(e(1), \ldots, e(I)) = \sum_{i=0}^{I-1} H\left(e(i+1) \,\middle|\, e(1), \ldots, e(i)\right). \tag{4.17}$$

Consider any $i \leq I$, and consider some possible outcomes $e_1, \ldots, e_i$ of $e(1), \ldots, e(i)$. By Fact 2.10, we always have the crude bound

$$H\left(e(i+1) \,\middle|\, e(1)=e_1, \ldots, e(i)=e_i\right) \leq \ln\left(\binom{n}{k}\right) = O(\ln n)\,.$$

On the other hand, suppose that $e_1, \ldots, e_i \neq *$. Then these outcomes of $e(1), \ldots, e(i)$ determine an outcome G_i of $G(i)$; let $\mu_i = \sum_{e \in E(G_i)} \boldsymbol{x}(e) \leq (1+o(1))p(i)^k(n/k)$ (here we are using (4.2)) and note that if $\mu_i > 0$ then (by the definition of the greedy matching process)

$$\begin{aligned} H\left(e(i+1) \,\middle|\, e(1)=e_1,\ \ldots,\ e(i)=e_i\right) &= \sum_{e \in E(G_i)} \frac{\boldsymbol{x}[e]}{\mu_i} \ln\left(\frac{\mu_i}{\boldsymbol{x}[e]}\right) \\ &= \frac{1}{\mu_i} \sum_{e \in E(G_i)} \boldsymbol{x}[e] \ln\left(\frac{1}{\boldsymbol{x}[e]}\right) + \ln \mu_i\,. \end{aligned}$$

If $G(i) = G_i$ satisfies (4.14) and (4.15), then we compute

$$H\left(e(i+1) \,\middle|\, e(1)=e_1,\ \ldots,\ e(i)=e_i\right) = \frac{h(G)}{n/k} + \ln\left(\frac{n}{k}\right) + k\ln\left(\frac{n/k-i}{n/k}\right) - o(1)\,.$$

Combining this with Section 4.2, and recalling that $\mathbb{P}[\mathcal{E}^c] = n^{-\omega(1)}$ (by Theorem 4.2), we have

$$\begin{aligned} &H\left(e(i+1) \,\middle|\, e(1), \ldots, e(i)\right) \\ &\geq \mathbb{P}[\mathcal{E}]\left(\frac{h(G)}{n/k} + \ln\left(\frac{n}{k}\right) + k\ln\left(\frac{n/k-i}{n/k}\right) - o(1)\right) + \mathbb{P}[\mathcal{E}^c] \cdot O(\ln n) \\ &\geq \frac{h(G)}{n/k} + \ln\left(\frac{n}{k}\right) + k\ln\left(\frac{n/k-i}{n/k}\right) - o(1)\,. \end{aligned}$$

Now, recall (4.17). Recalling that $I = (1-n^{-c/(2k)})n/k$, and computing

$$\sum_{i=1}^{I} k\ln\left(\frac{n/k-i}{n/k}\right) \geq k\ln\left(\frac{(n/k)!}{(n/k)^{n/k}}\right) = -n + o(n)$$

using Stirling's approximation, we have

$$H(e(1), \ldots, e(I)) = h(G) + \frac{n}{k}\ln\left(\frac{n}{k}\right) - n + o(n)\,.$$

Via (4.17), this implies the desired bound (4.13). $\square$

5 Upper Bound in Theorem 1.7

In this section we prove the upper bound in Theorem 1.7, as a consequence of the following technical theorem of Kahn [17, Theorem 4.2] (which is essentially a hypergraph generalisation of the main result of [3]). We introduce some notation that will be used throughout the section: fix $k \geq 2$, let G be a k-graph and let f be a perfect matching in G. We implicitly regard f as a set of edges.

- Let $\Lambda = (k-1)n/k$.
- For a vertex v, let f_v be the edge of f containing v, and let $f(v) = f_v \setminus \{v\}$.
- For a vertex v, a perfect matching f, and a $(k-1)$-set of vertices Y, let

$$T(v, f, Y) = \{B \in f : B \neq f_v,\ B \cap Y \neq \emptyset\}$$

  and $\tau(v, f, Y) = |T(v, f, Y)|$.
- For a vertex v and a $(k-1)$-set of vertices Y, let $\Gamma_v(Y)$ be the set of perfect matchings f satisfying $\tau(v, f, Y) < k-1$.

Also, given a (not necessarily uniform) random perfect matching $\mathbf{f}$ in G: for a vertex v and a $(k-1)$-set of vertices Y, define $p_v(Y) = \mathbb{P}[\mathbf{f}(v) = Y]$ and $\gamma_v(Y) = \mathbb{P}[\mathbf{f} \in \Gamma_v(Y)]$.

Theorem 5.1 *Fix $k \geq 2$, let G be a k-graph and let $\mathbf{f}$ be a (not necessarily uniformly) random perfect matching in G. Then*

$$H(\mathbf{f}) < \frac{1}{k} \sum_{v \in V(G)} H(\mathbf{f}(v)) - \Lambda + O\left(\sum_{v \in V(G)} \sum_{Y} p_v(Y) \gamma_v(Y)^{1/(k-1)} \right) + O(\ln n),$$

where, in the third sum, Y ranges over all $(k-1)$-subsets of $V(G)$.

In order to deduce the desired bound from Theorem 5.1 we use the following auxiliary lemma, appearing as [17, Lemma 4.3].

Lemma 5.2 *Consider $l, D, t \in \mathbb{N}$ and $\lambda, \varepsilon > 0$, and $p_i, \gamma_i \in [0, 1]$ for $i \in \{1, \dots, l\}$, satisfying*

- $l = nD$,
- $\sum_{i=1}^{l} p_i = n$,
- $n \ln(D) - \sum_{i=1}^{l} p_i \ln(1/p_i) < \lambda n$
- $\sum_{i=1}^{l} \gamma_i \leq \varepsilon n D$.

Then

$$\sum_{i=1}^{l} p_i \gamma_i^{1/(t-1)} = \delta_{k,\lambda}(\varepsilon) n,$$

for some $\delta_{k,\lambda}(\varepsilon) > 0$ that tends to zero as $\varepsilon \to 0$ (holding k, λ fixed).

In other words, if we can prove a $O(n)$ bound on the expression in the third bullet point, and an $o(nD)$ bound on the expression in the last bullet point, we deduce a $o(n)$ bound in the final expression.

We now deduce the upper bound in Theorem 1.7.

Proof of the upper bound in Theorem 1.7 Let $\mathbf{f}$ be a uniformly random perfect matching of G, and define a fractional matching $\boldsymbol{x}$ by $\boldsymbol{x}[e] = \mathbb{P}[e \in \mathbf{f}]$. Then, for each vertex v we have

$$H(\mathbf{f}(v)) = \sum_{e \ni v} \boldsymbol{x}[e] \ln\left(\frac{1}{\boldsymbol{x}[e]}\right),$$

so

$$\sum_{v \in V(G)} H(\mathbf{f}(v)) = kh(\boldsymbol{x}).$$

The desired result will follow from Theorem 5.1 if we can show that

$$\sum_{v \in V(G)} \sum_{S} p_v(S) \gamma_v(S)^{1/(k-1)} = o(n), \tag{5.1}$$

where, in the second sum (and all sums over S for the remainder of this proof), S ranges over all $(k-1)$-subsets of $V(G)$. Note that $p_v(S) = 0$ when $\{v\} \cup S$ is not an edge of G, so the sum in (5.1) can be viewed as a sum over the $k|E(G)|$ choices of v, S such that $\{v\} \cup S$ is an edge. Order these (v, S) pairs arbitrarily so that each $1 \le i \le k|E(G)|$ corresponds to some (v, S) pair. We will prove (5.1) by verifying the conditions in Lemma 5.2, with $t = k$, $l = k|E(G)|$, $D = l/n$, $p_i = p_v(S)$, $\gamma_i = \gamma_v(S)$, some $\varepsilon = o(1)$ and some $\lambda = O(1)$.

First, note that $\sum_S p_v(S) = 1$ by the definition of $p_v(S)$, so $\sum_v \sum_S p_v(S) = n$.

Second, writing $D = k|E(G)|/n = \Theta(n^{k-1})$, we also have

$$\begin{aligned}&\sum_{v \in V(G)} \sum_{S} p_v(S) \ln\left(\frac{1}{p_v(S)}\right)\\&\quad = kh(\boldsymbol{x}) \ge k \ln(\Phi(G)) \ge (k-1)n \ln n - O(n) = n \ln(D) - O(n).\end{aligned}$$

Indeed, in the first inequality, viewing $\mathbf{f}$ as a random vector on $\{0,1\}^{E(G)}$, we have

$$\begin{aligned}\ln(\Phi(G)) = H(\mathbf{f}) &= H(\mathbb{1}_{e_1 \in \mathbf{f}}, \cdots, \mathbb{1}_{e_{|E(G)|} \in \mathbf{f}}) \le \sum_{e \in E(G)} H(\mathbb{1}_{e \in \mathbf{f}})\\&= \frac{1}{k} \sum_{v \in V(G)} \sum_{e \ni v} \boldsymbol{x}[e] \ln(1/\boldsymbol{x}[e]) = h(\boldsymbol{x}),\end{aligned}$$

which follows from Fact 2.9 and Fact 2.11; in the last inequality, we are using a crude lower bound on $\Phi(G)$ which appears for example as [10, Corollary 1.7(i)] (it can also be easily deduced from results in [11] or [12], or from the lower bound in Theorem 1.7 proved in Section 4 together with Lemma 2.7). Thus, we may take $\lambda = O(1)$.

Then, for any vertex v, perfect matching f, and $(k-1)$-set of vertices S, note that $f \in \Gamma_v(S)$ if and only if $\{v\} \cup S$ intersects some edge of f in more than one vertex. So, for each perfect matching f,

$$|\{(v, S) : f \in \Gamma_v(S)\}| \leq \frac{n}{k}\binom{k}{2}\binom{n-2}{k-2}k = o(n^k)\,.$$

Indeed, there are at most $\frac{n}{k}\binom{k}{2}\binom{n-2}{k-2}$ sets of k vertices which intersect an edge of f in more than one vertex, and there are at most k ways to represent this set of k vertices as $\{v\} \cup S$. It follows that

$$\sum_{v \in V(G)} \sum_{S} \gamma_v(S) = \sum_{f} \mathbb{P}[\mathbf{f} = f] \cdot |\{(v, S) : f \in \Gamma_v(S)\}| = o(n^k) = o(nD)\,,$$

where the sum on the right-hand side ranges over all perfect matchings f of G. Thus, we may take $\varepsilon = o(1)$. So, (5.1) follows from Lemma 5.2. □

6 Lower Bound on $h(G)$ when $d \geq k/2$

The goal of this section is to give a tight lower bound on the entropy of a k-graph with a given minimum d-degree, when $d \geq k/2$. Specifically, we will prove the following theorem.

Theorem 6.1 *For $d, k \in \mathbb{N}$ satisfying $k/2 \leq d \leq k-1$ and any constant $\gamma > 0$, and let G be a (d, γ)-Dirac k-graph on n vertices. Then,*

$$h(G) \geq \frac{n}{k} \ln\left(\frac{k}{n} \cdot \frac{\binom{n}{d}}{\binom{k}{d}} \cdot \delta_d(G)\right)\,. \tag{6.1}$$

Note that when $G = K_n^{(k)}$ is the complete k-graph on n vertices, this lower bound exactly matches the upper bound given by Jensen's inequality (see Lemma 2.7). We first give the (quick) deduction of Theorem 1.8 from Theorem 6.1 and Theorem 1.7.

Proof of Theorem 1.8 given Theorem 6.1 Our goal is to show that

$$\ln \Phi(G) \geq \ln(\Phi_n^{(k)}) + \frac{n}{k} \ln p + o(n)\,.$$

From Theorem 1.7, we know that

$$\ln \Phi(G) \geq h(G) - (1 - 1/k)n + o(n)\,.$$

Applying Theorem 6.1 and recalling $p = \delta_d(G)/\binom{n-d}{k-d}$, and noting that $\ln\binom{n}{k} = k\ln n + o(1) - \ln(k!)$, we deduce

$$\begin{aligned}
\ln\Phi(G) &\geq \frac{n}{k}\ln\left(\frac{k}{n}\cdot\frac{\binom{n}{d}}{\binom{k}{d}}\cdot\binom{n-d}{k-d}\cdot p\right) - \left(1-\frac{1}{k}\right)n + o(n)\\
&= \frac{n}{k}\ln\left(\frac{k}{n}\cdot\binom{n}{k}p\right) - \left(1-\frac{1}{k}\right)n + o(n)\\
&= \frac{n}{k}\ln\left(\frac{k}{n}\right) + n\ln n - \frac{n}{k}\ln(k!) + \frac{n}{k}\ln p - \left(1-\frac{1}{k}\right)n + o(n)\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\ln\left(\Phi(K_n^{(k)})\right) &= \ln\left(\frac{n!}{(n/k)!\cdot(k!)^{n/k}}\right) = n\ln\left(\frac{n}{e}\right) - \frac{n}{k}\ln\left(\frac{n/k}{e}\right) - \frac{n}{k}\ln(k!)\\
&= n\ln n + \frac{n}{k}\ln\left(\frac{k}{n}\right) - \left(1-\frac{1}{k}\right)n - \frac{n}{k}\ln(k!)
\end{aligned}$$

by Stirling's approximation; the desired result follows. □

We will prove Theorem 6.1 by constructing an auxiliary bipartite graph G from our hypergraph H, and applying the following result of Cuckler and Kahn [4, Theorem 3.1] (proved via a linear-algebraic construction).

Lemma 6.2 *Let G be a balanced bipartite graph on $2n$ vertices with minimum degree $\delta(G) \geq n/2$. Then, $h(G) \geq n\ln(\delta(G))$.*

Proof of Theorem 6.1 Using the k-graph G, we construct a bipartite graph $\tilde{G}$, with parts $\tilde{A}$ and $\tilde{B}$, as follows. First, $\tilde{A}$ consists of all subsets of d vertices of $V(G)$, each duplicated exactly $\binom{n}{k-d}$ times, and $\tilde{B}$ consists of all subsets of $(k-d)$ vertices of $V(G)$, each duplicated $\binom{n}{d}$ times. This means that

$$|\tilde{A}| = |\tilde{B}| = \binom{n}{d}\binom{n}{k-d}\,.$$

Then, in $\tilde{G}$ we put an edge between $U \in \tilde{A}$ and $W \in \tilde{B}$ whenever $U \cup W$ forms an edge in G. Thus, each edge $e \in E(G)$ leads to $\binom{k}{d}\binom{n}{d}\binom{n}{k-d}$ corresponding edges in $E(\tilde{G})$. We write $\tilde{e} \sim e$ to denote that an edge $\tilde{e} \in E(\tilde{G})$ arose from an edge $e \in E(G)$.

Since $\tilde{B}$ has $\binom{n}{d}$ copies of each set of $k-d$ vertices, in the graph $\tilde{G}$ each $U \in \tilde{A}$ has degree at least $\binom{n}{d}\delta_d(G)$, and each $W \in \tilde{B}$ has degree at least $\binom{n}{k-d}\delta_{k-d}(G)$. Since $d \geq k/2$, Fact 2.4 implies that $\binom{n}{d}\delta_d(G) \leq \binom{n}{k-d}\delta_{k-d}(G)$, and therefore

$$\delta(\tilde{G}) = \binom{n}{d}\delta_d(G)\,. \tag{6.2}$$

Using Fact 2.5, we compute

$$\delta(\tilde{G}) = \binom{n}{d}\delta_d(G) \geq (1/2+\gamma)\binom{n}{d}\binom{n-d}{k-d} \geq (1/2)\binom{n}{d}\binom{n}{k-d} \geq \tilde{n}/2\,,$$

writing $\tilde{n} = |\tilde{A}| = |\tilde{B}|$. That is to say, $\tilde{G}$ is a balanced bipartite graph on $2\tilde{n}$ vertices with minimum degree at least $\tilde{n}/2$.

By Lemma 6.2, there exists a fractional perfect matching $\tilde{\boldsymbol{x}}$ in $\tilde{G}$ satisfying

$$\sum_{e\in E(\tilde{G})} \tilde{\boldsymbol{x}}[e]\ln(1/\tilde{\boldsymbol{x}}[e]) \geq \tilde{n}\ln(\delta(\tilde{G}))\,. \tag{6.3}$$

Now, define

$$L{:=}\frac{k}{n}\binom{n}{d}\binom{n}{k-d}, \quad \boldsymbol{x}[e] = \sum_{\tilde{e}\sim e}\tilde{\boldsymbol{x}}[\tilde{e}]/L \text{ for each edge } e\in E(G)\,.$$

We will show $\boldsymbol{x}\in\mathbb{R}^m$ is a fractional perfect matching of G, certifying (6.1).

First, it is a straightforward calculation to see that $\boldsymbol{x}$ is a fractional perfect matching: certainly each $\boldsymbol{x}[e]$ is nonnegative, and for every $v\in V(G)$ we have

$$\begin{aligned}
\sum_{e\ni v}\boldsymbol{x}[e] &= \frac{1}{L}\sum_{e\ni v}\sum_{\tilde{e}\sim e}\tilde{\boldsymbol{x}}[\tilde{e}] = \frac{1}{L}\left(\sum_{\substack{U\in\tilde{A}:\\ v\in U}}\sum_{\substack{\tilde{e}\in E(\tilde{G}):\\ U\in\tilde{e}}}\tilde{\boldsymbol{x}}[\tilde{e}] + \sum_{\substack{W\in\tilde{B}:\\ v\in W}}\sum_{\substack{\tilde{e}\in E(\tilde{G}):\\ W\in\tilde{e}}}\tilde{\boldsymbol{x}}[\tilde{e}]\right)\\
&= \frac{1}{L}\left(\sum_{\substack{U\in\tilde{A}:\\ v\in U}}1 + \sum_{\substack{W\in\tilde{B}:\\ v\in W}}1\right)\\
&= \frac{1}{L}\left(\binom{n}{k-d}\binom{n-1}{d-1} + \binom{n}{d}\binom{n-1}{k-d-1}\right) = 1\,,
\end{aligned}$$

recalling that $\tilde{A}$ contains $\binom{n}{k-d}$ copies of each d-set of vertices, and $\tilde{B}$ contains $\binom{n}{d}$ copies of each $(k-d)$-set of vertices. It remains to lower bound the entropy of $\boldsymbol{x}$.

Let $Q = \binom{k}{d}\binom{n}{d}\binom{n}{k-d} = \binom{k}{d}\tilde{n}$ be the number of edges $\tilde{e}\in E(G)$ arising from each $e\in E(G)$. We compute

$$\begin{aligned}
h(\boldsymbol{x}) &= \sum_{e\in E(G)}\boldsymbol{x}[e]\ln(1/\boldsymbol{x}[e])\\
&= \sum_{e\in E(G)}\left(\frac{1}{L}\sum_{\tilde{e}\sim e}\tilde{\boldsymbol{x}}[\tilde{e}]\right)\ln\left(\frac{L}{\sum_{\tilde{e}\sim e}\tilde{\boldsymbol{x}}[\tilde{e}]}\right)\\
&= \frac{\ln(L/Q)}{L}\sum_{e\in E(G)}\sum_{\tilde{e}\sim e}\tilde{x}[\tilde{e}] + \frac{Q}{L}\sum_{e\in E(G)}\left(\frac{\sum_{\tilde{e}\sim e}\tilde{x}[\tilde{e}]}{Q}\right)\ln\left(\frac{Q}{\sum_{\tilde{e}\sim e}\tilde{x}[\tilde{e}]}\right)
\end{aligned}$$

$$\geq \frac{1}{L}\ln\left(\frac{k}{n}\cdot\frac{1}{\binom{k}{d}}\right)\sum_{\tilde{e}\in E(\tilde{G})}\tilde{x}[\tilde{e}]+\frac{Q}{L}\sum_{e\in E(G)}\sum_{\tilde{e}\sim e}\frac{1}{Q}\tilde{x}[\tilde{e}]\ln\left(\frac{1}{\tilde{x}[\tilde{e}]}\right)$$
$$\geq \frac{1}{L}\ln\left(\frac{k}{n}\cdot\frac{1}{\binom{k}{d}}\right)\tilde{n}+\frac{1}{L}\tilde{n}\ln(\delta(\tilde{G}))$$
$$\geq \frac{n}{k}\ln\left(\frac{k}{n}\cdot\frac{\binom{n}{d}}{\binom{k}{d}}\cdot\delta_d(G)\right),$$

where for the third line we used Jensen's inequality with the concave function $t\mapsto t\log(1/t)$, for the fourth line we used (6.3), and for the last line we used (6.2). □

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