La Matematica (2026) 5:33
https://doi.org/10.1007/s44007-026-00217-w

ORIGINAL RESEARCH ARTICLE

Nonlinear Dirichlet Forms, Energy Spaces, and Calculus Rules

Giovanni Brigati[1]

Received: 7 September 2023 / Revised: 11 April 2026 / Accepted: 13 April 2026
© The Author(s) 2026

Abstract
I review recent contributions on nonlinear Dirichlet forms. Then, I specialise to the case of 2−homogeneous and local forms. Inspired by the theory of Finsler manifolds and metric measure spaces, I establish new properties of such nonlinear Dirichlet forms, which are reminiscent of differential calculus formulae.

Keywords Nonlinear Dirichlet form · Dirichlet form · Nonlinear semigroup · Energy space · Differential calculus

Mathematics Subject Classification Primary 31C45 · Secondary 47H20 · 31C25 · 46E36 · 35K55

1 Introduction

The theory of (bilinear) Dirichlet forms [13, 30, 36] is a rich topic at the interface between analysis and probability, in connection with Markov semigroups (see Theorem 2.1 below). Dirichlet forms were introduced in [12] as a class of bilinear/quadratic forms generalising the standard Dirichlet energy

$$\mathcal{E}(u) = \begin{cases} \int_{\mathbb{R}^d} |Du(x)|^2 \, dx, & \text{if } u \in \mathrm{W}^{1,2}(\mathbb{R}^d); \\ +\infty, & \text{otherwise.} \end{cases} \quad (1)$$

An abstract calculus (called Γ-calculus) has been developed in [8] to capture the hypercontractivity and decay properties of linear Markov semigroups induced by quadratic Dirichlet forms, establishing connections between linear diffusion processes, Riemannian geometry, and functional inequalities [6, 9, 27]. In particular, on Riemannian manifolds, the Γ-calculus applied to the $\mathrm{W}^{1,2}$-seminorm (which is a quadratic Dirichlet

To my late grandfather, Roberto, with love and gratitude.

✉ Giovanni Brigati
giovanni.brigati@ist.ac.at

1 Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria

Published online: 29 April 2026

form) links the long-time behaviour of the heat flow (the associated Markov semigroup) with the Bochner identity via an inequality called *Bakry–Emery condition* [8]. In [24], a differential calculus for general regular quadratic Dirichlet forms, available even when the Γ-calculus does not apply has been developed, see also [34]. In case of a lower bound on the *Ricci curvature*, the Bakry–Emery condition is satisfied. The converse implication is also true, yielding a definition of Ricci lower bounds in pure terms of Dirichlet forms [8]. A third notion of Ricci lower bounds, formulated via optimal transport, was introduced independently by Lott, Villani [35], and Sturm [42, 43].

This last definition makes sense even for metric measure spaces. However, unlike Riemannian manifolds, the analogous of (1) on metric measure spaces, called *Cheeger's energy* [2, 3, 19], is not a quadratic Dirichlet form in general, and the metric heat flow is nonlinear. So, until now, Γ-calculus and a Bakry–Emery condition could be established only in metric measure spaces whose Cheeger's energy is quadratic (*infinitesimally Hilbertian spaces*) [3, 4]. In this class, in infinite dimension, Ambrosio, Gigli, and Savaré could recover the equivalence between the Bakry–Emery condition and the Ricci lower bounds of [35, 42]. Analogous results in the finite-dimension setting were obtained in [5, 28]. Then, Gigli [31, 32] developed a comprehensive first- and second-order calculus on metric measure spaces. The study of geometry and functional inequalities on RCD spaces (i.e. infinitesimally Hilbertian spaces satisfying a Ricci lower bound) flourished, becoming a main subject in the last ten years [44].

Much less is known in case the Cheeger energy is nonquadratic. The only available notion of Ricci lower bounds for a general, non-infinitesimally Hilbertian, metric measure space is the one by Lott, Villani, and Sturm. The Cheeger's energy being nonquadratic is not exceptional, as it is the case in all (non-Riemannian) Finsler geometries. Finsler structures will be a source of inspiration for this paper, as Riemannian manifolds are model examples for the RCD case. I detail hereby the state of the art.

- Ricci lower bounds on Finsler manifolds appear in [37–39]. The equivalence between an *intrinsic definition*, and the Lott–Villani–Sturm approach of [35, 42] holds true. Moreover, a suitable equivalent Bakry–Emery condition has been found [37, 38]. This last condition looks very different from the standard one, as it is a comparison estimate between the nonlinear Finsler Laplacian and a linearisation of it.
- A definition of *nonlinear Dirichlet form* has been given in [23]. Properties of nonlinear Dirichlet forms have been studied in [25, 26] and [16, 17]. No analog of the Bakry–Emery Γ-calculus is available in this context up to the best of my knowledge.
- Nonquadratic Cheeger's energies belong to the class of nonlinear Dirichlet forms of [23]. The converse, i.e. a representation theorem of abstract nonlinear Dirichlet forms as metric Cheeger's energies, is missing in general, but available in the quadratic case [4].

Motivated by the last point, my long-term goal is to strengthen the link between nonlinear Dirichlet forms and nonquadratic Cheeger's energies. In this note, I study 2-homogeneous, and local nonlinear Dirichlet forms. I analyse the associated non-Hilbertian energy space, and recover some abstract calculus rules, which are reminiscent of *concrete* calculus in metric measure spaces [2, 31], capturing the analogies as much as I can.

The work is organised as follows. Section 2 contains a presentation of the main notion involved in the paper. Section 3 specialises to 2-homogeneous and local non-linear Dirichlet forms. Property of the associated energy spaces are collected in Section 4. Sections 5-6 contain first-order, and second-order calculus rules, respectively. In Section 7 I list some desirable results which are still missing in the theory. Finally, in Section 8, I review a few very recent papers on nonlinear Dirichlet forms.

2 Definitions and Tools

2.1 Quadratic Dirichlet Forms

Let $(X, \mathcal{F}, m)$ be a σ-finite measure space, such that $\mathcal{F}$ is in a bi-measurable correspondence with the Borel class on $\mathbb{R}$.

A (quadratic) Dirichlet form is a quadratic, lower-semicontinuous (l.s.c.) functional

$$\mathcal{E} : \mathrm{L}^2(X, m) \to [0, \infty],$$

whose domain

$$D(\mathcal{E}) := \left\{ u \in \mathrm{L}^2(X, m) \ : \ \mathcal{E}(u) \neq \infty \right\}$$

is a dense subspace of $\mathrm{L}^2(X, m)$, and such that

$$\forall u \in D(\mathcal{E}), \quad \mathcal{E}(0 \vee u \wedge 1) \leqslant \mathcal{E}(u). \tag{2}$$

The symbols $\vee$, $\wedge$ stand for the maximum and minimum operations, respectively. The quadratic form $\mathcal{E}$ induces an unbounded, positive semi-definite, self-adjoint linear operator $A : D(A) \to \mathrm{L}^2(X, m)$, such that

$$\forall u \in D(\mathcal{E}), \qquad \mathcal{E}(u) = \int_X \sqrt{A}u \, \sqrt{A}u \, dm,$$

and

$$\forall u \in D(A), \qquad 2\mathcal{E}(u) = \int_X -u \, Au \, dm.$$

By construction, one has that $t \mapsto T_t := \mathrm{e}^{At}$ is a linear and continuous semigroup of contractions on $\mathrm{L}^2(X, m)$ [22] such that

$$\partial_t T_t = A \, T_t.$$

Finally, one could introduce the bilinear Dirichlet form associated with $\mathcal{E}$ as follows

$$\Lambda(u, v) := \int_X \sqrt{A}u \, \sqrt{A}v, \qquad \forall u, v \in D(\mathcal{E}), \tag{3}$$

which is called simply *Dirichlet form* in the literature [13, 30, 36]. In this paper, I will use the expression *Dirichlet form*, depending on the context, both for a quadratic Dirichlet form $\mathcal{E}$ and a bilinear Dirichlet form Λ.

The interest of Dirichlet forms, especially in connection with linear semigroups [30], lies in the following.

Theorem 2.1 *Let $\mathcal{E}$ be a quadratic, l.s.c., densely defined, positive semi-definite quadratic form over $\mathrm{L}^2(X, m)$. Then, the following are equivalent.*

1. *$\mathcal{E}$ is a Dirichlet form.*
2. *The linear semigroup $(T_t)_t = \mathrm{e}^{At}$ is a continuous and self-adjoint semigroup of contractions in $\mathrm{L}^p(X, m)$ for all $p \in [1, \infty]$:*

$$\forall t \geqslant 0, \quad \forall p \in [1, \infty], \qquad \|T_t\|_{\mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d)} \leqslant 1. \tag{4}$$

Moreover, $(T_t)_t$ is positivity-preserving:

$$\forall u \in \mathrm{L}^2(X, m),\ t \geqslant 0, \qquad u \geqslant 0 \implies T_t u \geqslant 0.$$

3. *The form $\mathcal{E}$ verifies the normal contraction property*

$$\forall u \in \mathrm{W}^{1,2}(\mathbb{R}^d), \quad \forall \phi \in \Phi, \qquad \mathcal{E}(\phi(u)) \leqslant \mathcal{E}(u), \tag{5}$$

where

$$\Phi := \{\phi : \mathbb{R} \to \mathbb{R} \ :\ \phi(0) = 0,\ \phi \text{ is } 1\text{-Lipschitz}\}.$$

.

The implication from 2. to 1. should be understood in the following sense. Given a linear, self-adjoint semigroup $(T_t)_t$, satisfying the hypotheses of condition 2., one could always (uniquely) define an unbounded, linear, positive semi-definite, and self-adjoint operator

$$A := \lim_{t \to 0} \frac{T_t - \mathrm{Id}}{t},$$

then, $\sqrt{A}$ by functional calculus. This way, the functional

$$\mathcal{E}(u) := \begin{cases} \int_X \left|\sqrt{A}u\right|^2 dm, & \text{if } u \in D(\sqrt{A}), \\ +\infty, & \text{otherwise}, \end{cases}$$

is a Dirichlet form. Semigroups satisfying condition 2. of Theorem 2.1 are usually called linear *Markov semigroups*.

2.2 Nonlinear Dirichlet Forms

In [23], a possible extension of Dirichlet forms to the nonlinear setting has been established. I adopt this approach, in view of a nonlinear extension of Theorem 2.1, namely Theorems 2.2 and 2.3.

Let $\mathcal{E} : \mathrm{L}^2(X, m) \to [0, \infty]$ be a convex, l.s.c. functional with dense domain. Indicate with $\partial\mathcal{E} : D(\partial\mathcal{E}) \subset \mathrm{L}^2(X, m) \to 2^{\mathrm{L}^2(X,m)}$ its subdifferential operator:

$$\partial\mathcal{E}(u) = \left\{ \xi \in \mathrm{L}^2(X, m) \,:\, \forall z \in \mathrm{L}^2(X, m), \quad \mathcal{E}(z) - \mathcal{E}(u) \geqslant \int_X \xi\, u\, dm \right\},$$

see [14, 15]. Let $(T_t)_{t\geqslant 0}$ be the semigroup of nonlinear operators generated by $-\partial\mathcal{E}$ via the differential equation

$$\begin{cases} \partial_t T_t\, u \in -\partial\mathcal{E}(T_t\, u), & \forall t \in (0, \infty), \quad \forall u \in \mathrm{L}^2(X, m), \\ T_0\, u = u, & \forall u \in \mathrm{L}^2(X, m). \end{cases} \tag{6}$$

Equation (6) is well-posed for all $u \in \mathrm{L}^2(X, m)$. Its solution is usually called the gradient flow of $\mathcal{E}$ starting at u. See [1, 14].

The functional $\mathcal{E}$ is a nonlinear Dirichlet form if the associated semigroup $(T_t)_t$ is a contraction in all $\mathrm{L}^p(X, m)$ spaces (4), and if, in addition, it is order-preserving:

$$\forall u, v \in \mathrm{L}^2(X, m),\ t \geqslant 0, \qquad u \geqslant v \implies T_t u \geqslant T_t v. \tag{7}$$

A semigroup of nonlinear maps satisfying (4), and (7) is called a *nonlinear Markov semigroup*. Notice that a quadratic Dirichlet form is a special case of a nonlinear Dirichlet form. Moreover, if $\mathcal{E}$ is quadratic, we have that $\partial\mathcal{E} = A$.

Thanks to [11], once condition (4) is verified for the case $p = \infty$, it holds for any $p \in [1, \infty]$. After the results in [10, 14, 17, 23], conditions (4) and (7) can be characterised in terms of invariance of convex sets in $\mathrm{L}^2(X, m; \mathbb{R}^2)$ under the action of the (doubled) semigroup $(T_t, T_t)_t$.

Then [14, Chapter 4], the invariance of a closed, convex set under the action of a semigroup is equivalent to a certain functional inequality, satisfied by the functional $\mathcal{E}$ inducing the semigroup itself. For nonlinear Dirichlet forms, the inequalities satisfied by a nonlinear Dirichlet form $\mathcal{E}$, corresponding to (4) and (7), are one of the main points of [23]. In [17], we proved an equivalent characterisation, which we state below.

Theorem 2.2 ([17, 23]) *Let $\mathcal{E} : \mathrm{L}^2(X, m) \to [0, \infty]$ be a l.s.c. functional. Then, $\mathcal{E}$ is a nonlinear Dirichlet form if and only if, for all $u, v \in \mathrm{L}^2(X, m)$, and $\alpha \in [0, \infty)$, $\mathcal{E}$ verifies*

$$\mathcal{E}(u \vee v) + \mathcal{E}(u \wedge v) \leqslant \mathcal{E}(u) + \mathcal{E}(v), \tag{8}$$

$$\mathcal{E}(H_\alpha(u, v)) + \mathcal{E}(H_\alpha(u, v)) \leqslant \mathcal{E}(u) + \mathcal{E}(v), \tag{9}$$

with

$$H_\alpha(u, v)(x) = \begin{cases} v(x) - \alpha & u(x) - v(x) < -\alpha, \\ u(x) & u(x) - v(x) \in [-\alpha, \alpha], \\ v(x) + \alpha & u(x) - v(x) > \alpha. \end{cases} \tag{10}$$

Inequality (8) is known as *submodularity* in [33], and identified as the crucial hypothesis to prove Lewy–Stampacchia inequalities in the abstract context of topological vector lattices. In view of Theorem 2.2, energies $\mathcal{E}$ which generate an order-preserving semigroup in L^2 (then, in particular, nonlinear Dirichlet forms) are a class covered by [33, Theorem 2.4].

The *normal contraction property* (5) was recovered also for the nonlinear setting.

Theorem 2.3 ([17]) *Let $\mathcal{E}$ be a nonlinear Dirichlet form. Then $\mathcal{E}$ has the normal contraction property* (5) *if and only if*

$$\mathcal{E}(-f) \leqslant \mathcal{E}(f) \quad \forall f \in \mathrm{L}^2(X, m). \tag{11}$$

By homogeneity, condition (11) is always true in the quadratic setting.

2.3 Finsler and Metric Sobolev Spaces

Let $\mathcal{M}$ be a smooth, closed, boundary-free manifold. Let m be a Borel, σ-finite measure on $\mathcal{M}$. A reversible Finsler structure on $\mathcal{M}$ is a smooth map $x \mapsto |\cdot|_x$ associating to each point $x \in \mathcal{M}$ a norm on $T_x\mathcal{M}$, such that, for all $x \in \mathcal{M}$ and all $\xi \in T_x\mathcal{M}$, with $\xi \neq 0$, the bilinear form

$$(T_x\mathcal{M})^2 \ni (\theta, \sigma) \mapsto \frac{1}{2} \frac{\partial^2}{\partial t \, \partial s}\bigg|_{s=t=0} |\xi + t\theta + s\sigma|_x^2 =: g_{x,\xi}(\theta, \sigma)$$

is positive definite (so that, for all $x \in \mathcal{M}$, the norm $|\cdot|_x$ is strictly convex away from 0). Let $|\cdot|_x^\star$ be the dual norm, and consider the bijective map $F : T\mathcal{M} \to T^\star\mathcal{M}$ defined as follows:

$$F(x, \xi) = \left\{ (x, \zeta) \; : \; \zeta \in T_x^\star\mathcal{M}, \; |\zeta|_x^\star = |\xi|_x, \quad \langle \zeta, \xi \rangle_x = |\xi|_x^2 \right\},$$

where $\langle \cdot, \cdot \rangle_x$ is the duality product between $T_x^\star\mathcal{M}$ and $T_x\mathcal{M}$. If $u : \mathcal{M} \to \mathbb{R}$ is a regular function, then Du is a section of $T^\star\mathcal{M}$.

We have that the Sobolev seminorm

$$\mathcal{E}(u) = \begin{cases} \int_{\mathcal{M}} \left(|Du(x)|_x^\star\right)^2 dm = \int_{\mathcal{M}} \left|F^{-1}(Du)\right|_x^2 dm, & \text{if } u \in \mathrm{W}^{1,2}(\mathcal{M}, dm), \\ \infty, \quad \text{otherwise}, \end{cases} \tag{12}$$

is a nonlinear Dirichlet form. Moreover, $\mathcal{E}$ is a quadratic form if and only if F is a linear map, if and only if $|\cdot|_x$ is induced by a scalar product for all $x \in \mathcal{M}$, so that $\mathcal{M}$ is a Riemmanian manifold. One can compute

$$\partial\mathcal{E}(u) = -2\nabla \cdot F^{-1}(Du),$$

which plays the role of the Laplacian on $(\mathcal{M}, |\cdot|, m)$, but it is a nonlinear operator. If $u \in D(\partial\mathcal{E})$, and $v \in \mathrm{L}^2(\mathcal{M}, m)$, then the scalar product

$$\int_{\mathcal{M}} -\nabla \cdot F^{-1}(Du)\, v\, dm$$

makes sense. As Gigli observes in [31], one cannot hope to find an adjoint operator S of $-\nabla \cdot F^{-1}(Du)$ such that

$$\int_{\mathcal{M}} -\nabla \cdot F^{-1}(Du)\, v\, dm = \int_{\mathcal{M}} S(v)\, u\, dm,$$

as the right-hand-side is linear in u, while the left-hand-side is not. The best one could do is moving only one derivative on v and finding

$$\int_{\mathcal{M}} -\nabla \cdot F^{-1}(Du)\, v\, dm = \int_{\mathcal{M}} \langle F^{-1}(Du), Dv\rangle_x\, dm.$$

The form

$$\Lambda(u, v) := \int_{\mathcal{M}} \langle F^{-1}(Du), Dv\rangle_x\, dm \tag{13}$$

is defined for $u, v \in D(\mathcal{E})$. Even if they belong to the same functional space, the two entries of Λ play different, non-interchangeable, roles. Finally [31]

$$\forall u, v \in D(\mathcal{E}), \qquad \Lambda(u, v) = \lim_{\sigma \to 0} \frac{\mathcal{E}(u + \sigma v) - \mathcal{E}(u)}{\sigma}. \tag{14}$$

Remark that the last formula holds even at a pointwise level in the context of Finsler manifolds.

I conclude the section by recalling the definition of the Cheeger's energy, which extends (1) and (12) to metric measure spaces [2].

Let (X, τ) be a topological space such that it is homeomorphic to a complete and separable metric space. Then (X, τ) is called a Polish space. Let (X, τ) be a Polish space equipped with a function $d : X \times X \to [0, +\infty]$ such that

- d is an extended distance on X;
- d is $\tau-$l.s.c.;
- for all sequences $(x_n)_n \subset X$ such that $d(x_n, x) \to 0$, for an element $x \in X$, we have $x_n \to x$ in τ.
- the extended metric space (X, d) is complete.

Let m be a Borel, $\sigma-$finite measure on (X, d, τ) for which there exists a constant $C > 0$, such that for every $x \in X$ and $r > 0$, the following holds

$$m(B(x, r)) \leqslant \exp(Cr^2). \tag{15}$$

Let $f \in \mathrm{Lip}_b(X)$ and let $x \in X$. The local slope of f at x is

$$|Df|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)},$$

with the convention that $|Df|(x) := 0$ whenever x is an isolated point in X. Let $\bar{\mathrm{Ch}} : \mathrm{L}^2(X, m) \to [0, +\infty]$ be defined via

$$\bar{\mathrm{Ch}}(u) = \frac{1}{2} \int_X |Du|^2 dm \qquad \text{if } u \in \mathrm{Lip}_b(X),$$

and $+\infty$ otherwise.

The Cheeger's energy of (X, d, τ, m) is the functional Ch defined via

$$\mathrm{Ch} = sc^- \bar{\mathrm{Ch}},$$

where sc^- is the l.s.c. envelope in the L^2-topology.

We have that $D(\mathrm{Ch})$ is a vector space, known as the metric Sobolev space and indicated in the literature with the symbol $\mathrm{W}^{1,2}(X, d, m)$. In general, the Cheeger's energy Ch is not quadratic and $\mathrm{W}^{1,2}(X, d, m)$ is not a Hilbert space. Let $u \in D(\partial \mathrm{Ch})$. Then,

$$-\Delta_{d,m}(u) := \partial^0 \mathrm{Ch}(u),$$

is called the metric Laplacian of u, where ∂^0 indicates the unique element of minimal norm in the subdifferential. My analysis of nonlinear Dirichlet forms is motivated by the following.

Theorem 2.4 ([2]) *The Cheeger's energy* Ch *is a nonlinear Dirichlet form in the sense of Cipriani and Grillo [23].*

For calculus rules in metric spaces I generally refer to [2, 31]. Precise results will be cited wherever they are needed.

3 2-homogeneous and Local Functionals

Even when the Cheeger's energy is nonquadratic, it is a 2-homogeneous functional satisfying some locality properties [2]. Then, let me reduce to the class of nonlinear Dirichlet forms which are 2-homogeneous and local (in a sense defined below).

Let $\mathcal{E} : \mathrm{L}^2(X, m) \to [0, \infty]$ be a functional. Then, $\mathcal{E}$ is 2−homogeneous if

$$\forall \nu \in \mathbb{R},\ \forall u \in \mathrm{L}^2(X, m), \qquad \mathcal{E}(\nu u) = \nu^2 \mathcal{E}(u).$$

We say that $\mathcal{E}$ is local if, for all $u, v \in D(\mathcal{E})$ such that u is constant on the support of v, we have

$$\mathcal{E}(u + v) = \mathcal{E}(u) + \mathcal{E}(v).$$

Finally, I introduce the symbol $(\cdot, \cdot)$ as a short-hand notation for the $\mathrm{L}^2(X, m)-$scalar product.

Theorem 3.1 *Let $\mathcal{E}$ be a 2-homogeneous, convex, l.s.c. functional. Then, the following hold true.*

1. $\mathcal{E}(0) = 0$.
2. *The subdifferential $\partial\mathcal{E}$ is $1-$homogeneous. Moreover the set $D(\partial\mathcal{E})$ is invariant by multiplication by scalars. Conversely, if $\mathcal{E}$ is a nonnegative, convex, l.s.c. functional, such that its subdifferential $\partial\mathcal{E}$ is $1-$homogeneous and $\mathcal{E}(0) = 0$, then $\mathcal{E}$ is $2-$homogeneous.*
3. *For all $u \in D(A)$, and for all $\xi \in \partial\mathcal{E}(u)$ we have*

$$\int_X u\xi \, dm = 2\mathcal{E}(u). \tag{16}$$

Formula (16) is reminiscent of the integration by parts formula

$$\int_{\mathbb{R}^d} |Du|^2 \, dx = -\int_{\mathbb{R}^d} u \, \Delta u \, dx.$$

In the case of a Finsler structure, with $\mathcal{E}$ defined as in (12), the same statement holds

$$\mathcal{E}(u) = \int_{\mathcal{M}} \langle F^{-1}(Du), Du \rangle_x \, dm = -\int_{\mathcal{M}} \nabla \cdot F^{-1}(Du) \, u \, dm, \qquad \forall u \in D(\partial\mathcal{E}).$$

Proof of Theorem 3.1 The first property is trivial: $\mathcal{E}(0) = 0^2\mathcal{E}(0) = 0$.

For the second property, suppose that $u \in D(\mathcal{E})$ and let $\lambda > 0$. Let $w \in \partial\mathcal{E}(u)$, so that

$$\mathcal{E}(v) - \mathcal{E}(u) \geqslant (w, v - u),$$

for all $v \in \mathrm{L}^2(X, m)$. Then

$$\mathcal{E}(\lambda v) - \mathcal{E}(\lambda u) = \lambda^2(\mathcal{E}(v) - \mathcal{E}(u))$$

hence

$$\mathcal{E}(\lambda v) - \mathcal{E}(\lambda u) \geqslant \lambda^2(w, v - u) = (\lambda w, \lambda v - \lambda u).$$

Since v is arbitrary, so it is λv.

For the converse, suppose in addition that $\mathcal{E}$ is a $\mathrm{C}^{1,1}$ functional. For a fixed element u, it is sufficient to prove that $t \mapsto \mathcal{E}(tu)$ is $2-$homogeneous. This fact is straightforward, since a real C^1 function which vanishes in 0, and whose derivative is $1-$homogeneous, is bound to be $2-$homogeneous. In the general case, one can argue by Yosida regularisation, as below.

For the third property, let me use Yosida regularisation, following [29, Section 9.6]. Being the subdifferential of a convex, l.s.c. functional, the operator $\partial\mathcal{E}$ is such that for all $u \in \mathrm{L}^2$ and all $\lambda > 0$, there exist unique two elements $v, w \in \mathrm{L}^2$, with $w \in \partial\mathcal{E}(v)$ and $v + \lambda w = u$. Then, $A_\lambda(u) := w$ is the Yosida regularisation (of order λ) of $\partial\mathcal{E}$ at u. Moreover, the Yosida regularised $\mathcal{E}_\lambda$ of $\mathcal{E}$ is defined as $\mathcal{E}_\lambda(u) =$

$\inf_{w\in L^2}\left(\mathcal{E}(w)+\frac{1}{2\lambda}|u-w|^2\right)$, and it holds that $\mathcal{E}_\lambda$ is a $C^{1,1}$ functional, with $\nabla\mathcal{E}_\lambda = A_\lambda$, for any $\lambda > 0$. If $\mathcal{E}$ is 2−homogenous, so is $\mathcal{E}_\lambda$ (and vice-versa)

$$\mathcal{E}_\lambda(\mu u) = \inf_{v\in L^2(X,m)}\left\{\mathcal{E}(\mu v)+\frac{1}{2\lambda}|\mu u-\mu v|^2\right\} = \mu^2\mathcal{E}_\lambda(u),$$

for all $\mu, \lambda > 0$. As an intermediate result I give an explicit formula for $\mathcal{E}_\lambda$. Fix $u \in D(\partial\mathcal{E})$. Let $g : [0, +\infty) \to \mathbb{R}$ be defined via

$$g(x) = \mathcal{E}_\lambda(xu) - x^2\mathcal{E}_\lambda(u).$$

We have that g is C^1 by composition and $g = g' = 0$, which reads as:

$$(A_\lambda(xu), u) - 2x\mathcal{E}_\lambda(u)) = 0,$$

that is (16) for $\mathcal{E}_\lambda$, hence

$$\mathcal{E}_\lambda(u) = \frac{1}{2}(A_\lambda(u), u).$$

If one passes in the limit for $\lambda \downarrow 0$, one obtains

$$\mathcal{E}(u) = \frac{1}{2}(A^0(u), u).$$

Take any other element $\xi \in \partial\mathcal{E}(u)$. Then, there exists[1] a sequence $(u_n)_n$ s.t. $u_n \to u$ strongly in L^2 and $A_{\frac{1}{n}}(u_n) \to \xi$ strongly in L^2, so that

$$\frac{1}{2}\left(A_{\frac{1}{n}}(u_n), u_n\right) \to \frac{1}{2}(\xi, u).$$

At the same time,

$$\frac{1}{2}\left(A_{\frac{1}{n}}(u_n), u_n\right) = \mathcal{E}_{\frac{1}{n}}(u_n) \uparrow \mathcal{E}(u),$$

see [14, Proposition 2.11], which concludes the proof. □

4 Energy Spaces

Let $\mathcal{E}$ be a 2-homogeneous nonlinear Dirichlet form. Then, the space $D(\mathcal{E})$ is called *Dirichlet space*. Some properties of $D(\mathcal{E})$ have already been given in [25], but, under the current hypotheses, we have a simpler structure with new results.

Theorem 4.1 *Let $\mathcal{E}$ be a 2-homogeneous nonlinear Dirichlet form. Then, the following hold true.*

[1] thanks to the G-convergence of $(A_\lambda)_\lambda$ to A, see [7]

1. *The space $D(\mathcal{E})$ is a vector space. The functional*

$$u \mapsto \|u\|_{\mathcal{E}}^2 := \|u\|_{\mathrm{L}^2(X,m)}^2 + \mathcal{E}(u)$$

is the square of a norm on $D(\mathcal{E})$. Moreover, $(D(\mathcal{E}), \|u\|_{\mathcal{E}})$ is a Banach space.
2. *The pair $(D(\mathcal{E}), \|u\|_{\mathcal{E}})$ is a Hilbert space if and only if $\mathcal{E}$ is a quadratic Dirichlet form.*
3. *Lipschitz functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(0) = 0$ act on $D(\mathcal{E})$:*

$$\forall u \in D(\mathcal{E}), \qquad \phi \circ u \in D(\mathcal{E}).$$

Moreover, $D(\mathcal{E}) \cap \mathrm{L}^\infty(X, m)$ is an algebra.
4. *$(D(\mathcal{E}), \|u\|_{\mathcal{E}})$ is a dual space, as shown in [26].*

Proof Since $D(\mathcal{E})$ is homogeneous and convex, it is a vector space. The stability by multiplication by scalars is encoded in the 2−homogeneity. I shall now prove the triangle inequality for the functional $\|u\|_{\mathcal{E}}$, while the homogeneity and the condition $\|u\|_{\mathcal{E}} = 0$ if and only if $u = 0$ are clear. For any normed vector space, the triangle inequality is equivalent to the convexity of the closed unit ball. Let $u, v \in D(\mathcal{E})$ such that $\|u\|_{\mathcal{E}} \vee \|v\|_{\mathcal{E}} \leqslant 1$. Then, for any $\lambda \in [0, 1]$:

$$\begin{aligned}
\|\lambda(u) + (1-\lambda)v\|_{\mathcal{E}}^2 \leqslant \\
&= |\lambda u + (1-\lambda)v|^2 + \mathcal{E}(\lambda u + (1-\lambda)v) \leqslant \\
&\leqslant \lambda|u|^2 + (1-\lambda)|v|^2 + \lambda\mathcal{E}(u) + (1-\lambda)\mathcal{E}(v) = \\
&= \lambda(|u|^2 + \mathcal{E}(u)) + (1-\lambda)(|v|^2 + \mathcal{E}(v)) \leqslant \\
&\leqslant \lambda + 1 - \lambda = 1.
\end{aligned}$$

Let now $(u_n)_n$ be a Cauchy sequence in $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}}$. Then $(u_n)_n$ is a Cauchy sequence in L^2. Hence $u_n \to u$ in L^2, for an element u. For $m \in \mathbb{N}$, $\mathcal{E}(u_n - u_m)$ is bounded in $\mathbb{R}$. Thanks to l.s.c.

$$\mathcal{E}(u - u_m) \leqslant \liminf_{n\to\infty} \mathcal{E}(u_n - u_m) < \infty.$$

Hence, $\mathcal{E}(u) \leqslant 2\mathcal{E}(u - u_m) + 2\mathcal{E}(u_m) < \infty$, which implies $u \in D(\mathcal{E})$. Finally,

$$0 \leqslant \lim_m \mathcal{E}(u - u_m) \leqslant \lim_m \liminf_n \mathcal{E}(u_n - u_m) \downarrow 0.$$

The first statement is then proved. The second statement follows by definition of $\|u\|_{\mathcal{E}}$. For the third statement, if ϕ is 1-Lipschitz, then, Theorem 2.3 ensures $u \in D(\mathcal{E})$, so that $\phi(u) \in D(\mathcal{E})$. Otherwise, if ϕ is L-Lipschitz, then ϕ/L is 1-Lipschitz, so

$$\mathcal{E}(\phi(u)) = L^2\,\mathcal{E}(L^{-1}\phi(u)) \leqslant L^2\,\mathcal{E}(u).$$

If $u \in D(\mathcal{E}) \cap \mathrm{L}^\infty$, then u^2 is a Lipschitz transformation of u. Hence, the function $u^2 \in D(\mathcal{E})$. The computation $uv = 1/2\left((u+v)^2 - u^2 - v^2\right)$ concludes the proof of the third statement of the theorem. □

Notice that the third statement of the last theorem replicates a well-known result in the theory of Sobolev spaces.

5 First-order Calculus

Let $\mathcal{E}$ be a 2−homogeneous nonlinear Dirichlet form. My goal is to reconstruct an object which plays the role of the biviariate functional Λ (14). In case $\mathcal{E}$ is quadratic, the natural associated bi-variate object is the bilinear Dirichlet form (3). In the general case, a canonical bi-variate object lacks, so I have to take a choice.

Having (14) in mind, I define

$$\Lambda^{\pm}(u, v) := \lim_{\sigma \to 0^{\pm}} \frac{\mathcal{E}(u + \sigma v) - \mathcal{E}(u)}{\sigma}, \qquad \forall u, v \in D(\mathcal{E}),$$

as the left and right slopes of $\mathcal{E}$, at u, in the direction of v. Notice that the definition is well-given, as $\sigma \mapsto \mathcal{E}(u + \sigma v) : \mathbb{R} \to \mathbb{R}$ is convex.

Theorem 5.1 *Let $\mathcal{E}$ be a 2-homogeneous nonlinear Dirichlet form. Then, the slopes $\Lambda^{\pm}$ have the following properties.*

1. *For all $u, v \in D(\mathcal{E})$, $\Lambda^{\pm}(u, v)$ are finite, and*

$$\left|\Lambda^{\pm}(u, v)\right| \leqslant 2\sqrt{\mathcal{E}(u)}\sqrt{\mathcal{E}(v)}, \tag{17}$$

   which is sharp for $u = v$. Moreover,

$$\Lambda^{-}(u, v) \leqslant \Lambda^{+}(u, v), \qquad \forall u, v \in D(\mathcal{E}).$$

2. *For all $u \in D(\mathcal{E})$, we have $\Lambda^{\pm}(u, u) = 2\mathcal{E}(u)$.*
3. *For all $u \in D(\mathcal{E})$, $D^{\pm}(u, \cdot)$ is positively 1−homogeneous. Moreover $D^{+}(u, \cdot)$ is convex, while $D^{-}(u, \cdot)$ is concave.*
4. *For all $v \in D(\mathcal{E})$, we have that $D^{\pm}(\cdot, v)$ is positively 1−homogeneous.*
5. *For all $u, v \in D(\mathcal{E})$, it holds $D^{+}(u, -v) = -D^{-}(u, v)$.*
6. *If $\mathcal{E}$ is local and $u, v \in D(\mathcal{E})$ are such that u is constant on supp(v), we have*

$$\Lambda^{\pm}(u, v) = 0.$$

Proof 1. The inequality $\Lambda^- \leqslant \Lambda^+$ is a consequence of convexity for the map $t \mapsto \mathcal{E}(u+tv)$. Take now any $u, v \in D(\mathcal{E})$.

$$\begin{aligned}
\Lambda^+(u,v) = \\
&= \lim_{h\to 0^+} h^{-1}(\mathcal{E}(u+hv) - \mathcal{E}(u)) = \\
&= \lim_h h^{-1}\left(\left(\sqrt{\mathcal{E}(u+hv)}\right)^2 - \left(\sqrt{\mathcal{E}(u)}\right)^2\right) \leqslant \\
&\leqslant \lim_h h^{-1}\left(\left(\sqrt{\mathcal{E}(u)} + h\sqrt{\mathcal{E}(v)}\right)^2 - \left(\sqrt{\mathcal{E}(u)}\right)^2\right) \leqslant \\
&\leqslant 2\sqrt{\mathcal{E}(u)}\sqrt{\mathcal{E}(v)} + \lim_h h\mathcal{E}(v) = 2\sqrt{\mathcal{E}(u)}\sqrt{\mathcal{E}(v)}.
\end{aligned}$$

I used the monotonicity of $t \mapsto t^2$ and the fact that $\sqrt{\mathcal{E}}$ is a seminorm. With an analogous argument, one can prove that

$$\Lambda^-(u,v) \geqslant -2\sqrt{\mathcal{E}(u)}\sqrt{\mathcal{E}(v)}.$$

2. Compute

$$\Lambda^\pm(u,u) = \lim_{h\to 0^\pm} \frac{\mathcal{E}((1+h)u) - \mathcal{E}(u)}{h} = \lim_{h\to 0^\pm} 2h\,\mathcal{E}(u) + h^2\,\mathcal{E}(u) = 2\,\mathcal{E}(u).$$

3. and 4. Let $\lambda > 0$.

$$\begin{aligned}
\Lambda^\pm(u,\lambda v) &= \lim_{h\to 0^\pm} h^{-1}(\mathcal{E}(u+h\lambda v) - \mathcal{E}(u)) = \\
&= \lambda \lim_{h\to 0^\pm} (\lambda h)^{-1}(\mathcal{E}(u+h\lambda v) - \mathcal{E}(u)) = \\
&= \lambda\Lambda^\pm(u,v).
\end{aligned}$$

Let $u, v \in D(\mathcal{E})$, let $\lambda > 0$. Hence

$$\begin{aligned}
\Lambda^\pm(\lambda u,v) &= \lim_{h\to 0^\pm} h^{-1}(\mathcal{E}(\lambda u+hv) - \mathcal{E}(\lambda u)) = \\
&= \lim_{h\to 0^\pm} h^{-1}(\mathcal{E}(\lambda(u+h\lambda^{-1}v)) - \mathcal{E}(\lambda u)) = \\
&= \lambda \lim_{h\to 0^\pm} \lambda h^{-1}(\mathcal{E}(u+h\lambda^{-1}v)) - \mathcal{E}(u)) = \\
&= \lambda\Lambda^\pm(u,v).
\end{aligned}$$

I prove only the convexity of $\Lambda^+(u,\cdot)$, being the other proof very similar. Fix $u, v_1, v_2 \in D(\mathcal{E})$.

$$\begin{aligned}
&\Lambda^+(u, \lambda v_1 + (1-\lambda)v_2) = \\
&= \lim_{\sigma\to 0^+} \sigma^{-1}(\mathcal{E}(u + \sigma\lambda v_1 + \sigma(1-\lambda)v_2) - \mathcal{E}(u)) = \\
&= \lim_{\sigma\to 0^+} \sigma^{-1}(\mathcal{E}(\lambda(u+\sigma v_1) + (1-\lambda)(u+\sigma v_2) - \mathcal{E}(u)) \leqslant \\
&\leqslant \lim_{\sigma\to 0^+} \sigma^{-1}(\lambda\mathcal{E}(u+\sigma v_1) + (1-\lambda)\mathcal{E}(u+\sigma v_2) - \mathcal{E}(u)) = \\
&= \lambda\Lambda^+(u, v_1) + (1-\lambda)\Lambda^+(u, v_2).
\end{aligned}$$

5. It is sufficient to switch h with $-h$ and take limits.
6. The proof of the last statement is a direct calculation. Let me perform it for the two limits at once:

$$\lim_{h\to 0} \frac{\mathcal{E}(u+hv) - \mathcal{E}(u)}{h} = \lim_{h\to 0} \frac{\mathcal{E}(hv)}{h} = \lim_{h\to 0} \frac{h^2\mathcal{E}(v)}{h} = 0.$$

□

Note that $\Lambda^\pm(u,v)$ generalise the integrals $\int_X D^\pm u(\nabla v)\, dm$ introduced by Gigli in [31] for metric measure spaces. There, even the pointwise objects $D^\pm u(\nabla v)(x)$ make sense, while in my setting the slopes $\Lambda^\pm$ are not necessarily represented by a density w.r.t. dm.

In general, $\Lambda^- \neq \Lambda^+$. The reader may think of a Finsler-like structure, equipped with the canonical nonlinear Dirichlet form (12), but where strict convexity for the norm $|\cdot|_x$ does not hold. In case $\Lambda^+(u,\cdot) = \Lambda^-(u,\cdot)$, the form $\mathcal{E}$ is said to be *regular* at u, and more structure is available. We also say that $\mathcal{E}$ is *regular* if $\mathcal{E}$ is regular at all $u \in D(\mathcal{E})$. If $\mathcal{E}$ is Fréchet-differentiable, we have that $\mathcal{E}$ is regular and $\Lambda(u,v) = (\nabla E(u), v)$, for all $u, v \in D(\mathcal{E})$.

Proposition 5.2 *Let $\mathcal{E}$ be a 2-homogeneous, regular nonlinear Dirichlet form. Then, $\Lambda := \Lambda^+ = \Lambda^-$ is linear in the second argument. Moreover, for all $u \in D(\mathcal{E})$, we have that $\Lambda(u,\cdot) \in D(\mathcal{E})^\star$. Finally, $\mathcal{E}$ is quadratic if and only if $\mathcal{E}$ is regular and*

$$\forall u, v \in D(\mathcal{E}), \qquad \Lambda(u,v) = \Lambda(v,u). \tag{18}$$

Proof The first assertion is entailed by the fact that $\Lambda(u,\cdot)$ is both concave and convex, and continuous, see Theorem 5.1. If $\mathcal{E}$ is quadratic, we have

$$\Lambda(u,v) = \int_X \sqrt{A}(u)\sqrt{A}(v)\, dm,$$

which is symmetric in u, v. For the converse, assume $\mathcal{E}$ to be regular. Then, the maps $t \mapsto \mathcal{E}(u+tv)$ are differentiable, for all $u, v \in D(\mathcal{E})$. Then,

$$\mathcal{E}(u+v) - \mathcal{E}(u) = \int_0^1 \frac{d}{dt}\, \mathcal{E}(u+tv)\, dt = \\ = \int_0^1 \Lambda(u+tv, v)\, dt = \int_0^1 \Lambda(v,u) + 2t\, \Lambda(v,v) = \Lambda(v,u) + \mathcal{E}(v).$$

By exchanging v with $-v$, and adding up, we prove that $\mathcal{E}$ satisfies the parallelogram identity, hence $\|\cdot\|_{\mathcal{E}}$ is induced by a scalar product. Equivalently, $\mathcal{E}$ is quadratic. $\square$

The last point shows that the arguments u, v of Λ actually play non-interchangeable roles (unless Λ is a bilinear Dirichlet form). This is intuitive in the case of Finsler manifolds, see (14).

6 Second-order Calculus

In metric measure spaces, the role of the Laplacian is played by the minimal section of $\partial\mathrm{Ch}$, where Ch denotes the Cheeger energy. In this section, I investigate the properties of $\partial\mathcal{E}$, for a 2-homogeneous nonlinear Dirichlet form. Moreover, I introduce an extended subdifferential, in analogy with [31].

Let $\mathcal{E}$ be a $2-$homogeneous nonlinear Dirichlet form. Consider the space $D(\partial\mathcal{E})$, equipped with the distance

$$d_\partial(u,v) = \sqrt{\|u-v\|^2_{\mathrm{L}^2(X,m)} + \|\partial\mathcal{E}(u) - \partial\mathcal{E}(v)\|^2_{\mathrm{L}^2(X,m)}},$$

where the second contribution is a distance between closed subsets

$$\|\partial\mathcal{E}(u) - \partial\mathcal{E}(v)\|^2_{\mathrm{L}^2(X,m)} = \inf_{\xi\in\partial\mathcal{E}(u),\, \zeta\in\partial\mathcal{E}(v)} \|\xi - \zeta\|^2_{\mathrm{L}^2(X,m)}.$$

Proposition 6.1 *Let $\mathcal{E}$ be a $2-$homogeneous nonlinear Dirichlet form. Then, we have that $D(\partial\mathcal{E}) \subset D(\mathcal{E})$ is dense and continuous at 0. Moreover, $D(\partial\mathcal{E}) \subset \mathrm{L}^2(X,m)$ is dense and continuous. Finally,*

$$\forall u \in D(\partial\mathcal{E}), \qquad \mathcal{E}(u) \leqslant \frac{1}{2}\, \|\partial^0\mathcal{E}(u)\|_{\mathrm{L}^2(X,v)}\, \|u\|_{\mathrm{L}^2(X,m)}.$$

Proof The density of the two inclusions is proved as follows. Let $u \in \mathrm{L}^2(X,m)$. Then, $T_t u \in D(\partial\mathcal{E})$ for all $t > 0$ [29, Section 9.6.3], where $(T_t)_t$ is the gradient flow generated by $\mathcal{E}$. The convergence properties of $T_t u \to u$, as $t \to 0$ are sufficient to conclude. In addition, $d_\partial(u,v) \geqslant \|u-v\|_{\mathrm{L}^2(X,m)}$ shows the continuity of the inclusion $D(\partial\mathcal{E}) \subset \mathrm{L}^2(X,m)$. The inequality $\mathcal{E}(u) \leqslant \frac{1}{2}\, \|\partial^0\mathcal{E}(u)\|_{\mathrm{L}^2(X,v)}\, \|u\|_{\mathrm{L}^2(X,m)}$, is a combination of (16) and Cauchy-Schwarz's inequality. This implies also the continuity at 0 of $D(\partial\mathcal{E}) \subset D(\mathcal{E})$. $\square$

Some integration by parts rules, reminiscent of those given in [2, 31], link the subdifferential with the slopes of Section 5, as the next result shows.

Theorem 6.2 *Let $\mathcal{E}$ be a 2-homogenous nonlinear Dirichlet form. Then, the following rules hold.*

1. *For all $u \in D(\partial E)$, and all $v \in D(\mathcal{E})$, we have that*

$$\Lambda^-(u, v) \leqslant (\partial\mathcal{E}(u), v) \leqslant \Lambda^+(u, v).$$

In particular, if $\mathcal{E}$ is such that $\Lambda(u, \cdot)^+ = \Lambda^-(u, \cdot)$, and $\mathcal{E}$ is subdifferentiable at u, we have

$$\forall v \in D(\mathcal{E}), \qquad \Lambda^\pm(u, v) = (\partial\mathcal{E}(u), v),$$

and $\partial\mathcal{E}(u)$ contains only one element.

2. *For $\lambda > 0$, let A_λ be the Yosida regularisation of $\partial\mathcal{E}$. Then, $\forall u, v \in D(\mathcal{E})$,*

$$\Lambda^-(u, v) \leqslant \liminf_{\lambda\to 0}(A_\lambda(u), v) \leqslant \limsup_{\lambda\to 0}(A_\lambda(u), v) \leqslant \Lambda^+(u, v).$$

In particular, if $\mathcal{E}$ is regular, we have $\lim_{\lambda\to 0}(A_\lambda(u), v) = \Lambda(u, v)$, for all $u, v \in D(\mathcal{E})$.

Proof 1.
Let me perform the calculation for one side of the inequality, being the other one analogous. Fix $\xi \in \partial\mathcal{E}(u)$ and compute

$$\mathcal{E}(u + hv) - \mathcal{E}(u) \geqslant (\xi, hv),$$

which reads, if $h < 0$, as

$$h^{-1}(\mathcal{E}(u + hv) - \mathcal{E}(u)) \leqslant (\xi, v).$$

The inequality follows by taking the supremum for $h < 0$. If $\mathcal{E}$ is regular at u, we have that $(\partial\mathcal{E}(u), v)$ is prescribed for all $v \in D(\mathcal{E})$ by the values of $\Lambda(u, v)$, hence, $\partial\mathcal{E}(u)$ contains only one element.
2.

Let me start with the lim inf inequality. For all $\lambda > 0$, u, v as in the hypothesis and $h < 0$, consider

$$\mathcal{E}_\lambda(u + hv) - \mathcal{E}_\lambda(u) \geqslant (A_\lambda(u), hv),$$

which leads to

$$\liminf_{\lambda\to 0} h^{-1}(\mathcal{E}_\lambda(u + hv) - \mathcal{E}_\lambda(u)) \leqslant \liminf_{\lambda\to 0}(A_\lambda(u), v).$$

The l.h.s. admits a limit, hence,

$$h^{-1}(\mathcal{E}(u + hv) - \mathcal{E}(u)) \leqslant \liminf_{\lambda\to 0}(A_\lambda(u), v).$$

Taking the supremum over $h < 0$ yields the sought inequality. For the lim sup inequality, still consider $\lambda > 0$, u, v as in the hypotheses, and $h > 0$. Write

$$\mathcal{E}_\lambda(u + hv) - \mathcal{E}_\lambda(u) \geqslant (A_\lambda u, hv),$$

and take the lim sup in both sides to get

$$\limsup_{\lambda\to 0} \mathcal{E}_\lambda(u + hv) - \mathcal{E}_\lambda(u) \geqslant \limsup_{\lambda\to 0}(A_\lambda u, hv),$$

which reads

$$\mathcal{E}(u + hv) - \mathcal{E}(u) \geqslant \limsup_{\lambda\to 0}(A_\lambda u, hv),$$

hence,

$$h^{-1}\mathcal{E}(u + hv) - \mathcal{E}(u) \geqslant \limsup_{\lambda\to 0}(A_\lambda u, v).$$

The result follows by taking the infimum in h. □

The domain of $\partial\mathcal{E}$ generalises the domain of the Laplacian in the case of smooth manifolds, which for $X = \mathbb{R}^d$ corresponds to the space $\mathrm{W}^{2,2}(\mathbb{R}^d)$, and

$$\partial\mathcal{E} : D(\partial\mathcal{E}) \to \mathrm{L}^2(X, m).$$

Hereby, I give an extended definition, in the spirit of [31], mimicking the distributional Laplacian

$$\Delta : \mathrm{W}^{1,2}(\mathbb{R}^d) \to \mathrm{H}^{-1}(\mathbb{R}^d).$$

Let $\mathcal{E}$ be a 2-homogeneous nonlinear Dirichlet form. Then, let me say that a function $u \in D(\mathcal{E})$ is a point of extended subdifferentiability if there exists a measure μ on $(X, \mathcal{F})$ such that all $v \in D(\mathcal{E})$ are μ-measurable and the following holds

$$\Lambda^-(u, v) \leqslant \int_X v\, d\mu \leqslant \Lambda^+(u, v). \tag{19}$$

In this case, we write $u \in D(\bar{\partial}\mathcal{E})$ and $\mu \in \bar{\partial}\mathcal{E}(u)$. Notice that Theorem 5.1 implies $\mu \in D(\mathcal{E})^\star$.

Let me collect some properties of $\bar{\partial}\mathcal{E}$ in the next result, which concludes my analysis.

Proposition 6.3 *Let $\mathcal{E}$ be a 2-homogeneous nonlinear Dirichlet form. Then, the extended subdifferential $\bar{\partial}$ satisfies the following:*

1. *$\bar{\partial}\mathcal{E}(u)$ is convex and 1-homogeneous;*
2. *if $u \in D(\bar{\partial}\mathcal{E})$ and $\mathcal{E}$ is regular at u, then $\bar{\partial}\mathcal{E}(u)$ contains only one measure.*
3. *if $u \in D(\partial\mathcal{E})$, and $\xi \in \partial\mathcal{E}(u)$, then $\xi\, dm \in \bar{\partial}\mathcal{E}(u)$.*

Proof The first assertion is a consequence of the 1-homogeneity of $\Lambda^\pm$ and of the linearity of the integral with respect to the measure. The second assertion follows from the fact that the values of $\int_X v\, d\mu$ are prescribed by $\Lambda(u, v)$. Finally, the third statement holds by definition, after Theorem 6.2. □

The validity of a converse of the third statement in the last theorem has been discussed for metric spaces in [31], and looks unclear. I am not able to give an analog of [31, Proposition 4.11] as well. Such formula is one of the hypotheses of [4] for the quadratic case.

7 Perspectives and *desiderata*

A missing point in the theory is a pointwise object representing $\Lambda^{\pm}(u, v)$, which should play the same role as $D^{\pm}v(\nabla u)(x)$ in [31], hence be a gradient-like object. It is unclear whether the existence of a density for $\Lambda^{\pm}$ should be imposed, or if it is a consequence of some locality hypotheses on $\mathcal{E}$. In any case, heuristically, it corresponds to finer integration by parts formulae than those of the current paper. Also a Leibniz formula and some chain rules on $\Lambda^{\pm}(u, \cdot)$ would be advances in the theory. Still, I do not know if one should expect such properties or impose them. In general metric measure spaces, locality properties of gradient-like objects are equivalent to the Leibniz and chain rules [31, Theorem 2.2.6]. My hope is that—given the locality of $\mathcal{E}$—also a possible pointwise object representing $\Lambda^{\pm}$ would inherit locality, and that the results of Gigli can be further extended in the context of nonlinear Dirichelt forms.

Another desirable statement is some integral representation of $\mathcal{E}$ under locality assumptions. However, this is not available even if $X = \mathbb{R}^2$. Finally, a missing notion is that of *linearised* flow associated to the gradient flow of a nonlinear Dirichlet form. Equivalently, one would need a *linearisation of* $\partial\mathcal{E}$ *in the direction of the gradient*, as in [39].

8 Bibliographical Note

Since the first version of this paper was written, a few new works on the subject of nonlinear Dirichlet forms have appeared. Even if they are not directly relevant for the *calculus rules* I discussed in this paper, I briefly highlight them. In [18], a complete, systematic, and general treatment of Markovianity for nonlinear semigroups is discussed, in relation with the corresponding properties at the level of the associated form, resolvent, and sub-differential operator. Moreover, locality is treated rigorously, at the measure theoretical level, generalising the results of [13]. In [40], non-linear Beurling–Deny criteria are discussed, and more contraction properties are proved for nonlienar Dirichlet forms. In [41], a notion of extended domain, related with transience and invariance is introduced. The work [20] is dedicated to Sobolev inequalities satisfied by nonlinear Dirichlet forms, and the related question of isocapacitary inequalities. Finally, in [21], nonlinear forms and the potential theory stemming from them are instrumental to establish a unified framework to treat several classes of nonlinear PDEs.

Acknowledgements I am thankful to G. Savaré, for introducing me to the study of nonlinear Dirichlet forms and metric measure spaces, and to D. Manini for stimulating discussions.

Funding: Open access funding provided by Institute of Science and Technology (IST Austria). The author has been funded by the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 754362. Partial support has been obtained from the EFI ANR-17-CE40-0030 Project of the French National Research Agency.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Competing Interests The author has no competing interests to declare that are relevant to the content of this article.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel (2008)
2. Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. **195**, 289–391 (2014)
3. Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. **163**, 1405–1490 (2014)
4. Ambrosio, L., Gigli, N., Savaré, G.: Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds. Ann. Probab. **43**, 339–404 (2015)
5. Ambrosio, L., Mondino, A., Savaré, G.: Nonlinear Diffusion Equations and Curvature Conditions in Metric Measure Spaces, vol. 262. American mathematical society (2019)
6. Arnold, A., Markowich, P., Toscani, G., Unterreiter, A.: On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations (2001)
7. Attouch, H.: Variational convergence for functions and operators. Appl. Math. Ser. (1984)
8. Bakry, D., Émery, M.: Diffusions hypercontractives. In: Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Math., pp. 177–206. Springer, Berlin (1985)
9. Bakry, D., Gentil, I., Ledoux, M.: Analysis and Geometry of Markov Diffusion Operators, vol. 103. Springer (2014)
10. Barthélemy, L.: Invariance d'un convexe fermé par un semi-groupe associé à une forme non-linéaire. Abstr. Appl. Anal. **1**, 237–262 (1996)
11. Bénilan, P., Picard, C.: Quelques aspects non linéaires du principe du maximum, in Séminaire de Théorie du Potentiel, No. 4 (Paris, 1977/1978), vol. 713 of Lecture Notes in Math., pp. 1–37. Springer, Berlin (1979)
12. Beurling, A., Deny, J.: Dirichlet spaces. Proc. Nat. Acad. Sci. U.S.A. **45**, 208–215 (1959)
13. Bouleau, N., Hirsch, F.: Dirichlet Forms and Analysis on Wiener Space. De Gruyter Studies in Mathematics, vol. 14. Walter de Gruyter & Co., Berlin (1991)
14. Brezis, H.: Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematics Studies, vol. 5. North-Holland Publishing Co., American Elsevier Publishing Co., Inc., Amsterdam-London; New York (1973)
15. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York (2011)

16. Brigati, G.: Kinetic equations, functional inequalities, and distances between probability measures, PhD thesis (2023)
17. Brigati, G., Hartarsky, I.: The normal contraction property for non-bilinear Dirichlet forms. Potential Anal. **60**, 1–16 (2022)
18. Brigati, G., Schiavo, L.D.: Non-bilinear Dirichlet Functionals: Markovianity, locality, invariance (2025)
19. Cheeger, J.: Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9**, 428–517 (1999)
20. Chill, R., Claus, B.: Sobolev inequalities for nonlinear Dirichlet forms (2026)
21. Chill, R., Warma, M.: A unified view of nonlinear, nonlocal operators and qualitative properties of associated elliptic and parabolic problems (2026)
22. Çınlar, E.: Probability and Stochastics, vol. 261. Springer Science & Business Media (2011)
23. Cipriani, F., Grillo, G.: Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces. J. Reine Angew. Math. **562**, 201–235 (2003)
24. Cipriani, F., Sauvageot, J.-L.: Derivations as square roots of Dirichlet forms. J. Funct. Anal. **201**, 78–120 (2003)
25. Claus, B.: Nonlinear Dirichlet Forms, PhD thesis, Technischen Universität Dresden (2021)
26. Claus, B.: Energy spaces, Dirichlet forms and capacities in a nonlinear setting. Potential Anal. **58**, 159–179 (2023)
27. Dolbeault, J., Nazaret, B., Savaré, G.: On the Bakry-Emery criterion for linear diffusions and weighted porous media equations. Commun. Math. Sci. **6**, 477–494 (2008)
28. Erbar, M., Kuwada, K., Sturm, K.-T.: On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. Invent. Math. **201**, 993–1071 (2015)
29. Evans, L.C.: Partial Differential Equations, vol. 19. American Mathematical Society Providence, RI (1998)
30. Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet Forms and Symmetric Markov Processes, vol. 19, extended De Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin (2011)
31. Gigli, N.: On the Differential Structure of Metric Measure Spaces and Applications, vol. 236. American Mathematical Society (2015)
32. Gigli, N.: Nonsmooth differential geometry–an approach tailored for spaces with Ricci curvature bounded from below. Mem. Amer. Math. Soc. **251**, 161 (2018)
33. Gigli, N., Mosconi, S.: The abstract Lewy-Stampacchia inequality and applications. J. de Mathématiques Pures et Appliquées **104**, 258–275 (2015)
34. Lejan, Y.: Mesures associées à une forme de Dirichlet. Appl., Bulletin de la Soc. Math. de France **106**, 61–112 (1978)
35. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) **169**, 903–991 (2009)
36. Ma, Z.-M., Röckner, M.: Introduction to the Theory of (non-symmetric) Dirichlet Forms. Universitext. Springer, Berlin, Heidelberg (2012)
37. Ohta, S.-I.: Comparison Finsler Geometry. Springer (2021)
38. Ohta, S.-I., Sturm, K.-T.: Heat flow on Finsler manifolds. Commun. on Pure and Appl. Math.: A J. Issued by the Courant Inst. of Math. Sci. **62**, 1386–1433 (2009)
39. Ohta, S.-I., Sturm, K.-T.: Bochner-Weitzenböck formula and Li-Yau estimates on Finsler manifolds. Adv. Math. **252**, 429–448 (2014)
40. Puchert, S.: Nonlinear Beurling-Deny criteria (2025)
41. Schmidt, M., Zimmermann, I.: The extended Dirichlet space and criticality theory for nonlinear Dirichlet forms (2025)
42. Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. **196**, 65–131 (2006)
43. Sturm, K.-T.: On the geometry of metric measure spaces. II. Acta Math. **196**, 133–177 (2006)
44. Sturm, K.-T.: Metric measure spaces and synthetic Ricci bounds: fundamental concepts and recent developments. In: European Congress of Mathematics, pp. 125–159 (2023)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.