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DENSITY FLUCTUATIONS IN WEAKLY INTERACTING PARTICLE SYSTEMS VIA THE DEAN–KAWASAKI EQUATION

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The Dean–Kawasaki equation—one of the most fundamental SPDEs of fluctuating hydrodynamics—has been proposed as a model for density fluctuations in weakly interacting particle systems. In its original form, it is highly singular and fails to be renormalizable, even by approaches such as regularity structures and paracontrolled distributions, hindering mathematical approaches to its rigorous justification. It has been understood recently that it is natural to introduce a suitable regularization, for example, by applying a formal spatial discretization or by truncating high-frequency noise: This yields well-posed equations that should still precisely approximate the law of the particle density fluctuations.

In the present work, we prove that a regularization in the form of a formal discretization of the Dean–Kawasaki equation indeed accurately describes density fluctuations in systems of weakly interacting diffusing particles: We show that, in suitable weak metrics, the law of fluctuations as predicted by the discretized Dean–Kawasaki SPDE approximates the law of fluctuations of the original particle system, up to an error that is of arbitrarily high order in the inverse particle number and a discretization error. In particular, the Dean–Kawasaki equation provides a means for efficient and accurate simulations of density fluctuations in weakly interacting particle systems.

CONTENTS

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1. Introduction.

The theory of *Fluctuating Hydrodynamics* [64] describes the dynamics of large, finite-size particle systems subject to fluctuations. In the framework of this theory, the particle system being investigated is described via a suitable stochastic PDE (SPDE), which captures the fluctuations of the system on top of its deterministic limiting dynamics. The physics applications of this theory are numerous and diversified, and touch upon several different fields; see, for instance, [4, 8, 19, 20, 27, 30, 40, 57, 58, 69, 70].

This work is concerned with giving a fully quantitative justification to a pivotal SPDE from fluctuating hydrodynamics, the so-called *Dean–Kawasaki equation* [18, 49]

$$\partial_t \rho = \sigma \Delta \rho + \nabla \cdot (\rho \nabla V * \rho) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma\rho}\,\xi). \tag{1.1}$$

Here ρ denotes the density of particles, N is the number of particles, V is an interaction potential, $\sigma > 0$ is the diffusion coefficient, and ξ denotes space-time vector-valued white noise. The model (1.1) is proposed as a mesoscopic description for the law of the empirical density

$$\mu^N(\cdot, t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^N(t)}$$

of a system of $N \gg 1$ particles with positions $X_i^N(t) \in \mathbb{T}^d$ driven by i.i.d. Brownian motions $(B_i)_{i=1}^N$ and interacting weakly via a smooth potential V,

$$\mathrm{d}X_i^N(t) = -N^{-1} \sum_{j=1}^{N} \nabla V\big(X_i^N(t) - X_j^N(t)\big)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_i(t). \tag{1.2}$$

The purpose of the Dean–Kawasaki equation (1.1) is to correctly describe the law of particle density fluctuations, going beyond the (deterministic) mean-field description of the particle density

$$\partial_t \bar\rho = \sigma \Delta \bar\rho + \nabla \cdot (\bar\rho \nabla V * \bar\rho). \tag{1.3}$$

As we shall see, the Dean–Kawasaki equation (1.1) turns out to be even substantially superior in accuracy, as compared to the leading-order description of fluctuations by the process

$$\partial_t \hat\rho = \sigma \Delta \hat\rho + \nabla \cdot (\hat\rho \nabla V * \hat\rho) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma \bar\rho}\,\xi). \tag{1.4}$$

The Dean–Kawasaki equation (1.1) itself is a highly singular SPDE; it is not even renormalizable by approaches like regularity structures or paracontrolled distributions. As shown in the seminal work [51], the singular SPDE (1.2) turns out to be in fact a formal—mathematically equivalent—rewriting of the associated microscopic particle system (3.1): All of its martingale solutions are precisely given as the empirical density of an interacting particle system of the form (1.2), that is, the only solutions to (1.1) are of the form

$$\rho(\cdot, t) \equiv \mu^N(\cdot, t) := N^{-1} \sum_{i=1}^{N} \delta_{X_i^N(t)} \tag{1.5}$$

for some X_i^N satisfying the system of SDEs (1.2). While this might—at first glance—appear to imply that the representation (1.1) brings no additional insight over the associated particle dynamics (1.2), it has been observed in recent years that this is, in fact, not the case: Natural regularizations of the formal Dean–Kawasaki SPDE (1.1) are better-behaved objects and may be expected to provide meaningful approximations of density fluctuations. Note that in any practical use for physics simulations, the Dean–Kawasaki equation (1.1) is necessarily subjected to a regularization, either by applying a spatial numerical discretization (which implicitly truncates the high-frequency noise modes) or by introducing an explicit frequency cutoff in the noise term. While regularised models may differ from one another in terms of specific features and applicability, they are all usually much more tractable and versatile versions of (1.1) and capable of describing—at least some—features of the underlying microscopic systems up to a quantifiable, small error.

In the present work, we prove that upon regularizing the Dean–Kawasaki equation (1.9) by applying a formal spatial discretization, it is capable of approximating the law of density fluctuations of the weakly interacting particle system (3.1) up to *arbitrary* precision in N^{-1} (plus numerical errors). Note that any meaningful comparison of the empirical density μ^N—a sum of Dirac measures—to the continuous solution ρ_h of a regularized variant of the Dean–Kawasaki equation (1.9) must be formulated in terms of weak spatial norms, that is, in terms of testing the densities against a test function with sufficient regularity. Furthermore, as μ^N and ρ_h live on different probability spaces, any comparison of ρ_h to μ^N can only be phrased in terms of their law (and not as a pathwise statement). The informal statement of our result is as follows. Here the spatial order of the discretization scheme denotes its accuracy, for example, for a scheme of order $p+1$ the discrete gradient satisfies $|\nabla_h \varphi - \nabla \varphi| \lesssim \|\varphi\|_{C^{p+2}} h^{p+1}$.

MAIN RESULT (Informal statement for high-order approximation of density fluctuations in the single-species case of Theorem 2.1 below). *Let $T > 0$ be a fixed time. Let ρ_h denote the solution to a suitable finite-difference discretisation of* (1.1) *on $\mathbb{T}^d \times [0, T]$ with spatial discretisation parameter h and of spatial order $p + 1$. Let μ^N denote the empirical density of the particle system* (1.2) *consisting of N particles. Suppose that the initial density $\rho_h(\cdot, 0) \geq \inf_x \rho_h(x, 0) > 0$ is strictly positive and approximates* (*in law*) *the empirical density $\mu^N(\cdot, 0)$. Furthermore, let the overline symbol* ($\overline{}$) *denote suitable "mean-field" analogues of the densities. For the scaling regime, assume that*

$$N^{1-\delta_0} h^d \geq 1 \quad \textit{for some } \delta_0 > 0. \tag{1.6}$$

Then for every $0 < \epsilon < \delta_0/4$, there exists a stopping time $T_\oslash$ with

$$\mathbb{P}(T_\oslash < T) \lesssim \exp(-CN^{\epsilon/2}), \tag{1.7}$$

where, for every $\kappa > 0$, there exists $N_0 \in \mathbb{N}$ such that, for every $t \in [0, T]$ and $N \geq N_0$,

$$\begin{aligned} &d_{\mathrm{weak},2j+1}\big[N^{1/2}(\rho_h - \overline{\rho}_h)(t \wedge T_\oslash), N^{1/2}(\mu^N - \overline{\rho})(t)\big] \\ &\quad \lesssim N^{2\epsilon+\kappa}\big(h^{p+1} + \exp(-CN^{\epsilon/2}) + N^{-j(1/2-2\epsilon)}\big) \\ &\quad =: \mathrm{Err}_{\mathrm{num}} + \mathrm{Err}_\oslash + \mathrm{Err}_{\mathrm{fluct,rel}} \end{aligned} \tag{1.8}$$

holds for all $j \in \mathbb{N}$. Here, $d_{\mathrm{weak},2j+1}$ is a negative Sobolev-type distance of order $-2j - 1$.

This result is desirable for three reasons:

- The approximate scaling regime $Nh^d \gg 1$ (cfr. (1.6)) is extremely relevant, as it corresponds to an—on average—large amount of particles per grid cell. Hence, it is the regime in which the direct particle simulation is more expensive than the numerical SPDE model.
- The estimate (1.8) shows that the (discrete) Dean–Kawasaki equation provides an accurate description of the underlying particle system, in the sense that the error due to the finiteness of the number of particles is—in suitable weak norms—of arbitrarily high order in N^{-1} and thus typically dominated by the numerical error $N^{2\epsilon+\kappa}h^{p+1}$.
- By (1.7) the stopping time (whose role in the proof is to guarantee that the fluctuations $\rho_h - \overline{\rho}_h$ roughly stay in the regime of classical mean field fluctuations and that ρ_h stays positive) runs short of the final time $T > 0$ only with exponentially small probability.

The key novelty of our result is the derivation of an error estimate for an *interacting* particle system that is *of arbitrary order* in the inverse particle number N^{-1} (plus numerical error $\mathrm{Err}_{\mathrm{num}}$ and modelling error $\mathrm{Err}_\oslash$): In particular, our result is the first to show the Dean–Kawasaki equation is a far more accurate descriptor of fluctuations than the leading-order equation (1.4) in the case of weakly interacting particle systems.

In the case of independent Brownian particles, that is, the case $V \equiv 0$ in (1.1), the corresponding result has been proven in [13]. Again in the noninteracting particle case, a short proof of a quantitative error bound of the order $O(N^{-1/(d+2)})$ has recently been given in [25]: Notably, the result in [25] does not require a positivity lower bound on the initial density profile. Previously, a low-order error estimate $O(N^{-\beta})$ for some $\beta < \frac{1}{2}$ had been established in [32], covering also the case of weakly interacting particles.

Let us mention that, having in mind a future application to cross-diffusion limits, throughout the present work we, in fact, consider a somewhat more general interacting particle system: We allow for multiple species of diffusing particles that interact with each other via possibly mildly (singularly) rescaled potentials $V_{\alpha\beta}^{r_I}(\cdot) = {r_I}^{-d} V(\cdot/r_I)$ with interaction length scale $r_I = r_I(\log N)$. The associated particle dynamics is given by the system of SDEs (3.1) below, while the corresponding analogue of the Dean–Kawasaki equation is given by the system

$$\partial_t \rho_\alpha^{r_I} = \sigma_\alpha \Delta \rho_\alpha^{r_I} + \nabla \cdot \left(\sum_{\beta=1}^{n_S} \rho_\alpha^{r_I} \nabla V_{\alpha\beta}^{r_I} * \rho_\beta^{r_I} \right) + N^{-1/2} \nabla \cdot \left(\sqrt{2\sigma_\alpha \rho_\alpha^{r_I}} \xi_\alpha \right) \tag{1.9}$$

for $\alpha, \beta \in \{1, \dots, n_S\}$, n_S being the number of species. However, we emphasize that our results are already new and relevant in the single-species case $n_S = 1$ and for $r_I \equiv 1$.

1.1. *Comparison with the independent Brownian case* ($V \equiv 0$) *treated in* [13]. In contrast to [13], the distinctive trait of this work is the handling of the nonlinear convolution term $\nabla \cdot (\rho \nabla V * \rho)$ in (1.1), which corresponds to the mean-field interaction between particles (in this section we take $n_S = 1$ for notational simplicity). Doing this entails working around several difficulties, which we now list informally for the purpose of this introduction. A more thorough analytical discussions of these difficulties can be found in Section 4.2, directly after—and in relation to—the informal summary of the proof of our main result (Theorem 2.1) in Section 4.1:

Quantitative convergence to the mean-field limit. The presence of the potential V needs handling when proving the convergence of the empirical density μ^N to the mean-field limit $\overline{\rho}$ in (1.3). The same also applies for the convergence of the discretised counterparts $\rho_h - \overline{\rho}_h$, although this is a consequence of the next point.

Linearisation argument for convolutional nonlinearity. The most remarkable feature of (1.9) is the linear cross-variation structure of the noise (with respect to the density ρ). More precisely, as we will discuss thoroughly, the Itô differential of the process $\int_{\mathbb{T}^d} (\rho - \overline{\rho})\varphi_1 \int_{\mathbb{T}^d} (\rho - \overline{\rho})\varphi_2$, for suitably chosen, smooth test functions φ_1, φ_2, has martingale component with cross-variation given by

$$N^{-1} \int_{\mathbb{T}^d} \rho \nabla \varphi_1 \cdot \nabla \varphi_2. \tag{1.10}$$

The linearity of (1.10) (w.r.t. the density ρ) is the pivotal ingredient upon which most arguments in [13] are based. One of the main challenges of this work is—in a nutshell—to linearise the quadratic convolution term $\nabla \cdot (\rho \nabla V * \rho)$ so as to recover the paradigm of [13] as much as possible. This involves, crucially, performing a Fourier expansion of $\nabla \cdot (\rho \nabla V * \rho)$ in twice the space dimension: doing this effectively "detaches" the two instances of the density ρ and leads to a countable number of tractable linear objects of type (1.10).

Estimate for probability of large fluctuations. It is important to provide bounds on $\rho_h - \overline{\rho}_h$, as this, among other things, gives an indication of how often the discretised solution ρ_h becomes negative. While—in the absence of the potential—this could be handled with a stopping time argument exclusively in the L^∞-norm, the presence of the potential requires a lengthy technical argument to also keep track of a suitable negative Sobolev norm (this argument also makes use of the convergence of the continuous mean-field limit). Controlling this second norm effectively tackles the superlinear nature of the convolution term, ultimately allowing for the exponentially decaying bound (1.7).

Compatible discrete setting. The challenge of this point is twofold. First, in contrast to [13], the initial configurations $\mu^N(\cdot, 0)$ and $\rho_h(0)$ are random, as sufficient regularity of their expected value is required for the regularity of the mean-field limit (no mean-field limit argument is needed in [13]). For this point and in order to obtain suitable initial data which satisfies the required high-order (in h) fluctuation estimates for $\rho_h(0) - \overline{\rho}_h(0)$, we propose a scheme based on polynomial interpolation. Second, additional care is needed when performing high-order (in h) approximations of the "time-dual" of the discrete mean-field limit. The need for these approximations will be discussed in due course.

REMARK 1.1. In order to obtain the fluctuations bound (1.8), we require the potential V to be sufficiently smooth. As will become apparent in the proofs, the smoothness of V allows—among other things—for: (i) the application of the linearisation argument mentioned above and (ii) small errors entailed by replacing continuous operators with discretised counterparts (in particular, when replacing the continuous convolution $\nabla V * \rho$ with a discrete version $\nabla V *_h \rho$).

We refer the reader to Section 1.2.3 for a broader discussion concerning existing literature and future investigations in the case of singular potentials.

1.2. *Related literature.*

1.2.1. *The leading order description via the additive noise model* (1.4). As a description for the fluctuations of the particle density, (1.4) with the purely additive noise term had already been established in the 1970s and 1980s before Dean and Kawasaki proposed (1.1), usually in the sense of a central limit theorem, viewing the convergence to the mean-field limit as a law of large numbers. In [5], density fluctuations were characterized for a second order system with random initial conditions but dynamics given just via interaction. It was shown that the normalized fluctuations $N^{1/2}(\mu^N - \overline{\rho})$ converge to a Gaussian stochastic process in a suitable sense. Recently, for this same setting optimal quantitative bounds for this convergence were obtained in [29] as a byproduct.

There is a large amount of results in this direction for other particle systems, but here we focus on our specific setting, that is, first order dynamics driven by Brownian motion and interaction via a bounded C^2 interaction potential. In [68] the limit of the normalized fluctuations is characterized for the simplified version of a quadratic interaction potential V in dimension one: The limit process in our notation corresponds to $N^{1/2}(\hat{\rho} - \overline{\rho})$, thus suggesting (1.4) as a good approximation of the particle system. Later, under general assumptions similar to ours the same has been done in [66], showing convergence of finite dimensional marginals. In [35], also for more general initial conditions, it has been shown that the normalized fluctuations converge in law as elements of a path-space with values in a weighted negative Sobolev space.

1.2.2. *Dean–Kawasaki model.* As discussed around (1.5), the Dean–Kawasaki equation in its original form is a rather rigid mathematical object, as it only allows for the empirical particle system as solution: this result is given in [51] and, also, in the (earlier) analogue version for noninteracting particles [52]. The papers [51, 52] are related to a series of works [1, 21, 53–55, 59, 71] which—among others—shed light on the rigid interplay, which deterministic and stochastic components of the Dean–Kawasaki model have to abide to (in the context of a suitable stochastic Wasserstein gradient flow).

The aforementioned rigidity of the Dean–Kawasaki model can be broken once suitable regularisations (such as, for instance, noise smoothing or truncation or a discretization) are introduced: A substantial number of works belong to this ever-growing framework.

In [23] a rigorous justification of the SPDE of fluctuating hydrodynamics for the simple exclusion process is provided (with leading order dynamics, although the noise term in this case is nonlinear and, therefore, much more challenging), together with convergence results concerning the rate functions for large deviation principles. Rigorous links between Dean–Kawasaki type models and large-deviation principles for zero-range processes (and associated thermodynamic setting) are given in [24].

In [31] the well-posedness theory (in a suitable kinetic formulation) of stochastic porous media and fast diffusion equations driven by nonlinear, conservative noise is provided. The generation of a random dynamical system is also discussed. The latter topic is expanded and enriched with uniqueness of invariant measures and mixing for the associated Markov process in [34]. Similar results are also derived in the case of correlated noise [33]. For the same noise, derivation of underlying microscopic dynamics is given in [22].

Rates of convergence of the discretized Dean–Kawasaki dynamics toward the particle system in the case of independent Brownian particles are discussed in [13]. In the recent paper [25], the authors prove weak error estimates in the noninteracting particle case, but they use a suitable SPDE approximation of the Dean–Kawasaki model rather than a discrete numerical approximation of the same. Their mathematical approach (which is centered around Laplace duality arguments and Kolmogorov backward equation techniques) is somewhat complementary to ours, and it is safe to say that the current work and [25] have different points of strength. Unlike the results in this work, those in [25] allow for general initial particle profiles, and provide nonnegativity of the solution (in addition to other well-posedness properties, including a comparison principle and entropy estimates): However, the weak error accuracy in [25] in terms of N is capped (in terms of relative error) by $N^{-1/(d/2+1)}\log N$ (which gets worse with the spatial dimension d), while our fluctuation rate (1.8) can be arbitrarily high.

Recently, conservative stochastic PDEs sharing strong similarities with Dean–Kawasaki models have been proven to be limit of stochastic interacting particle systems in the mean-field limit (e.g., the case of stochastic gradient descent dynamics in overparametrised, shallow neural networks is covered in [39] with optimal convergence rates provided for both convergence and associated central limit theorem).

For regularised Dean–Kawasaki models of inertial type (i.e., models capturing both density and momentum density), high-probability well-posedness for both independent and weakly interacting particle systems is discussed in [15–17].

The Dean–Kawasaki model is becoming more and more widespread in physics applications (see, for instance, [8, 19, 20, 27, 30, 40, 57, 58, 69, 70]). Consequently, works devoted to numerical approximations of the Dean–Kawasaki model are on the rise. Among such contributions on the numerical side, we mention structure-preserving finite difference and finite element schemes for high-order fluctuation bounds of noninteracting diffusing particles [13], analysis of finite element discretisations in the context of reaction-diffusion (agent-based) systems models [43, 50], analysis of finite differences discretisations of agent-based models describing co-evolving opinion and social dynamics under the influence of multiplicative noise [26], finite-volume schemes for stochastic gradient flow equations [28, 62], full reconstruction of dissipative operators in gradient flow equations from particle fluctuations [56], convergence of finite element schemes for a weak formulation of suitable smoothed Dean–Kawasaki model [2], convergence analysis of discountinuous Galerkin scheme—and modelling—for the regularised inertial Dean–Kawasaki model [14].

1.2.3. *The mean-field limit of* (3.1) *and discussion of future investigations for singular potentials and cross-diffusion systems.* The analysis of the Dean–Kawasaki model (1.9) is naturally built on top of the mean-field dynamics of the particle system (3.1). The study of the mean-field dynamics of systems of SDEs goes back to the 1980s (see, e.g., the reviews [41, 45]). Results for a weakly interacting system with smooth potential were established early and are now considered classical; see, for example, [67] for lecture notes from the time.

In the present work, we focus on the case of arbitrarily regularity of the interaction potential V. In recent years there has been major progress concerning the dynamics of diffusive particle systems with a singular mean-field interaction. In particular, we refer to [6, 7, 12] concerning the mean-field limit of (1.2) with singular interaction potentials V and to [72] for a leading-order description of fluctuations via an SPDE related to (1.4). For the kinetic analogue of (1.2), we refer to [3]. To which degree our present analysis could be extended to the case of singular interaction potentials remains an interesting open problem. While we anticipate that (1.1) should still provide a leading-order description of fluctuations, a finer analysis may be needed to achieve higher-order accuracy in $N^{-1/2}$; for instance, as already hinted in Remark 1.1, the Fourier expansion approach for the interaction kernel may need to be replaced by a more careful argument, and caution would need to be taken in discretising the convolution operator (see Corollary B.5).

In the rest of this section, we give a brief account of works related to cross-diffusion-type mean field dynamics (this is in accordance with our intention to also address fluctuations in cross-diffusion systems in the future). In [60], Oelschläger derived reaction-diffusion systems from moderately interacting particle systems, in fact, the case of cross-diffusion is included, however, with a positive-definiteness assumption on the diffusion coefficients. Quadratic porous-medium-type equations are derived from moderately interacting particle systems in [61]. The methods of Oelschläger were significantly extended by Stevens in [65] to derive a chemotaxis system. In [46] the mean-field limit and fluctuations of a moderately interacting particle system with nonlinear diffusion coefficients is studied. Further contributions include, for example, the derivation of a two-phase Stefan problem as the mean-field limit of a master equation [44, 48], the two-species Maxwell–Stefan model for the diffusion of gaseous mixtures as the hydrodynamic limit of two (singularly) interacting Brownian motions [63], nonlocal Lotka–Volterra systems with cross-diffusion as limits of a suitable Markov process [36], as well as Shigesada–Kawasaki–Teramoto type cross-diffusion systems [9].

Daus, Chen, and Jüngel have shown in [10] that the limiting behavior of our SDE system (3.4) under the simultaneous limit $N \to \infty$ and $r_I \sim (\log N)^{-\beta} \to 0$ is captured by a cross-diffusion system [10], (1); we refer to [37] for an earlier more restrictive result and to [47] for a recent extension. In the present paper, we will only concern ourselves with the first limit $N \to \infty$, yielding a mean-field limit of the form (3.4); however, many intermediate results are formulated to be of use also in a future work focusing on this cross-diffusion limit, by being established, for example, uniformly in r_I for a suitable range $(\log N)^c \lesssim r_I \lesssim 1$.

2. Main result. In our main result, we rigorously quantify the distance between: (a) the law of the density fluctuations arising in the interacting particle system (3.1) and (b) the law of density fluctuations in a suitable numerical discretisation of the Dean–Kawasaki SPDE (1.9). In order to compare these laws, we make use of the family of distances between $\mathbb{R}^K$-valued random variables X, Y, given by

$$d_{-j}[X,Y] := \sup_{\psi:\ \max_{0\le \tilde{j}\le j} \|D^{\tilde{j}}\psi\|_{L^\infty(\mathbb{R}^K)}\le 1} \big|\mathbb{E}[\psi(X)] - \mathbb{E}[\psi(Y)]\big|, \qquad j\in\mathbb{N}. \tag{2.1}$$

In other words, the distance d_{-j} is a negative Sobolev distance acting on the probability distributions of the arguments. Note that the distance d_{-1} is the Fortet–Mourier norm, a Wasserstein-1-type metric. Up to the zero-th order bound $\|\psi\|_{L^\infty} \le 1$, this corresponds to the dual formulation of the Wasserstein-1 distance.

While the full notation is given below in Section 3, we briefly define the remaining minimal ingredients needed to state our main theorem, namely: $G_{h,d}$ (the uniformly spaced grid on $\mathbb{T}^d$ with grid-size parameter $h > 0$), $(\cdot,\cdot)_h$ (the standard inner product in $L^2(G_{h,d})$); $\langle\cdot,\cdot\rangle$ (the standard measure/function duality), and $\mathcal{I}_h$ (the operator interpolating continuous functions on the grid points of $G_{h,d}$).

Our main theorem reads as follows.

THEOREM 2.1 (High-order approximation of density fluctuations in weakly interacting particle systems). *Let $T > 0$ and $N \in \mathbb{N}$. On $[0, T]$, let:*

- $\mu^{r_I,N} = (\mu_\alpha^{r_I,N})_{\alpha=1}^{n_S}$ *as in (1.5) be the empirical measures of the cross-diffusing particle system (3.1) satisfying Assumption 3.7 (particle system),*

- $\overline{\rho}^{r_I}$ *be the intermediate mean-field limit as given in* (3.4), *and satisfying Assumption* 3.8 (*existence and regularity of continuous mean-field limit*),
- $\rho_h^{r_I}$ *be a solution to the discretised Dean–Kawasaki model* (h-DK) *below in the context of a standard finite-difference discretisation of order* $p+1$ *and spatial spacing* $h > 0$ *satisfying Assumption* 3.9 (*discrete finite-difference operators*),
- $\overline{\rho}_h^{r_I}$ *be the corresponding finite difference mean-field limit, as defined in* (h-MFL).

Let the initial conditions satisfy Assumption 3.10 *and the parameters* r_I, N, h *satisfy Assumption* 3.11 *with* $0 < \delta_0 < 1$ (*scaling regime for* N *and* h).

Let $0 < \epsilon < \delta_0/4$. *Then there exists a stopping time* $T_\oslash \in [0, T]$ *with*

$$\mathbb{P}[T_\oslash < T] \lesssim \exp(-CN^{\epsilon/2}) \tag{2.2}$$

such that the discrete Dean-Kawasaki solutions $\{\rho_{h,\alpha}(t \wedge T_\oslash)\}_{\alpha=1}^{n_S}$ *capture the fluctuations of the the empirical measures* $\mu_{\alpha,t}^{r_I,N}$ *in the following sense.*

Let $j \in \mathbb{N}$. *Assume that with* $s(d,p,j) := p + \frac{3d}{2} + 4 + j$ *the continuous mean-field limit* $\overline{\rho}^{r_I}$ *is in* $L^\infty(0,T; C^{s(d,p,j)})$ *and that the interaction potentials satisfy* $V \in [W^{s_I,1}(\mathbb{T}^d)]^{n_S \times n_S}$ *with* $s_I > 2^{j-1}3j^2 s(d,p,j) + 2d + 1$. *Then abbreviating* data $:= \{V, \overline{\rho}^{r_I}, \rho_{\min}, \rho_{\max}, d, T, n_S, p, j\}$, *for each* $\kappa > 0$, *there exists* $N_0 = N_0(\kappa, \epsilon, K, \text{data})$ *such that if* $N > N_0$,

$$d_{-(2j+1)}\left[N^{1/2}\begin{pmatrix} \sum_{\alpha=1}^{n_S} \big(\mathcal{I}_h[\varphi_{1,\alpha}], (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(T_1 \wedge T_\oslash)\big)_h \\ \vdots \\ \sum_{\alpha=1}^{n_S} \big(\mathcal{I}_h[\varphi_{K,\alpha}], (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(T_K \wedge T_\oslash)\big)_h \end{pmatrix}, \right.$$

$$\left. N^{1/2}\begin{pmatrix} \sum_{\alpha=1}^{n_S} \langle \varphi_{1,\alpha}, (\mu_\alpha^{r_I,N} - \overline{\rho}_\alpha^{r_I})(T_1)\rangle \\ \vdots \\ \sum_{\alpha=1}^{n_S} \langle \varphi_{K,\alpha}, (\mu_\alpha^{r_I,N} - \overline{\rho}_\alpha^{r_I})(T_K)\rangle \end{pmatrix} \right] \tag{2.3}$$

$$\begin{aligned} &\le C\big(K, \|\boldsymbol{\varphi}\|_{H^{s(d,p,j)}}, \text{data}\big) N^{2\epsilon+\kappa}\big(h^{p+1} + \exp(-CN^{\epsilon/2}) + N^{-j(1/2-2\epsilon)}\big) \\ &=: \mathrm{Err}_{\mathrm{num}} + \mathrm{Err}_\oslash + \mathrm{Err}_{\mathrm{fluct,rel}} \end{aligned} \tag{2.4}$$

holds for any $\boldsymbol{\varphi} = (\varphi_{k,\alpha})_{\alpha=1,\dots,n_S}^{k=1,\dots,K} \in [H^{s(d,p,j)}(\mathbb{T}^d, \mathbb{R}^{n_S})]^K$ *and any* $(T_1, \dots, T_K) \in [0,T]^K$, *where the distance* $d_{-j}[X, Y]$ *has been introduced in* (2.1).

REMARK 2.2. Throughout the paper the generic expression *data*, which may change from use to use, denotes a relevant subset of parameters $\{V, \overline{\rho}^{r_I}, \rho_{\min}, \rho_{\max}, d, T, n_S, p, j\}$. More specifically, dependencies on V, $\overline{\rho}^{r_I}$ come in terms of Sobolev norms. Note that r_I is not included. We assume that there is a uniform bound in r_I for the respective Sobolev norms of $\overline{\rho}^{r_I}$. This is a reasonable assumption due to the convergence to the solution of a suitable cross-diffusion system given regular enough initial data for $r_I \to 0$; see for example, [10].

REMARK 2.3. Theorem 2.1 captures the relative error in fluctuations, $\mathrm{Err}_{\mathrm{fluct,err}}$: this is due to the prefactor $N^{1/2}$ in the arguments of the metric d_{-j}, which balances out the natural order of fluctuations $N^{-1/2}$ of the inner products $\sum_{\alpha=1}^{n_S}(\mathcal{I}_h[\varphi_{k,\alpha}], (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(T_k \wedge T_\oslash))_h$ and $\sum_{\alpha=1}^{n_S}\langle \varphi_{k,\alpha}, (\mu_{\alpha,T_k}^{r_I,N} - \overline{\rho}_\alpha^{r_I})(T_k)\rangle$. The term $\mathrm{Err}_{\mathrm{num}}$ accounts for the intrinsic numerical error of the scheme. The term $\mathrm{Err}_{\oslash}$ accounts for the cases where the stopping time runs short of T. The definition of the stopping time (see Section 8) ensures positivity of $\rho_h^{r_I}$: This is due to fact that $T_{\oslash}$ incorporates an L^∞-bound for $\rho_h^{r_I} - \overline{\rho}_h^{r_I}$, and the fact that the mean-field limit is $\overline{\rho}_h^{r_I}$ is strictly positive by Assumption 3.8 below. Additionally, $T_{\oslash}$ ensures that the fluctuations $\rho_h^{r_I} - \overline{\rho}_h^{r_I}$ roughly stay within the natural regime $N^{-1/2}$. Hence, control of the stopping time is the discrete equivalent of quantifying the mean-field limit convergence.

The additional factors of N^{ϵ} appear since we are only able to control the stopping time if we relax the fluctuation bound from $N^{-1/2}$ to $N^{-1/2+\epsilon}$. The factor N^{κ} stems from the logarithmic scaling of the interaction radius r_I with respect to N; see Assumption 3.11. If r_I is constant and does not scale, N^{κ} can be replaced with a constant depending on *data*.

REMARK 2.4. We will often abbreviate the $\mathbb{R}^K$-valued random variables in (2.3) as

$$N^{1/2}\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle, \qquad N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_{\oslash})\big)_h, \tag{2.5}$$

where $\boldsymbol{T} := [T_1, \dots, T_K] \in [0, T]^K$.

2.1. *Structure of the paper.* The details of the weakly interacting particle systems we consider as well as the relevant discretised Dean–Kawasaki model are given in Section 3. Section 4 gives an informal—yet exhaustive—summary of the most important results needed to prove Theorem 2.1: In particular, Theorem 2.1 is of inductive type, and the small fluctuation error $\mathrm{Err}_{\mathrm{fluct,rel}} \propto N^{2\epsilon+\kappa-j(1/2-2\epsilon)}$ is obtained after j induction steps. Section 5 spells out the structure of one of these induction steps and provides all necessary building blocks (above all, Proposition 5.9 indicates how to quantitatively include the convolutional nonlinearity in the iteration, thus resolving the mismatch with the linear noise covariance). The proof of Theorem 2.1 (i.e., the quantitative performance of all j steps) is finalised in Section 6. The core technical lemmas are deferred to subsequent sections, namely: Quantitative convergence to the mean-field limit (Section 7); exponentially decaying bound for the probability of the stopping time $T_{\oslash}$ coming short of the final time horizon $T > 0$ (Section 8).

Finally, the Appendix contains the following: Regularity estimates for the continuous test functions (Appendix A), regularity estimates for the discretised mean-field limit and the discretised test functions as well as error bounds with respect to their continuous counterparts (Appendix B), explicit construction of a set of admissible initial conditions for the discrete mean-field limit (Appendix C).

3. Setting, notation, and assumptions. Throughout the paper we use C to denote a generic constant whose value may change from line to line. Relevant dependencies on specific parameters are highlighted whenever needed. Moreover, for generic functions $f, g\colon \mathbb{T}^d \to \mathbb{R}^m$ ($m \in \mathbb{N}$) and $s\colon \mathbb{T}^d \to \mathbb{R}$, we denote $(f * s)(x) := [\int_{\mathbb{T}^d} f_\ell(y)s(x-y)\,\mathrm{d}y]_{\ell=1}^m$ and $(f *_c g)(x) := \sum_{\ell=1}^m (f_\ell * g_\ell)(x)$.

We now give specific notation and relevant assumptions for the weakly interacting particle system we consider and its Dean–Kawasaki approximation.

3.1. *The continuous setting—the particle system.* The weakly interacting particle system we are interested in is given by

$$\begin{cases} \mathrm{d}X_{\alpha,i}^{r_I,N}(t) = -\displaystyle\sum_{\beta=1}^{n_S} N^{-1} \sum_{j=1}^{N} \nabla V_{\alpha\beta}^{r_I}\big(X_{\alpha,i}^{r_I,N}(t) - X_{\beta,j}^{r_I,N}(t)\big)\,\mathrm{d}t + \sqrt{2\sigma_\alpha}\,\mathrm{d}B_{\alpha,i}(t), \\ X_{\alpha,i}^{r_I,N}(0) = \eta_{\alpha,i}, \quad \alpha = 1, \dots, n_S, i = 1, \dots, N. \end{cases} \tag{3.1}$$

In (3.1), $\{X_{\alpha,i}^{r_I,N}\}_{i=1}^N \subset \mathbb{T}^d$ denote positions of particles of species α, $\{\sigma_\alpha\}_{\alpha=1}^{n_S} > 0$ are diffusion constants, $B_{\alpha,i}$ are independent Brownian motions (also independent of $\{X_{\alpha,i}^{r_I,N}(0)\}_{\alpha,i}$),

$\{\eta_{\alpha,i}\}_{i=1}^N$ are the particles' initial positions, and the potentials $\{V_{\alpha\beta}^{r_I}\}_{\alpha,\beta=1}^{ns}$ are defined as standard mass-preserving rescaling of smooth potentials $V_{\alpha\beta}$, namely,

$$V_{\alpha\beta}^{r_I}(\cdot) = {r_I}^{-d} V_{\alpha\beta}(\cdot/r_I). \tag{3.2}$$

We refer to Assumption 3.7 below for the regularity of the potentials $\{V_{\alpha\beta}^{r_I}\}_{\alpha,\beta=1}^{ns}$ and the law of the particles' initial positions $\{\eta_{\alpha,i}\}_{i=1}^N$.

For each species α, we define the empirical measure

$$\mu_{\alpha,t}^{r_I,N} := N^{-1}\sum_{i=1}^N \delta_{X_{\alpha,i}^{r_I,N}(t)}. \tag{3.3}$$

In the limit $N\to\infty$, the empirical densities (3.3) converge almost surely to the deterministic limit $\overline{\rho}_\alpha^{r_I}$ satisfying the PDE

$$\begin{cases} \partial_t \overline{\rho}_\alpha^{r_I} = \sigma_\alpha \Delta \overline{\rho}_\alpha^{r_I} + \nabla\cdot\left(\displaystyle\sum_{\beta=1}^{ns} \overline{\rho}_\alpha^{r_I} \nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\right), \\ \overline{\rho}_\alpha^{r_I}(0) = \overline{\rho}_\alpha^0, \end{cases} \tag{3.4}$$

where $\overline{\rho}_\alpha^0$ is a suitable deterministic approximation of the initial particle distribution; see Assumption 3.7.

Finally, the fluctuating hydrodynamics Dean–Kawasaki equation capturing the fluctuations of the particle system (3.1) on top of the mean-field limit (3.4) is precisely (1.9), the SPDE of interest for this work.

REMARK 3.1. As already mentioned, we view the current work in the weakly interacting particle setting (which translates in the mollified potentials $V_{\alpha\beta}^{r_I}$ via the parameter $r_I>0$) as laying the ground for future applications to the purely local cross-diffusion case (i.e., considering $r_I\to 0$).

3.2. *Discretisation of the Dean–Kawasaki model.* We work with the uniformly spaced grid on the d-dimensional torus $\mathbb{T}^d := [-\pi,\pi)^d$. Specifically, for $L\in 2\mathbb{N}$, we define the spatial discretisation parameter $h := 2\pi/L$ and set $G_{h,d} := h\mathbb{Z}^d \cap \mathbb{T}^d = \{-\pi, -\pi+h, \dots, \pi-h\}^d$. For $m\in\mathbb{N}$, we endow the space $[L^2(G_{h,d})]^m$ with the standard inner product

$$(u_h, v_h)_h := \sum_{x\in G_{h,d}} h^d u_h(x)\cdot v_h(x),$$

and the orthonormal basis $f_{x,\ell}^m(y) := h^{-d/2}\delta_{x,y} e_\ell$ for $(x,\ell)\in(G_{h,d},\{1,\dots,m\})$, where e_ℓ is the ℓth vector of the $\mathbb{R}^d$ canonical basis. If there is no ambiguity, the notation is simplified as $f_{x,\ell}^m \equiv f_x$. The natural discrete analogue of the continuous convolution operator $*$ (resp., $*_c$, see beginning of Section 3) with respect to the $L^2(G_{h,d})$-inner product is denoted by $*_h$ (resp., by $*_{c,h}$). Furthermore, we denote by $\mathcal{I}_h$ the interpolation operator of continuous functions onto $[L^2(G_{h,d})]^m$, meaning that $\mathcal{I}_h f(x) = f(x)$ for every $x\in G_{h,d}$.

As for the discrete differential operators, we use: (i) a discrete gradient ∇_h and divergence $\nabla_h\cdot$ based on suitable first-order discrete partial derivatives $[\partial_{h,x_1},\dots,\partial_{h,x_d}]$, and (ii) second-order discrete derivates D_{h,x_ℓ}^2 satisfying a standard integration by parts rule

$$\left(D_{h,x_\ell}^2 u_h, v_h\right)_h = -(D_{h,x_\ell} u_h, D_{h,x_\ell} v_h)_h \tag{3.5}$$

for some other first-order operators D_{h,x_ℓ}. We denote $\Delta_h := \sum_{\ell=1}^d D_{h,x_\ell}^2$ and $\nabla_{h,D} := [D_{h,x_1},\dots,D_{h,x_d}]$. In general, D_{h,x_ℓ} may differ from ∂_{h,x_ℓ}.

We can now define the discretized Dean–Kawasaki model.

DEFINITION 3.2 (Finite difference Dean–Kawasaki model of order $p+1$). We say that the $L^2(G_{h,d})$-valued processes $(\rho_{h,\alpha}^{r_I})_{\alpha=1,\dots n_S}$ solve the finite difference Dean–Kawasaki model if they solve the system of stochastic differential equations

$$\text{(h-DK)} \quad \begin{cases} \mathrm{d}(\rho_{h,\alpha}^{r_I}, f_x)_h = \left[\sigma_\alpha(\Delta_h \rho_{h,\alpha}^{r_I}, f_x)_h - \sum_{\beta=1}^{n_S}(\rho_{h,\alpha}^{r_I}(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \rho_{h,\beta}^{r_I}), \nabla_h f_x)_h\right]\mathrm{d}t \\ \qquad - \sqrt{2\sigma_\alpha}N^{-1/2} \sum_{\substack{y\in G_{h,d} \\ l\in\{1,\dots,d\}}} \left(\sqrt{(\rho_{h,\alpha}^{r_I})^+} f_{y,l}^d, \nabla_h f_x\right)_h \mathrm{d}W_{(y,l)}^\alpha, \quad \forall x, \\ \rho_{h,\alpha}^{r_I}(0) = \rho_{h,\alpha}^0 \end{cases}$$

on a finite-time horizon $T > 0$, where $(f_x)_{x\in G_{h,d}}$ is the basis from above and where $\{W_{(y,l)}^\alpha\}_{(y,l)\in G_{h,d}\times\{1,\dots,d\}}^{\alpha=1,\dots,n_S}$ are independent Brownian motions. The assumptions on the random initial datum $\{\rho_{h,\alpha}^0\}_{\alpha=1,\dots,n_S}$ will be given in Assumption 3.10 below. Moreover, the families $\{W_{(y,l)}^\alpha\}_{(y,l)\in G_{h,d}\times\{1,\dots,d\}}^{\alpha=1,\dots,n_S}$ and $\{\rho_{h,\alpha}^0\}_{\alpha=1,\dots,n_S}$ are independent.

Analogously to the continuous case, the model (h-DK) captures the fluctuations around the following discretised mean-field limit.

DEFINITION 3.3 (Mean-field limit for (h-DK)). We say that the $L^2(G_{h,d})$-valued functions $(\overline{\rho}_{h,\alpha}^{r_I})_{\alpha=1,\dots n_S}$ solve the discrete version of the mean-field limit of (3.1) if they solve the system of differential equations

$$\text{(h-MFL)} \quad \begin{cases} \partial_t \overline{\rho}_{h,\alpha}^{r_I} = \sigma_\alpha \Delta_h \overline{\rho}_{h,\alpha}^{r_I} + \nabla_h \cdot \left(\overline{\rho}_{h,\alpha}^{r_I} \sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \overline{\rho}_{h,\beta}^{r_I}\right) & \text{on } G_{h,d}\times(0,T), \\ \overline{\rho}_{h,\alpha}^{r_I}(0) = \overline{\rho}_{h,\alpha}^0, \quad \alpha = 1,\dots,n_S, \end{cases}$$

for given deterministic initial datum $\overline{\rho}_{h,\alpha}^0$; see Assumption 3.10 and Remark 3.13.

3.3. *Relevant functions spaces.*

3.3.1. *The spaces $\mathcal{L}_{\mathrm{pow},r}^q$ of functions with polynomially growing derivatives.* While the definition of the metric d_{-j} takes the supremum over "generalised moment functions" $\psi \in W^{j,\infty}\cap C^j(\mathbb{R}^K,\mathbb{R})$, the natural spaces for such functions will turn out to be

$$\mathcal{L}_{\mathrm{pow},r}^q(\mathbb{R}^K) := \left\{\psi \in C^q(\mathbb{R}^K); \|\psi\|_{\mathcal{L}_{\mathrm{pow},r}^q} := \max_{0\le\tilde{q}\le q} \left\|(1+|\cdot|^2)^{-r/2} D^{\tilde{q}}\psi\right\|_{L^\infty} < \infty\right\}$$

for $K\in\mathbb{N}$, $q,r\in\mathbb{N}_0$. In particular, it holds $W^{j,\infty}\cap C^j = \mathcal{L}_{\mathrm{pow},0}^j$.

3.3.2. *The discrete Sobolev norms.* We denote the L^2-norm induced by the discrete inner product $(\cdot,\cdot)_h$ on $L^2(G_{h,d})$ by either $\|\cdot\|_{L^2(G_{h,d})}$ or, more succinctly, $\|\cdot\|_{L_h^2}$. Analogously to the continuous setting, we also use the notation $\|g\|_{L^\infty(G_{h,d})} = \|g\|_{L_h^\infty} := \max_{x\in G_{h,d}}|g(x)|$.

We will also need a discrete version of (also negative) Sobolev norms: This version uses the first order one-sided finite differences given by

$$\partial_{h,1}^{e_\ell} g(x) = \frac{g(x+he_\ell) - g(x)}{h} \quad \text{for all } g\in L^2(G_{h,d}), \ell = 1,\dots,d.$$

DEFINITION 3.4 (The discrete Sobolev norms). For $s\in\mathbb{N}_0$ and $g\in L^2(G_{h,d})$, set

$$\|g\|_{H^s(G_{h,d})} = \|g\|_{H_h^s} := \sup_{|\nu|\le s} \left\|\partial_{h,1}^\nu g\right\|_{L^2(G_{h,d})} \tag{3.6}$$

with the supremum over multiindices $\nu\in\mathbb{N}_0^d$.

REMARK 3.5. The discrete H^s-norms can be equivalently characterized via the discrete Fourier basis: Let $(\vartheta_m)_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \subset L^2(G_{h,d})$ with $\vartheta_m(x) := (2\pi)^{-d/2}e^{\mathrm{i}m\cdot x}$. Then

$$\|g\|^2_{H^s(G_{h,d})} \approx \sum_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \left(1+|m|_2^2\right)^s (g,\vartheta_m)_h^2.$$

This stems from the observation that, due to the discrete version of the Plancherel theorem,

$$\|\partial^{e_\ell}_{h,1} g\|_{L^2(G_{h,d})} = \sum_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \frac{1}{h^2}\left|e^{\mathrm{i}m_\ell h}-1\right|^2 (g,\vartheta_m)_h^2.$$

This remark leads to a natural definition of the discrete negative Sobolev norms.

DEFINITION 3.6 (Discrete negative Sobolev norms). For $\mathbb{Z} \ni s < 0$ and $g \in G_{h,d}$, we set

$$\|g\|^2_{H^s(G_{h,d})} := \sum_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \left(1+|m|_2^2\right)^s (g,\vartheta_m)_h^2,$$

where $(\vartheta_m)_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \subset L^2(G_{h,d})$ with $\vartheta_m(x) := e^{\mathrm{i}m\cdot x}$ is the Fourier basis of $L^2(G_{h,d})$.

3.4. *Assumptions*.

ASSUMPTION 3.7 (Weakly interacting particle system and associated mean-field limit). We consider the weakly interacting particle system $\{X^{r_I,N}_{\alpha,i}\}^{i=1,\ldots,N}_{\alpha=1,\ldots,n_S}$, as given in (3.1), and its associated mean-field limit (3.4). In terms of regularity, we assume that

$$V_{\alpha\beta} \in C^{p+3}(\mathbb{T}^d), \quad \forall \alpha,\beta = 1,\ldots,n_S$$

and, furthermore, that $V_{\alpha\beta}$ is symmetric (i.e., $V_{\alpha\beta}(x) = V_{\alpha\beta}(-x)$). This regularity is passed on to the rescaled potentials $V^{r_I}_{\alpha\beta}$.

For the initial values of the particles, we assume that either they are i.i.d. according to some probability distribution or that they satisfy a spectral gap inequality in the sense that for any $F \in C^1((\mathbb{R}^d)^N)$ and all $\alpha = 1,\ldots,n_S$ there holds

$$\mathbb{E}\left[\left|F\left(\left(X^{r_I,N}_{\alpha,i}(0)\right)_i\right) - \mathbb{E}\left[F\left(\left(X^{r_I,N}_{\alpha,i}(0)\right)_i\right)\right]\right|^2\right] \le C\mathbb{E}\left[\left|\nabla F\left(\left(X^{r_I,N}_{\alpha,i}(0)\right)_i\right)\right|^2\right]. \tag{3.7}$$

For the initial value $\overline{\rho}^{r_I}(0) = \overline{\rho}^0$ of the mean field limit, we assume that

$$\left\|\mathbb{E}\left[\mu_0^{r_I,N}\right] - \overline{\rho}^0\right\|_{H^{-d/2-2}} \le CN^{-1/2}. \tag{3.8}$$

ASSUMPTION 3.8 (Continuous mean field limit). The solution $\{\overline{\rho}^{r_I}_\alpha\}^{n_S}_{\alpha=1}$ to (3.4) exists and belongs to $L^\infty(0,T;C^{p+3}(\mathbb{T}^d,\mathbb{R}^{n_S}))$. Furthermore, we assume that there exist $\rho_{\min},\rho_{\max} \in \mathbb{R}$ such that on $[0,T]$ it holds that $0 < \rho_{\min} \le \overline{\rho}^{r_I}_\alpha \le \rho_{\max}$ for all $\alpha = 1,\ldots,n_S$.

ASSUMPTION 3.9 (Discrete differential operators). Let $p \in \mathbb{N}$ be fixed. The discrete operators $\partial_{h,x_\ell}, D_{h,x_\ell}, D^2_{h,x_\ell}$ introduced in Section 3.2 are standard finite difference operators of order $p+1$ (meaning $|\partial_{h,x_\ell}\varphi - \partial_{x_\ell}\varphi| \le \|\varphi\|_{C^{p+2}}h^{p+1}$, and similarly for the other operators). In particular, being of finite difference type, these operators commute. Finally, the two first order differential operators $\partial_{h,x_\ell}, D_{h,x_\ell}$ satisfy the inequality

$$\|\partial^R_{h,x_\ell} g\|^2_{L^2(G_{h,d})} \le \frac{1}{C_D}\|D_{h,x_\ell} g\|^2_{L^2(G_{h,d})}, \quad \forall g \in L^2(G_{h,d}), \tag{3.9}$$

for some $C_D > 0$, where ∂^R_{h,x_ℓ} is the reflected version of ∂_{h,x_ℓ} (i.e., its adjoint), which appears when integrating by parts in the discrete setting.

ASSUMPTION 3.10 (Discrete initial conditions). The initial conditions $\{\rho^0_{h,\alpha}\}_\alpha$ for (h-DK) and $\{\overline{\rho}^0_{h,\alpha}\}_\alpha$ for (h-MFL) are chosen such that the following properties hold:

- *Initialization via interpolation:* We set $\overline{\rho}^0_h := \mathcal{I}_h[\overline{\rho}^{r_I}(0)] = \mathcal{I}_h[\overline{\rho}^0]$.
- *Positivity and mass restriction:* We assume that the random discrete initial data satisfies $\rho^0_{h,\alpha} \geq 0$ and $\|\rho^0_{h,\alpha}\|_{L^1_h} = h^d \sum_{x\in G_{h,d}} |\rho^0_{h,\alpha}(x)| \leq 2$ for all $\alpha = 1, \dots, n_S$.
- *High-order fluctuation bound:* For any $K, q \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $\psi \in \mathcal{L}^q_{\mathrm{pow},r}(\mathbb{R}^K)$, the bound

$$\begin{aligned} &\big|\mathbb{E}\big[\psi\big(N^{1/2}\langle \boldsymbol{\varphi}, \mu^{r_I,N}_0 - \overline{\rho}^{r_I}(0)\rangle\big)\big] \\ &\quad - \mathbb{E}\big[\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], \big(\rho^{r_I}_h - \overline{\rho}^{r_I}_h\big)\big)_h(0)\big)\big]\big| \leq C\|\psi\|_{\mathcal{L}^q_{\mathrm{pow},r}} h^{p+1}\|\boldsymbol{\varphi}\|^{r+1}_{C^{p+1}} \end{aligned} \tag{3.10}$$

  holds for any $\boldsymbol{\varphi} \in [C^{p+1}(\mathbb{T}^d, \mathbb{R}^{n_S})]^K$. In (3.10) we have used the vectorial notation convention specified in Remark 2.4.
- *Exponentially decaying bound for probability of observing "large fluctuations:"* We assume that for any $1 \leq R \leq N^{1/2}h^{d/2}$ it holds that

$$\mathbb{P}\big[\|\rho^0_h - \overline{\rho}^0_h\|_{L^\infty_h} \geq N^{-1/2}h^{-d/2}R\big] \leq C\exp(-CR), \tag{3.11}$$

$$\mathbb{P}\big[\|\rho^0_h - \overline{\rho}^0_h\|_{H_h^{-\lfloor d/2+1\rfloor}} \geq N^{-1/2}R\big] \leq C\exp(-CR). \tag{3.12}$$

ASSUMPTION 3.11 (Parameter scaling). The parameters (N, h, r_I) scale with respect to the following relations: For a chosen, arbitrarily small $0 < \delta_0 < 1$, we assume that

$$N^{1-\delta_0}h^d \geq 1, \tag{3.13}$$

$$N^{\delta_0(T+1)}h \leq 1, \tag{3.14}$$

$$r_I^{-2(d+2)} \leq \log N. \tag{3.15}$$

The scaling regime (3.15) is analogous to that of [9], Theorem 3.

REMARK 3.12 (Ad Assumptions 3.7–3.8). The regularity of $\{\overline{\rho}^{r_I}_\alpha\}_{\alpha=1}^{n_S}$ prescribed in Assumption 3.8 can be met on any time interval $[0, T]$, for instance, if the initial value for the mean-field limit $\overline{\rho}^0_\alpha$ belongs to H^s, $s > \max\{d/2 + 1; p + 3 + (d/2 + 1)\}$ and satisfies a smallness assumption $\|\overline{\rho}^0_\alpha\|_{H^s} \leq \{\min_\alpha \sigma_\alpha\}/[C(s,d)\sum_{\alpha,\beta}\|V_{\alpha\beta}\|_{L^1}]$ [10]; see also [9] for an alternative setting.

REMARK 3.13 (Ad Assumption 3.10). Since $\overline{\rho}^{r_I}_h(0) = \mathcal{I}_h[\overline{\rho}^{r_I}(0)]$, for order $p + 1$ finite difference operators $(\partial^{e_\ell}_h)_{\ell=1,\dots,d}$, we have

$$\sup_{|\nu|\leq s} \big\|\mathcal{I}_h\big[\partial^\nu \overline{\rho}^{r_I}(0)\big] - \partial^\nu_h \overline{\rho}^{r_I}_h(0)\big\|_{L^2_h} \leq C\big\|\overline{\rho}^{r_I}(0)\big\|_{C^{s+p+2}} h^{p+1}, \tag{3.16}$$

where the supremum is over multiindices $\nu \in \mathbb{N}^d_0$ provided $\overline{\rho}^{r_I}_h(0)$ is regular enough.

REMARK 3.14 (Positivity and Boundedness of the discrete mean field limit). Analogously to the continuous Sobolev inequality, in the discrete setting there exists $C_S > 0$ such that $\|g\|_{L^\infty_h} \leq C_S\|g\|_{H^s}$ for all $s > d/2$, $g \in L^2(G_{h,d})$. In particular, in light of Proposition B.7, as long as $\overline{\rho}^{r_I} \in L^\infty(0, T; C^{p+3+\lfloor d/2\rfloor+1})$, there exist $\rho_{\min,h}$, $\rho_{\max,h}$ such that for all $N > N_0$ and $h < h_0$ it holds that on $[0, T]$

$$0 < \rho_{\min,h} \leq \overline{\rho}^{r_I}_{h,\alpha} \leq \rho_{\max,h} \quad \text{for all } \alpha = 1, \dots, n_S,$$

for some $N_0 = N_0(p, \text{data})$, where $h_0 = h_0(\|\overline{\rho}^{r_I}\|_{C^{\lfloor d/2\rfloor+2}}, \rho_{\min})$.

REMARK 3.15 (Constructing the discrete initial data). The assumptions for the initial random distribution ρ_h^0 (Assumption 3.10) may seem extensive at first glance, but such discrete initial data can be rather naturally constructed from the continuous initial data. The most straightforward example is a pathwise approach based on an interpolation scheme—each realisation of ρ_h^0 is derived from the initial particle realisation $\mu_0^{r_I,N}$. Essentially, the mass of a particle starting at $X_{\alpha,i}^{r_I,N}(0)$ is split across surrounding grid points according to interpolation weights. These weights stem from an interpolation scheme of sufficiently high order that is used to approximate functions at $X_{\alpha,i}^{r_I,N}(0)$ based on the function values at the grid points.

4. Strategy of the proof of Theorem 2.1: An informal view. In this section we spell out the main ideas behind the proof of Theorem 2.1 in an informal way. All arguments will be made precise and rigorous later on.

4.1. *The induction step.* The proof of Theorem 2.1 is of inductive type, as we now detail.

In order the compare the fluctuations of the discrete Dean–Kawasaki solution ρ_h (see (h-DK)) and of the particle empirical density $\mu^{r_I,N}$ (see (3.3)), we choose a set of times $\boldsymbol{T} = (T_k)_{k=1}^K$, regular enough test functions $\boldsymbol{\varphi} = (\varphi_{k,\alpha})_{k=1,\dots,K}^{\alpha=1,\dots,n_S}$ and set

$$\begin{aligned} \boldsymbol{\zeta}^T &:= N^{1/2}\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle \in \mathbb{R}^K, \\ \boldsymbol{\zeta}_h^T &:= N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_\oslash)\big)_h \in \mathbb{R}^K, \end{aligned} \tag{4.1}$$

which is a shorthand vectorial notation (over the indexes $k = 1, \dots, K$ and $\alpha = 1, \dots, n_S$) for the random variables in (2.3), as anticipated in Remark 2.4. The role and definition of the stopping time $T_\oslash$ will be discussed in due course.

We measure the distance between $\boldsymbol{\zeta}^T$ and $\boldsymbol{\zeta}_h^T$ using the Wasserstein-type metric d_{-j}-metric (2.1), namely,

$$d_{-j}[\boldsymbol{\zeta}^T, \boldsymbol{\zeta}_h^T] := \sup_{\psi:\ \max_{1\le \tilde{j}\le j} \|D^{\tilde{j}}\psi\|_{L^\infty}\le 1} \big|\mathbb{E}[\psi(\boldsymbol{\zeta}^T)] - \mathbb{E}[\psi(\boldsymbol{\zeta}_h^T)]\big|.$$

Furthermore, we define the shorthand notation[1]

$$\mathcal{M}(\psi, \boldsymbol{\varphi}) := \mathbb{E}[\psi(\boldsymbol{\zeta}^T)] - \mathbb{E}[\psi(\boldsymbol{\zeta}_h^T)]. \tag{4.2}$$

The proof of Theorem 2.1 crucially revolves around essentially obtaining the following relation via the Itô calculus:

$$\mathbb{E}\big[\mathrm{d}\mathcal{M}(\psi, \boldsymbol{\varphi})\big] \propto N^{-1/2+2\epsilon} \cdot \mathbb{E}\big[\mathcal{M}(\tilde{\psi}, \tilde{\boldsymbol{\varphi}})\big]\,\mathrm{d}t + \mathrm{dErr}_\oslash + \mathrm{dErr}_{\mathrm{num}}, \tag{4.3}$$

where the parameter $\epsilon > 0$ is small (so that $2\epsilon < 1/2$). The bound (4.3) has an iterative component, as the right-hand-side contains—yet—another object of kind $\mathcal{M}$, with modified test functions $(\tilde{\psi}, \tilde{\boldsymbol{\varphi}})$ derived from the original pair $(\psi, \boldsymbol{\varphi})$. Additionally, there are a numerical error $\mathrm{Err}_{\mathrm{num}}$, which will show to be of the type

$$\mathrm{Err}_{\mathrm{num}} \propto N^{2\epsilon+\kappa} h^{p+1}, \tag{4.4}$$

and a modelling error

$$\mathrm{Err}_\oslash \propto N^{2\epsilon+\kappa} \exp(-CN^{\epsilon/2}), \tag{4.5}$$

where, again, the parameter $\kappa > 0$ is small enough. The role of the small parameters ϵ, κ will be clarified throughout the proofs, and we need not discuss it in this summary.

[1] We use a slight abuse of notation, as $\psi(\boldsymbol{\zeta}^T)$ and $\psi(\boldsymbol{\zeta}_h^T)$ live in different probability spaces.

By iterating (4.3) over $\mathcal{M}$, one cumulates as many (small) prefactors $N^{-1/2+2\epsilon}$ as steps performed, together with numerical and modelling errors (4.4)–(4.5). The number of iterative steps one performs is only capped by the regularity of the initial test functions ψ, $\boldsymbol{\varphi}$ (such regularity usually deteriorates from step to step). Once the regularity of the test functions is exhausted and (4.3) is inapplicable, one closes off the argument by obtaining the—final—fluctuation contribution in (2.4). The precise details concerning (4.3)–(4.5) are given in Theorem 5.1, which, in turn, relies on several other ingredients which we now list.

4.1.1. *Choice of dynamics for test functions $\boldsymbol{\varphi}$ and $\mathcal{I}_h\boldsymbol{\varphi}$.* The particle system (3.1) satisfies a crucial property: as the particles are driven by independent Brownian motions, it is easy to see that the cross-variation (denoted by square brackets) of the quantities $\langle \varphi_1, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle$ and $\langle \varphi_2, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle$ (these are the "building blocks" for $\boldsymbol{\zeta}^T$ in (4.1)) is, for sufficiently regular φ_1, φ_2, given by

$$
\begin{aligned}
&\big[\langle \varphi_1, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle, \langle \varphi_2, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle\big]\\
&\quad = N^{-1}\langle \nabla\varphi_1 \cdot \nabla\varphi_2, \mu^{r_I,N}\rangle \\
&\quad = N^{-1}\langle \nabla\varphi_1 \cdot \nabla\varphi_2, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle + N^{-1}\langle \nabla\varphi_1 \cdot \nabla\varphi_2, \overline{\rho}^{r_I}\rangle =: P_1 + P_2.
\end{aligned} \tag{4.6}
$$

With the exception of term P_2 (which we will deal with at a later stage), (4.6) amounts to saying that the cross-variation preserves linear functionals of $\mu^{r_I,N} - \overline{\rho}^{r_I}$: this fact is crucial, as it plays directly into the iterative structure of $\mathcal{M}$ in (4.2).

In light of (4.6), it is convenient to define time-dependent test functions $\boldsymbol{\phi}$ which:

- coincide with the original test functions $\boldsymbol{\varphi}$ at the evaluation times $\boldsymbol{T}$ (i.e., $\boldsymbol{\phi}^{\boldsymbol{T}} = \boldsymbol{\varphi}$) and
- reduce the deterministic drift of the Itô differential for $\boldsymbol{\zeta}^T$ in (4.1) "as much as possible," as such deterministic drift cannot easily be treated using (4.6).

The latter requirement leads to the derivation of the backward evolution for $\boldsymbol{\phi}$ in Lemma 5.2. Noticeably, the only deterministic drift term which survives this cancellation effort is

$$
Q_t^{r_I,N}(\phi^t) := \sum_{\alpha,\beta=1}^{n_S} \big\langle \nabla\phi_\alpha^t \cdot \big(\nabla V_{\alpha\beta}^{r_I} * (\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t))\big), \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\big\rangle. \tag{4.7}
$$

The structure of $Q^{r_I,N}$, which appears as a compensation term after linearising the convolution nonlinearity, is—at least not yet—compatible with the structure of $\mathcal{M}$, and will be dealt with in Section 4.1.4 below.

A totally analogous discussion also applies for suitable discretisations of (1.9): Being statistically equivalent to the particle system (3.1), the Dean–Kawasaki model (1.9) enjoys the "mesoscopic analogue" of (4.6). Namely, testing (1.9) with smooth enough φ_1, φ_2 and integrating by parts, one finds the noise cross-variation to be (1.10), which we recall here,

$$
\Big[N^{-1/2}\int_{\mathbb{T}^d} \sqrt{\rho_\alpha}\xi \cdot \nabla\varphi_1, N^{-1/2}\int_{\mathbb{T}^d} \sqrt{\rho_\alpha}\xi \cdot \nabla\varphi_2\Big] = N^{-1}\int_{\mathbb{T}^d} \rho_\alpha \nabla\varphi_1 \cdot \nabla\varphi_2.
$$

Our chosen discretisation for the Dean–Kawasaki model (h-DK) preserves (1.10) on the discrete level: Namely, for any $\alpha = 1, \dots, n_S$ and $\varphi_{1,h}, \varphi_{2,h} \in L^2(G_{h,d})$, we have

$$
\begin{aligned}
&\big[(\varphi_{1,h}, \rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})_h, (\varphi_{2,h}, \rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})_h\big]\\
&\quad = \Big[N^{-1/2} \sum_{y\in G_{h,d}, l\in\{1,\dots,d\}} \Big(\sqrt{(\rho_{h,\alpha}^{r_I})^+} f_{y,l}^d, \nabla_h\varphi_{1,h}\Big)_h \mathrm{d}W_{(y,l)}^\alpha,\\
&\qquad N^{-1/2} \sum_{y\in G_{h,d}, l\in\{1,\dots,d\}} \Big(\sqrt{(\rho_{h,\alpha}^{r_I})^+} f_{y,l}^d, \nabla_h\varphi_{2,h}\Big)_h \mathrm{d}W_{(y,l)}^\alpha\Big]
\end{aligned} \tag{4.8}
$$

$$= N^{-1}(\nabla_h\varphi_{1,h}\cdot\nabla_h\varphi_{2,h}, \rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})_h + N^{-1}(\nabla_h\varphi_{1,h}\cdot\nabla_h\varphi_{2,h}, \overline{\rho}^{r_I}_{h,\alpha})_h$$
$$+ N^{-1}(\nabla_h\varphi_{1,h}\cdot\nabla_h\varphi_{2,h}, (\rho^{r_I}_{h,\alpha})^-)_h =: M_1 + M_2 + M_3.$$

With the exception of the terms M_2 and M_3 (which will be treated in Sections 4.1.2 and 4.1.3), (4.8) shows that the cross-variation preserves linear functionals of $\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha}$. In the same way as in the continuous setting, we define a suitable backward evolution $\boldsymbol{\phi}_h$ for $\boldsymbol{\phi}_h^{\boldsymbol{T}} = \mathcal{I}_h\boldsymbol{\varphi}$; see Lemma 5.4. Taking all species and test functions into account in the definition of $\boldsymbol{\zeta}_h^T$ in (4.1), the only deterministic drift term which survives is given by

$$Q^{r_I,N}_{h,t}(\phi^t_h) := \sum_{\alpha,\beta=1}^{n_S} (\nabla_h\phi^t_{h,\alpha}\cdot(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t)), (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t))_h. \tag{4.9}$$

This term, which is the discrete analogous of $Q^{r_I,N}_t(\phi^t)$ above and which also does not yet fit the structure of $\mathcal{M}$, is treated in Section 4.1.4 below.

4.1.2. *The generalised moment structure* ψ. The Itô analysis conducted in Section 4.1.1 shows that—aside from the mean-field contributions P_2, M_2 and the mismatch $\rho^+_{h,\alpha} \leftrightarrow \rho_{h,\alpha}$ in M_3—the iterative structure in $\mathcal{M}$ is preserved at the level of noise cross-variations. In order to include the mean-field contributions P_2, M_2, we also need to define a suitable backward equation for the generalised moment function ψ: This analysis (see Lemma 5.5) is somewhat complementary to the derivation of the dynamics for the test functions $\boldsymbol{\phi}$ discussed earlier. The discussion is tailored to the continuous case associated with the particle system, within the adjustments needed for the discrete case treated in Lemma 5.6.

4.1.3. *Comparing the generalised moments*, *the iterative structure*. In this step we take the difference of the contributions of the Itô differential on microscopic and SPDE level (discussed in the Sections 4.1.1 and 4.1.2), and we derive the bound

$$\begin{aligned}\mathbb{E}[d\mathcal{M}(\psi^t,\boldsymbol{\phi}^t)] \propto N^{-1/2+2\epsilon}\cdot\mathbb{E}[\mathcal{M}(\widetilde{\psi}^t,\widetilde{\boldsymbol{\phi}}^t)]\,dt \\ + \mathrm{dErr}_{\mathrm{num}} + \mathrm{dErr}_{\oslash} + \mathrm{d}(\mathrm{Err}_{\mathrm{lin},a} - \mathrm{Err}_{\mathrm{lin},b}).\end{aligned} \tag{4.10}$$

The terms $\mathrm{Err}_{\mathrm{lin},a}$, $\mathrm{Err}_{\mathrm{lin},b}$ compensate for linearising the convolutional nonlinearity. Note, however, that in this form they are not yet suitable for the purposes of iterating. Estimating them without further treatment would yield an order $N^{-1/2}$ error: They are roughly given by

$$\mathrm{dErr}_{\mathrm{lin},a} \propto N^{1/2}\mathbb{E}[Q^{r_I,N}_t(\boldsymbol{\phi}^t)]\,\mathrm{d}t,$$
$$\mathrm{dErr}_{\mathrm{lin},b} \propto N^{1/2}\mathbb{E}[Q^{r_I,N}_{h,t\wedge T_\oslash}(\boldsymbol{\phi}^t_h)]\,\mathrm{d}t,$$

where $Q^{r_I,N}_t$ and $Q^{r_I,N}_{h,t}$ are the surviving drift terms from Section 4.1.1, as defined in (4.7) and (4.9), respectively. Crucially, in (4.10) the mismatch $\rho_h \leftrightarrow \rho_h^+$ (cfr. M_3 in (4.8)) has been resolved using the stopping time $T_\oslash$: the definition of such a stopping time and Remark 3.14 entail the nonnegativity of $\rho_h(t)$ for all $t \le T_\oslash$. A suitable estimate concerning the smallness of $P[T_\oslash < T]$ (i.e., (2.2)) is proved separately in Proposition 8.2 and justifies the bound (4.5). Additionally, the error term $\mathrm{Err}_{\mathrm{num}}$ in (4.10) keeps track of several numerical approximations, including:

- difference of continuous test functions $\boldsymbol{\phi}$ and discrete counterparts $\boldsymbol{\phi}_h$,
- difference of continuous mean-field limit $\overline{\rho}^{r_I}$ and discrete counterpart $\overline{\rho}^{r_I}_h$, and
- difference of initial conditions between the particle system (3.1) and the discrete Dean–Kawasaki model (h-DK), see Assumption 3.10 and Appendixes A–C.

The argument for this subsection is spelled out in Proposition 5.7.

4.1.4. *Linearising the convolution contributions.* Here we linearise the terms $\mathrm{Err}_{\mathrm{lin},a}$, $\mathrm{Err}_{\mathrm{lin},b}$ introduced above in order to make them suitable for the iteration. The linearisation is carried our using a $2d$-Fourier argument, which is proved in a separate result (Proposition 5.9), and discussed in more detail in Section 4.2.2 below.

4.1.5. *Closing the estimate.* All arguments carried out so far are used to cumulate as many iteration steps as possible (depending on the initial regularity of the initial test functions $\boldsymbol{\varphi}$, ψ). When the iteration can no longer be performed, the remaining terms (other than $\mathrm{Err}_{\mathrm{num}}$) are bounded using quantitative convergence bounds to the mean-field limit (proven in Section 7) and the properties of the stopping time $T_{\oslash}$ (proven in Section 8).

4.2. *Technical challenges.* We highlight the four main technical challenges, which we address in order to achieve the proof's building blocks sketched in Sections 4.1.1–4.1.5.

4.2.1. *Quantitative convergence to the mean-field limit.* In several points of the argument, we need a quantitative bound on the convergence to the mean-field limit in both continuous and discrete setting. Specifically, in Proposition 7.1 we prove bounds of the type

$$\begin{aligned}&\|\mu_{\alpha,t}^{r_I,N} - \overline{\rho}_{\alpha}^{r_I}(t)\|_{H^{-d/2-2}}\\&\quad\le C(T+1)\exp\bigl(CTr_I^{-(d+2)}\bigr)\bigl(\|\mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\|_{H^{-d/2-2}} + \mathcal{C}N^{-1/2}\bigr)\end{aligned}$$

for a random variable $\mathcal{C}$ with Gaussian moments $\mathbb{E}[\exp(C\mathcal{C}^2)] \le 3$. For the proof we follow the classical strategy of introducing a coupled auxiliary particle system, where the interaction is calculated with respect to the mean-field limit; see, for example, [67], Theorem 1.4. The corresponding estimate on the discrete level is a by-product of Section 8 below.

4.2.2. *Linearisation argument for convolutional nonlinearity.* Linearising the quadratic terms $Q_t^{r_I,N}(\phi^t)$, $Q_{h,t}^{r_I,N}(\phi_h^t)$ (given in (4.7) and (4.9)) essentially revolves around rewriting them into an infinite sum of suitable objects. If the interaction potential is regular enough, the Fourier expansion allows to rewrite the function $(x, y) \mapsto \partial_\ell V_{\alpha\beta}^{r_I}(x-y)\colon \mathbb{T}^{2d} \to \mathbb{R}$ as $\partial_\ell V_{\alpha\beta}^{r_I}(x-y) = \sum_{m,n\in\mathbb{Z}^d} \hat{F}_{k,\alpha\beta}^{\ell}[m,n]e^{\mathrm{i}m\cdot x}e^{in\cdot y}$, effectively separating the variables x, y in each addend of the sum. We use this to rewrite $Q_t^{r_I,N}(\phi^t)$ as

$$\begin{aligned}Q_t^{r_I,N}(\phi^t) \propto \sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d} \hat{F}_{k,\alpha\beta}^{\ell}[m,n]\Biggl(N^{1/2}\sum_{\gamma=1}^{n_S}\bigl\langle\delta^{\beta\gamma}\vartheta_m, \mu_{\gamma,t}^{r_I,N} - \overline{\rho}_{\gamma}^{r_I}(t)\bigr\rangle\Biggr)\\ \times\Biggl(N^{1/2}\sum_{\gamma=1}^{n_S}\bigl\langle\delta^{\alpha\gamma}\partial_\ell\phi_\alpha^t\vartheta_n, \mu_{\gamma,t}^{r_I,N} - \overline{\rho}_{\gamma}^{r_I}(t)\bigr\rangle\Biggr),\end{aligned}$$

where we have set $\vartheta_n(x) := \cos(n\cdot x)$ and where $\delta^{\cdot,\cdot}$ is the usual Kronecker delta. The above expression is now compatible with the structure of $\mathcal{M}$, as the nonlinearity has been "split"—at the expense of having an infinite sum—in two suitable stand-alone contributions (in the round brackets). Analogous discussions are applicable for $Q_{h,t\wedge T_{\oslash}}^{r_I,N}(\phi_h^t)$.

4.2.3. *Estimate for probability of large fluctuations.* The exponentially decaying estimate (2.2) for $P[T_{\oslash} < T]$ is centered around proving Gaussian type moment bounds for the quantities $\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I}$. For this purpose, two different stopping times are needed: (i) an L^∞-type stopping time, which controls the size of the stochastic noise, and (ii) an $H_h^{-2\lfloor d/2\rfloor-2}$-type stopping time (for which the estimates in Section 4.2.1 are needed), which allows to linearise the contribution of the quadratic convolutional nonlinearity.

4.2.4. *Compatible discrete setting.* A number of auxiliary results are needed in order to substantiate our numerical approximations. The two main difficulties here are related to:

- quantifying the difference of relevant continuous functions (namely, the nonlinear mean-field limit and test functions) and bounding their discretised counterparts in higher-order Sobolev norms, see Appendixes A–B, and
- constructing a compatible initial profile of fluctuations in the case of high order operators (i.e., $p > 1$): we use arguments from polynomial interpolation theory; see Appendix C.

5. The key step—setting up the iterative structure. In this section we formalize the arguments outlined in Section 4.1. That is, we compute and compare the Itô formulas (given by Itô's rule) for

$$\mathbb{E}[\psi(\boldsymbol{\zeta}^T)] = \mathbb{E}[\psi(N^{1/2}\langle\boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle)],$$
$$\mathbb{E}[\psi(\boldsymbol{\zeta}_h^T)] = \mathbb{E}[\psi(N^{1/2}(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_\oslash))_h)].$$

Theorem 5.1 below is the integrated, more precise version of (4.3), that also provides bounds on $\widetilde{\psi}$, $\widetilde{\boldsymbol{\phi}}$, $\mathrm{Err}_{\mathrm{num}}$, and $\mathrm{Err}_{\mathrm{neg}}$. On the first read of this section, the reader may wish to simply focus on the structure of the proofs and the comparison of the various blocks therein and skip the quantitative estimates (such estimates rely on the results from the subsequent Section 7, Section 8, Appendix A, and Appendix B).

THEOREM 5.1 (Iterative Structure). *Let* $0 < T$. *On* $[0, T]$, *let*:

- $\mu^{r_I,N} = (\mu_\alpha^{r_I,N})_{\alpha=1}^{n_S}$ *as in* (3.3) *be the empirical measures of the particle system* (3.1) *satisfying Assumption* 3.7,
- $\overline{\rho}^{r_I}$ *as in* (3.4) *be the intermediate mean-field limit satisfying Assumption* 3.8,
- $\rho_h^{r_I}$ *solve the discretised Dean–Kawasaki model* (h-DK) *satisfying Assumption* 3.9,
- $\overline{\rho}_h^{r_I}$ *be the finite difference mean-field limit given in* (h-MFL).

Let Assumption 3.10 (*concerning the initial conditions*) *hold. Let the parameters* r_I, N, h *obey the scaling in Assumption* 3.11. *Now, let* $K \in \mathbb{N}$, $q, r \in \mathbb{N}_0$, $q \geq 3$, *and* $s \in \mathbb{N}$, *satisfying*

$$s > p + 3d/2 + 5. \tag{5.1}$$

Let $\psi \in \mathcal{L}_{\mathrm{pow},r}^q(\mathbb{R}^K)$, $\boldsymbol{\varphi} = (\varphi_{k,\alpha})_{k=1,\dots,K}^{\alpha=1,\dots,n_S} \in [H^s(\mathbb{T}^d, \mathbb{R}^{n_S})]^K$, $\boldsymbol{T} = (T_1, \dots, T_K) \in [0, T]^K$. *Additionally, assume that* $\overline{\rho}^{r_I} \in L^\infty(0, T; [C^s(\mathbb{T}^d)]^{n_S})$ *and* $V \in W^{s_I,1}(\mathbb{T}^d)$ *for* $s_I \in \mathbb{N}$ *with*

$$s_I > 2d + 1 + (r+2)\lceil d/2 + 1\rceil. \tag{5.2}$$

Let $0 < \epsilon < \delta_0/4$, *and let* $T_\oslash = T_\oslash(\epsilon/2, \epsilon)$ *be the associated stopping time, as defined in* (8.2) *at the start of Section* 8 *for* $\delta = \epsilon/2$.

Then the following facts hold. First, for all $t \in (0, T)$, $k, \tilde{k} \in \{1, \dots, K\}$, $m, n \in \mathbb{Z}^d$, *and* $\alpha, \beta \in \{1, \dots, n_S\}$ *as well as* $\ell \in \{1, \dots, d\}$, *there exist generalised moment functions*

$$\psi^0 \in \mathcal{L}_{\mathrm{pow},r}^q(\mathbb{R}^K), \qquad \widetilde{\psi}_{k\tilde{k}}^t \in \mathcal{L}_{\mathrm{pow},r+1}^{q-2}(\mathbb{R}^{K+1}), \qquad \breve{\psi}_k^t \in \mathcal{L}_{\mathrm{pow},r+2}^{q-1}(\mathbb{R}^{K+2}),$$

and coefficients $\hat{F}_{k,\alpha\beta}^\ell[m,n] \in \mathbb{R}$ *with sets of test functions*

$$\boldsymbol{\phi}^0 \in [H^s(\mathbb{T}^d, \mathbb{R}^{n_S})]^K, \qquad \widetilde{\boldsymbol{\phi}}_{k\tilde{k}}^t \in [H^{s-1}(\mathbb{T}^d, \mathbb{R}^{n_S})]^{K+1},$$
$$\breve{\boldsymbol{\phi}}_{k,mn,\alpha\beta,\ell}^t \in [H^s(\mathbb{T}^d, \mathbb{R}^{n_S})]^{K+2},$$

and test times $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$, $\breve{\boldsymbol{T}}_k \in [0,T]^{K+2}$ *such that*

$$\begin{aligned}
&\mathbb{E}\big[\psi\big(N^{1/2}\langle\boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle\big)\big] \\
&\quad = \mathbb{E}\big[\psi^0\big(N^{1/2}\langle\boldsymbol{\phi}^0, \mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\rangle\big)\big] \\
&\qquad + N^{-1/2}\sum_{k,\tilde{k}=1}^{K}\int_0^{T_k\wedge T_{\tilde{k}}} \mathbb{E}\big[\widetilde{\psi}_{k\tilde{k}}^t\big(N^{1/2}\langle\widetilde{\boldsymbol{\phi}}_{k\tilde{k}}^t, \mu_{t\wedge\tilde{\boldsymbol{T}}_{k\tilde{k}}}^{r_I,N} - \overline{\rho}^{r_I}(t\wedge\tilde{\boldsymbol{T}}_{k\tilde{k}})\rangle\big)\big]\,\mathrm{d}t \qquad (5.3)\\
&\qquad + N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}_{k,\alpha\beta}^{\ell}[m,n] \\
&\qquad \times \mathbb{E}\big[\breve{\psi}_k^t\big(N^{1/2}\langle\breve{\boldsymbol{\phi}}_{k,mn,\alpha\beta,\ell}^t, \mu_{t\wedge\breve{\boldsymbol{T}}_k}^{r_I,N} - \overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\rangle\big)\big]\,\mathrm{d}t
\end{aligned}$$

as well as

(5.4)

$$\begin{aligned}
&\mathbb{E}\big[\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], \big(\rho_h^{r_I} - \overline{\rho}_h^{r_I}\big)(\boldsymbol{T}\wedge T_\oslash)\big)_h\big)\big] \\
&\quad = \mathbb{E}\big[\psi^0\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^0], \big(\rho_h^{r_I} - \overline{\rho}_h^{r_I}\big)(0)\big)_h\big)\big] \\
&\qquad + N^{-\frac{1}{2}}\sum_{k,\tilde{k}=1}^{K}\int_0^{T_k\wedge T_{\tilde{k}}} \mathbb{E}\big[\widetilde{\psi}_{k\tilde{k}}^t\big(N^{1/2}\big(\mathcal{I}_h[\widetilde{\boldsymbol{\phi}}_{k\tilde{k}}^t], \big(\rho_h^{r_I} - \overline{\rho}_h^{r_I}\big)(t\wedge\tilde{\boldsymbol{T}}_{k\tilde{k}}\wedge T_\oslash)\big)_h\big)\big]\,\mathrm{d}t \\
&\qquad + N^{-\frac{1}{2}}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}_{k,\alpha\beta}^{\ell}[m,n] \\
&\qquad \times \mathbb{E}\big[\breve{\psi}_k^t\big(N^{1/2}\big(\mathcal{I}_h[\breve{\boldsymbol{\phi}}_{k,mn,\alpha\beta,\ell}^t], \big(\rho_h^{r_I} - \overline{\rho}_h^{r_I}\big)(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\big)_h\big)\big]\,\mathrm{d}t \\
&\qquad + \mathrm{Err}_{\mathrm{neg}} + \mathrm{Err}_{\mathrm{num}} + \mathrm{Err}_\oslash.
\end{aligned}$$

Second, for any $\kappa > 0$, *there exists* $N_0 = N_0(\kappa, \epsilon, r, s, K, \text{data})$ *such that, for all* $N > N_0$, *the following estimates hold. The stopping time satisfies*

$$\mathbb{P}(T_\oslash < T) \le C\exp\big(-CN^{\epsilon/2}\big). \qquad (5.5)$$

For $t \in (0,T)$, $k, \tilde{k} \in \{1,\dots,K\}$, $m, n \in \mathbb{Z}^d$, *the bounds*

$$\|\psi^0\|_{\mathcal{L}_{\mathrm{pow},r}^q} \le CN^\kappa\big(1 + \|\boldsymbol{\varphi}\|_{H^s}^r\big)\|\psi\|_{\mathcal{L}_{\mathrm{pow},r}^q}, \qquad (5.6)$$

$$\|\widetilde{\psi}_{k\tilde{k}}^t\|_{\mathcal{L}_{\mathrm{pow},r+1}^{q-2}} \le CN^\kappa\big(1 + \|\boldsymbol{\varphi}\|_{H^s}^r\big)\|\psi\|_{\mathcal{L}_{\mathrm{pow},r}^q}, \qquad (5.7)$$

$$\|\breve{\psi}_k^t\|_{\mathcal{L}_{\mathrm{pow},r+2}^{q-1}} \le CN^\kappa\big(1 + \|\boldsymbol{\varphi}\|_{H^s}^r\big)\|\psi\|_{\mathcal{L}_{\mathrm{pow},r}^q} \qquad (5.8)$$

hold. For all $t \in (0,T)$, $k \in \{1,\dots,K\}$, $m, n \in \mathbb{Z}^d$, $\alpha, \beta \in \{1,\dots,n_S\}$, *and* $\ell \in \{1,\dots,d\}$ *the coefficients* $\hat{F}_{k,\alpha\beta}^t[m,n]$ *are subject to*

$$\big|\hat{F}_{k,\alpha\beta}^t[m,n]\big| \le CN^\kappa\big(|m|_2 + |n|_2\big)^{-s_I+1}. \qquad (5.9)$$

The test functions are subject to

$$\|\boldsymbol{\phi}^0\|_{H^s} \le CN^\kappa\|\boldsymbol{\varphi}\|_{H^s}, \qquad (5.10)$$

$$\|\widetilde{\boldsymbol{\phi}}_{k\tilde{k}}^t\|_{H^{s-1}} \le CN^\kappa\big(1 + \|\boldsymbol{\varphi}\|_{H^s}\big)\|\boldsymbol{\varphi}\|_{H^s}, \qquad (5.11)$$

$$\|\breve{\boldsymbol{\phi}}_{k,mn,\alpha\beta,\ell}^t\|_{H^{s-1}} \le CN^\kappa\big(1 + \|\boldsymbol{\varphi}\|_{H^s}\big)\big(1 + |m|_\infty^{s-1} \vee |n|_\infty^{s-1}\big). \qquad (5.12)$$

Furthermore, $\mathrm{Err}_{\oslash}$, $\mathrm{Err}_{\mathrm{neg}}$, *and* $\mathrm{Err}_{\mathrm{num}}$ *can be estimated via*

$$|\mathrm{Err}_{\oslash}| \leq C\|\psi\|_{\mathcal{L}^2_{\mathrm{pow},r}}(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r+2})N^{r\epsilon+\kappa}\exp(-CN^{\epsilon/2}), \tag{5.13}$$

$$|\mathrm{Err}_{\mathrm{neg}}| = 0, \tag{5.14}$$

$$|\mathrm{Err}_{\mathrm{num}}| \leq C\|\psi\|_{\mathcal{L}^3_{\mathrm{pow},r}}(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r+3})N^{(r+1)\epsilon+\kappa}h^{p+1}. \tag{5.15}$$

PROOF. The result will follow from Proposition 5.7 and Proposition 5.9. □

5.1. *The backward evolution equations for the test functions*. We first look at the dynamics of the empirical measure and then the solution of the discretized Dean–Kawasaki equation. This will lead to a choice for the backward evolution for the test functions in both the continuous and discrete setting, summarized in Lemma 5.2 and Lemma 5.4, respectively.

5.1.1. *Dynamics of the empirical measure*. Taking a set of generic time-dependent test functions $\eta_\alpha\colon [0,T]\times\mathbb{T}^d\to\mathbb{R}$, writing $\eta_\alpha^t = \eta_\alpha(t,\cdot)$, we apply the Itô rule and calculate

$$\begin{aligned}\mathrm{d}\langle\eta_\alpha^t, \mu_{\alpha,t}^{r_I,N}\rangle &= \left\langle \partial_t\eta_\alpha^t + \sigma_\alpha\Delta\eta_\alpha^t - \nabla\eta_\alpha^t\cdot\left(\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \mu_{\beta,t}^{r_I,N}\right), \mu_{\alpha,t}^{r_I,N}\right\rangle \mathrm{d}t \\ &\quad + \frac{\sqrt{2\sigma_\alpha}}{N}\sum_{i=1}^N \nabla\eta_\alpha^t\big(X_{\alpha,i}^{r_I,N}(t)\big)\,\mathrm{d}B_{\alpha,i}(t).\end{aligned}$$

We linearise the interaction induced term around the mean field limit, that is,

$$\begin{aligned}&\left\langle \nabla\eta_\alpha^t\cdot\left(\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \mu_{\beta,t}^{r_I,N}\right), \mu_{\alpha,t}^{r_I,N}\right\rangle \\ &\quad = \left\langle \nabla\eta_\alpha^t\cdot\left(\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}(t)\right), \mu_{\alpha,t}^{r_I,N}\right\rangle \\ &\qquad + \sum_{\beta=1}^{n_S}\langle\nabla\eta_\alpha^t\cdot\big(\nabla V_{\alpha\beta}^{r_I} * \big(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\big)\big), \overline{\rho}_\alpha^{r_I}(t)\rangle + Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t),\end{aligned}$$

where $Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t)$ is the quadratic linearisation compensation given by

$$Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t) := \sum_{\beta=1}^{n_S}\langle\nabla\eta_\alpha^t\cdot\big(\nabla V_{\alpha\beta}^{r_I} * \big(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\big)\big), \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\rangle.$$

Introducing the notation $U_\alpha^{r_I}(t) := \sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}(t)$ and by subtracting the mean-field limit equation (3.4), we obtain

$$\begin{aligned}\mathrm{d}\langle\eta_\alpha^t, \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\rangle &= \langle\partial_t\eta_\alpha^t + \sigma_\alpha\Delta\eta_\alpha^t - U_\alpha^{r_I}(t)\cdot\nabla\eta_\alpha^t, \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\rangle\,\mathrm{d}t \\ &\quad - \sum_{\beta=1}^{n_S}\langle\nabla\eta_\alpha^t\cdot\big(\nabla V_{\alpha\beta}^{r_I} * \big(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\big)\big), \overline{\rho}_\alpha^{r_I}(t)\rangle\,\mathrm{d}t \\ &\quad - Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t)\,\mathrm{d}t + \frac{\sqrt{2\sigma_\alpha}}{N}\sum_{i=1}^N\nabla\eta_\alpha^t\big(X_{\alpha,i}^{r_I,N}(t)\big)\,\mathrm{d}B_{\alpha,i}(t).\end{aligned} \tag{5.16}$$

Recalling the definition of $*_c$ (cfr. Section 3) and summing over all species, we obtain

$$\begin{aligned}&\sum_{\alpha=1}^{n_S}\sum_{\beta=1}^{n_S}\langle\nabla\eta_\alpha^t\cdot(\nabla V_{\alpha\beta}^{r_I}*(\mu_{\beta,t}^{r_I,N}-\overline{\rho}_\beta^{r_I}(t))),\overline{\rho}_\alpha^{r_I}(t)\rangle\\&\quad=-\sum_{\beta=1}^{n_S}\left\langle\sum_{\alpha=1}^{n_S}\nabla V_{\alpha\beta}^{r_I}*_c(\overline{\rho}_\alpha^{r_I}(t)\nabla\eta_\alpha^t),\mu_{\beta,t}^{r_I,N}-\overline{\rho}_\beta^{r_I}(t)\right\rangle,\end{aligned}$$

where we used $V_{\alpha\beta}^{r_I}(-x)=V_{\alpha\beta}^{r_I}(x)$ for $x\in\mathbb{T}^d$ (cfr. Assumption 3.7). Switching the species indices for this term and setting $Q_t^{r_I,N}(\eta^t)=\sum_{\alpha=1}^{n_S}Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t)$, we have

$$\begin{aligned}&\mathrm{d}\left(\sum_{\alpha=1}^{n_S}\langle\eta_\alpha^t,\mu_{\alpha,t}^{r_I,N}-\overline{\rho}_\alpha^{r_I}(t)\rangle\right)\\&\quad=\sum_{\alpha=1}^{n_S}\Big\langle\partial_t\eta_\alpha^t+\sigma_\alpha\Delta\eta_\alpha^t-U_\alpha^{r_I}(t)\cdot\nabla\eta_\alpha^t\\&\qquad+\sum_{\beta=1}^{n_S}\nabla V_{\beta\alpha}^{r_I}*_c(\overline{\rho}_\beta^{r_I}(t)\nabla\eta_\beta^t),\mu_{\alpha,t}^{r_I,N}-\overline{\rho}_\alpha^{r_I}(t)\Big\rangle\mathrm{d}t\\&\qquad-Q_t^{r_I,N}(\eta^t)\,\mathrm{d}t+\sum_{\alpha=1}^{n_S}\frac{\sqrt{2\sigma_\alpha}}{N}\sum_{i=1}^{N}\nabla\eta_\alpha^t(X_{\alpha,i}^{r_I,N}(t))\,\mathrm{d}B_{\alpha,i}(t).\end{aligned}\tag{5.17}$$

Thus, we can read off the choice for the backward evolution, which kills as much of the deterministic drift as possible, and obtain Lemma 5.2 below.

LEMMA 5.2 (The backward evolution equation for the test functions). *Under the assumptions of Theorem 5.1, let $\varphi=(\varphi_\alpha)_\alpha\in H^s(\mathbb{T}^d,\mathbb{R}^{n_S})$, $s>2+d/2$. Then there exists a unique $\phi\in C^1(0,T;H^s(\mathbb{T}^d,\mathbb{R}^{n_S}))$ solving*

$$\begin{cases}-\partial_t\phi_\alpha^t=\sigma_\alpha\Delta\phi_\alpha^t-U_\alpha^{r_I}(t)\cdot\nabla\phi_\alpha^t+\sum_{\beta=1}^{n_S}\nabla V_{\beta\alpha}^{r_I}*_c(\overline{\rho}_\beta^{r_I}(t)\nabla\phi_\beta^t) & \text{on }\mathbb{T}^d\times(0,T),\\ \phi_\alpha^T=\varphi_\alpha, & \alpha=1,\dots,n_S,\end{cases}\tag{5.18}$$

where $\phi^t=\phi(t,\cdot)$ and for $\alpha=1,\dots,n_S$, $t\in[0,T]$,

$$U_\alpha^{r_I}(t):=\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I}*\overline{\rho}_\beta^{r_I}(t).\tag{5.19}$$

Furthermore, it holds that

$$\begin{aligned}&d\left(\sum_{\alpha=1}^{n_S}\langle\phi_\alpha^t,\mu_{\alpha,t}^{r_I,N}-\overline{\rho}_\alpha^{r_I}(t)\rangle\right)\\&\quad=-Q_t^{r_I,N}(\phi^t)\,\mathrm{d}t+\sum_{\alpha=1}^{n_S}\frac{\sqrt{2\sigma_\alpha}}{N}\sum_{i=1}^{N}\nabla\phi_\alpha^t(X_{\alpha,i}^{r_I,N}(t))\,\mathrm{d}B_{\alpha,i}(t),\end{aligned}\tag{5.20}$$

where

$$Q_t^{r_I,N}(\phi^t):=\sum_{\alpha,\beta=1}^{n_S}\langle\nabla\phi_\alpha^t\cdot(\nabla V_{\alpha\beta}^{r_I}*(\mu_{\beta,t}^{r_I,N}-\overline{\rho}_\beta^{r_I}(t))),\mu_{\alpha,t}^{r_I,N}-\overline{\rho}_\alpha^{r_I}(t)\rangle.\tag{5.21}$$

PROOF. For existence, uniqueness, and regularity, see Lemma A.1. Then (5.20) immediately follows from (5.17), since the regularity is sufficient for the calculations to hold. □

REMARK 5.3. Note that the reason, for which we are not able to kill all of the deterministic drift by evolving the test functions backward, is the underlying nonlinearity of the system. Specifically, the linearization compensation $Q_t^{r_I,N}(\cdot)$ associated with the nonlinear interaction terms of the cross-diffusion system is the only part of the drift that survives.

5.1.2. *Dynamics of the solutions to the discretized Dean–Kawasaki model*. The following is the discrete version of Lemma 5.2 (recall the definitions of $*_h$, $*_{h,c}$ from Section 3).

LEMMA 5.4 (The backward evolution equation for the discrete test functions). *Under the assumptions of Theorem 5.1, let $\varphi_h = (\varphi_{h,\alpha})_\alpha \in L^2(G_{h,d}, \mathbb{R}^{n_S})$. Then there exists a unique $\phi_h \in C^1(0,T; L^2(G_{h,d}, \mathbb{R}^{n_S}))$ solving*

$$\begin{cases} -\partial_t \phi_{h,\alpha}^t = \sigma_\alpha \Delta_h \phi_{h,\alpha}^t - U_{h,\alpha}^{r_I} \cdot \nabla_h \phi_{h,\alpha}^t + \sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V_{\beta\alpha}^{r_I}] *_{h,c} (\overline{\rho}_{h,\beta}^{r_I} \nabla_h \phi_{h,\beta}^t), \\ \phi_{h,\alpha}^T = \varphi_{h,\alpha}, \quad \alpha = 1, \dots, n_S, \end{cases} \tag{5.22}$$

where for $\alpha = 1, \dots, n_S$, $t \in [0,T]$, we have set

$$U_{h,\alpha}^{r_I}(t) := \sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \overline{\rho}_{h,\beta}^{r_I}. \tag{5.23}$$

Furthermore, it holds for $t \in [0,T]$ that

$$\begin{aligned} &\mathrm{d}\left(\sum_{\alpha=1}^{n_S} (\phi_{h,\alpha}^t, (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(t))_h \right) \\ &\quad = -Q_h^{r_I,N}(\phi_h^t)\,\mathrm{d}t \\ &\qquad - \sum_{\alpha=1}^{n_S} \sqrt{2\sigma_\alpha} N^{-1/2} \sum_{\substack{y \in G_{h,d} \\ l \in \{1,\dots,d\}}} \left(\sqrt{(\rho_{h,\alpha}^{r_I}(t))^+} f_{y,l}^d, \nabla_h \phi_{h,\alpha}^t \right)_h \mathrm{d}W_{(y,l)}^\alpha, \end{aligned} \tag{5.24}$$

where $\{f_{y,l}^d\}$ is the basis of $L^2(G_{h,d}, \mathbb{R}^d)$ defined in Section 3.2 and we have set

$$Q_{h,t}^{r_I,N}(\phi_h^t) := \sum_{\alpha,\beta=1}^{n_S} (\nabla_h \phi_{h,\alpha}^t \cdot (\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h (\rho_{h,\beta}^{r_I} - \overline{\rho}_{h,\beta}^{r_I})(t)), (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(t))_h. \tag{5.25}$$

PROOF. We first take a set of generic discrete test functions $\eta_{h,\alpha} \colon [0,T] \times G_{h,d} \to \mathbb{R}$ which are differentiable in time. Since $\{f_x\}_{x \in G_{h,d}}$, as defined in Section 3.2, is an orthonormal basis of $L^2(G_{h,d})$, we expand $\eta_{h,\alpha} = \sum_{x \in G_{h,d}} (\eta_{h,\alpha}, f_x)_h f_x$. Applying the Itô rule to

$$(\eta_{h,\alpha}, \rho_{h,\alpha}^{r_I})_h(t) = \sum_{x \in G_{h,d}} (\eta_{h,\alpha}(t), f_x)_h (\rho_{h,\alpha}^{r_I}(t), f_x)_h$$

using (h-DK) and the expansion $\nabla_h \eta_{h,\alpha} = \sum_{x \in G_{h,d}} (\eta_{h,\alpha}, f_x)_h \nabla_h f_x$, we obtain

$$\mathrm{d}(\eta_{h,\alpha}, \rho_{h,\alpha}^{r_I})_h = \Big[(\partial_t \eta_{h,\alpha}, \rho_{h,\alpha}^{r_I})_h + \sigma_\alpha (\eta_{h,\alpha}, \Delta_h \rho_{h,\alpha}^{r_I})_h$$

$$- \sum_{\beta=1}^{n_S} \big(\nabla_h \eta_{h,\alpha}, \rho_{h,\alpha}^{r_I} \big(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \rho_{h,\beta}^{r_I}\big)\big)_h \Bigg] \mathrm{d}t$$

$$- \sqrt{2\sigma_\alpha} N^{-1/2} \sum_{\substack{y \in G_{h,d} \\ l \in \{1,\dots,d\}}} \Big(\sqrt{(\rho_{h,\alpha}^{r_I})^+} f_{y,l}^d, \nabla_h \eta_{h,\alpha}\Big)_h \mathrm{d}W_{(y,l)}^\alpha.$$

Following the same steps as in Lemma 5.2, we get the discrete analogue to (5.17). Inserting the test functions evolving according to the discrete backward evolution (5.22), we obtain (5.24). The existence and uniqueness of the solution to (5.22) follows from linearity. □

5.2. *The generalised moment structure.* In 5.2.1 we analyze the generalised moment structure in the continuous case, highlighting how we choose the backward evolution for the generalised moment functions. In 5.2.2 we calculate the resulting generalised moment structure for the discrete setting. Finally, in 5.2.3 we will compare the continuous and discrete setting, collecting and estimating the error terms from adjusting the discrete terms to fit the iterative structure. By defining as well as estimating $\widetilde{\psi}_{k\tilde{k}}^t$, $\widetilde{\boldsymbol{\phi}}_{k\tilde{k}}^t$, and $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$, we complete the first big step toward proving Theorem 5.1.

5.2.1. *Deriving the generalised moment structure in the continuous setting.*

LEMMA 5.5. *Let the assumptions of Theorem 5.1 hold. Then there exists a unique set of functions $\boldsymbol{\phi} = (\phi_1, \dots, \phi_K) \in [C^1(0,T; H^s(\mathbb{T}^d, \mathbb{R}^{n_S}))]^K$ such that*

$$\begin{cases} -\partial_t \phi_{k,\alpha}^t = \sigma_\alpha \Delta \phi_{k,\alpha}^t - U_\alpha^{r_I}(t) \cdot \nabla \phi_{k,\alpha}^t + \displaystyle\sum_{\beta=1}^{n_S} \nabla V_{\beta\alpha}^{r_I} *_c \big(\overline{\rho}_\beta^{r_I}(t) \nabla \phi_{k,\beta}^t\big), & t < T_k, \\ \phi_{k,\alpha}^t = \varphi_{k,\alpha}, & t \in [T_k, T] \end{cases} \tag{5.26}$$

for all $\alpha = 1, \dots, n_S$ and each $k = 1, \dots, K$. Further, there exists a unique backward evolution of ψ in $C^1(0,T; \mathcal{L}_{\mathrm{pow},r}^q(\mathbb{R}^K))$ satisfying

$$\begin{cases} -\partial_t \psi^t = \displaystyle\sum_{k,\tilde{k}=1}^K \chi_{t \le T_k \wedge T_{\tilde{k}}} \sum_{\alpha=1}^{n_S} \sigma_\alpha \langle \nabla \phi_{k,\alpha}^t \cdot \nabla \phi_{\tilde{k},\alpha}^t, \overline{\rho}_\alpha^{r_I}(t) \rangle \partial_k \partial_{\tilde{k}} \psi^t & \text{on } \mathbb{R}^K \times (0,T), \\ \psi^T = \psi. \end{cases} \tag{5.27}$$

Then it holds that

$$\begin{aligned} &\mathbb{E}\big[\psi\big(N^{1/2}\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle\big)\big] \\ &\quad = \mathbb{E}\big[\psi^0\big(N^{1/2}\langle \boldsymbol{\phi}^0, \mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\rangle\big)\big] \\ &\qquad + N^{-1/2} \sum_{k,\tilde{k}=1}^K \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\Bigg[\partial_k \partial_{\tilde{k}} \psi^t \big(N^{1/2}\langle \boldsymbol{\phi}^t, \mu_{t \wedge \boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(t \wedge \boldsymbol{T})\rangle\big) \\ &\qquad \times N^{1/2} \sum_{\alpha=1}^{n_S} \langle \sigma_\alpha \nabla \phi_{k,\alpha}^t \cdot \nabla \phi_{\tilde{k},\alpha}^t, \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\rangle \Bigg] \mathrm{d}t \\ &\qquad - N^{1/2} \sum_{k=1}^K \int_0^{T_k} \mathbb{E}\big[\partial_k \psi^t \big(N^{1/2}\langle \boldsymbol{\phi}^t, \mu_{t \wedge \boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(t \wedge \boldsymbol{T})\rangle\big) Q_t^{r_I,N}(\phi_k^t)\big] \mathrm{d}t. \end{aligned} \tag{5.28}$$

Additionally, there exists $N_0 = N_0(\kappa, r, K, \text{data})$ such that, for all $0 \le \tilde{q} \le q$ and $t \in [0,T]$,

$$\|\psi^t\|_{\mathcal{L}_{\mathrm{pow},r}^{\tilde{q}}} \le C N^\kappa \big(1 + \|\varphi\|_{H^s}^r\big) \|\psi\|_{\mathcal{L}_{\mathrm{pow},r}^{\tilde{q}}}. \tag{5.29}$$

PROOF. The existence, uniqueness, and regularity of $\boldsymbol{\phi}$ is settled as in Lemma 5.2. For the existence, uniqueness and regularity of ψ^t, see [13], Proof of Proposition 6, Step 1. These arguments applied for every $0 \le \tilde{q} \le q$ yield

$$\begin{aligned}\|\psi^t\|_{\mathcal{L}^{\tilde{q}}_{\mathrm{pow},r}} &\le C\big(1+\|\boldsymbol{\phi}\|^r_{L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))}\big)\|\psi\|_{\mathcal{L}^{\tilde{q}}_{\mathrm{pow},r}} \\ &\le C\big(1+\|\boldsymbol{\phi}\|^r_{L^\infty(0,T;H^{\lceil (d+3)/2\rceil}(\mathbb{T}^d))}\big)\|\psi\|_{\mathcal{L}^{\tilde{q}}_{\mathrm{pow},r}}.\end{aligned}$$

Now, (5.29) follows from $s \ge \lceil (d+3)/2\rceil$ and Lemma A.1. We now turn to poving (5.28). In what follows, the regularity of $\boldsymbol{\phi}$ and ψ^t is sufficient for the calculations to hold.

For the convenience of notation, we set

$$\boldsymbol{\zeta}^t := N^{1/2}\langle \boldsymbol{\phi}^t, \mu^{r_I,N}_{t\wedge \boldsymbol{T}} - \overline{\rho}^{r_I}(t\wedge \boldsymbol{T})\rangle.$$

Using this notation, we rewrite (5.20) from Lemma 5.2 for $k = 1, \dots, K$ as

$$\mathrm{d}\zeta^t_k = -\chi_{t\le T_k} N^{1/2} Q^{r_I,N}_t(\phi^t_k)\,\mathrm{d}t + \mathrm{d}M^t_k$$

with martingale part M^t_k given by

$$\mathrm{d}M^t_k := \chi_{t\le T_k}\sum_{\alpha=1}^{n_S}\frac{\sqrt{2\sigma_\alpha}}{N^{1/2}}\sum_{i=1}^{N}\nabla\phi^t_{k,\alpha}\big(X^{r_I,N}_{\alpha,i}(t)\big)\,\mathrm{d}B_{\alpha,i}(t).$$

Thus, as outlined in (4.6), we observe the linearity preserving property of the cross-variation (see Section 4.1.1, specifically (4.6)), which yields for $k, \tilde{k} \in \{1, \dots, K\}$ that

$$\begin{aligned}\mathrm{d}\big[M^t_k, M^t_{\tilde{k}}\big] &= \chi_{t\le T_k}\chi_{t\le T_{\tilde{k}}}\sum_{\alpha=1}^{n_S}\frac{2\sigma_\alpha}{N}\sum_{i=1}^{N}\nabla\phi^t_{k,\alpha}\big(X^{r_I,N}_{\alpha,i}(t)\big)\cdot\nabla\phi^t_{\tilde{k},\alpha}\big(X^{r_I,N}_{\alpha,i}(t)\big)\,\mathrm{d}t \\ &= \chi_{t\le T_k\wedge T_{\tilde{k}}}\sum_{\alpha=1}^{n_S}2\sigma_\alpha\langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}, \mu^{r_I,N}_{\alpha,t}\rangle\,\mathrm{d}t.\end{aligned}\tag{5.30}$$

Combining these observations with the Itô rule and taking the expected value yields

$$\begin{aligned}\mathrm{d}\mathbb{E}\big[\psi^t(\boldsymbol{\zeta}^t)\big] &= \mathbb{E}\big[\partial_t\psi^t(\boldsymbol{\zeta}^t)\big]\,\mathrm{d}t - \mathbb{E}\Bigg[\sum_{k=1}^{K}\partial_k\psi^t(\boldsymbol{\zeta}^t)\chi_{t\le T_k}N^{1/2}Q^{r_I,N}_t(\phi^t_k)\Bigg]\,\mathrm{d}t \\ &\quad + \mathbb{E}\Bigg[\sum_{k,\tilde{k}=1}^{K}\partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}^t)\chi_{t\le T_k\wedge T_{\tilde{k}}}\sum_{\alpha=1}^{n_S}\sigma_\alpha\langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}, \mu^{r_I,N}_{\alpha,t}\rangle\Bigg]\,\mathrm{d}t.\end{aligned}$$

Thinking of the term with $Q^{r_I,N}_t(\phi^t_k)$ as a linearisation error, we see that (5.27) is the clear choice for the backward evolution of ψ^t if we want the first two terms to fit the iterative structure, that is combining linear functionals of $\mu^{r_I,N} - \overline{\rho}^{r_I}$. Plugging (5.27) in and integrating the equation in time yields (5.28). □

5.2.2. *Deriving the generalised moment structure in the discrete setting*. Analogously to Lemma 5.5, we obtain the following result in the discrete case. However, since the backward evolution for ψ^t is tailored for the continuous case, the generalised moment structure does not immediately show the iterative structure. In Section 5.2.3 we will estimate the cost of the adjustments necessary in order to obtain this structure.

LEMMA 5.6. *Let the assumptions of Theorem 5.1 hold. Then there exists a unique set of functions* $\boldsymbol{\phi}_h = (\phi_{h,1}, \dots, \phi_{h,K}) \in [C^1(0,T; L^2(G_{h,d}, \mathbb{R}^{n_S}))]^K$ *solving the respective discrete backward evolution equation for test functions, that is,*

$$
\begin{cases} -\partial_t \phi_{h,k,\alpha}^t \\ \qquad = \sigma_\alpha \Delta_h \phi_{h,k,\alpha}^t - U_{h,\alpha}^{r_I} \cdot \nabla_h \phi_{h,k,\alpha}^t + \sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V_{\beta\alpha}^{r_I}] *_{h,c} (\overline{\rho}_{h,\beta}^{r_I} \nabla_h \phi_{h,k,\beta}) \\ \qquad \text{for } t < T_k, \\ \phi_{h,k,\alpha}^t = \mathcal{I}_h[\varphi_{k,\alpha}] \quad \text{for } t \in [T_k, T] \end{cases} \tag{5.31}
$$

for all $\alpha = 1, \dots, n_S$ *and each* $k = 1, \dots, K$. *Further, let* $t \to \psi^t$ *with* $\psi^T = \psi$ *be the solution of* (5.27) *from Lemma 5.5. As assumed in Theorem 5.1, let* $T_\oslash = T_\oslash(\epsilon/2, \epsilon)$ *be the stopping time, as defined in* (8.2). *Then*

$$
\begin{aligned}
&\mathbb{E}\big[\psi\big(N^{1/2}(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_\oslash))_h\big)\big] \\
&\quad = \mathbb{E}\big[\psi^0\big(N^{1/2}(\boldsymbol{\phi}_h^0, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0))_h\big)\big] \\
&\qquad + \sum_{k,\tilde{k}=1}^{K} \sum_{\alpha=1}^{n_S} \sigma_\alpha \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\big[\partial_k \partial_{\tilde{k}} \psi^t\big(N^{1/2}(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash))_h\big) \\
&\qquad \times \big(\big(\nabla_h \phi_{h,k,\alpha}^t \cdot \nabla_h \phi_{h,\tilde{k},\alpha}^t, (\rho_{h,\alpha}^{r_I})^+(t \wedge T_\oslash)\big)_h \\
&\qquad - \langle \nabla \phi_{k,\alpha}^t \cdot \nabla \phi_{\tilde{k},\alpha}^t, \overline{\rho}_\alpha^{r_I}(t \wedge T_\oslash)\rangle\big)\big]\,\mathrm{d}t \\
&\qquad - N^{1/2} \sum_{k=1}^{K} \int_0^{T_k} \mathbb{E}\big[\partial_k \psi^t\big(N^{1/2}(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash))_h\big) Q_{h,t\wedge T_\oslash}^{r_I,N}(\phi_{h,k}^t)\big]\,\mathrm{d}t. \\
&\qquad + \mathrm{Err}_\oslash.
\end{aligned} \tag{5.32}
$$

For any $\kappa > 0$, *there exists* $N_0 = N_0(\kappa, \epsilon, r, s, K, \textit{data})$ *such that* $\mathrm{Err}_\oslash$ *is subject to* (5.13) *for all* $N > N_0$.

PROOF. The existence, uniqueness and regularity for $\boldsymbol{\phi}_h$ is as in Lemma 5.4. The regularity is sufficient for the following calculations to hold:

Step 1: (5.32) *and defining the error term.* We denote

$$\boldsymbol{\zeta}_h^t := N^{1/2}(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash))_h.$$

Rewriting (5.24) in Lemma 5.4 with this notation yields

$$\mathrm{d}\zeta_{h,k}^{t\wedge T_\oslash} = -\chi_{t \le T_k \wedge T_\oslash} N^{1/2} Q_{h,t}^{r_I,N}(\phi_{h,k}^t)\,\mathrm{d}t + \mathrm{d}M_{h,k}^{t\wedge T_\oslash}$$

with martingale part $M_{h,k}$ given by

$$\mathrm{d}M_{h,k}^{t\wedge T_\oslash} := -\chi_{t \le T_k \wedge T_\oslash} \sum_{\alpha=1}^{n_S} \sqrt{2\sigma_\alpha} \sum_{\substack{y \in G_{h,d} \\ l \in \{1,\dots,d\}}} \Big(\sqrt{(\rho_{h,\alpha}^{r_I}(t))^+} f_{y,l}^d, \nabla_h \phi_{h,k,\alpha}^t\Big)_h \,\mathrm{d}W_{(y,l)}^\alpha(t).$$

The cross-variation terms with $k, \tilde{k} \in \{1, \dots, K\}$, as detailed in Section 4.1.1 (specifically (4.8)), are given by

$$\mathrm{d}\big[M_{h,k}^{t\wedge T_\oslash}, M_{h,\tilde{k}}^{t\wedge T_\oslash}\big] = \chi_{t \le T_k \wedge T_{\tilde{k}} \wedge T_\oslash} \sum_{\alpha=1}^{n_S} 2\sigma_\alpha \big(\nabla_h \phi_{h,k,\alpha}^t \cdot \nabla_h \phi_{h,\tilde{k},\alpha}^t, (\rho_{h,\alpha}^{r_I}(t))^+\big)_h.$$

Now, as in the continuous case, we have

$$
\begin{aligned}
\mathrm{d}\psi^t(\boldsymbol{\zeta}_h^{t\wedge T_\oslash}) = {} & \partial_t\psi^t(\boldsymbol{\zeta}_h^{t\wedge T_\oslash})\,\mathrm{d}t - \sum_{k=1}^K \partial_k\psi^t(\boldsymbol{\zeta}_h^{t\wedge T_\oslash})\chi_{t\le T_k\wedge T_\oslash} N^{1/2} Q_h^{r_I,N}(\phi_{h,k}^t)\,\mathrm{d}t \\
& + \sum_{k,\tilde{k}=1}^K \partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}_h^{t\wedge T_\oslash})\chi_{t\le T_k\wedge T_{\tilde{k}}\wedge T_\oslash} \\
& \times \sum_{\alpha=1}^{n_S}\sigma_\alpha\big(\nabla_h\phi_{h,k,\alpha}^t\cdot\nabla_h\phi_{h,\tilde{k},\alpha}^t,(\rho_{h,\alpha}^{r_I}(t))^+\big)_h\,\mathrm{d}t \\
& + \sum_{k=1}^K \partial_k\psi^t(\boldsymbol{\zeta}_h^{t\wedge T_\oslash})\,\mathrm{d}M_{h,k}^{t\wedge T_\oslash}.
\end{aligned}
\tag{5.33}
$$

Integrating in time from 0 to $T_\oslash$, plugging in (5.27), and then taking the expected value, we obtain (5.32) where the error term is given by $\mathrm{Err}_\oslash = \mathrm{Err}_{\oslash,1} + \mathrm{Err}_{\oslash,2}$. The first error term $\mathrm{Err}_{\oslash,1}$ stems from the correction $(\psi^{T_\oslash},\boldsymbol{\phi}_h^{T_\oslash}) \to (\psi^T,\boldsymbol{\phi}_h^T) = (\psi,\boldsymbol{\varphi})$, given by

$$
\begin{aligned}
\mathrm{Err}_{\oslash,1} := \mathbb{E}\big[\chi_{T_\oslash<T}\big(&\psi\big(N^{1/2}(\mathcal{I}_h[\boldsymbol{\varphi}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash))_h\big) \\
& - \psi^{T_\oslash}\big(N^{1/2}(\boldsymbol{\phi}_h^{T_\oslash},(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash))_h\big)\big)\big],
\end{aligned}
$$

while $\mathrm{Err}_{\oslash,2}$ comes from extending the integration to T as $\chi_{t\le T_\oslash}\to 1$, that is,

$$
\begin{aligned}
\mathrm{Err}_{\oslash,2} := {} & -\sum_{k,\tilde{k}=1}^K\sum_{\alpha=1}^{n_S}\sigma_\alpha\int_0^{T_k\wedge T_{\tilde{k}}}\mathbb{E}\big[\chi_{T_\oslash<t\le T}\,\partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}_h^t) \\
& \times\big(\big(\nabla_h\phi_{h,k,\alpha}^t\cdot\nabla_h\phi_{h,\tilde{k},\alpha}^t,(\rho_{h,\alpha}^{r_I})^+(t\wedge T_\oslash)\big)_h - \langle\nabla\phi_{k,\alpha}^t\cdot\nabla\phi_{\tilde{k},\alpha}^t,\overline{\rho}_\alpha^{r_I}(t\wedge T_\oslash)\rangle\big)\big]\,\mathrm{d}t \\
& + N^{1/2}\sum_{k=1}^K\int_0^{T_k}\mathbb{E}\big[\chi_{T_\oslash<t\le T}\,\partial_k\psi^t(\boldsymbol{\zeta}_h^t)\,Q_{h,t\wedge T_\oslash}^{r_I,N}(\phi_{h,k}^t)\big]\,\mathrm{d}t.
\end{aligned}
$$

Step 2: Estimating the error terms. For the following estimates, we assume N to be large enough to use the suitable auxiliary results. Additionally, we will rely on the definition of the stopping time $T_\oslash$; see (8.2) for $0<\epsilon<\delta_0/4$ and $\delta=\epsilon/2$ as fixed in the assumptions of Theorem 5.1. Due to the definition of the $\mathcal{L}^q_{\mathrm{pow},r}$-spaces, we have

$$
\begin{aligned}
& \big|\psi\big(N^{1/2}(\mathcal{I}_h[\boldsymbol{\varphi}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash))_h\big)\big| \\
& \quad\le \|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\big(1+\big|N^{1/2}(\mathcal{I}_h[\boldsymbol{\varphi}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash))_h\big|^2\big)^{r/2} \\
& \quad\le \|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\big(1+\|\mathcal{I}_h[\boldsymbol{\varphi}]\|^2_{H_h^{d+2}}N\big\|(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big\|^2_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big)^{r/2} \\
& \quad\le \|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\big(1+C\|\boldsymbol{\varphi}\|^2_{C^{d+2}}\big(N^{2\epsilon}+N\chi_{T_\oslash=0}\big\|(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(0)\big\|^2_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big)\big)^{r/2}
\end{aligned}
$$

where the last step follows from the definition of the stopping time $T_\oslash$. Additionally, plugging in (B.11) from Lemma B.8 to estimate $\|\boldsymbol{\phi}_h^{T_\oslash}\|_{H^{d+2}(G_{h,d})}$ and (5.29) from Lemma 5.5 to bound $\|\psi^{T_\oslash}\|_{\mathcal{L}^q_{\mathrm{pow},r}}$, we obtain that—as long as N is large enough—for any small $\kappa>0$,

$$
\begin{aligned}
& \psi^{T_\oslash}\big(N^{1/2}(\boldsymbol{\phi}_h^{T_\oslash},(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash))_h\big) \\
& \quad\le C\|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}N^\kappa\big(1+\|\boldsymbol{\varphi}\|^r_{H^s(\mathbb{T}^d)}\big) \\
& \qquad\times\big(1+N^\kappa\|\boldsymbol{\varphi}\|^r_{C^{d+2}(\mathbb{T}^d)}\big(N^{r\epsilon}+\chi_{T_\oslash=0}N^{r/2}\big\|(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(0)\big\|^r_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big)\big).
\end{aligned}
$$

Since Assumption (5.1) implies $s > 3d/2 + 2$, we have $\|\boldsymbol{\varphi}\|_{C^{d+3}} \le C\|\boldsymbol{\varphi}\|_{H^s}$ via Sobolev embedding and thus get

$$\begin{aligned}\mathrm{Err}_{\oslash,1} &\le \mathbb{P}(T_\oslash < T)C\|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)N^{r\epsilon+\kappa} \\ &\quad + \mathbb{E}\big[\chi_{T_\oslash=0}\big\|\big(\rho_h^{r_I} - \overline{\rho}_h^{r_I}\big)(0)\big\|^r_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big]C\|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)N^{r/2}.\end{aligned}\tag{5.34}$$

For now, ignoring the case $T_\oslash = 0$, plugging in the bound for $\mathbb{P}(T_\oslash < T)$ from Proposition 8.2, we obtain (5.13).

With respect to the case $T_\oslash = 0$, due to Assumption 3.10, the mass restriction and positivity of $\rho_h^{r_I}(0)$, we obtain the very rough estimate

$$\big\|\rho_h^{r_I}(0)\big\|_{H_h^{-2(\lfloor d/2\rfloor+1)}} \le \big\|\rho_h^{r_I}(0)\big\|_{L_h^2} \le \big\|\rho_h^{r_I}(0)\big\|_{L_h^\infty} \le h^{-d}\big\|\rho_h^{r_I}(0)\big\|_{L_h^1} \le Ch^{-d}.$$

Furthermore, since $\overline{\rho}_h^{r_I}(0) = \mathcal{I}_h[\overline{\rho}^{r_I}(0)]$, we also get

$$\big\|\overline{\rho}_h^{r_I}(0)\big\|_{H_h^{-2(\lfloor d/2\rfloor+1)}} \le \big\|\overline{\rho}_h^{r_I}(0)\big\|_{L_h^2} \le C\big\|\overline{\rho}^{r_I}(0)\big\|_{C^0}.$$

Both are controlled by the bound for $\mathbb{P}[T_\oslash = 0] \le \mathbb{P}[T_\oslash < 0]$: In particular,

$$\mathbb{P}[T_\oslash = 0] \le C\exp\big(-CN^{\epsilon/2}\big) \le CN^{-r/2}\big(h^{-d} + \|\overline{\rho}^{r_I}(0)\|^{-r}_{C^1}\big)^{-r}\exp\big(-C/2N^{\epsilon/2}\big),$$

where the second inequality is due to the scaling regime; see Assumption 3.11 for N large enough (depending on $\|\overline{\rho}^{r_I}(0)\|_{C^1}$). Plugging these estimates into (5.34), we obtain the bound given in (5.13).

Concerning $\mathrm{Err}_{\oslash,2}$ with the integration, we can ignore the case $T_\oslash = 0$. For $0 < t < T_\oslash$ via Remark 3.14, we have

$$\big\|\big(\rho_{h,\alpha}^{r_I}\big)^+(t\wedge T_\oslash)\big\|_{L_h^\infty} \le \big\|\big(\overline{\rho}_h^{r_I}\big)\big\|_{L_h^\infty} + N^{-\epsilon} \le C\rho_{\max,h},\tag{5.35}$$

and using Corollary B.2, Lemma B.8, and that $\|\nabla_h f\|_{L_h^2} \le C\|f\|_{H_h^1}$ for $f \in L^2(G_{h,d})$,

$$\|\nabla_h\phi^t_{h,k,\alpha}\cdot\nabla_h\phi^t_{h,\tilde{k},\alpha}\|_{L_h^1} \le C\|\boldsymbol{\phi}_h^t\|^2_{H_h^1} \le CN^\kappa\|\mathcal{I}_h[\boldsymbol{\varphi}]\|^2_{H_h^1} \le CN^\kappa\|\boldsymbol{\varphi}\|^2_{H^s}.\tag{5.36}$$

For the other term, note that

$$\begin{aligned}\big|N^{1/2}Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi^t_{h,k})\big| &\le CN^{\epsilon+\kappa}\|\nabla_h\boldsymbol{\phi}_h^t\|_{L_h^1}\big\|\big(\rho_h^{r_I} - \overline{\rho}_h^{r_I}\big)(t)\big\|_{L_h^\infty} \\ &\le C\|\boldsymbol{\varphi}\|_{H^s}N^\kappa,\end{aligned}\tag{5.37}$$

where for the first inequality we estimated the convolution pointwise via the definition of $T_\oslash$. Together with Assumption 3.8, we obtain

$$\mathrm{Err}_{\oslash,2} \le C\mathbb{P}(T_\oslash < T)\|\psi\|_{\mathcal{L}^2_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^r_{H^s}\big)\|\boldsymbol{\varphi}\|^r_{H^s}N^{r\epsilon+\kappa}\big(\|\boldsymbol{\varphi}\|^2_{H^s} + \|\boldsymbol{\varphi}\|_{H^s}\big).$$

Combining this with the bound for $\mathbb{P}(T_\oslash < T)$, the scaling from Assumption 3.11, and the estimate for $\mathrm{Err}_{\oslash,1}$, we obtain (5.13). □

5.2.3. *Comparing the generalised moments, the iterative structure.* Now, comparing (5.28) and (5.32), up to some small error terms we obtain the iterative structure, that is, the same type of terms we started with but with an additional small prefactor of $N^{-1/2}$.

PROPOSITION 5.7. *Under the assumptions of Theorem 5.1, for all $t \in (0, T)$, as well as $k, \tilde{k} \in \{1, \dots, K\}$, $\alpha, \beta \in \{1, \dots, n_S\}$ there exist functions*

$$\psi^0 \in \mathcal{L}^q_{\mathrm{pow},r}(\mathbb{R}^K), \qquad \widetilde{\psi}^t_{k\tilde{k}} \in \mathcal{L}^{q-2}_{\mathrm{pow},r+1}(\mathbb{R}^{K+1}),$$

sets of test functions

$$\boldsymbol{\phi}^0 \in [H^s(\mathbb{T}^d, \mathbb{R}^{ns})]^K, \qquad \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}} \in [H^{s-1}(\mathbb{T}^d, \mathbb{R}^{ns})]^{K+1},$$

and test times $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0, T]^{K+1}$ *such that*

$$\begin{aligned}
&\mathbb{E}\big[\psi\big(N^{1/2}\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle\big)\big] \\
&\quad = \mathbb{E}\big[\psi^0\big(N^{1/2}\langle \boldsymbol{\phi}^0, \mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\rangle\big)\big] \\
&\qquad + \frac{1}{N^{1/2}} \sum_{k,\tilde{k}=1}^{K} \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\big[\widetilde{\psi}^t_{k\tilde{k}}\big(N^{1/2}\langle \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, \mu_{t\wedge \tilde{\boldsymbol{T}}_{k\tilde{k}}}^{r_I,N} - \overline{\rho}^{r_I}(t \wedge \tilde{\boldsymbol{T}}_{k\tilde{k}})\rangle\big)\big]\,\mathrm{d}t \\
&\qquad + \mathrm{Err}_{\mathrm{lin},a}
\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}
&\mathbb{E}\big[\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_{\oslash})\big)_h\big)\big] \\
&\quad = \mathbb{E}\big[\psi^0\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^0], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\big)_h\big)\big] \\
&\qquad + N^{-1/2} \sum_{k,\tilde{k}=1}^{K} \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\big[\widetilde{\psi}^t_{k\tilde{k}}\big(N^{1/2}\big(\mathcal{I}_h[\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \tilde{\boldsymbol{T}}_{k\tilde{k}} \wedge T_{\oslash})\big)_h\big)\big]\,\mathrm{d}t \\
&\qquad + \mathrm{Err}_{\oslash} + \mathrm{Err}_{\mathrm{neg}} + \mathrm{Err}_{\mathrm{num}} + \mathrm{Err}_{\mathrm{lin},b}.
\end{aligned} \tag{5.39}$$

The linearization errors are constructed using $\boldsymbol{\phi}$ *as in* (5.26), ψ^t *as in* (5.27), $\boldsymbol{\phi}_h$ *as in* (5.31), $Q^{r_I,N}$ *as in* (5.21), *and* $Q_h^{r_I,N}$ *as in* (5.25): *explicitly, they read*

$$\mathrm{Err}_{\mathrm{lin},a} := -N^{1/2} \sum_{k=1}^{K} \int_0^{T_k} \mathbb{E}\big[\partial_k \psi^t\big(N^{1/2}\langle \boldsymbol{\phi}^t, \mu_{t\wedge \boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(t \wedge \boldsymbol{T})\rangle\big) Q_t^{r_I,N}(\phi_k^t)\big]\,\mathrm{d}t, \tag{5.40}$$

$$\begin{aligned}
\mathrm{Err}_{\mathrm{lin},b} := &-N^{1/2} \sum_{k=1}^{K} \int_0^{T_k} \mathbb{E}\big[\partial_k \psi^t\big(N^{1/2}\big(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_{\oslash})\big)_h\big) \\
&\times Q_{h,t\wedge T_{\oslash}}^{r_I,N}(\phi_{h,k}(t))\big]\,\mathrm{d}t.
\end{aligned} \tag{5.41}$$

For any $\kappa > 0$, *there exists* $N_0 = N_0(\kappa, \epsilon, r, s, K, data)$ *such that, for all* $N > N_0$, *the following estimates hold. The stopping time satisfies* (5.13), *the new generalised moment functions* ψ^0 *and* $\{\widetilde{\psi}^t_{k\tilde{k}}\}_{t,k,\tilde{k},m,n}$ *are subject to* (5.6) *and* (5.7), *respectively. The test function* $\boldsymbol{\phi}^0$ *satisfy* (5.10), *and the test functions* $\{\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}\}_{t,k,\tilde{k}}$ *satisfy* (5.11). *Finally, for the error terms, the estimates* (5.13), (5.14), *and* (5.15) *hold, respectively.*

REMARK 5.8. The terms $\mathrm{Err}_{\mathrm{lin},a}$ and $\mathrm{Err}_{\mathrm{lin},b}$ do not show the itarative structure yet due to the convolutional structure. In Section 5.3 below, we will follow 4.2.2 to separate the two contributions of the quadratic nonlinearity.

PROOF. In Steps 1 and 2, we will introduce the definitions which yield (5.38) and (5.39) from Lemma 5.5 and Lemma 5.6, respectively. In Steps 3, 4, and 5 we will prove the estimates:

Step 1: Defining $\widetilde{\psi}^t_{k\tilde{k}}$, $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}$, $\tilde{\boldsymbol{T}}_{k\tilde{k}}$ to obtain (5.38). We introduce $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}(x)\colon \mathbb{T}^d \to \mathbb{R}^{K+1}$ and $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$ with

$$\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}(x) = \begin{pmatrix} (\phi^t_{1,\alpha}(x))_{\alpha=1,\dots,n_S} \\ \vdots \\ (\phi^t_{K,\alpha}(x))_{\alpha=1,\dots,n_S} \\ (\sigma_\alpha \nabla\phi^t_{k,\alpha}(x)\cdot\nabla\phi^t_{\tilde{k},\alpha}(x))_{\alpha=1,\dots,n_S} \end{pmatrix}, \qquad \tilde{\boldsymbol{T}}_{k\tilde{k}} = \begin{pmatrix} T_1 \\ \vdots \\ T_K \\ T_k \wedge T_{\tilde{k}} \end{pmatrix}, \tag{5.42}$$

where $\boldsymbol{\phi}^t$ and thus particularly $\boldsymbol{\phi}^0$ come from (5.26). Further defining $\widetilde{\psi}^t_{k\tilde{k}}\colon \mathbb{R}^{K+1} \to \mathbb{R}$,

$$\widetilde{\psi}^t_{k\tilde{k}}(z) = \chi_{t\le T_k\wedge T_{\tilde{k}}}\partial_k\partial_{\tilde{k}}\psi^t(z_1,\dots,z_K)z_{K+1}$$

where ψ^t, and thus also ψ^0 are given by (5.27), we immediately obtain (5.38) from (5.28).

Step 2: (5.39) *and defining the remaining error terms.* From Lemma 5.6 we have (5.32). Comparing this with (5.39), there is a series of slight adjustments to obtain the appropriate form, for which we have to pay with the respective error term. Recall that

$$\boldsymbol{\zeta}^t_h := N^{1/2}(\boldsymbol{\phi}^t_h, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge \boldsymbol{T}\wedge T_\oslash))_h.$$

First, in order to adjust $(\rho^{r_I}_{h,\alpha})^+ \longrightarrow \rho^{r_I}_{h,\alpha}$, we have to pay

$$\begin{aligned}\mathrm{Err}_{\mathrm{neg}} := \sum_{k,\tilde{k}=1}^{K}\sum_{\alpha=1}^{n_S}\sigma_\alpha\int_0^{T_k\wedge T_{\tilde{k}}}&\mathbb{E}\big[\partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}^t_h)\\ &\times(\nabla_h\phi^t_{h,k,\alpha}\cdot\nabla_h\phi^t_{h,\tilde{k},\alpha}, (\rho^{r_I}_{h,\alpha})^-(t\wedge T_\oslash))_h\big]\,\mathrm{d}t.\end{aligned}$$

Second, to adjust $\nabla_h\phi^t_{h,k,\alpha}\cdot\nabla_h\phi^t_{h,\tilde{k},\alpha} \longrightarrow \mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}]$, we pay

$$\begin{aligned}\mathrm{Err}_{\mathrm{num},1} := \sum_{k,\tilde{k}=1}^{K}\sum_{\alpha=1}^{n_S}\sigma_\alpha\int_0^{T_k\wedge T_{\tilde{k}}}&\mathbb{E}\big[\partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}^t_h)\\ &\times((\nabla_h\phi^t_{h,k,\alpha}\cdot\nabla_h\phi^t_{h,\tilde{k},\alpha} - \mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}], \rho^{r_I}_{h,\alpha}(t\wedge T_\oslash))_h)\big]\,\mathrm{d}t.\end{aligned}$$

Third, for the adjustment $\overline{\rho}^{r_I}_\alpha \longrightarrow \overline{\rho}^{r_I}_{h,\alpha}$, we have to pay

$$\begin{aligned}\mathrm{Err}_{\mathrm{num},2} := \sum_{k,\tilde{k}=1}^{K}\sum_{\alpha=1}^{n_S}\sigma_\alpha\int_0^{T_k\wedge T_{\tilde{k}}}&\mathbb{E}\big[\partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}^t_h)\big]\\ &\times((\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}], \overline{\rho}^{r_I}_{h,\alpha}(t\wedge T_\oslash))_h - \langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}, \overline{\rho}^{r_I}_\alpha(t\wedge T_\oslash)\rangle)\,\mathrm{d}t.\end{aligned}$$

For adjusting $\boldsymbol{\zeta}^t_h \longrightarrow N^{1/2}(\mathcal{I}_h[\boldsymbol{\phi}^t], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T}))_h$ as in the argument of ψ^t, we pay

$$\begin{aligned}\mathrm{Err}_{\mathrm{num},3} := \sum_{k,\tilde{k}=1}^{K}\sum_{\alpha=1}^{n_S}\sigma_\alpha\int_0^{T_k\wedge T_{\tilde{k}}}&\mathbb{E}\big[\big(\partial_k\partial_{\tilde{k}}\psi^t(N^{1/2}(\boldsymbol{\phi}^t_h, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T}\wedge T_\oslash))_h)\\ &-\partial_k\partial_{\tilde{k}}\psi^t(N^{1/2}(\mathcal{I}_h[\boldsymbol{\phi}^t], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T}\wedge T_\oslash))_h)\big)\\ &\times(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}], (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash))_h\big]\,\mathrm{d}t.\end{aligned}$$

Finally, in order to adjust $\boldsymbol{\phi}^0_h \to \mathcal{I}_h[\boldsymbol{\phi}^0]$, we pay

$$\begin{aligned}\mathrm{Err}_{\mathrm{num},4} := \mathbb{E}\big[&\psi^0(N^{1/2}(\mathcal{I}_h[\boldsymbol{\phi}^0], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(0))_h)\\ &-\psi^0(N^{1/2}(\boldsymbol{\phi}^0_h, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(0))_h)\big].\end{aligned}$$

Setting

$$\mathrm{Err}_{\mathrm{num}} := \mathrm{Err}_{\mathrm{num},1} + \mathrm{Err}_{\mathrm{num},2} + \mathrm{Err}_{\mathrm{num},3} + \mathrm{Err}_{\mathrm{num},4},$$

we obtain (5.39).

Step 3: Estimates for ψ^0, $\widetilde{\psi}^t_{k\tilde{k}}$, $\boldsymbol{\phi}^0$, $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}$. Estimates for $\boldsymbol{\phi}^0$ and ψ^0 immediately follow from Lemma 5.2 and Lemma 5.5, respectively. With respect to $\widetilde{\psi}^t_{k\tilde{k}}$, we observe that

$$\|\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}\|_{H^{s-1}} \le 2C(s)\sum_{l=0}^{\lfloor (s-1)/2\rfloor}\|D^{l+1}\boldsymbol{\phi}^t\|_{L^\infty}\|D^{s-l}\boldsymbol{\phi}^t\|_{L^2} \le C(s)\|\boldsymbol{\phi}^t\|^2_{H^s},$$

where we use $s > d+2$ for the Sobolev embedding $H^s \subset\subset C^{\lfloor (s+1)/2\rfloor}$. With the definition of $\widetilde{\psi}^t_{k\tilde{k}}$ and Lemma A.1, we thus obtain (5.11). We obtain (5.7) for $\widetilde{\psi}^t_{k\tilde{k}}$ by using its definition and (5.29), thus getting

$$\|\widetilde{\psi}^t_{k\tilde{k}}\|_{\mathcal{L}^{q-2}_{\mathrm{pow},r+1}} \le \|\psi^t\|_{\mathcal{L}^q_{\mathrm{pow},r}} \le CN^\kappa(1+\|\boldsymbol{\varphi}\|^r_{H^s})\|\psi\|_{\mathcal{L}^q_{\mathrm{pow},r}}.$$

Step 4: Estimates for the negativity and numerical errors. Estimate (5.13) for $\mathrm{Err}_\oslash$ corresponds to the estimates in the proof of Lemma 5.6. With respect to (5.14), we have $\mathrm{Err}_{\mathrm{neg}} = 0$ since for $T_\oslash > 0$, all $\alpha = 1,\dots,n_S$

$$\rho^{r_I}_{h,\alpha}(t\wedge T_\oslash) \ge \overline{\rho}^{r_I}_{h,\alpha}(t\wedge T_\oslash) - N^{-\epsilon} \ge \rho_{\min,h} - N^{-\epsilon} \ge 0 \tag{5.43}$$

due to the definition of $T_\oslash$ and Remark 3.14 for N large enough.

For the rest of this step, to prove the bound on $\mathrm{Err}_{\mathrm{num}}$, take an arbitrary $\kappa > 0$. The following estimates hold for N large enough with the κ-depending bound corresponding to the, respectively, used results. To simplify the notation, we allow for the specific value of κ to change from line to line here. Note that, for $T_\oslash > 0$,

$$|\boldsymbol{\zeta}^t_h| \le N^{1/2}\|\boldsymbol{\phi}^t_h\|_{H^{d+2}_h}\|(\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge \boldsymbol{T}\wedge T_\oslash)\|_{H_h^{-2\lfloor d/2\rfloor+2}} \le N^{\epsilon+\kappa}\|\boldsymbol{\varphi}\|_{H^s} \tag{5.44}$$

due to (B.11) from Lemma B.8 paired with $\|\mathcal{I}_h[\cdot]\|_{H^{d+2}_h} \lesssim \|\cdot\|_{C^{d+2}} \lesssim \|\cdot\|_{H^s}$ as well as the definition of $T_\oslash$.

With respect to $\mathrm{Err}_{\mathrm{num},1}$, if $T_\oslash > 0$, we have

$$\begin{aligned}\|\rho^{r_I}_h(t\wedge T_\oslash)\|_{L^2_h} &\le \|\overline{\rho}^{r_I}_h\|_{L^\infty(L^2_h)} + C\|(\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge T_\oslash)\|_{L^\infty_h}\\ &\le C\|\overline{\rho}^{r_I}\|_{L^\infty(C^0)} + \|\overline{\rho}^{r_I}_h - \mathcal{I}_h[\overline{\rho}^{r_I}]\|_{L^\infty(L^2_h)} + N^{-\epsilon} \le C\end{aligned}$$

by the definition of $T_\oslash$, Proposition B.6, and the scaling from Assumption 3.11. Thus, with (5.29) to bound $\|\psi^t\|_{\mathcal{L}^2_{\mathrm{pow},r}}$ and (B.17) from Lemma B.9, we obtain

$$|\mathrm{Err}_{\mathrm{num},1}| \le C(\|\overline{\rho}^{r_I}\|_{L^\infty})\|\psi\|_{\mathcal{L}^2_{\mathrm{pow},r}}(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s})\|\boldsymbol{\varphi}\|^2_{H^s}N^{r\varepsilon+\kappa}h^{p+1}. \tag{5.45}$$

With respect to $\mathrm{Err}_{\mathrm{num},2}$, applying (B.1) from Proposition B.6 as well as the Euler–Maclaurin formula—see Lemma B.4—we have

$$\begin{aligned}&\big|(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}], \overline{\rho}^{r_I}_{h,\alpha}(t))_h - \langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}, \overline{\rho}^{r_I}_\alpha(t)\rangle\big|\\ &\le \big|(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}], (\overline{\rho}^{r_I}_{h,\alpha} - \mathcal{I}_h[\overline{\rho}^{r_I}_\alpha])(t))_h\big|\\ &\quad + \big|(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}], \mathcal{I}_h[\overline{\rho}^{r_I}_\alpha])(t))_h - \langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}, \overline{\rho}^{r_I}_\alpha(t)\rangle\big|\\ &\le C\|\boldsymbol{\phi}^t\|^2_{C^2}N^\kappa h^{p+1} + C\|\boldsymbol{\phi}^t\|^2_{C^{p+3}}\|\overline{\rho}^{r_I}(t)\|_{C^{p+2}}h^{p+1}.\end{aligned}$$

Since (5.1) implies $s > p+3+d/2$ and, therefore, $\|\cdot\|_{C^{p+3}} \lesssim \|\cdot\|_{H^s}$, with (B.11), it follows that

$$|\mathrm{Err}_{\mathrm{num},2}| \leq C\big(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+2})}\big)\|\psi\|_{\mathcal{L}^2_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r}\big)\|\boldsymbol{\varphi}\|_{H^s}^2 N^{r\epsilon+\kappa}h^{p+1}. \tag{5.46}$$

Concerning $\mathrm{Err}_{\mathrm{num},3}$, for $T_\oslash > 0$ by the definition of $T_\oslash$, we have

$$\big|\big(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}],(\rho^{r_I}_{h,\alpha}-\overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash)\big)_h\big| \leq \|\boldsymbol{\phi}^t\|^2_{C^{d+3}}N^{-1/2+\epsilon}, \tag{5.47}$$

and using (B.15), we get

$$\begin{aligned}\big|\boldsymbol{\zeta}^t_h - N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^t],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T})\big)_h\big| &\leq C\|\mathcal{I}_h[\boldsymbol{\phi}^t]-\boldsymbol{\phi}^t_h\|_{H^{d+2}_h}N^\epsilon \\ &\leq \|\boldsymbol{\varphi}\|_{H^s}N^\kappa h^{p+1}N^\epsilon,\end{aligned} \tag{5.48}$$

where we also used $\|\cdot\|_{C^{d+2+p+3}} \lesssim \|\cdot\|_{H^s}$ since $s > p+3d/2+5$. In particular, for the connecting segment $I := [\boldsymbol{\zeta}^t_h, N^{1/2}(\mathcal{I}_h[\boldsymbol{\phi}^t],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T}))_h] \subset \mathbb{R}^K$, it holds that $\sup_{u\in I}|u| \leq CN^{\epsilon+\kappa}\|\boldsymbol{\varphi}\|_{H^s}$. Thus, by mean value theorem and the definition of $\|\cdot\|_{\mathcal{L}^q_{\mathrm{pow},r}}$,

$$|\mathrm{Err}_{\mathrm{num},3}| \leq C\|\psi\|_{\mathcal{L}^3_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)\|\boldsymbol{\varphi}\|^3_{H^s}N^{-1/2+(r+2)\epsilon+\kappa}h^{p+1}. \tag{5.49}$$

As for $\mathrm{Err}_{\mathrm{num},4}$, including the case $T_\oslash = 0$ in (5.48) and dealing with it as in the estimate of $\mathrm{Err}_\oslash$ in the proof of Lemma 5.6 via the mean value theorem, we obtain

$$|\mathrm{Err}_{\mathrm{num},4}| \leq C\|\psi\|_{\mathcal{L}^1_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)\|\boldsymbol{\varphi}\|_{H^s}N^{(r+1)\epsilon+\kappa}h^{p+1}. \tag{5.50}$$

Combining (5.45), (5.46), (5.49), and (5.50) yields (5.15) for $\mathrm{Err}_{\mathrm{num}}$. □

5.3. *Including the linearisation errors in the iterative structure.* Proposition 5.7 leaves us with the task to fit $\mathrm{Err}_{\mathrm{lin},a}$ and $\mathrm{Err}_{\mathrm{lin},b}$ to the iterative structure. These are the terms induced by the linearisation compensations $Q^{r_I,N}_t(\cdot)$ and $Q^{r_I,N}_{h,t}(\cdot)$, which stem from linearising the respective interaction terms in the continuous and discrete setting,

$$Q^{r_I,N}_t(\phi^t_k) = \sum_{\alpha,\beta=1}^{n_S}\big\langle\nabla\phi^t_{k,\alpha}\cdot\big(\nabla V^{r_I}_{\alpha\beta}*\big(\mu^{r_I,N}_{\beta,t}-\overline{\rho}^{r_I}_\beta(t)\big)\big),\mu^{r_I,N}_{\alpha,t}-\overline{\rho}^{r_I}_\alpha(t)\big\rangle,$$

$$Q^{r_I,N}_{h,t}(\phi^t_{h,k}) = \sum_{\alpha,\beta=1}^{n_S}\big(\nabla_h\phi^t_{h,k,\alpha}\cdot\big(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}]*_h(\rho^{r_I}_{h,\beta}-\overline{\rho}^{r_I}_{h,\beta})(t)\big),(\rho^{r_I}_{h,\alpha}-\overline{\rho}^{r_I}_{h,\alpha})(t)\big)_h.$$

We now rewrite $Q^{r_I,N}_t(\phi^t_k)$ and $Q^{r_I,N}_{h,t}(\phi^t_{h,k})$ (and thus $\mathrm{Err}_{\mathrm{lin},a}$ and $\mathrm{Err}_{\mathrm{lin},b}$) to conform them to the iterative structure as outlined in Section 4.2.2.

PROPOSITION 5.9 (Iterative structure for the linearisation compensations). *Under the assumptions of Theorem 5.1, let* $\mathrm{Err}_{\mathrm{lin},a}$ *and* $\mathrm{Err}_{\mathrm{lin},b}$ *be defined as in Proposition 5.7, (5.40), and (5.41). Then for all* $t\in(0,T)$, $k\in\{1,\dots,K\}$, $m,n\in\mathbb{Z}^d$, $\alpha,\beta\in\{1,\dots,n_S\}$, *and* $\ell\in\{1,\dots,d\}$, *there exist generalised moment functions* $\breve{\psi}^t_k\in\mathcal{L}^{q-1}_{\mathrm{pow},r+2}(\mathbb{R}^{K+2})$, *coefficients* $\hat{F}^\ell_{k,\alpha\beta}[m,n]$ *with sets of test functions* $\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}\in[H^s(\mathbb{T}^d,\mathbb{R}^{n_S})]^{K+2}$, *and test times* $\breve{\boldsymbol{T}}_k\in[0,T]^{K+2}$ *such that*

$$\begin{aligned}\mathrm{Err}_{\mathrm{lin},a} = N^{-1/2}\sum_{k=1}^K\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^d\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^\ell_{k,\alpha\beta}[m,n]\\ \times\mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\langle\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell},\mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k}-\overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\rangle\big)\big]\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\mathrm{Err}_{\mathrm{lin},b} = N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\\ \times\mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\big(\mathcal{I}_h\big[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}\big],\big(\rho_h^{r_I}-\overline{\rho}_h^{r_I}\big)(t\wedge\breve{\boldsymbol{T}}_k\wedge T_{\oslash})\big)_h\big)\big]\,\mathrm{d}t\\ +\mathrm{Err}_{\mathrm{num,lin}}.\end{aligned}$$

For any $\kappa>0$, there exists $N_0=N_0(\kappa,\epsilon,r,s,K,data)$ such that, for $N>N_0$, the following estimates hold. The generalised moment functions $\{\breve{\psi}^t_k\}_{t,k}$ are subject to (5.8), the coefficients $\{\hat{F}_{\alpha\beta}[m,n]\}_{\alpha,\beta,m,n}$ to (5.9). The test functions $\{\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}\}$ satisfy (5.12). For the error term, the estimate (5.15) holds.

PROOF. *Step 1: Rewriting $Q_t^{r_I,N}(\phi^t_k)$.* We have $Q_t^{r_I,N}(\phi^t_k)=\sum_{\alpha,\beta=1}^{n_S}I_{\alpha,\beta}$ with

$$I_{\alpha,\beta}=\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\nabla\phi^t_{k,\alpha}(x)\cdot\nabla V^{r_I}_{\alpha\beta}(x-y)\,\mathrm{d}\big(\mu^{r_I,N}_{\beta,t}-\overline{\rho}^{r_I}_{\beta}(t)\big)(y)\,\mathrm{d}\big(\mu^{r_I,N}_{\alpha,t}-\overline{\rho}^{r_I}_{\alpha}(t)\big)(x).$$

For $\ell=1,\dots,d$, we use the $2d$-dimensional Fourier series representation

$$\partial_\ell V^{r_I}_{\alpha\beta}(x-y)=\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\cos(m\cdot y)\cos(n\cdot x),\tag{5.51}$$

with

$$\hat{F}^{\ell}_{k,\alpha\beta}[m,n]:=\frac{1}{(2\pi)^{2d}}\int_{\mathbb{T}^d\times\mathbb{T}^d}\partial_\ell V^{r_I}_{\alpha\beta}(x-y)\cos(m\cdot y)\cos(n\cdot x)\,\mathrm{d}x\,\mathrm{d}y.$$

Here we replaced $e^{in\cdot x}$ with $\cos(n\cdot x)$ since V is real-valued. Since $(x,y)\to\partial_\ell V^{r_I}_{\alpha\beta}(x-y)$ is in $W^{1,s_I-1}(\mathbb{T}^d\times\mathbb{T}^d)$, we have pointwise convergence of the Fourier series (e.g., see [42], Theorem 3.3.9 and Corollaries 3.4.9/3.4.10) with coefficients estimated by

$$\big|\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\big|\le C(s_I)\frac{\|\nabla V^{r_I}_{\alpha\beta}\|_{W^{s_I-1,1}(\mathbb{T}^d)}}{(|m|_2+|n|_2)^{s_I-1}},$$

which yields (5.9) after applying the scaling relation (3.15) from Assumption 3.11 to $\|\nabla V^{r_I}_{\alpha\beta}\|_{W^{s_I-1,1}(\mathbb{T}^d)}\le Cr_I^{-s_I}$. Thus, since $s_I>2d+1$, we also have uniform convergence of the Fourier series, and, therefore, abbreviating $\vartheta_n(x):=\cos(n\cdot x)$ for $n\in\mathbb{Z}^d$, we obtain

$$I_{\alpha,\beta}=\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\big\langle\vartheta_m,\mu^{r_I,N}_{\beta,t}-\overline{\rho}^{r_I}_{\beta}(t)\big\rangle\big\langle\partial_\ell\phi^t_{k,\alpha}\vartheta_n,\mu^{r_I,N}_{\alpha,t}-\overline{\rho}^{r_I}_{\alpha}(t)\big\rangle.$$

Adjusting for N-prefactors, this leads to

$$\begin{aligned}NQ_t^{r_I,N}(\phi^t_k)=\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\Bigg(N^{1/2}\sum_{\gamma=1}^{n_S}\big\langle\delta^{\beta\gamma}\vartheta_m,\mu^{r_I,N}_{\gamma,t}-\overline{\rho}^{r_I}_{\gamma}(t)\big\rangle\Bigg)\\ \times\Bigg(N^{1/2}\sum_{\gamma=1}^{n_S}\big\langle\delta^{\alpha\gamma}\partial_\ell\phi^t_{k,\alpha}\vartheta_n,\mu^{r_I,N}_{\gamma,t}-\overline{\rho}^{r_I}_{\gamma}(t)\big\rangle\Bigg),\end{aligned}$$

where $\delta^{\beta\gamma}$ is the Kronecker-Delta.

Step 2: Rewriting $Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi^t_{h,k})$. First, we get an error for switching from $\nabla_h\phi^t_{h,k,\alpha}$ to $\mathcal{I}_h[\nabla\phi^t_{k,\alpha}]$. That is, we have

$$Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi^t_{h,k})=\sum_{\alpha,\beta=1}^{n_S}I^h_{\alpha,\beta}+\overline{\mathrm{Err}}^{k,t}_{\mathrm{num},1}$$

with

$$\begin{aligned}\overline{\mathrm{Err}}^{k,t}_{\mathrm{num},1} &:= \sum_{\alpha,\beta=1}^{n_S} \big((\nabla_h \phi^t_{h,k,\alpha} - \mathcal{I}_h[\nabla \phi^t_{k,\alpha}]) \\ &\qquad \cdot (\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t\wedge T_\oslash)), \\ &\qquad (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash)\big)_h, \\ I^h_{\alpha,\beta} &:= h^{2d} \sum_{x,y\in G_{h,d}} \nabla\phi^t_{k,\alpha}(x)\cdot \nabla V^{r_I}_{\alpha\beta}(x-y)(\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t\wedge T_\oslash, y) \\ &\qquad \times (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash, x).\end{aligned} \tag{5.52}$$

Plugging in (5.51), we obtain

$$I^h_{\alpha,\beta} = \sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d} \hat{F}^\ell_{k,\alpha\beta}[m,n](\mathcal{I}_h[\vartheta_m], \rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})_h (\mathcal{I}_h[\partial_\ell \phi^t_{k,\alpha}\vartheta_n], \rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})_h (t\wedge T_\oslash),$$

and, therefore,

$$\begin{aligned} N Q^{r_I,N}_{h,t}(\phi^t_{h,k}) &= \sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d} \hat{F}^\ell_{k,\alpha\beta}[m,n]\Bigg(N^{1/2}\sum_{\gamma=1}^{n_S}(\mathcal{I}_h[\delta^{\beta\gamma}\vartheta_m], \rho^{r_I}_{h,\gamma} - \overline{\rho}^{r_I}_{h,\gamma})_h\Bigg) \\ &\quad \times \Bigg(N^{1/2}\sum_{\gamma=1}^{n_S}(\mathcal{I}_h[\delta^{\alpha\gamma}\partial_\ell\phi^t_{k,\alpha}\vartheta_n], \rho^{r_I}_{h,\gamma} - \overline{\rho}^{r_I}_{h,\gamma})_h\Bigg)(t\wedge T_\oslash) + N\overline{\mathrm{Err}}^{k,t}_{\mathrm{num},1}.\end{aligned}$$

Step 3: Defining the iterated functions. Defining $\breve{\psi}^t_k : \mathbb{R}^{K+2}\to\mathbb{R}$ as

$$\breve{\psi}^t_k(z) := -\chi_{t\le T_k}\partial_k \psi^t(z_1,\dots,z_K) z_{K+1} z_{K+2},$$

we have via the dominated convergence theorem (see Step 5 for a full justification)

$$\begin{aligned}\mathrm{Err}_{\mathrm{lin},a} &= N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^\ell_{k,\alpha\beta}[m,n] \\ &\qquad \times \mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\langle \breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}, \mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k} - \overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\rangle\big)\big]\,\mathrm{d}t\end{aligned} \tag{5.53}$$

with

$$\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}(x) = \begin{pmatrix} (\phi^t_{1,\gamma}(x))_{\gamma=1,\dots,n_S} \\ \vdots \\ (\phi^t_{K,\gamma}(x))_{\gamma=1,\dots,n_S} \\ (\delta^{\beta\gamma}\vartheta_m(x))_{\gamma=1,\dots,n_S} \\ (\delta^{\alpha\gamma}\partial_\ell\phi^t_{k,\alpha}(x)\vartheta_n(x))_{\gamma=1,\dots,n_S}\end{pmatrix}, \qquad \breve{\boldsymbol{T}}_k = \begin{pmatrix} T_1 \\ \vdots \\ T_K \\ T_k \\ T_k\end{pmatrix}. \tag{5.54}$$

On the discrete side, we analogously have again by dominated convergence (see Step 5)

$$\begin{aligned}\mathrm{Err}_{\mathrm{lin},b} &= N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^\ell_{k,\alpha\beta}[m,n] \\ &\qquad \times \mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}(\mathcal{I}_h[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash))_h\big)\big]\,\mathrm{d}t \\ &\qquad + \mathrm{Err}_{\mathrm{num},5} + \mathrm{Err}_{\mathrm{num},6},\end{aligned} \tag{5.55}$$

where $\mathrm{Err}_{\mathrm{num},5}$ stems from the errors $\overline{\mathrm{Err}}^{k,t}_{\mathrm{num},1}$ defined in (5.52) of Step 2 and is given by

$$\begin{aligned}\mathrm{Err}_{\mathrm{num},5} := -\sum_{k=1}^{K}\int_0^{T_k} \mathbb{E}\big[\partial_k \psi^t\big(N^{1/2}\big(\boldsymbol{\phi}^t_h, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash)\big)_h\big) \\ \times N^{1/2}\overline{\mathrm{Err}}^{k,t}_{\mathrm{num},1}\big]\,\mathrm{d}t,\end{aligned} \tag{5.56}$$

while $\mathrm{Err}_{\mathrm{num},6}$ is obtained from replacing $\boldsymbol{\zeta}^t_h \longrightarrow N^{1/2}(\mathcal{I}_h[\boldsymbol{\phi}^t], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \check{\boldsymbol{T}}_k \wedge T_\oslash))_h$, leading to

$$\begin{aligned}\mathrm{Err}_{\mathrm{num},6} := -\sum_{k=1}^{K}\int_0^{T_k} \mathbb{E}\big[\big(\partial_k \psi^t\big(N^{1/2}\big(\boldsymbol{\phi}^t_h, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash)\big)_h\big) \\ - \partial_k \psi^t\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^t], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash)\big)_h\big)\big) \\ \times N^{1/2} Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi^t_{h,k})\big]\,\mathrm{d}t.\end{aligned} \tag{5.57}$$

Step 4: Collecting the estimates. With respect to the test functions, we recall that $\vartheta_n(x) := \cos(n \cdot x)$. Thus, for $n \in \mathbb{Z}^d$, $\tilde{s} \in \mathbb{N}_0$, we have

$$\|\vartheta_n\|_{H^{\tilde{s}}(\mathbb{T}^d)} \le C|n|_\infty^{\tilde{s}}. \tag{5.58}$$

This, based on (5.54) together with Lemma A.1 and the same argument from Step 3 in the proof of Proposition 5.7, yields (5.12). Using the definition of the new generalised moment function and (5.29), we obtain

$$\|\check{\psi}^t_k\|_{\mathcal{L}^{q-1}_{\mathrm{pow},r+2}} \le \|\psi^t\|_{\tilde{\mathcal{L}}^q_{\mathrm{pow},r}} \le CN^\kappa\big(1 + \|\varphi\|^r_{H^s}\big)\|\psi\|_{\mathcal{L}^q_{\mathrm{pow},r}}.$$

For estimating the error terms, let $\kappa > 0$ be an arbitrary, small exponent. The following estimates hold for N large enough with the κ-depending bound corresponding to the, respectively, used results. To simplify the notation, we allow for the value of κ to change from line to line. For any $k = 1, \ldots, K$, if $T_\oslash > t > 0$, the discrete Hölder inequality (Corollary B.2) entails

$$\big|N^{1/2}\overline{\mathrm{Err}}^{k,t}_{\mathrm{num},1}\big| \le CN^{\epsilon+\kappa}\|\nabla_h \boldsymbol{\phi}^t_h - \mathcal{I}_h[\nabla \boldsymbol{\phi}^t]\|_{L^1_h}\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L^\infty_h} \le C\|\boldsymbol{\varphi}\|_{H^s}N^\kappa h^{p+1},$$

where for the second inequality we used the definition of $T_\oslash$ that $\|\cdot\|_{L^1_h} \lesssim \|\cdot\|_{L^2_h}$ and (B.16); for the first inequality, we used that, for $T_\oslash > t > 0$, it holds that

$$\begin{aligned}&N^{1/2}\|\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t)\|_{L^\infty_h} \\ &\quad\le N^{1/2}\|\mathcal{I}_h[\nabla V^{r_I}]\|_{H^{2\lfloor d/2\rfloor+2}}\|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{H^{-2\lfloor d/2\rfloor-2}}(t) \\ &\quad\le N^\epsilon\|\nabla V^{r_I}\|_{C^{d+3}} \le N^\epsilon {r_I}^{-(2d+4)} \le N^{\epsilon+\kappa}.\end{aligned}$$

Here the last two inequalities are due to the definition of V^{r_I}, given in (3.2), and Assumption 3.11. Combining this estimate with $|\boldsymbol{\zeta}^t_h|$-estimate (5.44) from the proof of Proposition 5.7 and (5.29) from Lemma 5.6 yields

$$|\mathrm{Err}_{\mathrm{num},5}| \le C\|\psi\|_{\mathcal{L}^1_{\mathrm{pow},r}}\big(1 + \|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)\|\boldsymbol{\varphi}\|_{H^s}N^{r\epsilon+\kappa}h^{p+1}. \tag{5.59}$$

With respect to $\mathrm{Err}_{\mathrm{num},6}$ using (5.37) for $0 < t < T_\oslash$ from the proof of Lemma 5.6, analogously to the $|\mathrm{Err}_{\mathrm{num},3}|$-estimate (5.49) from the proof of Proposition 5.7 via the mean value theorem, we obtain

$$C\|\psi\|_{\mathcal{L}^2_{\mathrm{pow},r}}\big(1 + \|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)\|\boldsymbol{\varphi}\|^2_{H^s}N^{(r+1)\epsilon+\kappa}h^{p+1}. \tag{5.60}$$

Combining (5.59) and (5.60) yields (5.15) for $\mathrm{Err}_{\mathrm{num},5} + \mathrm{Err}_{\mathrm{num},6}$.

Step 5: Commuting infinite sum and expected value for (5.53) *and* (5.55). We have with (5.9) and (5.12) (or more precisely, (5.58)) that

$$
\begin{aligned}
&\big|\hat{F}^t_{k,\alpha\beta}[m,n]\breve{\psi}^t_k\big(N^{1/2}\langle\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}, \mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k} - \overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\rangle\big)\big| \\
&\quad\le CN^\kappa\big(|m|_2+|n|_2\big)^{-s_I+1}\|\breve{\psi}^t_k\|_{\mathcal{L}^0_{\mathrm{pow},r+2}} \\
&\qquad\times N^{(r+2)/2}\|\breve{\boldsymbol{\phi}}^t_{k,mn,\ell}\|^{r+2}_{H^{\lceil d/2+1\rceil}}\|\mu^{r_I,N}-\overline{\rho}^{r_I}\|^{r+2}_{H^{-\lceil d/2+1\rceil}} \\
&\quad\le CN^\kappa\big(1+\|\boldsymbol{\varphi}\|^{r+2}_{H^s}\big)\big(|m|_2+|n|_2\big)^{-s_I+1}\big(|m|_\infty+|n|_\infty\big)^{(r+2)\lceil d/2+1\rceil} \\
&\qquad\times\|\breve{\psi}^t_k\|_{\mathcal{L}^0_{\mathrm{pow},r+2}}N^{(r+2)/2}\|\mu^{r_I,N}-\overline{\rho}^{r_I}\|^{r+2}_{H^{-\lceil d/2+1\rceil}},
\end{aligned}
$$

which is a converging sum in n, m since $s_I > 2d+1+(r+2)\lceil d/2+1\rceil$ with (5.2), while the expected value is controlled due to Proposition 7.1. Thus, the application of the dominated convergence theorem in (5.53) is justified. In the discrete case, with $\|\mathcal{I}_h[\cdot]\|_{L^1_h} \lesssim \|\cdot\|_{C^0}$,

$$
\begin{aligned}
&\big|\hat{F}^\ell_{k,\alpha\beta}[m,n]\breve{\psi}^t_k\big(N^{1/2}\big(\mathcal{I}_h[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\big)_h\big| \\
&\quad\le CN^\kappa\|\boldsymbol{\varphi}\|_{H^s}\big(|m|_2+|n|_2\big)^{-s_I+1}\|\breve{\psi}^t_k\|_{\mathcal{L}^0_{\mathrm{pow},r+2}} \\
&\qquad\times N^{(r+2)/2}\|\breve{\boldsymbol{\phi}}^t_{k,mn,\ell}\|^{r+2}_{C^0}\|(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\|^{r+2}_{L^\infty_h} \\
&\quad\le CN^\kappa\big(1+\|\boldsymbol{\varphi}\|^{r+3}_{H^s}\big)\big(|m|_2+|n|_2\big)^{-s+1}N^{(r+2)/2-\epsilon(r+2)}
\end{aligned}
$$

due to the definition of the stopping time $T_\oslash$ in (8.2) and that $\|\breve{\boldsymbol{\phi}}^t_{k,mn,\ell}\|_{C^0} \le 1+\|\boldsymbol{\phi}^t\|_{H^s}$. Since $s_I > 2d+1$, the infinite sum other these terms converges and (5.55) is valid. □

6. Proof of Theorem 2.1. Our main result follows from the iterative application of Theorem 5.1, where Proposition 7.1, Proposition 8.2, and the closeness of the initial fluctuations provided by Assumption 3.10 are used to close the estimate after a sufficient number of iterations.

PROOF. We choose the stopping time $T_\oslash = T_\oslash(\epsilon/2,\epsilon)$, as defined in (8.2). With this choice, Proposition 8.2 directly yields (2.2). We further assume that, in the following, N is always large enough for the used auxiliary results to hold. The precise dependencies are summarized in the result. For this proof we allow for the specific value of $\kappa > 0$ to change from line to line, since all estimates hold for arbitrary small κ, given N is large enough (this can be easily adjusted down the line):

Step 1: The case $j = 1$. Since some of the following considerations will be needed for the case $j > 1$, we proceed under slightly more general assumptions than necessary for just the case $j = 1$. That is, let $\psi \in \mathcal{L}^q_{\mathrm{pow},r}$, $\boldsymbol{\varphi} \in [H^s(\mathbb{T}^d,\mathbb{R}^{n_S})]^K$ and $V \in [W^{s_I,1}(\mathbb{T}^d)]^{n_S\times n_S}$ with $q, r, s, s_I \in \mathbb{N}_0$. As in (4.2), we set

$$
\begin{aligned}
\mathcal{M}(\psi,\boldsymbol{\varphi},\boldsymbol{T}) :=\ &\mathbb{E}\big[\psi\big(N^{1/2}\langle\boldsymbol{\varphi},\mu^{r_I,N}_{\boldsymbol{T}}-\overline{\rho}^{r_I}(\boldsymbol{T})\rangle\big)\big] \\
&-\mathbb{E}\big[\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(\boldsymbol{T}\wedge T_\oslash)\big)_h\big)\big].
\end{aligned}
$$

Now, if $q \ge 3$, $s > p + 3d/2 + 5$ and $s_I > (d+2)(r+2)+2d+1$, Theorem 5.1 yields

$$
\mathcal{M}(\psi,\boldsymbol{\varphi},\boldsymbol{T}) = N^{-1/2}\sum_{k,\tilde{k}}^{K}\int_0^{T_k\wedge T_{\tilde{k}}}\mathcal{M}\big(\widetilde{\psi}^t_{k\tilde{k}=1},\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}},t\wedge\tilde{\boldsymbol{T}}_{k\tilde{k}}\big)
$$

$$
\begin{aligned}
\text{(6.1)} \qquad & + N^{-1/2} \sum_{k=1}^{K} \sum_{\alpha,\beta=1}^{n_S} \sum_{\ell=1}^{d} \int_0^{T_k} \sum_{m\in\mathbb{Z}^{2d}} \hat{F}^{\ell}_{k,\alpha\beta}[m] \\
& \times \mathcal{M}\big(\breve{\psi}^t_k, \breve{\boldsymbol{\phi}}^t_{k,m,\alpha\beta,\ell}, t\wedge \breve{\boldsymbol{T}}_k\big) + \mathrm{Err}(\psi, \boldsymbol{\varphi}, \boldsymbol{T}),
\end{aligned}
$$

with

$$
\mathrm{Err}(\psi, \boldsymbol{\varphi}, \boldsymbol{T}) := \mathcal{M}\big(\psi^0, \boldsymbol{\phi}^0, 0\big) + \mathrm{Err}_{\mathrm{num}} + \mathrm{Err}_{\mathrm{neg}} + \mathrm{Err}_{\oslash} \tag{6.2}
$$

for new ψ^0, $\widetilde{\psi}^t_{k\tilde{k}}$, $\breve{\psi}^t_k$, coefficients $\hat{F}^{\ell}_{k,\alpha\beta}[m]$, test functions $\boldsymbol{\phi}^0$, $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}$, $\breve{\boldsymbol{\phi}}^t_{k,m,\alpha\beta,\ell}$ as well as test times $\tilde{\boldsymbol{T}}_{k\tilde{k}}$, $\breve{\boldsymbol{T}}_k$ and errors $\mathrm{Err}_{\mathrm{num}}$, $\mathrm{Err}_{\mathrm{neg}}$, $\mathrm{Err}_{\oslash}$, as described in Theorem 5.1.

With (3.10) from Assumption 3.10, (5.6), (5.10), and $\|\cdot\|_{C^{p+1}} \lesssim \|\cdot\|_{H^{(d/2+p+1)+}}$, we have

$$
\big|\mathcal{M}\big(\psi^0, \boldsymbol{\phi}^0, 0\big)\big| \le C\|\psi\|_{\mathcal{L}^1_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{r+1}_{H^s}\big)h^{p+1},
$$

and thus with (5.15), (5.14), and (5.13),

$$
\big|\mathrm{Err}(\psi, \boldsymbol{\varphi}, \boldsymbol{T})\big| \le C\|\psi\|_{\mathcal{L}^3_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{2r+3}_{H^s}\big)N^{(r+1)\epsilon+\kappa}\big(h^{p+1}+\exp(-CN^{\epsilon/2})\big). \tag{6.3}
$$

Due to Proposition 7.1, definition (8.2), and $\|\mathcal{I}_h(\cdot)\|_{H^{d+2}_h} \lesssim \|\cdot\|_{C^{d+2}}$, we obtain the estimate

$$
\big|\mathcal{M}(\psi, \boldsymbol{\varphi}, \boldsymbol{T})\big| \le C\|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\|\boldsymbol{\varphi}\|^r_{H^{d/2+2}}N^{\kappa} + C\|\psi\|_{\mathcal{L}^0_{\mathrm{pow},r}}\|\boldsymbol{\varphi}\|^r_{C^{d+2}}N^{r\epsilon}.
$$

Recalling the definition (5.7) for $\widetilde{\psi}^t_{k\tilde{k}}$ and (5.11) for $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}$ and using $\|\cdot\|_{C^{d+2}} \lesssim \|\cdot\|_{H^s}$,

$$
\big|\mathcal{M}\big(\widetilde{\psi}^t_{k\tilde{k}}, \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, t\wedge\tilde{\boldsymbol{T}}_{k\tilde{k}}\big)\big| \le C\|\psi\|_{\mathcal{L}^2_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{3r+2}_{H^s}\big)N^{(r+1)\epsilon+\kappa}. \tag{6.4}
$$

Furthermore, with (5.8) and $\|\breve{\boldsymbol{\phi}}^t_m\|_{C^{d+2}} \le (1+|m|^{d+2})(1+N^{\kappa}\|\boldsymbol{\phi}\|_{H^s})$ due to (5.54), we get an inequality concerning the terms from the linearization compensation, namely,

$$
\begin{aligned}
\text{(6.5)} \qquad & \big|\mathcal{M}\big(\breve{\psi}^t_k, \breve{\boldsymbol{\phi}}^t_{k,m,\alpha\beta,\ell}, t\wedge\breve{\boldsymbol{T}}_k\big)\big| \\
& \le C\|\psi\|_{\mathcal{L}^1_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^r_{H^s}\big)\big(1+|m|^{(r+2)(d+2)}_{\infty}\big)\big(1+\|\boldsymbol{\varphi}\|^{r+2}_{H^s}\big)N^{(r+2)\epsilon+\kappa}.
\end{aligned}
$$

Plugging these back into (6.1) and applying (5.9) to $\hat{F}^{\ell}_{k,\alpha\beta}[m]$, we obtain

$$
\begin{aligned}
\text{(6.6)} \qquad & \big|\mathcal{M}(\psi, \boldsymbol{\varphi}, \boldsymbol{T})\big| \\
& \le C\|\psi\|_{\mathcal{L}^3_{\mathrm{pow},r}}\big(1+\|\boldsymbol{\varphi}\|^{3r+3}_{H^s}\big)\big(N^{-1/2}+h^{p+1}+\exp(-CN^{\epsilon/2})\big)N^{(r+2)\epsilon+\kappa},
\end{aligned}
$$

where the infinite sum converges since we assumed $s_I > (d+2)(r+2)+2d+1$.

These considerations, together with the definition of $s(d,p,j)$ and the assumption on s_I as given in Theorem 2.1, yield (2.3) (specifically, the assumptions for $j=1$ correspond to $s = s(d,p,1) = p+3d/2+5$ and $s_I > 3(p+3d/2+5)+2d+1$, while the use of the norm d_{-3} corresponds to $q=3$ and $r=0$).

Step 2: The case $j>1$. We iteratively apply Theorem 5.1 j-times starting with arbitrary $\boldsymbol{\varphi} \in [H^{s_0}(\mathbb{T}^d, \mathbb{R}^{n_S})]^K$, $s_0 = s(d,p,j)$, and arbitrary $\psi \in \mathcal{L}^{q_0}_{\mathrm{pow},r_0}$ corresponding to the metric $d_{-(2j+1)}$, that is, $q_0 = 2j+1$, $r_0 = 0$.

This results in a tree structure: Starting from $(\psi, \boldsymbol{\phi})$, the first application of Theorem 5.1 spawns three successors: The error term $\mathrm{Err}(\psi, \boldsymbol{\phi})$ as defined in (6.2), the *regular* successor $(\widetilde{\psi}, \widetilde{\boldsymbol{\phi}})$, and the *compensation* successor $(\breve{\psi}, \breve{\boldsymbol{\phi}}_{m,\alpha\beta,\ell})_{m\in\mathbb{Z}^{2d}}$—for simplicity, we do not explicitly keep track of indices t, k, $\tilde{k}$, α, β, and the test times; all estimates are independent of these anyways. The edge to the regular successor is labeled with $C_{K,T}N^{-1/2}$, the edge to the compensation successor with $C_{K,T}N^{-1/2}\sum_{m\in\mathbb{Z}^{2d}}\hat{F}[m]$. The error term stays untouched and forms a leaf node of the tree. Then the same process is repeated for the regular and compensation successor. We iterate until we reach depth j.

Each path from the root node to a leaf node corresponds to one distinct group of terms resulting from the iterations. If we add up the bounds for these groups, we obtain a bound for $\mathcal{M}(\psi, \boldsymbol{\phi})$. Naturally, each path either ends in an error term or is of length j. Thus, with the considerations for the case $j = 1$, the corresponding terms are bounded by either $C(\|\psi\|_{C^{2j+1}}, \|\boldsymbol{\varphi}\|_{s_0})h^{p+1}N^{-\breve{j}/2+\breve{r}\epsilon+\kappa}$ or $C(\|\psi\|_{C^{2j+1}}, \|\boldsymbol{\varphi}\|_{s_0})N^{-j/2+\tilde{r}\epsilon+\kappa}$ for suitable $\breve{j}, \breve{r}, \tilde{r} \in \mathbb{N}_0$. We only need to show three things:

- The remaining regularity of the test functions and generalized moment functions at the second to last node of all paths is sufficient to close the estimates as in Step 1.
- All appearing combinations of $\breve{j}, \breve{r}, \tilde{r}$ fit the scaling in (2.4).
- s_I is chosen large enough so that along all paths each infinite sum converges.

Each path is represented by a sequence $(\psi_{0,\vec{m}_0}, \boldsymbol{\varphi}_{0,\vec{m}_0})_{\vec{m}_0}, \ldots, (\psi_{\tilde{j},\vec{m}_{\tilde{j}}}, \boldsymbol{\varphi}_{\tilde{j},\vec{m}_{\tilde{j}}})_{\vec{m}_{\tilde{j}}}, (\mathrm{leaf}_{\vec{m}_{\tilde{j}}})_{\vec{m}_{\tilde{j}}}$ with $\tilde{j} \leq j-1$, $(\psi_{0,\vec{m}_0}, \boldsymbol{\varphi}_{0,\vec{m}_0})_{\vec{m}_0} = (\psi, \boldsymbol{\varphi})$, and $\vec{m}_i$ for $0 \leq i \leq \tilde{j}$ being the vector of the collected $\mathbb{Z}^{2d}$-indices for all compensation nodes up until i.

The necessary regularity is given by sequences $(q_i)_{i=1,\ldots,\tilde{j}}, (r_i)_{i=1,\ldots,\tilde{j}}, (s_i)_{i=1,\ldots,\tilde{j}} \subset \mathbb{N}_0$ such that we control $\|\psi_{i,\vec{m}_i}\|_{\mathcal{L}^{q_i}_{\mathrm{pow},r_i}}$ and $\|\boldsymbol{\varphi}_{i,\vec{m}_i}\|_{H^{s_i}}$ in terms of their predecessors. With respect to $(q_i)_i$, from Step 1 we know that we need $q_{\tilde{j}} \geq 3$. For $i = 1, \ldots, \tilde{j}$, we have

$$q_i = \begin{cases} q_{i-1} - 2 & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ regular successor, corresponding to (5.7),} \\ q_{i-1} - 1 & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ compensation successor, see (5.8).} \end{cases}$$

Since by definition $q_0 = 2j+1$, we satisfy $q_{\tilde{j}} \geq 3$. For $(r_i)_i$, by definition $r_0 = 0$ and

$$r_i = \begin{cases} r_{i-1} + 1 & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ regular successor, corresponding to (5.7),} \\ r_{i-1} + 2 & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ compensation successor, see (5.8).} \end{cases}$$

Concerning $(s_i)_i$, with (5.11) and (5.12), for both types of successors we have $s_i = s_{i-1} - 1$. Since $s(d, p, j) = p + 3d/2 + 4 + j$, we guarantee $s_{\tilde{j}} > p + 3d/2 + 5$, as assumed in Step 1: In summary, for all i we have

$$r_i \leq 2i, \qquad s_i = p + \frac{3d}{2} + 4 + j - i. \tag{6.7}$$

It now remains to check that N^ϵ only appears as given in (2.4) and that s_I is chosen large enough to guarantee the convergence of the infinite sums. Let $((\psi_{i_\ell,\vec{m}_{i_\ell}}, \boldsymbol{\varphi}_{i_\ell,\vec{m}_{i_\ell}})_{\vec{m}_{i_\ell}})_{\ell=1,\ldots,l}$ with $(i_\ell)_\ell \subset \{1, \ldots, \tilde{j}\}$ be the compensation nodes on the path. Let $\mathcal{G}$ be the group of terms corresponding to this path. Respectively, (6.3), (6.4), and (6.5) yield an estimate for the leaf term as in (6.6). Adding the edge labels, we obtain

$$\begin{aligned} |\mathcal{G}| \leq C_{K,T} \sum_{\vec{m}_{\tilde{j}}} N^{-\tilde{j}/2} \Bigg(\prod_{\ell=1}^{l} \hat{F}[m_{i_\ell}] \Bigg) \|\psi_{\tilde{j},\vec{m}_{\tilde{j}}}\|_{\mathcal{L}^{q_{\tilde{j}}}_{\mathrm{pow},r_{\tilde{j}}}} \\ \times \big(1 + \|\boldsymbol{\varphi}_{\tilde{j}}\|^{3r_{\tilde{j}}+3}_{H^{s_{\tilde{j}}}}\big)\big(N^{-(j-\tilde{j})/2} + h^{p+1} + \exp(-CN^{\epsilon/2})\big)N^{(r_{\tilde{j}}+2)\epsilon+\kappa}. \end{aligned}$$

With respect to the N^ϵ-scaling, note that the term $N^{-(j-\tilde{j})/2}$ actually only appears if the leaf does not correspond to an error term, in particular if $\tilde{j} = j-1$. Plugging in $r_{\tilde{j}} \leq 2\tilde{j}$ and using that $N^{-\tilde{j}/2}N^{(2\tilde{j}+2)\epsilon} < N^{2\epsilon}$ since $\epsilon < 1/4$, we obtain the scaling from (2.4).

With respect to the convergence of the sums, we iteratively replace $((\psi_{\tilde{j},\vec{m}_{\tilde{j}}}, \boldsymbol{\varphi}_{\tilde{j},\vec{m}_{\tilde{j}}})_{\vec{m}_{\tilde{j}}})$ with their predecessors via (5.7), (5.8), (5.11), and (5.12). Assuming convergence for the previous steps, we obtain a sequence $(\lambda_i)_{i=1,\dots,\tilde{j}} \subset \mathbb{N}$ such that

$$
\begin{aligned}
|\mathcal{G}| \le C_{K,T} \sum_{\vec{m}_i} \Big(\prod_{i_\ell \le i} \hat{F}[m_{i_\ell}] \Big) \|\psi_{i,\vec{m}_i}\|_{\mathcal{L}^{q_i}_{\mathrm{pow},r_i}} \\
\times \big(1 + \|\boldsymbol{\varphi}_i\|^{\lambda_i}_{H^{s_i}}\big)\big(N^{-(j-\tilde{j})/2} + h^{p+1} + \exp(-CN^{\epsilon/2})\big)N^{-\tilde{j}/2+(2\tilde{j}+2)\epsilon+\kappa}.
\end{aligned} \tag{6.8}
$$

This sequence is given by $\lambda_{\tilde{j}} := 3r_{\tilde{j}} + 3$ and from (5.7)/(5.8) for $\|\psi_{i,\vec{m}_i}\|_{\mathcal{L}^{q_i}_{\mathrm{pow},r_i}}$ and (5.11)/(5.12) for $\|\boldsymbol{\varphi}_i\|$, we can read off that

$$
\lambda_{i-1} = \begin{cases} 2\lambda_i + r_{i-1} & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ regular successor,} \\ \lambda_i + r_{i-1} & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ compensation successor.} \end{cases}
$$

Plugging in (5.9) for $\hat{F}[m_i]$ (assuming that $(\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i}$ is a compensation successor) and (5.12) for $\|\boldsymbol{\varphi}_i\|_{H^{s_i}}$ into (6.8), we see that the convergence of the sum over m_i can be guaranteed if for each $\ell = 1, \dots, l$, we have $s_I > s_{i_\ell}\lambda_{i_\ell} + 2d + 1$. With (6.7) we roughly estimate $\lambda_i \le (j-i)2^{j-i}3j$. Hence, due to our choice of s_I, we indeed have for $i \ge 1$ that

$$
s_I > 2^{j-1}3j^2 s(d,p,j) + 2d + 1 \ge \lambda_i s_i + 2d + 1.
$$

Thus, all infinite sums along the path converge. This completes the proof. □

7. Quantitative convergence to the mean-field limit.

PROPOSITION 7.1. *Let $\{X^{r_I,N}_{\alpha,i}\}^{\alpha=1,\dots,n_S}_{i=1,\dots,N}$ be a solution to the interacting particle system* (3.1) *satisfying Assumption* 3.7 *with mean-field limit $\overline{\rho}^{r_I}$ satisfying Assumption* 3.8. *Let $\{\tilde{X}^{r_I,N}_{\alpha,i}\}$ be a solution to the diffusing particle system with mean-field forces*

$$
\begin{cases} \mathrm{d}\tilde{X}^{r_I}_{\alpha,i}(t) = -\sum_{\beta=1}^{n_S} \big(\nabla V^{r_I}_{\alpha\beta} * \overline{\rho}^{r_I}_\beta\big)\big(\tilde{X}^{r_I,N}_{\alpha,i}(t), t\big)\,\mathrm{d}t, + \sqrt{2\sigma_\alpha}\,\mathrm{d}B_{\alpha,i}(t), \\ \tilde{X}^{r_I}_{\alpha,i}(0) = X^{r_I}_{\alpha,i}(0) \end{cases} \tag{7.1}
$$

(*cfr.* [10], *System* (3)). *Then for all $\alpha \in \{1, \dots, n_S\}$ and all $i \in \{1, \dots, N\}$, we obtain*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \big|X^{r_I,N}_{\alpha,i} - \tilde{X}^{r_I,N}_{\alpha,i}\big|(t) \\
&\quad \le C(T+1)\exp\big(CTr_I^{-(d+2)}\big)\big(\|\mu^{r_I,N}_0 - \overline{\rho}^{r_I}(0)\|_{H^{-d/2-2}} + \mathcal{C}N^{-1/2}\big),
\end{aligned} \tag{7.2}
$$

where $\mathcal{C}$ is a random variable with Gaussian moments $\mathbb{E}[\exp \frac{1}{C}\mathcal{C}^2] \le 3$. Furthermore, for any $\kappa > 0$, also using (3.15) *from Assumption* 3.11, *there exists $N_0 = N_0(\kappa, \mathrm{data})$ such that, for all $N \ge N_0$, we have the estimate*

$$
\begin{aligned}
&\|\mu^{r_I,N}_{\alpha,t} - \overline{\rho}^{r_I}_\alpha(t)\|_{H^{-d/2-2}} \\
&\quad \le C(T+1)\exp\big(CTr_I^{-(d+2)}\big)\big(\|\mu^{r_I,N}_0 - \overline{\rho}^{r_I}(0)\|_{H^{-d/2-2}} + \mathcal{C}N^{-1/2}\big) && (7.3)\\
&\quad \le \mathcal{C}N^{\kappa T}N^{-1/2}. && (7.4)
\end{aligned}
$$

To prove Proposition 7.1, we first establish the following bound.

LEMMA 7.2. *Suppose that the assumptions of Proposition 7.1 hold. Additionally, set* $\tilde{\mu}_{\alpha,t}^{r_I,N} := \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{X}_{\alpha,i}^{r_I}(t)}$. *Then the estimate*

$$\sup_{t\in[0,T]} |X_{\alpha,i}^{r_I,N} - \tilde{X}_{\alpha,i}^{r_I,N}|(t) \le C(T+1)\exp\big(CTr_I^{-(d+2)}\big)\int_0^T \|\tilde{\mu}_t^{r_I,N} - \overline{\rho}^{r_I}(t)\|_{H^{-d/2-2}}\,\mathrm{d}t$$

holds for all all $\alpha \in \{1,\dots,n_S\}$ *and all* $i \in \{1,\dots,N\}$.

PROOF. By subtracting (3.1) and (7.1), we obtain

$$\begin{aligned}
&\mathrm{d}\big(X_{\alpha,i}^{r_I,N} - \tilde{X}_{\alpha,i}^{r_I,N}\big)(t)\\
&\quad = -\sum_{\beta=1}^{n_S} N^{-1}\sum_{j=1}^N \big(\nabla V_{\alpha\beta}^{r_I}\big(X_{\alpha,i}^{r_I,N}(t) - X_{\beta,j}^{r_I,N}(t)\big) - \big(\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big(\tilde{X}_{\alpha,i}^{r_I,N}(t)\big)\big)\,\mathrm{d}t\\
&\quad = -\sum_{\beta=1}^{n_S} N^{-1}\sum_{j=1}^N \big(\nabla V_{\alpha\beta}^{r_I}\big(X_{\alpha,i}^{r_I,N}(t) - X_{\beta,j}^{r_I,N}(t)\big) - \nabla V_{\alpha\beta}^{r_I}\big(X_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\big)\big)\,\mathrm{d}t\\
&\qquad -\sum_{\beta=1}^{n_S} N^{-1}\sum_{j=1}^N \big(\nabla V_{\alpha\beta}^{r_I}\big(X_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\big) - \nabla V_{\alpha\beta}^{r_I}\big(\tilde{X}_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\big)\big)\,\mathrm{d}t\\
&\qquad -\sum_{\beta=1}^{n_S}\left(N^{-1}\sum_{j=1}^N \nabla V_{\alpha\beta}^{r_I}\big(\tilde{X}_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\big) - \big(\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big(\tilde{X}_{\alpha,i}^{r_I,N}(t)\big)\right)\mathrm{d}t,
\end{aligned}$$

which yields

$$\begin{aligned}
\big|X_{\alpha,i}^{r_I,N} - \tilde{X}_{\alpha,i}^{r_I,N}\big|(T) &\le C\|\nabla^2 V^{r_I}\|_{L^\infty}\int_0^T \sum_{\beta=1}^{n_S}\frac{1}{N}\sum_{j=1}^N \big|X_{\beta,j}^{r_I,N} - \tilde{X}_{\beta,j}^{r_I,N}\big|(t)\,\mathrm{d}t\\
&\quad + C\int_0^T \sum_{\beta=1}^{n_S}\|\nabla V^{r_I}\|_{H^{d/2+2}}\|\tilde{\mu}_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}\|_{H^{-d/2-2}}\,\mathrm{d}t.
\end{aligned}$$

The Gronwall inequality implies an estimate of the desired order for the averaged and time-integrated error $\int_0^T \sum_{\beta=1}^{n_S}\frac{1}{N}\sum_{j=1}^N |X_{\beta,j}^{r_I,N} - \tilde{X}_{\beta,j}^{r_I,N}|(t)\,\mathrm{d}t$; plugging this back into the previous inequality, we arrive at our claim using $\|\nabla^2 V^{r_I}\|_{L^\infty} \le C{r_I}^{-(d+2)}$ and absorbing ${r_I}^{-1}$ factors (coming from the potential V^{r_I}) into the exponential. $\square$

REMARK 7.3 (Initial distribution of the particles). We only prove Proposition 7.1 for the case that the particle positions initially satisfy the spectral gap inequality (3.7). The proof for the case of i.i.d. positions is analogous, replacing the spectral gap inequality with Hoeffdings inequality and using that $\{\tilde{X}_{\alpha,i}^{r_I,N}(t)\}_{i=1}^N$ stay independent also for $t > 0$.

PROOF OF PROPOSITION 7.1. In view of the bound from Lemma 7.2, we need to bound the term $\int_0^T \|\tilde{\mu}_t^{r_I,N} - \overline{\rho}^{r_I}\|_{H^{-d/2-2}}\,\mathrm{d}t$. Note that $w_\alpha := \mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}] - \overline{\rho}_\alpha^{r_I}$ solve the PDEs

$$\partial_t w_\alpha = \sigma_\alpha \Delta w_\alpha + \nabla\cdot\left(\sum_{\beta=1}^{n_S} w_\alpha \nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\right).$$

Testing this PDE with w with respect to the $H^{-d/2-2}$-inner product, with a standard energy estimate, and using the smoothness of V^{r_I} and $\overline{\rho}^{r_I}$ (cfr. Assumptions 3.7 and 3.8), we get

$$\|\mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}] - \overline{\rho}_\alpha^{r_I}\|_{H^{-\frac{d}{2}-2}} \lesssim \exp\big(Cr_I^{-(d+1)}t\big)\|\mathbb{E}[\tilde{\mu}_{\alpha,0}^{r_I,N}] - \overline{\rho}_\alpha^{r_I}(0)\|_{H^{-\frac{d}{2}-2}}. \tag{7.5}$$

With (3.8) from Assumption 3.7, it only remains to bound $\|\tilde{\mu}^{r_I,N}_{\alpha,t} - \mathbb{E}[\bar{\mu}^{r_I,N}_{\alpha,t}]\|_{H^{-d/2-2}}$. Testing $\tilde{\mu}^{r_I,N}_{\alpha,t}$ with elements of the Fourier basis $\exp(i\xi \cdot x)$ with $\xi \in \mathbb{Z}^d$, we have

$$\begin{aligned}\int_{\mathbb{T}^d} \exp(i\xi \cdot x)\, \mathrm{d}\tilde{\mu}^{r_I,N}_{\alpha,t}(x) &= N^{-1} \sum_{j=1}^{N} \exp\big(i\xi \cdot \tilde{X}^{r_I,N}_{\alpha,j}(t)\big) \\ &=: F_t = F_t\big(\big(X^{r_I,N}_{\alpha,j}(0)\big)_j\big).\end{aligned} \tag{7.6}$$

Note that by (7.1) we have $\partial X^{r_I,N}_{\alpha,j}(t)/\partial X^{r_I,N}_{\alpha,j}(0) \le C{r_I}^{-(d+1)}$. Applying the spectral gap inequality (3.7) (with the usual extension to general exponents ≥ 2), we obtain, for any $p \ge 1$,

$$\begin{aligned}&\mathbb{E}\bigg[\bigg|\int_{\mathbb{T}^d} \big(\tilde{\mu}^{r_I,N}_{\alpha,t} - \mathbb{E}[\tilde{\mu}^{r_I,N}_{\alpha,t}]\big) \exp(i\xi \cdot x)\, dx\bigg|^{2p}\bigg]^{1/p} \\ &\quad \le Cp\mathbb{E}\bigg[\bigg(\sum_{j=1}^{N} \bigg|\frac{\partial F}{\partial X^{r_I,N}_{\alpha,j}(0)}\bigg|^2\bigg)^p\bigg]^{1/p} \le Cp|\xi|^2 {r_I}^{-2(d+1)} N^{-1},\end{aligned}$$

where F is defined in (7.6), which entails

$$\bigg|\int_{\mathbb{T}^d} \big(\tilde{\mu}^{r_I,N}_{\alpha,t} - \mathbb{E}[\tilde{\mu}^{r_I,N}_{\alpha,t}]\big) \exp(i\xi \cdot x)\, \mathrm{d}x\bigg| \le \mathcal{C}|\xi| {r_I}^{-(d+1)} N^{-1/2}$$

for a random variable $\mathcal{C}$ subject to a Gaussian moment bound. We thus deduce

$$\begin{aligned}&\|\tilde{\mu}^{r_I,N}_{\alpha,t} - \mathbb{E}[\tilde{\mu}^{r_I,N}_{\alpha,t}]\|^2_{H^{-d/2-2}} \\ &\quad = \sum_{\xi \in \mathbb{Z}^d} (1+|\xi|^2)^{-d/2-2} \bigg|\int_{\mathbb{T}^d} \big(\bar{\mu}^{r_I,N}_{\alpha,t} - \mathbb{E}[\tilde{\mu}^{r_I,N}_{\alpha,t}]\big) \exp(i\xi \cdot x)\, \mathrm{d}x\bigg|^2 \\ &\quad \le \mathcal{C}{r_I}^{-2(d+1)} N^{-1}.\end{aligned} \tag{7.7}$$

Plugging this estimate as well as (7.5) into Lemma 7.2, we arrive at (7.2). Combining (7.7) with (7.5) and using the fact that the left-hand side of (7.2) controls the 1-Wasserstein distance between $\mu^{r_I,N}$ and $\tilde{\mu}^{r_I,N}$—which is precisely the $W^{-1,\infty}$ distance—and a Sobolev embedding, we arrive at (7.3). Then (7.4) follows via another application of the spectral gap inequality and the bound on $\mathbb{E}[\mu^{r_I,N}_0] - \overline{\rho}^0$ (cfr. Assumption 3.7). □

8. Exponentially decaying estimate for the probability that $T < T_\oslash$. The stopping time $T_\oslash$ acts as a discrete analogue to the quantitative convergence of the empirical measure to the mean-field limit proven in Section 7. That is, it guarantees that also in the discrete case the fluctuations are roughly of order $N^{-1/2}$. Additionally, together with Remark 3.14, it ensures that the solution $\rho_h^{r_I}$ of the discretized Dean–Kawasaki equation (h-DK) stays positive. The stopping time is defined as follows. First, for this section set

$$l := \lfloor d/2 \rfloor + 1. \tag{8.1}$$

Given $\rho_h^{r_I}$, the discrete mean-field limit, $\overline{\rho}_h^{r_I}$, and small parameters $\epsilon, \delta > 0$, we define

$$T_\oslash = T_\oslash(\delta, \epsilon) := T_\oslash^\infty(\delta, \epsilon) \wedge T_\oslash^l(\delta, \epsilon) \wedge T, \tag{8.2}$$

where, with discrete L^p-norms as fixed in Definition B.1 and negative Sobolev norms as fixed in Definition 3.6,

$$T_\oslash^\infty(\delta, \epsilon) := \begin{cases} \inf\{0 < t \le T : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \ge N^{-\epsilon}\} \\ \qquad \text{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{L_h^\infty} \le N^{-\epsilon-\delta}, \\ 0 \quad \text{otherwise,} \end{cases}$$

$$T^l_\oslash(\delta,\epsilon) := \begin{cases} \inf\{0 < t \le T : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2\ell}} \ge N^{-1/2+\epsilon}\} \\ \quad \text{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{H_h^{-\ell}} \le N^{-1/2+\epsilon-\delta}, \\ 0 \quad \text{otherwise.} \end{cases}$$

However, for the main result and hence in Sections 5 and 6, for given $\epsilon > 0$, we always choose $\delta = \epsilon/2$. In any way this stopping time is only meaningful if we can bound the probability of the stopping time triggering, that is, $T_\oslash < T$, which results from the corresponding bounds being broken. Such control of $\mathbb{P}(T_\oslash < T)$ via stretched exponential bounds is established in the following proposition.

REMARK 8.1.

(1) Note that the conditional definition of the stopping time ensures that if $T_\oslash > 0$, the size of the fluctuations at $t = 0$ is guaranteed to be smaller than the stopping condition by a factor of $N^{-\delta}$ (in a stronger norm too for $T^l_\oslash(\delta,\epsilon)$). If r_I does not scale but remains constant, then replacing $N^{-\delta}$ with a small enough constant would be sufficient.

(2) While the bounds satisfy $N^{-\epsilon}N^{-1/2+\epsilon} = N^{-1/2}$, the parameters for the two auxiliary stopping times are synced just for convenience: The arguments below can be adapted for $T_\oslash := T^\infty_\oslash(\delta,\epsilon) \wedge T^l_\oslash(\tilde{\delta},\tilde{\epsilon})$. However, we can not hope to get good probability estimates if we choose stricter bounds in the definition of $T_\oslash$. The range for which we get good probability bounds is limited by Assumption 3.10 for the size of the fluctuations at $t = 0$. This assumption is justified by the dicussion in Section C.

(3) With our method we can not obtain probability estimates for the auxiliary stopping times separately due to the nonlinearity. In the proof the nonlinearity appears in form of the linearization compensation $Q^{r_I,N}_{h,t}(\phi^t_h)$ for which to effectively bound, we need an L^∞-bound as well as an order $\backsim N^{-1/2}$ bound.

PROPOSITION 8.2. *Let $\rho_h^{r_I}$ solve (h-DK) and $\overline{\rho}_h^{r_I}$ solve (h-MFL). Suppose Assumption 3.7 (regularity of the potential), Assumption 3.8 (regularity of the continuous mean field limit), Assumption 3.9 (discrete differential operators), Assumption 3.10 (discrete initial conditions), and Assumption 3.11 (scaling regime) hold.*

Let the stopping time $T_\oslash$ be defined as in (8.2) with $\epsilon,\delta \in (0,\delta_0/4)$. Then there exists $N_0 = N_0(\delta,\epsilon,\text{data})$ such that, for all $N > N_0$, there holds

$$\mathbb{P}[T_\oslash < T] \le C\exp(-CN^{\epsilon-\delta}). \tag{8.3}$$

REMARK 8.3. Despite the definition of $T^\infty_\oslash$, larger ϵ yields better bounds. This is since contributions from $T^\infty_\oslash$ are given and—with Assumption 3.11 and $\epsilon < \delta_0/4$—bounded by

$$\exp(-CN^{1/2-\epsilon-\delta}h^{d/2}) \le \exp(-CN^{\delta_0/2-\epsilon-\delta}) \le \exp(-CN^{\epsilon-\delta}).$$

REMARK 8.4 (Strategy for the proof of Proposition 8.2 in Section 8.2). We will show that, with high probability for all $0 < t \le T_\oslash$, we continuously have stricter bounds than guaranteed just by $t \le T_\oslash$. However, by definition this can only be the case if $T_\oslash = T$. Hence, as long as $T_\oslash > 0$, with high probability we obtain $T_\oslash = T$. Since Assumption 3.10 yields $T_\oslash > 0$ with high probability, this proves the proposition (here, "with high probability" means that we obtain stretched exponential bounds as in (8.3) for the complement).

More precisely, we will proceed in seven steps as follows:

- For a fixed time, $\|\cdot\|_{L_h^\infty}$ and $\|\cdot\|_{H_h^{-2l}}$ can be represented via suitable test functions. Hence, we can calculate them with the evolutional scheme from Section 5.1 (Steps 1 and 2).

- Via moment bounds for tested fluctuations (see Lemma 8.5 below), we obtain stricter bounds at a fixed time T_0 with high probability assuming $T_0 \le T_\oslash$ (Steps 3 and 4).
- Given a discretization $0 \le t_1 < t_2 < \cdots \le T$ of $[0, T]$ with step size $\backsim h^\theta$ for arbitrary $\theta > 0$, we have stricter bounds for all $t_k \le T_\oslash$ with high probability (Step 5).
- If θ is large enough, then with high probability we have strict bounds on the difference between the fluctuations at any time $t < T_\oslash$ and at the closest previous time step (Step 6).
- We now cover all $t \le T_\oslash$ and conclude thanks to the bound on $\mathbb{P}[T_\oslash = 0]$ (Step 7).

8.1. *Moment bounds for the discrete fluctuations.* Here, given a stopping time with the same structure as $T_\oslash$, we provide a moment bound crucial for the proof of Proposition 8.2.

LEMMA 8.5 (Moment bounds for the discrete fluctuations). *Let $\rho_h^{r_I}$ solve* (h-DK) *and $\overline{\rho}_h^{r_I}$ solve* (h-MFL). *Suppose Assumptions* 3.7, 3.8, *and* 3.9 *hold. Let $\overline{\rho}_h^{r_I}(0) = \mathcal{I}_h[\overline{\rho}^{r_I}(0)]$, and with respect to the scaling regime of r_I, assume that* (3.15) *from Assumption* 3.11 *holds. Define the stopping time $\widetilde{T}_\oslash := \widetilde{T}_\oslash^\infty \wedge \widetilde{T}_\oslash^l \wedge T$ for some $B_\infty, B_l, B_\infty^0, B_l^0 \in \mathbb{R}$ as*

$$\widetilde{T}_\oslash^\infty := \begin{cases} \inf\{t > 0 : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \ge B_\infty\} & \textit{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{L_h^\infty} \le B_\infty^0, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\widetilde{T}_\oslash^l := \begin{cases} \inf\{t > 0 : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2l}} \ge B_l\} & \textit{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{H_h^{-l}} \le B_l^0, \\ 0 & \textit{otherwise}. \end{cases}$$

Let $\varphi_h \in L^2(G_{h,d}, \mathbb{R}^{ns})$ with backward evolution $\phi_h^t \in C^1(L^2(G_{h,d}, \mathbb{R}^{ns}))$, as in (5.22).

Then for each $\kappa > 0$, there exists $N_0 = N_0(\text{data}, \kappa) \in \mathbb{N}$ independent of j such that, for all $N \ge N_0$ and each even $j \in \mathbb{N}$, there holds

$$\begin{aligned} &\mathbb{E}\Big[\chi(\widetilde{T}_\oslash > 0) \sup_{0 \le t \le \widetilde{T}_\oslash} (\phi_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^j\Big] \\ &\quad \le j^j \big(\max\{CN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}, j^{-1}CN^\kappa \min\{\|\varphi_h\|_{L_h^1} B_\infty^0, \|\varphi_h\|_{H_h^l} B_l^0\}, \\ &\qquad j^{-1}CN^\kappa \min\{h^{-1}\|\varphi_h\|_{L_h^1}, \|\varphi_h\|_{L_h^2}\} B_\infty B_l\}\big)^j. \end{aligned} \tag{8.4}$$

PROOF. Lemma 5.4 implies that

$$\mathcal{M}_t := \chi(\widetilde{T}_\oslash > 0)(\phi_h^{t\wedge\widetilde{T}_\oslash}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \widetilde{T}_\oslash))_h - \int_0^{t\wedge\widetilde{T}_\oslash} Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s$$

is a martingale with Dean–Kawasaki-type noise as in (5.24). Hence, Doob's inequality yields

$$\begin{aligned} &\mathbb{E}\Big[\chi(\widetilde{T}_\oslash > 0) \sup_{0 \le t \le \widetilde{T}_\oslash} \Big((\phi_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h - \int_0^t Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s\Big)^j\Big] \\ &\quad \le \Big(\frac{j}{j-1}\Big)^j \\ &\qquad \times \mathbb{E}\Big[\chi(\widetilde{T}_\oslash > 0)\Big((\phi_h^{\widetilde{T}_\oslash}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\widetilde{T}_\oslash))_h - \int_0^{\widetilde{T}_\oslash} Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s\Big)^j\Big]. \end{aligned} \tag{8.5}$$

Note that $(j/(j-1))^j \le 4$. For the right-hand side, up to this constant, via the Itô rule with (5.24) we obtain for large enough N that

$$\mathbb{E}[(\mathcal{M}_{\widetilde{T}_\oslash})^j] = \mathbb{E}[\chi(\widetilde{T}_\oslash > 0)(\phi_h^0, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0))_h^j]$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\int_0^{\widetilde{T}_\oslash} j(j-1)(\mathcal{M}_t)^{j-2}\sum_{\alpha=1}^{n_S} N^{-1}\sigma_\alpha\big(|\nabla_h\phi_{h,\alpha}^t|^2, (\rho_{h,\alpha}^{r_I})^+(t)\big)_h\,\mathrm{d}t\Big].\\
&\le \min\{\|\phi_h^t\|_{L_h^1}B_\infty^0, \|\phi_h^t\|_{H_h^l}B_l^0\}^j\\
&\quad + j^2N^{-1}|\sigma|\rho_{\max,h}\mathbb{E}\Big[\sup_{0\le t\le\widetilde{T}_\oslash}(\mathcal{M}_t)^{j-2}\Big]\int_0^T\|\nabla_h\phi_h^t\|_{L_h^2}^2\,\mathrm{d}t\\
&\le \big(CN^\kappa\min\{\|\varphi_h\|_{L_h^1}B_\infty^0, \|\varphi_h\|_{H_h^l}B_l^0\}\big)^j\\
&\quad + \big(CjN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}\big)^2\mathbb{E}\Big[\sup_{0\le t\le\widetilde{T}_\oslash}(\mathcal{M}_t)^j\Big]^{(j-2)/j}.
\end{aligned}
$$

For the first inequality, we applied Hölder's inequality (Corollary B.2), used the definitions of $\|\cdot\|_{H_h^s}$ and the stopping time, and that we can bound $\|(\rho^{r_I}(t\wedge\widetilde{T}_\oslash))^+\|_{L_h^\infty}\le\rho_{\max,h}$, as in Remark 3.14. The second inequality follows from bounds for the backward evolution of test functions in Lemma B.8 and Lemma B.10 as well as an application of Jensen's inequality.

Plugging this inequality back into (8.5) and applying Young's inequality to absorb the moment-term (this is necessary only for $j>2$), we obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\chi(\widetilde{T}_\oslash>0)\sup_{0\le t\le\widetilde{T}_\oslash}\Big(\big(\phi_h^t, (\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t)\big)_h-\int_0^t Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s\Big)^j\Big]\\
&\quad\le\big(CN^\kappa\min\{\|\varphi_h\|_{L_h^1}B_\infty^0, \|\varphi_h\|_{H_h^l}B_l^0\}\big)^j + j^j\big(CN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}\big)^j.
\end{aligned}
$$

Taking the jth root, applying the reverse triangle inequality on the left, moving the Q-term to the other side, taking the jth power, and applying $(a+b)^j\le 2^ja^j+2^jb^j$ yields

$$
\begin{aligned}
&\mathbb{E}\Big[\chi(\widetilde{T}_\oslash>0)\sup_{0\le t\le\widetilde{T}_\oslash}\big(\phi_h^t, (\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t)\big)_h^j\Big]\\
&\quad\le j^j\big(CN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}\big)^j\\
&\qquad+\big(CN^\kappa\min\{\|\varphi_h\|_{L_h^1}B_\infty^0, \|\varphi_h\|_{H_h^l}B_l^0\}\big)^j+2^j\mathbb{E}\Big[\Big(\int_0^{\widetilde{T}_\oslash}|Q_{h,t}^{r_I,N}(\phi_h^t)|\,\mathrm{d}t\Big)^j\Big].
\end{aligned}
$$

Hence, to finish the proof, it is enough to show that

$$
\mathbb{I}_Q:=\int_0^{\widetilde{T}_\oslash}|Q_{h,t}^{r_I,N}(\phi_h^t)|\,\mathrm{d}t\le CN^\kappa\min\{\|\varphi_h\|_{L_h^2}, h^{-1}\|\varphi_h\|_{L_h^1}\}B_\infty B_l. \tag{8.6}
$$

This follows immediately from the two following estimates by applying the test function estimates from Lemma B.8 and Lemma B.10, the definition of the stopping time, and the observation $\|\mathcal{I}_h[\nabla V^{r_I}\|_{H_h^{2l}}\lesssim\|V^{r_I}\|_{C^{2l+1}}\lesssim r_I^{-(d+2l+1)}$: Hölder's inequality yields

$$
\begin{aligned}
\mathbb{I}_Q&=\int_0^{\widetilde{T}_\oslash}\Big|\sum_{\alpha,\beta=1}^{n_S}\big(\nabla_h\phi_{h,\alpha}^t\cdot(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}]*_h(\rho_{h,\beta}^{r_I}-\overline{\rho}_{h,\beta}^{r_I})(t)), (\rho_{h,\alpha}^{r_I}-\overline{\rho}_{h,\alpha}^{r_I})(t)\big)_h\Big|\,\mathrm{d}t\\
&\le\begin{cases}\Big(\int_0^T\|\nabla_h\phi_h^t\|_{L_h^2}^2\,\mathrm{d}t\Big)^{\frac12}\Big(\int_0^{\widetilde{T}_\oslash}\|\mathcal{I}_h[\nabla V^{r_I}\|_{H_h^{2l}}^2\|\rho_h^{r_I}-\overline{\rho}_h^{r_I}\|_{H_h^{-2l}}^2\|\rho_h^{r_I}-\overline{\rho}_h^{r_I}\|_{L_h^\infty}^2\,\mathrm{d}t\Big)^{\frac12},\\ h^{-1}\int_0^{\widetilde{T}_\oslash}\|\phi_h^t\|_{L_h^1}\|\mathcal{I}_h[\nabla V^{r_I}\|_{H_h^{2l}}\|\rho_h^{r_I}-\overline{\rho}_h^{r_I}\|_{H_h^{-2l}}\|\rho_h^{r_I}-\overline{\rho}_h^{r_I}\|_{L_h^\infty}\,\mathrm{d}t.\end{cases}
\end{aligned}
$$

Here for the first option, we used that $\|\cdot\|_{L_h^2}\le C\|\cdot\|_{L_h^\infty}$ before applying Young's convolution inequality (Corollary B.3). For the second option, we used that $\|\nabla_h f\|_{L_h^1}\le Ch^{-1}\|f\|_{L_h^1}$ for any $f\in L^2(G_{h,d})$. $\square$

8.2. *Proof of the bound for* $\mathbb{P}(T_\oslash < T)$.

PROOF OF PROPOSITION 8.2. The proof consists of seven steps, outlined in Remark 8.4:

Step 1: $\|\cdot\|_{L_h^\infty}$ *via test functions.* We define $\varphi_{h,x_0}^\gamma \in [L^2(G_{h,d})]^{n_S}$ for any $\gamma \in \{1,\dots,n_S\}$ and $x_0 \in G_{h,d}$ as $\varphi_{h,x_0,\alpha}^\gamma := h^{-d/2}\delta_{\gamma,\alpha} f_{x_0}$ where $\delta_{\gamma,\alpha}$ is the Dirac delta and f_{x_0}, as defined in Section 3.2. Hence, for any $\eta_h \in [L^2(G_{h,d})]^{n_S}$, we have $(\varphi_{h,x_0}^\gamma, \eta_h)_h = \eta_{h,\gamma}(x_0)$, and, therefore,

$$\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} = \max_{x_0 \in G_{h,d}} \max_{\gamma=1,\dots,n_S} \big|(\varphi_{h,x_0}^\gamma, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h\big|. \tag{8.7}$$

Note that, by the definition of φ_{h,x_0}^γ, we have

$$\|\varphi_{h,x_0}^\gamma\|_{L_h^1} = 1, \qquad \|\varphi_{h,x_0}^\gamma\|_{L_h^2} = h^{-d/2}. \tag{8.8}$$

Step 2: $\|\cdot\|_{H_h^{-2l}}$ *via test functions.* We define $\varphi_{h,m}^\gamma \in [L^2(G_{h,d})]^{n_S}$ for any $\gamma \in \{1,\dots,n_S\}$ and $m \in \mathbb{Z}^d \cap [-\frac{\pi}{h}, \frac{\pi}{h})^d$ as $\varphi_{h,m,\alpha}^\gamma := \delta_{\gamma,\alpha}\vartheta_m$, where $(\vartheta_m)_m$ is the discrete Fourier basis as introduced in Remark 3.5. Hence, with Definition 3.6 we have

$$\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2l}}^2 = \sum_{\gamma=1}^{n_S} \sum_m \frac{1}{(1+|m|^2)^{2l}} (\varphi_{h,m}^\gamma, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^2, \tag{8.9}$$

where the sum is over $m \in \mathbb{Z}^d \cap [-\frac{\pi}{h}, \frac{\pi}{h})^d$. For $\varphi_{h,m}^\gamma$, we have

$$\|\varphi_{h,m}^\gamma\|_{L_h^1} = (2\pi)^d \qquad \|\varphi_{h,m}^\gamma\|_{L_h^2} = (2\pi)^{d/2} \qquad \|\varphi_{h,m}^\gamma\|_{H_h^l} \le (1+|m|^2)^{l/2}, \tag{8.10}$$

where the last equation is an immediate consequence of Remark 3.5.

Step 3: Stricter L_h^∞*-bounds for a single time point* $0 < T_0 \le T_\oslash$. Let $0 < \kappa < \delta$. In this step we will show that if N is large enough, for any fixed $T_0 \in [0,T]$, we have

$$\mathbb{P}\Big[T_0 \le T_\oslash \text{ and } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(T_0)\|_{L_h^\infty} \ge \frac{N^{-\epsilon}}{4}\Big] \le n_S h^{-d} \exp\big(-C N^{1/2-\epsilon-\kappa} h^{d/2}\big). \tag{8.11}$$

In order to achieve this, we will apply Lemma 8.5 to the backward evolution of the test functions φ_{h,x_0}^γ from Step 1 with $j = 2\lfloor \frac{1}{4} C_0^{-1} \frac{N^{-\epsilon}}{4} N^{1/2-\kappa} \|\varphi_{h,x_0}^\gamma\|_{L_h^2}^{-1} \rfloor$ where C_0 is the constant in Lemma 8.5. The reasoning for this choice will be explained at the end of this step. Note that we have $j \in 2\mathbb{N}$ for large enough N since with (8.8) and (3.13) from Assumption 3.11

$$N^{1/2-\kappa-\epsilon} \|\varphi_{h,x_0}^\gamma\|_{L_h^2}^{-1} = N^{1/2-\kappa-\epsilon} h^{d/2} \ge N^{\delta_0/2-\kappa-\epsilon}.$$

We will now show that in our setting the maximum on the right-hand side of (8.4) is dominated by $C_0 N^{-1/2+\kappa} \|\varphi_{h,x_0}^\gamma\|_{L_h^2}$. Note that for our choice of j

$$N^{-\epsilon}/16 \le j\big(C_0 N^{-1/2+\kappa} \|\varphi_{h,x_0}^\gamma\|_{L_h^2}\big) \le N^{-\epsilon}/8. \tag{8.12}$$

Regarding the middle term on the right-hand side of (8.4), with (8.8) and $\kappa < \delta$, we have

$$C_0 N^\kappa \|\varphi_{h,x_0}^\gamma\|_{L_h^1} N^{-\epsilon-\delta} = C_0 N^{-\epsilon} N^{-(\delta-\kappa)} \le N^{-\epsilon}/16$$

for large enough N. With respect to the last term, with (3.13) from Assumption 3.11 and large enough N

$$C_0 N^\kappa \|\varphi_{h,x_0}^\gamma\|_{L_h^2} N^{-1/2} = C_0 h^{-d/2} N^{-1/2} N^\kappa \le C_0 N^{-(\delta_0/2-\kappa)} \le N^{-\epsilon}/16.$$

Now, we apply Lemma 8.5. Denoting with $\phi_{h,x_0}^{\gamma,t}$ the backward evolution according to (5.22) with $\phi_{h,x_0}^{\gamma,t} = \varphi_{h,x_0}^{\gamma}$ for $t \geq T_0$, we obtain

$$\mathbb{E}\Big[\chi(T_\oslash > 0) \sup_{0\leq t\leq T_\oslash \wedge T_0} (\phi_{h,x_0}^{\gamma,t}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^j\Big] \leq j^j (C_0 N^{-1/2+\kappa} \|\varphi_{h,x_0}^{\gamma}\|_{L_h^2})^j.$$

With Chebyshev's inequality, then plugging in the choice of j and applying (8.12), this yields

$$\begin{aligned}
&\mathbb{P}\Big[T_\oslash > 0 \text{ and } T_0 \leq T_\oslash \text{ and } |(\varphi_{h,x_0}^{\gamma}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(T_0))_h| \geq \frac{N^{-\epsilon}}{4}\Big] \\
&\leq \mathbb{P}\Big[\chi(T_\oslash > 0) \sup_{0\leq t\leq T_\oslash \wedge T_0} |(\phi_{h,x_0}^{\gamma,t}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h| \geq \frac{N^{-\epsilon}}{4}\Big] \\
&\leq \exp\big(j\big[\log(j) + \log(4N^{\epsilon} C_0 N^{-1/2+\kappa} \|\varphi_{h,x_0}^{\gamma}\|_{L_h^2})\big]\big) \\
&\leq \exp(-C N^{1/2-\epsilon-\kappa} h^{d/2}).
\end{aligned}$$

This last inequality dictates the choice of j: Up to constants, it is the largest j such that the logarithms combine to a negative constant. Via (8.7) from Step 1, repeating the same argument for all different $\gamma = 1, \ldots, n_S$ and the h^{-d} grid points $x_0 \in G_{h,d}$ yields (8.11).

Step 4: Stricter H_h^{-2l}-bounds for a single time point $T_0 \leq T_\oslash$. Again, let $0 < \kappa < \delta$. In this step we show that, for any fixed $0 < T_0 \leq T$, if N is large enough, it holds that

$$\mathbb{P}\Big[T_0 \leq T_\oslash \text{ and } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(T_0)\|_{H_h^{-2l}} \geq \frac{N^{-1/2+\epsilon}}{4}\Big] \leq n_S \exp(-C N^{\epsilon-\kappa}). \tag{8.13}$$

As in Step 3, the main part of the argument is an application of Lemma 8.5, here to the functions $\varphi_{h,m}^{\gamma}$ from Step 2. We choose $j = 2\lfloor \frac{1}{4} C_d^{-1} C_0^{-1} N^{\epsilon-\kappa}\rfloor$ independent of m, where C_0 denotes the constant in Lemma 8.5 and C_d is a constant, which we will fix later in this step. Clearly, $j \in 2\mathbb{N}$ for N large enough.

With these choices the maximum in (8.4) is dominated by $C_0(1 + |m|^2)^{l/2} N^{-1/2+\kappa}$ for large enough N. Indeed, with our choice of j and (8.9), we have

$$\begin{aligned}
j^{-1} C_0 N^{\kappa} \|\varphi_{h,m}^{\gamma}\|_{L_h^2} N^{-1/2} &\leq C_0 N^{-1/2+\kappa} \|\varphi_{h,m}^{\gamma}\|_{L_h^2} = C_0 N^{-1/2+\kappa}, \\
j^{-1} C_0 N^{\kappa} \|\varphi_{h,m}^{\gamma}\|_{H_h^l} N^{-1/2+\epsilon-\delta} &\leq 4 C_d C_0^2 (1 + |m|^2)^{l/2} N^{-1/2+\kappa} N^{-(\delta-\kappa)}.
\end{aligned}$$

Thus, applying Lemma 8.5 with $\phi_{h,m}^{\gamma,t}$ denoting the backward evolution according to (5.22) with $\phi_{h,m}^{\gamma,t} = \varphi_{h,m}^{\gamma}$ for $t \geq T_0$, we obtain

$$\mathbb{E}\Big[\chi(T_\oslash > 0) \sup_{0\leq t\leq T_\oslash \wedge T_0} (\phi_{h,m}^{\gamma,t}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^j\Big] \leq j^j (C_0 (1 + |m|^2)^{l/2} N^{-1/2+\kappa})^j.$$

Thus, already (8.9) for $\|\cdot\|_{H_h^{-2l}}$ in mind, after applying the triangle inequality we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\chi(T_\oslash > 0) \sup_{0\leq t\leq T_\oslash \wedge T_0} \Big(\sum_m \frac{1}{(1 + |m|^2)^{2l}} (\phi_{h,m}^{\gamma,t}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^2\Big)^{j/2}\Big]^{2/j} \\
&\leq \sum_m \frac{C}{(1 + |m|^2)^{2l}} \mathbb{E}\Big[\chi(T_\oslash > 0) \sup_{0\leq t\leq T_\oslash \wedge T_0} (\phi_{h,m}^{\gamma,t}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^j\Big]^{2/j} \\
&\leq \sum_m \frac{1}{(1 + |m|^2)^l} (j C_0 N^{-1/2+\kappa})^2.
\end{aligned}$$

With Chebyshev's inequality, choosing C_d only depending on the dimension and n_S such that $\sum_m \frac{n_S}{(1+|m|^2)^l} \leq C_d^2$ (recall that $l := \lfloor d/2 \rfloor + 1$), with the above choice of j we obtain

$$\mathbb{P}\Big[T_0 \leq T_\oslash \text{ and } \sum_m \frac{1}{(1+|m|^2)^{2l}} (\varphi_{h,m}^\gamma, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t))_h^2 \geq \frac{(N^{-1/2+\epsilon})^2}{n_S 4^2}\Big]$$
$$\leq \exp\big(j\big[\log(j) + \log\big(4C_d C_0 N^{-\epsilon+\kappa}\big)\big]\big) \leq \exp\big(-CN^{\epsilon-\kappa}\big),$$

as in Step 3. Iterating through all species yields (8.13) via (8.9).

Step 5: Extension to finitely many time points in $[0, T_\oslash]$ *with step size* h^θ. We discretize the time interval $[0, T]$ with step size h^θ for some large $\theta > 1$, which we will determine in the next step. Adding up the bounds in (8.11) from Step 3 as well as (8.13) from Step 4 for $T_0 = h^\theta, 2h^\theta, \ldots, \lfloor T/h^\theta \rfloor h^\theta$, for any $0 < \kappa < \delta$ and large enough N, we obtain

$$\begin{aligned}(8.14)\qquad &\mathbb{P}\Big[T_\oslash > 0 \text{ and } \exists i \in \mathbb{N}, ih^\theta \leq T_\oslash \colon \big\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\big\|_{L_h^\infty} \geq \frac{N^{-\epsilon}}{4}\Big] \\ &\leq Ch^{-d-\theta} \exp\big(-CN^{1/2-\epsilon-\kappa} h^{d/2}\big),\end{aligned}$$

$$\begin{aligned}(8.15)\qquad &\mathbb{P}\Big[T_\oslash > 0 \text{ and } \exists i \in \mathbb{N}, ih^\theta \leq T_\oslash \colon \big\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\big\|_{H_h^{-2l}} \geq \frac{N^{-1/2+\epsilon}}{4}\Big] \\ &\leq Ch^{-\theta} \exp\big(-CN^{\epsilon-\kappa}\big).\end{aligned}$$

Step 6: Strict bounds for changes of $\rho_h^{r_I} - \overline{\rho}_h^{r_I}$ *in small time increments.* In this step we show that, given any $\kappa > 0$ and $j > 1$, for large enough N and any $i = 1, \ldots, \lfloor T/h^\theta \rfloor$, we have

$$\begin{aligned}(8.16)\qquad &\mathbb{E}\Big[\chi(T_\oslash > 0) \sup_{t \in [ih^\theta, (i+1)h^\theta \wedge T_\oslash]} \big\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t) - (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\big\|_\infty^j\Big]^{1/j} \\ &\leq jCh^{\theta/2-5d/2-4} N^{-1/2+\epsilon+\kappa}.\end{aligned}$$

This implies (via Chebyshev's inequality and optimization in j, as in Steps 3 and 4) that

$$\mathbb{P}\Big[T_\oslash > 0 \text{ and } \sup_{t \in [ih^\theta, (i+1)h^\theta \wedge T_\oslash]} \big\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t) - (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\big\|_\infty \geq \frac{N^{-\frac{1}{2}+\epsilon}}{4}\Big]$$
$$\leq \exp\big(-Ch^{-\theta/2+5d/2+4} N^{-\kappa}\big).$$

With (3.14) from Assumption 3.11, choosing $\kappa = \delta_0$,

$$h^{-\theta/2+5d/2-4} N^{-\kappa} \geq h^{-\theta/2+5d/2+5}$$

and hence, choosing $\theta = 5d + 12$ and adding up the bounds for all i,

$$\begin{aligned}&\mathbb{P}\Big[T_\oslash > 0 \text{ and } \exists i \in \mathbb{N}, ih^\theta \leq T_\oslash \colon \\ (8.17)\qquad &\qquad \sup_{t \in [ih^\theta, (i+1)h^\theta \wedge T_\oslash]} \big\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t) - (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\big\|_\infty \geq \frac{N^{-1/2+\epsilon}}{4}\Big] \\ &\leq Ch^{-\theta} \exp\big(-Ch^{-1}\big).\end{aligned}$$

The exact estimate to (8.17) holds when replacing L_h^∞ with H_h^{-2l}, since $\|\cdot\|_{H_h^{-2l}} \leq C\|\cdot\|_{L_h^\infty}$ for a constant C independent of h (for L_h^∞ we can replace $N^{-1/2+\epsilon}$ with $N^{-\epsilon}$ since $\epsilon < 1/2$).

We now prove (8.16) by bounding the increments for each species $\gamma = 1, \dots, n_S$ and $x_0 \in G_{h,d}$. With the basis functions f_{x_0} as in Section 3.3, following the calculations in Section 5.1, the discrete version of (5.16) yields

$$
\begin{aligned}
&(\rho^{r_I}_{h,\gamma} - \overline{\rho}^{r_I}_{h,\alpha})(t, x_0) - (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(ih^\theta, x_0) \\
&\quad = h^{-d/2}\big((f_{x_0}, (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t))_h - (f_{x_0}, (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(ih^\theta))_h\big) \\
&\quad = h^{-d/2}\int_{ih^\theta}^t \big(\sigma_\alpha \Delta_h f_{x_0} - U^{r_I}_{h,\alpha}(s)\cdot\nabla_h f_{x_0}, (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(s)\big)\,\mathrm{d}s \\
&\qquad - h^{-d/2}\sum_{\beta=1}^{n_S}\int_{ih^\theta}^t \big(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_{h,c} (\overline{\rho}^{r_I}_{h,\alpha}(s)\nabla_h f_{x_0}), (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(s)\big)\,\mathrm{d}s \\
&\qquad + h^{-d/2}\int_{ih^\theta}^t \mathrm{d}\mathcal{W}_h\big((\rho^{r_I}_{h,\alpha})^+, f_{x_0}\big)(s),
\end{aligned}
\tag{8.18}
$$

where $U^{r_I}_{h,\alpha}$ is defined as in (5.23) and where $\mathcal{W}_h((\rho^{r_I}_{h,\alpha})^+, f_{x_0})$ is a real-valued martingale corresponding to the Dean–Kawasaki noise with quadratic variation satisfying

$$
\mathrm{d}\big[\mathcal{W}((\rho^{r_I}_{h,\alpha})^+, f_{x_0}), \mathcal{W}((\rho^{r_I}_{h,\alpha})^+, f_{x_0})\big](t) = 2\sigma_\alpha N^{-1}\big((\rho^{r_I}_{h,\alpha})^+(t), \nabla_h f_{x_0}\cdot\nabla_h f_{x_0}\big)_h.
$$

Applying the Burkholder–Davis–Gundy inequality (for $j > 1$) then gives

$$
\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[ih^\theta,(i+1)h^\theta\wedge T_\oslash]}\bigg|\int_{ih^\theta}^t \mathrm{d}\mathcal{W}_h\big((\rho^{r_I}_{h,\alpha})^+, f_{x_0}\big)(s)\bigg|^j\bigg]^{1/j} \\
&\quad \le Cj\,\mathbb{E}\bigg[\bigg(\int_{ih^\theta}^{(i+1)h^\theta\wedge T_\oslash} 2\sigma_\alpha N^{-1}\big((\rho^{r_I}_{h,\alpha})^+(t), \nabla_h f_{x_0}\cdot\nabla_h f_{x_0}\big)_h\,\mathrm{d}t\bigg)^{j/2}\bigg]^{1/j} \\
&\quad \le Cjh^{\theta/2}N^{-1/2}\sqrt{\rho_{\max,h}}\,h^{-1},
\end{aligned}
\tag{8.19}
$$

where we used that $\|(\rho^{r_I}_{h,\alpha})^+(t)\|_{L^\infty_h} \le \rho_{\max,h}$ as in Remark 3.14 and $\|\nabla_h f_{x_0}\|_{L^2_h} \le Ch^{-1}$.

With respect to the first term in (8.18), with $\|\cdot\|_{H^{2l}_h} \le Ch^{-2l}\|\cdot\|_{L^2_h}$, we have

$$
\begin{aligned}
&\bigg|\int_{ih^\theta}^t \big(\sigma_\alpha \Delta_h f_{x_0} - U^{r_I}_{h,\alpha}(s)\cdot\nabla_h f_{x_0}, (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(s)\big)\,\mathrm{d}s\bigg| \\
&\quad \le Ch^{-2l}\int_{ih^\theta}^{(i+1)h^\theta\wedge T_\oslash} \big(\|\Delta_h f_{x_0}\|_{L^2_h} + \|U^{r_I}_h\|_{L^\infty_h}\|\nabla_h f_{x_0}\|_{L^2_h}\big)\|\rho^{r_I}_h - \overline{\rho}^{r_I}_h\|_{H^{-2l}_h}\,\mathrm{d}s \\
&\quad \le Ch^{-2l+\theta}\big(h^{-2} + r_I{}^{-d-1}h^{-1}\big)N^{-1/2+\epsilon},
\end{aligned}
\tag{8.20}
$$

where we used the definition of $T_\oslash$ and that $\|U^{r_I}\|_{L^\infty_h} \le \|\mathcal{I}_h[V^{r_I}]\|_{L^\infty}\|\overline{\rho}^{r_I}\|_{L^1_h} \le Cr_I{}^{-d-1}$ due to Young's inequality; see Corollary B.3. Note that if $\overline{\rho}^{r_I}(t) \ge 0$, in particular for $t \le T_\oslash$ with Remark 3.14, it holds that $\|\overline{\rho}^{r_I}(t)\|_{L^1_h} = \|\overline{\rho}^{r_I}(0)\|_{L^1_h}$.

For the second term in (8.18), an analogous argument yields

$$
\begin{aligned}
&\bigg|\int_{ih^\theta}^t \big(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_{h,c} (\overline{\rho}^{r_I}_{h,\alpha}(s)\nabla_h f_{x_0}), (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(s)\big)\,\mathrm{d}s\bigg| \\
&\quad \le Ch^{-2l}\int_{ih^\theta}^{(i+1)h^\theta\wedge T_\oslash} \|\mathcal{I}_h[V^{r_I}]\|_{L^2_h}\|\overline{\rho}^{r_I}_h\|_{L^2_h}\|\nabla_h f_{x_0}\|_{L^2_h}\|\rho^{r_I}_h - \overline{\rho}^{r_I}_h\|_{H^{-2l}_h}\,\mathrm{d}s \\
&\quad \le Ch^{-2l+\theta}r_I{}^{-d-1}\rho_{\max,h}h^{-1}N^{-1/2+\epsilon}.
\end{aligned}
\tag{8.21}
$$

Plugging the sum of (8.18) over all $\gamma = 1, \dots, n_S$ and the h^{-d} different $x_0 \in G_{h,d}$ into the left-hand side of (8.16), applying the triangle inequality and then (8.20), (8.21), and (8.19), we obtain the right-hand side of (8.16) with l chosen as in (8.1) for large enough N with (3.15) from Assumption 3.11.

Step 7: Conclusion from Steps 5 and 6. Combining (8.14) and (8.15) for the discrete time points with (8.17) for the increments in-between with respect to $\|\cdot\|_{L_h^\infty}$ and $\|\cdot\|_{H_h^{-2l}}$ yields

$$\begin{aligned}
\mathbb{P}[T_\oslash = T] \geq\ & \mathbb{P}\Big[T_\oslash > 0 \text{ and } \forall 0 \leq t \leq T_\oslash : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \leq \frac{N^{-\epsilon}}{2} \\
& \quad \text{and } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2\ell}} \leq \frac{N^{-1/2+\epsilon}}{2}\Big] \\
\geq\ & 1 - \mathbb{P}[T_\oslash = 0] - \mathbb{P}\Big[T_\oslash > 0 \text{ and } \exists 0 \leq t \leq T_\oslash : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} > \frac{N^{-\epsilon}}{2}\Big] \\
& - \mathbb{P}\Big[T_\oslash > 0 \text{ and } \exists 0 \leq t \leq T_\oslash : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2\ell}} > \frac{N^{-1/2+\epsilon}}{2}\Big] \\
\geq\ & 1 - C\exp(-CN^{1/2-\epsilon-\delta}h^{d/2}) - C\exp(-CN^{\epsilon-\delta}) \\
& - Ch^{-\theta}\big(h^{-d}\exp(-CN^{1/2-\epsilon-\kappa}h^{d/2}) + \exp(-CN^{\epsilon-\kappa}) \\
& + \exp(-Ch^{-1})\big),
\end{aligned}$$

where we used (3.11) and (3.12) from Assumption 3.10 to bound $\mathbb{P}[T_\oslash = 0]$. After absorbing the h^{-1} prefactors and applying Assumption 3.11 (e.g., as in Remark 8.3), we obtain (8.3). $\square$

APPENDIX A: REGULARITY OF CONTINUOUS TEST FUNCTIONS

LEMMA A.1 (Bounds on Sobolev norms of continuous test functions). *Let $\varphi_\alpha \in H^s(\mathbb{T}^d)$ for all $\alpha \in 1, \dots, n_S$, for some $s \in \mathbb{N}_0$. Assume $\overline{\rho}^{r_I} \in C^s$. Then there exists a unique H^s-valued strong solution ϕ_α^t to the backward evolution equation* (5.18). *Additionally, given an arbitrary $\kappa > 0$, there exists $N_0 = N_0(\kappa, \text{data})$, such that, for all $N > N_0$, we have the bound*

$$\|\phi^t\|_{H^s} \leq N^{\kappa(T-t)}\|\varphi\|_{H^s}. \tag{A.1}$$

The proof of Lemma A.1 is straightforward, and we refer the reader to the arXiv version of this paper (https://arxiv.org/abs/2303.00429).

APPENDIX B: RELEVANT ESTIMATES IN THE DISCRETE SETTING

We prove several auxiliary results relating to the discretised setting, namely: error bounds for the difference of continuous mean-field limit $\overline{\rho}^{r_I}$ and discrete mean-field limit $\overline{\rho}_h^{r_I}$ (Section B.1), error bounds for the difference of continuous test functions (5.18) and discrete test functions (5.22), and L^1-bound for the discrete test functions (5.22) (Section B.2).

In what follows, we will use a discrete version of Hölder's inequality and Young's convolution inequality. To state these, we need a discrete notion of L^p-norms.

DEFINITION B.1 (Discrete L^p-norms). For all $p \in [1, \infty)$ and $f \in L^2(G_{h,d})$, we write

$$\|f\|_{L^p(G_{h,d})} = \|f\|_{L_h^p} = \Big(\sum_{x \in G_{h,d}} h^d |f(x)|^p\Big)^{1/p}, \qquad \|f\|_{L^\infty(G_{h,d})} = \max_{x \in G_{h,d}} |f(x)|.$$

In particular, this is consistent with the previous definition of $\|\cdot\|_{L^2(G_{h,d})}$.

COROLLARY B.2 (Discrete Hölder inequality). *Let $q, r \in [1, \infty]$ with $\frac{1}{q} + \frac{1}{r} = 1$. Then for all $f, g \in L^2(G_{h,d})$, it holds that*

$$(f, g)_h \le \|f\|_{L^q(G_{h,d})} \|g\|_{L^r(G_{h,d})}.$$

With the validity of a Hölder inequality, Young's convolution inequality automatically holds in the same framework.

COROLLARY B.3 (Discrete Young's convolution inequality). *Assume $q, \widetilde{q}, r \in [1, \infty]$ satisfy $\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{1}{r} + 1$ and $f, g \in L^2(G_{h,d})$. Then*

$$\|f *_h g\|_{L^r(G_{h,d})} \le \|f\|_{L^q(G_{h,d})} \|g\|_{L^{\widetilde{q}}(G_{h,d})}.$$

We obtain a quantitative error bound for comparing continuous and discrete convolutions based on the classical Euler–Maclaurin formula for numerical integration.

LEMMA B.4 (A multidimensional Euler-Maclaurin formula). *Let $s \in \mathbb{N}_0$. Then there exists $C_s > 0$ such that, for all $f \in C^{2s+2}(\mathbb{T}^d)$,*

$$\left| \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x - \sum_{x \in G_{h,d}} h^d f(x) \right| \le C_s h^{2s+2} \|f\|_{C^{2s+2}(\mathbb{T}^d)}.$$

PROOF. For $d = 1$, see, for instance, [11], Chapter 1. The lemma then follows with Fubini's theorem by induction. □

These facts imply the following approximation property of the discrete convolution.

COROLLARY B.5 (Approximation order of convolutions on periodic grids). *Let $s \in \mathbb{N}_0$. Then there exists $C_s > 0$ such that, for all $f, g \in C^{2s+2}(\mathbb{T}^d)$,*

$$\|\mathcal{I}_h[f * g] - \mathcal{I}_h[f] *_h \mathcal{I}_h[g]\|_{L^\infty(G_{h,d})} \le C_s h^{2s+2} \|f\|_{C^{2s+2}(\mathbb{T}^d)} \|g\|_{C^{2s+2}(\mathbb{T}^d)}.$$

B.1. Error bounds for difference of continuous and discrete mean field limit. In this section we compare the continuous and discretised mean-field limits. First, we obtain an L^2-type bound of order h^{p+1} via an energy estimate. Since we need to assume the difference to be small to deal with the quadratic nonlinearity, the estimate only holds up to a finite positive time. Second, building upon the L^2-estimate, we will derive higher order estimates difference.

PROPOSITION B.6 (L^2-Estimate for the continuous-discrete mean-field limit difference). *Let Assumption 3.7 hold. Let $\overline{\rho}^{r_I}$ be the continuous mean-field limit, as in (3.4), satisfying Assumption 3.8 with discrete counterpart $\overline{\rho}_h^{r_I}$, as in (h-MFL). Assume that the initial conditions satisfy Assumption 3.10 while the parameters r_I, N, h are subject to Assumption 3.11. Set*

$$\overline{T}_h := \inf\{t \in [0, \infty); \|\mathcal{I}_h[\overline{\rho}^{r_I}](t) - \overline{\rho}_h^{r_I}(t)\|_{L^2(G_{h,d})} > 1\} \overset{\text{Assumption 3.10}}{>} 0.$$

Then for all $\kappa > 0$, there exists $N_0 = N_0(p, \kappa, data)$ such that, for all $t \in [0, \overline{T}_h]$ and $N > N_0$,

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}(t) \le C h^{2(p+1)} N^{\kappa t} \tag{B.1}$$

In particular, $\overline{T}_h \ge T$, since for small $\kappa > 0$ and $h > 0$ with Assumption 3.11, it holds that

$$C h^{2(p+1)} N^{\kappa t} \le C h^{2(p+1)} N^{\frac{\kappa}{\delta_0}\delta_0 T} \le C h^{2(p+1)} h^{-\kappa/\delta_0} \le 1.$$

PROOF. Testing (h-MFL), the definition of the discrete mean-field limit, with $\eta_h \in L^2(G_{h,d})$ and applying integration by parts for ∇_h, we obtain its adjoint/reflected version ∇_h^R

$$\big(\partial_t \overline{\rho}_{h,\alpha}^{r_I}, \eta_h\big)_h = \sigma_\alpha \big(\Delta_h \overline{\rho}_{h,\alpha}^{r_I}, \eta_h\big)_h - \sum_{\beta=1}^{n_S} \big(\overline{\rho}_{h,\alpha}^{r_I}\big(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \overline{\rho}_{h,\beta}^{r_I}\big), \nabla_h^R \eta_h\big)_h$$

for all species $\alpha = 1, \dots, n_S$. After applying $\mathcal{I}_h$ to (3.4) and testing with η_h, for the continuous mean-field limit we have

$$\big(\mathcal{I}_h[\partial_t \overline{\rho}_\alpha^{r_I}], \eta_h\big)_h = \sigma_\alpha \big(\mathcal{I}_h[\Delta \overline{\rho}_\alpha^{r_I}], \eta_h\big)_h + \sum_{\beta=1}^{n_S} \big(\mathcal{I}_h\big[\nabla \cdot \big(\overline{\rho}_\alpha^{r_I}\big(\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big)\big], \eta_h\big)_h. \tag{B.2}$$

In order to make both evolutions comparable, we want to rephrase (B.2) in terms of discrete differential operators. For the diffusion term, with Assumption 3.9—the order of the differential operators—and the discrete Hölder inequality, we have

$$\big|\big(\mathcal{I}_h[\Delta \overline{\rho}_\alpha^{r_I}], \eta_h\big)_h - \big(\Delta_h \overline{\rho}_\alpha^{r_I}, \eta_h\big)_h\big| \le C \|\eta_h\|_{L^1(G_{h,d})} \|\overline{\rho}_\alpha^{r_I}\|_{C^{p+3}(\mathbb{T}^d)} h^{p+1}.$$

Using integration by parts, we rewrite (for all $l = 1, \dots, d$ and $\alpha, \beta = 1, \dots, n_S$)

$$\begin{aligned}
&\big(\mathcal{I}_h\big[\partial_{x_\ell}\big(\overline{\rho}_\alpha^{r_I}\big(\partial_{x_\ell} V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big)\big], \eta_h\big)_h \\
&\quad = \big(\partial_{h,x_\ell}\mathcal{I}_h\big[\overline{\rho}_\alpha^{r_I}\big(\partial_{x_\ell} V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big], \eta_h\big)_h + \widetilde{R}_\nabla \\
&\quad = -\big(\mathcal{I}_h\big[\overline{\rho}_\alpha^{r_I}\big(\partial_{x_\ell} V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big], \partial_{h,x_\ell}^R \eta_h\big)_h + \widetilde{R}_\nabla \\
&\quad = -\big(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]\big(\mathcal{I}_h[\partial_{x_\ell} V_{\alpha\beta}^{r_I}] *_h \mathcal{I}_h[\overline{\rho}_\beta^{r_I}]\big), \partial_{h,x_\ell}^R \eta_h\big)_h + \widetilde{R}_\nabla + \widetilde{R}_*,
\end{aligned}$$

where for $\nabla \to \nabla_h$ with Assumption 3.9 and the discrete Hölder inequality, we pay

$$\begin{aligned}
|\widetilde{R}_\nabla| &\le C \|\eta_h\|_{L^1(G_{h,d})} \big\|\overline{\rho}_\alpha^{r_I}\big(\partial_{x_\ell} V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}\big)\big\|_{C^{p+2}(\mathbb{T}^d)} h^{p+1} \\
&\le C \|\eta_h\|_{L^1(G_{h,d})} \|\overline{\rho}^{r_I}\|_{C^{p+2}(\mathbb{T}^d)}^2 \|\nabla V_{\alpha\beta}^{r_I}\|_{L^1(\mathbb{T}^d)} h^{p+1},
\end{aligned}$$

while for $* \to *_h$ due to Lemma B.5 with the discrete Hölder inequality, we have

$$|\widetilde{R}_*| \le C \|D_\ell^h \eta_h\|_{L^1(G_{h,d})} \|\overline{\rho}_\alpha^{r_I}\|_{L^\infty} \|\nabla V_{\alpha\beta}^{r_I}\|_{C^{p+2}} \|\overline{\rho}_\beta^{r_I}\|_{C^{p+2}} h^{p+1}.$$

Thus, combining these two observations, we obtain

$$\begin{aligned}
&\big(\partial_t\big(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}] - \overline{\rho}_{h,\alpha}^{r_I}\big), \eta_h\big)_h \\
&\quad = \sigma_\alpha \big(\Delta_h\big(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}] - \overline{\rho}_{h,\alpha}^{r_I}\big), \eta_h\big)_h \\
&\qquad - \sum_{\beta=1}^{n_S} \big(\big(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}] - \overline{\rho}_{h,\alpha}^{r_I}\big)\big(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \mathcal{I}_h[\overline{\rho}_\beta^{r_I}]\big), \nabla_h^R \eta_h\big)_h \\
&\qquad - \sum_{\beta=1}^{n_S} \big(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]\big(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \big(\mathcal{I}_h[\overline{\rho}_\beta^{r_I}] - \overline{\rho}_{h,\beta}^{r_I}\big)\big), \nabla_h^R \eta_h\big)_h \\
&\qquad + \sum_{\beta=1}^{n_S} \big(\big(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}] - \overline{\rho}_{h,\alpha}^{r_I}\big)\big(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \big(\mathcal{I}_h[\overline{\rho}_\beta^{r_I}] - \overline{\rho}_{h,\beta}^{r_I}\big)\big), \nabla_h^R \eta_h\big)_h \\
&\qquad + R_{\Delta,\nabla} \|\eta_h\|_{L^1(G_{h,d})} + R_* \|\nabla_h^R \eta_h\|_{L^1(G_{h,d})},
\end{aligned} \tag{B.3}$$

with

$$|R_{\Delta,\nabla}| \le C\big(\|\overline{\rho}^{r_I}\|_{C^{p+3}} + \|\overline{\rho}^{r_I}\|^2_{C^{p+2}}\|\nabla V^{r_I}\|_{L^1}\big)h^{p+1},$$
$$|R_*| \le C\|\overline{\rho}^{r_I}\|_{L^\infty}\|\overline{\rho}^{r_I}\|_{C^{p+2}}\|\nabla V^{r_I}\|_{C^{p+2}}h^{p+1}.$$

The remainder of the proof will follow the steps of a standard energy estimate. Thus, we will choose $\eta_h = \mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha}$ as a test function. Note that with Hölder

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha}\|_{L^1(G_{h,d})} \le |\mathbb{T}^d|^{1/2}\|\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha}\|_{L^2(G_{h,d})},$$
$$\|\nabla^R_h(\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha})\|_{L^1(G_{h,d})} \le |\mathbb{T}^d|^{1/2}\|\nabla^R_h(\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha})\|_{L^2(G_{h,d})}.$$

Applying (3.5), comparing the first-order operators via (3.9) from Assumption 3.9 and integrating by parts, we obtain

$$\begin{aligned}\sigma_\alpha\big(\Delta_h(\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha}), \mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha}\big)_h &= -\sigma_\alpha\|\nabla_{D,h}(\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha})\|^2_{L^2(G_{h,d})}\\ &\le -\sigma_\alpha C_D\|\nabla^R_h(\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha})\|^2_{L^2(G_{h,d})}.\end{aligned}$$

Now, applying a combination of the discrete Hölder inequality, Young's discrete convolution inequality and Young's inequality for products to (B.3), we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\|\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha}\|^2_{L^2(G_{h,d})} + \frac{C_D\underline{\sigma}}{5}\|\nabla^R_h(\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \overline{\rho}^{r_I}_{h,\alpha})\|^2_{L^2(G_{h,d})}\\ &\quad\le 2\frac{5}{C_D\underline{\sigma}}\|\mathcal{I}_h[\nabla V]\|^2_{L^2(G_{h,d})}\|\mathcal{I}_h[\overline{\rho}^{r_I}]\|^2_{L^2(G_{h,d})}\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})}\\ &\qquad+\frac{5}{C_D\underline{\sigma}}\|\mathcal{I}_h[\nabla V]\|^2_{L^2(G_{h,d})}\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^4_{L^2(G_{h,d})}\\ &\qquad+\frac{5}{C_D\underline{\sigma}}|\mathbb{T}^d||R_*|^2 + |\mathbb{T}^d||R_{\Delta,\nabla}|^2 + \|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})},\end{aligned}$$

where $\underline{\sigma} := \min_\alpha \sigma_\alpha$. For $t \in [0, \overline{T}_h]$, which implies

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^4_{L^2(G_{h,d})}(t) \le \|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})}(t),$$

summing over all species results in

$$\begin{aligned}&\partial_t\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})} + \|\nabla^R_h(\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h)\|^2_{L^2(G_{h,d})}\\ &\quad\le C\big(1 + (1 + \|\mathcal{I}_h[\overline{\rho}^{r_I}]\|^2_{L^2(G_{h,d})})\|\mathcal{I}_h[\nabla V^{r_I}]\|^2_{L^2(G_{h,d})}\big)\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})} + |R|^2,\end{aligned}$$

with

$$|R| \le C\big(1 + \|\nabla V^{r_I}\|_{C^{p+2}}\|\overline{\rho}^{r_I}\|_{C^{p+2}}\big)\|\overline{\rho}^{r_I}\|_{C^{p+3}}h^{p+1}.$$

Now, Assumption 3.10 yields

$$\|\mathcal{I}_h[\overline{\rho}^0] - \overline{\rho}^{r_I}_h(0)\|_{L^2(G_{h,d})} \le Ch^{p+1}.$$

Further, using $\|\mathcal{I}_h[f]\|_{L^2(G_{h,d})} \le C\|f\|_{C(\mathbb{T}^d)}$ and that $\|\nabla V^{r_I}\|_C \le C{r_I}^{-(d+1)}$ as well as $\|\nabla V^{r_I}\|_{C^{p+2}} \le C{r_I}^{-(d+p+3)}$, the Gronwall lemma then yields

$$\begin{aligned}&\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})}(t)\\ &\quad\le \big(Ch^{2(p+1)} + tC(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})})h^{2(p+1)}{r_I}^{-2(d+p+3)}\big)\\ &\qquad\times\exp\big(tC(\rho_{\max}){r_I}^{-2(d+1)}\big).\end{aligned}\tag{B.4}$$

In order to simplify the structure, we estimate

$$tC(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})}){r_I}^{-2(d+p+3)} \le \exp(tC(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})}, p){r_I}^{-2(d+1)}).$$

With the scaling from Assumption 3.11, we have, for any chosen $\kappa > 0$,

$${r_I}^{-2(d+1)} \le \log(N)^{\frac{2d+2}{2d+4}} \le \kappa\big(C(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})}, p) + C(\rho_{\max})\big)^{-1} \log(N)$$

for N large enough. Combining these observations with (B.4) yields (B.1). $\square$

B.1.1. *Higher order bounds for the mean-field limit difference.* We will obtain higher order bounds for the mean-field limit difference via induction. Opposed to Proposition B.6 in this proof, we do not need additional smallness assumptions to deal with the quadratic nonlinearity because—when taking derivatives—at most one factor is of the highest order. The other one is controlled by the induction assumption.

The first order one-sided finite differences, given by

$$\partial_{h,1}^{e_\ell} f(x) = \frac{f(x+he_\ell) - f(x)}{h} \tag{B.5}$$

for $\ell = 1, \dots, d$, satisfy the product rule

$$\partial_{h,1}^{e_\ell}(fg) = (\tau_{he_\ell} f)(\partial_{h,1}^{e_\ell} g) + (\partial_{h,1}^{e_\ell} f)g \tag{B.6}$$

for any $f, g \in L^2(G_{h,d})$ with the shift operators $\tau_{h\nu}$, $\nu \in \mathbb{Z}^d$, given by $\tau_{h\nu} f(x) = f(x+h\nu)$.

PROPOSITION B.7 (Higher order bound for the discrete mean-field limit). *Let $s \in \mathbb{N}$. In the setting of Proposition B.6 with the additional assumption $\overline{\rho}^{r_I} \in L^\infty(0, T; C^{s+p+3}(\mathbb{T}^d))$, for every $\kappa > 0$ there exists $N_0 = N_0(\kappa, s, data)$ such that, for all $N > N_0$,*

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}_h^{r_I}\|_{H^s(G_{h,d})}^2(t) \le Ch^{2(p+1)} N^{\kappa t}, \quad \forall t \in [0, T]. \tag{B.7}$$

PROOF. We use induction over $s \in \mathbb{N}_0$. The base case is settled by Proposition B.6. For the induction step, assume that (B.7) holds for $s-1$, $s \ge 1$, and large enough N. Let $\nu \in \mathbb{N}_0^d$ with $|\nu| = s$. With (B.6), similarly to the proof of Proposition B.6, for $\eta_h \in L^2(G_{h,d})$, we get

$$\begin{aligned} &\big(\partial_t(\partial_{h,1}^\nu \overline{\rho}_{h,\alpha}^{r_I}), \eta_h\big)_h \\ &\quad = -\sigma_\alpha\big(\nabla_{D,h}\partial_{h,1}^\nu \overline{\rho}_{h,\alpha}^{r_I}, \nabla_{D,h}\eta_h\big)_h \\ &\qquad - \sum_{\gamma \le \nu} \sum_{\beta=1}^{n_S} b_{\nu,\gamma}\big(\tau_{h\gamma}\partial_{h,1}^{\nu-\gamma} \overline{\rho}_{h,\alpha}^{r_I}(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \partial_{h,1}^\gamma \overline{\rho}_{h,\beta}^{r_I}), \nabla_h^R \eta_h\big)_h, \end{aligned} \tag{B.8}$$

where ∇_h^R is the adjoint/reflected version of ∇_h while $(b_{\nu,\gamma})_\gamma$ are the typical product rule prefactors (hence binomial coefficients), and

$$\begin{aligned} &\big(\partial_t\partial_{h,1}^\nu \mathcal{I}_h[\overline{\rho}_\alpha^{r_I}], \eta_h\big)_h \\ &\quad = -\sigma_\alpha\big(\nabla_{D,h}\partial_{h,1}^\nu \mathcal{I}_h[\overline{\rho}_\alpha^{r_I}], \nabla_{D,h}\eta_h\big)_h \\ &\qquad - \sum_{\gamma \le \nu} \sum_{\beta=1}^{n_S} b_{\nu,\gamma}\big(\tau_{h\gamma}\partial_{h,1}^{\nu-\gamma} \mathcal{I}_h[\overline{\rho}_\alpha^{r_I}](\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \partial_{h,1}^\gamma \mathcal{I}_h[\overline{\rho}_\beta^{r_I}]), \nabla_h^R \eta_h\big)_h \\ &\qquad + R_{\Delta,\nabla}\|\eta_h\|_{L^2(G_{h,d})} + R_*\|\nabla_h^R \eta_h\|_{L^2(G_{h,d})} \end{aligned} \tag{B.9}$$

for all species $\alpha = 1, \dots, n_S$ with consistency errors

$$|R_{\Delta,\nabla}|, |R_*| \le C\big(1 + \|\nabla V^{r_I}\|_{C^{p+2}}\|\overline{\rho}^{r_I}\|_{C^{s+p+2}}\big)\|\overline{\rho}^{r_I}\|_{C^{s+p+3}} h^{p+1}. \tag{B.10}$$

To obtain this bound for $R_{\Delta,\nabla}$, we use that

$$\|\partial_{h,1}^{\nu}(\Delta_h-\Delta)\overline{\rho}_\alpha^{r_I}\|_{L^\infty}\le Ch^{p+1}\|\partial_{h,1}^{\nu}\overline{\rho}_\alpha^{r_I}\|_{C^{p+3}}\le Ch^{p+1}\|\overline{\rho}_\alpha^{r_I}\|_{C^{p+s+3}}$$

and the analogous estimate for ∇_h, which hold since $\partial_{h,1}^{\nu}$ is an order $|\nu|$ finite difference operator and with the mean value theorem. To bound R_*, note that with Lemma B.5

$$\begin{aligned}&\|\partial_{h,1}^{\nu}(\overline{\rho}_\alpha^{r_I}(\nabla V^{r_I}*\overline{\rho}_\beta^{r_I}))-\partial_{h,1}^{\nu}(\overline{\rho}_\alpha^{r_I}(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}]*_h\overline{\rho}_\beta^{r_I}))\|_{L^\infty}\\&\quad=\Big\|\sum_{\gamma\le\nu}b_{\nu,\gamma}\tau_{h\gamma}\partial_{h,1}^{\nu-\gamma}\overline{\rho}_\alpha^{r_I}(\nabla V_{\alpha\beta}^{r_I}*\partial_{h,1}^{\gamma}\overline{\rho}_\beta^{r_I}-\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}]*_h\partial_{h,1}^{\gamma}\overline{\rho}_\beta^{r_I})\Big\|_{L^\infty}\\&\quad\le Ch^{p+1}\sum_{\gamma\le\nu}\|\partial_{h,1}^{\nu-\gamma}\overline{\rho}^{r_I}\|_{L^\infty}\|\nabla V^{r_I}\|_{C^{p+2}}\|\partial_{h,1}^{\gamma}\overline{\rho}^{r_I}\|_{C^{p+2}}.\end{aligned}$$

Taking the difference of (B.8) and (B.9), we obtain a structure corresponding to (B.3). Further proceeding as in the previous proof, we choose $\eta_h=\partial_{h,1}^{\nu}\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\partial_{h,1}^{\nu}\overline{\rho}_{h,\alpha}^{r_I}$, apply a combination of Young's discrete convolution inequality, the discrete Hölder inequality as well as Young's inequality for products, and then absorb the $\|\nabla_h^R(\mathcal{I}_h[\partial^\nu\overline{\rho}^{r_I}]-\partial_{h,1}^{\nu}\overline{\rho}_h^{r_I})\|_{L^2}$-terms. Note that in the interaction terms always at least one of the differences only has derivatives of order at most $s-1$. Evoking the induction assumption, we thus obtain

$$\begin{aligned}\partial_t\|\partial_{h,1}^{\nu}\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\partial_{h,1}^{\nu}\overline{\rho}_{h,\alpha}^{r_I}\|_{L^2}\le{}&C\|\mathcal{I}_h[\nabla V^{r_I}]\|_{L_h^2}^2\Big(\sup_{|\gamma|\le s}\|\partial_{h,1}^{\gamma}\mathcal{I}_h[\overline{\rho}^{r_I}]\|_{L^2}^2+h^{2(p+1)}N^{\kappa t}\Big)\\&\times(\|\partial_{h,1}^{\nu}\mathcal{I}_h[\overline{\rho}^{r_I}]-\partial_{h,1}^{\nu}\overline{\rho}_h^{r_I})\|_{L^2}^2+h^{2(p+1)}N^{\kappa t})\\&+\|\partial_{h,1}^{\nu}\mathcal{I}_h[\overline{\rho}^{r_I}]-\partial_{h,1}^{\nu}\overline{\rho}_h^{r_I}\|_{L^2}^2+|R_{\Delta,\nabla}|^2+C|R_*|^2.\end{aligned}$$

Assumption 3.11 implies $h^{2(p+1)}N^{\kappa T}\le 1$ for $\kappa\le\delta_0$, while as for the $R_{\Delta,\nabla}$-bound

$$\sup_{|\gamma|\le s}\|\partial_{h,1}^{\gamma}\mathcal{I}_h[\overline{\rho}^{r_I}]\|_{L^2}^2\le C\|\overline{\rho}^{r_I}\|_{L^\infty(0,T;C^s(\mathbb{T}^d))}^2.$$

Thus, after summing over all species and replacing the V^{r_I}-norms with the respective r_I-scaling, since $\partial_{h,1}^{\nu}(\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I})(0)=0$ via the Gronwall inequality, we obtain

$$\begin{aligned}\|\mathcal{I}_h[\partial^\nu\overline{\rho}^{r_I}]-\partial_{h,1}^{\nu}\overline{\rho}_h^{r_I}\|_{L^2(G_{h,d})}^2(t)\le{}&Ct\big(1+\|\overline{\rho}^{r_I}\|_{L^\infty(C^{s+p+3})}^4\big)h^{2(p+1)}N^{\kappa t}{r_I}^{-2(d+p+2)}\\&\times\exp\big(Ct\big(1+\|\overline{\rho}^{r_I}\|_{L^\infty(C^s)}^2\big){r_I}^{-2(d+1)}\big).\end{aligned}$$

Now, we plug in the scaling of r_I and simplify as we did at the end of the proof for Proposition B.6. Looping over all $|\nu|=s$ leads to (B.7) for N large enough (and a slightly larger κ than chosen for the induction assumption), thus finishing the proof by induction. □

B.2. Regularity of discrete test functions.

LEMMA B.8 (Bounds on Sobolev norms of discrete test functions). *Let ϕ_h^t be the solution to (5.22), and let the assumptions of Proposition B.7 be satisfied. Let $\kappa>0$. Then for every $s\in\mathbb{N}_0$, for every $t\in[0,T]$, and for N large enough, we have the bound*

$$\|\phi_h^t\|_{H_h^s}^2+\int_t^T\|\nabla_h\phi_h^\tau\|_{H_h^s}^2\,\mathrm{d}\tau\le N^{\kappa(T-t)}\|\mathcal{I}_h\varphi\|_{H_h^s}^2\le N^{\kappa(T-t)}\|\varphi\|_{C^s}^2. \tag{B.11}$$

We omit the proof, a straightforward adaption of Lemma A.1 combined with (B.7).

LEMMA B.9 (High-order bounds on Sobolev norms for difference of continuous and discrete test functions). *Let ϕ^t_α be the solution to* (5.18) *with final datum $\phi^T = \varphi$. Let $\phi^t_{h,\alpha}$ be the solution to* (5.22). *Furthermore, let all the assumptions in Proposition* B.6 *be satisfied. Let $\kappa > 0$. Then the following estimates hold for large enough N*:

- Part I: Bound for L^2-difference (analogue of Lemma B.6). *If the regularity requirements*

$$V_{\beta\alpha} \in C^{p+3}(\mathbb{T}^d), \qquad \phi_\alpha \in C^{p+3}(\mathbb{T}^d), \qquad \overline{\rho}^{r_I} \in C^{p+3} \tag{B.12}$$

  are satisfied, we have the bound

$$\|\phi^t_{h,\alpha} - \mathcal{I}_h\phi^t_\alpha\|^2_h \le Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|_{C^{p+3}}. \tag{B.13}$$

- Part II: Bound for discrete Sobolev norm of difference (analogue of Proposition B.7). *If the following stricter regularity requirement*

$$V_{\beta\alpha} \in C^{s+(p+3)}(\mathbb{T}^d), \qquad \phi_\alpha \in C^{s+(p+3)}(\mathbb{T}^d), \qquad \overline{\rho}^{r_I} \in C^{s+(p+3)} \tag{B.14}$$

  is satisfied, we have the bound

$$\|\phi^t_{h,\alpha} - \mathcal{I}_h\phi^t_\alpha\|^2_{H^s(G_{h,d})}(t) \le Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|_{C^{s+p+3}}. \tag{B.15}$$

- Part III: Bound for gradient difference. *The assumptions of Part II) being satisfied for $s = 1$, we have the bound*

$$\|\nabla_h\phi^t_{h,\alpha} - \mathcal{I}_h\nabla\phi^t_\alpha\|_h \le Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|_{C^{p+4}}. \tag{B.16}$$

- Part IV: Bound for difference of product of gradients. *Assuming the validity of the hypotheses in Parts I), II), III), we obtain, as a special case, that*

$$\|\nabla_h\phi^t_{h,\alpha_1}\cdot\nabla_h\phi^t_{h,\alpha_2} - \mathcal{I}_h[\nabla\phi^t_{\alpha_1}\cdot\nabla\phi^t_{\alpha_2}]\|^2_h \le Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|^2_{C^{\lceil d/2\rceil+p+5}}. \tag{B.17}$$

The proof of Lemma B.9 closely follows the proofs of Lemma B.6, Proposition B.7, and makes use of Lemma A.1, and Lemma B.8. Such a proof—which we omit here—can be found in the arXiv version of this paper (https://arxiv.org/abs/2303.00429).

LEMMA B.10 (L^1 bound for discrete test functions). *Let ϕ^t_h be the solution to* (5.22). *Let $\kappa > 0$. Then for large enough N, ϕ^t_h satisfies the L^1 bound*

$$\sup_{t\in[0,T]}\|\phi^t_h\|_{L^1(G_{h,d})} \le N^{\kappa T}\|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})}. \tag{B.18}$$

PROOF. We expand the solution ϕ_h using the discrete Green's function $\mathcal{G}_h$ associated with the discrete Laplace Δ_h, obtaining

$$\begin{aligned}\phi^{T-t}_{h,\alpha}(x) &= (\mathcal{G}^t_h *_h \mathcal{I}_h\varphi)(x) + \int_{T-t}^T (\mathcal{G}^{T-s}_h(x-\cdot), [-U^{r_I}_{h,\alpha}(s)\cdot\nabla_h\phi^s_{h,\alpha}])_h\,\mathrm{d}s \\ &\quad + \int_{T-t}^T\left(\mathcal{G}^{T-s}_h(x-\cdot), \sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\overline{\rho}^{r_I}_{h,\beta}\nabla_h\phi^s_{h,\beta})\right)_h\mathrm{d}s.\end{aligned}$$

Using integration by parts so as to remove all first derivatives from ϕ_h, we get

$$\begin{aligned}\phi^{T-t}_{h,\alpha}(x) &= (\mathcal{G}^t_h *_h \mathcal{I}_h\varphi)(x) + \int_{T-t}^T (\phi^s_{h,\alpha}, \nabla_h\cdot[\mathcal{G}^{T-s}_h(x-\cdot)U^{r_I}_{h,\alpha}(s)])_h\,\mathrm{d}s \\ &\quad - \int_{T-t}^T\left(\nabla_h\mathcal{G}^{T-s}_h(x-\cdot), \sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\overline{\rho}^{r_I}_{h,\beta}\phi^s_{h,\beta})\right)_h\mathrm{d}s \\ &\quad - \int_{T-t}^T\left(\mathcal{G}^{T-s}_h(x-\cdot), \sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\nabla_h[\overline{\rho}^{r_I}_{h,\beta}]\phi^s_{h,\beta})\right)_h\mathrm{d}s.\end{aligned}$$

Using the inequality

$$\big\|\nabla_h \mathcal{G}_h^{T-s}(\cdot - y)\big\|_{L^1(G_{h,d})} \lesssim (T-s)^{-1/2},$$

the bound $\|U_{h,\alpha}^{r_I}\|_{W^{1,\infty}} \le \|V_{\beta\alpha}^{r_I}\|_{H^2}\|\overline{\rho}_h^{r_I}\|_{W^{1,\infty}(G_{h,d})}$, the discrete Young's convolution inequality given in Corollary B.3), and summing up over all species, we obtain

$$\sum_{\alpha=1}^{n_S}\|\phi_{h,\alpha}^{T-t}\|_{L^1(G_{h,d})} \le \|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})} + \|V_{\beta\alpha}^{r_I}\|_{H^2}\|\overline{\rho}_h^{r_I}\|_{W^{1,\infty}(G_{h,d})} \times \int_{T-t}^{T}\big(1+(T-s)^{-1/2}\big)\sum_{\alpha=1}^{n_S}\|\phi_{h,\alpha}^{s}\|_{L^1(G_{h,d})}\,\mathrm{d}s.$$

Taking the supremum for $t \in [0,\tilde{t}]$, we deduce

$$\begin{aligned}&\sup_{t\in[0,\tilde{t}]}\sum_{\alpha=1}^{n_S}\|\phi_{h,\alpha}^{T-t}\|_{L^1(G_{h,d})}\\ &\quad\le \|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})} + \|V_{\beta\alpha}^{r_I}\|_{H^2}\|\overline{\rho}_h^{r_I}\|_{W^{1,\infty}(G_{h,d})}\\ &\qquad\times\left[\int_{T-\tilde{t}}^{T}\big(1+(T-s)^{-\frac{1}{2}}\big)\,\mathrm{d}s\right]\sup_{t\in[0,\tilde{t}]}\sum_{\alpha=1}^{n_S}\|\phi_{h,\alpha}^{T-t}\|_{L^1(G_{h,d})}.\end{aligned}\tag{B.19}$$

Choosing $\tilde{t} \le C\|V_{\beta\alpha}^{r_I}\|_{H^2}^{-2}\|\overline{\rho}_h^{r_I}\|_{W^{1,\infty}(G_{h,d})}^{-2}$ with small enough C allows to perform an absorption argument in (B.19), leading to

$$\sup_{t\in[0,\tilde{t}]}\sum_{\alpha=1}^{n_S}\|\phi_{h,\alpha}^{T-t}\|_{L^1(G_{h,d})} \le 2\|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})}.$$

Repeating the analysis over all time intervals $[j\tilde{t},(j+1)\tilde{t}]$ up to saturation of $[0,T]$ (there are $\lceil T/\tilde{t}\rceil \lesssim \|V_{\beta\alpha}^{r_I}\|_{H^2}^2\|\overline{\rho}_h^{r_I}\|_{W^{1,\infty}(G_{h,d})}^2 T$ of such intervals), we get

$$\sup_{t\in[0,T]}\|\phi_h(t)\|_{L^1(G_{h,d})} \le 2^{T\|V_{\beta\alpha}^{r_I}\|_{H^2}^2\|\overline{\rho}_h^{r_I}\|_{W^{1,\infty}(G_{h,d})}^2}\|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})},$$

and the proof is now concluded bounding $\|\overline{\rho}_{h,\beta}^{r_I}\|_{W^{1,\infty}}$ using (B.7), and bounding $\|V_{\beta\alpha}^{r_I}\|_{H^2}^2$ using the scaling (3.15). $\square$

APPENDIX C: CONSTRUCTING DISCRETE INITIAL DATA VIA INTERPOLATION

We substantiate Remark 3.15: specifically, we show how to construct discrete initial data satisfying Assumption 3.10 from given continuous initial data (i.i.d.) by redistributing mass to an appropriate set of surrounding grid points according to interpolation weights.

C.1. The multivariate interpolation scheme. We briefly discuss the polynomial multivariate interpolation scheme of order $p+1$. First, for each $y \in \mathbb{T}^d$, we have to fix the set of surrounding grid points $B_h^p(y)$ onto which to distribute mass from the continuous initial data. For schemes of order $p+1$ (interpolating polynomials of order at most p exactly), we need $\binom{p+d}{d}$ points in a suitable formation. As a subset of our rectangular grid, we choose, for example, the "lower left triangular cone," defined as

$$B_h^p := \{x \in G_{h,d};\ \forall = 1,\dots,d : 0 \le x_i,\ x_1+\cdots+x_d \le ph\};$$

see [38], Section 2.2. Thus, for $y \in [0,h)^d$, we choose $B_h^p(y) = B_h^p$. In case y is in another grid section, we appropriately shift the set of interpolation points: $B_h^p(y) = B_h^p + h\lfloor y/h \rfloor$. With this choice, for $f \in C^{p+1}$ and $y \in [0,h)^d$, we have the error estimate

$$f(y) = \sum_{x \in B_h^p} f(x)\Lambda_x(y) + \mathcal{O}\big(\|\varphi\|_{C^{p+1}} h^{p+1}\big), \tag{C.1}$$

where the Λ_x are the Lagrange fundamental polynomials with $\Lambda_x(\tilde{x}) = \delta_{x\tilde{x}}$ for $\tilde{x} \in B_h^p$; see [38], Section 4.1. Note though that we can rephrase $\Lambda_x(y)$ as samples $\Lambda(x-y)$ from a continuously defined convolution kernel

$$\Lambda : \mathbb{T}^d \to \mathbb{R}, \qquad \Lambda(z) := \begin{cases} \Lambda_x(x-z) & \text{for } x - z \in [0,h)^d, x \in B_h^p, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the interior of each grid section the kernel Λ is a polynomial and thus continuous. Due to the definition of the Lagrange polynomials, Λ is further continuous at all grid points, but in general discontinuous on the remaining contact areas between grid sections.

With this definition we can extend (C.1) to all $y \in \mathbb{T}^d$ as

$$f(y) = \sum_{x \in G_{h,d}} f(x)\Lambda(x-y) + \mathcal{O}\big(\|\varphi\|_{C^{p+1}} h^{p+1}\big). \tag{C.2}$$

C.2. Constructing the discrete initial data. To construct discrete initial data from the continuous one, for $\alpha = 1, \dots, n_S$ we would like to straight-up distribute the mass of a particle starting at $X_{\alpha,i}^{r_I,N}(0)$ according to the interpolation weights, that is, for $x \in G_{h,d}$

$$\rho_{h,\alpha}^{0,*}(x) := h^{-d} \int_{\mathbb{T}^d} \Lambda(x-y)\, \mathrm{d}\mu_{0,\alpha}^{r_I,N}(y) = h^{-d}\big(\Lambda * \mu_{0,\alpha}^{r_I,N}\big)(x),$$

where the volume factor h^{-d} comes from the interpretation as a density. In order to abide by the wanted properties of the fluctuations (Assumption 3.10), we have to adjust for the fact that the discrete mean-field limit is initialized as $\overline{\rho}_h^0 = \mathcal{I}_h[\overline{\rho}^0]$. Thus, for $x \in G_{h,d}$, we set

$$\rho_{h,\alpha}^0(x) := h^{-d}\big(\Lambda * \mu_{0,\alpha}^{r_I,N}\big)(x) - h^{-d}\big(\Lambda * \overline{\rho}_\alpha^0\big)(x) + \mathcal{I}_h\big[\overline{\rho}_\alpha^0\big](x).$$

From (C.2) it follows that this definition satisfies (3.10).

With respect to (3.11) and (3.12), note that since the particles are initially i.i.d., Hoeffding's inequality implies for any $m \in \mathbb{Z}^d \cap [-\frac{h}{\pi}, \frac{h}{\pi})^d$

$$\left| h^d \sum_{x \in G_{h,d}} \big(\rho_h^0 - \overline{\rho}_h^0\big)(x) \exp(im \cdot x) \right|^2 \le \mathcal{C}^2 N^{-1},$$

where $\mathcal{C}$ is subject to a Gaussian moment bound. Multiplying with $(1+|m|)^{-d-1}$ and summing in m, we deduce $\|\rho_h^0 - \overline{\rho}_h^0\|_{H_h^{-(\lfloor d/2 \rfloor+1)}} \le \mathcal{C} N^{-1/2}$, which immediately implies that (3.12) is satisfied. Similarly, by Bernstein's inequality we obtain

$$\mathbb{P}\left[\big|\rho_h^0(x) - \overline{\rho}_h^0(x)\big| \ge \sqrt{\frac{CN \operatorname{Var} \rho_h^0(x)}{N}} R + \frac{C|N \max_\Omega \rho_h^0(x)| R^2}{N} \right] \le \exp(-R^2),$$

for every $x \in G_{h,d}$ and any $R \ge 1$, and thus by bounding the local mollifier $h^{-d}\Lambda$

$$\mathbb{P}\big[\big|\rho_h^0(x) - \overline{\rho}_h^0(x)\big| \ge CN^{-1/2}h^{-d/2}R + Ch^{-d}N^{-1}R^2\big] \le \exp(-R^2).$$

This implies that this set of initial conditions satisfies (3.11). The only parts of Assumption 3.10 left are positivity and the mass restriction. Due to $\overline{\rho}^0 \ge \rho_{\min}$ and (3.11), the probability of negative initial data decays exponentially; hence, for large enough N, we can restrict to positive data at the cost of only an additional insignificant error in (3.10). Since the interpolation weights preserve mass, for positive initial data we have $\|\rho_{h,\alpha}^0\|_{L_h^1} = \|\mathcal{I}_h[\overline{\rho}_\alpha^0]\|_{L_h^1} \approx 1$ for h small enough. Thus, restricting to $\rho_h^0 \geq 0$, this construction satisfies Assumption 3.10.

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