

Synchronizing Fields with Singularities

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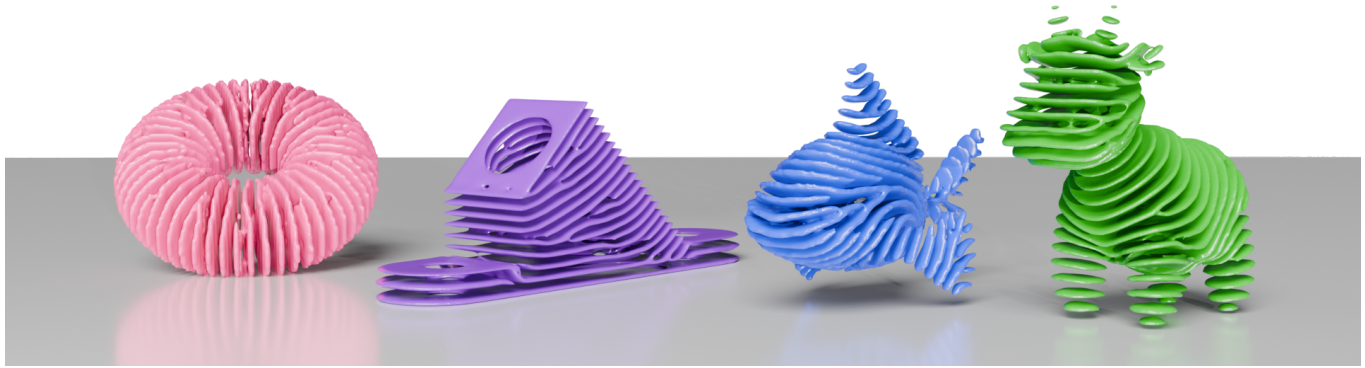


Fig. 1. Many geometry processing tasks can be formulated as synchronization problems. Here, $U(1)$ synchronization over a volumetric domain produces evenly spaced volumetric stripe patterns with explicit boundary control, e.g., for curved-layer 3D printing.

A variety of problems in geometry processing boil down to finding the most parallel field relative to a connection. Instances of this prototypical problem show up in computing direction fields and stripe patterns, quadrilateral meshing, and visualization of fluid flows. When the class of allowed fields includes those with topological defects, a relaxation is required to make the problem well-posed. We observe that these problems can be viewed as *synchronization problems*, which admit a natural semidefinite relaxation. We propose a unified method of solving all these problems via the efficient Burer-Monteiro factorization method. Geometrically, this amounts to lifting the field values to a higher-dimensional manifold, naturally resolving the singular nature of defects. Practically, we show that our convex relaxation method achieves better and more reliable optima than previous work employing alternative relaxations.

CCS Concepts: • **Computing methodologies** → **Shape analysis**; • **Mathematics of computing** → **Semidefinite programming**; *Convex optimization*; *Nonconvex optimization*; *Algebraic topology*.

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1 Introduction

Many optimization problems in graphics and geometry processing amount to finding a "most parallel" section of a bundle, i.e., one that minimizes a covariant Dirichlet energy with respect to a connection on the bundle. This prototypical problem often turns up when computing sections that feature *topological defects*, and it can be viewed as a way of *rounding* the curvature of the connection to a discrete set of defects.

Such problems, once discretized, can be recast as instances of a so-called *synchronization problem*. Synchronization problems are non-convex but admit relaxations to convex problems. Previous work in geometry processing has invariably resorted to either non-convex optimization or *spectral relaxation*. We propose to use the natural relaxation to a semidefinite program (SDP) instead. Semidefinite relaxation has several advantages. First, it is tighter than spectral relaxation (dropping fewer constraints), a fact we confirm empirically. Second, the solution to the relaxation provides both a feasible solution to the un-relaxed problem and a lower bound on its value, thus quantifying how close it is to global optimality. Third, combining relaxation with a randomized rounding procedure provides a whole family of near-optimal solutions with different topological defect configurations.

Finally, there is a convenient and efficient factorization algorithm, known as Burer-Monteiro factorization, that amounts to partially reversing the relaxation. Geometrically, the bundle fibers are lifted to higher-dimensional analogues. The factorization problem is then treated by manifold optimization on this lifted space. Remarkably, this low-dimensional non-convex problem is equivalent to the full

SDP, without ever constructing the full semidefinite matrix. Moreover, in the lifted setting, topological defects are no longer an issue, making the energy well-defined. The factorization problems provide a sequence of well-defined geometric relaxations that have the advantage of implicit convexity.

We demonstrate the application of these ideas to several problems in geometry processing, from computing optimal direction fields, to stripe and quad patterns on surfaces and in volumes, to Clebsch map regression for visualizing vorticity in fluids. In comparison with the spectral relaxations of previous work, our SDP-based approach produces results with lower energy and better-placed defects.

Contributions. In summary, we:

- unify a host of geometry processing problems featuring topological defects in a common synchronization framework;
- solve them with SDP relaxation and connect its geometry to that of defects; and
- enable new applications such as volumetric stripe patterns by allowing flexible imposition of constraints in a convex relaxation framework.

2 Related Work

2.1 Synchronization

Group synchronization problems, also referred to as generalized phase retrieval, non-unique games [Bandeira et al. 2020], or the little Grothendieck problem [Bandeira et al. 2016], seek to recover unknown group elements from noisy or incomplete relative measurements. They are widely used in areas such as sensor network localization [Cucuringu et al. 2012a]; computer vision [Tzeneva 2011]; structural biology [Bandeira et al. 2020; Cucuringu et al. 2012b; Singer and Shkolnisky 2011]; robotics [Rosen et al. 2019]; and machine learning [Singer and Wu 2012].

Singer [2011] proposed spectral and SDP relaxations of angular synchronization (the $U(1)$ case). Bandeira et al. [2013] established bounds on the performance of spectral relaxation. Bandeira et al. [2016] generalize this bound to synchronization over the higher-dimensional orthogonal and unitary groups. For $U(1)$ synchronization, SDP relaxation with randomized rounding is known to give a $(\pi/4)$ -approximation ratio in expectation under a positivity assumption on the cost matrix [Ben-Tal et al. 2003; So et al. 2007; Zhang and Huang 2006]. Bandeira et al. [2017] show tightness of an SDP relaxation, and Wang and Singer [2013] show exactness of an SDP relaxation for a “robust” version of the problem, both under statistical assumptions.

2.2 Semidefinite Programming

Semidefinite programs (SDPs) are efficiently-solvable convex cone optimization problems generalizing linear programs. We refer the reader to Blekherman et al. [2013]; Gärtner and Matousek [2012] for general background material.

SDPs have enjoyed outsized success in approximating the solutions to NP-hard problems, most famously in the case of MAX-CUT [Goemans and Williamson 1995]. MAX-CUT can be viewed as a synchronization problem over the group $\mathbb{Z}_2$, presaging the use of SDP relaxation for other synchronization problems.

In computer graphics, SDP relaxation powers methods for rigid [Dym and Lipman 2017; Maron et al. 2016] and non-rigid [Huang

and Guibas 2013] shape matching, hexahedral mesh correction [Marschner et al. 2020], and polynomial geometry kernels [Marschner et al. 2021]. Closest to our work is that of Palmer et al. [2020], which also features ideas of topological defects and semidefinite programming.

The first provably polynomial-time algorithms for solving SDPs employed ellipsoid [Grötschel 1993; Khachiyan 1979] and interior-point methods [Alizadeh 1995; Nesterov and Nemirovskii 1994], extending earlier work on solving linear programs. The factorization technique of Burer and Monteiro [2003] (BM), further developed by Boumal [2016]; Boumal et al. [2020]; Cifuentes [2021]; Cifuentes and Moitra [2022], has led to an alternative polynomial-time method for solving SDPs that produces low-rank solutions. It is this method we exploit in our work. In doing so, we introduce a new perspective on the geometry of the BM factorizations in the case of synchronization-type problems.

2.3 Topological Defects

The theory of topological defects has a long history in condensed matter physics, tracing its roots to the seminal work of Landau [1936, 1937] on symmetry breaking and phase transitions. For an introduction, we recommend the excellent expository paper of Mermin [1979]. Canonical examples include magnetic monopoles in gauge fields [Castelnovo et al. 2008], vortices in superfluids, and disclinations in liquid crystals [Machon and Alexander 2016]. An explanation of phase transitions in terms of statistics of defects [Kosterlitz and Thouless 1973] won the Nobel Prize in physics in 2016. Other applications of defects range from quantum error correction [Kitaev 2003; Pretko et al. 2019] to biology [Copenhagen et al. 2021; Fard et al. 2022; Guillamat et al. 2022; Maroudas-Sacks et al. 2021]. This sampling only hints at the ubiquity of topological defects in the natural world.

In geometry processing, *optimization* of topological defects is important to applications including direction field design [Couplet et al. 2024; Diamanti et al. 2014; Knöppel et al. 2013; Palmer et al. 2020] and surface parameterization [Bommes et al. 2009; Fang et al. 2018]. These problems are intimately tied to quadrilateral and hexahedral meshing, in which the placement of disclination defects (also known as *singularities*) is of paramount importance. Optimization techniques have included spectral relaxation [Knöppel et al. 2013]; nonconvex optimization based on the Ginzburg-Landau functional [Beaufort et al. 2017; Couplet et al. 2024; Golovaty et al. 2021; Viertel and Osting 2019]; explicit optimization of defect positions [Farchi and Ben-Chen 2018; Ma et al. 2024; Myles and Zorin 2013]; and lifting [Palmer et al. 2024]. The above citations only sample the extensive literature [Bommes et al. 2013; Pietroni et al. 2022; Vaxman et al. 2016].

Periodic patterns have been used in graphics for texturing [Knöppel et al. 2015] and quad meshing [Fang et al. 2018; Ray et al. 2006]. Computational fabrication applications like curved-layer 3D printing [Zhang et al. 2022], machine knitting [Mitra et al. 2025; Narayanan et al. 2018], and topology optimization [Maestre et al. 2023; Stutz et al. 2022] also seek striped or laminar structures on curved surfaces or volumetric domains optimized for geometric, structural, or functional criteria. These prominently feature dislocation defects that must be placed intelligently for optimal object performance.

3 Geometric Synchronization Problems

We introduce the language of synchronization first in the prototypical case of optimal unit direction fields on surfaces. We then walk through several additional examples from graphics to build intuition before describing the general case.

3.1 Smooth Direction Field Design

A unit vector field $\mathbf{v}$ on a smooth Riemannian manifold M is a section of the tangent bundle $\mathbf{v} \in \Gamma(TM)$ that additionally satisfies the normalization constraint $|\mathbf{v}|^2 = 1$ everywhere. Equivalently, we can describe $\mathbf{v}$ as a section of the *unit tangent bundle STM*, consisting of the unit vectors in each tangent space. The fibers of *STM* are spheres. In particular, when M is a two-dimensional surface, *STM* is a circle bundle. The *covariant Dirichlet energy* $E_\nabla[\mathbf{v}] := \|\nabla \mathbf{v}\|_{L^2(M)}^2$ measures the smoothness of $\mathbf{v}$ relative to the Levi-Civita connection ∇ . A typical field design problem seeks the globally smoothest direction field:

$$\min_{\mathbf{v} \in \Gamma(STM)} \|\nabla \mathbf{v}\|_{L^2(M)}^2. \quad (1)$$

We refer to this as a *smooth synchronization problem*.

3.1.1 Topological Defects. Unfortunately, the formulation (1) is ill-posed on most surfaces: the Poincaré-Hopf index theorem rules out non-vanishing tangent vector fields on surfaces of nonzero Euler characteristic.

Thus, we are led to consider *singular* unit fields $\mathbf{v} \in \Gamma(STM|_{M \setminus \Sigma})$, defined away from a set Σ of *singularities* or *disclinations*, also known as *topological defects*. A defect is an isolated point x around which $\mathbf{v}$ winds in an irreducible way, such that it cannot be extended continuously over x . This winding is characterized by an integer index associated to x , the same index that appears in the Poincaré-Hopf theorem. Beyond these topological considerations, all the “interesting” behavior of a direction field is concentrated at its singularities, and positioning these singularities is the main challenge of direction field design.

3.1.2 Ill-posedness and Relaxation. Consider a singular unit vector field $\mathbf{v}$ with a singularity at a point $x \in M$. On a small neighborhood $U \ni x$, we may choose a complex coordinate so that locally $\mathbf{v}|_{U \setminus \{x\}}$ may be represented as $e^{i\theta}$, and the connection as $\nabla := d - i\eta$, for $\eta \in \Omega^1(U)$ a local connection 1-form (in particular, $d\eta = \kappa$, the Gauss curvature 2-form). The phase $\theta \in \mathbb{R}$ is multi-valued on $U \setminus \{x\}$, winding by an integer multiple of 2π on any loop enclosing the defect. Now consider a loop γ of length $2\pi\epsilon$ bounding a region $V \subset U$ about x . Then on γ ,

$$\begin{aligned} 2\pi\epsilon \int_\gamma |\nabla \mathbf{v}|^2 ds &= \left(\int_\gamma ds \right) \left(\int_\gamma |d\theta - \eta|^2 ds \right) \\ &\geq \left| \int_\gamma d\theta - \eta \right|^2 \quad (\text{Cauchy-Schwarz}) \\ &= \left| 2\pi - \int_V \kappa \right|^2 \xrightarrow{\epsilon \rightarrow 0} 4\pi^2 \quad (\text{Stokes; smooth } \kappa) \end{aligned}$$



Fig. 2. Dislocations in 2D and 3D.

Integrating over the radial direction, one finds that $\nabla \mathbf{v}$ is not square-integrable in the neighborhood of a defect. To get a well-posed problem, one must change either the objective functional or the constraint in (1). This is known as *relaxation*. We will discuss relaxation in more detail in Section 4.2. First, we discuss the discretized version of (1) as a synchronization problem.

3.2 Discrete Direction Field Design

We now recall the standard approach to discretizing smooth synchronization problems of the form (1).

To discretize (1) on a triangulated surface, one represents $\mathbf{v}$ by its values on either vertices or faces. The connection is represented by parallel transport operators along or across edges, respectively. We summarize this state of affairs by a graph $\Gamma = (V, E)$, which will be either the vertex-edge or face-dual edge graph of the triangle mesh. After making an arbitrary choice of frame at each graph node $i \in V$, unit tangent vectors can be represented by unit complex numbers v_i . The connection is represented by parallel transport operators $u_{ij} \in \text{SO}(2) \cong \text{U}(1)$, i.e., also unit complex numbers. With these choices, we can immediately define the discrete covariant derivative and Dirichlet energy as:

$$E_\nabla[\mathbf{v}] := \sum_{i \sim j} w_{ij} \|(\nabla \mathbf{v})_{ij}\|_F^2, \quad (\nabla \mathbf{v})_{ij} := v_j - u_{ij} v_i. \quad (2)$$

Here $w_{ij} > 0$ are geometric weights, such as the standard cotangent weights in the case where Γ is the primal graph.

Thus, we arrive at the discrete synchronization problem

$$\min_v \left\{ v^\dagger L_u v : |v_i|^2 = 1 \ \forall i \in V \right\}, \quad (3)$$

where L_u is the *connection Laplacian* associated to the weights w_{ij} and connection u_{ij} , and $v \in \mathbb{C}^n$ is a vector whose i th entry is v_i . The constraint manifold, being a product of n copies of $\text{U}(1) \cong \text{S}^1$, is nonlinear and nonconvex, and in practice, (3) is difficult to solve to global optimality. This is a second reason why relaxation of the problem is essential.

3.3 Stripe Patterns

In stripe pattern generation as formulated by Knöppel et al. [2015], one aims to produce a periodic stripe pattern over a surface M guided by an input vector field $\mathbf{v}$. Just as direction fields feature disclination defects, stripe patterns in the real world include topological defects called *dislocations* at which it is impossible to choose a well-defined phase θ (see Figure 2). Allowing automatic placement of dislocations is essential for flexible design of stripe patterns.

Formally, away from dislocations, a circle-valued field $\varphi : M \rightarrow \text{S}^1$ encodes the phase of the periodic pattern. Ideally, we want to find φ

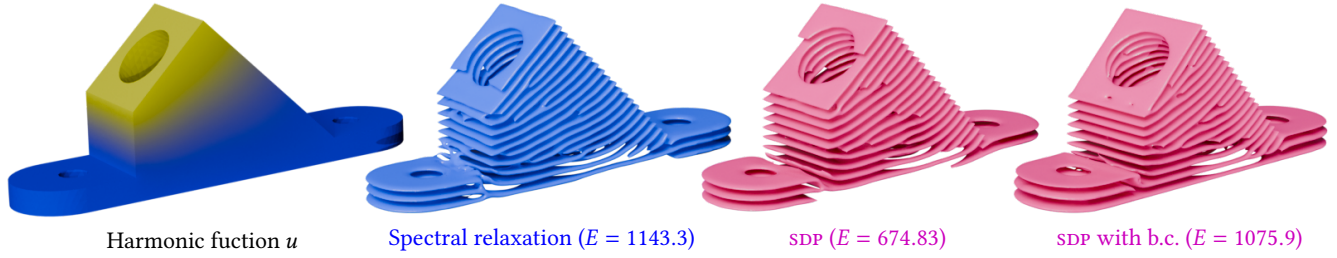


Fig. 3. The gradient vector field of a harmonic function (left) guides the formation of volumetric stripes with spectral relaxation (blue) and SDP relaxation (pink). In the SDP framework, we may impose boundary conditions that give us better control over alignment of the layers (right).

whose phase rotates at a rate $\hat{\mathbf{v}} = \mathbf{v}/\hbar$, where $\hbar$ is a global parameter controlling the period of the stripes. In other words, we want $\nabla\theta \approx \hat{\mathbf{v}}$, where locally $\varphi = e^{i\theta}$. With an input vector field $\mathbf{v}$ of constant norm, the level sets of φ (stripes) will then be approximately evenly spaced.

Relaxing this requirement to a least-squares objective yields an optimization problem of the form

$$\min_{\varphi: M \rightarrow S^1} \|\nabla\theta - \hat{\mathbf{v}}\|_{L^2(M)}^2. \quad (4)$$

It is convenient to work with the 1-form $\hat{\eta} = \hat{\mathbf{v}}^\flat$. The objective can then be rewritten as follows:

$$E_{\hat{\eta}}[\varphi] := \|\nabla\theta - \hat{\mathbf{v}}\|^2 = \|d\theta - \hat{\eta}\|^2 = \|(id\theta - i\hat{\eta})\varphi\|^2 = \|D_{\hat{\eta}}\varphi\|^2, \quad (5)$$

where $D_{\hat{\eta}} := d - i\hat{\eta}$ is the covariant derivative associated to the $U(1)$ -connection form $\hat{\eta}$. Like (1), this is an instance of a $U(1)$ -synchronization problem.

Discretely, the underlying graph comprises the vertices and edges of a triangulation. The discrete connection values are calculated as

$$u_{ij} = \exp\left(i \int_{ij} \hat{\eta}\right) = \exp\left(\frac{i}{\hbar} \sum_{t \sim ij} \alpha_{ij}^t \mathbf{v}_t \cdot \mathbf{e}_{ij}^t\right), \quad (6)$$

where $\mathbf{v}_t$ is the (constant) value of $\mathbf{v}$ on face t , $\mathbf{e}_{ij}^t$ is the vector representing the edge (i, j) in the local frame associated to the face t , and α_{ij}^t is an averaging weight.

3.4 Volumetric Stripe Patterns

It is straightforward to generalize the stripe pattern problem to a volumetric setting, with synchronization taking place on the vertices and edges of a tetrahedral mesh. The resulting evenly-spaced laminar structures could, e.g., model layer deposition in curved-layer 3D prints [Zhang et al. 2022]. For this application, optimal surface finish requires that layers align to boundary surfaces, necessitating boundary constraints (see discussion in Sections 4.4 and 5). An example is shown in Figure 3.

To visualize volumetric stripe patterns, we generalize the interpolant from Knöppel et al. [2015] to singular tetrahedra, pictured on the right of Figure 2. When a tetrahedron is punctured by a dislocation curve, any cross-section through the pattern transverse to the curve features a 2D dislocation like the one on the left of Figure 2. Details on this volumetric interpolant are provided in Supplementary Section 2.

3.5 Quad Patterns

Fang et al. [2018] compute bidirectionally periodic patterns, driven by cross fields, for application in quad meshing (see Figure 4). This generalizes the synchronization problems we have examined so far to higher-dimensional fibers and groups.

These “quad patterns” have two independent directions of periodicity at each point, encoded in two independent phases. So locally, they can be represented by functions valued in $S^1 \times S^1 = \mathbb{T}^2$. However, around a disclination of the cross field, these two phases cannot be treated independently. As a cross field experiences a rotation of $\pi/2$ upon encircling an index-1 disclination, so must the quad pattern phases experience a corresponding transformation taking $(e^{i\theta_x}, e^{i\theta_y}) \mapsto (e^{i\theta_y}, e^{-i\theta_x})$, whose matrix is

$$A = \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix}, \quad (7)$$

where K is the complex conjugation map.

To set up the discrete synchronization problem on a triangulated surface M , given a cross field χ with disclinations Σ , the surface must first be cut to separate $M \setminus \Sigma$ into a topological disk $\bar{M}$. On $\bar{M}$, χ can be lifted to a pair of unit vector fields, which exchange places and signs according to the gluing relations along the seams. Finally, $S^1 \times S^1$ fibers are attached to each vertex of $\bar{M}$, with constraints of the form $\varphi_i = A^n \varphi_j$ for each pair of vertices $i \sim j$ identified across a seam. The synchronization problem can then be constructed as for a pair of ordinary stripe patterns driven by a pair of vector fields on each triangle. Since A preserves the product of real parts of the phases, a globally-defined real scalar oscillatory field can be extracted, as visualized in Figure 4. Note, however, that due to the anti-unitarity of K , we cannot use complex number computations and must instead represent the circular phases in real coordinates.

Similarly, computing a non-orientable stripe pattern guided by a line field amounts to synchronization on a cover branched at disclinations of the line field [Knöppel et al. 2015] (see Figure 6). Both quad and stripe patterns optimize dislocations against a background of fixed disclinations. Automatically placing both kinds of defects remains an open problem.

3.6 Fluid Visualization: Clebsch Map Regression

In fluid dynamics, a generalized Clebsch map is a map $\varphi : \Omega \rightarrow X$ from the fluid domain $\Omega \subseteq \mathbb{R}^3$ to a model manifold X , such that the vorticity 2-form ω of the fluid on Ω is the pullback of a fixed 2-form

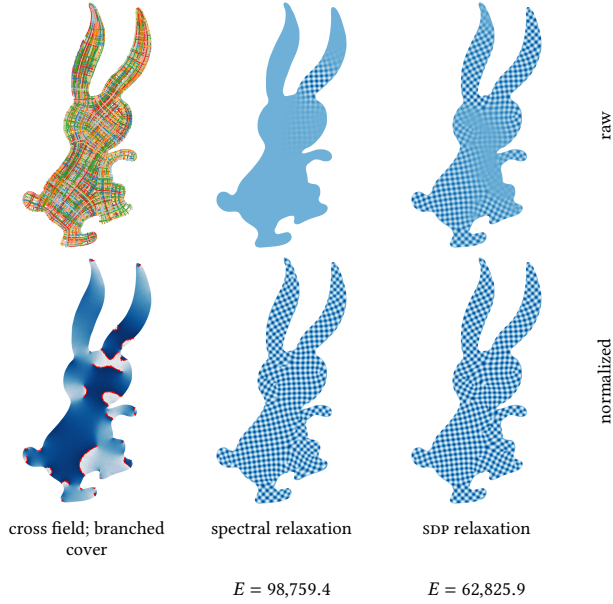


Fig. 4. Synchronization on a nontrivial torus bundle produces oscillatory fields suitable for quad meshing.

β on X ,

$$\omega = \varphi^* \beta. \quad (8)$$

In computer graphics, Clebsch maps are used to visualize vortex structures such as filaments and tubes [Chern et al. 2017; Weißmann et al. 2014]. First, given the velocity 1-form η and corresponding vorticity $\omega = d\eta$ of a fluid flow, one finds a map φ approximately satisfying (8). Next, vortex structures are extracted from φ . We refer to such approaches as *Clebsch map regression* methods.

3.6.1 Vortex Filaments. Weißmann et al. [2014] set $X = \mathbb{C}$ and $\beta = h\delta_0$, where h is a quantization parameter and δ_0 is the Dirac delta measure at the origin. With these choices, a Clebsch map has the interpretation of cutting out *vortex filaments*: given a test surface patch $S \subset \Omega$,

$$\int_S \omega = \int_S \varphi^* \beta = \int_{\varphi(S)} \beta = \#(\varphi(S) \cap \{0\}) = \#(S \cap \varphi^{-1}(0)), \quad (9)$$

where $\#$ denotes oriented count. In other words, the vorticity through any surface is determined by its intersection with a set of oriented curves comprising $\varphi^{-1}(0)$, each carrying vorticity h . One can think of the Clebsch map regression as a way of *quantizing* or *rounding* ω to such a singular vorticity field. As such, we refer to this problem as *singular Clebsch mapping*.

To find φ , one might naïvely pose the optimization problem

$$\min_{\varphi: \Omega \rightarrow \mathbb{C}} \|\varphi^* \beta - \omega\|_{L^2}^2. \quad (10)$$

However, since β is singular, it is more natural to work with its integral, $d\theta$, where θ is the multiple-valued argument function on $\mathbb{C}$. Recall that $d^2\theta \neq 0$, but is instead equal to $2\pi\delta_0$. Now we want

$$d\eta = \omega = \varphi^* \beta = \frac{h}{2\pi} d(\varphi^* d\theta) =: h d(\varphi^* d\theta). \quad (11)$$

It is equivalent to seek φ such that

$$\hat{\eta} := \frac{\eta}{h} = \varphi^* d\theta = d(\theta \circ \varphi) \quad (12)$$

φ now appears only via its composition with θ , so we might as well treat it (away from vortex filaments) as valued in $S^1 \subset \mathbb{C}$. Thus, writing $\varphi = e^{i\theta}$, observing that $d\varphi = i\varphi d\theta$, and enforcing (12) in least squares, we obtain the optimization problem

$$\min_{\varphi: \Omega \rightarrow S^1} \|d\varphi - i\hat{\eta}\varphi\|_{L^2}^2, \quad (13)$$

whose objective is identical to (5). For direction fields and stripe patterns, the topological defects were *disclinations* and *dislocations*, respectively; here they encode the vortex filaments (see Figure 5).

3.6.2 Vortex Tubes. Chern et al. [2017] use Clebsch map regression to visualize *vortex tubes*. They set $X = S^2$ and β the area form on X , scaled by h . Because the model manifold is a sphere, they refer to these as *spherical Clebsch maps*. To see that spherical Clebsch maps encode vortex tubes, let $R \subseteq S^2$ and consider again a test surface $S \subset \Omega$.

$$\int_{S \cap \varphi^{-1}(R)} \omega = \int_{S \cap \varphi^{-1}(R)} \varphi^* \beta = \int_{\varphi(S) \cap R} \beta = h \text{area}(\varphi(S) \cap R), \quad (14)$$

counted with orientation and multiplicity. That is, wherever the tube $\varphi^{-1}(R)$ passes through S , it carries a constant vorticity flux $h \text{area}(R)$.

Unlike in the singular Clebsch case, β is not exact. But it is equal to the (scaled) curvature 2-form of the sphere. So it becomes exact after lifting to the unit tangent bundle of S^2 , i.e., the Hopf fibration $\pi_H: S^3 \rightarrow S^2$:

$$\pi_H^* \beta = h d\alpha, \quad (15)$$

where α is the Levi-Civita connection form on S^3 viewed as STS^2 . Thus Chern et al. [2017] compute a lifted Clebsch map $\psi: \Omega \rightarrow S^3$, with $\varphi := \pi_H \circ \psi$. With this choice, $\varphi^* \beta = \omega$ if and only if $\psi^* \alpha = \hat{\eta}$.

After adding a regularization term, Chern et al. [2017] arrive at the objective:

$$\|d\psi - i\hat{\eta}\psi\|_{\epsilon}^2 \quad (16)$$

where ϵ is a regularization parameter, and $\|\cdot\|_{\epsilon}$ denotes the L^2 metric incorporating the Berger metric on the sphere. We restrict to the case $\epsilon = 1$, which corresponds to the ordinary isotropic metric. Note the extreme similarity to (13), (5), and (1). The only difference is that the $U(1)$ connection $\hat{\eta}$ now acts on ψ valued in S^3 rather than S^1 .

3.7 General Synchronization Problems

It should be evident that the previous examples fit a template, which we summarize in the following definition:

Definition 3.1. Let $\Gamma = (V, E, w)$ be a weighted graph. Let $F \subseteq \mathbb{R}^n$ and let G be a subgroup of $O(n)$. Given a discrete G -connection u , consisting of one element $u_{ij} \in G$ for each edge $(i, j) \in E$, the G -synchronization problem on Γ with fiber F and connection u is defined as follows:

$$\begin{aligned} \min_{\varphi} \quad & \sum_{i,j} w_{ij} \|\varphi_j - u_{ij}\varphi_i\|^2 \\ \text{subject to} \quad & \varphi_i \in F \quad \forall i. \end{aligned} \quad (17)$$

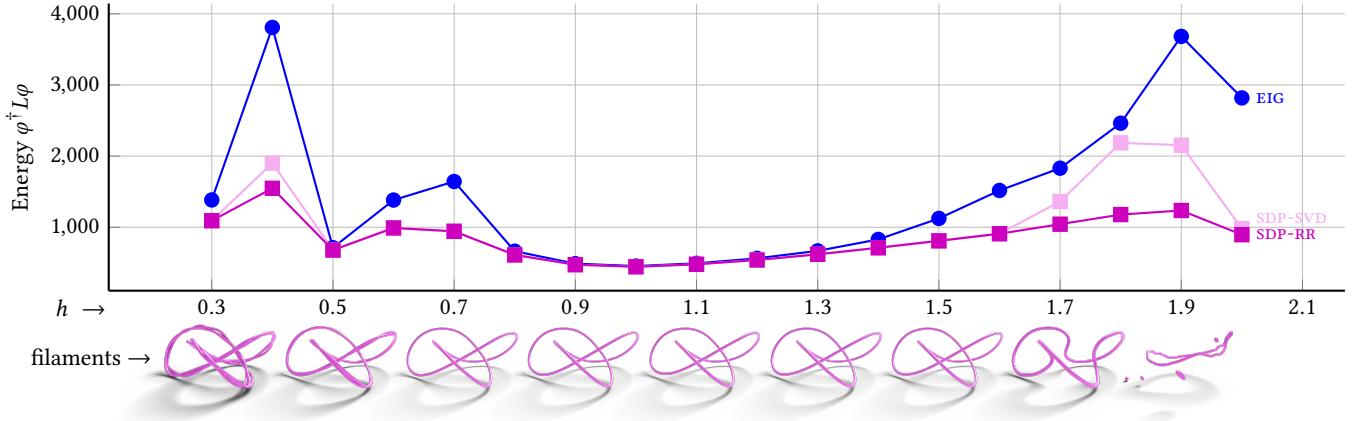


Fig. 5. Weißmann et al. [2014] recover vortex filaments by solving a synchronization problem. Here we compare several relaxations of their covariant Dirichlet energy on a $40 \times 40 \times 20$ grid. **EIG** shows the energy attained by the smallest eigenvector of L after normalization (spectral relaxation). **SDP-SVD** corresponds to the energy of the (normalized) leading eigenvector extracted from the SDP relaxation via singular value decomposition. **SDP-RR** shows the further reduction in energy obtained by randomized rounding: when the SDP solution has rank greater than one, we sample multiple random projections from the SDP solution space and retain the lowest-energy realization.

Note that the right action of G , $\varphi_i \mapsto \varphi_i g$, preserves the objective function. So the best one can hope for is to find the orbit under this symmetry, i.e., a point in F^n/G .

In the literature, synchronization is often given a statistical interpretation. The u_{ij} are viewed as noisy observations of relative orientation or phase, and the goal of synchronization is to stitch together these partial, incompatible observations to infer global information about absolute orientations [Singer 2011; Singer and Sigworth 2020; Wang and Singer 2013]. However, the more geometric language of connections has also shown up in the literature [Bandeira et al. 2013; Gao et al. 2021], in line with concepts of discrete connections in geometry processing [Berwick-Evans et al. 2023; Braune et al. 2025; Knöppel et al. 2025] and lattice gauge theory in physics.

In the smooth setting, we view φ as a (possibly singular) section of a fiber bundle $\pi : E \rightarrow M$ with fiber F and connection D . For example, cross fields on surfaces are sections of $S(TM)^{\otimes 4}$, the sphere bundle of the fourth tensor power of the tangent bundle, expressing the well-known fact that a cross field can be represented by its complex fourth power in an appropriate basis.

4 Method

In this section, we describe and motivate our use of SDP relaxation and factorization to solve synchronization problems like those detailed above. We focus here on the case of $U(1)$ synchronization for convenience, though the constructions are substantially similar in the more general setting. The discrete $U(1)$ synchronization problem with connection Laplacian L

$$s^*(L) = \min_{\varphi \in \mathbb{C}^n} \left\{ \varphi^\dagger L \varphi : |\varphi_i|^2 = 1 \forall i \right\} \quad (18)$$

encompasses the cases of directional fields, stripe patterns, and singular Clebsch maps from Section 3.

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4.1 Detour: Spectral Relaxation

We have seen in Sections 3.1.2 and 3.2 that synchronization problems require relaxation. Knöppel et al. [2013, 2015]; Weißmann et al. [2014] all employ *spectral relaxation*, also known as *eigenvalue relaxation*. In this approach, one replaces the vertex-wise normalization constraint in (18) with a global L^2 normalization constraint:

$$\lambda_1(L, M) := \min_{\varphi} \left\{ \varphi^\dagger L \varphi : \varphi^\dagger M \varphi = 1 \right\}, \quad (19)$$

where M is a diagonal mass matrix, and L is a discrete connection Laplacian derived from a triangle mesh.

The resulting problem is equivalent to finding the minimum generalized eigenvalue and corresponding eigenspace of the pair (L, M) :

$$L\varphi_1 = \lambda_1(L, M)M\varphi_1 \quad (20)$$

This is indeed a relaxation—after rescaling by a factor $\sqrt{\text{tr } M}$, any feasible solution of (18) is a solution of (19). After solving the eigenproblem (20), vertex-wise normalization recovers an approximate solution to the original problem (18). Unfortunately, this is a loose relaxation, replacing n constraints with a single constraint.

4.2 Semidefinite Relaxation

To motivate a tighter *semidefinite relaxation* of (18), we first rephrase the spectral relaxation (19) as an SDP:

$$\lambda_1(L, M) = \min_{\Phi} \{ \langle L, \Phi \rangle : \Phi \succeq 0, \langle M, \Phi \rangle = 1 \} \quad (21)$$

where Φ is Hermitian, and $\langle L, \Phi \rangle := \text{tr}(L^\dagger \Phi)$ denotes the Frobenius inner product between Hermitian matrices. We state the following result and defer the (short) proof to the supplement:

PROPOSITION 4.1. *The optimal value of (21) is the minimum generalized eigenvalue of (L, M) . The columns of the optimal Φ^* sit in the corresponding eigenspace. Thus (21) is equivalent to (19).*

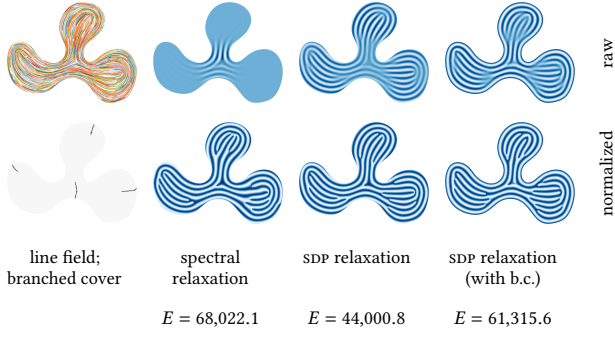


Fig. 6. While eigenfunctions decay to zero over much of the domain, SDP relaxation is more faithful to the original constraints, leading to lower objective values after normalization. Unlike with spectral relaxation, it is easy to impose boundary conditions.

A tighter semidefinite relaxation, due to Singer [2011], is

$$S^*(L) := \min_{\Phi} \{ \langle L, \Phi \rangle : \Phi \succeq 0, \Phi_{ii} = 1 \forall i \} \quad (22)$$

To see that (22) is a relaxation, note that if Φ is a feasible solution of rank 1, we can decompose it as $\Phi = \varphi\varphi^\dagger$, whence $1 = \Phi_{ii} = |\varphi_i|^2$ for all i . In other words, the rank-one feasible solutions of (22) are exactly the feasible solutions of (18).

On the other hand, (22) is tighter than (21): any feasible solution of (22) is, after rescaling by $\text{tr } M$, a feasible solution of (21). As such,

$$(\text{tr } M) \lambda_1(L, M) \leq S^*(L) \leq s^*(L). \quad (23)$$

Moreover, if the optimum Φ^* of (22) has rank 1, the second inequality becomes an equality, and its factor φ^* is a global optimum of (18). This certifiable optimality is known as *exact recovery* (see [Blekherman et al. 2013]).

4.3 Burer-Monteiro Factorization

SDPs like (22) can be solved to global optimality by a variety of efficient methods. As we have seen in Section 4.2, when an SDP arises from a relaxation, a low-rank solution is desirable. It is thus convenient to use an optimization algorithm that exploits low rank.

We employ the factorization approach to solving SDPs, due to Burer and Monteiro [2003, 2005]. The Burer-Monteiro (BM) method searches for a low-rank solution by partially inverting the relaxation procedure. For example, given the problem (22), pick a rank $r \leq n$ and restrict to solutions that factor as $\Phi = YY^\dagger$, where $Y \in \mathbb{C}^{n \times r}$. The SDP (22) becomes:

$$\min_Y \left\{ \text{tr}(Y^\dagger LY) : |Y_j|^2 = 1 \forall j \right\} \quad (24)$$

where Y_j is the j th row of Y . Note the similarity between (24) and the original, unrelaxed problem (18). The only difference is that Y_j is now a unit complex vector rather than a unit complex scalar. Geometrically, we have lifted a problem valued in $(S^1)^n$ to one valued in $(S^{2r-1})^n$.

Though (24) is a nonconvex quadratically constrained quadratic program (QCQP), it turns out that, under mild conditions, it is actually *equivalent* to the SDP (22) for high enough candidate rank r .

To wit, Barvinok [1995]; Pataki [1998] showed that an SDP with a generic objective will have a solution of rank $r^* \in O(\sqrt{m})$, where m is the number of linear constraints. Moreover, any *rank-deficient* local optimum of (24) is actually a global optimum of (22). More recently, it has been established that for a generic objective L and high enough rank $r \geq r^*$, any second-order local optimum will be rank-deficient, making (24) equivalent to (22) [Boumal et al. 2020]. Putting these ideas together, BM factorization provides a polynomial-time algorithm that finds low-rank $(\Theta(\sqrt{m}))$ solutions [Cifuentes and Moitra 2022]. In practice, the rank of solutions is often constant.

BM factorization is applicable more generally for SDP relaxations of problems over smooth homogeneous spaces like spheres or Lie groups [Boumal et al. 2020; Cifuentes 2021], which include synchronization problems of the form (17). Specifically, the nonconvex factored problem must satisfy a linear independence constraint qualification (LICQ). For such SDPs featuring block-diagonal constraints, the corresponding BM problem will—as we saw for (24)—also be an optimization problem over a product of low-dimensional homogeneous spaces. The SDP can be solved efficiently by solving BM problems at increasing ranks via Riemannian optimization. This is known as the Riemannian staircase algorithm [Boumal 2016]. We sketch it in Algorithm 1 in the case of $U(1)$. We solve the subproblem (24) by the Riemannian trust regions method (RTR) [Absil et al. 2007] as implemented in MANOPT [Boumal et al. 2014].¹

Algorithm 1: $U(1)$ Synchronization with Riemannian Staircase

```

function  $Y = \text{staircase}(L)$ :
  for  $r = 1, 2, \dots, R$  do
    solve (24) for  $Y$  via RTR starting from  $Y_0$ 
    if  $\text{rank } Y < r$  then // rank deficient - done!
      return  $Y$ 
    else // increase rank
       $Y_0 \leftarrow \begin{pmatrix} Y & 0 \end{pmatrix}$ 
      // find minimum eigenpair of Riemannian Hessian
       $Z \leftarrow \arg \min \{ \text{tr}(Z^\dagger LZ) \mid Z \in T_{Y_0} \mathcal{M}_r, \|Z\|_F^2 = 1 \}$ 
       $\lambda \leftarrow \text{tr}(Z^\dagger LZ)$ 
      if  $\lambda \geq 0$  then // rank deficient - done!
        return  $Y_0$ 
      else // found a descent direction
         $Y_0 \leftarrow \text{linesearch}(Y_0, Z)$ 
    end
  end

function  $\varphi = \text{synch}(L)$ :
   $Y \leftarrow \text{staircase}(L)$ 
  begin // svd rounding
     $USV^\dagger \leftarrow \text{svd}(Y)$ 
     $\varphi_i \leftarrow U_{i1}/|U_{i1}| \quad \forall i$ 
  return  $\varphi$ 
end

```

¹Code for this paper is available at <https://github.com/dpa1mer/synchfields>.

4.4 Boundary Conditions

Our BM formulation easily extends to Dirichlet boundary conditions (see Figure 6). Decomposing φ into interior and fixed boundary terms $\varphi = I\varphi_I + B\varphi_B$, the unrelaxed objective becomes

$$\min_{\varphi_I \in \mathbb{C}^n} \left\{ \varphi_I^\dagger L_{II} \varphi_I + 2\Re(b^\dagger \varphi_I) : |\varphi_i|^2 = 1 \ \forall i \in V_I \right\}, \quad (25)$$

where $b := L_{IB}\varphi_B$ and $L_{II} = I^\dagger L I$, etc. The SDP relaxation of this inhomogeneous problem takes the form

$$\begin{aligned} \min_{\tilde{\Phi}} \{ \langle C, \tilde{\Phi} \rangle : \tilde{\Phi} \succeq 0, \tilde{\Phi}_{ii} = 1 \ \forall i \}, \\ \tilde{\Phi} = \begin{pmatrix} 1 & \varphi^\dagger \\ \varphi & \Phi \end{pmatrix}, \quad C = \begin{pmatrix} 0 & b^\dagger \\ b & L_{II} \end{pmatrix}. \end{aligned} \quad (26)$$

Choosing a factorization of $\tilde{\Phi}$ and dropping constant terms, the BM objective becomes

$$\text{tr } Y_I^\dagger L_{II} Y_I + \frac{2}{\sqrt{r}} \Re(b^\dagger Y_I \mathbf{1}), \quad (27)$$

where $\mathbf{1}$ is the all-ones vector and $Y_I \in \mathbb{C}^{n \times r}$.

4.5 Rounding

Just as the spectral relaxation (19) required entrywise normalization to recover a feasible point of (18), the SDP method requires a *rounding procedure*.

The solution of (22) is an $n \times n$ Hermitian semidefinite matrix Φ . One way to look at such a matrix is as the *moment matrix* of a probability distribution. Suppose $\varphi \sim \mathcal{N}(0, \Phi)$ is a random variable. Then

$$\mathbb{E}[\varphi^\dagger L \varphi] = \text{tr}(L\Phi) = \langle L, \Phi \rangle. \quad (28)$$

Similarly, $\mathbb{E}[|\varphi_i|^2] = \Phi_{ii} = 1$. In other words, a solution to the SDP (22) can be viewed as a Gaussian random variable that optimizes the original problem (18) *in expectation*. This motivates a natural randomized rounding scheme: sample φ and normalize its entries to obtain a feasible solution of (18). Practically, given the BM solution $Y \in \mathbb{C}^{n \times r}$, sampling φ can be done as follows:

$$\mu, v \sim \mathcal{N}(0, I_{r \times r}) \implies \tilde{\varphi} = Y(\mu + iv) \implies \varphi_k = \tilde{\varphi}_k / |\tilde{\varphi}_k|. \quad (29)$$

Alternatively, a simple deterministic rounding scheme computes the singular value decomposition (SVD) of Y and sets $\tilde{\varphi}$ to the dominant singular vector. Note that as Y approaches rank-one, the randomized and SVD approaches produce increasingly similar solutions.

4.6 Geometric Interpretation

The composition of SDP relaxation and BM factorization lifts a synchronization problem to a higher-dimensional manifold. We have already seen how in the $U(1) \cong S^1$ case, the discrete problem (18), valued in $(S^1)^n$, gets transformed into (24), valued in $(S^{2r-1})^n$. Interpreted in continuous language, this means that we have lifted our circle bundle $E \rightarrow M$ to a new, higher-dimensional bundle $\tilde{E} \rightarrow M$ and pulled back the connection to $\tilde{E}$. The connection remains a $U(1)$ connection operating on the higher-dimensional bundle.

A solution to the lifted problem is a section $\tilde{\varphi} \in \Gamma(\tilde{E})$. Rounding converts this into a section $\varphi \in \Gamma(E)$. The rounding procedure ρ

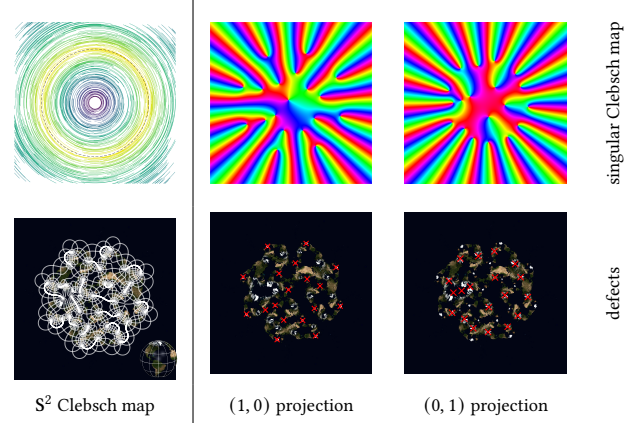


Fig. 7. An S^3 -valued Clebsch map yields an S^2 -valued Clebsch map via the Hopf fibration, visualized here with an earth texture map. Since S^3 synchronization is the first Burer-Monteiro lift of S^1 synchronization, each point on $S^2 \cong \mathbb{CP}^1$ also gives a different singular Clebsch map with a corresponding defect configuration.

consists of projection Π onto a complex vector $\xi \in \mathbb{C}^r$, followed by normalization N . However, the solution φ is defined only up to a global phase (see Section 3.7), so the rounding procedure is parametrized by the complex line $\tilde{\xi} = \text{span } \xi \subset \mathbb{C}^r$. Globally, rounding is parametrized by a choice of complex line bundle $\ell \subset \tilde{E}$, i.e., by a section of the projectivized bundle $\text{proj } \tilde{E}$. In the typical case, where $E \simeq M \times S^1$ is trivial, this reduces to the choice of a constant point $\tilde{\xi} \in \mathbb{CP}^{r-1}$.

The Clebsch mapping application supplies geometric insight. The first BM lift of the $U(1)$ -synchronization problem coming from singular Clebsch map regression is an S^3 -synchronization problem with a $U(1)$ connection—precisely the problem of spherical Clebsch map regression!

In the other direction, this means that we can round a spherical Clebsch map $\psi : M \rightarrow S^3$ to a singular Clebsch map $\varphi : M \setminus \Sigma \rightarrow S^1$. The rounding procedure is parametrized by a choice of point $\xi \in \mathbb{CP}^1 \cong S^2$. Whenever $\tilde{\varphi}(x) \perp \xi \subset \mathbb{C}^2$ for some $x \in M$, the projection is ill-defined, producing a topological defect at x in the resulting φ . This is true exactly when $\pi_H \circ \psi(x)$ is antipodal to ξ in S^2 . Hence, every time the spherical Clebsch map $\pi_H \circ \psi$ covers the 2-sphere, one vortex will appear in the singular Clebsch map φ , precisely at the inverse image of $-\xi$. Varying $\xi \in S^2$ will simply move these defects around the base domain M . Some examples of singular Clebsch maps derived from one spherical Clebsch map are depicted in Figure 7.

Generalizing this perspective, we can view the first BM lift of any $U(1)$ synchronization problem as optimizing a “spherical Clebsch map.” The lifted map is defect-free because $\pi_1(S^3) = 0$. But it encodes defects in an approximate sense: for any choice of projection, every inverse image of the 2-sphere contains a defect somewhere.

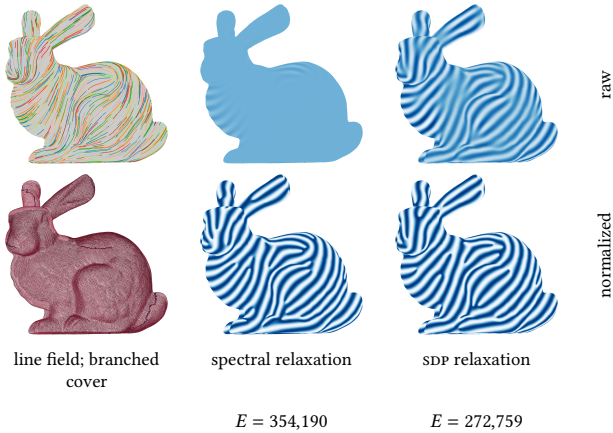


Fig. 8. Aligning to a line field derived from the principal curvature directions, SDP relaxation finds a significantly lower-energy stripe pattern on the bunny.

5 Results

Figure 6 demonstrates the faithfulness of SDP relaxation relative to spectral relaxation. Whereas the lowest eigenfunction of the connection Laplacian decays to zero over large regions of the domain, the SDP-based solution has a consistent magnitude across the domain, decaying only where necessary near singularities. Consequently, normalization is much more faithful when applied to the SDP solution compared to the eigenfunction, explaining the lower energy of the normalized SDP solution. To put it another way, the tighter relaxation means that the SDP-relaxed energy measures phase variation more faithfully.

Moreover, unlike with spectral relaxation, it is easy to impose additional linear constraints, such as boundary conditions, when solving the SDP. In Figure 6, imposing a constant phase along the boundary results in stripes that precisely match the boundary curve. The line field input was also computed via SDP relaxation.

Results on a curved surface are shown in Figure 8. Note that the line fields displayed are rotated by 90° relative to the ones used for computations, which are always normal to the stripes. This is to aid visual comparison of alignment.

Figure 4 depicts quadrilateral oscillatory patterns computed with SDP and spectral relaxation.

In Figure 3, we show a comparison between spectral and SDP relaxation in computing volumetric stripes. Figure 3 also depicts a volumetric stripe pattern with fixed boundary conditions. Notice how the top surfaces of the model line up exactly with the top stripe layer. This is important for curved-layer 3D printing as aliasing at object boundaries would lead to imperfect surface finish and higher surface friction (see Section 3.4).

In Figure 9 we compare the energy values of SDP relaxation to two other methods on the task of computing unit vector fields on a dataset of 116 surface models from Myles et al. [2014], all rescaled to have unit total area. All three methods were implemented using the same connection Laplacian (and mass matrix), which was also used to measure the covariant Dirichlet energy E . For MBO [Merriman et al. 1994; Viertel and Osting 2019], which takes a

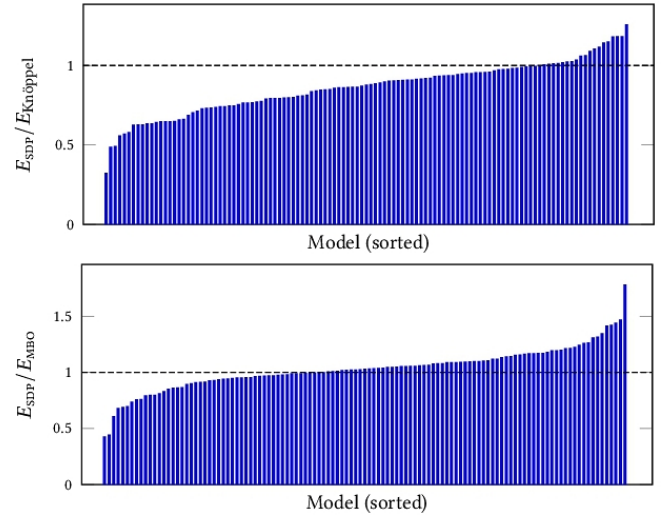


Fig. 9. Our SDP relaxation improves on the method of Knöppel et al. [2013] and performs comparably to the nonconvex MBO method from Viertel and Osting [2019] on computing unit vector fields over a dataset of surface meshes.

timescale parameter, we use the parameter 0.1 for all models. For the method of Knöppel et al. [2013], we use spectral relaxation on closed surfaces and solve a linear system with Dirichlet boundary conditions otherwise since spectral relaxation is incompatible with boundary conditions.

This experiment was conducted on a MacBook Pro (M1 Max, 64 GB RAM). On 67% of meshes, solving our SDP took less than 60 seconds. 39% of the time, our method was faster than MBO. However, on 22% of examples, it was at least an order of magnitude slower than MBO, with the very slowest run taking over 7800 seconds (two orders of magnitude slower than MBO). The timing variation is not wholly explained by mesh size. For example, we observe a negative correlation between runtime and minimum triangle area (as low as 10^{-13} of total area on this dataset). Similarly, we observe positive correlations between runtime and both maximum (up to 5×10^8 on this dataset) and RMS vertex curvature. As usual in geometry processing, quality and performance depend to a large degree on the quality of the input geometry.

Figure 5 compares the two relaxation methods on computing singular Clebsch maps at different values of h for the trefoil knot vorticity field from Weißmann et al. [2014]. The resulting quantized vortices are shown below. SDP relaxation out-competes spectral relaxation with either rounding procedure.

Figure 7 displays a two-dimensional spherical Clebsch map computed from the planar velocity field from Chern et al. [2017], whose curl is constant inside the unit circle and zero elsewhere. As the spherical Clebsch problem is the first Burer-Monteiro factorization of the singular Clebsch problem, in computing the spherical map we implicitly compute a family of singular maps, also shown in Figure 7.

6 Conclusion

Our synchronization framework unifies direction fields, stripe patterns on surfaces and in volumes, quad oscillatory fields, and Clebsch maps. We anticipate our SDP relaxation and BM factorization approach will be adaptable to many other problems that can be phrased as synchronization. For example, the full quad meshing problem (with free disclinations) can be viewed as finding a singular section of a twisted torus bundle (see [Hocking et al. 2022]). Volumetric frame field optimization for hexahedral meshing involves synchronization over a quotient of $SO(3)$ [Palmer et al. 2020].

Another avenue for future work is improving the efficiency of our method. The Riemannian trust regions algorithm that forms the backbone of our method benefits from preconditioning, so one direction to pursue might be a multigrid preconditioner that works for connection Laplacians [Kalkreuter 1995]. Alternatively, MBO could be used to solve the BM factorizations, fusing the efficiency of MBO with the reliability of a convex method.

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