Spectral Rigidity and Nonrigidity of Dynamical Systems

by

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Abstract

This thesis studies spectral rigidity and nonrigidity phenomena in dynamical systems. The central question is whether a dynamical system can be determined, up to a natural conjugacy, from its spectrum. We consider three related spectra: the length spectrum, the action spectrum, and the Lyapunov spectrum.

The first part of the thesis concerns Liouville metrics on the two-dimensional torus. It is a long-standing folklore conjecture that Liouville metrics are the only integrable metrics on the torus. We prove a length-spectral rigidity result for linear conformal deformations of Liouville metrics by exploiting the dynamical properties of the rational tori – analogues of the resonant convex caustics in billiards. We also establish a complementary classification result showing that marked-length-isospectral Liouville metrics are characterized by rearrangements of the one-dimensional functions appearing in their conformal factors, generalizing a theorem of Abbondandolo-Mazzucchelli. In particular, the second result gives nonrigidity examples within the class of Liouville metrics.

The second part of the thesis studies the standard map from the viewpoint of action and Lyapunov spectra. We construct nontrivial deformations of the standard map which preserve the symplectic actions (respectively, the Lyapunov exponents) of infinitely many periodic orbits accumulating on an invariant curve. The proof combines a resonant normal form construction with Picard iteration schemes to obtain a sequence of periodic orbits accumulating on an invariant curve with a Liouville rotation number. Within the resonant normal forms we capture the dependence of these periodic orbits on the resonant Fourier coefficients of the dynamics on the invariant curve and, using the contraction mapping principle, obtain a suitable deformation achieving the prescribed spectral data associated with this sequence of orbits. The result can be viewed as a symplectic twist-map analogue of a length-spectral nonrigidity phenomenon for Riemannian manifolds and convex billiards, and it motivates the existence problem for similar 'partially length-isospectral' deformations of strictly convex billiard tables.

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About the Author

Yunzhe Li completed a B.Sc. and an M.Sc. in Mathematics at the National University of Singapore (NUS), as well as a Diplôme d'Ingénieur at École Polytechnique in Palaiseau, France.

His interest in mathematical research began during a 2018 research internship with Serge Troubetzkoy at Aix-Marseille Université, where he worked on the symbolic dynamics of polygonal billiards. He also worked with Dinh Tien-Cuong on homogeneous dynamics and Ratner theory while serving as a full-time teaching assistant at NUS, before joining the Kaloshin group at ISTA as a Ph.D. student.

His main research interests lie in the spectral rigidity and integrable rigidity of Hamiltonian dynamical systems, including billiards and geodesic flows. He is also interested in public science education and co-developed science communication projects with the VISTA team during his Ph.D. studies.

List of Collaborators and Publications

Chapter 2 of this thesis is contained in a joint work with Joscha Henheik, Vadim Kaloshin and Amir Vig. The following preprint is reused in part in Chapter 2.

- J. Henheik, V. Kaloshin, Y. Li, and A. Vig. Spectral rigidity of Liouville tori, 2025. doi:10.48550/arXiv.2511.10398

The following preprint is reused in full in Chapter 3.

- Y. Li. Deformations of the Standard Map with prescribed actions and Lyapunov exponents, 2025. doi:10.48550/arXiv.2512.03865

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CHAPTER 1

Introduction

This thesis is devoted to the study of *spectral rigidity* in dynamical systems. The guiding question is whether a dynamical system can be recovered from data attached to its periodic orbits.

Broadly speaking, in many classes of dynamical systems one can define a collection of data, called a "spectrum," associated with the periodic orbits of the system. Depending on the types of dynamical systems considered, different kinds of spectra may be considered. In this thesis, we focus on the following three examples of spectra.

- In the setting of geodesic flows we consider the *length spectrum*, which is defined as the collection of the Riemannian length of all closed orbits under the geodesic flow.
- In the setting of exact symplectic maps, we consider the *action spectrum* of the map. This is defined as the collection of the *symplectic actions* of all periodic orbits of the map.
- Lastly, we also consider the *Lyapunov spectrum*, here defined as the collection of *Lyapunov exponents* of closed orbits of a differentiable dynamical system.

These spectra often encode important information about the underlying dynamics. The problem of spectral rigidity asks to what extent one can determine the system, up to an appropriate notion of conjugacy, from its spectrum.

Let us emphasize that, throughout this thesis, we focus especially on *deformational* spectral rigidity, namely, we study whether a dynamical system can be continuously deformed without changing the spectrum.

1.1 An overview of the main results

This thesis contains two groups of results, corresponding to two distinct settings.

- The first concerns Liouville metrics on the two-dimensional torus: we prove a rigidity theorem for linear conformal deformations and classify length-isospectral Liouville metrics in terms of rearrangements.

- The second concerns standard maps: we construct nontrivial deformations preserving the actions (resp., the Lyapunov exponents) of infinitely many selected periodic orbits.

We now formulate these results more precisely.

1.1.1 Spectral rigidity of Liouville metrics

Let (X, g) be a compact Riemannian manifold. We define the *length spectrum*

$$\mathcal{L}(X, g) \subset \mathbb{R}^+$$

to be the collection of lengths of all closed geodesics of (X, g). Suppose $(g_\epsilon)_{|\epsilon|<1}$ is a one-parameter family of Riemannian metrics on X. We say that the family (g_ϵ) is *length-isospectral*, or *isospectral* for short, if $\mathcal{L}(X, g_\epsilon) = \mathcal{L}(X, g_0)$ for all ϵ.

In the first part of this thesis, we study spectral rigidity of *Liouville metrics* on the two-dimensional torus. That is, we consider the case $X = \mathbb{T}^2$, and take g_0 to be a Riemannian metric whose line element is of the form

$$\mathrm{d}s^2 = \Big(1 + f_1(x_1) + f_2(x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2). \tag{1.1}$$

Throughout this thesis, we use the notation $\mathbb{T}^d$ for the standard d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$.

Our consideration of Liouville metrics is motivated by a well-known "folklore conjecture" that Liouville metrics are the only *integrable* metrics on $\mathbb{T}^2$. This conjecture can be thought of as an analogue of the famous *Birkhoff conjecture* in the setting of billiard dynamics.

We now formulate our first main result.

Theorem A. *Let $(g_\epsilon)_{|\epsilon|<1}$ be a family of smooth Riemannian metrics on $\mathbb{T}^2$ of the form*

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon\, U(x_1, x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)\,. \tag{1.2}$$

Suppose $(g_\epsilon)_{|\epsilon|<1}$ is length-isospectral. Then $U \equiv 0$.

We emphasize that the *linear dependence* on ϵ of the deformation term $\epsilon\, U(x_1, x_2)$ in (1.2) is not just an artificial limitation. In fact, we also establish Theorem B which, roughly speaking, shows that two Liouville metrics are length-isospectral if and only if they are *rearrangements* of each other. In particular, this result exhibits a large class of length-isospectral deformations of (1.1) *within* the class of Liouville metrics.

To give a precise formulation of this result, we recall that for a measurable function $f : \mathbb{T} \to \mathbb{R}$, the *superlevel measure* at $t \in \mathbb{R}$ is defined to be

$$L_f(t) := \mathrm{Leb}\{x \in \mathbb{T} : f(x) > t\}, \tag{1.3}$$

where Leb is the Lebesgue measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We say that $\tilde{f}$ is a *rearrangement* of f if $L_{\tilde{f}} \equiv L_f$. See Figure 1.1 for an illustration.

Let $\mathcal{L}^{m,n}(\mathbb{T}^2, g)$ denote the subset of $\mathcal{L}(\mathbb{T}^2, g)$ consisting of the lengths of geodesics in the homology class $(m, n) \in \mathbb{Z}^2 \simeq H_1(\mathbb{T}^2, \mathbb{Z})$. We define the *marked length spectrum* of $(\mathbb{T}^2, g)$ to be the map

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{R}^+ \\ (m, n) & \longmapsto & \min \mathcal{L}^{m,n}(\mathbb{T}^2, g). \end{array}$$

We now state a result accompanying Theorem A.

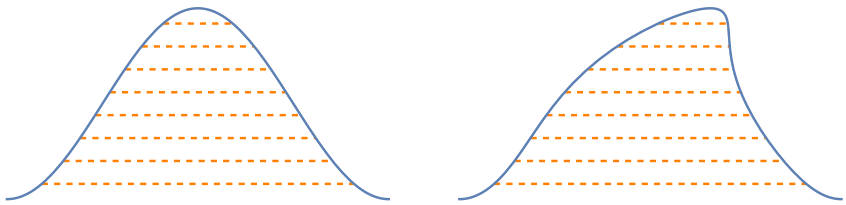

Figure 1.1: A rearrangement preserves the superlevel measure. The blue curves are the graphs of a function before and after a rearrangement.

Theorem B. *Let g and $\widetilde{g}$ be smooth Liouville metrics on $\mathbb{T}^2$ with conformal factors $1 + f_1(x_1) + f_2(x_2)$ and $1 + \widetilde{f}_1(x_1) + \widetilde{f}_2(x_2)$ respectively.*

Then the following are equivalent:

1. *The metrics g and $\widetilde{g}$ have the same marked length spectrum, i.e.*
$$\min \mathcal{L}^{m,n}(\mathbb{T}^2, g) = \min \mathcal{L}^{m,n}(\mathbb{T}^2, \widetilde{g}) \qquad \text{for all } (m,n) \in \mathbb{Z}^2.$$
2. *There exists a constant $c \in \mathbb{R}$ such that $\widetilde{f}_1$ is a rearrangement of $f_1 + c$ and $\widetilde{f}_2$ is a rearrangement of $f_2 - c$.*

Moreover, if each of the functions $f_1, f_2, \widetilde{f}_1, \widetilde{f}_2$ has exactly two critical points on $\mathbb{T}$, then either of the above conditions (and hence both) implies that $\mathcal{L}(\mathbb{T}^2, g) = \mathcal{L}(\mathbb{T}^2, \widetilde{g})$, i.e., the metrics g and $\widetilde{g}$ are length-isospectral.

In particular, within the two-critical-point class, the marked length spectrum, which is defined only with minimal geodesics, determines the full length spectrum, which also contains nonminimal geodesics.

The statement and the proof of Theorem B are inspired by a recent work of Abbondandolo and Mazzucchelli [AM26] on analogous properties of spheres of revolution.

Theorem B exhibits substantial spectral *nonrigidity* of Liouville metrics: one can easily construct many length-isospectral deformations of (1.1) by individual rearrangements of f_1 and f_2, at least when f_1 and f_2 have only two critical points. On the other hand, Theorem B can be viewed as a *rigidity* result under additional symmetry conditions. Indeed, if $f : \mathbb{T} \to \mathbb{R}$ satisfies $f(x) = f(1-x)$ and has exactly two critical points, then the superlevel measures (1.3) completely fix f. Thus, by the claim (1) $\Longrightarrow$ (2) in Theorem B, the (marked) length spectrum uniquely determines the Liouville metric under such symmetry assumptions on f_1 and f_2.

1.1.2 Weak spectral nonrigidity of standard maps

We now briefly describe the main result in the second part of this thesis, which focuses on the different setting of standard maps. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let $V : \mathbb{T} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Recall that the *standard map* $\mathscr{F}$ associated with the *potential* V is the dynamical system on $\mathbb{T} \times \mathbb{R}$ defined by

$$\begin{aligned} \mathscr{F} &: \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}, \\ &(x,y) \mapsto \big(x + y + \dot{V}(x),\, y + \dot{V}(x)\big), \end{aligned} \tag{1.4}$$

where $\dot{V} := \frac{\mathrm{d}V}{\mathrm{d}x}$.

Suppose (x, y) is periodic under $\mathscr{F}$ with (minimal) period $q \geq 1$ and write $(x_k, y_k) = \mathscr{F}^k(x, y)$ for $k = 0, 1, \ldots, q-1$. Then the *(symplectic) action* of this periodic orbit is defined as

$$\sum_{k=0}^{q-1}\left[\frac{1}{2}\Big(x_k - x_{k+1}\Big)^2 + V(x_k)\right]$$

with indices taken modulo q, after choosing a lift of the periodic orbit to $\mathbb{R} \times \mathbb{R}$. Let $\mathrm{d}_{(x,y)}\mathscr{F}^q$ denote the differential of the iterated map $\mathscr{F}^q$ at (x, y). We encode the linearized dynamics near the periodic point by the quantity

$$\det\Big(\mathrm{d}_{(x,y)}\mathscr{F}^q - \mathrm{Id}\Big). \tag{1.5}$$

This quantity will be called the *Lyapunov exponent* of the periodic orbit (x, y).

The primary goal of the second part of this thesis is to construct nontrivial deformations of the potential V that preserve the symplectic actions (respectively, the Lyapunov exponents) of infinitely many periodic orbits. In particular, we prove:

Theorem C. *There exists a nontrivial analytic family of standard maps $(\mathscr{F}_\tau)_\tau$ such that, for each τ, the map $\mathscr{F}_\tau$ has infinitely many dynamically distinct*[1] *periodic orbits. These orbits can be chosen to depend analytically on τ, and their symplectic actions are independent of τ. The same conclusion holds with "symplectic action" replaced by "Lyapunov exponent".*

The standard map is a prototypical example of an *exact symplectic twist map*. It is a classical fact that for every coprime pair (p, q) of integers with $q > 0$, a (p, q)-periodic orbit of an exact symplectic twist map can be given by a minimum of the *action functional*

$$(x_k)_{k=0}^q \mapsto \sum_{k=0}^{q-1} S(x_k, x_{k+1})$$

over the space of sequences $(x_k)_{k=0,\ldots,q-1}$ satisfying $x_q = x_0 + p$, where S is a generating function of the exact symplectic map. The minimizing sequence (x_k) corresponds to the first coordinate of a periodic orbit of period q lifted to a suitable universal cover. Such orbits are called *minimal orbits*.

For the standard map with potential V, the generating function can be taken as

$$S(x, x') = \frac{1}{2}(x - x')^2 + V(x). \tag{1.6}$$

We refer to Section 3.2 for more details on exact symplectic maps and the definition of (p, q)-periodic orbits.

Given an exact symplectic map F with a choice of generating function, *Mather's beta function* is defined to be the continuous function $\beta : \mathbb{R} \to \mathbb{R}$ such that

$$\beta\Big(\frac{p}{q}\Big) = \frac{1}{q}\min\{\text{ symplectic actions of } (p, q)\text{-periodic orbits }\}. \tag{1.7}$$

Mather's beta function encodes important dynamical properties of the symplectic twist map. We refer to Section 1.2.6 for more details. For a potential $V : \mathbb{T} \to \mathbb{R}$, let β_V denote Mather's beta function associated with the standard map defined by the generating function (1.6). We now formulate a refined version of Theorem C.

[1] We say that two periodic orbits are *dynamically distinct* if the orbits do not coincide as sets (they are not just reparametrizations of one another, and one is not a repetition of the other).

Theorem D. *Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ be such that*

$$\left|\omega - \frac{p}{q}\right| \leq q^{-70} \mathrm{e}^{-2\pi q}$$

for infinitely many coprime pairs of integers (p, q). Then there exists a nontrivial real-analytic family of potentials $(V_\tau : \mathbb{T} \to \mathbb{R})_\tau$ such that, for a sequence $p_j/q_j \to \omega$, the value

$$\beta_{V_\tau}\left(\frac{p_j}{q_j}\right)$$

is constant in τ for each j.

A major motivation for Theorem C and Theorem D arises from the better-known spectral rigidity questions of the *billiard map* of a strictly convex domain in $\mathbb{R}^2$, and the close analogy between the standard map and the billiard map. We refer to Section 1.2 for more details.

1.2 Background on spectral rigidity

1.2.1 "Hearing the shape of the drum"

The question of spectral rigidity was popularized by Mark Kac's famous question *Can one hear the shape of a drum?* [Kac66]. Concretely, Kac considered the Dirichlet Laplacian operator Δ on a bounded planar domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. The spectrum of $-\Delta$ consists of a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,$$

where each eigenvalue is repeated according to its multiplicity. The question asks if one can determine Ω up to isometry from the knowledge of the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$.

More generally, one asks how much geometric information about a Riemannian manifold (M, g) is encoded in the spectrum of its Laplace operator.

It is now known that the answer to Kac's question, in full generality, is negative: for instance, Gordon, Webb and Wolpert [GWW92], using the method of Sunada [Sun85], constructed Laplace-isospectral nonisometric pairs of planar domains. However, the question remains open for smooth strictly convex planar domains.

The dynamical spectra considered below are different from the Laplace spectrum, but the same rigidity question motivates them: how much geometry is encoded by a dynamical spectrum?

1.2.2 Spectral rigidity in dynamical systems

Unlike the Laplace spectrum, the spectra considered in this thesis are defined directly from the periodic orbits of a dynamical system. This makes them more geometric and arguably more accessible.

In the setting of billiards and geodesic flows, a dynamical counterpart of the Laplace spectrum is the (marked) *length spectrum* of closed geodesics. The length spectrum can be generalized to the *action spectrum* in the setting of exact symplectic maps. Another dynamically relevant object for a general differentiable dynamical system is the *Lyapunov spectrum*. These three types of spectrum will be the focus of this thesis.

We now provide some motivations for considering these spectra.

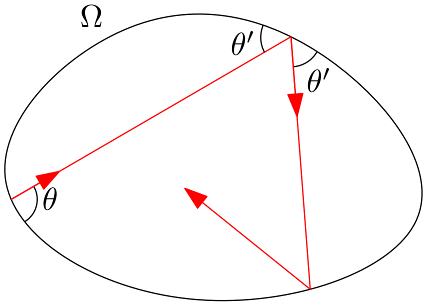

Figure 1.2: An illustration of the billiard dynamics

Spectral rigidity of billiards and geodesic flows

A smooth convex domain $\Omega \subset \mathbb{R}^2$ can be associated with a natural dynamical system: the billiard flow. We consider a frictionless billiard ball moving in a straight line inside the domain Ω, which, when hitting the boundary $\partial\Omega$, is bounced back according to the law of elastic reflection, i.e., the angle of reflection equals the angle of incidence. See Figure 1.2.

We define the length spectrum associated with Ω as the subset of $\mathbb{R}^+$ consisting of the (Euclidean) lengths of all closed billiard trajectories. The dynamical analogue of Kac's question can be formulated as

$$\text{Does the length spectrum uniquely determine } \Omega \text{ up to isometry?} \tag{1.8}$$

We say a smooth family of domains $(\Omega_\tau)_{|\tau|<1}$ is *(length-)isospectral* if the length spectrum of Ω_τ is independent of τ. A domain Ω is said to be *spectrally rigid* if every (continuous) isospectral family $(\Omega_\tau)_{|\tau|<1}$ with $\Omega_0 = \Omega$ must in fact be an isometric family. A deformational version of Question (1.8) is

$$\text{Is every } \Omega \text{ spectrally rigid?} \tag{1.9}$$

For smooth strictly convex planar billiards, both Question (1.8) and its deformational version remain widely open in general. However, several important partial results are known. De Simoi–Kaloshin–Wei [DSKW17] proved deformational length-spectral rigidity for sufficiently smooth $\mathbb{Z}_2$-symmetric strictly convex domains sufficiently close to a circle. Their argument uses a carefully chosen sequence of periodic orbits with rotation numbers tending to zero and analyzes the first variation of their lengths.

There are also several results concerning marked length spectra. For generic strictly convex billiard tables, Huang–Kaloshin–Sorrentino [HKS18] showed that the marked length spectrum determines the eigendata associated with Aubry–Mather periodic orbits. Here, the marked length spectrum is defined by associating each rotation number $\rho \in (0, 1/2]$ with the maximal length of periodic billiard orbits with rotation number ρ. For open dispersing billiards, the marking is given by the symbolic sequence of scatterers hit by the orbit over one period. De Simoi–Kaloshin–Leguil [DSKL23] studied such billiards with three convex scatterers having an axial symmetry and proved that the marked length spectrum determines the geometry of the scatterers.

Another important source of rigidity results comes from integrability. The billiard in an ellipse is the basic integrable example, and the Birkhoff conjecture predicts that ellipses are the only integrable strictly convex planar billiards. Avila–De Simoi–Kaloshin [ADSK16] and Kaloshin–Sorrentino [KS18] proved local versions of this conjecture near ellipses, and, as corollaries, deduced spectral-rigidity results for elliptic billiards. More recently, Fierobe–Kaloshin–Sorrentino [FKS26] proved deformational spectral rigidity for domains close to ellipses under dihedral symmetry. The results of Avila–De Simoi–Kaloshin and Kaloshin–Sorrentino have been extended by Koval [Kov26], where he showed that, for almost every ellipse, a small perturbation that preserves integrability just near the boundary must again be an ellipse. On the other hand, Bialy-Mironov [BM22] proved a nonperturbative version of the conjecture within the class of centrally symmetric and Kaloshin–Koudjinan–Zhang [KKZ24] extended this result to show that an integrable domain sufficiently close to a centrally symmetric one must be an ellipse. We refer to Section 1.2.3 for further discussions of the Birkhoff conjecture.

The same type of question can be posed for geodesic flows on Riemannian manifolds, with or without boundary. In this setting, the relevant spectrum is the collection of lengths of closed geodesics, often with additional marking. Marked length-spectral rigidity has been especially successful in settings where the geodesic flow is uniformly hyperbolic. For example, for closed surfaces of negative curvature, Croke and Otal [Cro90, Ota90] independently showed that the marked length spectrum determines the metric up to isometry. Recently, Guillarmou and Lefeuvre [GL19] established deformational marked length-spectral rigidity for a broad class of manifolds with non-positive curvature and Anosov geodesic flows in arbitrary dimension. The results of Croke and Otal have also been extended by Guillarmou–Lefeuvre–Paternain [GLP25], who established global marked length-spectral rigidity for surfaces with Anosov geodesic flows, not just negatively curved surfaces. We also mention a more recent development by Cekić–Lefeuvre [CL25]. These results show that, in uniformly hyperbolic dynamics, the marked length spectrum is a particularly robust invariant.

On the other hand, the first part of the thesis studies *nearly-integrable* geodesic flows – a fairly different situation from the uniformly hyperbolic cases discussed above. More precisely, we study Liouville metrics on $\mathbb{T}^2$ and prove a spectral rigidity result for a class of linear conformal deformations. The proof uses the special integrable structure of Liouville metrics, and we return to the role of integrability in Section 1.2.3.

The Poisson relation

One reason that the length spectrum is a natural dynamical counterpart of the Laplace spectrum is the *Poisson relation*. This relation says, roughly, that singularities of the wave trace, which is a Laplace-spectral invariant, occur only at points in the length spectrum.

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and let $(\lambda_j)_{j\geq 1}$ be the eigenvalues of the Dirichlet Laplacian on Ω. The associated wave trace is the distribution

$$w_\Omega(t) = \sum_{j\geq 1} \cos\Big(t\sqrt{\lambda_j}\Big).$$

If $\mathcal{L}(\Omega)$ denotes the length spectrum of the billiard flow in Ω, then the Poisson relation states that

$$\operatorname{sing\,supp} w_\Omega \subset \pm\mathcal{L}(\Omega) \cup \{0\}. \tag{1.10}$$

Under suitable nondegeneracy and genericity assumptions, this inclusion is an equality. Thus the Laplace spectrum generically detects the lengths of closed billiard trajectories, which are purely dynamical objects.

More is true: the singularity of w_Ω at a length L contains refined information about the closed billiard trajectories of length L. For example, if γ is an isolated nondegenerate closed billiard trajectory, then its leading contribution to the wave trace involves the linearized Poincaré map P_γ. Schematically, the leading coefficient has the form

$$\frac{L_\gamma^\#}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \tag{1.11}$$

up to phase factors depending on the Maslov index and the number of reflections. Here $L_\gamma^\#$ denotes the primitive length of γ.

The Poisson relation for manifolds without boundary goes back to Chazarain [Cha74] and Duistermaat–Guillemin [DG75]. For billiards, the corresponding results were developed by Andersson–Melrose [AM77] and Guillemin–Melrose [GM79]. See also [PS92] for a systematic introduction of these results.

This connection motivated Guillemin and Melrose [GM81] to study the length spectrum jointly with the quantities

$$|\det(\mathrm{Id} - P_\gamma)|$$

attached to closed orbits. Thus the Poisson relation motivates not only the length spectrum itself, but also the Lyapunov spectrum, which consists of the quantity $|\det(\mathrm{Id} - P_\gamma)|$ associated with periodic orbits in the system.

Symplectic framework and the action spectrum

We now turn from geodesic and billiard flows to exact symplectic maps, where the role of length is played by the symplectic action.

We first recall the general setting. Let $I \subset \mathbb{R}$ be an interval, and let

$$F : \mathbb{T} \times I \to \mathbb{T} \times I$$

be an exact symplectic map with respect to the standard one-form $y\,\mathrm{d}x$. Thus there exists a generating function $S : \mathbb{T} \times I \to \mathbb{R}$ such that

$$F^*(y\,\mathrm{d}x) - y\,\mathrm{d}x = \mathrm{d}S.$$

After choosing such a generating function, one can assign a symplectic action to each periodic orbit by summing S along the orbit. The *action spectrum* is the collection of these actions over all periodic orbits. As usual, this spectrum depends on the additive constant in the choice of the generating function.

For exact symplectic twist maps, it is often more convenient to use a generating function $S(x, x')$ whose variables are given by the coordinate on $\mathbb{T}$ of consecutive points along an orbit. In that case the action of a q-periodic orbit is written as

$$\sum_{j=0}^{q-1} S(x_j, x_{j+1}),$$

where the indices are taken modulo q, after choosing the appropriate lift.

The billiard map of a strictly convex planar domain gives a basic example of this framework. In suitable coordinates, it is an exact symplectic twist map on $\mathbb{T}\times[0,\pi]$, and its generating function is, up to sign, the Euclidean distance between consecutive reflection points. Consequently, the action of a periodic billiard orbit agrees, up to sign, with its Euclidean length. Thus the action spectrum of the billiard map is essentially the length spectrum of periodic billiard trajectories.

The standard map (1.4) is another prototype of exact symplectic twist maps with a generating function

$$S(x, x') = \frac{1}{2}(x - x')^2 + V(x). \tag{1.12}$$

Therefore, if $(x_j, y_j)_{j=0}^{q-1}$ is a periodic orbit and the coordinates x_j are lifted consistently to $\mathbb{R}$, its action is

$$\sum_{j=0}^{q-1} \left[\frac{1}{2}(x_j - x_{j+1})^2 + V(x_j)\right].$$

This analogy between billiards and standard maps is summarized in Table 1.1.

	Billiards	Standard maps
Geometry	Domain $\Omega \subset \mathbb{R}^2$	Potential $V : \mathbb{T} \to \mathbb{R}$
Natural equivalence	Isometry of Ω	Rotation and reflection of V
Spectrum	Length spectrum	Action spectrum

Table 1.1: Analogy between billiards and standard maps.

The natural rigidity questions for standard maps are therefore analogous to the length-spectral rigidity questions for billiards:

Does the action spectrum determine V up to rotation and reflection?

and, in deformational form,

Can V be deformed nontrivially while preserving the action spectrum?

The results in the second part of the thesis address a weak version of the second question. Namely, we construct nontrivial deformations for which infinitely many selected points in the action spectrum are preserved, even though the full action spectrum is not shown to remain fixed.

The Lyapunov spectrum

Consider a differentiable dynamical system $f : M \to M$. Suppose that x is periodic with minimal period $q > 0$, so that $f^q(x) = x$. The Lyapunov exponents of this periodic orbit are the quantities

$$\frac{1}{q} \log |\lambda|, \tag{1.13}$$

where λ ranges over the eigenvalues of the linearized return map

$$\mathrm{d}_x f^q : T_x M \to T_x M.$$

In the case $\dim M = 2$ and f preserves a symplectic form, the Lyapunov exponents of a q-periodic orbit x can be conveniently encoded by the quantity

$$\det(\mathrm{d}_x f^q - \mathrm{Id}).$$

Throughout this thesis, we shall slightly abuse terminology and refer to the above quantity as the "Lyapunov exponent" of the periodic orbit. With this convention, the *Lyapunov spectrum* of f means the collection of these quantities over all periodic orbits of f.

Remark. In the broader dynamical systems literature, the term "Lyapunov exponent" has a more general meaning. Let $f : M \to M$ be a $\mathcal{C}^r$-diffeomorphism, with $r \geq 1$, of a compact manifold M, and let μ be an f-invariant probability measure. The integrability conditions

$$\int_M \log^+ \|\mathrm{d}_x f\| \,\mathrm{d}\mu(x) < \infty, \qquad \int_M \log^+ \|(\mathrm{d}_x f)^{-1}\| \,\mathrm{d}\mu(x) < \infty$$

where $\log^+(s) = \max\{\log s, 0\}$, are automatic in this compact smooth setting. By the multiplicative ergodic theorem of Oseledets [Ose68] (see also [KH95a], [CM06] or [Via14]), there exists an f-invariant Borel set $H \subset M$ of full μ-measure such that for every $x \in H$ there are real numbers

$$\lambda_x^{(1)} > \cdots > \lambda_x^{(m)}$$

with $m = m(x)$ and a f-invariant splitting

$$T_x M = E_x^{(1)} \oplus \cdots \oplus E_x^{(m)}$$

and, for every nonzero vector $v \in E_x^{(j)}$,

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|\mathrm{d}_x f^n(v)\| = \lambda_x^{(j)}.$$

The numbers $\lambda_x^{(j)}$ are called the *Lyapunov exponents* of f at x. In particular, if x is periodic with minimal period $q > 0$, then the Lyapunov exponents along the periodic orbit of x are precisely of the form (1.13). If μ is ergodic, then the numbers $\lambda_x^{(j)}$ and the dimensions of the spaces $E_x^{(j)}$ are independent of x for μ-almost every x; in this case one often speaks of the Lyapunov spectrum of the measure μ. In this thesis, however, we focus only on the Lyapunov exponents of periodic orbits. In fact, our result, Theorem C, only concerns a subset of such periodic-orbit Lyapunov data.

Similar to the action spectrum, the Lyapunov spectrum can also be regarded as an analogue, in spirit, of the length spectrum for geodesic flows and billiards.

The analogy between the length spectrum and the Lyapunov spectrum appears explicitly in holomorphic dynamics. McMullen [McM10] considers proper holomorphic maps of the unit disk and studies, for a periodic cycle of period q, the logarithmic multiplier

$$\log |(f^q)'(z)|.$$

These logarithmic multipliers are shown to exhibit properties closely analogous to the lengths of closed geodesics on hyperbolic surfaces.

In the setting of expanding maps on the circle, a classical result of Shub and Sullivan [SS85] shows that the *marked* Lyapunov spectrum determines the map up to *smooth* conjugacy. It is well known that every expanding map on the circle is topologically conjugate to the linear model $x \mapsto dx$ for some degree $d \in \mathbb{Z} \setminus \{0\}$. Such a conjugacy provides a natural symbolic coding of periodic orbits, and hence a canonical *marking* of the Lyapunov spectrum. We also refer to the work of Drach and Kaloshin [DK25] for recent progress on an *unmarked* version of this rigidity phenomenon and for a detailed historical review of the problem therein.

The settings considered in this thesis, namely geodesic flows on $\mathbb{T}^2$ and standard maps, are quite different from these uniformly hyperbolic examples. One of the goals of the second part of this thesis is to exhibit a weak form of *nonrigidity* for the Lyapunov spectrum in the sense that the Lyapunov exponents from certain sequences of periodic orbits of the standard map can be preserved under nontrivial deformations.

1.2.3 The role of integrability

Our argument of Theorem A follows a common strategy in spectral rigidity: first show that an isospectral deformation preserves an integrable structure, and then use the rigidity of that integrable structure to trivialize the deformation. This connects spectral rigidity with the problem of *integrable rigidity*. The latter question asks to what extent an integrable dynamical system can be deformed while preserving integrability.

An integrable system is characterized by the presence of sufficiently many conserved quantities, so that a large part of phase space is foliated by invariant submanifolds on which the dynamics is conjugate to simple models. A classical definition of integrability uses the framework of Arnold-Liouville, which we briefly recall in Section 2.3.1.

Integrable systems are expected to be very rare, and therefore one often expects them to be rigid under deformation. This belief features prominently in the study of billiards. The long-standing *Birkhoff conjecture* states:

Conjecture 1 (Birkhoff)**.** The boundary of a strictly convex integrable billiard domain is necessarily an ellipse.

There is strong evidence for this conjecture. Among many recent developments, we mention the works of Avila–De Simoi–Kaloshin [ADSK16] and Kaloshin–Sorrentino [KS18], which prove that ellipses are isolated among integrable billiard domains, and the work of Bialy–Mironov [BM22], which establishes the nonperturbative version of this conjecture within the class of centrally symmetric domains. Recently, Koval [Kov26] obtained a remarkable generalization of the results of Avila–De Simoi–Kaloshin and Kaloshin–Sorrentino: for almost every ellipse, a small perturbation that preserves integrability just near the boundary must again be an ellipse. On the other hand, building on the work of Bialy–Mironov, Kaloshin–Koudjinan–Zhang [KKZ24] showed that an integrable domain sufficiently close to a centrally symmetric one must be an ellipse. We remark in particular that Kaloshin–Sorrentino [KS18] and Koval [Kov26] both deduced spectral rigidity results of ellipses from their integrable rigidity as a simple corollary.

We also mention the work of Hezari–Zelditch [HZ22], which builds on earlier work of Avila–De Simoi–Kaloshin [ADSK16] and shows that nearly circular ellipses are uniquely determined by their Laplace spectrum among all simply connected domains.

A similar picture appears for standard maps. Suris [Sur89] discovered a class of potentials for which the corresponding standard maps are integrable. A recent work of Fierobe–Tsodikovich [FT26] proves a version of integrable rigidity for Suris-type potentials and obtains action-spectral rigidity of these potentials as a corollary.

In the setting of geodesic flows on $\mathbb{T}^2$, the analogue of the Birkhoff conjecture is the following folklore conjecture.

Conjecture 2. Up to isometry, Liouville metrics are the only integrable metrics on $\mathbb{T}^2$.

Henheik [Hen25] proved the following deformational result in this direction.

Theorem 1.1 (Henheik, 2025)**.** *Consider a family of metrics (g_ϵ) on $\mathbb{T}^2$ of the form*

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon U(x_1, x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2),$$

where f_1 and f_2 belong to a generic class of analytic functions satisfying $1+\min f_1+\min f_2 > 0$, and $U(x_1, x_2)$ is a trigonometric polynomial. Assume that g_ϵ is integrable for all ϵ. Then there exist functions $U_1, U_2 : \mathbb{T} \to \mathbb{R}$ such that

$$U(x_1, x_2) = U_1(x_1) + U_2(x_2).$$

Our proof of Theorem A follows a similar route: we first show that length-isospectral deformations preserve rational integrability of the geodesic flow, and then use this preserved integrable structure to prove that a linear conformal deformation must vanish.

The following table summarizes the analogy among billiards, standard maps, and geodesic flows on $\mathbb{T}^2$.

	Billiards	Standard maps	Geodesics on $\mathbb{T}^2$
Geometry	Domain $\Omega \subset \mathbb{R}^2$	Potential $V : \mathbb{T} \to \mathbb{R}$	Riemannian metric
Natural equivalence	Isometry	Rotation & reflection of V	Isometry
Spectrum	Length spectrum	Action spectrum	Length spectrum
Integrable examples	Ellipses	Suris potentials	Liouville metrics

Table 1.2: Comparison among billiards, standard maps, and geodesic flows. It remains open whether the examples listed in the last row exhaust all integrable cases in their respective settings.

We remark that a precise connection between elliptical billiards and Liouville metrics can be given by the work of Popov–Topalov [PT03], where they introduced *Liouville billiard tables*. Their construction starts from the cylinder $C = \mathbb{T} \times [-N, N]$, equipped with a Liouville-type metric

$$\mathrm{d}s^2 = (f(x) - q(y))(\mathrm{d}x^2 + \mathrm{d}y^2),$$

where f and q satisfy suitable symmetry and compatibility conditions. The billiard table is obtained as a quotient of C by a branched covering map $C \to \widetilde{C}$ where $\widetilde{C}$ is homeomorphic to a disc. Elliptical billiards in the Euclidean plane arise as the special case in which this covering is given by the standard elliptic-coordinate map.

Lastly, to conclude our discussion of integrability-related rigidity results, we also mention a separate question about the integrable rigidity in a neighborhood of an elliptic periodic orbit. As a prototypical example of this problem, Treschev [Tre13, Tre15, Tre17] considered a period-2 billiard orbit in a domain and constructed examples of formal power series expansions of the billiard boundary such that the billiard dynamics near the period-2 orbit is formally conjugate to a rigid rotation of the type

$$(x, y) \mapsto (x + a, y) \qquad \text{for some constant} \qquad a \in \mathbb{R}.$$

Wang and Zhang [WZ25] proved a Gevrey regularity of Treschev's examples in the case where a is Diophantine. We also mention Koval's recent extension [Kov24] of the results of Wang-Zhang. Currently, the analyticity of Treschev's examples remains an open question.

1.2.4 Flexibility of geodesic flows

The analogy between geodesic flows and billiards is useful, but it should not be taken too literally. Geodesic flows actually appear to be more flexible than billiard maps. In particular, there is strong evidence suggesting that Conjecture 2 is false.

For example, Corsi–Kaloshin [CK18] constructed a non-Liouville metric that is locally integrable in a p-cone in the cotangent bundle. Agapov-Bialy-Mironov [ABM17] constructed an example of a non-Liouville integrable magnetic geodesic flow. On the spectral side, the recent work of Abbondandolo–Mazzucchelli [AM26] shows that length-isospectral spheres of revolution with one equator are abundant and classifies them by rearrangements.

Theorem B gives an analogous flexibility result for Liouville metrics on $\mathbb{T}^2$: length-isospectral Liouville metrics can be constructed by rearranging the one-dimensional functions f_1 and f_2. Thus Liouville metrics are not rigid within the full class of Liouville metrics. Particularly, the rigidity result of Theorem A is specific to linear-in-ϵ conformal deformations of the form (2.2).

Two natural questions remain. The first asks whether rearrangements account for all possible length-isospectral deformations of Liouville metrics.

Question 1. Consider a family of metrics (g_ϵ) on $\mathbb{T}^2$ of the form

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon U(x_1, x_2, \epsilon)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2). \tag{1.14}$$

Suppose that the family is length-isospectral. Does it follow that g_ϵ is again a Liouville metric obtained by rearrangements of g_0?

The second question asks whether imposing reflection symmetries eliminates the rearrangement-type flexibility. Reflection-symmetric functions on $\mathbb{T}$ with exactly two critical points are rigid under rearrangement, so one may ask the following analogue of the symmetric billiard problem studied in [DSKW17].

Question 2. Consider a family of metrics (g_ϵ) on $\mathbb{T}^2$ of the form (1.14), and assume that f_1 and f_2 each have exactly two critical points on $\mathbb{T}$. Suppose that the family is length-isospectral and that each g_ϵ is invariant under reflections about the axes

$$\{x_1 = 1/2\} \qquad \text{and} \qquad \{x_2 = 1/2\}.$$

Must the family be trivial, i.e. $g_\epsilon = g_0$ for all sufficiently small ϵ?

1.2.5 Rigidity from accumulating periodic orbits

The second part of this thesis is concerned with a rigidity question of a slightly different flavor. Instead of asking whether the entire spectrum determines the system, one may ask which parts of the spectrum are responsible for rigidity.

In many recent rigidity results, the full spectrum is not used; instead, one studies the spectral data of a sequence of periodic orbits accumulating on a distinguished object, such as a boundary, an invariant curve, or a periodic orbit. Our results in the second part point in the opposite direction: we construct nontrivial deformations preserving the spectral data of infinitely many selected orbits.

We briefly recall some rigidity results based on such accumulating sequences of periodic orbits. First, the theorem of De Simoi, Kaloshin and Wei [DSKW17] mentioned earlier proves *deformational* length-spectral rigidity within the class of nearly circular, $\mathbb{Z}^2$-symmetric domains. Their proof relies on a detailed analysis of the first-order variation, under boundary deformations, of the lengths of a sequence of symmetric periodic orbits with rotation number $1/q$, $q \to \infty$, together with a delicate inversion of the resulting linearized operator.

In the work of Huang, Kaloshin and Sorrentino [HKS18], they show that, for a generic billiard domain, the eigendata of minimal orbits can be recovered from the marked length spectrum. In fact, one recovers the Lyapunov exponent of a minimal orbit of rotation number p/q using only a sequence of lengths of minimal orbits whose rotation numbers converge to p/q.

We also mention work of De Simoi, Kaloshin and Leguil [DSKL23], which considers dispersing billiard systems of three scatterers with axial symmetry and shows that the marked length spectrum determines the geometry of the scatterers. In that work, the authors consider the lengths of a sequence of periodic orbits converging to a homoclinic orbit of a particular 2-periodic orbit. Using the spectral data from this sequence, one reconstructs the Taylor series of the boundaries of the scatterers at the endpoints of the 2-periodic orbit. This result has been extended by Osterman [Ost23], who considers the same setting without symmetry assumptions and shows that the marked length spectrum determines the analytic conjugacy class of the billiard map.

Notably, these positive results mentioned above make use not of the *full* spectrum, but only of a countable subset corresponding to a sequence of orbits accumulating on some invariant "boundary" or "axis of symmetry" of the dynamics. On the other hand, our main result shows that, in the standard-map setting, spectral data associated with certain accumulating sequences of periodic orbits are *not* deformationally rigid. This motivates the following subtle question about which countable subsets of the spectrum suffice to determine the underlying dynamics.

Question 3. Does there exist a nonisometric deformation of a smooth strictly convex planar billiard domain which preserves the lengths of infinitely many periodic orbits accumulating to the boundary, to a periodic orbit, or to an invariant curve?

There is a positive answer to a *nondeformative* version of Question 3: Buhovsky and Kaloshin [BK18] constructed a pair of billiard domains such that the marked length spectra of the two domains coincide along a sequence of rotation numbers $1/q_j$ with $q_j \to \infty$.

The methods in this thesis do not answer Question 3. The key issue is that the proof of our main theorem, Theorem 3.1, relies crucially on the special structure of the standard map: one can explicitly construct potentials for a standard map admitting an invariant curve with prescribed dynamics restricted to that curve. In the setting of billiards, the standard tool for obtaining invariant curves is KAM theory (Section 1.2.7), but the Liouville-type arithmetic condition (3.5) required in our construction is incompatible with the usual Brjuno-type or Diophantine-type conditions needed for KAM theory.

1.2.6 Rigidity of Mather's beta function

Mather's beta function is another spectral invariant that features prominently in the rigidity problems.

Given a general exact symplectic twist map and a choice of generating function, Mather's beta function β is defined, as in (1.7), to be a continuous function whose value at a rational number p/q is $1/q$ times the action of the minimal orbits of rotation number p/q. It is known that β is convex on $\mathbb{Q}$ and therefore extends continuously, by density, to an interval of $\mathbb{R}$ called the *rotation interval* of the map.

If Mather's beta function admits an asymptotic expansion at ω of the form

$$\beta(\alpha) \sim \sum_{k\geq 0} c_k(\alpha-\omega)^k, \tag{1.15}$$

we call the sequence (c_k) the *infinite jet* of β at ω, by analogy with the jets of differentiable functions. More precisely, the asymptotics (1.15) means that for any $N \in \mathbb{N}$ and any sequence $\alpha_j \to \omega$,

$$\frac{1}{(\alpha_j-\omega)^N}\left(\beta(\alpha_j) - \sum_{k=0}^{N} c_k(\alpha_j-\omega)^k\right) \to 0 \quad \text{as } j \to \infty.$$

We refer to [Sib04, MF94] for more details on Mather's beta function.

In the billiard setting, Mather's beta function admits an infinite jet at 0, which is commonly known as the Marvizi–Melrose invariants [MM82]. Hence, if a deformation of a billiard domain preserves a sequence of orbits accumulating to the boundary, then this deformation preserves all Marvizi–Melrose invariants. In particular, Buhovsky and Kaloshin [BK18] used this observation to construct nonisometric pairs of domains with identical Marvizi–Melrose invariants.

We recall that a *rotational invariant curve (RIC)* of a map $F : \mathbb{T}\times I \to \mathbb{T}\times I$ is the graph

$$C = \{(x, h(x)) : \ x \in \mathbb{T}\},$$

of a continuous function $h : \mathbb{T} \to \mathbb{R}$, such that C is invariant under F. Using Birkhoff normal forms, one can show that if the symplectic map is sufficiently smooth and it has an RIC whose restricted dynamics conjugates to a rigid rotation $x \mapsto x+\omega$ by a *Diophantine* number (i.e. ω is irrational and $\sup_{p\in\mathbb{Z}} |q\omega - p|^{-1}$ grows at most polynomially in q), then Mather's beta function admits an infinite jet at ω. Thus, if a deformation of the system preserves a sequence of orbits accumulating to such an invariant curve, then this deformation preserves the infinite jet of Mather's beta function at ω.

On the other hand, in a forthcoming work of Mathieu Helfter, the finite order jet at a fixed Diophantine frequency of the beta function of nearly integrable standard maps is shown to be deformationally rigid with respect to deformations within generic finite-dimensional subspaces of potential functions.

Hence, we are led to the following question, which is in some sense weaker than Question 3.

Question 4. Does there exist a nonisometric deformation of a smooth strictly convex planar billiard domain which preserves the infinite jet of Mather's beta function (when it exists) at some $\omega \in \mathbb{R}$?

We remark that Mather [Mat90] proved that Mather's beta function is differentiable at every *irrational* ω and generically fails to be differentiable at rational ω. In fact, Mather's beta function is differentiable at p/q if and only if there exists an RIC conjugate to a rigid rotation by p/q. However, it is not clear in general whether Mather's beta function has an infinite jet at a frequency ω that is irrational but not Diophantine. Hence, our result Theorem D does not provide information relevant to Question 4, due to the strong arithmetic condition (3.5). This motivates a further regularity question for Mather's beta function.

Question 5. Under what conditions does Mather's beta function admit an infinite jet at a given $\omega \in \mathbb{R}$?

1.2.7 The role of KAM theory

As another strategy for rigidity problems, one can analyze the spectral invariants associated with the persistence of quasiperiodic dynamics in the system obtained from Kolmogorov–Arnold–Moser (KAM) theory. One example of such KAM-related spectral invariants is the jet of Mather's beta function at a Diophantine frequency discussed earlier. In this section, we discuss several prior results that exploit KAM theory to study rigidity problems.

KAM theory deals with the persistence of quasiperiodic motions in Hamiltonian dynamical systems under small perturbations. In a prototypical setup, we consider a perturbation of an integrable Hamiltonian H_0 by H_1 on the phase space $\mathbb{T}^d \times \mathbb{R}^d$:

$$H_0(I) + \epsilon\, H_1(\theta, I), \qquad (\theta, I) \in \mathbb{T}^d \times \mathbb{R}^d,\ |\epsilon| \ll 1. \tag{1.16}$$

KAM theory shows that, under suitable nondegeneracy conditions on H_0, a large-measure set of invariant tori of the integrable part H_0 persists under the perturbation (1.16), and the restricted dynamics on each such torus is conjugate to a linear flow on $\mathbb{T}^d$:

$$x \longmapsto x + t\omega$$

with a *Diophantine* direction $\omega \in \mathbb{R}^d$, i.e., there exist $c > 0$ and $\tau > d-1$ such that

$$|\omega \cdot k| \geq \frac{c}{\|k\|^\tau} \qquad \text{for all} \quad k \in \mathbb{Z}^d \setminus \{0\}. \tag{1.17}$$

Such invariant tori are customarily called *KAM tori*, or *KAM curves* in the case of billiards. Following the original works of Kolmogorov [Kol54], Arnold [Arn63a, Arn63b] and Moser [Mö62], there is by now a vast literature on KAM theory. We refer to [CM10] and [Pö09], and to the references therein, for further general details. In the particular setting of billiard dynamics, a foundational work is due to Lazutkin [Laz73], who showed that, for sufficiently smooth strictly convex billiard tables, there exists a positive-measure Cantor family of invariant curves accumulating at the boundary of phase space. We also refer to Moser–Levi [LM01] for a particularly elegant KAM scheme adapted to symplectic twist maps, including billiard maps and standard maps.

From the spectral-rigidity point of view, it is natural to study what happens to the KAM tori when the perturbation (1.16) is assumed to be *isospectral*. In this direction, Popov [Pop94] considered the family of KAM curves of the billiard map and showed that, under length-isospectral deformations, the billiard map restricted to the union of KAM curves remains symplectically conjugate to the undeformed case. On the other hand, Popov and Topalov studied the KAM curves in [PT12] and subsequently in [PT19] from the perspective of the Laplace spectrum via the notion of *quasimodes*. More precisely, each KAM curve that persists under a deformation can be associated with a *continuous quasimode* of order $M \geq 1$ parametrized by the deformation parameter t. This is a sequence of pairs of approximate Laplace eigenvalues and eigenfunctions $(\mu_q(t), u_q(t))_{q\geq 1}$ depending continuously on t such that

$$\|\Delta_t u_q(t) - \mu_q(t)^2 u_q(t)\|_{L^2} \leq C\mu_q(t)^{-M} \tag{1.18}$$

for some $C > 0$. Moreover, the approximate eigenvalue has an expansion

$$\mu_q(t) = \mu_q^0 + c_{q,0}(t) + \frac{c_{q,1}(t)}{\mu_q^0} + \frac{c_{q,2}(t)}{(\mu_q^0)^2} + \cdots + \frac{c_{q,M}(t)}{(\mu_q^0)^M}.$$

Popov and Topalov related the coefficients $c_{q,j}(t)$ above to geometric information defining the system. As an application, they proved in [PT12] the spectral rigidity for the function $K : \partial\Omega \to \mathbb{R}$ appearing in the Robin boundary condition under symmetry assumptions, for a class of *Liouville billiard tables* (see Section 1.2.3). This approach is substantially developed in their subsequent work [PT19], where they proved infinitesimal Laplace spectral rigidity of Liouville billiard tables and a result for $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric deformations of ellipses that satisfy a nondegeneracy condition on its elliptic period-2 orbit.

An important advantage of the Popov–Topalov approach is that it extracts certain length-spectral invariants, including values of Mather's beta function at Diophantine frequencies and Birkhoff normal form data, without relying on the wave trace (see Section 1.2.2). Although the wave trace can detect periodic orbits through its singular support, its use is often complicated by possible cancellations of singularities and by the clean-intersection hypotheses needed to describe the corresponding singular expansions. We refer to Section 2.7 for a discussion of how these difficulties are addressed in our setting.

Our approach to Theorem A is rather different from the KAM-based strategies discussed above. The dynamical objects responsible for rigidity in our argument are rational tori. Since rational tori are well-known to be readily destroyed by generic perturbations, their preservation imposes strong constraints on the allowed deformations, in sharp contrast with the robustness of the KAM tori. We use length-spectral rigidity to deduce the preservation of rational tori, and then use their preservation to trivialize the deformation. To pass from length-spectral rigidity to Laplace-spectral rigidity, we still use wave-trace methods, combining a Marvizi–Melrose-type parametrix with a theorem of Soga on oscillatory integrals to recover the relevant length-spectral information. See Section 2.7 for details.

1.3 Organization of the thesis

In this introductory chapter, we have presented the general motivation behind spectral rigidity problems, stated the main results of the thesis, and discussed several related rigidity and nonrigidity questions. The rest of the thesis consists of two main parts, contained in Chapter 2 and Chapter 3, respectively.

Chapter 2 is devoted to the length-spectral rigidity and nonrigidity of Liouville metrics on $\mathbb{T}^2$. In Section 2.1, we outline the overall two-step scheme of the proof of Theorem A and we present a formal proof of this theorem assuming two intermediate results. We also explain the main idea underlying the proof of Theorem B. Next, we introduce some definitions related to the length spectrum and the marked length spectrum in Section 2.2, and then recall the relevant integrability notions in Section 2.3. In particular, we introduce the notion of rational integrability, which is the main dynamical structure used in the proof. The technical Section 2.4 establishes several geometric properties of rational tori, including the twist property needed to construct families of connecting orbits, which play a crucial role in the proof of Theorem A. Section 2.5 proves that length-isospectral deformations preserve rational tori. These results are then combined in Section 2.6 to prove the rigidity theorem for linear conformal deformations of Liouville metrics. Section 2.7 discusses a Laplace-spectral analogue of the rigidity statement. Finally, Theorem B, which gives the rearrangement classification of marked-length-isospectral Liouville metrics, is proved in Section 2.8. The final section collects auxiliary facts used throughout the chapter.

Chapter 3 studies a weak form of action-spectral nonrigidity for standard maps. After introducing notation and basic definitions for exact symplectic maps in Section 3.2, we describe in Section 3.3 the coordinates adapted to a rotational invariant curve. Sections 3.4 and 3.5 contain the main technical tools: a resonant normal-form construction near an invariant curve and a Picard iteration scheme which allows us to prescribe the resonant data associated with a sequence of rational approximations. In Section 3.6, these tools are used to construct periodic orbits accumulating on the invariant curve and to compute their symplectic actions and Lyapunov exponents, completing the proof of the main nonrigidity theorem. The proof of the resonant normal-form theorem, stated in Sections 3.4, is given in Section 3.7, and the remaining auxiliary estimates are collected in the final section of the chapter.

CHAPTER 2

Spectral Rigidity of Liouville Metrics

This chapter appears in part in the following preprint.

- J. Henheik, V. Kaloshin, Y. Li, and A. Vig. Spectral rigidity of Liouville tori, 2025. doi:10.48550/arXiv.2511.10398

2.1 Introduction

In this chapter we study length-spectral rigidity for Liouville metrics on $\mathbb{T}^2$. The undeformed metrics have the form

$$\mathrm{d}s^2 = \Big(1 + f_1(x_1) + f_2(x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \tag{2.1}$$

where $1 + f_1 + f_2 > 0$. The main result of this chapter, Theorem A, concerns linear conformal deformations of the form

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon U(x_1, x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2). \tag{2.2}$$

We prove that if the length spectrum of g_ϵ is independent of ϵ, then $U \equiv 0$. Thus Liouville metrics are spectrally rigid under this class of linear conformal deformations.

The proof uses the integrability of Liouville metrics in an essential way. The key point is that length-isospectral deformations preserve rational tori, and the preservation of even one such rational torus forces a linear conformal deformation to vanish.

The results in this chapter are joint work with Joscha Henheik, Vadim Kaloshin and Amir Vig, available in the preprint article [HKLV25]. Our focus in this thesis is the length-spectral aspect of this work, which corresponds to sections 3 and 5 of the cited preprint article.

2.1.1 Outline of the proof

We now outline the proof of Theorem A. The argument has two main steps:

Step 1: isospectrality preserves rational tori,
Step 2: one preserved rational torus with fixed length forces $U \equiv 0$.

The first step is the following general result. It does not require the initial metric to be Liouville.

Theorem 2.1 (Isospectral deformations preserve rational integrability)**.** *Let $(g_\epsilon)_{|\epsilon|<1}$ be a smooth family of Riemannian metrics on $\mathbb{T}^2$, and suppose that g_0 is rationally integrable. If (g_ϵ) is length-isospectral, then g_ϵ is rationally integrable for every $|\epsilon| < 1$.*

Roughly speaking, rational integrability means that every closed geodesic belongs to a *rational torus*. Such a rational torus is a family of closed geodesics that foliates an invariant torus in the unit tangent bundle, and the invariant torus projects diffeomorphically onto $\mathbb{T}^2$.

While the precise definition is given in Section 2.3.2, here we emphasize that our definition requires a rational torus to be a *graph* over $\mathbb{T}^2$. This makes the rational torus a close analogue of a *resonant convex caustic* in billiards.

We explain the main idea of the proof of Theorem 2.1. While a rational torus generally breaks up under perturbations, a smooth family of *connecting orbits* naturally arises from the perturbed rational torus as a consequence of a twist property established in Proposition 2.11. More precisely, a connecting orbit based at $x \in \mathbb{T}^2$ is a (not necessarily periodic) geodesic segment in a fixed homology class which starts and ends at x. We observe that closed geodesics are characterized by the critical points of the length $\overline{\tau}_\epsilon(x)$ of the connecting orbit as a function of the base point x.[1] The isospectrality assumption then forces $x \mapsto \overline{\tau_\epsilon}(x)$ to be a constant, implying the preservation of a rational torus. The main technical difficulty lies in the verification of the twist property of the rational torus in Proposition 2.11, where the arguments rely on transversality properties of certain intersecting curves in the plane.

We refer to Sections 2.4-2.5 for the detailed proof.

The idea of using the twist property and the implicit function theorem to construct families of connecting orbits is not new. This has been used, for example, in the works [RR06] and [PdCRR13] to study the first-order variation of the rational caustics in elliptical billiard tables.

We now turn to the second step. In this step, we specialize to conformal deformations of the form (2.2). It shows that the preservation of a single rational torus already forces the deformation to be trivial.

Theorem 2.2 (Rigidity of a rational torus)**.** *Let $(g_\epsilon)_{|\epsilon|<1}$ be a conformal family of Riemannian metrics on $\mathbb{T}^2$ of the form*

$$\mathrm{d}s_\epsilon^2 = \Big(\Lambda(x_1, x_2) + \epsilon U(x_1, x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \tag{2.3}$$

where $\Lambda, U : \mathbb{T}^2 \to \mathbb{R}$ are smooth. Assume that, for all sufficiently small ϵ, the metric g_ϵ admits a rational torus, and that the common length of the closed geodesics on this torus is independent of ϵ. Then $U \equiv 0$.

The basic idea underlying the proof of Theorem 2.2 is to consider the *energy* $E_\epsilon(\gamma_\epsilon)$ of the unique closed geodesic γ_ϵ in the preserved rational torus which starts and ends at a fixed $x \in \mathbb{T}^2$, as a function of ϵ:

$$\epsilon \mapsto E_\epsilon(\gamma_\epsilon)$$

and observe that

$$E_\epsilon(\gamma_\epsilon) = \mathrm{const} \quad \text{and} \qquad \mathrm{d}_{\gamma_\epsilon} E_\epsilon \equiv 0.$$

[1] The function $\overline{\tau}_\epsilon(x)$ is an analogue of the so-called *loop functions* in billiards literature.

The first identity follows from the isospectrality assumption and the second identity holds because geodesics are critical points of the energy functional. Expanding the above conditions in powers of ϵ and combining (see (2.34)–(2.35)), we obtain a second-order condition:

$$d^2_{\gamma^{(0)}} E^{(0)}(\gamma^{(1)}, \gamma^{(1)}) = 0, \tag{2.4}$$

where we have used the notation

$$\gamma_\epsilon = \gamma^{(0)} + \epsilon\gamma^{(1)} + \epsilon^2\gamma^{(2)} + \mathcal{O}(\epsilon^3), \qquad E_\epsilon = E^{(0)} + \epsilon E^{(1)}.$$

This is the *only*, but crucial, use of the linearity assumption of the deformation, because a general deformation will produce a nontrivial term on the left side of (2.4) involving $E^{(2)}$. On the other hand, by minimality of $\gamma^{(0)}$ (see Proposition 2.11), the Hessian $d^2_{\gamma^{(0)}} E^{(0)}$ is positive semi-definite. This in turn forces $\mathrm{d}_{\gamma^{(0)}} E^{(1)} \equiv 0$. Comparing the Euler-Lagrange equations resulting from $\mathrm{d}_{\gamma^{(0)}} E^{(0)} \equiv 0$ and $\mathrm{d}_{\gamma^{(0)}} E^{(1)} \equiv 0$, we conclude that $U = 0$ along the unperturbed geodesic $\gamma^{(0)}$. Varying $x \in \mathbb{T}^2$ and using the foliation of $\mathbb{T}^2$ by closed geodesics of the rational torus, we obtain $U \equiv 0$.

The reader will find the detailed proof in Section 2.6.

Having discussed the intermediate Theorem 2.1 and Theorem 2.2, we now explain how Theorem A follows from these results.

Proof of Theorem A. Assume that a family $(g_\epsilon)_{|\epsilon|<1}$ of the form (1.2) is length-isospectral. Since g_0 is a Liouville metric, it is rationally integrable; see Section 2.3.3. By Theorem 2.1, the metric g_ϵ is rationally integrable for every $|\epsilon| < 1$.

Fix a pair $(m, n) \in (\mathbb{Z}^\times)^2$ of coprime integers. Then $(\mathbb{T}^2, g_\epsilon)$ has an (m, n)-rational torus. By Proposition 2.11 part 1, every geodesic in this rational torus is length-minimizing within the homology class. Thus, the length is equal to

$$\min \mathcal{L}^{m,n}(\mathbb{T}^2, g_\epsilon).$$

By Corollary 2.22, length-isospectrality implies that this quantity is independent of ϵ. Therefore the hypotheses of Theorem 2.2 are satisfied with

$$\Lambda(x_1, x_2) = 1 + f_1(x_1) + f_2(x_2).$$

It follows that $U \equiv 0$. This proves Theorem A. $\square$

Lastly, we discuss the idea underlying the proof of Theorem B, which characterizes marked-length-isospectral classes of Liouville metrics in terms of rearrangements.

As pointed out in the introductory section, our results are similar in spirit to the prior results of Abbondandolo-Mazzucchelli [AM26]. In the cited work, the authors proved, among other results, that the marked length spectrum determines the sphere of revolution with a single equator up to a rearrangement of the *profile function*. The authors of [AM26] consider a symplectic twist map given by a return map associated with the unique equator, and use its marked spectral data to determine its Mather's beta function which, in turn, determines the profile function via Abel's transform.

The method of [AM26] does not immediately extend to the setting of Liouville metrics, even in the rather similar setting where $f_2 = 0$ in (2.1). By the method of [AM26], the spectral data of geodesics that oscillate around a fixed elliptic closed geodesic (corresponding to a local maximum of f_1) can, in principle, determine the metric over a subregion of $\mathbb{T}^2$ bounded by certain hyperbolic geodesics (corresponding to a local minimum of f_1). However, unless f_1 has only two critical points, one needs to partition the length spectrum appropriately according to the oscillation intervals of such geodesics. But this information is generally unavailable from the usual definition of the length spectrum for $\mathbb{T}^2$ with or without marking.

We shall modify the idea of [AM26] and adopt a more "global" approach which uses only geodesics of homology classes (m, n) with $mn \neq 0$. In particular, the issues with multiple critical points and with nonconstant f_1 and f_2 can be handled in a unified way.

The key observation is that, via a separable Hamiltonian obtained from the *Maupertuis principle* (Section 2.3.3), the marked length-spectral data of a Liouville metric $(1 + f_1(x_1) + f_2(x_2))(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ can be encoded by a real-analytically embedded curve $\Gamma_{f_1,f_2} \subset (\mathbb{R}_{>0})^2$ that is given by the image of

$$(-\min f_1, 1 + \min f_2) \ni \; e \longmapsto \left(\int_{\mathbb{T}} \sqrt{e + f_1(x)} \mathrm{d}x, \int_{\mathbb{T}} \sqrt{1 - e + f_2(x)} \mathrm{d}x \right) \; \in \mathbb{R}^2 \quad (2.5)$$

It is straightforward to see that individual rearrangements of f_1 and f_2 do not change the curve Γ_{f_1,f_2}, and consequently, are always isospectral deformations. Conversely, if $(1 + \tilde{f}_1(x_1) + \tilde{f}_2(x_2))(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ is another, marked-length-isospectral, Liouville metric, then $\Gamma_{f_1,f_2} = \Gamma_{\tilde{f}_1,\tilde{f}_2}$. In particular, we show that the analogous map of (2.5) with f_j replaced by $\widetilde{f}_j$ must coincide with the map (2.5) up to precomposing a real-analytic reparametrization $\varphi : \mathbb{R} \to \mathbb{R}$. A closer analysis of φ shows that $z \mapsto \varphi(z)/z$ can be analytically extended to $\mathbb{C}$ as a bounded function, forcing it to be constant, and then the asymptotics at $|z| \to \infty$ shows $\varphi(z) = z$. It then follows from an application of Laplace transform that the function $\tilde{f}_j$ rearranges f_j up to an additive constant for $j = 1, 2$. We refer to Section 2.8 for the detailed proof.

2.1.2 Some further consequences

Trigonometric deformations Theorem 2.1 also has consequences for deformations with more general dependence on ϵ, at the cost of imposing additional structures in the variables (x_1, x_2). The following result combines Theorem 2.1 with Henheik's integrable-rigidity theorem 1.1.

Theorem 2.3. *Consider a family $(g_\epsilon)_{|\epsilon|<1}$ of metrics on $\mathbb{T}^2$ of the form*

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon U(x_1, x_2, \epsilon)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \quad (2.6)$$

where:

1. *f_1 and f_2 belong to the generic class of analytic functions appearing in Theorem 1.1;*
2. *U is analytic in (x_1, x_2, ϵ), and its Taylor expansion in ϵ,*

$$U(x_1, x_2, \epsilon) = \sum_{n \geq 0} \epsilon^n U_n(x_1, x_2),$$

has the property that each U_n is a trigonometric polynomial on $\mathbb{T}^2$.

Suppose that the family (g_ϵ) is length-isospectral. Then, for each sufficiently small ϵ, the metric g_ϵ is again a Liouville metric (obtained by rearrangements of g_0).

Indeed, the length-isospectral assumption implies, by Theorem 2.1, that the deformation preserves rational integrability. Under the additional trigonometric-polynomial assumptions, the argument of Henheik's theorem can be applied order by order in ϵ to obtain

$$U_n(x_1, x_2) = U_{n,1}(x_1) + U_{n,2}(x_2)$$

for every n. Hence g_ϵ remains within the class of Liouville metrics. Theorem B then describes g_ϵ in terms of rearrangements. We refer to [HKLV25] for further details.

Laplace spectral rigidity We also record a Laplace-spectral analogue of Theorem A. This result is closer in spirit to Kac's question discussed in Section 1.2. Let Δ_g denote the Laplace–Beltrami operator of $(\mathbb{T}^2, g)$, and let

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the spectrum of $-\Delta_g$, counted with multiplicity.

Theorem 2.4. *Let $(g_\epsilon)_{|\epsilon|<1}$ be a family of Riemannian metrics on $\mathbb{T}^2$ of the form*

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon U(x_1, x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2).$$

Suppose that g_0 belongs to an open dense class of Liouville metrics satisfying a noncoincidence condition described below. If the Laplace spectrum of $(\mathbb{T}^2, g_\epsilon)$ is independent of ϵ, then $U \equiv 0$.

The additional open dense condition on the original metric g_0 is imposed in order to rule out possible cancellations in the wave trace. Namely, we require that g_0 admit at least one rational torus whose length T does not coincide with the length of any closed geodesic in an axial homology class $(k, 0)$ or $(0, k)$. [2]

The proof of Theorem 2.4 combines the preceding rigidity argument with results obtained by microlocal analysis. The analytical part of the proof goes beyond the expertise of the author and the scope of this thesis. We shall nevertheless explain in Section 2.7 some dynamical and geometrical ingredients in the proof. We refer to [HKLV25] instead for the full proof.

2.2 Preliminaries on the length spectrum

Throughout this chapter, we use Φ_g^t to denote the geodesic flow on $T\mathbb{T}^2$ associated with $(\mathbb{T}^2, g)$. The flow Φ_g^t naturally restricts to a flow on the unit tangent bundle $S\mathbb{T}^2$ with respect to the given metric g.

We begin with some basic notions about the length spectrum.

Recall that our basic object is the collection of the lengths of all closed geodesics of $(\mathbb{T}^2, g)$, which we denote by $\mathcal{L}(\mathbb{T}^2, g)$. We also note that every closed geodesic of $(\mathbb{T}^2, g)$ can be uniquely associated with a homology class parametrized by the homology group $H_1(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}^2$. Therefore, we may write

$$\mathcal{L}(\mathbb{T}^2, g) = \bigcup_{m,n \in \mathbb{Z}} \mathcal{L}^{m,n}(\mathbb{T}^2, g), \tag{2.7}$$

where $\mathcal{L}^{m,n}(\mathbb{T}^2, g) \subset \mathbb{R}^+$ denotes the collection of the lengths of all closed geodesics belonging to the homology class $(m, n) \in \mathbb{Z}^2$.

[2] We remind again that, by our terminology, a rational torus is required to be a graph over $\mathbb{T}^2$. For Liouville metrics, geodesics not lying in a rational torus must be in homology classes $(k, 0)$ and $(0, k)$. These geodesics also form invariant tori in the tangent bundle but, in general, such tori do not project diffeomorphically to $\mathbb{T}^2$.

Remark. The labelling of homology classes by integer pairs (m, n) is *not* canonical. This labelling depends on the choice of the generators of the homology group $H_1(\mathbb{T}^2, \mathbb{Z})$. For concreteness, however, we will consider $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and fix the labelling such that a closed curve in homology class (m, n) lifts to the covering space $\mathbb{R}^2$ which connects a point $(x, y) \in \mathbb{R}^2$ to $(x + m, y + n) \in \mathbb{R}^2$.

Generally, even for a fixed pair (m, n), the subset $\mathcal{L}^{m,n}(\mathbb{T}^2, g)$ could already exhibit rather complicated structure. However, by a classical result in the calculus of variations, the infimum

$$\ell_g^{m,n} := \inf \mathcal{L}^{m,n}(\mathbb{T}^2, g)$$

is always attained by at least one closed geodesic in the homology class, which is customarily called a *minimal geodesic*.[3]

We may thus define the *marked length spectrum* of $(\mathbb{T}^2, g)$ to be a map

$$(m, n) \mapsto \ell_g^{m,n}. \tag{2.8}$$

In fact, the marked length spectrum is determined by its restriction to the *primitive* elements of the lattice $\mathbb{Z}^2$: it follows from a standard argument of the Aubry-Mather theory on $\mathbb{T}^2$ (see for example [Ban88, Ch. 3]) that every minimal geodesic with homology class (km, kn) is given by k-times iteration of a minimal closed geodesic with homology class (m, n). Hence, the map (2.8) is fully determined by its values on the coprime pairs of integers (m, n). [4]

We finish this section by noting the following elementary continuity property of the marked length spectrum.

Lemma 2.5. *For each fixed homology class (m, n), the minimal length $\ell_g^{m,n}$ depends continuously on g in the C^0-topology.*

2.3 Integrability and rational integrability

As a classical result, the geodesic flow of a Liouville metric on $\mathbb{T}^n$ is *integrable* in the sense of Arnold-Liouville. For readers' convenience, we recall this classical notion in the setting of Hamiltonian dynamical systems.

2.3.1 Integrable Hamiltonian dynamics

A general case. Hamiltonian dynamical systems were first introduced by Hamilton [Ham33] as an abstract reformulation of classical mechanics. In modern language, a Hamiltonian dynamical system consists of a symplectic manifold (M, ω) and a *Hamiltonian function* $H : M \to \mathbb{R}$. The pair (M, ω) is given by an even-dimensional differentiable manifold M together with an everywhere nondegenerate differential 2-form ω on M, which is usually called the *symplectic form*. The *Hamiltonian vector field* X_H generated by H is defined by the following equation

$$\mathrm{d}H = i_{X_H}\omega \tag{2.9}$$

[3]When $m = n = 0$, we may formally define $\ell_g^{0,0} = 0$ since a point may be regarded as a closed geodesic with length 0.

[4]We remark that the set of coprime pairs of integers includes $(0, \pm 1)$ and $(\pm 1, 0)$.

where $\mathrm{d}H$ is the exterior derivative of H and i_{X_H} stands for the tensor contraction by the vector field X_H. The *Hamiltonian flow* generated by H is defined to be the flow on M under the vector field X_H.

It is easy to show from the definition that H is always invariant under the Hamiltonian flow it generates. Thus, the level sets of H, which are usually called the *energy surfaces*, are invariant under the dynamics.

An integrable system is characterized by having sufficiently many independent conserved quantities such that the dynamics restricted to the joint level sets of these conserved quantities is conjugate to simple models. We formulate this notion precisely.

Definition 2.6. A Hamiltonian dynamical system on the $2n$-dimensional symplectic manifold (M, ω) is called *integrable in the sense of Arnold-Liouville*, or just *integrable* for short, if there exist n functions $F_1, \dots, F_n$ on M (called *first integrals*) such that

1. they are functionally independent on M, i.e., for points x belonging to an open and dense set of M, the 1-forms $\mathrm{d}F_j$, $j = 1, \dots, n$ form a linearly independent set in the cotangent space $T^*_x M$;

2. they are pairwise in involution, i.e., for $1 \leq i \leq j \leq n$,

$$\{F_i, F_j\} := \omega(X_{F_i}, X_{F_j}) = 0. \tag{2.10}$$

Remark. We may always choose $F_1 = H$. The quantity $\{F_i, F_j\}$ in the condition (2.10) is usually called the *Poisson bracket* of F_i and F_j with respect to ω.

The following well-known theorem by Arnold-Liouville (see for example [Arn89, Ch. 10]) gives a simple description of the dynamics restricted to the joint invariant set of the first integrals in Definition 2.6.

Theorem 2.7 (Arnold-Liouville)**.** *Consider an integrable Hamiltonian dynamical system in the sense of Definition 2.6 with $F_1 = H$. For $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{R}^n$, set*

$$M_{\boldsymbol{f}} = \{x \in M \ : \ F_i(x) = f_i,\, i = 1, \cdots, n\}.$$

Suppose the 1-forms $\mathrm{d}F_j$, $j = 1, \dots, n$, are linearly independent at each point of $M_{\boldsymbol{f}}$. Then,

1. *The level set $M_{\boldsymbol{f}}$ is a smooth submanifold of dimension n that is invariant under the Hamiltonian flow;*

2. *If $M_{\boldsymbol{f}}$ is compact and connected, then it is diffeomorphic to an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$;*

3. *In the case of 2., there exist local coordinates $(\theta, I) : U \to \mathbb{T}^n \times \mathbb{R}^n$ defined in a neighborhood U of $M_{\boldsymbol{f}}$ which is symplectic with respect to ω and the standard symplectic form $\sum_{j=1}^n \mathrm{d}\theta_j \wedge \mathrm{d}I_j$ on $\mathbb{T}^n \times \mathbb{R}^n$. Moreover, in these coordinates the Hamiltonian function H depends only on I and the Hamiltonian flow is given by the system of differential equations*

$$\dot{I} = 0, \qquad \dot{\theta} = \frac{\partial H}{\partial I}(I), \tag{2.11}$$

where the dot denotes the derivative with respect to time.

In particular, for an integrable Hamiltonian dynamical system, an open dense subset of M is foliated by invariant submanifolds of the form $M_{\boldsymbol{f}}$ as in Theorem 2.7, and the restriction of the dynamics on compact connected invariant submanifolds is conjugate to a simple rotation on $\mathbb{T}^n$ described by (2.11).

The case of geodesic flows. Recall that the geodesic flow of a Riemannian manifold (X, g) can be formulated as a Hamiltonian flow on the cotangent bundle T^*X. We briefly recall this standard construction; see, for instance, Abraham–Marsden [AM87, Ch. 3].

Let

$$L : TX \to \mathbb{R}, \qquad L(x, v) = \frac{1}{2}|v|_g^2$$

be the kinetic-energy Lagrangian. Its fibre derivative, or Legendre transform, is the map

$$\mathcal{L} : TX \to T^*X, \qquad \mathcal{L}(x, v) = g_x(v, \cdot).$$

Since g is positive definite, $\mathcal{L}$ is a smooth fibrewise linear diffeomorphism. The Hamiltonian associated with L is obtained by the Legendre transform:

$$H(x, \xi) = \sup_{v \in T_xX} \Big(\xi(v) - L(x, v)\Big) = \frac{1}{2}|\xi|_{g^{-1}}^2,$$

where g^{-1} denotes the dual metric on T^*X.

Equipping T^*X with its canonical symplectic form, the Hamiltonian flow generated by H is conjugate, via $\mathcal{L}$, to the geodesic flow on TX. In other words, if $(x(t), v(t))$ is a trajectory of the geodesic-flow on TX, then

$$(x(t), \xi(t)) := \mathcal{L}(x(t), v(t))$$

is a trajectory of the Hamiltonian flow of H on T^*X, and conversely.

In local coordinates $(x^1, \ldots, x^n)$ on X, writing

$$g = g_{ij}\, dx^i \otimes dx^j, \qquad (g^{ij}) = (g_{ij})^{-1},$$

the Hamiltonian takes the form

$$H(x, \xi) = \frac{1}{2}\sum_{i,j} g^{ij}(x)\xi_i\xi_j. \tag{2.12}$$

Thus the geodesic flow may be studied as the Hamiltonian flow generated by this function. In view of this correspondence, we shall say that a Riemannian metric g is integrable if its associated Hamiltonian system is integrable in the sense of Arnold-Liouville.

2.3.2 Rational integrability

While we have recalled the notion of Arnold-Liouville integrability in Section 2.3.1 for motivational purposes, we nevertheless shall not deal with this classical notion directly in the current chapter. Instead, we shall work with a slightly different notion of "rational integrability".

Roughly speaking, the geodesic flow is rationally integrable if every closed orbit is contained in a *rational torus*, namely, a family of closed geodesics that smoothly foliates an invariant torus in the unit tangent bundle with the condition that the invariant torus projects diffeomorphically onto the base manifold. This notion has been employed in the works on Birkhoff conjecture such as [ADSK16], [KS18] and [Kov26], where the notion of rational tori corresponds to *resonant convex caustics*.

Let us now formally define this notion in the setting of geodesic flows on $\mathbb{T}^2$.

Definition 2.8. Let g be a Riemannian metric on $\mathbb{T}^2$ and $s : \mathbb{T}^2 \to S\mathbb{T}^2$ be a continuous section of the unit tangent bundle of $(\mathbb{T}^2, g)$. We call the image $\mathcal{T} = \mathrm{Im}(s)$ a *rational torus* of $(\mathbb{T}^2, g)$ if $\mathcal{T}$ is invariant under the geodesic flow Φ_g^t and there exists $T > 0$ such that every point of $\mathcal{T}$ is periodic under the geodesic flow with period T. We define the *period* of $\mathcal{T}$ to be the minimal $T > 0$ for this to hold. We define the *homology class* of $\mathcal{T}$ to be the homology class $[\gamma] \in H_1(\mathbb{T}^2, \mathbb{Z})$ of any closed geodesic $\gamma : [0, T] \to \mathbb{T}^2$ which corresponds to a period-T closed orbit in $\mathcal{T}$.

We remark that since all closed geodesics of a fixed rational torus are homotopy equivalent as closed curves in $\mathbb{T}^2$, the homology class of a rational torus $\mathcal{T}$ specified in Definition 2.8 does not depend on the particular choice of the closed orbit in $\mathcal{T}$.

Although we do not *a priori* assume any primitivity of the homology class of the rational torus in Definition 2.8, it will later follow from Proposition 2.11 (particularly the minimality property in part 1) that the homology class of $\mathcal{T}$ is always a *primitive* element of $H_1(\mathbb{T}^2, \mathbb{Z})$, i.e., it is not a positive integer multiple of a different element in $H_1(\mathbb{T}^2, \mathbb{Z})$. Under the identification $\mathbb{Z}^2 \simeq H_1(\mathbb{T}^2, \mathbb{Z})$, the primitive elements (m, n) are precisely the coprime pairs of integers.

Definition 2.9. We say $(\mathbb{T}^2, g)$ is *rationally integrable* if there is an identification $H_1(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}^2$ such that for every coprime pair of integers (m, n) with $mn \neq 0$ there exists a rational torus of $(\mathbb{T}^2, g)$ with homology class (m, n).

For convenience, when a rational torus has homology class (m, n), we shall call it an (m, n)-*rational torus* or simply an (m, n)-*torus*. We will show in Proposition 2.11 that if an (m, n)-rational torus exists then it is always unique.

Remark. Definition 2.9 deliberately excludes rational tori of homology classes $(1, 0)$ and $(0, 1)$. This is because, for a typical Liouville metric (and with the usual identification $H_1(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}^2$ described in the remark in Section 2.2), the invariant tori in these homology classes do not project diffeomorphically onto $\mathbb{T}^2$. In our later treatment of the deformations of Liouville metrics, it is therefore convenient to consider these geodesics separately from those in the rational tori in the sense of Definition 2.8.

Remark. To the author's best knowledge, there is no clear relation between rational integrability and integrability in the sense of Arnold-Liouville. We explain the subtleties underlying these two notions. Consider an integrable metric in the sense of Arnold-Liouville. Assume for simplicity that an open dense set of TX is foliated by invariant tori diffeomorphic to $\mathbb{T}^2$ and a dense collection of these tori are further foliated by closed orbits. The problem, however, is that it is unclear under what conditions these invariant tori project diffeomorphically onto the base space $\mathbb{T}^2$ and hence constitute *rational tori* in the sense of Definition 2.8. Conversely, if a metric is rationally integrable, then it is not hard to show that a "large" open subset of TX is foliated by invariant tori. However, this foliation is *a priori* only $\mathcal{C}^0$-regular. Indeed, if the foliation fails to be smooth, then it cannot be formed by regular level sets of first integrals in the sense of Theorem 2.7.

2.3.3 Maupertuis principle and the integrability of Liouville metrics

The main goal of this section is to illustrate the dynamical properties of Liouville metrics. In particular, we explain a close analogy between Liouville metrics and the pendulum – a standard model of integrable systems. In addition, we derive some useful formulae for the lengths of closed geodesics of a Liouville metric which will be used later on.

Maupertuis principle. To this end, we shall first recall the *Maupertuis principle*, a useful tool to describe geodesics of Liouville metrics. We begin with a general statement. Take a compact Riemannian manifold $(X, \widetilde{g})$ and consider a mechanical Hamiltonian function on T^*M as follows.

$$H(x,\lambda) = \frac{1}{2}|\lambda|^2_{\widetilde{g}^{-1}} - V(x), \qquad (\lambda \in T^*_x X), \tag{2.13}$$

where $V \in \mathcal{C}^2(X)$ and $|\cdot|_{\widetilde{g}^{-1}}$ is the dual metric induced by $\widetilde{g}$. Hamiltonian functions of the form (2.13) are customarily referred to as *mechanical Hamiltonians* and the function V is called a potential function. In physics, the first term in (2.13) corresponds to the kinetic energy and the second term to the potential energy. Now, let $T_h = H^{-1}(\{h\})$ be an energy surface for some $h > -\min_x V(x)$. We then note that T_h is also an energy surface for another Hamiltonian function given by

$$\widetilde{H}(x,\lambda) = \frac{1}{2}\frac{|\lambda|^2_{\widetilde{g}^{-1}}}{h + V(x)}. \tag{2.14}$$

That is, we have $T_h = \widetilde{H}^{-1}(\{1\})$. Now, the *Maupertuis principle* states that the Hamiltonian flows generated by $\widetilde{H}$ and H have the same integral curves on T_h. More precisely, the orbits of the two Hamiltonian flows on T_h can differ at most by reparametrizations of time.

Case of Liouville metrics. To apply the Maupertuis principle to geodesic flows, we note that the vector field generated by (2.14) gives rise to the geodesic flow of the Riemannian metric g with

$$g = (h + V)\widetilde{g}\,.$$

See (2.12). We will use this for

$$X = \mathbb{T}^2, \qquad h = 1, \qquad V(x) = f_1(x_1) + f_2(x_2)$$

and $\widetilde{g}$ given by the standard Euclidean metric $\mathrm{d}x_1^2 + \mathrm{d}x_2^2$ on $\mathbb{T}^2$ induced from $\mathbb{R}^2$. We may identity $T^*\mathbb{T}^2 \simeq \mathbb{T}^2_x \times \mathbb{R}^2_\xi$ with the standard symplectic form given by $\omega = \sum_{j=1,2} \mathrm{d}x_j \mathrm{d}\xi_j$. By the Maupertuis principle, the geodesics of the Liouville metric

$$(1 + f_1(x_1) + f_2(x_2))(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$$

are determined up to time reparametrization by the mechanical Hamiltonian

$$H(x,\xi) = \frac{|\xi|^2}{2} - f_1(x_1) - f_2(x_2) \tag{2.15}$$

restricted to the energy surface $T_h = \{H(x,\xi) = 1\}$.

Having the Hamiltonian function (2.15) in hand, we can easily analyze the geodesics on $\mathbb{T}^2$ with a Liouville metric. In fact, the Hamiltonian (2.15) is the sum of two Hamiltonians $H_i(x_i,\xi_i) = \xi_i^2/2 - f_i(x_i)$ for $i = 1, 2$, each defined on a phase cylinder $\mathbb{T} \times \mathbb{R}$.

Analogy with the pendulum. The Liouville geodesic flow has a close analogy with the classical pendulum-type system. Recall that, in classical mechanics, the phase portrait of a pendulum is described by a Hamiltonian of the form

$$k(x, \xi) = \frac{\xi^2}{2} + a\cos(2\pi x), \qquad a > 0,$$

defined on the phase cylinder $\mathbb{T} \times \mathbb{R}$. There is a unique hyperbolic fixed point $(0, 0)$ of energy a. The energy surface $\{k = a\}$ is called the *separatrix* and consists of two orbits asymptotic to $(0, 0)$. The other energy levels of k have two qualitatively different types. For energies above the separatrix, the coordinate $x(t)$ winds monotonically around the circle; these are the rotational motions. For energies below the separatrix, the coordinate $x(t)$ oscillates between two turning points; these are the so-called librational motions. See Figure 2.1

The one-dimensional Hamiltonians

$$H_i(x_i, \xi_i) = \frac{\xi_i^2}{2} - f_i(x_i), \qquad i = 1, 2,$$

which arise from the Liouville metric have exactly the same phase-space structure, with the function $-f_i$ playing the role of the pendulum potential $a\cos(2\pi x)$. Thus the Hamiltonian

$$H(x, \xi) = H_1(x_1, \xi_1) + H_2(x_2, \xi_2)$$

may be regarded as the product of two uncoupled pendulum-type systems. The conserved quantities H_1 and H_2 determine the motion in each coordinate separately, while the constraint

$$H_1 + H_2 = 1$$

selects the energy surface corresponding, via the Maupertuis principle, to the unit-speed geodesic flow of the Liouville metric.

This viewpoint explains the two types of behavior that appear below. If the energy level of H_i lies above the maximum of the effective potential $-f_i$, then $x_i(t)$ winds monotonically around $\mathbb{T}$. If the energy level lies between the minimum and maximum of $-f_i$, then $x_i(t)$ oscillates between turning points. Consequently, closed geodesics of a Liouville metric are obtained by matching the periods of the two one-dimensional motions: either both coordinates rotate, giving geodesics in homology classes (m, n) with $mn \neq 0$, or one coordinate oscillates while the other rotates, giving geodesics in a homology class of the type $(0, n)$ or $(m, 0)$.

Guided by this heuristic picture, we give a more precise description of the closed geodesics in the next paragraph.

Rational integrability of Liouville metrics and some formulae Recall that the Hamiltonian (2.15) is the sum of two Hamiltonians $H_1 + H_2$ with $H_i(x_i, \xi_i) = \xi_i^2/2 - f_i(x_i)$. The Hamiltonian flow on $\mathbb{T} \times \mathbb{R}$ generated by an individual Hamiltonian H_i is described by the following ODE.

$$\begin{cases} x_i'(t) = \xi_i(t) \\ \xi_i'(t) = f_i'(x_i(t)) \end{cases} \qquad \text{for} \quad i = 1, 2. \tag{2.16}$$

Hence, the projections of orbits of (2.15) onto $\mathbb{T}^2$ are solutions of the uncoupled ODEs

$$\begin{cases} x_1'(t) = \pm\sqrt{2(e + f_1(x_1(t)))} \\ x_2'(t) = \pm\sqrt{2(1 - e + f_2(x_2(t)))} \end{cases} \tag{2.17}$$

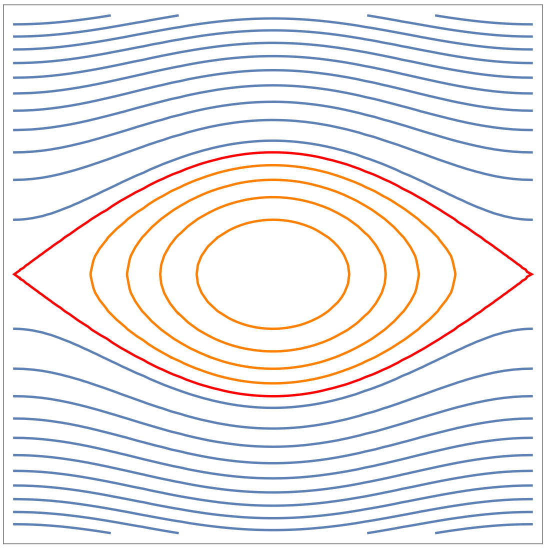

Figure 2.1: The phase portrait of a pendulum. The separatrix is shown in red. The energy levels corresponding to the rotational motions are displayed in blue whereas those corresponding to the librational motions are displayed in orange.

where $e \in \mathbb{R}$ is a conserved quantity (with $\xi_1^2/2 - f_1(x_1) = e$).

In particular, for geodesics in a (m_1, m_2)-rational torus, the corresponding periodic solution $(x_1(t), x_2(t))$ of (2.17) satisfies $x_j'(t) \neq 0$ and $x_j(\widetilde{T}) - x_j(0) = m_j$ (for any lift to the universal cover), where

$$\widetilde{T} = |m_1| \int_{\mathbb{T}} \frac{\mathrm{d}x_1}{\sqrt{2(e + f_1(x_1))}} = |m_2| \int_{\mathbb{T}} \frac{\mathrm{d}x_2}{\sqrt{2(1 - e + f_2(x_2))}} \tag{2.18}$$

is the period of the solution $(x_1(t), x_2(t))$ to the ODE (2.17). The relation (2.18) uniquely determines $e \in \mathbb{R}$ in terms of (m_1, m_2).

On the other hand, for the periodic solution $(x_1(t), x_2(t))$ of (2.17) corresponding to a geodesic in the homology class $(0, m)$, the coordinate $x_1(t)$ oscillates through a connected component $[x_1^{\min}, x_1^{\max}]$ of $\{x \in \mathbb{T} \mid e + f_1(x) > 0\}$. In this case, either $x_1^{\min} = x_1^{\max}$, forcing $x_1'(t) = 0$ and thus $f_1'(x_1) = 0$ by (2.16); or else $x_1^{\min} < x_1^{\max}$ and we have, similar to (2.18), the time-matching condition

$$\widetilde{T} = 2n \int_{x_1^{\min}}^{x_1^{\max}} \frac{\mathrm{d}x_1}{\sqrt{e + f_1(x_1)}} = |m| \int_{\mathbb{T}} \frac{\mathrm{d}x_2}{\sqrt{1 - e + f_2(x_2)}}$$

for the number $n \in \mathbb{Z}_{>0}$ of oscillations of $x_1(t)$ between $x_1^{\min}$ and $x_1^{\max}$ over one period $\widetilde{T}$. An analogous formula holds, of course, for homology class $(m, 0)$.

Next, we calculate the length and the energy of geodesics of Liouville metrics via the Maupertuis principle. Let us start with a general solution $(x_1(t), x_2(t))$ of (2.17) with some choice of e. From the definition, the Riemannian length of such orbits over an arbitrary time interval $[0, \widetilde{T}]$

is

$$\begin{aligned}&\int_0^{\widetilde{T}} \sqrt{(1+f_1(x_1(t))+f_2(x_2(t)))(x_1'(t)^2+x_2'(t)^2)}\mathrm{d}t\\ =&\int_0^{\widetilde{T}} \sqrt{2}(1+f_1(x_1(t))+f_2(x_2(t)))\mathrm{d}t\\ =&\int_0^{\widetilde{T}} \sqrt{2}(e+f_1(x_1(t)))\mathrm{d}t+\int_0^{\widetilde{T}} \sqrt{2}(1-e+f_2(x_2(t)))\mathrm{d}t.\end{aligned}$$

Suppose for example $x_j'(t)>0$. Then the changes of variables

$$\frac{\mathrm{d}x_1}{\sqrt{2(e+f_1(x_1))}}=\mathrm{d}t, \qquad \frac{\mathrm{d}x_2}{\sqrt{2(1-e+f_2(x_2))}}=\mathrm{d}t$$

give

$$\int_0^{\widetilde{T}} \sqrt{2}\left(e+f_1(x_1(t))\right)\mathrm{d}t=\sqrt{2}\int_{x_1(0)}^{x_1(\widetilde{T})} \frac{e+f_1(x_1)}{\sqrt{2(e+f_1(x_1))}}\,\mathrm{d}x_1=\int_{x_1(0)}^{x_1(\widetilde{T})} \sqrt{e+f_1(x_1)}\,\mathrm{d}x_1. \tag{2.19}$$

Similarly,

$$\int_0^{\widetilde{T}} \sqrt{2}\left(1-e+f_2(x_2(t))\right)\mathrm{d}t=\int_{x_2(0)}^{x_2(\widetilde{T})} \sqrt{1-e+f_2(x_2)}\,\mathrm{d}x_2. \tag{2.20}$$

It follows that the length of the orbit over $[0,\widetilde{T}]$ is given by

$$\int_{x_1(0)}^{x_1(\widetilde{T})} \sqrt{e+f_1(x_1)}\,\mathrm{d}x_1+\int_{x_2(0)}^{x_2(\widetilde{T})} \sqrt{1-e+f_2(x_2)}\,\mathrm{d}x_2.$$

If $(x_1(t),x_2(t))$ is a periodic solution lying in the homology class $(m_1,m_2)\in(\mathbb{Z}\setminus\{0\})^2$ with period $\widetilde{T}$, then its Riemannian length over one period is simply

$$|m_1|\int_{\mathbb{T}} \sqrt{e+f_1(x_1)}\,\mathrm{d}x_1+|m_2|\int_{\mathbb{T}} \sqrt{1-e+f_2(x_2)}\,\mathrm{d}x_2$$

where e is determined by (2.18). Analogous statements hold, of course, for homology $(m,0)$ and $(0,m)$. We now summarize.

Lemma 2.10. *Let $g=(1+f_1(x_1)+f_2(x_2))(\mathrm{d}x_1^2+\mathrm{d}x_2^2)$ be a Liouville metric. Then:*

- *For each $(m_1,m_2)\in(\mathbb{Z}\setminus\{0\})^2$, there exists a unique $e\in(-\min f_1, 1+\min f_2)$ satisfying*

$$|m_1|\int_0^1 \frac{\mathrm{d}x_1}{\sqrt{e+f_1(x_1)}}=|m_2|\int_0^1 \frac{\mathrm{d}x_2}{\sqrt{1-e+f_2(x_2)}} \tag{2.21}$$

  such that the length of every closed geodesic in an (m_1,m_2)-rational torus is

$$|m_1|\int_{\mathbb{T}} \sqrt{e+f_1(x_1)}\mathrm{d}x_1+|m_2|\int_{\mathbb{T}} \sqrt{1-e+f_2(x_2)}\mathrm{d}x_2. \tag{2.22}$$

- *For $m\in\mathbb{Z}\setminus\{0\}$, every closed geodesic in the homology class $(0,m)$ belongs to one of the two following types:*

– *There exists a critical point $x_1 \in \mathbb{T}$ of f_1 such that the closed geodesic is contained in $\{x_1\} \times \mathbb{T}$. In this case, the length of the closed geodesic is given by*

$$|m| \int_{\mathbb{T}} \sqrt{1 + f_1(x_1) + f_2(x_2)} \mathrm{d}x_2 \tag{2.23}$$

– *There exists $n \in \mathbb{Z}_{>0}$ and $e \in (-\max f_1, -\min f_1)$ satisfying*

$$2n \int_{x_1^{\min}}^{x_1^{\max}} \frac{\mathrm{d}x_1}{\sqrt{e + f_1(x_1)}} = |m| \int_{\mathbb{T}} \frac{\mathrm{d}x_2}{\sqrt{1 - e + f_2(x_2)}} \tag{2.24}$$

where $(x_1^{\min}, x_1^{\max})$ is a connected component of $\{x \mid e + f_1(x) > 0\}$. In particular, e is not a critical value of $-f_1$. In this case, the closed geodesic is contained in $[x_1^{\min}, x_1^{\max}] \times \mathbb{T}$ and its length is given by

$$2n \int_{x_1^{\min}}^{x_1^{\max}} \sqrt{e + f_1(x_1)} \mathrm{d}x_1 + |m| \int_{\mathbb{T}} \sqrt{1 - e + f_2(x_2)} \mathrm{d}x_2. \tag{2.25}$$

- *The symmetric statement holds for closed geodesics in the homology class $(m, 0)$.*

2.4 Dynamical properties of the rational torus

In this section, we collect some properties of rational tori, which will be used in the proof of Theorem A.

We point out that the results in this section do not require the Riemannian metric to be a Liouville metric. In particular, these results apply to *any* Riemannian metric on $\mathbb{T}^2$ admitting a rational torus.

On the other hand, the proof of the main result in this section, namely Proposition 2.11 below, relies crucially on the dimension of the torus being exactly 2. In fact, we do not know if our results can be generalized to $\mathbb{T}^d$ with $d \geq 3$. This is the main reason why we have to limit Theorem A to $\mathbb{T}^2$ instead of higher dimensional tori.

Proposition 2.11 is motivated by several well-known properties of convex caustics in billiards and, more generally, the invariant curves of symplectic twist maps. For readers' convenience, we recall the following properties for *essential* invariant curves of an exact symplectic twist map.[5]

1. all essential invariant curves are Lipschitz graphs over $\mathbb{T}$;
2. distinct essential invariant curves do not intersect and the set of all such curves is totally ordered by their *rotation numbers*;
3. if an essential invariant curve consists of periodic orbits with period $q > 0$, then the q-th iterate of the map has the twist property on the invariant curve and, moreover, the curve is as regular as the map itself.

[5] We say an invariant curve is *essential* if it is not homotopically trivial, i.e., not a contractible curve in the cylinder.

Statement 1. above is a classical theorem due to Birkhoff. Statement 2. and 3. are rather elementary. See for example [PdCRR13, Lemma 2.1] for a result in the context of billiard maps.

In the higher dimensional setting, Arnaud-Massetti-Sorrentino [AMS23] considered *Lagrangian periodic graphs* for symplectic twist maps on $\mathbb{T}^d \times \mathbb{R}^d$ for all $d \geq 2$ and, as intermediate results, established properties analogous to the invariant curves listed above.

The statement of Proposition 2.11 below mirrors [AMS23, Section 2], but these two results deal with slightly different scenarios. Particularly, the cited work assumes the symplectic map satisfies a *global twist* condition, while, in our set-up, the global twist does not hold unless the metric is flat. As mentioned above, it is also unclear if Proposition 2.11 can be generalized to higher dimensions.

Recall that we use Φ_g^t to denote the geodesic flow on $T\mathbb{T}^2$ associated with $(\mathbb{T}^2, g)$. The flow Φ_g^t naturally restricts to a flow on the unit tangent bundle $S\mathbb{T}^2$ with respect to the given metric g.

Proposition 2.11. *Suppose $\mathcal{T} = \mathrm{Im}(s)$ is a rational torus of $(\mathbb{T}^2, g)$ corresponding to a $\mathcal{C}^0$ section $s : \mathbb{T}^2 \to S\mathbb{T}^2$. Then the following properties hold.*

1. *$\mathcal{T}$ contains exactly all unit-speed closed geodesics in its homology class. In particular, every closed geodesic in $\mathcal{T}$ is length minimizing within that homology class.*

2. *The section s is Lipschitz; moreover, its Lipschitz constant depends upper semicontinuously on g with respect to the $\mathcal{C}^2$-topology.*

3. *("Twist property") Let $T > 0$ be the common period of orbits in $\mathcal{T}$. For every $x \in \mathbb{T}^2$, the derivative at $u = 0$ of the map*

$$\begin{aligned} T_x\mathbb{T}^2 &\longrightarrow \mathbb{T}^2, \\ u &\longmapsto \pi \circ \Phi_g^T\big(s(x) + u\big) \end{aligned} \tag{2.26}$$

has full rank (i.e. rank 2), where we have denoted by $T_x\mathbb{T}^2$ the tangent space of $\mathbb{T}^2$ at x and π the canonical projection $T\mathbb{T}^2 \to \mathbb{T}^2$.

Proof. Lift g to the universal cover $\mathbb{R}^2$; we keep the same letter g for the lifted $\mathbb{Z}^2$-periodic metric. Then s lifts to a $\mathbb{Z}^2$-periodic section of $S\mathbb{R}^2$. For $x \in \mathbb{R}^2$, let $\gamma_x : \mathbb{R} \to \mathbb{R}^2$ denote the (unit-speed) geodesic with $\gamma_x(0) = x$ and $\gamma_x'(0) = s(x)$. By the definition of $\mathcal{T}$, the closed geodesic γ_x of $(\mathbb{T}^2, g)$ lifts to a curve belonging to the rational torus $\mathcal{T}$ and there exist $T > 0$ and $(m, n) \in \mathbb{Z}^2$ such that $\gamma_x(T) = (x_1 + m, x_2 + n)$ for all $x \in \mathbb{R}^2$; the class (m, n) is the homology class of $\mathcal{T}$.

Proof of (1). We claim the following:

$$\text{If a geodesic } \gamma \text{ of } (\mathbb{R}^2, g) \text{ intersects } \gamma_x \text{ at two distinct points, then } \mathrm{Im}(\gamma) = \mathrm{Im}(\gamma_x). \tag{2.27}$$

This statement [6] will imply part (1) of the proposition because, if $\gamma : \mathbb{R} \to \mathbb{R}^2$ lifts a geodesic of $(\mathbb{T}^2, g)$ of the same homology class as $\mathcal{T}$, then, for any $x \in \mathrm{Im}(\gamma)$, the curve γ intersects

[6] We emphasize that γ is said to *intersect* $\tilde{\gamma}$ as long as $\mathrm{Im}(\gamma) \cap \mathrm{Im}(\tilde{\gamma}) \neq \varnothing$, even though γ and $\tilde{\gamma}$ may not reach the point of intersection at the same time.

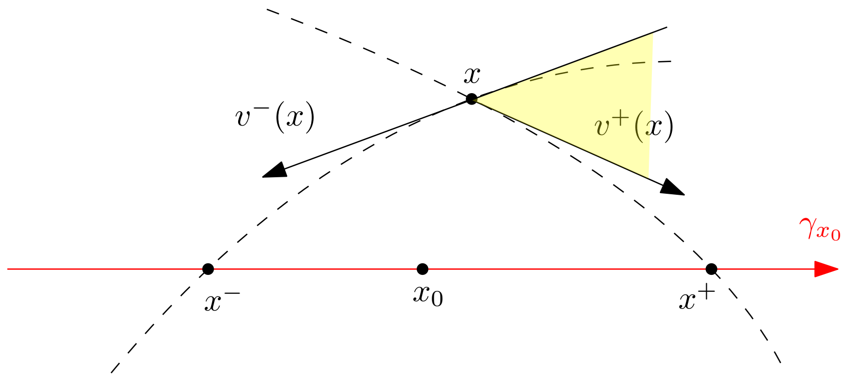

Figure 2.2: Proving the Lipschitz property of rational tori

γ_x at x. Since both $\mathrm{Im}(\gamma)$ and $\mathrm{Im}(\gamma_x)$ are invariant under the translation by (m,n), they intersect infinitely many times; hence by (2.27) they coincide up to reparametrization, so the closed geodesic belongs to $\mathcal{T}$.

We now prove (2.27). Let us suppose that a geodesic γ intersects γ_{x_0} more than once for some $x_0 \in \mathbb{R}^2$. If $\mathrm{Im}(\gamma) \neq \mathrm{Im}(\gamma_{x_0})$, then the two curves are not tangent at the points of their intersections, and by dimensionality their intersections must be transverse. We may hence assume that two adjacent intersections happen at $\gamma(T_1)$ and $\gamma(T_2)$ for some $T_1 < T_2$. By our choice, the segment $\gamma((T_1,T_2))$ does not intersect $\mathrm{Im}(\gamma_{x_0})$. Consider the following subsets of $\mathbb{R}^2$.

$$H_1 = \{x \in \mathbb{R}^2 \mid \gamma_x \text{ intersects } \gamma|_{(T_1,T_2)}\} \quad \text{and} \quad H_2 = \{x \in \mathbb{R}^2 \mid \gamma_x \text{ intersects } \gamma|_{[T_1,T_2]}\}.$$

By continuous dependence of geodesics on initial data and stability of transverse intersections, H_1 is open and H_2 is closed; hence $\overline{H_1} \subset H_2$. Moreover, $H_2 \setminus H_1 = \mathrm{Im}(\gamma_{x_0})$: indeed, if $x \in H_2 \setminus H_1$, then γ_x meets $\gamma([T_1,T_2])$ at an endpoint, and since γ_x and γ_{x_0} correspond to the same invariant torus $\mathcal{T}$, they cannot intersect transversely; thus they are tangent and coincide up to reparametrization, so $\mathrm{Im}(\gamma_x) = \mathrm{Im}(\gamma_{x_0})$. Hence H_1 is both open and closed in $\mathbb{R}^2 \setminus \mathrm{Im}(\gamma_{x_0})$, so it must be a union of connected components of $\mathbb{R}^2 \setminus \mathrm{Im}(\gamma_{x_0})$. But in any of the two connected components of $\mathbb{R}^2 \setminus \mathrm{Im}(\gamma_{x_0})$ there exist points x' such that the Riemannian distance between $\mathrm{Im}(\gamma_{x'})$ and $\gamma(T_1)$ is bigger than $T_2 - T_1$. In particular, we have $\mathrm{Im}(\gamma_{x'}) \cap \mathrm{Im}(\gamma|_{(T_1,T_2)}) = \varnothing$, contradicting the definition of H_1. Therefore (2.27) holds, completing the first half of (1).

By what we have shown, the rational torus $\mathcal{T}$ contains the minimal closed geodesic in the homology class of $\mathcal{T}$. Since all closed geodesics in $\mathcal{T}$ by definition have the common Riemannian length T, each is length minimizing within the homology class. The proof of part (1) is finished.

Proof of (2). To prove that the section $s : \mathbb{T}^2 \to S\mathbb{T}^2$ of a rational torus is Lipschitz, the guiding heuristic is that the initial directions of geodesics in a rational torus whose starting points are close cannot differ much. If they did, then under bounded curvature of the surface (uniform $\mathcal{C}^2$-control of g), the geodesics would be forced to intersect, contradicting the fact that a rational torus is a graph over $\mathbb{T}^2$. The following proof formalizes this heuristic. Let us denote by $B(x_0, r)$ the geodesic ball centered at x_0 with radius $r > 0$. By standard results in Riemannian geometry, there exists $\varrho > 0$, depending lower-semicontinuously on the $\mathcal{C}^2$-topology of g, such that for all $x_0 \in \mathbb{R}^2$ and all $x, y \in \overline{B(x_0,\varrho)}$, there exists a unique minimizing unit-speed geodesic segment joining x to y and depending smoothly on (x,y). [7]

[7]For example, it suffices to choose $\varrho < \frac{r_0}{2}$ where r_0 is the *injectivity radius* of $(\mathbb{T}^2, g)$. This is related

Let us fix an arbitrary $x_0 \in \mathbb{R}^2$ and fix two points $x^\pm := \gamma_{x_0}(\pm\varrho) \in \overline{B(x_0,\varrho)}$. Consider the following smooth map

$$\overline{B(x_0,\varrho)} \ni x \mapsto (v^+(x), v^-(x)) \in \mathbb{R}^2 \times \mathbb{R}^2$$

where $v^\pm(x)$ is the initial velocity of the unique unit-speed minimizing geodesic segment joining x to $x^\pm$. See Figure 2.2 for an illustration. Since the geodesic differential equation involves derivatives of g only up to first order, it follows that if g and g' are sufficiently close in the $\mathcal{C}^2$-topology (assuming both have the rational torus of the same homology class), then the corresponding functions $v^\pm$ are close in the $\mathcal{C}^1$-topology. It is clear from definition that $v^+(x_0) = -v^-(x_0)$.

We now claim that for all $x \in B(x_0,\varrho)$, the initial velocity $\gamma_x'(0) \in \mathbb{R}^2$ of the corresponding unit-speed geodesic lifted from $\mathcal{T}$ lies outside the (positive) cone generated by the tangent vectors $v^+(x)$ and $v^-(x)$. Indeed, if this does not hold, then $\gamma_x'(0)$ points towards the interior of the geodesic triangle formed by minimizing geodesic segments with vertices x, x^- and x^+. However, by uniqueness of minimizing geodesic segments, the geodesic γ_x does not intersect again the boundary of this geodesic triangle. This implies that $\gamma_x(t)$ is "trapped" in the interior of this triangle for all future times $t > 0$, contradicting the periodicity. Similarly, by time reversal, the same argument shows that $\gamma_x'(0)$ lies outside the cone generated by $-v^+(x)$ and $-v^-(x)$. By continuity and the fact that $\gamma_{x_0}'(0) = v^+(x_0) = -v^-(x_0)$, we rule out the cone generated by $-v^+(x)$ and $v^-(x)$. Hence, we deduce that $\gamma_x'(0) \in \mathbb{R}^2$ always lies in the positive cone generated by $v^+(x)$ and $-v^-(x)$ (indicated by the yellow region in Figure 2.2).

Note that $v^\pm(x)$ depends jointly differentiably on both x and x_0. Using $v^\pm(x)$ as bounding directions, the cone they span varies smoothly with x and x_0. This provides uniform two-sided bounds on the components of $s(x) = \gamma_x'(0)$. By $\mathbb{Z}^2$-periodicity (compactness), these local bounds yield a global Lipschitz estimate for the map

$$\mathbb{R}^2 \ni x \longmapsto s(x) = \gamma_x'(0) \in S\mathbb{R}^2$$

with a Lipschitz constant depending upper semicontinuously on the $\mathcal{C}^1$ norms of $v^\pm$. Since the $\mathcal{C}^1$ norms of $v^\pm$ depend continuously on the metric g in the $\mathcal{C}^2$-topology, the Lipschitz constant of s depends upper semicontinuously on g. The proof of point (2) is finished.

Proof of (3). Before entering the detailed argument, let us informally explain the guiding heuristic of the proof. We vary the initial direction of a geodesic in a rational torus and let the perturbed geodesic (the red curve in Figure 2.3) intersect a nearby closed geodesic of the rational torus (the blue curve). By the observation in part (1), after this intersection the red geodesic and the original geodesic must lie on opposite sides of the blue one. Using the Lipschitz regularity established in part (2), we then show that the red geodesic deviates sufficiently in the transverse direction. In the limit, this implies that the derivative of the map (2.26) at $u = 0$ in the transverse direction is nonvanishing, establishing the "twist property".

We now present the details. Infinitesimally, varying u in (2.26) corresponds to perturbing the initial direction $s(x)$ of the geodesic γ_x while keeping the base point fixed. Because γ_x is

to, but slightly different from the notion of the *convexity radius* of a Riemannian manifold. The latter is defined to be the maximum ϱ such that every geodesic ball $B(x_0,\varrho)$ is *geodesically convex*, i.e., for any pair $x, y \in B(x_0,\varrho)$ there is a unique length-minimizing geodesic segment joining x and y which is contained in $B(x_0,\varrho)$. Our argument, however, does not need the geodesic to be contained in the ball.

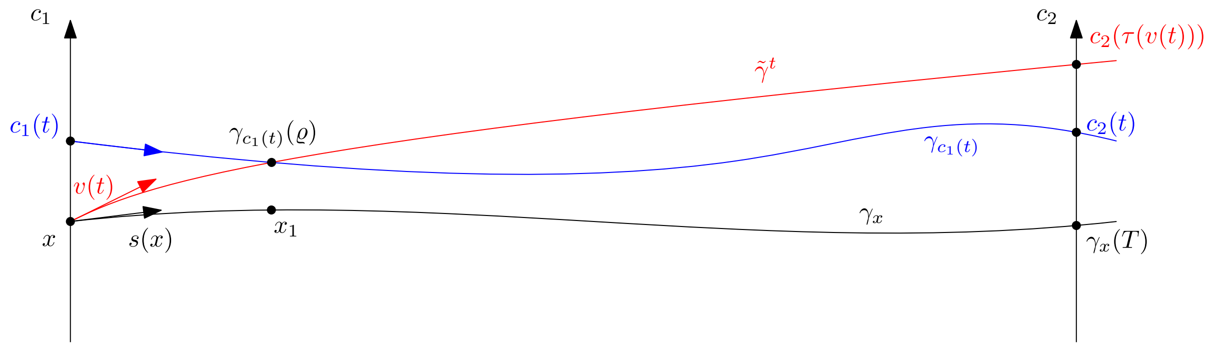

Figure 2.3: Proving the twist property of rational tori

periodic, the derivative of (2.26) at $u = 0$ preserves $s(x)$ as an eigenvector (of eigenvalue 1). Hence it suffices to study variations of u transverse to $s(x)$.

Fix $x \in \mathbb{R}^2$ and let $c_1 : \mathbb{R} \to \mathbb{R}^2$ be a $\mathcal{C}^1$ curve intersecting γ_x transversely at $c_1(0) = x$. Define $c_2(t) := c_1(t) + (m, n)$; then c_2 intersects γ_x transversely at $c_2(0) = \gamma_x(T)$. For $v \in S_x\mathbb{R}^2$ sufficiently close to $s(x)$, the geodesic with initial velocity v meets c_2 transversely at a unique point $c_2(\tau(v))$ with $\tau(v) \in \mathbb{R}$ near 0; we must show that $v \mapsto \tau(v)$ has nonzero derivative at $v = s(x)$. Take $\varrho > 0$ smaller than the injectivity radius and set $x_1 = \gamma_x(\varrho)$. For $|t|$ small, let $\tilde{\gamma}^t$ be the unique unit-speed minimizing geodesic from x to $\gamma_{c_1(t)}(\varrho)$, and write $v(t) \in T_x\mathbb{R}^2$ for its initial velocity. We recall that, by definition, the geodesic $\gamma_{c_1(t)}$ starts at the point $c_1(t) \in \mathbb{R}^2$ with the initial velocity $s(c_1(t))$ defined by the rational torus $\mathcal{T}$, thus lifting a closed geodesic in $\mathcal{T}$. By (2) and smooth dependence on endpoints, $t \mapsto v(t)$ is Lipschitz:

$$\|v(t) - v(0)\| \leq L|t| \qquad (|t| \text{ small}), \tag{2.28}$$

for some $L < \infty$, where $\|\cdot\|$ is the norm induced by g on $T_x\mathbb{R}^2$. For $|t|$ small, $v(t)$ stays close to $s(x) = v(0)$, so the geodesic with initial velocity $v(t)$ meets c_2 at $c_2(\tau(v(t)))$ with $\tau(v(t))$ near 0. The curves $\gamma_{c_1(t)}$ and $\tilde{\gamma}^t$ meet at $\gamma_{c_1(t)}(\varrho)$ and, by part (1), do not meet again; this forces $\tau(v(t))$ and t to have the same sign and $|\tau(v(t))| > |t|$. Using (2.28),

$$\frac{|\tau(v(t))|}{\|v(t) - v(0)\|} > \frac{|t|}{\|v(t) - v(0)\|} \geq \frac{1}{L} > 0.$$

It is easy to show using the intermediate value theorem that $t \mapsto v(t)$ is surjective onto an open neighborhood of $s(x) = v(0)$ in $S_x\mathbb{R}^2$. The above inequality then implies

$$|\tau(v)| > \frac{1}{L}\|v - s(x)\|$$

for v sufficiently close to $s(x)$. This in turn shows that $S_x\mathbb{R}^2 \ni v \mapsto \tau(v)$ has nonzero derivative at $s(x)$. The proof of (3) is finished. □

Let us apply Proposition 2.11 part (3) to study what happens to the rational torus under a perturbation of the metric. We show that, although in general the rational torus may not survive the perturbation, there exists nevertheless a family of "connecting orbits" joining any pair of points x and y in $\mathbb{T}^2$ sufficiently close to each other and these orbits depend smoothly on the end points (x, y) as well as the deformation parameter. In particular, for every $x \in \mathbb{T}^2$, there exists a unit-speed geodesic segment with respect to the perturbed metric joining x back to itself and having the same homology class as $\mathcal{T}$. In general, this geodesic segment is *not* periodic, in contrast to the case of a rational torus.

Corollary 2.12. *Let $(g_\epsilon)_{|\epsilon|\leq 1}$ be a smooth family of Riemannian metrics on $\mathbb{T}^2$ such that $(\mathbb{T}^2, g_0)$ has a rational torus $\mathcal{T}$. Then, there exists a neighborhood $\mathcal{U}$ of the diagonal of $\mathbb{T}^2 \times \mathbb{T}^2$ such that, for all ϵ in a sufficiently small neighborhood of zero, there exist smooth functions $s_\epsilon : \mathcal{U} \to S\mathbb{T}^2$ and $\tau_\epsilon : \mathcal{U} \to \mathbb{R}^+$ depending smoothly on ϵ and*

$$\pi \circ s_\epsilon(x, y) = x \quad \textit{and} \quad \pi \circ \Phi_{g_\epsilon}^{\tau_\epsilon(x,y)}(s_\epsilon(x, y)) = y \quad \textit{for all} \quad (x, y) \in \mathcal{U} \subset \mathbb{T}^2 \times \mathbb{T}^2 \tag{2.29}$$

with $\mathcal{T} = \{s_0(x, x) : x \in \mathbb{T}^2\}$.

Proof. Let T be the period of $\mathcal{T}$. By definition of rational tori, the map

$$x \mapsto \pi \circ \Phi_{g_0}^T(s(x))$$

is an identity on $\mathbb{T}^2$. By Proposition 2.11 statement (3), we may apply the implicit function theorem to obtain a unique tangent vector $u(\epsilon, x, y)$ in a neighborhood of 0 in $T_x\mathbb{R}^2$ for $|\epsilon|$ sufficiently small and y sufficiently near x such that $\pi \circ \Phi_{g_\epsilon}^T(s(x) + u(\epsilon, x, y)) = y$. Moreover, the vector $u(\epsilon, x, y)$ depends smoothly on ϵ, x and y and $u(0, x, x) = 0$. By rescaling the tangent vector $s(x) + u(\epsilon, x, y)$ into a vector $s_\epsilon(x, y)$ of unit length and defining $\tau_\epsilon(x, y)$ to be $T\|s(x) + u(\epsilon, x, y)\|_{g_\epsilon}$, we obtain (2.29) as desired. It is clear from the construction that $s(x) = s_0(x, x)$. It follows that $\mathcal{T} = \{s_0(x, x) : x \in \mathbb{T}^2\}$. □

Remark. The conclusion of Corollary 2.12 in particular implies that a rational torus of $(\mathbb{T}^2, g)$, if it exists, is automatically a smooth submanifold of $T\mathbb{T}^2$.

Geometrically, the tangent vector $s_\epsilon(x, x)$ in the preceding corollary is the initial condition of a connecting orbit based at x, i.e. it is the locally unique initial velocity of the unit-speed geodesic that starts at $x \in \mathbb{T}^2$, comes back to x after time $\tau_\epsilon(x, x)$ and thus forms a closed curve in the same homology class as the geodesics in the unperturbed rational torus $\mathcal{T}$.

We show that closed geodesics arising from the perturbation of the rational torus are characterized by the critical points of τ_ϵ restricted onto the diagonal of $\mathbb{T}^2 \times \mathbb{T}^2$.

Corollary 2.13. *Consider the same setup as Corollary 2.12 and take ϵ small such that the conclusion of the corollary holds. Put*

$$\begin{aligned} \overline{\tau}_\epsilon : \mathbb{T}^2 &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \tau_\epsilon(x, x). \end{aligned}$$

Then we have

$$\Phi_{g_\epsilon}^{\overline{\tau}_\epsilon(x)}(s_\epsilon(x, x)) = s_\epsilon(x, x)$$

if and only if x is a critical point of $\overline{\tau}_\epsilon$. In particular, the set of critical values of $\overline{\tau}_\epsilon$ is contained in the length spectrum $\mathcal{L}(\mathbb{T}^2, g_\epsilon)$.

Proof. Fix ϵ sufficiently small and let τ_ϵ and s_ϵ be as in Corollary 2.12. We need to prove that when $x \in \mathbb{T}^2$ is a critical point of $\overline{\tau}_\epsilon$ the unit-speed geodesic driven by the metric g_ϵ with initial velocity $s_\epsilon(x, x)$ is a closed geodesic. To this end, let $(-\eta_0, \eta_0) \ni \eta \mapsto x_\eta$ be a $\mathcal{C}^1$ path in $\mathbb{T}^2$ and set

$$\gamma_\eta(t) := \pi \circ \Phi_{g_\epsilon}^{t\,\overline{\tau}_\epsilon(x_\eta)}(s_\epsilon(x_\eta, x_\eta)).$$

By construction, $\gamma_\eta(0) = \gamma_\eta(1) = x_\eta$ and the length of $\gamma_\eta([0,1])$ is $\overline{\tau}_\epsilon(x_\eta)$. By Lemma 2.20 and Proposition 2.19 applied to the variation vector field $\delta(t) := \mathrm{d}(\gamma_\eta(t))/\mathrm{d}\eta|_{\eta=0}$ along γ_0, we have

$$\left.\frac{\mathrm{d}\overline{\tau}_\epsilon(x_\eta)}{\mathrm{d}\eta}\right|_{\eta=0} = \frac{1}{2}E(\gamma_0)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\eta}E(\gamma_\eta)\bigg|_{\eta=0} = \frac{1}{\overline{\tau}_\epsilon(x_0)}\left\langle \left.\frac{\mathrm{d}x_\eta}{\mathrm{d}\eta}\right|_{\eta=0}, \gamma_0'(1) - \gamma_0'(0)\right\rangle_g. \tag{2.30}$$

It follows that x_0 is a critical point of $\overline{\tau}_\epsilon$ if and only if $\gamma_0'(0) = \gamma_0'(1)$. After reparametrizing γ_0, this is equivalent to the geodesic with initial condition $s_\epsilon(x_0, x_0)$ being closed, which is the desired conclusion. □

2.5 Preservation of rational tori by isospectral deformations

With the results of the preceding section, especially Corollary 2.13, in hand, we can now easily establish the intermediate result Theorem 2.1, completing the first step in the overall scheme of the proof of Theorem A.

First, we reduce Theorem 2.1 to its "single-torus" version, stated as follows.

Proposition 2.14. *Let $(g_\epsilon)_{|\epsilon|<1}$ be a smooth family of Riemannian metrics on $\mathbb{T}^2$ and let $(m,n) \in (\mathbb{Z}\setminus\{0\})^2$ be such that g_0 has an (m,n)-rational torus. Suppose the family (g_ϵ) is length-isospectral. Then g_ϵ has an (m,n)-rational torus for all $|\epsilon| < 1$.*

Proof of Theorem 2.1 assuming Proposition 2.14. Let $(g_\epsilon)_{|\epsilon|<1}$ be an isospectral family of Riemannian metrics on $\mathbb{T}^2$ as in the statement of Theorem 2.1. Take any coprime pair of integers (m,n). Since g_0 is assumed to be rationally integrable, it admits an (m,n)-torus. Since (g_ϵ) is isospectral, by Proposition 2.14 the metric g_ϵ has an (m,n)-rational torus for all $|\epsilon| < 1$. Varying (m,n), we deduce that g_ϵ is rationally integrable. The proof is complete. □

The rest of this section is devoted to the proof of Proposition 2.14.

The first part of the proof relies on a classical argument using the fact that the length spectrum $\mathcal{L}(\mathbb{T}^2, g)$ is a zero measure set, which in turn implies that a length-isospectral deformation cannot give rise to continuous variations of points in the length spectrum. This argument is well-known in the context of billiard dynamics. It has been used to deduce that length-spectral rigidity implies marked length-spectral rigidity. See, for example, [Sib04, Proposition 3.2.2] and also Corollary 2.22. We combine this argument with the characterization of closed geodesics via critical points of $\overline{\tau}_\epsilon$ in Corollary 2.13.

The second part of the proof addresses an apparent limitation of Corollary 2.13, which requires the smallness of the deformation parameter ϵ. Indeed, using the Lipschitz property of the rational torus (Proposition 2.11 part 2.) and a bootstrapping argument, we show that, for a family (g_ϵ), rational tori are preserved for *any* range of values for ϵ as long as the length-isospectrality assumption holds over this range.

Proof of Proposition 2.14. Let $(g_\epsilon)_{|\epsilon|<1}$ be a length-isospectral family as in the statement. Fix arbitrary coprime pair of integers (m,n) and define $\Sigma \subset (-1,1)$ to be the set of ϵ for which $(\mathbb{T}^2, g_\epsilon)$ admits an (m,n)-rational torus. It is enough to show Σ is both open and closed.

Proof that Σ is open. Take $\epsilon_0 \in \Sigma$. For ϵ sufficiently close to ϵ_0, let $\tau_\epsilon(x,x)$ be given by Corollary 2.12 applied to the (m,n)-rational torus of g_{ϵ_0} and put $\overline{\tau}_\epsilon(x) = \tau_\epsilon(x,x)$. By Corollary 2.13 and isospectrality assumption, we have

$$\{\max(\overline{\tau}_\epsilon), \min(\overline{\tau}_\epsilon)\} \subset \mathcal{L}(\mathbb{T}^2, g_{\epsilon_0}).$$

On the other hand, since $\overline{\tau}_\epsilon$ depends smoothly on ϵ (Corollary 2.12), in particular, the quantities $\max(\overline{\tau}_\epsilon)$ and $\min(\overline{\tau}_\epsilon)$ are continuous over a neighborhood of ϵ_0. However, by Proposition 2.21, the set $\mathcal{L}(\mathbb{T}^2, g_{\epsilon_0})$ has zero measure and, in particular, does not contain an interval. It follows that $\max(\overline{\tau}_\epsilon)$ and $\min(\overline{\tau}_\epsilon)$ must both be constant in ϵ. Finally, note that we have $\overline{\tau}_{\epsilon_0} \equiv T$ where T is the period of the (m,n)-torus. It follows that $\max(\overline{\tau}_\epsilon) = \min(\overline{\tau}_\epsilon) = T$ for all ϵ sufficiently close to ϵ_0, which implies via Corollary 2.13 that $(\mathbb{T}^2, g_\epsilon)$ has an (m,n)-rational torus.

Proof that Σ is closed. Suppose $\Sigma \ni \epsilon_j \to \epsilon_\infty$ and for each $j \geq 1$ there exists an (m,n)-torus

$$\mathcal{T}_j = \mathrm{Im}(s_j) \subset S\mathbb{T}^2$$

of $(\mathbb{T}^2, g_{\epsilon_j})$, where s_j is a $\mathcal{C}^0$-section of the unit tangent bundle. We must show that $(\mathbb{T}^2, g_{\epsilon_\infty})$ also admits an (m,n)-torus.

By continuity of $\epsilon \mapsto g_\epsilon$ in the $\mathcal{C}^2$-topology, the $\mathcal{C}^2$-norm of g_ϵ is uniformly bounded for ϵ near ϵ_∞. Proposition 2.11 part 2 then implies that each s_j is Lipschitz with a Lipschitz constant bounded uniformly in j. Hence, by the Arzelà-Ascoli theorem, a subsequence of (s_j) converges uniformly to a $\mathcal{C}^0$-section s_∞ of $S\mathbb{T}^2$.

Set $\mathcal{T}_\infty = \mathrm{Im}(s_\infty)$. We claim that $\mathcal{T}_\infty$ is an (m,n)-torus of $(\mathbb{T}^2, g_{\epsilon_\infty})$. Indeed, by Proposition 2.11(1), the period T_j of $\mathcal{T}_j$ is the minimal Riemannian length among curves in the homology class (m,n). Since minimal lengths in fixed homology classes depend continuously on the metric, the sequence (T_j) converges to some limit $T_\infty > 0$.

Finally, by continuity of the geodesic flow with respect to the metric in the $\mathcal{C}^2$-topology, we may pass to the limit $j \to \infty$ in the identity

$$\Phi^{T_j}_{g_{\epsilon_j}}(s_j(x)) = s_j(x), \qquad x \in \mathbb{T}^2,$$

to obtain

$$\Phi^{T_\infty}_{g_{\epsilon_\infty}}(s_\infty(x)) = s_\infty(x), \qquad x \in \mathbb{T}^2.$$

Thus every geodesic determined by s_∞ is T_∞-periodic. It follows that $\mathcal{T}_\infty$ is an (m,n)-torus of $(\mathbb{T}^2, g_{\epsilon_\infty})$, completing the proof. $\square$

2.6 Spectral rigidity under linear-in-ϵ deformations

In this section, we present the proof of Theorem 2.2, which is the second step in our overall scheme for the proof of Theorem A.

The basic idea is to compute the second variation of the *energy functional* and show that, thanks to the dependence on ϵ being only linear, the unperturbed geodesic $\gamma^{(0)}$ is a critical point of both the unperturbed energy functional $E^{(0)}$ and the first-order expansion $E^{(1)}$; see (2.33) below. Combining the two associated Euler-Lagrange equations then allows us to conclude that $U \equiv 0$. We refer to Section 2.9.1 for a brief review of relevant notions in the calculus of variations used in this proof.

To start the proof, let us suppose, as in the statement of Theorem 2.2, that a deformation (g_ϵ) whose line element takes the form

$$\mathrm{d}s_\epsilon^2 = (\Lambda(x_1,x_2) + \epsilon U(x_1,x_2))(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) \tag{2.31}$$

preserves a rational torus and moreover the lengths of closed geodesics on that torus are constant in ϵ. Suppose the preserved rational torus has homology class (m_1,m_2) and the length of its closed geodesics is $T>0$. We consider the lift of g_ϵ to the universal cover $\mathbb{R}^2$ and still denote it by g_ϵ. The *energy* $E_\epsilon(\gamma)$ of any $\mathcal{C}^1$ curve $\gamma:[0,T]\to\mathbb{R}^2$ with respect to the metric g_ϵ is

$$E_\epsilon(\gamma) = \int_0^T [\Lambda(\gamma(t)) + \epsilon U(\gamma(t))]|\gamma'(t)|^2\mathrm{d}t. \tag{2.32}$$

Here and for the rest of this section, we use $|\cdot|$ to denote the standard Euclidean norm. The study of geodesics as critical points of the energy functional is classical. We refer to [Mil63] for more details.

Fix $a,b\in\mathbb{R}^2$ such that $b=a+(m_1,m_2)$. By Corollary 2.12 and a lift of geodesics from $\mathbb{T}^2$ to $\mathbb{R}^2$, for each ϵ, the plane $\mathbb{R}^2$ is foliated by a family of unit-speed geodesics of g_ϵ such that each geodesic γ in this foliating family satisfies $\gamma(0)+(m_1,m_2)=\gamma(T)$ and corresponds to a closed geodesic in the (m_1,m_2)-rational torus after projection. Therefore, for each ϵ we pick the unique unit-speed geodesic $\gamma_\epsilon:[0,T]\to\mathbb{R}^2$ of $(\mathbb{R}^2,g_\epsilon)$ joining a and b. Note that Corollary 2.12 implies γ_ϵ depends smoothly on ϵ. By Lemma 2.20, we have $E_\epsilon(\gamma_\epsilon)=T$. In particular, the energy of γ_ϵ is constant in ϵ.

Let us now write the ϵ-expansion of E_ϵ and γ_ϵ as follows

$$\gamma_\epsilon = \gamma^{(0)} + \epsilon\gamma^{(1)} + \epsilon^2\gamma^{(2)} + \mathcal{O}(\epsilon^3), \qquad E_\epsilon = E^{(0)} + \epsilon E^{(1)}.$$

Explicitly, we have

$$E^{(0)}(\gamma) = \int_0^T \Lambda(\gamma(t))|\gamma'(t)|^2\mathrm{d}t \quad \text{and} \quad E^{(1)}(\gamma) = \int_0^T U(\gamma(t))|\gamma'(t)|^2\mathrm{d}t. \tag{2.33}$$

Using definitions (2.50) and (2.51) with $\alpha(t,\epsilon)=\gamma_\epsilon(t)$ and $\alpha(t,\epsilon_1,\epsilon_2)=\gamma_{\epsilon_1+\epsilon_2}(t)$ respectively, we compute the ϵ-expansion of $E_\epsilon(\gamma_\epsilon)$:

$$\begin{aligned} E_\epsilon(\gamma_\epsilon) =& E^{(0)}(\gamma^{(0)}) + \epsilon\Big(E^{(1)}(\gamma^{(0)}) + \mathrm{d}_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)})\Big) \\ &+ \epsilon^2\Big(\mathrm{d}_{\gamma^{(0)}}E^{(1)}(\gamma^{(1)}) + \frac{1}{2}\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)},\gamma^{(1)}) + \mathrm{d}_{\gamma^{(0)}}E^{(0)}(\gamma^{(2)})\Big) + \mathcal{O}(\epsilon^3). \end{aligned} \tag{2.34}$$

On the other hand, since γ_ϵ is by construction a geodesic with respect to g_ϵ, Theorem 2.18 implies that $\mathrm{d}_{\gamma_\epsilon}E_\epsilon(\delta)=0$ for any variation vector field δ along γ_ϵ fixing endpoints. Expanding in ϵ yields

$$0 = \mathrm{d}_{\gamma_\epsilon}E_\epsilon(\delta) = \mathrm{d}_{\gamma^{(0)}}E^{(0)}(\delta) + \epsilon\Big(\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\delta,\gamma^{(1)}) + \mathrm{d}_{\gamma^{(0)}}E^{(1)}(\delta)\Big) + \mathcal{O}(\epsilon^2) \tag{2.35}$$

Using (2.35) with $\delta=\gamma^{(2)}$ and $\delta=\gamma^{(1)}$, we can simplify (2.34) into

$$E_\epsilon(\gamma_\epsilon) = E^{(0)}(\gamma^{(0)}) + \epsilon E^{(1)}(\gamma^{(0)}) - \frac{1}{2}\epsilon^2\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)},\gamma^{(1)}) + \mathcal{O}(\epsilon^3). \tag{2.36}$$

We arrive at the second-order condition

$$\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)},\gamma^{(1)}) = 0. \tag{2.37}$$

By Proposition 2.11 part (1), the geodesic $\gamma^{(0)}$ minimizes the g_0-distance between its endpoints. Therefore, since $\gamma^{(0)}(t)$ has unit-speed, it also minimizes the energy functional $E^{(0)}$ (with fixed endpoints). It follows that the Hessian $\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}$ is positive semi-definite. In particular, for any variation δ fixing endpoints and any $c \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leq d^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)} + c\delta, \gamma^{(1)} + c\delta) \\ &= \mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)}, \gamma^{(1)}) + 2c\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)}, \delta) + c^2\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\delta, \delta). \end{aligned}$$

Combining with (2.37), the above inequality for arbitrary $c \in \mathbb{R}$ forces $\mathrm{d}^2_{\gamma^{(0)}}E^{(0)}(\gamma^{(1)}, \delta) = 0$ for arbitrary variation δ fixing endpoints. Substituting into (2.35), we deduce that

$$\mathrm{d}_{\gamma^{(0)}}E^{(1)} = 0,$$

i.e., the undeformed geodesic $\gamma^{(0)}$ is also a critical point of the perturbing term $E^{(1)}$, in addition to being a critical point of the original energy functional $E^{(0)}$. In these two cases, the Euler-Lagrange equations satisfied by $\gamma^{(0)}$ write respectively

$$\frac{1}{2}|\dot{\gamma}^{(0)}(t)|^2\nabla_{\gamma^{(0)}(t)}U = \left(\nabla_{\gamma^{(0)}(t)}U \cdot \dot{\gamma}^{(0)}(t)\right)\dot{\gamma}^{(0)}(t) + U(\gamma^{(0)}(t))\ddot{\gamma}^{(0)}(t)$$

and

$$\frac{1}{2}|\dot{\gamma}^{(0)}(t)|^2\nabla_{\gamma^{(0)}(t)}\Lambda = \left(\nabla_{\gamma^{(0)}(t)}\Lambda \cdot \dot{\gamma}^{(0)}(t)\right)\dot{\gamma}^{(0)}(t) + \Lambda(\gamma^{(0)}(t))\ddot{\gamma}^{(0)}(t).$$

Here and for the rest of this section, we use $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ to denote the first- and second-order derivatives $\gamma'(t)$ and $\gamma''(t)$ of a curve γ. Taking the Euclidean inner product with $\dot{\gamma}^{(0)}(t)$ and rearranging, we obtain from the first equation

$$0 = \frac{1}{2}|\dot{\gamma}^{(0)}(t)|^2\frac{\mathrm{d}}{\mathrm{d}t}U(\gamma^{(0)}(t)) + U(\gamma^{(0)}(t))\ddot{\gamma}^{(0)}(t) \cdot \dot{\gamma}^{(0)}(t).$$

For $t \in \mathbb{R}$ such that $U(\gamma^{(0)}(t)) \neq 0$, the above equation gives

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log|U(\gamma^{(0)}(t))| = \frac{\ddot{\gamma}^{(0)}(t) \cdot \dot{\gamma}^{(0)}(t)}{|\dot{\gamma}^{(0)}(t)|^2}.$$

Since the same equation with U replaced by Λ also holds, we have $\frac{\mathrm{d}}{\mathrm{d}t}\log|U(\gamma^{(0)}(t))| = \frac{\mathrm{d}}{\mathrm{d}t}\log|\Lambda(\gamma^{(0)}(t))|$. It follows that $U(\gamma^{(0)}(t))$ is a constant multiple of $\Lambda(\gamma^{(0)}(t))$ over each connected component of $\{t \mid U(\gamma^{(0)}(t)) \neq 0\}$. Actually, since $\Lambda > 0$, by continuity of Λ and U there exists a constant c such that $U(\gamma^{(0)}(t)) = c\Lambda(\gamma^{(0)}(t))$ for all t. Now, using the first-order condition of (2.36), we have $E^{(1)}(\gamma^{(0)}) = 0$. Since $\gamma^{(0)}$ has unit-speed with respect to g_0, we obtain

$$0 = E^{(1)}(\gamma^{(0)}) = \int_0^T U(\gamma^{(0)}(t))|\dot{\gamma}^{(0)}(t)|^2\mathrm{d}t = c\int_0^T \Lambda(\gamma^{(0)}(t))|\dot{\gamma}^{(0)}(t)|^2\mathrm{d}t = cT. \tag{2.38}$$

Since $T > 0$, the above equation forces $c = 0$ and therefore U vanishes on the image of $\gamma^{(0)}$. Finally, since the family of closed geodesics $\gamma^{(0)}$ of a rational torus foliates $\mathbb{T}^2$, we can apply the same argument to every such geodesic $\gamma^{(0)}$ to deduce that $U = 0$ everywhere. The proof of Theorem 2.2 is finished.

Remark. We emphasize that the preceding argument requires the deformation to preserve not only a rational torus itself, but also the *period* of the rational torus (guaranteed by the isospectral assumption). The condition for preserving the period is necessary: one can of course deform the metric nontrivially within the class of Liouville metrics, however at the cost of nontrivially changing the spectrum.

2.7 Laplace spectral rigidity

In this section, we explain how Theorem 2.4 follows from the preceding rigidity results together with results from microlocal analysis.

Let $(g_\epsilon)_{|\epsilon|<1}$ be a smooth family of Riemannian metrics on $\mathbb{T}^2$, and suppose that g_0 is a Liouville metric. Assume that g_0 has a rational torus of length T, and that T does not coincide with the length of any closed geodesic in a homology class $(k, 0)$ or $(0, k)$. A key step is to show that, if the Laplace spectrum of $(\mathbb{T}^2, g_\epsilon)$ is independent of ϵ, then the length spectrum is locally fixed near T: more precisely, there exists an open interval $I \subset \mathbb{R}$ containing T such that

$$\mathcal{L}(\mathbb{T}^2, g_\epsilon) \cap I$$

is independent of ϵ, for all sufficiently small ϵ.

We briefly explain the microlocal idea behind this local length-rigidity. The Laplace spectrum determines the wave trace

$$t \longmapsto \sum_{j \geq 0} \cos\Big(t\sqrt{\lambda_j}\Big),$$

and the Poisson relation, discussed in Section 1.2.2, links the singular support of this distribution to the length spectrum. Hence Laplace isospectrality can, in principle, constrain how the lengths of closed geodesics vary.

The technical hurdle to use Laplace spectrum this way is that, if there are different closed geodesics with the same length, then their contributions to the wave trace may potentially cancel out each other and thus the corresponding length becomes undetectable from the singular support. Furthermore, the usual formula given by the works of Guillemin-Melrose [GM81] and Duistermaat-Guillemin [DG75] requires a generalized transversality condition called the *clean intersection* condition, which is difficult to check for deformations of Liouville metrics.

We overcome the above issues by employing a parametrix construction originally developed by Marvizi–Melrose [MM82] in the billiard setting, which gives a more robust description of the Poisson relation near a rational torus.

For a billiard table $\Omega \subset \mathbb{R}^2$, the contribution of periodic billiard trajectories with a fixed rotation number $1/q$ is represented by an oscillatory integral whose phase function is a *q-loop function* $\psi : \partial\Omega \to \mathbb{R}^+$, where $\psi(x)$ is defined to be the length of the billiard trajectory which starts at x and returns to x after bouncing q times and winding around the billiard table once. The critical values of ψ give the lengths of periodic trajectories.

In the present setting, the corresponding phase function is, analogously,

$$\bar{\tau}_\epsilon(x),$$

defined in Corollary 2.13. Its critical values are precisely the lengths of closed geodesics produced by perturbing the given rational torus.

Using this parametrix together with a theorem of Soga [Sog81], one shows that these critical values give genuine singularities of the wave trace, even when the critical points of $\bar{\tau}_\epsilon$ are degenerate. Moreover, the contributions coming from perturbations of different rational tori do not cancel one another.

Now, the only remaining possible cancellation would have to come from closed geodesics not arising from the perturbed rational torus. The noncoincidence assumption on T rules out such interference from the closed geodesics in the homology classes $(k, 0)$ and $(0, k)$. Consequently, if the Laplace spectrum is fixed, then the wave trace is fixed, and therefore the local length spectrum near T must also remain fixed.

The interested readers will find details of the above arguments in [HKLV25, Section 4].

We now apply this local length rigidity to the family in Theorem 2.4. Choose a rational torus of g_0 whose length T satisfies the noncoincidence condition. By the preceding discussion, there exists an open interval $I \ni T$ such that

$$\mathcal{L}(\mathbb{T}^2, g_\epsilon) \cap I$$

is independent of ϵ, for $|\epsilon|$ sufficiently small.

The proof of Proposition 2.14 uses global length-isospectrality only to keep the critical values of $\overline{\tau}_\epsilon$ inside a fixed zero-measure subset of the real line. The same argument therefore applies under the local length-isospectrality in the interval I. It follows that the chosen rational torus persists under the deformation, and that the common length of its closed geodesics remains equal to T.

We may then apply Theorem 2.2 to the family

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1(x_1) + f_2(x_2) + \epsilon U(x_1, x_2)\Big)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2).$$

The conclusion is that $U \equiv 0$, which proves Theorem 2.4.

2.8 Length-isospectral families of Liouville metrics

In this section, we give the proof of Theorem B, which states that, roughly speaking, Liouville metrics on $\mathbb{T}^2$ are length-isospectral if and only if they are obtained from one another by a "rearrangement".

We recall the notation $\mathcal{L}(\mathbb{T}^2, g)$ for the length spectrum and $\mathcal{L}^{m,n}(\mathbb{T}^2, g)$ for the set of lengths of closed geodesics in the homology class $(m, n) \in \mathbb{Z}^2$. For m, n not both zero, we have set $\ell_g^{m,n} := \min \mathcal{L}^{m,n}(\mathbb{T}^2, g)$ and defined the *marked length spectrum* to be the map

$$(m, n) \quad \longmapsto \quad \ell_g^{m,n}.$$

We also refer to Section 1.1.1 for the definition of "rearrangements". Although the notion of rearrangements applies to measurable functions, we shall nevertheless assume that, in the rest of this section, all functions defined on $\mathbb{T}$ are at least $\mathcal{C}^2$. This regularity is standard in the discussion of Riemannian metrics.

Before presenting the proof of Theorem B, we note the following classification of length-isospectral deformations *within* the class of Liouville metrics, which is a simple consequence of Theorem B.

Corollary 2.15. *Let $(g_\epsilon)_{|\epsilon|<1}$ be a family of Liouville metrics with line element*

$$\mathrm{d}s_\epsilon^2 = \Big(1 + f_1^\epsilon(x_1) + f_2^\epsilon(x_2)\Big)\,(\mathrm{d}x_1^2 + \mathrm{d}x_2^2), \qquad \epsilon \in (-\epsilon_0, \epsilon_0),$$

depending smoothly on ϵ. Assume that (g_ϵ) is length-isospectral*, i.e., we have $\mathcal{L}(\mathbb{T}^2, g_\epsilon) = \mathcal{L}(\mathbb{T}^2, g_0)$ for all ϵ. Then for each ϵ there exists $c \in \mathbb{R}$ such that f_1^ϵ is a rearrangement of $f_1^0 + c$ and f_2^ϵ is a rearrangement of $f_2^0 - c$.*

Proof. By the length-isospectrality assumption and Corollary 2.22, we deduce that the marked length spectrum of g_ϵ is independent of ϵ, i.e., for each (m,n) the function $\epsilon \mapsto \min \mathcal{L}^{m,n}(\mathbb{T}^2, g_\epsilon)$ is constant. The corollary follows immediately from Theorem B applied to the pairs (f_1^0, f_2^0) and $(f_1^\epsilon, f_2^\epsilon)$. □

The rest of the section is devoted to proving Theorem B.

For a $\mathcal{C}^2$ function $f : \mathbb{T} \to \mathbb{R}$, define

$$J_f(e) := \int_{\{x\in\mathbb{T}\ :\ e+f(x)>0\}} \sqrt{e+f(x)}\mathrm{d}x, \qquad (e\in\mathbb{R}). \tag{2.39}$$

Clearly J_f is strictly increasing, strictly concave and analytic over the interval $(-\min f, \infty)$. We emphasize that the analyticity holds even if f may not be analytic. Consider $\mathcal{C}^2$ functions $f_j : \mathbb{T} \to \mathbb{R}$ for $j = 1, 2$ such that $\min f_1 + \min f_2 > -1$ and define

$$\begin{aligned}
\Gamma_{f_1,f_2} =& \{(J_{f_1}(e), J_{f_2}(1-e))\ :\ -\min f_1 < e < 1+\min f_2\}\\
\Gamma^+_{f_1,f_2} =& \{(J_{f_1}(e), J_{f_2}(1-e))\ :\ 1+\min f_2 < e < 1+\max f_2\}\\
\Gamma^-_{f_1,f_2} =& \{(J_{f_1}(e), J_{f_2}(1-e))\ :\ -\max f_1 < e < -\min f_1\}.
\end{aligned} \tag{2.40}$$

Then Γ_{f_1,f_2} is a real-analytically embedded curve in the first quadrant of $\mathbb{R}^2$ and it is the graph of a strictly decreasing concave function. The end points of Γ_{f_1,f_2} are

$$\begin{gathered}
\Big(\int_{\mathbb{T}} \sqrt{f_1(x)-\min f_1}\mathrm{d}x,\ \int_{\mathbb{T}} \sqrt{1+\min f_1 + f_2(x)}\mathrm{d}x\Big) \quad \text{and}\\
\Big(\int_{\mathbb{T}} \sqrt{1+\min f_2 + f_1(x)}\mathrm{d}x,\ \int_{\mathbb{T}} \sqrt{f_2(x)-\min f_2}\mathrm{d}x\Big).
\end{gathered} \tag{2.41}$$

We refer to Figure 2.4 for an illustration.

Let μ_f denote the measure on $\mathbb{R}$ given by the push-forward $f_*\,\mathrm{Leb}_{\mathbb{T}}$ of the Lebesgue measure on $\mathbb{T}$ by $f : \mathbb{T} \to \mathbb{R}$. Naturally μ_f is supported within the interval $[\min f, \max f]$. Recall that L_f has been defined in (1.3). If L_f is differentiable a.e. then μ_f is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}t$ on $\mathbb{R}$ with a density $-L_f'(t)\mathrm{d}t$. Clearly, if $\widetilde{f}$ is a rearrangement of f then $\mu_f = \mu_{\widetilde{f}}$.

By the Layer-cake representation (see for example [LL01, Ch. 1.13]) and the definition of μ_f,

$$J_f(e) = \frac{1}{2}\int_{-e}^{\infty} \frac{L_f(t)}{\sqrt{t+e}}\mathrm{d}t = \int_{-e}^{\infty} \sqrt{t+e}\ \mathrm{d}\mu_f(t). \tag{2.42}$$

Clearly, if $\widetilde{f}$ is a rearrangement of f, then (2.42) shows that $J_f = J_{\widetilde{f}}$ as functions. It follows that the curves (2.40) are invariant under rearrangements of f_1 and f_2. Since $J_{f+c}(e) = J_f(e+c)$ for $c \in \mathbb{R}$ by definition of J_f, it is clear that the curves (2.40) are also invariant (as subsets of $\mathbb{R}^2$) under the transformation $(f_1, f_2) \mapsto (f_1 + c, f_2 - c)$. We show a converse of this observation.

Lemma 2.16. *Let $(\widetilde{f}_1, \widetilde{f}_2)$ and (f_1, f_2) be two pairs of $\mathcal{C}^2$ functions $\mathbb{T} \to \mathbb{R}$ with $\min f_1 + \min f_2 > -1$ and $\min \widetilde{f}_1 + \min \widetilde{f}_2 > -1$ Then $\Gamma_{f_1,f_2} = \Gamma_{\widetilde{f}_1,\widetilde{f}_2}$ if, and only if there exists $c \in \mathbb{R}$ such that $\widetilde{f}_1$ is a rearrangement of $f_1 + c$ and $\widetilde{f}_2$ is a rearrangement of $f_2 - c$.*

Proof. We have observed the "if" part. For the "only if" part, suppose $\Gamma_{\widetilde{f}_1,\widetilde{f}_2} = \Gamma_{f_1,f_2}$ as subsets of $\mathbb{R}^2$ for some $\mathcal{C}^2$ functions $f_1, f_2, \widetilde{f}_1$ and $\widetilde{f}_2$ satisfying the conditions of the lemma. Modulo transformations of the type $(f_1, f_2) \mapsto (f_1 + c, f_2 - c)$, we may assume that $\min f_1 = \min \widetilde{f}_1 = 0$. To show the lemma it is enough to prove that $\widetilde{f}_j$ is a rearrangement of f_j for $j = 1, 2$.

We begin by matching the endpoints (2.41) of the curves $\Gamma_{\widetilde{f}_1,\widetilde{f}_2}$ and Γ_{f_1,f_2}, which in particular implies

$$\int_{\mathbb{T}} \sqrt{f_j(x) - \min f_j}\,\mathrm{d}x = \int_{\mathbb{T}} \sqrt{\widetilde{f}_j(x) - \min \widetilde{f}_j}\,\mathrm{d}x \qquad (j = 1, 2).$$

Since $J_f(e)$ is real-analytic and strictly increasing for $e > -\min f$, it follows that

$$J_{\widetilde{f}_j}^{-1} \circ J_{f_j} : (-\min f_j, \infty) \to (-\min \widetilde{f}_j, \infty) \qquad (j = 1, 2)$$

are real-analytic diffeomorphisms. Define

$$\varphi(e) := \begin{cases} J_{\widetilde{f}_1}^{-1} \circ J_{f_1}(e) & (e > -\min f_1) \\ 1 - J_{\widetilde{f}_2}^{-1} \circ J_{f_2}(1 - e) & (e < 1 + \min f_2). \end{cases}$$

Since Γ_{f_1,f_2} projects injectively to x_j-coordinate for $j = 1, 2$, we can verify that the two definitions of $\varphi(e)$ are consistent for $-\min f_1 < e < 1 + \min f_2$ and

$$(J_{f_1}(e), J_{f_2}(1 - e)) = (J_{\widetilde{f}_1}(\varphi(e)), J_{\widetilde{f}_2}(1 - \varphi(e))) \qquad (-\min f_1 < e < 1 + \min f_2). \tag{2.43}$$

By analyticity of J_{f_j} and $J_{\widetilde{f}_j}$ over their respective domains, we deduce that $\varphi : \mathbb{R} \to \mathbb{R}$ is a real-analytic diffeomorphism. We remark that $\varphi(1 + \min f_2) = 1 + \min \widetilde{f}_2$ and

$$\varphi(0) = \varphi(-\min f_1) = -\min \widetilde{f}_1 = 0$$

since we have assumed $\min f_1 = \min \widetilde{f}_1 = 0$.

Let us denote $f = f_1$ and $\widetilde{f} = \widetilde{f}_1$. Consider

$$\widehat{J}_f(e) = \int_0^\infty \sqrt{t + e}\,\mathrm{d}\mu_f(t).$$

Choosing the principal branch of the square root, we define $\widehat{J}_f$ analytically on the domain $\mathbb{C} \setminus (-\infty, 0]$. Observe that $J_f|_{(0,\infty)}$ is exactly the restriction of $\widehat{J}_f$ on $(0, \infty)$. Hence J_f determines $\widehat{J}_f$ by analytic continuation.

Let $\mathbb{H}^\pm$ denote the upper/lower complex half-plane. If $e \in \mathbb{H}^\pm$, then $\sqrt{t + e}$ lies in the first/fourth quadrant of $\mathbb{C}$ for any $t > 0$. It follows from convexity that $\widehat{J}_f$ maps $\mathbb{H}^\pm$ into the first/fourth quadrant respectively. In fact, we show that $\widehat{J}_f$ restricts to biholomorphisms

$$\widehat{J}_f : \mathbb{H}^\pm \xrightarrow{\sim} \mathcal{D}^\pm, \tag{2.44}$$

where $\mathcal{D}^\pm$ are the unbounded domains in the first/fourth quadrant of $\mathbb{C}$ bounded by the curve

$$e \mapsto \lim_{\epsilon \to 0^\pm} \widehat{J}_f(e + i\epsilon) = \int_{-e}^{\max f} \sqrt{t + e}\,\mathrm{d}\mu_f(t) \pm i \int_0^{-e} \sqrt{|t + e|}\,\mathrm{d}\mu_f(t). \tag{2.45}$$

Notice that (2.45) parametrizes the image of the boundary of $\mathbb{H}^\pm$ under $\widehat{J}_f$, and it is understood that the first integral is zero if $-e > \max f$ and similarly the second integral is zero for $-e < 0$.

To show (2.44), we first observe that, since μ_f is a finite measure supported in the interval $[0, \max f]$, we have

$$|z|^{-1/2}|\widehat{J}_f(z)| \to 1 \quad \text{uniformly as } |z| \to \infty \tag{2.46}$$

Let $z_0 \in \mathcal{D}^+$. Then, for R sufficiently large, the point z_0 is enclosed by a closed loop formed by the image of the semicircle $\{z \in \mathbb{H}^+ : |z| = R\}$ under the function $\widehat{J}_f$ and the boundary (2.45). By shrinking R, the usual argument using contractible loops shows that $z_0 \in \widehat{J}_f(\mathbb{H}^+)$. Hence $\widehat{J}_f(\mathbb{H}^+) = \mathcal{D}^+$. Similarly $\widehat{J}_f(\mathbb{H}^-) = \mathcal{D}^-$. To see the injectivity of $\widehat{J}_f$, suppose $\widehat{J}_f(z_1) = \widehat{J}_f(z_2)$. Then

$$0 = \widehat{J}_f(z_1) - \widehat{J}_f(z_2) = \int_0^\infty \sqrt{t+z_1} - \sqrt{t+z_2}\mathrm{d}\mu_f(t) = (z_1 - z_2)\int_0^\infty \frac{\mathrm{d}\mu_f(t)}{\sqrt{t+z_1} + \sqrt{t+z_2}}.$$

Since $\Re\sqrt{t+z} > 0$ for all $t > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0]$, it is easy to see that the last integral above does not vanish. It follows that $z_1 = z_2$ and hence $\widehat{J}_f$ is injective. It is also clear that $\widehat{J}_f$ has nonvanishing derivative everywhere. It follows that (2.44) are indeed biholomorphisms. Similarly, we have $\widehat{J}_{\widetilde{f}} : \mathbb{H}^\pm \xrightarrow{\sim} \widetilde{\mathcal{D}}^\pm$ where $\widetilde{\mathcal{D}}^\pm$ are domains defined analogously with boundaries given by (2.45) with f replaced by $\widetilde{f}$.

Recall that we have $J_f(e) = J_{\widetilde{f}}(\varphi(e))$ for $e > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ real-analytic. Extending φ analytically over a complex neighborhood $\mathcal{U} \supset \mathbb{R}$, we have $\widehat{J}_f = \widehat{J}_{\widetilde{f}} \circ \varphi$ over $\mathcal{U} \setminus (-\infty, 0]$. Shrinking $\mathcal{U}$ if necessary we may assume $\varphi(\mathcal{U} \cap \mathbb{H}^\pm) \subset \mathbb{H}^\pm$. It follows by taking the limit $\varphi(e + i\epsilon) \to \varphi(e)$ for $e \in \mathbb{R}$ that

$$\lim_{\epsilon \to 0^\pm} \widehat{J}_f(e + i\epsilon) = \lim_{\epsilon \to 0^\pm} \widehat{J}_{\widetilde{f}}(\varphi(e) + i\epsilon).$$

Consequently, the boundary curves (2.45) defined by f and by $\widetilde{f}$ are identical up to a reparametrization by φ and hence both $\widehat{J}_{\widetilde{f}}$ and $\widehat{J}_f$ are biholomorphisms $\mathbb{H}^\pm \xrightarrow{\sim} \mathcal{D}^\pm$ onto the same image. We can therefore extend the original definition of φ to an analytic function $\varphi = \widehat{J}_{\widetilde{f}}^{-1} \circ \widehat{J}_f$ over $\mathbb{C} \setminus (-\infty, 0]$. Since φ is real-analytic over $\mathbb{R}$, by the Schwarz reflection principle, it is an entire function.

Since $\varphi(0) = 0$, the function $z \mapsto \varphi(z)/z$ is entire and bounded by the asymptotics (2.46). It then follows from Liouville's theorem that $\varphi(z) = az$ for some $a \in \mathbb{R}$. But (2.46) forces $a = 1$. Thus $\varphi(z) = z$ and we obtain accordingly $J_f = J_{\widetilde{f}}$ and $J_{f_2}(1-e) = J_{\widetilde{f}_2}(1-e)$. Since $L_f(e) = L_{\widetilde{f}}(e) = 1$ for $e < 0$, Equation (2.42) implies

$$\int_0^\infty \frac{L_f(t)}{\sqrt{t+e}}\mathrm{d}t = \int_0^\infty \frac{L_{\widetilde{f}}(t)}{\sqrt{t+e}}\mathrm{d}t \quad \text{for all } e > 0.$$

Using the identity $\int_0^\infty s^{-1/2}\mathrm{e}^{-(t+e)s}\mathrm{d}s = \pi^{1/2}(e+t)^{-1/2}$, we have

$$\int_0^\infty \frac{L_f(t)}{\sqrt{t+e}}\mathrm{d}t = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^\infty L_f(t)\frac{\mathrm{e}^{-(e+t)s}}{\sqrt{s}}\mathrm{d}s\mathrm{d}t = \frac{1}{\sqrt{\pi}}\mathscr{L}_s\left[\frac{\mathscr{L}_t[L_f(t)](s)}{\sqrt{s}}\right](e)$$

where $\mathscr{L}_t$ and $\mathscr{L}_s$ denote the Laplace transform in the variable t and s respectively. By injectivity of Laplace transform, we have $L_f = L_{\widetilde{f}}$. By the same argument applied to $J_{f_2}(1-e) = J_{\widetilde{f}_2}(1-e)$ for $e < 1 + \min f_2$, we deduce $L_{f_2} = L_{\widetilde{f}_2}$. Hence $\widetilde{f}_j$ is a rearrangement of f_j as desired. The proof is complete. $\square$

We now prove Theorem B.

Proof of Theorem B. Consider the Liouville metrics g and $\widetilde{g}$ given in the statement. The lengths of geodesics in an (m_1, m_2)-rational torus are described by Lemma 2.10. We prove (1) $\iff$ (2) by reformulating Lemma 2.10 using the geometric objects Γ_{f_1,f_2} and $\Gamma^{\pm}_{f_1,f_2}$.

Proof of (1) $\implies$ (2). By definition of J_f and Lemma 2.10, for each pair of nonnegative integers $(m_1, m_2) \neq (0,0)$, there exists a unique $e \in [-\min f_1, 1 + \min f_2]$ such that

$$m_1 J'_{f_1}(e) - m_2 J'_{f_2}(1-e) = 0 \tag{2.47}$$

and

$$\ell_g^{m_1,m_2} = m_1 J_{f_1}(e) + m_2 J_{f_2}(1-e). \tag{2.48}$$

We note in particular that when $m_1 = 0$, then (2.48) follows from the formula (2.23) and vice versa. Therefore, the marked length spectrum $(m_1, m_2) \mapsto \ell_g^{m_1,m_2}$ completely determines the tangent lines of the curve Γ_{f_1,f_2} with rational slope. See Figure 2.4. By density of $\mathbb{Q}$ in $\mathbb{R}$ and strict concavity, these tangent lines determine Γ_{f_1,f_2}, which is just the envelope of these lines. It follows from (1) that $\Gamma_{f_1,f_2} = \Gamma_{\widetilde{f}_1,\widetilde{f}_2}$, which in turn implies (2) via Lemma 2.16.

Proof of (2) $\implies$ (1). The strict concavity of Γ_{f_1,f_2} implies that for each coprime pair of positive integers (m_1, m_2) there is a unique tangent line orthogonal to (m_1, m_2) so that (2.47) holds. The distance of this tangent line to the origin determines $\ell_g^{\pm m_1, \pm m_2}$ via (2.48). Hence $\ell_g^{m_1,m_2}$ is determined by the curve Γ_{f_1,f_2}. For the cases $m_1 = 0$ and $m_2 = 0$, formula (2.23) from Lemma 2.10 implies ℓ_g^{0,m_2} and $\ell_g^{m_1,0}$ are given by the y-coordinate and the x-coordinate of the two endpoints of Γ_{f_1,f_2}, respectively. Since (2) implies $\Gamma_{f_1,f_2} = \Gamma_{\widetilde{f}_1,\widetilde{f}_2}$, it follows that g and $\widetilde{g}$ have the same rotational marked length spectrum. We have proved (2) $\implies$ (1).

Proof of length-isospectrality. Assume that each of the functions f_1 and f_2 has exactly two critical points on $\mathbb{T}$. Then, by part (2) of Lemma 2.10, the lengths of closed geodesics in the homology class $(0, m)$ are given by one of the following values:

$$|m| J_{f_2}(1 + \max f_1), \qquad |m| J_{f_2}(1 + \min f_1) \qquad \text{or} \qquad 2n J_{f_1}(e) + |m| J_{f_2}(1-e) \tag{2.49}$$

where $n \in \mathbb{Z}_{>0}$ is such that $2nJ'_{f_1}(e) - |m|J'_{f_2}(1-e) = 0$. Indeed, the first two values correspond to (2.23) whereas the last value corresponds to (2.25). We have used the assumption that f_1 has exactly 2 critical points, which implies that $\{x \in \mathbb{T} \,:\, f_1(x) + e > 0\}$ is connected for $-\max f_1 \leq e \leq -\min f_1$. Note that the quantities (2.49) are all determined by the curve $\Gamma^-_{f_1,f_2}$: the first two are integer multiples of the y-coordinates of its endpoints, whereas the values of the last type are determined by tangent lines of $\Gamma^-_{f_1,f_2}$ with rational slope in a way similar to the preceding proof of (1) $\iff$ (2). A symmetric argument applies to the homology class $(m, 0)$. Therefore $\bigcup_{m_1 m_2 = 0} \mathcal{L}^{m_1,m_2}(\mathbb{T}^2, g)$ is completely determined by $\Gamma^{\pm}_{f_1,f_2}$. It follows that, if $\widetilde{f}_1$ and $\widetilde{f}_2$ each have exactly two critical points, as do f_1 and f_2, and satisfy statement (2), then $\Gamma^{\pm}_{f_1,f_2} = \Gamma^{\pm}_{\widetilde{f}_1,\widetilde{f}_2}$ by the Layer-cake representation (2.42) and thus $\bigcup_{m_1 m_2 = 0} \mathcal{L}^{m_1,m_2}(\mathbb{T}^2, g) = \bigcup_{m_1 m_2 = 0} \mathcal{L}^{m_1,m_2}(\mathbb{T}^2, \widetilde{g})$. Since $\widetilde{g}$ and g also have the same marked length spectrum by part (1), and $\mathcal{L}^{m_1,m_2}(\mathbb{T}^2, g) = \{\ell_g^{m_1,m_2}\}$ for $m_1 m_2 \neq 0$ by rational integrability and Proposition 2.11 part 1, we deduce that $\widetilde{g}$ and g are length-isospectral. The proof is complete. □

Remark. It is clear from the proof that the assumption that f_1, f_2, $\widetilde{f}_1$ and $\widetilde{f}_2$ each has exactly two critical points on $\mathbb{T}$ can be weakened to the assumption that $\{x \in \mathbb{T} \,:\, f(x) + e > 0\}$ is connected for $f \in \{f_1, f_2, \widetilde{f}_1, \widetilde{f}_2\}$.

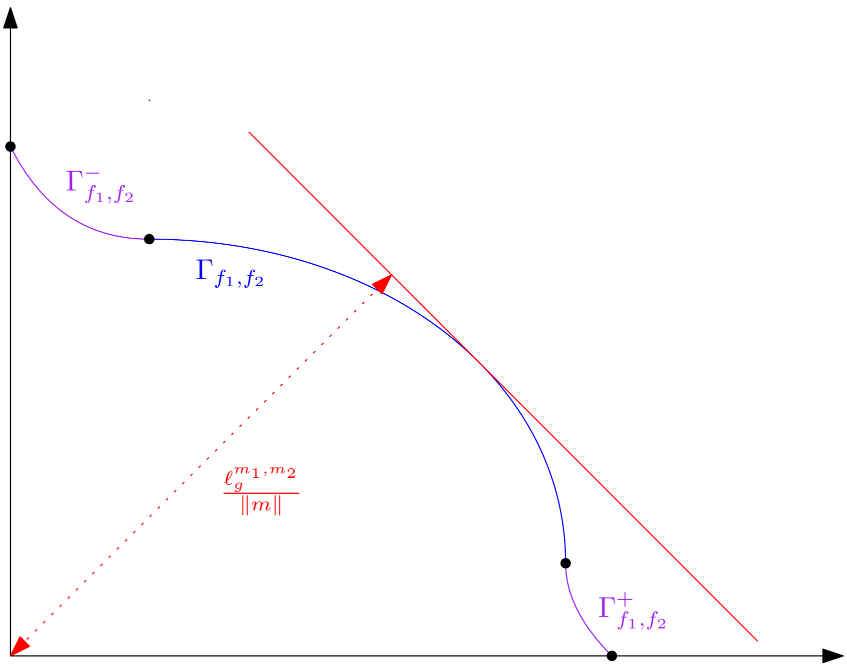

Figure 2.4: The curves Γ_{f_1,f_2} and $\Gamma^{\pm}_{f_1,f_2}$ in the first quadrant of $\mathbb{R}^2$. The red solid line is the unique tangent line to Γ_{f_1,f_2} with slope $-m_1/m_2$ for some fixed $m_1, m_2 \in \mathbb{Z}_{>0}$. The value of $\ell_g^{m_1,m_2}$ is equal to the distance of this tangent line to the origin times $\|m\| = \sqrt{m_1^2 + m_2^2}$.

Remark. The argument can be modified to show that, more generally, a deformation of a Liouville metric is length-isospectral if and only if it is a rearrangement-type deformation preserving a "branched" version of the superlevel measures (1.3). More precisely, the requirement is that the deformation should preserve the cyclic order of the connected components of $\{x \in \mathbb{T} \,:\, e + f_j(x) > 0\}$ for $j = 1, 2$ as well as the measure of each individual connected component. In order to make the idea of the proof transparent, we have nevertheless opted to stick to the simpler result with the assumptions that f_1 and f_2 each has two critical points.

2.9 Auxiliary results

2.9.1 Background on variational calculus

Let (X, g) be a Riemannian manifold. We denote by $\Phi_g^t : TX \to TX$ the geodesic flow associated with the metric g on the tangent bundle TX.

Recall that the *exponential map* $\exp_g : TX \to X$ is defined by

$$\exp_g(v) := \pi \circ \Phi_g^1(v),$$

where $\pi : TX \to X$ is the canonical projection. The following is a basic result in Riemannian geometry; see, for example, [Mil63, §10].

Proposition 2.17. *There exists a neighborhood $\mathcal{U}$ of the zero section of TX such that the map*

$$\mathcal{U} \ni v \mapsto (\pi(v), \exp_g(v)) \in X \times X$$

is a smooth diffeomorphism onto a neighborhood $\mathcal{U}'$ of the diagonal in $X \times X$. After shrinking $\mathcal{U}'$ if necessary, for every $(x, y) \in \mathcal{U}'$ there exists a unique minimizing geodesic joining x and y, depending smoothly on (x, y).

We now recall some basic notions in the calculus of variations applied to geodesics. Our exposition loosely follows Milnor's book [Mil63].

Let $\omega : [0,T] \to X$ be a piecewise smooth curve, i.e., there exist $n \geq 1$ and a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$$

such that $\omega|_{[t_j,t_{j+1}]}$ is smooth for $j = 0, \dots, n-1$. We define the *energy* of ω associated with g by

$$E(\omega) := \int_0^T |\omega'(t)|_g^2 \,\mathrm{d}t.$$

A *variation vector field* along ω is a continuous and piecewise smooth section

$$\delta : [0,T] \to \omega^* TX$$

of the pull-back bundle $\omega^* TX$. Explicitly, $\delta(t) \in T_{\omega(t)}X$ for each $t \in [0,T]$, and the restriction $\delta|_{[t_j,t_{j+1}]}$ is smooth for every j. We say that δ *fixes the endpoints* if $\delta(0) = 0$ and $\delta(T) = 0$.

Given such a variation vector field δ, we define the *first variation* of the energy functional at ω in the direction δ by

$$\mathrm{d}_\omega E(\delta) := \frac{\mathrm{d}}{\mathrm{d}\eta} E\Big(\alpha(\,\cdot\,,\eta)\Big)\Big|_{\eta=0}, \tag{2.50}$$

where $\alpha : [0,T] \times (-\eta_0, \eta_0) \to X$ is any map such that

$$\alpha(t,0) = \omega(t), \qquad \frac{\partial \alpha}{\partial \eta}(t,0) = \delta(t) \quad \text{for all } t \in [0,T].$$

One checks that the value of (2.50) does not depend on the choice of α. For instance, for $|\eta|$ sufficiently small, one may take $\alpha(t,\eta) = \exp_g(\eta\, \delta(t))$.

Similarly, given two variation vector fields δ_1, δ_2 along ω, we define the corresponding *second variation* of E by

$$\mathrm{d}^2_\omega E(\delta_1, \delta_2) := \frac{\partial^2}{\partial \eta_1\, \partial \eta_2} E\Big(\alpha(\,\cdot\,,\eta_1,\eta_2)\Big)\Big|_{\eta_1=\eta_2=0}, \tag{2.51}$$

where $\alpha : [0,T] \times (-\eta_0, \eta_0)^2 \to X$ is any map satisfying

$$\alpha(t,0,0) = \omega(t), \qquad \frac{\partial \alpha}{\partial \eta_1}(t,0,0) = \delta_1(t), \qquad \frac{\partial \alpha}{\partial \eta_2}(t,0,0) = \delta_2(t).$$

These notions give the usual characterization of geodesics; see, for example, [Mil63, §§12–13].

Theorem 2.18. *The curve ω is a geodesic if and only if $\mathrm{d}_\omega E(\delta) = 0$ for every variation vector field δ along ω fixing the endpoints. In addition, if ω is a minimizing geodesic segment, then the quadratic form*

$$\delta \mapsto \mathrm{d}^2_\omega E(\delta, \delta)$$

is positive semidefinite on the space of variation vector fields fixing the endpoints.

We also need the following endpoint formula for the first variation (2.50) along geodesics. The proof is a direct consequence of [Mil63, Theorem 12.2].

Proposition 2.19. *Suppose that $\omega : [0,T] \to X$ is a geodesic. Then for every variation vector field δ along ω,*

$$\mathrm{d}_\omega E(\delta) = 2\langle \delta(T), \omega'(T)\rangle_g - 2\langle \delta(0), \omega'(0)\rangle_g. \tag{2.52}$$

Finally, we recall the relation between the energy functional $E(\omega)$ and the length of ω. This follows from the Cauchy–Schwarz inequality; see, for example, [Mil63, §12].

Lemma 2.20. *The following statements are equivalent.*

- *The curve ω is parametrized proportionally to arc length, i.e., the speed $|\omega'(t)|_g$ is constant on $[0,T]$.*
- *The Riemannian length of ω is given by $\sqrt{E(\omega)\,T}$.*

In particular, the length of a unit-speed geodesic segment is equal to its energy.

2.9.2 Some properties of the length spectrum

In this section, we collect some elementary results concerning the properties of the length spectrum which are used in the main part of this chapter.

The first statement of Proposition 2.21 has a well-known analogue in the setting of billiards, which is customarily proved by applying Sard's lemma to the billiards' length functionals, which are defined over finite-dimensional spaces for fixed period. Although the space of closed geodesics is infinite dimensional, we can nevertheless reduce the setup to a finite-dimensional case and apply Sard's lemma the usual way.

Proposition 2.21. *Let (X,g) be a Riemannian manifold without boundary. Then the length spectrum $\mathcal{L}(X,g)$ has zero Lebesgue measure. If X is compact, then $\mathcal{L}(X,g)$ is closed.*

Proof. We first prove that $\mathcal{L}(X,g)$ has zero measure. By Proposition 2.17, there exists an open neighborhood $\mathcal{V}$ of the diagonal in $X \times X$ such that for every $(x,y) \in \mathcal{V}$, there exists a unique minimizing geodesic joining x and y, depending smoothly on (x,y). Let $\Lambda(x,y)$ denote the length of this geodesic.

For $n \geq 1$, define

$$\mathcal{V}_n := \left\{ (x_j)_{j=0}^n \in X^{n+1} : x_0 = x_n \text{ and } (x_j, x_{j+1}) \in \mathcal{V} \text{ for } j = 0, \dots, n-1 \right\},$$

and define

$$\Lambda_n : \mathcal{V}_n \to \mathbb{R}, \qquad \Lambda_n((x_j)_{j=0}^n) := \sum_{j=0}^{n-1} \Lambda(x_j, x_{j+1}).$$

Since Λ_n is smooth, Sard's theorem implies that the set of critical values of Λ_n has zero Lebesgue measure for each n.

Since the union over $n \geq 1$ is countable, it suffices to show that every $T \in \mathcal{L}(X,g)$ is a critical value of Λ_n for some $n \geq 1$. To this end, we first compute the derivative of Λ. Fix $(x,y) \in \mathcal{V}$ and write $\Lambda(x,y) = T_0$. Let $\eta \mapsto (x_\eta, y_\eta)$ be a smooth path in $\mathcal{V}$ with $(x_0, y_0) = (x,y)$, and let $\gamma_\eta : [0, T_0] \to X$ be the unique minimizing geodesic joining x_η and y_η, parametrized proportionally to arc length. By Lemma 2.20,

$$\Lambda(x_\eta, y_\eta) = \sqrt{E(\gamma_\eta)\, T_0},$$

where $E(\gamma_\eta)$ is the energy of γ_η. Differentiating at $\eta = 0$, using $E(\gamma_0) = T_0$, and applying Proposition 2.19 to the variation vector field

$$\delta(t) := \frac{\partial}{\partial \eta}\gamma_\eta(t)\Big|_{\eta=0},$$

we obtain, after a calculation similar to (2.30), that

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\Lambda(x_\eta, y_\eta)\Big|_{\eta=0} = \Big\langle \frac{\mathrm{d}y_\eta}{\mathrm{d}\eta}\Big|_{\eta=0}, \gamma_0'(T_0)\Big\rangle_g - \Big\langle \frac{\mathrm{d}x_\eta}{\mathrm{d}\eta}\Big|_{\eta=0}, \gamma_0'(0)\Big\rangle_g. \tag{2.53}$$

Now let $\gamma : [0, T] \to X$ be a closed unit-speed geodesic. By compactness, there exist $n \geq 1$ and a subdivision

$$0 = t_0 < t_1 < \cdots < t_n = T$$

such that $(\gamma(t_j), \gamma(t_{j+1})) \in \mathcal{V}$ for each $j = 0, \ldots, n-1$. By definition, the length of γ is given by $\Lambda_n(\gamma(t_0), \ldots, \gamma(t_n))$. We claim that $(\gamma(t_0), \ldots, \gamma(t_n))$ is a critical point of Λ_n. Indeed, let $(\xi_j)_{j=0}^n \in T_{(\gamma(t_0),\ldots,\gamma(t_n))}\mathcal{V}_n$ be a tangent vector to $\mathcal{V}_n$. Applying (2.53) to each segment $[t_j, t_{j+1}]$, we find that the differential of Λ_n in the direction $(\xi_j)_{j=0}^n$ is

$$\sum_{j=0}^{n-1}\Big(\langle \xi_{j+1}, \gamma'(t_{j+1})\rangle_g - \langle \xi_j, \gamma'(t_j)\rangle_g\Big).$$

These terms cancel telescopically, and the remaining endpoint contribution vanishes because $(\xi_j)_{j=0}^n \in T\mathcal{V}_n$ satisfies $\xi_0 = \xi_n$ and $\gamma'(0) = \gamma'(T)$. Thus $(\gamma(t_0), \ldots, \gamma(t_n))$ is a critical point of Λ_n, and hence T is a critical value of Λ_n. This proves that $\mathcal{L}(X, g)$ has zero Lebesgue measure.

Now assume that X is compact and consider the restriction of the geodesic flow Φ_g^t to the unit tangent bundle SX of X. Let $\Delta \subset SX \times SX$ denote the diagonal. Then

$$T \notin \mathcal{L}(X, g) \quad \text{if and only if} \quad \mathrm{Graph}(\Phi_g^T) \cap \Delta = \varnothing. \tag{2.54}$$

Since Δ and $\mathrm{Graph}(\Phi_g^T)$ are compact and $\mathrm{Graph}(\Phi_g^T)$ depends continuously on T, it follows that the set of T satisfying (2.54) is open, so $\mathcal{L}(X, g)$ is closed. □

As an application of the first statement of Proposition 2.21, we may use a well-known argument (see, for example, [Sib04, Proposition 3.2.2]) to show that the usual length-spectral rigidity implies marked length-spectral rigidity.

Corollary 2.22. *Suppose $(g_\epsilon)_{|\epsilon|<1}$ is a length-isospectral family of metrics on $\mathbb{T}^2$ with g_ϵ depending continuously on ϵ. Then the marked length spectrum of $(\mathbb{T}^2, g_\epsilon)$ is independent of ϵ, i.e., for each homology class (m, n), the function $\epsilon \mapsto \min \mathcal{L}^{m,n}(\mathbb{T}^2, g_\epsilon)$ is constant.*

Proof. Fix a homology class (m, n) and consider the length

$$\ell_{g_\epsilon}^{m,n} = \min \mathcal{L}^{m,n}(\mathbb{T}^2, g_\epsilon)$$

of its minimal geodesics as a function of ϵ. We may of course assume $(m,n) \neq (0,0)$. By Lemma 2.5, the value $\ell_{g_\epsilon}^{m,n}$ is continuous in ϵ. Since the family is isospectral, $\ell_{g_\epsilon}^{m,n}$ takes values in the set $\mathcal{L}(\mathbb{T}^2, g_0)$ which is independent of ϵ. Suppose for a contradiction that $\ell_{g_\epsilon}^{m,n}$ is not constant in ϵ. Then, by the intermediate value theorem, the set $\{\ell_{g_\epsilon}^{m,n} : |\epsilon| < 1\}$ must be a nontrivial interval which is in turn contained in $\mathcal{L}(\mathbb{T}^2, g_0)$. However, by Proposition 2.21, the length spectrum $\mathcal{L}(\mathbb{T}^2, g_0)$ has zero measure and thus does not contain nontrivial intervals, a contradiction. It follows that $\ell_{g_\epsilon}^{m,n}$ is a constant in ϵ. Since this argument holds for every (m,n), we have proved the desired result. $\square$

CHAPTER 3

Partially Isospectral Deformations of the Standard Map

This chapter appears in full in the following preprint.

- Y. Li. Deformations of the Standard Map with prescribed actions and Lyapunov exponents, 2025. doi:10.48550/arXiv.2512.03865

3.1 Introduction

In this chapter, we study action-spectral rigidity of standard maps. As introduced in Section 1.1.2, the standard map is a map on the cylinder $\mathbb{T} \times \mathbb{R}$ defined by

$$\mathscr{F} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \dot{V}(x) \\ y + \dot{V}(x) \end{pmatrix} \tag{3.1}$$

where $V : \mathbb{T} \to \mathbb{R}$ is called the *potential*.

As discussed in the introduction chapter, action-spectral rigidity is a direct analogue of length-spectral rigidity for geodesic flows and for convex billiards. See Table 1.1 and Table 1.2. The primary goal of this chapter is to construct examples of nontrivial deformations of the potential V that preserve the symplectic actions and, respectively, the Lyapunov exponents of infinitely many periodic orbits. In particular, our results show that specifying an infinite subset of the action spectrum does not necessarily fix the standard map.

3.1.1 The main technical result

Theorem C and Theorem D are immediate consequences of the more general Theorem 3.1 below.

To state this general theorem, let us recall an elementary observation that *a rotational invariant curve (RIC), together with its conjugacy to a rigid rotation of rotation number ω, determines the standard map*. More precisely, suppose there exists a circle diffeomorphism $u : \mathbb{T} \to \mathbb{T}$ and a smooth map $v : \mathbb{T} \to \mathbb{R}$ such that

$$\mathscr{F}(u(t), v(t)) = (u(t + \omega),\ v(t + \omega))$$

for all $t \in \mathbb{T}$. Writing $u = \mathrm{Id} + \phi$ with $\phi : \mathbb{T} \to \mathbb{R}$, and expressing the standard map in the coordinates (u, v), one immediately obtains an explicit relation

$$\dot{V}\Big(t + \phi(t)\Big) \;=\; \phi(t+\omega) - 2\phi(t) + \phi(t-\omega) \tag{3.2}$$

between the potential V and the function ϕ describing the RIC (see Section 3.3 for details). In particular, once the pair (ω, ϕ) is fixed, the potential V is determined by (3.2) up to an additive constant, and, consequently, the standard map is uniquely determined. We refer to Section 3.3 for more details on this point.

We now present the main technical theorem. This theorem is stated in terms of the standard maps determined by the pair (ω, ϕ) in the sense discussed above. Roughly speaking, after fixing $\omega \in \mathbb{R}$ which is sufficiently well-approximated by rationals, one can construct ϕ so that the standard map determined by (ω, ϕ) has a sequence of closed orbits whose spectral data attain certain prescribed values. We also refer to Figure 3.1 for an illustration of the theorem.

Theorem 3.1. *Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and let (p_j, q_j) be a sequence of coprime pairs of integers such that $p_j/q_j \to \omega$ as $j \to \infty$ and q_1 is sufficiently large. Then the following statements hold.*

1. (Lyapunov nonrigidity) *Suppose*

$$\left|\omega - \frac{p_j}{q_j}\right| \le q_j^{-70}. \tag{3.3}$$

Then there exists a real-analytic map

$$\alpha \mapsto \phi_\alpha$$

sending any sequence $\alpha = (\alpha_j)_{j\ge1}$ of real numbers with $|\alpha_j| < 1/4$ to a real-analytic function $\phi_\alpha : \mathbb{T} \to \mathbb{R}$. Let $\mathscr{F}_\alpha : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ denote the standard map determined by the pair (ω, ϕ_α) as above. Then the following holds.

   a) *We have a bound $\|\dddot{\phi}_\alpha\| \le 1$ uniformly in α, where $\dddot{\phi}_\alpha$ is the third order derivative of ϕ_α and $\|\cdot\|$ is the Fourier-weighted analytic norm defined in Section 3.2;*

   b) *the standard map $\mathscr{F}_\alpha$ has a sequence of nondegenerate (p_j, q_j)-periodic orbits $(\xi_j)_{j\ge1}$, each of which depends analytically on α*

   c) *the Lyapunov exponent of ξ_j is given by*

$$-(2+\alpha_j)\, q_j \left(\frac{p_j}{q_j} - \omega\right)^2 \mathrm{e}^{-2\pi q_j}. \tag{3.4}$$

2. (Action nonrigidity) *Suppose*

$$\left|\omega - \frac{p_j}{q_j}\right| \le q_j^{-70}\mathrm{e}^{-2\pi q_j}. \tag{3.5}$$

Then there exist $\epsilon_0 > 0$ and a real-analytic map

$$\alpha \mapsto \phi_\alpha$$

sending any sequence $\alpha = (\alpha_j)_{j\ge1}$ of real numbers with $|\alpha_j| < \epsilon_0$ to a real-analytic function $\phi_\alpha : \mathbb{T} \to \mathbb{R}$. Let $\mathscr{F}_\alpha$ denote the standard map determined by (ω, ϕ_α). Then *the analogous properties (a) and (b) in part 1 hold. Let* (ξ_j) *denote the sequence of* (p_j, q_j)*-periodic orbits provided by (b). Then, after potentially adjusting the generating function by an additive constant depending analytically on* α*, the symplectic action of* ξ_j *is*

$$\frac{q_j\left(\frac{p_j}{q_j}-\omega\right)^2}{2(1+f_1^*)}\left[1-\frac{(2+\alpha_j)\mathrm{e}^{-2\pi q_j}}{2\pi^2 q_j^3}\right]+\left[\omega-(\phi_\alpha\dot{\phi}_\alpha^+)^*\right]p_j, \tag{3.6}$$

where $\phi_\alpha^+ := \phi_\alpha(\cdot+\omega)$ *and* f_1^* *and* $(\phi_\alpha\dot{\phi}_\alpha^+)^*$ *denote the averages over* $\mathbb{T}$ *of the functions*

$$f_1 := (1+\dot{\phi}_\alpha)^{-1}(1+\dot{\phi}_\alpha^+)^{-1}-1 \quad \textit{and} \quad \phi_\alpha\dot{\phi}_\alpha^+,$$

respectively. Furthermore, the map $\alpha \mapsto \phi_\alpha$ *can be constructed so that the quantities* f_1^* *and* $(\phi_\alpha\dot{\phi}_\alpha^+)^*$ *are both constant in* α *and for each* j *the orbit* ξ_j *can be chosen as a minimal orbit, i.e., an orbit which minimizes the symplectic action among all* (p_j, q_j)*-orbits.*

We provide some informal clarification of the statement of Theorem 3.1. The analyticity of the map $\alpha \mapsto \phi_\alpha$ means the following: if $\alpha^\tau = (\alpha_j^\tau)_{j\geq 1}$ is a family of sequences parametrized by $\tau \in \mathbb{C}$ such that, for each j, the function $\tau \mapsto \alpha_j^\tau$ is analytic, then the resulting function ϕ_{α^τ} depends analytically on τ. Moreover, if $\alpha_j \in \mathbb{R}$ for all j, then ϕ_α is real-valued on $\mathbb{T}$. For complex-valued sequences (α_j), analogous results hold for the complexified standard map with complex-valued potentials and symplectic actions.

The dynamical meaning of ϕ_α is that the map $\mathrm{Id}+\phi_\alpha : \mathbb{T} \to \mathbb{T}$ conjugates the rigid rotation R_ω to the dynamics restricted to the RIC. In part (2) of Theorem 3.1, the adjustment made to the generating function is an additive constant determined by the *normalization* requirement that the generating function must vanish on the RIC. This additive constant has an analytic expression in terms of the pair (ω, ϕ) (Section 3.3).

In our construction, the function ϕ_α will be chosen as a *lacunary* Fourier series in the sense that, for all $|k| \geq q_1$, the kth Fourier coefficient $\hat{\phi}_k$ vanishes unless $k \in \pm\{q_j, 2q_j\}$ for some $j \geq 1$. The proof of Theorem 3.1 uses Picard iterations to construct the function ϕ_α as the fixed point of a certain contracting map iterated within a functional space consisting of such lacunary Fourier series. We give a more detailed outline of its proof in the next section.

3.1.2 An outline of the proof

We first explain how Theorem C and Theorem D follow from Theorem 3.1.

To obtain Theorem C from Theorem 3.1, we consider a one-parameter analytic family $\alpha^\tau = (\alpha_j^\tau)_{j\geq 1}$ of sequences such that α_j^τ is constant in τ for all $j \geq 2$, and only the first parameter α_1^τ varies. Applying Theorem 3.1 to this family with a choice of ϕ_α keeping the averaged quantities f_1^* and $(\phi_\alpha\dot{\phi}_\alpha^+)^*$ constant in α, we deduce from the explicit formula (3.6) that the action (or Lyapunov exponent) of each (p_j, q_j)-orbit with $j \geq 2$ is constant in τ. Since the pairs (p_j, q_j) are coprime and the q_j are strictly increasing, these periodic orbits are dynamically distinct, yielding Theorem C.

Finally, Theorem D follows by combining Theorem 3.1 with the definition of Mather's beta function. Indeed, by Theorem 3.1, the sequence of (p_j, q_j)-orbits can be chosen as minimal orbits. Hence, they control the value of Mather's beta function at p_j/q_j. For a suitable choice

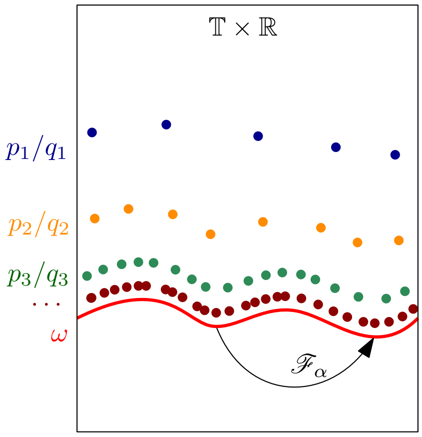

Figure 3.1: An illustration of Theorem 3.1. The standard map $\mathscr{F}_\alpha$ has a rotational invariant curve with rotation number ω. There exists a sequence of (p_j, q_j)-periodic orbit accumulating to this invariant curve and the action (resp., the Lyapunov exponent) of each orbit in the sequence can be explicitly prescribed in terms of α.

of the family α^τ as above (again varying only α_1^τ), the values of $\beta_{V_\tau}(p_j/q_j)$ remain constant in τ for the whole subsequence $p_j/q_j \to \omega$, which is precisely the statement of Theorem D.

We now outline the proof of the main result, Theorem 3.1.

Step 1. Coordinates adapted to a rotational invariant curve. Before Theorem 3.1, we recalled the elementary observation that the pair (ω, ϕ) describing a rotational invariant curve (RIC) completely determines the standard map. In particular, for fixed ω, the class of standard maps admitting an RIC of rotation number ω and whose dynamics on the RIC is conjugate to the rigid rotation R_ω is parametrized by the function $\phi : \mathbb{T} \to \mathbb{R}$. In the proof of Theorem 3.1 we work inside this class: we first construct a family of functions ϕ_α, and then define the corresponding potentials V_α by solving (3.2), choosing the additive constant appropriately. To facilitate computation, we first consider coordinates adapted to this RIC, where the standard map takes the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \omega + y + f(x, y) \\ y + g(x, y) \end{pmatrix} \tag{3.7}$$

where the RIC corresponds to $\{y = 0\}$ and the functions f and g depend implicitly on (ω, ϕ).

Step 2. The q-resonant normal form near the RIC

Fix a suitable $\omega \in \mathbb{R} \setminus \mathbb{Q}$. We choose a sequence of coprime integer pairs (p_j, q_j) with $q_j > 0$ and $p_j/q_j \to \omega$ as $j \to \infty$, subject to the arithmetic condition (3.3) in Theorem 3.1.

The guiding heuristic is that, for each such rational approximation p_j/q_j, the *q_j-th Fourier mode of ϕ should dominate the spectral data of (p_j, q_j)-periodic orbits* when p_j/q_j approximates ω sufficiently well. Here, by *spectral data* we mean either the symplectic actions or the Lyapunov exponents of these periodic orbits. The rough proof idea is that if q_j and q_k are very far apart, then the contribution of the q_k-th Fourier mode to the dynamics of (p_j, q_j)-orbits should be "averaged out" because the corresponding periods are incompatible. This viewpoint is inspired by resonant averaging techniques in Hamiltonian dynamics (see, for example, [BN12]). The heuristic above is made precise by a resonant normal-form construction described in Section 3.4. Adapting the method of [MRRTS15] to the standard map, we show that, in a complex neighborhood of the line $\{y = p/q - \omega\}$, the standard map (3.7) can be conjugated to a q-*resonant normal form*. More concretely, in these q-dependent coordinates the map is conjugate to a map (defined after re-centering the y-coordinate at $p/q - \omega$) of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + p/q + y \\ y + g_q(x, y) \end{pmatrix}, \tag{3.8}$$

where $g_q(x, 0)$ is *almost* $1/q$-periodic in x.

We consider the (dominant) *resonant part* $\langle g_q \rangle_q^N$ of g_q, obtained by truncating its Fourier series to modes of order at most N that are multiples of q. Theorem 3.3 shows that, for N not too large and $|\omega - p/q|$ small enough, the resonant part $\langle g_q \rangle_q^N$ can be well-approximated by the resonant part of an explicit expression in terms of the leading Taylor expansion $f_1(x)$ of $f(x, y)$ at $y = 0$, which in turn depends rather explicitly on ϕ.

Step 3. A fixed-point argument for the resonant data. In Section 3.5 we use a Picard iteration scheme to construct ϕ_α so that the resonant normal forms g_{q_j} associated with the approximants (p_j, q_j) have a prescribed structure.

Using the estimates from Theorem 3.3, we show that for each $q \gg 1$ the function $\langle \partial_x g_q \rangle_q^N$ is closely approximated by $\langle \dddot{\phi} \rangle_q^N$, i.e., the resonant part of the third derivative of ϕ. Because the periods q_j grow sufficiently fast, the influence of the q_k-th resonant block on the q_j-th resonant block is very small when $k \neq j$. This near-decoupling allows us to prescribe the resonant parts $\langle g_{q_j} \rangle_{q_j}^{N_j}$ *simultaneously* for the infinite sequence (p_j, q_j) by solving a fixed-point equation for ϕ.

This fixed-point problem is encoded in a nonlinear operator on a Banach space of analytic functions, and we prove that this operator is a contraction. The resulting contraction mapping theorem is formulated as Theorem 3.4 and yields an *inversion* of the map

$$\phi \mapsto \Big(\langle \partial_x g_{q_j}(\cdot, 0) \rangle_{q_j}^{N_j} \Big)_{j \geq 1}. \tag{3.9}$$

In other words, Theorem 3.4 allows us to control the resonant parts $\langle g_{q_j} \rangle_{q_j}^{N_j}$, $j \geq 1$, essentially independently by tuning ϕ.

Step 4. Construction of periodic orbits and computation of spectral data. In Section 3.6 we apply the general fixed-point scheme proved in Section 3.5 to construct and analyze (p_j, q_j)-periodic orbits explicitly. For a sequence of the parameters σ_j, we apply the inverse of (3.9) to the choices

$$N_j = 2q_j \quad \text{and} \quad \langle \partial_x g_{q_j}(\cdot, 0) \rangle_{q_j}^{N_j} \;=\; \sigma_j \, \mathrm{e}^{-2\pi q_j} \Big(p_j/q_j - \omega \Big)^2 \cos(2\pi q_j x).$$

Using $g_{q_j} \approx \langle g_{q_j} \rangle_{q_j}^{N_j}$, one can construct a (p_j, q_j)-periodic orbit that closely shadows the *equidistributed* orbits $(kp_j/q_j, 0)_{k=0,\ldots,q_j-1}$.

By analysing directly the Jacobian matrix of the normal form (3.8) along this periodic orbit, we obtain an explicit expression for the Lyapunov exponent in terms of the parameter σ_j. Furthermore, using the near-equidistribution property of the orbit together with the structure of the normal form, we derive a corresponding estimate for the symplectic action of the orbit, again expressed in terms of σ_j. In the final step, we perform a second Picard-type iteration, this time in a suitable Banach space of sequences (σ_j), to adjust the parameters σ_j so that the Lyapunov exponent (or, in the second part of Theorem 3.1, the action) of the j-th periodic orbit attains the prescribed value.

The rest of the chapter is organized as follows. After the preparatory sections 3.2-3.3, the main technical step is the construction of resonant normal forms (Theorem 3.3 in Section 3.4), which are suited to the study of (p,q)-periodic orbits when p/q approximates ω sufficiently well. The proof of the main theorem, Theorem 3.1, is formally completed in Section 3.6 modulo the proof of Theorem 3.3, which is in turn proved in the penultimate section 3.7. The last section 3.8 collects some auxiliary technical results that are used in the proofs.

3.2 Notations and basic definitions

For $b > 0$, let $\mathbb{D}_b$ denote the open disc in $\mathbb{C}$ of radius b centred at the origin. We write $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ for the real 1-torus. For $a > 0$, define the complex neighborhood

$$\mathbb{T}_a := \{\, z \in \mathbb{C}/\mathbb{Z} : |\Im z| < a \,\}.$$

If $x \in \mathbb{T}_a$ and $z \in \mathbb{C}$, we write $x + z$ for the class in $\mathbb{C}/\mathbb{Z}$ obtained by adding representatives of x and z. In particular, adding a *real* number preserves $\mathbb{T}_a$.

For $\omega \in \mathbb{R}$, the *rigid rotation by* ω is

$$\begin{aligned} R_\omega : \mathbb{T}_a &\longrightarrow \mathbb{T}_a, \\ x &\longmapsto x + \omega. \end{aligned}$$

For a function $u : \mathbb{T}_a \to \mathbb{C}$, we shall denote by

$$u^{\pm} := u \circ R_{\pm\omega}$$

the translations of u by an $\pm\omega$-rotation.

We write $A \lesssim B$ to mean that $A \leq c\, B$ for some universal constant $c > 0$ (independent of any parameters involved in A and B).

For a function $u : \mathbb{T}_a \to \mathbb{C}$, we denote by $\dot{u}$, $\ddot{u}$, and $\dddot{u}$ its first, second, and third derivatives with respect to x.

Analytic norms. Let $u : \mathbb{T}_a \to \mathbb{C}$ be analytic. Its supremum norm on $\mathbb{T}_a$ is

$$|u|_a := \sup\{\, |u(x)| : x \in \mathbb{T}_a \,\}.$$

Expanding u into its Fourier series

$$u(x) \;=\; \sum_{k\in\mathbb{Z}} \widehat{u}_k\, \mathrm{e}^{2\pi \mathrm{i} k x}, \qquad \widehat{u}_k \;=\; \int_{\mathbb{T}} u(x)\, \mathrm{e}^{-2\pi \mathrm{i} k x}\, \mathrm{d}x,$$

(where $\mathrm{d}x$ denotes the normalized Lebesgue measure on $\mathbb{T}$), we define the *Fourier-weighted norm*

$$\|u\|_a := \sum_{k\in\mathbb{Z}} |\widehat{u}_k|\, \mathrm{e}^{2\pi a |k|}.$$

The spaces

$$\{u : \mathbb{T}_a \to \mathbb{C} : |u|_a < \infty\} \quad \text{and} \quad \{u : \mathbb{T}_a \to \mathbb{C} : \|u\|_a < \infty\}$$

are Banach spaces. Moreover, the $|\cdot|_a$-space is a Banach algebra, i.e. $|fg|_a \leq |f|_a\, |g|_a$, and (see Lemma 3.24 in the Appendix) the $\|\cdot\|_a$-space is also a Banach algebra. We also introduce

$$\|u\|_a' := \|\, \dddot{u}\, \|_a.$$

Since we often take $a = 1$, we suppress the subscript in this case and write simply

$$|u| := |u|_1, \qquad \|u\| := \|u\|_1 \qquad \text{and} \qquad \|u\|' := \|u\|_1'.$$

For $f : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}$ analytic, set

$$|f|_{a,b} := \sup\{\, |f(x,y)| : x \in \mathbb{T}_a,\ y \in \mathbb{D}_b \,\}.$$

Resonant projections in Fourier space. Let $q \in \mathbb{N}$ and $u : \mathbb{T}_a \to \mathbb{C}$ be analytic. We write $\langle u \rangle_q$ for the projection of u onto the $q\mathbb{Z}$-modes (the $1/q$-periodization), and $\{u\}_q := u - \langle u \rangle_q$ for the complementary part:

$$\langle u \rangle_q (x) \;=\; \sum_{\substack{k\in\mathbb{Z}\\ q|k}} \widehat{u}_k \, \mathrm{e}^{2\pi \mathrm{i} kx} \;=\; \frac{1}{q}\sum_{j=0}^{q-1} u\Big(x + \frac{j}{q}\Big),$$

$$\{u\}_q (x) \;=\; \sum_{\substack{k\in\mathbb{Z}\\ q\nmid k}} \widehat{u}_k \, \mathrm{e}^{2\pi \mathrm{i} kx}.$$

For $N \in \mathbb{N}$, define the low/high-frequency truncations

$$\langle u \rangle^N (x) := \sum_{\substack{k\in\mathbb{Z}\\ |k|\le N}} \widehat{u}_k \, \mathrm{e}^{2\pi \mathrm{i} kx}, \qquad \langle u \rangle^{>N} (x) := \sum_{\substack{k\in\mathbb{Z}\\ |k|> N}} \widehat{u}_k \, \mathrm{e}^{2\pi \mathrm{i} kx},$$

and combine them as

$$\langle u \rangle_q^N (x) := \sum_{\substack{k\in\mathbb{Z}\\ q|k,\ |k|\le N}} \widehat{u}_k \, \mathrm{e}^{2\pi \mathrm{i} kx}.$$

We also use

$$u^* := \int_{\mathbb{T}} u(x)\,\mathrm{d}x, \qquad u^\bullet := u - u^*,$$

for the average and zero-average parts of u.

For $f : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}$ analytic and fixed $y \in \mathbb{D}_b$, let $\widehat{f}(y)_k$ be the kth Fourier coefficient of $x \mapsto f(x,y)$:

$$\widehat{f}(y)_k \;=\; \int_{\mathbb{T}} f(x,y)\, \mathrm{e}^{-2\pi \mathrm{i} kx}\,\mathrm{d}x.$$

We apply the same resonant projection notations $\langle f \rangle_q$, $\langle f \rangle^N$, f^*, $f^\bullet$ etc., in the x-variable (with y fixed), e.g.

$$\langle f \rangle_q^N (x,y) \;=\; \sum_{\substack{k\in\mathbb{Z}\\ q|k,\ |k|\le N}} \widehat{f}(y)_k \, \mathrm{e}^{2\pi \mathrm{i} kx}; \qquad f^*(y) = \int_{\mathbb{T}} f(x,y)\mathrm{d}x.$$

Exact symplectic maps and actions. We recall that a $\mathcal{C}^1$ map $\mathscr{F} : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ is said to be *exact symplectic* if the differential 1-form $\mathscr{F}^*(y\mathrm{d}x) - y\mathrm{d}x$ is exact, i.e.

$$\mathscr{F}^*(y\mathrm{d}x) - y\mathrm{d}x = \mathrm{d}\mathscr{S} \tag{3.10}$$

for some function $\mathscr{S}$ on $\mathbb{T}\times\mathbb{R}$. The function $\mathscr{S}$, which is unique up to an additive constant, is called a *generating function* of $\mathscr{F}$. In particular, taking the exterior derivative of (3.10) shows that $\mathscr{F}$ preserves the canonical area form $\mathrm{d}x\mathrm{d}y$. Let a generating function $\mathscr{S}$ be fixed.

Let $(\mathscr{F}^k(x_0, y_0))_{k=0}^{q-1}$ be a q-periodic orbit of $\mathscr{F}$. Then the *(symplectic) action* of this orbit is defined as

$$\sum_{k=0}^{q-1} \mathscr{S}(\mathscr{F}^k(x_0, y_0)).$$

Denote $(x', y') = \mathscr{F}(x, y)$. If x' is a monotone function in y for each fixed x, then $\mathscr{F}$ is called a *twist map* and, in this case, the lift of the map $\mathbb{T} \times \mathbb{R} \ni (x, y) \mapsto (x, x') \in \mathbb{T}^2$ is a diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. Hence, one can lift $\mathscr{S} : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ to $\mathbb{R}^2$ and then consider $\mathscr{S}$ as a function in the (x, x')-variables. The (x, x')-coordinates are sometimes called the *Lagrangian coordinates*. If $\mathscr{F}$ is the standard map associated with a potential $V(x)$, then its generating function can be given by

$$\mathscr{S}(x, x') = \frac{1}{2}(x - x')^2 + V(x). \tag{3.11}$$

We recall that the notions of (p, q)-periodic orbits and minimal orbits were defined in Section 1.1.2. Given a coprime pair (p, q), Birkhoff proved that there exist at least two dynamically distinct (p, q)-orbits. Besides a minimal orbit with rotation number p/q, another (p, q)-periodic orbit can be obtained using a "mountain pass" argument (see, e.g., [KH95b, Ch. 9]). It is customary to call this additional (p, q)-periodic orbit a *minimax* orbit.

3.3 Invariant curves and adapted coordinates

Recovering the potential from an RIC. Let $\mathscr{F}$ be the standard map (1.4) associated with a potential V. A special feature of (1.4) is that it is uniquely determined by any one of its rotational invariant curves that conjugate to rigid rotations (if it has any). We explain this now.

Fix $\omega \in \mathbb{R}$. Suppose $\mathscr{F}$ has a rotational invariant curve (RIC) in the sense defined in Section 1.2.6 and assume furthermore that the dynamics on C is conjugate to the rigid rotation $R_\omega : x \mapsto x + \omega$. Concretely, there exists a circle diffeomorphism $u : \mathbb{T} \to \mathbb{T}$ and a map $v : \mathbb{T} \to \mathbb{R}$ such that the map $t \mapsto (u(t), v(t))$ parametrizes C and

$$\mathscr{F}(u(t), v(t)) \;=\; \Big(u(t + \omega),\, v(t + \omega)\Big) \quad \text{for all } t \in \mathbb{T}. \tag{3.12}$$

Write $u(t) = t + \phi(t)$ with a $\mathcal{C}^1$ function $\phi : \mathbb{T} \to \mathbb{R}$; then u is an orientation-preserving circle diffeomorphism if and only if $\dot{\phi} > -1$. By choosing a lift, we consider u and v as functions on $\mathbb{R}$ with $u(t + 1) = u(t) + 1$ and $v(t + 1) = v(t)$, and interpret (3.12) using lifts so that equalities are in $\mathbb{R}$ for the x-component. From (1.4) and (3.12),

$$v(t + \omega) \;=\; u(t + \omega) - u(t), \qquad \dot{V}\Big(u(t)\Big) \;=\; v(t + \omega) - v(t). \tag{3.13}$$

Substituting $u = \mathrm{Id} + \phi$ and eliminating v, we obtain

$$\dot{V}\Big(t + \phi(t)\Big) \;=\; \phi(t + \omega) - 2\phi(t) + \phi(t - \omega)\,. \tag{3.14}$$

Thus, the pair (ω, ϕ) determines $\dot{V}$ on $\mathbb{T}$, and hence determines V up to an additive constant.

Existence of V for a given conjugacy. Let $u = \mathrm{Id} + \phi$ with $\dot{\phi} > -1$ and set $\phi^{\pm} := \phi \circ R_{\pm\omega}$ as in Section 3.2. Define

$$w(x) \;:=\; \Big(\phi^{+} - 2\phi + \phi^{-}\Big) \circ u^{-1}(x).$$

We claim $\int_{\mathbb{T}} w(x)\,\mathrm{d}x = 0$, which implies that there exists $V : \mathbb{T} \to \mathbb{R}$ (unique up to an additive constant) such that $\dot{V} = w$ and hence (3.14) holds.

Indeed, using the change of variables $x = u(t)$ and that $\int_{\mathbb{T}} (\phi^{+} - 2\phi + \phi^{-})\,\mathrm{d}t = 0$ and $\int_{\mathbb{T}} 2\phi(t)\dot{\phi}(t)\mathrm{d}t = 0$,

$$\begin{aligned}
\int_{\mathbb{T}} w(x)\,\mathrm{d}x &= \int_{\mathbb{T}} \Big(\phi^{+}(t) - 2\phi(t) + \phi^{-}(t)\Big)\,(1 + \dot{\phi}(t))\,\mathrm{d}t \\
&= \int_{\mathbb{T}} \Big(\phi^{+}(t) - 2\phi(t) + \phi^{-}(t)\Big)\,\dot{\phi}(t)\,\mathrm{d}t \\
&= \int_{\mathbb{T}} \phi(t+\omega)\,\dot{\phi}(t)\,\mathrm{d}t \;+\; \int_{\mathbb{T}} \phi(t-\omega)\,\dot{\phi}(t)\,\mathrm{d}t \\
&= -\int_{\mathbb{T}} \dot{\phi}(t+\omega)\,\phi(t)\,\mathrm{d}t \;+\; \int_{\mathbb{T}} \phi(t)\,\dot{\phi}(t+\omega)\,\mathrm{d}t \;=\; 0,
\end{aligned}$$

where in the last line we used an integration by parts on the first term.

We summarize:

Lemma 3.2. *For any $\omega \in \mathbb{R}$ and any $\phi : \mathbb{T} \to \mathbb{R}$ with $\dot{\phi} > -1$, there exists a potential V (unique up to an additive constant) such that the standard map associated with V admits an RIC whose restricted dynamics conjugates to R_ω by the diffeomorphism $u = \mathrm{Id} + \phi$.*

Standard maps in adapted coordinates. Let $\mathscr{F}$ be the unique standard map having an RIC whose restricted dynamics conjugates to R_ω by $\mathrm{Id} + \phi$ as in Lemma 3.2.

To facilitate later calculations, we introduce coordinates on $\mathbb{T} \times \mathbb{R}$ adapted to this RIC. We parametrize the RIC by $t \mapsto (u(t), v(t))$ as in (3.12). Consider the area-preserving change of coordinates $\mathscr{H}$ that "straightens" the invariant curve,

$$\mathscr{H}(x,y) = \Big(u(x), v(x) + \frac{y}{1 + \dot{\phi}(x)}\Big). \tag{3.15}$$

Hence $\mathscr{F} \circ \mathscr{H}(x,0) = \mathscr{H}(x+\omega, 0)$. Define $F := \mathscr{H}^{-1} \circ \mathscr{F} \circ \mathscr{H}$ and functions $f, g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ so that

$$F(x,y) = (x + \omega + y + f(x,y),\; y + g(x,y)). \tag{3.16}$$

A straightforward substitution of $v(\cdot + \omega) = u(\cdot + \omega) - u(\cdot)$ and $u = \mathrm{Id} + \phi$ yields a set of equations characterizing f and g:

$$\begin{cases} y + f(x,y) = \frac{y}{1+\dot{\phi}(x)} + \phi(x+\omega) - \phi(x + \omega + y + f(x,y)) \\ y + g(x,y) = \Big(y + f(x,y) + \phi(x + y + f(x,y)) - \phi(x)\Big)\Big(1 + \dot{\phi}(x + \omega + y + f(x,y))\Big). \end{cases} \tag{3.17}$$

Setting $y = 0$ in (3.17) gives $f(x,0) = g(x,0) = 0$, as expected since $F(x,0) = (x+\omega, 0)$.

In the sequel, we will call F the *standard map associated with (ω, ϕ) in the adapted coordinates.*

Actions of periodic orbits Next, we consider the generating function of the standard map $F = \mathscr{H}^{-1} \circ \mathscr{F} \circ \mathscr{H}$ associated with (ω, ϕ) in the adapted coordinates as defined in the preceding paragraph. Since $\mathscr{H}$ and $\mathscr{F}$ are area-preserving, the map F is area-preserving. Since F preserves $\mathbb{T} \times \{0\}$, the integral of the differential 1-form $F^*(y\mathrm{d}x) - y\mathrm{d}x$ over the circle $\mathbb{T} \times \{0\}$ vanishes. Thus $F^*(y\mathrm{d}x) - y\mathrm{d}x$ belongs to the zero cohomology class by Poincaré duality, and is therefore exact. It follows that F is an exact symplectic map with respect to the symplectic 1-form $y\mathrm{d}x$. Let us now find a generating function of F in terms of ϕ.

Note that the differential 1-form $\mathscr{H}^*(y\mathrm{d}x) - y\mathrm{d}x$ is closed by area preservation. Since the de Rham cohomology group $H^1(\mathbb{T} \times \mathbb{R})$ is generated by the class of $\mathrm{d}x$ where x parametrizes $\mathbb{T}$, we can write

$$\mathscr{H}^*(y\mathrm{d}x) = y\mathrm{d}x + \lambda\mathrm{d}x + \mathrm{d}\Sigma \tag{3.18}$$

for some $\lambda \in \mathbb{R}$ and $\Sigma : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$. Let $\mathscr{S}$ be a generating function of the standard map $\mathscr{F}$. Using $\mathscr{F}^*(y\mathrm{d}x) - y\mathrm{d}x = \mathrm{d}\mathscr{S}$ and (3.18), we compute:

$$\begin{aligned} F^*(y\mathrm{d}x) =& F^*(\mathscr{H}^*(y\mathrm{d}x) - (\lambda\mathrm{d}x + \mathrm{d}\Sigma)) \\ =& \mathscr{H}^*\mathscr{F}^*(y\mathrm{d}x) - F^*(\lambda\mathrm{d}x + \mathrm{d}\Sigma) \\ =& \mathscr{H}^*(y\mathrm{d}x) + \mathscr{H}^*(\mathrm{d}\mathscr{S}) - F^*(\lambda\mathrm{d}x + \mathrm{d}\Sigma) \\ =& y\mathrm{d}x + \lambda\mathrm{d}x + \mathrm{d}\Sigma + \mathscr{H}^*(\mathrm{d}\mathscr{S}) - F^*(\lambda\mathrm{d}x + \mathrm{d}\Sigma) \end{aligned} \tag{3.19}$$

whereas λ can be computed by integrating $\mathscr{H}^*(y\mathrm{d}x)$ over the circle $\mathbb{T} \times \{0\}$ and using (3.15), (3.13) and the relation $u = \mathrm{Id} + \phi$:

$$\begin{aligned} \lambda &= \int_{\mathbb{T}} \lambda\mathrm{d}x = \int_{\mathbb{T}\times\{0\}} \mathscr{H}^*(y\mathrm{d}x) = \int_{\mathbb{T}} v(x)(1 + \dot\phi(x))\mathrm{d}x \\ &= \int_{\mathbb{T}} (\omega + \phi(x) - \phi(x-\omega))(1 + \dot\phi(x))\mathrm{d}x = \omega - \int_{\mathbb{T}} \phi(x-\omega)\dot\phi(x)\mathrm{d}x = \omega - \int_{\mathbb{T}} \phi\dot\phi^+. \end{aligned}$$

Substituting into (3.19), we obtain a generating function S of F:

$$S = \mathscr{S} \circ \mathscr{H} + \left(\omega - \int_{\mathbb{T}} \phi\dot\phi^+\right)(x - x\circ F) + (\Sigma - \Sigma \circ F) \tag{3.20}$$

where $x - x \circ F$ is interpreted using a lift of F. By (3.16), we have simply $x - x \circ F = -(\omega + y + f(x,y))$. Suppose $(F^k(x_0,y_0))_{k=0}^{q-1}$ is a (p,q)-periodic orbit of F. Then the sum of $x - x \circ F$ over this orbit is exactly $-p$ by the definition of (p,q)-periodic orbits (see Section 3.2), and the sum of the third term in (3.20) vanishes by a telescopic sum. Hence, the symplectic action of this orbit with respect to the generating function (3.20) is

$$\sum_{k=0}^{q-1} S \circ F^k(x_0,y_0) = -\left(\omega - \int_{\mathbb{T}} \phi\dot\phi^+\right)p + \sum_{k=0}^{q-1} \mathscr{S} \circ \mathscr{H} \circ F^k(x_0,y_0). \tag{3.21}$$

Observe that the term $\sum_{k=0}^{q-1} \mathscr{S} \circ \mathscr{H}(F^k(x_0,y_0))$ is the symplectic action of the corresponding periodic orbit under the original standard map $\mathscr{F}$, with a generating function $\mathscr{S}$ of the form (3.11) for a potential V satisfying (3.14). Looking forward, we will actually evaluate via (3.21) the $\mathscr{S}$-action $\sum_{k=0}^{q-1} \mathscr{S} \circ \mathscr{H}(F^k(x_0,y_0))$ in terms of the S-action $\sum_{k=0}^{q-1} S(F^k(x_0,y_0))$ associated with the map F in adapted coordinates. To this end, we now derive a separate formula for S.

By taking the derivative of the first line of (3.17) in y, it is easy to see that $\partial f/\partial y > -1$ as long as $\dot\phi > -1$ on $\mathbb{T}$. Therefore, the map F is a twist map. After lifting S to $\mathbb{R}^2$ and transforming to the Lagrangian coordinates (x, x'), we deduce from $F^*(y\mathrm{d}x) - y\mathrm{d}x = \mathrm{d}S$ that, in a neighborhood of $\{y = 0\}$,

$$F(x, y) = (x', y') \quad \text{if and only if} \quad \frac{\partial S}{\partial x}(x, x') = -y \quad \text{and} \quad \frac{\partial S}{\partial x'}(x, x') = y'. \tag{3.22}$$

Using (3.22), and regarding $x'(x, y)$ as a function in terms of (x, y), we can calculate the derivatives of $S(x, x'(x, y))$ in x and y as follows.

$$\frac{\partial S}{\partial x}(x, y) = \frac{\partial S}{\partial x} + \frac{\partial S}{\partial x'}\frac{\partial x'}{\partial x} = y'\frac{\partial x'}{\partial x} - y \quad \text{and} \quad \frac{\partial S}{\partial y}(x, y) = \frac{\partial S}{\partial x'}\frac{\partial x'}{\partial y} = y'\frac{\partial x'}{\partial y}.$$

Substituting $x' = x + \omega + y + f(x, y)$ and $y' = y + g(x, y)$, we then obtain explicitly

$$\frac{\partial S}{\partial x}(x, y) = (y + g(x, y))\Big(1 + \frac{\partial f}{\partial x}(x, y)\Big) - y; \qquad \frac{\partial S}{\partial y}(x, y) = (y + g(x, y))\Big(1 + \frac{\partial f}{\partial y}(x, y)\Big). \tag{3.23}$$

Hence, the F-generating function S can be obtained by integrating the second equation of (3.23) in y.

In particular, since $g(x, 0) = 0$, we see from the first equation in (3.23) that S is constant along $\{y = 0\}$. By adjusting the additive constant of the potential V solving (3.14), we may adjust $\mathscr{S}$ (and hence S) by an additive constant so that (3.20) vanishes identically on the RIC. This choice fixes $\mathscr{S}$ and S for a given pair (ω, ϕ); we call $\mathscr{S}$ the *normalized generating function* associated with (ω, ϕ) in the original coordinates, and S the corresponding normalized generating function in the adapted coordinates.

3.4 Resonant coordinates: the statements

In this section, we state the properties of the resonant coordinates. The main result is Theorem 3.3, where we obtain a "q-resonant normal form" for the standard map F associated with (ω, ϕ) in the adapted coordinates constructed in Section 3.3.

The purpose of this normal form is to capture the dynamics of F in a small neighborhood of the circle

$$\mathbb{T} \times \{p/q - \omega\}$$

and to show that, roughly speaking, the dynamics there depends strongly on the resonant Fourier modes of ϕ, i.e. on the Fourier coefficients $\hat{\phi}_k$ with $q \mid k$, and is only weakly influenced by the nonresonant modes. Localization near $\mathbb{T} \times \{p/q - \omega\}$ is important because the dynamics there is close to a rigid rotation in x by the rational angle p/q and, at least when ϕ is small, all (p, q)-periodic orbits of F are expected to lie nearby. The normal form is valid when ω is sufficiently well approximated by a rational number p/q with large denominator q.

The following theorem is the technical core of the proof of Theorem 3.1. For technical reasons, we consider the complexified standard maps in this theorem: we can obviously define an analytic map $\mathscr{F}$ using the formula (1.4) with a complex analytic potential V, which can in turn be given by a real ω and a complex analytic function $\phi : \mathbb{T}_1 \to \mathbb{C}$ using the construction of Section 3.3. By abuse of notation we will still call the resulting map F the standard map associated with (ω, ϕ) in adapted coordinates, even though one may not be able to iterate F on the real cylinder $\mathbb{T} \times \mathbb{R}$ indefinitely unless ϕ is real-valued. The formulae (3.17) describing the map F still hold, of course. The corresponding normalized generating function S will be in general complex valued and the "symplectic actions" obtained by summing over periodic orbits are complex quantities. We refer to Section 3.2 for the notation.

Theorem 3.3. *Fix $\nu_0 := 64$, $\nu \geq \nu_0 + 6$ and $\kappa \in \mathbb{Z}_{>0}$. Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and let $\phi : \mathbb{T}_1 \to \mathbb{C}$ be analytic, of zero average, with $\| \dddot{\phi} \| \leq 1$. Let F be the standard map associated with (ω, ϕ) in adapted coordinates. Suppose q is sufficiently large and*

$$\left|\omega - \frac{p}{q}\right| \leq q^{-\nu} \qquad \text{for some } p \in \mathbb{Z}. \tag{3.24}$$

Then there exists an analytic change of coordinates

$$H_q : \mathbb{T}_a \times \mathbb{D}_b \longrightarrow \mathbb{T}_1 \times \mathbb{C}, \qquad a = 1 - \frac{3}{q}, \quad b = \frac{1}{5}\left|\omega - \frac{p}{q}\right|,$$

with the following properties.

1. *The map H_q is close to the translation by $p/q - \omega$ in y in the sense that*

$$H_q : (x, y) \longmapsto \Big(x + h_{q,1}(x, y),\, y + p/q - \omega + h_{q,2}(x, y)\Big) \tag{3.25}$$

where

$$|h_{q,1}|_{a,b} \lesssim q\left|\omega - \frac{p}{q}\right|; \quad |h_{q,2}|_{a,b} < \frac{1}{10}\left|\omega - \frac{p}{q}\right|. \tag{3.26}$$

2. *Defining $F_q = H_q^{-1} \circ F \circ H_q$, we have*

$$F_q : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x + p/q + y \\ y + g_q(x, y) \end{pmatrix} \tag{3.27}$$

for some analytic $g_q : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}$ such that, with $N := \kappa q$,

$$|g_q|_{a,b} \lesssim \left|\omega - \frac{p}{q}\right|^2, \tag{3.28}$$

$$\left|g_q - \langle g_q\rangle_q^N\right|_{1/q,b} \lesssim \mathrm{e}^{6\pi\kappa}\,\mathrm{e}^{-2\pi(\kappa+1)q}\left|\omega - \frac{p}{q}\right|^2. \tag{3.29}$$

3. *For $|y| < b$, the averaged part of $g_q(\cdot,\ y)$ satisfies*

$$|g_q^*(y)| \lesssim q^{\nu_0 - \nu} \sup_{\Im x = 0} |g_q^\bullet(x, y)|. \tag{3.30}$$

4. *Writing*

$$f_1(x) := \frac{1}{(1 + \dot{\phi}(x))(1 + \dot{\phi}(x + \omega))} - 1, \tag{3.31}$$

we have the comparison estimate

$$\left\|\left\langle g_q(\cdot, 0) + \frac{1}{2}\Big(\frac{p}{q} - \omega\Big)^2 \frac{\langle \dot{f}_1\rangle_q}{1 + \langle f_1\rangle_q}\right\rangle_q^N\right\| \lesssim \kappa\,\mathrm{e}^{4\pi\kappa}\, q^{\nu_0 - \nu}\left|\omega - \frac{p}{q}\right|^2. \tag{3.32}$$

5. *Let S be the normalized generating function associated with (ω, ϕ) in adapted coordinates (see the end of Section 3.3), and define*

$$S_q := S \circ H_q.$$

Then

$$\left|S_q(x, y) - \Big(\frac{p}{q} - \omega + y\Big)^2 \frac{1 + f_1(x)}{2\big(1 + \langle f_1\rangle_q(x)\big)^2}\right| \lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3, \qquad (x, y) \in \mathbb{T}_a \times \mathbb{D}_b. \tag{3.33}$$

Moreover, the change of coordinates H_q depends analytically on ϕ and is real-analytic when ϕ is real-valued.

Remark. The bound "q sufficiently large" depends only on ν and κ. In particular, it is independent of (ω, ϕ).

Remark. The analytic dependence of H_q on ϕ means more precisely as follows. Suppose $\phi^\tau(x)$ is jointly analytic in $x \in \mathbb{T}_1$ and a complex parameter τ such that $\phi = \phi^\tau$ satisfies the conditions of Theorem 3.3. Then the corresponding coordinate transform $H_q = H_q^\tau$ is such that $H_q^\tau(x, y)$ is jointly analytic in (τ, x, y).

In the rest of this chapter, we adopt the following terminology. Given a standard map F associated with (ω, ϕ) in adapted coordinates and a q satisfying the assumptions of Theorem 3.3, we call the coordinates defined by H_q the *q-resonant coordinates*. We call the function g_q in (3.27) the *q-resonant normal form* of F. More generally, for a function g we call $\langle g\rangle_q^N$ the *(q, N)-resonant part* (or simply the *resonant part*) of g, and we call $g - \langle g\rangle_q^N$ the *nonresonant part*. We call the map $F_q = H_q^{-1} \circ F \circ H_q$ the *standard map associated with (ω, ϕ) in q-resonant coordinates*.

The proof of Theorem 3.3 is rather technical and will be given in Section 3.7.

3.5 Independent control of resonant parts

The goal of this section is to show that, by choosing ϕ appropriately, one can independently control the zero-average resonant part $\langle g_q^\bullet\rangle_q^N$ of the resonant normal form g_q constructed by Theorem 3.3 for an infinite sequence of q's. This will in turn enable us to control a corresponding sequence of periodic orbits in the next section.

More precisely, we shall apply Theorem 3.3 to a sequence of coprime pairs of integers (p_j, q_j) with $q_j > 0$ such that p_j/q_j approximates ω sufficiently fast, obtaining accordingly a sequence of resonant normal forms g_{q_j} from Theorem 3.3. With $N_j = \kappa q_j$ for a fixed κ, we shall approximate the zero-average (q_j, N_j)-resonant part $\langle g_{q_j}^\bullet\rangle_{q_j}^{N_j}$ by the corresponding (q_j, N_j)-resonant part $\langle \phi\rangle_{q_j}^{N_j}$ of the function ϕ defining the standard map, which we can prescribe. Then, using a Picard iteration, we find a choice of ϕ for which the individual resonant parts $\langle g_{q_j}^\bullet\rangle_{q_j}^{N_j}$ of the map take the desired forms.

Remark. To prove the results in this section, the only conclusions needed from Theorem 3.3 are statement (4) and the analytic dependence of g_q on ϕ. The other conclusions of Theorem 3.3 are not used in this section.

We begin by defining certain Banach spaces suitably adapted to our problem. Let $\mathcal{A}$ denote the Banach space of analytic functions on $\mathbb{T}_1$ with zero average and finite $\|\cdot\|$-norm. For an integer $n > 0$, define a Banach subspace $\mathcal{A}_n \subset \mathcal{A}$ by

$$\mathcal{A}_n = \{u \in \mathcal{A} \mid \hat{u}_k = 0 \text{ for all } |k| < n\}.$$

Let $\mathcal{A}'$ be the subspace of $\mathcal{A}$ consisting of functions u such that $\ddot{u} \in \mathcal{A}$ and put $\|u\|' := \|\ddot{u}\|$. Since elements of $\mathcal{A}'$ have zero average, the space $\mathcal{A}'$ is a Banach space with respect to the $\|\cdot\|'$-norm. Similarly, we also define the Banach subspaces

$$\mathcal{A}'_n = \mathcal{A}_n \cap \mathcal{A}'$$

with induced $\|\cdot\|'$-norm.

We now state the main result of this section.

Theorem 3.4. *For any $\epsilon > 0$, $\kappa \in \mathbb{Z}_{>0}$ and $\nu \geq 70$, there exists $n \in \mathbb{Z}_{>0}$ such that the following holds. For any $\omega \in \mathbb{R}\setminus\mathbb{Q}$ and any sequence (p_j, q_j) of coprime integer pairs satisfying*

$$\left|\omega - \frac{p_j}{q_j}\right| \leq q_j^{-\nu}, \quad q_{j+1} > 2\kappa q_j \quad \text{and} \quad q_1 > n, \tag{3.34}$$

and for any $\phi_0 \in \mathcal{A}'_n$ and $h \in \mathcal{A}$ such that $h = \sum_j \langle h \rangle_{q_j}^{N_j}$ with $N_j = \kappa q_j$ and

$$\|h\| + 2\|\phi_0\|' < 1 - 2\epsilon, \tag{3.35}$$

there exists a unique analytic function $\phi_1 \in \mathcal{A}'_n$ depending analytically on (h, ϕ_0) such that $\phi_1 = \sum_j \langle \phi_1 \rangle_{q_j}^{N_j}$, $\|\phi_1\|' \leq 1 - \epsilon - \|\phi_0\|'$ and

$$\left\langle \frac{\partial g_{q_j}}{\partial x}(\cdot, 0) \right\rangle_{q_j}^{N_j} = \left(\frac{p_j}{q_j} - \omega\right)^2 \langle h \rangle_{q_j}^{N_j} \quad \text{for} \quad j \geq 1 \tag{3.36}$$

where g_{q_j} is the q_j-resonant normal form as constructed in Theorem 3.3 with $\phi := \phi_0 + \phi_1$ and the chosen ω, κ and ν.

Remark. Analytic dependence of ϕ_1 on (h, ϕ_0) is meant in the sense analogous to the second remark following Theorem 3.3, i.e., if $h^\tau(x)$ and $\phi_0^\tau(x)$ are both jointly analytic in $x \in \mathbb{T}_1$ and a complex parameter τ such that $h = h^\tau$ and $\phi_0 = \phi_0^\tau$ satisfy the conditions of Theorem 3.4, then the corresponding function $\phi_1^\tau(x)$ is jointly analytic in x and τ.

Remark. If n is sufficiently large, then the second condition of (3.34) is implied by the first condition. Indeed, the first condition implies

$$\frac{1}{q_j q_{j+1}} \leq \left|\frac{p_j}{q_j} - \frac{p_{j+1}}{q_{j+1}}\right| \leq q_j^{-\nu} + q_{j+1}^{-\nu}$$

which in turn gives

$$\frac{q_j^{\nu-1}}{1 + (q_j/q_{j+1})^\nu} \leq q_{j+1}.$$

Thus, taking n sufficiently large so that $q_j^{\nu-2} > 4\kappa$, we have

$$q_{j+1} > 2\kappa q_j \quad \text{for all} \quad j.$$

Theorem 3.4 will be proved by combining the usual Picard iteration (a.k.a. the Banach fixed point theorem or the contraction mapping principle) with the quantitative estimates from the following lemma.

Lemma 3.5. *For any $\epsilon > 0$, $\kappa \in \mathbb{Z}_{>0}$ and $\nu \geq 70$, there exists $n \in \mathbb{Z}_{>0}$ such that the following holds. Let ω and (p_j, q_j) be as in Theorem 3.4 such that* (3.34) *holds. For any $\phi \in \mathcal{A}'_n$ with $\|\phi\|' \leq 1$, let*

$$F_{q_j} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + p_j/q_j + y \\ y + g_{q_j}(x, y) \end{pmatrix}, \qquad g_{q_j} : \mathbb{T}_{1-3/q_j} \times \mathbb{D}_{|\omega - p_j/q_j|/5} \to \mathbb{C}$$

be the standard map (3.27) *associated with* (ω, ϕ) *in the* q_j*-resonant coordinates as constructed by applying Theorem 3.3 with* $(p,q) = (p_j, q_j)$ *and define*

$$\mathcal{P}(\phi) := \sum_{j=1}^{\infty} \left\langle \left(\frac{p_j}{q_j} - \omega\right)^{-2} \frac{\partial g_{q_j}}{\partial x}(\cdot, 0) - \dddot{\phi} \right\rangle_{q_j}^{N_j} \quad \textit{where} \quad N_j = \kappa q_j. \tag{3.37}$$

Then $\mathcal{P}(\phi) : \mathbb{T}_1 \to \mathbb{C}$ *is an analytic function with*

$$\|\mathcal{P}(\phi)\| < \epsilon, \tag{3.38}$$

and, thus defined, the map $\mathcal{P}$ *sends the unit ball of* $\mathcal{A}'_n$ *into* $\mathcal{A}$ *and is analytic. Moreover, for* $\|\phi\|' < 1$*, the operator norm of the derivative* $\mathrm{d}_\phi \mathcal{P} : \mathcal{A}'_n \to \mathcal{A}$ *satisfies*

$$\|\mathrm{d}_\phi \mathcal{P}\| < \frac{\epsilon}{1 - \|\phi\|'}. \tag{3.39}$$

We now present the proof of Theorem 3.4.

Proof of Theorem 3.4 assuming Lemma 3.5. Let n be given by Lemma 3.5 and fix ω and (p_j, q_j) satisfying the conditions of Theorem 3.4. Using $q_{j+1} > N_j$ and the requirement $\phi_1 = \sum_j \langle \phi_1 \rangle_{q_j}^{N_j}$, it is easy to verify that the required conclusion (3.36) is equivalent to

$$\dddot{\phi}_1 + \sum_{j \geq 1} \langle \dddot{\phi}_0 \rangle_{q_j}^{N_j} = h - \mathcal{P}(\phi_0 + \phi_1) \tag{3.40}$$

where $\mathcal{P}$ is as defined in (3.37). In order to cast the problem into a fixed point problem, let $\mathscr{D} : u \mapsto \dot{u}$ denote the derivative map. Then $\mathscr{D}^3 : \mathcal{A}' \to \mathcal{A}$ is a Banach space isometry. Take the inverse $\mathscr{D}^{-3} : \mathcal{A} \to \mathcal{A}'$ and define

$$\begin{aligned} \mathcal{H}_{h,\phi_0} : \mathcal{A}'_n \to& \mathcal{A}'_n \\ \phi_1 \mapsto& \mathscr{D}^{-3}\Big[h - \mathcal{P}(\phi_0 + \phi_1) - \sum_{j\geq 1} \langle \dddot{\phi}_0 \rangle_{q_j}^{N_j}\Big]. \end{aligned} \tag{3.41}$$

Then finding ϕ_1 satisfying (3.40) is equivalent to finding a fixed point of $\mathcal{H}_{h,\phi_0}$.

Fix $\phi_0 \in \mathcal{A}'_n$ and $h \in \mathcal{A}$ satisfying (3.35) and pick any r such that

$$\|h\| + \epsilon + \|\phi_0\|' < r < 1 - \epsilon - \|\phi_0\|'. \tag{3.42}$$

Then, for $\|\phi_1\|' \leq r$ we have $\|\phi_0 + \phi_1\|' < 1 - \epsilon < 1$. By the definition of $\|\cdot\|'$-norm, the estimate (3.38) of Lemma 3.5 and the assumption (3.35),

$$\|\mathcal{H}_{h,\phi_0}(\phi_1)\|' \leq \|h\| + \|\mathcal{P}(\phi_0 + \phi_1)\| + \|\phi_0\|' < \|h\| + \epsilon + \|\phi_0\|' < r.$$

Taking the derivative of $\mathcal{H}_{h,\phi_0}$ and using the linearity of $\mathscr{D}^{-3}$, we have $\mathrm{d}_{\phi_1}\mathcal{H}_{h,\phi_0} = -\mathscr{D}^{-3} \circ \mathrm{d}_{\phi_0+\phi_1}\mathcal{P}$ and hence we deduce from (3.39) and (3.42) that the operator norm of the derivative map $\mathrm{d}_{\phi_1}\mathcal{H}_{h,\phi_0} : \mathcal{A}'_n \to \mathcal{A}'_n$ satisfies, uniformly over $\|\phi_1\|' \leq r$, that

$$\begin{aligned} \|\mathrm{d}_{\phi_1}\mathcal{H}_{h,\phi_0}\| &= \sup_{\|\delta\|'=1} \|\mathscr{D}^{-3}[\mathrm{d}_{\phi_0+\phi_1}\mathcal{P}(\delta)]\|' = \|\mathrm{d}_{\phi_0+\phi_1}\mathcal{P}\| \\ &< \frac{\epsilon}{1 - \|\phi_0 + \phi_1\|'} \leq \frac{\epsilon}{1 - \|\phi_0\|' - r} < 1. \end{aligned}$$

Note also that by definition of $\mathcal{P}$ and the condition on h, we have $\mathcal{H}_{h,\phi_0}(\phi_1) = \sum_{j\geq 1}\langle \mathcal{H}_{h,\phi_0}(\phi_1)\rangle_{q_j}^{N_j}$. Hence, the contraction mapping principle applies to $\mathcal{H}_{h,\phi_0}$ on the closed set

$$\left\{u \in \mathcal{A}'_n \;:\; \|u\|' \leq r,\; u = \sum_{j\geq 1}\langle u\rangle_{q_j}^{N_j}\right\}$$

and yields a unique fixed point therein. Moreover, by analytic dependence of $\mathcal{P}$ on ϕ as shown in Lemma 3.5, the fixed point depends analytically on h and ϕ_0. The proof is finished. □

In the rest of this section, we focus on the proof of Lemma 3.5. Recall that Theorem 3.3 gives a comparison estimate (3.32). The main idea for the proof of Lemma 3.5 will be to give a similar comparison estimate between the resonant part $\langle \dddot{\phi}\rangle_{q_j}^{N_j}$ of $\dddot{\phi}$ and the function

$$-\frac{1}{2}\frac{\partial}{\partial x}\left\langle \frac{\langle \dot{f}_1\rangle_{q_j}}{1+\langle f_1\rangle_{q_j}}\right\rangle_{q_j}^{N_j}.$$

Since f_1 depends explicitly on ϕ via (3.31), we can compute an estimate directly.

Proof of Lemma 3.5. Let $\epsilon > 0$, $\kappa \in \mathbb{Z}_{>0}$ and $\nu \geq 70$ be given. Consider a sequence (p_j, q_j) of coprime integer pairs with $q_j > 0$ satisfying the assumptions (3.34) with some $n > 0$ to be fixed later. Taking n sufficiently large, we can apply Theorem 3.3 to any $\phi \in \mathcal{A}'_n$ with $\|\phi\|' \leq 1$ and $(p,q) = (p_j,q_j)$ and, accordingly, we can define $\mathcal{P}(\phi)$ as in (3.37) using the sequence $(g_{q_j})_{j\geq 1}$ of q_j-resonant normal forms furnished by Theorem 3.3.

To study the series on the right hand side of the definition (3.37), we shall take $M_j = 2N_j = 2\kappa q_j$ and decompose $\mathcal{P}(\phi) = \widetilde{\mathcal{P}}(\phi) + \sum_{j\geq 1}\Big(\mathcal{P}_j(\phi) + \mathcal{P}'_j(\phi)\Big)$ where

$$\begin{aligned}
\widetilde{\mathcal{P}}(\phi) &= -\sum_{j\geq 1}\left\langle \frac{1}{2}\ddot{f}_1 + \dddot{\phi}\right\rangle_{q_j}^{N_j},\\
\mathcal{P}_j(\phi) &= \frac{1}{2}\left\langle \ddot{f}_1 - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\langle \dot{f}_1\rangle_{q_j}^{M_j}}{1+\langle f_1\rangle_{q_j}^{M_j}}\right\rangle_{q_j}^{N_j},\\
\mathcal{P}'_j(\phi) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\langle \left(\frac{p_j}{q_j}-\omega\right)^{-2} g_{q_j}(\cdot,0) + \frac{1}{2}\frac{\langle \dot{f}_1\rangle_{q_j}^{M_j}}{1+\langle f_1\rangle_{q_j}^{M_j}}\right\rangle_{q_j}^{N_j}
\end{aligned}$$

and $f_1 : \mathbb{T}_1 \to \mathbb{C}$ is as defined in (3.31). We shall analyze $\widetilde{\mathcal{P}}$, $\mathcal{P}_j$ and $\mathcal{P}'_j$ separately.

We start with $\widetilde{\mathcal{P}}$. By definition of $\|\cdot\|'$-norm, the assumptions $\|\phi\|' \leq 1$ and $\phi \in \mathcal{A}'_n$ imply

$$\|\ddot{\phi}\| = \sum_{|k|\geq n}(2\pi|k|)^2|\hat{\phi}_k|\mathrm{e}^{2\pi|k|} \leq \frac{1}{2\pi n}\sum_{|k|\geq n}(2\pi|k|)^3|\hat{\phi}_k|\mathrm{e}^{2\pi|k|} = \frac{1}{2\pi n}.$$

Similarly, we have $\|\dot{\phi}\| \leq (2\pi n)^{-2}$. From the explicit expression $(1+f_1) = ((1+\dot{\phi})(1+\dot{\phi}^+))^{-1}$, a direct calculation shows that

$$\frac{1}{2}f_1 + \dot{\phi} + \frac{1}{2}\mathcal{D}_\omega\dot{\phi} = \frac{1}{2}(f_1 + \dot{\phi} + \dot{\phi}^+) = \frac{\dot{\phi}^2 + (\dot{\phi}^+)^2 + \dot{\phi}\dot{\phi}^+ + \dot{\phi}^2\dot{\phi}^+ + \dot{\phi}(\dot{\phi}^+)^2}{2(1+\dot{\phi})(1+\dot{\phi}^+)}.$$

By differentiating the above equation twice and using $\|\dot{\phi}\| \le (2\pi n)^{-2}$, $\|\ddot{\phi}\| \le (2\pi n)^{-1}$ and $\|\dddot{\phi}\| \le 1$, it is straightforward to verify that the second-order derivative of the right hand side has $\|\cdot\|$-norm bounded by $\lesssim 1/n$. On the other hand,

$$\|\langle \mathcal{D}_\omega \dddot{\phi} \rangle_{q_j}^{N_j}\| = \|\langle \mathcal{D}_{\omega - p_j/q_j} \dddot{\phi} \rangle_{q_j}^{N_j}\| \lesssim N_j \left|\omega - \frac{p_j}{q_j}\right| \|\dddot{\phi}\|.$$

Thus, using $N_j = \kappa q_j$ we obtain

$$\|\tilde{\mathcal{P}}(\phi)\| \lesssim \frac{1}{n} + \sum_j N_j \left|\omega - \frac{p_j}{q_j}\right| \|\dddot{\phi}\| \lesssim \frac{1}{n} + \kappa \sum_j q_j \left|\omega - \frac{p_j}{q_j}\right|.$$

Using (3.34), we can bound $q_j|\omega - p_j/q_j| < [(2\kappa)^{j-1} n]^{-\nu+1}$ and then bound the series above by a converging geometric series to deduce

$$\|\tilde{\mathcal{P}}(\phi)\| \lesssim \frac{1}{n} + \frac{\kappa n^{-\nu+1}}{1 - (2\kappa)^{-\nu+1}}. \tag{3.43}$$

Next, we consider $\mathcal{P}_j$. By the second inequality in (3.34) and the choice of M_j, we have $q_{j+1} > M_j > N_j$. Using the disjointness of the supports of Fourier series of $\mathcal{P}_j(\phi)$ for different j's (note that $\mathcal{P}_j(\phi)$ has zero average), we have

$$\Big\|\sum_{j\ge 1} \mathcal{P}_j(\phi)\Big\| = \sum_{j\ge 1} \|\mathcal{P}_j(\phi)\| \le \frac{1}{2}\sum_{j\ge 1} \left\| \langle \ddot{f}_1 \rangle_{q_j}^{M_j} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\langle \dot{f}_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}} \right\| \tag{3.44}$$

whereas a straightforward computation yields

$$\langle \ddot{f}_1 \rangle_{q_j}^{M_j} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\langle \dot{f}_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}} = \frac{(\langle \dot{f}_1 \rangle_{q_j}^{M_j})^2}{(1 + \langle f_1 \rangle_{q_j}^{M_j})^2} + \frac{\langle \ddot{f}_1 \rangle_{q_j}^{M_j} \langle f_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}}.$$

Using $\|\dot{\phi}\| \le (2\pi n)^{-2}$ and the formula (3.31) for f_1, we have

$$\|f_1\| \lesssim \|\dot{\phi}\| \lesssim \frac{1}{n^2}, \quad \|\dot{f}_1\| \lesssim \|\ddot{\phi}\| \lesssim \frac{1}{n} \quad \text{and} \quad \|\ddot{f}_1\| \lesssim \|\dddot{\phi}\| \le 1.$$

It follows that, with n sufficiently large,

$$\left\| \langle \ddot{f}_1 \rangle_{q_j}^{M_j} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\langle \dot{f}_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}} \right\| \lesssim \frac{1}{n} \|\langle \dot{f}_1 \rangle_{q_j}^{M_j}\| + \frac{1}{n^2} \|\langle \ddot{f}_1 \rangle_{q_j}^{M_j}\|.$$

Since $\dot{f}_1$ and $\ddot{f}_1$ have zero average, the Fourier supports of $\langle \dot{f}_1 \rangle_{q_j}^{M_j}$ and $\langle \ddot{f}_1 \rangle_{q_j}^{M_j}$ are disjoint for different j. Combining with (3.44), we obtain

$$\Big\|\sum_{j\ge 1} \mathcal{P}_j(\phi)\Big\| \lesssim \sum_{j\ge 1} \left(\frac{1}{n} \|\langle \dot{f}_1 \rangle_{q_j}^{M_j}\| + \frac{1}{n^2} \|\langle \ddot{f}_1 \rangle_{q_j}^{M_j}\| \right) \lesssim \frac{1}{n} \|\dot{f}_1\| + \frac{1}{n^2} \|\ddot{f}_1\| \lesssim \frac{1}{n^2}. \tag{3.45}$$

To analyze $\mathcal{P}_j'(\phi)$, we start with the following decomposition

$$\mathcal{P}_j'(\phi) = \frac{\mathrm{d}}{\mathrm{d}x} \left\langle \left(\omega - \frac{p_j}{q_j}\right)^{-2} g_{q_j}(\cdot, 0) + \frac{1}{2} \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \right\rangle_{q_j}^{N_j} + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \left\langle \frac{\langle \dot{f}_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}} - \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \right\rangle_{q_j}^{N_j}.$$

By the estimate (3.32) of Theorem 3.3 and $N_j = \kappa q_j$, the first term is bounded by

$$\begin{aligned}
&\left\| \frac{\mathrm{d}}{\mathrm{d}x} \left\langle \left(\omega - \frac{p_j}{q_j}\right)^{-2} g_{q_j}(\cdot, 0) + \frac{1}{2} \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \right\rangle_{q_j}^{N_j} \right\| \\
\leq & 2\pi N_j \left\| \left\langle \left(\omega - \frac{p_j}{q_j}\right)^{-2} g_{q_j}(\cdot, 0) + \frac{1}{2} \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \right\rangle_{q_j}^{N_j} \right\| \\
\lesssim & \kappa^2 \mathrm{e}^{4\pi\kappa} q_j^{\nu_0 - \nu + 1}
\end{aligned} \tag{3.46}$$

with $\nu_0 = 64$ as in Theorem 3.3, whereas the second term is

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \left\langle \frac{\langle \dot{f}_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}} - \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \right\rangle_{q_j}^{N_j} =& \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left\langle \log(1 + \langle f_1 \rangle_{q_j}^{M_j}) - \log(1 + \langle f_1 \rangle_{q_j}) \right\rangle_{q_j}^{N_j} \\
=& -\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_0^1 \left\langle \frac{\langle f_1 \rangle_{q_j}^{>M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j} + t \langle f_1 \rangle_{q_j}^{>M_j}} \right\rangle_{q_j}^{N_j} \mathrm{d}t.
\end{aligned}$$

By Lemma 3.26 with $a = 1$, $N = N_j$, $M = M_j$ and $u = \langle f_1 \rangle_{q_j}^{>M_j}$ and the definition $M_j = 2N_j = 2\kappa q_j$,

$$\begin{aligned}
\left\| \left\langle \frac{\langle f_1 \rangle_{q_j}^{>M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j} + t \langle f_1 \rangle_{q_j}^{>M_j}} \right\rangle_{q_j}^{N_j} \right\| \leq & (2\kappa + 1) \mathrm{e}^{-4\pi(M_j - N_j)} \left\| \frac{1}{1 + \langle f_1 \rangle_{q_j}^{M_j} + t \langle f_1 \rangle_{q_j}^{>M_j}} \right\| \| \langle f_1 \rangle_{q_j}^{>M_j} \| \\
\lesssim & \kappa \mathrm{e}^{-4\pi\kappa q_j}.
\end{aligned}$$

Substituting into the integral above, we obtain

$$\left\| \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} \left\langle \frac{\langle \dot{f}_1 \rangle_{q_j}^{M_j}}{1 + \langle f_1 \rangle_{q_j}^{M_j}} - \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \right\rangle_{q_j}^{N_j} \right\| \lesssim N_j^2 \kappa \mathrm{e}^{-4\pi\kappa q_j} \lesssim \kappa^3 q_j^2 \mathrm{e}^{-4\pi\kappa q_j}.$$

Combining with (3.46) and summing over $j \geq 1$, we obtain

$$\left\| \sum_{j \geq 1} \mathcal{P}_j'(\phi) \right\| \lesssim \sum_{j \geq 1} \kappa^2 \mathrm{e}^{4\pi\kappa} q_j^{\nu_0 - \nu + 1} + \kappa^3 q_j^2 \mathrm{e}^{-4\pi\kappa q_j}.$$

Using the assumption $\nu \geq \nu_0 + 6$ and $q_j > (2\kappa)^{j-1} n$ and bounding the series in the right hand side above by convergent geometric series,

$$\left\| \sum_{j \geq 1} \mathcal{P}_j'(\phi) \right\| \lesssim \frac{\kappa^2 \mathrm{e}^{4\pi\kappa}}{1 - (2\kappa)^{-\nu + \nu_0 + 1}} n^{-\nu + \nu_0 + 1} + \kappa^3 n^2 \mathrm{e}^{-4\pi\kappa n}. \tag{3.47}$$

Consolidating (3.43), (3.45) and (3.47), we arrive at

$$\|\mathcal{P}(\phi)\| \lesssim \frac{1}{n} + \frac{\kappa^2 \mathrm{e}^{4\pi\kappa}}{1 - (2\kappa)^{-\nu + \nu_0 + 1}} n^{-\nu + \nu_0 + 1} + \kappa^3 n^2 \mathrm{e}^{-4\pi\kappa n}. \tag{3.48}$$

In particular, the right hand side of (3.37) is an absolutely convergent series in $\mathcal{A}$. Thus $\mathcal{P}(\phi)$ is well-defined as an element in $\mathcal{A}$. Taking n sufficiently big, we can make the right hand side of (3.48) smaller than any given positive number and thus obtain (3.38). Since $\nu_0 = 64$ and we assume $\nu \geq 70$, we have $n^{-\nu + \nu_0 + 1} \leq n^{-5}$.

Next, we consider the analyticity $\phi \mapsto \mathcal{P}(\phi)$. Let $\phi^\tau \in \mathcal{A}'$ depend analytically on some complex parameter $\tau \in \mathbb{C}$. Suppose $\|\phi^\tau\|' \le 1$ for all $|\tau| \le R$ so that $\mathcal{P}(\phi^\tau)$ is defined. By the estimate (3.38), the map $\tau \mapsto \mathcal{P}(\phi^\tau)$ is locally bounded as a map into the Banach space $\mathcal{A}$. By definition (3.37) and the disjointness of the Fourier supports of the projections $\langle \cdot \rangle_{q_j}^{N_j}$, each Fourier coefficient of $\mathcal{P}(\phi^\tau)$ receives contribution from at most one index j. By the analytic dependence of g_{q_j} on ϕ^τ, it follows that each Fourier coefficient of $\mathcal{P}(\phi^\tau)$ is analytic in τ. Thus $\tau \mapsto \mathcal{P}(\phi^\tau)$ is locally bounded and very weakly holomorphic (with respect to the separating family of functionals on $\mathcal{A}$ sending $u \in \mathcal{A}$ to its Fourier coefficients $\hat{u}_k$). By [Are16, Theorem 2.1], it is a $\mathcal{A}$-valued holomorphic function.

Finally, we show (3.39). Put $\phi^\tau = \phi + \tau\delta$ for some $\phi, \delta \in \mathcal{A}'_n$ with $\|\phi\|' < 1$. Then we have that $\mathcal{P}(\phi^\tau)$ is analytic for $|\tau| < \frac{1-\|\phi\|'}{\|\delta\|'}$. Applying Cauchy's estimate to the derivative $\frac{d}{d\tau}\mathcal{P}(\phi^\tau)|_{\tau=0}$ and using (3.38), we obtain

$$\|\mathrm{d}_\phi \mathcal{P}(\delta)\| < \left(\frac{1-\|\phi\|'}{\|\delta\|'}\right)^{-1} \epsilon = \frac{\epsilon}{1-\|\phi\|'}\|\delta\|'.$$

Since δ can be chosen arbitrarily from an open ball, it follows that $\|\mathrm{d}_\phi \mathcal{P}\| \le \epsilon(1-\|\phi\|')^{-1}$ in operator norm. The proof of the lemma is finished. □

Remark. In the proof of Theorem 3.4, one can avoid dealing with analytic functions taking values in an infinite dimensional Banach space by employing a slightly different argument. Let $\hat{u}_k(z)$ be the k-th Fourier coefficient of $\mathcal{P}(\phi + z\delta)$ where $\phi, \delta \in \mathcal{A}'$ such that $\|\phi\|' + \|\delta\|' < 1$. By analytic dependence of g_{q_j}'s, each $\hat{u}_k(z)$ is an analytic function in z for $\|\phi + z\delta\|' < 1$. Let us denote

$$G(\phi,\delta) : x \mapsto \sum_k \hat{u}'_k(0) \mathrm{e}^{2\pi \mathrm{i} k x}.$$

We shall prove that $G(\phi, \cdot) : \mathcal{A}' \to \mathcal{A}$ is a bounded linear map and it is in fact equal to $\mathrm{d}_\phi \mathcal{P}$.

Fix $\theta_k \in \mathbb{R}$ such that $|\hat{u}_k(1) - \hat{u}_k(0) - \hat{u}'_k(0)| = (\hat{u}_k(1) - \hat{u}_k(0) - \hat{u}'_k(0))\mathrm{e}^{\mathrm{i}\theta_k}$. Consider

$$H(z) = \sum_{k\in\mathbb{Z}} \hat{u}_k(z) \mathrm{e}^{\mathrm{i}\theta_k} \mathrm{e}^{2\pi|k|}. \tag{3.49}$$

By the same argument using Montel's theorem and Ascoli's theorem as in the proof of Lemma 3.5, one can show that the series (3.49) converges uniformly on compact subsets of the open disc of radius $(1-\|\phi\|')/\|\delta\|'$ and therefore defines a complex analytic function in z, which is uniformly bounded by ϵ. Similarly, the function $z \mapsto \sum_k \hat{u}_k(z)\mathrm{e}^{2\pi|k|}$ is analytic and uniformly bounded by ϵ over the same domain. Then, by Cauchy's estimate we have $\|G(\phi,\delta)\| \le \epsilon(1-\|\phi\|')^{-1}\|\delta\|'$. Hence, $G(\phi,\cdot)$ is bounded in operator norm by $\epsilon(1-\|\phi\|')^{-1}$. Next, by the choice of θ_k, the mean value theorem and Cauchy's estimate,

$$\begin{aligned}
\|\mathcal{P}(\phi+\delta) - \mathcal{P}(\phi) - G(\phi,\delta)\| =& \sum_k |\hat{u}_k(1) - \hat{u}_k(0) - \hat{u}'_k(0)| \mathrm{e}^{2\pi|k|} \\
=& \sum_k (\hat{u}_k(1) - \hat{u}_k(0) - \hat{u}'_k(0)) \mathrm{e}^{\mathrm{i}\theta_k} \mathrm{e}^{2\pi|k|} \\
=& H(1) - H(0) - H'(0) \\
=& \int_0^1 H''(t)(1-t)dt \\
\le& \frac{\epsilon}{2(\frac{1-\|\phi\|'}{\|\delta\|'} - 1)^2} \\
\lesssim& \frac{\epsilon \|\delta\|'^2}{(1-\|\phi\|' - \|\delta\|')^2}.
\end{aligned}$$

Assuming $\|\delta\|'$ is sufficiently small ($\|\delta\|' < (1 - \|\phi\|')/2$ suffices), we obtain $\|\mathcal{P}(\phi + \delta) - \mathcal{P}(\phi) - G(\phi, \delta)\| \lesssim \epsilon(1 - \|\phi\|')^{-2}\|\delta\|'^2$. This shows that $G(\phi, \cdot)$ is indeed the derivative $\mathrm{d}_\phi\mathcal{P}$ and thus we have $\|\mathrm{d}_\phi\mathcal{P}\| \leq \epsilon(1 - \|\phi\|')^{-1}$ as required.

3.6 Analysis of almost equidistributed periodic orbits: the proof of Theorem 3.1

We will apply Theorem 3.4 to construct a sequence of (p_j, q_j)-periodic orbits of a standard map with well-controlled symplectic actions and eigendata, and then complete the proof of Theorem 3.1.

We start with the following lemma. We recall that the functional space $\mathcal{A}'_n$ and the norm $\|\cdot\|'$ have been defined before Theorem 3.4.

Lemma 3.6. *The following statements hold for all sufficiently large $n > 0$.*

Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and assume there exists a sequence $(p_j, q_j)_{j\geq 1}$ of coprime integer pairs such that, for some $\nu \geq 70$,

$$\left|\omega - \frac{p_j}{q_j}\right| \leq q_j^{-\nu}, \qquad q_{j+1} > 4q_j, \qquad q_1 > n. \tag{3.50}$$

Then, for every $\phi_0 \in \mathcal{A}'_n$ and every sequence $(\sigma_j)_{j\geq 1}$ of complex numbers satisfying

$$|\sigma_j| q_j \geq 1 \quad \textit{for all } j \qquad \textit{and} \qquad \sum_{j\geq 1} |\sigma_j| + 2\|\phi_0\|' \leq \frac{1}{2},$$

there exists $\phi_1 \in \mathcal{A}'_n$, depending analytically on ϕ_0 and on $(\sigma_j)_{j\geq 1}$, with the following properties.

Let $\phi := \phi_0 + \phi_1$ and let F be the standard map associated with (ω, ϕ) in the adapted coordinates.

For each $j \geq 1$ there exist two distinguished (p_j, q_j)-periodic orbits of F in a complex neighborhood of $\mathbb{T} \times \mathbb{R}$. These orbits depend analytically on ϕ_0 and $(\sigma_j)_{j\geq 1}$ and satisfy:

1. *The two orbits are geometrically distinct. If $(\sigma_j)_j$ and ϕ_0 are real-valued, then the two orbits lie in $\mathbb{T} \times \mathbb{R}$ and they are the* only *(p_j, q_j)-periodic orbits of F in $\mathbb{T} \times \mathbb{R}$. In particular, one of them is a (globally) minimal orbit and the other is a minimax orbit.*

2. *Let S be the normalized generating function associated with F in adapted coordinates, and define the rescaled symplectic actions of the two (p_j, q_j)-orbits by*

$$\mathcal{A}_{j,\pm} := \frac{1}{q_j(\omega - p_j/q_j)^2} \sum_{k=0}^{q_j - 1} S \circ F^k(\xi_{j,\pm}), \tag{3.51}$$

where $\xi_{j,+}$ and $\xi_{j,-}$ are any points on the two orbits, respectively. Then

$$\left|\mathcal{A}_{j,\pm} - \frac{1}{2(1 + f_1^*)}\left(1 \mp \frac{2\sigma_j \mathrm{e}^{-2\pi q_j}}{(2\pi q_j)^2}\right)\right| \lesssim q_j^{\nu_0}\left|\omega - \frac{p_j}{q_j}\right| + q_j^{\nu_0 - \nu}\mathrm{e}^{-2\pi q_j}, \tag{3.52}$$

where $f_1 : \mathbb{T}_1 \to \mathbb{C}$ is defined in terms of $\phi = \phi_0 + \phi_1$ as in (3.31)*, f_1^* denotes its average over $\mathbb{T}$, and $\nu_0 = 64$.*

3. *Define the rescaled Lyapunov exponents of these orbits by*

$$\Lambda_{j,\pm} := -\frac{1}{q_j^2(\omega - p_j/q_j)^2} \det\Big(\mathrm{d}_{\xi_{j,\pm}} F^{q_j} - \mathrm{Id}\Big), \tag{3.53}$$

where $\xi_{j,\pm}$ are as in (2)*. Then*

$$\Big|\Lambda_{j,\pm} \mp \sigma_j \mathrm{e}^{-2\pi q_j}\Big| < \frac{1}{50}|\sigma_j|\mathrm{e}^{-2\pi q_j}. \tag{3.54}$$

The main theorem will be deduced from Lemma 3.6 together with the following elementary lemma.

To state Lemma 3.7, we denote by ℓ^∞ the Banach space of bounded sequences $\beta = (\beta_j)_{j\geq 1}$ of complex numbers with the supremum norm

$$|\beta|_{\ell^\infty} := \sup_{j\geq 1} |\beta_j|.$$

Lemma 3.7. *Let B be the unit open ball of ℓ^∞ centered at 0. Let $T : B \to \ell^\infty$ be an analytic map. Suppose*

$$|T(\beta) - \beta|_{\ell^\infty} < \epsilon \qquad \text{for all} \quad \beta \in B. \tag{3.55}$$

Then, for any $\alpha \in \ell^\infty$ such that $|\alpha|_{\ell^\infty} < (1-4\epsilon)/3$, there exists a unique $\beta_* \in \ell^\infty$ depending analytically on α such that $|\beta_*|_{\ell^\infty} < (1-\epsilon)/3$ and $T(\beta_*) = \alpha$. Furthermore, if $\mathcal{U}$ is an open subset of a complex Banach space and the map $T = T_u$ is analytically parametrized by $u \in \mathcal{U}$, i.e., the value $T_u(\beta)_j$ is analytic in u and $\beta \in B$ for each j, then the corresponding β_* such that $T_u(\beta_*) = \alpha$ also depends analytically on u.

Remark. We say that a map $T : U \to \ell^\infty$ defined on an open subset $U \subset \ell^\infty$ is *analytic* if the following property holds: for any family (b^τ) of elements in U parametrized by a complex parameter τ such that, for each index $j \geq 1$, the function $\tau \mapsto b_j^\tau$ is analytic, the function $\tau \mapsto T(b^\tau)_j$ is analytic for every index j.

Proof of Lemma 3.7. We prove this lemma by reducing it to a fixed-point problem and solving it by the usual Picard iteration. Let $\alpha \in \ell^\infty$ be given. Define an auxiliary map

$$\widetilde{T}_\alpha : \beta \mapsto \alpha - T(\beta) + \beta.$$

Then finding β solving the equation $T(\beta) = \alpha$ is equivalent to finding a fixed point of $\widetilde{T}_\alpha$.

By the assumption (3.55), we have $|\widetilde{T}_\alpha(\beta)|_{\ell^\infty} < |\alpha|_{\ell^\infty} + \epsilon$. Hence, for $|\alpha|_{\ell^\infty} + \epsilon \leq r < 1$, the map $\widetilde{T}_\alpha$ preserves the closed ball of radius r centered at 0. Pick any $\beta, \beta' \in \ell^\infty$ such that $|\beta|_{\ell^\infty} + |\beta'|_{\ell^\infty} \leq 1$. Then by the mean value theorem applied to the function $\tau \mapsto \widetilde{T}_\alpha((1-\tau)\beta + \tau\beta')_j$ for an arbitrary $j \geq 1$,

$$\widetilde{T}_\alpha(\beta')_j - \widetilde{T}_\alpha(\beta)_j = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{T}_\alpha((1-\tau)\beta + \tau\beta')_j d\tau.$$

By (3.55) and Cauchy's estimate over the domain $|\tau| \leq (1 - |\beta|_{\ell^\infty})/|\beta - \beta'|_{\ell^\infty}$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{T}_\alpha((1-\tau)\beta + \tau\beta')_j\right| \leq \frac{\epsilon|\beta - \beta'|_{\ell^\infty}}{1 - |\beta|_{\ell^\infty} - |\beta' - \beta|_{\ell^\infty}} \qquad \text{for} \quad |\tau| \leq 1.$$

Since the above bound is independent of j, we deduce

$$|\widetilde{T}_\alpha(\beta') - \widetilde{T}_\alpha(\beta)|_{\ell^\infty} \leq \frac{\epsilon|\beta - \beta'|_{\ell^\infty}}{1 - |\beta|_{\ell^\infty} - |\beta' - \beta|_{\ell^\infty}}.$$

In particular, for $r < 1/3$, the Lipschitz constant of $\widetilde{T}_\alpha$ is at most $\epsilon/(1-3r)$. By the condition on α, there exists r such that $|\alpha| + \epsilon \leq r < (1-\epsilon)/3$. It follows that the map $\widetilde{T}_\alpha$ preserves the closed ball of radius r and is a contracting map there. By the Banach fixed point theorem, we obtain a unique fixed point for $\widetilde{T}_\alpha$ therein.

It remains to show the analytic dependence of β on α and on $u \in \mathcal{U}$, if $T = T_u$ is analytically parametrized by u belonging to an open subset of a complex Banach space. Recall that, by the usual proof of the Banach fixed point theorem, the fixed point β_* can be given by the limit $\beta_* = \lim_{n\to\infty} \widetilde{T}_\alpha^n(0)$, where the contracting property of $\widetilde{T}_\alpha$ implies that $(\widetilde{T}_\alpha^n(0))_n$ is a Cauchy sequence in ℓ^∞. On the other hand, if α^τ depends analytically on a complex parameter τ in a domain of $\mathbb{C}$ and has small $|\cdot|_{\ell^\infty}$-norm uniformly in τ, and T is analytically parametrized by u, then, by the assumed analyticity of T, for fixed j the sequence $\{\widetilde{T}_{\alpha^\tau}^n(0)_j \;:\; n \geq 1\}$ is a Cauchy sequence (with respect to uniform norm in τ) of analytic functions in τ and u. By standard results in complex analysis, the limit $\lim_{n\to\infty} \widetilde{T}_{\alpha^\tau}^n(0)_j$ is an analytic function in τ and u. It follows that β_* is analytic in α and u. The proof of the lemma is finished. $\square$

3.6.1 Proof of the main theorem.

We now complete the proof of the main theorem using Lemma 3.6 and Lemma 3.7. Let ω and (p_j, q_j) be given as in the statement of Theorem 3.1. In particular, we may assume $q_1 > n$ so that Lemma 3.6 applies to the sequence (p_j, q_j). We consider

$$\sigma_j = \frac{2 + \widetilde{\sigma}_j}{q_j} \quad \text{with} \quad |\widetilde{\sigma}_j| \leq 1.$$

This choice of σ_j automatically ensures that $q_j|\sigma_j| \geq 1$. Moreover, for q_1 sufficiently large we have $\sum_j |\sigma_j| < 1/10$. It follows that the condition $\sum_j |\sigma_j| + 2\|\phi_0\|' \leq 1/2$ is satisfied whenever $\|\phi_0\|' \leq 1/5$. In this case, by Lemma 3.6 there exists $\phi_1 \in \mathcal{A}_n'$ depending analytically on $(\widetilde{\sigma}_j)$ and ϕ_0 such that the conclusions of Lemma 3.6 hold. In particular, for each $j \geq 1$ there is a pair of (p_j, q_j)-orbits depending analytically on $(\widetilde{\sigma}_j)$ and ϕ_0. Let $\xi_{j,\pm}$ be points picked from the two distinguished (p_j, q_j)-orbits furnished by Lemma 3.6, respectively. We shall verify that the conclusions of Theorem 3.1 hold with respect to the sequence of orbits $\xi_j = \xi_{j,+}$ provided by Lemma 3.6.

Proof of part (1). Let $\Lambda_j = \Lambda_{j,+}$ with $\Lambda_{j,+}$ defined as in (3.53). We provisionally define

$$\widetilde{\Lambda}_j = q_j \Lambda_j \mathrm{e}^{2\pi q_j} - 2.$$

By analytic dependence of ϕ_1 and ξ_j on (σ_j) and ϕ_0, the map $T : (\widetilde{\sigma}_j)_j \mapsto (\widetilde{\Lambda}_j)_j$ is analytic and is analytically parametrized by ϕ_0. On the other hand, the estimate (3.54) and the choice of σ_j imply

$$\left|\widetilde{\Lambda}_j - \widetilde{\sigma}_j\right| < \frac{1}{16}.$$

We may in turn apply Lemma 3.7 with $\epsilon = 1/16$ and $T : (\widetilde{\sigma}_j)_j \mapsto (\widetilde{\Lambda}_j)_j$ to conclude that, for all sequences $(\alpha_j)_j$ with $|\alpha_j| < 1/4$ and all ϕ_0 such that $\|\phi_0\|' \leq 1/5$, there exists a unique sequence $(\widetilde{\sigma}_j)_j$ with $|\widetilde{\sigma}_j| < 5/16$ that depends analytically on (α_j) and ϕ_0 and satisfies $\widetilde{\Lambda}_j = \alpha_j$. By Lemma 3.6, the function ϕ_1 realizing the prescribed values of $\widetilde{\Lambda}_j$ can be chosen to depend analytically on (α_j) and ϕ_0. It now follows from (3.54) that the q_j-periodic orbits (ξ_j) have a Lyapunov exponent given by

$$\det(\mathrm{d}_{\xi_j} F^{q_j} - \mathrm{Id}) = -(2+\alpha_j) q_j \Big(\frac{p_j}{q_j} - \omega\Big)^2 \mathrm{e}^{-2\pi q_j}.$$

Since the Lyapunov exponent is obviously independent of coordinates, we deduce (3.4), where the potential V_α is defined by (3.14) with $\phi = \phi_0 + \phi_1$ and the choice of additive constant fixed by the normalization condition on S.

Proof of part (2). We impose the following arithmetic condition as in (3.5)

$$\left|\omega - \frac{p_j}{q_j}\right| \le q_j^{-\nu} \mathrm{e}^{-2\pi q_j} \quad \text{with} \quad \nu = 70. \tag{3.56}$$

Let us write $\mathcal{A}_j = \mathcal{A}_{j,+}$ with $\mathcal{A}_{j,+}$ as in (3.51) and provisionally define

$$\widetilde{\mathcal{A}}_j := 2\pi^2 q_j^3 \Big[1 - 2(1+f_1^*)\mathcal{A}_j\Big] \mathrm{e}^{2\pi q_j} - 2.$$

By analyticity we have similarly that $T : (\widetilde{\sigma}_j)_j \mapsto (\widetilde{\mathcal{A}}_j)_j$ is an analytic map. Combining the arithmetic condition (3.56) with (3.52),

$$\left|\widetilde{\mathcal{A}}_j - \widetilde{\sigma}_j\right| \lesssim q_j^{\nu_0 + 3 - \nu} \lesssim \frac{1}{q_j^3}$$

In particular, if q_1 is large enough, then the right hand side above can be bounded uniformly in j by $< 1/16$. We may in turn apply Lemma 3.7 as in the preceding proof of part (2) to conclude that, for all sequences $(\alpha_j)_j$ with $|\alpha_j| < 1/4$ and all ϕ_0 such that $\|\phi_0\|' \le 1/5$, there exists a unique sequence $(\widetilde{\sigma}_j)_j$ with $|\widetilde{\sigma}_j| < 5/16$ such that $\widetilde{\mathcal{A}}_j = \alpha_j$. By Lemma 3.6, there exists ϕ_1 depending analytically on (α_j) and ϕ_0 such that the standard map F associated with $(\omega, \phi = \phi_0 + \phi_1)$ in adapted coordinates has a sequence of q_j-periodic orbits (ξ_j) whose symplectic action with respect to the normalized generating function S in adapted coordinates is

$$\sum_{k=0}^{q_j - 1} S \circ F^k(\xi_j) = \frac{q_j \big(\frac{p_j}{q_j} - \omega\big)^2}{2(1+f_1^*)} \Big[1 - \frac{(2+\alpha_j)\mathrm{e}^{-2\pi q_j}}{2\pi^2 q_j^3}\Big].$$

By (3.21), the symplectic action of (ξ_j) with respect to the normalized generating function $\mathscr{S}$ in the original coordinates is

$$\sum_{k=0}^{q_j-1} \mathscr{S} \circ \mathscr{H} \circ F^k(\xi_j) = \sum_{k=0}^{q_j-1} S \circ F^k(\xi_j) + \Big(\omega - \int_{\mathbb{T}} \phi \dot{\phi}^+\Big) p_j$$

where $\mathscr{H}$ is the coordinate transform defined in (3.15). Hence, we have deduced (3.6).

If (σ_j) and ϕ_0 are real-valued as assumed in Theorem 3.1, then by Lemma 3.6 part (1), the standard map has only two geometrically distinct (p_j, q_j)-orbits, containing $\xi_{j,+}$ and $\xi_{j,-}$, respectively. Since $\sigma_j > 0$ by our choice, it follows easily from (3.52) and the estimate (3.56) that the orbit of $\xi_{j,+}$ has smaller symplectic action between the two, and hence $\xi_j = \xi_{j,+}$ is a (globally) minimal orbit of rotation number p_j/q_j.

Finally, it remains to show that one can choose V_α in such a way that the quantities f_1^* and $(\phi_\alpha \dot{\phi}_\alpha^+)^*$ are both constant in α. The main idea is that both quantities are dominated by the low-frequency components in the Fourier series of ϕ. Hence, if q_1 is sufficiently large compared to the order of nonzero Fourier coefficients of ϕ_0, then the choice of ϕ_0 will have a dominant effect on f_1^* and $(\phi_\alpha \dot{\phi}_\alpha^+)^*$. By adjusting ϕ_0, we can keep the two quantities constant while varying α.

Let us formalize this argument. We define a map

$$\mathcal{G} : (\phi_0, \phi_1) \mapsto \left(f_1^*, (\phi \dot{\phi}^+)^* \right)$$

where $\phi = \phi_0 + \phi_1$ and f_1 is defined as in (3.31). In the preceding steps, we have constructed an analytic map $(\alpha, \phi_0) \mapsto \phi_1$ where the standard map associated with $(\omega, \phi = \phi_0 + \phi_1)$ has a sequence of closed orbits with specified symplectic actions. Since ϕ_1 is determined by (α, ϕ_0), we consider $\mathcal{G}(\phi_0, \phi_1)$ as a function in terms of (α, ϕ_0) and we apply the implicit function theorem. In fact, we pick a nonzero $\widetilde{\phi}_0 \in \mathcal{A}_n'$ and consider the derivative

$$\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\phi_0} = \frac{\partial \mathcal{G}}{\partial \phi_0} + \frac{\partial \mathcal{G}}{\partial \phi_1}\frac{\partial \phi_1}{\partial \phi_0}.$$

Pick $\widetilde{\phi}_0 \in \mathcal{A}_n'$ such that the derivative $\frac{\partial \mathcal{G}}{\partial \phi_0}$ is surjective onto $\mathbb{C}^2$ and take $q_1 \gg n$. Since

$$\sup_{\Im x = 0} |\phi_1(x)| \le \mathrm{e}^{-2\pi q_1},$$

it is easy to establish by Cauchy's estimate that $\frac{\partial \mathcal{G}}{\partial \phi_1}\frac{\partial \phi_1}{\partial \phi_0}$ has arbitrarily small operator norm as $q_1 \to \infty$. Hence, for q_1 sufficiently large, $\mathrm{d}\mathcal{G}/\mathrm{d}\phi_0$ is surjective and we can apply the implicit function theorem to obtain $\epsilon_0 > 0$ and an analytic map $\alpha \mapsto \phi_0$ such that $\mathcal{G}(\phi_0, \phi_1)$ is constant in α for $|\alpha|_{\ell^\infty} < \epsilon_0$.

We have completed the proof of Theorem 3.1 using Lemma 3.6. We proceed to prove Lemma 3.6 now.

3.6.2 Proof of Lemma 3.6.

From now on, we fix

$$\kappa = 2.$$

Let $n > 0$ be given by Theorem 3.4 with $\epsilon = 1/4$ and $\kappa = 2$. Take $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and a sequence (p_j, q_j) such that (3.50) holds for some $\nu \ge 70$. Let $\phi_0 \in \mathcal{A}_n'$ and $(\sigma_j)_j$ be given as in Lemma 3.6. In particular, we assume

$$|\sigma_j| q_j \ge 1$$

and

$$\sum_j |\sigma_j| + 2\|\phi_0\|' \le \frac{1}{2}. \tag{3.57}$$

Application of Theorem 3.4. We apply Theorem 3.4 to the function $h \in \mathcal{A}$ defined by

$$h(x) := \sum_{j \ge 1} \sigma_j \mathrm{e}^{-2\pi q_j} \cos(2\pi q_j x). \tag{3.58}$$

Note that, by the definition of the Fourier-weighted norm,

$$\|h\| = \sum_{j=1}^{\infty} |\sigma_j|.$$

By (3.57), the condition (3.35) of Theorem 3.4 is satisfied with the choice $\epsilon = 1/4$.

Fix $N_j := \kappa q_j = 2q_j$. Then, by Theorem 3.4, there exists $\phi_1 \in \mathcal{A}'_n$ depending analytically on h and ϕ_0 such that the standard map associated with $(\omega, \phi := \phi_0 + \phi_1)$ in the q_j-resonant coordinates can be written as

$$F_{q_j}(x, y) = \Big(x + p_j/q_j + y,\ y + g_{q_j}(x, y)\Big),$$

where the resonant normal form g_{q_j} satisfies (3.36).

For notational simplicity, put

$$\mathfrak{g}_{q_j} := \langle g_{q_j} \rangle_{q_j}^{N_j}.$$

By (3.36) and our choice of h,

$$\frac{\partial \mathfrak{g}_{q_j}}{\partial x}(x, 0) = \sigma_j \mathrm{e}^{-2\pi q_j} \Big(\frac{p_j}{q_j} - \omega\Big)^2 \cos(2\pi q_j x). \tag{3.59}$$

Integrating (3.59) in x gives

$$\mathfrak{g}_{q_j}(x, 0) = \frac{\sigma_j \mathrm{e}^{-2\pi q_j}}{2\pi q_j} \Big(\frac{p_j}{q_j} - \omega\Big)^2 \sin(2\pi q_j x) \ +\ g^*_{q_j}(0), \tag{3.60}$$

where $g^*_{q_j}(0)$ denotes the average of $x \mapsto g_{q_j}(x, 0)$ over $\mathbb{T}$.

By Theorem 3.3, the functions $\mathfrak{g}_{q_j}$ and g_{q_j} are defined on $\mathbb{T}_{a_j} \times \mathbb{D}_{b_j}$ with

$$a_j = 1 - \frac{3}{q_j}, \qquad b_j = \frac{1}{5}\Big|\omega - \frac{p_j}{q_j}\Big|.$$

The conclusions (3.29) and (3.32) respectively read as

$$|g_{q_j} - \mathfrak{g}_{q_j}|_{1/q_j, b_j} \ \lesssim\ \mathrm{e}^{-6\pi q_j} \Big|\omega - \frac{p_j}{q_j}\Big|^2, \tag{3.61}$$

$$\Big\| \Big\langle g_{q_j}(\cdot, 0) + \frac{1}{2}\Big(\frac{p_j}{q_j} - \omega\Big)^2 \frac{\langle \dot{f}_1 \rangle_{q_j}}{1 + \langle f_1 \rangle_{q_j}} \Big\rangle_{q_j}^{N_j} \Big\| \ \lesssim\ q_j^{\nu_0 - \nu} \Big|\omega - \frac{p_j}{q_j}\Big|^2, \tag{3.62}$$

where f_1 is as in (3.31) with $\phi = \phi_0 + \phi_1$, and we have absorbed the factor $\kappa \mathrm{e}^{4\pi\kappa}$ into the implicit constant because $\kappa = 2$ is fixed.

Moreover, by (3.30), for $|y| < b_j$ we have

$$|g^*_{q_j}(y)| \ \lesssim\ q_j^{\nu_0 - \nu} \Big(\sup_{\Im x = 0} |(g_{q_j} - \mathfrak{g}_{q_j})(x, y)| + \sup_{\Im x = 0} |\mathfrak{g}^\bullet_{q_j}(x, y)| \Big), \tag{3.63}$$

since $g^\bullet_{q_j} = g_{q_j} - g^*_{q_j} = \Big(g_{q_j} - \mathfrak{g}_{q_j}\Big) + \Big(\mathfrak{g}_{q_j} - \mathfrak{g}^*_{q_j}\Big) = \Big(g_{q_j} - \mathfrak{g}_{q_j}\Big) + \mathfrak{g}^\bullet_{q_j}$ and $\mathfrak{g}^*_{q_j} = g^*_{q_j}$.

For $y = 0$, we have an explicit formula (3.60) for $\mathfrak{g}_{q_j}(x, 0)$. Hence, using (3.61) and the fact that $|\sigma_j| \geq 1/q_j \geq \mathrm{e}^{-4\pi q_j}$ for q_j sufficiently large, we obtain

$$|g^*_{q_j}(0)| \ \lesssim\ |\sigma_j|\, q_j^{\nu_0 - \nu - 1} \mathrm{e}^{-2\pi q_j} \Big|\omega - \frac{p_j}{q_j}\Big|^2. \tag{3.64}$$

For general $|y| < b_j$, we have, by $\kappa = 2$ and the estimate for Fourier coefficients,

$$\begin{aligned} |\mathfrak{g}^\bullet_{q_j}|_{1/q_j,b_j} &\le \sum_{k\in\pm\{q_j,2q_j\}} \sup_{|y|<b_j} |\widehat{g}_{q_j}(y)_k| \, \mathrm{e}^{2\pi\frac{|k|}{q_j}} \\ &\le \sum_{k\in\pm\{q_j,2q_j\}} \mathrm{e}^{-2\pi(a_j-\frac{1}{q_j})|k|} \, |g_{q_j}|_{a_j,b_j} \\ &\lesssim \mathrm{e}^{-2\pi q_j} |g_{q_j}|_{a_j,b_j} \ \lesssim\ \mathrm{e}^{-2\pi q_j}\Big|\omega - \frac{p_j}{q_j}\Big|^2, \end{aligned}$$

where in the last inequality we used (3.28). Combining this with (3.63) and (3.61), and noting that

$$\sup_{\Im x=0} |(g_{q_j} - \mathfrak{g}_{q_j})(x,y)| \ \le\ |g_{q_j} - \mathfrak{g}_{q_j}|_{1/q_j,b_j},$$

we obtain

$$\sup_{|y|<b_j} |g^*_{q_j}(y)| \ \lesssim\ \mathrm{e}^{-2\pi q_j}\Big|\omega - \frac{p_j}{q_j}\Big|^2.$$

It follows from $\mathfrak{g}_{q_j} = g^*_{q_j} + \mathfrak{g}^\bullet_{q_j}$ that

$$\Big|\mathfrak{g}_{q_j}\Big|_{1/q_j,b_j} \ \lesssim\ \mathrm{e}^{-2\pi q_j}\Big|\omega - \frac{p_j}{q_j}\Big|^2. \tag{3.65}$$

Almost equidistributed orbits. We now show that F_{q_j} has two geometrically distinct (p_j, q_j)-periodic orbits and they are *almost equidistributed* in the sense that they closely shadow the sequence of points $(x^* + kp_j/q_j, 0)$, $k = 0, 1, \dots, q_j - 1$ for some $x^* \in \mathbb{T}_{1/q_j}$.

To ease the notation, we will suppress the subscripts j and q_j in the following two paragraphs and write simply $(p, q) = (p_j, q_j)$, $N = N_j$, $\sigma = \sigma_j$, $g = g_{q_j}$, $\mathfrak{g} = \mathfrak{g}_{q_j}$, $a = a_j = 1 - 3/q$ and $b = b_j = |\omega - p/q|/5$. We will apply the estimates (3.61), (3.62), (3.64) and (3.65) obtained by the preceding paragraph. Also recall that, by (3.60),

$$\mathfrak{g}(x, 0) = \frac{\sigma \mathrm{e}^{-2\pi q}}{2\pi q}\Big(\frac{p}{q} - \omega\Big)^2 \sin(2\pi q x) + g^*(0).$$

The first term on the right hand side has exactly $2q$ simple zeros at $x = k/(2q)$ for $k = 0, \dots, 2q - 1$. By (3.64), we may view the second term $g^*(0)$ as a small perturbation of the first term and use Rouché's theorem and periodicity of $\mathfrak{g}$ to conclude that $\mathfrak{g}(\cdot, 0)$ has exactly $2q$ simple zeros in $\mathbb{T}_{1/q}$ which are equidistributed and close to the points $k/(2q)$. More precisely, for some $x^* \in \mathbb{T}_a$ with $|x^*| \lesssim q^{\nu_0-\nu-1}$, we have

$$\mathfrak{g}\Big(x^*_\pm + \frac{k}{q}, 0\Big) = 0, \quad \text{where } k = 0, \dots, q - 1\ ,\ x^*_+ = x^* \text{ and } x^*_- = \frac{1}{2q} - x^*. \tag{3.66}$$

By $1/q$-periodicity of $\mathfrak{g}(\cdot, 0)$, it is readily verified that the points $(x^*_\pm + k/q, 0)$ are (p, q)-periodic under the "reduced" map

$$\mathfrak{F}_q(x, y) = (x + p/q + y, y + \mathfrak{g}(x, y)). \tag{3.67}$$

In view of (3.61), we may consider the map

$$F_q(x, y) = (x + p/q + y, y + g(x, y))$$

as a small perturbation of (3.67). We will show in the following lemma that the periodic orbits of $\mathfrak{F}_q$ starting from $(x^*_\pm + k/q, 0)$ persist under this perturbation.

Lemma 3.8. *Let $x^*_\pm + k/q$ be the zeros of $\mathfrak{g}(\cdot, 0)$ given by* (3.66)*. Set*

$$a' := \frac{1}{2q}, \qquad b' := \frac{1}{6}\left|\omega - \frac{p}{q}\right|, \qquad \alpha^k_\pm := \frac{\partial \mathfrak{g}}{\partial x}\Big(x^*_\pm + \frac{k}{q}, 0\Big).$$

Then, for q sufficiently large (depending only on the universal implicit constants in the estimates below), the following holds.

There exist exactly $2q$ distinct points

$$(\tilde{x}^k_+, \tilde{y}^k_+), \qquad (\tilde{x}^k_-, \tilde{y}^k_-), \qquad k = 0, \dots, q-1,$$

lying in $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$ which are (p,q)-periodic under the map F_q. More precisely, these $2q$ points form exactly two geometrically distinct (p,q)-periodic orbits of F_q: for each sign $\pm$,

$$F^n_q(\tilde{x}^k_\pm, \tilde{y}^k_\pm) = \Big(\tilde{x}^{k+np}_\pm, \tilde{y}^{k+np}_\pm\Big), \qquad n \in \mathbb{Z},$$

where the index $k + np$ is taken modulo q.

Furthermore, the points $(\tilde{x}^k_\pm, \tilde{y}^k_\pm)$ depend analytically on the parameters $(\sigma_j)_j$ and on ϕ_0, and the following estimates hold:

$$\left|\tilde{x}^k_\pm - (x^*_\pm + k/q)\right| \lesssim q\,\mathrm{e}^{-4\pi q}, \qquad |\tilde{y}^k_\pm| \lesssim q^3 \mathrm{e}^{-6\pi q}\left|\omega - \frac{p}{q}\right|^2, \tag{3.68}$$

$$\left|\det\Big(\mathrm{d}_{(\tilde{x}^k_\pm, \tilde{y}^k_\pm)} F^q_q - \mathrm{Id}\Big) + q^2 \alpha^k_\pm\right| \ll q^2 |\alpha^k_\pm|. \tag{3.69}$$

Remark. In the above lemma and its proof, we use the notation $A \ll B$ to mean that for all $\varepsilon > 0$ there exists $Q > 0$ depending only on ε such that $|A/B| < \varepsilon$ for all $q \geq Q$.

Proof of Lemma 3.8. Consider the interpolation map

$$F^\tau_q(x, y) = (x + p/q + y, y + \mathfrak{g}(x, y) + \tau(g(x, y) - \mathfrak{g}(x, y))).$$

Since $b \ll q^{-2}$ and $(3q-1)(|\mathfrak{g}|_{1/q,b} + |g - \mathfrak{g}|_{1/q,b}) \ll b$ by (3.65) and (3.61), we may apply Lemma 3.23 (with $f = 0$ and $a = 1/q$) to F^τ_q for $|\tau| \leq R$ with some $R > 1$ to obtain $\gamma^\tau, \Gamma^\tau : \mathbb{T}_{a'} \to \mathbb{C}$ depending analytically on τ. Using Cauchy's estimate in τ and choosing R such that, for example, $(3q-1)(|\mathfrak{g}|_{1/q,b} + R|\mathfrak{g} - g|_{1/q,b}) = b$, we have

$$\left|\Gamma^1 - \Gamma^0\right|_{a'} \leq \frac{\sup_{|\tau| \leq R} |\Gamma^\tau|_{a'}}{R - 1} \lesssim q|\mathfrak{g} - g|_{1/q,b}. \tag{3.70}$$

We first consider $k = 0$ and look for the periodic orbit near the point $(x^*_+, 0)$. The cases for x^*_- and other k can be treated similarly. Observe in particular that by the explicit formula (3.60) for $\mathfrak{g}(\cdot, 0)$ and the equality $x^*_- = (2q)^{-1} - x^*_+$, the quantity $\alpha^k_\pm$ is independent of k and $\alpha^0_+ = -\alpha^0_-$.

From now on until the end of the proof, we write $x^* = x^*_+$ for simplicity of notation. Let us put $\alpha = \alpha^0_+ = \partial\mathfrak{g}/\partial x(x^*, 0)$. By the explicit formula (3.60),

$$\alpha = \sigma \mathrm{e}^{-2\pi q}\Big(\frac{p}{q} - \omega\Big)^2 \cos(2\pi q x^*). \tag{3.71}$$

Since $(x^*, 0)$ is (p, q)-periodic under $\mathfrak{F}_q$, it follows from Lemma 3.23 that $\Gamma^0(x^*) = 0$ and $\gamma^0(x^*) = 0$. By (3.200) and periodicity of $\mathfrak{g}(\cdot, 0)$ as well as the estimates (3.65) and $b \ll q^{-2}$

$$\left|\Gamma^0 - q\mathfrak{g}(\cdot, 0)\right|_{a'} \lesssim \frac{q^2|\mathfrak{g}|^2_{1/q,b}}{b}. \tag{3.72}$$

By Cauchy's estimate and $|x^*| \lesssim q^{\nu_0-\nu-1} \ll a'$, we have

$$\left|\dot{\Gamma}^0(x^*) - q\alpha\right| \lesssim \frac{q^3|\mathfrak{g}|^2_{1/q,b}}{b} \ll q|\alpha|. \tag{3.73}$$

The last $\ll$ can be obtained from the explicit (3.71). We apply Lemma 3.9 with $R = a' - |x^*|$, $\eta_0 = \Gamma^0$ and $\eta_1 = \Gamma^1 - \Gamma^0$. To verify the condition (3.78), observe that, by (3.70), (3.61) and (3.65),

$$r_0 r_1 \leq \frac{|\Gamma^0|_{a'}|\Gamma^0 - \Gamma^1|_{a'}}{(a' - |x^*|)^2} \lesssim q^4|\mathfrak{g}|_{1/q,b}|g - \mathfrak{g}|_{1/q,b} \ll q^2|\alpha|^2.$$

Combining with (3.73), we readily verify the condition (3.78) for q sufficiently large. Hence, the lemma applies and we obtain a zero $\widetilde{x}$ of Γ^1 satisfying

$$|\widetilde{x} - x^*| \lesssim \frac{|\Gamma^1 - \Gamma^0|_{a'}}{|\dot{\Gamma}^0(x^*)|} \lesssim \frac{|\mathfrak{g} - g|_{1/q,b}}{|\alpha|} \lesssim q\mathrm{e}^{-4\pi q}. \tag{3.74}$$

Let us put $\widetilde{y} := \gamma^1(\widetilde{x})$. Then, since $\gamma^0(x^*) = 0$, by (3.70), (3.74) and the triangle inequality,

$$\begin{aligned}
|\widetilde{y}| \leq& |\gamma^1(\widetilde{x}) - \gamma^0(\widetilde{x})| + |\gamma^0(\widetilde{x}) - \gamma^0(x^*)| \\
\lesssim& q|\mathfrak{g} - g|_{1/q,b} + \frac{q^2|\mathfrak{g}|_{1/q,b}|\mathfrak{g} - g|_{1/q,b}}{|\alpha|} \\
\lesssim& \frac{q^2|\mathfrak{g}|_{1/q,b}|\mathfrak{g} - g|_{1/q,b}}{|\alpha|} \\
\lesssim& q^3\mathrm{e}^{-6\pi q}|\omega - p/q|^2.
\end{aligned}$$

In the penultimate line, we have used $|\alpha| \lesssim q|\mathfrak{g}|_{1/q,b}$, which follows from Cauchy's estimate. We have established the estimates (3.68) for $k = 0$.

For (3.69), we decompose

$$\begin{aligned}
&|\det(\mathrm{d}_{(\widetilde{x},\widetilde{y})}F_q^q - \mathrm{Id}) + q^2\alpha| \\
\leq& |\det(\mathrm{d}_{(\widetilde{x},\widetilde{y})}F_q^q - \mathrm{Id}) + q\dot{\Gamma}^1(\widetilde{x})| + |q\dot{\Gamma}^1(\widetilde{x}) - q\dot{\Gamma}^0(x^*)| + |q\dot{\Gamma}^0(x^*) - q^2\alpha|.
\end{aligned} \tag{3.75}$$

By (3.73), the last term on the right hand side is $\ll q^2|\alpha|$. By the second inequality of (3.79), the second term on the right hand side of (3.75) can be estimated as

$$|q\dot{\Gamma}^1(\widetilde{x}) - q\dot{\Gamma}^0(x^*)| \lesssim \frac{q^4|\mathfrak{g}|_{1/q,b}|g - \mathfrak{g}|_{1/q,b}}{|\alpha|} \ll q^2|\alpha|.$$

Finally, by (3.201), the first term on the right hand side of (3.75) can be estimated as

$$|\det(\mathrm{d}_{(\widetilde{x},\widetilde{y})}F_q^q - \mathrm{Id}) + q\dot{\Gamma}^1(\widetilde{x})| \lesssim \frac{q^4|g|^2_{1/q,b}}{b} \ll q^2|\alpha|.$$

This completes the proof of (3.69) for $(\widetilde{x}, \widetilde{y}) = (\widetilde{x}^0_+, \widetilde{y}^0_+)$. The cases for other k can be treated similarly by replacing x^* with $x^*_\pm + k/q$ in the above argument.

Finally, we show the uniqueness of the (p,q)-periodic points in $\mathbb{T}_{a'}\times\mathbb{D}_{b'}$, where $b' = |\omega - p/q|/6$ as in the statement. Since $q^{-4} \ll |\alpha|/(q^2|\mathfrak{g}|_{1/q,b}) \lesssim |\dot{\Gamma}^0(x^*)|(a' - |x^*|)^2/|\Gamma^0|_{a'}$, Lemma 3.9 implies that $\tilde{x}^k_\pm$ is the unique zero of Γ^1 in the q^{-4}-neighborhood of $x^*_\pm + k/q$. Hence, any (p,q)-periodic point of F_q lying in a q^{-4}-neighborhood of some $x^*_\pm + k/q$ in the x-coordinate must coincide with the point $(\tilde{x}^k_\pm, \tilde{y}^k_\pm)$ constructed above. We need to show that there is no (p,q)-periodic point in $\mathbb{T}_{a'}\times\mathbb{D}_{b'}$ which lies outside these neighborhoods. By the explicit expression for $\mathfrak{g}(\cdot,0)$,

$$q^{-6}\mathrm{e}^{-2\pi q}|\omega - p/q|^2 \ll |\mathfrak{g}(x,0)| \tag{3.76}$$

for $x \in \mathbb{T}_{a'}$ lying outside the union of the q^{-4}-neighborhoods of the points $x^*_\pm + k/q$. On the other hand, by (3.72) and (3.70), we have

$$\Big|\Gamma^1 - q\mathfrak{g}(\cdot,0)\Big|_{a'} \lesssim \frac{q^2|\mathfrak{g}|^2_{1/q,b}}{b} + q|\mathfrak{g} - g|_{1/q,b} \lesssim q^2\mathrm{e}^{-4\pi q}|\omega - p/q|^3 + q\mathrm{e}^{-6\pi q}|\omega - p/q|^2. \tag{3.77}$$

Combining (3.76) and (3.77), we deduce that Γ^1 does not vanish outside the q^{-4}-neighborhoods of the points $x^*_\pm + k/q$, which implies via Lemma 3.23 that the points $(\tilde{x}^k_\pm, \tilde{y}^k_\pm)$ constructed above are the only (p,q)-periodic points of F_q in $\mathbb{T}_{a'}\times\mathbb{D}_{b'}$. It is also simple to verify, for example using the formula for F_q and $|\tilde{y}^k_\pm| \ll q^{-4}$, that we must have $F^n_q(\tilde{x}^k_\pm, \tilde{y}^k_\pm) = (\tilde{x}^{k+np}_\pm, \tilde{y}^{k+np}_\pm)$. Finally, the analyticity of the points $(\tilde{x}^k_\pm, \tilde{y}^k_\pm)$ in terms of g follows from the construction using Lemma 3.9 and Lemma 3.23. The proof is complete. ☐

The following lemma was used in the proof of Lemma 3.8. This is an elementary result in complex analysis, but we nevertheless give a proof for clarity.

Lemma 3.9. *Let $R > 0$ and $\eta_0, \eta_1 : \mathbb{D}_R \to \mathbb{C}$ be bounded analytic functions such that $\eta_0(0) = 0$ and $\eta_0'(0) \neq 0$. Let us denote $r_j = R^{-1}\sup\{|\eta_j(z)| : |z| < R\}$ for $j = 0, 1$. Suppose*

$$r_0 r_1 < c|\eta_0'(0)|^2 \tag{3.78}$$

for some universal constant c. Then there exists a unique zero x of $\eta_0 + \eta_1$ within the open disc $\mathbb{D}_{\bar{R}}$ with

$$\bar{R} = 2c\frac{|\eta_0'(0)|}{r_0}R.$$

Moreover, we have

$$\frac{|x|}{R} \lesssim \frac{r_1}{|\eta_0'(0)|} \quad \textit{and} \quad |\eta_0'(x) + \eta_1'(x) - \eta_0'(0)| \lesssim \frac{r_0 r_1}{|\eta_0'(0)|}. \tag{3.79}$$

Proof. After replacing $\eta_j(x)$ by $\eta_j(Rx)/(R\eta_0'(0))$ we may assume $R = 1$ and $\eta_0'(0) = 1$ so that r_j is the uniform norm of η_j over the unit disc. By the mean value theorem we can write

$$\eta_0(x) = x + x^2\int_0^1 \eta_0''(tx)(1-t)dt.$$

By Cauchy's estimate applied to $|x| < 1/2$, we have $|\eta_0(x) - x| < |x|/2$ if $|x| < (16r_0)^{-1}$. Hence, if $r_1 < (32r_0)^{-1}$, then Rouché's theorem implies that there exists a unique simple zero x^* of $\eta_0 + \eta_1$ in the disc $|x| < (16r_0)^{-1}$. In fact, Rouché's theorem applies to the circle of radius $2r_1$ and we deduce $|x^*| \lesssim r_1$. By the mean value theorem and Cauchy's estimate, we have $|\eta_0'(x^*) + \eta_1'(x^*) - \eta_0'(0)| \lesssim r_0|x^*| + r_1 \lesssim r_0 r_1$. In the last $\lesssim$ we used the fact $r_0 \geq 1$, which follows from Cauchy's estimate of $|\eta_0'(0)|$. After scaling η_0 and η_1 back to the original functions, we obtain the desired conclusions with $c = 1/32$. ☐

We complement Lemma 3.8 with the following corollary showing the uniqueness of the orbits that it constructs.

Corollary 3.10. *Suppose that ϕ is real-valued, so that the standard map F acts on the real locus $\mathbb{T} \times \mathbb{R}$ in the adapted coordinates. Then the (p, q)-periodic points $(\widetilde{x}^k_\pm, \widetilde{y}^k_\pm)$ of F_q obtained in Lemma 3.8 lie in the real locus, and their images under H_q give the only (p, q)-periodic orbits of F contained in $\mathbb{T} \times \mathbb{R}$.*

Proof of Corollary 3.10. Let ϕ be real-valued so that F acts on $\mathbb{T} \times \mathbb{R}$. By Lemma 3.8, the points $(\widetilde{x}^k_\pm, \widetilde{y}^k_\pm)_{k=0}^{q-1}$ in Lemma 3.8 exhaust all (p, q)-periodic points in the domain $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$ of F_q. To prove the required statement it is enough to show that all (p, q)-orbits of F must intersect the subset $H_q(\mathbb{T}_{a'} \times \mathbb{D}_{b'})$. For this, it suffices to show that every (p, q)-periodic orbit of F (in the real locus $\mathbb{T} \times \mathbb{R}$) must intersect the real annulus A defined as

$$A := \mathbb{T} \times \left\{ y \in \mathbb{R} \ : \ \left| y - \left(\frac{p}{q} - \omega\right) \right| < \frac{1}{15} \left| \omega - \frac{p}{q} \right| \right\}.$$

Indeed, the estimate (3.26) clearly implies $A \subset H_q(\mathbb{T}_{a'} \times \mathbb{D}_{b'})$.

We also recall that the conditions $\| \dddot{\phi} \| \leq 1$ and $\phi^* = 0$ imposed by Theorem 3.3 automatically guarantee that $\|\dot{\phi}\| \leq (2\pi)^{-2}$ by definition of the $\| \cdot \|$-norm. Then, by the mean value theorem applied to (3.17) and the bound on $\|\dot{\phi}\|$, it is straightforward to show (for example using the formula (3.137)) that $\max(2|f(x, y)|, |g(x, y)|) < |y|/8$.

Now, let $(x_k, y_k) = F^k(x_0, y_0)$ be a (p, q)-periodic orbit under F lifted to the universal cover. In particular, $x_q = x_0 + p$ and $y_q = y_0$. Assume for a contradiction that (x_k, y_k) does not intersect A for any k. Then we observe that $y_k - (p/q - \omega)$ must have the same sign for all k. For, if not, then we have $|y_k - y_{k+1}| \geq \frac{2}{15}|\omega - p/q|$ and $|y_k| \leq \frac{14}{15}|\omega - p/q|$ for some k. By (3.16) and (3.17), we deduce $\frac{2}{15}|\omega - p/q| \leq |g(x_k, y_k)|$. However, this implies $\frac{2}{15}|\omega - p/q| < |y_k|/8$, which contradicts $|y_k| < |\omega - p/q|$.

We have shown that all points (x_k, y_k) of the orbit lie on the same side of A. Suppose for example $p/q - \omega > 0$ and $y_k \geq \frac{16}{15}(p/q - \omega)$ for all k. Since $x_q - x_0 = p$, we have $x_{k+1} - x_k \leq p/q$ for some k, which implies $y_k + f(x_k, y_k) \leq p/q - \omega$. On the other hand, since $|f(x_k, y_k)| < |y_k|/16$ we have $y_k < (p/q - \omega)/(1 - 1/16)$, which contradicts $y_k > \frac{16}{15}(p/q - \omega)$. Hence, we have shown that any (p, q)-orbit of F must intersect A, which by $A \subset H_q(\mathbb{T}_{a'} \times \mathbb{D}_{b'})$ implies that it intersects $H_q(\mathbb{T}_{a'} \times \mathbb{D}_{b'})$. The cases $y_k \leq \frac{14}{15}(p/q - \omega)$ for all k and the cases with $p/q - \omega < 0$ can be treated similarly. It follows from this argument that this orbit must intersect A and it must be one of the (p, q)-orbits obtained by Lemma 3.8. $\square$

End of proof of Lemma 3.6. Finally, we show that for each j the two periodic orbits constructed in Lemma 3.8 satisfy the conclusions of Lemma 3.6, thereby completing its proof.

The uniqueness statement in part (1) of Lemma 3.6 follows immediately from Corollary 3.10. We now verify the estimates (3.54) and (3.52) for these periodic orbits.

We begin with (3.54). In the case of the zero $x^*_- + k/q$, the corresponding quantity α^k_- in (3.69) is given by

$$\alpha^k_- = -\sigma \mathrm{e}^{-2\pi q} \left(\frac{p}{q} - \omega\right)^2 \cos(2\pi q x^*).$$

Thus,

$$\left| \frac{\det\Big(\mathrm{d}_{(\widetilde{x}^k_\pm, \widetilde{y}^k_\pm)} F^q_q - \mathrm{Id}\Big)}{q^2(p/q-\omega)^2} \pm \sigma \mathrm{e}^{-2\pi q} \cos(2\pi q x^*) \right| \ll |\sigma| \mathrm{e}^{-2\pi q}.$$

From the estimate $|x^*| \lesssim q^{\nu_0-\nu-1} \ll 1/q$ we obtain $|\cos(2\pi q x^*) - 1| \ll 1$. The desired estimate (3.54) now follows from the definition of $\Lambda_{j,\pm}$ by taking q sufficiently large.

We next turn to (3.52). Recall that, by (3.33), the function S_q satisfies

$$\left| S_q(x,y) - \Big(\frac{p}{q} - \omega + y\Big)^2 \frac{1+f_1(x)}{2\Big(1+\langle f_1\rangle_q(x)\Big)^2} \right| \lesssim q^{\nu_0} \left| \omega - \frac{p}{q} \right|^3, \tag{3.80}$$

with $\nu_0 = 64$. Substituting $x = \widetilde{x}^k_\pm$, $y = \widetilde{y}^k_\pm$ and using the mean value theorem together with (3.68), we obtain

$$\begin{aligned}
&\left| \Big(\frac{p}{q} - \omega + \widetilde{y}^k_\pm\Big)^2 \frac{1+f_1(\widetilde{x}^k_\pm)}{2\Big(1+\langle f_1\rangle_q(\widetilde{x}^k_\pm)\Big)^2} - \Big(\frac{p}{q} - \omega\Big)^2 \frac{1+f_1(x^*_\pm + k/q)}{2\Big(1+\langle f_1\rangle_q(x^*_\pm + k/q)\Big)^2} \right| \\
&\quad \lesssim \left\| \frac{1+f_1}{2(1+\langle f_1\rangle_q)^2} \right\| \left| \omega - \frac{p}{q} \right| |\widetilde{y}^k_\pm| + \left\| \frac{\mathrm{d}}{\mathrm{d}x} \Big(\frac{1+f_1}{2(1+\langle f_1\rangle_q)^2} \Big) \right\| \left| \omega - \frac{p}{q} \right|^2 \left| \widetilde{x}^k_\pm - \Big(x^*_\pm + \frac{k}{q} \Big) \right| \\
&\quad \lesssim \left| \omega - \frac{p}{q} \right| |\widetilde{y}^k_\pm| + \left| \omega - \frac{p}{q} \right|^2 \left| \widetilde{x}^k_\pm - \Big(x^*_\pm + \frac{k}{q} \Big) \right| \lesssim q \mathrm{e}^{-4\pi q} \left| \omega - \frac{p}{q} \right|^2.
\end{aligned} \tag{3.81}$$

In the last line we have used the basic estimates $\|f_1\| < 1/2$ and $\|\dot{f}_1\| \lesssim \|\ddot{\phi}\|$, which follow from the definition (3.31).

Summing over $k = 0, \ldots, q-1$ yields

$$\sum_{k=0}^{q-1} \frac{1+f_1(x^*_\pm + k/q)}{2(1+\langle f_1\rangle_q(x^*_\pm + k/q))^2} = q \Big\langle \frac{1+f_1}{2(1+\langle f_1\rangle_q)^2} \Big\rangle_q (x^*_\pm) = \frac{q}{2\Big(1+f_1^* + \langle f_1^\bullet\rangle_q(x^*_\pm)\Big)}. \tag{3.82}$$

Recall that $x^*_+ = x^*$ and $x^*_- = (2q)^{-1} - x^*$ with $|x^*| \lesssim q^{\nu_0-\nu-1} \ll 1/q$. In particular, we have $|\Im x^*_\pm| \ll 1$. By Cauchy's estimate and the mean value theorem we obtain

$$\Big| \langle f_1^\bullet\rangle_q(x^*_+) - \langle f_1^\bullet\rangle_q(0) \Big| \lesssim q^{\nu_0-\nu} |\langle f_1^\bullet\rangle_q|_{1/q} \lesssim q^{\nu_0-\nu} \Bigg(\sum_{\substack{|k|\geq q \\ q|k}} |f_1| \mathrm{e}^{-2\pi(1-1/q)|k|} \Bigg) \lesssim q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}. \tag{3.83}$$

The second $\lesssim$ follows by estimating the Fourier coefficients of f_1 individually using analyticity over $\mathbb{T}_1$. Similarly

$$\left| \langle f_1^\bullet\rangle_q(x^*_-) - \langle f_1^\bullet\rangle_q\Big(\frac{1}{2q}\Big) \right| \lesssim q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}. \tag{3.84}$$

Moreover, for $\Im x = 0$ we have $|\langle f_1\rangle_q^{>q}(x)| \leq \sum_{|k|>q,\, q|k} |f_1| \mathrm{e}^{-2\pi|k|} \lesssim \mathrm{e}^{-4\pi q}$. Combining this with (3.83) and (3.84) gives

$$\Big| \pm\Big((\hat{f}_1)_q + (\hat{f}_1)_{-q} \Big) - \langle f_1^\bullet\rangle_q(x^*_\pm) \Big| \lesssim q^{\nu_0-\nu} \mathrm{e}^{-2\pi q},$$

where $(\hat{f}_1)_{\pm q}$ are the Fourier coefficients of f_1 of order $\pm q$. It follows that

$$\left| \frac{1}{1+f_1^* \pm ((\hat{f}_1)_q + (\hat{f}_1)_{-q})} - \frac{1}{1+f_1^* + \langle f_1^\bullet\rangle_q(x^*_\pm)} \right| \lesssim q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}. \tag{3.85}$$

Let $\mathcal{A}_\pm$ denote the rescaled action defined by the right-hand side of (3.51) with ξ_j taken to be the (p,q)-orbit of $(\tilde{x}^k_\pm, \tilde{y}^k_\pm)$ under F_q, i.e.

$$\mathcal{A}_\pm := \frac{1}{q(\omega - p/q)^2} \sum_{k=0}^{q-1} S_q(\tilde{x}^k_\pm, \tilde{y}^k_\pm).$$

Combining (3.80), (3.81), (3.82), and (3.85) yields

$$\left| \mathcal{A}_\pm - \frac{1}{2\Big(1 + f_1^* \pm ((\hat{f}_1)_q + (\hat{f}_1)_{-q})\Big)} \right| \lesssim q^{\nu_0} \left|\omega - \frac{p}{q}\right| + q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}.$$

Since $|(\hat{f}_1)_{\pm q}| \lesssim \mathrm{e}^{-2\pi q}$, up to an additional error $\lesssim q^{\nu_0-\nu}\mathrm{e}^{-2\pi q}$ we may write

$$\left| \mathcal{A}_\pm - \frac{1}{2(1+f_1^*)}\left(1 \mp \frac{(\hat{f}_1)_q + (\hat{f}_1)_{-q}}{1+f_1^*}\right) \right| \lesssim q^{\nu_0} \left|\omega - \frac{p}{q}\right| + q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}. \tag{3.86}$$

We now control $(\hat{f}_1)_q + (\hat{f}_1)_{-q}$ via the choice of the coefficients σ_j in (3.58). Recall from (3.62) and the definition of $\mathfrak{g}$ that

$$\left\| \mathfrak{g}^\bullet(\cdot, 0) + \frac{1}{2}\left(\frac{p}{q} - \omega\right)^2 \left\langle \frac{\langle \dot{f}_1 \rangle_q}{1 + \langle f_1 \rangle_q} \right\rangle_q^N \right\| \lesssim q^{\nu_0-\nu} \left| \omega - \frac{p}{q} \right|^2.$$

Since $N = 2q$, the kth Fourier coefficient of the expression inside the $\|\cdot\|$-norm is nonzero only for $|k| = q$ or $2q$, and analyticity in $\mathbb{T}_1$ gives a bound $q^{\nu_0-\nu}\mathrm{e}^{-2\pi q}|\omega - p/q|^2$ for such coefficients. Hence the supremum norm over the real torus $\mathbb{T}$ is bounded by $\lesssim q^{\nu_0-\nu}\mathrm{e}^{-2\pi q}|\omega - p/q|^2$. A similar argument shows that the supremum norm over $\mathbb{T}$ of $\langle \langle \dot{f}_1 \rangle_q (1 + \langle f_1 \rangle_q)^{-1} \rangle^{>N}$ is bounded by $\lesssim \mathrm{e}^{-6\pi q}$. For $\Im x = 0$, the explicit formula (3.60) then yields

$$\left| \frac{\sigma \mathrm{e}^{-2\pi q}}{2\pi q} \sin(2\pi q x) + \frac{1}{2} \frac{\langle \dot{f}_1 \rangle_q(x)}{1 + \langle f_1 \rangle_q(x)} \right| \lesssim q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}. \tag{3.87}$$

We decompose $\langle f_1 \rangle_q = f_1^* + \langle f_1 \rangle_q^q + \langle f_1 \rangle_q^{>q}$. Analyticity and $\Im x = 0$ imply $|\langle \dot{f}_1 \rangle_q^{>q}(x)| \lesssim q\mathrm{e}^{-4\pi q}$, $|\langle f_1 \rangle_q^q| \lesssim \mathrm{e}^{-2\pi q}$, and $|\langle \dot{f}_1 \rangle_q^q| \lesssim q\mathrm{e}^{-2\pi q}$, hence

$$\left| \frac{\langle \dot{f}_1 \rangle_q(x)}{1 + \langle f_1 \rangle_q(x)} - \frac{\langle \dot{f}_1 \rangle_q^q(x)}{1 + f_1^*} \right| \lesssim q\mathrm{e}^{-4\pi q} \lesssim q^{\nu_0-\nu}\mathrm{e}^{-2\pi q}.$$

Substituting this into (3.87) and comparing the Fourier coefficients of order $\pm q$, we obtain

$$\left| \frac{\sigma \mathrm{e}^{-2\pi q}}{(2\pi q)^2} - \frac{1}{2} \frac{(\hat{f}_1)_q + (\hat{f}_1)_{-q}}{1 + f_1^*} \right| \lesssim q^{\nu_0-\nu}\mathrm{e}^{-2\pi q}.$$

Combining this with (3.86), we arrive at

$$\left| \mathcal{A}_\pm - \frac{1}{2(1+f_1^*)}\left(1 \mp \frac{2\sigma \mathrm{e}^{-2\pi q}}{(2\pi q)^2}\right) \right| \lesssim q^{\nu_0} \left|\omega - \frac{p}{q}\right| + q^{\nu_0-\nu} \mathrm{e}^{-2\pi q}.$$

This is precisely (3.52). The proof of Lemma 3.6 is complete.

We have now completed all steps in the proof of Theorem 3.1, with the sole exception of the technical intermediate result Theorem 3.3. The remainder of this chapter will be devoted to the proof of Theorem 3.3.

3.7 Construction of the resonant coordinates: the proof of Theorem 3.3

Before entering the proof of Theorem 3.3, we broadly outline the main steps.

We start with the standard map (3.16) in adapted coordinates, where the RIC is given by $\{y=0\}$. For a fixed pair (p,q), we focus on a complex neighborhood of the line $\{y=p/q-\omega\}$. On this neighborhood, we construct two successive changes of coordinates, denoted by Θ and $\mathrm{Id}+\Sigma$, in Section 3.7.1 and Section 3.7.2, respectively.

The common idea behind the construction of these two coordinate changes is to consider a near-identity change of variables

$$\Phi = \mathrm{Id}+\Psi \quad \text{where} \quad \Psi(x,y) = (\psi(x,y), \varphi(x,y)) \tag{3.88}$$

for some functions ψ and φ. Put $A(x,y)=(x+\omega+y,y)$ and $G(x,y)=(f(x,y),g(x,y))$. Let

$$\Phi \circ F \circ \Phi^{-1} = \widetilde{F} = A + \widetilde{G}$$

where $\widetilde{G}(x,y) = (\widetilde{f}(x,y), \widetilde{g}(x,y))$. Expanding $\widetilde{F}\circ\Phi = \Phi\circ F$, we have

$$\widetilde{G}\circ(\mathrm{Id}+\Psi) = G + \Psi\circ F + A - A\circ(\mathrm{Id}+\Psi).$$

Componentwise, this is

$$\begin{cases} \widetilde{f}\circ(\mathrm{Id}+\Psi) = f + \psi\circ F - \psi - \varphi, \\ \widetilde{g}\circ(\mathrm{Id}+\Psi) = g + \varphi\circ F - \varphi. \end{cases} \tag{3.89}$$

To simplify $\widetilde{F}$ as much as possible, we shall make $\widetilde{f}=0$ by fixing

$$\varphi = f + \psi\circ F - \psi. \tag{3.90}$$

Then $\widetilde{F}(x,y) = (x+\omega+y, y+\widetilde{g}(x,y))$ with

$$\widetilde{g}\circ(\mathrm{Id}+\Psi) = g + (f\circ F - f) + (\psi\circ F^2 - 2\psi\circ F + \psi). \tag{3.91}$$

Our goal is to choose ψ so that $\widetilde{g}(\cdot,0)$ is closely approximated by its resonant part $\langle\widetilde{g}\rangle_q^N$, as formalized by the estimate (3.29).

In the first step, we base the construction of ψ on a formal Taylor-series computation, and the coordinate transform (3.88) is accordingly defined over a neighborhood of $\{y=0\}$ which contains the circle $\{y=p/q-\omega\}$. More precisely, we start in Section 3.7.1 by analyzing the Taylor expansion

$$\widetilde{g}(x,y) = \widetilde{g}_1(x)y + \widetilde{g}_2(x)y^2 + \mathcal{O}(y^3) \tag{3.92}$$

where $\widetilde{g}$ is as given in (3.91), and we take

$$\psi(x,y) = \psi_1(x)y + \psi_2(x)y^2$$

with suitable choices of ψ_1 and ψ_2 so that $\widetilde{g}_1$ and $\widetilde{g}_2$ are almost $1/q$-periodic. The resulting coordinate transform Φ will not be symplectic in general.

In order to evaluate the symplectic action of any orbit found in these new coordinates, we also compute the leading term $\widetilde{s}_2$ in the Taylor expansion

$$\widetilde{S}(x,y) = \widetilde{s}_2(x)y^2 + \mathcal{O}(y^3) \tag{3.93}$$

of the pullback $\widetilde{S} = S \circ \Phi^{-1}$ of the generating function S.

Then, we restrict everything to a neighborhood of $\{y = p/q - \omega\}$. This requires evaluating the Taylor expansion around $y = 0$ at a finite $y \approx p/q - \omega$, and hence we also need a quantitative estimate of the $\mathcal{O}(y^3)$ remainders in (3.92) and (3.93). This estimate will be the main content of Section 3.7.1. The coordinate transform resulting from Step 1 will be constructed as $\Theta = \Phi^{-1} \circ \tau$ where τ is a translation that re-centers the domain at $y = p/q - \omega$.

Roughly speaking, at $y \approx p/q - \omega$ the Taylor remainders in (3.92) and (3.93) have size $\sim |\omega - p/q|^3$, whereas the q-resonant part of $\widetilde{g}$ restricted to the real torus $\mathbb{T}$ has size $\sim |\omega - p/q|^2 \mathrm{e}^{-2\pi q}$. Since $\widetilde{g}$ has a rather explicit dependence on ϕ, this informally indicates that the eigendata of the (p, q)-orbit can be controlled by the $q\mathbb{Z}$-Fourier modes of ϕ over a range of size at most $\sim |\omega - p/q|^2 \mathrm{e}^{-2\pi q}$. It can be verified, although we do not do so here, that if we impose the stronger arithmetic condition $|\omega - p/q| \leq \mathrm{e}^{-2\pi C q}$ for some $C > 1$, then the size $|\omega - p/q|^3$ of the error term will be small enough and the first coordinate change already suffices to deduce the main results.

If, however, the size of $|\omega - p/q|$ is only polynomially small in q, as assumed in part (1) of the main theorem, then the error contributed by the Taylor remainder outweighs the size of the q-resonant part of $\widetilde{g}$. Therefore, we need an iterative scheme in Step 2 to reduce the non-resonant errors further.

In Section 3.7.2, we use a similar type of coordinate transform given by (3.88)-(3.91) but with $\omega = p/q$, since we have re-centered the coordinates at $y = p/q - \omega$ in Step 1. The coordinate transform obtained in Step 2 is near-identity and is denoted $\mathrm{Id} + \Sigma$. The iteration scheme is similar in spirit to the works of [BN12] and [MRRTS15]. Unlike [BN12], the coordinates we obtain are not symplectic.

We remark that, in contrast to Step 1, the construction in Step 2 (Section 3.7.2) is nonexplicit because it involves $\sim q$ iterations of coordinate transforms. Hence, while we can make the non-resonant part (3.156) of $\widetilde{g}$ exponentially small in q, we do not have a better approximation of the resonant part in (3.157) — we do not have sufficient information on the contribution to the resonant part from this iteration scheme.

For technical reasons, we cannot skip Step 1 and directly begin with the iteration scheme in Step 2. This is because, without Step 1, the first iteration of the scheme in Step 2 will introduce an error of comparable size to the resonant part of the original map, and it seems difficult to analyze this error term explicitly. On the other hand, Step 1 ensures that the non-resonant part of the map is small enough so that errors introduced by each iteration in Step 2 are always much smaller than the resonant part. This situation appears similar to Lazutkin's proof of existence of invariant curves near the boundary of a convex billiard table: in his proof, one has to transform the original coordinates into higher-order "Lazutkin coordinates" so that the non-integrable part of the billiard map vanishes to sufficiently high order near the boundary before applying the KAM iteration.

We also remark that, while the method in Step 2 yields the weaker arithmetic condition (3.3) for the results on Lyapunov exponents, it does not improve the condition (3.5) for the results on symplectic actions. The key issue is that, whereas Lyapunov exponents can be computed in *any* coordinates, the symplectic action is invariant only under exact symplectic coordinate transforms. Although the coordinate transform in Step 1 is non-symplectic, it is fairly explicit, and we can approximate the symplectic action using just the leading term (3.93), *provided* the Taylor remainder is much smaller than the leading term, as ensured by the strong condition (3.5). If only weaker conditions of the type (3.3) are assumed, then the relevant information on the symplectic actions of q-periodic orbits is contained in the first $\sim q$ terms in the Taylor expansion (3.93). In other words, it is unclear how to analyze the dependence of the q-resonant part of the restricted generating function $S(\cdot, y)$ on ϕ when $|y-\omega|$ is only *polynomially* small in q.

We now begin the proof of Theorem 3.3 proper.

3.7.1 Step 1: coordinate changes via Taylor expansion

Proposition 3.11. *Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $\phi : \mathbb{T}_1 \to \mathbb{C}$ be analytic with zero average and $\| \dddot{\phi} \| \leq 1$. Let F be the standard map associated with (ω, ϕ) in the adapted coordinates. Then the following statements hold. For $q \in \mathbb{Z}_{>0}$ sufficiently large, if*

$$\left|\omega - \frac{p}{q}\right| \leq \frac{q^{-6}}{6} \qquad \text{for some} \quad p \in \mathbb{Z}, \tag{3.94}$$

then there exists an analytic change of coordinates Θ on the domain $\mathbb{T}_{1-q^{-1}} \times \mathbb{D}_{|\omega-p/q|/2}$ which is close to a translation by $p/q - \omega$ in y in the sense that

$$\Theta(x,y) = (x + \theta_1(x,y),\ y + p/q - \omega + \theta_2(x,y)) \tag{3.95}$$

with

$$|\theta_1|_{1-1/q, |\omega-p/q|/2} \lesssim q\left|\omega - \frac{p}{q}\right|; \quad |\theta_2|_{1-1/q, |\omega-p/q|/2} < \frac{1}{11}\left|\omega - \frac{p}{q}\right|, \tag{3.96}$$

and the map $\widetilde{F}_q := \Theta^{-1} \circ F \circ \Theta$ in these new coordinates is

$$\widetilde{F}_q : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{p}{q} + y \\ y + \widetilde{g}_q(x,y) \end{pmatrix}.$$

Define $f_1 : \mathbb{T}_1 \to \mathbb{C}$ as in (3.31) *and $\widetilde{S}_q := S \circ \Theta$, where S is the normalized generating function associated with (ω, ϕ) in adapted coordinates. Then for $\nu_0 = 64$ and for all $(x,y) \in \mathbb{T}_{1-1/q} \times \mathbb{D}_{|\omega-p/q|/2}$,*

$$\begin{aligned} \left|\widetilde{g}_q(x,y) + \frac{1}{2}\Big(\frac{p}{q} - \omega + y\Big)\Big(\frac{p}{q} - \omega - y\Big)\frac{\langle \dot{f}_1\rangle_q(x)}{1 + \langle f_1\rangle_q(x)}\right| &\lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3 \\ \left|\widetilde{S}_q(x,y) - \Big(\frac{p}{q} - \omega + y\Big)^2 \frac{1 + f_1}{2(1 + \langle f_1\rangle_q)^2}(x)\right| &\lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3. \end{aligned} \tag{3.97}$$

Moreover, the coordinate transform Θ depends analytically on ϕ and is real-analytic when ϕ is real-valued. In this case, the map $\widetilde{F}_q$ preserves the area form

$$\Theta^* \mathrm{d}x\mathrm{d}y = \Big(\frac{1}{1 + \langle f_1\rangle_q} + \widetilde{\eta}\Big)\mathrm{d}x\mathrm{d}y \tag{3.98}$$

for some real-analytic $\widetilde{\eta}$ such that

$$|\widetilde{\eta}|_{1-\frac{1}{q}, \frac{1}{2}|\omega - \frac{p}{q}|} \lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|. \tag{3.99}$$

The rest of Section 3.7.1 aims to prove Proposition 3.11.

Preliminary Taylor series calculations

Recall that the standard map associated with (ω, ϕ) in adapted coordinates F is given by (3.16) - (3.17). Consider a change of coordinates $\Phi = \mathrm{Id} + \Psi$ as described in (3.88) - (3.91). We shall choose ψ to be of the form $\psi(x,y) = \psi_1(x)y + \psi_2(x)y^2$ for some ψ_1 and ψ_2 such that in a neighborhood of $\{y = p/q - \omega\}$ the function $\widetilde{g}$ in (3.91) is almost $1/q$-periodic in x.

Taylor series of F and the generating function. In this paragraph we obtain the Taylor expansion at $y = 0$ of f and g as well as the generating function S up to order $\mathcal{O}(y^2)$. Let us begin by fixing some notations. Denote

$$f(x,y) = f_1(x)y + f_2(x)y^2 + \mathcal{O}(y^3) \quad \text{and} \quad g(x,y) = g_1(x)y + g_2(x)y^2 + \mathcal{O}(y^3).$$

We adopt similar notations for the Taylor expansions of ψ, φ and $\widetilde{g}$. Then, a straightforward computation using the Taylor series of (3.17) yields

$$1 + f_1 = \frac{1}{(1+\dot{\phi})(1+\dot{\phi}^+)}; \quad f_2 = -\frac{\ddot{\phi}^+}{2(1+\dot{\phi})^2(1+\dot{\phi}^+)^3}; \quad g_1 = 0; \quad g_2 = -\frac{1}{2}\dot{f}_1. \tag{3.100}$$

We remark that the last equality $g_2 = -\dot{f}_1/2$ can be computed from the area-preservation property $F^*(\mathrm{d}x\mathrm{d}y) = \mathrm{d}x\mathrm{d}y$ and $g_1 = 0$. Next, we consider the normalized generating function S associated with (ω, ϕ) in adapted coordinates (see Section 3.3). Integrating the Taylor expansion of the second equation in (3.23) along y and using the Taylor remainder formula and the normalization condition $S(x,0) = 0$, we get

$$S(x,y) = \frac{1 + f_1(x)}{2}y^2 + \mathcal{O}(y^3). \tag{3.101}$$

Taylor series after coordinate changes. To express the Taylor expansion of (3.90) and (3.91) in y, let us define a finite-difference operator $\mathcal{D}_\omega$ by

$$\mathcal{D}_\omega u := u \circ R_\omega - u \quad \text{for any} \quad u : \mathbb{T}_a \to \mathbb{C}. \tag{3.102}$$

Also recall the notation $u^\pm := u \circ R_{\pm\omega}$. Then, Taylor expanding the right hand sides of (3.90) and of the second line of (3.89) and substituting $g_1 = 0$, $g_2 = -\dot{f}_1/2$,

$$\begin{aligned} \varphi =& (f_1 + \mathcal{D}_\omega\psi_1)y + \left(f_2 + \mathcal{D}_\omega\psi_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+\right)y^2 + \mathcal{O}(y^3). \\ g + \varphi\circ F - \varphi =& (\mathcal{D}_\omega\varphi_1)y + \left(\mathcal{D}_\omega\varphi_2 - \frac{1}{2}(1+\varphi_1^+)\dot{f}_1 + (1+f_1)\dot{\varphi}_1^+\right)y^2 + \mathcal{O}(y^3), \end{aligned} \tag{3.103}$$

On the other hand, Taylor expanding $\widetilde{g} \circ (\mathrm{Id} + \Psi)$ at $y = 0$ yields

$$\widetilde{g} \circ (\mathrm{Id} + \Psi) = (1+\varphi_1)\widetilde{g}_1 y + \left((1+\varphi_1)^2\widetilde{g}_2 + \varphi_2\widetilde{g}_1 + (1+\varphi_1)\psi_1\dot{\widetilde{g}}_1\right)y^2 + \mathcal{O}(y^3). \tag{3.104}$$

From the first line of (3.103),

$$\varphi_1 = f_1 + \mathcal{D}_\omega\psi_1, \quad \varphi_2 = f_2 + \mathcal{D}_\omega\psi_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+. \tag{3.105}$$

Equating the second line of (3.104) with (3.103),

$$\begin{aligned} (1+\varphi_1)\widetilde{g}_1 =& \mathcal{D}_\omega\varphi_1, \\ (1+\varphi_1)^2\widetilde{g}_2 + \varphi_2\widetilde{g}_1 + (1+\varphi_1)\psi_1\dot{\widetilde{g}}_1 =& \mathcal{D}_\omega\varphi_2 - \frac{1}{2}(1+\varphi_1^+)\dot{f}_1 + (1+f_1)\dot{\varphi}_1^+. \end{aligned}$$

We may differentiate the first line in x and multiply throughout by ψ_1 to obtain $(1+\varphi_1)\psi_1\dot{\tilde{g}}_1 + \psi_1\dot\varphi_1\tilde g_1 = \psi_1\mathcal{D}_\omega\dot\varphi_1$, which can be used to eliminate the $(1+\varphi_1)\psi_1\dot{\tilde{g}}_1$ term on the second line above and obtain

$$\begin{aligned}(1+\varphi_1)\tilde g_1 =&\mathcal{D}_\omega\varphi_1,\\ (1+\varphi_1)^2\tilde g_2 + (\varphi_2-\psi_1\dot\varphi_1)\tilde g_1 + \psi_1\mathcal{D}_\omega\dot\varphi_1 =&\mathcal{D}_\omega\varphi_2 - \frac{1}{2}(1+\varphi_1^+)\dot f_1 + (1+f_1)\dot\varphi_1^+.\end{aligned} \tag{3.106}$$

The formulae (3.105) and (3.106) together give the dependence of the leading Taylor coefficients $\tilde g_1$ and $\tilde g_2$ of $\tilde g$ in terms of ψ_1 and ψ_2. Our next task is to choose ψ_1 and ψ_2 appropriately so that $\tilde g_1$ and $\tilde g_2$ are almost $1/q$-periodic on $\mathbb{T}$.

The choice of ψ_1, ψ_2. From now on, let $q>1$ and $p\in\mathbb{Z}$ be fixed. Analogous to (3.102), we define a finite-difference operator $\mathcal{D}_{p/q}$ by

$$\mathcal{D}_{p/q}u := u\circ R_{p/q} - u \quad \text{for any} \quad u:\mathbb{T}_a\to\mathbb{C}.$$

Recall that we denote by $\langle u\rangle_q$ and $\{u\}_q$ to be the $1/q$-periodization of u and the complement part respectively (see Section 3.2). Given $v:\mathbb{T}_1\to\mathbb{C}$, there is a *unique* $u:\mathbb{T}_1\to\mathbb{C}$, denoted $\mathcal{D}_{p/q}^{-1}v$, such that $\mathcal{D}_{p/q}u=v$ and $\langle u\rangle_q=0$. The function u can be defined via its Fourier series as

$$\hat u_k = \begin{cases}\frac{\hat v_k}{\mathrm{e}^{2\pi\mathrm{i}k\frac{p}{q}}-1} & \text{if } q\nmid k\\ 0 & \text{otherwise.}\end{cases}$$

Recall that $\tilde g_1$, $\tilde g_2$ and $\tilde s_2$ denote the leading terms of the Taylor expansions of $\tilde g$ and $\tilde S$ as in (3.92) and (3.93). In the sequel, we shall choose ψ_1 and ψ_2 so as to make $\{\tilde g_1\}$ and $\{\tilde g_2\}_q$ as small as possible assuming p/q is sufficiently close to ω. Informally, we will obtain

$$\tilde g_1 \approx \left(\omega-\frac{p}{q}\right)\frac{\langle\dot f_1\rangle_q}{1+\langle f_1\rangle_q}, \quad \tilde g_2\approx\frac{1}{2}\frac{\langle\dot f_1\rangle_q}{1+\langle f_1\rangle_q} \tag{3.107}$$

and

$$\tilde s_2 \approx \frac{1+f_1}{2(1+\langle f_1\rangle_q)^2}. \tag{3.108}$$

In this section, we obtain the terms on the right hand sides of "$\approx$" above. The rigorous estimates of the differences between the two sides will be left to the next section. The guiding heuristics of the choice of ψ that follows is that $|\omega-p/q|$ shall be regarded as a "small" quantity so that $\mathcal{D}_\omega\langle u\rangle_q$ and $(\mathcal{D}_\omega-\mathcal{D}_{p/q})u$ generally have "smaller" orders of magnitude than $\langle u\rangle_q$ and u respectively. In fact, to achieve (3.107) we take $\psi(x,y)=\psi_1(x)y+\psi_2(x)y^2$ with

$$\begin{aligned}\psi_1 &= (\mathcal{D}_\omega-\mathcal{D}_{p/q})\mathcal{D}_{p/q}^{-2}\{f_1\}_q - \mathcal{D}_{p/q}^{-1}\{f_1\}_q;\\ \psi_2 &= -\mathcal{D}_{p/q}^{-2}\left\{\mathcal{D}_\omega\left(f_2-\frac{1}{2}\dot f_1\psi_1^+ + (1+f_1)\dot\psi_1^+\right) - \frac{1}{2}(1+\varphi_1^+)\dot f_1 + (1+f_1)\dot\varphi_1^+\right\}_q.\end{aligned} \tag{3.109}$$

By the choice of ψ_1 and the first line of (3.103),

$$\begin{aligned}\varphi_1 =&f_1 + (\mathcal{D}_\omega-\mathcal{D}_{p/q})\psi_1 + \mathcal{D}_{p/q}\psi_1\\ =&f_1 + (\mathcal{D}_\omega-\mathcal{D}_{p/q})\psi_1 + (\mathcal{D}_\omega-\mathcal{D}_{p/q})\mathcal{D}_{p/q}^{-1}\{f_1\}_q - \{f_1\}_q\\ =&\langle f_1\rangle_q + (\mathcal{D}_\omega-\mathcal{D}_{p/q})(\psi_1+\mathcal{D}_{p/q}^{-1}\{f_1\}_q)\\ =&\langle f_1\rangle_q + (\mathcal{D}_\omega-\mathcal{D}_{p/q})^2\mathcal{D}_{p/q}^{-2}\{f_1\}_q.\end{aligned} \tag{3.110}$$

Thus $\varphi_1 = \langle f_1 \rangle_q + \{\varphi_1\}_q$ and we may consider $\{\varphi_1\}_q$ and $\mathcal{D}_\omega \varphi_1$ as small errors. It follows from the first line of (3.106) that

$$\begin{aligned} \widetilde{g}_1 =& \frac{\mathcal{D}_\omega \langle f_1 \rangle_q}{1+\langle f_1 \rangle_q} + \frac{\mathcal{D}_\omega \{\varphi_1\}_q - \{\varphi_1\}_q \widetilde{g}_1}{1+\langle f_1 \rangle_q} \\ =& \Big(\omega - \frac{p}{q}\Big)\frac{\langle \dot{f}_1 \rangle_q}{1+\langle f_1 \rangle_q} + \frac{\mathcal{D}_\omega \langle f_1 \rangle_q - (\omega - \frac{p}{q})\langle \dot{f}_1 \rangle_q}{1+\langle f_1 \rangle_q} + \frac{\{\varphi_1^+\}_q - (1+\widetilde{g}_1)\{\varphi_1\}_q}{1+\langle f_1 \rangle_q}. \end{aligned} \tag{3.111}$$

On the other hand, substituting (3.105) into the right hand side of the second line of (3.106) and using the choice of ψ_2,

$$\begin{aligned} &\mathcal{D}_\omega^2 \psi_2 + \mathcal{D}_\omega\Big(f_2 - \frac{1}{2}\dot{f}_1 \psi_1^+ + (1+f_1)\dot{\psi}_1^+\Big) - \frac{1}{2}(1+\varphi_1^+)\dot{f}_1 + (1+f_1)\dot{\varphi}_1^+ \\ =&(\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 + \mathcal{D}_\omega \Big\langle f_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+ \Big\rangle_q + \Big\langle (1+f_1)\dot{\varphi}_1^+ - \frac{1}{2}(1+\varphi_1^+)\dot{f}_1 \Big\rangle_q \\ =&(\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 + \mathcal{D}_\omega \Big\langle f_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+ \Big\rangle_q + \Big\langle (1+f_1)\dot{\varphi}_1 - \frac{1}{2}(1+\varphi_1)\dot{f}_1 \Big\rangle_q \\ &+ \Big\langle (1+f_1)\mathcal{D}_\omega\dot{\varphi}_1 - \frac{1}{2}\dot{f}_1\mathcal{D}_\omega\varphi_1 \Big\rangle_q \\ =&(\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 + \mathcal{D}_\omega \Big\langle f_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+ \Big\rangle_q + \Big\langle (1+f_1)\langle\dot{f}_1\rangle_q - \frac{1}{2}(1+\langle f_1\rangle_q)\dot{f}_1 \Big\rangle_q \\ &+ \Big\langle (1+f_1)\{\dot{\varphi}_1\}_q - \frac{1}{2}\{\varphi_1\}_q\dot{f}_1 \Big\rangle_q + \Big\langle (1+f_1)\mathcal{D}_\omega\dot{\varphi}_1 - \frac{1}{2}\dot{f}_1\mathcal{D}_\omega\varphi_1 \Big\rangle_q \\ =&\frac{1}{2}(1+\langle f_1\rangle_q)\langle\dot{f}_1\rangle_q + (\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 + \mathcal{D}_\omega \Big\langle f_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+ \Big\rangle_q \\ &+ \Big\langle (1+f_1)\{\dot{\varphi}_1\}_q - \frac{1}{2}\{\varphi_1\}_q\dot{f}_1 \Big\rangle_q + \Big\langle (1+f_1)\mathcal{D}_\omega\dot{\varphi}_1 - \frac{1}{2}\dot{f}_1\mathcal{D}_\omega\varphi_1 \Big\rangle_q. \end{aligned} \tag{3.112}$$

In the last line, we used the simple observation that $\langle (1+f_1)\langle \dot{f}_1\rangle_q\rangle_q = (1+\langle f_1\rangle_q)\langle\dot{f}_1\rangle_q$. Using $(1+\varphi_1)^2 = (1+\langle f_1\rangle_q)^2 + (2+\langle f_1\rangle_q + \varphi_1)\{\varphi_1\}$, the left hand side of the second line of (3.106) is

$$(1+\langle f_1\rangle_q)^2\widetilde{g}_2 + (2+\langle f_1\rangle_q + \varphi_1)\{\varphi_1\}\widetilde{g}_2 + (\varphi_2 - \psi_1\dot{\varphi}_1)\widetilde{g}_1 + \psi_1\mathcal{D}_\omega\dot{\varphi}_1. \tag{3.113}$$

Equating (3.113) with (3.112), we obtain

$$\begin{aligned} \widetilde{g}_2 = \frac{1}{2}\frac{\langle\dot{f}_1\rangle_q}{1+\langle f_1\rangle_q} + \frac{1}{(1+\langle f_1\rangle_q)^2}\Big[&(\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 + \mathcal{D}_\omega \Big\langle f_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+ \Big\rangle_q \\ &+ \Big\langle (1+f_1)\{\dot{\varphi}_1\}_q - \frac{1}{2}\{\varphi_1\}_q\dot{f}_1 \Big\rangle_q + \Big\langle (1+f_1)\mathcal{D}_\omega\dot{\varphi}_1 - \frac{1}{2}\dot{f}_1\mathcal{D}_\omega\varphi_1 \Big\rangle_q \\ &- (2+\langle f_1\rangle_q+\varphi_1)\{\varphi_1\}\widetilde{g}_2 - (\varphi_2 - \psi_1\dot{\varphi}_1)\widetilde{g}_1 - \psi_1\mathcal{D}_\omega\dot{\varphi}_1\Big]. \end{aligned} \tag{3.114}$$

We now compute an approximation of the pullback $\widetilde{S} = S \circ \Phi^{-1}$ of the generating function to the new coordinates using (3.101):

$$\widetilde{S}(x,y) = \frac{1+f_1}{2(1+\varphi_1)^2}y^2 + \mathcal{O}(y^3). \tag{3.115}$$

Using $(1+\varphi_1)^2 = (1+\langle f_1\rangle_q)^2 + (2+\langle f_1\rangle_q+\varphi_1)\{\varphi_1\}$, we may write

$$\widetilde{s}_2 = \frac{1+f_1}{2(1+\langle f_1\rangle_q)^2} - \frac{(1+f_1)(2+\langle f_1\rangle_q+\varphi_1)\{\varphi_1\}_q}{2(1+\langle f_1\rangle_q)^2(1+\varphi_1)^2}. \tag{3.116}$$

Quantitative estimates: the leading terms of $\widetilde{g}$.

The goal of this section is to give a precise justification of the approximations (3.107) and (3.108) by showing that all but the first terms on the right hand sides of (3.111), (3.114) and (3.116) are relatively small. More precisely, we prove

Lemma 3.12. *Suppose*

$$\|\dddot{\phi}\| \leq 1 \quad \textit{and} \quad \left|\omega - \frac{p}{q}\right| \leq \frac{1}{q^3}. \tag{3.117}$$

Then

$$\left\|\widetilde{g}_1 - \left(\omega - \frac{p}{q}\right)\frac{\langle \dot{f}_1\rangle_q}{1+\langle f_1\rangle_q}\right\| \lesssim q^2\left|\omega - \frac{p}{q}\right|^2, \qquad \left\|\widetilde{g}_2 - \frac{1}{2}\frac{\langle \dot{f}_1\rangle_q}{1+\langle f_1\rangle_q}\right\| \lesssim q^4\left|\omega - \frac{p}{q}\right| \tag{3.118}$$

and

$$\left\|\widetilde{s}_2 - \frac{1+f_1}{2(1+\langle f_1\rangle_q)^2}\right\| \lesssim \left|\omega - \frac{p}{q}\right|. \tag{3.119}$$

Before proceeding to the proof of Lemma 3.12, let us note some elementary facts. Since

$$k \nmid q \quad \Longrightarrow \quad |\mathrm{e}^{2\pi\mathrm{i}kp/q} - 1| \geq |\mathrm{e}^{2\pi\mathrm{i}/q} - 1| = 2\sin\frac{\pi}{q} \geq \frac{4}{q},$$

we have, for $u : \mathbb{T}_1 \to \mathbb{C}$ analytic, that

$$\|\mathcal{D}_{p/q}^{-1}\{u\}_q\| = \sum_{q\nmid k}\left|\frac{\hat{u}_k}{\mathrm{e}^{2\pi\mathrm{i}kp/q}-1}\right|\mathrm{e}^{2\pi|k|} \leq \frac{q}{4}\|u\| \tag{3.120}$$

where $\mathcal{D}_{p/q}^{-1}\{u\}_q$ is uniquely defined as the function v such that $\langle v\rangle_q = 0$ and $\mathcal{D}_{p/q}v = u$. On the other hand, since $(\mathcal{D}_\omega - \mathcal{D}_{p/q})u = u(\cdot+\omega) - u(\cdot+p/q) = \mathcal{D}_{\omega-p/q}(u\circ R_{p/q})$, where $R_{p/q} : x \mapsto x + p/q$ is a rotation on $\mathbb{T}_1$, by Lemma 3.25 with $\delta = \omega - p/q$,

$$\|(\mathcal{D}_\omega - \mathcal{D}_{p/q})u\| = \|\mathcal{D}_{\omega-p/q}(u\circ R_{p/q})\| \leq \left|\omega - \frac{p}{q}\right|\|\dot{u}\|. \tag{3.121}$$

Finally, we also remark that, after integrating the Fourier series of $\dddot{\phi}$ and using the definition of $\|\cdot\|$-norm, the condition $\|\dddot{\phi}\| \leq 1$ implies the following convenient estimates.

$$\|\ddot{\phi}\| \leq \frac{1}{2\pi} \quad \text{and} \quad \|\dot{\phi}\| \leq \frac{1}{(2\pi)^2}. \tag{3.122}$$

This observation has already been used previously in the proof of Corollary 3.10.

Let us begin to prove Lemma 3.12.

By (3.122) and the explicit expression (3.100), we first have

$$\|f_1\| \leq \frac{(2+\|\dot{\phi}\|)\|\dot{\phi}\|}{(1-\|\dot{\phi}\|)^2} < \frac{1}{18}. \tag{3.123}$$

In the above, we used the inequality $\|(1+f_1)^{-1}\| \leq (1-\|f_1\|)^{-1}$, which follows from the fact that the space of analytic functions $\mathbb{T}_1 \to \mathbb{C}$ with finite $\|\cdot\|$-norm is a Banach algebra (Lemma 3.24).

Furthermore, it is straightforward to deduce from $\|\dot{\phi}\| \leq (2\pi)^{-2}$ and the explicit expressions (3.100) that

$$\|\dot{f}_1\|,\ \|\ddot{f}_1\|,\ \|f_2\|,\ \|\dot{f}_2\|,\ \|g_2\|,\ \|\dot{g}_2\| \lesssim 1. \tag{3.124}$$

Recall that by (3.110) we have $\varphi_1 = \langle f_1\rangle_q + (\mathcal{D}_\omega - \mathcal{D}_{p/q})^2\mathcal{D}_{p/q}^{-2}\{f_1\}_q$. Hence, by (3.120) - (3.124) and assumption 3.117,

$$\begin{aligned}\|\varphi_1\| \leq& \|\langle f_1\rangle_q\| + \|(\mathcal{D}_\omega - \mathcal{D}_{p/q})^2\mathcal{D}_{p/q}^{-2}\{f_1\}_q\| \\ \leq& \|f_1\| + \left(\frac{q}{4}\right)^2\left|\omega - \frac{p}{q}\right|\|(\mathcal{D}_\omega - \mathcal{D}_{p/q})\dot{f}_1\| < \frac{1}{18}\end{aligned} \tag{3.125}$$

for q sufficiently large. Similarly, using (3.124)

$$\begin{aligned}\|\dot{\varphi}_1\| =& \|\langle\dot{f}_1\rangle_q + (\mathcal{D}_\omega - \mathcal{D}_{p/q})^2\mathcal{D}_{p/q}^{-2}\{\dot{f}_1\}_q\| \lesssim \|\dot{f}_1\| + q^2\left|\omega - \frac{p}{q}\right|\|\ddot{f}_1\| \lesssim 1; \\ \|\ddot{\varphi}_1\| =& \|\langle\ddot{f}_1\rangle_q + (\mathcal{D}_\omega - \mathcal{D}_{p/q})^2\mathcal{D}_{p/q}^{-2}\{\ddot{f}_1\}_q\| \lesssim \|\ddot{f}_1\| + q^2\|\ddot{f}_1\| \lesssim q^2.\end{aligned} \tag{3.126}$$

Remark. We give more precise bounds in (3.123) and (3.125) instead of just an implicit universal constant $\lesssim 1$ as in (3.124) and (3.126). The reason is that we will need to provide lower bounds on the denominators in (3.111), (3.114) and (3.116).

Since $\{\varphi_1\}_q = (\mathcal{D}_\omega - \mathcal{D}_{p/q})^2\mathcal{D}_{p/q}^{-2}\{f_1\}_q$ and $\langle\varphi_1\rangle_q = \langle f_1\rangle_q$, by periodicity of $\langle f_1\rangle$ and (3.120), (3.121), (3.124),

$$\begin{aligned}\|\{\varphi_1\}_q\| \lesssim& q^2\left|\omega - \frac{p}{q}\right|^2\|\ddot{f}_1\| \lesssim q^2\left|\omega - \frac{p}{q}\right|^2 \qquad \text{(applying (3.121) twice);} \\ \|\{\dot{\varphi}_1\}_q\| \lesssim& q^2\left|\omega - \frac{p}{q}\right|\|\ddot{f}_1\| \lesssim q^2\left|\omega - \frac{p}{q}\right| \qquad \text{(applying (3.121) once);} \\ \|\mathcal{D}_\omega\varphi_1\| =& \|(\mathcal{D}_\omega - \mathcal{D}_{p/q})\langle f_1\rangle_q\| + \|\mathcal{D}_\omega\{\varphi_1\}_q\| \lesssim \left|\omega - \frac{p}{q}\right|\|\dot{f}_1\| + \|\{\varphi_1\}_q\| \lesssim \left|\omega - \frac{p}{q}\right|; \\ \|\mathcal{D}_\omega\dot{\varphi}_1\| =& \|(\mathcal{D}_\omega - \mathcal{D}_{p/q})\langle\dot{f}_1\rangle_q\| + \|\mathcal{D}_\omega\{\dot{\varphi}_1\}_q\| \lesssim \left|\omega - \frac{p}{q}\right|\|\ddot{f}_1\| + \|\{\dot{\varphi}_1\}_q\| \lesssim q^2\left|\omega - \frac{p}{q}\right|;\end{aligned} \tag{3.127}$$

Using the first line of (3.106), we have $\widetilde{g}_1 = (1+\varphi_1)^{-1}\mathcal{D}_\omega\varphi_1$. By (3.125) and (3.127),

$$\|\widetilde{g}_1\| \lesssim \|\mathcal{D}_\omega\varphi_1\| \lesssim \left|\omega - \frac{p}{q}\right|.$$

We may now estimate the last two terms of (3.111). Using Lemma 3.25 with $\delta = \omega - p/q$ and the preceding bounds (3.123), (3.124) on f_1 as well as the first line of (3.127), we have

$$\begin{aligned}\left\|\widetilde{g}_1 - \left(\omega - \frac{p}{q}\right)\frac{\langle\dot{f}_1\rangle_q}{1+\langle f_1\rangle_q}\right\| \leq& \left\|\frac{\mathcal{D}_\omega\langle f_1\rangle_q - (\omega - \frac{p}{q})\langle\dot{f}_1\rangle_q}{1+\langle f_1\rangle_q}\right\| + \left\|\frac{\{\varphi_1^+\}_q - (1+\widetilde{g}_1)\{\varphi_1\}_q}{1+\langle f_1\rangle_q}\right\| \\ \lesssim& \left|\omega - \frac{p}{q}\right|^2\|\ddot{f}_1\| + q^2\left|\omega - \frac{p}{q}\right|^2 \lesssim q^2\left|\omega - \frac{p}{q}\right|^2.\end{aligned}$$

We have established the first inequality of (3.118). To approximate $\widetilde{g}_2$, it is necessary to bound the second term of (3.114), which in turn requires bounds on ψ_1 and ψ_2 as well as φ_2. To this end, let us first use definition (3.109) of ψ_1, inequalities (3.120), (3.121), (3.124) and assumption (3.117) to obtain

$$\|\psi_1\| \lesssim q^2 \left|\omega - \frac{p}{q}\right| \|\dot{f}_1\| + q\|f_1\| \lesssim q; \qquad \|\dot{\psi}_1\| \lesssim q^2 \left|\omega - \frac{p}{q}\right| \|\ddot{f}_1\| + q\|\dot{f}_1\| \lesssim q \quad (3.128)$$

and

$$\|\ddot{\psi}_1\| \lesssim q^2\|\ddot{f}_1\| + q\|\ddot{f}_1\| \lesssim q^2. \quad (3.129)$$

Similarly, by definition (3.109) and (3.126), (3.128),

$$\|\psi_2\| \lesssim q^2\Big(\|f_2\| + \|\dot{f}_1\psi_1^+\| + \|(1+f_1)\dot{\psi}_1^+\| + \|(1+\varphi_1^+)\dot{f}_1\| + \|(1+f_1)\dot{\varphi}_1^+\|\Big) \lesssim q^3.$$

By a direct computation, we find that $\dot{\psi}_2$ is given by

$$-\mathcal{D}_{p/q}^{-2}\left\{\mathcal{D}_\omega\Big(\dot{f}_2 - \frac{1}{2}\ddot{f}_1\psi_1^+ + (1+f_1)\ddot{\psi}_1^+ + \frac{1}{2}\dot{f}_1\dot{\psi}_1^+\Big) - \frac{1}{2}(1+\varphi_1^+)\ddot{f}_1 + (1+f_1)\ddot{\varphi}_1^+ + \frac{1}{2}\dot{\varphi}_1^+\dot{f}_1\right\}_q.$$

Thus, by (3.126) and (3.129),

$$\begin{aligned}\|\dot{\psi}_2\| \lesssim & q^2\Big(\|\dot{f}_2\| + \|\ddot{f}_1\psi_1^+\| + \|(1+f_1)\ddot{\psi}_1^+\| + \|\dot{f}_1\dot{\psi}_1^+\| + \|(1+\varphi_1^+)\ddot{f}_1\| + \|(1+f_1)\ddot{\varphi}_1^+\| + \|\dot{\varphi}_1^+\dot{f}_1\|\Big)\\ \lesssim & q^4.\end{aligned}$$

Then, from the definition (3.105) of φ_2, a similar estimate gives

$$\|\varphi_2\| = \left\|f_2 + \mathcal{D}_\omega\psi_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+\right\| \lesssim q^3.$$

It now follows from (3.125) the second line of (3.106) that

$$\|\widetilde{g}_2\| \lesssim \left\|\mathcal{D}_\omega\varphi_2 - \frac{1}{2}(1+\varphi_1^+)\dot{f}_1 + (1+f_1)\dot{\varphi}_1^+ - \Big((\varphi_2 - \psi_1\dot{\varphi}_1)\widetilde{g}_1 + \psi_1\mathcal{D}_\omega\dot{\varphi}_1\Big)\right\| \lesssim q^3.$$

To recapitulate, we have obtained

$$\begin{gathered}\|\psi_1\|, \|\dot{\psi}_1\| \lesssim q; \quad \|\ddot{\psi}_1\| \lesssim q^2; \quad \|\psi_2\| \lesssim q^3; \quad \|\dot{\psi}_2\| \lesssim q^4; \quad \|\varphi_1\| < \frac{1}{18}; \quad \|\dot{\varphi}_1\| \lesssim 1;\\ \|\ddot{\varphi}_1\| \lesssim q^2; \quad \|\varphi_2\| \lesssim q^3; \quad \|\{\varphi_1\}_q\| \lesssim q^2\left|\omega - \frac{p}{q}\right|^2; \quad \|\{\dot{\varphi}_1\}_q\| \lesssim q^2\left|\omega - \frac{p}{q}\right|;\\ \|\mathcal{D}_\omega\varphi_1\| \lesssim \left|\omega - \frac{p}{q}\right|; \quad \|\mathcal{D}_\omega\dot{\varphi}_1\| \lesssim q^2\left|\omega - \frac{p}{q}\right|; \quad \|\widetilde{g}_1\| \lesssim \left|\omega - \frac{p}{q}\right|; \quad \|\widetilde{g}_2\| \lesssim q^3.\end{gathered} \quad (3.130)$$

Now we are ready to approximate $\widetilde{g}_2$ by estimating the last term of (3.114). Since $\|(1 + \langle f_1\rangle_q)^{-2}\| \lesssim 1$, we may focus on the terms within the square bracket of (3.114). Since $(\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 = (\mathcal{D}_\omega - \mathcal{D}_{p/q})(\mathcal{D}_\omega + \mathcal{D}_{p/q})\psi_2$, we may use (3.121) together with the inequalities summarized in (3.130) to bound the first two terms as follows.

$$\begin{aligned}&\left\|(\mathcal{D}_\omega^2 - \mathcal{D}_{p/q}^2)\psi_2 + \mathcal{D}_\omega\left\langle f_2 - \frac{1}{2}\dot{f}_1\psi_1^+ + (1+f_1)\dot{\psi}_1^+\right\rangle_q\right\|\\ \lesssim &\left|\omega - \frac{p}{q}\right|\left(\|\dot{\psi}_2\| + \left\|\left\langle\dot{f}_2 - \frac{1}{2}\ddot{f}_1\psi_1^+ + (1+f_1)\ddot{\psi}_1^+ + \frac{1}{2}\dot{f}_1\dot{\psi}_1^+\right\rangle_q\right\|\right) \lesssim q^4\left|\omega - \frac{p}{q}\right|.\end{aligned}$$

The rest of the terms can be controlled by a routine application of (3.130):

$$\left\| \left\langle (1+f_1)\{\dot\varphi_1\}_q - \frac{1}{2}\{\varphi_1\}_q \dot f_1 \right\rangle_q \right\| \lesssim q^2 \left|\omega - \frac{p}{q}\right|$$
$$\left\| \left\langle (1+f_1)\mathcal{D}_\omega \dot\varphi_1 - \frac{1}{2}\dot f_1 \mathcal{D}_\omega \varphi_1 \right\rangle_q \right\| \lesssim q^2 \left|\omega - \frac{p}{q}\right|$$
$$\|(2+\langle f_1\rangle_q + \varphi_1)\{\varphi_1\}\widetilde{g}_2 + (\varphi_2 - \psi_1\dot\varphi_1)\widetilde{g}_1 + \psi_1 \mathcal{D}_\omega \dot\varphi_1\| \lesssim q^3 \left|\omega - \frac{p}{q}\right|.$$

We have shown

$$\left\| \widetilde{g}_2 - \frac{1}{2}\frac{\langle \dot f_1\rangle_q}{1+\langle f_1\rangle_q} \right\| \lesssim q^4 \left|\omega - \frac{p}{q}\right|$$

and completed the proof of (3.118). Finally, we turn to the estimate (3.119) of $\widetilde{s}_2$. By the second line of (3.115) and the bounds on $\|f_1\|$, $\|\varphi_1\|$ and $\|\{\varphi_1\}_q\|$ collected in (3.130) as well as the assumption (3.117),

$$\left\| \widetilde{s}_2 - \frac{1+f_1}{2(1+\langle f_1\rangle_q)^2} \right\| = \left\| \frac{(1+f_1)(2+\langle f_1\rangle_q + \varphi_1)\{\varphi_1\}_q}{2(1+\langle f_1\rangle_q)^2(1+\varphi_1)^2} \right\|$$
$$\lesssim \|\{\varphi_1\}_q\| \lesssim q^2 \left|\omega - \frac{p}{q}\right|^2 \lesssim \left|\omega - \frac{p}{q}\right|.$$

The proof of Lemma 3.12 is finished.

Quantitative estimates: the Taylor remainder

Having obtained approximations of the leading Taylor coefficients $\widetilde{g}_1$ and $\widetilde{g}_2$ of $\widetilde{g}$ and $\widetilde{s}_2$ of the pullback generating function $\widetilde{S}$, we now need to estimate the corresponding Taylor remainders of $\widetilde{g}$. We need to introduce some additional definitions in the following.

For $\lambda > 0$, we define the complex domain $\mathcal{D}_\lambda \subset \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ to be

$$\mathcal{D}_\lambda = \{(x,y) \in \mathbb{T}_1 \times \mathbb{C} \; : \; |\Im x| + \lambda^{-1}|y| < 1\}.$$

If u is a function defined on $\mathcal{D}_\lambda$, we denote by $|u|^{(\lambda)}$ the supremum norm of u over $\mathcal{D}_\lambda$. If u is of class $\mathcal{C}^r$ on $\mathcal{D}_\lambda$, we set

$$|u|^{(\lambda,r)} := \sup\left\{ \left|\frac{\partial^{j+k}u}{\partial x^j \partial y^k}(x,y)\right| \; : \; (x,y)\in\mathcal{D}_\lambda, \; 0 \le j+k \le r \right\}. \tag{3.131}$$

The elementary properties of the $|\cdot|^{(\lambda,r)}$-norms that we will need are collected in Section 3.8.1.

The following Lemma is the main result of the current section.

Lemma 3.13. *Suppose*

$$\|\dddot\phi\| \le 1 \quad \textit{and} \quad \left|\omega - \frac{p}{q}\right| \le \frac{q^{-6}}{6}.$$

Then $\widetilde{g}$ and $\widetilde{S}$ are analytic on $\mathcal{D}_\lambda$ with $\lambda = q^{-5}/4$. Furthermore, for $(x,y)\in\mathcal{D}_\lambda$, we have

$$\left|\widetilde{g}(x,y) - \left(\widetilde{g}_1(x)y + \widetilde{g}_2(x)y^2\right)\right| \lesssim q^{\nu_0}|y|^3 \tag{3.132}$$

and

$$\left|\widetilde{S}(x,y) - \widetilde{s}_2(x)y^2\right| \lesssim q^{\nu_0}|y|^3 \tag{3.133}$$

where $\nu_0 = 64$.

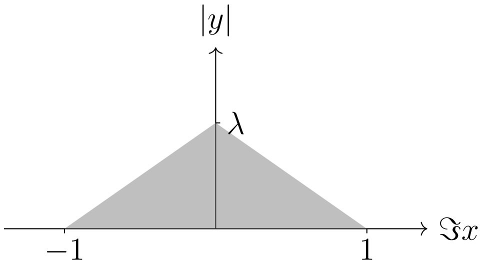

Figure 3.2: An illustration of $\mathcal{D}_\lambda$.

Before proceeding to the proof, we explain the technical issue that necessitates the introduction of the domains of the form $\mathcal{D}_\lambda$. The key issue is that the function $\widetilde{g}(\cdot, y)$ for a fixed $y \neq 0$ is in general less regular than $\phi : \mathbb{T}_1 \to \mathbb{C}$, in the sense that it extends analytically only to a complex domain $\mathbb{T}_a$ for some $a < 1$. [1] The loss of regularity implies that the Taylor remainder in (3.92) has smaller analyticity strip in x whereas the leading terms $\widetilde{g}_j, j = 1, 2$ are analytic over the larger domain $\mathbb{T}_1$. By analyticity, the k-th Fourier coefficients of $\widetilde{g}_j$ decay exponentially faster in k than those of the Taylor remainder. This is potentially a problem because, with $y \approx p/q - \omega$, the $\pm q$-th Fourier coefficients of $\widetilde{g}(\cdot, y)$, which are presumably the largest Fourier coefficients of the resonant part of $\widetilde{g}$, have size $\sim |\omega - p/q|^2 \mathrm{e}^{2\pi(a-1)q}$ whereas those of the Taylor remainder have size $\sim |\omega - p/q|^3$. If $\mathrm{e}^{2\pi(a-1)q} \ll |\omega - p/q|$ then the uncontrolled error from the Taylor remainder will be much larger than the approximating functions in the estimates (3.118) and (3.119). Therefore, it is key to keep track of how fast a decreases as $|y|$ increases from 0. More precisely, we aim to establish the analyticity of $\widetilde{g}$ over a domain $\mathcal{D}_\lambda$, which by definition has a "triangular shape" in the sense that its y-slice is a complex neighborhood $\mathbb{T}_a$ such that a decreases linearly from 1 with a speed λ^{-1} as $|y|$ increases from 0. Then, we restrict $\widetilde{g}$ and $\widetilde{S}$ on the slab $\mathcal{D}_\lambda \cap \{|y - (p/q - \omega)| < |\omega - p/q|/2\}$, where the loss of analyticity width in the x-variable is uniformly bounded in q by $\lesssim q^5|\omega - p/q| \ll 1$. This allows us to show that the first N Fourier coefficients (with N not too big relative to q) of the Taylor remainders of (3.92) and (3.115) are much smaller than the counterparts of the leading terms, and hence they can be regarded truly as "error terms".

Another technical issue arises when we consider the Taylor remainders over $\mathcal{D}_\lambda$: it seems not straightforward to show that the remainders vanish to the order y^3. If a function is analytic and bounded over a "rectangular" domain like $\mathbb{T}_1 \times \mathbb{C}$, then it is standard to use Cauchy's estimate to obtain that the N-th order Taylor remainder of the expansion at $y = 0$ vanishes to the order y^N. For functions defined on $\mathcal{D}_\lambda$, however, Cauchy's estimate is not applicable. Therefore, we resort to a more rudimentary approach by directly bounding the $\mathcal{C}^r$ norms (3.131) of $\widetilde{g}$ and $\widetilde{S}$ viewed as a $\mathcal{C}^\infty$ function on $\mathcal{D}_\lambda$. The lengthy but elementary estimates about functions on domains of the form $\mathcal{D}_\lambda$ are collected in Section 3.8.1. This approach is probably far from optimal and it is partially responsible for the large size of the exponent ν in the condition (3.24) of Theorem 3.3.

We now begin the proof of Lemma 3.13.

Chain of domains. Recall that, by construction,

$$\widetilde{g} = \Big(g + \varphi \circ F - \varphi\Big) \circ (\mathrm{Id} + \widetilde{\Sigma}), \qquad \varphi = f + \psi \circ F - \psi$$

[1] This is analogous to the fact in billiard dynamics that the billiard map on the phase space is generally less smooth than the boundary of the convex billiard domain.

where $\mathrm{Id}+\widetilde{\Sigma}$ is the right inverse of $\mathrm{Id}+\Psi$. We shall consider the chain of inclusion of domains schematized as follows

$$\begin{tikzcd} & \widetilde{g} & g+\varphi\circ F-\varphi & \varphi, f, g & \psi \\ \mathcal{D}_{\lambda_4} \arrow[r, hook] & \mathcal{D}_{\lambda_3} \arrow[u, dotted] \arrow[r, hook, "\mathrm{Id}+\widetilde{\Sigma}"] & \mathcal{D}_{\lambda_2} \arrow[u, dotted] \arrow[r, "F"] & \mathcal{D}_{\lambda_1} \arrow[u, dotted] \arrow[r, "F"] & \mathcal{D}_{\lambda_0} \arrow[u, dotted] \end{tikzcd} \tag{3.134}$$

where we shall fix

$$\lambda_0 = 1, \quad \lambda_1 = \frac{1}{q^5}, \quad \lambda_2 = \frac{1}{2q^5}, \quad \lambda_3 = \frac{1}{3q^5} \quad \text{and} \quad \lambda_4 = \frac{1}{4q^5}.$$

In the diagram (3.134), the vertical arrows with dotted lines denote domains of definition. We will show $\widetilde{g}$ and $\widetilde{S}$ are well-defined analytic functions on $\mathcal{D}_{\lambda_3}$ and establish a uniform $\mathcal{C}^3$ bound

$$|\widetilde{g}|^{(\lambda_4,3)} \lesssim q^{\nu_0} \quad \text{and} \quad |\widetilde{S}|^{(\lambda_4,3)} \lesssim q^{\nu_0}. \tag{3.135}$$

with a universal constant $\nu_0 = 64$.

In fact, the estimate (3.135) is sufficient for us to deduce (3.132) and (3.133) of Lemma 3.13. Indeed, by the usual Taylor remainder formula,

$$\widetilde{g}(x,y) = \widetilde{g}_1(x)y + \widetilde{g}_2(x)y^2 + \mathcal{Q}_1(x,y); \qquad \mathcal{Q}_1(x,y) = \frac{y^3}{2}\int_0^1 \frac{\partial^3 \widetilde{g}}{\partial y^3}(x,ty)(1-t)^2 dt;$$

$$\widetilde{S}(x,y) = \tilde{s}_2(x)y^2 + \mathcal{Q}_2(x,y), \qquad \mathcal{Q}_2(x,y) = \frac{y^3}{2}\int_0^1 \frac{\partial^3 \widetilde{S}}{\partial y^3}(x,ty)(1-t)^2 dt.$$

For $(x,y)\in\mathcal{D}_{\lambda_4}$, by the explicit expressions above and (3.135),

$$|\mathcal{Q}_j(x,y)| \lesssim q^{\nu_0}|y|^3, \qquad (j=1,2).$$

We readily obtain (3.132) and (3.133).

In the rest of this section, we focus on proving that all the compositions in (3.134) are well-defined and the estimate (3.135) holds.

Estimates of $\overline{f}$ and $\overline{g}$. Recall that the functions f and g in the formula $F(x,y) = (x+\omega+y+f(x,y), y+g(x,y))$ are determined by (ω,ϕ) via (3.17) and that $f(x,0)=g(x,0)=0$ for all x so that F preserves the real torus $\mathbb{T}\times\{0\}$. By our choice of ψ in the preceding section and by (3.90), we clearly have $\varphi(x,0)=0$ as well. We may therefore put

$$f(x,y) = y\overline{f}(x,y), \quad g(x,y) = y\overline{g}(x,y), \quad \psi(x,y) = y\overline{\psi}(x,y), \quad \text{and} \quad \varphi(x,y) = y\overline{\varphi}(x,y)$$

for some $\overline{f}$, $\overline{g}$, $\overline{\psi}$ and $\overline{\varphi}$ analytic in a complex neighborhood of $\mathbb{T}\times\{0\}$.

Let us first focus on $\overline{f}$ and $\overline{g}$. Substituting the definitions of $\overline{f}$ and $\overline{g}$ into (3.17) and rearranging,

$$1+\overline{f}(x,y) + \frac{\phi(x+\omega+y(1+\overline{f}(x,y))) - \phi(x+\omega)}{y} = \frac{1}{1+\dot{\phi}(x)}$$

$$1+\overline{g}(x,y) = \left[1+\dot{\phi}(x+\omega+y(1+\overline{f}(x,y)))\right]\left[1+\overline{f}(x,y) + \frac{\phi(x+y(1+\overline{f}(x,y))) - \phi(x)}{y}\right]. \tag{3.136}$$

By the Mean Value Theorem,

$$\begin{aligned}\frac{\phi(x+\omega+y(1+\overline{f}(x,y)))-\phi(x+\omega)}{y} =&(1+\overline{f}(x,y))\int_0^1\dot{\phi}(x+\omega+t(y+f(x,y)))dt\\ \frac{\phi(x+y(1+\overline{f}(x,y)))-\phi(x)}{y} =&(1+\overline{f}(x,y))\int_0^1\dot{\phi}(x+t(y+f(x,y)))dt.\end{aligned}$$

Hence, Equation (3.136) can be written as

$$\begin{aligned}\overline{f}(x,y) =&\left[(1+\dot{\phi}(x))\Big(1+\int_0^1\dot{\phi}(x+\omega+t(y+f(x,y)))dt\Big)\right]^{-1}-1\\ \overline{g}(x,y) =&\frac{(1+\dot{\phi}(x+\omega+y+f(x,y)))\Big(1+\int_0^1\dot{\phi}(x+t(y+f(x,y)))dt\Big)}{(1+\dot{\phi}(x))\Big(1+\int_0^1\dot{\phi}(x+\omega+t(y+f(x,y)))dt\Big)}-1.\end{aligned} \tag{3.137}$$

Using (3.136) and (3.137), we control the norms of $\overline{f}$ and $\overline{g}$ over domains $\mathcal{D}_\lambda$ for a sufficiently small λ.

Lemma 3.14. *Under the condition $\|\dddot{\phi}\|\le 1$, the functions $\overline{f}$ and $\overline{g}$ are well-defined on the domain $\mathcal{D}_\lambda$ with $\lambda=(1-|\dot{\phi}|)^2$, and*

$$|\overline{f}|^{(\lambda)}\le\frac{|\dot{\phi}|(2-|\dot{\phi}|)}{(1-|\dot{\phi}|)^2},\quad |1+\overline{f}|^{(\lambda)}\le\frac{1}{(1-|\dot{\phi}|)^2}\quad \textit{and}\quad |\overline{g}|^{(\lambda)}\le\frac{2|\dot{\phi}|(2+|\dot{\phi}|)}{(1-|\dot{\phi}|)^2}. \tag{3.138}$$

Proof. We use (3.136) and the contraction mapping principle to solve for $\overline{f}$. Let $\lambda=(1-|\dot{\phi}|)^2$ and define $r=\lambda^{-1}-1$. Fix $(x,y)\in\mathcal{D}_\lambda$ and consider the map

$$\begin{aligned}&\mathcal{P}_{x,y}:\overline{\mathbb{D}}_r\to\mathbb{C}\\ &\qquad z\mapsto-\frac{\dot{\phi}(x)}{1+\dot{\phi}(x)}-\frac{\phi(x+\omega+y(1+z))-\phi(x+\omega)}{y}.\end{aligned}$$

By definitions of $\mathcal{D}_\lambda$ and r, when $|z|\le r$ we have $|\Im(x+\omega+y(1+z))|\le|\Im x|+\lambda^{-1}|y|<1$. Hence, the map $\mathcal{P}_{x,y}$ is well-defined on $\overline{\mathbb{D}}_r$, and $\mathcal{P}'_{x,y}(z)=-\dot{\phi}(x+\omega+y(1+z))$. Since the assumption $\|\dddot{\phi}\|\le 1$ implies *a fortiori* that $|\dot{\phi}|<1$, the map $\mathcal{P}_{x,y}$ is contracting over $\overline{\mathbb{D}}_r$. On the other hand, by the mean value theorem and the definitions of λ and r, for $|z|\le r$,

$$|\mathcal{P}_{x,y}(z)|\le\frac{|\dot{\phi}|}{1-|\dot{\phi}|}+(1+r)|\dot{\phi}|=\Big(\frac{1}{1-|\dot{\phi}|}+\frac{1}{\lambda}\Big)|\dot{\phi}|=\frac{2-|\dot{\phi}|}{(1-|\dot{\phi}|)^2}|\dot{\phi}|=\frac{1}{(1-|\dot{\phi}|)^2}-1=r.$$

It follows that $\mathcal{P}_{x,y}(\overline{\mathbb{D}}_r)\subset\overline{\mathbb{D}}_r$. Hence, by the contraction mapping principle there exists a unique fixed point $\mathcal{P}_{x,y}(z)=z$ in $\overline{\mathbb{D}}_r$ depending analytically on (x,y). We shall define $\overline{f}(x,y)$ to be this fixed point. In particular, we have

$$|\overline{f}(x,y)|\le r=\frac{|\dot{\phi}|(2-|\dot{\phi}|)}{(1-|\dot{\phi}|)^2}\quad\text{and}\quad|1+\overline{f}(x,y)|\le 1+r=\frac{1}{\lambda}=\frac{1}{(1-|\dot{\phi}|)^2}.$$

We have obtained the first two inequalities of (3.138). Finally, from the second line of (3.137), we deduce that g is also well-defined on $\mathcal{D}_\lambda$ and satisfies

$$|\overline{g}(x,y)|\le\frac{2|\dot{\phi}|(2+|\dot{\phi}|)}{(1-|\dot{\phi}|)^2}.$$

The proof of the lemma is finished. □

Next, we apply Proposition 3.20 to estimate the norms $|\overline{f}|^{(\lambda',2)}$ and $|\overline{g}|^{(\lambda',2)}$ by shrinking $\mathcal{D}_\lambda$ to a smaller domain $\mathcal{D}_{\lambda'}$.

Corollary 3.15. *Suppose* $\|\dddot{\phi}\| \le 1$. *Then*

$$|\overline{f}|^{(\lambda',2)} + |\overline{g}|^{(\lambda',2)} \lesssim 1 \quad \textit{for} \quad \lambda' = (1-|\dot{\phi}|)^2/2.$$

Proof. To estimate $|\overline{f}|^{(\lambda',2)}$ and $|\overline{g}|^{(\lambda',2)}$, let us define $\Sigma(x,y) = (0, f(x,y))$ and

$$\begin{aligned} w_1(x,y) =& \left[(1+\dot{\phi}(x))\Big(1+\int_0^1 \dot{\phi}(x+\omega+ty)dt\Big)\right]^{-1} - 1, \\ w_2(x,y) =& \frac{(1+\dot{\phi}(x+\omega+y))\big(1+\int_0^1\dot{\phi}(x+ty)dt\big)}{(1+\dot{\phi}(x))\big(1+\int_0^1\dot{\phi}(x+\omega+ty)dt\big)} - 1. \end{aligned}$$

By (3.137),

$$\overline{f} = w_1 \circ (\mathrm{Id}+\Sigma) \quad \text{and} \quad \overline{g} = w_2 \circ (\mathrm{Id}+\Sigma).$$

Observe that for $0 \le t \le 1$, the expression $\dot{\phi}(x+\omega+ty)$ is well-defined for $|\Im x|+|y| < 1$. Hence w_1 and w_2 are well-defined on $\mathcal{D}_1$. On the other hand, for $\lambda = (1-|\dot{\phi}|)^2$ and $(x,y)\in\mathcal{D}_\lambda$, the results of Lemma 3.14 imply that $|\Im x|+|y+f(x,y)| \le |\Im x|+(|1+\overline{f}|^{(\lambda)})|y| \le |\Im x|+\lambda^{-1}|y| < 1$ with $0\le t\le 1$, and thus $(\mathrm{Id}+\Sigma)(\mathcal{D}_\lambda) \subset \mathcal{D}_1$ and the map $w_j\circ(\mathrm{Id}+\Sigma)$ is well-defined on $\mathcal{D}_\lambda$. Moreover, since the assumption $\|\dddot{\phi}\| \le 1$ implies $|\dot{\phi}| \le (2\pi)^{-2}$, computing the derivatives of $w_j,\ j=1,2$ shows that the following crude bounds hold

$$|w_j|^{(1,1)} \lesssim 1 + \|\ddot{\phi}\| \lesssim 1 \quad \text{and} \quad |w_j|^{(1,2)} \lesssim 1 + \|\dddot{\phi}\| \lesssim 1. \tag{3.139}$$

Let us now apply Proposition 3.20 part 3 with $v_1 = 0$, $v_2 = w_1$, $\bar{\sigma}_1 = 0$, $\bar{\sigma}_2 = \overline{f}$ and $r = 2$. We have accordingly that for all $\lambda' < \lambda = (1-|\dot{\phi}|)^2$, [2]

$$|\overline{f}|^{(\lambda',2)} \le C_2\Big(\frac{2+\lambda}{1-(\lambda'(1-|\dot{\phi}|)^{-2})^{\frac{1}{2}}}\Big)^{A_2}(1+|w_1|^{(1,1)})^{B_2}|w_1|^{(1,2)}.$$

Fixing $\lambda' = \frac{1}{2}(1-|\dot{\phi}|)^2$ and using (3.139), we have

$$|\overline{f}|^{(\lambda',2)} \lesssim 1. \tag{3.140}$$

At last, applying Proposition 3.20 part 2 to $\overline{g} = w_2\circ(\mathrm{Id}+\Sigma)$ and using (3.139), (3.140) and the assumption $\|\dddot{\phi}\| \le 1$, we have

$$|\overline{g}|^{(\lambda',2)} \lesssim |w_2|^{(1,2)}(1+|\overline{f}|^{(\lambda',2)})^2 \lesssim 1.$$

The proof is finished. □

We now consider the last two arrows of (3.134).

Since $\lambda_1 < (1-|\dot{\phi}|)^2$, it follows from Lemma 3.14 that F is well-defined on $\mathcal{D}_{\lambda_1}$.

We first derive an extension to Proposition 3.20 part (1) and part (2) in order to verify that the last two arrows of (3.134) make sense and thus the compositions $\psi\circ F$ and $\varphi\circ F$ are well-defined. Let decompose $F = \tau_\omega \circ F_0$ where $F_0(x,y) = (x+y+f(x,y), y+g(x,y))$ and

[2] The proposition is applied with $(\lambda,\lambda',\lambda'') \leftarrow (1,\lambda,\lambda')$.

$\tau_\omega(x,y) = (x+\omega, y)$. Since $\tau_\omega(\mathcal{D}_\lambda) = \mathcal{D}_\lambda$ for any $\lambda > 0$, we just need to verify $F_0(\mathcal{D}_{\lambda_2}) \subset \mathcal{D}_{\lambda_1}$ and $F_0(\mathcal{D}_{\lambda_1}) \subset \mathcal{D}_{\lambda_0}$.

Fix $q \geq 2$. By the estimates in Lemma 3.14 for $|\overline{f}|^\lambda$ and $|\overline{g}|^\lambda$ and the smallness of $|\dot{\phi}|$, one easily verifies that the condition (3.184) of Proposition 3.20 is satisfied with $\bar{\sigma}_1 = 1 + \overline{f}$, $\bar{\sigma}_2 = \overline{g}$ and $(\lambda', \lambda) \in \{(q^{-5}, 1),\ (q^{-5}/2, q^{-5})\}$. Thus, we may apply Proposition 3.20 part (1) with $\mathrm{Id} + \Sigma = F_0$, which implies the desired inclusion.

Having verified that compositions $\psi \circ F$ and $\varphi \circ F$ are well-defined on $\mathcal{D}_{\lambda_1}$ and on $\mathcal{D}_{\lambda_2}$ respectively, we apply Proposition 3.20 part (2) with $\sigma_1(x,y) = y + f(x,y)$, $\sigma_2(x,y) = g(x,y)$, $v = \psi \circ \tau_\omega$ and $v = \varphi \circ \tau_\omega$ to get

$$\begin{cases} |\psi \circ F|^{(\lambda_1,3)} = |\psi \circ \tau_\omega \circ F_0|^{(\lambda_1,3)} \lesssim |\psi \circ \tau_\omega|^{(\lambda_0,3)}(1 + |y+f|^{(\lambda_1,3)} + |g|^{(\lambda_1,3)})^3 \\ |\varphi \circ F|^{(\lambda_2,3)} = |\varphi \circ \tau_\omega \circ F_0|^{(\lambda_2,3)} \lesssim |\varphi \circ \tau_\omega|^{(\lambda_1,3)}(1 + |y+f|^{(\lambda_2,3)} + |g|^{(\lambda_2,3)})^3. \end{cases}$$

Using Lemma 3.19 with $(\lambda', \lambda) = (\lambda_1, 2\lambda_1)$, we have

$$1 + |y+f|^{(\lambda_1,3)} + |g|^{(\lambda_1,3)} \leq \frac{3+\lambda_1}{1-1/2}(1 + |1+\overline{f}|^{(2\lambda_1,2)} + |\overline{g}|^{(2\lambda_1,2)}),$$

where, since $2\lambda_1 < (1-|\dot{\phi}|)^2/2$, Corollary 3.15 implies that the terms inside the parentheses in the preceding expression is $\lesssim 1$. We also have the obvious equalities $|\psi \circ \tau_\omega|^{(\lambda_0,3)} = ||\psi|^{(\lambda_0,3)}$ and $|\varphi \circ \tau_\omega|^{(\lambda_0,3)} = ||\varphi|^{(\lambda_0,3)}$. It follows that

$$|\psi \circ F|^{(\lambda_1,3)} \lesssim |\psi|^{(\lambda_0,3)} \quad \text{and} \quad |\varphi \circ F|^{(\lambda_2,3)} \lesssim |\varphi|^{(\lambda_1,3)}. \tag{3.141}$$

By the same argument using Lemma 3.19,

$$|f|^{(\lambda_1,3)} + |g|^{(\lambda_1,3)} \lesssim 1. \tag{3.142}$$

Estimates of ψ and φ. Recall that $\bar{\psi}(x,y) = \psi_1(x) + \psi_2(x)y$ by construction. Applying Lemma 3.19 twice and using $\|\ddot{\psi}_1\| \lesssim q^2$ and $\|\dot{\psi}_2\| \lesssim q^4$ from (3.130), we have

$$|\psi|^{(\lambda_0,3)} \lesssim |\bar{\psi}|^{(2\lambda_0,2)} \lesssim \|\ddot{\psi}_1\| + \|\dot{\psi}_2\| \lesssim q^4. \tag{3.143}$$

Using the definition $\varphi = f + \psi \circ F - \psi$ and applying (3.141) - (3.143), we obtain

$$|\varphi|^{(\lambda_1,3)} \lesssim |f|^{(\lambda_1,3)} + |\psi|^{(\lambda_0,3)} \lesssim q^4. \tag{3.144}$$

Since $\varphi(x,y) = y\bar{\varphi}(x,y)$, by the mean value theorem,

$$\bar{\varphi}(x,y) = \varphi_1(x) + y\int_0^1 \frac{\partial^2 \varphi}{\partial y^2}(x,ty)(1-t)dt. \tag{3.145}$$

When $(x,y) \in \mathcal{D}_{\lambda_1}$, we have $|y| \leq q^{-5}$. Therefore, the estimate (3.144) together with $\|\varphi_1\| < 1/18$ and $\|\ddot{\varphi}_1\| \lesssim q^2$ from (3.130) implies that, for some universal constant $c > 0$,

$$\begin{aligned} |\bar{\varphi}|^{(\lambda_1)} &\leq \|\varphi_1\| + \frac{1}{2q^5}|\varphi|^{(\lambda_1,2)} < \frac{1}{18} + \frac{c}{q}; \\ |\bar{\varphi}|^{(\lambda_2,2)} &\lesssim \|\ddot{\varphi}_1\| + \left|\frac{\partial^2 \varphi}{\partial y^2}\right|^{(\lambda_1,1)} \lesssim q^2 + |\varphi|^{(\lambda_1,3)} \lesssim q^4. \end{aligned} \tag{3.146}$$

In the second line, we have applied Lemma 3.19 to the integral term of (3.145).

Estimates of $\mathrm{Id}+\widetilde{\Sigma}$**.** Next, we consider the map $\mathrm{Id}+\widetilde{\Sigma}$ in (3.134). To apply Proposition 3.21 with $\bar{\xi}_1 \leftarrow \bar{\psi}$, $\bar{\xi}_2 \leftarrow \bar{\varphi}$, $\Xi \leftarrow \Psi$ and $\Sigma \leftarrow \widetilde{\Sigma}$, we need to verify the condition (3.188) holds with $\lambda' = \lambda_3$, $\lambda = \lambda_2$ and q sufficiently large, but this follows easily from (3.143) and (3.146) as well as the definitions of λ_3 and λ_2.

Accordingly, Proposition 3.21 applies and the right inverse $\mathrm{Id}+\widetilde{\Sigma}$ of $\mathrm{Id}+\Psi$ exists and is defined on $\mathcal{D}_{\lambda_3}$. By Proposition 3.20 part (1) one also verifies that $(\mathrm{Id}+\Psi)(\mathcal{D}_{\lambda_4}) \subset \mathcal{D}_{\lambda_3}$ for q sufficiently large. Hence, Proposition 3.21 conclusion 3.190 implies that $(\mathrm{Id}+\widetilde{\Sigma})\circ(\mathrm{Id}+\Psi) = \mathrm{Id}$ on $\mathcal{D}_{\lambda_4}$. If we put $\widetilde{F} = (\mathrm{Id}+\Psi) \circ F \circ (\mathrm{Id}+\widetilde{\Sigma})$, then in view of (3.134) the map $\widetilde{F}$ is well-defined on $\mathcal{D}_{\lambda_3}$ and we have $(\mathrm{Id}+\Psi) \circ F = \widetilde{F} \circ (\mathrm{Id}+\Psi)$ on $\mathcal{D}_{\lambda_4}$. Write $\widetilde{F}(x,y) = (x+\omega+y+\widetilde{f}(x,y), y+\widetilde{g}(x,y))$ for analytic $\widetilde{f},\widetilde{g} : \mathcal{D}_{\lambda_3} \to \mathbb{C}$. Then by the discussion in the beginning of Section 3.7 we have $\widetilde{f} \circ (\mathrm{Id}+\Psi) = 0$ and $\widetilde{g} = (g+\varphi\circ F-\varphi)\circ(\mathrm{Id}+\widetilde{\Sigma})$. By analyticity $\widetilde{f}$ vanishes identically.

Write $\widetilde{\Sigma}(x,y) = (y\bar{\sigma}_1(x,y), y\bar{\sigma}_2(x,y))$. By the conclusion (3.191) of Proposition 3.21 with $r=2$ and $\lambda'' = 7\lambda_4/6$,

$$1+|\bar{\sigma}_1|^{(\frac{7}{6}\lambda_4,2)}+|\bar{\sigma}_2|^{(\frac{7}{6}\lambda_4,2)} \le C_2\Big(\frac{2+\lambda_3}{1-\beta}\Big)^{A_2}\Big(\frac{1+|\bar{\psi}|^{(\lambda_2,1)}+|\bar{\varphi}|^{(\lambda_2,1)}}{1-|\bar{\varphi}|^{(\lambda_2)}}\Big)^{B_2}\Big(1+|\bar{\psi}|^{(\lambda_2,2)}+|\bar{\varphi}|^{(\lambda_2,2)}\Big)$$

where $\beta = (\frac{7}{6}\frac{\lambda_4}{\lambda_3})^{1/2} = (7/8)^{1/2} < 1$. By (3.143) and (3.146), we arrive at a crude bound:

$$1+|\bar{\sigma}_1|^{(\frac{7}{6}\lambda_4,2)}+|\bar{\sigma}_2|^{(\frac{7}{6}\lambda_4,2)} \lesssim q^{4(B_2+1)}. \tag{3.147}$$

Estimate of $\widetilde{g}$**.** Finally, we may bound $|\widetilde{g}|^{(\lambda_4,3)}$ as follows.

$$\begin{aligned}
|\widetilde{g}|^{(\lambda_4,3)} &= |(g+\varphi\circ F-\varphi)\circ(\mathrm{Id}+\widetilde{\Sigma})|^{(\lambda_4,3)} \\
&\lesssim |g+\varphi\circ F-\varphi|^{(\lambda_2,3)}(1+|\sigma_1|^{(\lambda_4,3)}+|\sigma_2|^{(\lambda_4,3)})^3 && \text{by Proposition 3.20 part (2)} \\
&\lesssim |g+\varphi\circ F-\varphi|^{(\lambda_2,3)}(1+|\bar{\sigma}_1|^{(\frac{7}{6}\lambda_4,2)}+|\bar{\sigma}_2|^{(\frac{7}{6}\lambda_4,2)})^3 && \text{by Lemma 3.19} \\
&\lesssim q^{12(B_2+1)}|g+\varphi\circ F-\varphi|^{(\lambda_2,3)} && \text{by (3.147)} \\
&\lesssim q^{12(B_2+1)}(|g|^{(\lambda_2,3)}+|\varphi|^{(\lambda_1,3)}) && \text{by (3.141)} \\
&\lesssim q^{12(B_2+1)}q^4 && \text{by (3.142) and (3.144).}
\end{aligned}$$

By the explicit recursive formula for B_r at the end of the proof of 3.20, one can readily compute that $B_2 = 4$. Therefore, we have obtained $|\widetilde{g}|^{(\lambda_4,3)} \lesssim q^{64}$, which is the first inequality of (3.135).

Estimate of $\widetilde{S}$**.** We start by estimating the $|\cdot|^{(\lambda_2,3)}$-norm of the normalized generating function S associated with (ω,ϕ) in adapted coordinates. By (3.23),

$$\frac{\partial S}{\partial x} = y\Big[(1+\overline{g})\Big(1+\frac{\partial f}{\partial x}\Big)-1\Big] \quad \text{and} \quad \frac{\partial S}{\partial y} = y(1+\overline{g})\Big(1+\frac{\partial f}{\partial y}\Big).$$

It follows from Lemma 3.19, Corollary 3.15 and (3.142) that

$$\begin{aligned}
\Big|\frac{\partial S}{\partial x}\Big|^{(\lambda_2,2)} &\lesssim \Big|(1+\overline{g})\Big(1+\frac{\partial f}{\partial x}\Big)-1\Big|^{(\lambda_1,1)} \lesssim (1+|\overline{g}|^{(\lambda_1,1)})\Big(1+\Big|\frac{\partial f}{\partial x}\Big|^{(\lambda_1,1)}\Big) \lesssim 1 \\
\Big|\frac{\partial S}{\partial y}\Big|^{(\lambda_2,2)} &\lesssim \Big|(1+\overline{g})\Big(1+\frac{\partial f}{\partial y}\Big)\Big|^{(\lambda_1,1)} \lesssim (1+|\overline{g}|^{(\lambda_1,1)})\Big(1+\Big|\frac{\partial f}{\partial y}\Big|^{(\lambda_1,1)}\Big) \lesssim 1.
\end{aligned}$$

Applying the second estimate above to the Taylor remainder of

$$S(x,y) = \frac{1+f_1(x)}{2}y^2 + y^3\int_0^1 \frac{\partial^3 S}{\partial y^3}(x,ty)(1-t)^2 \mathrm{d}t$$

and using $\|f_1\| \lesssim 1$, we get $|S|^{(\lambda_2)} \lesssim 1$ and thus,

$$|S|^{(\lambda_2,3)} \le \max\left\{|S|^{(\lambda_2)}, \left|\frac{\partial S}{\partial x}\right|^{(\lambda_2,2)}, \left|\frac{\partial S}{\partial y}\right|^{(\lambda_2,2)}\right\} \lesssim 1. \tag{3.148}$$

It follows that

$$\begin{aligned}
|\widetilde{S}|^{(\lambda_4,3)} &\lesssim |S|^{(\lambda_2,3)}(1+|\sigma_1|^{(\lambda_4,3)}+|\sigma_2|^{(\lambda_4,3)})^3 && \text{by Proposition 3.20 part (2)}\\
&\lesssim |S|^{(\lambda_2,3)}(1+|\bar\sigma_1|^{(7\lambda_4/6,\ 2)}+|\bar\sigma_2|^{(7\lambda_4/6,\ 2)})^3 && \text{by Lemma 3.19}\\
&\lesssim q^{12(B_2+1)} && \text{by (3.147) and (3.148).}
\end{aligned}$$

Using $B_2 = 4$ we obtain $|\widetilde{S}|^{(\lambda_4,3)} \lesssim q^{60}$, which obviously implies the second inequality of (3.135). In view of the discussion immediately following (3.135), we have finished the proof of Lemma 3.13.

End of the proof of Proposition 3.11

We now complete the proof of Proposition 3.11. Let us fix from now on that $\lambda = q^{-5}/4$ and let $\mathcal{D}_\lambda$ and the $|\cdot|^{(\lambda,r)}$-norm be defined as in the beginning of Section 3.7.1. Take $(x,y) \in \mathcal{D}_\lambda$. Combining Lemma 3.12 and Lemma 3.13,

$$\begin{aligned}
\left|\widetilde{g}(x,y) - y\Big(\omega - \frac{p}{q} + \frac{y}{2}\Big)\frac{\langle \dot f_1\rangle_q(x)}{1+\langle f_1\rangle_q(x)}\right| &\lesssim q^2\left|\omega-\frac{p}{q}\right||y| + q^5\left|\omega-\frac{p}{q}\right||y|^2 + q^{\nu_0}|y|^3\\
\left|\widetilde{S}(x,y) - \frac{1+f_1(x)}{2(1+\langle f_1\rangle_q(x))^2}y^2\right| &\lesssim \left|\omega-\frac{p}{q}\right||y|^2 + q^{\nu_0}|y|^3.
\end{aligned} \tag{3.149}$$

Recall that in the preceding paragraph we have shown $(\mathrm{Id}+\Psi)\circ(\mathrm{Id}+\widetilde{\Sigma}) = \mathrm{Id}$ over $\mathcal{D}_\lambda$ and

$$(\mathrm{Id}+\Psi)\circ F\circ(\mathrm{Id}+\widetilde{\Sigma}) : \begin{pmatrix} x\\ y\end{pmatrix} \mapsto \begin{pmatrix} x+\omega+y\\ y+\widetilde{g}(x,y)\end{pmatrix}$$

is analytic on $\mathcal{D}_\lambda$. Define a translation $\tau(x,y) = (x, p/q-\omega+y)$. Then we have

$$\tau^{-1}\circ(\mathrm{Id}+\Psi)\circ(\mathrm{Id}+\widetilde{\Sigma})\circ\tau = \mathrm{Id} \quad \text{over} \quad \tau^{-1}\mathcal{D}_\lambda. \tag{3.150}$$

Using assumption (3.94), one readily checks that $\mathbb{T}_{1-q^{-1}} \times \mathbb{D}_{|\omega-p/q|/2} \subset \tau^{-1}\mathcal{D}_\lambda$. We prove the conclusion of Proposition 3.11 with the following choices.

$$\Theta := (\mathrm{Id}+\widetilde{\Sigma})\circ\tau; \quad \widetilde{F}_q := \tau^{-1}\circ(\mathrm{Id}+\Psi)\circ F\circ\Theta; \quad \widetilde{S}_q := S\circ\Theta; \quad \widetilde{g}_q := \widetilde{g}\circ\tau. \tag{3.151}$$

Then $\widetilde{F}_q(x,y) = (x+p/q+y, y+\widetilde{g}_q(x,y))$ for $\widetilde{g}_q : \mathbb{T}_{1-q^{-1}} \times \mathbb{D}_{|\omega-p/q|/2} \to \mathbb{C}$ analytic.

Let us fix $(x,y) \in \mathbb{T}_{1-q^{-1}} \times \mathbb{D}_{|\omega-p/q|/2}$.

Define $\theta_1, \theta_2 : \mathbb{T}_{1-q^{-1}} \times \mathbb{D}_{|\omega-p/q|/2} \to \mathbb{C}$ as in (3.95). By (3.150) and the definition of Θ, we have $(\mathrm{Id}+\Psi)\circ\Theta = \tau$. Componentwise, this means that

$$\begin{cases}\theta_1(x,y) + \psi(x+\theta_1(x,y),\ y+p/q-\omega+\theta_2(x,y)) = 0\\ \theta_2(x,y) + \varphi(x+\theta_1(x,y),\ y+p/q-\omega+\theta_2(x,y)) = 0.\end{cases} \tag{3.152}$$

Since $\Theta(x,y) \in \mathcal{D}_{\lambda_2}$, by the first line of (3.146) and $\|\phi_1\| < 1/18$, we have

$$|\varphi(x+\theta_1(x,y),\ y+p/q-\omega+\theta_2(x,y))| < \frac{1}{18}|y+p/q-\omega+\theta_2(x,y)|$$

for q sufficiently large. Combining with the second line of (3.152) and using $|y| < |\omega - p/q|/2$,

$$|\theta_2(x,y)| < \frac{1}{18}\Big(\frac{3}{2}\left|\omega - \frac{p}{q}\right| + |\theta_2(x,y)|\Big).$$

The second estimate of (3.96) follows from above. Recall that $\psi(x,y) = y\psi_1(x) + y^2\psi_2(x)$ and from (3.130) we have $\|\psi_1\| \lesssim q$ and $\|\psi_2\| \lesssim q^3$. Therefore the first line of (3.152) and $|\theta_2(x,y)| \lesssim |\omega - p/q|$ imply

$$|\theta_1(x,y)| \lesssim q\left|\omega - \frac{p}{q}\right|.$$

We have used $|\omega - p/q| \leq q^{-6}$. The proof of (3.96) is complete.

Using $|p/q - \omega + y| \lesssim |\omega - p/q|$ and (3.149),

$$\left|\widetilde{g}_q(x,y) + \frac{1}{2}\Big(\frac{p}{q} - \omega + y\Big)\Big(\frac{p}{q} - \omega - y\Big)\frac{\langle \dot{f}_1\rangle_q(x)}{1+\langle f_1\rangle_q(x)}\right| \lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3$$
$$\left|\widetilde{S}_q(x,y) - \Big(\frac{p}{q} - \omega + y\Big)^2\frac{1+f_1(x)}{2(1+\langle f_1\rangle_q(x))^2}\right| \lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3,$$

which give (3.97). It is clear from the construction that the change of coordinates $\mathrm{Id}+\widetilde{\Sigma}$ depends analytically on ϕ and it is real-analytic when ϕ is so.

Recall that $\Sigma(x,y) = (\sigma_1(x,y), \sigma_2(x,y)) = (y\bar{\sigma}_1(x,y), y\bar{\sigma}_2(x,y))$. By (3.147), Lemma 3.19 and $B_2 = 4,\ \lambda = \lambda_4$,

$$|\bar{\sigma}_j|^{(\lambda,2)} \lesssim q^{20} \qquad (j = 1,2). \tag{3.153}$$

Since the original map F preserves the area form $\mathrm{d}x\mathrm{d}y$, the map $\widetilde{F}_q$ defined by (3.151) preserves the area form $\Theta^*\mathrm{d}x\mathrm{d}y = \tau^*(\mathrm{Id}+\widetilde{\Sigma})^*\mathrm{d}x\mathrm{d}y$, where

$$(\mathrm{Id}+\widetilde{\Sigma})^*\mathrm{d}x\mathrm{d}y = \Big(1 + \frac{\partial\sigma_1}{\partial x} + \frac{\partial\sigma_2}{\partial y} + \frac{\partial\sigma_1}{\partial x}\frac{\partial\sigma_2}{\partial y} - \frac{\partial\sigma_1}{\partial y}\frac{\partial\sigma_2}{\partial x}\Big)\mathrm{d}x\mathrm{d}y.$$

Hence we may write

$$\Theta^*\mathrm{d}x\mathrm{d}y = \Big(\frac{1}{1+\langle f_1\rangle_q} + \widetilde{\eta}\Big)\mathrm{d}x\mathrm{d}y$$

where

$$\widetilde{\eta} = \frac{1}{1+\varphi_1} - \frac{1}{1+\langle f_1\rangle_q} + \tau^*\left[\Big(\frac{\partial\sigma_2}{\partial y}(x,y) + 1 - \frac{1}{1+\varphi_1(x)}\Big) + \Big(\frac{\partial\sigma_1}{\partial x}\Big(1 + \frac{\partial\sigma_2}{\partial y}\Big) - \frac{\partial\sigma_1}{\partial y}\frac{\partial\sigma_2}{\partial x}\Big)\right].$$

Recall that the construction in Section 3.7.1 gives $\varphi_1 = \langle f_1\rangle_q + \{\varphi_1\}_q$ and we have an upper bound $\|\{\varphi_1\}_q\| \lesssim q^2|\omega - p/q|^2$ from (3.130). Therefore, it can be verified with mean value theorem that

$$\left\|\frac{1}{1+\varphi_1} - \frac{1}{1+\langle f_1\rangle_q}\right\| \lesssim q^2\left|\omega - \frac{p}{q}\right|^2.$$

On the other hand, Taylor expanding the y-component of the equality $\widetilde{\Sigma}+(\mathrm{Id}+\Psi)\circ(\mathrm{Id}+\widetilde{\Sigma})=0$ we have $\partial\sigma_2/\partial y(x,0)=(1+\varphi_1(x))^{-1}-1$. By (3.153) and $\sigma_j(x,0)=0$, we have that, for $(x,y)\in\mathcal{D}_\lambda$,

$$\left|\frac{\partial\sigma_2}{\partial y}(x,y)+1-\frac{1}{1+\varphi_1(x)}\right|\lesssim|y||\sigma_2|^{(\lambda,2)}\lesssim q^{20}|y|,$$
$$\left|\frac{\partial\sigma_j}{\partial x}(x,y)\right|\lesssim|y||\sigma_j|^{(\lambda,2)}\lesssim q^{20}|y|\quad\text{and}\quad\left|\frac{\partial\sigma_j}{\partial y}(x,y)\right|\lesssim|\sigma_j|^{(\lambda,1)}\lesssim q^{20}.$$

Since the translation τ maps $(\mathbb{T}_{1-q^{-1}}\times\mathbb{D}_{|\omega-p/q|/2})$ into $\mathcal{D}_\lambda$, applying the above estimates to $\widetilde{\eta}$ over the domain $\mathbb{T}_{1-q^{-1}}\times\mathbb{D}_{|\omega-p/q|/2}$, we get

$$|\widetilde{\eta}|_{1-\frac{1}{q},\frac{1}{2}|\omega-\frac{p}{q}|}\lesssim q^{40}\left|\omega-\frac{p}{q}\right|.$$

Since $40<\nu_0$, we have proved (3.98). The proof of Proposition 3.11 is finished.

3.7.2 Step 2: coordinate changes by an iterative scheme

Proposition 3.16. *We consider the same setup as Proposition 3.11. Fix $\nu>\nu_0$, $M\geq 1$ and $\kappa\in\mathbb{Z}_{>0}$. Then for all $q\in\mathbb{Z}_{>0}$ sufficiently large and satisfying*

$$\left|\omega-\frac{p}{q}\right|\leq q^{-\nu},\tag{3.154}$$

and with $\widetilde{F}_q=\Theta^{-1}\circ F\circ\Theta:(x,y)\mapsto(x+p/q+y,y+\widetilde{g}_q(x,y))$ denoting the standard map in the coordinates constructed in Proposition 3.11, the following conclusion holds true.

There exists a further change of coordinates

$$\mathrm{Id}+\Sigma:\mathbb{T}_{1-\frac{2}{q}}\times\mathbb{D}_{\frac{1}{3}|\omega-\frac{p}{q}|}\to\mathbb{T}_{1-\frac{1}{q}}\times\mathbb{D}_{\frac{1}{2}|\omega-\frac{p}{q}|}$$

such that the map $F_q:=(\mathrm{Id}+\Sigma)^{-1}\circ\widetilde{F}_q\circ(\mathrm{Id}+\Sigma)$ takes the form $(x,y)\mapsto(x+p/q+y,y+g_q(x,y))$ for some analytic g_q satisfying

$$|g_q|_{1-\frac{2}{q},\frac{1}{3}|\omega-\frac{p}{q}|}\lesssim\left|\omega-\frac{p}{q}\right|^2\tag{3.155}$$

$$|\{g_q\}_q|_{1-\frac{2}{q},\frac{1}{3}|\omega-\frac{p}{q}|}\lesssim\mathrm{e}^{-2\pi Mq}\left|\omega-\frac{p}{q}\right|^2\tag{3.156}$$

$$\|\langle g_q(\cdot,0)\rangle_q^N-\langle\widetilde{g}_q(\cdot,0)\rangle_q^N\|\lesssim\kappa\mathrm{e}^{4\pi\kappa}q^{\nu_0-\nu}\left|\omega-\frac{p}{q}\right|^2\quad\textit{where}\quad N=\kappa q.\tag{3.157}$$

Moreover, writing $\Sigma(x,y)=(\sigma_1(x,y),\sigma_2(x,y))$ for analytic $\sigma_1,\sigma_2:\mathbb{T}_{1-2/q}\times\mathbb{D}_{|\omega-p/q|/3}\to\mathbb{C}$, we have

$$|\sigma_j|_{1-\frac{2}{q},\frac{1}{3}|\omega-\frac{p}{q}|}\lesssim q^{5+\nu_0-\nu}\left|\omega-\frac{p}{q}\right|^2,\qquad(j=1,2).\tag{3.158}$$

Remark. The "sufficiently large" condition on q depends only on ν, M and κ. In particular, it is independent of the map F as long as $\|\dddot{\phi}\|\leq 1$. If the irrationality exponent of ω is $\geq\nu$, then there exist infinitely many q satisfying (3.154).

An iterative lemma for Proposition 3.16

Let us consider generally a map

$$F : (x, y) \mapsto (x + p/q + y, y + g(x, y)) \tag{3.159}$$

for analytic $g : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}$. We consider a change of coordinates $\Phi = \mathrm{Id} + \Psi$ where $\Psi(x, y) = (\psi(x, y), \varphi(x, y))$ and $\varphi = \psi \circ F - \psi$. A straightforward calculation analogous to that in Section 3.7.1 shows that, in the new coordinates, the transformed map $\Phi \circ F \circ \Phi^{-1}$ is

$$\Phi \circ F \circ \Phi^{-1}(x, y) = (x + p/q + y, y + \widetilde{g}(x, y))$$

where $\widetilde{g}$ is an analytic function satisfying

$$\widetilde{g} \circ \Phi(x, y) = g + \psi \circ F^2 - 2\psi \circ F + \psi. \tag{3.160}$$

In the following lemma, we show that a judicious choice of ψ can the map almost $1/q$-periodic in the new x-coordinate in the sense that $\widetilde{g}$ has a smaller nonresonant part compared with g.

To state this result, we need to introduce a new norm on the space of analytic functions: for $u : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}$ analytic, let us define

$$|||u|||_{a,b} = \sup_{|y|<b,\ k\in\mathbb{Z}} |\hat{u}(y)_k| \mathrm{e}^{2\pi a|k|}.$$

Recall that $\hat{u}(y)_k$ is the k-th Fourier coefficient of $u(\cdot, y)$. Since $|\hat{u}(y)_k| \le |u|_{a,b}\mathrm{e}^{-2\pi a|k|}$,

$$|||u|||_{a,b} \le |u|_{a,b}.$$

Also observe that for $N \in \mathbb{Z}_{\ge 1}$ and $a' < a$,

$$\begin{aligned} |\left\langle u\right\rangle^{>N}|_{a',b} \le& \sum_{|k|\ge N+1} \sup_{|y|<b} |\hat{u}(y)_k| \mathrm{e}^{2\pi a'|k|} \le \sum_{|k|\ge N+1} |||u|||_{a,b} \mathrm{e}^{-2\pi(a-a')|k|} \\ \le& \frac{\mathrm{e}^{-2\pi(a-a')(N+1)}}{1-\mathrm{e}^{-2\pi(a-a')}} |||u|||_{a,b} \le \frac{\mathrm{e}^{-2\pi(a-a')N}}{2\pi(a-a')} |||u|||_{a,b}. \end{aligned} \tag{3.161}$$

In the last line, we used the elementary inequality $\mathrm{e}^{-x}(1-\mathrm{e}^{-x})^{-1} < x^{-1}$ for $x \in \mathbb{R}$.

Lemma 3.17. *Let $\epsilon > 0$, $q \in \mathbb{Z}_{>1}$, $N \in \mathbb{Z}_{\ge q}$, $5b < \alpha < a$ and $5\epsilon < \beta < b$ such that*

$$qNb \le \frac{1}{2} \quad \textit{and} \quad \frac{q^2 N\epsilon}{\beta} \le \frac{1}{2}. \tag{3.162}$$

Let F be a map of the form (3.159) *with $|g|_{a,b} \le \epsilon$. Then there exists a near-identity analytic change of coordinates*

$$\mathrm{Id} + \Sigma : \mathbb{T}_{a-\alpha} \times \mathbb{D}_{b-\beta} \to \mathbb{T}_{a-\alpha/2} \times \mathbb{D}_{b-\beta/2},$$

where $\Sigma(x, y) = (\sigma_1(x, y), \sigma_2(x, y))$ for some $\sigma_1, \sigma_2 : \mathbb{T}_{a-\alpha} \times \mathbb{D}_{b-\beta} \to \mathbb{C}$ analytic and

$$|\sigma_1|_{a-\alpha,b-\beta} + |\sigma_2|_{a-\alpha,b-\beta} < q^2 N \left|\left|\left| g - \langle g\rangle_q^N \right|\right|\right|_{a,b}, \tag{3.163}$$

and the map $(\mathrm{Id} + \Sigma) \circ F \circ (\mathrm{Id} + \Sigma)^{-1}$ in the new coordinates is

$$(x, y) \mapsto (x + p/q + y, y + \widetilde{g}(x, y))$$

where $\widetilde{g} : \mathbb{T}_{a-\alpha} \times \mathbb{D}_{b-\beta} \to \mathbb{C}$ *is real-analytic and*

$$|\widetilde{g} - \langle g\rangle_q^N|_{a-\alpha,b-\beta} < \lambda \Big|\Big|\Big| g - \langle g\rangle_q^N \Big|\Big|\Big|_{a,b} \quad \textit{and} \quad |\widetilde{g}|_{a-\alpha,b-\beta} < |g|_{a,b} + \lambda \Big|\Big|\Big| g - \langle g\rangle_q^N \Big|\Big|\Big|_{a,b}, \tag{3.164}$$

where

$$\lambda = 12\Big(\frac{q^2N\epsilon}{\beta} + \frac{\mathrm{e}^{-\pi\alpha N}}{\alpha}\Big).$$

Proof. We shall construct $\mathrm{Id} + \Sigma$ as the inverse of the map $\Phi = \mathrm{Id} + \Psi$ where $\Psi(x, y) = (\psi(x, y), \varphi(x, y))$ and $\varphi := \psi \circ F - \psi$. Let $p, q, a, b, \epsilon, N, \alpha, \beta$ be given as in the statement. Set the k-th Fourier coefficient of $\psi(\cdot, y)$ to be

$$\hat{\psi}(y)_k = \frac{\hat{g}(y)_k}{2\mathrm{e}^{2\pi \mathrm{i}k(p/q+y)}[1 - \cos(2\pi k(p/q + y))]} \qquad (q \nmid k \text{ and } |k| \leq N),$$

and $\hat{\psi}(y)_k = 0$ for all other k. By Fourier expansion, this choice of ψ is equivalent to

$$\psi \circ A^2 - 2\psi \circ A + \psi = -\{g\}_q^N.$$

Hence, by (3.160),

$$\widetilde{g} \circ \Phi = \langle g\rangle_q^N + \{g\}_q^{>N} + (\psi \circ F^2 - \psi \circ A^2) + 2(\psi \circ A - \psi \circ F).$$

Let us denote

$$\mathcal{R}' = (\psi \circ F^2 - \psi \circ A^2) + 2(\psi \circ A - \psi \circ F)$$

and

$$\mathcal{R} = \langle g\rangle_q^{>N} + \Big(\langle g\rangle_q \circ \Phi^{-1} - \langle g\rangle_q\Big) + \{g\}_q^{>N} \circ \Phi^{-1} + \mathcal{R}' \circ \Phi^{-1}$$

so that

$$\widetilde{g} = \langle g\rangle_q^N + \mathcal{R}.$$

Step 1: estimate of ψ *and* φ. By assumptions $Nb \leq 1/(2q)$ and $q \geq 2$, for $|y| < b$, $|k| \leq N$ and $q \nmid k$,

$$\Big|1 - \cos 2\pi k\Big(\frac{p}{q} + y\Big)\Big| \geq 1 - \cos\frac{\pi}{q} \geq \frac{4}{q^2}.$$

We have used the elementary inequality $1 - \cos(2\pi x) \geq 16x^2$ for $|x| \leq 1/4$. Thus, by definition of ψ, we have

$$\begin{aligned} |\psi|_{a,b} &\leq \sum_{\substack{q\nmid k,\\ |k|\leq N}} \frac{q^2}{8} \sup_{|y|<b} |\hat{g}(y)_k| \leq \sum_{\substack{q\nmid k,\\ |k|\leq N}} \frac{q^2}{8} \Big|\Big|\Big| \{g\}_q^N \Big|\Big|\Big|_{a,b} \mathrm{e}^{-2\pi|k|a} \\ &< 2N\frac{q^2}{8} \Big|\Big|\Big| \{g\}_q^N \Big|\Big|\Big|_{a,b} \mathrm{e}^{-2\pi a} < \frac{q^2N}{4} \Big|\Big|\Big| \{g\}_q^N \Big|\Big|\Big|_{a,b} \leq \frac{q^2N}{4} \Big|\Big|\Big| g - \langle g\rangle_q^N \Big|\Big|\Big|_{a,b}. \end{aligned} \tag{3.165}$$

Since $\Big|\Big|\Big| g - \langle g\rangle_q^N \Big|\Big|\Big|_{a,b} \leq |g|_{a,b}$, we have in particular that

$$|\psi|_{a,b} < \frac{q^2N\epsilon}{4}. \tag{3.166}$$

Since $|g|_{a,b} \leq \epsilon$, we also have $F(\mathbb{T}_{a-b+\epsilon} \times \mathbb{D}_{b-\epsilon}) \subset \mathbb{T}_a \times \mathbb{D}_b$. It follows from $\varphi = \psi \circ F - \psi$ that

$$|\varphi|_{a-b+\epsilon,b-\epsilon} \leq 2|\psi|_{a,b}. \tag{3.167}$$

Step 2: estimate of $\mathcal{R}'$ *over* $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$. Let $a' = a - \alpha/2$ and $b' = b - \beta/2$. Using the explicit expression of F and the bound $|g|_{a,b} \leq \epsilon$ as well as the assumptions $5\epsilon < \beta$ and $5b < \alpha$, we deduce that the images of $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$ under the maps F^2, A^2, F and A are all contained in $\mathbb{T}_{a-\alpha/10} \times \mathbb{D}_{b-\beta/10}$. [3] By the mean value theorem and Cauchy's estimate,

$$\begin{aligned}
&|\psi \circ F^2 - \psi \circ A^2|_{a',b'} \\
=&\sup_{|\Im x|<a',\ |y|<b'} \left|\psi\Big(x + \frac{2p}{q} + 2y + g(x,y), y + g(x,y) + g\circ F(x,y)\Big) - \psi\Big(x+\frac{2p}{q}+2y, y\Big)\right| \\
=&|g|_{a,b}\left|\frac{\partial \psi}{\partial x}\right|_{a-\alpha/10,b-\beta/10} + 2|g|_{a,b}\left|\frac{\partial \psi}{\partial y}\right|_{a-\alpha/10,b-\beta/10} \\
\leq&\Big(\frac{10}{\alpha}+\frac{20}{\beta}\Big)|\psi|_{a,b}|g|_{a,b} < \frac{22}{\beta}|\psi|_{a,b}|g|_{a,b}.
\end{aligned}$$

In the last inequality, we used $\alpha > 5b > 5\beta$. Similarly,

$$|\psi \circ F - \psi \circ A|_{a',b'} = |g|_{a,b}\left|\frac{\partial \psi}{\partial y}\right|_{a-\alpha/10,b-\beta/10} = \frac{10}{\beta}|\psi|_{a,b}|g|_{a,b}.$$

We have proved

$$|\mathcal{R}'|_{a',b'} \leq \frac{42}{\beta}|\psi|_{a,b}|g|_{a,b}.$$

Step 3: inverting Φ. Observe that since $|g|_{a,b} \leq \epsilon$, the function $\varphi = \psi \circ F - \psi$ is well-defined on $\mathbb{T}_{a-b+\epsilon} \times \mathbb{D}_{b-\epsilon}$ and $|\varphi|_{a-b+\epsilon,b-\epsilon} \leq 2|\psi|_{a,b}$. We shall apply Lemma 3.18 with $a_0 = a - b + \epsilon$, $b_0 = b - \epsilon$, $\delta_1 = a_0 - a'$ and $\delta_2 = b_0 - b'$. Since $a' = a - \alpha/2$ and $b' = b - \beta/2$, we have $\delta_1 > 3\alpha/10$ and $\delta_2 > 3\beta/10$ using the assumptions $5\epsilon < \beta$ and $5b < \alpha$. By (3.166) and $\alpha > 5\beta$, one verifies that the second assumption in (3.162) ensures the condition (3.169). It now follows from Lemma 3.18 that Φ is a diffeomorphism from $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$ onto the image, which contains $\mathbb{T}_{a'-\delta_1} \times \mathbb{D}_{b'-\delta_2}$. In particular, we may define $\Phi^{-1} = \mathrm{Id} + \Sigma : \mathbb{T}_{a-\alpha} \times \mathbb{D}_{b-\beta} \to \mathbb{T}_{a'} \times \mathbb{D}_{b'}$ where $\Sigma(x,y) = (\sigma_1(x,y), \sigma_2(x,y))$ with

$$|\sigma_1|_{a-\alpha,b-\beta} \leq |\psi|_{a',b'}; \qquad |\sigma_2|_{a-\alpha,b-\beta} \leq |\varphi|_{a',b'}. \tag{3.168}$$

Combining with (3.165) and (3.167), we obtain (3.163). Since we have shown that $\widetilde{g} \circ \Phi$ is well-defined on $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$, by precomposing with $(\mathrm{Id} + \Sigma)$, we deduce that $\widetilde{g}$ is well-defined on $\mathbb{T}_{a-\alpha} \times \mathbb{D}_{b-\beta}$ as desired.

Step 4: estimate of $\mathcal{R}'$ *over* $\mathbb{T}_{a-\alpha} \times \mathbb{D}_{b-\beta}$. Similar to step 2, using the mean value theorem and Cauchy's estimate as well as (3.168), we obtain

$$\begin{aligned}
\left|\langle g\rangle_q \circ \Phi^{-1} - \langle g\rangle_q\right|_{a-\alpha,b-\beta} \leq& \left|\frac{\partial \langle g\rangle_q}{\partial x}\right|_{a',b'} |\sigma_1|_{a-\alpha,b-\beta} + \left|\frac{\partial \langle g\rangle_q}{\partial y}\right|_{a',b'} |\sigma_2|_{a-\alpha,b-\beta} \\
\leq&\Big(\frac{2|\psi|_{a,b}}{\alpha} + \frac{4|\psi|_{a,b}}{\beta}\Big)|\langle g\rangle_q|_{a,b} \\
\leq&\frac{22}{5\beta}|\psi|_{a,b}|g|_{a,b}.
\end{aligned}$$

[3] These images are actually contained in an even smaller domain, but we will not need this stronger fact.

On the other hand, applying (3.161) to $u = \langle g\rangle_q$ and $u = \{g\}_q$,

$$
\begin{aligned}
|\,\langle g\rangle_q^{>N}\,|_{a-\alpha,b-\beta} \leq& \frac{\mathrm{e}^{-2\pi\alpha N}}{2\pi\alpha}\left|\left|\left|\langle g\rangle_q^{>N}\right|\right|\right|_{a,b}\\
|\{g\}_q^{>N}\circ\Phi^{-1}|_{a-\alpha,b-\beta} \leq& |\{g\}_q^{>N}|_{a',b'} \leq \frac{\mathrm{e}^{-\pi\alpha N}}{\pi\alpha}\left|\left|\left|\{g\}_q^{>N}\right|\right|\right|_{a,b}.
\end{aligned}
$$

Lastly, we have the obvious bound $|\mathcal{R}'\circ\Phi^{-1}|_{a-\alpha,b-\beta}\leq|\mathcal{R}'|_{a',b'}$ Thus,

$$
\begin{aligned}
|\mathcal{R}|_{a-\alpha,b-\beta} \leq& \frac{\mathrm{e}^{-2\pi\alpha N}}{2\pi\alpha}\left|\left|\left|\langle g\rangle_q^{>N}\right|\right|\right|_{a,b} + \frac{\mathrm{e}^{-\pi\alpha N}}{\pi\alpha}\left|\left|\left|\{g\}_q^{>N}\right|\right|\right|_{a,b} + \frac{22}{5\beta}|\psi|_{a,b}|g|_{a,b} + \frac{42}{\beta}|\psi|_{a,b}|g|_{a,b}\\
\leq& \left[\left(\frac{1}{\pi}+\frac{\mathrm{e}^{-\pi\alpha N}}{2\pi}\right)\frac{\mathrm{e}^{-\pi\alpha N}}{\alpha} + \left(\frac{22}{5}+42\right)\frac{q^2N\epsilon}{4\beta}\right]\left|\left|\left|g-\langle g\rangle_q^{N}\right|\right|\right|_{a,b}\\
<& 12\left(\frac{q^2N\epsilon}{\beta}+\frac{\mathrm{e}^{-\pi\alpha N}}{\alpha}\right)\left|\left|\left|g-\langle g\rangle_q^{N}\right|\right|\right|_{a,b}.
\end{aligned}
$$

We have proven the first line of (3.164). To prove the second line of (3.164), it is enough to apply $|\,\langle g\rangle_q\,|_{a,b}\leq|g|_{a,b}$ and the preceding estimates to

$$
\widetilde{g} = \langle g\rangle_q + \left(\langle g\rangle_q\circ\Phi^{-1} - \langle g\rangle_q\right) + \{g\}_q^{>N}\circ\Phi^{-1} + \mathcal{R}'\circ\Phi^{-1}.
$$

□

Lemma 3.18. *Suppose $\psi,\varphi:\mathbb{T}_{a_0}\times\mathbb{D}_{b_0}\to\mathbb{C}$ are analytic and, for some $\delta_1,\delta_2>0$,*

$$
\frac{|\psi|_{a_0,b_0}}{\delta_1}+\frac{|\varphi|_{a_0,b_0}}{\delta_2}<1. \tag{3.169}
$$

Then the map $\Phi=\mathrm{Id}+\Psi$ with $\Psi(x,y)=(\psi(x,y),\varphi(x,y))$ is an analytic diffeomorphism from $\mathbb{T}_{a_0-\delta_1}\times\mathbb{D}_{b_0-\delta_2}$ onto its image and surjects onto $\mathbb{T}_{a_0-2\delta_1}\times\mathbb{D}_{b_0-2\delta_2}$. The inverse $\mathrm{Id}+\Sigma:\mathbb{T}_{a_0-2\delta_1}\times\mathbb{D}_{b_0-2\delta_2}\to\mathbb{T}_{a_0-\delta_1}\times\mathbb{D}_{b_0-\delta_2}$ of Φ satisfies $\Sigma(x,y)=(\sigma_1(x,y),\sigma_2(x,y))$ and

$$
|\sigma_1|_{a_0-2\delta_1}\leq|\psi|_{a_0-\delta_1};\quad |\sigma_2|_{a_0-2\delta_1}\leq|\varphi|_{a_0-\delta_1}
$$

Proof of Proposition 3.16

Proof. By Proposition 3.11, there exists a universal constant c such that, for sufficiently large q, we have

$$
|\widetilde{g}_q|_{1-1/q,|\omega-p/q|/2}\leq c(1-\mathrm{e}^{-2\pi M})\left|\omega-\frac{p}{q}\right|^2.
$$

Let us pick an integer $n\in(2,\nu-3)$ and define

$$
\epsilon=c\left|\omega-\frac{p}{q}\right|^2;\quad \mathfrak{N}=q^n;\quad \alpha=\frac{1}{q^2};\quad \beta=\frac{1}{6q}\left|\omega-\frac{p}{q}\right|;\quad b=\frac{1}{2}\left|\omega-\frac{p}{q}\right|.
$$

By the choice of n, there exists $q_0 \gg 1$ such that for all $q \geq q_0$ satisfying (3.154),

$$\begin{aligned}
&\frac{b}{\alpha} = \frac{q^2}{2}|\omega - p/q| \leq \frac{q^{2-\nu}}{2} < \frac{1}{5};\\
&\frac{\epsilon}{\beta} = 6cq\left|\omega - \frac{p}{q}\right| \leq 6cq^{1-\nu} < \frac{1}{5};\\
&q\mathfrak{N}b = \frac{1}{2}q^{n+1}\left|\omega - \frac{p}{q}\right| \leq \frac{1}{2}q^{n+1-\nu} \leq \frac{1}{2};\\
&\frac{q^2\mathfrak{N}\epsilon}{\beta} = 6cq^{3+n}\left|\omega - \frac{p}{q}\right| \leq 6cq^{3+n-\nu} < \frac{\mathrm{e}^{-2\pi M}}{24};\\
&\frac{1}{\alpha}\mathrm{e}^{-\pi\alpha\mathfrak{N}} = q^2\mathrm{e}^{-\pi q^{n-2}} < \frac{\mathrm{e}^{-2\pi M}}{24};\\
&\mathrm{e}^{-2\pi q^{n-1}} \leq q^{\nu_0-\nu} \quad \text{and} \quad q \geq \kappa.
\end{aligned} \tag{3.170}$$

Fix such a q and define

$$a = 1 - \frac{1}{q}; \qquad a' = 1 - \frac{2}{q}; \qquad b' = \frac{1}{3}\left|\omega - \frac{p}{q}\right|.$$

The first five lines of (3.170) ensure that for any $\tilde{a} \in (a'+\alpha, a)$ and $\tilde{b} \in (b'+\beta, b)$, Lemma 3.17 applies to any map F of the form (3.159) with analytic $g : \mathbb{T}_{\tilde{a}} \times \mathbb{D}_{\tilde{b}} \to \mathbb{C}$ such that $|g|_{\tilde{a},\tilde{b}} \leq \epsilon$. The lemma produces a coordinate transform $\mathrm{Id} + \Sigma$ such that the function $\widetilde{g}$ in the new coordinates satisfies

$$\begin{aligned}
|\widetilde{g} - \langle g\rangle_q^{\mathfrak{N}}|_{\tilde{a}-\alpha,\tilde{b}-\beta} <&\mathrm{e}^{-2\pi M}\left|\left|\left|g - \langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{\tilde{a},\tilde{b}}\\
|\widetilde{g}|_{\tilde{a}-\alpha,\tilde{b}-\beta} <&|g|_{\tilde{a},\tilde{b}} + \mathrm{e}^{-2\pi M}\left|\left|\left|g - \langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{\tilde{a},\tilde{b}}.
\end{aligned}$$

For notational convenience, we define

$$\mathcal{T}g := \widetilde{g}; \qquad \mathcal{R}g := \widetilde{g} - \langle g\rangle_q^{\mathfrak{N}}$$

so that the preceding inequalities write

$$\begin{aligned}
|\mathcal{R}g|_{\tilde{a}-\alpha,\tilde{b}-\beta} <&\mathrm{e}^{-2\pi M}\left|\left|\left|g - \langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{\tilde{a},\tilde{b}}\\
|\mathcal{T}g|_{\tilde{a}-\alpha,\tilde{b}-\beta} <&|g|_{\tilde{a},\tilde{b}} + \mathrm{e}^{-2\pi M}\left|\left|\left|g - \langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{\tilde{a},\tilde{b}}.
\end{aligned} \tag{3.171}$$

Let us also put $a_j = a - j\alpha$ and $b_j = b - j\beta$. Then $a_q = a'$ and $b_q = b'$. In order to iterate Lemma 3.17 multiple times starting from an analytic $g : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}$, we suppose

$$|g|_{a,b} \leq (1 - \mathrm{e}^{-2\pi M})\epsilon. \tag{3.172}$$

Then we show by induction on $1 \leq j \leq q$ that, after repeatedly applying Lemma 3.17,

$$\mathcal{T}^j g = \langle g\rangle_q^{\mathfrak{N}} + \sum_{k=1}^{j-1}\left\langle \mathcal{R}\mathcal{T}^{k-1}g\right\rangle_q^{\mathfrak{N}} + \mathcal{R}\mathcal{T}^{j-1}g, \tag{3.173}$$

$$|\mathcal{R}\mathcal{T}^{j-1}g|_{a_j,b_j} < \mathrm{e}^{-2\pi Mj}\left|\left|\left|g - \langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b} \quad \text{and} \quad |\mathcal{T}^j g|_{a_j,b_j} < \sum_{k=0}^{j}\mathrm{e}^{-2\pi Mk}|g|_{a,b}. \tag{3.174}$$

The case of $j=1$ follows from definitions of $\mathcal{T}g$ and $\mathcal{R}g$ as well as (3.171). Let us assume (3.173) and (3.174) hold for some $1\leq j<q$. Then, the assumption (3.172) ensures that $|\mathcal{T}^j g|_{a_j,b_j}<(1-\mathrm{e}^{-2\pi M})^{-1}|g|_{a,b}\leq\epsilon$ and hence Lemma 3.17 applies to $\mathcal{T}^j g$. Notice that $\mathcal{T}^j g-\langle\mathcal{T}^j g\rangle_q^{\mathfrak{N}}=\mathcal{R}\mathcal{T}^{j-1}g-\langle\mathcal{R}\mathcal{T}^{j-1}g\rangle_q^{\mathfrak{N}}$. Therefore, by (3.171) with g replaced by $\mathcal{T}^j g$ and the induction hypothesis,

$$\begin{aligned}
|\mathcal{R}\mathcal{T}^j g|_{a_{j+1},b_{j+1}} <&\mathrm{e}^{-2\pi M}\left|\left|\left|\mathcal{T}^j g-\langle\mathcal{T}^j g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a_j,b_j}\\
=&\mathrm{e}^{-2\pi M}\left|\left|\left|\mathcal{R}\mathcal{T}^{j-1}g-\langle\mathcal{R}\mathcal{T}^{j-1}g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a_j,b_j}\\
\leq&\mathrm{e}^{-2\pi M}|\mathcal{R}\mathcal{T}^{j-1}g|_{a_j,b_j}\\
<&\mathrm{e}^{-2\pi M(j+1)}\left|\left|\left|g-\langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}
\end{aligned}\tag{3.175}$$

Thus, we obtain the first inequality of (3.174) for the case $j+1$. To obtain the second inequality of (3.174), we apply the second inequality of (3.171) with g replaced by $\mathcal{T}^j g$ and use the induction hypothesis and (3.175) to obtain

$$\begin{aligned}
|\mathcal{T}^{j+1}g|_{a_{j+1},b_{j+1}} <&|\mathcal{T}^j g|_{a_j,b_j}+\mathrm{e}^{-2\pi M}\left|\left|\left|\mathcal{T}^j g-\langle\mathcal{T}^j g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a_j,b_j}\\
<&\sum_{k=0}^{j}\mathrm{e}^{-2\pi Mk}|g|_{a,b}+\mathrm{e}^{-2\pi M(j+1)}\left|\left|\left|g-\langle g\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}\leq\sum_{k=0}^{j+1}\mathrm{e}^{-2\pi Mk}|g|_{a,b}.
\end{aligned}$$

The induction is finished. Let us now take $g=\widetilde{g}_q$ for the analytic function $\widetilde{g}_q$ obtained from Proposition 3.11 and take $j=q$ and $g_q:=\mathcal{T}^q\widetilde{g}_q$. We readily get

$$\begin{aligned}
|\{g_q\}_q|_{a',b'} =&|\{\mathcal{R}\mathcal{T}^{q-1}\widetilde{g}_q\}_q|_{a',b'}\leq 2|\mathcal{R}\mathcal{T}^{q-1}\widetilde{g}_q|_{a',b'}<2\mathrm{e}^{-2\pi Mq}\left|\left|\left|\widetilde{g}_q-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}\\
\leq&2\mathrm{e}^{-2\pi Mq}|\widetilde{g}_q|_{a,b}\lesssim\mathrm{e}^{-2\pi Mq}\left|\omega-\frac{p}{q}\right|^2
\end{aligned}$$

and

$$|g_q|_{a',b'}<\frac{|\widetilde{g}_q|_{a,b}}{(1-\mathrm{e}^{-2\pi M})}\lesssim\left|\omega-\frac{p}{q}\right|^2.$$

We have proved (3.155) and (3.156). We now show (3.157). Since $N=\kappa q\leq\mathfrak{N}$ by the last inequality in (3.170), Equation (3.173) implies

$$\langle g_q\rangle_q^{\mathfrak{N}}-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}=\sum_{j=1}^{q}\langle\mathcal{R}\mathcal{T}^{j-1}\widetilde{g}_q\rangle_q^{\mathfrak{N}}.$$

By the first inequality in (3.174), we have $|\mathcal{R}\mathcal{T}^{j-1}\widetilde{g}_q|_{a_j,b_j}<\mathrm{e}^{-2\pi Mj}\left|\left|\left|\widetilde{g}_q-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}$. Thus, fixing $y=0$ and substituting the definition of a_j, we find

$$\begin{aligned}
\|\langle\mathcal{R}\mathcal{T}^{j-1}\widetilde{g}_q(\cdot,0)\rangle_q^{\mathfrak{N}}\|\leq&\sum_{|k|\leq\mathfrak{N},\ q|k}|\mathcal{R}\mathcal{T}^{j-1}\widetilde{g}_q|_{a_j,b_j}\mathrm{e}^{-2\pi a_j|k|}\mathrm{e}^{2\pi|k|}\\
=&\sum_{|k|\leq\mathfrak{N},\ q|k}\mathrm{e}^{-2\pi Mj}\left|\left|\left|\widetilde{g}_q-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}\mathrm{e}^{2\pi(1/q+j/q^2)|k|}\\
\leq&\Big(\frac{2\mathfrak{N}}{q}+1\Big)\mathrm{e}^{-2\pi Mj}\left|\left|\left|\widetilde{g}_q-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}\mathrm{e}^{2\pi\frac{2\mathfrak{N}}{q}}\\
\leq&(2\kappa+1)\mathrm{e}^{-2\pi Mj}\left|\left|\left|\widetilde{g}_q-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}\mathrm{e}^{4\pi\kappa}\\
\lesssim&\kappa\mathrm{e}^{4\pi\kappa}\mathrm{e}^{-2\pi Mj}\left|\left|\left|\widetilde{g}_q-\langle\widetilde{g}_q\rangle_q^{\mathfrak{N}}\right|\right|\right|_{a,b}.
\end{aligned}$$

Summing over $1 \le j \le q$ and using $M \ge 1$, we obtain

$$\|\langle g_q(\cdot,0)\rangle_q^{\mathfrak{N}} - \langle \widetilde{g}_q(\cdot,0)\rangle_q^{\mathfrak{N}}\| \lesssim \kappa \mathrm{e}^{4\pi\kappa} \left|\!\left|\!\left| \widetilde{g}_q - \langle \widetilde{g}_q\rangle_q^{\mathfrak{N}} \right|\!\right|\!\right|_{a,b}. \tag{3.176}$$

On the other hand, we may write

$$\widetilde{g}_q - \langle \widetilde{g}_q\rangle_q^{\mathfrak{N}} = \{\widetilde{g}_q\}_q + \langle \widetilde{g}_q + h\rangle_q^{>\mathfrak{N}} - \langle h\rangle^{>\mathfrak{N}},$$

where

$$h(x,y) = \frac{1}{2}\Big(\frac{p}{q} - \omega + y\Big)\Big(\frac{p}{q} - \omega - y\Big)\frac{\langle \dot{f}_1\rangle_q(x)}{1 + \langle f_1\rangle_q(x)}.$$

Then, by the first line in (3.97), we have

$$\left|\!\left|\!\left| \{\widetilde{g}_q\}_q + \langle \widetilde{g}_q + h\rangle_q^{>\mathfrak{N}} \right|\!\right|\!\right|_{a,b} \le |\widetilde{g}_q + h|_{a,b} \lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3 \lesssim q^{\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2$$

and, since the $\|\cdot\|$-norm of $\langle \dot{f}_1\rangle_q(1+\langle f_1\rangle_q)^{-1}$ is bounded, by the choice of $\mathfrak{N}$ and the last line of (3.170),

$$\left|\!\left|\!\left| \langle h\rangle^{>\mathfrak{N}} \right|\!\right|\!\right|_{a,b} \lesssim \mathrm{e}^{-2\pi(1-a)\mathfrak{N}}\left|\!\left|\!\left| \langle h\rangle^{>\mathfrak{N}} \right|\!\right|\!\right|_{1,b} \lesssim \mathrm{e}^{-2\pi q^{n-1}}\left|\omega - \frac{p}{q}\right|^2 \lesssim q^{\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2.$$

It follows that

$$\left|\!\left|\!\left| \langle \widetilde{g}_q\rangle_q^{\mathfrak{N}} - \widetilde{g}_q \right|\!\right|\!\right|_{a,b} \lesssim q^{\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2. \tag{3.177}$$

Combining with (3.176) we obtain (3.157).

It now remains to show (3.158). Let us denote by $\mathrm{Id} + \Sigma_j : \mathbb{T}_{a_j} \times \mathbb{D}_{b_j} \to \mathbb{T}_{a_{j-1}} \times \mathbb{D}_{b_{j-1}}$ the coordinate transform obtained from the j-th iteration of the lemma. Let us write $\Sigma_j(x,y) = (\sigma_{1,j}(x,y), \sigma_{2,j}(x,y))$ for some analytic $\sigma_{1,j}, \sigma_{2,j} : \mathbb{T}_{a_j} \times \mathbb{D}_{b_j} \to \mathbb{C}$. By (3.163), (3.173), (3.174) and (3.177),

$$\begin{aligned}
|\sigma_{1,j}|_{a_j,b_j} + |\sigma_{2,j}|_{a_j,b_j} &< q^2\mathfrak{N}\left|\!\left|\!\left| \mathcal{T}^{j-1}g_q - \langle \mathcal{T}^{j-1}g_q\rangle_q^{\mathfrak{N}} \right|\!\right|\!\right|_{a_{j-1},b_{j-1}} \\
&\le q^2\mathfrak{N}|\mathcal{R}\mathcal{T}^{j-2}g_q|_{a_{j-1},b_{j-1}} \\
&\le q^2\mathfrak{N}\mathrm{e}^{-2\pi M(j-1)}\left|\!\left|\!\left| g_q - \langle g_q\rangle_q^{\mathfrak{N}} \right|\!\right|\!\right|_{a,b} \\
&\le \mathrm{e}^{-2\pi M(j-1)} q^{2+n+\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2.
\end{aligned}$$

On the other hand, by composing $(\mathrm{Id} + \Sigma) := (\mathrm{Id} + \Sigma_1) \circ (\mathrm{Id} + \Sigma_2) \circ \cdots \circ (\mathrm{Id} + \Sigma_q)$, we find that

$$\Sigma = \sum_{j=1}^{q} \Sigma_j \circ (\mathrm{Id} + \Sigma_{j+1}) \circ (\mathrm{Id} + \Sigma_{j+2}) \circ \cdots \circ (\mathrm{Id} + \Sigma_q).$$

Therefore, writing $\Sigma(x,y) = (\sigma_1(x,y), \sigma_2(x,y))$, we have

$$|\sigma_1|_{a',b'} + |\sigma_2|_{a',b'} \le \sum_{j=1}^{q} |\sigma_{1,j}|_{a_j,b_j} + |\sigma_{2,j}|_{a_j,b_j} < \sum_{j=1}^{q} \mathrm{e}^{-2\pi M(j-1)} q^{2+n+\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2.$$

For concreteness, we may fix $n = 3$. Then the proof of (3.158) is finished by summing the geometric series in j. $\square$

3.7.3 Step 3: control g_q^* using exactness and the end of the proof

We are now in position to prove Theorem 3.3. So far, we have essentially obtained all the required results in the statement of Theorem 3.3 with the exception of part (3). The proof of part (3) will be the main focus of this section. The main idea for the proof of part (3) is to use the exactness property of the original standard map and the fact that the resonant coordinates do not "distort" the standard area form by much. Then, using an elementary geometric property of exact symplectic map, we can make use of the estimate of the zero-average part $g_q^\bullet$ to estimate the average part g_q^*. We remark that a similar idea was also used in [MRRTS15].

Proof of Theorem 3.3. Fix $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $\phi : \mathbb{T}_1 \to \mathbb{C}$ analytic with $\| \dddot{\phi} \| \leq 1$. Let $\kappa \geq 2$ and $\nu > \nu_0 := 64$ be given as in the statement of Theorem 3.3. Consider $q \gg 1$ such that the arithmetic condition (3.24) holds and let p be the corresponding minimizer of $|q\omega - p|$ over $p \in \mathbb{Z}$. Since the condition (3.24) is stronger than the arithmetic condition (3.94) of Proposition 3.11, taking q sufficiently large ensures that Proposition 3.11 applies and we accordingly obtain a coordinate transform Θ and a map

$$\begin{aligned}\widetilde{F}_q = \Theta^{-1} \circ F \circ \Theta : \mathbb{T}_{1-1/q} \times \mathbb{D}_{|\omega - p/q|/2} \to & \mathbb{T}_1 \times \mathbb{C} \\ (x,y) \mapsto & \Big(x + \frac{p}{q} + y,\ y + \widetilde{g}_q(x,y) \Big)\end{aligned}$$

satisfying the conclusions of Proposition 3.11.

Increasing q if necessary, we may then apply Proposition 3.16 with $M = \kappa + 1$ (and ν, κ as given in the theorem statement) to obtain a second coordinate transform $\mathrm{Id} + \Sigma$ and a map

$$\begin{aligned}F_q = (\mathrm{Id} + \Sigma)^{-1} \circ \widetilde{F}_q \circ (\mathrm{Id} + \Sigma) : \mathbb{T}_{1-2/q} \times \mathbb{D}_{|\omega - p/q|/3} \to & \mathbb{T}_1 \times \mathbb{C} \\ (x,y) \mapsto & \Big(x + \frac{p}{q} + y,\ y + g_q(x,y) \Big)\end{aligned}$$

satisfying the conclusions of Proposition 3.16. We now prove that the change of coordinates $H_q := \Theta \circ (\mathrm{Id} + \Sigma)$, the map F_q and the function g_q satisfy the conclusions of Theorem 3.3.

Let $a = 1 - 3/q$ and $b = |\omega - p/q|/5$ as in the theorem statement.

Let $h_{q,1}$ and $h_{q,2}$ be defined as in (3.25). By (3.95) and $\Sigma(x,y) = (\sigma_1(x,y), \sigma_2(x,y))$, we have

$$h_{q,1} = \sigma_1 + \theta_1 \circ (\mathrm{Id} + \Sigma), \qquad h_{q,2} = \sigma_2 + \theta_2 \circ (\mathrm{Id} + \Sigma).$$

Since $(\mathrm{Id} + \Sigma)(\mathbb{T}_a \times \mathbb{D}_b) \subset \mathbb{T}_{1-1/q} \times \mathbb{D}_{|\omega - p/q|/2}$, it follows from (3.96) and (3.158) as well as the choice $\nu \geq \nu_0 + 6$ that

$$|h_{q,1}|_{a,b} \lesssim q \left| \omega - \frac{p}{q} \right|; \qquad |h_{q,2}|_{a,b} < \frac{1}{10} \left| \omega - \frac{p}{q} \right|$$

for q sufficiently large. We have obtained (3.26).

Next, from the conclusion (3.155) of Proposition 3.16 we have immediately (3.28). Next, to obtain (3.29), we use (3.155), (3.156) and the choices $N = \kappa q$, $M = \kappa + 1$ to get

$$\begin{aligned}
|g_q - \langle g_q\rangle_q^N|_{1/q,b} \leq& |\{g_q\}_q|_{1/q,b} + |\langle g_q\rangle_q^{>N}|_{1/q,b} \\
\lesssim& |\{g_q\}_q|_{1-2/q,|\omega-p/q|/3} + \sum_{|j|>N,\ q|j} |g_q|_{1-2/q,|\omega-p/q|/3} \mathrm{e}^{-2\pi(1-2/q-1/q)|j|} \\
\lesssim& \mathrm{e}^{-2\pi Mq}\left|\omega - \frac{p}{q}\right|^2 + \sum_{|j|>N,\ q|j} \mathrm{e}^{-2\pi(1-3/q)|j|}\left|\omega - \frac{p}{q}\right|^2 \\
\lesssim& \mathrm{e}^{-2\pi Mq}\left|\omega - \frac{p}{q}\right|^2 + \mathrm{e}^{6\pi\kappa}\mathrm{e}^{-2\pi(\kappa+1)q}\left|\omega - \frac{p}{q}\right|^2 \lesssim \mathrm{e}^{6\pi\kappa}\mathrm{e}^{-2\pi(\kappa+1)q}\left|\omega - \frac{p}{q}\right|^2.
\end{aligned}$$

We have proven (3.29).

We shall postpone the proof of (3.30) to the end of this section. To prove (3.32), we denote

$$\mathcal{Q}(x,y) = \widetilde{g}_q(x,y) + \frac{1}{2}\Big(\frac{p}{q} - \omega + y\Big)\Big(\frac{p}{q} - \omega - y\Big)\frac{\langle \dot{f}_1\rangle_q(x)}{1 + \langle f_1\rangle_q(x)}.$$

Then, by the first inequality of (3.97), we have $|\mathcal{Q}|_{1-1/q,|\omega-p/q|/2} \lesssim q^{\nu_0}|\omega - p/q|^3$. It follows from the definition of the $\|\cdot\|$-norm and the choice $N = \kappa q$ that

$$\begin{aligned}
\left\|\langle \mathcal{Q}(\cdot,0)\rangle_q^N\right\| =& \sum_{\substack{q|k \\ |k|\leq N}} |\hat{\mathcal{Q}}(0)_k|\mathrm{e}^{2\pi|k|} \leq \sum_{\substack{q|k \\ |k|\leq N}} |\mathcal{Q}|_{1-1/q,|\omega-p/q|/2}\mathrm{e}^{-2\pi(1-1/q)|k|}\mathrm{e}^{2\pi|k|} \\
\leq& \Big(\frac{2N}{q} + 1\Big)|\mathcal{Q}|_{1-1/q,|\omega-p/q|/2}\mathrm{e}^{2\pi\frac{N}{q}} \\
\lesssim& \kappa\mathrm{e}^{2\pi\kappa}q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3.
\end{aligned}$$

Combining with (3.157), we obtain

$$\begin{aligned}
\left\|\left\langle g_q(\cdot,0) + \frac{1}{2}\Big(\frac{p}{q} - \omega\Big)^2\frac{\langle \dot{f}_1\rangle_q}{1 + \langle f_1\rangle_q}\right\rangle_q^N\right\| \leq& \left\|\langle g_q(\cdot,0)\rangle_q^N - \langle \widetilde{g}_q(\cdot,0)\rangle_q^N\right\| + \left\|\langle \mathcal{Q}(\cdot,0)\rangle_q^N\right\| \\
\lesssim& \kappa\mathrm{e}^{4\pi\kappa}q^{\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2 + \kappa\mathrm{e}^{2\pi\kappa}q^{\nu_0}\left|\omega - \frac{p}{q}\right|^3 \\
\lesssim& \kappa\mathrm{e}^{4\pi\kappa}q^{\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^2.
\end{aligned}$$

We have proven (3.32). To prove (3.33), we first note that, by combining the second inequality of (3.97) with the estimate $\|f_1\| < 1/18$ from (3.123), we have $|\widetilde{S}_q|_{1-1/q,|\omega-p/q|/2} \lesssim |\omega - p/q|^2$. Since $S_q = S \circ H_q = \widetilde{S}_q \circ (\mathrm{Id} + \Sigma)$, by (3.158) and Cauchy's estimate we have

$$\begin{aligned}
|S_q - \widetilde{S}_q|_{a,b} \leq& |\sigma_2|_{a,b}\left|\frac{\partial \widetilde{S}_q}{\partial y}\right|_{1-2/q,|\omega-p/q|/3} + |\sigma_1|_{a,b}\left|\frac{\partial \widetilde{S}_q}{\partial x}\right|_{1-2/q,|\omega-p/q|/3} \\
\lesssim& q|\sigma_2|_{a,b}\left|\widetilde{S}_q\right|_{1-1/q,|\omega-p/q|/2} + \left|\omega - \frac{p}{q}\right|^{-1}|\sigma_1|_{a,b}\left|\widetilde{S}_q\right|_{1-1/q,|\omega-p/q|/2} \\
\lesssim& q^{5+\nu_0-\nu}\left|\omega - \frac{p}{q}\right|^3.
\end{aligned}$$

We have assumed q is sufficiently large so that the map $\operatorname{Id}+\Sigma$ sends $\mathbb{T}_a \times \mathbb{D}_b$ into $\mathbb{T}_{1-2/q} \times \mathbb{D}_{|\omega-p/q|/3}$. Since $\nu > 5$, the conclusion (3.33) follows from the above estimate and the second line of (3.97) by a triangle inequality.

Finally, it remains to prove (3.30). Informally, the claim states that $g_q(\cdot, 0)$ is dominated by its zero average (oscillating) part $g_q^\bullet$. This is expected from the fact that the map F_q preserves an area form close to the standard form $\mathrm{d}x\mathrm{d}y$: if the preserved area form were exactly $\mathrm{d}x\mathrm{d}y$, then g_q^* would be identically zero, whereas for an area form close to $\mathrm{d}x\mathrm{d}y$, the average part g_q^* is expected to be small compared to the oscillatory part $g_q^\bullet$.

We make this argument precise as follows. Observe that, if ϕ is real-analytic, then the map F_q is real-analytic and exact symplectic with the (nonstandard) symplectic 1-form $H_q^*(y\mathrm{d}x)$, whose exterior derivative is an area form $H_q^*(\mathrm{d}x\mathrm{d}y)$ preserved by F_q. By (3.98), the area form $H_q^*(\mathrm{d}x\mathrm{d}y)$ is

$$\begin{aligned}
&(\Theta \circ (\operatorname{Id}+\Sigma))^*(\mathrm{d}x\mathrm{d}y)\\
=&(\operatorname{Id}+\Sigma)^*\Big(\Big(\frac{1}{1+\langle f_1\rangle_q} + \widetilde{\eta}\Big)\mathrm{d}x\mathrm{d}y\Big)\\
=&\Big(\frac{1}{1+\langle f_1\rangle_q \circ (\operatorname{Id}+\Sigma)} + \widetilde{\eta}\circ(\operatorname{Id}+\Sigma)\Big)\Big(1 + \frac{\partial\sigma_1}{\partial x} + \frac{\partial\sigma_2}{\partial y} + \frac{\partial\sigma_1}{\partial x}\frac{\partial\sigma_2}{\partial y} - \frac{\partial\sigma_1}{\partial y}\frac{\partial\sigma_2}{\partial x}\Big)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

where $\widetilde{\eta}$, σ_1 and σ_2 are analytic functions satisfying (3.99) and (3.158) respectively. This expression holds on the domain $\mathbb{T}_{1-2/q} \times \mathbb{D}_{|\omega-p/q|/3}$. In particular,

$$|\widetilde{\eta}\circ(\operatorname{Id}+\Sigma)|_{1-2/q,|\omega-p/q|/3} \lesssim q^{\nu_0}\left|\omega - \frac{p}{q}\right|.$$

By Cauchy's estimate and (3.158),

$$\left|\frac{\partial\sigma_1}{\partial x} + \frac{\partial\sigma_2}{\partial y} + \frac{\partial\sigma_1}{\partial x}\frac{\partial\sigma_2}{\partial y} - \frac{\partial\sigma_1}{\partial y}\frac{\partial\sigma_2}{\partial x}\right|_{1-3/q,|\omega-p/q|/4} \lesssim q^{5+\nu_0-\nu}\left|\omega - \frac{p}{q}\right|.$$

Since $\langle f_1\rangle_q = f_1^* + \langle f_1^\bullet\rangle_q$, we may write

$$\frac{1}{1+\langle f_1\rangle_q\circ(\operatorname{Id}+\Sigma)} = \frac{1}{1+f_1^*} - \frac{\langle f_1^\bullet\rangle_q\circ(\operatorname{Id}+\Sigma)}{(1+f_1^*)(1+\langle f_1\rangle_q\circ(\operatorname{Id}+\Sigma))}.$$

Since the smallest order of nonzero Fourier coefficients of $\langle f_1^\bullet\rangle_q$ is at least q, using $\|f_1\| < 1/18$ from (3.123) and the smallness of σ_1, the quantity $\langle f_1^\bullet\rangle_q \circ (\operatorname{Id}+\Sigma)(x,y)$ for $\Im x = 0$ can be bounded by $\lesssim \mathrm{e}^{-2\pi q}$. The second term on the right-hand side in the above equation can then be estimated as

$$\left|\frac{\langle f_1^\bullet\rangle_q\circ(\operatorname{Id}+\Sigma)}{(1+f_1^*)(1+\langle f_1\rangle_q\circ(\operatorname{Id}+\Sigma))}\right|_{0,|\omega-p/q|/4} \lesssim \mathrm{e}^{-2\pi q}.$$

Here and henceforth, we denote by $|\cdot|_{0,b}$ the supremum norm over the domain $\mathbb{T}\times\mathbb{D}_b$. Combining the above estimates and using $\mathrm{e}^{-2\pi q} \ll q^{-\nu}$ and $|\omega - p/q| \le q^{-\nu}$, we find that

$$H_q^*(\mathrm{d}x\mathrm{d}y) = \Big(\frac{1}{1+f_1^*} + \eta\Big)\mathrm{d}x\mathrm{d}y, \qquad \text{with} \qquad |\eta|_{0,|\omega-p/q|/4} \lesssim q^{\nu_0-\nu}. \tag{3.178}$$

By construction, the function $\eta : \mathbb{T}_{1-3/q}\times\mathbb{D}_{|\omega-p/q|/4} \to \mathbb{C}$ depends analytically on ϕ.

If ϕ is real-analytic, then the changes of coordinates are all real-analytic and therefore η is real-analytic. By the standard property of exact symplectic maps, the *signed* area of the region bounded by the curve $\mathbb{T} \times \{y\}$ and its image under F_q is zero for a fixed *real* y, i.e.,

$$\int_{\mathbb{T}} \int_{y}^{y+g_q(x,y)} \frac{1}{1+f_1^*} + \eta(x,y) \mathrm{d}y \mathrm{d}x = 0. \tag{3.179}$$

Since η is analytic on $\mathbb{T}_{1-3/q} \times \mathbb{D}_{|\omega-p/q|/4}$ and $|g_q|_{1-3/q,|\omega-p/q|/4} \ll |\omega - p/q|$ by (3.28), the left-hand side of the equation above is well-defined and analytic for y in the complex disc $\mathbb{D}_b$ with $b = |\omega - p/q|/5$.

Fix $y \in \mathbb{D}_b \cap \mathbb{R}$. Consider now the Fourier expansion of ϕ of the following form

$$\phi(x) = \sum_{k \geq 1} a_k \sin(2\pi kx) + b_k \cos(2\pi kx)$$

where $a_k, b_k \in \mathbb{C}$. If $a_k, b_k \in \mathbb{R}$ for all k, then ϕ is real-analytic and the equation (3.179) holds. Suppose $a_k = b_k = 0$ for all but finitely many k's. Then we may regard the left hand side of (3.179) as an analytic function in the finitely many variables a_k, b_k. Since the equation (3.179) holds for all real values of a_k, b_k, it must hold for all complex values of a_k, b_k by analytic continuation. By approximating a general ϕ by its Fourier truncations and using the continuous dependence of η and g_q on ϕ, we conclude that (3.179) holds for all analytic $\phi : \mathbb{T}_1 \to \mathbb{C}$ such that $\| \dddot{\phi} \| \leq 1$.

Applying the above argument for each fixed real y and using analytic continuation again, we deduce that (3.179) holds for all $y \in \mathbb{D}_b$.

We now use (3.179) to control $g_q^*(y)$. Fixing $|y| < b$ and rewriting the left hand side of (3.179), we have

$$\frac{g_q^*(y)}{1+f_1^*} + \int_{\mathbb{T}} \int_0^1 g_q(x,y) \eta(x, y + t g_q(x,y)) \mathrm{d}t \mathrm{d}x = 0$$

The second term on the left hand side can be bounded as

$$\left| \int_{\mathbb{T}} \int_0^1 g_q(x,y) \eta(x, y + t g_q(x,y)) \mathrm{d}t \mathrm{d}x \right| \leq |\eta|_{0,|\omega-p/q|/4} \sup_{x \in \mathbb{T}} |g_q(x,y)|$$

Therefore, we obtain from (3.178) that

$$|g_q^*(y)| \lesssim q^{\nu_0 - \nu} \sup_{x \in \mathbb{T}} |g_q(x,y)|.$$

Note that $g_q(x,y) = g_q^*(y) + g_q^\bullet(x,y)$. Since $\nu_0 < \nu$, for q sufficiently large the quantity $q^{\nu_0-\nu}$ is arbitrarily small. Hence we may write

$$|g_q^*(y)| \lesssim q^{\nu_0 - \nu} \sup_{x \in \mathbb{T}} |g_q^\bullet(x,y)|.$$

We have obtained (3.30) and hence completed the proof of the theorem. □

3.8 Auxiliary results

3.8.1 Functions defined on $\mathcal{D}_\lambda$

Recall that for $\lambda > 0$ we have defined the domain $\mathcal{D}_\lambda$ to be

$$\mathcal{D}_\lambda = \{(x,y) \in \mathbb{T}_1 \times \mathbb{C} \mid |\Im x| + \lambda^{-1}|y| < 1\}. \tag{3.180}$$

If u is a function defined on $\mathcal{D}_\lambda$, we denote by $|u|^{(\lambda)}$ the supremum norm of u over $\mathcal{D}_\lambda$. If u is of class $\mathcal{C}^r$ on $\mathcal{D}_\lambda$, we set

$$|u|^{(\lambda,r)} := \sup\left\{\left|\frac{\partial^{j+k}u}{\partial x^j \partial y^k}(x,y)\right| \;:\; (x,y) \in \mathcal{D}_\lambda,\ 0 \le j+k \le r\right\}.$$

Lemma 3.19 (Cauchy's estimate)**.** *Let $\lambda > 0$ and $\bar{u} : \mathcal{D}_\lambda \to \mathbb{C}$ be analytic, and put $u(x,y) = y\,\bar{u}(x,y)$. Then, for any $r \ge 1$ and any $\lambda' \in (0,\lambda)$,*

$$|u|^{(\lambda',r)} \;\le\; \frac{r+\lambda'}{1-\lambda'/\lambda}\,|\bar{u}|^{(\lambda,\,r-1)}. \tag{3.181}$$

In particular,

$$\left|\frac{\partial u}{\partial x}\right|^{(\lambda')} \le \frac{\lambda\lambda'}{\lambda-\lambda'}\,|\bar{u}|^{(\lambda)}, \qquad \left|\frac{\partial u}{\partial y}\right|^{(\lambda')} \le \frac{\lambda}{\lambda-\lambda'}\,|\bar{u}|^{(\lambda)}. \tag{3.182}$$

Proof. Throughout the proof, let us fix $(x,y) \in \mathcal{D}_{\lambda'}$.

Step 1: first derivatives. By (3.180), for fixed y the available x-strip in $\mathcal{D}_\lambda$ has half-width $1-|y|/\lambda$ and in $\mathcal{D}_{\lambda'}$ has half-width $1-|y|/\lambda'$. Cauchy's estimate in x gives

$$\left|y\,\frac{\partial\bar{u}}{\partial x}(x,y)\right| \le\; |y|\Big(\Big(1-\frac{|y|}{\lambda}\Big)-\Big(1-\frac{|y|}{\lambda'}\Big)\Big)^{-1}|\bar{u}|^{(\lambda)} \;\le \frac{\lambda\lambda'}{\lambda-\lambda'}\,|\bar{u}|^{(\lambda)}. \tag{3.183}$$

For fixed x, the available y-radii are $\lambda(1-|\Im x|)$ in $\mathcal{D}_\lambda$ and $\lambda'(1-|\Im x|)$ in $\mathcal{D}_{\lambda'}$. Cauchy's estimate in y yields

$$\left|y\,\frac{\partial\bar{u}}{\partial y}(x,y)\right| \le\; \frac{|y|}{\lambda(1-|\Im x|)-\lambda'(1-|\Im x|)}\,|\bar{u}|^{(\lambda)} = \frac{\lambda'}{\lambda-\lambda'}\,|\bar{u}|^{(\lambda)}.$$

Since $\partial_y u = \bar{u} + y\,\partial_y\bar{u}$, we obtain

$$\left|\frac{\partial u}{\partial y}\right|^{(\lambda')} \le |\bar{u}|^{(\lambda')} + \frac{\lambda'}{\lambda-\lambda'}\,|\bar{u}|^{(\lambda)} \le \frac{\lambda}{\lambda-\lambda'}\,|\bar{u}|^{(\lambda)},$$

and together with (3.183) this proves (3.182).

Step 2: higher derivatives. Let $1 \le j+k \le r$. Applying the previous estimates to $\partial_x^{j-1}\partial_y^k\bar{u}$ (if $k=0$) or to $\partial_x^j\partial_y^{k-1}\bar{u}$ (if $k \ge 1$) gives

$$\left|y\,\frac{\partial^{j+k}\bar{u}}{\partial x^j\partial y^k}(x,y)\right| \le \begin{cases} \dfrac{\lambda\lambda'}{\lambda-\lambda'}\left|\dfrac{\partial^{j-1}\bar{u}}{\partial x^{j-1}}\right|^{(\lambda)} & \text{if } k=0,\\[2ex] \dfrac{\lambda'}{\lambda-\lambda'}\left|\dfrac{\partial^{j+k-1}\bar{u}}{\partial x^j\partial y^{k-1}}\right|^{(\lambda)} & \text{if } k \ge 1. \end{cases}$$

By the Leibniz rule,

$$\left|\frac{\partial^{j+k}u}{\partial x^j\partial y^k}(x,y)\right| = \left|y\,\frac{\partial^{j+k}\bar{u}}{\partial x^j\partial y^k}(x,y)+k\,\frac{\partial^{j+k-1}\bar{u}}{\partial x^j\partial y^{k-1}}(x,y)\right| \le \max\Big\{\frac{\lambda\lambda'}{\lambda-\lambda'},\ \frac{\lambda'}{\lambda-\lambda'}+r\Big\}\,|\bar{u}|^{(\lambda,\,r-1)}.$$

Since $\frac{\lambda\lambda'}{\lambda-\lambda'} = \frac{\lambda'}{1-\lambda'/\lambda}$ and $\frac{\lambda'}{\lambda-\lambda'}+r < \frac{1}{1-\lambda'/\lambda}+r < \frac{r}{1-\lambda'/\lambda}$, we get

$$\sup_{1\le j+k\le r}\left|\frac{\partial^{j+k}u}{\partial x^j\partial y^k}\right|^{(\lambda')} < \frac{r+\lambda'}{1-\frac{\lambda'}{\lambda}}\,|\bar{u}|^{(\lambda,\,r-1)}.$$

Step 3: zeroth order. For $j = k = 0$, using $|y| < \lambda'$ on $\mathcal{D}_{\lambda'}$,

$$|u(x,y)| \leq |y|\,|\bar{u}(x,y)| \leq \lambda'\,|\bar{u}|^{(\lambda')} \leq \frac{r+\lambda'}{1-\frac{\lambda'}{\lambda}}\,|\bar{u}|^{(\lambda,\,r-1)}.$$

Taking the supremum over $(x,y) \in \mathcal{D}_{\lambda'}$ and $0 \leq j+k \leq r$ yields (3.181). $\square$

Proposition 3.20 (Composition). *Let $\lambda > \lambda' > 0$ and suppose $\bar{\sigma}_j : \mathcal{D}_{\lambda'} \to \mathbb{C}$ is analytic for $j = 1, 2$. Define $\sigma_j(x,y) := y\,\bar{\sigma}_j(x,y)$ for $j = 1, 2$, set $\Sigma(x,y) := (\sigma_1(x,y), \sigma_2(x,y))$, and consider the map $\mathrm{Id} + \Sigma : \mathcal{D}_{\lambda'} \to \mathbb{C}^2$. Then:*

1. *If*
$$|\bar{\sigma}_1|^{(\lambda')} + \frac{1+|\bar{\sigma}_2|^{(\lambda')}}{\lambda} \;\leq\; \frac{1}{\lambda'}, \tag{3.184}$$
then $\mathrm{Id}+\Sigma$ maps $\mathcal{D}_{\lambda'}$ into $\mathcal{D}_\lambda$.

2. *For all $r \geq 0$, there exists a combinatorial constant $\tilde{C}_r > 0$ depending only on r such that, whenever the composition $v \circ (\mathrm{Id}+\Sigma)$ is well-defined on $\mathcal{D}_{\lambda'}$ for an analytic $v : \mathcal{D}_\lambda \to \mathbb{C}$,*
$$|v \circ (\mathrm{Id}+\Sigma)|^{(\lambda',r)} \;\leq\; \tilde{C}_r\,|v|^{(\lambda,r)}\Big(1 + |\sigma_1|^{(\lambda',r)} + |\sigma_2|^{(\lambda',r)}\Big)^r. \tag{3.185}$$

3. *Let $v_j : \mathcal{D}_\lambda \to \mathbb{C}$ be analytic for $j = 1, 2$ such that $v_j \circ (\mathrm{Id}+\Sigma)$ is well-defined on $\mathcal{D}_{\lambda'}$, and suppose $\bar{\sigma}_j$ satisfies*
$$\bar{\sigma}_j \;=\; v_j \circ (\mathrm{Id}+\Sigma) \qquad (j = 1, 2). \tag{3.186}$$
Then, for any $r \in \mathbb{Z}_{\geq 0}$, there exist constants $A_r, B_r \in \mathbb{Z}_{\geq 0}$ and $C_r > 0$ depending only on r such that, for all $\lambda'' \in (0, \lambda')$,
$$|\bar{\sigma}_j|^{(\lambda'',r)} \;\leq\; C_r\Big(\frac{r+\lambda'}{1-\beta}\Big)^{A_r}\Big(1 + |v_1|^{(\lambda,r-1)} + |v_2|^{(\lambda,r-1)}\Big)^{B_r}|v_j|^{(\lambda,r)}, \qquad \beta := \Big(\frac{\lambda''}{\lambda'}\Big)^{1/r}, \tag{3.187}$$
with the convention that $|v_j|^{(\lambda,-1)} := 0$.

Proof. (1) Let $(x,y) \in \mathcal{D}_{\lambda'}$, so $|\Im x| < 1 - |y|/\lambda'$. Using $|\Im(x+\sigma_1)| \leq |\Im x| + |\sigma_1|$ and $|\sigma_j| \leq |y|\,|\bar{\sigma}_j|^{(\lambda')}$,

$$\begin{aligned}
|\Im(x+\sigma_1(x,y))| + \frac{|y+\sigma_2(x,y)|}{\lambda} \;&\leq\; |\Im x| + |y|\Big(|\bar{\sigma}_1|^{(\lambda')} + \frac{1+|\bar{\sigma}_2|^{(\lambda')}}{\lambda}\Big) \\
&<\; 1 - \frac{|y|}{\lambda'} + |y|\Big(|\bar{\sigma}_1|^{(\lambda')} + \frac{1+|\bar{\sigma}_2|^{(\lambda')}}{\lambda}\Big).
\end{aligned}$$

If (3.184) holds, then the term between the parentheses is $\leq \frac{1}{\lambda'}$, and the right-hand side is < 1. Hence $(x+\sigma_1(x,y), y+\sigma_2(x,y)) \in \mathcal{D}_\lambda$.

(2) Differentiate $v(x+\sigma_1(x,y),\, y+\sigma_2(x,y))$ up to total order r in (x,y). By the multivariate chain rule (Faà di Bruno), each derivative is a finite linear combination (with coefficients depending only on r) of terms

$$\frac{\partial^{\alpha+\beta} v}{\partial x^\alpha \partial y^\beta}\Big(\varsigma_1(x,y), \varsigma_2(x,y)\Big)\prod_{\ell=1}^{\alpha}\frac{\partial^{j_\ell+k_\ell}\varsigma_1}{\partial x^{j_\ell}\partial y^{k_\ell}}(x,y)\;\prod_{\ell=1}^{\beta}\frac{\partial^{j'_\ell+k'_\ell}\varsigma_2}{\partial x^{j'_\ell}\partial y^{k'_\ell}}(x,y),$$

where $\varsigma_j(x,y) = x + \sigma_j(x,y)$, and $\sum_{\ell=1}^{\alpha}(j_\ell + k_\ell) + \sum_{\ell=1}^{\beta}(j'_\ell + k'_\ell) \le r$. Since the composition is well-defined, we have $(\varsigma_1(x,y), \varsigma_2(x,y)) \in \mathcal{D}_\lambda$ and hence $\left|\partial^{\alpha+\beta} v\right|^{(\lambda)} \le |v|^{(\lambda,r)}$. On the other hand, each derivative of ς_j is bounded by $1 + |\sigma_j|^{(\lambda',r)}$. Taking suprema and absorbing the number of terms into $\tilde{C}_r$ yields (3.185).

(3) We argue by induction on $r \ge 0$.

Base case $r = 0$. From (3.186) we have $|\bar\sigma_j|^{(\lambda'')} \le |v_j|^{(\lambda)}$, so (3.187) holds with $C_0 = 1$ and $A_0 = B_0 = 0$.

Induction step. Fix $r \ge 1$ and $\lambda'' \in (0, \lambda')$, and set $\beta = (\lambda''/\lambda')^{1/r} \in (0,1)$. Applying (3.185) with $v = v_j$ and using Lemma 3.19 to estimate $\sigma_j = y\bar\sigma_j$ on $\mathcal{D}_{\lambda''}$ from $\bar\sigma_j$ on the larger domain $\mathcal{D}_{\lambda''/\beta} \subset \mathcal{D}_{\lambda'}$, we get

$$\begin{aligned} |\bar\sigma_j|^{(\lambda'',r)} &\le \tilde{C}_r\, |v_j|^{(\lambda,r)} \Big(1 + |\sigma_1|^{(\lambda'',r)} + |\sigma_2|^{(\lambda'',r)}\Big)^r \\ &\le \tilde{C}_r\, |v_j|^{(\lambda,r)} \left(\frac{r+\lambda''}{1-\beta}\right)^r \Big(1 + |\bar\sigma_1|^{(\lambda''/\beta,\, r-1)} + |\bar\sigma_2|^{(\lambda''/\beta,\, r-1)}\Big)^r. \end{aligned}$$

Note that $\lambda'' < \lambda''/\beta \le \lambda'$ and $((\lambda''/\beta)/\lambda')^{1/(r-1)} = \beta$. Thus the induction hypothesis at level $r-1$ (applied with λ'' replaced by λ''/β) yields

$$1 + |\bar\sigma_1|^{(\lambda''/\beta,\, r-1)} + |\bar\sigma_2|^{(\lambda''/\beta,\, r-1)} \le C_{r-1} \left(\frac{r+\lambda''/\beta}{1-\beta}\right)^{A_{r-1}} \Big(1 + |v_1|^{(\lambda,r-1)} + |v_2|^{(\lambda,r-1)}\Big)^{B_{r-1}+1}.$$

Combining and using $\lambda''/\beta < \lambda'$ gives

$$|\bar\sigma_j|^{(\lambda'',r)} \le \tilde{C}_r\, C_{r-1}^r \left(\frac{r+\lambda'}{1-\beta}\right)^{r + rA_{r-1}} \Big(1 + |v_1|^{(\lambda,r-1)} + |v_2|^{(\lambda,r-1)}\Big)^{r(B_{r-1}+1)} |v_j|^{(\lambda,r)}.$$

Thus (3.187) holds with the recursive choices

$$A_r := r + rA_{r-1}, \qquad B_r := r(B_{r-1}+1), \qquad C_r := \tilde{C}_r\, C_{r-1}^r,$$

and the stated base values. □

Remark (Explicit choice of $\tilde{C}_r$ in (3.185)). One may take

$$\tilde{C}_0 := 1, \qquad \tilde{C}_r := r!\, 2^r \quad (r \ge 1).$$

This follows from the multivariate Faà di Bruno expansion for $v \circ (\mathrm{Id} + \Sigma)$: an r-th (x,y)-derivative is a sum of at most $r!\,2^r$ terms, each bounded by $|v|^{(\lambda,r)}(1 + |\sigma_1|^{(\lambda',r)} + |\sigma_2|^{(\lambda',r)})^r$.

Proposition 3.21 (Inversion). *Let $\bar\xi_1, \bar\xi_2 : \mathcal{D}_\lambda \to \mathbb{C}$ be analytic and define $\xi_j(x,y) := y\,\bar\xi_j(x,y)$ and $\Xi(x,y) := (\xi_1(x,y), \xi_2(x,y))$. Suppose*

$$0 < \lambda' < \frac{\Big(1 - |\bar\xi_2|^{(\lambda)}\Big)^2}{\frac{1}{\lambda} + |\bar\xi_1|^{(\lambda)}\Big(2 - |\bar\xi_2|^{(\lambda)}\Big)}. \tag{3.188}$$

Then there exist analytic functions $\bar\sigma_1, \bar\sigma_2 : \mathcal{D}_{\lambda'} \to \mathbb{C}$ such that, with $\sigma_j(x,y) := y\,\bar\sigma_j(x,y)$ and $\Sigma(x,y) := (\sigma_1(x,y), \sigma_2(x,y))$, we have

$$(\mathrm{Id} + \Xi) \circ (\mathrm{Id} + \Sigma) = \mathrm{Id} \quad \text{on } \mathcal{D}_{\lambda'} \tag{3.189}$$

and, for any $\lambda'' \in (0, \lambda')$ such that $(\mathrm{Id} + \Xi)(\mathcal{D}_{\lambda''}) \subset \mathcal{D}_{\lambda'}$,

$$(\mathrm{Id} + \Sigma) \circ (\mathrm{Id} + \Xi) = \mathrm{Id} \quad \text{on } \mathcal{D}_{\lambda'}. \tag{3.190}$$

Moreover, for any $r \in \mathbb{Z}_{\geq 0}$ there exist constants $A_r, B_r \in \mathbb{Z}_{\geq 0}$ and $C_r > 0$ (depending only on r) such that, for any $\lambda'' \in (0, \lambda')$,

$$\begin{aligned} 1 &+ |\bar{\sigma}_1|^{(\lambda'',r)} + |\bar{\sigma}_2|^{(\lambda'',r)} \\ &\leq C_r \Big(\frac{r + \lambda'}{1 - \beta} \Big)^{A_r} \left(\frac{1 + |\bar{\xi}_1|^{(\lambda, r-1)} + |\bar{\xi}_2|^{(\lambda, r-1)}}{1 - |\bar{\xi}_2|^{(\lambda)}} \right)^{B_r} \Big(1 + |\bar{\xi}_1|^{(\lambda, r)} + |\bar{\xi}_2|^{(\lambda, r)} \Big), \end{aligned} \tag{3.191}$$

where $\beta := (\lambda''/\lambda')^{1/r}$.

Proof. Set $a := |\bar{\xi}_1|^{(\lambda)}$ and $b := |\bar{\xi}_2|^{(\lambda)}$ (so $b < 1$ is implicit in (3.188)). From (3.188) one checks there exists $\lambda_* > 0$ such that

$$\lambda' \leq \frac{1 - b}{\frac{1}{\lambda_*} + a} \qquad \text{and} \qquad \lambda_* < \frac{1 - b}{\frac{1}{\lambda} + a}. \tag{3.192}$$

(Indeed, the first inequality is equivalent to $\lambda_* \geq ((1 - b)/\lambda' - a)^{-1}$, and the second one gives an upper bound; (3.188) is precisely the condition that the interval is nonempty.)

Equip $\mathbb{C}^2$ with the anisotropic norm $\|(x, y)\|^{(\lambda_*)} := \lambda_* |x| + |y|$. For $(x, y), (x', y') \in \mathcal{D}_{\lambda_*}$, Lemma 3.19 (with $(\lambda', \lambda) \leftarrow (\lambda_*, \lambda)$) and the mean value theorem give, for $j = 1, 2$,

$$\begin{aligned} |\xi_j(x, y) - \xi_j(x', y')| \leq& |\xi_j(x, y) - \xi_j(x, y')| + |\xi_j(x, y') - \xi_j(x', y')| \\ \leq& |y - y'| \left| \frac{\partial \xi_j}{\partial y} \right|^{(\lambda_*)} + |x - x'| \left| \frac{\partial \xi_j}{\partial x} \right|^{(\lambda_*)} \\ \leq& |y - y'| \frac{\lambda}{\lambda - \lambda_*} |\bar{\xi}_j|^{(\lambda)} + |x - x'| \frac{\lambda \lambda_*}{\lambda - \lambda_*} |\bar{\xi}_j|^{(\lambda)} \\ =& \frac{\lambda}{\lambda - \lambda_*} |\bar{\xi}_j|^{(\lambda)} \|(x - x', y - y')\|^{(\lambda_*)}. \end{aligned}$$

Hence

$$\begin{aligned} \|\Xi(x, y) - \Xi(x', y')\|^{(\lambda_*)} =& \lambda_* |\xi_1(x, y) - \xi_1(x', y')| + |\xi_2(x, y) - \xi_2(x', y')| \\ \leq& \frac{\lambda}{\lambda - \lambda_*} \Big(\lambda_* a + b \Big) \|(x - x', y - y')\|^{(\lambda_*)} \end{aligned}$$

By the second inequality in (3.192), $\lambda_* a + b < 1 - \lambda_*/\lambda$, so the last line is $< \|(x - x', y - y')\|^{(\lambda_*)}$. Thus Ξ is a contraction on $\mathcal{D}_{\lambda_*}$ for the distance induced by $\| \cdot \|^{(\lambda_*)}$ and, in particular, $\mathrm{Id} + \Xi$ is injective there. Moreover, since the determinant of the Jacobian matrix of $\mathrm{Id} + \Xi$ at (x, y) is

$$\det(\mathrm{d}_{(x,y)}(\mathrm{Id} + \Xi)) = \Big(1 + \frac{\partial \xi_1}{\partial x}(x, y) \Big) \Big(1 + \frac{\partial \xi_2}{\partial y}(x, y) \Big) - \frac{\partial \xi_1}{\partial y}(x, y) \frac{\partial \xi_2}{\partial x}(x, y),$$

by (3.182) again, we have for $(x, y) \in \mathcal{D}_{\lambda_*}$ that

$$\begin{aligned} |\det(\mathrm{d}_{(x,y)}(\mathrm{Id} + \Xi))| \geq& \Big(1 - \left| \frac{\partial \xi_1}{\partial x} \right|_{\lambda_*} \Big) \Big(1 - \left| \frac{\partial \xi_2}{\partial y} \right|_{\lambda_*} \Big) - \left| \frac{\partial \xi_1}{\partial y} \frac{\partial \xi_2}{\partial x} \right|_{\lambda_*} \\ \geq& \Big(1 - \frac{\lambda \lambda_*}{\lambda - \lambda_*} |\bar{\xi}_1|_\lambda \Big) \Big(1 - \frac{\lambda}{\lambda - \lambda_*} |\bar{\xi}_2|_\lambda \Big) - \frac{\lambda}{\lambda - \lambda_*} |\bar{\xi}_1|_\lambda \frac{\lambda \lambda_*}{\lambda - \lambda_*} |\bar{\xi}_2|_\lambda \\ =& 1 - \frac{\lambda}{\lambda - \lambda_*} (\lambda_* |\bar{\xi}_1|_\lambda + |\bar{\xi}_2|_\lambda) > 0. \end{aligned}$$

Hence, $\mathrm{Id}+\Xi$ is a diffeomorphism from $\mathcal{D}_{\lambda_*}$ onto its image.

Next, we show $\mathcal{D}_{\lambda'} \subset (\mathrm{Id}+\Xi)(\mathcal{D}_{\lambda_*})$. It suffices to check that $(\mathrm{Id}+\Xi)$ maps the boundary $\partial\mathcal{D}_{\lambda_*}$ outside the interior of $\mathcal{D}_{\lambda'}$ and then appeal to degree/invariance-of-domain for the injective homotopy $\mathrm{Id}+t\Xi$, $t \in [0,1]$. Let $(x,y) \in \partial\mathcal{D}_{\lambda_*}$, so $|\Im x| + |y|/\lambda_* = 1$. Using $|\xi_1| \leq |y|\, a$ and $|\xi_2| \leq |y|\, b$, we estimate

$$\begin{aligned} |\Im(x+\xi_1)| + \frac{|y+\xi_2|}{\lambda'} &\geq |\Im x| + \frac{|y|}{\lambda'} - |y|\, a - \frac{|y|}{\lambda'}\, b \\ &\geq |\Im x| + \frac{|y|}{\lambda_*}\Big(\frac{\lambda_*}{\lambda'} - \lambda_* a - \frac{\lambda_*}{\lambda'}\, b\Big). \end{aligned}$$

Writing $s := |\Im x|$ and $t := |y|/\lambda_*$ (so $s+t=1$) gives

$$s + (1-s)\,\lambda_*\Big(\frac{1}{\lambda'} - a - \frac{b}{\lambda'}\Big) \;\geq\; \min\Big\{1,\, \lambda_*\Big(\frac{1}{\lambda'} - a - \frac{b}{\lambda'}\Big)\Big\}.$$

By the first inequality in (3.192), $\lambda_*\big(\frac{1}{\lambda'} - a - \frac{b}{\lambda'}\big) \geq 1$, hence the minimum is 1. Thus $(\mathrm{Id}+\Xi)(\partial\mathcal{D}_{\lambda_*})$ lies outside the interior of $\mathcal{D}_{\lambda'}$, and consequently $\mathcal{D}_{\lambda'} \subset (\mathrm{Id}+\Xi)(\mathcal{D}_{\lambda_*})$. Since $\mathrm{Id}+\Xi$ is injective on $\mathcal{D}_{\lambda_*}$, we can define $\mathrm{Id}+\Sigma : \mathcal{D}_{\lambda'} \to \mathcal{D}_{\lambda_*}$ satisfying (3.189). By construction, $\Sigma = (\sigma_1, \sigma_2)$ satisfies $\sigma_j(x,0) = 0$, so there are analytic $\bar\sigma_j$ with $\sigma_j(x,y) = y\,\bar\sigma_j(x,y)$.

Suppose $0 < \lambda'' < \lambda'$ such that $(\mathrm{Id}+\Xi)(\mathcal{D}_{\lambda''}) \subset \mathcal{D}_{\lambda'}$. Then restricting (3.189) to $(\mathrm{Id}+\Xi)(\mathcal{D}_{\lambda''})$ yields $(\mathrm{Id}+\Xi)\circ(\mathrm{Id}+\Sigma)\circ(\mathrm{Id}+\Xi) = (\mathrm{Id}+\Xi)$ on $\mathcal{D}_{\lambda''}$. Since $(\mathrm{Id}+\Sigma)\circ(\mathrm{Id}+\Xi)(\mathcal{D}_{\lambda''}) \subset \mathcal{D}_{\lambda_*}$ by construction of Σ and $(\mathrm{Id}+\Xi)$ injective on $\mathcal{D}_{\lambda_*}$, we obtain (3.190).

It remains to prove (3.191). From $(\mathrm{Id}+\Xi)\circ(\mathrm{Id}+\Sigma) = \mathrm{Id}$ we get $\Sigma + \Xi\circ(\mathrm{Id}+\Sigma) = 0$, which yields

$$\bar\sigma_j \;=\; -\,\frac{\bar\xi_j \circ (\mathrm{Id}+\Sigma)}{1 + \bar\xi_2 \circ (\mathrm{Id}+\Sigma)} \qquad (j = 1,2).$$

Define $v_j(x,y) := -\bar\xi_j(x,y)/(1+\bar\xi_2(x,y))$. By repeated differentiation (quotient rule) one obtains, for each $r \geq 0$,

$$|v_1|^{(\lambda,r)} + |v_2|^{(\lambda,r)} \;\leq\; D_r \left(\frac{1 + |\bar\xi_1|^{(\lambda,r-1)} + |\bar\xi_2|^{(\lambda,r-1)}}{1 - |\bar\xi_2|^{(\lambda)}}\right)^{r+1} \Big(|\bar\xi_1|^{(\lambda,r)} + |\bar\xi_2|^{(\lambda,r)}\Big)$$

for some constant $D_r > 0$ depending only on r. Since $\bar\sigma_j = v_j \circ (\mathrm{Id}+\Sigma)$, Proposition 3.20(3) gives, for $\lambda'' \in (0,\lambda')$ and $\beta = (\lambda''/\lambda')^{1/r}$,

$$1+|\bar\sigma_1|^{(\lambda'',r)}+|\bar\sigma_2|^{(\lambda'',r)} \;\leq\; C_r\Big(\frac{r+\lambda'}{1-\beta}\Big)^{A_r}\Big(1+|v_1|^{(\lambda,r-1)}+|v_2|^{(\lambda,r-1)}\Big)^{B_r}\Big(1+|v_1|^{(\lambda,r)}+|v_2|^{(\lambda,r)}\Big).$$

Substituting the bounds for v_j and enlarging constants completes the proof of (3.191). □

3.8.2 A characterization of periodic orbits.

Let $\omega \in \mathbb{R}$ and $a, b > 0$. Consider an analytic map

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \omega + y + f(x,y) \\ y + g(x,y) \end{pmatrix}, \qquad f, g : \mathbb{T}_a \times \mathbb{D}_b \to \mathbb{C}. \tag{3.193}$$

For iterates we write

$$F^j(x,y) \;=\; \begin{pmatrix} x + j\omega + jy + f_j(x,y) \\ y + g_j(x,y) \end{pmatrix}, \tag{3.194}$$

whenever the right-hand side is defined. From (3.193),

$$f_0 = g_0 = 0, \quad f_1 = f, \quad g_1 = g, \qquad \begin{cases} f_{j+1} = f_j + g_j + f \circ F^j, \\ g_{j+1} = g_j + g \circ F^j. \end{cases} \tag{3.195}$$

The next lemma gives a domain of definition and sup-norm bounds for f_j, g_j.

Lemma 3.22. *Let f, g be analytic on $\mathbb{T}_a \times \mathbb{D}_b$ and let f_j, g_j be as in* (3.194)*. Fix $a' < a$, $b' < b$ and $q \in \mathbb{Z}_{>0}$. If*

$$\begin{aligned} a' \;+\; q\,b' \;+\; q\,|f|_{a,b} \;+\; \frac{q(q-1)}{2}\,|g|_{a,b} \;&\le a, \\ b' \;+\; q\,|g|_{a,b} \;&\le b, \end{aligned} \tag{3.196}$$

then, for $0 \le j \le q$, the functions f_j, g_j are well-defined on $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$ and

$$|g_j|_{a',b'} \le j\,|g|_{a,b}, \qquad |f_j|_{a',b'} \le j\,|f|_{a,b} + \frac{j(j-1)}{2}\,|g|_{a,b}. \tag{3.197}$$

Proof. Induction on j. The case $j = 0$ is trivial since $F^0 = \mathrm{Id}$. Assume $0 \le j < q$ and $(x_j, y_j) := F^j(x,y)$ is defined for every $(x,y) \in \mathbb{T}_{a'} \times \mathbb{D}_{b'}$ with (3.197). Then

$$|y_j| \le b' + j\,|g|_{a,b}, \qquad |\Im x_j| \le a' + j\,b' + j\,|f|_{a,b} + \frac{j(j-1)}{2}\,|g|_{a,b}.$$

By (3.196), $|y_j| < b$ and $|\Im x_j| < a$, so the compositions in (3.195) are well-defined; hence f_{j+1} and g_{j+1} are well-defined there. Taking suprema in (3.195) and using $|f \circ F^j|_{a',b'} \le |f|_{a,b}$, and $|g \circ F^j|_{a',b'} \le |g|_{a,b}$, we obtain

$$\begin{gathered} |g_{j+1}|_{a',b'} \le |g_j|_{a',b'} + |g|_{a,b} \le (j+1)\,|g|_{a,b}, \\ |f_{j+1}|_{a',b'} \le |f_j|_{a',b'} + |g_j|_{a',b'} + |f|_{a,b} \le (j+1)\,|f|_{a,b} + \frac{j(j+1)}{2}\,|g|_{a,b}, \end{gathered}$$

which is exactly (3.197) with j replaced by $j+1$. This completes the induction. □

From now on, set

$$\omega = \frac{p}{q}$$

for some coprime (p,q) with $q > 0$, and assume $|f|_{a,b}$, $|g|_{a,b}$ are small enough that Lemma 3.22 applies for some $a' < a$, $b' < b$. In particular, f_q and g_q are defined on $\mathbb{T}_{a'} \times \mathbb{D}_{b'}$ via (3.195). By (3.194), a point (x,y) belongs to a (p,q)-periodic orbit of F iff

$$\begin{cases} qy + f_q(x,y) = 0, \\ g_q(x,y) = 0 \end{cases}$$

for some lift of f_q to the universal cover. To solve the first equation we look for $\gamma : \mathbb{T}_{a'} \to \mathbb{C}$ such that

$$\gamma(x) = -\frac{1}{q}\, f_q(x, \gamma(x)).$$

Then it suffices to find x_0 with
$$g_q(x_0,\gamma(x_0))=0.$$
The point $(x_0,\gamma(x_0))$ will be a (p,q)-orbit under F. Let us formalize this argument with quantitative details in the next lemma.

Lemma 3.23. *Consider a map F of the form* (3.193) *with $\omega=p/q$ and suppose, for some $a'<a$, that*
$$b+|f|_{a,b}\le\frac{a-a'}{q}\quad\textit{and}\quad|f|_{a,b}+\frac{3q-1}{2}|g|_{a,b}<b. \tag{3.198}$$
Put $b':=b-q|g|_{a,b}$ and $b''=|f|_{a,b}+\frac{q-1}{2}|g|_{a,b}$. Then all (p,q)-periodic orbits of $F|_{\mathbb{T}_{a'}\times\mathbb{D}_{b'}}$ lie in the strip $\mathbb{T}_{a'}\times\mathbb{D}_{b''}$. More precisely, there exist analytic functions $\gamma,\Gamma:\mathbb{T}_{a'}\to\mathbb{C}$ depending analytically on f and g with
$$|\gamma|_{a'}\le|f|_{a,b}+\frac{q-1}{2}|g|_{a,b},\qquad|\Gamma|_{a'}\le q\,|g|_{a,b} \tag{3.199}$$
such that the following holds. For $(x,y)\in\mathbb{T}_{a'}\times\mathbb{D}_{b'}$, the point (x,y) is a (p,q)-periodic orbit of F iff
$$y=\gamma(x)\quad\textit{and}\quad\Gamma(x)=0.$$
Moreover, with $\bar a=2q|f|_{a,b}+q(q-1)|g|_{a,b}$ and $\bar b=|f|_{a,b}+(3q-1)|g|_{a,b}/2$, we have
$$\left|\Gamma-\sum_{k=0}^{q-1}g\Big(\cdot+\frac{k}{q},0\Big)\right|_{a'}\le q\Big(\frac{\bar a}{a-a'-\bar a}+\frac{\bar b}{b-\bar b}\Big)|g|_{a,b} \tag{3.200}$$
and
$$\Big|\det\Big(\mathrm{d}_{(x,\gamma(x))}F^q-\mathrm{Id}\Big)+q\,\dot\Gamma(x)\Big|\le\frac{\bar a}{2}\frac{|\dot\Gamma(x)|}{b'-|\gamma(x)|}. \tag{3.201}$$

Proof. Define $b'=b-q|g|_{a,b}$ as in the statement. The second inequality of condition (3.198) implies $q|g|_{a,b}<b$, hence $b'>0$. Using the first inequality in (3.198) and definition of b',
$$a'+qb'+q|f|_{a,b}+\frac{q(q-1)}{2}|g|_{a,b}\le a'+qb+q|f|_{a,b}\le a,$$
so Lemma 3.22 applies, and f_q,g_q are defined on $\mathbb{T}_{a'}\times\mathbb{D}_{b'}$ with
$$|f_q|_{a',b'}\le q|f|_{a,b}+\frac{q(q-1)}{2}|g|_{a,b},\qquad|g_q|_{a',b'}\le q|g|_{a,b}. \tag{3.202}$$
By the second inequality in (3.198),
$$|f|_{a,b}+\frac{q-1}{2}|g|_{a,b}\;<\;b-q|g|_{a,b}\;=\;b',$$
so for each fixed $x\in\mathbb{T}_{a'}$, on $|z|=b'$ we have
$$\Big|\frac{1}{q}f_q(x,z)\Big|\;\le\;\frac{1}{q}|f_q|_{a',b'}\;<\;b'\;=\;|z|.$$
By Rouché's theorem, $z\mapsto z+\frac{1}{q}f_q(x,z)$ has exactly one zero in $\mathbb{D}_{b'}$, denoted $\gamma(x)$, depending analytically on x. The bound (3.199) on $|\gamma|_{a'}$ follows from $\gamma(x)=-f_q(x,\gamma(x))/q$ and (3.202). Define $\Gamma(x):=g_q(x,\gamma(x))$, so $|\Gamma|_{a'}\le|g_q|_{a',b'}\le q|g|_{a,b}$.

By construction, the lift of F and $(x,\gamma(x))$ into the universal cover satisfies

$$F^q(x,\gamma(x)) = \Big(x+p,\ \gamma(x)+\Gamma(x)\Big), \tag{3.203}$$

and uniqueness of the zero implies the first coordinate of $F^q(x,y')$ equals $x+p$ iff $y'=\gamma(x)$; thus (x,y) is a (p,q)-orbit under F iff $y=\gamma(x)$ and $\Gamma(x)=0$.

For (3.200), we have by (3.195) that

$$\Gamma(x) = g_q(x,\gamma(x)) = \sum_{j=0}^{q-1} g\circ F^j(x,\gamma(x))$$

where, by (3.194) with $\omega = p/q$,

$$g\circ F^j(x,\gamma(x)) = g(x+jp/q+j\gamma(x)+f_j(x,\gamma(x)),\gamma(x)+g_j(x,\gamma(x))).$$

Since $j\le q$, the estimates (3.199) and (3.197) imply that

$$|j\gamma(x)+f_j(x,\gamma(x))|\le \bar a; \qquad |\gamma(x)+g_j(x,\gamma(x))|\le \bar b.$$

Hence, by the mean value theorem and Cauchy's estimate,

$$|g\circ F^j(x,\gamma(x)) - g(x+jp/q,0)| \le \left|\frac{\partial g}{\partial x}\right|_{a'+\bar a,\bar b}\bar a + \left|\frac{\partial g}{\partial y}\right|_{a'+\bar a,\bar b}\bar b \le \Big(\frac{\bar a}{a-a'-\bar a}+\frac{\bar b}{b-\bar b}\Big)|g|_{a,b}.$$

Summing over j, we obtain (3.200).

For (3.201), differentiate (3.203) to get

$$\frac{d}{dt}F^q(x+t,\gamma(x+t))\Big|_{t=0} = (1,\dot\gamma(x)+\dot\Gamma(x)).$$

Also,

$$\frac{\partial F^q}{\partial y}(x,\gamma(x)) = \Big(q+\frac{\partial f_q}{\partial y}(x,\gamma(x)), 1+\frac{\partial g_q}{\partial y}(x,\gamma(x))\Big)$$

In the basis $\{(1,\dot\gamma(x)),(0,1)\}$ the Jacobian is

$$\mathrm{d}_{(x,\gamma(x))}F^q = \begin{pmatrix} 1 & q+\frac{\partial f_q}{\partial y}(x,\gamma(x)) \\ \dot\Gamma(x) & 1+\frac{\partial g_q}{\partial y}(x,\gamma(x)) - \dot\gamma(x)\Big(q+\frac{\partial f_q}{\partial y}(x,\gamma(x))\Big)\end{pmatrix}.$$

A direct computation yields

$$\det\Big(\mathrm{d}_{(x,\gamma(x))}F^q-\mathrm{Id}\Big)+q\,\dot\Gamma(x) \;=\; -\,\dot\Gamma(x)\,\partial_y f_q(x,\gamma(x)).$$

Cauchy's estimate and (3.202) give

$$|\partial_y f_q(x,\gamma(x))| \;\le\; \frac{|f_q|_{a',b'}}{b'-|\gamma(x)|} \;\le\; \frac{q|f|_{a,b}+\frac{q(q-1)}{2}|g|_{a,b}}{b'-|\gamma(x)|},$$

which implies (3.201). □

3.8.3 Properties of the Fourier-weighted norm

Recall that for $u : \mathbb{T}_a \to \mathbb{C}$ analytic, the Fourier-weighted norm $\|u\|_a$ is defined to be

$$\|u\|_a := \sum_{k\in\mathbb{Z}} |\hat{u}_k| e^{2\pi a|k|},$$

where $\hat{u}_k$ is the kth Fourier coefficient of u. In this section, we collect some elementary properties of this norm used in the thesis.

Lemma 3.24. *For fixed $a > 0$, the space of analytic functions $f : \mathbb{T}_a \to \mathbb{C}$ with $\|f\|_a < \infty$ is a Banach algebra under the norm $\|\cdot\|_a$.*

Proof. It suffices to show $\|fg\|_a \le \|f\|_a \|g\|_a$. Recall that the Fourier coefficient $\widehat{fg}_n$ of the product function fg can be given in terms of a convolution $\sum_{k\in\mathbb{Z}} \widehat{f}_{n-k}\widehat{g}_k$. To prove the lemma, the key observation is the triangle inequality $|n| \le |n-k| + |k|$ and the following computation:

$$\begin{aligned}
\|fg\|_a &= \sum_{n\in\mathbb{Z}} |\widehat{fg}_n|\, e^{2\pi a|n|} \le \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{f}_{n-k}|\, |\widehat{g}_k|\, e^{2\pi a|n|} \\
&\le \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\widehat{f}_{n-k}|\, e^{2\pi a|n-k|}\, |\widehat{g}_k|\, e^{2\pi a|k|} = \sum_{k\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} |\widehat{f}_{n-k}|\, e^{2\pi a|n-k|} \Big) |\widehat{g}_k|\, e^{2\pi a|k|} \\
&\le \sum_{k\in\mathbb{Z}} \Big(\sum_{m\in\mathbb{Z}} |\widehat{f}_m|\, e^{2\pi a|m|} \Big) |\widehat{g}_k|\, e^{2\pi a|k|} = \|f\|_a\, \|g\|_a.
\end{aligned}$$

The interchange of sums is justified since all series involve non-negative terms and converge absolutely. This completes the proof of the lemma. □

Lemma 3.25. *Let $\delta \in \mathbb{R}$. For $u : \mathbb{T}_a \to \mathbb{C}$ analytic, define $\mathcal{D}_\delta u(x) := u(x+\delta) - u(x)$. Then the function $\mathcal{D}_\delta u$ satisfies*

$$\left\|\mathcal{D}_\delta u\right\|_a \le |\delta|\, \left\|\dot{u}\right\|_a \tag{3.204}$$

$$\left\|\mathcal{D}_\delta u - \delta\dot{u}\right\|_a \le \frac{1}{2}\, |\delta|^2\, \left\|\ddot{u}\right\|_a. \tag{3.205}$$

Proof. Since $|e^{i\theta} - 1| \le |\theta|$ for all $\theta \in \mathbb{R}$ and the kth Fourier coefficient of $\dot{u}$ is $2\pi i k \hat{u}_k$,

$$\left\|\mathcal{D}_\delta u\right\|_a = \sum_{k\in\mathbb{Z}} \left| \left(e^{2\pi i k\delta} - 1\right) \hat{u}_k \right| e^{2\pi a|k|} \le \sum_{k\in\mathbb{Z}} |\delta|\, |2\pi k \hat{u}_k|\, e^{2\pi a|k|} = |\delta|\, \left\|\dot{u}\right\|_a.$$

On the other hand, since $|e^{i\theta} - 1 - i\theta| \le \theta^2/2$ for all $\theta \in \mathbb{R}$ and the kth Fourier coefficient of $\ddot{u}$ is $(2\pi i k)^2 \hat{u}_k$,

$$\begin{aligned}
\left\|\mathcal{D}_\delta u - \delta\dot{u}\right\|_a &= \sum_{k\in\mathbb{Z}} \left| \left(e^{2\pi i k\delta} - 1 - 2\pi i k\delta\right) \hat{u}_k \right| e^{2\pi a|k|} \\
&\le \sum_{k\in\mathbb{Z}} \frac{1}{2}\, |\delta|^2\, |2\pi k|^2\, |\hat{u}_k|\, e^{2\pi a|k|} = \frac{1}{2}\, |\delta|^2\, \left\|\ddot{u}\right\|_a.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.26 (Fourier separation lemma)**.** *Suppose $M > N$ and $u, f : \mathbb{T}_a \to \mathbb{C}$ are analytic functions. Then we have*

$$\left\| \left\langle f \left\langle u \right\rangle_q \right\rangle_q^N \right\|_a \le \|f\|_a \left\| \left\langle u \right\rangle_q^M + \Big(\frac{2N}{q} + 1\Big) e^{-4\pi a(M-N)} \left\langle u \right\rangle_q^{\ge M} \right\|_a. \tag{3.206}$$

Proof. Using $|k|-|n| \leq |n-k| \leq |k|+|n|$, we have

$$
\begin{aligned}
&\left\|\left\langle f\,\langle u\rangle_q^{\geq M}\right\rangle_q^N\right\|_a = \sum_{\substack{q|n\\|n|\leq N}} \sum_{\substack{q|k\\|k|\geq M}} \left|\hat{f}_{n-k}\hat{u}_k\right| \mathrm{e}^{2\pi a|n|} \leq \sum_{\substack{q|n\\|n|\leq N}} \sum_{\substack{q|k\\|k|\geq M}} |f|_a\, \mathrm{e}^{-2\pi a|n-k|}\,|\hat{u}_k|\,\mathrm{e}^{2\pi a|n|}\\
&\leq \sum_{\substack{q|n\\|n|\leq N}} \sum_{\substack{q|k\\|k|\geq M}} |f|_a\, \mathrm{e}^{-2\pi a|n-k|}\mathrm{e}^{2\pi a(|n|-|k|)}\,|\hat{u}_k|\,\mathrm{e}^{2\pi a|k|} \leq \sum_{\substack{q|n\\|n|\leq N}} \sum_{\substack{q|k\\|k|\geq M}} |f|_a\, \mathrm{e}^{-4\pi a(|k|-|n|)}\,|\hat{u}_k|\,\mathrm{e}^{2\pi a|k|}\\
&\leq\left(\frac{2N}{q}+1\right) \sum_{\substack{q|k\\|k|\geq M}} |f|_a\, \mathrm{e}^{-4\pi a(M-N)}\,|\hat{u}_k|\,\mathrm{e}^{2\pi a|k|} \leq \left(\frac{2N}{q}+1\right)\mathrm{e}^{-4\pi a(M-N)}\,|f|_a \left\|\langle u\rangle_q^{\geq M}\right\|_a .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left\|\left\langle f\,\langle u\rangle_q\right\rangle_q^N\right\|_a &\leq \left\|\left\langle f\,\langle u\rangle_q^{M}\right\rangle_q^N\right\|_a + \left\|\left\langle f\,\langle u\rangle_q^{\geq M}\right\rangle_q^N\right\|_a\\
&\leq \|f\|_a\left(\left\|\langle u\rangle_q^{M}\right\|_a + \left(\frac{2N}{q}+1\right)\mathrm{e}^{-4\pi a(M-N)}\left\|\langle u\rangle_q^{\geq M}\right\|_a\right)\\
&\leq \|f\|_a\left\|\langle u\rangle_q^{M} + \left(\frac{2N}{q}+1\right)\mathrm{e}^{-4\pi a(M-N)}\,\langle u\rangle_q^{\geq M}\right\|_a
\end{aligned}
$$

which proves (3.206). This completes the proof of the lemma. □

Declaration of the Use of Generative AI

Generative AI tools, specifically ChatGPT, were used in the preparation of this thesis to support the literature review process and to refine academic writing. In particular, these tools assisted in paraphrasing complex content to improve clarity while preserving the integrity of the scientific content.

The author acknowledges the limitations of such tools, including the potential for misinterpretation or misrepresentation of scientific information. To mitigate these risks, all AI-generated content was carefully reviewed, validated, and revised by the author to the best of their ability.

The author takes full responsibility and accountability for all content submitted in this thesis, regardless of the involvement of AI tools.

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