Order Statistics in Population Protocols via Simple Dynamics

Niccolò D'Archivio
COATI, Inria d'Université Côte d'Azur & I3S
Sophia Antipolis, France
niccolo.darchivio@inria.fr

Hind AlMahmoud
King's College London
London, UK
hind.almahmoud@kcl.ac.uk

Emanuele Natale
CNRS, COATI, I3S, Université Côte d'Azur
Sophia Antipolis, France
emanuele.natale@inria.fr

Frederik Mallmann-Trenn
King's College London
London, UK
Institute of Science & Technology Austria
Klosterneuburg, Austria
frederik.mallmann-trenn@kcl.ac.uk

Abstract

We study simple dynamics in the population protocol model, in which n agents start with totally ordered initial opinions $x_1, x_2, \dots, x_n$ and, in each round, a randomly chosen agent changes its opinion as a function of the opinion of other randomly chosen agents. Such dynamics often converge to consensus on a single *fixation* value $\hat{X}$. This paper asks how to *control* the distribution of $\hat{X}$ as a randomised choice among the initial opinions by designing suitable simple dynamics. Writing the sorted initial values as $x_{(1)} \leq \cdots \leq x_{(n)}$, we design two protocols that realise natural target laws over order statistics.

First, for a parameter $p \in (0, 1)$, our *geometric protocol* biases toward larger opinions and satisfies $\mathbb{P}\left[\hat{X} = x_{(k)}\right] \propto p^{n-k}$, for $k = 1, \dots, n$. Equivalently, $\mathbb{P}\left[\hat{X} = x_{(k)}\right] = (1-p)p^{n-k}/(1-p^n)$. Second, our *binomial protocol* assigns a shifted binomial law to the ranks in ascending order: if $K - 1 \sim \mathrm{Bin}(n-1, 1-p)$, then $\hat{X} = x_{(K)}$, i.e., $\mathbb{P}\left[\hat{X} = x_{(k)}\right] = \binom{n-1}{k-1} p^{n-k}(1-p)^{k-1}$, for $k = 1, \dots, n$.

Applications of this include computing the Top-k values for small k on general interaction graphs. A central contribution of this work is that, in contrast to most population protocols, we can characterise the fixation distribution in closed form. This is enabled by a novel analysis technique, which also yields applications: we derive new results for the Median protocol that extend the state of the art.

CCS Concepts

• **Computing methodologies → Distributed algorithms**; *Multi-agent systems*; • **Mathematics of computing → Probabilistic algorithms**; **Markov processes**; • **Theory of computation → Distributed algorithms**.

Keywords

multi-agent system, population protocol, order statistics, ranking, discrete Markov chains, voter model, Top-k

ACM ISBN 979-8-4007-2512-8/26/07
https://doi.org/10.1145/3796701.3815922

ACM Reference Format:
Niccolò D'Archivio, Hind AlMahmoud, Emanuele Natale, and Frederik Mallmann-Trenn. 2026. Order Statistics in Population Protocols via Simple Dynamics . In *ACM Symposium on Principles of Distributed Computing (PODC '26), July 6–10, 2026, Egham, United Kingdom.* ACM, New York, NY, USA, 12 pages. https://doi.org/10.1145/3796701.3815922

1 Introduction

The study of extremely simple randomised distributed algorithms in which n agents repeatedly interact at random and update their opinions using a fixed, memoryless rule based on sampled neighbours is a rich area of research, motivated by the study of real systems such as sensor networks, peer-to-peer networks, mobile networks of vehicles and social networks, among others. Classical examples include the Voter model, where an activated agent adopts a sampled neighbour's opinion [31], and majority-type rules such as 3-Majority and 2-Choices [8]. In the distributed computing community, this theoretical area is mainly known as population protocols [2]; other names for it include computational dynamics [6], opinion exchange dynamics [36] and gossip algorithms [38].

In this work, we ask whether simple dynamics can be designed so that the final consensus value (i.e., the absorbing state of the associated Markov chain) follows a predetermined distribution over the order statistics of the agent opinions in the initial configuration.

Formally, writing the initial values in sorted order $x_{(1)} \leq \cdots \leq x_{(n)}$, can one design a memoryless protocol whose eventual consensus value $\hat{X}$ satisfies $\mathbb{P}\left[\hat{X} = x_{(i)}\right] = f(i)$ for a prescribed probability density function f over the opinion ranking (e.g., geometric or binomial)? We answer this in the affirmative by introducing a new analysis technique which allows us to design dynamics whose fixation distributions admit exact closed forms.

Let $X^t_{(i)}$ denote the i-th order statistic at time t, i.e., the ith smallest value. We show that the vector of expectations $\mathbb{E}[(X^t_{(1)}, \dots, X^t_{(n)})]$ evolves linearly under a fixed tri-diagonal operator P on the rank set $\{1, \dots, n\}$: For $2 \leq i \leq n-1$,

$$\mathbb{E}[X^{t+1}_{(i)} \mid \mathcal{F}_t] = f_i X^t_{(i-1)} + g_i X^t_{(i)} + h_i X^t_{(i+1)}, \quad g_i = 1 - f_i - h_i,$$

with boundary adjustments at $i = 1, n$, where f_i and h_i are explicit functions of i and n. Beyond expectations, we identify a rank-ancestry process on $\{1, \dots, n\}$ whose t-step kernel is P^t: the entry $(P^t)_{j,i}$ equals the probability that the value at rank i at time t

descends from the initial rank j. Despite their non-linear update functions, this makes it possible to analyse our dynamics through a linear process, in a way reminiscent of max-plus algebra [40]. In the following, we present the results that our technique has enabled.

1.1 First Contribution: Copy-or-Max Dynamics and Geometric Distribution

Our first contribution is the introduction of a simple dynamic that enables sampling from a geometric distribution over the order statistics, which we name Copy-or-Max dynamics.

We first study it in the standard population protocol model, where at each round a node becomes active and samples a neighbour: with probability p it copies the neighbour's value (this update coincides with the Voter model in the population protocol literature), while with probability $1-p$ it adopts the maximum between its own value and that of the neighbour. We show that the resulting dynamics induces a *geometric* distribution over the order statistics of the initial configuration, converging to a consensus on opinion $\hat{X}$ such that $\mathbb{P}\left[\hat{X}=x_{(k)}\right] \propto p^{n-k}$, for $k=1,\ldots,n$.

We then show that the same phenomenon persists in a significantly more general framework, which we introduce under the name *weighted population protocols*. In this setting, at each round a directed edge (i,j) is activated with probability proportional to a weight $w_{i,j}$ (i.e., with probability $w_{i,j}/W$, where $W=\sum_{i,j} w_{i,j}$); agent i then copies the value of agent j with probability p, and otherwise adopts the maximum between its own value and that of j. In this setting, the fixation probabilities of the process are governed by a simple balance condition on the weights: whether they form a *circulation* [34], that is, whether every agent has equal total incoming and outgoing weight, i.e., $\sum_j w_{i,j} = \sum_j w_{j,i}$ for $i=1,\ldots,n$.

Theorem 1. *The Copy-or-Max dynamics with parameter $p \in (0,1)$ satisfies the following.*

1. ***Population protocol.*** *The dynamics converges to the k-th smallest initial value, $x_{(k)}$, with geometric probability*
$$\frac{(1-p)p^{n-k}}{1-p^n}.$$
Moreover, convergence occurs in $O(\log n)$ parallel rounds with high probability.
2. ***Weighted population protocol.*** *The dynamics converges to the k-th smallest initial value, $x_{(k)}$, with geometric probability $\frac{(1-p)p^{n-k}}{1-p^n}$ if and only if the weights form a circulation and the underlying graph is connected.*

A formal definition of the edge-activation model, together with a precise statement of the theorem, is given in Section 4.2. This framework strictly generalises well-studied models and reveals a surprising robustness of geometric fixation probabilities. In particular, for any undirected graph, sampling edges uniformly at random induces a circulation, and hence the fixation probabilities follow a geometric distribution over the order statistics, independently of the graph structure. Moreover, when restricting to the classical node-activation model, where we first pick an agent i who then picks a neighbour j u.a.r., the circulation condition reduces to regularity of the graph, implying that on all regular graphs the Copy-or-Max dynamics exhibits geometric fixation probabilities [1].

1.2 Second Contribution: Biased-Median Dynamics and Binomial Distribution

Our second contribution is the design of a simple protocol that samples from a *binomial* distribution, obtained by revisiting the classical *Median* dynamics [23]. We introduce the *Biased-Median* dynamics, in which an activated agent samples two neighbours [2] and computes the median of their values together with its own; if the median is larger than its current value, the agent adopts it; if the median is smaller, the agent adopts it with probability $p/(1-p)$, where $p \in (0,1/2]$.

This combination of the median rule with a bias towards larger opinions achieves the binomial law, and provides a complete characterisation of the fixation probabilities for the *Median* dynamics (setting $p=1/2$). In contrast, prior state-of-the-art results only show that the dynamics converges with high probability to an approximation of the median with few standard deviations (i.e., of order $\sqrt{n\log n}$) [23].

Theorem 2. *The Biased-Median dynamics of parameter p on the clique converges to the k-th smallest initial value, $x_{(k)}$, with binomial probability $\binom{n-1}{k-1}p^{n-k}(1-p)^{k-1}$. In particular, the Median dynamics ($p=1/2$) converges to the k-th smallest initial value with binomial probability $\binom{n-1}{k-1}2^{-(n-1)}$.*

Since the Median dynamics is equivalent to the well-known 2-Choices protocol when the population holds binary opinions, our result on the Median dynamics implies the following Corollary, which again strengthens previous results [16].

Corollary 3. *Consider a binary starting configuration (i.e., $x_i \in \{0,1\}$ for all $i \in [n]$), with n_1 agents holding value 1. The 2-Choices dynamics on the clique converges to value 1 with probability equal to $\mathbb{P}[X<n_1]$, with $X \sim$ Binomial$(n-1,1/2)$.*

1.3 Other Contributions: Beta-Binomial Distribution, Limitations and Applications

By applying our technique to the variant of the Median dynamics in which each agent samples three opinions uniformly at random from the system and adopts their median (without necessarily including its own current opinion among the samples), we obtain a different behaviour: the fixation probability is no longer governed by a binomial distribution, but by a Beta–Binomial law. Considering binary configurations, we obtain the following Corollary.

Corollary 4. *Let the starting configuration be binary, with n_1 agents holding value 1. The 3-Majority dynamics on the clique converges to value 1 with probability equal to $\mathbb{P}[X<n_1]$, with $X \sim$ Beta-Binomial$(n-1,n/2+1,n/2+1)$.*

Interestingly, Corollaries 3 and 4 shed light for the first time on the difference in the exact fixation probabilities of 3-Majority and 2-Choices. See Figures 1 and 2 for an illustration.

[1] If lower values are of interest, the Copy-or-Min protocol can be easily derived as an extension of the Copy-or-Max protocol.

[2] Some readers may note that the *Median* dynamics involves three-way interactions rather than the classical pairwise model of population protocol; any such interaction can be simulated by pairwise interactions at the cost of additional states (see [14]).

We remark that Theorem 2, Corollaries 3 and 4 do not extend to arbitrary graph structures. In fact, for dynamics that sample more than one agent (such as the median protocol) on general graphs, fixation probabilities are sensitive to the geometry in the initial configuration, and cannot be characterised solely in terms of order statistics. This limitation is discussed in section 3.8 of the full version of the paper [3].

Having established that it is possible to sample from a geometric distribution over the order statistics of X, we exploit this property to design an algorithm for computing the Top-k values in the system. By running multiple independent instances of Copy-or-Max, each agent performs $O(\log(1/\delta))$ samples drawn from a geometric distribution over the order statistics. This allows the agents to identify, with probability at least $1-\delta$, the k largest values in the system, for constant k. Although the time complexity of our algorithm is suboptimal compared to the state-of-the-art [30], it is worth emphasising that our approach solves this problem on more general graph topologies.

	Uniform	Geometric	Binomial
Name	Voter	Copy-or-Max	Biased-Median
Update rule	Sample u, v u.a.r. $X_u \leftarrow X_v$	Sample u, v u.a.r. w.p. p: $X_u \leftarrow X_v$; otherwise: $X_u \leftarrow \max\{X_u, X_v\}$	Sample u, v, s u.a.r. $m = \mathrm{median}(X_u, X_v, X_s)$ If $m > X_u$: $X_u \leftarrow m$; otherwise ($m \leq X_u$): $X_u \leftarrow m$ w.p. $\frac{p}{1-p}$
$\mathbb{P}(\hat{X} = x_{(k)})$	$\frac{1}{n}$	$\frac{(1-p)p^{n-k}}{1-p^n}$	$\binom{n-1}{k-1}p^{n-k}(1-p)^{k-1}$
Reference	Hassin and Peleg [31]	Theorem 1	Theorem 2

Table 1: Population protocols inducing uniform, geometric, and binomial distributions over order statistics in complete graphs. See also Figures 1 and 2 for a depiction.

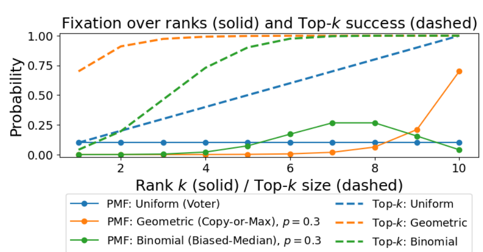

Figure 1: $n = 10$. **Solid curves show the fixation distribution** $\mathbb{P}\left[\hat{X} = x_{(k)}\right]$ **over order-statistic ranks** k. **Dashed curves (same colour) show the cumulative probability of selecting one of the** k **largest initial values,** $\mathbb{P}(\hat{X} \in \{x_{(n-k+1)}, \ldots, x_{(n)}\})$.

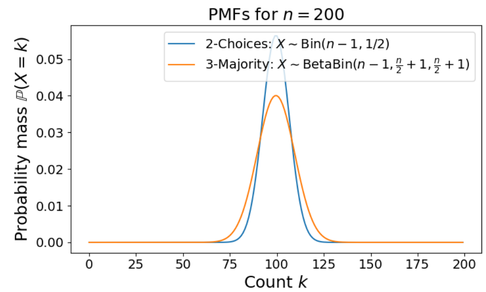

Figure 2: $n = 200$. **2-Choices vs 3-Majority PMFs. The former is more strongly concentrated around** $n/2$.

1.4 Related Work

In the context of population protocols and computational dynamics, different forms of consensus have been studied, including average consensus [5, 12], median consensus [23], plurality consensus [7, 24] and min–max consensus [26]. These have been studied in many different models, ranging from random networks [1] to systems in which agents can join or leave during the process [10], from models with noisy communication channels [22] to models in which agents' value is influenced by external signals [33]. The literature on population protocols and computational dynamics is vast across different communities; here, we focus on works most pertinent to the theory of distributed computing [6].

The simplest stochastic consensus protocol is the Voter model, where a random agent samples a neighbour uniformly at random and adopts its value [31]. [31] famously proved that the probability that Voter dynamics reaches consensus on a given opinion is proportional to the volume of the nodes initially holding that opinion. The Voter dynamics has been extensively studied, leading to elegant arguments on its convergence time [15] and the introduction of many variants [4, 18, 20]. In relation to our Copy-or-Max dynamics, we mention here [27], which analyses a Voter-like population protocol where not all agents initially hold a value. This process combines features of the Voter and the Max dynamics, but in a different way from our Copy-or-Max dynamics. In particular, valueless agents act like low-valued agents in the Max protocol, gradually losing influence, while the valued agents follow Voter dynamics, which decides the final outcome. As for the Max dynamics, several previous works leverage the Max-Plus Algebra to transform non-linear update rules into linear ones, enabling the study of more complex models in a linear way similar to our proposed technique in this paper. An example of Max-Plus algebra can be seen in [40], which investigates max-consensus in discrete-time multi-agent systems over directed and time-varying random networks.

The Median dynamics has been famously investigated in [23], where tight convergence bounds are established. In [13] they consider the settings where each agent has a private and public opinion and use a median based method to update agents' opinions. Several generalisations of the Median dynamics have been investigated in other communities, such as weighted generalisations (see [35] and the references therein), continuous-time models where agents may join or leave the network over time [25]. When restricted to two possible opinions, the MEDIAN dynamics coincides with the 2-CHOICES dynamics, famously studied in [16] and later on expander graphs [19]. Another closely related dynamics is the 3-MAJORITY [8, 9] dynamics. A general analysis of the convergence time of 2-CHOICES and 3-MAJORITY was first provided in [28] and refined in [24, 39]. Substantial work has been devoted to analysing the convergence time of these dynamics when the number of initial opinions is very large, see e.g. [11] and [21]. We emphasise that for more than two opinions, the equivalence that we leverage in this paper with the median rule does not hold. Many of the aforementioned works investigate these processes in synchronous time models in which nodes update their opinion at the same time; recent work, we consider instead the classical Population Protocols setting in which only one node updates at each step, which simplifies the transitions of the associated Markov chain but makes it more challenging to obtain concentration of probability [17].

In this work we leverage the COPY-OR-MAX dynamics to compute the top-k values in the system. This problem is closely related to the quantile problem. Several papers studied it such as [30, 32]. In [30], Haeupler et al. present an optimal algorithm for solving both the approximate and exact quantile problems in the GOSSIP model in complete graph. Their exact algorithm completes in $\log n$ parallel rounds. Since the quantile problem is closely related to computing the k-th ranked value, their approach can be directly applied to compute the top-k values in $k \log n$ parallel rounds. The algorithm relies on the ability of agents to push information: for example, one of its subroutines computes the sum of all agents' values, a task that becomes significantly more challenging if agents can only pull information. Furthermore, the algorithm can be adapted to be adversarially robust [29]. In [41], the authors study the quantile problem under noisy communication channels over general graphs. They reduce the task to a convex optimisation problem and prove convergence of their algorithm.

2 Organisation

The remainder of the paper is organised as follows. In Section 4.1, we analyse the fixation probabilities of the Copy-or-Max dynamics on the complete graph within the standard population protocol model. In Section 4.2, we extend this analysis to arbitrary graphs in the weighted population protocol setting. In Section 4.4, we introduce Order-Based Dynamics, a broad class of protocols for which our rank-ancestry process yields exact fixation probabilities. Next, Sections 4.5 and 4.7 establish the exact fixation probabilities for the Biased Median Dynamics and its variant. Finally, Section 5 concludes the paper with directions for future work. Due to space constraints, several proofs from Section 4.4 onwards have been omitted; these proofs, alongside convergence time proofs and an application to compute Top-k rankings, are deferred to the full version of this paper [3].

3 Preliminaries

We denote the value of agent i at round t as X_i^t. Let $X_{(i)}^t$ be the value of the rank i element (ith smallest) at time t, and $X_{\text{ord}}^t = \left(X_{(i)}^t\right)_{i\in[n]}$.

4 Analysis: Fixation Probabilities

4.1 Geometric Distribution in Population Protocol

We analyse a starting configuration $\left\{X_i^0\right\}_{i\in[n]}$ s.t. $X_i^0 \neq X_j^0$ for all $i \neq j \in [n]$. The generalisation to any starting configuration will trivially follow by summing up the fixation probabilities of two identical values.

We first compute the probability that an agent at rank i adopts the value at rank j.

LEMMA 4.1. *Let $i, j \in [n]$ and denote by $S_{i,j}^t$ the event "At round t, the agent with i-th rank value activates and adopts $X_{(j)}^t$". Consider a configuration X_{ord}^t s.t. $X_{(j)}^t \neq X_{(\ell)}^t$ for all $\ell \neq j$. We have that*

$$q_j(i) := \mathbb{P}\left[S_{i,j}^t \mid X_{ord}^t\right] = \begin{cases} \frac{1}{n^2} & \text{if } j > i \\ \frac{p}{n^2} & \text{if } j < i \end{cases}$$

PROOF. The agent at rank i is activated with probability $\frac{1}{n}$. For all $t \in \mathbb{N}$, let $C_t \sim \text{Bernoulli}(1-p)$ be the random variable that is 1 if the activated agent does a Max update at round t, and 0 otherwise. If $C_t = 0$, the activated agent samples uniformly over all values. Hence,

$$\mathbb{P}\left[S_{i,j}^t \mid C_t = 0, X_{\text{ord}}^t\right] = \frac{1}{n^2}.$$

If $C_t = 1$, the activated agent adopts the maximum between its own opinion and the sampled one. Then

- If $i > j$, we have that $\mathbb{P}\left[S_{i,j}^t \mid C_t = 1, X_{\text{ord}}^t\right] = 0$, as the agent will never adopt a smaller rank opinion.
- If $j > i$, $\mathbb{P}\left[S_{i,j}^t \mid C_t = 1, X_{\text{ord}}^t\right] = \frac{1}{n^2}$, as the agent has to sample opinion $X_{(j)}^t$ to adopt it, and, by hypothesis, it has only one occurrence in the system.

We can therefore conclude by the law of total probability. □

In the next lemma, we write the expectation of the k-th value at the next round as a linear combination of the $(k-1)$-th, (k)-th, and $(k+1)$-th value at the current round. In the proof, we obtain more as we compute the density distribution of $X_{(k)}^{t+1}$ conditional on X_{ord}^t, but only under the assumption that $X_{(k+1)}^t > X_{(k)}^t > X_{(k-1)}^t$. To maintain generality and avoid treating ties separately, it is more convenient to work with expectations, which allow us to handle ties in a unified and elegant manner.

LEMMA 4.2. *Let*

$$f_k = p\frac{(n-k+1)(k-1)}{n^2},$$

$$h_k = \frac{k(n-k)}{n^2},$$

and

$$g_k = 1 - f_k - h_k.$$

For all $k \in [n]$[3]*, we have*

$$\mathbb{E}\left[X_{(k)}^{t+1} \mid X_{ord}^t\right] = f_k X_{(k-1)}^t + g_k X_{(k)}^t + h_k X_{(k+1)}^t$$

[3] Assuming the boundary convention $X_{(k)}^t = 0$ for $k = 0$ or $k = n$.

Proof. Let's first assume that $X^t_{(k+1)} > X^t_{(k)} > X^t_{(k-1)}$. The event $\left\{X^{t+1}_{(k)} = X^t_{(k+1)}\right\}$ happens if and only if an agent with rank smaller or equal to k adopts the value of rank strictly larger than k (see Figure 3 for a clear visual explanation). Therefore, by Lemma 4.1, we have

$$\mathbb{P}\left[X^{t+1}_{(k)} = X^t_{(k+1)} \mid X^t\right] = \sum_{i=1}^{k}\sum_{j=k+1}^{n} q_j(i) = \frac{k(n-k)}{n^2} = h_k. \quad (1)$$

On the other hand, the event $\left\{X^{t+1}_{(k)} = X^t_{(k-1)}\right\}$ happens if and only if an agent with rank larger or equal to k adopts the value of rank strictly smaller than k. We obtain

$$\begin{aligned}\mathbb{P}\left[X^{t+1}_{(k)} = X^t_{(k-1)} \mid X^t\right] &= \sum_{i=k}^{n}\sum_{j=1}^{k-1} q_j(i) = p\frac{(n-k+1)(k-1)}{n^2} \\ &= f_k. \end{aligned} \quad (2)$$

Since only one agent updates per round, we have that

$$\mathbb{P}\left[X^{t+1}_{(k)} = X^t_{(k)}\right] = 1 - f_k - h_k = g_k.$$

This concludes the proof of Lemma 4.2 if $X^t_{(k-1)} < X^t_{(k)} < X^t_{(k+1)}$. If instead we have $X^t_{(k-1)} = X^t_{(k)} < X^t_{(k+1)}$ the event

$$\left\{X^{t+1}_{(k)} = X^t_{(k+1)}\right\}$$

happens if and only if an agent with rank smaller or equal to k adopts the value of rank strictly larger than k, as before. Otherwise we will always have $X^{t+1}_{(k)} = X^t_{(k-1)}$. Therefore, we obtain

$$\begin{aligned}\mathbb{E}\left[X^{t+1}_{(k)} \mid \mathcal{F}_t\right] &= (1-h_k)X^t_{(k)} + h_k X^t_{(k+1)} \\ &= f_k X^t_{(k-1)} + g_k X^t_{(k)} + h_k X^t_{(k+1)}.\end{aligned}$$

Other cases can be computed similarly. This concludes the proof of Lemma 4.2 in the general case. □

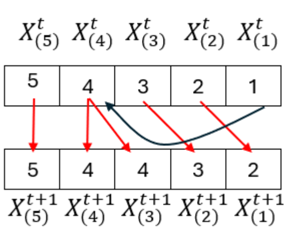

Figure 3: The agent at rank 1 is activated and adopts the value $X^t_{(4)}$. We observe the event $\left\{X^{t+1}_{(k)} = X^t_{(k+1)}\right\}$ for $k = 1, 2, 3$.

In the next lemma, we use the tower law for conditional expectation in order to compute the expected value of the order configuration after t rounds.

Lemma 4.3. *Let P be the following tridiagonal stochastic matrix*

$$P = \begin{bmatrix} g_1 & f_2 & 0 & \cdots & 0 \\ h_1 & g_2 & f_3 & \ddots & \vdots \\ 0 & h_2 & g_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & f_n \\ 0 & \cdots & 0 & h_{n-1} & g_n \end{bmatrix}$$

We have

$$\mathbb{E}\left[X^t_{ord}\right] = X^0_{ord}P^t.$$

Proof. We prove the claim by induction on $t \geq 0$. For $t = 0$, the statement is clearly true. Assume $\mathbb{E}\left[X^t_{ord}\right] = X^0_{ord}P^t$. We obtain

$$\begin{aligned}\mathbb{E}\left[X^{t+1}_{ord}\right] &= \mathbb{E}\left[\mathbb{E}\left[X^{t+1}_{ord} \mid X^t_{ord}\right]\right] && \text{(by law of total expectation)} \\ &= \mathbb{E}\left[X^t_{ord} \cdot P\right] && \text{(by Lemma 4.2)} \\ &= \mathbb{E}\left[X^t_{ord}\right] \cdot P && \text{(by linearity expected value)} \\ &= X^0_{ord} \cdot P^{t+1}. && \text{(by inductive hypothesis)}\end{aligned}$$

This concludes the proof of Lemma 4.3. □

In the following lemma, we fully characterise the density distribution of the ordered distribution after t rounds as function of the matrix P. Recall that we assume that there are no ties in the initial configuration.

Lemma 4.4. *For all i, j, t, $\mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right] = P^t_{j,i}$.*

Proof. Fix i and t. We prove that, $\mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right] = P^t_{j,i}$. for all $j \in [n]$. Define $e_j := \mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right] - P^t_{j,i}$. Since the dynamics never creates new values, $X^t_{(i)} \in \{X^0_{(1)}, \ldots, X^0_{(n)}\}$. Therefore,

$$\mathbb{E}[X^t_{(i)}] = \sum_{j=1}^{n} X^0_{(j)}\mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right].$$

On the other hand, by Lemma 4.3,

$$\mathbb{E}[X^t_{(i)}] = \sum_{j=1}^{n} X^0_{(j)}(P^t)_{j,i}.$$

Subtracting the two identities gives

$$\sum_{j=1}^{n} X^0_{(j)} e_j = 0. \quad (1)$$

Now let Y^0 be another initial vector, and let Y^t be the corresponding evolution. Coupling the two processes with the same random choices, the rank ancestry of the process is the same for X^0 and Y^0, since the update rule depends only on comparisons and copying. Hence

$$\mathbb{P}\left[Y^t_{(i)} = Y^0_{(j)}\right] = \mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right]$$

for all $j \in [n]$. Applying the same expectation identity to Y^0 gives

$$\sum_{j=1}^{n} Y^0_{(j)} e_j = 0. \quad (2)$$

Subtracting (2) from (1), we obtain

$$\sum_{j=1}^{n}\big(X^0_{(j)} - Y^0_{(j)}\big)e_j = 0. \tag{3}$$

Fix $j \in [n]$. Choose Y^0 such that

$$Y^0_{(\ell)} = X^0_{(\ell)} \qquad \text{for all } \ell \neq j,$$

and choose $Y^0_{(j)} \neq X^0_{(j)}$ while preserving the strict order. Namely, if $1 < j < n$, choose $X^0_{(j-1)} < Y^0_{(j)} < X^0_{(j+1)}$; if $j = 1$, choose $Y^0_{(1)} < X^0_{(2)}$; and if $j = n$, choose $Y^0_{(n)} > X^0_{(n-1)}$.
Then (3) becomes

$$\big(X^0_{(j)} - Y^0_{(j)}\big)e_j = 0.$$

Since $X^0_{(j)} \neq Y^0_{(j)}$, we get $e_j = 0$.

As j was arbitrary, this shows that

$$\mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right] = (P^t)_{j,i}$$

for all i, j, t. □

In order to explicitly compute the fixation probabilities we need to compute $P^t_{j,i}$ for $t \to \infty$. In the next lemma we analyse the Markov chain associated to the matrix P.

Lemma 4.5. *A Markov chain with transition matrix $P^\top$ is aperiodic and irreducible. Its unique stationary distribution π is*

$$\pi_k = \frac{(1-p)p^{n-k}}{1-p^n},$$

for all $k \in [n]$.

Proof. Since $p \in (0,1)$, $h_k > 0$ for all $k \in [n-1]$. It also holds that $f_k > 0$ for all $k = 2, \dots, n$. Therefore, the off-diagonal entries of the transition matrix are positive and the chain is irreducible. We have that

$$\begin{aligned} g_k = 1 - f_k - h_k &= \frac{n^2 - (1+p)k(n-k) + p(n-2k+1)}{n^2} \\ &\geq \frac{n^2 - 2k(n-k) + p(n-2k+1)}{n^2}, \end{aligned}$$

where we used that the expression is decreasing in $p \in [0,1]$. The latter expression is minimised for $k = (n+1)/2$, therefore

$$g_k \geq \frac{n^2+1}{2n^2} > 0$$

for all $k \in [n]$. This implies that the transition matrix has positive diagonal entries and therefore that it is aperiodic. If π satisfies the detailed balance equation, it holds $\pi_{k+1} = \frac{h_k}{f_{k+1}}\pi_k$. Note that $\frac{h_k}{f_{k+1}} = \frac{1}{p}$. Unfolding the recurrence yields a product formula: $\pi_k = \pi_1 \prod_{j=1}^{k-1} \frac{1}{p} = \pi_1 \left(\frac{1}{p}\right)^{k-1} = \pi_1\, p^{-(k-1)}$. Since π is a probability distribution, we have $1 = \sum_{k\in[n]} \pi_k = \pi_1 \sum_{k\in[n]} \left(\frac{1}{p}\right)^{k-1}$, and therefore $\pi_1 = \frac{(1-p)p^{n-1}}{1-p^n}$ and $\pi_k = \frac{(1-p)p^{n-k}}{1-p^n}$. □

Now we have all the tools to compute the fixation probabilities.

Proposition 5. *The Copy-or-Max dynamics converges to the k-th smallest initial value with probability $\frac{(1-p)p^{n-k}}{1-p^n}$.*

Proof. In Lemma 4.5 we obtained that the left 1-eigenvalue of $P^\top$ is π. By the fact that $P^\top$ is stochastic, its spectral radius satisfies $\rho(P) \leq 1$. Since by Lemma 4.5 $P^\top$ is aperiodic and irreducible, we can apply the Perron-Frobenius theorem to obtain that the eigenvalue 1 is simple and strictly dominant, and therefore we obtain $P^t_{j,i} \xrightarrow{t\to\infty} \pi_j$, for all $i, j \in [n]$. By applying Lemmas 4.4 and 4.5, we conclude that $\mathbb{P}\left[X^t_{(i)} = X^0_{(j)}\right] \xrightarrow{t\to\infty} \pi_j$. Note that the Markov chain describing the opinion dynamics of the whole system has n absorbing states, each corresponding to the convergence to one of the initial values. As there exists a sequence of events with non-zero probability leading to consensus to the k-th value from the initial configuration, the probability of reaching consensus to the k-th value is π_k, concluding the proof of Proposition 5. □

4.2 Geometric Distribution in Weighted Population Protocol

Now consider the weighted graph G, with non-negative weights matrix $\big(w_{i,j}\big)_{i,j\in[n]}$. In this setting, in each round $t \in \mathbb{N}$, an oriented edge (i, j) is activated with probability proportional to $w_{i,j}$. From the dynamics perspective, this activation represents node i sampling node j.

In this section, we extend the results of the previous section by showing that the same fixation probabilities hold when the underlying graph is connected and is a circulation, which we define below.

Definition 4.1 (Circulation [37]). *A weighted directed graph is called a **Circulation** if for all nodes $v \in V$ the incoming flow equals the outgoing flow of v, i.e. it holds*

$$\sum_{u\in V} w_{u,v} = \sum_{u\in V} w_{v,u}.$$

This includes, in particular, all undirected weighted graphs. From the definition of circulation we deduce the following stronger property.

Lemma 4.6. *G is a circulation if and only if for all $S \subset V$, the incoming flow equals the outgoing flow of S, i.e. it holds*

$$\sum_{u\in S}\sum_{v\notin S} w_{u,v} = \sum_{u\in S}\sum_{v\notin S} w_{v,u}.$$

Proof. For all $v \in V$, by setting $S = \{v\}$, we have that

$$\sum_{u\in V} w_{u,v} = \sum_{u\in V} w_{v,u} \iff \sum_{u\in S}\sum_{v\notin S} w_{u,v} = \sum_{u\in S}\sum_{v\notin S} w_{v,u}, \tag{3}$$

proving that if for all $S \subset V$ the incoming flow equals the outgoing flow of S, G is a circulation. We prove the reverse by induction on $|S| \geq 1$. The case $|S| = 1$ follows from Equation (3). Let $z \in S$, we have

$$\begin{aligned}
&\sum_{u\in S}\sum_{v\notin S}\left(w_{u,v}-w_{v,u}\right)\\
&=\sum_{u\in S\setminus\{z\}}\sum_{v\notin S}\left(w_{u,v}-w_{v,u}\right)+\sum_{v\notin S}\left(w_{z,v}-w_{v,z}\right)\\
&=\sum_{u\in S\setminus\{z\}}\sum_{v\notin S\setminus\{z\}}\left(w_{u,v}-w_{v,u}\right)\\
&\quad-\sum_{u\in S\setminus\{z\}}\left(w_{u,z}-w_{z,u}\right)\\
&\quad+\sum_{v\notin S}\left(w_{z,v}-w_{v,z}\right)\\
&\overset{(i)}{=}\sum_{u\in V\setminus\{z\}}w_{z,u}-\sum_{u\in V\setminus\{z\}}w_{u,z}\\
&\overset{(ii)}{=}\sum_{u\in V}w_{z,u}-\sum_{u\in V}w_{u,z}\\
&\overset{(iii)}{=}0.
\end{aligned}$$

where in (i) we used the inductive hypothesis, in (ii) we added and subtracted $w_{z,z}$, while in (iii) we applied the definition of circulation. This concludes the proof of Lemma 4.6. □

The following lemma can be interpreted as a generalisation of the ergodic theorem for non-homogeneous Markov chains. While the classical theorem guarantees convergence to a stationary distribution for a single fixed transition matrix, this result extends that property to a dynamics system in which the transition rules may change over time.

Lemma 4.7. *Let $\{P^{(t)}\}_{t\in\mathbb{N}}$ be a set of stochastic matrices such that it satisfies the following properties:*

i) For all $t\in\mathbb{N}$, $P^{(t)}$ is aperiodic and irreducible.
ii) The cardinality of $\{P^{(t)}\}_{t\in\mathbb{N}}$ is finite.
iii) There exists a vector π which is a left 1-eigenvector of $P^{(t)}$ for all $t\in\mathbb{N}$.
iv) There exist a non-singular matrix D such that

$$H^{(t)}=D^{-1}P^{(t)}D$$

is a symmetric matrix for all $t\in\mathbb{N}$.

For all probability vector $v\in\mathbf{R}^n$, it holds

$$\lim_{T\to+\infty}v\prod_{t=1}^{T}P^{(t)}=\pi.$$

Proof. By Item iii), we have that $\pi P^{(t)}D=\pi D$, and therefore $\pi DH^{(t)}=\pi DD^{-1}P^{(t)}D=\pi D$, or in other words, πD is left 1-eigenvector of $H^{(t)}$ for all $t\in\mathbb{N}$. Given a probability vector $v\in\mathbf{R}^n$, we can write $vD=\pi D+\delta$, with $\delta\in\mathbf{R}^n$ s.t. $(\pi D,\delta)=0$. This implies that, by the spectral theorem, $\delta\prod_{t=1}^{T-1}H^{(t)}$ is orthogonal to πD for all $T\in\mathbb{N}$. Therefore, if $v_1,v_2,\dots,v_n$ are orthogonal eigenvectors of $H^{(T)}$ (corresponding to $|\lambda_1^{(T)}|\ge|\lambda_2^{(T)}|\ge\dots\ge|\lambda_n^{(T)}|$ real eigenvalues), then $\delta\prod_{t=1}^{T-1}H^{(t)}=\sum_{i=2}^{n}c_iv_i$ for some coefficient $c_i\in\mathbf{R}$,and we obtain

$$\begin{aligned}
\left\|\delta\prod_{t=1}^{T}H^{(t)}\right\|&=\left\|\left(\sum_{i=2}^{n}c_iv_i\right)H^{(T)}\right\|\\
&=\left\|\sum_{i=2}^{n}c_i\lambda_i^{(T)}v_i\right\|\\
&\le|\lambda_2^{(T)}|\left\|\delta\prod_{t=1}^{T-1}H^{(t)}\right\|.
\end{aligned}\tag{4}$$

Since D is non singular, $v=\pi+\delta D^{-1}$, and we write

$$\begin{aligned}
\left\|v\left(\prod_{t=0}^{T}P^{(t)}\right)D-\pi D\right\|&\overset{(a)}{=}\left\|\delta D^{-1}\left(\prod_{t=0}^{T}P^{(t)}\right)D\right\|\\
&\overset{(b)}{=}\left\|\delta\left(\prod_{t=1}^{T}H^{(t)}\right)\right\|\\
&\overset{(c)}{\le}|\lambda_2^{(T)}|\left\|\delta\left(\prod_{t=1}^{T-1}H^{(t)}\right)\right\|\\
&\overset{(d)}{\le}\|\delta\|\prod_{t=1}^{T}|\lambda_2^{(t)}|,
\end{aligned}\tag{5}$$

where in equation (a) we used Item iii), in equation (b) follows from Item iv), (c) holds by Equation (4), and (d) follows iterating the step. By Item iv), $H^{(t)}$'s eigenvalues are the same as $P^{(t)}$. By Item i), $\lambda_2^{(t)}<1$ for all $t\in\mathbb{N}$. Moreover, by Item ii), we have that $\overline{\lambda}_2:=\sup_{t\ge 0}\lambda_2^{(t)}<1$. This fact, together with Equation (5), implies that

$$\lim_{T\to+\infty}\left\|v\left(\prod_{t=0}^{T}P^{(t)}\right)D-\pi D\right\|\le\lim_{T\to+\infty}|\overline{\lambda}_2|^T\|\delta\|=0.$$

This implies that $v\left(\prod_{t=0}^{T}P^{(t)}\right)D-\pi D\to 0$ and, since D is non-singular, that $v\prod_{t=0}^{T}P^{(t)}\to\pi$. □

Proposition 6. *The Copy-or-Max dynamics over a graph G converges to the k-th smallest initial value with probability $\frac{(1-p)p^{n-k}}{1-p^n}$ if and only if G is connected and is a circulation.*

Proof. Given a configuration X^t_{ord}, relabel at every round the nodes so that node i holds the value $X^t_{(i)}$. By using the same notation of Lemma 4.1,

$$q_{i,j}^{(t)}=\begin{cases}w_{i,j}^t\frac{1}{W} & \text{if } j>i\\ w_{i,j}^t\frac{p}{W} & \text{if } j<i,\end{cases}$$

where we define $W=\sum_{i,j\in[n]}w_{i,j}^t$. We can now compute, as in Lemma 4.2,

$$f_k^{(t)}=\sum_{i=k}^{n}\sum_{j=1}^{k-1}w_{i,j}^t\frac{p}{W},\qquad h_k^{(t)}=\sum_{i=1}^{k}\sum_{j=k+1}^{n}w_{i,j}^t\frac{1}{W},$$

and $g_k^{(t)} = 1 - f_k^{(t)} - h_k^{(t)}$. For all rounds $t \geq 0$, let $P^{(t)}$ be the following tridiagonal stochastic matrix

$$P^{(t)} = \begin{bmatrix} g_1^{(t)} & f_2^{(t)} & 0 & \cdots & 0 \\ h_1^{(t)} & g_2^{(t)} & f_3^{(t)} & \ddots & \vdots \\ 0 & h_2^{(t)} & g_3^{(t)} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & f_n^{(t)} \\ 0 & \cdots & 0 & h_{n-1}^{(t)} & g_n^{(t)} \end{bmatrix}.$$

Similarly to Lemmas 4.3 and 4.4 we obtain that , for all i, j, t,

$$\mathbb{P}\left[X_{(i)}^t = X_{(j)}^0\right] = \left(\prod_{s=0}^{t-1} P^{(s)}\right)_{j,i}. \tag{6}$$

We prove the two implications separately.

Connectivity and circulation imply geometric fixation probabilities.

Since the probability distribution of the final state is expressed as a product of matrices, our goal is to show we can apply Lemma 4.7 to $\left\{{P^{(t)}}^T\right\}_{t\in\mathbb{N}}$.

Item i) To prove that the matrix is aperiodic we show that $g_k^{(t)} > 0$ for all $t \in \mathbb{N}, k \in [n]$. By the fact G is a circulation and by Lemma 4.6, we have that

$$2\sum_{i=k}^{n}\sum_{j=1}^{k-1} w_{i,j}^t = \sum_{i=k}^{n}\sum_{j=1}^{k-1} w_{i,j}^t + \sum_{i=1}^{k-1}\sum_{j=k}^{n} w_{i,j}^t \leq \sum_{i,j\in[n]} w_{i,j}^t = W. \tag{7}$$

Therefore, using again the fact G is a circulation, we obtain that

$$\begin{aligned} g_k^{(t)} &= 1 - f_k^{(t)} - h_k^{(t)} \\ &= 1 - \sum_{i=k}^{n}\sum_{j=1}^{k-1} w_{i,j}^t \frac{1+p}{W} \\ &\geq 1 - \frac{1+p}{2} \\ &> 0, \end{aligned}$$

where in the last two inequalities we used Equation (7) and the fact $p < 1$.

To prove that ${P^{(t)}}^T$ is irreducible for all $t \geq 0$, we show that $f_{k+1}^{(t)}, h_k^{(t)} > 0$ for all $t \geq 0, k \in [n-1]$. By contradiction, assume that there exist $t \geq 0$ and $k \in [n-1]$, such that $h_k^{(t)} = 0$. By definition of $h_k^{(t)}$, since $p > 0$, this implies that $\sum_{i=1}^{k}\sum_{j=k+1}^{n} w_{i,j}^t = 0$. By the fact G is a circulation and by Lemma 4.6, it follows that $\sum_{i=1}^{k}\sum_{j=k+1}^{n} w_{j,i}^t = 0$. Since $w_{i,j}^t \geq 0$ for all $i, j \in [n]$, this implies that $w_{i,j}^t = w_{j,i}^t = 0$ for all $i \in [k], j \in n \setminus [k]$, which means that $[k]$ and $n \setminus [k]$ are two disconnected components of the graph G, contradicting the fact G is connected. The same argument can be applied to $f_{k+1}^{(t)}$.

Item ii) $\left\{{P^{(s)}}^T\right\}_{s=0,\ldots,t-1}$ is finite as each ${P^{(s)}}^T$ is uniquely determined by the relabelling of the nodes, which are at most $n!$.

Item iii) For all $t \geq 0$, by the fact the matrix w is a circulation and by Equation (3) with $S = 1, \ldots, k$, we obtain

$$\sum_{i=1}^{k}\sum_{j=k+1}^{n} w_{i,j}^t = \sum_{i=k+1}^{n}\sum_{j=1}^{k} w_{i,j}^t.$$

Therefore we obtain that for all $t \geq 0$, ${P^{(t)}}^T$ admits a solution to the detailed balance equation

$$\frac{\pi_{k+1}}{\pi_k} = \frac{h_k^{(t)}}{f_{k+1}^{(t)}} = \frac{\sum_{i=1}^{k}\sum_{j=k+1}^{n} w_{i,j}^t \frac{1}{W}}{\sum_{i=k+1}^{n}\sum_{j=1}^{k} w_{i,j}^t \frac{p}{W}} = \frac{1}{p}, \tag{8}$$

with solution the vector π that satisfies $\pi_k = \frac{(1-p)p^{n-k}}{1-p^n}$. π is a 1-left eigenvector for $P^{(t)}$ for all $t \geq 0$ (Item iii)).

Item iv) Let D the diagonal matrix such that $D_{k,k} = \sqrt{\pi_k}$ for all $k \in [n]$. By using Equation (8), we can check that $D^{-1}{P^{(t)}}^T D$ is symmetric for all $t \geq 0$.

We can now apply Lemma 4.7, and we obtain that

$$\lim_{t\to+\infty} \mathbb{P}\left[X_{(i)}^t = X_{(j)}^0\right] = \lim_{t\to+\infty}\left(\prod_{s=0}^{t-1} P^{(s)}\right)_{j,i} = \pi_j.$$

Note that the Markov chain describing the opinion dynamics of the whole system has n absorbing states, each corresponding to convergence to one of the initial values. As there exists a sequence of events with non-zero probability leading to consensus to the k-th value from the initial configuration, the probability of reaching consensus to the k-th value is π_k, concluding the first part of the proof of Proposition 6.

Geometric fixation probabilities imply connectivity and circulation. If the graph is disconnected, convergence on a single value can't be reached starting from a configuration with all different values, meaning that geometric fixation probabilities imply that G is connected. By Equation (6) and since we are assuming that the process has geometric fixation probabilities, we have that there exists a $N \in \mathbb{N}$ such that, for all $t \geq N$

$$\mathbb{P}\left[X_{(i)}^t = X_{(j)}^0\right] = \left(\prod_{s=0}^{t-1} P^{(s)}\right)_{j,i} = \pi_j := \frac{(1-p)p^{n-j}}{1-p^n}.$$

This implies that we have

$$\begin{bmatrix} | & | & \\ \pi & \pi & \ldots \\ | & | & \end{bmatrix} = P^{(N)} \cdot \left(\prod_{s=0}^{N-1} P^{(s)}\right) = \begin{bmatrix} | & | & \\ P^{(N)}\pi & P^{(N)}\pi & \ldots \\ | & | & \end{bmatrix},$$

which means that π is a 1-left eigenvector of ${P^{(N)}}^T$.

Moreover, we have that a 1-left eigenvector of ${P^{(N)}}^T$ must be a solution of the detailed balance equation, Equation (8). In fact, the eigenvector equation $\pi^T {P^{(N)}}^T = \pi^T$ implies, for every $k \in [n]$,

$$\pi_k = \pi_{k-1}h_{k-1}^{(T)} + \pi_k g_k^{(T)} + \pi_{k+1}f_{k+1}^{(T)},$$

with the convention that $\pi_0, h_0^{(T)}, f_1^{(T)}, \pi_{n+1}, h_n^{(T)}, f_{n+1}^{(T)} = 0$. Since $g_k = 1 - f_k^{(T)} - h_k^{(T)}$, we rewrite this as

$$\pi_k(f_k^{(T)} + h_k^{(T)}) = \pi_{k-1}h_{k-1}^{(T)} + \pi_{k+1}f_{k+1}^{(T)}.$$

Now define for all $k \in [n]$ the local flux $J_k := \pi_k h_k^{(T)} - \pi_{k+1} f_{k+1}^{(T)}$. We can rearrange the previous equation and obtain that

$$J_{k-1} = J_k \qquad \text{for all } k \in [n],$$

showing that the flux J_k is constant across all edges. At the boundary, by definition we have $J_0 = J_n = 0$, which forces $J_k = 0$ for all $k \in [n]$. Therefore, $\pi_k h_k^{(T)} = \pi_{k+1} f_{k+1}^{(T)}$ for all $k \in [n-1]$, which is exactly the detailed balance condition:

$$\frac{\pi_{k+1}}{\pi_k} = \frac{h_k^{(T)}}{f_{k+1}^{(T)}} = \frac{\sum_{i=1}^{k}\sum_{j=k+1}^{n} w_{i,j}^t \frac{1}{W}}{\sum_{i=k+1}^{n}\sum_{j=1}^{k} w_{i,j}^t \frac{p}{W}}.$$

Since $\frac{\pi_{k+1}}{\pi_k} = \frac{1}{p}$, we deduce that

$$\sum_{i=1}^{k}\sum_{j=k+1}^{n} w_{i,j}^t = \sum_{i=k+1}^{n}\sum_{j=1}^{k} w_{i,j}^t,$$

for all $k \in [n]$, which, by Lemma 4.6, implies that G is a circulation. □

4.3 Proof of Theorem 1

Theorem 1's statement about fixation probabilities follows trivially from Proposition 5 and Proposition 6. The proof of the convergence time is deferred to the full version of the paper [3].

4.4 Fixation Probabilities for Order-Based Dynamics

In this section, we show that the technique used in Section 4.1 can be applied to a more general class of dynamics.

Definition 4.2 (Order-Based Dynamics). *Let X be an arbitrary configuration of the system with all values distinct. Let $i, j \in [n]$, and denote by $S_{i,j}$ the event "the agent holding the i-th ranked value is activated and adopts the j-th ranked value in X". We define an* Order-Based dynamics *as a protocol for which there exists a function $q : [n] \times [n] \to [0,1]$ s.t.*

$$q_j(i) = \mathbb{P}\left[S_{i,j}\right], \quad \sum_{i\in[n]}\sum_{j\in[n]} q_j(i) = 1.$$

In other words, the first conditions means that $\mathbb{P}\left[S_{i,j}\right]$ is a function of the ranks i and j only, and not of the actual values in the system. The second condition guarantees that the agent adopts a value already present in X.

This class of dynamics encompasses a wide range of protocols. In the following statement, we present several representative examples, without claiming to exhaust all possible protocols that fall within this class.

Fact 4.1. *Consider any protocol P where, at each round, an agent is activated uniformly at random and observes its own value Y_{self} and that of $h-1$ values from the current configuration chosen independently and uniformly at random. Let $Y_1 \le Y_2 \le \cdots \le Y_{self} \le \cdots \le Y_h$ be the observed values, ordered increasingly (with ties broken arbitrarily), and assume that the agent adopts opinion Y_j with probability depending only on j. Then, P is an Order-Based Dynamics.*

Theorem 7. *Consider an Order-Based dynamics such that, for all $k \in [n-1]$, satisfies*

$$f_{k+1} = \sum_{i=k+1}^{n}\sum_{j=1}^{k} q_j(i) > 0 \qquad h_k = \sum_{i=1}^{k}\sum_{j=k+1}^{n} q_j(i) > 0.$$

We have that the configuration eventually converges to the k-th smallest value of the initial configuration with probability π_k defined as

$$\pi_1 = \left(\sum_{k=1}^{n}\prod_{i=1}^{k-1}\frac{h_i}{f_{i+1}}\right)^{-1}, \quad \pi_k = \pi_1\prod_{i=1}^{k-1}\frac{h_i}{f_{i+1}}, \quad \text{for all } k = 2,\dots,n.$$

4.5 Binomial Distribution

In this section, we apply the general result obtained for Order-Based Dynamics in Section 4.4 to the *Biased Median Dynamics.* Given n agents with initial values $X_1^0 \le \cdots \le X_n^0$, in round t, an agent u is drawn uniformly at random. This agent samples uniformly at random an unordered pair of agents (v, s) (v and s may be equal): agent u updates based on $m = \text{median}(X_u, X_v, X_s)$. The update rule is as follows for $p \in (0, 1/2]$:

- **Larger median:** If $m > X_u^t$, set $X_u^t \leftarrow m$.
- **Smaller median:** Otherwise ($m \le X_u^t$), w.p. $\frac{p}{1-p}$, set $X_u^t \leftarrow m$.

Lemma 4.8. *Let $i, j \in [n]$ and denote by $S_{i,j}^t$ the event "At round t, the agent with i-th rank value activates and adopts $X_{(j)}^t$". Consider a configuration X_{ord}^t s.t. $X_{(j)}^t \neq X_{(\ell)}^t$ for all $\ell \neq j$. We have that*

$$q_j(i) := \mathbb{P}\left[S_{i,j}^t \mid X_{ord}^t\right] = \begin{cases} \frac{2(n-j)+1}{n^3} & \text{if } j > i \\ \left(\frac{p}{1-p}\right)\frac{2j-1}{n^3} & \text{if } j < i. \end{cases}$$

4.6 Proof of Theorem 2

Proof. We can compute, as in Lemma 4.2,

$$f_k^{(t)} = \sum_{i=k}^{n}\sum_{j=1}^{k-1} q_j(i), \qquad h_k^{(t)} = \sum_{i=1}^{k}\sum_{j=k+1}^{n} q_j(i),$$

and $g_k^{(t)} = 1 - f_k^{(t)} - h_k^{(t)}$. Using the results of Lemma 4.8 we obtain

$$f_k^{(t)} = \sum_{i=k}^{n}\sum_{j=1}^{k-1}\frac{p}{1-p}\left(\frac{2j-1}{n^3}\right) = \left(\frac{p}{1-p}\right)\frac{(n-k+1)(k-1)^2}{n^3}$$

$$h_k^{(t)} = \sum_{i=1}^{k}\sum_{j=k+1}^{n}\frac{2(n-j)+1}{n^3} = \frac{k(n-k)^2}{n^3}.$$

We compute, for all $k \in [n-1]$

$$\frac{\pi_{k+1}}{\pi_k} = \frac{h_k}{f_{k+1}} = \left(\frac{1-p}{p}\right)\frac{n-k}{k},$$

which implies that

$$\pi_k = \pi_1\left(\frac{1-p}{p}\right)^{k-1}\prod_{i=1}^{k-1}\frac{n-i}{i},$$

whose only solution s.t. $\sum_{k=1}^{n}\pi_k = 1$ is $\pi_k = \binom{n-1}{k-1}p^{n-k}(1-p)^{k-1}$. In order to show that this is the unique stationary distribution for a Markov chain with transition matrix P, we prove that P is aperiodic and irreducible by showing that for all $i, j s.t. |i-j| \le 1$, $P_{i,j} > 0$, as in Lemma 4.5. Since $p \in (0, 1/2]$, $f_{k+1}, h_k > 0$ for all $k \in [n-1]$. We can assume that $g_k > 0$ for all $k \in [n]$, as if it wasn't the case we can consider a lazy version of the biased median and obtain the same fixation probabilities. As in Proposition 5, we can conclude that the Biased Median dynamics converges to the k-th initial value with probability $\pi_k = \binom{n-1}{k-1} p^{n-k}(1-p)^{k-1}$. □

4.7 3-Majority Fixation Probabilities in Binary Setting

In this section, we prove Corollary 4. We introduce and analyse a variant of the Median dynamics, that we call *Self-Excluding Median*, in which each agent samples three opinions uniformly at random from the system and adopts their median (without necessarily including its own current opinion among the samples). Because this dynamic coincides with 3-Majority in the binary setting, establishing its fixation probabilities directly yields the proof of Corollary 4. Given n agents with initial values $X_1^0 \leq \cdots \leq X_n^0$, in each round t, an agent u is drawn uniformly at random. This agent samples uniformly at random three agents (s, v, w) (s, v, w may be equal): agent u updates to $m = \text{median}(X_s, X_v, X_w)$.

Lemma 4.9. *Let $i, j \in [n]$ and denote by $S_{i,j}^t$ the event "At round t, the agent with i-th rank value activates and adopts $X_{(j)}^t$". Consider a configuration X_{ord}^t s.t. $X_{(j)}^t \neq X_{(\ell)}^t$ for all $\ell \neq j$. We have that*

$$q_j(i) := \mathbb{P}\left[S_{i,j}^t \mid X_{ord}^t\right] = \frac{6nj - 3n - 6j^2 + 6j - 2}{n^4}.$$

Theorem 8. *The Self-Excluding Median dynamics on the clique converges to the k-th smallest initial value with probability*

$$\pi_k = \frac{\Gamma(n+2)}{\Gamma(2n+1)\Gamma\left(\frac{n}{2}+1\right)^2}\binom{n-1}{k-1}\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{3n}{2}-k+1\right).$$

Proof. We can compute, as in Lemma 4.2,

$$f_k^{(t)} = \sum_{i=k}^{n}\sum_{j=1}^{k-1} q_j(i), \qquad h_k^{(t)} = \sum_{i=1}^{k}\sum_{j=k+1}^{n} q_j(i),$$

and $g_k^{(t)} = 1 - f_k^{(t)} - h_k^{(t)}$. Using the results of Lemma 4.9, we obtain

$$\begin{aligned} f_k^{(t)} &= (n-k+1)\sum_{j=1}^{k-1} q_j(i) \\ &= \frac{(n-k+1)(k-1)^2(3n-2k+2)}{n^4}, \end{aligned}$$

$$\begin{aligned} h_k^{(t)} &= k\sum_{j=k+1}^{n} q_j(i) \\ &= \frac{k(n-k)^2(n+2k)}{n^4}. \end{aligned}$$

We compute for all $k \in [n-1]$

$$\frac{\pi_{k+1}}{\pi_k} = \frac{h_k}{f_{k+1}} = \frac{k \cdot (n-k)^2(n+2k)}{(n-k)\cdot k^2(3n-2k)} = \frac{(n-k)(n+2k)}{k(3n-2k)}.$$

Then for $k \in [n]$,

$$\pi_k = \pi_1 \prod_{j=1}^{k-1}\frac{n-j}{j} \cdot \prod_{j=1}^{k-1}\frac{n+2j}{3n-2j}.$$

The products simplify to

$$\prod_{j=1}^{k-1}\frac{n-j}{j} = \binom{n-1}{k-1},$$

$$\prod_{j=1}^{k-1}\frac{n+2j}{3n-2j} = \prod_{j=1}^{k-1}\frac{\frac{n}{2}+j}{\frac{3n}{2}-j} = \frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2}+1)}\cdot\frac{\Gamma(\frac{3n}{2}-k+1)}{\Gamma(\frac{3n}{2})}.$$

Therefore, for $k \in [n]$, we have

$$\pi_k = \frac{1}{Z}\binom{n-1}{k-1}\Gamma\left(\frac{n}{2}+k\right)\Gamma\left(\frac{3n}{2}-k+1\right),$$

where $Z = \frac{\Gamma(2n+1)\Gamma(\frac{n}{2}+1)^2}{\Gamma(n+2)}$ is the normalising constant. □

5 Conclusion and Future Work

In this work, we have explored the interplay between computational dynamics and order statistics, demonstrating that simple memoryless protocols can be designed to "sample" from specific distributions over initial values. We introduced two such protocols: *Copy-or-Max*, which follows a geometric law, and *Biased Median*, which achieves a binomial distribution. Furthermore, our analysis showed that variants of these dynamics, such as 3-Majority, can even realise more complex distributions like the Beta-Binomial law.

Our central technical contribution is the development of a novel rank-indexed analysis framework. By identifying a linear *rank-ancestry process* underlying these non-linear dynamics, we provided exact closed-form characterisations of fixation probabilities. This technique not only bridges the gap between distributed consensus and order statistics but also extends the state-of-the-art for classical models like the Median protocol. Furthermore, our findings on weighted population protocols reveal a surprising generality: geometric fixation probabilities persist on any graph satisfying a simple circulation condition, including all regular and undirected weighted graphs.

From a practical perspective, we leveraged these dynamics to design a broadly applicable algorithm for the Top-k problem, which operates correctly even in general graphs where prior gossip-based approaches may not apply. Several questions naturally arise from our work. First, it is natural to ask which other families of distributions can be realised through simple, memoryless local interactions. In particular, are one-sample dynamics fundamentally limited to producing uniform or geometric fixation distributions (or variants thereof)? Second, what is the robustness of these protocols under noisy communication or in the presence of Byzantine agents?

We introduce a rank-based analysis framework that turns local update rules into analysable rank dynamics, and we believe that it has the potential to uncover further connections between local update rules and emergent global statistical properties.

Acknowledgments

This work has been supported by the AID INRIA-DGA project n°2023000872 "BioSwarm", the French government National Research Agency (ANR) through the UCA JEDI (ANR-15-IDEX-01), the EUR DS4H (ANR-17-EURE-004) and the 3IA Cote d'Azur Investments ANR-23-IACL-0001, and EPSRC grant EP/W005573/1.

References

[1] Mohammed Amin Abdullah and Moez Draief. 2015. Global Majority Consensus by Local Majority Polling on Graphs of a given Degree Sequence. *Discrete Applied Mathematics* 180 (Jan. 2015), 1–10. https://doi.org/10.1016/j.dam.2014.07.026

[2] D. Angluin, J. Aspnes, and D. Eisenstat. 2008. A Simple Population Protocol for Fast Robust Approximate Majority. *Distributed Computing* 21, 2 (2008), 87–102. https://doi.org/10.1007/s00446-008-0059-z

[3] Niccolò D Archivio, Hind Almahmoud, Emanuele Natale, and Frederik Mallmann-Trenn. 2026. Order Statistics in Population Protocols via Simple Dynamics. (March 2026). https://hal.science/hal-05548829 working paper or preprint.

[4] Luca Becchetti, Vincenzo Bonifaci, Emilio Cruciani, and Francesco Pasquale. 2023. On a Voter Model with Context-Dependent Opinion Adoption. In *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems* (London, United Kingdom) *(AAMAS '23)*. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 2766–2768.

[5] Luca Becchetti, Andrea Clementi, Pasin Manurangsi, Emanuele Natale, Francesco Pasquale, Prasad Raghavendra, and Luca Trevisan. 2018. Average Whenever You Meet: Opportunistic Protocols for Community Detection. In *26th Annual European Symposium on Algorithms (ESA 2018) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 112)*, Yossi Azar, Hannah Bast, and Grzegorz Herman (Eds.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 7:1–7:13. https://doi.org/10.4230/LIPIcs.ESA.2018.7

[6] Luca Becchetti, Andrea Clementi, and Emanuele Natale. 2020. Consensus Dynamics: An Overview. *ACM SIGACT News* 51, 1 (March 2020), 58–104. https://doi.org/10.1145/3388392.3388403

[7] Luca Becchetti, Andrea Clementi, Emanuele Natale, Francesco Pasquale, and Riccardo Silvestri. 2015. Plurality Consensus in the Gossip Model. In *Proceedings of the 26th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA) (SODA '15)*. SIAM, San Diego, California, 371–390.

[8] Luca Becchetti, Andrea Clementi, Emanuele Natale, Francesco Pasquale, Riccardo Silvestri, and Luca Trevisan. 2017. Simple Dynamics for Plurality Consensus. *Distributed Computing* 30, 4 (Aug. 2017), 293–306. https://doi.org/10.1007/s00446-016-0289-4

[9] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, and L. Trevisan. 2016. Stabilizing consensus with many opinions. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms* (Arlington, Virginia) *(SODA '16)*. Society for Industrial and Applied Mathematics, USA, 620–635.

[10] Luca Becchetti, Andrea Clementi, Francesco Pasquale, Luca Trevisan, and Isabella Ziccardi. 2023. Expansion and Flooding in Dynamic Random Networks with Node Churn. *Random Structures & Algorithms* 63, 1 (2023), 61–101. https://doi.org/10.1002/rsa.21133

[11] Petra Berenbrink, Andrea Clementi, Robert Elsässer, Peter Kling, Frederik Mallmann-Trenn, and Emanuele Natale. 2017. Ignore or Comply? On Breaking Symmetry in Consensus. In *Proceedings of the ACM Symposium on Principles of Distributed Computing* (Washington, DC, USA) *(PODC '17)*. Association for Computing Machinery, New York, NY, USA, 335–344. https://doi.org/10.1145/3087801.3087817

[12] Petra Berenbrink, Colin Cooper, Cristina Gava, David Kohan Marzagão, Frederik Mallmann-Trenn, Tomasz Radzik, and Nicolas Rivera. 2023. Distributed Averaging in Opinion Dynamics. In *Proceedings of the 2023 ACM Symposium on Principles of Distributed Computing* (Orlando, FL, USA) *(PODC '23)*. Association for Computing Machinery, New York, NY, USA, 211–221. https://doi.org/10.1145/3583668.3594593

[13] Petra Berenbrink, Martin Hoefer, Dominik Kaaser, Marten Maack, Malin Rau, and Lisa Wilhelmi. 2025. Opinion Dynamics with Median Aggregation. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems* (Detroit, MI, USA) *(AAMAS '25)*. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 271–279.

[14] Michael Blondin, Javier Esparza, and Stefan Jaax. 2018. Large Flocks of Small Birds: on the Minimal Size of Population Protocols. In *35th Symposium on Theoretical Aspects of Computer Science (STACS 2018) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 96)*, Rolf Niedermeier and Brigitte Vallée (Eds.). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 16:1–16:14. https://doi.org/10.4230/LIPIcs.STACS.2018.16

[15] Colin Cooper, Robert Elsasser, Hirotaka Ono, and Tomasz Radzik. 2013. Coalescing random walks and voting on connected graphs. *SIAM Journal on Discrete Mathematics* 27, 4 (2013), 1748–1758.

[16] Colin Cooper, Robert Elsässer, and Tomasz Radzik. 2014. The Power of Two Choices in Distributed Voting. In *Automata, Languages, and Programming*, Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 435–446.

[17] Colin Cooper, Frederik Mallmann-Trenn, Tomasz Radzik, Nobutaka Shimizu, and Takeharu Shiraga. 2025. *Asynchronous 3-Majority Dynamics with Many Opinions.* Society for Industrial and Applied Mathematics, 4095–4131. https://doi.org/10.1137/1.9781611978322.140 arXiv:https://epubs.siam.org/doi/pdf/10.1137/1.9781611978322.140

[18] Colin Cooper, Frederik Mallmann-Trenn, Tomasz Radzik, Nobutaka Shimizu, and Takeharu Shiraga. 2026. Undecided State Dynamics with Many Opinions. arXiv:2603.02636 [cs.DC] https://arxiv.org/abs/2603.02636

[19] Colin Cooper, Tomasz Radzik, Nicolás Rivera, and Takeharu Shiraga. 2017. Fast Plurality Consensus in Regular Expanders. In *31st International Symposium on Distributed Computing (DISC 2017) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 91)*, Andréa W. Richa (Ed.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 13:1–13:16. https://doi.org/10.4230/LIPIcs.DISC.2017.13

[20] Colin Cooper and Nicolás Rivera. 2016. The Linear Voting Model. *LIPIcs, Volume 55, ICALP 2016* 55 (2016), 144:1–144:12. https://doi.org/10.4230/LIPICS.ICALP.2016.144

[21] Francesco d'Amore, Niccolò D'Archivio, George Giakkoupis, and Emanuele Natale. 2025. On the H-Majority Dynamics with Many Opinions. *LIPIcs, Volume 356, DISC 2025* 356 (2025), 27:1–27:24. https://doi.org/10.4230/LIPICS.DISC.2025.27

[22] Francesco d'Amore and Isabella Ziccardi. 2025. Phase Transition of the 3-Majority Opinion Dynamics with Noisy Interactions. *Theoretical Computer Science* 1028 (Feb. 2025), 115030. https://doi.org/10.1016/j.tcs.2024.115030

[23] Benjamin Doerr, Leslie Ann Goldberg, Lorenz Minder, Thomas Sauerwald, and Christian Scheideler. 2011. Stabilizing consensus with the power of two choices. In *Proceedings of the twenty-third annual ACM symposium on Parallelism in algorithms and architectures (SPAA '11)*. Association for Computing Machinery, New York, NY, USA, 149–158. https://doi.org/10.1145/1989493.1989516

[24] Robert Elsässer, Tom Friedetzky, Dominik Kaaser, Frederik Mallmann-Trenn, and Horst Trinker. 2017. Rapid Asynchronous Plurality Consensus. arXiv:1602.04667 [cs.DC] https://arxiv.org/abs/1602.04667

[25] Mauro Franceschelli, Alessandro Giua, and Alessandro Pisano. 2016. Finite-time consensus on the median value with robustness properties. *IEEE Trans. Automat. Control* 62, 4 (2016), 1652–1667.

[26] Antonio Furchì, Martina Lippi, Alessandro Marino, and Andrea Gasparri. 2025. A finite-time distributed dynamic consensus protocol for tracking maximum (minimum) reference signals. *Automatica* 179 (2025), 112390.

[27] Marcelo Matheus Gauy, Anna Abramishvili, Eduardo Colli, Tiago Madeira, Frederik Mallmann-Trenn, Vinícius Franco Vasconcelos, and David Kohan Marzagao. 2025. Voter Model Meets Rumour Spreading: A Study of Consensus Protocols on Graphs with Agnostic Nodes. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems* (Detroit, MI, USA) *(AAMAS '25)*. International Foundation for Autonomous Agents and Multiagent Systems, Richland, SC, 848–857.

[28] Mohsen Ghaffari and Johannes Lengler. 2018. Nearly-Tight Analysis for 2-Choice and 3-Majority Consensus Dynamics. In *Proceedings of the 2018 ACM Symposium on Principles of Distributed Computing (PODC '18)*. ACM, New York, NY, USA, 305–313. https://doi.org/10.1145/3212734.3212738

[29] Bernhard Haeupler, Marc Kaufmann, Raghu Raman Ravi, and Ulysse Schaller. 2025. Adversarially-Robust Gossip Algorithms for Approximate Quantile and Mean Computations. https://doi.org/10.48550/arXiv.2502.15320 arXiv:2502.15320 [cs.DS]

[30] Bernhard Haeupler, Jeet Mohapatra, and Hsin-Hao Su. 2018. Optimal Gossip Algorithms for Exact and Approximate Quantile Computations. In *Proceedings of the 2018 ACM Symposium on Principles of Distributed Computing* (Egham, United Kingdom) *(PODC '18)*. Association for Computing Machinery, New York, NY, USA, 179–188. https://doi.org/10.1145/3212734.3212770

[31] Yehuda Hassin and David Peleg. 2001. Distributed Probabilistic Polling and Applications to Proportionate Agreement. *Information and Computation* 171, 2 (Dec. 2001), 248–268. https://doi.org/10.1006/inco.2001.3088

[32] David Kempe, Alin Dobra, and Johannes Gehrke. 2003. Gossip-based computation of aggregate information. In *Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2003)*. IEEE, Cambridge, MA, USA, 482–491. https://doi.org/10.1109/SFCS.2003.1238221

[33] Hicham Lesfari, Frédéric Giroire, and Stéphane Pérennes. 2022. Biased Majority Opinion Dynamics: Exploiting Graph k-Domination. In *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence*. International Joint Conferences on Artificial Intelligence Organization, Vienna, Austria, 377–383. https://doi.org/10.24963/ijcai.2022/54

[34] Erez Lieberman, Christoph Hauert, and Martin A. Nowak. 2005. Evolutionary dynamics on graphs. *Nature* 433, 7023 (Jan. 2005), 312–316. https://doi.org/10.1038/nature03204 Publisher: Nature Publishing Group.

[35] Wenjun Mei, Julien M. Hendrickx, Ge Chen, Francesco Bullo, and Florian Dörfler. 2024. Convergence, Consensus, and Dissensus in the Weighted-Median Opinion Dynamics. *IEEE Trans. Automat. Control* 69, 10 (2024), 6700–6714. https://doi.org/10.1109/TAC.2024.3376752

[36] Elchanan Mossel and Omer Tamuz. 2017. Opinion Exchange Dynamics. *Probability Surveys* 14 (2017), 155–204. https://doi.org/10.1214/14-PS230

[37] Martin A. Nowak and Robert M. May. 1992. Evolutionary games and spatial chaos. *Nature* 359, 6398 (Oct. 1992), 826–829. https://doi.org/10.1038/359826a0 Publisher: Nature Publishing Group.

[38] Devavrat Shah. 2007. Gossip Algorithms. *Foundations and Trends® in Networking* 3, 1 (2007), 1–125. https://doi.org/10.1561/1300000014

[39] Nobutaka Shimizu and Takeharu Shiraga. 2025. 3-Majority and 2-Choices with Many Opinions. In *Proceedings of the ACM Symposium on Principles of Distributed Computing (PODC '25)*. Association for Computing Machinery, New York, NY, USA, 207–217. https://doi.org/10.1145/3732772.3733511

[40] Jianing Yang, Liqi Zhou, Bohui Wang, and Yuanshi Zheng. 2024. Max-consensus of multi-agent systems in random networks. *Journal of the Franklin Institute* 361, 6 (2024), 106712. https://doi.org/10.1016/j.jfranklin.2024.106712

[41] Xu Zhang and Marcos M. Vasconcelos. 2023. Top-k data selection via distributed sample quantile inference. In *Proceedings of The 5th Annual Learning for Dynamics and Control Conference (Proceedings of Machine Learning Research, Vol. 211)*, Nikolai Matni, Manfred Morari, and George J. Pappas (Eds.). PMLR, Philadelphia, PA, USA, 813–824. https://proceedings.mlr.press/v211/zhang23d.html