Equivariant K-theory of affine Grassmannians in representation theory and arithmetic

by

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Abstract

We develop and employ techniques from equivariant algebraic K-theory and related invariants in the context of geometric representation theory, in both arithmetic and topological situations. We showcase the use of such techniques on the affine Grassmannian $\mathcal{G}\mathrm{r}$, a space of fundamental interest in the geometric Langlands program.

It is a deep development of mathematics of the last century that many concrete, yet combinatorially complex algebraic problems may be effectively studied through the lens of algebraic geometry. The objects of interest can be often realized as cohomological invariants of algebraic varieties, and good understanding of their geometry sheds light into the original questions. Such techniques have seen immense applications in the Langlands program, where they go under the label of geometric representation theory.

One source of powerful invariants in algebraic geometry comes from algebraic K-theory, Hochschild homology, and their relatives. These localizing invariants contain large amount of information, but are quite hard to compute. For this reason, their usage in geometric representation theory has been limited.

The aim of this thesis is to showcase how to control such invariants in the situations of interest and use them to obtain new insights. We start by reinterpreting equivariant Hochschild homology in terms of functions on certain fixed-point schemes, which are of independent interest. We compare it to equivariant K-theory via the trace map. We give new computations and comparisons of such invariants of affine Schubert varieties in $\mathcal{G}\mathrm{r}$, including arithmetic situations. We show that they behave much better than expected.

We finally utilize this circle of ideas in a purely topological setting. We describe the varying fixed points of the extended torus action on the affine Grassmannian, and use it to compute its equivariant topological K-theory ring. The answer is nontrivial and verifies an outstanding conjecture in the subject.

We compare, partly conjecturally, the resulting K-theory ring to the completed center of an integral even hybrid quantum group and its deformed quantum category $\mathcal{O}$. This gives a genuine application of our computations in pure representation theory.

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About the Author

Jakub Löwit completed his Bachelor's degree in 2019 at Charles University Prague and Master's degree in 2021 at the University of Bonn. He likes to think about the interplay of different areas of pure mathematics, such as algebraic and arithmetic geometry, algebraic topology, and representation theory.

During his PhD, Jakub was awarded the DOC Fellowship of the Austrian Academy of Sciences. He published several papers [Löw25; Löw26a; Löw26b], some of which are a part of this thesis. Apart for shorter seminar invitations, he spent extended periods of time visiting Laboratoire de Mathematiques d'Orsay and Stanford University. After finishing his PhD, Jakub will move to a postdoctoral position at Max Planck Institute for Mathematics in Bonn. Apart from research, he also co-organized the annual preparatory competition CAPS for the math olympiad teams of neighboring countries.

List of Collaborators and Publications

The content of this thesis is due to the author. As listed in the acknowledgements, the author is indebted to many people for ideas, discussions and suggestions regarding this work.

Chapters I and II of this thesis contain, respectively, the following journal publications of the author. Up to minor modification (such as extensions of remarks, uniformizing notation, renumbering for overall coherence), these are included in the published form.

- [Löw26a] Jakub Löwit. "Equivariant K-theory, affine Grassmannian and perfection". In: *Documenta Mathematica* (2026). DOI: 10.4171/DM/1064
- [Löw26b] Jakub Löwit. "Equivariant localizing invariants of simple varieties". In: *International Mathematics Research Notices* 7 (2026), rnag058. DOI: 10.1093/imrn/rnag058

Chapter III is new. Its main focus is to compute the equivariant topological K-theory ring of the affine Grassmannian and outline its representation-theoretical meaning. It is expected to appear as a part of author's joint work with Tamás Hausel, where we relate it to centers of hybrid quantum groups.

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CHAPTER 01

Preface

The aim of this work is to develop methods in equivariant K-theory and employ them in the context of geometric representation theory and Langlands program, in both topological and arithmetic situations. In particular, we study K-theory and related localizing invariants of the affine Grassmannian $\mathcal{G}\mathrm{r}$, a space of utmost interest in this context.

Selected pieces of history

Let us go back and recall some history. This account is not meant to be exhaustive. We emphasize what is relevant for us, and we try to do so from a non-expert viewpoint.

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In classical representation theory, one studies linear representation of groups and related algebras. These naturally appear in many contexts, ranging from number theory to physics. In terms of definitions, they are quite explicit. However, in their study, highly complex combinatorial phenomena arise. These complicated phenomena often lack any apparent structure, which makes their study challenging.

In the course of last century, it was realized that one can often supply this missing structure by looking through the conceptual lens of algebraic geometry. The explicit objects of interest can be often realized as cohomological invariants of algebraic varieties, and good understanding of their geometry sheds light into the original questions. Such techniques have seen immense applications throughout mathematics.

A major playground for such thinking originated from the ideas of Langlands, formulated around 1967. Originally, Langlands found an unexpected connection between representations of arithmetic groups and representations of the corresponding Galois groups. Analogous patterns were found in physics. His philosophy was successfully employed to predict true statements between otherwise unrelated subjects, and turned into a vast program active still today.

Many of Langland's predictions were eventually geometrized and this served as an important driving force in the development of modern algebraic geometry. Except for providing new tools, it resulted in a new interplay of algebraic geometry with classical representation theory. Many representation-theoretical questions may be approached through cohomological methods, and this paradigm goes under the label of geometric representation theory.

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There are much more refined invariants than cohomology. One source of exceptionally powerful invariants in algebraic geometry comes from algebraic K-theory, build out of algebraic vector bundles on the spaces of interest.

Since the invention of K_0 by Grothendieck [Gro57] in 1957, K-theory undertook a long development. It was soon realized that one can extend the theory to higher K-groups and study all of them simultaneously. Throughout the 1960's, it was realized that such a theory would intimately relate to Chow groups and motivic cohomology; several ad-hoc approaches were developed. The study of algebraic K-theory was then set on a firm ground by Quillen throughout the 1970's [Qui73; Qui74; Qui75]. He in particular showed how to control K-theory of smooth varieties through G-theory, its variant built out of coherent sheaves.

Not long afterwards, K-theoretical techniques were successfully used to tackle some open problems in geometric representation theory. For these applications, it was sufficient to use equivariant G-theory and its topological counterparts along the lines of [Seg68]. Most notably, building on earlier ideas of [Lus85; KL85] and Ginzburg, Kazhdan–Lusztig [KL87] realized affine Hecke algebras through a suitable version of equivariant G-theory, whose missing foundations were supplied by [Tho88a; Tho86; Tho88b]. In a similar setup, equivariant G-theory of the affine Grassmannian $\mathcal{G}\mathrm{r}_G$ was related to the affine Hecke algebra in [KK90].

Building on Thomason's foundations, this K-theoretic circle of ideas in representation theory was further refined and nicely exposed in [CG97]. We also refer the reader to the excellent historical overview there. In the representation theory context, the setup from [CG97] became the baseline for further endeavors as well as the canonical reference. In this framework, the difference between G-theory and cohomology is relatively small. The general belief therefore was that considering the latter is usually sufficient – when K-theory was used, then only in the simple setup from [CG97].

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After this early interplay, the two fields have seen rapid independent development.

In algebraic K-theory, structural and computational tools were clearly needed. While the existing theory was well-applicable to smooth varieties, no techniques were available beyond. This was turned around by Thomason in [TT90], where he managed to use Waldhausen's construction to set up K-theory of schemes as an invariant of their categories of perfect complexes. He established general structural properties such as projective bundle formula, proto-localization and so on, applicable in arbitrary singular situations. The same methods directly work in the equivariant case of [Tho88b] and from today's viewpoint are the correct ones for any derived stack.

This opened the door to actual study of K-theory in singular situations beyond ad hoc computations. Nevertheless, K-theory of singular varieties has complicated behavior, often pathological from the topological perspective. Among other things, it may not be homotopy-invariant under taking $\mathbb{A}^1$-bundles. In parallel with [TT90], Weibel [Wei89] proposed a systematic way to enforce such invariance. The resulting homotopy-invariant algebraic K-theory KH turned out to keep many appealing features, without the previous pathologies. A different systematic framework to $\mathbb{A}^1$-invariance was then realized by Voevodsky.

A completely different technique for K-theoretical computations was, in a spoken form, found out by Dennis in 1976 and developed by others afterwards [Wal79; KM00]. Namely, Dennis realized the existence of a trace map $K \to HH$ from K-theory to Hochschild homology. This Dennis trace turned out to be extremely useful – Hochschild homology is fully controlled by algebraic differential forms and often computable. For example, when R is smooth $\mathbb{Q}$-algebra, it is simply the sum of all $\Omega^j_{\mathbb{Q}}R$ by the Hochschild–Kostant-Rosenberg theorem.

Understanding of the Dennis trace then allows to effectively transport information back to K-theory. To approximate K-theory better, refinements of Hochschild homology were introduced [Mad95; McC94; KM00; DGM12; NS18]. Abstracting the common properties of these invariants led to the notion of localizing invariants [BGT13], which can be considered in the same categorical framework as K-theory. Trace maps became one of the major computations tools.

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Meanwhile, representation theory continued to bloom.

Motivated by physics, representations of infinite dimensional Lie groups and Lie algebras were developed in earnest. Extending techniques from the finite dimensional case, the flag varieties of these Kac–Moody algebras became a natural object of study [Kac84; Kum02]. These were further studied from a purely topological viewpoint [PS88]. The most important example is the affine Grassmannian $\mathcal{G}\mathrm{r}_G$, a homogeneous space for the infinite-dimensional loop group $\mathrm{L}G$ of a given reductive group G. Working over $\mathbb{C}$, the homotopy type of $\mathcal{G}\mathrm{r}_G$ is given by the based loop space of the maximal compact subgroup of G, and many cohomological questions can be treated from this perspective alone. Nevertheless, its algebro-geometric realization carries a lot of further information.

Soon afterwards [BL94] realized and employed the relationship of the affine Grassmannian to the moduli stack of principal G-bundles on a curve. Building on this, a systematic geometrization of Langland's ideas was outlined in the manuscript [BD95]. This approach proved to be very powerful and strongly shaped all further developments in the geometric Langlands program. Several important cases of these conjectures were turned into theorems soon afterwards.

The very same ideas of [BD95] then saw a direct application in arithmetic Langlands program via [Ngô08], who managed to reduce conjectures on trace formulas of arithmetic groups to geometric statements about the Hitchin fibration, and consequently about affine Springer fibers inside the affine Grassmannian. This was recently reproved in the more natural, yet technically deeper, group-theoretic setup of the multiplicative Hitchin fibration [Wan24].

The geometrization philosophy of [BD95] was exported to other tiers of the Langland program. The geometric conjecture was recently proved by heavy technical machinery. The p-adic analogues of the affine Grassmannian were introduced using perfect geometry in [Zhu17a; BS17], and eventually led to the geometrization of the local Langlands program [FS21].

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In any case, let us move back to $\mathcal{G}r_G$. As we said, its algebraic geometry contains much more information than its bare homotopy type. One manifestation is the geometric Satake equivalence, originating in [Gin95] and [MV07], which allows to realize representations of a split reductive group G over rather general base rings purely from the geometry of $\mathcal{G}r_G$. While this is an essential input for the geometric Langlands program, it also found direct applications in representation theory.

Many extensions of the geometric Satake isomorphism followed. To emphasize one, let us discuss its quantum aspect. During 1980's, [Dri88; Jim85] realized that classical representation theory of Lie algebras may be deformed via a formal variable q, called the quantum parameter. In particular, deforming the usual universal enveloping algebra U gives rise to the associated quantum group U_q. Throughout the 1990's, quantum groups became a useful tool in classical representation theory [CP94; Jan96; Lus10] and low dimensional topology [RT90].

It was soon realized that in the context of equivariant K-theory, the quantum parameter q matches the equivariant parameter of a specific $\mathbb{G}_m$-action on the varieties of interest. In the context of [KL85], this was already seen in [CG97].

Motivated by [Gin95], Bezrukavnikov–Finkelberg [BFM05; BF08] studied the equivariant cohomology and G-theory of the affine Grassmannian, including the loop rotation actions on it. They obtained a deformed version of the geometric Satake equivalence, relating to representation theory of quantum groups. While this quantization procedure turned into a widely believed principle, its actual scope has not been fully understood.

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The realization of representation theory from geometry of $\mathcal{G}r_G$ is now a standard technique, but not much work has been done beyond cohomology. Moreover, the categorical nature of modern geometric representation theory often asks for interesting decategorification – K-theory and related invariants provide very useful candidates.

This said, let us mention two other recent ventures in such directions, independent of ours. In order to formulate suitable generalizations of the geometric Satake equivalence, a supplement for intersection cohomology is needed. While there is no general approach to "intersection K-theory", one such algebraic notion was tailored in [SW18; SVW18; EK19; Ebe22; ES22; ES23; Ebe24b; Ebe24a; EE24; EE25]. At the same time, from purely topological perspective, generalizations of the geometric Satake equivalence to other cohomology theories were established in [Dev23; Dev24b; Dev24a; Dev25].

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As the above history suggests, while K-theory has substantially evolved, not many of its mature techniques were considered in the context of geometric representation theory. If there is an overall aim of this work, it is to contribute towards bridging this gap and find some new interesting connections.

Aims and structure of this work

This thesis binds together three closely related projects of its author. The projects covered by Chapters I and II have been already published elsewhere. Chapter III appears for the first time.

Although closely related, the three chapters of this thesis can be read largely independently. Each chapter has its own introduction with a detailed discussion of its motivation and results. Notation is also set internally to each chapter in detail.

For convenience, we overview the contents and their relations now.

Chapter I. We first define and develop the notion of fixed-point schemes of group actions on algebraic varieties. We show that this construction outputs varieties of independent interest in geometric representation theory. To give one simple example: when fed with partial flag varieties, it outputs partial multiplicative Springer alterations. Our fixed-point schemes provide a convenient model of the corresponding derived loop space, and we interpret their global functions conceptually via equivariant Hochschild homology.

Furthermore, we study the Dennis trace map from algebraic K-theory to Hochschild homology in this equivariant context. In terms of the fixed-point schemes, it acquires a clear geometric meaning. While the Dennis trace has been extensively used for the study of algebraic K-theory of rings over last fifty years, its ubiquity in the equivariant setup has not been explored. As opposed to the non-equivariant case, the equivariant Dennis trace is highly interesting already in degree zero alone.

To showcase the usefulness of this map, we develop it in the context of perfect algebraic geometry. This is both the natural setup for the Witt-vector affine Grassmannian and also leads to some technical simplifications regarding K-theory. In particular, we study torus-equivariant algebraic K-theory of affine Schubert varieties in the perfect affine Grassmannians of GL_n over $\mathbb{F}_p$. We prove that $\mathbb{F}_p$-linearly, this comparison is an isomorphism. Our approach is constructive, resulting in new computations of these K-theory rings.

We establish various structural results for equivariant perfect algebraic K-theory on the way; we believe these are of independent interest. Among other things, we give a counterexample to a question posed by Adeel Khan using our setup.

As a final application, we consider equivariant K-theory of toric varieties over $\mathbb{F}_p$. Reinterpreting the notion of dilation, we first reprove some results of Cortiñas–Haesemeyer–Walker–Weibel though perfect geometry. More importantly, we deduce a non-vanishing statement on negative equivariant K-theory of singular proper toric varieties. This result is new and contrasts other known computations for toric varieties; our method also gives a clear geometric interpretation.

Chapter II. We then generalize the methods from Chapter I to prove other isomorphism between localizing invariants on the examples of our interest, without perfection and in arbitrary characteristic. More precisely, we define a certain class of *simple varieties* over a field k by a constructive recipe and show how to control their equivariant truncating invariants. This class is set up so that it contains both finite and affine Schubert varieties for GL_n, which are the main examples of our interest.

This has two kinds of applications. On one hand, we can concretely control and compute such truncating invariants, most notably KH. We show that these are equivariantly formal in a rather strong sense – while such formality statements were expected, they were missing in the literature until now. Similarly, we show that for simple varieties over complex numbers, KH matches topological K-theory.

On the other hand, we can establish isomorphisms between different localizing invariants (without really computing them) as long as their difference is truncating – most notably between K-theory and variants of topological cyclic homology. We show that on simple varieties, such comparisons work surprisingly well. For example, we prove a mod p isomorphism between K-theory and topological cyclic homology. Such examples are quite surprising even from the viewpoint of K-theory alone. We also show similar behavior over $\mathbb{Q}$ for the degree zero trace map between rationalized K-theory and negative cyclic homology.

Chapter III. Finally, we discuss some purely topological results and their consequences in representation theory. While we stick to the topological picture, results from Chapter II show that, at least for GL_n, homotopy-invariant K-theory KH would give the same answer.

We first revisit classical equivariant localization techniques in topological K-theory from the viewpoint of the Dennis trace and our fixed-point schemes. While much of the groundwork is laid out elsewhere, we offer a satisfying form of localization, describing fibers of torus-equivariant K-theory as singular cohomology of the fixed points, applicable in large generality.

We then use this to describe the fibers of the equivariant K-theory of $\mathcal{G}\mathrm{r}_G$ over varying points (x, ζ) in the equivariant base $T \times \mathbb{G}_m^{\mathrm{rot}}$ in terms of cohomology of the corresponding fixed point schemes. Extending known results, we compute these fixed points explicitly. In fact, we run the same computations separately for any affine Schubert variety. The resulting fixed-points have an explicit description in terms of closed unions of finite Schubert varieties resp. closed union of Iwahori orbits, both with respect to a suitable resonant Levi subgroup. These refined computations are new and completely match known phenomena in representation theory.

Using the above comparison to cohomology of fixed-points, we then deduce a description of the topological K-theory ring $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C})$ of the affine Grassmannian in terms of certain affine blowups over all roots of unity. This extends and proves a conjecture of Roman Bezrukavnikov. While this K-theory ring is fairly large, we provide a preferred countably infinite set of topological generators. This result clearly illustrates the amount of information in equivariant K-theory, which is lost in cohomology. We expect an analogous picture integrally.

We then move on to compare, partly conjecturally, the resulting ring to the center of an integral even hybrid quantum group and the center of its deformed quantum category $\mathcal{O}$. For example – in the classical case – we get a refined interpretation of the K-theory of each affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ as the algebra of universal central endomorphisms of the tensor product of a Verma module with a finite dimensional irreducible representation.

CHAPTER I

Equivariant K-theory, affine Grassmannian and perfection

1. Introduction

The contents of this chapter cover the results of [Löw26a] Jakub Löwit. "Equivariant K-theory, affine Grassmannian and perfection". In: *Documenta Mathematica* (2026). DOI: 10.4171/DM/1064.

1.1. Motivation and context

1.1.1. Affine Grassmannians and multiplicative Higgs moduli stacks. Motivated by the geometric Langlands program, [Hau22; Hau24] formulated a conjectural picture for mirror symmetry between Higgs moduli spaces for Langlands dual groups. The supports of his mirrors are modelled by families of certain affine Springer fibers. The global functions on these families are conjecturally given by the equivariant cohomology of affine Schubert varieties, the case of partial flag varieties being worked out in [HR25].

This comparison conceptually clarifies if we replace equivariant cohomology with equivariant algebraic K-theory, which relates to functions on families of multiplicative affine Springer fibers, and consequently to multiplicative Higgs moduli spaces [Wan24] by Beauville–Laszlo gluing.

The aim of this chapter is to study these K-groups and their relationship to the rings of functions on multiplicative affine Springer fibers independently in the setup of perfect algebraic geometry. In particular, we prove the expected isomorphism in this setting – see Theorem 1.1.

1.1.2. Equivariant algebraic K-theory and trace maps. Given a group scheme G acting on a qcqs scheme X, we have the equivariant algebraic K-theory spectrum $K^G(X)$ of [Tho88b; TT90] built from the category of G-equivariant perfect complexes on X. Its zeroth homotopy group $K_0^G(X)$ is a commutative ring via tensor product.

Starting from the same datum, we can record the fixed points of each $g \in G$ on X and glue them to a family over G denoted $\mathrm{Fix}_G(X)$, see §2.1 for details. Doing this for the loop group action on the affine Grassmannian returns certain families of multiplicative affine Springer fibers [Yun17, §2.7.3].

There is a natural comparison map from equivariant algebraic K-theory to G-invariant global functions on $\mathrm{Fix}_G(X)$, see §2.4. This is an isomorphism in interesting cases, allowing us to describe such rings by $K_0^G(X)$. Reinterpreting $\mathrm{Fix}_G(X)$ in terms of equivariant Hochschild homology, this is precisely the Dennis trace map [McC94; KM00; DGM12; BGT13; BFN10; Toë14; HSS17; AMR17], which was put to great use in algebraic K-theory and arithmetic geometry over last fifty years – however, its use for equivariant algebraic K-theory of projective varieties has not been explored yet.

For singular schemes, algebraic K-theory is hard to compute. In geometric representation theory, this is mostly bypassed by instead computing algebraic G-theory – algebraic K-theory of coherent sheaves [Tho88b; TT90]. Unfortunately, this simplification discards a lot of important information: for example, G-theory does not have a natural ring structure, making it useless for our purpose.

We believe that it is actually possible to compute equivariant algebraic K-theory – or at least its homotopy-invariant version KH, see [Wei89; Hoy20; Kha22] – for the examples of our interest, and link these computations to geometric representation theory.

1.1.3. Perfect geometry. Algebraic geometry over $\mathbb{F}_p$ has the feature of carrying the Frobenius endomorphism φ and there is a canonical procedure called perfection which turns φ into an automorphism. It was realized in [Zhu17a; BS17] that perfection forces desirable homological properties (such as descent for vector bundles along proper maps); this was in particular used to construct the Witt-vector affine Grassmannian as a perfect ind-scheme.

In fact, perfection simplifies equivariant algebraic K-theory and the above-discussed trace map, while preserving a good amount of interesting information. In the non-equivariant case, the result of perfection on algebraic K-theory is also considered in [KM21; Cou23; AMM22]. It is thus very inviting to study our trace map in the perfect setup: perfect affine Grassmannians provide important examples in their own right, while many technical issues (present in characteristic zero) simplify.

The representation-theoretic phenomena occurring in perfect geometry are also of considerable interest: they encode Frobenius twisting in positive characteristic representation theory. The effect of perfection on reductive group schemes and their representations was studied by [CW24].

1.2. Results

Our main aim in this chapter is to study the trace map (2.11), (2.13) for affine Schubert varieties $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ in the perfect affine Grassmannian $\mathcal{G}\mathrm{r}$ for GL_n over $k = \mathbb{F}_p$ discussed in §6.1. Let T be the perfected diagonal torus of GL_n. We prove the following.

Theorem 1.1 (Theorem 6.5). *Let μ be a dominant coweight of T and $X_{\leqslant\mu}$ the corresponding perfect affine Schubert variety. Then the trace map gives an equivalence*

$$\mathrm{tr} : K^T(\mathcal{G}\mathrm{r}_{\leqslant\mu}; k) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(\mathcal{G}\mathrm{r}_{\leqslant\mu}), \mathcal{O}). \tag{1.1}$$

Moreover:

- *both sides are supported in homological degree zero,*
- *integrally,* $K^T(\mathcal{G}\mathrm{r}_{\leqslant\mu}) \simeq KH^T(\mathcal{G}\mathrm{r}_{\leqslant\mu})$.

The proof of Theorem 1.1 goes as follows.

We first interpret $\mathrm{R\Gamma}(\mathrm{Fix}_{\frac{T}{T}}(\mathcal{G}\mathrm{r}_{\leqslant\mu}), \mathcal{O})$ in terms of equivariant k-linear Hochschild homology $HH^T(\mathcal{G}\mathrm{r}_{\leqslant\mu}/k)$ in the sense of §2.3. This puts both sides of (1.1) on the equal footing of k-linear localizing invariants [BGT13; HSS17; Tam18; LT19]. In doing so, we take the opportunity to discuss the trace map, record its properties, relate it to existing literature and show some examples. Most of this is known to experts, but the literature is a bit sparse: the case of global quotients was not exploited in detail and some of our statements even seem to be new. To give some intuition, we discuss partial flag varieties where the trace map gives an isomorphism in degree zero in Example 2.38. This identifies their K_0^G with global invariant functions on multiplicative Grothendieck–Springer resolutions.

Secondly, we discuss equivariant algebraic K-theory and equivariant Hochschild homology for perfect schemes. After starting this project, we noted that [KM21; AMM22; Cou23] studied non-equivariant algebraic K-theory of perfect schemes and we take advantage of their methods. We establish basic structural properties, noting that both sides of the trace map behave compatibly under perfection. Most notably, we check descent for localizing invariants on T-equivariant perfect abstract blowup squares.

Theorem 1.2 (Theorem 4.11)**.** *Localizing invariants satisfy proper excision on T-equivariant perfect abstract blowup squares. This in particular applies to $K^T(-)$ and $HH^T(-/k)$.*

We further observe that $K_i^G(X)$ are $\mathbb{Z}[\frac{1}{p}]$-modules for $i \geqslant 1$ in Lemma 4.5. We also prove in Observation 4.7 that equivariant K-theory and G-theory agree for perfect schemes – consequently, G-theory of perfect schemes can be non-connective by Note 8.4.

We then deduce Theorem 1.1 inductively on the affine Schubert stratification by descent from partial affine Demazure resolutions, the main input being Theorem 1.2 and semi-orthogonal decompositions for stratified Grassmannian bundles [Jia23].

Finally, it is not only the case that the trace map is an abstract isomorphism, but we can employ Theorem 1.2 to obtain explicit presentations for the rings in question. To showcase this, we compute K_0^T for some singular Schubert varieties in the GL_2 and GL_3 affine Grassmannians – see Examples 7.1, 7.2 for details.

We expect direct analogues for GL_n-equivariant (or even L^+GL_n-equivariant) algebraic K-theory. Also, our computations can be run for equivariant homotopy K-theory KH^{GL_n} in any characteristic (without perfection); see Chapter II covering [Löw26b]. We further believe the rings in question are then given by the base-change of equivariant cohomology along the composed map $H^\bullet_{GL_n}(\mathrm{pt}) \to K_0^{GL_n}(\mathrm{pt}) \to K_\bullet^{GL_n}(\mathrm{pt})$, where the first arrow is given in coordinates as $\mathbb{Z}[t_1, \dots, t_n]^{S_n} \to \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm n}]^{S_n}$. However, we refrain from discussing these questions here.

We close up the chapter with more computations of perfect K-theory. In particular, we use the existing literature together with Theorem 1.2 to understand equivariant K-theory and the trace map for all perfectly proper toric varieties.

Theorem 1.3 (Theorem 8.9). *Let T be a perfect split torus and X any perfectly proper toric variety. Then the trace map induces an equivalence*

$$K^T(X; k) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}).$$

and both sides are supported in homological degrees $\leqslant 0$. Moreover, $K^T(X) \simeq KH^T(X)$.

However, there can be negative degrees for singular X. In particular, we deduce that proper singular toric varieties over k can have nontrivial negative $K^T(-)$ by exhibiting cohomology classes on the right-hand side; this is the case even before perfection – see Note 8.11 for details. We did not find similar computations in the literature.

1.3. Structure

We start in §2 by introducing fixed-point schemes of group actions and their basic properties. We continue by discussing the trace map from equivariant algebraic K-theory to their global functions in elementary terms. We explain the relationship to the existing literature on Hochschild homology and derived loop spaces. The contents of this section work in any characteristic.

In §3 we give a short overview of perfect algebraic geometry, which is the setup for the rest of the chapter. We discuss equivariant K-theory of perfect schemes in §4, explaining some of its main structural properties. In §5 we show how the previous two sections fit together via the trace map, allowing us to control its behaviour.

We focus on perfect affine Grassmannians in §6, proving that the trace map is an isomorphism for the T-equivariant K-theory of affine Schubert varieties in the GL_n-affine Grassmannian – see Theorem 6.5. In §7 we explicitly compute presentations for these rings in some small examples. In fact, these computations work equally well for equivariant homotopy K-theory KH in the classical setup (including characteristic zero).

We conclude with more examples in §8. These should give intuition about perfect K-theory, but are orthogonal to our original motivation. We in particular discuss the trace map for perfect toric varieties in Theorem 8.9.

2. Fixed-point schemes and traces from equivariant K-theory

We work over a fixed base ring k. We denote by Sch_k the category of k-schemes. Let G be a group scheme over k and denote Sch_k^G the category of G-equivariant k-schemes. Unless specified otherwise, quotients are taken as stack quotients in the fpqc topology.

2.1. Fixed-point schemes of group actions

2.1.1. Definition of fixed point families. Suppose G acts on $X \in \mathrm{Sch}_k^G$ via $m : G \times X \to X$. We define the associated *fixed-point scheme* as the fiber product of schemes

$$\begin{tikzcd} \mathrm{Fix}_G(X) \arrow[r] \arrow[d] & G \times X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "m \times \mathrm{pr}_2"] & G \\ X \arrow[r, "\Delta_X"] & X \times X & \end{tikzcd} \tag{2.1}$$

Its functor of points is given by

$$\mathrm{Fix}_G(X) : R \mapsto \{(g, x) \in G(R) \times X(R) \mid gx = x\}.$$

The horizontal composition in (2.1) gives a map $\pi : \mathrm{Fix}_G(X) \to G$. For any $g \in G(k)$, the fiber $\pi^{-1}(g)$ parametrizes those $x \in X(R)$ which are fixed by g. Given any $S \in \mathrm{Sch}_k$ with a map $S \to G$, we obtain

$$\mathrm{Fix}_S(X) := S \underset{G}{\times} \mathrm{Fix}_G \tag{2.2}$$

and view it as the fixed point family of the restriction of the G-action to S.

Example 2.1. For instance, when applied to the full flag variety $X = G/B$ with the obvious left G-action, $\mathrm{Fix}_G(X)$ returns the multiplicative Grothendieck–Springer resolution; the fibers of $\pi : \mathrm{Fix}_G(G/B) \to G$ are the multiplicative Springer fibers. For other parabolic subgroups P, it is a parabolic version thereof. See Example 2.38 for more. These play an important role in geometric representation theory.

2.1.2. Functoriality and equivariant structure. If $X \xrightarrow{f} Y$ is a G-equivariant morphism, the defining diagram of $\mathrm{Fix}_G(X)$ naturally maps to the defining diagram $\mathrm{Fix}_G(Y)$ via f on each occurrence of X and id_G on the group. By the universal property of fiber products, this induces a map

$$\mathrm{Fix}_G(f) : \mathrm{Fix}_G(X) \to \mathrm{Fix}_G(Y),$$

which we also call f by abusing notation.

Now let $g \in G(R)$ and note the following: a point $x \in X(R)$ is fixed by an element $h \in G(R)$ if and only if its translate $gx \in X(R)$ is fixed by $ghg^{-1} \in G(R)$. In other words, $\mathrm{Fix}_G(X)$ carries a functorial G-action

$$\begin{aligned} (\mathrm{Ad}_G \times m) : G \times \mathrm{Fix}_G(X) &\to \mathrm{Fix}_G(X), \\ (g, (h, x)) &\mapsto (ghg^{-1}, gx). \end{aligned}$$

In particular, $\mathrm{Fix}_G(\mathrm{pt}) \in \mathrm{Sch}_k^G$ returns G with the adjoint action $G \times G \xrightarrow{\mathrm{Ad}} G$, $(g, h) \mapsto ghg^{-1}$. We denote the stack quotient by $\frac{G}{G}$. For general $X \in \mathrm{Sch}_k^G$ we denote the stack quotient of $\mathrm{Fix}_G(X)$ by G as $\mathrm{Fix}_{\frac{G}{G}}(X)$. This has a canonical structure map

$$\mathrm{Fix}_{\frac{G}{G}}(X) \xrightarrow{\pi} \frac{G}{G}.$$

Taking quotients of (2.1) further gives the following fiber square of global quotient stacks.

$$\begin{tikzcd} \mathrm{Fix}_{\frac{G}{G}}(X) \arrow[r] \arrow[d] & \frac{G\times X}{G} \arrow[r, "\pi"] \arrow[d, "m\times \mathrm{pr}_2"] & \frac{G}{G} \\ G\backslash X \arrow[r, "\Delta_X"] & \frac{X\times X}{G} & \end{tikzcd} \tag{2.3}$$

As in (2.2), for any algebraic stack S over k with a map $S \to \frac{G}{G}$ we write $\mathrm{Fix}_S(X)$ for the base change of $\mathrm{Fix}_{\frac{G}{G}}(X)$ to S. In particular, base change along the quotient map $G \to \frac{G}{G}$ returns back $\mathrm{Fix}_G(X)$, making our notation consistent.

2.1.3. Steinberg section. Let G be a reductive group over a field and denote $\mathfrak{s} := \frac{G}{G}$ the GIT quotient of G by its adjoint action. There is the canonical affinization map $\frac{G}{G} \to \mathfrak{s}$. Denote further $G^{\mathrm{reg}} \hookrightarrow G$ the regular locus of G consisting of points whose centralizer has the minimal possible dimension (equal to the rank of G) [Hum95, §4]; this is an open subvariety of codimension $\geqslant 2$ by [Hum95, §4.13]. Let $\rho : G^{\mathrm{reg}} \hookrightarrow G \to \mathfrak{s}$ be the natural map. A *Steinberg section* is a section of this map ρ. Such a section is not unique, but always exists at least when G is split, semisimple, simply connected group or GL_n by [Hum95, §4.15].

Via such a Steinberg section, $\mathfrak{s} \hookrightarrow G$ is a closed subscheme intersecting every regular conjugacy class of G in a single geometric point. It is often very useful to restrict attention to $\mathrm{Fix}_{\mathfrak{s}}(X)$. This is a scheme whose global functions often agree with global functions on the stack $\mathrm{Fix}_{\frac{G}{G}}(X)$.

2.1.4. Open and closed immersions.

Proposition 2.2. *Assume that a G-equivariant morphism $f : U \to X$ has one of the following properties. Then the same is true for $f : \mathrm{Fix}_G(U) \to \mathrm{Fix}_G(X)$.*

(i) monomorphism
(ii) open immersion
(iii) closed immersion

In particular, $\mathrm{Fix}_G(-)$ preserves Zariski covers and locally closed stratifications.

Proof. If $f : U \to X$ is a G-equivariant monomorphism, one can immediately check that on functors of points

$$\mathrm{Fix}_G(U)(R) = \{(g,u) \in G(R) \times U(R) \mid gu = u\} = (\mathrm{Fix}_G(X) \times_X U)(R)$$

so that the map on fixed point schemes is also a monomorphism, proving (i). Moreover, the above computation shows that the diagram

$$\begin{tikzcd} \mathrm{Fix}_G(U) \arrow[r] \arrow[d] & \mathrm{Fix}_G(X) \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \tag{2.4}$$

is fibered. To prove (ii), (iii), note that these properties are stable under base change; the result thus follows from the fiber square (2.4); similarly for Zariski covers. The case of locally closed stratifications is now immediate. □

2.1.5. Properness and global functions.

Proposition 2.3. *The fixed point scheme satisfies the following.*

(i) If $X \in \mathrm{Sch}_k$ is separated, then the $\mathrm{Fix}_G(X) \to G \times X$ is a closed immersion.
(ii) If $X \in \mathrm{Sch}_k$ is proper, then $\mathrm{Fix}_G(X) \to G$ is also proper.
(iii) If $X \in \mathrm{Sch}_k$ is proper, then $\mathrm{Fix}_{\frac{G}{G}}(X) \to \frac{G}{G}$ is also proper.

Proof. When X is separated over k, Δ_X is a closed immersion, so the same is true for its base change $\mathrm{Fix}_G(X) \to G \times X$, proving (i). When X is proper over k, then also $G \times X \to G$ is proper. Since X is in particular separated over k, the map $\mathrm{Fix}_G(X) \to G \times X$ is a closed immersion by part (i), so in particular proper. Therefore also the composition $\mathrm{Fix}_G(X) \to G \times X \to G$ is proper, proving (ii). Part (iii) follows from (ii) by fpqc descent for proper maps. □

Notation 2.4. We will use the homological grading on derived global sections $\mathrm{R}\Gamma(X, -)$ of coherent sheaves. We denote it by lower indices as

$$H_i(X, \mathcal{O}) = \mathrm{R}\Gamma_i(X, \mathcal{O}) := \mathrm{R}\Gamma^{-i}(X, \mathcal{O}), \qquad \forall i \in \mathbb{Z}\,.$$

The reason for this notation is compatibility with the homological grading on algebraic K-theory.

Remark 2.5. Given a G-action on a scheme X, we tautologically have

$$H_0(\mathrm{Fix}_{\frac{G}{G}}(X), \mathcal{O}) = H_0(\mathrm{Fix}_G(X), \mathcal{O})^G.$$

We have seen that whenever X is proper, the structure map $\mathrm{Fix}_G(X) \to G$ is proper as well, so the global functions $H_0(\mathrm{Fix}_G(X), \mathcal{O})$ form a finitely generated $H_0(G, \mathcal{O})$-module.

2.2. Comparison to derived loop spaces

2.2.1. Reduced and derived versions. There are the following versions. On the one hand, $\mathrm{Fix}_{\frac{G}{G}}(X)$ will be often non-reduced, and we can take its reduction $\mathrm{Fix}^{\mathrm{red}}_{\frac{G}{G}}(X)$. On the other hand, we may take the defining fiber product (2.1) in derived schemes, giving a derived enhancement which we denote $\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X)$. We get closed immersions

$$\mathrm{Fix}^{\mathrm{red}}_{\frac{G}{G}}(X) \hookrightarrow \mathrm{Fix}_{\frac{G}{G}}(X) \hookrightarrow \mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X). \tag{2.5}$$

2.2.2. Derived loop spaces. The above construction is not new: it is a special case of the construction of derived loop stacks of [BFN10, §2.4], [Toë14, §4.4]; also see [KP20]. This interpretation has useful structural properties, so we spell out the (well-known) relationship.[1]

For any derived stack $\mathcal{X}$ over k, its derived loop space $\mathcal{L}\mathcal{X}$ is defined by the derived fiber product

$$\begin{tikzcd} \mathcal{L}\mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] & \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_k \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ & \mathcal{X} \arrow[r] & \mathrm{pt} \end{tikzcd} \tag{2.6}$$

[1] We thank Quoc Ho for helpful discussions regarding this.

Alternatively, this is the mapping space $\mathcal{L}(\mathcal{X}) = \mathrm{Maps}(S^1, \mathcal{X})$ of derived stacks. In other words, it is the derived inertia stack of $\mathcal{X}$. The same definition can be done in a non-derived way, giving the classical substack $\mathcal{L}(\mathcal{X})^{\mathrm{cl}} \hookrightarrow \mathcal{L}(\mathcal{X})$.

2.2.3. Identification of (derived) fixed-point schemes and (derived) loop spaces. In order to compare our fixed-point schemes to loop spaces, consider the following (non-derived) fiber squares.

$$\begin{tikzcd}
\mathrm{Fix}_{\frac{G}{G}}(X) \arrow[r] \arrow[d] & \frac{G\times X}{G} \arrow[r] \arrow[d, "m\times \mathrm{pr}_2"] & G\backslash X \arrow[d, "\Delta_{G\backslash X}"] \\
G\backslash X \arrow[r, "\Delta_X"] & \frac{X\times X}{G} \arrow[r] & G\backslash X \times G\backslash X
\end{tikzcd} \tag{2.7}$$

The left-hand square was constructed in (2.3). The right-hand square can be easily constructed on the level of functors of points as follows. Let $R \in \mathrm{Sch}_k$ be a test scheme. Then we obtain the groupoids:

$$\begin{aligned}
(G\backslash X)(R) &= \left\{(\mathcal{P}, f) \;\middle|\; \substack{\mathcal{P} \text{ principal } G\text{-bundle on } R, \\ f:\mathcal{P}\to X \ G\text{-equivariant map}}\right\}, \\
\left(\frac{X\times X}{G}\right)(R) &= \left\{(\mathcal{P}, f_1, f_2) \;\middle|\; \substack{\mathcal{P} \text{ principal } G\text{-bundle on } R, \\ f_1,f_2:\mathcal{P}\to X \ G\text{-equivariant maps}}\right\}, \\
(G\backslash X \times G\backslash X)(R) &= \left\{(\mathcal{P}_1, \mathcal{P}_2, f_1, f_2) \;\middle|\; \substack{\mathcal{P}_1,\mathcal{P}_2 \text{ principal } G\text{-bundles on } R, \\ f_1:\mathcal{P}_1\to X,\ f_2:\mathcal{P}_2\to X \ G\text{-equivariant maps}}\right\}, \\
\left(\frac{G\times X}{G}\right)(R) &= \left\{(\mathcal{P}, f_1, f_2) \;\middle|\; \substack{\mathcal{P} \text{ principal } G\text{-bundle on } R, \\ f_1:\mathcal{P}\to G,\ f_2:\mathcal{P}\to X \ G\text{-equivariant maps}}\right\} \\
&= \left\{(\mathcal{P}, f, \psi) \;\middle|\; \substack{\mathcal{P} \text{ principal } G\text{-bundle on } R, \\ \psi:\mathcal{P}\to\mathcal{P} \text{ automorphism}, \\ f:\mathcal{P}\to X \ G\text{-equivariant map}}\right\}.
\end{aligned}$$

Here, the first line is the definition of a stack quotient through its functor of points. The next three lines follow from this definition by elementary manipulations. For the last line, fix a principal G-bundle $\mathcal{P}$ over R. We only need to identify G-equivariant maps $f_1 : \mathcal{P} \to G$ with automorphisms $\psi : \mathcal{P} \to \mathcal{P}$. We construct this bijection on functors of points. Let R' be any test R-algebra and $s \in \mathcal{P}(R')$ any section of $\mathcal{P}$ over R'. In the forward direction, if $f_1 : s \mapsto f_1(s)$, we define $\psi : s \mapsto f_1(s) \cdot s$. In the backward direction, given $\psi : s \mapsto \psi(s)$, we define $f_1(s)$ to be the unique element of $G(R')$ sending s to $\psi(s)$. These maps are inverse bijections, so the final line follows.

Now, this last expression is indeed given by the fiber product of three preceding groupoids along the obvious maps. Altogether, the right-hand square in (2.7) is fibered as claimed.

Finally, let us remark that the horizontal maps in the right-hand square are flat – indeed, they are quotient maps by the group scheme G over k. Hence the right-hand square is fibered in the derived sense as well.

Lemma 2.6. *We have the following isomorphism of stacks:*

$$\mathrm{Fix}_{\frac{G}{G}}(X) = \mathcal{L}(G\backslash X)^{\mathrm{cl}} \qquad \text{and} \qquad \mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X) = \mathcal{L}(G\backslash X).$$

Proof. As both small squares in the (2.7) are fiber squares of classical stacks, the big rectangle is fibered also, proving the first part.

The derived version is analogous. Namely, replace $\mathrm{Fix}_{\frac{G}{G}}(X)$ by $\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X)$ in (2.7). Then the left-hand square is fibered in derived stacks by definition; the same is true for the right-hand square by the non-derived case and flatness of the horizontal maps. We again conclude by the two-out-of three property. Also see [Che20, Proposition 2.1.8] for related discussion (in characteristic zero). $\square$

2.2.4. Compatibility with twisted products.

Lemma 2.7. *Taking $\mathcal{L}(-)$ commutes with all limits in derived stacks. Taking $\mathcal{L}^{\mathrm{cl}}(-)$ commutes with all limits in stacks.*

Proof. This is clear since $\mathcal{L}$ and $\mathcal{L}^{\mathrm{cl}}$ are defined in two steps using limit diagrams (2.6) in derived stacks resp. stacks. $\square$

Setup 2.8. Let X_1 be a G_1-scheme, $\rho_1 : \mathcal{P}_1 \to X_1$ a G_1-equivariant principal G_2-bundle. Let X_2 be a G_2-scheme. Then we have the associated twisted product, which is again a G_1-scheme:

$$X_1 \widetilde{\times} X_2 = \mathcal{P}_1 \overset{G_2}{\times} X_2.$$

This formally amounts to the following fiber product of global quotient stacks:

$$\begin{tikzcd} G_1\backslash \left(X_1 \widetilde{\times} X_2\right) \arrow[r] \arrow[d] & G_2\backslash X_2 \arrow[d] \\ G_1\backslash X_1 \arrow[r, "\mathcal{P}_1"] & G_2\backslash \mathrm{pt} \end{tikzcd} \tag{2.8}$$

The next useful lemma follows easily from the loop space interpretation of fixed-point schemes (but is less obvious from the fixed-point scheme definition).

Lemma 2.9. *The fixed-point stack of the twisted product fits into the fibered square*

$$\begin{tikzcd} \mathrm{Fix}_{\frac{G_1}{G_1}}(X_1 \widetilde{\times} X_2) \arrow[r] \arrow[d] & \mathrm{Fix}_{\frac{G_2}{G_2}}(X_2) \arrow[d, "\pi_2"] \\ \mathrm{Fix}_{\frac{G_1}{G_1}}(X_1) \arrow[r, "\rho_1"] \arrow[d, "\pi_1"] & \frac{G_2}{G_2} \\ \frac{G_1}{G_1} & \end{tikzcd}$$

Proof. Apply $\mathcal{L}(-)$ to (2.8). The resulting square is a pullback by Lemma 2.7. We conclude by Lemma 2.6 together with $\mathrm{Fix}_{\frac{G}{G}}(\mathrm{pt}) = \frac{G}{G}$. $\square$

2.3. Two viewpoints on Hochschild homology

We recall how derived fixed-point schemes relate to equivariant Hochschild homology. This interpretation has useful structural properties such as the projective bundle formula.

We first fix some standard notation. We write $\mathrm{Vect}(-)$ for the category of vector bundles, $\mathrm{QCoh}(-)$ for the category of quasi-coherent sheaves, $\mathrm{Perf}(-)$ for the ∞-category of perfect complexes, $D_{\mathrm{QCoh}}(-)$ for the ∞-category of quasi-coherent sheaves. Given a group scheme G over k, we denote the categories of G-equivariant objects by the superscript $(-)^G$; this matches the corresponding category on the associated quotient stack.

2.3.1. Flawless stacks. We write $\mathrm{Ind}(-)$ for the functor of ind-completion of ∞-categories. The following definition is due to [BFN10, Definition 3.2].

Definition 2.10 (Flawless stacks)**.** A derived algebraic stack $\mathcal{X}$ over k is called *flawless*[2] iff it has affine diagonal and the canonical comparison map gives an equivalence of ∞-categories

$$\mathrm{Ind}(\mathrm{Perf}(\mathcal{X})) \simeq D_{\mathrm{QCoh}}(\mathcal{X}).$$

There are many flawless stacks [BFN10, p.912]. In characteristic zero, a quotient of a quasi-projective derived scheme by an affine group scheme is flawless. In characteristic p, a quotient of a quasi-projective derived scheme by a linearly reductive group (such as $T = \mathbb{G}_m^r$) is flawless.

On the other hand, reductive groups in characteristic p are usually not linearly reductive, and global quotients by such are usually not flawless.

2.3.2. Categorical and geometric models for Hochschild homology. There are the following two models of Hochschild homology over k, defined for any derived stack over k. They agree on flawless stacks. For concreteness, we stick to the case of a global quotient $\mathcal{X} = G\backslash X$.

- Take the small stable k-linear ∞-category of perfect complexes $\mathrm{Perf}(\mathcal{X})$. Taking the geometric realization of the k-linear cyclic nerve produces the algebra object

  $$HH^G(X/k) := HH(\mathcal{X}/k) := HH(\mathrm{Perf}(\mathcal{X})/k) = |N^{\mathrm{cyc}}_{/k}\,\mathrm{Perf}(\mathcal{X})| \in D(k).$$

  See [Hoy18], [Che20, Definition 2.2.18]. This fits into the framework of k-linear localizing invariants of [BGT13; HSS17].
- Take the derived loop space $\mathcal{L}(\mathcal{X}) = \mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X)$ and consider the algebra object given by the global sections of its structure sheaf

  $$\mathrm{R}\Gamma(\mathcal{L}\,\mathcal{X}, \mathcal{O}) = \mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X), \mathcal{O}) \in D(k).$$

Remark 2.11. Recall that Hochschild homology of the small stable k-linear ∞-category $\mathrm{Perf}(\mathcal{X})$ can be equivalently computed as the image of the monoidal identity k in $\mathrm{Perf}(k)$ under the categorical trace in dualizable presentable stable k-linear ∞-categories given by the Ind-completions:

$$\begin{array}{ccccc} \mathrm{Ind}(\mathrm{Perf}(k)) & \xrightarrow{\mathrm{coev}} & \mathrm{Ind}(\mathrm{Perf}(\mathcal{X})) \otimes \mathrm{Ind}(\mathrm{Perf}(\mathcal{X}))^\vee & \xrightarrow{\mathrm{ev}} & \mathrm{Ind}(\mathrm{Perf}(k)) \\ k & \longmapsto & HH(\mathcal{X}/k). & & \end{array}$$

See [Che20, §2.2].

[2]The authors of [BFN10] call this property *perfect* (as it has to do with perfect complexes), but there is an unfortunate clash of terminology with the notion of a *perfect stack* in positive characteristic arithmetic geometry (referring to Frobenius being automorphism). Since both notions appear in this chapter, we invented the word *flawless* to avoid confusion.

Discussion 2.12 (Comparison and agreement)**.** There is a natural comparison map

$$HH(\mathcal{X}/k) \xrightarrow{\text{comp}} \mathrm{R}\Gamma(\mathcal{L}\mathcal{X}, \mathcal{O}) \tag{2.9}$$

which is an equivalence when $\mathcal{X}$ is flawless.

The existence of the comparison map $\varkappa$ follows from [HSS17, Theorem 6.5] – to bootstrap this from their finer results, one needs to compose with the map from global sections of their Tannakian loop space to the global sections of the derived loop space, and furthermore with the inclusion of homotopy fixed-points to all functions.

If $\mathcal{X}$ is a flawless derived algebraic stack over k, the comparison map $\varkappa$ is an equivalence by the discussion [Che20, Example 2.2.20]. Strictly speaking, [Che20] works in characteristic zero, but the only input is [BFN10, Theorem 1.2] valid over any base ring k. (Also note that the case of qcqs k-schemes is classical: it follows by Zariski descent from the case of k-algebras.)

For reader's convenience, we recall the arguments from [Che20; BFN10] in more detail. Assume $\mathcal{X}$ is flawless. By Remark 2.11, we can then compute $HH(\mathcal{X}/k)$ as the image of the monoidal identity under

$$D_{\mathrm{QCoh}}(k) \xrightarrow{\mathrm{coev}} D_{\mathrm{QCoh}}(\mathcal{X}) \otimes D_{\mathrm{QCoh}}(\mathcal{X})^\vee \xrightarrow{\mathrm{ev}} D_{\mathrm{QCoh}}(k). \tag{2.10}$$

Using [BFN10, Theorem 4.7 and Corollary 4.8], we can identify $D_{\mathrm{QCoh}}(\mathcal{X}) \otimes D_{\mathrm{QCoh}}(\mathcal{X})^\vee$ with $D_{\mathrm{QCoh}}(X \times X)$. Under this identification, coevaluation and evaluation are given by $\mathrm{coev} = R\Delta_* \circ Lp^*$ and $\mathrm{ev} = Rp_* \circ L\Delta^*$ along the correspondences

$$\begin{tikzcd}
& & \mathcal{L}(\mathcal{X}) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & & \\
& \mathcal{X} \arrow[dl, "p"'] \arrow[dr, "\Delta"'] & & \mathcal{X} \arrow[dl, "\Delta"] \arrow[dr, "p"] & \\
\mathrm{pt} & & \mathcal{X} \times \mathcal{X} & & \mathrm{pt}\,.
\end{tikzcd}$$

Since all terms in this diagram are flawless by [BFN10, §3.3], we can use derived base-change [BFN10, Proposition 3.10] in the middle square to get the desired identification

$$HH(\mathcal{X}/k) = \mathrm{coev} \circ \mathrm{ev}(k) = Rp_* \circ L\Delta^* \circ R\Delta_* \circ Lp^*(k) = R(pq_2)_* \circ L(pq_1)^*(k) = \mathrm{R}\Gamma(\mathcal{L}(\mathcal{X}), \mathcal{O}).$$

Question 2.13. Let $G = GL_n$ over $k = \mathbb{F}_p$. Take $X \in \mathrm{Sch}_k^G$. The stacks $\mathcal{X} = G \backslash X$ are usually not flawless. For example, taking $X = \mathrm{pt}$, the classifying stack $BG = G \backslash \mathrm{pt}$ is not flawless: its structure sheaf is perfect but not compact. Is the comparison map (2.9) still an isomorphism?

To the best of our knowledge, this is an open question. The standard approach to such comparison is built on flawlessness, which does not appear to be the optimal assumption. See also [Che20, Remark 2.2.21] for similar questions in characteristic zero.

2.4. Trace maps from equivariant K-theory

We now recall the trace map from equivariant algebraic K-theory to functions on the fixed-point scheme and its relationship to the Dennis trace map.

2.4.1. Conventions for equivariant K-theory. Let $\mathrm{Sch}_k^{\mathrm{qcqs}}$ be the category of quasi-compact quasi-separated k-schemes.

Notation 2.14. Given $X \in \mathrm{Sch}_k^{\mathrm{qcqs}}$, we denote by $K(X)$ the non-connective algebraic K-theory spectrum of the category of perfect complexes $\mathrm{Perf}(X)$ on X in the sense of [TT90]. We denote $K_i(X)$, $i \in \mathbb{Z}$ its homotopy groups.

Let G be an affine algebraic group over k. If $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$, we denote $K^G(X)$ its equivariant algebraic K-theory spectrum [Tho88b]. This is the algebraic K-theory spectrum of the category $\mathrm{Perf}^G(X)$ of G-equivariant perfect complexes on X. Equivalently, it is the K-theory spectrum of perfect complexes $\mathrm{Perf}(G\backslash X)$ on the global quotient stack $G\backslash X$. We denote by $K_i^G(X)$, $i \in \mathbb{Z}$ its homotopy groups.

Notation 2.15. We write $G(X)$ for the algebraic K-theory spectrum of cohomologically bounded pseudo-coherent complexes on X [TT90, Definition 3.3], similarly $G^G(X)$ for the equivariant version [Tho88b].

Notation 2.16. We denote $KH(X)$ the homotopy K-theory spectrum in the sense of [Wei89] and $KH^G(X)$ its equivariant version defined via the simplicial spectrum $K^G(\Delta^\bullet \times X)$.

Notation 2.17. We write $\otimes_R$ for tensor product over R. We denote $K_i(X)_k := K_i(X) \otimes_{\mathbb{Z}} k$. We write $\otimes_R^{\mathbf{L}}$ for derived tensor product over R and $\otimes_R^i := \mathrm{Tor}_i^R(-,-)$ for the Tor-groups.

Given a coefficient ring Λ, we denote $K(X;\Lambda)$ the K-theory spectrum with coefficients in Λ. Taking $\Lambda = \mathbb{F}_p$, this has homotopy groups

$$K_i(X;\mathbb{F}_p) \cong \left(K_i(X) \otimes_{\mathbb{Z}}^0 \mathbb{F}_p\right) \oplus \left(K_{i-1}(X) \otimes_{\mathbb{Z}}^1 \mathbb{F}_p\right).$$

2.4.2. The equivariant trace map in degree zero. Let $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$ and $\mathcal{E} \in \mathrm{Vect}^G(X)$ be a G-equivariant vector bundle on X. Let $x \in \mathrm{Fix}_g(X)(k)$ be a fixed point of a given $g \in G(k)$. Then g acts on the fiber $\mathcal{E}_x$ and we may take the trace $\mathrm{tr}_g(\mathcal{E}_x) \in k$. This idea plays out as follows.

Construction 2.18. Let $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$. Then we define the ring homomorphism

$$\begin{aligned} \mathrm{tr} : K_0^G(X) &\to H_0(\mathrm{Fix}_{\frac{G}{G}}(X), \mathcal{O}) \\ [\mathcal{E}] &\mapsto ((g,x) \mapsto \mathrm{tr}_g(\mathcal{E}_x)) \end{aligned} \tag{2.11}$$

Indeed, let $\mathcal{E} \in \mathrm{Perf}^G(X)$. Pick an affine test scheme $\mathrm{Spec}\, R \in \mathrm{Sch}_k$ and let $(g,x) \in \mathrm{Fix}_G(X)(R)$ be an R-valued point. Then the pullback $\mathcal{E}_x \in \mathrm{Perf}(R)$ is strictly perfect – it is represented by a bounded complex of projective R-modules acted on by g. Passing to an affine Zariski cover of $\mathrm{Spec}\, R$, we may even assume all of its terms are free. Then one can take the usual trace of g on this complex

$$\mathrm{tr}_{(g,x)} \mathcal{E} := \sum_i (-1)^i \, \mathrm{tr}_g \, \mathcal{E}_{x,i}$$

and obtain a global function on $\mathrm{Spec}\, R$.

This association is clearly additive: given a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ in $\mathrm{Perf}^G(X)$, one sees $\mathrm{tr}_{(g,x)} \mathcal{E} = \mathrm{tr}_{(g,x)} \mathcal{E}' + \mathrm{tr}_{(g,x)} \mathcal{E}''$. Moreover, if $h : \mathcal{E}' \to \mathcal{E}$ is a quasi-isomorphism in $\mathrm{Perf}^G(X)$, then $\mathrm{tr}_{(g,x)} \mathcal{E}' = \mathrm{tr}_{(g,x)} \mathcal{E}$. Indeed, the mapping cone $\mathcal{E}'' :=$

$\mathrm{cone}(h) \in \mathrm{Perf}^G(X)$ of h is acyclic, hence it suffices to see that its trace is zero by additivity. This can be easily seen by induction on the degrees in which $\mathcal{E}''$ is nonzero by splitting off the top degree: one can always produce a short exact sequence $0 \to \mathcal{G}' \to \mathcal{E}'' \to \mathcal{G}'' \to 0$ in $\mathrm{Perf}^G(X)$ where $\mathcal{G}'$ is supported in a shorter interval and $\mathcal{G}''$ is of the form $(\cdots \to 0 \to R^{\oplus a} \xrightarrow{=} R^{\oplus a} \to 0 \dots)$.

Altogether, this assembles into the trace map (2.11): we land in algebraic functions, as we can test on arbitrary R-points of $\mathrm{Fix}_G(X)$. The resulting function is also G-invariant, as taking traces is invariant under the adjoint action of G on itself. Finally, the trace factors through $K_0^G(X)$ by [TT90, §1.5.6] and the above.

Observation 2.19. *The trace map* tr *is a homomorphism of rings.*

Proof. The trace of a tensor product is the product of traces of factors. □

This observation is essential for our applications: we can describe rings of functions on interesting spaces via equivariant K-theory. Moreover, it explains why we need equivariant K-theory (and not G-theory, which doesn't have a natural ring structure). In fact, the above construction of the trace map would not even work for G-theory.

Remark 2.20. Since X is defined over k, the trace extends to

$$\mathrm{tr} : K_0^G(X)_k \to H_0(\mathrm{Fix}_{\frac{G}{G}}(X), \mathcal{O}).$$

See Discussion 2.23 for a finer version. Reformulating, the trace map amounts to a morphism

$$\mathrm{tr}_X : \mathrm{Fix}_{\frac{G}{G}}(X) \to \mathrm{Spec}\, K_0^G(X)_k \tag{2.12}$$

which factors through the affinization of the left-hand side.

2.4.3. The Dennis trace map. The above-discussed degree zero definition lifts to the level of spectra, and even factors through finer versions of Hochschild homology. This is a classical construction in algebraic K-theory. For explicit formulas, see [McC94; KM00] or the encyclopedic treatments [Mad95; DGM12]. Also see [BGT13] for a modern approach via localizing invariants. It was further interpreted geometrically in terms of derived loop stacks [BFN10; Toë14; HSS17; AMR17]. We recall the simplest version sufficient for our purposes.

Recollection 2.21. Let $\mathrm{Cat}_k^{\mathrm{ex}}$ be the ∞-category of small, stable, idempotent complete, k-linear ∞-categories and exact functors. We will use the notion of localizing invariants $E(-)$ on $\mathrm{Cat}_k^{\mathrm{ex}}$ in the sense of [HSS17, §4–5]. The following are notable examples relevant for us.

(i) non-connective algebraic K-theory $K(-)$ and its version with coefficients,
(ii) homotopy K-theory $KH(-)$ and its version with coefficients,
(iii) Hochschild homology $HH(-/k)$ relative to k.

In fact, $K(-)$ is a universal localizing invariant, see [HSS17; BGT13]. The case of $KH(-)$ follows formally, see [LT19]. The versions with coefficients are also clear. Finally, $HH(-/k)$ is a localizing invariant on $\mathrm{Cat}_k^{\mathrm{ex}}$ in the sense of [HSS17] by the discussion in [LT19, p.914 after 3.8].

Notation 2.22. We evaluate localizing invariants on derived stacks through their ∞-categories of perfect complexes $\mathrm{Perf}(-)$: given a derived stack $\mathcal{X}$ over k and a localizing invariant E, we denote

$$E(\mathcal{X}) := E(\mathrm{Perf}(\mathcal{X})).$$

Given $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$, we denote $E^G(X) := E(G\backslash X) = E(\mathrm{Perf}(G\backslash X)) = E(\mathrm{Perf}^G(X))$.

Discussion 2.23 (Traces)**.** There are the following trace maps:

$$\begin{tikzcd} K^G(X;k) \arrow[r, "\mathrm{tr}_{\mathrm{cat}}"'] \arrow[rr, bend left, "\mathrm{tr}_{\mathsf{geom}}"] & HH^G(X/k) \arrow[r, "\mathrm{comp}"'] & \mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X), \mathcal{O}). \end{tikzcd} \tag{2.13}$$

The categorical trace $\mathrm{tr}_{\mathrm{cat}}$ exists as $K^G(X;k)$ and $HH^G(X/k)$ fit in the framework of k-linear localizing invariants of [BGT13; HSS17]. The geometric trace $\mathrm{tr}_{\mathsf{geom}} = \mathrm{tr}_{\mathsf{geom}} \circ \mathrm{comp}$ extends the degree zero definition presented above.

Remark 2.24. This gives a natural map $K_0^G(X;k) \to H_0(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}, \mathcal{O})$ extending Remark 2.20.

Recollection 2.25. Localizing invariants are in particular additive [HSS17, Definitions 5.11 and 5.16] or [Kha20, Definition 2.6]. They send semi-orthogonal decompositions to direct sums by [Kha20, Lemma 2.8].

In particular, this is the case for $K(-)$, $KH(-)$, $K(-;k)$ and $HH(-/k)$; such direct sum decompositions are compatible with the trace map.

2.4.4. Local constancy along fibers. The T-equivariant trace map from Construction 2.18 lands in functions which are locally constant along fibers. Therefore, it can be close to isomorphism only in situations when all function of $\mathrm{Fix}_{\frac{T}{T}}(X)$ have this property, e.g. when X is proper. We include this observation for context.

Notation 2.26. Let S be a base scheme and $Z \in \mathrm{Sch}_S$ with structure map $\pi : Z \to S$. A function $f \in H_0(Z, \mathcal{O})$ is called *locally constant relatively to* S if it Zariski locally on Z comes as a pullback of a function on S along π.

Lemma 2.27. *Let k be a base field, T a torus, $X \in \mathrm{Sch}_k^{\mathrm{qcqs},T}$. Then the image of* $\mathrm{tr} : K_0^T(X)_k \to H_0(\mathrm{Fix}_T(X), \mathcal{O})$ *lands in functions which are locally constant relatively to T.*

Proof. Let $\mathcal{E} \in \mathrm{Perf}^T(X)$ be a perfect complex and $\mathrm{tr}\,\mathcal{E} \in H_0(\mathrm{Fix}_T(X), \mathcal{O})$ the corresponding function. We only need to prove the claim Zariski locally on $\mathrm{Fix}_T(X)$. Fix a test algebra $R \in \mathrm{Alg}_k$ with $S = \mathrm{Spec}\, R$ and let $(t, x) \in \mathrm{Fix}_T(X)(R)$. Such (t, x) form a basis of the Zariski topology on $\mathrm{Fix}_T(X)$, so it is enough to see that $\mathrm{tr}_t(\mathcal{E})$ is a pullback of a function along the projection $\mathrm{Fix}_T(X) \to T$.

$$\begin{tikzcd} S \arrow[r, "{(t,x)}"] \arrow[ddr, "t"'] & \mathrm{Fix}_T(X) \arrow[d] \\ & T \times X \arrow[d] \\ & T \end{tikzcd}$$

But $\mathcal{E}_{(t,x)}$ is a T-equivariant perfect complex on S and we may without loss of generality assume it is a vector bundle. Restricting to a single connected component of S, we may further assume that $\mathcal{E}_{(t,x)}$ has constant rank. Equivalently, it is a $\mathbb{Z}^d$-graded projective module $M = \bigoplus_{i \in \mathbb{Z}^d} M_i \cdot t^i$ with each M_i projective of constant rank. After choosing an isomorphism $T \cong \mathbb{G}_m^d$, the map t identifies with a d-tuple $(r_1, \ldots, r_d)$ of invertible elements in R. With this notation, the trace of M at the given $t = (r_1, \ldots, r_d) \in T(R)$ is given by $\sum_{i \in \mathbb{Z}^d} (\operatorname{rank} M_i) \cdot r^i$ where $r^i := r_1^{i_1} \cdots r_d^{i_d}$. Hence the function $\operatorname{tr}_t \mathcal{E}_x$ on S indeed comes as a pullback along $t : S \to T$. Since this holds for any (t, x), we have proved the lemma for T. □

Corollary 2.28. *For a reductive group G and $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$, the image of $\operatorname{tr} : K_0^G(X)_k \to H_0(\operatorname{Fix}_G^{\mathrm{red}}(X), \mathcal{O})$ lands in functions which are locally constant relatively to G.*

Proof. Let $\mathcal{E} \in \operatorname{Perf}^G(X)$ and pick any k-valued point (g, x) of $\operatorname{Fix}_G(X)$. Then $\mathcal{E}_{(g,x)}$ is equivariant for the whole commutative subgroup scheme C_g of G over k generated by g. This sits in a short exact sequence between its unipotent radical and maximal torus:

$$1 \to U_g \to C_g \to T_g \to 1$$

The trace does not depend on the unipotent part of g, while the torus case was handled above. The lemma follows. □

2.5. Projective bundle formulas

We now recall the projective bundle formula and its more elaborate variants. This is essential for controlling algebraic K-theory. All of these results ultimately rely on suitable semi-orthogonal decompositions.

2.5.1. Projective bundle formula and consequences. Since both K-theory and Hochschild homology are localizing invariants, they automatically satisfy the equivariant projective bundle formula going back to [TT90; Tho88b].

Proposition 2.29. *Let G act on X. Let $\mathcal{E} \in \operatorname{Vect}^G(X)$ be a G-equivariant vector bundle of rank r and $\mathbb{P}(\mathcal{E})$ its projectivization. Let $E(-)$ be a localizing invariant. Then $E^G(-)$ satisfies the projective bundle formula – there is a natural equivalence*

$$E^G(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{i=0}^{r-1} E^G(X) \tag{2.14}$$

compatible with natural maps between localizing invariants.

Proof. In the view of Recollection 2.25, localizing invariants satisfy the projective bundle formula by the semi-orthogonal decomposition [Kha20, Theorem 3.3 and Corollary 3.6]. □

One consequence is the well-known splitting principle.

Lemma 2.30. *Let $c \in K_0^G(X)$ be the class of a G-equivariant vector bundle $\mathcal{E}$ on X of rank n. Then there exists a G-equivariant morphism $f : Y \to X$ such that $f^* : K_0^G(X) \to K_0^G(Y)$ is injective and $f^*[E] = \sum_i^n [\mathcal{L}_i]$ for some G-equivariant line bundles $\mathcal{L}_i$ on Y. Similarly for $K_0^G(-; k)$.*

Proof. Such Y can be constructed by induction on n. Let $Y_0 := X$, $E_0 := E$. Consider $Y_1 = \mathbb{P}(E_0)$ with its canonical map $f_1 : Y_1 \to Y_0$. The tautological line bundle $\mathcal{L}_1$ on Y_1 is a G-equivariant subbundle of $f_1^*E_0$. The quotient $E_1 := f^*E_0/\mathcal{L}_1$ is a rank $(n-1)$ bundle on Y_1 and we have $f_1^*[E] = [\mathcal{L}_1] + [E_1]$. Moreover, the pullback $f^* : K_0^G(Y_0)_k \to K_0^G(Y_1)_k$ is injective by the projective bundle theorem. We can continue by induction. □

The following standard trick extends the projective bundle formula to flag varieties (alternatively, one can deduce this from a suitable semi-orthogonal decomposition).

Discussion 2.31. Take $d \in \mathbb{N}$, let μ be a dominant cocharacter of the diagonal torus $T \leqslant GL_d$ and consider the corresponding Levi subgroup L_μ with block sizes $(d_1, \dots, d_l)$.

Let $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$ and $\mathcal{E} \in \mathrm{Vect}^G(X)$. Let $Y = \mathrm{Flag}_X(\mathcal{E}, \mu)$ be the associated partial flag variety bundle of type μ over X, whose fibers parametrize flags of vector bundle quotients of $\mathcal{E}$ with successive graded pieces of dimensions $(d_1, \dots, d_{l-1})$ – for example, if $\mu = \omega_j$ is the j-th fundamental coweight, then $\mathrm{Flag}_X(\mathcal{E}, \omega_j) = \mathrm{Grass}_X(\mathcal{E}, j)$ is the relative Grassmannian of rank j quotients of $\mathcal{E}$. Take $W = \mathrm{Flag}_X(\mathcal{E})$ to be the corresponding full flag variety bundle. Then we have natural maps

$$\begin{tikzcd} W \arrow[r, "g"'] \arrow[rr, bend left, "h"] & Y \arrow[r, "f"'] & X \end{tikzcd} \tag{2.15}$$

and both g and h factor as towers of equivariant projective bundles.

Corollary 2.32. *Let $X \in \mathrm{Sch}_k^{\mathrm{qcqs},G}$ and $\mathcal{E} \in \mathrm{Vect}^G(X)$. Let $Y = \mathrm{Flag}_X(\mathcal{E}, \mu)$ be the associated partial flag variety bundle. Let $E(-)$, $E'(-)$ be localizing invariants and $\tau : E(-) \to E'(-)$ a natural map between them. Pick $i \in \mathbb{Z}$. Then the following are equivalent:*

(i) $\tau_i : E_i^G(X) \to E_i'^G(X)$ *is an isomorphism,*
(ii) $\tau_i : E_i^G(Y) \to E_i'^G(Y)$ *is an isomorphism.*

Proof. For projective bundles, this follows immediately from (2.14). Applying the case of projective bundles multiple times for the bundles appearing in the factorizations of g and h from (2.15), we deduce the result for f. □

Remark 2.33. We will later apply Corollary 2.32 to

- the trace map $\mathrm{tr} : K^G(-;k) \to HH^G(-/k)$,
- the natural map $K^G(-) \to KH^G(-)$.

To phrase the first case in words: the trace map $\mathrm{tr} : K^G(-;k) \to HH^G(-/k)$ is an isomorphism for X if and only if it is an isomorphism for Y.

2.5.2. Stratified Grassmannian bundles. We will employ a similar statement for *stratified* projective bundles. To make this work, we need to use their derived enhancements studied in [Jia25; Jia22; Jia23].[3]

Recollection 2.34. Let $\mathcal{E} \in \mathrm{Perf}^G(X)$ be a perfect complex supported in homological degrees $\geqslant 0$. Denote by $\mathcal{H}_0(\mathcal{E}) \in \mathrm{QCoh}^G(X)$ its zero homology group. Consider the relative derived Grassmannian of rank j of $\mathcal{E}$ on X in the sense of [Jia22, Definition 4.3], denoted

$$Y = \mathrm{Grass}_X(\mathcal{E}, j) \xrightarrow{f} X. \tag{2.16}$$

[3]We thank Andres Fernandez Herrero for pointing out the relevance of these results to us.

This is a G-equivariant derived scheme. By [Jia22, Proposition 4.7], its classical truncation is the usual relative Grassmannian

$$Y^{\mathrm{cl}} = \mathrm{Grass}_X(\mathcal{H}_0(\mathcal{E}), j) \xrightarrow{f^{\mathrm{cl}}} X. \tag{2.17}$$

Assume further that $\mathcal{E}$ has Tor-amplitude $[1,0]$ and $j = 1$. The category $\mathrm{Perf}^G(Y)$ then has an explicit semi-orthogonal decomposition in terms of a flattening stratification for $\mathcal{E}$ by [Jia23, Theorem 3.2 and Remark 1.1]. In particular, $\mathrm{Perf}^G(X)$ is a semi-orthogonal summand of $\mathrm{Perf}^G(Y)$ via f^*.

In fact, we only need the following consequence.

Proposition 2.35. *Let $\mathcal{E} \in \mathrm{Perf}^G(X)$ of Tor-amplitude $[1,0]$. Let $Y = \mathrm{Grass}_X(\mathcal{E},1) \to X$ be the associated relative derived Grassmannian of rank 1. Let $E(-)$ be a localizing invariant. Then $E^G(X)$ is a direct summand of $E^G(Y)$, compatibly with natural maps of localizing invariants.*

Proof. Follows from Recollection 2.34 and Recollection 2.25. ☐

In particular, $K^G(X;k) \xrightarrow{\mathrm{tr}_X} HH^G(X/k)$ is a direct summand of $K^G(Y;k) \xrightarrow{\mathrm{tr}_Y} HH^G(Y/k)$. Similarly for the comparison map from K^G to KH^G.

2.6. First examples

The trace map is an isomorphism in degree zero for interesting varieties arising in geometric representation theory. Let us give the most basic examples of this phenomenon: first for the point and then for partial flag varieties. An analogous trace map for equivariant cohomology in these examples (and beyond) was studied in [HR25], which was an important motivation for us.

2.6.1. Equivariant point. Let G be a connected split reductive group of rank n over a field k and T its maximal torus. Let W be the Weyl group. We start by noting how the above works out for the equivariant point.

Example 2.36 (Torus-equivariant point)**.** Let $X = \mathrm{pt}$ with the trivial T-action. Then the k-linearized trace map gives an isomorphism

$$\mathrm{tr}_X : K_0^T(\mathrm{pt};k) \xrightarrow{\cong} H_0(\mathrm{Fix}_{\frac{T}{T}}(\mathrm{pt}), \mathcal{O}).$$

Both sides are given by the localized polynomial ring $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Proof. Follows from the identification of the left-hand side with the k-linearized representation ring $R(T;k)$ and the right-hand side with $H_0(\frac{T}{T}, \mathcal{O}) = H_0(T, \mathcal{O})$. The trace map is then given by taking characters of representations, giving the desired isomorphism. ☐

Example 2.37 (Equivariant point)**.** Let $X = \mathrm{pt}$ with the trivial G-action. Then the k-linearized trace map gives an isomorphism

$$\mathrm{tr}_X : K_0^G(\mathrm{pt}; k) \xrightarrow{\cong} H_0(\mathrm{Fix}_{\frac{G}{G}}(\mathrm{pt}), \mathcal{O}).$$

Both sides are given by the localized polynomial ring $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^W$, whose spectrum naturally identifies with $\mathfrak{s} = \frac{G}{G}$.

Proof. Under the identification of the left-hand side with the k-linearized representation ring $R(G;k)$ of G and the right-hand side with the ring of conjugacy-invariant functions on G, this is the classical isomorphism [Hum95, §3.2], [Mil17, §22.a–22.b] valid in any characteristic. □

2.6.2. Partial flag varieties. We illustrate the above notions for partial flag varieties. For the sake of clarity, we stick to the case $k = \mathbb{C}$ and $G = GL_n$. Given a dominant coweight μ of the diagonal torus T of G, we write $P = P_\mu$ for the associated parabolic subgroup of G and W_μ the corresponding Weyl group. Let $X = G/P$ be the quotient of G by P equipped with the left translation by G. Recall from Example 2.1 that $\mathrm{Fix}_G(X)$ amounts to the multiplicative Grothendieck–Springer resolution.

We now show that the degree zero trace map is an isomorphism in this case, realizing (reduced) rings of invariant functions on multiplicative Grothendieck–Springer resolutions via equivariant algebraic K-theory. In fact, the derived and infinitesimal structures of the fixed-point scheme are irrelevant.

Example 2.38 (Partial flag variety)**.** Take $k = \mathbb{C}$ and consider $X = G/P$ with G acting by left translations. Then we get isomorphisms

$$\mathrm{tr}_X\colon K_0^G(X;k) \xrightarrow{\cong} H_0(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X),\mathcal{O}) \xrightarrow{\cong} H_0(\mathrm{Fix}_{\frac{G}{G}}(X),\mathcal{O}) \xrightarrow{\cong} H_0(\mathrm{Fix}^{\mathrm{red}}_{\frac{G}{G}}(X),\mathcal{O}) \xrightarrow{\cong} H_0(\mathrm{Fix}^{\mathrm{red}}_{\mathfrak{s}}(X),\mathcal{O})$$

and these are explicitly given by the polynomial ring $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{W_\mu}$.

Proof. Since the stacks in question are flawless, we can interpret $H_0(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X), \mathcal{O})$ as $\mathbb{C}$-linear Hochschild homology. The trace map is an isomorphism in degree 0 for the point by Example 2.37, so the first map is an isomorphism by the projective bundle formula from Proposition 2.29.

Further note that

$$\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X) = \mathcal{L}(G\backslash(G/P)) = \mathcal{L}(\mathrm{pt}/P) = \tfrac{P}{P}$$

which is already classical and reduced. Hence $\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X) = \mathrm{Fix}_{\frac{G}{G}}(X) = \mathrm{Fix}^{\mathrm{red}}_{\frac{G}{G}}(X)$ giving the next two isomorphisms. See also [BN13, §3.1] for related discussion; the classicality can be alternatively deduced through the criterion [BN13, Proposition 3.3].

Now note that

$$K_0^G(G/P) = K_0(G\backslash G/P) = K_0(\mathrm{pt}/P) = R(P) = R(L).$$

is the representation ring of the Levi quotient $L = L_\mu$ of P. This indeed matches the ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{W_\mu}$. Thus $K_0^G(X;k) = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{W_\mu}$, giving the desired presentation. In particular, this ring is reduced and normal.

For the final term, we argue by classical algebraic geometry. Over a geometric point $s \in \mathfrak{s}$, the reduced fiber of $K_0^G(X;k)$ consists of finitely many points labelled by the set Eig_s^μ of μ-partitions of the multiset of roots of s (regarded as a symmetric polynomial). Similarly, the reduced fiber of $\mathrm{Fix}_{\mathfrak{s}}(X)$ over s consists of finitely many points labelled by the same set Eig_s^μ, as these are the fixed points of (any) regular lift $g_s \in G$ of s. Now, $\mathrm{tr}_{\mathfrak{s}}^{\mathrm{red}} : \mathrm{Fix}_{\mathfrak{s}}^{\mathrm{red}}(X) \to \operatorname{Spec} K_0^G(X)$ is a well-defined map of (affine) schemes over $\mathfrak{s}$. It induces a bijection on geometric points, as this is the case fiberwise over $\mathfrak{s}$. Since both sides are reduced and the target is normal (it is the spectrum of a localized polynomial ring), we deduce that it is an isomorphism. Hence the whole composition in the statement is an isomorphism, finishing the proof.

The isomorphism between the last two terms can be alternatively deduced as follows. Note $\mathfrak{s}$ intersects all regular conjugacy classes in G and $G\backslash G^{\mathrm{reg}}$ has codimension at least 2. We deduce that the restriction map $H_0(\mathrm{Fix}_{\frac{G}{G}}^{\mathrm{red}}(X), \mathcal{O}) \to H_0(\mathrm{Fix}_{\mathfrak{s}}^{\mathrm{red}}(X), \mathcal{O})$ is injective. It is also surjective by the above, hence an isomorphism. (In particular, one can proceed this way to prove everything without appeal to semiorthogonal decompositions.) $\square$

2.7. Equivariant KH and proper excision

We now recall the $\mathbb{A}^1$-homotopy invariant version of algebraic K-theory KH in the equivariant setup. This in particular satisfies *proper excision*, which is one of our main computational tools later.

Let us recall from Notation 2.16 that $KH^G(X)$ is defined via the simplicial spectrum $K^G(\Delta^\bullet \times X)$. Note that KH is a localizing invariant on $\mathrm{Cat}_k^{\mathrm{ex}}$ by Recollection 2.21 and we have a natural map $K(-) \to KH(-)$.

Let T be a torus over k. Then $KH^T(-)$ agrees with other commonly used definitions by [Hoy20, Theorem 3.1.(4)]. Moreover, $KH^T(-)$ is $\mathbb{A}^1$-homotopy invariant and the natural map $K^T(-) \to KH^T(-)$ is an isomorphism on smooth schemes [Hoy20, Theorem 1.3]. Furthermore, $KH^T(-)$ is indifferent to derived structures [LT19], [KR24, Theorem F]. Finally $KH^T(-)$ satisfies proper excision on equivariant abstract blowups in any characteristic, which we now recall.

Recollection 2.39. A pullback square in $\mathrm{Sch}_k^{\mathrm{qcqs},T}$

$$\begin{tikzcd} Y \arrow[d, "f"] & E \arrow[l, hook'] \arrow[d] \\ X & Z \arrow[l, hook', "i"'] \end{tikzcd} \tag{2.18}$$

is called an *equivariant abstract blowup square* if f is proper and i is a finitely presented closed immersion such that f is an isomorphism over the open complement $X \backslash Z$.

Proposition 2.40. *Applying $KH^T(-)$ to* (2.18) *gives a homotopy fiber square*

$$\begin{tikzcd} KH^T(Y) \arrow[r] & KH^T(E) \\ KH^T(X) \arrow[r] \arrow[u] & KH^T(Z) \arrow[u] \end{tikzcd}$$

Proof. This holds by [KR24, Theorem G]. □

Remark 2.41. Given an invariant F with values in spectra, we say that F satisfies *(equivariant) proper excision* if it sends any (equivariant) abstract blowup square (2.18) to a homotopy fiber square. If this is the case, (2.18) in particular induces the long exact sequence on homotopy groups of Mayer-Vietoris type

$$\dots \to F_1(E) \to F_0(X) \to F_0(Y) \oplus F_0(Z) \to F_0(E) \to F_{-1}(X) \to \dots$$

(and similarly in the equivariant situation).

3. Perfect schemes

In the rest of the chapter, we restrict our attention to perfect schemes in characteristic p, which we presently recall. See [BS17, §3], [Zhu17a, Appendix A] for a detailed introduction.

3.1. Perfect schemes and their basic properties

3.1.1. Perfect schemes in characteristic p. Let k be a perfect field; we take $k = \mathbb{F}_p$. A scheme $X \in \mathrm{Sch}_k$ is called perfect, if the Frobenius morphism $\varphi : X \to X$ is an isomorphism. The full subcategory of perfect schemes is denoted $\mathrm{Sch}_k^{\mathrm{perf}} \subseteq \mathrm{Sch}_k$. Given a scheme $X \in \mathrm{Sch}_k$, its perfection $X_{\mathrm{perf}} \in \mathrm{Sch}_k^{\mathrm{perf}}$ is defined as the $\mathbb{N}$-indexed inverse limit along φ

$$X_{\mathrm{perf}} := \lim_{\varphi} X.$$

Note that φ is an affine map; in particular $\lim_{\varphi} X$ exists as a scheme [Sta, Tag 01YV]. On affines, the perfection is simply given by iteratively adjoining p-th roots of all functions. This gives a functor

$$\begin{aligned}(-)_{\mathrm{perf}} : \mathrm{Sch}_k &\to \mathrm{Sch}_k^{\mathrm{perf}} \\ X &\mapsto X_{\mathrm{perf}}.\end{aligned}$$

More generally, the same formula defines the perfection $\mathcal{X}_{\mathrm{perf}} := \lim_{\varphi} \mathcal{X}$ of any derived stack $\mathcal{X}$ over k.

Setup 3.1. We denote $\mathrm{Sch}_k^{\mathrm{fp}} \subseteq \mathrm{Sch}_k$ the full subcategory of finitely presented k-schemes. We denote its objects by a subscript "0" as X_0, Y_0, $f_0 : Y_0 \to X_0$ and so on.

We work with the category $\mathrm{Sch}_k^{\mathrm{pfp}}$ of perfectly finitely presented perfect k-schemes (and perfectly finitely presented morphisms) – see [BS17, §3]: we only consider objects and morphisms coming as perfections of objects and morphism from $\mathrm{Sch}_k^{\mathrm{fp}}$. We denote such perfection by dropping the subscript: unless specified otherwise, X stands for the perfection of X_0, f stands for the perfection of f_0, and so on.

3.1.2. Basic properties. We recall some facts from [BS17, §3] about perfect schemes.

(i) the functor $(-)_{\mathrm{perf}} : \mathrm{Sch}_k \to \mathrm{Sch}_k^{\mathrm{perf}}$ is right adjoint to the inclusion $\mathrm{Sch}_k^{\mathrm{perf}} \subseteq \mathrm{Sch}_k$,
(ii) the functor $(-)_{\mathrm{perf}}$ commutes with arbitrary limits, in particular with fiber products,

Indeed, the first line is clear from the definition; the second line follows formally. Many standard properties of schemes are preserved under taking perfection – see [BS17, Lemma 3.4] for a list.

3.1.3. Functions and coherent sheaves.

(iii) perfect schemes are reduced,
(iv) global functions on perfect schemes satisfy proper excision, i.e. taking $\mathrm{R\Gamma}(-, \mathcal{O})$ of a perfection of an abstract blowup square induces long exact sequence on cohomology groups.
(v) for a coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X_0)$ it holds that $H^i(X, \mathcal{F}) = \mathrm{colim}_{\varphi^*} H^i(X_0, \varphi^{*,(i)}(\mathcal{F}))$,
(vi) all perfect derived schemes are classical.

For (iii) note that all non-reducedness disappears in the colimit along Frobenius on functions. Part (iv) is [BST17, Lemma 3.9], [BS17, Lemma 4.6]. Part (v) is a standard fact about the compatibility of coherent cohomology with inverse limits along affine morphisms (by affineness of bonding maps this follows from [Sta, Tag 073D or Tag 0GQU]). Part (vi) is [BS17, Corollary 11.9].

3.1.4. Rationality and perfection. Rational singularities are preserved under perfection.

Definition 3.2. A map of schemes $f : Y \to X$ is called *rational* if $Rf_* \mathcal{O}_Y = \mathcal{O}_X$.

Observation 3.3. *Let $f_0 : Y_0 \to X_0$ be a rational map in $\mathrm{Sch}_k^{\mathrm{fp}}$. Then its perfection is rational as well.*

Proof. Working Zariski-locally on X_0, the rationality of f_0 amounts to the following: For each open affine $U_0 \subseteq X_0$, $\mathrm{R\Gamma}((Y_U)_0, \mathcal{O}_{(Y_U)_0}) = \mathcal{O}_{U_0}$. Equivalently, this reads as

$$\mathrm{R\Gamma}^i((Y_U)_0, \mathcal{O}_{(Y_U)_0}) = \begin{cases} 0 & \text{if } i \neq 0, \\ \mathcal{O}_{U_0} & \text{if } i = 0. \end{cases}$$

We need to prove that for each open affine $U \subseteq X$, $\mathrm{R\Gamma}(Y_U, \mathcal{O}_{Y_U}) = \mathcal{O}_U$. This can be checked on each cohomology group separately, where it follows from (v):

$$\mathrm{R\Gamma}^i(Y_U, \mathcal{O}_{Y_U}) = \operatorname*{colim}_{\varphi^*} \mathrm{R\Gamma}^i((Y_U)_0, \mathcal{O}_{(Y_U)_0}) = \begin{cases} 0 & \text{if } i \neq 0, \\ \mathcal{O}_U & \text{if } i = 0. \end{cases}$$

□

3.1.5. Topological invariance.

(vii) the map $X_{\mathrm{perf}} \to X$ is a universal homeomorphism,
(viii) a universal homeomorphism of perfect schemes is an isomorphism,
(ix) the functor $(-)_{\mathrm{perf}}$ agrees with absolute weak normalization.

For (vii) and (viii), see [BS17, Theorem 3.7, Lemma 3.8]. It is straightforward to deduce (ix) – we do this now for completeness.

Observation 3.4. *Let $Y_0 \in \mathrm{Sch}_k$. Then its perfection Y is the absolute weak normalization of Y_0.*

Proof. We recall the notion of absolute weak normalization [Sta, Tag 0EUK] (also see [GT80; LV81; Man80]). The absolute weak normalization $Y_0^{\mathrm{awn}} \to Y_0$ of Y_0 can be characterized as the initial scheme equipped with a universal homeomorphism to Y_0 [Sta, Tag 0EUS]. Let $f_0 : X_0 \to Y_0$ be the absolute weak normalization of Y_0. Taking perfections, we obtain the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[dl, dashed] \\ X_0 \arrow[r, "f_0"] & Y_0 \end{tikzcd}$$

Now f_0 is universal homeomorphism by design, so f is an isomorphism by (viii). Thus we get the factorization $Y \to X_0 \to Y_0$ as indicated by the dashed arrow. Since Y is perfect, it is reduced by (iii), so $Y \to X_0$ is an isomorphism by [Sta, Tag 0H3H, (3)]. □

3.1.6. Perfection of global quotient stacks. Global quotient stacks are compatible with perfections in the following manner.

Proposition 3.5. *Let G be a smooth algebraic group acting on a scheme X over k. Then*

$$(X/G)_{\mathrm{perf}} = X_{\mathrm{perf}}/G_{\mathrm{perf}}$$

Proof. This is a special case of [Zhu17a, Lemma A.11]. □

3.1.7. Perfect Stein factorization. We first recall the classical Stein factorization [Sta, Tag 03GX]. Let $f_0 : X_0 \to S_0$ be a morphism. Setting $S_0' = \underline{\mathrm{Spec}}_{\mathcal{O}_{S_0}}(f_{0,*}\,\mathcal{O}_X)$, we can factor f_0 as follows.

$$\begin{tikzcd} X_0 \arrow[rr, "f_0'"] \arrow[dr, "f_0"'] & & S_0' \arrow[dl, "\pi_0"] \\ & S_0 & \end{tikzcd}$$

Assuming $f_0 : X_0 \to S_0$ is proper and S_0 is locally noetherian, [Sta, Tag 03H0] shows that f_0' is proper with geometrically connected fibers, $f'_{0,*}\,\mathcal{O}_{X_0} = \mathcal{O}_{S_0'}$, π_0 is finite, S_0' is the normalization of S_0 in X_0. In particular, for any geometric point $s_0 \in S_0$ it holds that $\pi_0(X_{s_0}) = \pi_0(S'_{s_0})$ as finite sets.

Assume now $X, S \in \mathrm{Sch}_k^{\mathrm{pfp}}$ and let $f : Y \to S$ be a perfectly proper map. Let $S' = \underline{\mathrm{Spec}}_{\mathcal{O}_S}(f_*\,\mathcal{O}_X)$.

Lemma 3.6. *The map $S' \to S$ is an isomorphism if and only if all geometric fibers of $f : X \to S$ are connected.*

Proof. Without loss of generality, we may fix finite type models X_0, Y_0, f_0 and S_0'. Stein factorization is compatible with base change [Sta, Tag 03GY].

Consider the Stein factorization $X \to S' \to S$. For each geometric point $s \in S$, S_s' is given by finitely many points, which are in bijection with $\pi_0(X_s)$. By (viii), $S' \to S$ is an isomorphism iff it is a universal homeomorphism. But the latter happens iff each $S_s = \pi_0(X_s)$ has a single point. $\square$

Lemma 3.7. *Consider a diagram of pfp perfect schemes*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & T \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

Assume that $X \to S$ is perfectly proper and $T \to S$ is affine. Assume that for each geometric point s of S, $\pi_0(g) : \pi_0(X_s) \to \pi_0(T_s)$ is bijective. Then $g : X \to T$ induces an isomorphism on global sections.

Proof. Let $S' = \underline{\mathrm{Spec}}_{\mathcal{O}_S}(f_* \mathcal{O}_X)$. As T is affine over S, the Stein factorization S' naturally fits into

$$X \to S' \to T \to S.$$

We will prove that the corresponding map $\mathcal{O}_T \to \mathcal{O}_{S'}$ of $\mathcal{O}_S$-algebras is an isomorphism, which yields the result after pushforward. To this end, we can without loss of generality assume that S and hence T is affine.

Since $X \to T$ is perfectly proper, by Lemma 3.6 it is enough to check that its geometric fibers are connected. But this can be checked separately over each geometric point $s \in S$, where it boils down to our assumption. $\square$

4. Perfect equivariant K-theory

We now describe the main structural properties of equivariant K-theory of perfect schemes. Perfection changes K-theory in a controlled way by "inverting the Frobenius" and this forces good descent properties. In fact, [Kra80a; KM21; AMM22; Cou23] have already studied such questions for non-equivariant algebraic K-theory (see also [EK20] for motivic results). We take advantage of their methods.

Let $k = \mathbb{F}_p$. Unless stated otherwise, let G_0 be a group scheme over k and G its perfection. Note that G is still a group scheme over k. It is affine if and only if G_0 is.

4.1. Effect of perfection on equivariant K-theory

Perfection changes K-theory by inverting the Frobenius.

Lemma 4.1. *Let $X_0 \in \mathrm{Sch}_k^{\mathrm{qcqs},G_0}$. Let G and X be the perfections. Then*

$$K^G(X) = \operatorname*{colim}_{\varphi^*} K^{G_0}(X_0).$$

Proof. Algebraic K-theory $K(-)$ sends filtered inverse limits of qcqs schemes along affine maps to colimits by [TT90, Proposition 3.20], giving the result for the non-equivariant K-theory of $X = \lim X_0$. The case of equivariant K-theory follows by the same argument, using [TT90, Theorem 3.20.1] first and only then passing to $K^G(-)$. $\square$

Taking exterior powers furnishes equivariant K-groups with λ-operations λ_j for $j \in \mathbb{N}_0$. We equip the sum of non-negative K-groups $\bigoplus_{i \geqslant 0} K_i^G(X)$ with the ring structure coming from the usual module structure over $K_0^G(X)$ and zero multiplication on the positive part. This is a standard definition in this context, making $\bigoplus_{i \geqslant 0} K_i^G(X)$ into a λ-ring by [Köc98, §2] and [KZ25, §5]. We denote its Adams operations by ψ^j for $j \in \mathbb{N}_0$, see [Wei13, p. 102]. We then formally extend these Adams operations to negative K-groups by Bass delooping [TT90, Theorem 6.6].

The following statement is well-known for K-theory of rings by [Kra80b; Kra80a; Hil81] or [Cou23, Lemma 3.1.5]; the case of qcqs schemes follows by Zariski descent. We provide the arguments for the equivariant extension.[4]

Lemma 4.2. *For each $i \in \mathbb{Z}$, the action of Frobenius pullback φ^* on $K_i^{G_0}(X_0)$ agrees with the p-th Adams operation ψ^p.*

Before starting the proof, we recall the formalism of polynomial functors (over k) from [HKT17, §8], we refer to their nice exposition for details. The category $\mathrm{Vect}(k)$ is enriched in schemes by regarding each $\mathrm{Hom}_k(V, W)$ for $V, W \in \mathrm{Vect}(k)$ as the affine space on the underlying vector space. A *polynomial functor over* k is an enriched endofunctor of $\mathrm{Vect}(k)$. In other words, it is a functor $F : \mathrm{Vect}(k) \to \mathrm{Vect}(k)$ such that each of the maps $\mathrm{Hom}_k(V, W) \to \mathrm{Hom}_k(F(V), F(W))$ on morphism spaces is given by a tuple of polynomials [HKT17, Definition 8.1]. Polynomial functors form an exact category $\mathrm{Pol}(k)$, equipped with tensor product and exterior power operations. There is a notion of polynomial functors of *homogeneous degree* d and we write $\mathrm{Pol}(k)_d$ for the corresponding full subcategory; we further write $\mathrm{Pol}(k)_{<\infty}$ for the full subcategory spanned by finite direct sums of homogeneous polynomial functors. Finally, $\mathrm{Pol}(k)^0$ stands for the full subcategory of polynomial functors sending the zero vector space to itself (and similarly for the variants).

In particular, the Frobenius pullback φ^* as well as the exterior powers Λ^j for $j \in \mathbb{N}_0$ can be regarded as objects of $\mathrm{Pol}(k)$. This can be done by hand from the definition: φ^* is the identity on objects, but on morphisms it raises the polynomial coordinates to p-th powers (it is the actual Frobenius when we view the morphism spaces as schemes). For Λ^j, see [HKT17, Example 8.2]. Furthermore, φ^* has degree p, while Λ^j has degree j and all of these preserve the zero object.

The Grothendieck group $K_0(\mathrm{Pol}(k)_{<\infty})$ is naturally a λ-ring, graded by the homogeneous degree. It in particular contains the elements φ^* and λ_j, $j \in \mathbb{N}$ corresponding to the Frobenius and exterior powers. It is explicitly given as the free λ-ring on one variable (also known as the ring of symmetric functions)

$$K_0(\mathrm{Pol}(k)_{<\infty}) \cong \mathbb{Z}[s_j \mid j \in \mathbb{N}], \qquad \lambda_j \mapsto s_j$$

by [HKT17, Theorem 8.10]. It is a free polynomial ring on the generators s_j with $\deg s_j = j$, while $K_0(\mathrm{Pol}(k)^0_{<\infty})$ is its augmentation ideal. The degree d part $K_0(\mathrm{Pol}(k)_d) \cong \mathbb{Z}[s_j \mid j \in \mathbb{N}]_d$ can be naturally described as the representation ring of the Schur algebra $\Gamma^d(\mathrm{Mat}(n, k))$ as in [HKT17, §8.B, Theorem 8.8]. This latter representation ring receives a natural surjection from $R_k(GL_n)$ once $n \geqslant d$ so that λ_j comes from the class of the j-th fundamental representation of GL_n, see [HKT17, Remark 8.17], [Ser68, §3.8].

[4] We thank Bernhard Köck for a detailed discussion of these arguments.

Proof of Lemma 4.2. Consider the λ-ring $K_0(\mathrm{Pol}(k)^0_{<\infty})$. The Adams operation ψ^j for $j \in \mathbb{N}$ is by definition a linear combination of degree j monomials in λ's, so the elements φ^* and ψ^p lie in the graded degree p. They are equal: indeed, it is enough to check this on the level of operations on the representation ring $R_k(GL_n)$, which naturally surjects onto $\mathbb{Z}[s_j \mid j \in \mathbb{N}]_d$ once $n \geqslant d$. On $R_k(GL_n)$, the desired equality $\varphi^* = \psi^p$ holds by the classical results [Kra80b, Corollary 3.7], also see [Kra80a, proof of Proposition 5.4], [Hil81, Theorem 5.1].

We conclude via arguments from [KZ25; HKT17], who develop a method for reducing statements about λ-operations on higher equivariant K-groups to computations in the λ-ring $K_0(\mathrm{Pol}(k)^0_{<\infty})$. In fact, we can completely follow the reasoning from [KZ25, proof of Theorem 5.1], which employs this method (to prove a different equality).

In more detail, to prove an equality of operations on $K_i^G(X)$ for any given $i \geqslant 0$, it is enough to prove it inside $K_0(\mathrm{End}((B^q)^i \,\mathrm{Vect}^G(X)))$, the zeroth K-group of the exact category of endofunctors of a suitable exact category $(B^q)^i \,\mathrm{Vect}^G(X)$ of i-dimensional bounded acyclic binary multicomplexes, defined from $\mathrm{Vect}^G(X)$ inductively on i – see [HKT17, Definition 1.1] for the notation. This reduction step is explained in [KZ25, proof of Theorem 5.1] or [HKT17, proof of Theorem 8.18] (where the notation $\mathcal{P}(G, X) = \mathrm{Vect}^G(X)$ is used): there is a natural surjection $K_0((B^q)^i \,\mathrm{Vect}^G(X)) \twoheadrightarrow K_i^G(X)$; testing on classes of objects in the left-hand side we reduce to checking the equality inside $K_0(\mathrm{End}((B^q)^i \,\mathrm{Vect}^G(X)))$.

Furthermore, by [KZ25, (5.3)], there is an exact functor

$$\mathrm{Pol}^0_{<\infty}(k) \to \mathrm{End}((B^q)^i \,\mathrm{Vect}^G(X)).$$

Consider the induced homomorphism on K_0. Both φ^* and ψ^p exist and are equal as elements in $K_0(\mathrm{Pol}_{<0}(k))$ by above; they realize the desired operations under this map. Therefore, the same holds in K_0 of the right-hand side and we conclude that $\varphi^* = \psi^p$ on all higher equivariant K-groups by the previous paragraph. The case of negative K-groups then follows by Bass delooping (by our definition of ψ^p). $\square$

In other words, equivariant K-theory of X is obtained from $K^{G_0}(X_0)$ by localizing ψ^p:

$$K^G(X) = \operatorname*{colim}_{\psi^p} K^{G_0}(X_0).$$

The following is a special case of Lemma 4.2 in degree zero after k-linearization.

Lemma 4.3 (Commuting with perfections)**.** *Given $X_0 \in \mathrm{Sch}_k^{\mathrm{qcqs}, G_0}$, we have*

$$K_0^G(X)_k = \left(K_0^{G_0}(X_0)_k\right)_{\mathrm{perf}}$$

Proof. The left adjoint $(-)_k = (- \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p)$ commutes with colimits, so by Lemma 4.1 we can rewrite the left-hand side as

$$\operatorname*{colim}_{\varphi^*} \left(K_0^{G_0}(X_0)_k\right).$$

We now claim that the maps in this colimit are given by taking p-th powers. If this is the case, the colimit becomes $\left(K_0^{G_0}(X_0)_k\right)_{\mathrm{perf}}$ by definition. It thus suffices to prove Claim 4.4. $\square$

Claim 4.4. *The Frobenius pullback φ^* on $K_0^{G_0}(X_0)_k$ is given by taking p-th powers $c \mapsto c^p$.*

Proof of claim. Let c be the class of a G_0-equivariant vector bundle $\mathcal{E}$ of rank n on X_0. Then we use Lemma 2.30: there exists a G_0-equivariant morphism $f : Y_0 \to X_0$ such that $f^* : K_0^{G_0}(X_0)_k \to K_0^{G_0}(Y_0)_k$ is injective and $f^*[\mathcal{E}] = \sum_i^n [\mathcal{L}_i]$ for some G_0-equivariant line bundles $\mathcal{L}_i$. To prove $\varphi^*[\mathcal{E}] = [\mathcal{E}]^p$, we can thus wlog assume that $[\mathcal{E}]$ is a sum of classes of equivariant line bundles.

First assume that c is a class of a G_0-equivariant line bundle $\mathcal{L}$. Then $\varphi^* \mathcal{L} \cong \mathcal{L}^{\otimes p}$ with its canonical G_0-equivariant structure. Indeed, this can be seen explicitly on the level of transition functions; see [BS17, Lemma 3.5] for the non-equivariant statement. Since we have base changed to $k = \mathbb{F}_p$, the same holds true for sums of classes of line bundles. Indeed,

$$\varphi^* : (c_1 + \cdots + c_n) \mapsto (c_1^p + \cdots + c_n^p) = (c_1 + \cdots + c_n)^p.$$

□

4.2. Kratzer's p-divisibility of perfect higher K-theory

The interpretation of the Frobenius pullback on K-theory as the p-th Adams operation ψ^p implies p-divisibility of higher K-theory of perfect schemes [Kra80a]. This is an important structural result, which immediately generalizes to equivariant K-theory.

Lemma 4.5. *Let $X \in \mathrm{Sch}_k^{\mathrm{pfp},G}$. Then for all $i \geqslant 1$, $K_i^G(X)$ is a $\mathbb{Z}[\frac{1}{p}]$-module.*

Proof. This is a direct generalization of the non-equivariant statement [Kra80a, Corollary 5.5 and Example (1) below it]; see also [Wei13, §II.4 and §IV.5]. We review Kratzer's proof.

Since the multiplication on $\bigoplus_{i\geqslant 0} K_i^G(X)$ compatible with the λ-ring structure is trivial in positive degrees, Lemma 4.2 and [Kra80a, Proposition 5.3.(i)] show that on $K_i^G(X)$ with $i \geqslant 1$ we have

$$\varphi^* = \psi^p = (-1)^{p-1} \cdot p \cdot \lambda^p. \tag{4.1}$$

Since $G\backslash X$ is perfect, its Frobenius φ is an isomorphism, hence φ^* is an isomorphism. By (4.1) and additivity of λ^p, the multiplication by p needs to be isomorphism as well. Hence $K_i^G(X)$ is a $\mathbb{Z}[\frac{1}{p}]$-module for all $i \geqslant 1$. □

Lemma 4.5 in particular shows the following vanishing after k-linearization.

Corollary 4.6. *Let $X \in \mathrm{Sch}_k^{\mathrm{pfp},G}$, then*

- $K_i^G(X;k) = 0$ *for* $i \geqslant 2$,
- $K_1^G(X;k) = K_0^G(X) \otimes_{\mathbb{Z}}^1 k$.

Proof. For all $i \geqslant 1$, we know that $K_i^G(X)$ is a $\mathbb{Z}[\frac{1}{p}]$-module by Lemma 4.5. The result follows by the universal coefficient theorem $K_j^G(X;k) = (K_j^G(X) \otimes_{\mathbb{Z}}^0 k) \oplus (K_{j-1}^G(X) \otimes_{\mathbb{Z}}^1 k)$. □

In fact, we don't know examples where $K_1^G(X;k) \neq 0$. In the non-equivariant setup, this is always zero for weight reasons – see Observation 8.1.

4.3. Perfect K-theory and G-theory agree

After perfection, K-theory agrees with G-theory.

Observation 4.7 (Perfect K-theory and G-theory)**.** *Let $X \in \mathrm{Sch}_k^{\mathrm{pfp},G}$. Then there is a homotopy equivalence of K-theory spectra*

$$K^G(X) \simeq G^G(X).$$

Proof. By [BS17, Proposition 11.31, Remark 11.32], any complex in $D^b_{\mathrm{QCoh}}(X)$ has finite Tor amplitude. By [TT90, Proposition 2.2.12], it follows that the notions of perfect complexes and pseudo-coherent complexes with locally bounded cohomology agree on X. Hence perfect complexes of globally finite Tor-amplitude and pseudo-coherent complexes with globally bounded cohomology agree, so [TT90, Definition 3.1] and [TT90, Definition 3.3] agree, proving the statement for the trivial group G. Also see [TT90, Theorem 3.21]. The same reasoning works with any G, adding "G-equivariant complex" everywhere in the proof. □

4.4. Equivariant perfect proper excision

Algebraic K-theory usually doesn't enjoy proper excision, the best general supplement being the pro-excision of [Mor16; KST18]. Forcing it on all $\mathrm{Sch}_k^{\mathrm{qcqs}}$ results in KH by [KST18, Theorem 6.3].

However, for pfp perfect schemes, K-theory satisfies proper excision [KM21, Proof of Theorem 4.3]. We conjecture that perfect proper excision works G-equivariantly, and we prove it in the T-equivariant case in Theorem 4.11. This is one of the main computational tools for our results.

Setup 4.8. Let $k = \mathbb{F}_p$. Let G_0 be a reductive group over k and suppose we are given an abstract blowup square of G_0-equivariant schemes in $\mathrm{Sch}_k^{\mathrm{fp}}$:

$$\begin{tikzcd} Y_0 \arrow[d, "f_0"] & E_0 \arrow[l, hook'] \arrow[d, "f_0'"] \\ X_0 & Z_0 \arrow[l, hook'] \end{tikzcd} \tag{4.2}$$

Passing to perfections (by dropping the subscript "0") we obtain a *perfect abstract blowup square* of G-equivariant pfp perfect schemes over k:

$$\begin{tikzcd} Y \arrow[d, "f"] & E \arrow[l, hook'] \arrow[d, "f'"] \\ X & Z \arrow[l, hook'] \end{tikzcd} \tag{4.3}$$

Let $T_0 = \mathbb{G}_m^n$ and T its perfection. Before we embark on proving T-equivariant proper excision, we record the following useful lemma.

Lemma 4.9. *Let $k = \mathbb{F}_p$. Let $T_0 = \mathbb{G}_m^n$ be a split torus acting on $X_0 \in \mathrm{Sch}_k^{\mathrm{qcqs}}$. Let T and X be the corresponding perfections. Then the stack $T\backslash X$ is flawless.*

Proof. Let us first prove the case of quasi-projective X_0 following [BFN10, Corollary 3.22]. First note that $BT = T\backslash\mathrm{pt}$ is flawless. Clearly, it has affine diagonal T. Both dualizable (perfect) and compact objects are given by finite-dimensional representations of the affine multiplicative group scheme T. These also generate, so BT is flawless by say [BFN10, Proposition 3.9].

Since X_0 is a quasi-projective k-variety, it is qcqs and possesses an ample family of line bundles; the same follows for X by [BS17, Lemmas 3.4 and 3.6]. Now apply [BFN10, Proposition 3.21] to the morphism $T\backslash X \to T\backslash\mathrm{pt}$ of algebraic stacks as in the proof of [BFN10, Corollary 3.22].

More generally, this works for any qcqs X_0. Since T_0 is of multiplicative type, [Kha22, Theorem 1.40, Example 1.41.(ii)] show that the global quotient $T_0\backslash X_0$ of a qcqs scheme X_0 by T_0 is flawless. Since $\mathrm{Perf}(-)$, $D_{\mathrm{QCoh}}(-)$ turn the Frobenius limit to a colimit and $\mathrm{Ind}(-)$ commutes with colimits (being left adjoint), we deduce $T\backslash X$ is flawless as well. □

Remark 4.10. If $G_0 = GL_n$ with $n \geqslant 2$ over k, then $BG_0 = G_0\backslash\mathrm{pt}$ is not flawless: its structure sheaf is non-compact. This is the technical point that causes problems.

We are now ready to prove the T-equivariant case.

Theorem 4.11. *Consider a T-equivariant perfect abstract blowup square of pfp schemes from Setup 4.8. Then the following square of equivariant K-theory spectra is a homotopy fiber square.*

$$\begin{array}{ccc} K^T(Y) & \longrightarrow & K^T(E) \\ \uparrow & & \uparrow \\ K^T(X) & \longrightarrow & K^T(Z) \end{array} \tag{4.4}$$

In particular, we get the long exact sequence:

$$\cdots \to K_1^T(E) \to K_0^T(X) \to K_0^T(Y) \oplus K_0^T(Z) \to K_0^T(E) \to K_{-1}^T(X) \to \dots \tag{4.5}$$

The same conclusion holds if we replace K-theory by any localizing invariant E.

Proof. It is enough to verify the following three conditions:

(i) $\mathrm{Ind}(\mathrm{Perf}(T\backslash S)) = D_{\mathrm{QCoh}}(T\backslash S)$ as ∞-categories for the spaces $S = X, Y, Z, E$ featuring above,
(ii) the induced square of ∞-categories given by applying $D_{\mathrm{QCoh}}(-)$ is a pullback,
(iii) the functor $Ri_* : D_{\mathrm{QCoh}}(T\backslash E) \to D_{\mathrm{QCoh}}(T\backslash Y)$ is fully faithful.

Assuming (i), (ii), (iii), we follow the reasoning in [KM21, Proof of Theorem 4.3]. To apply [Tam18, Theorem 18], we need to check that the square given by $\mathrm{Perf}^T(-)$ is excisive [Tam18, Definition 14]. By (i), the Ind-completion of this square agrees with the analogous square of $D_{\mathrm{QCoh}}(-)$, which satisfies the desired conditions by (ii) and (iii). We prove (i), (ii), (iii) below, concluding the proof. □

Proof of (i). This follows from Lemma 4.9. □

Proof of (ii). We prove that the following square is a pullback of ∞-categories:

$$\begin{array}{ccc} D_{\mathrm{QCoh}}(T\backslash Y) & \longrightarrow & D_{\mathrm{QCoh}}(T\backslash E) \\ \uparrow & & \uparrow \\ D_{\mathrm{QCoh}}(T\backslash X) & \longrightarrow & D_{\mathrm{QCoh}}(T\backslash Z) \end{array} \tag{4.6}$$

For perfect schemes, this is proved in [BS17, Theorem 11.2.(1)]. To deduce the same result for perfect global quotient stacks, note that perfection commutes with the formation of global quotients by Proposition 3.5. Moreover, the diagram

$$T\backslash X \leftarrow X \leftleftarrows T \times X \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow}} T \times T \times X \mathrel{\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}} \dots \tag{4.7}$$

allows to compute

$$D_{\mathrm{QCoh}}(T\backslash X) = \lim \left(D_{\mathrm{QCoh}}(X) \rightrightarrows D_{\mathrm{QCoh}}(T \times X) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} D_{\mathrm{QCoh}}(T \times T \times X) \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}} \dots \right) \tag{4.8}$$

by faithfully flat descent. More precisely, $T\backslash X \leftarrow X$ is a v-cover by [BS17, Definition 11.1 and Example 2.3] so this also follows from [BS17, Theorem 11.2.(1)] and the definition of $D_{\mathrm{QCoh}}(T\backslash X)$. Taking iterated self-products of this cover precisely returns (4.7).

Each of the terms in the limit (4.8) is a scheme. Taking $(T \times \cdots \times T \times (-))$ of the original abstract blowup is an abstract blowup, so the corresponding square of ∞-categories after applying $D_{\mathrm{QCoh}}(-)$ is a pullback. Commuting limits, we deduce that the limit square (4.6) is a pullback as well. $\square$

Proof of (iii). Since $i : E \to Y$ is a closed immersion of perfect schemes, $Li^* Ri_* = \mathrm{id}$ on $D_{\mathrm{QCoh}}(E)$ by [BS17, Lemma 3.16]. The equivariant structure carries through, so the same holds on $D_{\mathrm{QCoh}}(T\backslash E)$. Hence Ri_* is fully faithful as $\mathrm{Hom}(Ri_*(\mathcal{F}), Ri_*(\mathcal{G})) = \mathrm{Hom}(Li^* Ri_*(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. $\square$

Remark 4.12. Note that claims (ii) and (iii) hold without change for any perfectly reductive group G instead of T. The technical issue in generalizing Theorem 4.11 from T to G is the failure of (i). To apply [Tam18, Theorem 18], we would need to work with the square given by $\mathrm{Ind}(\mathrm{Perf}^G(-))$ directly.

In general, we conjecture the following.

Conjecture 4.13. *Perfect proper excision holds G-equivariantly.*

Regarding the validity of this conjecture, we know:

(a) In the non-equivariant setup, it holds by [KM21, Proof of Theorem 4.3].
(b) For $K^T(-)$, we proved it in Theorem 4.11.
(c) For $KH^T(-)$, it holds by standard techniques (without perfection) by Proposition 2.40.
(d) For Borel-type equivariant K-theory $K^G_{\triangleleft}(-)$, it holds by Kan-extension from (a).

4.5. Homotopy K-theory

Let us record some immediate properties of $KH^G(-)$ for perfect schemes.

Lemma 4.14.

$$KH_i^G(X) = \operatorname*{colim}_{\varphi^*} KH_i^{G_0}(X_0).$$

Proof. Formally follows from Lemma 4.1 and the definition of $KH(-)$ by commuting colimits. □

Lemma 4.15. *$KH^T(-)$ satisfies perfect proper excision.*

Proof. Take the perfection of Proposition 2.40. □

It is reasonable to expect the following.

Conjecture 4.16. *The canonical map $K^G(X) \to KH^G(X)$ is an equivalence.*

Non-equivariantly, the map $K(X) \to KH(X)$ is actually an equivalence for any pfp perfect scheme X by [AMM22, Proposition 5.1]. However, their argument needs proper excision as an input.

Remark 4.17. If X is T-equivariant and perfectly smooth, then $K^T(X) \to KH^T(X)$ is an equivalence. By the above theorems, both sides satisfy descent on equivariant perfect abstract blowup squares. We conclude that $K^T(X) \to KH^T(X)$ is an equivalence whenever there exists a sequence of equivariant abstract blowups reducing everything to perfectly smooth schemes. We will see this happen in situations where explicit resolutions are known – see Theorem 6.7 and Theorem 8.8.

5. Perfect fixed-point schemes and trace maps

We now record how the fixed-point schemes and trace map from §2 play out in the perfect setup of §3, §4. We discuss how to control whether the trace map is an isomorphism; we also provide some direct geometric intuition.

5.1. Perfect fixed-point schemes

The fixed-point schemes and trace maps from §2 specialize to the perfect setup.

5.1.1. Immediate observations. In Setup 3.1, we can apply the fixed-point scheme functor $\mathrm{Fix}_{\frac{G}{G}}(-)$ and its variants. Since perfect schemes are reduced (iii) and perfect derived schemes are classical (iv), the natural maps (2.5) give isomorphisms

$$\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X) \cong \mathrm{Fix}_{\frac{G}{G}}(X) \cong \mathrm{Fix}^{\mathrm{red}}_{\frac{G}{G}}(X).$$

Since $(-)_{\mathrm{perf}}$ commutes with fiber products (ii), it formally follows that

$$\mathrm{Fix}_G(X) \cong (\mathrm{Fix}_{G_0}(X_0))_{\mathrm{perf}}.$$

Compatibility with stack quotients from Proposition 3.5 further shows

$$\mathrm{Fix}_{\frac{G}{G}}(X) \cong \mathrm{Fix}_{\frac{G_0}{G_0}}(X_0)_{\mathrm{perf}}.$$

and similarly for any restriction $\mathrm{Fix}_S(X)$. Passing through the limit along Frobenius, we in particular note that our trace map

$$\mathrm{tr}_X : K_0^G(X)_k \to H_0(\mathrm{Fix}_{\frac{G}{G}}(X), \mathcal{O})$$

is given by the perfection of

$$\mathrm{tr}_{X_0} : K_0^{G_0}(X_0)_k \to H_0(\mathrm{Fix}_{\frac{G_0}{G_0}}(X_0), \mathcal{O}).$$

In particular, if tr_{X_0} is an isomorphism, the same is true for tr_X.

5.1.2. Perfect tori. Let T be a perfect torus; then the situation simplifies further. Since global quotients by T are flawless by Lemma 4.9, we have

$$HH_i^T(X/k) = \mathrm{R}\Gamma_i(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}). \tag{5.1}$$

In particular,

$$HH_i^T(X/k) = \mathrm{R}\Gamma_i(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}) = 0, \qquad i \geqslant 1. \tag{5.2}$$

Since $(-)^T$ is exact on functions, we can commute it with taking cohomology to get

$$\mathrm{R}\Gamma_i(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}) = H_i(\mathrm{Fix}_T(X), \mathcal{O})^T, \qquad i \in \mathbb{Z}. \tag{5.3}$$

In particular, this can be nonzero only in those degrees where coherent cohomology of $\mathrm{Fix}_T(X)$ is nonzero. Such vanishing can be further checked fiberwise over T.

Lemma 5.1. *Let $i \in \mathbb{Z}$. Then $\mathrm{R}\Gamma_i(\mathrm{Fix}_T(X), \mathcal{O}) = 0$ if and only if $\mathrm{R}\Gamma_i(\mathrm{Fix}_t(X), \mathcal{O}) = 0$ for all geometric points t of T.*

Proof. Considering the structure map $\mathrm{Fix}_T(X) \to T$ with T affine, the result holds by perfect base-change [BS17, Lemma 3.18]. ☐

5.2. Trace map for equivariant points

Given a split reductive group G_0 over k, we discuss perfect equivariant K-theory of the point. This closely relates to the representation theory of perfect reductive groups studied in [CW24]. In classical terms, the Frobenius map corresponds to the multiplication by p on characters and consequently to Frobenius twisting on $\mathrm{Rep}_k(G_0)$.

Lemma 5.2. *Let $X_0 = \mathrm{pt}$ and $T_0 = \mathbb{G}_m^n$. Then*

$$\mathrm{tr} : K^T(\mathrm{pt}; k) \to \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(\mathrm{pt}), \mathcal{O})$$

is an isomorphism of spectra. Both sides are supported in homotopy degree 0 with value

$$k[t_1^{\pm\frac{1}{p^\infty}}, \ldots, t_n^{\pm\frac{1}{p^\infty}}].$$

Proof. Since Rep_T is equivalent to the category of $\mathbb{Z}[\frac{1}{p}]^n$-graded vector spaces, we can compute $K^T(\mathrm{pt})$ from $K(\mathrm{pt})$ by additivity. But the latter was computed by Quillen [Wei13, Corollary IV.1.13]. In particular, the k-linear version $K^T(\mathrm{pt}; k)$ is supported in degree 0 with value given by the representation ring of T.

On the other hand, $\mathrm{Fix}_{\frac{T}{T}}(\mathrm{pt}) = \frac{T}{T}$, so the right-hand side is $\mathrm{R}\Gamma(T, \mathcal{O})^T$. Since T is affine, this is concentrated in degree 0 by Serre's vanishing theorem. There it is simply given by $H_0(T, \mathcal{O})$ which agrees with the above representation ring via the trace map. □

Lemma 5.3. *Let $X = \mathrm{pt}$ with the trivial G-action. Then the trace map gives an isomorphism*

$$\mathrm{tr}_X : K_0^G(\mathrm{pt}; k) \xrightarrow{\cong} H_0(\mathrm{Fix}_{\frac{G}{G}}(\mathrm{pt}), \mathcal{O}) \xrightarrow{\cong} H_0(\mathfrak{s}, \mathcal{O}).$$

Both sides are given by

$$k[t_1^{\pm\frac{1}{p^\infty}}, \ldots, t_n^{\pm\frac{1}{p^\infty}}]^W.$$

Proof. This is already the case before perfection for $\mathfrak{s}_0$ and G_0 by Example 2.37. Hence tr is also isomorphism after perfecting. □

For example if $G_0 = GL_n$, the above ring is explicitly given by $k[c_1^{\frac{1}{p^\infty}}, \ldots, c_{n-1}^{\frac{1}{p^\infty}}, c_n^{\pm\frac{1}{p^\infty}}]$.

Remark 5.4. In the current setup, $K^G(\mathrm{pt}; k)$ is supported in degree zero. Indeed, equivariant K-theory of a point is always supported in nonnegative degrees; taking k-coefficients and perfection at the same time forces all the positive K-groups to vanish by §4.2 (there is no contribution from $(-) \otimes_1^{\mathbb{Z}} k$ because $K_0^G(\mathrm{pt})$ which is torsionfree).

On the other hand, the adjoint representation of G on its ring of functions does not have any higher group cohomology. Indeed, since G is assumed reductive, its adjoint representation has a good filtration [Jan03, II.4.20.Proposition and II.4.21.Remark] and the claim follows [Jan03, II.4.16.Proposition]. In other words, $\mathrm{R}\Gamma(\mathrm{Fix}_{\frac{G}{G}}(\mathrm{pt}), \mathcal{O})$ is also supported in cohomological degree zero (even before perfecting). It matches $H_0(\mathfrak{s}, \mathcal{O})$, the ring of functions on the affine scheme $\mathfrak{s}$.

5.3. Perfect projective bundles and traces

We now record how T-equivariant K-theory and fixed-point schemes behave under passage to perfect projective bundles. The easiest argument works for any localizing invariant, but we also sketch simple geometric arguments for fixed-point schemes valid in degree zero in §5.4.

Lemma 5.5. *Let T act on X and $\mathcal{E} \in \mathrm{Vect}^T(X)$ of rank r. Let $Y_0 = \mathbb{P}(\mathcal{E})$ be the corresponding projective space over X and Y its perfection. Let $\mathrm{tr} : K^T(-; k) \to HH^T(-/k)$ be the trace map. Then the following are equivalent:*

(i) tr *is isomorphism for X,*
(ii) tr *is isomorphism for Y.*

Proof. By Proposition 2.29 we have natural identifications $K^T(\mathbb{P}(\mathcal{E}); k) = \prod_{i=0}^{r-1} K^T(X; k)$ and $HH^T(\mathbb{P}(\mathcal{E})/k) = \prod_{i=0}^{r-1} HH^T(X/k)$. Hence tr is an isomorphism on X if and only if it is an isomorphism on $\mathbb{P}(\mathcal{E})$. Now, $K_T(Y;k) = \operatorname{colim}_{\varphi^*} K_T(\mathbb{P}(\mathcal{E});k)$ and $HH_T(Y/k) = \operatorname{colim}_{\varphi^*} HH_T(\mathbb{P}(\mathcal{E})/k)$ compatibly with these decompositions.

(i) $\implies$ (ii). If tr is an isomorphism for X, it is an isomorphism for $\mathbb{P}(\mathcal{E})$ and hence for Y by exactness of filtered colimits.

(ii) $\implies$ (i). Assume tr is an isomorphism for Y. Since X is perfect and $K^T(X;k) \to HH^T(X/k)$ is a retract of $K^T(\mathbb{P}(\mathcal{E});k) \to HH^T(\mathbb{P}(\mathcal{E})/k)$ compatibly with the Frobenius, we know that $K^T(X;k) \to HH^T(X/k)$ is a retract of $K^T(Y;k) \to HH^T(Y/k)$. Hence tr is an isomorphism for X. □

Lemma 5.6. *Let T act on X and $\mathcal{E} \in \mathrm{Vect}^T(X)$ of rank r. Let $Y = \mathrm{Flag}_X(\mathcal{E},\mu)_{\mathrm{perf}}$ be the corresponding perfect flag variety of type μ over X. Let $\mathrm{tr} : K^T(-;k) \to HH^T(-/k)$ be the trace map. Then the following are equivalent:*

- *(i)* tr *is isomorphism for X,*
- *(ii)* tr *is isomorphism for Y.*

Proof. Take perfection of (2.15). Applying Lemma 5.5 multiple times for the bundles appearing in the factorizations of g and h, we deduce the result for f. □

Example 5.7. Let $\mathcal{E} \in \mathrm{Vect}^T(\mathrm{pt})$. Denote $\lambda_1, \ldots, \lambda_m$ the set of distinct weights of T on $\mathcal{E}$. Let $Y = \mathbb{P}(\mathcal{E})_{\mathrm{perf}}$. Then the trace map

$$K^T(Y) \xrightarrow{\simeq} \mathrm{R\Gamma}(\mathrm{Fix}_{\frac{T}{T}}(Y), \mathcal{O})$$

is an equivalence supported in degree zero. In degree zero, both sides are given by the ring

$$\left(k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}, x^{\pm 1}] / \left(\prod_{i=1}^{m} (x - \lambda_i)\right)\right)_{\mathrm{perf}}.$$

Proof. For the point, the trace map is an isomorphism concentrated in degree zero. The same follows for $\mathbb{P}(\mathcal{E})$ by the projective bundle formula for $K(-;k)$ and $HH(-/k)$. This isomorphism is compatible with Frobenius pullback and taking the colimit along it is exact. Hence $K^T(Y) \to HH^T(Y/k)$ is still an isomorphism supported in degree zero.

The explicit presentation follows by the projective bundle formula for $\mathbb{P}(\mathcal{E})$ and by noting that perfection discards nilpotence (so weight multiplicities don't contribute). □

Finally, let us record how this plays out for perfect stratified projective bundles.

Lemma 5.8. *Let $X \in \mathrm{Sch}_k^{\mathrm{pfp},T}$ and $\mathcal{F} \in \mathrm{QCoh}^T(X)$. Assume there is a finitely presented model $X_0 \in \mathrm{Sch}_k^{\mathrm{fp},T_0}$ with an ample family of line bundles so that $\mathcal{F}$ is a pullback of a coherent sheaf $\mathcal{F}_0 \in \mathrm{Coh}^{T_0}(X_0)$. Let $Y = \mathrm{Grass}_X(\mathcal{F},1)_{\mathrm{perf}}$ be the associated perfect stratified Grassmannian. Then $K^T(X)$ is a natural direct summand of $K^T(Y)$; similarly for other localizing invariants and natural maps between them.*

Proof. By the assumptions on X_0, the quotient stack $T_0\backslash X_0$ has the resolution property [Tho87, §2]. We can thus write $\mathcal{F}_0$ as the zeroth homology $\mathcal{H}_0$ of a T_0-equivariant perfect complex on X_0 of Tor-amplitude $[1,0]$. Pulling this back to X presents $\mathcal{F}$ on X as $\mathcal{H}_0$ of a T-equivariant perfect complex $\mathcal{E}$ of Tor-amplitude $[1,0]$. Now we can form the derived enhancement $\mathrm{Grass}_X(\mathcal{E},1)$ of $\mathrm{Grass}_X(\mathcal{F},1)$ of [Jia25; Jia22] and apply Recollection 2.25 and

Proposition 2.35. This remains true after perfection by exactness of the Frobenius colimit. Since perfection discards the derived structure, the final statement indeed features the classical $Y = \mathrm{Grass}_X(\mathcal{F},1)_{\mathrm{perf}} = \mathrm{Grass}_X(\mathcal{E},1)_{\mathrm{perf}}$. □

5.4. Global functions on perfect fixed-point schemes

We record a few more auxiliary results on global functions on perfect fixed-point schemes. We hope this gives a clear geometric intuition about their behaviour, independent of the categorical construction of Hochschild homology above. Let G be the perfection of GL_n (or alternatively any perfect, split, semisimple, simply connected group).

5.4.1. Perfect proper excision for functions on fixed-point schemes. Consider an abstract blowup square (4.3). Applying the functor $\mathrm{Fix}_{\frac{G}{G}}(-)$ we get the following pullback square.

$$\begin{tikzcd} \mathrm{Fix}_{\frac{G}{G}}(Y) \arrow[d] & \mathrm{Fix}_{\frac{G}{G}}(E) \arrow[l, hook'] \arrow[d] \\ \mathrm{Fix}_{\frac{G}{G}}(X) & \mathrm{Fix}_{\frac{G}{G}}(Z) \arrow[l, hook'] \end{tikzcd} \tag{5.4}$$

The corresponding proper excision can be seen geometrically as follows.

Lemma 5.9. *There is a distinguished triangle*

$$\mathrm{R}\Gamma(\mathrm{Fix}_{\frac{G}{G}}(X),\mathcal{O}) \to \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{G}{G}}(Y),\mathcal{O}) \oplus \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{G}{G}}(Z),\mathcal{O}) \to \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{G}{G}}(E),\mathcal{O}).$$

and similarly for the versions with $\mathrm{Fix}_S(-)$. *In particular, there is an exact sequence*

$$0 \to H_0(\mathrm{Fix}_{\frac{G}{G}}(X),\mathcal{O}) \to H_0(\mathrm{Fix}_{\frac{G}{G}}(Y),\mathcal{O}) \oplus H_0(\mathrm{Fix}_{\frac{G}{G}}(Z),\mathcal{O}) \to H_0(\mathrm{Fix}_{\frac{G}{G}}(E),\mathcal{O}) \to \cdots$$

Proof. We first treat the fixed-point scheme $\mathrm{Fix}_G(-)$. Since perfections are compatible with open and closed immersions by Proposition 2.2, the square (5.4) is still an abstract blowup. The induced sequence is exact by (iv). To pass to $\mathrm{Fix}_{\frac{G}{G}}(-)$, apply the right derived functor of $(-)^G$. The version for $\mathrm{Fix}_S(-)$ works by the same reasoning. □

5.4.2. A fiberwise criterion. We record the following simple fiberwise criterion for checking whether the trace map is an isomorphism in degree zero. Let $\mathfrak{s} \hookrightarrow G$ be the perfect Steinberg section. For $X = \mathrm{pt}$ it holds that $\mathrm{Fix}_{\mathfrak{s}}(\mathrm{pt}) \cong \mathfrak{s}$ and the trace map induces canonical isomorphisms

$$\mathrm{Fix}_T(\mathrm{pt}) \xrightarrow{\cong} \mathrm{Spec}\, K_0^T(\mathrm{pt})_k, \qquad \text{and} \qquad \mathrm{Fix}_{\mathfrak{s}}(\mathrm{pt}) \xrightarrow{\cong} \mathrm{Spec}\, K_0^G(\mathrm{pt})_k.$$

The degree zero trace map can be understood fiberwise over the equivariant base by the following simple criterion.

Lemma 5.10. *Let $X \in \mathrm{Perf}_k^{\mathrm{pfp},G}$ and assume that for each closed geometric point $s \in \mathfrak{s}$, the trace map*

$$\begin{tikzcd} \mathrm{Fix}_{\mathfrak{s}}(X) \arrow[rr, "\mathrm{tr}_X"] \arrow[dr] & & \mathrm{Spec}\, K_0^G(X)_k \arrow[dl] \\ & \mathfrak{s} & \end{tikzcd}$$

induces a bijection on π_0 of the fibers over s. Then $K_0^G(X)_k \to H_0(\mathrm{Fix}_{\mathfrak{s}}(X), \mathcal{O})$ is an isomorphism.

Proof. Follows from Lemma 3.7. □

5.4.3. Geometric viewpoint on the projective bundle formula. The degree zero part of Lemma 5.5 for G can be also seen directly from the geometry of the fixed-point scheme as follows. Let $X \in \mathrm{Sch}_k^{\mathrm{pfp},G}$, $\mathcal{E} \in \mathrm{Vect}^G(X)$ and $Y = \mathbb{P}(\mathcal{E})_{\mathrm{perf}}$. We claim that

$$\begin{gathered} \mathrm{tr} : K_0^G(X;k) \to H_0(\mathrm{Fix}_{\mathfrak{s}}(X), \mathcal{O}) \text{ isomorphism} \\ \Longleftrightarrow \\ \mathrm{tr} : K_0^G(Y;k) \to H_0(\mathrm{Fix}_{\mathfrak{s}}(Y), \mathcal{O}) \text{ isomorphism.} \end{gathered}$$

Indeed, consider the diagram

$$\begin{tikzcd} \mathrm{Fix}_{\mathfrak{s}}(Y) \arrow[r, "\mathrm{tr}"] \arrow[d] & \mathrm{Spec}\, K_0^G(Y;k) \arrow[d] \\ \mathrm{Fix}_{\mathfrak{s}}(X) \arrow[r, "\mathrm{tr}"] & \mathrm{Spec}\, K_0^G(X;k) \end{tikzcd} \tag{5.5}$$

Let $(g,x)\colon \mathrm{pt} \to \mathrm{Fix}_G(X)$ be a geometric point and $(g,x)'\colon \mathrm{pt} \to \mathrm{Fix}_G(X) \to \mathrm{Spec}\, K_0^G(X;k)$ the corresponding geometric point of $\mathrm{Spec}\, K_0^G(X;k)$. We restrict attention to the fiber $\mathrm{Fix}_G(Y)_{(g,x)} = \mathrm{Fix}_g(Y_x)$ of $\mathrm{Fix}_G(Y)$ over (g,x), whose points are given by the 1-dimensional subspaces of $\mathcal{E}_x$ fixed by g. Hence $\pi_0(\mathrm{Fix}_G(Y)_{(g,x)})$ identifies with the set $\mathrm{Eig}_g(\mathcal{E}_x)$ of distinct eigenvalues of g on $\mathcal{E}_x$.

By the projective bundle formula for equivariant K-theory, we also have:

$$\mathrm{Spec}(K_0^G(Y;k))_{(g,x)'} = \mathrm{Spec}\left(k[\xi] \,/\, \left(\sum_{i=0}^{r} (-1)^i \cdot \mathrm{tr}_g[\wedge^i \mathcal{E}_x] \cdot \ \xi^{r-i}\right)\right)_{\mathrm{perf}} \tag{5.6}$$

Factoring this polynomial, it follows that $\pi_0(\mathrm{Spec}(K_0^G(Y;k))_{(g,x)'})$ is also given by the set $\mathrm{Eig}_g(\mathcal{E}_x)$ of distinct eigenvalues of g on $\mathcal{E}_x$. The restricted trace map is a bijection

$$\mathrm{tr} : \pi_0(\mathrm{Fix}_{\mathfrak{s}}(Y)_{(g,x)}) \xrightarrow{\cong} \pi_0(\mathrm{Spec}(K_0^G(Y;k))_{(g,x)'}) \tag{5.7}$$

given by identity on $\mathrm{Eig}_g(\mathcal{E}_x)$ under the above identifications. We then conclude by Lemma 5.10.

6. Trace map for perfect affine Grassmannians

Our main examples of interest are the affine Grassmannians and affine Schubert varieties, in both classical and perfect setups. As a proof of concept, we prove our main result for T-equivariant K-theory of affine Schubert varieties $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ in the perfect affine Grassmannian $\mathcal{G}\mathrm{r}$ for GL_n.

We recall affine Grassmannians and their perfect versions in §6.1. We prove our conjecture for the GL_2 affine Grassmannian in §6.2 using equivariant proper excision from Theorem 4.11 and usual projective bundle formula; this allows for explicit computations. We then in §6.3 use the semi-orthogonal decomposition for derived projective bundles [Jia23] to prove the general case of the GL_n affine Grassmannian.

6.1. Affine Grassmannians and their perfections

We make a brief overview of affine Grassmannians and their perfect versions. This is a standard material; up to perfection, we follow the excellent introduction [Zhu17b]. For perfection and the Witt-vector affine Grassmannian, see [Zhu17a; BS17].

6.1.1. Notation. Let k be a perfect field of positive characteristic; we take $k = \mathbb{F}_p$. As above, we denote $\mathrm{Sch}_k^{\mathrm{pfp}}$ the category of perfectly finitely presented perfect schemes over k. Consider the reductive group $G_0 = GL_n$ over k and let G be its perfection. Let T_0 be its split diagonal torus and T its perfection. Let $\mathbf{X}_* := \mathbf{X}_*(T_0)$ be the cocharacter lattice of T_0; we denote by $\mathbf{X}_*^+ := \mathbf{X}_*^+(T_0)$ the monoid of dominant cocharacters (with respect to the upper-triangular Borel subgroup). If $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_d)$ is a sequence of cocharacters, we denote by $\mu = |\mu_\bullet| = \mu_1 + \mu_2 + \cdots + \mu_d$ its sum.

6.1.2. Affine Grassmannian and its perfection. We mostly follow the notation from [Zhu17b]. By base change, G_0 gives a reductive group over $k(\!(t)\!)$ with an integral model over $k[\![t]\!]$. Associated to these we have the loop groups with functors of points sending a test algebra $R \in \mathrm{Alg}_k$ to

$$\mathrm{L}G_0 : R \mapsto G_0(R(\!(t)\!))), \qquad \mathrm{L}^+G_0 : R \mapsto G_0(R[\![t]\!]), \qquad \mathrm{L}^hG_0 : R \mapsto G_0(R[\![t]\!]/(t^{h+1})).$$

We call these respectively *loop group* $\mathrm{L}G_0$, *positive loop group* L^+G_0 and *truncated loop groups* L^hG_0. The quotient

$$(\mathcal{G}\mathrm{r})_0 = \mathrm{L}G_0/\mathrm{L}^+G_0$$

gives a well-defined ind-scheme called the *affine Grassmannian* of G_0. In the case of GL_n, its functor of points on k-algebras is given by

$$R \mapsto \{\Lambda \subseteq R(\!(t)\!)^{\oplus n} \mid \Lambda \text{ full } R[\![t]\!]\text{-lattice}\}.$$

We denote $\mathrm{L}G$, L^+G, L^hG, $\mathcal{G}\mathrm{r}$ the perfections of these objects. We call $\mathcal{G}\mathrm{r}$ the *(equal characteristic) perfect affine Grassmannian*; it is the quotient

$$\mathcal{G}\mathrm{r} = \mathrm{L}G/\mathrm{L}^+G.$$

From now on, everything is implicitly perfect. For the sake of readability, we omit this adjective. Since the same results hold before perfection, there is no risk of confusion.

6.1.3. Affine Schubert cells. For $\mu \in \mathbf{X}_*$, we let t^μ be the image of the point $t \in \mathrm{L}\mathbb{G}_m(k)$ under $\mathrm{L}\mu : \mathrm{L}\mathbb{G}_m(k) \to \mathrm{L}G(k)$. The map $\mu \mapsto t^\mu$ gives a group homomorphism $\mathbf{X}_* \to \mathrm{L}G(k)$. The L^+G (double) cosets induce a scheme theoretic stratification

$$\mathrm{L}G(k) = \bigsqcup_{\mu \in \mathbf{X}_*^+} \mathrm{L}^+G(k) \cdot t^\mu \cdot \mathrm{L}^+G(k) \qquad \text{and} \qquad \mathcal{G}\mathrm{r}(k) = \bigsqcup_{\mu \in \mathbf{X}_*^+} \mathrm{L}^+G(k) \cdot t^\mu$$

of $\mathrm{L}G$ resp. $\mathcal{G}\mathrm{r}$ into locally closed pieces $\mathrm{L}G_\mu$ resp. $\mathcal{G}\mathrm{r}_\mu$ labelled by $\mu \in \mathbf{X}_*^+$ and called the *affine Schubert cells*.

Explicitly, $\mathcal{G}\mathrm{r}_\mu$ is an affine bundle of dimension $(2\rho, \mu)$ over the partial flag variety G/P_μ, where P_μ is the parabolic subgroup associated to μ. The structure map is given by setting the formal parameter t equal to zero as follows:

$$\begin{tikzcd}
\mathrm{L}G_\mu \arrow[d] & \mathrm{L}^+G \cdot t^\mu \arrow[l, hook'] \arrow[r, "(g \mapsto g \cdot t^{-\mu})"] \arrow[d] & \mathrm{L}^+G \arrow[r, two heads, "(t \mapsto 0)"] \arrow[d] & G \arrow[d] \\
\mathcal{G}\mathrm{r}_\mu \arrow[r, equal] & \mathcal{G}\mathrm{r}_\mu \arrow[r, equal] & \mathcal{G}\mathrm{r}_\mu \arrow[r, two heads] & G/P_\mu
\end{tikzcd} \tag{6.1}$$

The vertical maps in (6.1) are right torsors for the following groups and comparison maps between them:

$$\mathrm{L}^+G \hookleftarrow \mathrm{L}^+G \cap (t^{-\mu} \cdot \mathrm{L}^+G \cdot t^\mu) \xrightarrow{g \mapsto t^\mu g t^{-\mu}} (t^\mu \cdot \mathrm{L}^+G \cdot t^{-\mu}) \cap \mathrm{L}^+G \xrightarrow{t \mapsto 0} P_\mu$$

The map $\mathcal{G}\mathrm{r}_\mu \to G/P_\mu$ is an isomorphism if and only if μ is minuscule.

6.1.4. Affine Schubert varieties. The closures $X_{\leqslant \mu} = \mathcal{G}\mathrm{r}_{\leqslant \mu}$ of the affine Schubert cells $\mathcal{G}\mathrm{r}_\mu$ respect this stratification: there is a set-theoretic decomposition $\mathcal{G}\mathrm{r}_{\leqslant \mu} = \bigsqcup_{\lambda \leqslant \mu} \mathcal{G}\mathrm{r}_\lambda$. These $\mathcal{G}\mathrm{r}_{\leqslant \mu}$ are called the *affine Schubert varieties*. In particular, they carry an action of L^+G.

In contrast to the ind-scheme $\mathcal{G}\mathrm{r}$, affine Schubert varieties are of finite type. The singular locus of $\mathcal{G}\mathrm{r}_{\leqslant \mu}$ is precisely $\mathcal{G}\mathrm{r}_{\leqslant \mu} \setminus \mathcal{G}\mathrm{r}_\mu = \bigsqcup_{\lambda < \mu} \mathcal{G}\mathrm{r}_\lambda$. In particular, $\mathcal{G}\mathrm{r}_{\leqslant \mu}$ is smooth if and only if μ is minimal in the dominance order.

The singularities of $\mathcal{G}\mathrm{r}_{\leqslant \mu}$ are rational by [Zhu17b, Theorem 2.1.21] and Observation 3.3.

In terms of the moduli description, $\mathcal{G}\mathrm{r}_{\leqslant \mu}$ is the closed subfunctor of $\mathcal{G}\mathrm{r}$ parametrizing those Λ which are in relative position $\leqslant \mu$ with respect to the standard lattice $\Lambda_0 = k[\![t]\!]^{\oplus n}$.

6.1.5. Affine Demazure resolutions. For any d-tuple $\mu_\bullet = (\mu_1, \ldots, \mu_d)$ of dominant cocharacters $\mu_1, \ldots\ldots, \mu_d \in \mathbf{X}_*^+$, the associated *affine Demazure variety* is the iterated twisted product

$$\begin{aligned} Y_{\leqslant \mu_\bullet} = \mathcal{G}\mathrm{r}_{\leqslant \mu_\bullet} &:= \mathrm{L}G_{\leqslant \mu_1} \overset{\mathrm{L}^+G}{\times} \mathrm{L}G_{\leqslant \mu_2} \overset{\mathrm{L}^+G}{\times} \ldots \overset{\mathrm{L}^+G}{\times} \mathrm{L}G_{\leqslant \mu_d} \overset{\mathrm{L}^+G}{\times} \mathrm{pt} \\ &= \mathcal{G}\mathrm{r}_{\leqslant \mu_1} \tilde{\times} \mathcal{G}\mathrm{r}_{\leqslant \mu_2} \tilde{\times} \ldots \tilde{\times} \mathcal{G}\mathrm{r}_{\leqslant \mu_d}. \end{aligned}$$

with respect to the L^+G–torsors $\mathrm{L}G_{\leqslant \mu_i} \to \mathcal{G}\mathrm{r}_{\leqslant \mu_i}$ for $i = 1, 2, \ldots, d$.

Note that $\mathcal{G}\mathrm{r}_{\leqslant \mu_\bullet}$ carries a natural action of $\mathrm{L}_{\leqslant \mu_1}G$ induced from the left translation on the first factor. In particular, it has an action of L^+G. The $(d-1)$-fold multiplication

$$\mathrm{L}G_{\leqslant \mu_1} \times \mathrm{L}G_{\leqslant \mu_2} \times \cdots \times \mathrm{L}G_{\leqslant \mu_d} \xrightarrow{m} \mathrm{L}G_{\leqslant |\mu_\bullet|}$$

factors through the affine Demazure variety, giving rise to the convolution map

$$\mathcal{G}r_{\leqslant\mu_\bullet} \xrightarrow{m} \mathcal{G}r_{\leqslant|\mu_\bullet|}\,. \tag{6.2}$$

This is a proper surjective L^+G-equivariant map. It restricts to an an isomorphism over the open cell $\mathcal{G}r_{|\mu_\bullet|} \hookrightarrow \mathcal{G}r_{\leqslant|\mu_\bullet|}$ in the target.

In terms of the moduli description, $\mathcal{G}r_{\leqslant\mu_\bullet}$ parametrizes chains of lattices $(\Lambda_0, \Lambda_1, \ldots, \Lambda_d)$ with Λ_i and Λ_{i-1} in relative position μ_i and Λ_0 the standard lattice. The convolution map sends this chain to Λ_d.

In particular, assume all μ_i are minuscule. Since each $\mathcal{G}r_{\leqslant\mu_i} = G/P_{\mu_i}$ is smooth, $\mathcal{G}r_{\leqslant\mu_\bullet}$ is smooth as well. In fact, it is an iterated Grassmannian bundle, coming from a sequence of equivariant algebraic vector bundles, which we now describe.

6.1.6. Affine Demazure resolutions as Grassmannian bundles. Given $\mu \in \mathbf{X}_\bullet^+$, the variety $X_{\leqslant\mu} := \mathcal{G}r_{\leqslant\mu}$ carries a natural vector bundle $\mathcal{E} = \mathcal{E}_{\leqslant\mu}$ defined as follows. We write $\mathrm{mod}^{\mathrm{fl}}(R[\![t]\!])$ for the abelian category of finite length $R[\![t]\!]$-modules.

Let $\omega_1, \omega_2, \ldots, \omega_n$ be the fundamental dominant coweights of GL_n given in coordinates as $\omega_j = (1, \ldots, 1, 0, \ldots, 0)$ with j ones and $n-j$ zeroes. Let $\omega_1^*, \ldots, \omega_n^*$ be given as $\omega_j^* = \omega_{n-j} - \omega_n = (0, \ldots, 0, -1, \ldots, -1)$ with $n-j$ zeroes and j minus ones. For any $\mu \in \mathbf{X}_*^+$, there is a unique integer m such that $\mu = b\omega_n + \mu_1 + \mu_2 + \cdots + \mu_d$ with each μ_i among $\omega_1^*, \ldots, \omega_{n-1}^*$. This decomposition is unique up to reordering the terms; we call $\mathrm{lg}(\mu) := d$ the length of μ.

Note that b is the smallest integer such that any $\Lambda \in X_{\leqslant\mu}(R)$ contains $t^b\Lambda_0$. Then define

$$\begin{aligned}\mathcal{E} = \mathcal{E}_{\leqslant\mu} : X_{\leqslant\mu}(R) &\to \mathrm{mod}^{\mathrm{fl}}(R[\![t]\!])\\ \Lambda &\mapsto \Lambda/t^b\Lambda_0.\end{aligned}$$

This assembles into a L^+G-equivariant vector bundle on $X_{\leqslant\mu}$, which moreover carries a nilpotent operator $t : \mathcal{E} \to \mathcal{E}$ making it an object of $\mathrm{mod}^{\mathrm{fl}}(R[\![t]\!])$. Consider now

$$\mathcal{E}_\bullet := [\mathcal{E} \xrightarrow{t} \mathcal{E}] \qquad \text{and} \qquad \mathcal{F} = \mathcal{F}_{\leqslant\mu} := \mathcal{H}_0(\mathcal{E}_\bullet) = \mathrm{coker}(\mathcal{E} \xrightarrow{t} \mathcal{E}).$$

Then $\mathcal{E}_\bullet$ is a perfect complex of Tor amplitude $[1, 0]$ resolving the coherent sheaf $\mathcal{F}$. The affine Schubert stratification of $X_{\leqslant\mu}$ is a flattening stratification for $\mathcal{F}$, meaning that the restrictions of $\mathcal{F}$ to affine Schubert cells are vector bundles.

The affine Demazure resolution is given inductively as follows. Write $\mu_\bullet = (\lambda, \mu_d)$ with μ_d coming from above and $\lambda = \mu - \mu_d$. Say $\mu_d = \omega_j^*$ for some $1 \leqslant j \leqslant n-1$. Then the convolution map (6.2) is given by the structure map

$$Y_{\leqslant\mu_\bullet} = \mathrm{Grass}_{X_{\leqslant\mu}}(\mathcal{F}, j) \xrightarrow{f} X_{\leqslant\mu}.$$

Denote $\mathcal{E}'$ the preimage under $\mathcal{E} \twoheadrightarrow \mathcal{F}$ of the tautological quotient of $\mathcal{F}$ on $Y_{\leqslant\mu_\bullet}$. The left-hand term of the tautological exact sequence on $Y_{\leqslant\mu_\bullet}$

$$0 \to \mathcal{E}' \to f^*\,\mathcal{E} \to \mathcal{E}'' \to 0$$

is an equivariant coherent sheaf on $Y_{\leqslant\mu_\bullet}$ with an action of t and one can continue inductively.

On the other hand, $Y_{\leqslant\mu_\bullet}$ is an honest Grassmannian bundle over $X_{\leqslant\lambda}$. For this, consider the vector bundle $\mathcal{G} = \mathcal{G}_{\leqslant\lambda}$ on $X_{\leqslant\lambda}$ given by

$$\begin{aligned} \mathcal{G} : X_{\leqslant\lambda}(R) &\to \mathrm{mod}(R) \hookrightarrow \mathrm{mod}^{\mathrm{fl}}(R[\![t]\!]) \\ \Lambda &\mapsto t^{-1}\Lambda/\Lambda. \end{aligned}$$

This is a L^+G-equivariant vector bundle on $X_{\leqslant\lambda}$, carrying the zero action of t. We then have

$$Y_{\leqslant\mu_\bullet} = \mathrm{Grass}_{X_{\leqslant\lambda}}(\mathcal{G}, j).$$

Altogether, we obtain the convolution diagram

$$\begin{tikzcd} Y_{\leqslant\mu_\bullet} \arrow[r, "g"] \arrow[d, "f"] & X_{\leqslant\lambda} \\ X_{\leqslant\mu} & \end{tikzcd} \tag{6.3}$$

6.1.7. Witt-vector affine Grassmannian. The analogs of the above spaces for mixed characteristic groups were constructed in [Zhu17a; BS17]. These objects are defined only in the perfect setup.

Given a non-archimedean local field of mixed characteristic $\mathcal{K}$ with ring of integers $\mathcal{O}$ and residue field k, base change of G_0 gives a reductive group over $\mathcal{K}$ with an integral model over $\mathcal{O}$. Let $\mathbb{W}(-)$ be the functor of ramified Witt vectors on perfect k-algebras, with a uniformizer ϖ. Associated to this we have the p-adic loop groups with functors of points sending a test algebra $R \in \mathrm{Alg}_k^{\mathrm{perf}}$ to

$$\mathrm{L}G_0 : R \mapsto G_0(\mathbb{W}(R)[\tfrac{1}{\varpi}]), \qquad \mathrm{L}^+G_0 : R \mapsto G_0(\mathbb{W}(R)), \qquad \mathrm{L}^hG_0 : R \mapsto G_0(\mathbb{W}_h(R)).$$

The quotient

$$\mathcal{G}\mathrm{r} = \mathrm{L}G/\mathrm{L}^+G$$

is a well-defined perfect ind-scheme called the *Witt-vector affine Grassmannian*. In the case of GL_n, its functor of points on perfect k-algebras is given by

$$R \mapsto \{\Lambda \subseteq \mathbb{W}(R)[\tfrac{1}{\varpi}]^{\oplus n} \mid \Lambda \text{ full } \mathbb{W}(R)\text{-lattice}\}.$$

The discussion of the above paragraphs can be repeated in the Witt vector case upon replacing $R[\![t]\!]$ by $\mathbb{W}(R)$ and t by ϖ; see [Zhu17a; BS17]. The resulting affine Schubert varieties are rational by [BS17, Remark 8.5].

We still have the sheaf

$$\begin{aligned} \mathcal{E} : X_{\leqslant\mu}(R) &\to \mathrm{mod}^{\mathrm{fl}}(\mathbb{W}(R)) \\ \Lambda &\mapsto \Lambda/\varpi^b\Lambda_0. \end{aligned}$$

Strictly speaking, this is not a vector bundle on $X_{\leqslant\mu}$ since its fibers are not $\mathbb{F}_p$-modules. However, $\mathcal{F} = \mathrm{coker}(\mathcal{E} \xrightarrow{\varpi} \mathcal{E})$ is a quasi-coherent sheaf on $X_{\leqslant\mu}$ and $\mathcal{G} : \Lambda \mapsto \varpi^{-1}\Lambda/\Lambda$ is a vector bundle, making the rest of the discussion preceding (6.3) valid; see [BS17, §7-8].

6.2. Trace map for perfect GL_2 affine Grassmannian

To convey the main idea, we now prove Theorem 1.1 for the perfect GL_2 affine Grassmannian. This only needs equivariant proper excision and the projective bundle formula. Furthermore, proper excision gives a constructive recipe for computing the ring $K^T(X_{\leqslant\mu})$ integrally.

Let $\mathcal{G}\mathrm{r}$ be either the perfect affine Grassmannian or the Witt-vector affine Grassmannian.

Theorem 6.1. *Let $G_0 = GL_2$ with its diagonal torus T_0 over k. Then for each affine Schubert variety $X_{\leqslant\mu}$ of $\mathcal{G}\mathrm{r}$, the trace*

$$\mathrm{tr} : K^T(X_{\leqslant\mu}; k) \to \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(X_{\leqslant\mu}), \mathcal{O}) \tag{6.4}$$

is an equivalence.

Proof. Note that $\mathbf{X}_\bullet^+ = \mathbb{N}_{\geqslant 0}[\omega_1, \omega_2]$ is the free commutative monoid on the set of fundamental coweights. Each affine Schubert variety is isomorphic to one of the form $X_{\leqslant k\omega_1}$. Indeed, there is an equivariant isomorphism $X_{\leqslant(a\omega_1+b\omega_2)} \cong X_{\leqslant a\omega_1}$. We prove the result by induction on a.

The base cases of the induction are $\mu = 0 \cdot \omega_1$ and $\mu = 1 \cdot \omega_1$. In the first case, $X_{\leqslant 0} = \mathrm{pt}$ so the trace map is indeed an isomorphism supported in homological degree zero by Lemma 5.2. In the second case, $X_{\leqslant\omega_1}$ is the perfection of $\mathbb{P}^1$ with the natural T-action, so the trace map is also an isomorphism supported in homological degree zero by Example 5.7.

Let $a \geqslant 1$ and assume that we know the result for all $j\omega_1$ with $j \leqslant a$; we prove it for $\mu := (a+1)\omega_1$. Denote $\mu' := (a-1)\omega_1 + \omega_2$ the maximal element strictly below μ in the dominance order.

Put $\mu_\bullet := (\lambda, \omega_1)$ and consider (6.3). This gives the affine Demazure variety $Y_{\leqslant\mu_\bullet}$ with a map $g : Y_{\leqslant\mu_\bullet} \to X_{\leqslant\lambda}$. We also have the affine Demazure morphism $f : Y_{\leqslant\mu_\bullet} \to X_{\leqslant\mu}$ which leads to an abstract blowup square

$$\begin{tikzcd} Y_{\leqslant\mu_\bullet} \arrow[d, "f"] & E \arrow[l] \arrow[d, "f'"] \\ X_{\leqslant\mu} & X_{\leqslant\mu'} \arrow[l] \end{tikzcd} \tag{6.5}$$

- By construction, $Y_{\leqslant\mu_\bullet}$ is a perfect $\mathbb{P}^1$-bundle over $X_{\leqslant\lambda}$ via g. Hence the claim holds for $Y_{\leqslant\mu_\bullet}$ by induction and Lemma 5.5.
- Since $X_{\leqslant\mu'} \cong X_{\leqslant(a-1)\omega_1}$, we know the claim by induction.
- The exceptional part E is a perfect $\mathbb{P}^1$-bundle over $X_{\leqslant\mu'}$ under f'. Indeed, it is the projectivization of the restriction of the quasi-coherent sheaf $\mathcal{F}_{\leqslant\mu}$ to the subvariety $X_{\leqslant\mu'}$; this restriction is a rank two vector bundle on $X_{\leqslant\mu'}$ by the combinatorics of coweights in GL_2. Since we already know the claim for $X_{\leqslant\mu'}$, we deduce it for E by Lemma 5.5.

Now consider the following fiber sequences coming from (6.5) via Theorem 4.11.

$$\begin{tikzcd} K^T(X_{\leqslant\mu}; k) \arrow[r] \arrow[d] & K^T(Y_{\leqslant\mu_\bullet}; k) \oplus K^T(X_{\leqslant\mu'}; k) \arrow[r] \arrow[d] & K^T(E; k) \arrow[d] \\ HH^T(X_{\leqslant\mu}/k) \arrow[r] & HH^T(Y_{\leqslant\mu_\bullet}/k) \oplus HH^T(X_{\leqslant\mu'}/k) \arrow[r] & HH^T(E/k) \end{tikzcd}$$

Combining the previous three items and the associated long exact sequences, we deduce by 5-lemma that $\mathrm{tr} : K^T(X_{\leqslant\mu}; k) \to HH^T(X_{\leqslant\mu}/k)$ is an isomorphism. □

Corollary 6.2. *Taking the zeroth homotopy group in* (6.4) *returns the natural isomorphism*

$$\mathrm{tr} : K_0^T(X_{\leqslant\mu}; k) \cong H_0(\mathrm{Fix}_{\frac{T}{T}}(X_{\leqslant\mu}), \mathcal{O}) \cong H_0(\mathrm{Fix}_T(X_{\leqslant\mu}), \mathcal{O}). \tag{6.6}$$

The above also shows that both sides of (6.4) are supported in homological degrees $\leqslant 0$. We will later see that they are actually supported in the homological degree zero only.

Corollary 6.3. *Both* $K^T(X_{\leqslant\mu}; k)$ *and* $\mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(X_{\leqslant\mu}), \mathcal{O})$ *are supported in homological degrees* $\leqslant 0$. *In particular,* $K_0^T(X_{\leqslant\mu})$ *has no* p*-torsion.*

Proof. Note that $(-)^T$ on functions is exact, so the right-hand side of (6.4) is supported in homological degrees $\leqslant 0$. Since tr is an isomorphism, the same is true for the left-hand side. The final line follows from $K_1^T(X; k) = K_0^T(X) \otimes^1_{\mathbb{Z}} k$. $\square$

Theorem 6.4. *Integrally, it holds that*

$$K^T(X_{\leqslant\mu}) \simeq KH^T(X_{\leqslant\mu}).$$

Proof. The proof of Theorem 6.1 in particular shows how to describe $K^T(X_{\leqslant\mu})$ from $K^T(-)$ of smooth T-equivariant schemes via homotopy fiber sequences associated to perfect abstract blowups. The same sequences exist for $KH^T(-)$ by Proposition 4.15, and there is a natural comparison map $K^T(-) \to KH^T(-)$. This is an isomorphism on smooth T-equivariant pfp schemes, so the result follows by induction. $\square$

6.3. Trace map for perfect GL_n affine Grassmannian

In general, it is demanding to control the K-theory of the exceptional fibers by proper excision only. The extra ingredient is the following: we also get split homotopy fiber sequences associated to stratified Grassmannian bundles $Y \to X$ by the semi-orthogonal decomposition of [Jia23]. With this extra knowledge, we prove the general case of Theorem 1.1. Let $\mathcal{G}\mathrm{r}$ be either the perfect affine Grassmannian or the Witt-vector affine Grassmannian for GL_n.

Theorem 6.5. *Let* $\mu \in \mathbf{X}_\bullet^+$ *and* $X_{\leqslant\mu}$ *the corresponding perfect affine Schubert variety in* $\mathcal{G}\mathrm{r}$. *Then*

$$\mathrm{tr} : K^T(X_{\leqslant\mu}; k) \to \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(X_{\leqslant\mu}), \mathcal{O}) \tag{6.7}$$

is an equivalence supported in homological degree zero.

Proof. We prove the theorem by induction on $\mu \in \mathbf{X}_\bullet^+$ with respect to $\lg(\mu)$. If $\mu = 0$, then $X_{\leqslant\mu} = \mathrm{pt}$ and the statement is clear by Lemma 5.2.

Assume now $\mu \neq 0$ and decompose it as $\mu = b\omega_n + \mu_1 + \cdots + \mu_d$ with $d \geqslant 1$ and each μ_i among $\omega_1^*, \ldots, \omega_{n-1}^*$. Say $\mu_d = \omega_j^*$. Let $\lambda = \mu - \mu_d$ and put $\mu_\bullet = (\lambda, \mu_d)$. Consider the following diagram building on (6.3):

$$\begin{tikzcd} \widetilde{Y}_{\leqslant\mu_\bullet} \arrow[r, "h"] \arrow[dr] & Y_{\leqslant\mu_\bullet} \arrow[r, "g"] \arrow[d, "f"] & X_{\leqslant\lambda} \\ & X_{\leqslant\mu} & \end{tikzcd} \tag{6.8}$$

Here $\widetilde{Y}_{\leqslant\mu_\bullet}$ parametrizes the data of $Y_{\leqslant\mu_\bullet}$ together with a full flag in the rank j quotient corresponding to μ_d. The map h is thus a perfected full flag variety bundle (for the group GL_j).

Now, the map fh is an iterated perfected relative $\mathrm{Grass}(-,1)$-bundle. Indeed, having $\mathcal{F} = \mathcal{H}_0(\mathcal{E} \to \mathcal{E})$ on $X_{\leqslant\mu}$, first take $\mathrm{Grass}_{X_{\leqslant\mu}}(\mathcal{F},1)$. The tautological exact sequence on $\mathrm{Grass}_{X_{\leqslant\mu}}(\mathcal{F},1)$ then reads as

$$0 \to \mathcal{F}' \to f^*\mathcal{F} \to \mathcal{F}'' \to 0$$

and we can continue the process with $\mathcal{F}'$. After j steps we arrive at $\widetilde{Y}_{\leqslant\mu_\bullet}$. Note that although these $\mathrm{Grass}(-,1)$-bundles may carry a nontrivial derived structure, their perfections are classical by (vi).

We already know the statement of the theorem holds for $X_{\leqslant\lambda}$ by induction. By stability on perfected partial flag variety bundles from Lemma 5.6, we deduce it for $Y_{\leqslant\mu_\bullet}$ and subsequently for $\widetilde{Y}_{\leqslant\mu_\bullet}$ as well.

Applying Lemma 5.8 to fh, the trace map $K^T(X_{\leqslant\mu};k) \to HH^T(X_{\leqslant\mu}/k)$ is a direct summand of the trace map $K^T(\widetilde{Y}_{\leqslant\mu_\bullet};k) \to HH^T(\widetilde{Y}_{\leqslant\mu_\bullet}/k)$, which is an isomorphism supported in homological degree zero by the above. We deduce the same for the former, completing the inductive step. $\square$

Remark 6.6. Note that at least in the equal characteristic case, $\mathcal{E}_\bullet$ gives a preferred presentation of $\mathcal{F}$ with Tor-amplitude $[1,0]$, simplifying the application of Lemma 5.8.

Theorem 6.7. *Integrally, the natural map is an equivalence*

$$K^T(X_{\leqslant\mu}) \simeq KH^T(X_{\leqslant\mu}).$$

Proof. Both $K(-)$ and $KH(-)$ are localizing invariants and there is a natural map $K(-) \to KH(-)$. Note that $K^T(-)$ and $KH^T(-)$ agree on smooth T-equivariant schemes (in particular on a point). The inductive argument from the proof of Theorem 6.5 thus applies. $\square$

7. Sample computations in affine Grassmannians

The proof of Theorem 6.1 is constructive: it yields an inductive way to compute the ring $K_0^T(X_{\leqslant\mu})$ integrally in terms of generators and relations. We provide a few concrete instances of this computation in practice.

Example 7.1. Assume k has $\operatorname{char} k \geqslant 3$. Let $G_0 = GL_2$ and T_0 its diagonal torus. Let $\mu = 2\omega_1$ and $X = \mathcal{G}\mathrm{r}_{\leqslant\mu}$. We get

$$K_0^T(X;k) \cong \left(k[t_1^{\pm1}, t_2^{\pm1}, e_1]/\left((e_1 - 2t_1)(e_1 - 2t_2)(e_1 - (t_1+t_2))\right)\right)_{\mathrm{perf}}.$$

Proof. Let $\mu_\bullet = (\omega_1, \omega_1)$ and $Y = \mathcal{G}\mathrm{r}_{\leqslant\mu_\bullet}$. The associated abstract blowup square induces a long exact sequence in K-theory, which splits into short exact sequences by the description of the K-theory of projective bundles used for $Y \to E$. We thus get a Mayer–Vietoris square

$$\begin{array}{ccc} K_0^T(Y;k) & \longrightarrow & K_0^T(E;k) \\ \uparrow & & \uparrow \\ K_0^T(X;k) & \longrightarrow & K_0^T(\mathrm{pt};k) \end{array}$$

and we can compute $K_0^T(X;k)$ from the associated short exact sequence as a kernel. In particular $K_0^T(X;k) = K_0^T(X)\otimes_{\mathbb{Z}}k$. We can explicitly write:

$$\begin{aligned} K_0^T(\mathrm{pt};k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}]\right)_{\mathrm{perf}} \\ K_0^T(E;k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, \xi]/((\xi - t_1)(\xi - t_2))\right)_{\mathrm{perf}} \\ K_0^T(Y;k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, \xi_0, \xi_1]/\left(\begin{smallmatrix}(\xi_0-t_1)(\xi_0-t_2)\\ (\xi_1-t_1)(\xi_1-t_2)\end{smallmatrix}\right)\right)_{\mathrm{perf}} \end{aligned}$$

The map $K_0^T(Y;k) \to K_0^T(E;k)$ is given by $\xi_0 \mapsto \xi$, $\xi_1 \mapsto (-\xi + t_1 + t_2)$. The above Mayer–Vietoris short exact sequence then shows $K_0^T(X;k) \hookrightarrow K_0^T(Y;k)$ is the subring generated by

$$\begin{aligned} e_1 &:= \xi_0 + \xi_1 \\ e_2 &:= \xi_0\xi_1 \end{aligned}$$

The computation of this kernel can be done before perfection by exactness of filtered colimits. Explicitly, $K_0^T(X;k)$ is given by the rank three flat $K_0^T(\mathrm{pt};k)$-algebra

$$\begin{aligned} K_0^T(X;k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, e_1, e_2] \,/\, \left(\begin{smallmatrix} e_1^2-(t_1+t_2)e_1-2e_2+2t_1t_2 \\ e_2^2-(t_1+t_2)^2e_2+t_1t_2(t_1+t_2)e_1-t_1^2t_2^2 \\ e_1e_2-2(t_1+t_2)e_2+t_1t_2e_1 \end{smallmatrix}\right)\right)_{\mathrm{perf}} \\ &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, e_1] \,/\, \left(e_1^3 - 3(t_1+t_2)e_1^2 + (4t_1t_2 + 2(t_1+t_2)^2)e_1 - 4(t_1+t_2)t_1t_2\right)\right)_{\mathrm{perf}} \\ &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, e_1] \,/\, ((e_1 - 2t_1)(e_1 - 2t_2)(e_1 - (t_1+t_2)))\right)_{\mathrm{perf}}. \end{aligned}$$

The first line holds by checking these relations for e_1 and e_2; these are all of them by counting ranks in the Mayer–Vietoris sequence. To get the second line, we get rid of e_2 from the first relation (this is the only point where we use $\mathrm{char}\, k \neq 2$). Substituting for e_2 into the third relation, we obtain the desired cubic relation for e_1. Substituting for e_2 into the second relation and using the above cubic relation for e_1, the second relation becomes tautological. The passage to the third line is clear. $\square$

Example 7.2. Let k be of $\mathrm{char}\, k \neq 3$. Let $G_0 = GL_3$ and T_0 its diagonal torus. Let $\mu = \omega_2 + \omega_1$ and $X \cong \mathcal{G}\mathrm{r}_{\leqslant\mu}$. We get

$$K_0^T(X;k) \cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, m_1, m_2] / \left(\begin{smallmatrix} 3m_1^5+(15c_2-5c_1^2)m_1^3+(2c_1^3-9c_1c_2+9c_3)m_2+(14c_2^2-4c_1^2c_2-6c_1c_3)m_1 \\ 3m_1^2m_2-2c_1m_1^3+(3c_2-c_1^2)m_2+(c_1c_2-9c_3)m_1 \\ 3m_2^2+m_1^4-4c_1m_1m_2+4c_2m_1^2 \end{smallmatrix}\right)\right)_{\mathrm{perf}}$$

where we denote elementary symmetric polynomials in t_1, t_2, t_3 by

$$c_1 := t_1 + t_2 + t_3, \qquad c_2 := t_1t_2 + t_2t_3 + t_1t_3, \qquad c_3 := t_1t_2t_3.$$

This is rank 7 finite free algebra over $K_0^T(\mathrm{pt};k)$.

Proof. Let $\mu_\bullet = (\omega_2, \omega_1)$ and $Y = \mathcal{G}\mathrm{r}_{\leqslant \mu_\bullet}$. The associated abstract blowup square has Y a perfected $\mathbb{P}^2$-bundle over $\mathbb{P}^2$, $Z = \mathrm{pt}$ and E perfect $\mathbb{P}^2$ as a section. One immediately gets

$$\begin{aligned}
K_0^T(Y;k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, \chi_0, \chi_1] \,/\, \left(\begin{smallmatrix}(\chi_0 - t_1)(\chi_0 - t_2)(\chi_0 - t_3) \\ (\chi_1 - t_1)(\chi_1 - t_2)(\chi_1 - t_3)\end{smallmatrix}\right)\right)_{\mathrm{perf}} \\
K_0^T(Z;k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]\right)_{\mathrm{perf}} \\
K_0^T(E;k) &\cong \left(k[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, \chi] \,/\, ((\chi - t_1)(\chi - t_2)(\chi - t_3))\right)_{\mathrm{perf}}
\end{aligned}$$

with the map $K_0^T(Y;k) \to K_0^T(E;k)$ given by $\chi_0 \mapsto \chi$, $\chi_1 \mapsto \chi$. Since the induced long exact sequence in K-theory splits into short exact sequences (by the shape of Y and E), we can explicitly compute $K_0^T(X;k)$ from the short exact sequence as a kernel. Moreover, since the remaining terms are projective $K_0^T(\mathrm{pt};k)$-modules of ranks $9, 1, 3$, it follows that $K_0^T(X;k)$ is a projective $K_0^T(\mathrm{pt};k)$-module of rank 7. The elements

$$\begin{aligned}
m_1 &:= \chi_0 - \chi_1 \\
m_2 &:= \chi_0^2 - \chi_1^2
\end{aligned}$$

are visibly in this kernel. A lengthy computation shows that they satisfy precisely the above ideal of relations inside $K_0^T(Y;k)$. For rank reasons, it follows that they generate $K_0^T(X;k)$ as an algebra. □

Remark 7.3. The above computations work the same for $KH^T(-)$ and $KH^G(-)$ in any characteristic (without perfection). In particular, this applies to characteristic zero.

Remark 7.4. The presentations from Examples 7.1 and 7.2 can be compared to their cohomological analogs studied in [Hau24] by different methods: see [Hau24, (4.6)] and [Hau24, (4.10)].

We want to briefly mention how the trace map looks for affine Demazure resolutions, as the above rings for $X_{\leqslant \mu}$ embed into those for $Y_{\leqslant \mu_\bullet}$.

Example 7.5. Let $\mu_\bullet = (\mu_1, \ldots, \mu_d)$ be a sequence of fundamental coweights and $Y_{\leqslant \mu_\bullet}$ the corresponding affine Demazure resolution. Then

$$\mathrm{tr} : K^T(Y_{\leqslant \mu_\bullet}; k) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(Y_{\leqslant \mu_\bullet}), \mathcal{O})$$

is an equivalence supported in homological degree zero. Explicitly,

$$K_0^T(Y_{\leqslant \mu_\bullet}) = K_0^T(G/P_{\mu_1}) \underset{K_0^T(\mathrm{pt})}{\otimes} K_0^T(G/P_{\mu_2}) \underset{K_0^T(\mathrm{pt})}{\otimes} \cdots \underset{K_0^T(\mathrm{pt})}{\otimes} K_0^T(G/P_{\mu_k})$$

Proof. This follows from Lemma 5.2 by iterating Lemma 5.6.

Alternatively, the trace map in degree zero for affine Demazure resolutions can be understood directly from the twisted product description by Lemma 2.9 and Lemma 5.10; these arguments work G-equivariantly. □

8. More examples

We now record a few more examples to convey a feeling about perfect trace maps. These examples are interesting and relate to other results in the literature. However, since they are rather unrelated to our original motivation, we confine them to this section.

8.1. Trivial group

If $G = e$ is the trivial group over k and X is perfectly proper, the trace map in degree zero is the following isomorphism encoding simple topological information.

Observation 8.1. *Assume X is perfectly proper over k. Then the trace map is an isomorphism*

$$\mathrm{tr}_X : K_0(X)_k \xrightarrow{\cong} H_0(\mathrm{Fix}_e(X), \mathcal{O}).$$

Both sides agree with the free k-module $k[\pi_0(X)]$ on the set of connected components of X. For $i \geqslant 1$, both $K_i(X; k)$ and $H_i(\mathrm{Fix}_e(X), \mathcal{O})$ are zero.

Proof. Let X_0 be a proper model of X. The reduced global sections $H_0(X_0, \mathcal{O}^{\mathrm{red}})$ have to be constant on each connected component by properness. Since X_0 is quasi-compact, there are finitely many connected components, so $H_0(X_0, \mathcal{O}^{\mathrm{red}}) = k[\pi_0(X_0)]$. Then

$$\begin{aligned} H_0(\mathrm{Fix}_e(X), \mathcal{O}) = H_0(X, \mathcal{O}) &= \operatorname*{colim}_{\varphi^*} H_0(X_0, \mathcal{O}) \\ &= \operatorname*{colim}_{\varphi^*} H_0(X_0, \mathcal{O}^{\mathrm{red}}) = k[\pi_0(X_0)] = k[\pi_0(X)] \end{aligned}$$

because k and hence $k[\pi_0(X_0)]$ is already perfect.

Concerning K-theory, it holds that $K_0(X) = \widetilde{K}_0(X) \oplus H_0(X, \mathbb{Z})$ with $\widetilde{K}_0(X)$ being a $\mathbb{Z}[\frac{1}{p}]$-module. Indeed, [Cou23, Theorem 3.1.2 and Remark 3.1.3] show that this is the case on affines, where it follows from the finite γ-filtration [Wei13, Theorem II.4.6]. Moreover, all $K_i(X)$ with $i > 0$ are also $\mathbb{Z}[\frac{1}{p}]$-modules. The general case of non-affine pfp X now follows by the Zariski descent spectral sequence [TT90, Proposition 8.3, (8.3.2)]. See also [Wei13, Proposition II.8.8.4].

Tensoring $K_0(X)$ with k, the first term $\widetilde{K}_0(X)$ vanishes, while the second becomes $k[\pi_0(X_0)]$. Since the trace map is induced by locally taking the rank, it induces the identity under the above identifications.

For the final statement, $(K_0(X) \otimes^{\mathrm{L}}_{\mathbb{Z}} k) \cong (\widetilde{K}_0(X) \otimes^{\mathrm{L}}_{\mathbb{Z}} k) \oplus (H_0(X, \mathbb{Z}) \otimes^{\mathrm{L}}_{\mathbb{Z}} k) \cong 0$ together with Corollary 4.6 shows that $K_i(X; k)$ vanishes for all $i \geqslant 1$. The vanishing of $H_i(\mathrm{Fix}_e(X), \mathcal{O})$ for $i \geqslant 1$ is clear. ☐

Note 8.2. In the non-equivariant case, the perfected trace map is an isomorphism in homological degrees $\geqslant 0$ by Observation 8.1. However, this may still fail in negative degrees – the following example is based on [BS17, Remark 11.7].

Example 8.3. Let Y_0 be a smooth ordinary elliptic curve over k. Take Y to be its perfection. Then $K_{-1}(Y; k) = 0$ by smoothness, but $H_{-1}(Y, \mathcal{O}) \neq 0$ by ordinarity. In particular, the trace map is not an isomorphism in degree -1.

8.2. Nodal curve and negative K-theory

Note 8.4. Algebraic K-theory of perfect schemes can be non-connective: the negative K-groups can be non-zero, see Example 8.5 below. At the same time, we know by Observation 4.7 that $K(X) = G(X)$ for perfect schemes. In other words, G-theory of perfect schemes can be non-connective. This gives a negative answer to a question posed by [Kha22, Remark 3.2].

We now record a concrete example of this non-connectivity; see also [AMM22].

Example 8.5 (Nodal cubic curve)**.** Let $k = \mathbb{F}_p$. Consider the affine nodal cubic curve $C_0 = \mathrm{Spec}\,(k[x,y]/\,(y^2 - x^2(x+1)))$. This has a resolution of singularities by the affine line $\mathbb{A}^1_0$; the preimage of the singular point $(0,0)$ in C_0 is a disjoint union of two points. Perfecting, we get a perfect abstract blowup square

$$\begin{tikzcd} \mathbb{A}^1 \arrow[d] & \mathrm{pt} \sqcup \mathrm{pt} \arrow[l, hook'] \arrow[d] \\ C & \mathrm{pt} \arrow[l, hook'] \end{tikzcd}$$

The final piece of the associated long exact sequence in K-theory

$$\to K_0(\mathbb{A}^1) \oplus K_0(\mathrm{pt}) \to K_0(\mathrm{pt} \sqcup \mathrm{pt}) \to K_{-1}(C) \to 0$$

looks like

$$\to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to K_{-1}(C) \to 0$$

with the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It follows that $K_{-1}(C) = \mathbb{Z}$. Since K-theory in lower degrees vanishes, $K_{-1}(C;k) = k$.

8.3. Toric varieties

We illustrate how K^T and the trace map behave on perfectly proper toric varieties without any smoothness assumptions: the trace map is an isomorphism; moreover K^T and KH^T agree. This easily follows from our setup and available literature [VV03] and [AHW09; CHWW09; CHWW14; CHWW15; CHWW18]. Also see [Ryc24, Theorem 6.3.2] for related discussion of the comparison between $K_0^T(-,\mathbb{C})$ and fixed-point schemes for complex smooth projective toric varieties.

In fact, the notion of *dilation* on the K-groups appearing in [CHWW14] is given by the Frobenius and their main results – such as [CHWW14, Theorem 0.2] over a field of positive characteristic – can be reinterpreted and easily proved through perfect geometry. Moreover, we can use the perfect setup to exhibit examples of singular toric varieties with nonzero negative equivariant K-groups.

Notation 8.6. Let T_0 be a split torus over k. Let X_0 be a toric variety for T_0 corresponding to a fan Δ. Given any cone $\sigma \in \Delta$, there is the corresponding subtorus $T_{\sigma,0} \leqslant T_0$ and the associated T_0-orbit $T_0/T_{\sigma,0}$ on X_0. Note that in our conventions, toric varieties are assumed to be *split* – we do not consider any non-split forms over k.

Discussion 8.7 (Toric resolutions)**.** Let X_0 be a toric variety with respect to a split torus T_0. Then one combinatorially defines a resolution by a smooth toric variety with respect to the same torus T_0 as in [Cox00, §5]. Furthermore, such a resolution is obtained from a sequence of partial resolutions of the form [CHWW09, (2.2)] – these fit into abstract blowup squares

$$\begin{array}{ccc} Y_0 & \longleftarrow & E_0 \\ \big\downarrow & & \big\downarrow \\ X_0 & \longleftarrow & Z_0 \end{array}$$

This square is a T_0-equivariant abstract blowup of toric varieties for quotients of T_0.

Observation 8.8. *Let T be a perfect split torus and X any perfect toric variety. Then integrally*

$$K^T(X) \simeq KH^T(X).$$

Proof. For a perfectly smooth toric variety X we clearly have $K^T(X) \simeq KH^T(X)$. In general, work by induction on the Krull dimension of X. Since both $K^T(-)$ and $KH^T(-)$ satisfy proper excision by Theorem 4.11 and Lemma 4.15, looking at Discussion 8.7, the statement for X reduces to the statement for Y, Z, E. Since Z and E have smaller Krull dimension and the statement is stable under enlarging the torus, we have reduced the claim to the partial resolution Y. In finitely many steps, we reduce to the smooth situation. □

The irreducible components of fixed-point schemes for toric varieties have a combinatorial description. First, $\mathrm{Fix}_T(X)$ over T has several horizontal components isomorphic to T and labelled by the finite set of torus fixed-points $|X^T|$. Over more special subvarieties of T, we get further vertical components which can be read off from the fan combinatorics. More precisely, the fixed point scheme – together with its structure map – is given by the commutative diagram

$$\begin{array}{ccc} \bigcup_{\sigma\in\Delta} T_\sigma \times X^{T_\sigma} & \xrightarrow{\cong} & \mathrm{Fix}_T(X) \\ \big\downarrow{\scriptstyle \mathrm{pr}_{T_\sigma}} & & \big\downarrow \\ \bigcup_{\sigma\in\Delta} T_\sigma & \longrightarrow & T \end{array} \tag{8.1}$$

compatibly with the residual T-action. By definition, the *horizontal* components are the irreducible components labeled by the σ of maximal dimension. We call the other components *vertical*.

Assuming X is perfectly proper, the vertical components are also perfectly proper and thus carry only constant global functions. Hence $H_0(\mathrm{Fix}_{\frac{T}{T}}, \mathcal{O})$ injects into the direct sum $\bigoplus_{|X^T|} H_0(\frac{T}{T}, \mathcal{O})$ of functions on the horizontal components. It is cut out by the equalizer conditions given by restriction to the vertical components.

Theorem 8.9. *Let T be a perfect split torus and X any perfectly proper toric variety. Then the trace map induces an equivalence*

$$K^T(X;k) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}). \tag{8.2}$$

and both sides are supported in homological degrees $\leqslant 0$.

Proof. First assume X is smooth. Note that $\mathrm{R}\Gamma(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O})$ is supported in homological degrees $\leqslant 0$ by perfectness; we claim that it vanishes in degrees $\leqslant -1$ as well. Since $(-)^T$ is exact, it is enough to check this on $\mathrm{Fix}_T(X)$. By perfect base change [BS17, Lemma 3.18], it is enough to check it on $\mathrm{Fix}_t(X)$ for each geometric point t of T. But now it follows from the fan combinatorics that $\mathrm{Fix}_t(X)$ is a disjoint union of (perfectly proper smooth) toric varieties[5]. Indeed, $\mathrm{Fix}_t(X)$ is the subvariety given by the set of those cones in the fan of X whose linear span contains t. We need to prove that these cones decompose into a disjoint union of stars. Since X is smooth, the fan of X is in particular simplicial. So if t lies in some cones σ_1, σ_2, it also lies in the intersection of their linear spans, which is itself the linear span of a cone by simpliciality. We thus succeed by decomposing the fixed cones according to the non-emptiness of their intersections. Since toric varieties have no higher cohomology by [Cox00, end of proof of Theorem 5.1], we deduce that $\mathrm{R\Gamma}(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O})$ is supported in degree zero.

Moreover, the desired isomorphism in degree zero follows from the description [VV03, Theorem 6.2] together with the properness of X. Indeed, the equalizer condition in [VV03] matches the condition on global sections of the fixed-point scheme imposed by its vertical fibers via proper excision (using the description from (8.1)). Hence the trace map is an isomorphism in degree zero. Since X is smooth, $K^T(X;k)$ is supported in homological degrees $\geqslant 0$. Moreover, by [VV03, Theorem 6.9] we in particular know that $K^T(X)$ has no p-torsion, so $K^T(X;k)$ is supported in degree zero by Lemma 4.5. This finishes the proof for smooth X; we in fact showed that the isomorphism (8.2) is supported in degree zero.

Now, both $K^T(-;k)$ and $\mathrm{R\Gamma}(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}) = HH^T(-/k)$ satisfy perfect proper excision by Theorem 4.11. By Discussion 8.7 and induction on dimension, the trace map is an isomorphism. Both sides are then supported in homological degrees $\leqslant 0$ by (5.2). $\square$

Remark 8.10. In the above theorem, it is not necessary to take the quotient by T – we have an equivalence

$$\mathrm{R\Gamma}(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}) \xrightarrow{\simeq} \mathrm{R\Gamma}(\mathrm{Fix}_T(X), \mathcal{O}).$$

Indeed, the action of T on $H_0(\mathrm{Fix}_T(X), \mathcal{O})$ is trivial because of the embedding into a direct sum of copies of $H_0(T, \mathcal{O})$. In particular, the claim holds for perfectly smooth toric varieties. Since both sides satisfy proper excision, we deduce the singular case as above by toric resolutions.

In particular, we can use the above results to find examples of proper singular toric varieties X_0 with non-connective $K^{T_0}(X_0)$. Note that this statement does not contain any perfection; we did not find similar computations elsewhere. Let us record this in more detail.

Note 8.11. When X is smooth (or more generally when the fan of X is simplicial), the proof of Theorem 8.9 actually shows that both sides are supported in homological degree 0.

However, negative degrees may occur in general. For an example, consider the fan in $\mathbb{Z}^{\oplus 3}$ spanned by the eight vertices $(\pm 2, \pm 2, -1)$, $(\pm 1, \pm 1, 1)$ and $t \in T$ spanning the third coordinate axis. Then $\mathrm{Fix}_t(X) \cong C_4 \sqcup \mathrm{pt} \sqcup \mathrm{pt}$, where C_4 denotes a cyclic chain of four perfect $\mathbb{P}^1$'s. Resolving C_4 by the disjoint union of four perfect $\mathbb{P}^1$'s and computing $\mathrm{R\Gamma}(C_4, \mathcal{O})$ from the associated abstract blowup square, we deduce that $H_{-1}(C_4, \mathcal{O}) \cong k \neq 0$. It follows from Remark 8.10 and Lemma 5.1 that $K^T_{-1}(X;k) \cong \mathrm{R\Gamma}_{-1}(\mathrm{Fix}_{\frac{T}{T}}(X), \mathcal{O}) \cong \mathrm{R\Gamma}_{-1}(\mathrm{Fix}_T(X), \mathcal{O}) \neq 0$.

In conclusion, perfect singular toric varieties may have nontrivial negative equivariant algebraic K-theory after perfection. In particular, this happens classically (before perfection) by Lemma 4.1.

[5] The following argument for this claim was explained to us by Kamil Rychlewicz.

CHAPTER III

Equivariant localizing invariants of simple varieties

9. Introduction

The contents of this chapter cover the results of [Löw26b] Jakub Löwit. "Equivariant localizing invariants of simple varieties". In: *International Mathematics Research Notices* 7 (2026), rnag058. DOI: 10.1093/imrn/rnag058.

9.1. Motivation

The aim of this note is to record a technique for comparing and computing equivariant localizing invariants on a simple class of (often singular) varieties arising in geometric representation theory.

Let k be a base field and G be an algebraic group over it. We define classes $\mathcal{B}_k^G$ and $\mathcal{C}_k^G$ of certain proper G-equivariant k-schemes by a simple constructive recipe and call them *simple varieties*. (For the main results, we will assume that G is nice – for example an n-dimensional split torus $T = \mathbb{G}_m^n$. More generally, any linearly reductive group G qualifies.)

Although such varieties are rather special, we nevertheless recover singular varieties of interest in geometric representation theory: *finite* and *affine Schubert varieties* for GL_n are the most prominent examples. In fact, these varieties are the original source of our motivation. The understanding of their (equivariant) cohomological invariants is an important topic in geometric representation theory; the most delicate information is controlled by their singularities. For instance, their intersection cohomology governs the representation theory of the algebraic group GL_n through the geometric Satake equivalence; this has further equivariant extensions. Such phenomena are expected to generalize to more complicated topological invariants.

One source of interesting invariants appearing in algebraic geometry are the *localizing invariants*. The most important examples come either from K-theory – such as algebraic K-theory $K(-)$ itself, its homotopy invariant version $KH(-)$, or its topological realization $K_{\mathbf{top}}(-)$ over $\mathbb{C}$; or from Hochschild homology – such as Hochschild homology $HH(-/k)$, negative cyclic homology $HC^-(-/k)$, topological cyclic homology $TC(-)$, and other variants. Localizing invariants can be evaluated on any derived stack through its category of perfect complexes. In particular, we can evaluate them on a global quotient X/G of any G-equivariant variety X, obtaining their *equivariant* versions.

9.2. Main results

The starting point of this project is to understand equivariant K-theory of affine Schubert varieties and trace maps from it, with a view towards geometric representation theory. In Chapter I covering [Löw26a] we described the relationship between T-equivariant K-theory and T-equivariant Hochschild homology of affine Schubert varieties after perfection in characteristic p. It turns out that our technique can be pushed further to control any truncating invariant (without perfection, in any characteristic).

This has two kinds of applications. On one hand, we can concretely control and compute such truncating invariants, most notably $KH(-)$. These are in particular equivariantly formal in a rather strong sense – while such formality statements were expected, they were missing until now. On the other hand, we can establish isomorphisms between different localizing invariants (without really computing them) as long as their difference is truncating – most notably between K-theory and variants of topological cyclic homology.

An abridged and simplified version of our present results goes as follows.

Theorem 9.1. *Let k be a field and G a nice (or linearly reductive) group over it*[1]*. We construct classes $\mathcal{B}_k^G \subseteq \mathcal{C}_k^G$ of G-equivariant projective k-schemes, which for example contain:*

- *projective spaces, Grassmannians, partial flag varieties,*
- *classical Schubert varieties in Grassmannians,*
- *affine Schubert varieties in the GL_n affine Grassmannian.*

For such varieties, the following statements hold:

- *If k is an algebraically closed field of characteristic p and $X \in \mathcal{C}_k^G$, then the mod-p cyclotomic trace induces an equivalence of ring spectra*

$$K^G(X;\mathbb{F}_p) \xrightarrow{\simeq} TC^G(X;\mathbb{F}_p).$$

  Similarly for the p-adic version.
- *If $k = \mathbb{Q}$ and $X \in \mathcal{B}_{\mathbb{Q}}^G$, then the Goodwillie–Jones trace induces a ring isomorphism*

$$K_0^G(X;\mathbb{Q}) \xrightarrow{\cong} (HC^-)_0^G(X/\mathbb{Q}).$$

  We further get a decomposition $K_i^G(X) \xrightarrow{\cong} KH_i^G(X) \oplus (HC^-)_i^G(X/\mathbb{Q})$ for all $i \geqslant 1$.
- *For any k and $X \in \mathcal{B}_k^G$, there is a natural ring isomorphism*

$$KH_0^G(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \xrightarrow{\cong} KH_\bullet^G(X)$$

  and $KH_0^G(X)$ is finite projective over $K_0^G(\mathrm{pt})$. In other words, the ring spectrum $KH^G(X)$ is connective and finite flat over $K^G(\mathrm{pt})$. When $k = \mathbb{C}$, we can further compare to topological K-theory: $KH_0^G(X) \cong K_{\mathbf{top},0}^G(X)$.

[1] See Remark 9.2 and refereces there for discussion of these assumptions.

Proof. See Example 12.1, Lemma 12.11 and Lemma 12.15 for proofs that the mentioned varieties are simple. The postulated equivalences are then respectively given by Theorem 11.2, Theorem 11.11 (with Proposition 11.15) and Theorem 11.18 (with Proposition 11.24). The variant for linearly reductive G is discussed in the supplementary §14. □

Remark 9.2 (Assumptions on G)**.** The above theorem works cleanest for *nice* groups G over k, see §10.3. This covers algebraic tori $T = \mathbb{G}_m^n$, groups of roots of unity μ_ℓ and finite groups of order coprime to $\operatorname{char} k$ (as well as their extensions, subquotients and forms). The technical restriction boils down to [ES21, Corollaries 5.2.3 and 5.2.6] and its variants, which are applicable to quotients by nice groups only.

However, the machinery of [LS25] together with [KR18; BKRS22; Kha20] allows to extend the aforementioned results to quotients by *linearly reductive groups*; we outline the arguments in the supplementary §14. With this extension, all our arguments go through without change for any linearly reductive group G over k. This does not make a difference if $\operatorname{char} k = p$ (where the notions of nice and linearly reductive coincide), but considerably extends the applicability if $\operatorname{char} k = 0$ to any reductive group (such as $G = GL_n$).

Remark 9.3 (Globality)**.** Let us highlight the following: our results are global. All the considered schemes are proper over k and our arguments stay within that realm. Even though localizing invariants satisfy Zariski (even Nisnevich) gluing, the desired isomorphisms do not hold Zariski locally. It is interesting that such isomorphisms nevertheless hold globally in naturally occurring singular examples.

9.3. Discussion and context

Let us emphasize that localizing invariants are highly sensitive to singularities – to the extent that their values are known only in the simplest singular cases. Even if the varieties in question have affine pavings (i.e. are cellular), there is no obvious method for the computations and comparisons of the localizing invariants we care about. The only way of attack is by detailed understanding of resolutions; this can be quite challenging. In effect, not many computations of equivariant K-theory nor KH in singular situations are known. We hope to contribute towards filling in this gap.

9.3.1. Discussion of results. Let us discuss the three comparisons from our main theorem in more detail. We emphasize that all these comparisons are compatible with the ring structures on the invariants at hand.

- The p-adic cyclotomic trace §11.1.
  The cyclotomic trace $K(-) \to TC(-)$ is the closest known approximation of algebraic K-theory by differential forms. In characteristic p with mod-p coefficients, this approximation is rather good. Nevertheless, it is known to be an equivalence only in specific local examples – see Remark 11.4 – and it is not an equivalence in general. Theorem 11.2 produces interesting (equivariant) singular projective examples where this is the case: for any $\mathcal{C}_k^G$, the map
  $$K^G(X;\mathbb{F}_p) \xrightarrow{\simeq} TC^G(X;\mathbb{F}_p)$$
  is still an equivalence – we are not aware of other nontrivial results of this form.
  While the above generality is completely sufficient to cover the examples of our interest, the theorem easily generalizes to a slightly bigger class $\mathcal{C}_p$ of algebraic stacks, including

some mixed-characteristic situations. We record this into a separate discussion in Theorem 11.8.

- Goodwillie–Jones trace §11.2.
When working over $\mathbb{Q}$, the cyclotomic trace specializes to the historically older Goodwillie–Jones trace $K(-) \to HC^-(-/\mathbb{Q})$ and we prove the isomorphisms
$$K_0^G(X;\mathbb{Q}) \xrightarrow{\cong} (HC^-)_0^G(X/\mathbb{Q})$$
for $X \in \mathcal{B}_k^G$ in Theorem 11.11. We further get a somewhat counterintuitive decomposition
$$K_i^G(X) \xrightarrow{\cong} KH_i^G(X) \oplus (HC^-)_i^G(X/\mathbb{Q}), \qquad i \geqslant 1.$$
in Proposition 11.15; this latter argument requires to use the supplementary truncating invariant $L_{\mathrm{cdh}}HC^-(-/\mathbb{Q})$ reviewed in the supplementary §13. We illustrate that this decomposition can be nontrivial in Example 12.7.
- Homotopy invariant K-theory §11.3.
The closest well-behaved approximation of algebraic K-theory of singular varieties is its homotopy invariant version KH. A reader interested in explicit ring presentations of equivariant invariants may want to focus on this part. Although no computations of KH in our setup appear in the literature, one expects to get reasonable results. This is indeed what we obtain: the natural identification
$$KH_0^G(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \xrightarrow{\cong} KH_\bullet^G(X)$$
from Theorem 11.18 shows that $KH^G(X)$ is equivariantly formal in a strong sense for any $X \in \mathcal{B}_k^G$, see also Remark 11.20. More conceptually, we prove that the ring spectrum $KH^G(X)$ is connective and finite flat over $K^G(\mathrm{pt})$. Equivariant formality statements of this form for KH are missing at the moment; we expect them to be useful for computations by Remark 11.23. We emphasize that $KH^G(X)$ is often explicitly computable – we do not seriously discuss this here, but see Remark 7.3, Examples 7.1 and 7.2 covering [Löw26a, §6] for some formulas.
The above formality has some immediate elementary consequences.
Firstly, for simple varieties from $\mathcal{B}_{\mathbb{C}}^G$, we deduce an isomorphism
$$KH_0^G(X) \cong K_{\mathbf{top},0}^G(X)$$
with topological K-theory of the analytification in Proposition 11.24. A posteriori, in these cases KH glues from affine pavings (since topological K-theory does). The whole $KH_\bullet^G(X)$ is determined by the independent geometric contribution of $K_{\mathbf{top},0}^G(X)$ and arithmetic contribution of $K_\bullet(\mathbb{C})$.
Secondly, simple varieties from $\mathcal{B}_{\mathbb{F}_q}^G$ satisfy Parshin's property by Example 11.30.

9.3.2. Strategy of proofs. In all three applications, our proof works as follows. We first express the desired statement in terms of some truncating invariant: in the first two cases, the fiber of the comparison map is truncating; in the third case KH is truncating to start with. We then check the statement on the point and consequently on the equivariant point BG; this can be usually done “by hand".

Once the above is settled, we are in a good shape. Indeed, we can control truncating invariants throughout the constructive steps defining simple varieties, so the results reduce to the case of BG. The subtle difference between the classes $\mathcal{B}_k^G$ and $\mathcal{C}_k^G$ morally depends on whether we wish to control the outputted homotopy groups one by one (as in the latter two applications), or if we are able to get a statement on the level of spectra (as in the first application).

To deduce our results for specific varieties (such as finite or affine Schubert varieties for GL_n), we just need to argue that they are simple. These geometric arguments are completely independent from the homotopical machinery of localizing invariants.

We would like to point out that the presentation of the proof we are offering goes backwards – the ideas originate from abstracting the case of affine Schubert varieties.

9.3.3. Outlook. The above results should not be viewed as a definitive list of applications; it should rather give a taste of what is possible.

In one direction, one could try to come up with other naturally occurring varieties which happen to be simple. Such endeavors can stay entirely within the realm of classical algebraic geometry via Definition 10.3.

However, one can further optimize the definition. To start with, one can directly add varieties X such that $\mathrm{Perf}(X)$ has a *full exceptional collection* to our class $\mathcal{B}_k$ – this immediately imports further examples where our results hold. For example, this applies to partial flag varieties of arbitrary type [SK25]. Also split smooth projective toric varieties qualify [Kaw06] (and one can generate singular ones via cdh descent in our bigger class $\mathcal{C}_k$). One may also cleanly formulate our constructive arguments on rather general derived stacks.

We do think that there are further examples coming from geometric representation theory where our results hold. Specifically, it is reasonable to expect that our methods may be adapted to cover finite and affine Schubert varieties of arbitrary types.

In another direction, one could try to carry through more refined computations and comparisons of localizing invariants along the same recipe. For example, following the constructive definition of simple varieties, KH (as well as other truncation invariants) are actually computable.

Furthermore, all the comparison maps are compatible with motivic filtrations and our results should refine to this level – for instance, our results on KH should refine to the level of cdh-local motivic cohomology $\mathbb{Z}(j)^{\mathrm{cdh}}$ of [BEM25]. However, some technical care would be needed, as motivic cohomology complexes do not apriori exist as localizing invariants.

9.3.4. Other work. To close the introduction, we would like to give an idea of the complexity of K-theory of the singular varieties we are dealing with.

Remark 9.4. Consider the (affine) conic singularity $\{z^2 - xy = 0\} \subseteq \mathbb{A}_k^3$ over a field k of characteristic zero. The rather nontrivial computation of its (non-equivariant) K-theory is carried out in [CHWW13, Theorem 4.3]; also see [PS21, §2.3, §2.4] for complementary results on finite quotient singularities.

The projective conic coincidentally appears as the smallest singular affine Schubert variety for GL_2 – it lies in the class $\mathcal{B}_k$ where all our results apply. See Example 12.7.

Remark 9.5. There is a notable amount of recent activity on equivariant K-theoretic and motivic study of finite and affine flag varieties, with serious applications in geometric representation theory – [SW18; SVW18; EK19; Ebe22; ES22; ES23; Ebe24b; Ebe24a; EE24; EE25], to name a few. In particular, the formalism of *reduced K-motives*, developed and successfully employed in [ES23; Ebe24a; EE24; EE25], provides a categorified variant of equivariant algebraic K-theory. However, in order to control equivariant formality, this variant enforces (by hand) the higher K-groups of the base to vanish.

As far as we know, equivariant formality statements such as our Theorem 11.18 were not available before. This result suggests that some equivariant formality holds directly, without modifying the definitions (and including higher K-theory). We hope that our methods could help to clarify such issues.

Remark 9.6. Let us also emphasize the difference between our approach and standard gluing techniques for smooth cellular varieties.

Let G be a linearly reductive group over k; assume that $X \in \mathrm{Sch}_k^G$ is smooth and posseses an equivariant paving by affine spaces $\mathbb{A}^n$. By smoothness, K-theory, KH and G-theory agree; their common value on equivariant $\mathbb{A}^n$ then matches its value on the equivariant point, which is explicit by §10.3.3. Furthermore, these invariants have the closed-open localization sequence, inherited from G-theory. A skeletal induction then allows to control $K^G(X)$ at least as a module – indeed, one may inductively argue that the long exact sequence associated to a gluing of an open cell splits, ultimately deducing equivariant formality results analogous to (the additive part) of Theorem 11.18.

While the main examples of our interest also have affine pavings, their singularities render the above techniques useless. To start with, even the additive computation outlined above does not seem obvious for equivariant KH in the singular case, not to even mention the ring structures.

Even more severe discrepancy between the methods occurs for our results about traces §11.1, §11.2: the statements in question do not hold for the affine spaces $\mathbb{A}^n$. Moreover, the Hoschschild-type invariants appearing as the target do not enjoy the relevant localization sequences. For these two reasons, there seems to be no hope for deducing our results by gluing from the affine strata.

Finally, the class $\mathcal{C}_k^G$ also covers some varieties without affine paving, such as Example 12.4.

9.4. Plan of the chapter

This chapter is structured as follows, separating the arguments into three logically independent parts.

In §10 we define the classes of simple varieties $\mathcal{B}_k^G \subseteq \mathcal{C}_k^G$ by a constructive recipe, recall the necessary background on localizing invariants, and prove the main technical results – Theorems 10.15, 10.18, 10.19 – which allow to control equivariant truncating invariants on simple varieties.
In §11 we discuss, one by one, the promised applications to various localizing invariants and equivalences between them – see in particular Theorems 11.2, 11.11, 11.18. These three applications can be read independently.
In §12, we supply explicit examples of simple varieties, starting from elementary ones and ending with finite and affine Schubert varieties for GL_n in Lemmata 12.11, 12.15. These final examples are truly interesting in geometric representation theory; all the results from the second section apply to them.

The chapter contains two supplementary sections. In §13 we review the supplementary localizing invariant $L_{\mathrm{cdh}}HC^{-}(-/k)$, relevant in §11.2. In §14 we outline the variant of our results for actions of a linearly reductive group G.

10. Localizing and truncating invariants of simple varieties

We start by giving a constructive definition of the classes $\mathcal{B}_k^G \subseteq \mathcal{C}_k^G$ of simple varieties in §10.1, which are the main object of study in this chapter. We continue by recalling the notion of localizing and truncating invariants in §10.2 as well as some group-theoretic preliminaries in §10.3. We then prove the technical result, Theorem 10.15 in §10.4, allowing us to control equivariant truncating invariants on simple varieties from $\mathcal{C}_k^G$. We close the section by useful variants valid on the smaller class $\mathcal{B}_k^G$, namely Theorems 10.18 and 10.19 in §10.5.

10.1. A class of simple varieties

Notation 10.1. Let R be an arbitrary base ring, we write $S = \operatorname{Spec} R$ and denote Sch_R the category of quasi-compact separated schemes over R. Let G be an affine group scheme over R. We then denote Sch_R^G the category of G-equivariant quasi-compact separated schemes over k and equivariant maps between them. Any object X of Sch_R^G gives rise to the global quotient stack X/G. Then $\mathrm{Vect}(X/G)$ and $\mathrm{Perf}(X/G)$ denote the categories of G-equivariant vector bundles and G-equivariant perfect complexes on X. We denote $BG = S/G$ the classifying stack of G over S.

Using the above notation, we will mostly work over a base field k. We then write $\mathrm{pt} = \operatorname{Spec} k$.

Notation 10.2. Given $X \in \mathrm{Sch}_k$, $\mathcal{F} \in \mathrm{QCoh}(X)$ and a dimension vector $d_\bullet = (d_1, \ldots, d_m)$ of length m and total dimension $d := d_1 + \cdots + d_m$, we write $\mathrm{Flag}_X(\mathcal{F}, d_\bullet)$ for the relative flag bundle of type $d_\bullet$ parametrizing flags of vector bundle quotients of $\mathcal{F}$ with successive subquotients of ranks $d_1, \ldots, d_m$. When $m = 1$, we get $\mathrm{Grass}_X(\mathcal{F}, d)$.

Let G be any affine group scheme over k. We give the constructive definition of simple G-equivariant varieties straightaway. We emphasize that this definition stays within the realm of classical algebraic geometry.

Definition 10.3 (Simple varieties $\mathcal{B}_k^G$)**.** Let $\mathcal{B}_k^G \subseteq \mathrm{Sch}_k^G$ be the smallest full subcategory satisfying the following closure properties:

(1) (the point) The point pt with the trivial G action lies in $\mathcal{B}_k^G$. Moreover, $\mathcal{B}_k^G$ is closed under finite disjoint unions.
(2) (projective bundles) Let $X \in \mathrm{Sch}_k^G$ and $\mathcal{E} \in \mathrm{Vect}(X/G)$ be an equivariant vector bundle.
    a) If $\mathcal{E}$ has rank $\geqslant 1$ everywhere[2], then

$$X \in \mathcal{B}_k^G \iff \mathbb{P}_X(\mathcal{E}) \in \mathcal{B}_k^G.$$

[2]The rank conditions here and below are equivalent to the associated projectivization having nonempty fibers.

b) More generally, let $d_\bullet$ be any dimension vector of total dimension d and assume that $\mathcal{E}$ has rank $\geqslant d$ everywhere. Then

$$X \in \mathcal{B}_k^G \iff \mathrm{Flag}_X(\mathcal{E}, d_\bullet) \in \mathcal{B}_k^G$$

(3) (stratified projective bundles with nonempty fibers) Let $X \in \mathrm{Sch}_k^G$ and $\mathcal{F} \in \mathrm{QCoh}(X/G)$ be an equivariant quasi-coherent sheaf on X that can be written as a cokernel of a map of equivariant vector bundles.

a) Assume that $\mathcal{F}$ has rank $\geqslant 1$ everywhere. Then

$$\mathbb{P}_X(\mathcal{F}) \in \mathcal{B}_k^G \implies X \in \mathcal{B}_k^G.$$

b) More generally, let $d_\bullet$ be any dimension vector of total dimension d and assume that $\mathcal{F}$ has rank $\geqslant d$ everywhere. Then

$$\mathrm{Flag}_X(\mathcal{F}, d_\bullet) \in \mathcal{B}_k^G \implies X \in \mathcal{B}_k^G.$$

(4) (3-out-of-4 for split abstract blowups) Take a noetherian abstract blowup square (X, Y, Z, E) in Sch_k^G, meaning an equivariant pullback square of noetherian schemes over k

$$\begin{tikzcd} Y \arrow[d] & E \arrow[l, hook'] \arrow[d] \\ X & Z, \arrow[l, hook'] \end{tikzcd}$$

where $X \hookleftarrow Z$ is a closed immersion while $Y \to X$ is proper and isomorphism over $X \backslash Z$. Assume that the square is *split* in the sense that either $E \to Y$ admits a retraction or $Y \to X$ admits a section. If three of its terms lie in $\mathcal{B}_k^G$, then the fourth term lies in $\mathcal{B}_k^G$ as well.

Definition 10.4 (Simple varieties $\mathcal{C}_k^G$)**.** We define a bigger class $\mathcal{C}_k^G \supseteq \mathcal{B}_k^G$ by the closure properties (1), (2), (3) and:

(4') (3-out-of-4 for abstract blowups) Take a noetherian abstract blowup square (X, Y, Z, E) in Sch_k^G. If three of its terms lie in $\mathcal{C}_k^G$, then the fourth term lies in $\mathcal{C}_k^G$ as well.

Notation 10.5. In particular, for trivial G we get the subcategories $\mathcal{B}_k \subseteq \mathcal{C}_k$ of Sch_k. Already this case is interesting.

Remark 10.6. If $G' \to G$ is a group homomorphism and $X \in \mathcal{B}_k^G$, we get $X \in \mathcal{B}_k^{G'}$ by restricting the action. In particular, $X \in \mathcal{B}_k$. Similarly for the bigger class $\mathcal{C}_k^G$.

In fact, the above definitions may be shortened by the following purely geometric argument.

Lemma 10.7. *The closure property* (2a) *implies* (2b)*. The closure properties* (2a) *and* (3a) *jointly imply* (3b)*. Consequently,* (2b) *and* (3b) *are automatic and can be omitted from the definition.*

Proof. To show that (2a) implies (2b), consider the corresponding full flag variety $\mathrm{Flag}_X(\mathcal{E}, d'_\bullet)$ where $d'_\bullet = (1, \ldots, 1)$ of total dimension d. Then we have natural maps

$$\begin{tikzcd} \mathrm{Flag}_X(\mathcal{E}, d'_\bullet) \arrow[r, "g"'] \arrow[rr, bend left, "h"] & \mathrm{Flag}_X(\mathcal{E}, d_\bullet) \arrow[r, "f"'] & X \end{tikzcd}$$

and both g and h factor as towers of equivariant projective bundles, reducing to that case.

To see that (2a) and (3a) jointly imply (3b), consider the diagram

$$\begin{tikzcd} \mathrm{Flag}_X(\mathcal{F}, d'_\bullet) \arrow[r, "g"'] \arrow[rr, bend left, "h"] & \mathrm{Flag}_X(\mathcal{F}, d_\bullet) \arrow[r, "f"'] & X \end{tikzcd}$$

where $\mathrm{Flag}_X(\mathcal{F}, d'_\bullet)$ is the stratified full flag variety bundle for $d' = (1, \ldots, 1)$ of total dimension d. Now g is a tower of honest projective bundles, so we know $\mathrm{Flag}_X(\mathcal{F}, d_\bullet) \in \mathcal{B}_k^G$ if and only if $\mathrm{Flag}_X(\mathcal{F}, d'_\bullet) \in \mathcal{B}_k^G$ by iterative use of (2a) alone. Moreover, h is a tower of stratified projective bundles, so we already know that $\mathrm{Flag}_X(\mathcal{F}, d'_\bullet) \in \mathcal{B}_k^G \implies X \in \mathcal{B}_k^G$ by iterative use of (3a) alone. Hence we conclude. $\square$

Remark 10.8. Given a noetherian scheme $X \in \mathrm{Sch}_k$ with underlying reduced subscheme $X_{\mathrm{red}} \hookrightarrow X$, we have $X \in \mathcal{B}_k \iff X_{\mathrm{red}} \in \mathcal{B}_k$. Similarly for the class $\mathcal{C}_k$. Indeed, this is a direct consequence of (4) – the square $(X, Y, Z, E) = (X, \varnothing, X_{\mathrm{red}}, \varnothing)$ is an abstract blowup; it is split with respect to the map $\varnothing \to \varnothing$.

(In the more general setup of derived stacks, the independence on nilpotent and derived structures may be conveniently built directly into the definition; see Definition 11.6 for such a formulation.)

Remark 10.9. Since Definitions 10.3, 10.4 are sufficient for our applications, we in general stick to them to minimize the technical load.

Nevertheless, they can be further optimized. Depending on the concrete application, we can even add more base cases to (1). For example, see Definition 11.6 in §11.1.2 for an absolute class $\mathcal{C}_p$ depending only on a prime p relevant for the cyclotomic trace, or Remark 11.21 for a discussion of KH of simple varieties over an arbitrary base ring R.

10.2. Localizing and truncating invariants

We now recall the invariants we wish to control. Let E be any k-linear *localizing invariant* valued in spectra in the sense of [LT19, Definition 1.2 and Remark 1.18]. By definition, $E(-)$ is a functor of ∞-categories

$$E(-) : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$$

which sends exact sequences to fiber sequences. In particular, if $\mathcal{D} \in \mathrm{Cat}_k^{\mathrm{perf}}$ has a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_j \rangle_{j=0}^m$, we get a natural decomposition $E(\mathcal{D}) \simeq \bigoplus_{j=0}^m E(\mathcal{D}_j)$.

A localizing invariant is called *finitary* if it further commutes with filtered colimits. Many important localizing invariants have this property (and sometimes it is included into the definition). Also see [Tam18; LT19; HSS17; BGT13].

A localizing invariant E is called *truncating* if it satisfies [LT19, Definition 3.1]: for every connective $\mathbb{E}_1$-ring spectrum A, the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence. This further forces nilinvariance and cdh descent in various degrees of generality [LT19; KR24; ES21].

By convention, we evaluate E on any derived stack X via its category of perfect complexes

$$E(X) := E(\mathrm{Perf}(X)).$$

In particular, given any G-equivariant scheme $X \in \mathrm{Sch}_k^G$, we denote

$$E^G(X) := E(X/G) := E(\mathrm{Perf}(X/G))$$

and freely switch between these notations.

For any derived stack X, the category $\mathrm{Perf}(X)$ carries a symmetric monoidal structure $\otimes$. All the localizing invariants we consider are lax monoidal: they in particular turn this into an $\mathbb{E}_\infty$-ring structure on the outputted spectrum $E(X)$, whose homotopy groups thus form a graded-commutative ring $E_\bullet(X)$. For brevity, we call such localizing invariants *multiplicative*.

Recollection 10.10. The following are k-linear localizing invariants:

- non-connective algebraic K-theory $K(-)$, it is finitary,
- topological Hochschild homology $THH(-)$, it is finitary,
- Hochschild homology of k-linear categories $HH(-/k)$, it is finitary,
- negative cyclic homology of k-linear categories $HC^-(-/k)$,
- topological cyclic homology $TC(-)$.

Furthermore, the following are k-linear truncating invariants:

- homotopy K-theory $KH(-)$ of k-linear categories, it is finitary,
- the fiber of the cyclotomic trace $K^{\mathrm{inf}}(-)$,
- periodic cyclic homology of k-linear categories $HP(-/k)$ if $\mathrm{char}\, k = 0$,
- topological K-theory of $\mathbb{C}$-linear categories $K_{\mathbf{top}}(-)$, it is finitary.

Proof. For $K(-)$, $THH(-)$, see [BGT13], [Tam18, Example 17]. For $HH(-/k)$, see [Hoy18], [HSS17]; it commutes with filtered colimits by definition. Further see [LT19, p. 914] for a discussion of $HH(-/k)$, $HC^-(-/k)$, $HP(-/k)$ as localizing invariants and [LT19, proof of Corollary 3.6] for $TC(-)$ and the truncating invariant $K^{\mathrm{inf}}(-)$. For $K_{\mathbf{top}}(-)$ as a finitary localizing invariant see [Bla16, Theorem 1.1]; it is truncating by [Kon21].

Also $KH(-)$ is a truncating invariant by [LT19, Proposition 3.14]; it commutes with filtered colimits as $K(-)$ does by swapping colimits in its definition. $\square$

Notation 10.11. We use the following notation:

- $K(-;\mathbb{F}_p) := K(-)/p$ for the mod-p algebraic K-theory,
- $K(-;\mathbb{Z}/p^j) := K(-)/p^j$ and $K(-;\mathbb{Z}_p) := \lim_j K(-;\mathbb{Z}/p^j)$ for its mod-p^j and p-adic versions,
- $K(-;\mathbb{Q}) := K(-)_{\mathbb{Q}}$ for rationalized K-theory.

See [TT90, §9.3] for details. Given any localizing invariant E in place of K-theory, we use the analogous notations $E(-;\mathbb{F}_p)$, $E(-;\mathbb{Z}/p^j)$, $E(-;\mathbb{Z}_p)$ and $E(-;\mathbb{Q})$.

The above discussion also works over an arbitrary base ring R in place of k; we will use this in a few side remarks.

10.3. An interlude on groups and representations

We now discuss the groups that we eventually want to consider. The results of this chapter work best for nice groups over fields. They further generalize to linearly reductive groups over fields (giving broader applicability in characteristic zero). In a different direction, restricting to diagonalizable groups only, our results work over more general base rings.

10.3.1. Nice and diagonalizable groups. Let M be an abstract abelian group. Then its group algebra $R[M]$ is naturally a commutative and cocommutative Hopf algebra, and we write $D(M) = \mathrm{Spec}(R[M])$ for the corresponding affine group scheme. An affine group scheme G over R is called *diagonalizable* if it is of the form $D(M)$ for some abelian group M.

The diagonalizable group $D(M)$ is of finite type if and only if M is finitely generated. By the classification of finitely generated abelian groups, diagonalizable groups of finite type are given by finite products of the multiplicative group $\mathbb{G}_m$ and the groups of finite order roots of unity μ_{p^a} of a prime power p^a.

An affine group scheme G over R is of *multiplicative type* if it is diagonalizable fpqc locally on R.

An affine group scheme G of finite type over R is called *nice*, if it is an extension of a closed and open normal subgroup G^0 of multiplicative type by an étale group scheme of order invertible in R. It is called *linearly reductive* if the functor $(-)^G$ of invariants is exact. Nice groups are always linearly reductive.

10.3.2. Linearly reductive groups over fields. Let G be a linearly reductive affine group scheme of finite type over a field k. In concrete terms [AHR19, Remark 2.3], such G is given as follows:

(i) if $\mathrm{char}\, k = 0$, this is equivalent to saying that G is an extension of a finite group by a connected reductive group,
(ii) if $\mathrm{char}\, k = p$ is positive, this is equivalent to saying that G is nice, i.e. an extension of an étale group whose order is not divisible by p by a group of multiplicative type.

In characteristic p, the notions of nice and linearly reductive groups coincide. In characteristic zero, reductive groups form a bigger class (e.g. GL_n is linearly reductive but not nice).

10.3.3. Representations and decomposability. In order to compute equivariant localizing invariants of the point, we need to control the category $\mathrm{Perf}(BG)$. We denote by $\mathrm{Rep}_k(G)$ the category of finite dimensional algebraic representations of G; it is tautologically equivalent to the category $\mathrm{Vect}(BG)$ of vector bundles on its classifying stack. We write $R(G) = K_0(BG)$ for the representation ring of G.

The above categories become very easy in the following naturally occurring situation.

Discussion 10.12. We say that an affine group scheme G over a ring R has *decomposable representation theory* if there is an index set I and an equivalence in $\mathrm{Cat}_R^{\mathrm{perf}}$

$$\mathrm{Perf}(R/G) \simeq \bigoplus_I \mathrm{Perf}(R). \tag{10.1}$$

The following groups give the most notable examples.

(i) Let G be a linearly reductive group over a field k. Then the category $\mathrm{Rep}_k(G)$ is semisimple; it is an infinite direct sum of copies of $\mathrm{Vect}(\mathrm{pt})$ labeled by the irreducibles in $\mathrm{Rep}_k(G)$. Consequently, G has decomposable representation theory with $\mathrm{Perf}(BG) \simeq \bigoplus_{\mathrm{Irr}(G)} \mathrm{Perf}(k)$.
(ii) Let $G = D(M)$ is a diagonalizable group over $S = \mathrm{Spec}\, R$. Representations of G correspond to M-graded quasi-coherent sheaves on S. The latter abelian category

decomposes as a direct product of copies of quasi-coherent sheaves on R over the index set M. Consequently, G has decomposable representation theory with $\mathrm{Perf}(S/G) \simeq \bigoplus_M \mathrm{Perf}(S)$.

In these examples, the natural monoidal structure on the left-hand side corresponds to the convolution on the right-hand side.

Decomposition (10.1) has the following simple consequence: the value of finitary localizing invariants on the equivariant base is determined by its value on R.

Lemma 10.13. *Let G be a group over R with decomposable representation theory and E a finitary localizing invariant. Then*

$$E^G(R) \simeq \bigoplus_I E(R).$$

Proof. Since $E(-)$ commutes with finite direct sums and filtered colimits, it is compatible with infinite direct sums, so we conclude from (10.1). □

Remark 10.14. In applications, one can often check the outcome of Lemma 10.13 even in cases when E is not finitary per se. More precisely, it is often sufficient to know that $E(\mathrm{pt})$ is sufficiently bounded. See Lemmata 11.5 and 11.12.

10.4. Truncating invariants of simple varieties

Let k be a field and G a nice group over it. The following theorem then gives control over truncating invariants on $\mathcal{C}_k^G$. It is one of the main technical observations of this chapter. Also see §10.5 for further degreewise variations, and the supplementary §14 for the case of linearly reductive G.

Theorem 10.15. *Let E be a k-linear truncating invariant with $E^G(\mathrm{pt}) \simeq 0$. Then for all $X \in \mathcal{C}_k^G$ we have*

$$E^G(X) \simeq 0.$$

Proof. We check that $E^G(X) \simeq 0$ is stable under the closure properties from Definition 10.4.

(1) By assumption, $E^G(\mathrm{pt}) \simeq 0$; the compatibility with finite disjoint unions is clear as E commutes with finite direct sums.

(2) The case (2a) of $\mathbb{P}_X(\mathcal{E})$ follows from the projective bundle formula for localizing invariants. More precisely, since X is quasi-compact by design, it has only finitely many connected components, so by (1) we may assume X is connected; we then denote $n := \operatorname{rank} \mathcal{E} \geqslant 1$. Now, $\mathrm{Perf}(\mathbb{P}_{X/G}(\mathcal{E}))$ has a semi-orthogonal decomposition into n copies of $\mathrm{Perf}(X/G)$ by the classical results of [Tho88b]; see also [Kha20, Theorem 3.3 and Corollary 3.6]. Since localizing invariants send semi-orthogonal decompositions to direct sums, we have $E^G(\mathbb{P}_X(\mathcal{E})) \simeq \bigoplus_{i=0}^{n-1} E^G(X)$, giving the desired equivalence. The more general case (2b) of $\mathrm{Flag}_X(\mathcal{E}, d_\bullet)$ is then automatic by Lemma 10.7.

(3) By the assumption on $\mathcal{F}$, we can find a two term complex of equivariant vector bundles $\mathcal{E}_\bullet = [\mathcal{E}_1 \to \mathcal{E}_0]$ with $\mathrm{coker}(\mathcal{E}_1 \to \mathcal{E}_0) \cong \mathcal{F}$. To prove (3a), take its derived projectivization $Y = \mathbb{P}_X(\mathcal{E}_\bullet)$ in the sense of [Jia25; Jia22]. This is a derived scheme with classical truncation

$Y_{\mathrm{cl}} = \mathbb{P}_X(\mathcal{F})$. Since $\mathcal{E}_\bullet$ has Tor-amplitude $[1,0]$, we obtain the semi-orthogonal decomposition of [Jia23, Theorem 3.2 and Remark 1.1], [Jia25, Theorem 7.5]. In particular, as the rank of $\mathcal{F}$ is $\geqslant 1$ everywhere, the pullback realizes $\mathrm{Perf}(X/G)$ as a semiorthogonal summand in $\mathrm{Perf}(Y/G)$ – note that this is independent of the characteristic of k (in fact works over any base) by [Jia23, Remark 1.1], [Jia25, Theorem 7.5] since we are treating only derived projectivizations at the moment. Since localizing invariants send semi-orthogonal decompositions to direct sums, $E^G(Y) \simeq 0 \implies E^G(X) \simeq 0$. Moreover, as $E(-)$ is truncating, we have $E^G(Y_{\mathrm{cl}}) \simeq E^G(Y)$ by [ES21, Corollary 5.2.3]. Altogether, this proves the first statement. The more general case (3b) of a stratified partial flag variety $\mathrm{Flag}_X(\mathcal{F}, d_\bullet)$ is again automatic by Lemma 10.7.

(4') Any truncating invariant sends G-equivariant abstract blowup squares to homotopy fiber squares by [ES21, Corollary 5.2.6] – the statement follows. □

Remark 10.16. The discussion of (1) and (2) clearly works for any localizing invariant. However, for both (3) and (4'), it is crucial that E is truncating. The closure properties (3) and (4') are important, as they allow us to go down and reach singular varieties.

Corollary 10.17. *Let $E \to F$ be a map of k-linear finitary truncating invariants. If it induces an equivalence $E(\mathrm{pt}) \simeq F(\mathrm{pt})$, then it induces an equivalence $E^G(X) \simeq F^G(X)$ for any $X \in \mathcal{C}_k^G$.*

Proof. The fiber $\mathrm{fib}(E \to F)$ is a k-linear finitary truncating invariant which vanishes on pt. We conclude by Lemma 10.13 and Theorem 10.15. □

The assumption that E, F are finitary is used solely to commute E through the decomposition (10.1) via Lemma 10.13. As we already mentioned in Remark 10.14, this can often be done by hand in more general situations.

10.5. Degreewise versions

If we restrict attention only to the smaller class $\mathcal{B}_k^G$, Theorem 10.15 works degreewise. This is a very useful variant – there are interesting maps of localizing invariants which induce isomorphisms only in certain degrees.

As before, assume G is a nice group over a field k. Also see §14 for the case of linearly reductive G.

Theorem 10.18. *Fix $i \in \mathbb{Z}$ and let E be a k-linear truncating invariant with $E_i^G(\mathrm{pt}) \simeq 0$. Then for all $X \in \mathcal{B}_k^G$ we have*

$$E_i^G(X) \simeq 0.$$

More generally, fix $i \in \mathbb{Z}$ and let E and F be k-linear truncating invariants together with a map $E \to F$ which induces an isomorphism $E_i^G(\mathrm{pt}) \simeq F_i^G(\mathrm{pt})$. Then for all $X \in \mathcal{B}_k^G$ we have

$$E_i^G(X) \simeq F_i^G(X).$$

Proof. We focus on the latter statement, the first being a special case for F trivial. It is enough to check that $E_i^G(X) \simeq F_i^G(X)$ is stable under the closure properties from Definition 10.3. This follows by the arguments from the proof of Theorem 10.15. Indeed, the desired isomorphism is stable under taking direct sums and direct summands in $\mathrm{Cat}_k^{\mathrm{perf}}$, so it is stable under (1) and (2). Since $E_i(-)$ and $F_i(-)$ disregard derived structures on stacks in the necessary generality [ES21, Corollary 5.2.3], the compatibility with direct summands shows the stability under (3). Since $E_i(-)$ and $F_i(-)$ turn split abstract blowup squares of stacks into split homotopy fiber squares in the necessary generality [ES21, Corollary 5.2.6], we deduce the stability on (4). □

In a similar vein, the value of any multiplicative truncating invariant E on some $X \in \mathcal{B}_k^G$ is determined by the degree zero part $E_0^G(X)$, which is well-controlled.

Theorem 10.19. *Let E be a k-linear finitary truncating invariant, which is multiplicative. Then for all $X \in \mathcal{B}_k^G$ the natural map induces a graded ring isomorphism*

$$E_0^G(X) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \xrightarrow{\cong} E_\bullet^G(X). \tag{10.2}$$

Furthermore, the commutative ring $E_0^G(X)$ is a finite projective module over $E_0^G(\mathrm{pt})$.

Proof. Since E is multiplicative, the universal property of tensor products induces is a natural ring homomorphism

$$E_0(-) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \to E_\bullet(-) \tag{10.3}$$

and we only need to check that it is an isomorphism on the underlying abelian groups. We check that this statement is stable under the closure properties from Definition 10.3.

For non-equivariant pt, the map (10.3) is tautologically an isomorphism. Since E is finitary, both sides of (10.3) commute with the infinite direct sum (10.1), giving

$$E_0^G(\mathrm{pt}) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \xrightarrow{\cong} E_\bullet^G(\mathrm{pt}). \tag{10.4}$$

For the same reason, the statement is stable under disjoint unions. Altogether, we have checked the closure property (1).

Given $\mathcal{D} \in \mathrm{Cat}_k^{\mathrm{perf}}$ with a semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_j \rangle_{j=0}^m$ and $i \in \mathbb{Z}$, we have

$$E_i(\mathcal{D}) = \bigoplus_{j=0}^m E_i(\mathcal{D}_j), \qquad \text{hence} \qquad E_0(\mathcal{D}) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) = \bigoplus_{j=0}^m E_0(\mathcal{D}_j) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}),$$

so both sides of (10.3) send semi-orthogonal decompositions to direct sums.

In particular, stability under (2) follows from the semi-orthogonal decompositions for projective bundles. Since E is truncating, the closure property (3) follows from the semi-orthogonal decompositions of derived projectivizations [Jia23] together with the independence of $E_i(-)$ on derived structures [ES21, Corollary 5.2.3].

Finally, since E is truncating, any noetherian split abstract blowup (X, Y, Z, E) induces a long exact sequence on homotopy groups of E by [ES21, Corollary 5.2.3], which falls apart into split short exact sequences

$$0 \to E_i^G(X) \to E_i^G(Y) \oplus E_i^G(Z) \to E_i^G(E) \to 0, \qquad i \in \mathbb{Z}\,.$$

In particular, we get the split short exact sequence

$$0 \to E_0^G(X) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \to (E_0^G(Y) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt})) \oplus (E_0^G(Z) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt})) \to E_0^G(E) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \to 0.$$

Altogether, both sides of (10.3) behave compatibly under the closure property (4).

Finally, in each step (1)-(4) we are only passing to finite direct sums or retracts. It is thus clear that the property of $E_0^G(X)$ being finite projective over $E_0^G(\mathrm{pt})$ is preserved. □

Remark 10.20. The statements of Theorems 10.18, 10.19 are not true for the bigger class $\mathcal{C}_k^G$ by Example 12.4.

Remark 10.21. The isomorphism from Theorem 10.19 refines to

$$E_0^G(X) \underset{E_0(\mathrm{pt})}{\otimes} E_\bullet(\mathrm{pt}) \xrightarrow{\cong} E_0^G(X) \underset{E_0^G(\mathrm{pt})}{\otimes} E_\bullet^G(\mathrm{pt}) \xrightarrow{\cong} E_\bullet^G(X).$$

Indeed, we have natural maps as indicated; the first one is also an isomorphism by (10.4).

For the next remark, recall that a map of $\mathbb{E}_\infty$-ring spectra $A \to B$ is called *(finite) faithfully flat* if the map of commutative rings $\pi_0(A) \to \pi_0(B)$ is (finite) faithfully flat and $\pi_\bullet(B) \cong \pi_0(B) \otimes_{\pi_0(A)} \pi_\bullet(A)$. In this language, we can rephrase the above results as follows.

Remark 10.22. Concisely, $E^G(X)$ is faithfully flat over $E(\mathrm{pt})$. It is finite faithfully flat over $E^G(\mathrm{pt})$.

11. Applications to concrete localizing invariants

We now present several instances of the above theorems applied to concrete localizing invariants. The most interesting applications concern global situations in positive and mixed characteristic where the p-adic cyclotomic trace becomes an equivalence §11.1; situations over the rationals where the Goodwillie–Jones trace induces an isomorphism in degree zero §11.2; and situations where homotopy-invariant K-theory becomes equivariantly formal §11.3. The lastly mentioned formality gives, for instance, a way to compute KH of complex simple varieties from their topological K-theory $K_{\mathbf{top}}$.

11.1. The p-adic cyclotomic trace

As a first application, we show in Theorem 11.2 that the mod-p cyclotomic trace is an equivalence on simple varieties with nice group actions; this already covers the geometric examples of our interest.

We then explain a mild generalization of the result including certain henselian situations in Theorem 11.8 – a reader interested in varieties over a field can skip this variant.

Setup 11.1. Throughout this section, assume k is an algebraically closed field of characteristic p. Let G be a nice group scheme over k.

11.1.1. The p-adic cyclotomic trace on simple varieties. We give the following application of Theorem 10.15 to simple varieties from $\mathcal{C}_k^G$.

Theorem 11.2 (Mod-p cyclotomic trace)**.** *For any $X \in \mathcal{C}_k^G$, the mod-p cyclotomic trace induces an equivalence*

$$K^G(X; \mathbb{F}_p) \xrightarrow{\simeq} TC^G(X; \mathbb{F}_p).$$

Proof. The fiber $K^{\mathrm{inf}}(-;\mathbb{F}_p)$ is a truncating invariant. We check that $K^{\mathrm{inf}}(BG;\mathbb{F}_p) \simeq 0$ in Lemma 11.5 below, hence $K^{\mathrm{inf}}(X/G;\mathbb{F}_p) \simeq 0$ for any $X \in \mathcal{C}_k^G$ by Theorem 10.15 as desired. □

Remark 11.3 (p-adic cyclotomic trace)**.** The same conclusion formally follows for mod-p^j and p-adic versions: for any $X \in \mathcal{C}_k^G$, we have

$$\forall j \in \mathbb{N},\ K^G(X;\mathbb{Z}/p^j) \xrightarrow{\simeq} TC^G(X;\mathbb{Z}/p^j) \qquad \text{and} \qquad K^G(X;\mathbb{Z}_p) \xrightarrow{\simeq} TC^G(X;\mathbb{Z}_p).$$

Remark 11.4 (Non-equivariant case is interesting)**.** The above statement is interesting already non-equivariantly: for any $X \in \mathcal{C}_k$, it gives an equivalence $K(X;\mathbb{F}_p) \xrightarrow{\simeq} TC(X;\mathbb{F}_p)$. In general, it is known that the cyclotomic trace $K(X;\mathbb{F}_p) \to TC(X;\mathbb{F}_p)$ realizes the target as the étale sheafification of the source by [CMM21, Theorem 6.3]; also see [GH99] for results in the smooth case. In other words, our Theorem 11.2 asserts that the étale sheafification is not necessary after evaluation on $X \in \mathcal{C}_k$. To us, this is far from obvious: we do not know how to deduce our result by studying the étale site of X.

To complete the proof of the theorem, we still need to supply the case of the equivariant point.

Lemma 11.5. *We have an equivalence*

$$K^G(\mathrm{pt};\mathbb{F}_p) \xrightarrow{\simeq} TC^G(\mathrm{pt};\mathbb{F}_p).$$

In other words, $K^{\mathrm{inf}}(BG;\mathbb{F}_p) \simeq 0$.

Proof. Since k is an algebraically closed field of characteristic p, the cyclotomic trace

$$K(\mathrm{pt};\mathbb{F}_p) \xrightarrow{\simeq} TC(\mathrm{pt};\mathbb{F}_p) \tag{11.1}$$

is an equivalence. Indeed, $K(\mathrm{pt})$ lives in degrees $\geqslant 0$ and all the higher K-groups are $\mathbb{Z}[\frac{1}{p}]$-modules [Kra80a, Corollary 5.5. and Example (1) below it]. Since $K_0(\mathrm{pt}) \cong \mathbb{Z}$, we see that $K(\mathrm{pt};\mathbb{F}_p)$ is concentrated in degree 0 with value $\mathbb{F}_p$. At the same time, since k is algebraically closed, $TC(\mathrm{pt};\mathbb{F}_p)$ also lives in degree 0 with value $\mathbb{F}_p$ by [KN18, Example 7.4 and Remark 7.6]. The trace map clearly induces an isomorphism between them.

To treat the equivariant case, consider the decomposition (10.1). Since $THH(-)$ is finitary, we get an equivalence

$$THH(BG) \simeq \bigoplus_{\mathrm{Irr}(G)} THH(\mathrm{pt}). \tag{11.2}$$

Note that $THH(k)$ is connective (this being the case for THH of any connective ring spectrum in place of k), so both sides of (11.2) are connective. By general theory, $THH(-)$ carries a natural structure of a cyclotomic spectrum; the forgetful map from cyclotomic spectra to spectra preserves colimits [CMM21, §2.1, p. 416]. Altogether, (11.2) holds in the category CycSp of cyclotomic spectra, and hence in the category $\mathrm{CycSp}_{\geqslant 0}$ of connective cyclotomic spectra. Now, by [CMM21, Theorem 2.7], the functor $TC(-;\mathbb{F}_p) : \mathrm{CycSp}_{\geqslant 0} \to \mathrm{Sp}$ commutes with colimits. Altogether,

$$TC(BG;\mathbb{F}_p) \simeq \bigoplus_{\mathrm{Irr}(G)} TC(\mathrm{pt};\mathbb{F}_p).$$

Since K-theory is finitary, the decomposition (10.1) also gives

$$K(BG;\mathbb{F}_p) \simeq \bigoplus_{\mathrm{Irr}(G)} K(\mathrm{pt};\mathbb{F}_p).$$

All the above identifications are compatible with the trace map, so we reduce to the case of pt treated in (11.1) above. □

11.1.2. Strictly henselian base rings. In fact, Theorem 11.2 can be easily generalized to include certain strictly henselian situations in both equal and mixed characteristic as base cases. To this end, we introduce the following absolute class of simple stacks $\mathcal{C}_p$ depending only on a prime number p, which is given by allowing more general base examples in Definition 10.3. We use this as an opportunity to rephrase the rest of the construction in a stacky way.

We denote $\mathrm{dStk}^{\mathrm{ans}}$ the category of derived algebraic stacks with nice stabilizers, see [BKRS22, §A.1], [KR24, §2.4 and §2.5]. We also recall that the notions of abstract blowup squares [BKRS22, Definition 2.1.3], Nisnevich squares [BKRS22, Definition 2.1.1], derived projectivizations [Jia25; Jia22; Jia23] work in this generality. We then define the class $\mathcal{C}_p \subseteq \mathrm{dStk}^{\mathrm{ans}}$ as follows.

Definition 11.6. Let p be a fixed prime. Let $\mathcal{C}_p \subseteq \mathrm{dStk}^{\mathrm{ans}}$ be the class of derived algebraic stacks given by the following closure properties:

(0) (independence on nilpotence) Let $\mathcal{X} \in \mathrm{dStk}^{\mathrm{ans}}$. Then
$$\mathcal{X} \in \mathcal{C}_p \iff \mathcal{X}_{\mathrm{cl}} \in \mathcal{C}_p \iff \mathcal{X}_{\mathrm{red}} \in \mathcal{C}_p\,.$$
(1) (base cases) Let $S = \operatorname{Spec} R$ be the spectrum of a strictly henselian, weakly regular, stably coherent ring R of residue characteristic p. Let G be a group scheme over S with decomposable representation theory. Then $S/G \in \mathcal{C}_p$. Moreover, $\mathcal{C}_p$ is closed on finite disjoint unions.
(2) (projective bundles) Let $\mathcal{X} \in \mathrm{dStk}^{\mathrm{ans}}$ and $\mathcal{E} \in \mathrm{Vect}(\mathcal{X})$ of rank $\geqslant 1$ everywhere, then $\mathbb{P}_{\mathcal{X}}(\mathcal{E}) \in \mathcal{C}_p \iff \mathcal{X} \in \mathcal{C}_p$. More generally, let $d_\bullet$ be a dimension vector of total dimension d and assume that $\mathcal{E}$ has rank $\geqslant d$ everywhere, then $\mathcal{X} \in \mathcal{C}_p \iff \mathrm{Flag}_{\mathcal{X}}(\mathcal{E}, d_\bullet) \in \mathcal{C}_p$.
(3) (stratified projective bundles) Let $\mathcal{X} \in \mathrm{dStk}^{\mathrm{ans}}$ and $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ be of the form $\mathcal{F} = H_0(\mathcal{E})$ for some $\mathcal{E} \in \mathrm{Perf}^{\geqslant 0}(\mathcal{X})$. Assume $\mathcal{F}$ has rank $\geqslant 1$ everywhere, then $\mathbb{P}_{\mathcal{X}}(\mathcal{F}) \in \mathcal{C}_p \implies \mathcal{X} \in \mathcal{C}_p$. More generally, let $d_\bullet$ be a dimension vector of total dimension d and assume that $\mathcal{F}$ has rank $\geqslant d$ everywhere, then $\mathrm{Flag}_{\mathcal{X}}(\mathcal{F}, d_\bullet) \in \mathcal{C}_p \implies \mathcal{X} \in \mathcal{C}_p$.
(4) (cdh descent) Given a noetherian abstract blowup square $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{E})$ in $\mathrm{dStk}^{\mathrm{ans}}$ such that three of its terms lie in $\mathcal{C}_p$, the fourth one does as well. Similarly for arbitrary Nisnevich squares.

Example 11.7. Let us mention some examples of stacks in $\mathcal{C}_p$.

(i) If G is a nice group over any algebraically closed field k of characteristic p and $X \in \mathcal{C}_k^G$, then $\mathcal{X} := X/G \in \mathcal{C}_p$.
(ii) We allow more general affine base rings R to lie in $\mathcal{C}_p$ in the closure property (1). In particular, R can be any regular noetherian strictly henselian ring such as the power series ring $k[\![t]\!]$ or the ring of integers $\breve{\mathbb{Z}}_p$ of the completion of the maximal unramified extension of $\mathbb{Q}_p$. More generally, R can be any strictly henselian valuation ring by [AMM22, Corollary 2.3].
(iii) The remaining closure properties then allow to combine the above: for example, the blowup of such regular noetherian strictly henselian R in its special point again lies in $\mathcal{C}_p$.

We then have the following generalization of Theorem 11.2 and Lemma 11.5 to the class $\mathcal{C}_p$.

Theorem 11.8. *For any $\mathcal{X} \in \mathcal{C}_p$, the cyclotomic trace induces an equivalence*
$$K(\mathcal{X}; \mathbb{F}_p) \xrightarrow{\simeq} TC(\mathcal{X}; \mathbb{F}_p).$$
Similarly for the mod-p^j and p-adic variants.

Proof. The independence on nilpotent structures (0) is [ES21]. The base cases (1) follow from Lemma 11.9 below. Then (2) and (3) are same as in Theorem 10.19 using that the semiorthogonal decompositions for projective bundles and derived projectivizations work over $\mathbb{Z}$, see [Jia25, Theorem 7.5] and [Jia23, Remark 1.1]. Finally, (4) follows from [ES21, Corollary 5.2.6]. □

Lemma 11.9. *Let $S = \operatorname{Spec} R$ be the spectrum of a strictly henselian, weakly regular, stably coherent ring R of residue characteristic p. Consider a group G over S with decomposable representation theory. Then the cyclotomic trace induces an equivalence*

$$K(S/G;\mathbb{F}_p) \xrightarrow{\simeq} TC(S/G;\mathbb{F}_p).$$

In other words, $K^{\mathrm{inf}}(S/G;\mathbb{F}_p) \simeq 0$.

Proof. For weakly regular, stably coherent rings R, the map $K^{\mathrm{cn}}(S) \to K(S)$ from connective K-theory to K-theory is an equivalence by [AMM22, Proposition 2.4]. Moreover, for strictly henselian rings R of residue characteristic p, the map $K^{\mathrm{cn}}(S;\mathbb{F}_p) \to TC(S;\mathbb{F}_p)$ is an equivalence by [CMM21, Theorem 6.1]. Altogether, we have an equivalence

$$K(S;\mathbb{F}_p) \xrightarrow{\simeq} TC(S;\mathbb{F}_p).$$

The rest of the proof of Lemma 11.5 then works word for word with k replaced by R, using that G has decomposable representation theory by assumption. □

11.2. Rational Goodwillie–Jones trace

Consider the trace map $K(-;\mathbb{Q}) \to HC^-(-/\mathbb{Q})$ from rational K-theory to $\mathbb{Q}$-linear negative cyclic homology. We prove that on $\mathcal{B}_{\mathbb{Q}}$, the rational Goodwillie–Jones trace induces an isomorphism in degree zero in Theorem 11.11. We also deduce a counterintuitive direct sum decomposition for positive degrees in Proposition 11.15.

Setup 11.10. Throughout this subsection we take $k = \mathbb{Q}$. Let G be a nice group over $\mathbb{Q}$. Via the supplementary §14, the results generalize to any reductive group G over $\mathbb{Q}$.

11.2.1. Rational Goodwillie–Jones trace in degree zero. We start by discussing the following degree zero isomorphism.

Theorem 11.11 (Rational Goodwillie–Jones trace in degree zero)**.** *For any $X \in \mathcal{B}_{\mathbb{Q}}^G$, the trace induces a ring isomorphism*

$$K_0^G(X;\mathbb{Q}) \xrightarrow{\cong} (HC^-)_0^G(X/\mathbb{Q}).$$

Proof. The fiber $K^{\mathrm{inf}}(-;\mathbb{Q}) = \operatorname{fib}(K(-;\mathbb{Q}) \to HC^-(-/\mathbb{Q}))$ is a truncating invariant [LT19, proof of Corollary 3.9]. We check that $K_0^{\mathrm{inf}}(BG;\mathbb{Q}) \cong 0 \cong K_{-1}^{\mathrm{inf}}(BG;\mathbb{Q})$ in Lemma 11.12 below. We then apply Theorem 10.18 to $K^{\mathrm{inf}}(-;\mathbb{Q})$ for $i = 0, -1$ separately. We deduce that for all $X \in \mathcal{B}_k^G$ it holds that $K_0^{\mathrm{inf}}(X/G;\mathbb{Q}) \cong 0 \cong K_{-1}^{\mathrm{inf}}(X/G;\mathbb{Q})$, so we get back the desired claim from the long exact sequence associated to the fiber sequence $K^{\mathrm{inf}}(-;\mathbb{Q}) \to K(-;\mathbb{Q}) \to HC^-(-/\mathbb{Q})$ on X/G. □

Again, we still need to supply the case of an equivariant point.

Lemma 11.12. *We have $K^{\mathrm{inf}}_0(BG;\mathbb{Q}) \simeq 0 \simeq K^{\mathrm{inf}}_{-1}(BG;\mathbb{Q})$.*

Proof. Since $HH(\mathrm{pt}/\mathbb{Q}) \simeq \mathbb{Q}$ sitting in degree zero and $HH(-/\mathbb{Q})$ is finitary, we deduce from (10.1) that $HH(BG/\mathbb{Q}) \simeq R(G;\mathbb{Q})$ is the rationalized representation ring sitting in degree zero. Now by the classical construction of $HC^-(-/\mathbb{Q})$ from $HH(-/\mathbb{Q})$ as in [Lod97, §5.1.7], which is compatible with the categorical definition by [Hoy18], we see that $HC^-_\bullet(BG/\mathbb{Q}) \cong HC^-_0(BG/\mathbb{Q})[u]$ as rings with the free generator u sitting in degree -2. In other words, $HC^-(-/\mathbb{Q})$ commutes with the direct sum decomposition (10.1). The same is true for $K(-;\mathbb{Q})$, as it is finitary.

We display the following piece of the associated long exact sequence on BG:

$$HC^-_1(BG/\mathbb{Q}) \to K^{\mathrm{inf}}_0(BG;\mathbb{Q}) \to K_0(BG;\mathbb{Q}) \to HC^-_0(BG/\mathbb{Q}) \to K^{\mathrm{inf}}_{-1}(BG;\mathbb{Q}) \to K_{-1}(BG;\mathbb{Q})$$

Now, $HC^-_1(BG/\mathbb{Q}) \cong 0$ by above, the trace induces an isomorphism $K_0(BG;\mathbb{Q}) \cong R(G;\mathbb{Q}) \cong HC^-_0(BG/\mathbb{Q})$, and $K_{-1}(BG;\mathbb{Q}) \cong 0$ by smoothness. Therefore $K^{\mathrm{inf}}_0(BG;\mathbb{Q}) \cong 0 \cong K^{\mathrm{inf}}_{-1}(BG;\mathbb{Q})$ as desired. ☐

Remark 11.13. We also deduce the corresponding claim for the cdh-sheaffified theories from the supplementary §13: for all $X \in \mathcal{B}^G_k$, we have an equivalence

$$KH^G_0(X;\mathbb{Q}) \xrightarrow{\cong} (L_{\mathrm{cdh}}HC^-)^G_0(X/\mathbb{Q}).$$

This is a weaker claim, where both sides are more computable.

Remark 11.14. Under the geometric interpretation of equivariant negative cyclic homology via derived loop stacks [BFN10; Che20; HSS17; Toë14] using the fixed point scheme notation from §2.1 covering [Löw26a, §1], we can rewrite Theorem 11.11 as

$$K^G_0(X;\mathbb{Q}) \xrightarrow{\cong} \pi_0(\mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X), \mathcal{O})^{hS^1}).$$

In words, it gives an isomorphism between the zeroeth equivariant K-group of X and homotopy-S^1-invariant global functions on the associated derived fixed-point scheme.

11.2.2. A decomposition for positive degrees. Let us complement Theorem 11.11 with a slightly counterintuitive result on the positive homotopy degrees. In general, algebraic K-theory can be described by gluing the homotopy-invariant contribution of $KH(-)$ with the contribution of $HC^-(-/\mathbb{Q})$ along their maps to the cdh-sheafification $L_{\mathrm{cdh}}HC^-(-/\mathbb{Q})$, which can be regarded as a truncating invariant. We recall this thoroughly in the supplementary §13.

It turns out that for simple varieties, the gluing condition is vacuous in positive degrees – their positive K-groups decompose into the direct sum of $KH(-)$ and $HC^-(-/\mathbb{Q})$.

Proposition 11.15. *Let $X \in \mathcal{B}^G_{\mathbb{Q}}$, then*

$$K^G_i(X) \cong KH^G_i(X) \oplus (HC^-)^G_i(X/\mathbb{Q}), \qquad \forall i \geqslant 1$$

Proof. From the homotopy fiber square of Construction 13.1 defining $L_{\mathrm{cdh}}HC^{-}(-)$ it suffices to prove

$$(L_{\mathrm{cdh}}HC^{-})_i^G(X/\mathbb{Q}) \cong 0, \qquad \forall i \geqslant 1$$

This is the case for $X = \mathrm{pt}$ since $L_{\mathrm{cdh}}HC_\bullet^{-}(BG/\mathbb{Q}) \cong HC_\bullet^{-}(BG/\mathbb{Q}) \cong HC_0^{-}(BG/\mathbb{Q})[u]$ vanishes for all $i \geqslant 1$. We conclude by applying Theorem 10.18 to the $\mathbb{Q}$-linear truncating invariant $L_{\mathrm{cdh}}HC^{-}(-/\mathbb{Q})$. ☐

Remark 11.16. In [CHWW09, Proposition 5.6], a related phenomenon appears for non-equivariant K-theory of toric varieties.

11.3. Equivariant formality for homotopy invariant K-theory

We now move towards applications to homotopy invariant algebraic K-theory KH. We first record a strong equivariant formality statement in Theorem 11.18. We then discuss two easy consequences: the comparison to its topological counterpart $K_{\mathbf{top}}$ over $\mathbb{C}$ in §11.3.2 and an elementary instance of Parshin vanishing over $\mathbb{F}_q$ in §11.3.3.

Setup 11.17. Throughout this subsection, let k be a base field and G a nice group over it. Via the supplementary §14, the results generalize to any linearly reductive group G over k.

11.3.1. Equivariant formality for KH.

Let us spell out the results from §10.5 for KH. This yields a strong equivariant formality statement for homotopy invariant K-theory of simple varieties.

Theorem 11.18 (KH of simple varieties)**.** *Let k be any base field. For any $X \in \mathcal{B}_k^G$ we have a natural ring isomorphism*

$$KH_0^G(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \xrightarrow{\cong} KH_\bullet^G(X).$$

Furthermore, $KH_0^G(X)$ is a finite projective module over the representation ring $KH_0^G(\mathrm{pt}) \cong R(G)$.

Proof. Follows from Theorem 10.19 as $KH(-)$ is a finitary truncating invariant which is multiplicative. ☐

Remark 11.19 (Faithfull flatness)**.** Concisely, the above theorem shows that $KH^G(X)$ is connective and finite faithfully flat over $K^G(\mathrm{pt})$.

Remark 11.20 (Change of group and equivariant formality)**.** The equivalence of Theorem 11.18 is compatible with changing the group through any group homomorphism $G' \to G$. In particular, restricting to the trivial group $1 \to G$ specializes to non-equivariant K-theory by base change along the augmentation map

$$R(G) \to \mathbb{Z}\,.$$

This is an *equivariant formality* statement for KH of varieties $X \in \mathcal{B}_k^G$. It allows us to compute non-equivariant KH directly from the equivariant one, where more techniques are available.

Remark 11.21 (Arbitrary base)**.** Let G be a group with decomposable representation theory over an arbitrary base ring R (e.g. over $\mathbb{Z}$). The same arguments show that for any $X_R \in \mathrm{Sch}_R^G$ we have the relative statement

$$KH_0^G(X_R) \underset{KH_0(R)}{\otimes} KH_\bullet(R) \xrightarrow{\cong} KH_\bullet^G(X_R)$$

functorially in base change along ring maps $R' \to R$. Indeed, the non-equivariant base case of $X_R = \operatorname{Spec} R$ is tautological, while the rest of our arguments goes through without change.

For the final remarks, let us focus on the case of a split torus $T = \mathbb{G}_m^n$ over k.

Remark 11.22 (Freeness)**.** For any $X \in \mathcal{B}_k^T$, we deduce that the rationalization $KH_0^T(X;\mathbb{Q})$ is a finite free module over $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

Indeed, it is a finite projective module over $KH_0^T(\mathrm{pt};\mathbb{Q}) \cong \mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ by Corollary 11.18. Any finite projective module over $\mathbb{Q}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is free by [Swa78] or [Lam06, §V.4, Corollaries V.4.10 and V.4.11].

Remark 11.23 (Equivariant localization)**.** We expect Theorem 11.18 to be useful in conjunction with equivariant localization from [KR24, Theorem C, §11] for explicit computations of the rings $KH_\bullet^T(X)$ where $X \in \mathrm{Sch}_k^T$.

Indeed, given a T-equivariant variety X, equivariant localization postulates that the natural restriction map between $KH^T(X)$ and $KH^T(X^T)$ becomes an equivalence after localization on the base $K_0^T(\mathrm{pt})$; the latter object is often easy to describe in terms of the T-equivariant geometry. Once equivariant formality holds, $KH^T(X)$ in particular embeds into its localization. The computational question then becomes to describe its image.

For cohomology, the above recipe is well-known and useful. For KH of singular varieties, not much has been done. For example, to the best of our knowledge, there are basically no results on equivariant formality of KH for singular varieties in the literature to start with.

11.3.2. Isomorphisms with topological theories in degree zero. If we work over the complex numbers $k = \mathbb{C}$, we can compare to topological K-theory. More precisely, we have maps of localizing invariants

$$K(-) \to KH(-) \to K_{\mathbf{st}}(-) \to K_{\mathbf{top}}(-)$$

of $\mathbb{C}$-linear finitary localizing invariants [Bla16]. Moreover, $KH(-)$, $K_{\mathbf{st}}(-)$, $K_{\mathbf{top}}(-)$ are truncating.

These three latter theories are believed to be reasonably close to each other, but again examples do not come in plenty. They are indeed close for simple varieties: their degree zero parts[3] are actually isomorphic (while the rest is determined by their value on the point).

Proposition 11.24 (KH and $K_{\mathbf{top}}$ of complex varieties)**.** *Let $X \in \mathcal{B}_{\mathbb{C}}^G$, then the natural maps induce ring isomorphisms*

$$KH_0^G(X) \to (K_{\mathbf{st}})_0^G(X) \to (K_{\mathbf{top}})_0^G(X).$$

[3] To avoid any potential confusion, we emphasize that in our conventions, $K_{\mathbf{top}}(X)$ matches the topological K-*cohomology* built up from vector bundles. In particular, $K_{\mathbf{top},0}(X)$ matches the zeroeth topological K-cohomology group (which is usually denoted with a superscript).

Proof. This is clearly the case for the non-equivariant point $\mathrm{pt} = \operatorname{Spec} \mathbb{C}$, where all three terms take the value $\mathbb{Z}$. We conclude by Theorem 10.18 and Lemma 10.13. □

Remark 11.25 (Equivariant formality)**.** Note that Theorem 10.19 applies equally well to the truncating finitary multiplicative invariants $K_{\mathbf{st}}(-)$, $K_{\mathbf{top}}(-)$ as to $KH(-)$ in §11.3. All of them are equivariantly formal for any $X \in \mathcal{B}^G_{\mathbb{C}}$, compatibly with the comparison maps. Similarly for the non-finitary truncating invariant $HP(-/\mathbb{C})$.

Remark 11.26 (Topological and arithmetic parts of KH)**.** Paired with the equivariant formality from previous remark, Proposition 11.24 allows to compute $KH^G_\bullet(X)$ by tensoring two independent contributions: the topological contribution of $K^G_{\mathbf{top},0}(X)$ and the arithmetic contribution of $K_\bullet(\mathbb{C})$.

To free the careful reader of any potential doubts, we would like to emphasize that $K^G_{\mathbf{top}}(-)$ is the classical invariant one hopes for.

Remark 11.27 (Blanc vs. Segal)**.** The equivariant topological K-theory $K^G_{\mathbf{top}}$ agrees with the classical construction [Seg68] on any $X \in \mathrm{Sch}^G_{\mathbb{C}}$.

Denote $M \subseteq G(\mathbb{C})$ the maximal compact subgroup of the topological group of complex points of G with the analytic topology. Then M is a compact real Lie group and it acts on the topological space $X(\mathbb{C})$ of complex points of X with the analytic topology. In this situation, [Seg68] considered the equivariant topological K-theory spectrum

$$K^M_{\mathrm{top}}(X(\mathbb{C})).$$

For X in $\mathrm{Sch}^G_{\mathbb{C}}$, we have a natural comparison map

$$K^G_{\mathbf{top}}(X) \to K^M_{\mathrm{top}}(X(\mathbb{C})).$$

constructed in [HP20, §2.1.3].

Now, both sides have proper excision: for $K^G_{\mathbf{top}}$ this holds because it is a truncating invariant via [ES21; LS25], while for K^M_{top} this follows from the long exact sequence of a pair. By equivariant resolution of singularities in characteristic zero (see [EKS25, §6.1] for a discussion), it is thus enough to check that the comparison map is an equivalence on smooth G-equivariant schemes. This has been done in [HP20], [Bla16].

For the sake of completeness, let us mention the following fact: the topological realization of the trace map is always an isomorphism.

Remark 11.28 (Topological Chern character)**.** On topological K-theory, the trace map induces the topological Chern character

$$K_{\mathbf{top}}(-) \otimes \mathbb{C} \to HP(-/\mathbb{C}).$$

For any $X \in \mathrm{Sch}^G_{\mathbb{C}}$ we have an equivalence

$$K^G_{\mathbf{top}}(X) \otimes \mathbb{C} \xrightarrow{\simeq} HP^G(X/\mathbb{C}).$$

For smooth schemes, this is due to [Kon21]. It was extended to smooth quotient stacks by [HP20]. Also see [ES21, Theorem 5.3.2]. For possibly singular stacks with nice stabilizers, this follows by cdh descent; see [Kha23]. This works for quotients of quasi-projective varieties by linearly reductive G via the supplementary §14.

11.3.3. Examples of the equivariant singular Parshin property. We also mention the following silly consequence of the formality of KH from Theorem 11.18. If $k = \mathbb{F}_q$ is a finite field of characteristic p with q elements, each $X \in \mathcal{B}_{\mathbb{F}_q}$ satisfies Parshin's property:

$$K_i(X;\mathbb{Q}) \cong 0, \qquad \forall i \geqslant 1.$$

This property is usually conjectured for smooth projective varieties over $\mathbb{F}_q$ and such claims are open in general. It is certainly expected for singular projective varieties over $\mathbb{F}_q$ – the statement is compatible with cdh descent, so the singular case would follow from the smooth case if one had resolution of singularities for projective varieties over $\mathbb{F}_q$.

In fact, this argument works even equivariantly.

Observation 11.29. *Let G be a nice group over $\mathbb{F}_q$ and (X, Y, Z, E) be an abstract blowup square in $\mathrm{Sch}^G_{\mathbb{F}_q}$. If the Parshin's property*

$$K_i^G(-;\mathbb{Q}) \cong 0, \qquad \forall i \geqslant 0$$

holds for Y, Z, E, then it holds for X as well.

Proof. Since $K^G(X;\mathbb{Q}) \simeq KH^G(X;\mathbb{Q})$ by [KR18, Theorem 1.3.(2)] and [Hoy20, Theorem 1.3.(4)], it satisfies cdh descent, so we conclude from the associated long exact sequence. □

However, not many instances of this method are available in practice. The equivariant formality of KH from Theorem 11.18 now easily implies that varieties from $\mathcal{B}^G_k$ qualify.

Example 11.30. Let G be a nice group over $\mathbb{F}_p$. For any $X \in \mathcal{B}^G_{\mathbb{F}_q}$, we have

$$K_i^G(X;\mathbb{Q}) \cong 0, \qquad \forall i \neq 0.$$

In particular, X satisfies Parshin's property.

Proof. To this end, compute

$$K_\bullet^G(X;\mathbb{Q}) \cong KH_\bullet^G(X;\mathbb{Q}) \cong KH_0^G(X) \underset{K_0(\mathrm{pt})}{\otimes} K_\bullet(\mathrm{pt}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong KH_0^G(X;\mathbb{Q}).$$

Here, the first isomorphism holds by [KR18, Theorem 1.3.(2)] and [Hoy20, Theorem 1.3.(4)] already with $\mathbb{Z}[\frac{1}{p}]$-coefficients; in the non-equivariant case this goes back to [Wei89; TT90]. The second isomorphism comes from Theorem 11.18 and the final isomorphism from the knowledge that $K_\bullet(\mathrm{pt};\mathbb{Q})$ is concentrated in degree zero with value $\mathbb{Q}$ by Quillen's work [Wei13, Corollary IV.1.13]. □

On the other hand, we can illustrate Observation 11.29 on the following example, which is not directly covered by the class $\mathcal{B}^G_{\mathbb{F}_q}$ (although see §9.3.3). Recall [VV03] for a quick overwiew of split toric varieties over k.

Example 11.31. Split toric varieties over $\mathbb{F}_p$ satisfy the equivariant Parshin's property.

Proof. Note that this is the case for the point; it then holds for smooth split toric varieties by [VV03, Theorem 6.2]. Since the statement is compatible with enlarging T along precomposition with a group homomorphism by Remark 10.6, the case of singular split toric varieties reduces to the smooth case via toric resolutions and cdh descent from Observation 11.29. □

Remark 11.32. The Parshin's property also holds for the so called *broken toric varieties*, defined by closed gluing of toric varieties along toric subvarieties. Indeed, the statement is clear from Example 11.31 via Observation 11.29 – see Example 12.5 below for the relevant abstract blowups. Broken toric varieties (with smooth irreducible components) naturally appear in the mirror symmetry literature, see [Sun25] and references there.

12. Examples of simple varieties

We now pay our debt to the reader by providing examples of varieties for which the above theory applies. Although such varieties are rather special, we recover examples of interest in geometric representation theory. After some basic examples in §12.1, we discuss both finite and affine Schubert varieties §12.2, §12.3 for GL_n. The singularities of such varieties are of deep interest in geometric representation theory.

12.1. Some basic examples

We start with a few elementary examples. Let k be a base field and G a linearly reductive group over it.

Example 12.1 (Partial flag varieties)**.** Projective spaces $\mathbb{P}^n$, Grassmannians $\mathrm{Gr}(n,d)$ and all partial flag varieties $\mathrm{Flag}(n,d_\bullet)$ for GL_n lie in $\mathcal{B}_k$ and hence in $\mathcal{C}_k$ by (1) and (2). Picking any G-action on the underlying vector space V, these G-equivariant varieties lie in $\mathcal{B}_k^G$ and hence in $\mathcal{C}_k^G$.

Example 12.2 (Hirzerbruch surfaces)**.** Let $m,d \geqslant 1$. Consider a direct sum of line bundles $\mathcal{E} = \mathcal{O}(i_1) \oplus \ldots \mathcal{O}(i_m)$ on $\mathbb{P}_k^d$ and let $X := \mathbb{P}_{\mathbb{P}^d}(\mathcal{E})$ be its projectivization. All of this further carries an obvious action of the torus $T = \mathbb{G}_m^{d+1}$. Then $X \in \mathcal{B}_k^T$ by (1) and (2).

It is known that this construction yields all smooth projective toric varieties of Picard rank 2. For example, specializing to $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-n)$ on $\mathbb{P}^1$ yields Hirzerbruch surfaces Σ_n.

Example 12.3 (Cusp)**.** Let X be the projective cuspidal curve over k. Then $X \in \mathcal{C}_k$. Indeed, this follows from the abstract blowup square

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[d] & \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \arrow[l, hook'] \arrow[d] \\
X & \mathrm{pt} \arrow[l, hook']
\end{tikzcd}$$

whose other terms lie in $\mathcal{C}_k$ by (1), (2) and Remark 10.8. We conclude via (4').

Example 12.4 (Node)**.** Let X be the projective nodal curve over k. Then $X \in \mathcal{C}_k$, but not in $\mathcal{B}_k$. Indeed, there is an abstract blowup square

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[d] & \mathrm{pt} \sqcup \mathrm{pt} \arrow[l, hook'] \arrow[d] \\
X & \mathrm{pt} \arrow[l, hook']
\end{tikzcd}$$

whose other terms lie in $\mathcal{C}_k$ by (1) and (2). Hence also X lies in $\mathcal{C}_k$ by Definition 10.4.

On the other hand, if X was in $\mathcal{B}_k$, then we would also have $KH_{-1}(X) \cong 0$ by Theorem 11.18. However, the long exact sequence associated with the abstract blowup square above ends with

$$\begin{tikzcd} {} \arrow[r] & KH_0(\mathbb{P}^1) \oplus KH_0(\mathrm{pt}) \arrow[r, "\mathrm{rank}"] \arrow[d, equal] & KH_0(\mathrm{pt} \sqcup \mathrm{pt}) \arrow[r] \arrow[d, equal] & KH_{-1}(C) \arrow[r] \arrow[d, equal] & 0 \\ {} \arrow[r] & \mathbb{Z}^{\oplus 3} \arrow[r, "{(a,b,c) \mapsto (a+b+c, a+b+c)}"] & \mathbb{Z}^{\oplus 2} \arrow[r] & KH_{-1}(C) \arrow[r] & 0 \end{tikzcd}$$

showing $KH_{-1}(X) \cong \mathbb{Z}$. Altogether, $X \notin \mathcal{B}_k$.

Example 12.5 (Closed gluing)**.** If X is covered by two closed subvarieties Z_1 and Z_2 with (reduced) intersection $Z_3 := Z_1 \cap Z_2$ such that $Z_1, Z_2, Z_3 \in \mathcal{C}_k$, then also $X \in \mathcal{C}_k$. Similarly for the equivariant case. Indeed, this immediately follows from the associated abstract blowup square featuring $(X, Z_1 \sqcup Z_2, Z_3, Z_3 \sqcup Z_3)$ and from Definition 10.4.

Example 12.6 (Closure on projective cones)**.** Let k be any base field – or even any base ring – and E a projective scheme over k. Assume that $E \in \mathcal{B}_k$ (resp. $\mathcal{C}_k$). Let $i : E \hookrightarrow \mathbb{P}^n_k$ be any projective embedding, $\mathcal{L} = i^* \mathcal{O}(1)$ the corresponding line bundle on E, and X the associated projective cone. Then $X \in \mathcal{B}_k$ (resp. $\mathcal{C}_k$) as well. Similarly for the equivariant case.

Indeed, blowing up X in the cone point, we obtain an abstract blowup

$$\begin{tikzcd} Y \arrow[d] & E \arrow[l, hook', "0"'] \arrow[d] \\ X & \mathrm{pt} \arrow[l, hook'] \end{tikzcd}$$

where $Y = \mathbb{P}_E(\mathcal{O} \oplus \mathcal{L})$ and the upper horizontal map is the inclusion of the zero section, which is split by the structure map $\mathbb{P}_E(\mathcal{O} \oplus \mathcal{L}) \to E$.

The above statement is covered by [EGAII, §8]: under the notation from [EGAII, (8.3.1) and (8.3.2)], the relevant commutative square is [EGAII, (8.7.1)]. It is indeed a pullback square by [EGAII, (8.7.8)]; the description of Y and the horizontal inclusion follows from [EGAII, (8.7.8) and (8.4.2)]. Alternatively see [EGAII, (8.8.2), (8.8.3), (8.8.4)] for discussion of the relevant affine chart away from infinity.

Altogether, we have $E \in \mathcal{B}_k$ by assumption, hence $Y = \mathbb{P}_E(\mathcal{O} \oplus \mathcal{L}) \in \mathcal{B}_k$ by (2). Also $\mathrm{pt} \in \mathcal{B}_k$ by (1). Since the abstract blowup square is split, we deduce that $X \in \mathcal{B}_k$ by (4). The discussion for the class $\mathcal{C}_k$ is similar; the equivariant versions hold by naturality of the arguments.

Example 12.7 (A projective cone of $\mathbb{P}^1$)**.** Let $k = \mathbb{Q}$. To illustrate that the decomposition from Proposition 11.15 can be nontrivial, we recall the example of the projective cone X of $\mathbb{P}^1$ under the embedding by $\mathcal{O}(2)$, heavily based on the results of [CHWW13] for the affine cone. Note that X accidentally matches the affine Schubert variety $X_{\leqslant 2\omega_1} \hookrightarrow \mathcal{G}r_{GL_2}$ by [Zhu17b, Lemma 2.1.14]. Also compare to [PS21].

We claim that

$$K_i(X) = \begin{cases} KH_i(X) \oplus HC_i^-(X/\mathbb{Q}) & \text{if } i \geqslant 1 \\ KH_0(X) & \text{if } i = 0 \\ 0 & \text{if } i \leqslant -1 \end{cases}$$

where

$$KH_i(X) \cong KH_i(\mathbb{Q})^{\oplus 3} \quad \text{if } i \geqslant 0 \qquad \text{and} \qquad HC_i^-(X/\mathbb{Q}) \cong \begin{cases} 0 & \text{if } i \geqslant 1 \text{ even,} \\ \mathbb{Q} & \text{if } i \geqslant 1 \text{ odd.} \end{cases}$$

To get this, write $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(2))$ for the resolution of X. This matches the affine Demazure resolution $Y_{\leqslant(\omega_1,\omega_1)}$, but we do not need this fact. We stratify X into the affine cone X_1 and the complement of the singular point X_2, and write $X_3 = X_1 \cap X_2$ for their intersection. Pulling back this stratification to Y, we get the diagram

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y_1 \sqcup Y_2 \arrow[l] \arrow[d, "f_1 \sqcup f_2"'] & Y_3 \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "f_3"'] \\ X & X_1 \sqcup X_2 \arrow[l] & X_3 \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

where rows are Zariski covers. The maps $f_2 \colon Y_2 \to X_2$ and $f_3 \colon Y_3 \to X_3$ are isomorphisms. We now apply $K(-)$ to this diagram; rows will then induce long exact sequences by Zariski descent.

$$\begin{tikzcd} K(Y) \arrow[r] & K(Y_1) \oplus K(Y_2) \arrow[r] & K(Y_3) \\ K(X) \arrow[r] \arrow[u, "f^*"] & K(X_1) \oplus K(X_2) \arrow[r] \arrow[u, "f_1^* \oplus \mathrm{id}"] & K(X_3) \arrow[u, "\mathrm{id}"] \end{tikzcd}$$

The upper row consists of smooth varieties, and we have $K(Y) = K(\mathbb{Q})^{\oplus 4}$ by repeated use of the projective bundle formula. From the split abstract blowup square associated to the resolution $f : Y \to X$, we deduce $KH(X) = K(\mathbb{Q})^{\oplus 3}$ is a direct summand in $KH(Y) = K(Y)$. Since X_2 and X_3 are smooth, the remaining difference between $K(X)$ and $K(Y)$ comes from $K(X_1)$, which is described by [CHWW13, Theorem 4.3 and Remark 4.3.1] – it gives $\mathbb{Q}$ in odd positive degrees $2j+1 \geqslant 1$ and zero else. However, each of these $\mathbb{Q}$ maps to 0 under $f_1^* : K_{2j+1}(X_1) \to K_{2j+1}(Y_1)$ – indeed, $\mathbb{Q}$ is uniquely divisible, while $K_{2j+1}(Y_1) \cong K_{2j+1}(\mathbb{Q})^{\oplus 2}$ is torsion for $2j+1 \geqslant 3$ by [Wei13] and $(\mathbb{Q}^\times)^{\oplus 2}$ for $2j+1 = 1$ (alternatively one can keep track of the weights). From the commutativity the right-hand square and the fact that the right vertical map is the identity, we now see that the whole $K_{2j+1}(X_1) \cong \mathbb{Q}$ maps to zero under $K_{2j+1}(X_1) \to K_{2j+1}(X_3)$ – hence it contributes to $K_{2j+1}(X)$ by long exact sequence induced by the bottom row. We have thus matched $K_i(X) = KH_i(X) = KH(\mathbb{Q})^{\oplus 3}$ when i is not odd positive; there is an extra $\mathbb{Q}$ appearing if i is odd positive.

On the other hand, we already know that $K_i(X) \cong KH_i(X) \oplus HC_i^-(X/\mathbb{Q})$ for all $i \geqslant 1$ by Proposition 11.15. Altogether, we deduce the result.

Remark 12.8. Consider an affine Demazure resolution $f : Y \to X$ over k. Since X has rational singularities, the pullback $f^* : \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$ is a fully faithful embedding. One could wishfully ask whether f^* makes $\mathrm{Perf}(X)$ into a semi-orthogonal summand of $\mathrm{Perf}(Y)$. This is not the case.

In fact, it fails by the following general principle. We work over a field k. Under mild assumptions, regularity of a scheme Z is equivalent to the existence of a *strong generator* of $\mathrm{Perf}(Z)$ via [Orl16, Theorem 3.27]. Furthermore, strong generators are inherited by semi-orthogonal summands [Orl16, Proposition 3.20]. Put together, if $\mathrm{Perf}(X)$ is a semi-orthogonal summand of $\mathrm{Perf}(Y)$ for a regular scheme Y, then X is regular as well.

In the case of affine Schubert varieties, the above failure can be seen already on the level of localizing invariants. Consider the map $f : Y \to X$ over $\mathbb{Q}$ from Example 12.7, which accidentally matches the affine Demazure resolution $f : Y_{\leqslant(\omega_1,\omega_1)} \to X_{\leqslant 2\omega_1}$ of the affine Schubert variety $X_{\leqslant 2\omega_1} \hookrightarrow \mathcal{G}\mathrm{r}_{GL_2}$ by [Zhu17b, Lemma 2.1.14]. If $\mathrm{Perf}(X)$ were a semi-orthogonal summand of $\mathrm{Perf}(Y)$, then the natural map $E_\bullet(X) \to E_\bullet(Y)$ would be injective for any localizing invariant E. This is wrong for negative cyclic homology $HC^-(-/\mathbb{Q})$. By the case of a point and projective bundle formula, $HC^-(Y/\mathbb{Q})$ is supported in non-positive degrees, but $HC^-(X/\mathbb{Q})$ is nonzero in each odd positive degree (it is a copy of $\mathbb{Q}$) by Example 12.7.

12.2. Finite Schubert varieties

We prove that finite Schubert varieties in usual Grassmannians are simple varieties in Lemma 12.11 – they lie in the class $\mathcal{B}_k^B$, and in particular in $\mathcal{B}_k^T$. Already here, our computations seem to be new.

We start by quickly recalling the definitions; see [Bri05] or [Oet21, §2.2, §2.3] for these standard facts. For the sake of readability, we use the convention dual to Notation 10.2: given $X \in \mathrm{Sch}_k$, $\mathcal{F} \in \mathrm{QCoh}(X)$ and a dimension vector $d_\bullet = (d_1, \ldots, d_m)$, we denote $\mathrm{Flag}_X^*(\mathcal{F}, d_\bullet)$ the scheme parametrizing flags of *subbundles* – as opposed to quotient bundles – with successive subquotients of ranks $d_1, \ldots, d_m$. In particular, $\mathrm{Grass}^*(\mathcal{F}, d)$ parametrizes rank d locally free *subbundles*, compatibly with the above references.

Setup 12.9. Let k be a base field and U a k-vector space of dimension n. Let $F_\bullet U = (0 = F_0U \subseteq F_1U \subseteq F_2U \subseteq \cdots \subseteq F_nU = U)$ be a fixed full reference flag in U. We denote by $B \leqslant GL_n$ the Borel subgroup stabilizing $F_\bullet U$ and by $T = \mathbb{G}_m^n$ its maximal torus. We choose an embedding $T \leqslant B$.

Recollection 12.10 (Finite Schubert varieties)**.** Fix a sequence of non-negative integers $j_\bullet = (0 = j_0 \leqslant j_1 \leqslant \cdots \leqslant j_n = d)$ with $j_i \leqslant i$ for each $i = 0, \ldots, n$. Denote $d_\bullet = (d_1, \ldots, d_n)$ the corresponding difference sequence with $d_i := j_i - j_{i-1}$ for each $i = 1, \ldots, n$. The associated *finite Schubert variety* $X_{\leqslant j_\bullet}$ is the reduced closed subvariety of $\mathrm{Grass}_k^*(U, d)$ classifying

$$X_{\leqslant j_\bullet} = \{V \subseteq U \mid \dim_k V = d,\ \forall i :\ \dim_k(V \cap F_iU) \geqslant j_i\}.$$

It is a projective variety with a natural action of $T \leqslant B$. Set-theoretically, it is a closed union of B-orbits in $\mathrm{Grass}_k^*(U, d)$. It is often singular; the smooth cases are determined by explicit combinatorial criteria.

Lemma 12.11 (Finite Schubert varieties are simple)**.** *For any d and $j_\bullet$ as above, we have $X_{\leqslant j_\bullet} \in \mathcal{B}_k^B$. In particular, $X_{\leqslant j_\bullet} \in \mathcal{B}_k^T$.*

Proof. Consider the Bott–Samelson resolution $Y_{\leqslant j_\bullet}$, which classifies

$$Y_{\leqslant j_\bullet} = \{(0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \mid V_i \subseteq F_iU,\ \forall i :\ \dim_k V_i = j_i\}$$

The map $f : Y_{\leqslant j_\bullet} \to X_{\leqslant j_\bullet}$ given by $(V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n) \mapsto V_n$ is proper birational and B-equivariant.

On one hand, $Y_{\leqslant j_\bullet}$ is a tower of honest Grassmannian bundles over pt, defined inductively as follows: in the i-th step, we consider the Grassmannian classifying d_i-dimensional subspaces in F_iU modulo the tautological bundle with fiber V_{i-1} from the previous step.

On the other hand, the map $f : Y_{\leqslant j_\bullet} \to X_{\leqslant j_\bullet}$ can be constructed as a tower of stratified Grassmannian bundles, defined inductively as follows. Denote $\mathcal{V}$ the tautological bundle on $X_{\leqslant j_\bullet}$ with fiber V. In the i-th step, we consider the Grassmannian classifing d_i-dimensional subspaces in the pullback of $\mathcal{V} \cap F_\bullet U_i$ modulo the tautological subbundle from the previous step (also note that by definition of $X_{\leqslant j_\bullet}$ and $\mathcal{V}$, this coherent sheaf has rank $\geqslant d_i$ everywhere).

Altogether, we get a B-equivariant diagram

$$X_{\leqslant j_\bullet} \xleftarrow{f} Y_{\leqslant j_\bullet} \xrightarrow{g} \mathrm{pt}$$

where f is an iterated stratified Grassmannian bundle with nonempty fibers and g is an honest iterated Grassmannian bundle. Since $\mathrm{pt} \in \mathcal{B}_k^B$ by (1), also $Y_{\leqslant j_\bullet} \in \mathcal{B}_k^B$ by iterative use of (2). Hence $X_{\leqslant j_\bullet} \in \mathcal{B}_k^B$ by iterative use of (3), the relevant rank condition being satisfied by the above. In particular, $X_{\leqslant j_\bullet} \in \mathcal{B}_k^T$ by Remark 10.6. □

12.3. Affine Schubert varieties

Let k be any base field. Consider the split group GL_n over k with its diagonal torus T. The motivating examples of our interest are the affine Schubert varieties in the GL_n affine Grassmannian $\mathcal{G}\mathrm{r}$. These are certain equivariant singular projective varieties; they are of considerable interest in geometric representation theory. We show in Lemma 12.15 that they lie in $\mathcal{B}_k^{GL_n}$, and in particular in $\mathcal{B}_k^T$.

Before going to the proof, we quickly recall the relevant geometric input. See [Zhu17b] for details. Also [BS17] is helpful (although the geometric setup is different, their description of the Demazure morphism is applicable and very clear).

Recollection 12.12 (Affine Schubert varieties)**.** The GL_n affine Grassmannian [Zhu17b, §1.1] is the ind-scheme representing the moduli problem

$$\mathcal{G}\mathrm{r} : R \mapsto \{\Lambda \subseteq R(\!(t)\!)^{\oplus n} \mid \Lambda \text{ full } R[\![t]\!]\text{-lattice}\}.$$

Let $\Lambda_0 = R[\![t]\!]^{\oplus n} \subseteq R(\!(t)\!)^{\oplus n}$ be the standard lattice. Given any dominant coweight $\mu \in X_\bullet^+(T)$, we obtain the corresponding affine Schubert variety $X_{\leqslant\mu}$. By definition [Zhu17b, §2.1], this is the reduced subscheme of $\mathcal{G}\mathrm{r}$ on the closed subfunctor

$$R \mapsto \{\Lambda \subseteq R(\!(t)\!)^{\oplus n} \mid \Lambda \text{ full } R[\![t]\!]\text{-lattice, the relative position of } \Lambda \text{ to } \Lambda_0 \text{ is } \leqslant \mu\}.$$

It is a projective variety over k with an action of the positive loop group L^+GL_n. Hence it carries an action of GL_n and in particular of T.

Recollection 12.13 (The tautological bundle)**.** The variety $X_{\leqslant\mu}$ carries a natural vector bundle $\mathcal{E} = \mathcal{E}_{\leqslant\mu}$ defined as follows (see also [Zhu17b, §1.1] and [BS17, §7]). We write $\mathrm{mod}^{\mathrm{fl}}(R[\![t]\!])$ for the abelian category of finite length $R[\![t]\!]$-modules. Without loss of generality, assume that we can write $\mu = \mu_1 + \cdots + \mu_\ell$ as a sum of ℓ fundamental dominant coweights $\mu_i \in X_\bullet^+(T)$ and call $\ell = \lg(\mu)$ the length of μ. Under this assumption, any $\Lambda \in X_{\leqslant\mu}(R)$ lies inside Λ_0. (The assumption is indeed harmless – all $\mu \in X_\bullet^+(T)$ are of this form up to adding a negative multiple $-m\omega_n$ of the determinant coweight, i.e. replacing Λ_0 by $t^{-m}\Lambda_0$.) We then define

$$\begin{aligned} \mathcal{E} : X_{\leqslant\mu}(R) &\to \mathrm{mod}^{\mathrm{fl}}(R[\![t]\!]) \\ \Lambda &\mapsto \Lambda_0/\Lambda. \end{aligned}$$

This gives a vector bundle on $X_{\leqslant\mu}$, carrying a nilpotent operator $t : \mathcal{E} \to \mathcal{E}$. It is naturally L^+GL_n-equivariant. Consider now

$$\mathcal{E}_\bullet := [\mathcal{E} \xrightarrow{t} \mathcal{E}] \qquad \text{and} \qquad \mathcal{F} := \mathcal{H}_0(\mathcal{E}_\bullet) = \operatorname{coker}(\mathcal{E} \xrightarrow{t} \mathcal{E}).$$

Then $\mathcal{E}_\bullet$ is a perfect complex of Tor amplitude $[1,0]$ resolving the coherent sheaf $\mathcal{F}$.

Recollection 12.14 (Affine Demazure resolutions)**.** Given any sequence $\mu_\bullet = (\mu_1, \ldots, \mu_\ell)$ summing up to μ, we get a corresponding (partial) affine Demazure resolution $Y_{\leqslant\mu_\bullet}$, see [Zhu17b, (2.1.17)]. This is the underlying reduced scheme of the moduli problem

$$R \mapsto \{(\Lambda_\ell, \ldots, \Lambda_0) \mid \Lambda_i \subseteq R(\!(t)\!)^{\oplus n} \text{ full } R[\![t]\!]\text{-lattice, relative position of } \Lambda_i \text{ to } \Lambda_{i-1} \text{ is } \leqslant \mu_i\}.$$

The *convolution map* $f : Y_{\leqslant\mu_\bullet} \to X_{\leqslant\mu}$ given by $(\Lambda_\ell, \ldots, \Lambda_0) \mapsto \Lambda_\ell$ is equivariant, proper, birational.

In particular, write $\mu = \mu_1 + \lambda$ with μ_1 minuscule and $\lg(\lambda) < \lg(\mu)$. Then the convolution map f is given by the structure map of the stratified Grassmannian bundle

$$Y_{\leqslant(\mu_1,\lambda)} = \operatorname{Grass}_{X_{\leqslant\mu}}(\mathcal{F}, \mu_1) \xrightarrow{f} X_{\leqslant\mu}.$$

On the other hand, since μ_1 is minuscule, $Y_{\leqslant\mu_\bullet}$ is an honest Grassmannian bundle given by $\operatorname{Grass}_{X_{\leqslant\lambda}}(V, \mu_1)$ over $X_{\leqslant\lambda}$ associated to an n-dimensional vector space V with a L^+GL_n-action.

Altogether, we obtain the convolution diagram

$$X_{\leqslant\mu} \xleftarrow{f} Y_{\leqslant\mu_\bullet} \xrightarrow{g} X_{\leqslant\lambda} \tag{12.1}$$

which is L^+GL_n-equivariant, and in particular GL_n-equivariant.

We are now ready to deduce that affine Schubert varieties are simple.

Lemma 12.15 (Affine Schubert varieties are simple)**.** *Let $X_{\leqslant\mu}$ be any affine Schubert variety in the GL_n affine Grassmannian, then*

$$X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}.$$

In particular, $X_{\leqslant\mu} \in \mathcal{B}_k^T$.

Proof. We will prove that $X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}$ by induction on the length of μ.

If $\lg(\mu) \leqslant 1$, then $X_{\leqslant\mu}$ is either the point pt or an honest Grassmannian $\operatorname{Grass}_k(V, \mu_i)$ associated to an n-dimensional vector space V with an action of GL_n. We thus have $X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}$ in this case by the closure properties (1) and (2).

Now assume $\lg(\mu) \geqslant 2$ and write $\mu = \mu_i + \lambda$ with μ_i minuscule and $\lg(\lambda) < \lg(\mu)$. We get the convolution diagram (12.1)

By induction, we already know that $X_{\leqslant\lambda} \in \mathcal{B}_k^{GL_n}$. Since $Y_{\leqslant\mu_\bullet}$ is an honest Grassmannian bundle over it, we deduce $Y_{\leqslant\mu_\bullet} \in \mathcal{B}_k^{GL_n}$ by the closure property (2). But now $Y_{\leqslant\mu}$ is a stratified Grassmannian bundle over $X_{\leqslant\mu}$, so we deduce $X_{\leqslant\mu} \in \mathcal{B}_k^{GL_n}$ by the closure property (3). This completes the inductive step.

In particular, X lies in $\mathcal{B}_k^T$ and $\mathcal{B}_k$ by Remark 10.6 $\square$

Remark 12.16. The same argument goes through with respect to the groups $T \times \mathbb{G}_m \leqslant GL_n \times \mathbb{G}_m^{\mathrm{rot}}$ extended by the loop rotation action of $\mathbb{G}_m^{\mathrm{rot}}$.

13. Cdh-sheafified (negative) cyclic homology as truncating invariant

For this section, let $k = \mathbb{Q}$ or a finite field extension thereof. It is an important consequence of [KST18; LT19; CHSW08] that on schemes, $K(-)$ is given by gluing the homotopy-invariant contribution of $KH(-)$ with $HC^{-}(-/k)$ along their maps to the cdh-sheaffification $L_{\mathrm{cdh}}HC^{-}(-/k)$; this works more generally in any characteristic using $TC(-)$. See [EM23, Diagrams (3.9) and (4.1)] for an overview. Consequently, $L_{\mathrm{cdh}}HC^{-}(-/k)$ extends to a truncating invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ as in [EM23, Lemma 5.7]; also see [EKS25, §8.1].

This not only yields a clear extension to the equivariant setup, but further allows arguments based on semi-orthogonal decompositions (whose pieces do not a priori need to come from geometry). Such arguments are useful in practice, so we record the details. The same trick works for cyclic homology; we include this as well.

Construction 13.1. Define the localizing invariant $L_{\mathrm{cdh}}HC^{-}(-/k) : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$ by the pushout

$$\begin{tikzcd}
K(-) \arrow[r] \arrow[d] & HC^{-}(-/k) \arrow[d] \\
KH(-) \arrow[r] & L_{\mathrm{cdh}}HC^{-}(-/k).
\end{tikzcd}$$

We use the standard conventions on (co)homological shifts. Namely, the shift $[1]$ corresponds to the suspension Σ; it increases homological degree (and decreases cohomological degree) by 1.

Construction 13.2. We define $L_{\mathrm{cdh}}HC(-/k) : \mathrm{Cat}_k^{\mathrm{perf}} \to \mathrm{Sp}$ by the $[-2]$-shift of the bottom fiber sequence in

$$\begin{tikzcd}
HC^{-}(-/k) \arrow[r] \arrow[d] & HP(-/k) \arrow[r] \arrow[d] & HC(-/k)[2] \arrow[d] \\
L_{\mathrm{cdh}}HC^{-}(-/k) \arrow[r] & HP(-/k) \arrow[r] & L_{\mathrm{cdh}}HC(-/k)[2].
\end{tikzcd}$$

Here, the map $L_{\mathrm{cdh}}HC^{-}(-/k) \to HP(-/k)$ comes from the defining pushout for the invariant $L_{\mathrm{cdh}}HC^{-}(-/k)$ by noting that that the composition $K(-) \to HC^{-}(-/k) \to HP(-/k)$ factors through $K(-) \to KH(-)$ since $KH(-)$ is the initial $\mathbb{A}^1$-invariant localizing invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ by [EKS25, Example 3.2.6.(2)] and $HP(-/k)$ is $\mathbb{A}^1$-homotopy invariant on $\mathrm{Cat}_k^{\mathrm{perf}}$ by the discussion [EKS25, Example 3.2.7.(3)].

Remark 13.3. It is an interesting question whether one can similarly define $L_{\mathrm{cdh}}HH(-/k)$ as a truncating invariant of small stable k-linear categories.

Observation 13.4. *The above defined $L_{\mathrm{cdh}}HC^{-}(-/k)$ and $L_{\mathrm{cdh}}HC(-/k)$ are truncating invariants on $\mathrm{Cat}_k^{\mathrm{perf}}$.*

Proof. First note that both $KH(-)$ and $HP(-/k)$ are truncating invariants on $\mathrm{Cat}_k^{\mathrm{perf}}$ by [LT19, Proposition 3.14 and Corollary 3.11] and [EM23, proof of Lemma 5.7]. Furthermore, looking at the first pushout diagram, the fiber $K^{\mathrm{inf}}(-) = \mathrm{fib}(K(-) \to HC^{-}(-/k)) \simeq$

$\mathrm{fib}(KH(-) \to L_{\mathrm{cdh}}HC^-(-/k))$ is a truncating invariant as well. From the fiber sequence $K^{\mathrm{inf}}(-) \to KH(-) \to L_{\mathrm{cdh}}HC^-(-/k)$ we deduce that $L_{\mathrm{cdh}}HC^-(-/k)$ is truncating. Since $HP(-/k)$ is truncating, the shifted cofiber $L_{\mathrm{cdh}}HC(-/k)$ of $L_{\mathrm{cdh}}HC^-(-/k) \to HP(-/k)$ is also truncating. □

Observation 13.5. *If $\mathcal{X} = X/G$ is a global quotient of a regular scheme X by a nice group G, the natural maps induce equivalences*

$$HC^-(\mathcal{X}/k) \xrightarrow{\simeq} L_{\mathrm{cdh}}HC^-(\mathcal{X}/k),$$
$$HC(\mathcal{X}/k) \xrightarrow{\simeq} L_{\mathrm{cdh}}HC(\mathcal{X}/k).$$

Proof. If $\mathcal{X} = X/G$ is a global quotient of a regular scheme by a nice group, the map $K(\mathcal{X}) \to KH(\mathcal{X})$ is an equivalence by [Hoy20, Theorem 1.3, (1) and (4)]; the result then propagates through the defining pushouts and fiber sequences. □

On schemes, the above defined invariants $L_{\mathrm{cdh}}HC^-(-/k)$ and $L_{\mathrm{cdh}}HC(-/k)$ recover the usual cdh-sheaffifications of $HC^-(-/k)$ and $HC(-/k)$. In fact, this works equivariantly with respect to nice groups.

Observation 13.6. *Let G be a nice group. When restricted to invariants of Sch_k^G, the above defined truncating invariants $(L_{\mathrm{cdh}}HC^-)^G(-/k)$ and $L_{\mathrm{cdh}}HC^G(-/k)$ are given by the actual cdh sheafifications of $(HC^-)^G(-/k)$ and $HC^G(-/k)$.*

Proof. Since the newly defined invariants $L_{\mathrm{cdh}}HC^-(-/k)$, $L_{\mathrm{cdh}}HC(-/k)$ are truncating by Observation 13.4, they in particular satisfy cdh descent on Sch_k^G by [LT19; ES21], hence they get natural maps from the actual cdh-sheaffifications of $HC^-(-/k)$ and $HC(-/k)$ on Sch_k^G. These maps are equivalences on regular schemes by Observation 13.5. Since we are working over a field of characteristic zero (a finite field extension of $\mathbb{Q}$), we conclude via cdh descent from the existence of G-equivariant resolutions of singularities – see [EKS25, §6.1] for a brief overview; this argument already appears in [Hoy20, Corollary 1.4]. □

14. Variant for actions of linearly reductive groups

Under further mild technical assumptions, our Theorems 10.15, 10.18, 10.19 and their consequences hold with respect to any linearly reductive group G over k. As explained in Remark 9.2: this is irrelevant if $\mathrm{char}\, k = p$, but considerably extends the applicability of our results to the case of any reductive group if $\mathrm{char}\, k = 0$. For instance, this applies to the action of GL_n on affine Schubert varieties, which are simple by Lemma 12.15.

The aim of this section is to outline this variant, fully relying on the methods developed in [LS25; KR24; Kha20; BKRS22]. Since the arguments need extra technical care, we decided to discuss them separately. Since the only extra content appears when $\mathrm{char}\, k = 0$, we will assume this throughout.

Setup 14.1. Throughout this section, let G be a linearly reductive group over a field k of $\mathrm{char}\, k = 0$.

For simplicity, we will state our results for the subcategory of quasi-projective G-equivariant varieties $\mathrm{Sch}_k^{G,\mathrm{qp}}$ (but see Recollections 14.2, 14.4 for relevant weaker conditions). We denote by $\mathcal{B}_k^{G,\mathrm{qp}} := \mathcal{B}_k^G \cap \mathrm{Sch}_k^{G,\mathrm{qp}}$ and $\mathcal{C}_k^{G,\mathrm{qp}} := \mathcal{C}_k^G \cap \mathrm{Sch}_k^{G,\mathrm{qp}}$ the classes of quasi-projective G-equivariant simple varieties over k.

Recollection 14.2. Firstly, let us recall the notion of stacks satisfying *connective perfect generation* from [LS25, Definition 6.21]. In particular, suppose X is a G-equivariant qcqs derived scheme over k such that X/G satisfies the resolution property [LS25, Definition 6.15]; then X/G satisfies connective perfect generation. Indeed, it is a quasi-geometric stack [LS25, Definition 6.1] with affine stabilizers over a field of characteristic zero by our setup, hence of finite cohomological dimension [LS25, Definition 6.10.(1), Example 6.13], so we conclude by [LS25, Example 6.19]. Let us further remark that X/G satisfies perfect generation [LS25, Definition 6.10.(2), Example 6.19].

Specifically, for any equivariant quasi-projective variety $X \in \mathrm{Sch}_k^{G,\mathrm{qp}}$, the quotient stack X/G satisfies connective perfect generation (as in Example [LS25, Example 6.22.(2)]). Also see [LS25, end of Example 6.22] for the expected generality.

Furthermore, the resolution property for a quasi-geometric derived stack $\mathcal{X}$ with affine stabilizers only depends on its classical truncation $\mathcal{X}_{\mathrm{cl}}$. Specifically, any derived enhancement of a quotient stack X/G for some $X \in \mathrm{Sch}_k^{G,\mathrm{qp}}$ as above satisfies connective perfect generation. (In this way, connective perfect generation holds for derived projectivizations [Jia25; Jia22; Jia23] of complexes over such X/G, since their underlying classical truncations are again of the form Y/G for some $Y \in \mathrm{Sch}_k^{G,\mathrm{qp}}$.)

Proposition 14.3. *Let $E(-)$ be a truncating k-linear localizing invariant and let $\mathcal{X}$ be a noetherian quasi-geometric stack satisfying connective perfect generation. Then $E(\mathcal{X}) \simeq E(\mathcal{X}_{\mathrm{cl}})$.*

Proof. This is [LS25, Theorem 6.35]. □

Recollection 14.4. Further recall the notion of *linearly scalloped stacks* from [KR24, Definition A.1]. Again, a quotient X/G of a quasi-projective variety $X \in \mathrm{Sch}_k^{G,\mathrm{qp}}$ by a linearly reductive group G falls into this class [KR24, Example A.4].

Proposition 14.5. *Let $E(-)$ be a truncating k-linear localizing invariant. Then it satisfies cdh descent on $\mathrm{Sch}_k^{G,\mathrm{qp}}$.*

Proof. We appeal to the cdh descent criterion [Kha20, Theorem 5.6], which is applicable in the current stacky setup by [Kha20, Remark 5.11.(iii) and §5.3.4]. The value of E on the empty scheme is trivial, giving (i). It moreover satisfied Nisnevich descent in the generality of stacks satisfying perfect generation by [BKRS22, Corollary 4.1.2], giving (ii). The closed descent condition (iii) follows by the finite case of [BKRS22, Theorem C], valid in the generality of linearly scalloped stacks by [BKRS22, Remark 0.0.9]; note that closed squares are finite abstract blowup squares by design [Kha20, Example 5.4]. The blow-up condition (iv) follows by combining [Kha20, Theorem A] with the derived nilinvariance from Proposition 14.3, analogously to [Kha20, Example 5.8]; the derived blowups in question will satisfy connective perfect generation by the analysis from Recollection 14.2. □

Remark 14.6. In the special case of $E = KH$, Propositions 14.3, 14.5 for linearly scalloped stacks are fully covered by [KR24, Theorem F, Theorem G, §A.8.1(b)]. In their treatment, the arguments covering derived nilinvariance are more specific to KH.

Remark 14.7. We may further bootstrap to obtain the above for any affine group scheme G of finite type over k of $\operatorname{char} k = 0$. Indeed, such G embeds into some GL_n for n sufficiently large. We may then formally switch from G-varieties to GL_n-varieties through $\mathrm{Sch}_k^{G,\mathrm{qp}} \to \mathrm{Sch}_k^{GL_n,\mathrm{qp}}$, $X \mapsto X \times^G GL_n$ and then apply the above results with respect to the linearly reductive group GL_n.

With the above facts in hand, we can now state the variant of our results for actions of any linearly reductive G.

Theorem 14.8. *Let G be a linearly reductive group over k. Then:*

- *Theorem 10.15 holds for the class $\mathcal{C}_k^{G,\mathrm{qp}}$,*
- *Theorems 10.18, 10.19 holds for the class $\mathcal{B}_k^{G,\mathrm{qp}}$.*

Proof. We need to prove stability under the closure properties defining simple varieties. The proofs of (1) and (2) work as before.

For (3), consider X, $Y = \mathbb{P}_X(\mathcal{E}_\bullet)$, $Y_{\mathrm{cl}} = \mathbb{P}_X(\mathcal{F})$. We have $E^G(Y) \simeq 0 \implies E^G(X) \simeq 0$ by semi-orthogonal decompositions as before. Moreover, as $E(-)$ is truncating, we have $E^G(Y_{\mathrm{cl}}) \simeq E^G(Y)$ by Proposition 14.3, which is applicable since all derived stacks in question satisfy connective perfect generation by Recollection 14.2. This proves (3a), while (3b) follows by Lemma 10.7.

For (4) and (4') we need to see that E sends G-equivariant abstract blowup squares to homotopy fiber squares; this is covered by Proposition 14.5 $\square$

Through this variant, the results of §11.2 and §11.3 work for actions of linearly reductive groups on quasi-projective varieties. For example, this applies to affine Schubert varieties from Lemma 12.15 with respect to $G = GL_n$ if $\operatorname{char} k = 0$.

CHAPTER III

Centers of quantum groups as fibers of equivariant K-theory of $\mathcal{G}r$

15. Introduction

15.1. Aims of this chapter

The aim of this chapter is to revisit fixed-point localization techniques in equivariant topological K-theory, employ them for an effective description of the equivariant K-theory ring of the affine Grassmannian, and relate the results to representation theory and quantum geometric Langlands program.

Ultimately, the fibers of this K-theory ring describe the cohomology rings of varying fixed-points, which further match centers of various categories $\mathcal{O}$ in pure representation theory. Remarkably, all this information is packaged in a single K-theory ring, which we completely describe in terms of topological generators.

Let us now explain our main results in more detail.

15.1.1. Localization techniques in equivariant K-theory via derived loop stacks. To set up the stage, we revisit the relationship between equivariant topological K-theory, equivariant periodic cyclic homology, functions on derived fixed-point schemes and singular cohomology of the varying fixed points. This stems from classical fixed-point localization techniques in equivariant topology and their reinterpretation in terms of equivariant Hochschild homology along the lines of [Che20].

Motto. *The fibers of torus-equivariant K-theory are given by singular cohomology of the fixed-points. In other words, equivariant K-theory puts the cohomologies of varying fixed points into an algebraic family. Globally, equivariant K-theory is given by equivariant periodic cyclic homology, which has a geometric model via functions on the derived fixed-point scheme.*

Combining different techniques from literature, we arrive at the following comprehensive statement.

Theorem 15.1. *Let $T \circlearrowright X$ be a qcqs scheme over $\mathbb{C}$ with an action of a torus T. Then we have natural multiplicative $\mathcal{O}(T)(\!(u)\!)$-linear identifications*

$$K_T^{\mathrm{top}}(X;\mathbb{C}) \xrightarrow{\simeq} HP_T(X/\mathbb{C}) \xrightarrow{\simeq} \mathrm{R\Gamma}(\mathrm{Fix}^{\mathbf{L}}_{\frac{T}{T}}(X), \mathcal{O})^{tS^1}.$$

Taking derived fibers over a closed point $t \in T$, this equivalence specializes to

$$K_T^{\mathrm{top}}(X;\mathbb{C})_{\mathbf{L}t} \xrightarrow{\simeq} H_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!).$$

If X is equivariantly formal, the final identification is underived, and in particular gives an isomorphism $K_T^{\mathrm{top},0}(X;\mathbb{C})_t \cong H^{2\bullet}_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})$ of commutative rings.

Proof. §17.4; Theorem 17.20 and Corollary 17.30. □

15.1.2. Fixed-points on affine Schubert varieties and affine Grassmannian. In particular, we will be interested in the action of the extended torus $T \times \mathbb{G}_m^{\mathrm{rot}}$ on the affine Grassmannian $\mathcal{G}\mathrm{r}_G$ associated to a reductive group G over $k = \mathbb{C}$.

We can fully compute the varying fixed-points of $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ in $\mathcal{G}\mathrm{r}_G$, and in fact in any affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ inside it. These examples are equivariantly formal – via Theorem 15.1 above, we can effectively compute the corresponding fibers of equivariant K-theory.

Motto. *The fixed points of an arbitrary element (x, ζ) in the extended torus on the affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ in the affine Grassmannian are given for ζ generic (resp. root of unity) by a closed union of Borel orbits (resp. Iwahori orbits) on a disjoint union of partial flag varieties (resp. affine flag varieties) for a resonant Levi subgroup $L_{[x,\zeta]} \leqslant G$.*

In detail, generalizing well-known computations [RW22; BASV22], we first describe the fixed-points $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_G)$ on the whole affine Grassmannian. We construct an associated resonant Levi subgroup $L = L_{[x,\zeta]} \leqslant G$ and show that these fixed-points are given as follows.

Proposition 15.2. *The fixed points of $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ on $\mathcal{G}\mathrm{r}_G$ are given as*

(i) an infinite disjoint union $\coprod_{\vartheta \in X_\bullet/W_L} L/P_\vartheta$ of finite flag varieties for L, if ζ is generic,
(ii) a finite disjoint union of dilated affine flag varieties for L, namely $\coprod_{\vartheta \in X_\bullet/(W_L, \bullet^\ell)} \mathrm{L}_\ell L/\mathcal{P}^\vartheta_{\ell,L}$, if ζ is a root of unity of primitive order ℓ.

Proof. Lemmata 19.8 and 19.14. □

Carefully intersecting the above with given $\mathcal{G}\mathrm{r}_{\leqslant\mu}$, we arrive at the following result.

Theorem 15.3. *Take any affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$. Let $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ and $L = L_{[x,\zeta]} \leqslant G$ the corresponding resonant Levi. Then the reduced fixed point scheme underlying $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$ is isomorphic to*

(i) a closed union of of finite Schubert varieties for L, if ζ is generic,
(ii) a closed union of Iwahori orbits on ℓ-dilated affine flag varieties $\mathcal{F}\mathrm{l}^\vartheta_{\ell,L}$ of L, if ζ is a root of unity of primitive order ℓ.

In both cases, we can concretely describe which finite Schubert varieties resp. closure of Iwahori orbits actually appear.

Proof. §19; Theorems 19.11 and 19.17. □

As we explain later, this geometric statement perfectly matches known representation-theoretical phenomena in classical and quantum categories $\mathcal{O}$. This was, in fact, our original motivation for conjecturing such a description in the first place. It is remarkable how accurately the resulting comparison holds.

15.1.3. Description of equivariant K-theory ring of the affine Grassmannian. Using fixed-point localization together with the description of fixed-points on the whole $\mathcal{G}\mathrm{r}_G$, we compute the $T \times \mathbb{G}_m^{\mathrm{rot}}$-equivariant complexified K-theory ring of the affine Grassmannian in terms of the completed ring of functions on a certain scheme $\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times (T /\!\!/ W))$ given by a gluing of deformations to normal cones of quasi-diagonals Z_ℓ inside $T \times T /\!\!/ W$ over each root of unity ζ.

Motto. *The complexified equivariant K-theory of the affine Grassmannian is given by the completed ring of functions on infinitely many affine blowups of $T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W$ over each root of unity ζ, centered at the ℓ-torsion subscheme Z_ℓ of $T /\!\!/ W$.*

Such a description was originally conjectured by Roman Bezrukavnikov; it is motivated by the analogous description of equivariant cohomology from [BF08, Theorem 1]. After properly accounting for the completion, we arrive at the following.

Theorem 15.4. *Let G be a semi-simple, simply connected group over $k = \mathbb{C}$. Then there is a functorial ring isomorphism*

$$\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times (T /\!\!/ W))) \xrightarrow{\cong} K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C}). \tag{15.1}$$

Proof. Theorem 20.40. □

In particular, the ring $K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C})$ is not finitely generated, even before completing. Moreover, the description (15.1) is explicit, as the following remark shows.

Remark 15.5. Unraveling Theorem 15.4 in terms of commutative algebra, the K-theory ring of the affine Grassmannian $K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C})$ is given by the completion of

$$\mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}][s_1^{\pm 1}, \dots, s_r^{\pm 1}]^W \big[\tfrac{e_j(t_1^\ell, \dots, t_r^\ell) - e_j(s_1^\ell, \dots, s_r^\ell)}{1 - q^\ell} \mid \ell \in \mathbb{N}, j = 1, \dots, r\big], \tag{15.2}$$

with respect to an explicit Schubert filtration. More precisely, the uncompleted ring (15.2) naturally surjects onto the equivariant K-theory of each affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant \mu}$; the equivariant K-theory of the whole $\mathcal{G}\mathrm{r}_G$ is then given by completing along the induced filtration by kernels.

In particular, we get a concrete countable infinite set of topological generators,

$$b_{j,\ell} := \tfrac{e_j(t_1^\ell, \dots, t_r^\ell) - e_j(s_1^\ell, \dots, s_r^\ell)}{1 - q^\ell}, \qquad j = 1, \dots, r,\ \ell \in \mathbb{N}$$

which we call *fundamental Bezrukavnikov classes*. These generators have an interpretation in terms of the tautological vector bundle and are related by Adams operations. We deduce that $K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C})$ is finitely generated as a complete topological λ-ring, see Corollary 20.43.

Remark 15.6. The above description goes against some naive analogies with cohomology: the equivariant cohomology of the affine Grassmannian with complex coefficients is a polynomial (or power-series) ring.

Let us explain this apparent discrepancy. In their computation of cohomology, [BF08] pass to one affine blowup over the origin. (In our terms, this is due to the fact that equivariant cohomology sees only the formal completion of equivariant K-theory at the origin $(1,1) \in T \times \mathbb{G}_m^{\mathrm{rot}}$.) The result of this single blowup is still coincidentally isomorphic to a polynomial ring (on shifted generators). This final coincidence breaks down once we are forced to blow up infinitely many times.

Remark 15.7. We want to emphasize that many interesting classes – such as the class of the *determinant line bundle* $\mathcal{L}_{\det}$ – are only present in the completion; see Lemma 22.6. In other words, the Schubert completion is important and nontrivial.

Remark 15.8. Our method is quite robust. In particular, it can be used to describe

(1) the corresponding result for any reductive G with respect to $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}}$,
(2) the result with respect to an isogenous torus T to T^{sc},
(3) the $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$-equivariant variant,
(4) equivariant K-theory of arbitrary affine flag varieties.

We address these generalizations in §20.8, see in particular Theorem 20.45.

Remark 15.9. We believe that our method may be further adapted to describe

(5) the equivariant K-theory with integral coefficients.

We comment more on this in §15.2.1. Here, let us also remark that integrally, K-theory behaves better than cohomology (where one needs to be very careful in small characteristics). For example, the GKM presentation of K-theory works integrally – as opposed to cohomology, where small primes have to be inverted.

Remark 15.10. Even for reductive G, it is sometimes preferable to consider the action of G^{sc} on $\mathcal{G}\mathrm{r}_G$. We feel obliged to address why.

- The equivariant K-theory with respect to this action is what appears as centers in representation theory of quantum groups. This is also compatible with the same appearance in [CK18].
- When G is not simply connected, the quotient $T /\!\!/ W$ is often singular [BZ00, §7]. On the other hand, $T^{\mathrm{sc}} /\!\!/ W$ is always isomorphic to the affine space $\mathbb{A}^d = \mathfrak{t} /\!\!/ W$.
- The simply-connected variant is compatible with the theory of Kac–Moody groups. In particular, the ample generator of the Picard group of $\mathcal{G}\mathrm{r}_G$ may not be G-equivariant otherwise [CK18, §3.2]; see [YZ11, Remarks 4.3 and 5.8] for more details.

15.1.4. Geometric approach to endomorphisms in classical and quantum category $\mathcal{O}$. We now turn to the interpretation in representation theory. Here, we work with the Langlands dual $\check{G}$ over k. For somewhat miraculous reasons, the categorical centers of various block in the classical Bernstein–Gelfand–Gelfand category $\mathcal{O}$ are given by cohomology rings of partial flag varieties. While this observation is well-known to geometric-representation theorists [Soe90; Str09], it is only understood by an explicit computation of these rings. One aim of this work is to shed some geometric light on this unexpected appearance.

Motto. *The fibers of equivariant K-theory of $\mathcal{G}\mathrm{r}_G$ over varying points (x,ζ) give categorical centers of $[x,\zeta]$-resonant parts $\mathcal{O}^{\mathrm{hb}}_{[x,\zeta]}$ of the hybrid quantum categoris $\mathcal{O}^{\mathrm{hb}}_{\zeta}$ associated to $\check{G}$ at ζ. The fiber of equivariant K-theory of an affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ over (x,ζ) gives the image of the above categorical center in the endomorphism ring of the tensor product $M_x \otimes V_\mu$ of a Verma module M_x with an irreducible highest weight representation V_μ.*

We will give our expected conjectural picture in §15.2.2. For the moment, we illustrate this motto in a special case, which can be directly read-off from our work and existing literature.

Assume ζ to be generic, and consider the classical BGG category $\mathcal{O}_\zeta$ over $\mathsf{k} = \mathbb{C}$ with arbitrary (not necessarily integral) weights. We denote $\mathcal{O}_{[x,\zeta]}$ its part whose non-integrality is determined by x. It is then easy to match the blocks of $\mathcal{O}_{[x,\zeta]}$ with cohomologies of connected components of the fixed-points $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_G)$. Indeed, these are just partial flag varieties for $L_{[x,\zeta]}$ by above. Altogether, this interprets each fiber of K-theory of $\mathcal{G}\mathrm{r}_G$ as a categorical center:

$$K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{C})_{(x,\zeta)} = \mathcal{Z}(\mathcal{O}_{[x,\zeta]}).$$

Our original question was about the meaning of the K-theory ring of a single affine Schubert variety $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathbb{C})$ and its fibers. These rings have an interpretation as centers of endomorphism rings of certain concrete objects. More precisely, for an object $M \in \mathcal{O}_\zeta$, denote $\mathcal{Z}^{\mathrm{cat}}(\mathrm{End}_{\mathcal{O}_\zeta}(M))$ the image of the categorical center $\mathcal{Z}(\mathcal{O}_\zeta)$ of $\mathcal{O}_\zeta$ inside the endomorphism algebra $\mathrm{End}_{\mathcal{O}_\zeta}(M)$ of M; this is a commutative ring. In particular, denote by M_x the Verma module labeled by a closed point x of $T /\!\!/ W$ and V_μ the finite dimensional module of highest weigh μ.

Theorem 15.11. *For (x,ζ) with ζ generic, we have*

$$K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathbb{C})_{(x,\zeta)} = \mathcal{Z}^{\mathrm{cat}}(\mathrm{End}_{\mathcal{O}_\zeta}(M_x\otimes V_\mu)).$$

Proof. Corollary 21.6. □

In the course of our geometric study of the categories $\mathcal{O}$ above, it naturally turns out that the discussion extends to hybrid quantum categories $\mathcal{O}^{\mathrm{hb}}_\zeta$ at a root of unity ζ. Here, the corresponding computation of centers were pioneered much more recently by [BASV22; Sit22; Sit23; Sit24; Los23] – in all known cases, these centers again match the cohomologies of the corresponding fixed-points on $\mathcal{G}\mathrm{r}_G$. Our discussion extends over this torsion locus, giving a uniform geometric treatment, be ζ generic or a root of unity. Also see Remark 21.11.

Remark 15.12. It seems that the image of the categorical center $\mathcal{Z}^{\mathrm{cat}}(-)$ may be replaced by the actual center $\mathcal{Z}(-)$ of the endomorphism ring in question. However, this may be a hard question; we also discuss this further in §15.2.3.

15.2. Expectations, questions and conjectures

We have a clear conjectural picture how the parallel between K-theory of the affine Grassmannian and centers of quantum groups extends to its natural generality – that is over the equivariant base for integral equivariant K-theory. We formulate our expected results in §15.2.1, §15.2.2 below; these are interesting even over $\mathbb{C}$. We then record a few more closely related questions and conjectures in §15.2.3 §15.2.4, §15.2.5.

15.2.1. Integral K-theory. We expect that our description of equivariant K-theory of $\mathcal{G}\mathrm{r}_G$ can be extended to work integrally. We start from from case of $G = GL_n$. Afterwards, we will take G to be split reductive, simply connected group. (To omit the simply connected assumption, one may further bootstrap following §20.8.)

Conjecture 15.13. *The integral K-theory of $\mathcal{G}\mathrm{r}_G$ is given as follows.*

(i) For $G = GL_n$, our $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}^1_G;\mathbb{Z})$ is generated as a topological $\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}})$-algebra by the exterior powers $\lambda^j(b_{1,1})$ for $j\in\mathbb{N}$ of the first Bezrukavnikov class $b_{1,1}$.

(ii) For general simply connected G, the integral equivariant K-theory ring $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{Z})$ is generated, as a complete topological λ-ring over the integral base $\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}})$, by pullbacks of the first Bezrukavnikov class $b_{1,1}$ along representations $\rho_V : G\to GL_n$.

The coordinates on the affine blowup are naturally given by Adams operations $\psi^\ell(b_{1,1})$. By classical Newton's identities, λ-operations generate ψ-operations over $\mathbb{Z}$, but the reverse statement is true only over $\mathbb{Q}$. For this reason, we have to replace ψ-operations (power sums in the universal λ-ring) by the λ-operations (elementary symmetric polynomials in the universal λ-ring) in the expected description over $\mathbb{Z}$.

However – assuming the underlying ring is torsionfree – there is an intrinsic characterization of λ-rings purely in terms of ψ-operations. The last conjecture may be then equivalently rephrased via affine blowups as follows.

Conjecture 15.14. *Denote $b := b_{1,1}$ and $b_\ell := b_{\ell,1}$. The integral K-theory ring of $\mathcal{G}\mathrm{r}^1_{GL_n}$ is given by the Schubert completion of the $\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}})$-module*

$$\mathbb{Z}[t_1^{\pm1},\dots,t_n^{\pm1}][q^{\pm1}][b_\ell \mid \ell\in\mathbb{N}][\tfrac{1}{p}(\psi^p(b_{p\ell})-b_\ell^p)\mid \forall\ell\in\mathbb{N}, p\ \mathrm{prime}]. \tag{15.3}$$

In particular, the integral K-theory ring of $\mathcal{G}\mathrm{r}^1_{GL_n}$ is generated by (15.3) *as a topological algebra over $\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}})$.*

Remark 15.15. Regarding the methods to approach this.

- We have access to an integral GKM description of K-theory by [HHH05]. This K-theory ring is torisionfree as abelian group.
- The class $b_{1,1}$ and all its relatives are realized from vector bundles, and hence lie in the integral K-theory ring.
- The whole construction of the affine blowup makes sense over $\mathbb{Z}$. Here, we need to blowup each subscheme Z_ℓ over several roots of unity at once, or more precisely over the cyclotomic subscheme $\mu_\ell^{\mathrm{prim}}\subseteq\mathbb{G}_m^{\mathrm{rot}}$ of primitive roots of unity of order ℓ.
- On top of the affine blowup in the complex case, we need to carefully blow up over usual primes (p) of $\mathbb{Z}$ to account for the gap between λ-operations and ψ-operations. This introduces the elements $\frac{1}{p}(\psi^p(b_{1,1})-b_{1,1}^p)$, resp, their pullbacks along representations.

We denote the affine scheme given by these affine blowups by $\mathcal{N}_{\wp,\mu_\infty}$. Its ring of functions is given by (15.3) in the case of GL_n. In general, for simple connected G, it is given by pullbacks of these functions along representations.

15.2.2. The correct quantum group and its deformed category $\mathcal{O}$. Throughout the literature, many choices are involved in constructing quantum groups. On the other hand, the affine Grassmannian of G is unique.

Motto. *The correct integral quantum group of $\check{G}$ should be such that its completed deformed center (resp. the center of its deformed category $\mathcal{O}$) matches the equivariant K-theory ring of the corresponding affine Grassmannian $\mathcal{G}\mathrm{r}_G$.*

The aim of this section is to outline our uniform expectations on the correct quantum group corresponding to the K-theory of the affine Grassmannian. These expectations are joint with Tamás Hausel. We discussed the deformed category $\mathcal{O}$ variant from Conjecture 15.16 jointly with Quan Situ.

We will formulate our expectations in their natural generality over $\mathbb{Z}$. However, we emphasize that these are interesting already over $\mathbb{C}$.

Let us explain this in more detail. Over the function field $\mathbb{Q}(v)$, quantum groups are labeled by reductive groups – we will denote the Drinfeld–Jimbo quantum group of the root system of $\check{G}$ as $U_v(\check{G})$. However, this has many integral forms over $A = \mathbb{Z}[v^{\pm 1}]$. The most common choices are the De Concini–Processi form U_A^{DP} and the Lusztig form U_A^{Lus}. These can be combined into a hybrid form U_A^{hb}, originally introduced in the context of knot invariants by [Hab07; Hab08; HL16]; it was rediscovered in the context of geometric representation theory by [Gai21] and further studied in [BASV22; Sit24; Sit22; Sit23; Los23]. The hybrid form itself has several modifications.

Given any such form, one can consider its category $\mathcal{O}$ and its specializations. Moreover, this category may be further deformed in the toral direction – we denote $R = \mathcal{O}(T' \times \mathbb{G}_m)$ for brevity, where $T' \times \mathbb{G}_m$ is a double-cover of the extended torus $T \times \mathbb{G}_m^{\mathrm{rot}}$. In particular, we obtain a deformed hybrid category $\mathcal{O}_R^{\mathrm{hb}}$, which is linear over the A-algebra R.

We expect the correct quantum group is given as follows. First take the hybrid quantum group, whose triangular decomposition

$$U_A^{\mathrm{hb}} = U_A^{\mathrm{DP}} \otimes_A U_A^{\mathrm{Lus}} \otimes U_A^{\mathrm{Lus}}$$

has the Lusztig's divided powers on the negative part, Lusztig's toral part, and the De-Concini–Processi positive part.

To incorporate the toral directions, we consider the deformation U_R^{hb}, defined as a Drinfeld double of integral Borel subalgebra; see [Lac20] for the rational version of this construction.

Finally, the above quantum groups carry and extra grading, and we have to take their *even parts*. To this end, let $q = v^2$ and $A_q := \mathbb{Z}[q^{\pm}]$. We denote the *even part* of the hybrid quantum group by $U_{A_q}^{\mathrm{hb,ev}}$ over A_q. Furthermore, we denote $U_{R_q}^{\mathrm{hb,ev}}$ the even part of the deformed hybrid form; it lives over the ring of functions on the extended torus $R_q := \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$.

Except for the bigger toral part and the extra deformation, this is precisely the definition from [Hab07, §3], [HL16, §8.7]; the correct toral part appears in [BASV22, §3.1.2].

We then expect the following to hold with $\mathbb{Z}$-coefficients.

Conjecture 15.16 (K-theory as center of deformed quantum category $\mathcal{O}$)**.** *There is a natural isomorphism of algebras over $\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$*

$$\mathcal{Z}(\mathcal{O}_{R_q}^{\mathrm{hb,ev}}(\check{G})) \xrightarrow{=} K_{T \times \mathbb{G}_m^{\mathrm{rot}}}^{\mathrm{top},0}(\mathcal{G}\mathrm{r}_G; \mathbb{Z}). \tag{15.4}$$

Both sides (are expected to) have an integral GKM description; we expect to directly match these descriptions. This may be done without computing the resulting rings explicitly.

Nevertheless, at least over $\mathbb{C}$, we know how the K-theory ring looks like in terms of distinguished topological generators – we may thus ask whether these can be realized already in the center $\mathcal{Z}(U_{R_q}^{\mathrm{hb,ev}})$ of the quantum group. This requirement pins down the form $U_{R_q}^{\mathrm{hb,ev}}$ rather well.

Conjecture 15.17 (Decompleted conjecture)**.** *We have a ring isomorphism*

$$\mathcal{Z}(U_{R_q}^{\mathrm{hb,ev}}(\check{G})) \xrightarrow{=} \mathcal{O}(\mathcal{N}_{\wp,\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \tag{15.5}$$

between the center of the even part of the hybrid quantum group and the ring of functions on our deformation.

We consider this a more refined statement – it directly tackles the center of the integral form $U_{R_q}^{\mathrm{hb,ev}}(\check{G})$ before completing. The centers of integral quantum groups are notoriously difficult to compute. The above conjecture is the reason for including the whole Lusztig toral part in our quantum group, deforming in the toral direction, as well as taking its even part.

Combining the above expectations, we obtain the following.

Conjecture 15.18 (Completed center of the quantum group)**.** *The natural map induces an R_q-algebra isomorphism*

$$\widehat{\mathcal{Z}}(U_{R_q}^{\mathrm{hb,ev}}(\check{G})) \xrightarrow{=} \mathcal{Z}(\mathcal{O}_{R_q}^{\mathrm{hb,ev}}). \tag{15.6}$$

Remark 15.19. Here, note that $\check{G}$ can be arbitrary reductive group (not necessarily simply connected nor adjoint). Using the naming conventions from [Jan96; Lus10], we indeed see a nice Langlands duality between the geometry and representation theory. This contrasts other formulations in the literature.

The hybrid quantum group carries the structure of a ribbon Hopf algebra – it contains a distinguished invertible central element r satisfying certain conditions.

Conjecture 15.20 (Determinant line bundle as ribbon element)**.** *Under the above conjecture, the determinant line bundle corresponds to the ribbon element*

$$\mathcal{L}_{\det} \leftrightarrow r$$

up to an explicit correction from the base.

Indeed, results of this form may be checked directly by matching localization formulas for the action of the ribbon element [ST09, Definition 3.15], [Gou92, equation (6)], [LG92, equation (25)], [GZB91, §II], [HL16, (73)] with the corresponding formulas for $\mathcal{L}_{\det}$ from Claim 22.3.

We find it astonishing that the correct definition of the integral quantum group – the one whose center matches $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G,\mathbb{Z})$ – was written almost completely in the quantum topology context [Hab07; Hab08; HL16]. Indeed, the only modifications we have to do is to enlarge the toral part to Lusztig's and deform in the toral direction – the correct combination of DP and Lus parts, its even part, as well as the completion already appear in Habiro's work.

15.2.3. Specialization and evaluation conjectures. The above conjecture can be then specialized to fibers over (x, ζ) of the deformed category. In both cases, taking $\mathcal{Z}(-)$ may a priori not commute with the specialization: we have a natural map, but the center of the specialization may be a priori bigger then the specialization of the center. Nevertheless, one can make the following conjecture.

Conjecture 15.21 (Specialization)**.** *Specializing the above, we have isomorphisms*

$$\mathcal{Z}(\mathcal{O}^{\mathrm{hb,ev}}_{[x,\zeta]}) = K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{Z})_{(x,\zeta)}$$

In all cases where the center is understood, this is the case. Namely, it always holds for complex points $(x, \zeta,)$ when either ζ is generic or $x = 1$. Indeed, the first case holds by comparison to classical BGG category $\mathcal{O}$; the second case holds by by [Sit22].

Similarly, we may conjecture that the K-theory of a given affine Schubert variety corresponds to the actual center of the endomorphism algebra of a concrete object – in other words that the categorical center evaluates to the whole center of its endomorphism algebra. To state this, denote M_x the Verma module of U^{hb}_ζ associated to x, and V^ζ_μ the Weyl module of U^{hb}_ζ labeled by μ.

Conjecture 15.22 (Evaluation)**.** *We have an algebra isomorphism*

$$K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathbb{Z})_{(x,\zeta)} = \mathcal{Z}^{\mathrm{cat}}(\mathrm{End}_{\mathcal{O}_\zeta}(M_x\otimes V^\zeta_\mu)) = \mathcal{Z}(\mathrm{End}_{\mathcal{O}_\zeta}(M_x\otimes V^\zeta_\mu)).$$

Here, recall that the middle term stands for the image of the categorical center of $\mathcal{O}_\zeta$ inside the endomorphism ring $\mathrm{End}_{\mathcal{O}_\zeta}(M_x \otimes V^\zeta_\mu)$, while the last term stand for the actual center of this endomorphism ring.

Remark 15.23. Let us work over $\mathsf{k} = \mathbb{C}$ and $\zeta \in \mathbb{G}_m^{\mathrm{rot}}$ generic The evaluation Conjecture 15.22 can then be rephrased as follows. For $x \in T$, $\mu \in X^+_\bullet$, there should be an algebra isomorphism

$$\bigoplus_\vartheta H^\bullet(X^\vartheta_{(x,\zeta)};\mathsf{k}) \cong \mathcal{Z}(\mathrm{End}_{\mathcal{O}_\zeta}(M_x\otimes V_\mu)).$$

where $X^\vartheta_{(x,\zeta)}$ are the connected components of $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$ – each of them is either empty or a union of finite Schubert varieties inside $L_{[x,\zeta]}/P_\vartheta$, determined by those fixed points λ which appear as weights in the corresponding summand of the tensor product $M_x \otimes V_\mu$.

Over $\mathbb{C}$ and for generic ζ, this is an unpublished expectation of Catharina Stroppel. Again, the nontrivial work is to see whether the categorical center surjects onto the whole center of the given endomorphism ring.

15.2.4. Hopf structure on K-theory. Note that $\mathcal{G}\mathrm{r}_G(\mathbb{C})$ has the homotopy type of the loop group ΩK, where K is the maximal compact subgroup of the topological group $G(\mathbb{C})$. This identification is compatible with the group structure on the nose: we can identify $\mathcal{G}\mathrm{r}_G \widetilde{\times} \mathcal{G}\mathrm{r}_G \to \mathcal{G}\mathrm{r}_G$ with $\Omega K \times \Omega K \to \Omega K$ via [Zhu17b, Theorem 1.6.1]. Since K is a topological group, ΩK is a H_1-space [Bro60]. Upon applying any multiplicative generalized cohomology theory, we thus obtain a Hopf algebra. (Alternatively, this induces the Pontryagin product in homology.) Since $\mathcal{G}\mathrm{r}_G(\mathbb{C})$ is homotopy-commutative, this Hopf algebra is co-commutative.; its Spec is a group scheme. For ordinary cohomology, the answer is described in [YZ11, §6.3].

The multiplication extends to the level of the quantum Hecke stack, giving a relative comultiplication over the equivariant base. This can be modeled explicitily on our cocharacter graphs. However, note that the resulting comultiplication will not be cocommutative; it deforms in q.

This should be also rephrased in terms of $\mathcal{Z}(U^{\mathrm{hb,ev}}_{R_q}(\check{G}))$ and braided Hopf algebra structure of $U^{\mathrm{hb,ev}}_{R_q}(\check{G})$. Also compare to the cohomological case from [BF08, §3.5].

15.2.5. Algebraic questions on K-theory. We would like to close up with a few more refined questions of the equivariant K-theory of $\mathcal{G}\mathrm{r}_G$ related to its algebraic realization.

Question 15.24. What is the meaning of the *decompletion*

$$\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \tag{15.7}$$

of the complete equivariant K-theory ring $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{C})$ in terms of K-theory? Note that the answer should not be trivial: the class of the *determinant line bundle* is *not* present in this decompletion by Lemma 22.6.

Let $X = \mathrm{colim}_i\, X_i$ be an ind-scheme, presented as a colimit of varieties X_i along closed immersions (or the equivariant analogue). As we already saw, classical topological K-theory over $\mathbb{C}$ will turn this into a limit – this follows from its representability. The same applies to KH defined via motivic homotopy theory, or any variant of K-theory which is Kan extended from qcqs schemes.

On the other hand, consider actual algebraic K-theory, or KH, or Blanc's K^{top} as localizing invariants. These are evaluated directly on $\mathrm{Perf}(X)$. While $\mathrm{Perf}(X) = \lim_i \mathrm{Perf}(X_i)$ holds by design, localizing invariants do not behave well with limits of categories – we only get a natural map

$$K(X) \to \lim_i K(X_i)$$

and similarly for the variants. In particular, taking KH or K^{top}, it might happen that this map is injective, realizing the target as a completion of the source. If so, it could be that $K^{\mathrm{top}}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{C})$ gives the desired decompletion (15.7).

A different context where such a picture is known to arise is for stacks (e.g. already for classifying stacks). There are colimit presentations for the classifying stack BG of a reductive group G by varieties. The limit is the Kan extended equivariant K-theory – a power series ring. The actual K-theory of $\mathrm{Perf}(BG)$ maps into it injectively, realizing it as the completion along the augmentation ideal.

In fact, we expect that we might as well work with homotopy invariant algebraic K-theory KH throughout. This has the feature of working in any base characteristic for the geometry.

Conjecture 15.25. *Our description works integrally for $KH^{\bullet}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{Z})$. Namely, $KH^{\bullet}_{T\times\mathbb{G}_m^{\mathrm{rot}}}$ of $\mathcal{G}\mathrm{r}_G$ is equivariantly formal, its degree zero part agrees with topological K-theory, and hence is described by the normal cone construction of this chapter.*

Remark 15.26. We have already answered this question affirmatively in type A in Chapter II covering [Löw26b]. More precisely, we have proved that for affine Schubert varieties in the GL_n-affine Grassmanian we have $KH^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathbb{Z}) = K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathbb{Z})$.

In other words, we know in type A (and conjecture in general) that the computations of this chapter work on the algebraic level of KH over any base field k in the best possible sense. Consequently, all refined structures of KH – such as the motivic filtration and its relation to cdh algebraic differential forms – should appear in our picture.

15.3. Plan of the chapter

After the current §15, this chapter is structured as follows.

In §16 we recall some preliminary material on algebraic geometry, reductive groups, and affine flag varieties.

In §17 we set up notations regarding equivariant K-theory, related invariants and their algebraic counterparts. The new material is our interpretation of equivariant localization in §17.4, in particular the formulation in Theorem 17.20.

In §18 we introduce the notion of resonance for $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ and consequently the resonant Levi sugroup $L_{[x,\zeta]} \leqslant G$. We further define the cocharacter graphs Γ inside $T \times \mathbb{G}_m^{\mathrm{rot}} \times T$ and explain how they put the resonant combinatorics into a family over $T \times \mathbb{G}_m^{\mathrm{rot}}$. There are no deep results, but the discussion in this section serves as a useful combinatorial tool in the rest of the chapter.

In §19, we fully describe the varying fixed-points $\mathrm{Fix}_{(x,\zeta)}(-)$ for the whole affine Grassmannian $\mathcal{G}\mathrm{r}_G$ in Lemmata 19.8 and 19.14, as well as after restriction to affine Schubert varieties $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ inside it in Theorems 19.11 and 19.17. This is later contrasted with the block decomposition for quantum categories $\mathcal{O}$.

In §20 we move on to compute the equivariant K-theory ring $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C})$ in terms of the infinitely many affine blowups at roots of unity $\zeta \in \mathbb{G}_m^{\mathrm{rot}}$. The main computation for simply connected G is done in Theorem 20.40. Generalizations of this result are then covered in Theorem 20.45.

In §21 we recall the hybrid quantum group $U_{\mathsf{k}}^{\mathrm{hb}}$ associated to $\check{G}$ and its categories $\mathcal{O}$. In the known cases, we recall their block decompositions and centers, and contrast this with the components of fixed-points on $\mathcal{G}\mathrm{r}_G$ and affine Schubert varieties in Corollary 21.6 and Remark 21.11.

The supplementary §22 provides some concrete formulas in GL_n. The reader may use this for sanity checking.

16. Groups and schemes

This section serves as a recollection of concepts from algebraic geometry and representation theory which we freely use later in the chapter.

16.1. Schemes

16.1.1. Base rings. Let k be a base ring for our geometry. The most relevant case is when k is an algebraically closed field of $\operatorname{char} k = 0$, for example $k = \mathbb{C}$. The reader is invited to focus on this case.

On the other hand, let k be an arbitrary ring of coefficients. Again, one may want to take $\mathsf{k} = \mathbb{C}$ for simplicity. Nevertheless, a majority of our results works over fields k of arbitrary characteristic. Even better, most of our constructions work over $\mathsf{k} = \mathbb{Z}$.

16.1.2. General conventions. Given a scheme X, we write $\mathcal{O}(X)$ for the ring of global sections of its structure sheaf $\mathcal{O}_X$. We write $|X|$ for the set of closed points of X – we prefer to use this concise notation for closed points only for the sake of readability; this should not lead to any confusion.

In general, we write $\coprod$ for actual coproduct (disjoint union) of schemes. On the other hand, we write $\bigsqcup$ for locally closed stratifications.

16.1.3. Closed subschemes and fibers. Given a closed subscheme $Z \hookrightarrow X$, we write $X - Z$ for the open complement. The ideal sheaf of Z in X is denoted by $\mathcal{I}_Z$; it sits in the short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

We further write $X_{Z,n} = \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^n) \hookrightarrow X$ for the n-th infinitesimal thickening of Z inside X. We denote $X_Z^\wedge := \lim_n X_{Z,n}$ the formal completion of X along Z.

Given a map of schemes $X \to Y$ and a point $y \in Y$, we write X_y for the fiber. Furthermore, we denote by $X_y^\wedge$ the formal completion of X around y. By definition, $X_y^\wedge := X_{X_y}^\wedge$ is the completion of X along the fiber X_y.

16.1.4. Quotients. Let G be a group scheme. Suppose $G \circlearrowright X$. We write X/G for the stack quotient (taken in the fpqc topology).

On the other hand, we denote by $X /\!\!/ G$ the GIT quotient (in general this depends on further choices). We only consider GIT quotients for $X = \mathrm{Spec}\, R$ affine, when $X /\!\!/ G$ is also affine with ring of functions $\mathcal{O}(X /\!\!/ G) = \mathcal{O}(X/G) = \mathcal{O}(X)^G = R^G$.

16.1.5. Ind-schemes. Consider the Yoneda embedding $\mathrm{Sch}_k \hookrightarrow \mathrm{Sh}(\mathrm{Sch}_k)$ into the category of sheaves (say of sets) on Sch_k (say in the étale topology). This embedding commutes with all limits, but not with colimits. The category Sch_k of k-schemes is in fact complete, but not cocomplete. In other words, colimits of schemes may only exists as object of the bigger category $\mathrm{Sh}(\mathrm{Sch}_k)$.

In particular, suppose X is given as a filtered colimit $X = \mathrm{colim}_{i \in I} X_i$ of schemes X_i along closed immersions. Such X are called *ind-schemes*, and span a full subcategory IndSch_k. Any filtered system $(X_i, i \in I)$ as above together with a chosen isomorphism $X = \mathrm{colim}_{i \in I} X_i$ is called an *ind-presentation* of X. Clearly, $\mathrm{Sch}_k \subseteq \mathrm{IndSch}_k$ via constant systems.

Many construction and properties of schemes and their morphisms extend to ind-schemes by checking on a ind-presentation. In particular, the notions of ind-proper and ind-closed immersion make sense.

16.1.6. Functions on ind-schemes. The global functions on an ind-scheme are given by the inverse limit of global functions on any ind-presentation

$$\mathcal{O}(X) := \mathrm{Hom}(X, \mathbb{A}^1) = \mathrm{Hom}(\operatorname*{colim}_{i \in I} X_i, \mathbb{A}^1) = \lim_{i \in I}(X_i, \mathbb{A}^1) = \lim_{i \in I} \mathcal{O}(X_i).$$

Similarly for other representable functors in place of $\mathbb{A}^1$.

Remark 16.1. Given a closed immersion of ind-schemes $Z \hookrightarrow X$, the induced map on global functions $\mathcal{O}(X) \to \mathcal{O}(Z)$ need not be surjective. For an illustrative example, let $X = \mathbb{A}^1_k$ over a field k of characteristic zero and $Z = \coprod_{n\in\mathbb{Z}} \mathrm{pt} = \mathrm{colim}_{i\in\mathbb{N}} \coprod_{n\in[-i,i]} \mathrm{pt}$, with the embedding $Z \hookrightarrow X$ given by the discrete set of integer points in $\mathbb{A}^1(k)$. This is clearly a closed ind-subscheme, since the map at each finite stage is a closed immersions of schemes. However, the induced map on global functions $\mathcal{O}(\mathbb{A}^1) \to \prod_{n\in\mathbb{Z}} k$ is not surjective: not every function $\mathbb{Z} \to k$ may be interpolated by a polynomial.

Given a closed immersion $Z \hookrightarrow X$ of ind-schemes such that Z is a scheme, the map $\mathcal{O}(X) \to \mathcal{O}(Z)$ is surjective.

16.2. Deformations to normal cones and affine blowups

16.2.1. Blowups. Given a scheme X with a closed subscheme $Z \hookrightarrow X$, we denote $\mathrm{Bl}_Z(X)$ the blowup of X at Z. It has a natural map $\pi : \mathrm{Bl}_Z(X) \to X$. The fiber $E := \pi^{-1}(Z) \subseteq \mathrm{Bl}_Z(X)$ is an effective Cartier divisor. The pair $(\mathrm{Bl}_Z(X), E)$ is the universal pair mapping to X characterized by the property that $E \hookrightarrow \mathrm{Bl}_Z(X)$ is an effective Cartier divisor [Sta, Tag 085U].

In particular $\mathrm{Bl}_X(X) = \varnothing$. In an ambient scheme X, suppose we are given closed subschemes $Z \hookrightarrow X_1, X_2 \hookrightarrow X$. Inside the ambient blowup $\mathrm{Bl}_Z(X)$, we then have

$$\mathrm{Bl}_Z(X_1) \cap \mathrm{Bl}_Z(X_2) = \mathrm{Bl}_Z(X_1 \cap X_2).$$

16.2.2. Strict transforms. See [Sta, Tag 080C] for a general discussion. Consider a closed subscheme $Z \hookrightarrow X$, and the associated blowup $\mathrm{Bl}_Z(X)$. Given any other closed subscheme $Y \hookrightarrow X$, we have its strict transform

$$Y^{\mathrm{st}} \hookrightarrow \mathrm{Bl}_Z(X).$$

which may be described as the scheme-theoretic closure of the preimage of $Y \cap (X - Z)$, namely

$$Y^{\mathrm{st}} = \overline{\pi^{-1}(Y \cap (X - Z))} \hookrightarrow \mathrm{Bl}_Z(X).$$

In terms of Y, the strict transform is given as

$$Y^{\mathrm{st}} = \mathrm{Bl}_{Y\cap Z} Y \hookrightarrow \mathrm{Bl}_Z X.$$

Here, the embedding comes from the functoriality of blowups. See [Sta, Tag 080E].

16.2.3. Deformations to normal cones. See [Ful84, §5]. Let $Z \hookrightarrow X$ be a closed subscheme. Consider the embedding $Z \times \{0\} \hookrightarrow X \times \mathbb{A}^1$. We can then consider the blowup $\mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{A}^1)$. The *deformation of X to the normal cone of Z* is then given by the open complement to the strict transform of $X \times \{0\}$, namely

$$N(Z, X) := \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{A}^1) - (X \times \{0\})^{\mathrm{st}}.$$

It is an affine flat scheme over $\mathbb{A}^1$. Denoting by $C(Z,X)$ the normal cone of $Z \subseteq X$, meaning the relative spectrum of the sheaf of the sheaf of algebraic differentials $\Omega_{Z/X}$, the deformation $N(Z,X)$ fits in the pullback diagram

$$\begin{array}{ccccc} C(Z,X) & \longrightarrow & N(Z,X) & \longleftarrow & X \times (\mathbb{A}^1 - \{0\}) \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \mathbb{A}^1 - \{0\}. \end{array}$$

In words, $N(Z,X)$ carries a structure map $N(Z,X) \to \mathbb{A}^1 = \mathrm{Spec}(k[h])$, whose fibers over $h \neq 0$ are isomorphic to X, while the fiber over $h = 0$ is given by the normal cone $C(Z,X)$ of Z inside X.

More explicitly, the structure sheaf $\mathcal{O}_X$ has a descending filtration $(\mathcal{I}_Z^j)_{j=0}^{\infty}$ by powers of the ideal sheaf $\mathcal{I}_Z$. The associated Rees construction

$$\mathcal{O}_X[\tfrac{f}{h^j} \mid f \in \mathcal{I}_Z^j]$$

turns it into a graded algebra, whose relative spectrum gives the above deformation:

$$N(Z,X) = \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X[\tfrac{f}{h^j} \mid f \in \mathcal{I}_Z^j]).$$

16.2.4. Basic properties. The deformation to the normal cone is functorial in the input morphism $Z \to X$. It is further compatible with pullbacks: given $Z \hookrightarrow X$ and $Y \to X$, we have a natural identification $N(Z,X) \times_X Y = N(Z \times_X Y, Y)$. If a finite group W acts on $Z \hookrightarrow X$, it also acts on $N(Z,X)$ and we have $N(Z /\!\!/ W, X /\!\!/ W) = N(Z,X) /\!\!/ W$.

Suppose we are given a finite set of generators of the ideal sheaf $\mathcal{I}_Z = \langle f_1, \dots, f_m \rangle$. Then

$$\mathcal{O}_X[\tfrac{f}{h^j} \mid f \in \mathcal{I}_Z^j] = \mathcal{O}_X[\tfrac{f_i}{h} \mid i = 1, \dots, m].$$

In other words, to generate the structure sheaf of $N(Z,X)$, we only need to add finitely many ring generators $\frac{f_i}{h}$ for $i = 1, \dots, m$.

16.2.5. Affine blowups. More generally, suppose we have a family of schemes $X \to \mathbb{A}^1$ over the base $\mathbb{A}^1$. Let $\zeta \in \mathbb{A}^1$ be a point, and suppose $Z \hookrightarrow Y_\zeta$ is a closed subscheme of the fiber of X at ζ. Then the *affine blowup of X at Z with respect to $X \to \mathbb{A}^1$*, denoted $\mathrm{Bl}_Z^\circ(X)$ is the open complement in $\mathrm{Bl}_Z(X)$ of the strict transform of X_ζ, namely

$$\mathrm{Bl}_Z^\circ(X) := \mathrm{Bl}_Z(X) - X_\zeta^{\mathrm{st}}.$$

We have a natural affine morphism $\mathrm{Bl}_Z^\circ(X) \to X$. By design, the deformation to the normal cone is a special case: $N(Z,X) = \mathrm{Bl}^\circ_{Z \times \{0\}}(X \times \mathbb{A}^1)$.

16.2.6. Strict transforms inside affine blowups. Assume that we have another closed subscheme $Y \hookrightarrow X$ such that $Z = Y \cap X_\zeta$. Then the strict transform of Y inside $\mathrm{Bl}_Z(X)$ lies inside the affine blowup, namely

$$Y^{\mathrm{st}} \hookrightarrow \mathrm{Bl}_Z^\circ(X).$$

Indeed, it is enough to show that Y^{st} does not intersect X_ζ^{st}. Using the properties from §16.2.1, 16.2.2, this follows from

$$Y^{\mathrm{st}} \cap X_\zeta^{\mathrm{st}} = \mathrm{Bl}_Z(Y) \cap \mathrm{Bl}_Z(X_\zeta) = \mathrm{Bl}_Z(Y \cap X_\zeta) = \mathrm{Bl}_Z(Z) = \varnothing.$$

16.2.7. Deformations centered at a set of points. We will often perform the affine blowup construction over a set of closed points of $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$. Namely, suppose we have a scheme X and consider $X \times \mathbb{G}_m$. Suppose we are given a set $(z_i \mid i \in I)$ of pairwise distinct closed points of $\mathbb{G}_m$ and a set of closed subschemes $\mathcal{Z} = (Z_i \subseteq X \mid i \in I)$. We regard $Z_i = Z_i \times \{z_i\} \hookrightarrow X \times \mathbb{G}_m$ as a closed subscheme.

We the may blow up independently each Z_i inside $X \times \mathbb{G}_m$; we denote the result by

$$N_{q=z_i}(Z_{z_i}, X) := \mathrm{Bl}^{\circ}_{Z_i \times \{z_i\}}(X \times \mathbb{G}_m^{\mathrm{rot}})$$

where the subscript $q = z_i$ emphasizes that the deformation is centered at the point z_i of $\mathbb{G}_m = \mathrm{Spec}(k[q^{\pm 1}])$.

We denote the associated gluing of deformations over the set of points $(z_i \mid i \in I)$ by

$$\mathcal{N}_I(\mathcal{Z}, X) := \mathrm{Bl}^{\circ}_{\mathcal{Z}}(X).$$

This is defined as the affine blowup of $X \times \mathbb{A}^1$ centered in $\coprod_{i \in I} Z_i \times \{z_i\} \hookrightarrow X \times \mathbb{A}^1$. However, note the when I is infinite, the center of the blowup is merely an ind-scheme.

Explicitly, if I is finite, this gluing is given by inductively blowing up at $Z_i \times \{z_i\}$ in some order; the order does not matter as these are disjoint. For infinite I, one takes the limit over all finite subsets of I – these are schemes, hence their limit also is.

16.3. Reductive groups

16.3.1. Reductive groups. Let G be a connected reductive group over an algebraically closed field k. Denote by T its universal maximal torus; we also write $T = T_G$ whenever we need to stress the ambient group. For the sake of concreteness, we fix an embedding $T \leqslant G$. We write $X_\bullet = X_\bullet(T)$ resp. $X^\bullet = X^\bullet(T)$ for the cocharacter resp. character lattice of T. There is a natural identification $T(k) = X_\bullet(T) \otimes_{\mathbb{Z}} k^\times$.

We choose a Borel $B \leqslant G$ containing T and denote by $X_\bullet^+ = X_\bullet^+(G)$ the set of dominant cocharacters with respect to B. We denote $B' \leqslant G$ the Borel subgroup opposite to B; all parabolics and parahorics will be taken with respect to B'.

16.3.2. Chevalley forms. We use the same notation as in §16.3.1 over the ring of coefficients k for our cohomology theories; for example we may take $\mathsf{k} = \mathbb{Z}$. In particular, we denote by $G := G_{\mathsf{k}}$ the Chevalley form over k. We further denote by $T := T_{\mathsf{k}}$ its universal maximal torus, $B := B_{\mathsf{k}}$ a chosen Borel subgroup and so on. We omit the subscript k when there is no risk of confusion.

16.3.3. Root data. We write $\Phi_\bullet \subseteq X_\bullet$ resp. $\Phi^\bullet \subseteq X^\bullet$ for the coroots resp. roots of G. They respectively generate the coroot lattice $Q_\bullet := \mathbb{Z}\,\Phi_\bullet \leqslant X_\bullet$ and root lattice $Q^\bullet := \mathbb{Z}\,\Phi^\bullet \leqslant X^\bullet$. Altogether, $(X_\bullet, \Phi_\bullet, X^\bullet, \Phi^\bullet)$ denotes the root datum of G; its automorphism group is the Weyl group W.

We write $\langle -, - \rangle$ for the natural pairing $X_\bullet \times X^\bullet \to \mathbb{Z}$. We denote by $(-)^\vee : \Phi^\bullet \to \Phi_\bullet$, $\beta \mapsto \check{\beta}$ the unique bijection such that $\langle \check{\beta}, \beta \rangle = 2$ for each root $\beta \in \Phi^\bullet$. Denote further $\Phi_\bullet^+$ resp. $\Phi^\bullet_+$

the sets of positive coroots and positive roots with respect to B. We let $\Phi_{\bullet}^{\mathrm{s}} = \{\check{\beta}_1, \ldots, \check{\beta}_d\}$ resp. $\Phi_{\mathrm{s}}^{\bullet} = \{\beta_1, \ldots, \beta_d\}$ denote the subsets of simple coroots and simple roots.

We write $\leqslant$ for the usual partial order on $X_{\bullet}$. By definition, $\lambda' \leqslant \lambda$ if and only if $\lambda - \lambda' \in \mathbb{N}_0\, \Phi_{\bullet}^{\mathrm{s}}$. Similarly in the dual case of $X^{\bullet}$.

16.3.4. Weights. Denote $X_{\bullet,\mathbb{Q}} := X^{\bullet}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $X_{\mathbb{Q}}^{\bullet} := X^{\bullet}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, $P_{\bullet} \leqslant X_{\bullet,\mathbb{Q}}$ denotes the coweight lattice, defined as the dual of $Q^{\bullet} \leqslant X^{\bullet}$. Similarly, $P^{\bullet} \leqslant X_{\mathbb{Q}}^{\bullet}$ denotes the weight lattice, defined as the dual of $Q_{\bullet} \leqslant X_{\bullet}$. We have the inclusions $Q_{\bullet} \leqslant X_{\bullet} \leqslant P_{\bullet}$ and $Q^{\bullet} \leqslant X^{\bullet} \leqslant P^{\bullet}$.

16.3.5. Langlands dual and its representations. If G is a reductive group with a root datum $(X_{\bullet}, \Phi_{\bullet}, X^{\bullet}, \Phi^{\bullet})$, its Langlands dual is the reductive group $\check{G}$ with the flipped root datum given by $(\check{X}_{\bullet}, \check{\Phi}_{\bullet}, \check{X}^{\bullet}, \check{\Phi}^{\bullet}) := (X^{\bullet}, \Phi^{\bullet}, X_{\bullet}, \Phi_{\bullet})$.

Given any $\mu \in X_{\bullet}^{+} = \check{X}_{+}^{\bullet}$, we denote V_{μ} the corresponding irreducible representation of $\check{G}$ of highest weight μ. We further denote $\mathrm{wt}(\mu) \subseteq X_{\bullet} = \check{X}^{\bullet}$ the subset of characters of nonzero weight spaces in V_{μ}. Then $\mathrm{wt}(\mu)$ is given by the union of W-orbits of elements $\lambda \in X_{\bullet}^{+}$ with $\lambda \leqslant \mu$.

16.3.6. Weyl groups. Denote $W = N_G(T)/T$ the Weyl group of T in G. Equivalently, it is the Weyl group of the root system $(X_{\bullet}, \Phi_{\bullet}, X^{\bullet}, \Phi^{\bullet})$. It acts on T and all the above objects in the standard way.

Given any root $\beta \in \Phi^{\bullet}$, we denote the corresponding reflection

$$s_{\beta} : X_{\bullet} \to X_{\bullet}, \qquad \lambda \mapsto \lambda - \langle \beta, \lambda \rangle \check{\beta}.$$

It induces an automorphism of T, in multiplicative notation given by

$$s_{\beta} : T \to T, \qquad s_{\beta}(c) = c \cdot \check{\beta}(\beta(c))^{-1}.$$

The reflections s_{β} induce elements of the Weyl group W. Moreover, W has the structure of a Coxeter group, generated by the simple reflections s_{β} for $\beta \in \Phi_{s}^{\bullet}$.

16.3.7. Shifted actions. Given $\alpha \in X_{\bullet}$, we denote $\bullet_{\alpha}$ the α-shifted action

$$\begin{aligned} &(W, \bullet_{\alpha}) \circlearrowright X_{\bullet} \\ &w \bullet_{\alpha} \mu = w(\mu - \alpha) + \alpha, \qquad \forall\, w \in W. \end{aligned}$$

When $\alpha = \rho$, this is the *dot action*. We use the same notation for the induced action on T.

16.3.8. Simply connected covers and adjoint groups. Denote G^{der} the derived group of G. This is a semisimple group. It is often useful to consider the finitely many (semisimple) groups isogeneous to G^{der}. In particular, let G^{sc} be the simply connected cover of G^{der}; its maximal torus is denoted T^{sc}. Similarly, let G^{ad} be the adjoint group of G^{der}, with maximal torus T^{ad}. Breaking this symmetry, the adjoin group is given directly by the quotient of G by its center: $G^{\mathrm{ad}} = G/Z(G)$. Concisely, we have the diagram

$$\begin{tikzcd}
G^{\mathrm{sc}} \arrow[d] \arrow[dr] & \\
G^{\mathrm{der}} \arrow[r, hook] \arrow[d] & G \arrow[dl, two heads] \\
G^{\mathrm{ad}} &
\end{tikzcd}$$

The groups in the left column are all semisimple; the vertical maps between them are central isogenies. Each connected reductive group isogenous to G^{der} sits in between G^{sc} and G^{ad}.

The above diagram also induces corresponding maps on the universal maximal tori, and consequently on other root theoretic data. In particular, these maps naturally identify the root resp. coroot lattices $\mathbb{Z}\,\Phi^{\bullet}$ resp. $\mathbb{Z}\,\Phi_{\bullet}$ of all the groups; we will implicitly use these identifications.

16.3.9. Algebraic fundamental group. Denote

$$\pi_1(G) := X_\bullet(T)/X_\bullet(T^{\mathrm{sc}}) = X_\bullet(T)/\mathbb{Z}\,\Phi_\bullet(T) \tag{16.1}$$

the *algebraic fundamental group of* G, see [Bor98, Definition 1.3]. It recovers the usual fundamental group of the topological space $G(\mathbb{C})$ over $\mathbb{C}$. We have $G = G^{\mathrm{sc}}$ if and only if $\pi_1(G) = 1$. The natural action of W on $\pi_1(G)$ is trivial and the fundamental group $\pi_1(G)$ is functorial in G; see [Bor98, Lemma 1.2 and its proof]. Given $\lambda \in X_\bullet$, we denote by $\overline{\lambda} \in \pi_1(G)$ the corresponding class.

We choose a family of representatives for elements of the fundamental group under the surjection $X_\bullet(T) \twoheadrightarrow \pi_1(G)$: given an element $\sigma \in \pi_1(G)$, let $\chi(\sigma) \in X_\bullet(T)$ be the chosen representative.

16.3.10. Levi subgroups. Given a cocharacter $\lambda : \mathbb{G}_m \to G$, we denote L_λ the centralizer of the image of λ inside G. Then $L_\lambda \leqslant G$ is a Levi subgroup; it is in particular connected reductive. We denote by $P_\lambda \leqslant G$ the corresponding parabolic subgroup with respect to B' (containing L_λ and B'). See [Mil17, §21.i and Theorem 13.33].

We call $L_\lambda \leqslant P_\lambda$ the Levi and parabolic associated to λ; note that this construction depends on the ambient reductive group G. Note that $T \leqslant L_\lambda$ is a maximal torus and $X_\bullet = X_\bullet(T)$ is the cocharacter lattice of L. The intersections of B resp. B' with L define a pair of opposite Borel subgroups B_L resp. B'_L of L.

16.3.11. Coordinates and invariant polynomials. Let G be an affine group scheme over a field k. The representation ring of G is denoted $R(G)$. Given a coefficient ring k, its k-linearization is denoted $R(G;\mathsf{k}) := R(G) \otimes_{\mathbb{Z}} \mathsf{k}$.

Assume $G = G$ is a split reductive group over k of rank n with maximal torus T. We then have its Chevalley form G_{k} over k, and we denote T_{k} is maximal torus. We then have

$$R(G) = \mathcal{O}(T_{\mathbb{Z}} /\!\!/ W) \qquad \text{and} \qquad R(G;\mathsf{k}) = \mathcal{O}(T_{\mathsf{k}} /\!\!/ W).$$

As a module, $R(G)$ is generated by classes of irreducible representations of G, denoted V_μ for $\mu \in X^\bullet_+$. For ring generators, one may take the $[V_{\mu_j}]$ with μ_j for $j = 1, \dots, n$, where V_j are chosen generators of $X^\bullet_+$. We denote these n elements by

$$e_j := e_j^G := [V_{\mu_j}] \qquad j = 1, \dots, n.$$

In particular:

- If $G = T$ to start with, we denote these coordinates by $\mathcal{O}(T_\mathbb{Z}) = \mathbb{Z}[X^\bullet] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In a coordinate-free way, we denote by t_ν the function coming from the character ν.
- If $G = GL_n$, we have $\mathcal{O}(T_\mathbb{Z} /\!\!/ W) = \mathbb{Z}[e_1, \dots, e_{n-1}, e_n^{\pm 1}]$. Here, e_j is the j-th elementary symmetric polynomial on $T_\mathbb{Z}$.
- For general G, interpreting any V_μ as a group morphism $\rho_\mu : G \to GL_n$ for a suitable n, we get the above classes as pullbacks of the standard representation: $e_j^G = r^*_{\mu_j}(e_1)$.
- For G simply connected, we take $e_j := [V_{\omega_j}]$ the classes of the fundamental representations with highest weights ω_j, for for $j = 1, \dots, n$.

Remark 16.2. For reductive G, the ring $\mathcal{O}(T_\mathbb{Z} /\!\!/ W)$ may be subtle. Namely, when $\pi_1(G) \neq 1$, it will usually acquire finite quotient singularities – see [BZ00, §7] for the case of $G = PGL_3$.

On the other hand, for G simply connected, the above ring is a polynomial ring: $\mathcal{O}(T_\mathbb{Z} /\!\!/ W) = \mathbb{Z}[e_1, \dots, e_n]$ with distinguished coordinates given by the fundamental representations $e_j \in \mathcal{O}(T /\!\!/ W)$.

16.3.12. Lie algebras. Given a split reductive group G over k, we denote its Lie algebra by $\mathfrak{g}$. In particular, the Lie algebra of its maximal torus T is denoted $\mathfrak{t}$. When G is semisimple and simply connected, we have $T /\!\!/ W = \mathfrak{t} /\!\!/ W$ as above.

16.4. Loop groups and affine flag varieties

16.4.1. Affine root data. We denote by $\Phi^\bullet \oplus \mathbb{Z}\,\hbar$ the set of *affine (real) roots* associated to the root system of G. We write elements of this set as $\beta + m\hbar$ with $\beta \in \Phi^\bullet$ and $m \in \mathbb{Z}$.

16.4.2. Affine and extended Weyl groups. Let $\mathbb{Z}\,\Phi_\bullet \leqslant X_\bullet(T)$ be the coroot lattice inside the cocharacter lattice. We denote by

$$W_{\mathrm{aff}} := \mathbb{Z}\,\Phi_\bullet \rtimes W \qquad \text{and} \qquad W_{\mathrm{ext}} := X_\bullet \rtimes W$$

the *affine Weyl group* and *extended affine Weyl group* associated to the root system of G. These semi-direct products are constructed from the natural action of W on $\Phi_\bullet$ resp. $X_\bullet$. They thus have preferred splittings, allowing us to regard W as a subgroup. Clearly, $W_{\mathrm{aff}} \leqslant W_{\mathrm{ext}}$ is a normal subgroup; the quotient identifies as $W_{\mathrm{ext}}/W_{\mathrm{aff}} = \pi_1(G)$. In particular $W_{\mathrm{ext}} = W_{\mathrm{aff}}$ if G is simply connected. Also note that W_{aff} only depends on the isogeny class of G, since this is the case for both W and $\Phi_\bullet$. On the other hand, W_{ext} depends on the actual group G.

The image of a coroot $\check{\beta} \in \Phi_\bullet$ is denoted $\tau_{\check{\beta}} := (\check{\beta}, 1) \in W_{\mathrm{aff}}$. We also have the finite reflections $s_\beta \in W \leqslant W_{\mathrm{aff}}$ labeled by $\beta \in \Phi^\bullet$. The affine Weyl group W_{aff} has the structure of a Coxeter group, with simple reflections given by

$$s_{\beta,m} := (\tau_{m\check{\beta}}, s_\beta) \in \mathbb{Z}\,\Phi_\bullet \rtimes W, \qquad \beta \in \Phi_\bullet,\ m \in \mathbb{Z}\,.$$

Similar discussion applies to W_{ext}.

16.4.3. Shifted affine Weyl group actions. Consider the following family of actions of W_{aff} on $X_\bullet$. Let $\ell \in \mathbb{N}$ be a natural number and $\alpha \in X_\bullet$ any cocharacter. Then we define the *ℓ-dilated α-shifted action*

$$\begin{aligned}&(W_{\mathrm{aff}}, \bullet_\alpha^\ell) \circlearrowright X_\bullet \\ &(\lambda, w) \bullet_\alpha^\ell \mu = w(\mu - \alpha - \ell\lambda) + \alpha, \qquad \forall \lambda \in \mathbb{Z}\,\Phi_\bullet,\ w \in W\end{aligned}$$

For $\ell = 1$ and α trivial, this is the natural of W_{aff} on $X_\bullet$. For general ℓ, it is the *ℓ-dilated action*. For $\alpha = \rho$ and general ℓ, we get the *ℓ-dilated dot action*. We use the same notation for the induced action on T.

16.4.4. Loop rotation torus. We denote by $\mathbb{G}_m^{\mathrm{rot}}$ the one dimensional torus with coordinate ring $\mathcal{O}(\mathbb{G}_m^{\mathrm{rot}}) = k[q^{\pm 1}]$ and call it the *loop rotation torus*. It plays a special role, so we include the superscript rot to distinguish from other tori. We denote its closed points by $\zeta \in |\mathbb{G}_m^{\mathrm{rot}}|$.

For $\ell \in \mathbb{N}$, we write $\mu_\ell \leqslant \mathbb{G}_m^{\mathrm{rot}}$ for the subgroup scheme of order ℓ roots of unity. We write $\mu_\infty := \mathrm{colim}_{\ell \in \mathbb{N}}\, \mu_\ell \leqslant \mathbb{G}_m^{\mathrm{rot}}$ for the ind-subgroup scheme of all roots of unity. This gives a decomposition $\mathbb{G}_m^{\mathrm{rot}} = (\mathbb{G}_m^{\mathrm{rot}} - \mu_\infty) \sqcup \mu_\infty$.

A closed point $\zeta \in \mathbb{G}_m^{\mathrm{rot}}$ is called *generic* if it is not a root of unity, i.e. if it is torsionfree element of the underlying multiplicative group. Otherwise, $\zeta \in \mu_\infty$ is a root of unity; we denote $\ell = \mathrm{ord}(\zeta) \in \mathbb{N}$ its primitive order. We write $\Sigma_\ell \hookrightarrow \mu_\ell$ for the cyclotomic subscheme of roots of unity of primitive order ℓ.

Denoting $\hbar$ the real affine root in the Lie algebra sense, our group coordinate q is given as $q = e^\hbar$. Since we work multiplicatively throughout, we do not need the exponential notation.

16.4.5. Loop groups with loop rotation. Given a connected reductive group G over k, we denote by $\mathrm{L}G$ its loop group and by L^+G its positive loop group, with functors of points

$$\mathrm{L}G : R \mapsto G(R(\!(t)\!)) \qquad \text{and} \qquad \mathrm{L}^+G : R \mapsto G(R[\![t]\!]).$$

We have $\mathrm{L}^+G \leqslant \mathrm{L}G$. The group of connected components of the loop group $\mathrm{L}G$ identifies with $\pi_1(G)$, while the positive loop group L^+G is connected. We have an embedding $G \leqslant \mathrm{L}^+G \leqslant \mathrm{L}G$ of the original reductive group G via constant loops.

The torus $\mathbb{G}_m^{\mathrm{rot}}$ acts on $\mathrm{L}G$ by the loop rotation

$$\begin{aligned}\mathbb{G}_m^{\mathrm{rot}} \times \mathrm{L}G &\to \mathrm{L}G, \\ (\zeta, g(t)) &\mapsto g(\zeta t).\end{aligned}$$

This action clearly restricts to L^+G. The extended loop groups are given by the semidirect products along loop rotation

$$\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}} \leqslant \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}.$$

These semidirect products become trivial after restriction to the constant loops $G \leqslant \mathrm{L}^+G$.

Given $\ell \in \mathbb{N}$, we write $\mathrm{L}_\ell G \leqslant \mathrm{L}G$ and $\mathrm{L}_\ell^+G \leqslant \mathrm{L}^+G$ the ℓ-dilated subgroups; they represent the subfunctors

$$\mathrm{L}_\ell G : R \mapsto G(R(\!(t^\ell)\!)) \qquad \text{and} \qquad \mathrm{L}_\ell^+G : R \mapsto G(R[\![t^\ell]\!]).$$

While abstractly isomorphic to the usual loop groups, the geometry of their embeddings is useful.

16.4.6. Extended tori. In particular, we have the extended torus $T \times \mathbb{G}_m^{\mathrm{rot}} \leqslant \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$. Here, we embed $T \leqslant G \leqslant \mathrm{L}G$ through its fixed embedding to G, composed with the embedding as constant loops to $\mathrm{L}G$. All computations will take place within this ambient extended loop group. For notational convenience, we pick coordinates

$$\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}}) = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}][q^{\pm 1}].$$

Any character $X^\bullet(T)$, and in particular any root $\beta \in \Phi^\bullet \leqslant X^\bullet(T)$, can be tautologically regarded as a function $\beta \in \mathcal{O}(T)$ given by the composition $\iota \circ \beta : T \to \mathbb{G}_m \to \mathbb{A}^1$, where $\iota : \mathbb{G}_m^{\mathrm{rot}} \hookrightarrow \mathbb{A}^1$ is the obvious inclusion. Similarly, any character of $X^\bullet(T \times \mathbb{G}_m^{\mathrm{rot}})$, and in particular any affine root $\beta + m\hbar$, can be tautologically regarded as a function $\beta + m\hbar \in \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ given by the composition $\iota \circ (\beta + m\hbar) : T \times \mathbb{G}_m^{\mathrm{rot}} \to \mathbb{G}_m \to \mathbb{A}^1$. It is sometimes customary to denote these functions as

$$e^\beta = \beta \in \mathcal{O}(T) \qquad \text{and} \qquad e^{\beta + m\hbar} = e^\beta \cdot e^{m\hbar} = \beta \cdot q^m \in \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}}).$$

Since we are working multiplicatively, we will avoid this exponential notation.

On the other hand, given any cocharacter $\lambda \in X_\bullet(T)$, regarded as a homomorphism $\lambda : \mathbb{G}_m \to T$, we denote $t^\lambda := \lambda(t) \in \mathrm{L}T$ the image of the formal variable t under the map $\mathrm{L}\lambda : \mathrm{L}\mathbb{G}_m \to \mathrm{L}T$. More generally, given any element $a \in k[\![t]\!]$, we denote $a^\lambda := \lambda(a) \in \mathrm{L}T$ its image under $\mathrm{L}\lambda$.

16.4.7. Root spaces in loop groups. Given an affine root $\beta + m\hbar \in \Phi^\bullet \oplus \mathbb{Z}\,\hbar$, we get the homomorphism $\mathbb{G}_a \to \mathrm{L}G$ given by $x \mapsto u_\beta(t^m x)$ and denote $U_{\beta + m\hbar} \leqslant \mathrm{L}G$ the corresponding root subgroup of $\mathrm{L}G$ given by its image; see [RW22, §4.3].

Lemma 16.3. *The adjoint action of $T \times \mathbb{G}_m^{\mathrm{rot}}$ restricts to each $U_{\beta+m\hbar}$, where it has weight*

$$(x, \zeta) \mapsto \beta(x) \cdot \zeta^m.$$

Proof. Given any point $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$, compute

$$(x,\zeta) \cdot (u_\beta(zt^m), 1) \cdot (x,\zeta)^{-1} = (x, 1) \cdot (u_\beta(\zeta^m \cdot zt^m), 1) \cdot (x^{-1}, 1) = (u_\beta(\beta(x)\zeta^m \cdot zt^m), 1).$$

□

Lemma 16.4. *For any $\lambda \in X_\bullet$ we have*

$$t^{-\lambda} \cdot U_{\beta+m\hbar} \cdot t^\lambda = U_{\beta + (m - \langle \lambda, \beta \rangle)\hbar}$$

Proof. See [RW22, Lemma 4.3. (2)]. □

16.4.8. Parahoric subgroups. The preimage of the Borel subgroup $B' \leqslant G$ under the evaluation map $\mathrm{L}^+G \to G$ gives rise to the standard Iwahori subgroup $\mathrm{Iw} \leqslant \mathrm{L}^+G$. Given any proper subset J of the extended Dynkin diagram of G, we denote the associated standard parahoric subgroup $\mathcal{P}_J$. We have $\mathrm{Iw} \leqslant \mathcal{P}_J \leqslant \mathrm{L}^+G$. To avoid too many subscript, we sometimes put J in the superscript and write $\mathcal{P}^J := \mathcal{P}_J$.

A parahoric subgroup $\mathcal{P}_J$ sits in a canonical short exact sequence

$$1 \to \mathcal{P}_J^{\mathrm{uni}} \to \mathcal{P}_J \to L_J \to 1$$

where $\mathcal{P}_J$ is its *pro-unipotent radical* and L_J is its reductive *Levi quotient*. For all questions regarding equivariant topological invariants, the difference between $\mathcal{P}_J$ and L_J is irrelevant.

For more details on parahoric subgroups, see [Yun17, §2.1.5] and references there.

16.4.9. Affine flag varieties. Given a parahoric subgroup $\mathcal{P} \leqslant \mathrm{L}^+G$, the associated affine flag variety is the fpqc quotient

$$\mathcal{F}\mathrm{l}_{\mathcal{P}} = \mathrm{L}G/\,\mathcal{P} = \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}/\,\mathcal{P} \rtimes \mathbb{G}_m^{\mathrm{rot}}.$$

It is an ind-projective ind-scheme. It carries a left transitive action of $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$; we often restrict the above action to various subgroups of $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$, such as $T \times \mathbb{G}_m^{\mathrm{rot}}$.

When $\mathcal{P} = \mathcal{P}_J$ is the standard parahoric, we also write $\mathcal{F}\mathrm{l}^J := \mathcal{F}\mathrm{l}_{\mathcal{P}_J}$. For $\mathcal{P} = \mathrm{L}^+G$, we have the affine Grassmannian

$$\mathcal{G}\mathrm{r}_G = \mathrm{L}G/\mathrm{L}^+G = \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}/\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}.$$

16.4.10. Local Hecke stacks. Taking the quotient by the above action, we obtain

$$\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}\backslash\,\mathcal{F}\mathrm{l}_{\mathcal{P}} = \mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}\backslash \mathrm{L}G/\,\mathcal{P} = \mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}\backslash \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}/\,\mathcal{P} \rtimes \mathbb{G}_m^{\mathrm{rot}}.$$

In the case of affine Grassmannian, we call this quotient the *quantum Hecke stack*

$$\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}\backslash\,\mathcal{G}\mathrm{r}_G = \mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}\backslash \mathrm{L}G/\mathrm{L}^+G = \mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}\backslash \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}/\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}.$$

16.4.11. Affine Schubert varieties. The $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ orbits on $\mathcal{G}\mathrm{r}_G$ are the affine Schubert cells $\mathcal{G}\mathrm{r}_\mu$, equipped with the reduced scheme structure. By the Iwasawa decomposition, they are labeled by the discrete set $X_\bullet^+$. The affine Schubert cell $\mathcal{G}\mathrm{r}_\mu$ is an affine bundle over the corresponding partial flag variety G/P_μ.

The affine Schubert varieties $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ are the closures of the affine Schubert cells, equipped with the reduced scheme structure. They have a locally closed decomposition into a disjoint union of the affine Schubert cells

$$\mathcal{G}\mathrm{r}_{\leqslant\mu} = \bigsqcup_{\mu' \leqslant \mu} \mathcal{G}\mathrm{r}_{\mu'}\,.$$

Each $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ is a singular projective variety, acted on by $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$. We refer to [Zhu17b] for an excellent introduction.

Remark 16.5. The affine Grassmannian $\mathcal{G}\mathrm{r}_G$ has a natural affine paving. Indeed, we have an action $\mathrm{Iw} \circlearrowright \mathcal{G}\mathrm{r}_G$ of the Iwahori group scheme. Its fixed points are precisely t^λ with $\lambda \in X_\bullet$. The action of Iw on t^λ factors through a finite-dimensional quotient. Since the reductive quotient T of Iw stabilizes t^λ, its orbit under Iw is an affine space. Consequently, we deduce that Iwahori orbits provide an affine paving of $\mathcal{G}\mathrm{r}_G$, and in particular of each affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$.

Also see [HHH05, (5.2)] and references there.

16.4.12. Dilated variants. We will use the above constructions in the ℓ*-dilated* context. Also, the ambient reductive group will in fact be a suitable Levi subgroup L of G. We spell this out in detail.

Let $\vartheta \in X_\bullet = X_\bullet(T)$. Consider the subset $I_{\ell,\vartheta}$ of affine reflections fixing ϑ. Denote $\mathcal{P}_{L,\ell}^{\vartheta} \leqslant \mathrm{L}_\ell L$ the standard parahoric subgroup corresponding to $I_{\ell,\vartheta}$ with respect to the opposite Borel B' – up to passage to the opposite Borel, see [BASV22, §2.1.4]). In terms of Bruhat–Tits theory,

$\mathcal{P}^{\vartheta}_{L,\ell}$ is the parahoric subgroup of $\mathrm{L}_\ell L$ corresponding to the facet f containing ϑ, see [RW22]. Explicitly, by [BASV22, equation (2.8)] together with our opposite Borel convention, it is given as

$$\mathcal{P}^{\vartheta}_{L,\ell} = \mathrm{L}_\ell L \cap t^{\vartheta} \cdot \mathrm{L}^+ L \cdot t^{-\vartheta}.$$

See [RW22, §4.2 – §4.3] and [BASV22, §2.1]. We denote

$$\mathcal{F}\mathrm{l}^{\vartheta}_{L,\ell} := \mathrm{L}_\ell L / \mathcal{P}^{\vartheta}_{L,\ell}$$

the corresponding partial affine flag variety of the ℓ-dilated loop group of L. Note that these constructions depend only on the stabilizer of ϑ in $W_{\ell,\mathrm{aff}}$.

16.4.13. Functoriality. The constructions in §16.4 are functorial in group homomorphisms $G \to G'$ of reductive groups. Specifically, we have an induced map $\mathcal{G}\mathrm{r}_G \to \mathcal{G}\mathrm{r}_{G'}$ compatible with the equivariance, Schubert stratifications, and so on. Consequently, many constructions can be reduced to the case of GL_n.

17. Equivariant K-theory and fixed-points

We now set conventions regarding equivariant topological K-theory. Here, we recall both the usual topological approach through homotopy theory §17.1, as well as the algebraic approach via localizing invariants §17.2. We also take this as an opportunity to recall how these approaches relate – they are equivalent on finite type schemes over $k = \mathbb{C}$. We emphasize that we work with genuine equivariance.

After recalling our notion of fixed-point schemes in §17.3, we present in §17.4 our reinterpretation of classical fixed-point localization in terms of the trace to equivariant periodic cyclic homology. The main statement is Theorem 17.20. We find this formulation both conceptually clear and useful in practice. In particular, the result holds without restrictions for any (possibly singular) X with respect to an arbitrary action of T. This formulation further gives the localization map a clear global meaning.

17.1. Generalized cohomology theories in topology

17.1.1. Generalized (co)homology theories. First recall the notion of a generalized (co)homology theory in topology [May96]. These are representable by spectra. Multiplicative generalized cohomology theories are represented by ring spectra. Given such spectrum E, one may evaluate it on any topological space X by taking the mapping space $\mathrm{map}(E, -)$ resp. $\mathrm{map}(-, E)$ and collecting its homotopy groups into a $\mathbb{Z}$-graded abelian group:

$$E_\bullet(X) := \pi_\bullet(\mathrm{map}(E, X)) \qquad \text{resp.} \qquad E^\bullet(X) := \pi_\bullet(\mathrm{map}(X, E)).$$

These respectively give the E-homology resp E-cohomology of X. Our exposition is biased towards the contravariant cohomological picture, which carries a natural ring structure when E is multiplicative. We focus on this case from now on.

Remark 17.1. To regard the above $E^\bullet(X)$ as actual ring, one needs to choose how to interpret the grading direction: for example, it is a matter of convention whether to take $\bigoplus_{\bullet\in\mathbb{Z}} \pi_\bullet(-)$ or $\prod_{\bullet\in\mathbb{Z}} \pi_\bullet(-)$.

In many situations – such as ordinary cohomology of finite CW complexes – this distinction is vacuous. However, in infinite-dimensional contexts and more general cohomology theories, one needs to treat this seriously. The correct cohomological convention is to take $\prod_{\bullet\in\mathbb{Z}}$.

Given a Lie group G, one can axiomatically consider multiplicative G-equivariant generalized cohomology theories [May96, §XIII]. These are representable by G-equivariant ring spectra.

17.1.2. Generalized cohomology of colimits. Recall the following principle [May99, §19.4], valid for any generalized cohomology theory.

Recollection 17.2. Let $j \in \mathbb{Z}$ and let X be a topological space, given as an $\mathbb{N}_0$-indexed colimit

$$X = \operatorname*{colim}_i X_i.$$

Then there is a short exact sequence

$$0 \to \lim_i{}^1(E^{j-1}(X_i)) \to E^j(X) \to \lim_i E^j(X_i) \to 0.$$

Moreover, if the system $(E^{j-1}(X_i))_i$ satisfies the Mittag–Leffler condition, the $\lim^1$-term vanishes and we deduce

$$E^j(X) \xrightarrow{\cong} \lim_i E^j(X_i).$$

Remark 17.3. Under the Mittag–Leffler assumption and using the correct (product) convention from Remark 17.1, we deduce a ring isomorphism

$$E^\bullet(X) \cong \lim_i E^\bullet(X_i).$$

Remark 17.4. In particular, assume that X is a CW complex with even cells only, together with its filtration by skeleta $\operatorname{sk}_i X$. Then, for any fixed j, all the maps in the inverse system $(E^j(X_i), i \in \mathbb{N}_0)$ are surjective, and the Mittag–Leffler condition is trivially satisfied.

Remark 17.5. For a scheme X over $\mathbb{C}$, consider its complex points $X(\mathbb{C})$ with analytic topology. Given a generalized cohomology theory E, we then denote $E(X) := E(X(\mathbb{C}))$. We use similar convention in the equivariant case.

Assume that X is a proper scheme over $\mathbb{C}$ that possesses a paving by affine spaces. Then its underlying topological space $X(\mathbb{C})$ is a CW complex with even cells only, and Remark 17.4 applies.

17.1.3. Cohomology. In particular, we denote by $H\mathbb{Z}$ the Eilenberg–Mac-Lane spectrum representing singular cohomology, and $H\mathsf{k}$ the version with coefficients. It represents singular cohomology; for a topological space X, we denote it $H(X) := H_{\mathrm{sing}}(X) := \mathrm{map}(X, H\mathbb{Z})$. Similarly with other coefficients.

Convention 17.6. Given a topological space X, we write

$$\widehat{H}^\bullet(X) := \prod_{i \geqslant 0} H^i(X).$$

Here, the convention is that we take the direct product of all cohomology groups rather then their direct sum. The product convention is the more natural one in the context of generalized cohomology theories (Remarks 17.1, 17.3). It would be for the best to write simply $H^\bullet$ without any adornments; however, since the opposite convention is quite pervasive in geometric representation theory, we write $\widehat{H}^\bullet$ to avoid confusion.

Cohomology extends to the equivariant setup through the Borel construction. Given a topological group G, denote $EG \to BG$ the classifying space of G with its universal G-bundle. For any G-space X, its equivariant cohomology is defined by $H_G(X) := H(EG \times^G X)$.

17.1.4. Topological K-theory. We denote by BU the complex topological K-theory spectrum, i.e. the $\mathbb{E}_\infty$-ring spectrum representing topological K-theory. Given a topological space X, we denote its topological K-theory

$$K^{\mathrm{top}}(X) := KU(X) := \mathrm{map}(X, BU).$$

We write $K_G^{\mathrm{top},\bullet}(X) := \pi_\bullet(K_G^{\mathrm{top}}(X))$ with $\bullet \in \mathbb{Z}$ for its homotopy groups, which form a graded-commutative ring.

Topological K-theory genuinely extends to the equivariant context.

Recollection 17.7 (Genuine equivariant topological K-theory)**.** Let G be a compact topological group. The genuine equivariant K-theory was firstly defined by [Seg68]. These ideas were further developed in terms of the representing spectrum BU_G, see [May96, §XIV, in particular §XIV.3 and §XIV.4]. Given any G-space X, we thus have equivariant topological K-theory

$$K_G^{\mathrm{top}}(X) := KU_G(X) := \mathrm{map}_G(X, BU_G).$$

We will use this genuine definition. The discussion of previous paragraphs works in this context. The definition extends to the case when the group G is a complex Lie group by passing to its maximal compact subgroup M; see also [HP20, §2.1.2]. For finite G-CW complexes, the above can be also defined directly from the category of equivariant topological vector bundles on X as in [Wei13]. However, this later interpretation does not hold when X is not finite (and does not behave like a generalized cohomology theory).

Notation 17.8. Given a coefficient ring k, we write $K_G^{\mathrm{top}}(X; \mathsf{k})$ for K-theory with coefficients, defined using the corresponding Moore space. See [TT90, §9.3] for the relevant cases. Compatibly with this convention, we sometimes write $K_G^{\mathrm{top}}(X; \mathbb{Z}) = K_G^{\mathrm{top}}(X)$ to emphasize integral coefficients.

Remark 17.9 (Borel-type theories)**.** There are also cheaper ways of extending topological K-theory to the equivariant context – one can use the non-equivariant definition on the Borel construction. The resulting theory is simpler, but carries considerably less information. Already for the equivariant point, it returns only the completion of its representation ring at the origin [AS69]. Also compare to [Tho86] and the discussion [Tho88a, §5]. We will not use the Borel-type version.

17.1.5. Goresky–Kottwitz–McPherson descriptions. Let G be a topological group and E_G a multiplicative G-equivariant cohomology theory. Let X be a topological space with $G \circlearrowright X$. Under favorable assumptions, the equivariant generalized cohomology $E_G^\bullet(X)$ can be computed combinatorially from the fixed points of $G \circlearrowright X$ by solving a specific system of congruences. For T-equivariant singular cohomology $H_T^\bullet(X;\mathbb{C})$ of a complex algebraic variety X, this was realized in [GKM97, §7]. It was then extended to generalized cohomology theories in [HHH05]; we will follow their excellent treatment, to which we refer for details.

Assumption 17.10. Let $G \circlearrowright X$ be a topological group acting on a topological space. Assume the following [HHH05, §3, Assumptions 1-4]:

(1) The space X admits a G-equivariant stratification $X = \bigsqcup_{i\in I} X_i$ labeled by a poset I, whose graded pieces $X_i/X_{<i}$ are Thom spaces of G-equivariant vector bundles V_i over some $G \circlearrowright F_i$.
(2) The bundles V_i are E-orientable and admit a decompositions $V_i = \bigoplus_{j<i} V_{i,j}$ by G-equivariant E-orientable vector bundles.
(3) There is a system of G-equivariant maps $a_{i,j} : F_i \to F_j$, such that for each $j < i$, the following diagram commutes

$$\begin{tikzcd} S(V_{i,j}) \arrow[r] \arrow[d] & X_{<i} \\ F_i \arrow[r, "a_{i,j}"] & F_j \arrow[u, hook] \end{tikzcd}$$

In words, the attaching map $S(V_i) \to X_{<i}$ from the sphere bundle of V_i is, on each subbundle $S(V_{i,j})$, induced by $a_{i,j}$.
(4) Fix any $i \in I$. As $j < i$ varies, the Euler classes $e(V_{i,j})$ are not zero divisors and pairwise relatively prime in $E_G^\bullet(F_i)$.

The main result of [HHH05] then reads as follows.

Proposition 17.11. *Assuming (1), the following restriction map is injective*

$$E_G^\bullet(X) \hookrightarrow \prod_{i\in I} E_G^\bullet(F_i).$$

Assuming (1)-(4), its image is given by the GKM ring

$$\mathrm{GKM}_G^\bullet(X) := \left\{(f_i) \in \prod_{i\in I} E_G^\bullet(F_i) \mid f_i - a_{i,j}^*(f_j) \in e(V_{i,j})E_G^\bullet(F_i)\ \forall j < i\right\}.$$

Proof. See [HHH05, Theorems 3.1 and 2.3]. □

Remark 17.12. The most common use case is that of a torus action on a complex algebraic variety $T \circlearrowright X$. It induces a Białynicki-Birula decomposition of X by attractors of fixed points. Assume that the fixed points X^T are isolated and the strata are affine spaces. One then takes $(F_i, i \in I)$ to be the set of fixed-points, V_i the attractor cells, $V_{i,j}$ their weight decompositions, $a_{i,j}$ the obvious maps. In many real-life examples – namely when the T-action is sufficiently non-degenerate – the conditions (3) and (4) hold.

We have followed [HHH05]. Also see [RK03, Appendix A], [HHRW16], [Zho25], [BS25], [KR24] for related subsequent work.

17.2. Algebraic cohomology theories as localizing invariants

17.2.1. Localizing invariants. Certain generalized (co)homology theories do have a purely algebraic counterpart as localizing invariants $E(-)$ of stable categories. Given a space X of algebro-geometric origin, one can then compute $E(X)$ via its derived category. More precisely, such theories again come in pairs: the covariant version $E(\mathrm{Coh}(X))$ and the contravariant version $E(\mathrm{Perf}(X))$.

Again, we are biased towards the latter. As in the topological case, one good reason for this is the ring structure. Namely, assume $E(-)$ is lax monoidal. Since $\mathrm{Perf}(X)$ has a symmetric monoidal structure via $\otimes$, the output $E(\mathrm{Perf}(X))$ is an $\mathbb{E}_\infty$-ring spectrum.

Given an affine group scheme G acting on a qcqs scheme X, we put

$$E_G(X) := E(\mathrm{Perf}(G\backslash X)),$$

in line with [Tho88b; TT90]. Taking homotopy groups, we obtain the $\mathbb{Z}$-graded abelian group $\pi_\bullet(E_G(X))$. If $E(-)$ was lax monoidal, this is a graded-commutative ring.

Convention 17.13. In this chapter, we use the following nonstandard convention. Given a localizing invariant E, we denote

$$E^i(\mathcal{X}) := \pi_i(E(\mathcal{X})) := \pi_i E(\mathrm{Perf}(\mathcal{X}))),$$

in other words, we put the homotopy degree to the upper index. While this is non-standard from the viewpoint of localizing invariants, we are forced to do so in order to match the homology-cohomology conventions from algebraic topology: our invariants are computed from $\mathrm{Perf}(\mathcal{X})$ and hence compare to cohomological versions in topology. (The lower-index convention would be given by $\mathrm{Coh}(\mathcal{X})$ and correspond to homology.)

The known localizing invariants either come from vector bundles (e.g. algebraic K-theory and its variants) or from differential forms (e.g Hochschild homology HH and its variants).

We write $K(\mathcal{X}) := K(\mathrm{Perf}(\mathcal{X}))$ for actual algebraic K-theory.

Perhaps surprisingly, purely topological invariants of qcqs schemes over $\mathbb{C}$ can be recovered from this machinery. The relevant examples are Blanc's topological K-theory $K^{\mathbf{top}}$ and periodic cyclic homology HP.

17.2.2. Equivariant topological K-theory. We work over $k = \mathbb{C}$. We denote by $K^{\mathbf{top}}$ Blanc's non-connective topological K-theory [Bla16]. This is a multiplicative localizing invariant of $\mathbb{C}$-linear categories. In particular, given any affine group G action on a qcqs scheme X over $\mathbb{C}$, we obtain

$$K_G^{\mathbf{top}}(X) := K^{\mathbf{top}}(\mathrm{Perf}(X/G)).$$

It is an $\mathbb{E}_\infty$-ring spectrum.

Remark 17.14. For X qcqs scheme over $\mathbb{C}$, Blanc's topological K-theory $K^{\mathbf{top}}(X)$ agrees with the equivariant topological K-theory $K^{\mathrm{top}}(X) \simeq KU(X)$ of the analytification from §17.1. More generally, for an action of an algebraic group $G \circlearrowright X$ where X is qcqs, it agrees with the genuine equivariant topological K-theory $K_G^{\mathrm{top}}(X) = KU_G(X)$ of on the analytification $X(\mathbb{C})$ in the sense of §17.7. See [Bla16], [HP20], and Remark 11.27 covering [Löw26b] for this identification.

Consequently, we may (and will) identify $K_G^{\mathbf{top}}(X) \simeq K_G^{\mathrm{top}}(X)$ in such situations, in particular when X is a G-equivariant projective variety. However, note that for ind-schemes we use the purely topological K_G^{top}; also see Question 15.24.

Notation 17.15. We denote by $u \in K_2(\mathrm{pt}) = K^2(\mathrm{pt})$ the Bott element in algebraic K-theory.

It maps into topological K-theory, giving an an invertible element $u \in K^{\mathbf{top},2}(\mathrm{pt})$ with inverse $u^{-1} \in K^{\mathbf{top},-2}(\mathrm{pt})$. In particular, multiplication by u is an isomorphism on homotopy groups shifting degree by 2, making $K^{\mathbf{top},\bullet}(X)$ 2-periodic. We write $K^{\mathbf{top},*}(X)$ with $* \in \mathbb{Z}/2$ for this $\mathbb{Z}/2$-graded, graded-commutative ring.

17.2.3. Equivariant periodic cyclic homology. Consider now the localizing invariant $HP(-/k)$ of k-linear categories. Following our conventions, we write $HP^G(X/k) := HP(\mathrm{Perf}(X/G)/k)$ for the G-equivariant periodic cyclic homology of $X \in \mathrm{Sch}_k^G$. We write $u \in HP_2(\mathrm{pt}/k) = HP^2(\mathrm{pt}/k)$ for the periodicity operator with inverse $u^{-1} \in HP^{-2}(\mathrm{pt})$ as in [Lod97, §5.1.4].

Let us specialize to $k = \mathbb{C}$. Then the Dennis trace induces a map of localizing invariants $K^{\mathbf{top}}(-) \to HP(-/\mathbb{C})$ called the topological Chern character. The trace map $K^{\mathbf{top}}(\mathrm{pt};\mathbb{C}) \to HP(\mathrm{pt}/\mathbb{C})$ identifies the Bott element on the left-hand side with the periodicity operator on the right-hand side. See [Lod97] for classical exposition; see [Che20; EKS25] for the equivariant case. We will recall its geometric incarnation in terms of fixed points in Proposition 17.25 below.

17.2.4. Equivariant base. For any group scheme G over k, we tautologically identify the equivariant K-theory of a point with the representation ring

$$K_G^0(\mathrm{pt}) = K_G^{\mathbf{top},0}(\mathrm{pt}) = R(G)$$

ans similarly with coefficients. Working over $k = \mathbb{C}$, the topological Chern character identifies

$$K_G^0(\mathrm{pt};\mathbb{C}) = K_G^{\mathbf{top},0}(\mathrm{pt};\mathbb{C}) = R(G;\mathbb{C}) = HP_G^0(\mathrm{pt}/\mathbb{C}).$$

17.2.5. Derived fibers over T. Working with a torus T over k, the equivariant base identifies with its ring of functions.

$$K_T^0(\mathrm{pt};k) = K_T^{\mathbf{top},0}(\mathrm{pt};k) = \mathcal{O}(T).$$

Given any closed point $t \in T$, we denote $t : \mathcal{O}(T) \to k$ the corresponding algebra homomorphism. To emphasize the $\mathcal{O}(T)$-algebra structure on the right-hand side, we sometimes write $k = k_t$. We further denote $(-)_{\mathbf{L}t} = (-)\otimes^{\mathbf{L}}_{\mathcal{O}(T)} k_t$ the derived tensor product along this map.

In particular, the functor $(-)_{\mathbf{L}t}$ can be applied to $K_T^{\mathbf{top}}(-;k)$ and $HP_T^{\mathbf{top}}(-/k)$, both regarded as algebra objects in the derived category of $\mathcal{O}(T)$-modules, getting

$$K_T^{\mathbf{top}}(X;k)_{\mathbf{L}t} \qquad \text{and} \qquad HP_T(X/k)_{\mathbf{L}t}.$$

17.2.6. Remarks. Let us outline how the current section parallels the topological discussion.

Remark 17.16 (Comparisons)**.** Comparisons of algebraic K_G and topological K_G^{top} have a long history – algebraic invariants usually encode the topological ones. Algebraic K-theory of coherent sheaves (usually called G-theory) relates to topological K-homology; algebraic K-theory of perfect complexes relates to topological K-cohomology. See [Tho88b; Tho88a].

Remark 17.17 (Kan extended theories)**.** In the same way as in topology, there is a cheaper way of producing equivariant invariants. Instead of using $\mathrm{Perf}(G\backslash X)$, one can Kan-extend from the non-equivariant case. The result then parallels the Borel-type variant in topology: it is much simpler and sees only the completion at the augmentation ideal. See [Tho86], or also [TV24] Again, we do not use these variants.

Remark 17.18 (Motivic filtrations)**.** Algebraic K-theory and its variants of any $X \in \mathrm{Sch}_k$ carry the *motivic filtration* [EM23]. In characteristic zero and after taking rational coefficients, this filtration naturally splits. In particular, $K^0(X;\mathbb{C})$ is naturally a graded ring. There is a compatible filtration on $HP(-/\mathbb{C})$. Interpreting the latter in terms of de Rham cohomology, this is precisely the de Rham filtration. Again, it naturally splits, making $HP^0(X/\mathbb{C})$ into a graded ring. The trace map is compatible with these filtrations.

Such filtrations also appear in the equivariant setup. For example, one can get a filtration on $K_G(X)$ by Kan extension from the motivic filtration on non-equivariant K-theory as in [EKS25]. Also see [TV24, (2.6)], [KN25], [EKS25] for more discussion of Atyiah–Segal completion phenomena in this context. However, it is not understood whether such a filtration exists genuinely [EKS25, Question 1.0.2], in other words “relatively over BG".

17.3. Fixed-point schemes

17.3.1. Fixed point schemes. Let G be a group scheme over k and denote Sch_k^G the category of G-equivariant k-schemes. Recall the notion of the associated fixed-point scheme $\mathrm{Fix}_G(X)$ from §2.1 and [Löw26a, §1], parametrizing pairs $(g,x) \in G \times X$ satisfying $gx = x$.

It carries a natural action $G \circlearrowright \mathrm{Fix}_G(X)$ induced from the original action on $G \circlearrowright X$ and conjugation action $G \circlearrowright G$ on the defining diagram (2.1).

In particular, suppose $h, g \in G$ and denote $g' = hgh^{-1}$. Then we have a natural identification

$$\mathrm{Fix}_g(X) = \mathrm{Fix}_{hgh^{-1}}(h \cdot X) \xrightarrow{=} \mathrm{Fix}_{g'}(X) \tag{17.1}$$

where the latter arrow is induced by the translation $h^{-1} : X \to X$.

We further have the reduced and derived variants

$$\mathrm{Fix}^{\mathrm{red}}_{\frac{G}{G}}(X) \hookrightarrow \mathrm{Fix}_{\frac{G}{G}}(X) \hookrightarrow \mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X). \tag{17.2}$$

For many applications in this chapter, the reduced version will be sufficient. Nevertheless, the derived variant allows to compare to periodic cyclic homology, which is important for our localization theorem; see Proposition 17.25.

If G is an affine group action on a quasi-projective variety X, the derived structure on $\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X)$ is trivial if and only if $G \circlearrowright X$ acts with finitely many orbits by the criterion [BN13, Proposition 3.3].

17.3.2. Fixed-point schemes preserve abstract blowups. We will need the following elementary observation about fixed-point schemes.

Lemma 17.19. *Let G be a group over k, and $S \to G$ any map from some $S \in \mathrm{Sch}_k$. Then the classical fixed-point scheme functor*

$$\mathrm{Fix}_S(-) : \mathrm{Sch}_k^G \to \mathrm{Sch}_S$$

preserves abstract blowup squares. In particular, this applies to $\mathrm{Fix}_g(-)$ for any point $g \in G$.

Proof. This directly follows from [Löw26a, §1]. The classical fixed-point scheme functor $\mathrm{Fix}_S(-)$ preserves isomorphisms, closed immersions, open immersions, closed-open decompositions, proper maps by Propositions 2.2 and 2.3. It commutes with fiber products, in fact with all limits 2.7. Therefore, it also preserves abstract blowup squares. □

17.4. K-theory, periodic cyclic homology, and cohomology of fixed points

17.4.1. Localization theorem. We work over the base field $k = \mathbb{C}$. Let $T = \mathbb{G}_m^n$ be an algebraic torus over k. Throughout this section, we write $\mathrm{Sch}_{\mathbb{C}}^T := \mathrm{Sch}_{\mathbb{C}}^{T,\mathrm{qcqs}}$ for the category of T-equivariant, quasi-compact quasi separated schemes over $\mathbb{C}$. We omit the superscript qcqs for readability.

Let $X \in \mathrm{Sch}_{\mathbb{C}}^T$ be any T-equivariant qcqs scheme over $\mathbb{C}$; it is in particular allowed to be singular. The purpose of this section is to recall the (maybe not so well) known relationship between its complexified equivariant topological K-theory $K_T^{\mathbf{top}}(X;\mathbb{C})$, equivariant periodic cyclic homology $HP^T(X/\mathbb{C})$ and the singular cohomologies of the varying fixed-point subschemes $\mathrm{Fix}_t(X)$ for $t \in T$.

Theorem 17.20. *Let $X \in \mathrm{Sch}_{\mathbb{C}}^T$. Then we have an equivalence*

$$K_T^{\mathbf{top}}(X;\mathbb{C}) \xrightarrow{\simeq} HP_T(X/\mathbb{C}) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{T}{T}}(X), \mathcal{O})^{tS^1} \tag{17.3}$$

in $\mathrm{CAlg}(\mathcal{O}(T)[u^{\pm 1}])$. Taking fibers over a closed point $t \in T$, this equivalence specializes to

$$K_T^{\mathbf{top}}(X;\mathbb{C})_{\mathbf{L}t} \xrightarrow{\simeq} H_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!) \tag{17.4}$$

in $\mathrm{CAlg}(D(\mathbb{C}))$.

Proof. Proposition 17.26, Proposition 17.25 and Proposition 17.29. □

Remark 17.21. In plain english: the fibers of equivariant topological K-theory over various closed points t of the equivariant base T are isomorphic with the singular cohomologies $H_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!)$ of the corresponding scheme-theoretical fixed-points $\mathrm{Fix}_t(X)$. Reformulating, T-equivariant topological K-theory of X puts the singular cohomologies of the fixed-point varieties $\mathrm{Fix}_t(X)$ into an algebraic family. This algebraic family is geometrically computed by

$$\mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{T}{T}}(X), \mathcal{O}))^{tS^1},$$

the Tate fixed points of the circle action on global functions on the derived fixed-point stack $\mathrm{Fix}^{\mathbf{L}}_{\frac{T}{T}}(X)$.

We also have the following formally deformed variant on the final assertion of the above theorem.

Proposition 17.22. *Let $X \in \mathrm{Sch}^T_{\mathbb{C}}$. Taking completions $(-)^\wedge_{\mathbf{L}t}$ along $t \in T$, we obtain*

$$K^{\mathbf{top}}_T(X;\mathbb{C})^\wedge_{\mathbf{L}t} \xrightarrow{\simeq} H_T(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!)^\wedge \tag{17.5}$$

Here, the completion of the right-hand side it with respect to the derived Hodge filtration in the Cartan model of equivariant cohomology.

Proof. Take Proposition 17.24, then take the formal derived completion $(-)^\wedge_{\mathbf{L}t}$ over $t \in T$, and finally apply Proposition 17.27 □

17.4.2. Case of reductive groups.

Remark 17.23 (Case of reductive groups.)**.** Let G be any reductive group acting on a quasi-projective variety X over $\mathbb{C}$. Then the equation (17.3) holds G-equivariantly:

$$K^{\mathbf{top}}_G(X;\mathbb{C}) \xrightarrow{\simeq} HP^G(X/\mathbb{C}) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{G}{G}}(X), \mathcal{O})^{tS^1}.$$

Moreover, let $g \in G$ be an element with semisimple part $\overline{g} \in \mathrm{Spec}(R_{\mathbb{C}}(G))$. Then the identification (17.6) goes through as:

$$K^{\mathbf{top}}_T(X;\mathbb{C})_{\mathbf{L}\overline{g}} \xrightarrow{\simeq} H_{\mathrm{sing}}(\mathrm{Fix}_{\overline{g}}(X);\mathbb{C})(\!(u)\!)$$

Note that this does not work without semisimplicifation.

Proof of Remark. This version follows by the same arguments, combining results of [Che20] in the smooth case with the recent progress on equivariant cdh descent [LS25] with respect to reductive groups G. □

In particular, even if we consider G-equivariant K-theory instead of T-equivariant one, there is no extra information in its fibers.

17.4.3. K-theory and periodic cyclic homology. Let us first recall the relationship between equivariant topological K-theory and equivariant periodic cyclic homology.

Proposition 17.24. *For any $X \in \mathrm{Sch}^T_{\mathbb{C}}$ we have an equivalence*

$$K^{\mathbf{top}}_T(X;\mathbb{C}) \xrightarrow{\simeq} HP_T(X/\mathbb{C})$$

as algebras over $K^{\mathbf{top}}_T(\mathrm{pt};\mathbb{C}) \simeq HP_T(\mathrm{pt}/\mathbb{C})$.

Proof. For a sufficient reference covering (possibly singular) stacks with nice stabilizers, see [Kha23]. This heavily builds on the previous works [Bla16; Kon21; HP20; ES21]. □

17.4.4. Periodic cyclic homology and fixed-points. Periodic cyclic homology $HP(-/\mathbb{C})$ over $\mathbb{C}$ is further geometrically modeled by suitable functions on the derived fixed-point scheme.

Proposition 17.25. *There is a natural equivalence*

$$HP_T(X/\mathbb{C}) \xrightarrow{\simeq} \mathrm{R}\Gamma(\mathrm{Fix}^{\mathbf{L}}_{\frac{T}{T}}(X), \mathcal{O})^{tS^1}.$$

Here, $(-)^{tS^1}$ refers to the Tate construction with respect to the natural S^1-action on the derived fixed-point scheme.

Proof. See Discussion 2.12 covering [Löw26a]; also see [Che20, Example 2.2.20]. Then apply the Tate construction. □

17.4.5. Periodic cyclic homology computes cohomology of fixed-points. Let us now explain how fibers of equivariant periodic cyclic homology relate to the singular cohomology of fixed-points.

Proposition 17.26. *Let $t \in T$ be a closed point. We then have an equivalence*

$$HP_T(X/\mathbb{C})_{\mathbf{L}t} \xrightarrow{\simeq} H_{\mathrm{sing}}(\mathrm{Fix}_t(X); \mathbb{C})(\!(u)\!). \tag{17.6}$$

Proof. First recall the identification of derived de Rham cohomology with singular cohomology of the analytification [Che20, Theorem 4.2.9 and Corollary 4.2.11] and [Bha12, Theorem 4.10], valid for any derived stack of finite type over $\mathbb{C}$ (the case of quasi-projective varieties is sufficient for us).

We first assume that the T-variety X is smooth. Then (17.6) holds by [Che20, Corollary 1.0.4] (also see [Che20, Theorem D] for a finer statement), using the above discussion to identify the right-hand side with singular cohomology. We use here the semisimplicity assumption on t.

Now, both sides of (17.6) satisfy proper excision on T-equivariant abstract blowup squares. Indeed, the left-hand side is given by the composition

$$\mathrm{Sch}^T_{\mathbb{C}} \xrightarrow{HP_T(-/\mathbb{C})} D(HP_T(\mathrm{pt}/\mathbb{C})) \xrightarrow{(-)_{\mathbf{L}t}} D(\mathbb{C})$$

Here, $HP(-/\mathbb{C})$ is a truncating invariant, so $HP^T(-/\mathbb{C})$ has proper excision by [ES21]. The same holds after composition with the derived functor $(-)_{\mathbf{L}t} = (-) \otimes^{\mathbf{L}}_{HP_T(\mathrm{pt}/\mathbb{C})} \mathbb{C}_t$. On the other hand, the right-hand side can be factored as

$$\mathrm{Sch}^T_{\mathbb{C}} \xrightarrow{\mathrm{Fix}_t(-)} \mathrm{Sch}_{\mathbb{C}} \xrightarrow{H_{\mathrm{sing}}(-;\mathbb{C})(\!(u)\!)} D(\mathbb{C})$$

Here, the functor $\mathrm{Fix}_t(-)$ preserves abstract blowup squares by Lemma 17.19. Moreover, singular cohomology (with any coefficients) has proper excision (this being the case for any generalized cohomology theory with open-closed long exact sequence). Alternatively, this last claim can be seen on the level of derived de Rham cohomology, which is a canonical direct summand of $HP(-/\mathbb{C})$ and hence has cdh descent [EM23, Lemma 4.5].

Altogether, by equivariant resolution of singularities in characteristic zero and induction on dimension, the singular case of (17.6) follows from the smooth one. □

17.4.6. Completions of equivariant cohomology. The above proposition has the following formally deformed variant.

Proposition 17.27. *Let $X \in \mathrm{Sch}_{\mathbb{C}}^T$. For any closed point $t \in T$, we have*

$$HP_T(X;\mathbb{C})^\wedge_{\mathbf{L}t} \xrightarrow{\simeq} H_{T,\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!)^\wedge. \tag{17.7}$$

Remark 17.28. Here, the completion of the left-hand side is the derived formal completion over $t \in T$. The completion of the right-hand side is with respect to the derived Hodge filtration in the Cartan model of equivariant cohomology – see [Che20, Example 2.1.10, Definition 4.2.3] for a discussion (using the cotangent complex in general).

Proof. If X is smooth, the statement is (a special case of) [Che20, Theorem D, Theorem 4.3.2]. We again deduce the general case by proper excision. For the left-hand side, $HP(-/\mathbb{C})^\wedge_{\mathbf{L}t}$ satisfies proper excision – it is given by composing the truncating invariant $HP(-/\mathbb{C})$ with $(-)^\wedge_{\mathbf{L}t}$. For the right-hand side, we again use that $\mathrm{Fix}_t(-)$ preserves abstract blowups. Moreover, since the derived de Rham complex satisfies proper excision, the associated filtrations on $H_{\mathrm{sing}}(-;\mathbb{C})(\!(u)\!)$ are also compatible with proper excision – so taking $H_{\mathrm{sing}}(-;\mathbb{C})(\!(u)\!)^\wedge$ satisfies proper excision. By existence of equivariant resolutions in characteristic zero, we thus reduce to the smooth case. □

17.4.7. Fibers of equivariant K-theory and cohomology of fixed-points. Putting the above discussion together, we obtain the second part of the theorem.

Proposition 17.29. *Let $X \in \mathrm{Sch}_{\mathbb{C}}^T$ and $t \in T$ be a closed point. Then we have an equivalence*

$$K_T^{\mathbf{top}}(X;\mathbb{C})_{\mathbf{L}t} \xrightarrow{\simeq} H_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!)$$

in $\mathrm{CAlg}(D(\mathbb{C}))$.

Proof. Let $t : \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathbb{C}$ be a classical point of the equivariant base. Taking the fiber $(-)_t$ of the equivalence from Proposition 17.24 and composing with the equivalence from Proposition 17.26, we deduce

$$K_T^{\mathbf{top}}(X;\mathbb{C})_{\mathbf{L}t} \xrightarrow{\simeq} HP_T(X/\mathbb{C})_{\mathbf{L}t} \xrightarrow{\simeq} H_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})(\!(u)\!).$$

□

17.4.8. Equivariant formality. Under extra flatness assumptions, the above derived identifications simplify. In particular, this happens under suitable equivariant formality hypothesis. Let us hide the formal parameter u by regarding $K_T^{\mathbf{top},*}(X)$ as graded by $* \in \mathbb{Z}/2$.

Corollary 17.30. *Assume that $K_T^{\mathbf{top},*}(X)$ is a flat module over the equivariant base and let $t \in T$. Then we have an isomorphism of $\mathbb{Z}/2$-graded-commutative rings*

$$K_T^{\mathbf{top},*}(X;\mathbb{C}) \underset{\mathcal{O}(T)}{\otimes} \mathbb{C}_t = H^\bullet_{\mathrm{sing}}(\mathrm{Fix}_t(X);\mathbb{C})$$

where $* \in \mathbb{Z}/2$ and $\bullet \in \mathbb{N}_{\geqslant 0}$.

Proof. Take $\pi_\bullet(-)$ in Proposition 17.29. By the flatness assumption, the base change along $\mathcal{O}(T) \to \mathbb{C}_t$ is underived, hence commutes with the passage to homotopy groups on the left-hand side.

Both sides are 2-periodic; the trace map is compatible with the natural filtrations; the element u has degree -2. This explicitly identifies

$$K_T^{\mathbf{top},0}(X;\mathbb{C}) \underset{\mathcal{O}(T)}{\otimes} \mathbb{C}_t = \bigoplus_{i\geqslant 0} H^{2i}_{\text{sing}}(\text{Fix}_t(X);\mathbb{C}) \cdot u^i$$

$$K_T^{\mathbf{top},1}(X;\mathbb{C}) \underset{\mathcal{O}(T)}{\otimes} \mathbb{C}_t = \bigoplus_{i\geqslant 0} H^{2i+1}_{\text{sing}}(\text{Fix}_t(X);\mathbb{C}) \cdot u^i$$

□

Remark 17.31. Even more specifically, suppose that $K_T^{\mathbf{top},*}(X)$ is concentrated in the even part, where it is given by a flat module over the equivariant base. Then we have a ring isomorphism

$$K_T^{\mathbf{top},0}(X;\mathbb{C}) \underset{\mathcal{O}(T)}{\otimes} \mathbb{C}_t = H^{\bullet}_{\text{sing}}(\text{Fix}_t(X);\mathbb{C}),$$

the right-hand size is concentrated in even degrees, and the motivic grading matches the cohomological grading. This is the case relevant for our applications.

Aside 17.32. An appropriate definition of the equivariant motivic filtration on the left-hand side would match the cohomological filtration on the right-hand side. For example, this is the case over $1 \in T$ via the Kan-extended motivic filtration.

17.4.9. Some references. The groundwork for the presented statements is contained in the literature. Moreover, weaker versions of some of these claims can be recovered from the classical literature on completion and localization techniques in equivariant topology, pioneered respectively by Segal and Goresky–Kottwitz-MacPherson. These weaker statements would be also sufficient for our main applications. Nevertheless, we hope our exposition gives a conceptually clear and useful viewpoint.

Our discussion above is based on the following two lines of development.

- The relationship between topological K-theory $K^{\mathbf{top}}$ and periodic cyclic homology HP comes, in various degrees of generality, from [Bla16], [HP20], [Kon21], [ES21], [Kha23], [LS25].
- The relationship between HH and functions on derived fixed-point schemes originates in [BFN10]; the topological version with HP follows by taking the Tate construction. We refer to [Che20] for a throughout and highly recommendable discussion of equivariant HP with its relation to fixed-point schemes and ultimately to singular cohomology of fixed-points. Also see [Bha12], [ACH19], [HSS17], [BN12; BN13], [Toë14] for more context.

18. Cocharacter graphs and resonace

We first introduce and develop in §18.1 the notion of *resonance* inside $T \times \mathbb{G}_m^{\text{rot}}$. Intuitively, two points in T are resonant with respect to a scalar $\zeta \in \mathbb{G}_m^{\text{rot}}$ if they multiplicatively differ by integer powers of ζ. This gives further rise to the *resonant Weyl group* $W_{[x,\zeta]}$ and the associated *resonant Levi* $L_{[x,\zeta]}$. The notion of resonance appears when describing fixed-points of (x, ζ) in the affine Grassmannian and computing the corresponding fibers of its equivariant K-theory. At the same time, resonance describes the linkage principles in (classical and quantum) categories $\mathcal{O}$.

We further discuss in §18.2 the *cocharacter graphs* Γ inside $T \times \mathbb{G}_m^{\mathrm{rot}} \times T$. We explain how Γ puts the resonant combinatorics into a family over $T \times \mathbb{G}_m^{\mathrm{rot}}$. Ultimately, Γ turns out to be essential for our description of the whole K-theory ring of the affine Grassmannian and affine Schubert varieties inside it, because the restriction to Γ realizes equivariant localization to fixed points.

Altogether, the discussion in this section serves as a useful combinatorial tool in the rest of the chapter.

18.1. Points in the equivariant base and resonance

18.1.1. Points in equivariant base. Let $T \times \mathbb{G}_m^{\mathrm{rot}}$ be the extended torus. Consider any k-valued point

$$(x, \zeta) := ((x_1, \ldots, x_n), \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$$

where $x_1, \ldots, x_n, \zeta \in k^\times$ via the chosen coordinates from §16.4.6. In particular, we have the point

$$(1, \zeta) := (\mathrm{id}, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}.$$

Note that ζ may be arbitrary: either a root of unity or generic. We fix ζ throughout this section.

18.1.2. Root coordinates on tori. Given $T \leqslant G$, each simple root $\beta \in \Phi_{\mathrm{s}}^{\bullet}$ gives a coordinate $\beta : T \to \mathbb{G}_m$. Denoting r the number of simple roots (the semi-simple rank of G), consider the homomorphism

$$\begin{aligned}(-)^{\mathrm{ad}} : T &\twoheadrightarrow \mathbb{G}_m^r \\ x &\mapsto x^{\mathrm{ad}} = (\beta_1(x), \ldots, \beta_r(x)).\end{aligned}$$

Since the root lattices of T and T^{ad} are canonically identified, the above map $(-)^{\mathrm{ad}}$ factors as

$$T \to T^{\mathrm{ad}} \xrightarrow{=} \mathbb{G}_m^r.$$

Since the lattice $X_\bullet(T^{\mathrm{ad}}) = P_\bullet(T)$ is dual to the root lattice $Q^\bullet(T^{\mathrm{ad}})$, the latter map is an isomorphism. Under this identification, $(-)^{\mathrm{ad}}$ becomes the natural projection $T \to T^{\mathrm{ad}}$. In particular, if T is of adjoint type to start with, $(-)^{\mathrm{ad}}$ is an isomorphism.

18.1.3. Resonance. A nonzero scalar $a \in \mathbb{G}_m(k)$ is called ζ-*resonant* if $a = \zeta^m$ for some $m \in \mathbb{Z}$. (One could alternatively call this ζ-*integral*, but we prefer less overloaded terminology.) Let $a', a'' \in \mathbb{G}_m(k)$ two nonzero scalars. We say that a', a'' are ζ-*resonant*, denoted $a' \sim_\zeta a''$, iff their ratio is ζ-resonant: $a'(a'')^{-1} = \zeta^m$ for some $m \in \mathbb{Z}$.

Note that when ζ is a primitive ℓ-th root of unity, the set $\log_\zeta(a'(a'')^{-1}) \subseteq \mathbb{Z}$ of such m is either empty or a coset for the subgroup $\ell\mathbb{Z} \leqslant \mathbb{Z}$. Otherwise, when ζ is not a root of unity, $\log_\zeta(a'(a'')^{-1})$ has at most one element.

Let $x', x'' \in T$ be two k-valued points. Let β be a root. We say that x', x'' are *ζ-resonant at β* iff their images under $\beta : T \to \mathbb{G}_m$ are ζ-resonant in the above sense: $\beta(x') \sim_\zeta \beta(x'')$. We say x', x'' are ζ-resonant, denoted $x' \sim_\zeta x''$ iff it is so at every (simple) root β.

A single point $x \in T$ is called *ζ-resonant* iff $x \sim_\zeta 1$. Equivalently, x is ζ-resonant if its image under $(-)^{\mathrm{ad}}$ is of the form $\lambda(\zeta)$ for some cocharacter $\lambda \in X_\bullet(\mathbb{G}_m^d)$.

18.1.4. Resonating coordinates. Given a point $a \in \mathbb{G}_m$, write

$$[a, \zeta] := [a]_\zeta := \begin{cases} a & \text{if } a \text{ is } \zeta\text{-resonant} \\ 1 & \text{else.} \end{cases}$$

Given $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$, denote

$$[x, \zeta] := [x]_\zeta := ([\beta_1(x), \zeta], \ldots, [\beta_d(x), \zeta]) \in \mathbb{G}_m^d.$$

Here, the notation $[x]_\zeta$ is meant to suggest the equivalence class of x under the ζ-resonance relation. For fixed ζ, we call it the *ζ-resonance class of x*. To avoid too many subscripts, we denote it $[x, \zeta]$.

18.1.5. Resonant Levi and its Weyl group. Depending on $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$, consider the subgroups

$$W_x \leqslant W_{[x,\zeta]} \leqslant W \tag{18.1}$$

given as

$$W_x := \{w \in W \mid wx = x\} \qquad \text{and} \qquad W_{[x,\zeta]} := \{w \in W \mid wx \sim_\zeta x\}. \tag{18.2}$$

Denote the corresponding Levi subgroups by

$$L_x \leqslant L_{[x,\zeta]} \leqslant G.$$

In words, W_x is the stabilizer of $x \in T$, while $W_{[x,\zeta]}$ is the stabilizer of its resonance class $[x, \zeta]$. Clearly, the resonant Weyl group $W_{[x,\zeta]}$ depends, up to canonical isomorphism, only on the resonance class $[x, \zeta]$ of (x, ζ). When the point (x, ζ) is clear from the context, we abbreviate

$$L := L_{[x,\zeta]} \leqslant G \qquad \text{and} \qquad W_L := W_{[x,\zeta]} \leqslant W.$$

and call it the *resonant Levi* and *resonant Weyl group* associated to (x, ζ).

18.1.6. Independence on W-action. Fix ζ throughout. Given a closed point $x \in T$, denote by $\vartheta = \vartheta_x \in |T /\!\!/ W|$ its W-orbit. Since ζ is fixed, the resonant Weyl group $W_{[x,\zeta]}$ then depends only on ϑ by (18.2). Therefore, we allow ourselves to denote

$$W_{[\vartheta]} := W_{[\vartheta,\zeta]} := W_{[x,\zeta]}, \qquad L_{[\vartheta]} := L_{[\vartheta,\zeta]} := L_{[x,\zeta]}.$$

Similarly, the stabilizing Weyl group W_x, the stabilizing Levi L_x and the associated parabolic subgroup P_x of $L_{[x,\zeta]}$ depend only on the orbit ϑ, and we denote them

$$W_\vartheta := W_x, \qquad L_\vartheta := L_x, \qquad P_\vartheta \leqslant L_{[\vartheta]}.$$

18.1.7. The integral character and resonating decomposition. Given $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$, we need to decompose x into a non-resonant and resonant part with respect to ζ. To do so, we first project x to a point x^{ad} of the adjoint torus T^{ad}. Consider then a cocharacter

$$\alpha := \alpha_{[x,\zeta]} \in X_\bullet(T^{\mathrm{ad}})$$

given as an arbitrary preimage of $[x, \zeta]$ under the composite

$$\begin{array}{ccccc} X_\bullet(T^{\mathrm{ad}}) & \xrightarrow{(-)^{\mathrm{ad}}} & X_\bullet(\mathbb{G}_m^d) & \xrightarrow{\mathrm{ev}_\zeta} & (k^\times)^d \\ \alpha & \longmapsto & & \longrightarrow & [x, \zeta]\,. \end{array}$$

Such a cocharacter always exists. Indeed, the point $[x, \zeta]$ lies in the image of ev_ζ by construction. Moreover, the lattice $X_\bullet(T^{\mathrm{ad}}) = P_\bullet(T)$ is dual to the root lattice $Q^\bullet(T)$, so the map $(-)^{\mathrm{ad}}$ is an isomorphism.

Remark 18.1. We make some remarks on the uniqueness of α.

(1) The map ev_ζ is not injective unless ζ is generic. However, any choice of α will do – we make this choice now.
(2) It is not always possible to lift α to a cocharacter of T unless G^{der} is of adjoint type. Assuming G^{der} is of adjoint type, such lift of α in T exists, but is not unique since the map $(-)^{\mathrm{ad}}$ is not injective unless G is of adjoint type.

Notation 18.2. Given the above choice of α, there is a unique element $b \in T^{\mathrm{ad}}$ such that

$$x^{\mathrm{ad}} = b \cdot \alpha(\zeta) \tag{18.3}$$

inside T^{ad}. We call this the decomposition of $x \in T$ into the *non-resonant* and *resonant* parts with respect to ζ.

Remark 18.3. Note that roots of the resonant Levi L do not see the difference between x and its resonant part $\alpha(x)$. More precisely:

$$\beta(x) = \beta(b) \cdot \beta(\alpha(\zeta)) = \beta(\alpha(\zeta)), \qquad \forall \beta \in \Phi_L^\bullet.$$

Indeed, a root $\beta \in \Phi_G^\bullet$ lies in $\Phi_L^\bullet$ if and only if the corresponding simple reflection $s_\beta \in W_G$ lies in the subgroup W_L. Now, W_L is generated by the simple reflections s_β corresponding to the simple roots $\beta \in \Phi_{G,\mathrm{s}}^\bullet$ such that x is ζ-resonant at β; we deduce $\beta(x) = \beta(\alpha(\zeta))$ by the defining property of α. In particular,

$$\beta(b) = 1, \qquad \forall \beta \in \Phi_L^\bullet.$$

18.1.8. Vocabulary to Humphrey's notation. When G is of adjoint type and ζ is generic, the above constructions appear in the study of the classical category $\mathcal{O}$, see [Hum08; Jan79]. More precisely, [Hum08] works (under Langlands dual notation) with the BGG category $\mathcal{O}$ of the semisimple Lie algebra $\check{\mathfrak{g}}$. The genericity of ζ corresponds to the classical version of the category $\mathcal{O}$ (as opposed to its quantum version at a root of unity). In this setup, the character $\alpha \in X_\bullet(T)$ is unique (as we saw in Remark 18.1). For convenience, we provide a short vocabulary to the notation from [Hum08] in this special case – compare:

- the definition of $W_L = W_{[\vartheta]} = W_{[x,\zeta]}$ to $W_{[c]}$ of [Hum08, §3.4],
- the definition of $\alpha = \alpha_{[x,\zeta]}$ to the definition of $c^\natural$ of [Hum08, §7.4],
- the definition of W_x to the definition of W_c° of [Hum08, §7.4].

18.2. Cocharacter graphs for simply connected G

The above combinatorics is conveniently encoded by the *cocharacter graphs* $\overline{\Gamma} \subseteq T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W$ introduced in this section.

To simplify the exposition, we will assume G to be simply connected throughout §18.2. With extra bookkeeping, the same discussion works for any reductive group; the necessary modifications are described afterwards in §18.3.

18.2.1. Cocharacter graphs for simply connected groups. Assume G is simply connected. For any element $(\lambda, w) \in W_{\mathrm{ext}} = W_{\mathrm{aff}}$ with $\lambda \in X_\bullet$ and $w \in W$, consider the map

$$\begin{aligned}(\lambda, w) : T \times \mathbb{G}_m^{\mathrm{rot}} &\to T \\ (x, \zeta) &\mapsto \lambda(\zeta) \cdot w(x).\end{aligned}$$

and the closed subscheme $\Gamma_{\lambda,w} \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T$ given by its graph

$$\begin{aligned}\Gamma_{\lambda,w} :=& \{(x, \zeta, \lambda(\zeta) \cdot w(x)) \in T \times \mathbb{G}_m^{\mathrm{rot}} \times T\} \\ =& \{(x, \zeta, y) \in T \times \mathbb{G}_m^{\mathrm{rot}} \times T \mid y = \lambda(\zeta) \cdot w(x)\}.\end{aligned}$$

Taking the union of these closed subschemes over varying λ, we obtain the closed ind-subscheme

$$\Gamma := \{(x, \zeta, y) \in T \times \mathbb{G}_m^{\mathrm{rot}} \times T \mid \exists \lambda \in X_\bullet,\ y \cdot w(x)^{-1} = \lambda(\zeta)\} \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T.$$

Thus, the irreducible componets $\Gamma_{\lambda,w}$ of Γ are copies of $T \times \mathbb{G}_m^{\mathrm{rot}}$ labeled by W_{ext}, intersecting in various ways. In particular, we have the distinguished component $\Gamma_{1,1}$, parametrizing the points $\{(x, \zeta, x) \in T \times \mathbb{G}_m^{\mathrm{rot}} \times T\}$.

18.2.2. Actions. There is a natural action $W_{\mathrm{ext}} = X_\bullet \rtimes W \circlearrowright \Gamma$, given by $(\lambda, w) \cdot (x, \zeta, y) = (x, \zeta, \lambda(\zeta) \cdot w(y))$. In particular, it acts freely and transitively on the set of irreducible components $(\Gamma_{\lambda,w} \mid (\lambda, w) \in W_{\mathrm{ext}})$ of Γ.

18.2.3. Variants. Given any pair of subgroups $W_1, W_2 \leqslant W$, consider their action on $T \times T$. The ind-subscheme $\Gamma \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T$ is clearly $(W_1 \times W_2)$-equivariant. Taking the quotient, we get the sub-ind scheme

$$\Gamma^{T /\!\!/ W_1, T /\!\!/ W_2} \hookrightarrow T /\!\!/ W_1 \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W_2.$$

We will be particularly interested in the case $W_1 = 1$ and $W_2 = W$. We then write

$$\overline{\Gamma} := \Gamma^{T, T /\!\!/ W} \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W.$$

The action of $W_2 = W$ on the right copy of T permutes the irreducible components of Γ according to the left translation on the labeling set $W \circlearrowright (X_\bullet \rtimes W)$. In other words, $\forall w \in W$, we have $w \cdot \Gamma_{\lambda,v} = \Gamma_{w(\lambda),wv}$. The irreducible components of $\overline{\Gamma}$ are thus copies of $T \times \mathbb{G}_m^{\mathrm{rot}}$, labeled by the quotient set $X_\bullet = W \backslash (X_\bullet \rtimes W)$. They correspond to W-orbits of irreducible components of Γ. For each $\lambda \in X_\bullet$, we denote by $\overline{\Gamma}_\lambda$ the component corresponding to the W-orbit of $\Gamma_{\lambda,1}$.

18.2.4. Fibers. Given Γ, $\overline{\Gamma}$ and the other variants, we may consider their fibers over points in the base $T \times \mathbb{G}_m^{\mathrm{rot}}$. Given any $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$, we denote these fibers as

$$\Gamma_{(x,\zeta)} \hookrightarrow T \qquad \text{and} \qquad \overline{\Gamma}_{(x,\zeta)} \hookrightarrow T /\!\!/ W.$$

Similarly, we denote the fibers over some $\zeta \in \mathbb{G}_m^{\mathrm{rot}}$ by

$$\Gamma_\zeta \hookrightarrow T \times T \qquad \text{and} \qquad \overline{\Gamma}_\zeta \hookrightarrow T \times T /\!\!/ W.$$

These are closed sub-ind-schemes.

18.2.5. Specializations of the lattice. The ind-subscheme Γ is reduced by construction; its irreducible components are copies of $T \times \mathbb{G}_m^{\mathrm{rot}}$. Take some $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$. Since $W_{\mathrm{ext}} \circlearrowright \Gamma$ transitively, it also acts transitively on the discrete set $|\Gamma_{(x,\zeta)}|$. We denote by

$$\mathrm{Stab}_{(x,\zeta)} := \mathrm{Stab}_{W_{\mathrm{ext}}}((x, \zeta, x))$$

the stabilizer of the base point $(x, \zeta, x) \in |\Gamma_{(x,\zeta)}|$. We may thus label the discrete set $|\Gamma_{(x,\zeta)}|$ by

$$y \in |\Gamma_{(x,\zeta)}| = W_{\mathrm{ext}} / \mathrm{Stab}_{(x,\zeta)}\,.$$

For any such y, we denote the stabilizer inside the finite Weyl group W of the corresponding closed point $y \in |\Gamma_{(x,\zeta)}| \subseteq |T \times \pi_1(G)|$ by

$$W_y := \mathrm{Stab}_W(y) \leqslant W.$$

Up to a natural isomorphism given by conjugation, each W_y becomes a subgroup of the resonant Weyl group $W_{[x,\zeta]}$. Indeed, up to conjugation, $W_{[x,\zeta]}$ is the stabilizer of the resonance class of each point $y \in |\Gamma_{(x,\zeta)}|$, while W_y is the stabilizer of the actual point y.

Up to a natural isomorphism given by conjugation, W_y clearly depends only on the W-orbit $\vartheta = W \cdot y$ inside T. Therefore, we denote it by

$$W_\vartheta, \qquad \vartheta \in |\overline{\Gamma}_{(x,\zeta)}| = X_\bullet(T) / \mathrm{Stab}_{(x,\zeta)}\,. \tag{18.4}$$

The isomorphism class of W_ϑ varies as ϑ goes through $|\overline{\Gamma}_{(x,\zeta)}|$. This notation is also consistent with §18.1.6.

18.2.6. Stabilizer and resonant Weyl group. The stabilizer projects onto the resonant Weyl group as follows.

Lemma 18.4. *Let $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$. The image of the stabilizer $\mathrm{Stab}_{(x,\zeta)}$ under the projection $W_{\mathrm{ext}} \to W$ is given by the resonant subgroup $W_{[x,\zeta]} \leqslant W$.*

Proof. Since we are assuming G to be simply connected, we have $W_{\mathrm{ext}} = W_{\mathrm{aff}}$.

An element $(\lambda, w) \in \mathbb{Z}\,\Phi_\bullet \rtimes W$ lies in $\mathrm{Stab}_{(x,\zeta)}$ if and only if $x = w(x) \cdot \lambda(\zeta)$. Therefore, an element $w \in W$ lifts to an element of $\mathrm{Stab}_{(x,\zeta)} \leqslant W_{\mathrm{aff}}$ if and only if there exists some $\lambda \in \mathbb{Z}\,\Phi_\bullet(T)$ such that $x \cdot w(x)^{-1} = \lambda(\zeta)$.

Now, it is well-known that the stabilizer $\mathrm{Stab}_{(x,\zeta)} \leqslant W_{\mathrm{aff}}$ is always a Coxeter subgroup [Bou68, chap. V, §4, no 6, Corollaire] or [Kan01, p. 130]. Furthermore, since the projection $W_{\mathrm{aff}} \to W$

is clearly a morphism of Coxeter groups, its image $\mathrm{im}(\mathrm{Stab}_{(x,\zeta)}) \leqslant W$ is a Coxeter subgroup as well. It is thus generated by some of the simple reflections s_β, $\beta \in \Phi_{\mathrm{s}}^{\bullet}$ and we are left to determine this subset.

Therefore, assume $w = s_\beta$. By definition of s_β, we have $x \cdot s_\beta(x)^{-1} = \check{\beta}(\beta(x))$. Since $\check{\beta}$ is a basis element of the free abelian group $\mathbb{Z}\,\Phi_\bullet$, the right-hand side is of the desired form $\lambda(\zeta)$, $\lambda \in \mathbb{Z}\,\Phi_\bullet$ if and only if $\beta(x)$ is of the form ζ^m, $m \in \mathbb{Z}$. This is the case if and only if x is ζ-resonant at β.

Altogether, $\mathrm{im}(\mathrm{Stab}_{(x,\zeta)}) = W_{[x,\zeta]}$ inside W as desired. □

18.2.7. Specializations at generic ζ. We first describe $\Gamma_{(x,\zeta)}$ in the generic case $\zeta \in \mathbb{G}_m - \mu_\infty$. We have short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & X_\bullet \arrow[r] & W_{\mathrm{ext}} \arrow[r] & W \arrow[r] & 1 \\
1 \arrow[r] & 1 \arrow[r] \arrow[u, hook] & \mathrm{Stab}_{(x,\zeta)} \arrow[r, "="] \arrow[u, hook] & W_{[x,\zeta]} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd} \tag{18.5}$$

This is clear since $\mathrm{Stab}_{(x,\zeta)} \cap X_\bullet = 1$ is trivial, and the image is given by $W_{[x,\zeta]}$ by Lemma 18.4.

Remark 18.5. The irreducible components of Γ_ζ are labeled by W_{ext}; each of these is a copy of T. On the other hand, irreducible components of the fiber $\Gamma_{(x,\zeta)}$ are given by $W_{\mathrm{ext}}/\mathrm{Stab}_{(x,\zeta)}$. Their collision are described by the quotient map

$$W_{\mathrm{ext}} \twoheadrightarrow W_{\mathrm{ext}}/W_{[x,\zeta]}.$$

18.2.8. Specializations at ζ root of unity. Assume now $\zeta \in \mu_\infty$ is primitive root of unity of $\mathrm{ord}(\zeta) = \ell$. We have the short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & X_\bullet \arrow[r] & W_{\mathrm{ext}} \arrow[r] & W \arrow[r] & 1 \\
1 \arrow[r] & \ell\,\mathbb{Z}\,\Phi_\bullet \arrow[r] \arrow[u, hook] & \mathrm{Stab}_{(x,\zeta)} \arrow[r] \arrow[u, hook] & W_{[x,\zeta]} \arrow[r] \arrow[u, hook] & 1 \\
1 \arrow[r] & \ell\,\mathbb{Z}\,\Phi_\bullet \arrow[r] \arrow[u, hook] & \ell\,\mathbb{Z}\,\Phi_\bullet \rtimes W_x \arrow[r] \arrow[u, hook] & W_x \arrow[r] \arrow[u, hook] & 1
\end{tikzcd} \tag{18.6}$$

Here, the bottom row is given by the separate stabilizers of the base point inside $X_\bullet$ and W. (Note that the actual stabilizer in the middle row is bigger that their semidirect product.)

The exactness of the middle row is clear from the following. Since $\pi_1(G) = 1$ by assumption, $X_\bullet = \mathbb{Z}\,\Phi_\bullet$. The quotient is then given by $W_{[x,\zeta]}$ by Lemma 18.4.

18.2.9. Quotients of $\Gamma_{(x,\zeta)}$. Lets now again work with ζ arbitrary. Above, we have established

$$|\Gamma_{(x,\zeta)}| = W_{\mathrm{ext}}/\mathrm{Stab}_{(x,\zeta)} = \begin{cases} W_{\mathrm{ext}}/W_{[x,\zeta]} & \text{for } \zeta \text{ generic,} \\ W_{\mathrm{ext}}/\ell\,\mathbb{Z}\,\Phi_\bullet \rtimes W_{[x,\zeta]} & \text{for } \zeta \text{ of } \mathrm{ord}(\zeta) = \ell. \end{cases} \tag{18.7}$$

We now deduce a description of the quotient $\overline{\Gamma}_{(x,\zeta)}$.

The group W_{ext} acts on itself by left and right translations. In particular, we get the commuting actions

$$W \circlearrowright W_{\mathrm{ext}} \circlearrowleft \mathrm{Stab}_{(x,\zeta)}$$

Here, the left action is induced by the given embedding of W into the semidirect product W_{ext}. The quotient by this naturally identifies as $X_\bullet = W \backslash W_{\mathrm{ext}}$. This identification is compatible with the right actions of $\mathrm{Stab}_{(x,\zeta)} \leqslant W_{\mathrm{ext}}$.

With the above left W-action, the embedding $W_{\mathrm{ext}}/\mathrm{Stab}_{(x,\zeta)} = |\Gamma_{(x,\zeta)}| \hookrightarrow T$ becomes equivariant, and we deduce the following.

Corollary 18.6. *We have*

$$|\overline{\Gamma}_{(x,\zeta)}| = X_\bullet/\mathrm{Stab}_{(x,\zeta)} = \begin{cases} X_\bullet/W_{[x,\zeta]} & \text{for } \zeta \text{ generic}, \\ X_\bullet/\ell\,\mathbb{Z}\,\Phi_\bullet \rtimes W_{[x,\zeta]} & \text{for } \zeta \text{ of } \mathrm{ord}(\zeta) = \ell. \end{cases} \tag{18.8}$$

In the first case, $X_\bullet/W_{[x,\zeta]} = X_\bullet^+(T, L_{[x,\zeta]})$ are the dominant cocharacters for the resonant Levi. In the latter case, $X_\bullet/\ell\,\mathbb{Z}\,\Phi_\bullet \rtimes W_{[x,\zeta]}$ is the set of orbits of the ℓ-dilated action of $W_{L_{[x,\zeta]},\mathrm{aff}}$.

18.2.10. Group structure on cocharacter graphs. As an aside, we note that the cocharacter graphs have a group structure relative to $T \times \mathbb{G}_m^{\mathrm{rot}}$. Indeed, we may regard the second copy of T as a group scheme over $T \times \mathbb{G}_m^{\mathrm{rot}}$. Its natural group structure clearly restrict to the sub-ind scheme Γ.

Explicitly, Γ is a group ind-scheme over $T \times \mathbb{G}_m^{\mathrm{rot}}$ via the multiplication

$$\Gamma \times_{T\times\mathbb{G}_m^{\mathrm{rot}}} \Gamma \to \Gamma$$
$$(x,\zeta,y_1)\cdot(x,\zeta,y_2) = (x,\zeta,y_1\cdot y_2).$$

On the level of connected components, this is the multiplication inside $X_\bullet$. Indeed, if $y_1 = \lambda_1(\zeta)\cdot w_1(x)$ and $y_2 = \lambda_2(\zeta)\cdot w_2(x)$, then their product in the second copy of T is

$$y_1\cdot y_2 = \lambda_1(\zeta)\cdot w_1(x)\cdot\lambda_2(\zeta)\cdot w_2(x) = (\lambda_1\cdot\lambda_2)(\zeta)\cdot(w_1\cdot w_2)(x).$$

18.3. Cocharacter graphs for reductive groups

18.3.1. Comments. Assume now G to be reductive. For technical reasons, the above discussion needs to be modified. Namely, we need to replace the second copy of T by $T^{\mathrm{sc}} \times \pi_1(G)$, the product of its simply connected cover and the fundamental group of G. At first, we will also replace the first copy of T by T^{sc}. We show how to descend back to T in §18.3.4.

18.3.2. Cocharacter graphs for reductive groups. First, note that a character $\lambda : \mathbb{G}_m^{\mathrm{rot}} \to T$ lifts to a character $\mathbb{G}_m^{\mathrm{rot}} \to T^{\mathrm{sc}}$ if and only if its class in the fundamental group $\overline{\lambda} \in \pi_1(G)$ is trivial. As in §16.3.9, we denote $\chi(\overline{\lambda}) \in X_\bullet(T)$ a chosen representative of $\overline{\lambda}$ under the surjection $X_\bullet(T) \twoheadrightarrow \pi_1(G)$. Consequently, $\underline{\lambda} := \lambda \cdot \chi(\overline{\lambda})^{-1}$ lies in $X_\bullet(T^{\mathrm{sc}}) \leqslant X_\bullet(T)$. For any $(\lambda, w) \in W_{\mathrm{ext}}$ with $\lambda \in X_\bullet(T)$ and $w \in W$, we thus have the map

$$\begin{aligned}(\lambda, w) : T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} &\to T^{\mathrm{sc}} \times \pi_1(G)\\ (x, \zeta) &\mapsto (\underline{\lambda}(\zeta) \cdot w(x), \overline{\lambda}).\end{aligned}$$

In words, by taking $\underline{\lambda}$, we consider the cocharacter λ as a map into the copy of T^{sc} labeled by $\overline{\lambda} \in \pi_1(G)$. We then consider the closed subscheme given by its graph

$$\Gamma_{\lambda,w} := \{(x, \zeta, y, \sigma) \in T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G) \mid y = \underline{\lambda}(q) \cdot w(x),\ \sigma = \overline{\lambda}\} \hookrightarrow T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G).$$

Taking the union of these closed subschemes over all $\lambda \in X_\bullet$, $w \in W$, we obtain the closed ind-subscheme $\Gamma \hookrightarrow T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G)$ given by

$$\Gamma := \{(x, \zeta, y, \sigma) \in T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G) \mid \exists \lambda \in X_\bullet,\ y = \underline{\lambda}(\zeta) \cdot w(x),\ \sigma = \overline{\lambda}\}.$$

In words, the ind-subscheme Γ now lives inside a disjoint union of copies of $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}}$ labeled by the discrete set $\pi_1(G)$. For each $\sigma \in \pi_1(G)$, we denote $\Gamma^\sigma \hookrightarrow T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \{\sigma\}$ the corresponding component. While all these copies are abstractly isomorphic (via the choice of lifts $\chi(\sigma)$), these identifications are not canonical.

As before, the irreducible componets $\Gamma_{\lambda,w}$ of Γ are copies of $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}}$ labeled by W_{ext}, intersecting in various ways. In particular, we have the distinguished component $\Gamma_{1,1}$, parametrizing $\{(x, \zeta, y, 1) \in T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G)\}$.

There is again a natural action

$$W_{\mathrm{ext}} = X_\bullet \rtimes W \circlearrowright \Gamma, \qquad (\lambda, w) \cdot (x, \zeta, y, \sigma) = (x, \zeta, \underline{\lambda}(\zeta) \cdot w(y), \overline{\lambda} \cdot \sigma).$$

In particular, it acts freely and transitively on the set of irreducible components of Γ.

18.3.3. Variants and stabilizers for reductive G. We can repeat the discussion §18.2.2, 18.2.3, 18.2.4: after taking the quotient, we get the variants $\Gamma^{T^{\mathrm{sc}}/\!\!/W_1, T^{\mathrm{sc}}/\!\!/W_2} \hookrightarrow T^{\mathrm{sc}}/\!\!/W_1 \times \mathbb{G}_m \times T^{\mathrm{sc}} /\!\!/ W_2 \times \pi_1(G)$. We in particular denote $\overline{\Gamma} := \Gamma^{T^{\mathrm{sc}}, T^{\mathrm{sc}}/\!\!/W} \hookrightarrow T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} /\!\!/ W \times \pi_1(G)$. We can further take fibers, consider stabilizers, etc.

The results of §18.2.5, 18.2.6, 18.2.7, 18.2.8, 18.2.9 then follow, with the following extra argument. We again consider the stabilizer of the base point $\mathrm{Stab}_{(x,\zeta)} = \mathrm{Stab}_{W_{\mathrm{ext}}}((x, 1, x, 1))$. The extended affine Weyl group W_{ext} acts on the components of $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G)$ labeled by $\pi_1(G)$ through the surjection $W_{\mathrm{ext}} \twoheadrightarrow W_{\mathrm{ext}}/W_{\mathrm{aff}} = \pi_1(G)$. After fixing $\sigma = 1$, we are thus left with the previous action of W_{aff} (which is the extended affine Weyl group of G^{sc}).

18.3.4. Action of the fundamental group. The fundamental group $\pi_1(G)$ acts on T^{sc} through the central deck transformations

$$\mathrm{act} : \pi_1(G) \to \pi_1(G^{\mathrm{ad}}) = Z(G^{\mathrm{sc}}) \leqslant T^{\mathrm{sc}} \circlearrowright T^{\mathrm{sc}}.$$

Consequently, $\pi_1(G) \circlearrowright T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G)$, using the diagonal deck transformation $\mathrm{act} \times \mathrm{act}$ on $T^{\mathrm{sc}} \times T^{\mathrm{sc}}$ and the trivial action on $\mathbb{G}_m^{\mathrm{rot}} \times \pi_1(G)$.

This action preserves each of the closed subschemes $\Gamma_{\lambda,w}$ for $(\lambda, w) \in W_{\mathrm{ext}}$. Indeed, $\Gamma_{\lambda,w}$ parametrizes (x, ζ, y, σ) satisfying $y = \lambda(\zeta) \cdot w(x)$ and $\sigma = \overline{\lambda}$. An element $\tau \in \pi_1(G)$ acts by

$\tau \cdot (x, \zeta, y, \sigma) = (\tau x, \zeta, \tau y, \sigma)$. The latter condition is preserved trivially; for the former one note that

$$y = \lambda(\zeta) \cdot w(x) \iff \tau \cdot y = \tau \cdot \lambda(\zeta) \cdot w(x) \iff \tau \cdot y = \lambda(\zeta) \cdot w(\tau \cdot x)$$

since the action of W is trivial on $Z(G^{\mathrm{sc}})$ and compatible with the product on T^{sc}.

Altogether, $\pi_1(G)$ acts on $\Gamma \hookrightarrow T^{\mathrm{sc}} \times T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times \pi_1(G)$. This action is clearly free. We obtain the quotient

$$\Gamma/\pi_1(G) \hookrightarrow \tfrac{T^{\mathrm{sc}} \times T^{\mathrm{sc}}}{\pi_1(G)} \times \mathbb{G}_m^{\mathrm{rot}} \times \pi_1(G). \tag{18.9}$$

Remark 18.7. We may further use the following identification. Consider the group isomorphism $T^{\mathrm{sc}} \times T^{\mathrm{sc}} \xrightarrow{\cong} T^{\mathrm{sc}} \times T^{\mathrm{sc}}$ given by $(x, y) \mapsto (x, x^{-1}y)$. Under this isomorphism, the diagonal action $\mathrm{act} \times \mathrm{act}$ of $\pi_1(G)$ identifies with its action on the first factor $\mathrm{act} \times \mathrm{id}$. Taking the quotient identifies $\frac{T^{\mathrm{sc}} \times T^{\mathrm{sc}}}{\pi_1(G)} = \frac{T^{\mathrm{sc}}}{\pi_1(G)} \times T^{\mathrm{sc}} = T \times T^{\mathrm{sc}}$. We rewrite (18.9) as

$$\Gamma^{T, T^{\mathrm{sc}}} \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} \times \pi_1(G). \tag{18.10}$$

where the sub-ind scheme $\Gamma^{T, T^{\mathrm{sc}}}$ is given by transporting Γ through the above identifications and taking the quotient by $\pi_1(G)$.

19. Fixed points on affine Schubert varieties

We now concretely describe the fixed-points of an arbitrary element $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ on the affine Grassmannian $\mathcal{G}\mathrm{r}_G$, extending known results in the case $x = 1$. The qualitative answer depends on whether ζ is generic or a root of unity and on the notion of resonance.

We push this computation further to determine the fixed-points of (x, ζ) inside any given affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant \mu}$. The answer is a union of closed finite Schubert varieties (resp. closed unions of Iwahori orbits). We believe this geometric result has not been observed before. It gives a geometric counterpart of known representation theoretic phenomena.

19.1. Fibers of the fixed point scheme

19.1.1. Fixed-points. Let G be a reductive group and $\mathcal{G}\mathrm{r} := \mathcal{G}\mathrm{r}_G$ its affine Grassmannian over k. Consider the fixed-point family $\mathrm{Fix}_{\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r})$. Since $\mathcal{G}\mathrm{r}$ is an ind-scheme, so is $\mathrm{Fix}_{\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r})$. Its natural derived structure is trivial by the criterion [BN13, Proposition 3.3] applied to the ind-presentation by affine Schubert varieties. Our main interest lies in the classical restriction of this family to the extended torus

$$\mathrm{Fix}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}).$$

In the rest of this subsection, we explain how to effectively describe the fibers of this family as the point $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ varies.

19.1.2. Notation. As before, fix a point $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$. Let $L := L_{[x,\zeta]} \leqslant G$ be the resonant Levi with resonant Weyl group $W_L := W_{[x,\zeta]} \leqslant W$. Let $\alpha = \alpha_{[x,\zeta]}$ be the ζ-resonant cocharacter of x.

Since $\mathcal{G}\mathrm{r}_G = \coprod_{\pi_1(G)} \mathcal{G}\mathrm{r}_{G^{\mathrm{sc}}}$, we do not loose anything by assuming G to be semisimple of adjoint type: $G = G^{\mathrm{ad}}$. Without loss of generality, we can also work equivariantly with respect to this adjoint group and its torus $T = T^{\mathrm{ad}}$.

We denote the affine Grassmannian of G by $\mathcal{G}\mathrm{r} = \mathcal{G}\mathrm{r}_G$, omiting the subscript. Similarly, its affine Schubert varieties are denoted $\mathcal{G}\mathrm{r}_{\leqslant\mu} := \mathcal{G}\mathrm{r}_{G,\leqslant\mu}$. On the other hand, whenever referring the affine Grassmannian $\mathcal{G}\mathrm{r}_L$ of the Levi L, we always indicate L in the subscript.

19.1.3. Translating by $\alpha_{[x,\zeta]}$.

Lemma 19.1. *We have an $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$-equivariant isomorphism*

$$t^\alpha \cdot \mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}) \cong \mathrm{Fix}_{(b,\zeta)}(t^\alpha \cdot \mathcal{G}\mathrm{r}). \tag{19.1}$$

Proof. Let $p \in \mathcal{G}\mathrm{r}$. We need to see that $(x, \zeta) \cdot p = p$ happens if and only if $(b, \zeta) \cdot \alpha(t) \cdot p = \alpha(t) \cdot p$. By the general structure of fixed-point schemes (17.1), it is enough to check that inside $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ we have

$$(x, \zeta) = \alpha^{-1}(t) \cdot (b, \zeta) \cdot \alpha(t). \tag{19.2}$$

This is the case by direct computation within $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$:

$$\begin{aligned}
(x, \zeta) &= (b \cdot \alpha(\zeta), \zeta) \\
&= (b, 1) \cdot (\alpha(t^{-1}) \cdot 1 \cdot \alpha(\zeta t), 1 \cdot \zeta \cdot 1) \\
&= (b, 1) \cdot (\alpha(t^{-1}), 1) \cdot (1, \zeta) \cdot (\alpha(t), 1) \\
&= \alpha^{-1} \cdot (b, \zeta) \cdot \alpha.
\end{aligned}$$

The first equality is by definition of α and b; the second equality is formal. The third equality comes from the semidirect product $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ since conjugation by the loop rotation factor scales t with weight one. The fourth equation is clear since b commutes with $\alpha(t^{-1})$ as they both lie in $\mathrm{L}T$. ☐

19.1.4. Fixed-points of the non-resonant part. We further need to identify the fixed points of (b, ζ). Denote $\mathcal{G}\mathrm{r}_L \subseteq \mathcal{G}\mathrm{r}_G$ the affine Grassmanians of $L \leqslant G$ respectively.

Lemma 19.2. *Let (b, ζ) and $L \leqslant G$ be as above; then:*

$$\mathrm{Fix}_{(b,\zeta)}(\mathcal{G}\mathrm{r}_G) = \mathrm{Fix}_{(1,\zeta)}(\mathcal{G}\mathrm{r}_L) \tag{19.3}$$

Proof. Any point of $x \in \mathcal{G}\mathrm{r}_G$ can be decomposed into a product

$$t^\lambda \cdot u_1 \cdots u_d \tag{19.4}$$

with $\lambda \in X_\bullet$ and $u_i \in U_{\beta_i + m_i\hbar}$ nontrivial points in finitely many pairwise distinct affine root groups; see [RW22, proof of Lemma 4.4] for this well-known fact ([RW22] discusses a more refined statement for a given Iwahori orbit – our variant follows since the Iwahori orbits are affine spaces paving the affine Grassmannian).

First note that b lies in the center of L by Remark 18.3 and [Mil17, §21.b]; it thus acts trivially on $\mathcal{G}\mathrm{r}_L$, so in particular $\mathrm{Fix}_{(b,\zeta)}(\mathcal{G}\mathrm{r}_L) = \mathrm{Fix}_{(1,\zeta)}(\mathcal{G}\mathrm{r}_L)$. Hence $\mathrm{Fix}_{(b,\zeta)}(\mathcal{G}\mathrm{r}_G) \supseteq \mathrm{Fix}_{(1,\zeta)}(\mathcal{G}\mathrm{r}_L)$.

On the other hand, on any root subgroup $U_{\beta+m\hbar}$ which is not in $\mathrm{L}L$, the element b acts by a scalar which is not an integer power of ζ, while the loop rotation acts by an integer power of ζ – indeed, the weight is $\beta(b)\zeta^m$ by Lemma 16.3. Hence the action of (b,ζ) on this root subgroup fixes only the origin. Writing an element of $\mathcal{G}\mathrm{r}_G$ as (19.4), this shows that $\mathrm{Fix}_{(b,\zeta)}(\mathcal{G}\mathrm{r}_G) \subseteq \mathcal{G}\mathrm{r}_L$ and hence $\mathrm{Fix}_{(b,\zeta)}(\mathcal{G}\mathrm{r}_G) \subseteq \mathrm{Fix}_{(1,\zeta)}(\mathcal{G}\mathrm{r}_L)$. Altogether, we proved the desired equality. □

19.1.5. Centralizer and its action on fixed-points. Consider the centralizer

$$J := J_{(x,\zeta)} \leqslant \mathrm{L}G$$

of (x,ζ) inside the loop group $\mathrm{L}G$. Furthermore, denote

$$J^+ := J^+_{(x,\zeta)} := J \cap (\mathrm{L}^+G)$$

the centralizer of (x,ζ) inside the extended positive loop group. Finally, consider the conjugate subgroup

$$J^+_\alpha := t^\alpha J^+ t^{-\alpha} = t^\alpha J t^{-\alpha} \cap t^\alpha \mathrm{L}^+G t^{-\alpha} \leqslant \mathrm{L}G.$$

Remark 19.3. We can also consider such centralizers within the loop groups $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ resp. $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ extended by the loop rotation. However, given any $\zeta' \in \mathbb{G}_m^{\mathrm{rot}}$, the elements (x,ζ) and $(1,\zeta')$ lie in the same torus $T \times \mathbb{G}_m^{\mathrm{rot}}$ and thus commute. The bigger centralizers that we just mentioned are thus respectively given by

$$J \rtimes \mathbb{G}_m^{\mathrm{rot}} \qquad \text{and} \qquad J^+ \rtimes \mathbb{G}_m^{\mathrm{rot}}.$$

We do not really need the extra $\mathbb{G}_m^{\mathrm{rot}}$ for our applications.

With the above definitions, the following statements are obvious.

Lemma 19.4. *The subgroup $J \rtimes \mathbb{G}_m^{\mathrm{rot}}$ acts on $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r})$.*

Proof. Since J is pointwise fixed under the adjoint action of (x,ζ) on $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$, it acts on the corresponding fiber of the fixed-point scheme. □

Lemma 19.5. *The subgroup $J^+ \rtimes \mathbb{G}_m^{\mathrm{rot}}$ acts on $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$.*

Proof. Since $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ acts on $\mathcal{G}\mathrm{r}_{\leqslant\mu}$, this follows from Lemma 19.4. □

19.1.6. Key identifications. From §19.1.5 we get the following diagrams of equivariant closed immersions:

$$J^+ \circlearrowright \quad \begin{array}{ccc} \mathcal{G}\mathrm{r}_{\leqslant\mu} & \hookrightarrow & \mathcal{G}\mathrm{r} \\ \uparrow & & \uparrow \\ \mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu}) & \hookrightarrow & \mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}) \end{array} \tag{19.5}$$

Shifting by α, we get

$$\begin{tikzcd} & t^\alpha \cdot \mathcal{G}r_{\leqslant\mu} \arrow[r, hook] & t^\alpha \cdot \mathcal{G}r \\ J_\alpha^+ \circlearrowright & & \\ & t^\alpha \cdot \mathrm{Fix}_{(x,\zeta)}(\mathcal{G}r_{\leqslant\mu}) \arrow[r, hook] \arrow[uu, hook] & t^\alpha \cdot \mathrm{Fix}_{(x,\zeta)}(\mathcal{G}r) \arrow[uu, hook] \end{tikzcd} \tag{19.6}$$

Using Lemmata 19.1 and 19.2, this diagram identifies with

$$\begin{tikzcd} & t^\alpha \cdot \mathcal{G}r_{\leqslant\mu} \arrow[r, hook] & t^\alpha \cdot \mathcal{G}r & t^\alpha \cdot \mathcal{G}r_L \arrow[l, hook'] \arrow[r, "\cong"] & \mathcal{G}r_L \\ J_\alpha^+ \circlearrowright & & & & \\ & \mathrm{Fix}_{(b,\zeta)}(t^\alpha \cdot \mathcal{G}r_{\leqslant\mu}) \arrow[r, hook] \arrow[uu, hook] & \mathrm{Fix}_{(b,\zeta)}(t^\alpha \cdot \mathcal{G}r) \arrow[uu, hook] & \mathrm{Fix}_{(1,\zeta)}(t^\alpha \cdot \mathcal{G}r_L) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[uu, hook] & \mathrm{Fix}_{(1,\zeta)}(\mathcal{G}r_L) \arrow[uu, hook] \end{tikzcd} \tag{19.7}$$

All maps in this diagram are closed immersions. We equip the top right $\mathcal{G}r_L$ with the $t^{-\alpha}$-conjugate J_α^+-action. Under these conventions, all maps are J_α^+-equivariant. In total, we deduce the following.

Lemma 19.6. *The variety $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}r_{\leqslant\mu})$ of our interest identifies with a closed J_α^+-equivariant subvariety*

$$J_\alpha^+ \circlearrowright \mathrm{Fix}_{(1,\zeta)}(t^\alpha \cdot \mathcal{G}r_{\leqslant\mu}) \hookrightarrow \mathrm{Fix}_{(1,\zeta)}(\mathcal{G}r_L). \tag{19.8}$$

Proof. Combine the bottom rows of diagams (19.7). □

To conclude our computation of the fixed points, it remains to explicitly describe the players in Lemma 19.6. We do this in the next two sections, distinguishing whether ζ is generic or a root of unity.

19.2. ζ generic

19.2.1. Setup. Let $\zeta \in \mathbb{G}_m^{\mathrm{rot}}$ be closed point, which is not a root of unity. The purpose of this subsection is to describe the fibers of the fixed-point scheme $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}r_{\leqslant\mu})$ as a union of classical Schubert varieties inside partial flag varieties for the resonant Levi subgroup $L = L_{[x,\zeta]}$.

19.2.2. Fixed points of generic ζ. We first make explicit Lemma 19.2. Here, note that the orbits of $W_L \circlearrowright X_\bullet$ are parametrized by the dominant cocharacters of the Levi L.

Lemma 19.7. *Assume $\zeta \in \mathbb{G}_m^{\mathrm{rot}}(k)$ is generic. Then*

$$\mathrm{Fix}_{(1,\zeta)}(\mathcal{G}r_L) = \coprod_{\vartheta \in X_\bullet/W_L} L/P_\vartheta. \tag{19.9}$$

where ϑ runs over the orbits of the action of W_L on $X_\bullet$ and $P_\vartheta \leqslant L$ is the parabolic subgroup associated to ϑ.

Proof. This is standard, e.g. [Zhu17b, above Example 2.1.12]. Each affine Schubert cell is a geometric vector bundle over the corresponding partial flag variety, with nontrivial weights in the fibers. The loop rotation fixed-points are then given by the zero section. These are the same as the fixed-points of the single element ζ by the genericity assumption. □

In particular, we record the following consequence.

Lemma 19.8. *For $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ with ζ generic, we have*

$$\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_G) = \coprod_{\vartheta \in X_\bullet/W_L} L/P_\vartheta. \tag{19.10}$$

Proof. By Lemma 19.1 we can shift to replace (x, ζ) by (b, ζ). Then apply Lemmata 19.2 and 19.7. □

19.2.3. Centralizers over generic ζ. We now describe the centralizer J of (x, ζ). To this end, consider the subgroup $t^{-\alpha} \cdot L \cdot t^\alpha \hookrightarrow \mathrm{L}G$; it is a copy of L embedded into $\mathrm{L}G$ in a non-constant way.

Lemma 19.9. *Inside $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$ we have*

$$J = t^{-\alpha} \cdot L \cdot t^\alpha. \tag{19.11}$$

Proof. Direct computation by on the root spaces from §16.4.7. Namely, by Lemma 16.3, we know that (x, ζ) acts separately on each root space $U_{\beta+m\hbar}$ with weight $\beta(x) \cdot \zeta^m$. To see which root spaces occur in J, we are thus solving

$$\beta(x) \cdot \zeta^m = 1.$$

This can clearly happen only when x is ζ-resonant at β, in other words when $\beta \in \Phi_L^\bullet$. Under this additional constraint, the desired equation is – by Remark 18.3 – further equivalent to

$$\beta(\alpha(\zeta)) \cdot \zeta^m = 1.$$

This can be rewritten as

$$\zeta^{\langle \alpha, \beta \rangle + m} = 1.$$

Since ζ is generic, this happens if and only if

$$\langle \alpha, \beta \rangle + m = 0.$$

This is the case if and only if the original root space is of the form

$$U_{\beta+m\hbar} = U_{\beta - \langle \alpha, \beta \rangle \hbar} = t^{-\alpha} \cdot U_{\beta+0\hbar} \cdot t^\alpha, \qquad \beta \in \Phi_L^\bullet.$$

Here, the second equality is by Lemma 16.4. These are precisely the root spaces of $t^{-\alpha} \cdot L \cdot t^\alpha$. Also, an element of LT lies in J if and only if it is inside $T \leqslant \mathrm{L}T$. Altogether, we deduce $J = t^{-\alpha} \cdot L \cdot t^\alpha$. □

19.2.4. Positive centralizers as parabolics. In particular, we obtain a description of J_α^+.

Lemma 19.10. *Inside $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}}$,*

$$J_\alpha^+ = P_\alpha$$

where $P_\alpha \leqslant L$ is the opposite parabolic subgroup corresponding to the cocharacter $\alpha : \mathbb{G}_m \to L$.

Proof. We have already described J in Lemma 19.9. To describe J^+, keep only the root spaces of $t^{-\alpha} \cdot L \cdot t^{\alpha}$ that lie inside L^+G. Explicitly, these are

$$U_{\beta - \langle \alpha, \beta \rangle \hbar}, \quad \text{such that} \quad \beta \in \Phi_L^\bullet \quad \text{and} \quad \langle \alpha, \beta \rangle \leqslant 0.$$

The description of J_α^+ then follows by conjugating back via t^α – its roots spaces are precisely

$$U_{\beta + 0\hbar}, \quad \text{such that} \quad \beta \in \Phi_L^\bullet \quad \text{and} \quad \langle \alpha, \beta \rangle \leqslant 0,$$

giving the opposite parabolic $P_{L,\alpha}$. □

19.2.5. Description of fixed-points as finite Schubert varieties. We can now prove the desired description of fixed-points.

Theorem 19.11. *The reduced scheme underlying* $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant \mu})$ *is isomorphic to a union of of classical Schubert varieties for the connected reductive group* $L = L_{[x,\zeta]}$.

Proof. Altogether, rewriting the diagram (19.7) via Lemmata 19.7 and 19.10 identifies the variety

$$\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant \mu})$$

of our interest as the closed P_α-equivariant subvariety

$$P_\alpha \circlearrowright \mathrm{Fix}_{(1,\zeta)}(t^\alpha \cdot \mathcal{G}\mathrm{r}_{\leqslant \mu}) \hookrightarrow \coprod_{\vartheta \in X_\bullet / W_L} L/P_\vartheta \tag{19.12}$$

where $P_\alpha \leqslant L$ acts in by left translation on each of the partial flag varieties L/P_ν.

By the above identification, it suffices to see that the scheme $\mathrm{Fix}_{(1,\zeta)}(t^\alpha \cdot \mathcal{G}\mathrm{r}_{\leqslant \mu})$ is P_α-equivariantly isomorphic to a union of of classical Schubert varieties for the group L.

To see his, we can restrict to a single component L/P_ν of the right-hand side. Then the fixed point scheme $\mathrm{Fix}_{(1,\zeta)}(t^\alpha \cdot X_{\leqslant \mu})$ cuts out a closed P_α-equivariant subvariety – therefore, it is a downwards closed union of finitely many Schubert cells. Altogether, we get a (not necessarily disjoint) union of classical Schubert varieties in L/P_ν. □

19.2.6. Combinatorial description. Finally, the classical Schubert varieties appearing in Theorem 19.11 can be described combinatorially. Indeed, to get such a description, all we need to do is to describe the set of relevant torus fixed-points. Given any $\mu \in X_\bullet^+$, write

$$\mathrm{wt}(\mu) := W \cdot \{\lambda \in X_\bullet^+ \mid \lambda \leqslant \mu\} \subseteq X_\bullet \tag{19.13}$$

for the union of the Weyl orbits of the dominant weights λ below μ. Equivalently, this is the set of weights of the highest weight representation V_μ of the Langlands dual group $\check{G}$.

By standard computations on the affine Grassmannian $\mathcal{G}\mathrm{r}_G$, the fixed points of T on $\mathcal{G}\mathrm{r}_{\leqslant \mu}$ are precisely given by the finite set $\mathrm{wt}(\mu)$. All of them are further fixed by the rotation torus $\mathbb{G}_m^{\mathrm{rot}}$ and lie in $\mathcal{G}\mathrm{r}_L$. Since conjugation by t^α preserves T, translation by t^α then T-equivariantly identifies $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant \mu})$ with $\mathrm{Fix}_{(1,\zeta)}(t^\alpha \cdot \mathcal{G}\mathrm{r}_{\leqslant \mu} \cap \mathcal{G}\mathrm{r}_L)$ via (19.8). The T-fixed points of the latter thus correspond to the α-shift

$$\alpha \cdot \mathrm{wt}(\mu). \tag{19.14}$$

Proposition 19.12. *The T-fixed points in the image of*

$$\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu}) \hookrightarrow \coprod_{\vartheta\in X_\bullet/W_L} L/P_\nu \tag{19.15}$$

are combinatorially given by

$$\alpha\cdot \mathrm{wt}(\mu) := \alpha\cdot W\cdot\{\lambda\in X_\bullet^+ \mid \lambda\leqslant\mu\}. \tag{19.16}$$

Two T-fixed points t^λ, $t^{\lambda'}$ in $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$ lie in the same connected component if and only if λ, λ' lie in the same orbit of the α-shifted action $(W_L,\bullet_\alpha)\circlearrowright X_\bullet^+$.

Proof. The first part is clear from the discussion above.

For the second part, note that two points t^λ, $t^{\lambda'}$ can be connected by a $\mathbb{P}^1$ whenever $\lambda+\alpha$ and $\lambda'+\alpha$ are related by a simple reflection of W_L. By induction, we deduce that whenever there is an element $w\in W_L$ relating $\lambda+\alpha$ and $\lambda'+\alpha$, the two points t^λ, $t^{\lambda'}$ lie in the same connected component.

For the converse, note that when $\lambda+\alpha$ and $\lambda'+\alpha$ lie in different W_L-orbit, t^λ, $t^{\lambda'}$ end up in different terms of the disjoint union $\coprod_{\nu\in X_\bullet/W_L} G/P_\nu$, hence lie in different connected components of its subscheme $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$. □

19.3. ζ root of unity

19.3.1. Setup. We now discuss the complementary case when ζ is a root of unity in k. Denote by $\ell=\mathrm{ord}(\zeta)\in\mathbb{N}$ its primitive order. We emphasize that ℓ is an arbitrary natural number (not necessarily prime). We describe the fibers $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$ in terms of closures of Iwahori orbits on affine flag varieties for the Levi subgroup $L=L_{[x,\zeta]}$.

19.3.2. Fixed points of loop rotation roots of unity. We now make explicit Lemma 19.2 for ζ primitive root of unity of order ℓ. This is developed in [RW22, §4].

Lemma 19.13. *Assume ζ is a primitive root of unity of order $\ell\in\mathbb{N}$. Then*

$$\mathrm{Fix}_{(1,\zeta)}(\mathcal{G}\mathrm{r}_L) = \coprod_{\vartheta} \mathrm{L}_\ell L/\mathcal{P}^\vartheta_{L,\ell} \tag{19.17}$$

where ϑ runs over the orbits of

$$(W_{\mathrm{aff}},\bullet_0^\ell)\circlearrowright X_\bullet$$

and $\mathcal{P}^\vartheta_{L,\ell}\leqslant \mathrm{L}_\ell G$ is the associated parahoric subgroup of the thin (ℓ-dilated) loop group.

Proof. Since we are working over $\mathbb{C}$, the group μ_ℓ is a finite cyclic group of order ℓ on the generator ζ, giving the first identification. For the decomposition, see [RW22, §1.6, §4.6] or [BASV22, §2.1.4]. □

For future reference, let us record the following consequence.

Lemma 19.14. *Let $(x,\zeta)\in T\times\mu_\infty$. Then*

$$\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_G) = \coprod_{\vartheta} \mathcal{F}\mathrm{l}^\vartheta_{\ell,L_{[x,\zeta]}}.$$

Proof. Combine Lemmata 19.1, 19.2, 19.13. □

19.3.3. Centralizers over roots of unity.

Lemma 19.15. *Inside* $\mathrm{L}G$*, we have* $J = t^{-\alpha} \cdot \mathrm{L}_\ell L \cdot t^\alpha$*. Hence* $t^\alpha \cdot J \cdot t^{-\alpha} = \mathrm{L}_\ell L$*.*

Proof. A direct computation. First, note that an element t^λ in $\mathrm{L}T$ lies in the centralizer of (x, ζ) if and only if $\lambda(\zeta) = 1$, i.e. if $\lambda \in \ell X_\bullet$.

We now restrict attention to root spaces. Since (x, ζ) acts separately on each root space $U_{\beta+m\hbar}$ where $\beta \in \Phi^\bullet$, $m \in \mathbb{Z}$, we can treat each of them separately.

Since the adjoint action of (x, ζ) on $U_{\beta+m\hbar}$ is given by the scalar $\beta(x) \cdot \zeta^m$ by Lemma 16.3, $U_{\beta+m\hbar}$ is contained in J if and only if

$$\beta(x) \cdot \zeta^m = 1.$$

For this to be possible, $\beta(x)$ needs to be ζ-resonant; it follows that $\beta \in \Phi_L^\bullet$. Under this extra assumption, the above equation becomes – by Remark 18.3 – equivalent to

$$\beta(\alpha(\zeta)) \cdot \zeta^m = 1$$

This can be further rewritten as

$$\zeta^{\langle \alpha, \beta \rangle + m} = 1$$

Since ζ is a primitive root of unity, this is the case if and only if

$$\langle \alpha, \beta \rangle + m \equiv 0 \mod \ell.$$

However, this is the case if and only the root space if of the form

$$U_{\beta+m\hbar} = U_{\beta+(\ell m' + \langle \alpha, \beta \rangle)}.$$

for some $m' \in \mathbb{Z}$. However, the right hand side equals to

$$t^{-\alpha} \cdot U_{\beta+\ell m'\hbar} \cdot t^\alpha$$

Letting β and m' vary, these are precisely the root spaces of $t^{-\alpha} \cdot \mathrm{L}_\ell L \cdot t^\alpha$. All the above manipulations were equivalent, so we conclude. □

19.3.4. Positive centralizers as parahoric subgroups.

Lemma 19.16. *Inside* L^+G*, we have*

$$J_\alpha^+ = \mathcal{P}_{\ell,L}^\alpha$$

given by the parahoric subgroup $\mathcal{P}_{\ell,L}^\alpha \leqslant \mathrm{L}_\ell L$ *corresponding to* α *with respect to* B'_L*.*

Proof. From the description [BASV22, equation (2.8)] applied to the group L with respect to our opposite Borel B'_L, we deduce the first equality in

$$\mathcal{P}_{\ell,L}^\alpha = \mathrm{L}_\ell L \cap t^\alpha \cdot \mathrm{L}^+L \cdot t^{-\alpha} = J_\alpha \cap t^\alpha \cdot \mathrm{L}^+G \cdot t^{-\alpha} = J_\alpha^+.$$

For the second equality, use that $J_\alpha = \mathrm{L}_\ell L$ by Lemma 19.15 (we can replace L^+L by L^+G since J_α already constrains the computation into $\mathrm{L}L$ anyway). The third equality is by definition. □

19.3.5. Description of fixed-points as parahoric orbits in dilated affine flag varieties.

Theorem 19.17. *The reduced scheme underlying* $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$ *is isomorphic to a closed union of* $\mathcal{P}^{\alpha}_{\ell,L}$*-orbits (in particular, a closed union of dilated Iwahori orbits) on affine flag varieties* $\mathcal{F}\mathrm{l}^{\vartheta}_{\ell,L}$ *for the dilated loop group* $\mathrm{L}_{\ell}L$ *of the resonant Levi* $L = L_{[x,\zeta]}$.

Proof. Rewriting the diagram (19.7) via Lemmata 19.13 and 19.16 identifies the variety

$$\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$$

of our interest as the closed $\mathcal{P}^{\alpha}_{\ell,L}$-equivariant subvariety

$$\mathcal{P}^{\alpha}_{\ell,L} \circlearrowright \mathrm{Fix}_{(1,\zeta)}(t^{\alpha}\cdot\mathcal{G}\mathrm{r}_{\leqslant\mu}) \hookrightarrow \coprod_{\vartheta\in X_{\bullet}/(W_L,\bullet^{\ell})} \mathrm{L}_{\ell}L/\mathcal{P}^{\vartheta}_{\ell,L} \tag{19.18}$$

where $\mathcal{P}^{\alpha}_{\ell,L} \leqslant \mathrm{L}_{\ell}L$ acts in by left translation on each of the affine flag varieties $\mathcal{F}\mathrm{l}^{\vartheta}_{\ell,L} = \mathrm{L}_{\ell}L/\mathcal{P}^{\vartheta}_{\ell,L}$.

Restricting attention to a single component $\mathcal{F}\mathrm{l}^{\vartheta}_{\ell,L}$ of the right-hand side, we get a closed union of $\mathcal{P}^{\alpha}_{\ell,L}$-orbits as desired. Since $\mathrm{Iw}_{\ell} \leqslant \mathcal{P}^{\alpha}_{\ell,L}$, it is in particular closed union of Iwahori orbits. □

19.3.6. Combinatorics of the fixed-points.

Proposition 19.18. *The T-fixed points in the image of*

$$\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu}) \hookrightarrow \coprod_{\vartheta\in X_{\bullet}/(W_L,\bullet^{\ell})} \mathrm{L}_{\ell}L/\mathcal{P}^{\vartheta}_{\ell,L} \tag{19.19}$$

are combinatorially given by

$$\alpha\cdot\mathrm{wt}(\mu) := \alpha\cdot W\cdot\{\lambda\in X^{+}_{\bullet}(G)\mid \lambda\leqslant\mu\}. \tag{19.20}$$

Two T-fixed points t^{λ}, $t^{\lambda'}$ of $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$ *lie in the same connected component if and only if λ, λ' are connected by the simple reflections $s_{i,\ell m}$. Equivalently, this is the case iff λ, λ' lie in the same orbit of*

$$(W_{L,\mathrm{aff}},\bullet^{\ell}_{\alpha}) \circlearrowright X_{\bullet}.$$

20. Equivariant K-theory ring of the affine Grassmannian

The aim of this section is to establish an algebro-geometric description of the equivariant K-theory ring $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{C})$, the main focus of this chapter. Our Theorem 20.40 presents this ring in terms of completed global functions on affine schemes of representation-theoretical origin. These are given by gluing together infinitely many deformations to normal cones at quasi-diagonals Z_{ℓ} in two copies of the torus T. We will see that this description can be made very explicit.

Here, let us point out some related literature. The historically first description [KK90] gives K-theory of the affine Grassmannian as a dual to its compactly supported variant. To further describe the ring structure, one can appeal to the GKM description [HHH05]. However, this only realizes $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathbb{C})$ as a subring of an infinite product of Laurent polynomial rings, cut out by an infinite set of equalizer conditions – to get a concrete understanding of the ring, one needs to solve this nontrivial system of congruences. Our result may be interpreted as indirectly doing so.

We first recall the case of cohomology in §20.1 and then the GKM description of K-theory in §20.2. We then move on to define the quasi-diagonals inside $T\times\mathbb{G}_m^{\mathrm{rot}}\times T\,/\!\!/\,W$ in §20.3 and compare them to specialization of cocharacter graphs. We blow up these quasi-diagonals in §20.4, obtaining the deformation to their normal cones. We then explain in §20.5, in purely combinatorial terms, how the completed ring of functions on this deformation maps to the GKM ring of the affine Grassmannian. We then recast this argument in terms of classes of vector bundles in §20.6. In fact, one may view these arguments independently. We then prove our main result, Theorem 20.40, with $\mathbb{C}$-coefficients for semisimple, simply connected group G in §20.7. We discuss its generalizations in §20.8 – this includes arbitrary reductive groups, arbitrary affine flag varieties, and change of equivariance.

20.1. Recollection of cohomology

Let $k=\mathbb{C}$. Before we embark to describe the K-theory, we recall the well-known case of cohomology of $\mathcal{G}\mathrm{r}_G$, following [Gin95; BF08; Yun10]. This argument has two parts: the purely topological computation of non-equivariant cohomology of $\mathcal{G}\mathrm{r}_G$, and the subsequent description of its equivariant version as a deformation to normal cone.

20.1.1. Cohomology of finite flag varieties. As a warmup, recall the equivariant cohomology of finite flag varieties, which will be useful later. Let k be a coefficient ring.

Lemma 20.1. *Let G be a reductive group and $T\leqslant G$ be a maximal torus over k. Let $P_J\leqslant G$ be a parabolic subgroup and $W_2\leqslant W$ corresponding Weyl group. Assume k is a field of $\operatorname{char}\mathsf{k}=0$. Then we have a graded ring isomorphism*

$$\mathcal{O}(\mathfrak{t}\times_{\mathfrak{t}/\!\!/W}\mathfrak{t}\,/\!\!/W_2)\xrightarrow{=}H^\bullet_T(G/P_J;\mathsf{k}),$$

where the grading on the left-hand side is by the weight two action on both copies of $\mathfrak{t}$. More generally, take a pair of Levi subgroups $L_1,L_2\leqslant G$ with Weyl groups $W_1,W_2\leqslant W$. Denote P_J the standard parabolic associated to L_2. Then we have a graded ring isomorphism

$$\mathcal{O}(\mathfrak{t}\,/\!\!/W_1\times_{\mathfrak{t}/\!\!/W}\mathfrak{t}\,/\!\!/W_2)\xrightarrow{=}H^\bullet_{L_1}(G/P_J;\mathsf{k}).$$

Proof. The G-equivariant cohomology is given by

$$H^\bullet_G(G/P_J;\mathsf{k})=H^\bullet(G\backslash G/P_J;\mathsf{k})=H^\bullet(\mathrm{pt}/P_J;\mathsf{k})=H^\bullet(\mathrm{pt}/L_J;\mathsf{k})=H^\bullet_{L_J}(\mathrm{pt};\mathsf{k}),$$

the equivariant base for L_J. Consequently, we have a natural identification

$$\mathcal{O}(\mathfrak{t}\,/\!\!/W_J)\xrightarrow{=}H^\bullet_G(G/P_J;\mathsf{k})$$

compatibly with the structure map to the equivariant base $\mathcal{O}(\mathfrak{t}\,/\!\!/W)\xrightarrow{=}H^\bullet_G(\mathrm{pt};\mathsf{k})$. By equivariant formality, the T-equivariant cohomology is given by base change along $H^\bullet_G(\mathrm{pt};\mathsf{k})\to H^\bullet_T(\mathrm{pt};\mathsf{k})$, so we get the desired

$$\mathcal{O}(\mathfrak{t}\times_{\mathfrak{t}/\!\!/W}\mathfrak{t}\,/\!\!/W_J)\xrightarrow{=}H^\bullet_T(G/P_J;\mathsf{k}).$$

Similarly, L_1-equivariant cohomology is given by base change along $H^\bullet_G(\mathrm{pt};\mathsf{k}) \to H^\bullet_L(\mathrm{pt};\mathsf{k})$, proving the more general statement. □

Remark 20.2. Consequently, adding the equivariance with respect to the trivial loop rotation, we have

$$\mathcal{O}(\mathbb{G}_a^{\mathrm{rot}} \times \mathfrak{t}/\!\!/W_1 \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}/\!\!/W_2) \xrightarrow{=} H^\bullet_{L_1\times\mathbb{G}_m^{\mathrm{rot}}}(G/P_J;\mathsf{k}).$$

In particular, this statement holds after completion $(-)^\wedge_{(0,0)}$ over the origin $(0,0) \in \mathfrak{t}\times\mathbb{G}_a^{\mathrm{rot}}$.

Remark 20.3. In terms of their GKM presentations, this description works as follows. Take the diagonal $\mathfrak{t}/\!\!/W \hookrightarrow \mathfrak{t}/\!\!/W \times \mathfrak{t}/\!\!/W$. Consider its base change $\Delta^J := \mathfrak{t}\times_{\mathfrak{t}/\!\!/W}\mathfrak{t}/\!\!/W_J \hookrightarrow \mathfrak{t}\times\mathfrak{t}/\!\!/W_J$. This defines a closed subscheme parametrizing $\{(x,y)\in\mathfrak{t}\times\mathfrak{t}/\!\!/W_J \mid x = y \mod W\}$. It has irreducible components isomorphic to copies of $\mathfrak{t}$ labeled by the finite set W/W_J. The GKM localization to fixed-points corresponds to restriction to these components.

Remark 20.4. Taking fiber over the origin in the equivariant base, we get the usual description of non-equivariant cohomology, namely

$$H^\bullet(G/P_J;\mathsf{k}) = \frac{\mathsf{k}[t_1,\dots,t_n]}{\mathsf{k}[t_1,\dots,t_n]^W}.$$

Remark 20.5. Upon taking the correct equivariant base, the discussion in this section works for any equivariant multiplicative generalized cohomology theory E_G. In particular, it works for integral K-theory: for any coefficient ring k, for example $\mathsf{k}=\mathbb{Z}$, we have

$$\mathcal{O}(T\times_{T/\!\!/W} T /\!\!/ W_J) = K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(G/P_J;\mathsf{k}).$$

All the above remarks are valid in this setting.

20.1.2. Based loop space description of ordinary cohomology. Assume G is a semisimple, simply connected group over $k=\mathbb{C}$. The non-equivariant cohomology of the affine Grassmannian $\mathcal{G}\mathrm{r}_G$ can be computed by the following classical argument, going back to [Gin95, §1.7].

Firstly, denote K a connected compact real Lie group with complexification $G(\mathbb{C})$. Then $K \leqslant G(\mathbb{C})$ is a maximal compact subgroup. Denote ΩK its based loop space. It is then a nontrivial fact that we can identify the homotopy type of the affine Grassmannian as $\mathcal{G}\mathrm{r}_G(\mathbb{C}) \simeq \Omega K$, see for example [YZ11, Introduction], [Gin95, §1.2], [HHH05, Remark 5.1], [Zhu17b, §1.6].

We are thus left to understand the graded ring $H^\bullet(\Omega K)$. Firstly, the cohomology $H^\bullet(K)$ is understood by [Bor53]. In particular, with coefficients in a field k of $\operatorname{char}\mathsf{k}=0$, the cohomology ring $H^\bullet(K;\mathsf{k}) = \Lambda_{\mathsf{k}}[x_1,\dots,x_n]$ is the exterior algebra on finitely many generators x_i, $i=1,\dots,d$ by [Bor53, Proposition 7.2]. Furthermore, $d=\operatorname{rank} G$ and $\deg x_i = a_i - 1$ where $a_1,\dots a_d$ are the exponents of G.

Consequently, the rational homotopy type of K is that of a product of odd dimensional spheres S^{a_i-1}. By the compatibility of the based loop space $\Omega(-)$ with Eilenberg-MacLane spaces, the rational homotopy type of ΩK is given by a product of even dimensional spheres S^{a_i-2}, and we deduce

$$H^\bullet(\Omega K;\mathsf{k}) = \mathsf{k}[y_1,\dots,y_n], \qquad \deg y_i = a_i - 2 \tag{20.1}$$

In a more hands-on way, the passage from K to ΩK can be also done directly through the path space fibration $\Omega K \to PK \to K$ and Serre's spectral sequence; the elements x_i are transgressions of y_i. See [Hat04, Example 1.16].

20.1.3. Bezrukavnikov–Finkelberg description of equivariant cohomology. We recall the description of the equivariant cohomology of affine flag varieties over $k = \mathbb{C}$ via certain deformations to normal cones, following [BF08; Yun10]. This crucially uses the topological input §20.1.2. We assume the coefficient ring k is a field of $\operatorname{char} \mathsf{k} = 0$.

Let us sketch this argument. The equivariant cohomology of $\mathcal{F}\mathrm{l}^J = \mathrm{L}^+G/\mathcal{P}_J$ is a module over the equivariant base via

$$H^\bullet_{G\times\mathbb{G}_m^{\mathrm{rot}}\times L_J}(\mathrm{pt};\mathsf{k}) \to H^\bullet_{\mathrm{L}^+G\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}^J;\mathsf{k}). \tag{20.2}$$

The map (20.2) is easily seen to be injective. The source is given by space $\mathcal{O}(\mathfrak{t}/\!\!/W \times \mathbb{G}_a^{\mathrm{rot}} \times \mathfrak{t}/\!\!/W_L)$. In fact, both sides are abstractly isomorphic polynomial rings.

However, (20.2) is not surjective. To correct for the missing functions, one needs to deform to the normal cone of the image of $\mathfrak{t}$ under the diagonal projection $\mathfrak{t} \to \mathfrak{t}/\!\!/W \times \mathfrak{t}/\!\!/W_L$, in other words to the normal cone of the closed subscheme $\mathfrak{t}/\!\!/W \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}/\!\!/W_L$. One directly checks that (20.2) factors injectively through this deformation. Finally, to prove surjectivity, one notes that (20.2) is a graded map with respect to the natural gradings on both sides. The graded pieces are finite-dimensional, so it is enough to compare their Hilbert series. These are easily computable from the deformation description for the left-hand side, and (20.1) together with equivariant formality for the right-hand side.

To state the result, consider Levi subgroups L_1, L_2 of G, with associated Weyl groups $W_1, W_2 \leqslant W$. Let $\mathcal{P}_J$ be the standard parahoric sugroup associated to L_2. Conisder the corresponding affine flag variety $\mathcal{F}\mathrm{l}^J = \mathrm{L}G/\mathcal{P}_J$, with the natural left action $\mathrm{L}^+L_1 \rtimes \mathbb{G}_m^{\mathrm{rot}} \circlearrowright \mathcal{F}\mathrm{l}^J$.

Proposition 20.6. *The $(\mathrm{L}^+L_1 \times \mathbb{G}_m^{\mathrm{rot}})$-equivariant cohomology ring of the affine flag variety $\mathcal{F}\mathrm{l}^J$ is given, together with its structure map, by*

$$\begin{tikzcd}
\mathcal{O}(N(\mathfrak{t}/\!\!/W_1 \times_{\mathfrak{t}/\!\!/W} \mathfrak{t}/\!\!/W_2, \mathfrak{t}/\!\!/W_1 \times \mathfrak{t}/\!\!/W_2)) \arrow[r, equal] & H_{\mathrm{L}^+L_1\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}^J;\mathsf{k}) \\
\mathcal{O}(\mathfrak{t}/\!\!/W_1 \times \mathbb{G}_a^{\mathrm{rot}} \times \mathfrak{t}/\!\!/W_2) \arrow[r, equal] \arrow[u, hook] & H_{L_1\times\mathbb{G}_m^{\mathrm{rot}}\times L_2}(\mathrm{pt};\mathsf{k}) \arrow[u, hook]
\end{tikzcd} \tag{20.3}$$

Proof. The case of $\mathrm{L}^+G \rtimes \mathbb{G}_m^{\mathrm{rot}} \circlearrowright \mathcal{G}\mathrm{r}_G$ is covered in [BF08, Theorem 1]. From there, the general case follows by the argument [Yun10, Theorem 4.3]. Strictly speaking, [Yun10] treats only the full flag variety with the Iwahori action, but the argument works in general. $\square$

Remark 20.7 (Informal GKM localization)**.** Working equivariantly with respect to $T \times \mathbb{G}_m^{\mathrm{rot}}$, the above identification can be unraveled in GKM terms. Let us give an informal account in the complex analytic setting now; we will give a detailed discussion in the K-theoretical context below.

On the complex analytic level, consider $\mathfrak{t}(\mathbb{C}) /\!\!/ W_{\mathrm{aff}} \hookrightarrow \mathfrak{t}(\mathbb{C}) /\!\!/ W_{\mathrm{aff}} \times \mathbb{G}_a \times \mathfrak{t}(\mathbb{C}) /\!\!/ W_{\mathrm{aff}}$. By pullback, we obtain the subspace $\overline{\boldsymbol{\Gamma}}(\mathbb{C}) \hookrightarrow \mathfrak{t}(\mathbb{C}) \times \mathbb{G}_a(\mathbb{C}) \times \mathfrak{t}(\mathbb{C}) /\!\!/ W_{\mathrm{aff}}$. Its irreducible components are copies of $\mathfrak{t}(\mathbb{C})$, labeled by the cocharacter lattice $X_\bullet$. Take the strict transform $\overline{\boldsymbol{\Upsilon}}(\mathbb{C})$ of $\overline{\boldsymbol{\Gamma}}(\mathbb{C})$ in the blowup $N(\mathfrak{t}(\mathbb{C}) /\!\!/ W_1 \times_{\mathfrak{t}(\mathbb{C})/\!\!/W} \mathfrak{t}(\mathbb{C}) /\!\!/ W_2, \mathfrak{t}(\mathbb{C}) /\!\!/ W_1 \times \mathfrak{t}(\mathbb{C}) /\!\!/ W_2)$. Its irreducible components are still copies of $\mathfrak{t}(\mathbb{C})$ labeled by $X_\bullet$, and the localization corresponds to restriction of functions to them. Also see [BF08, §3.2].

20.2. Goresky–Kottwitz–McPherson description of equivariant K-theory

We record some standard formality results on equivariant topological K-theory of $\mathcal{G}\mathrm{r}_G$, together with its well-known GKM description.

20.2.1. Limit along affine Schubert varieties. Recalling §17.1.2, consider the affine Grassmannian $\mathcal{G}\mathrm{r}_G$ with its $T \times \mathbb{G}_m^{\mathrm{rot}}$-equivariant Iwahori paving by affine cells. Consider the induced skeleta $\mathrm{sk}_i = \mathrm{sk}_i\,\mathcal{G}\mathrm{r}$, $i \in \mathbb{N}_0$. By induction, $K^{\mathrm{top},\bullet}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathrm{sk}_i)$ is a free module over $K^{\mathrm{top},\bullet}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathrm{pt})$; it is 2-periodic and concentrated in even degrees; the affine cells give a basis of the degree zero part. The restriction maps induced by the skeletal inclusions $\mathrm{sk}_i \hookrightarrow \mathrm{sk}_{i'}$ are thus surjective; the inverse system in particular satisfies the Mittag–Leffler condition. This paving restricts to each affine Schubert variety $\mathcal{G}\mathrm{r}_{\leqslant\mu}$, where the same results hold. We deduce the following.

Proposition 20.8. *We have an identification of rings*

$$K^{\mathrm{top},*}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}) = \lim_{\mu\in X_\bullet^+} K^{\mathrm{top},*}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu}).$$

All terms are concentrated in the even part.

Since we do not loose anything by focusing on the degree zero $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r})$ of the K-theory ring, we do so from now on.

20.2.2. GKM ring of affine Grassmannian. The action of the extended torus $T \times \mathbb{G}_m^{\mathrm{rot}}$ on $\mathcal{G}\mathrm{r}$ satisfies the GKM assumptions for integral topological K-theory; it thus may be described as an explicit subring of K-theory of the fixed points [HHH05].

Notation 20.9. Let k be a base ring. The GKM ring of the affine Grassmannian is the subring $\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}) \subseteq \prod_{\lambda\in X_\bullet} \mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}})$ is given by

$$\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}) = \Big\{(f_\lambda) \in \prod_{\lambda\in X_\bullet} \mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}) \;\Big|\; f_\lambda \equiv f_{s_{\beta+m\hbar}\cdot\lambda} \mod (1-\beta q^m), \forall\lambda, \forall s_{\beta+m\hbar}\Big\},$$

where the congruence conditions run over all cocharacters $\lambda \in X_\bullet$ and all affine reflections $s_{\beta+m\hbar}$ associated to affine roots $\beta + m\hbar \in \Phi^\bullet \oplus \mathbb{Z}\,\hbar$.

Proposition 20.10. *Over* $\mathsf{k} = \mathbb{Z}$, *we have*

$$K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}) = \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}).$$

Proof. The action of $T \times \mathbb{G}_m^{\mathrm{rot}} \circlearrowright \mathcal{G}\mathrm{r}_G$ satisfies the GKM assumptions for integral topological K-theory; this is the case for all Kac–Moody partial flag varieties by with respect to the natural torus [HHH05, §5]; let us sketch some details.

For $\mathcal{G}\mathrm{r}_G$, the fixed points of $T \times \mathbb{G}_m^{\mathrm{rot}}$ are given by $(t^\lambda \mid \lambda \in X_\bullet)$ and we claim

$$\begin{tikzcd}
\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}) \arrow[r, hook] \arrow[d, phantom, "\parallel"] & \prod_{\lambda\in X_\bullet} \mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}) \arrow[d, phantom, "\parallel"] \\
K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}) \arrow[r, hook] & \prod_{\lambda\in X_\bullet} K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(t^\lambda;\mathsf{k}).
\end{tikzcd}$$

Here, the Iwahori orbits on $\mathcal{G}r_G$ provide a $T \times \mathbb{G}_m^{\mathrm{rot}}$-equivariant affine paving. Via the construction from Remark 17.12, the Assumptions 17.10 are satisfied by [HHH05, §5, p.211]. The ring $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_G;\mathsf{k})$ thus has a GKM description by Proposition 17.11. Unraveling this gives precisely the ring from Notation 20.9 as in [HHH05]. □

Remark 20.11. Also see [BASV22, §2.2.3], [Par17], [Sit24, Appendix A], [Shi14] for the parallel overview of complexified cohomology of $\mathcal{G}r_G$ in our situation.

Remark 20.12. Given an affine Schubert variety $\mathcal{G}r_{\leqslant\mu}$, the corresponding GKM ring is given by passing to the subset $\mathrm{wt}(\mu) \subseteq X_\bullet$ as

$$\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_{\leqslant\mu};\mathsf{k}) = \Big\{(f_\lambda) \in \prod_{\lambda\in\mathrm{wt}(\mu)} \mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}) \quad\Big|\quad f_\lambda \equiv f_{\lambda'} \mod (1-\beta q^m)\Big\},$$

where the conditions run over all pairs $\lambda, \lambda' \in \mathrm{wt}(\mu)$ such that $\lambda' = s_{\beta+m\hbar}\cdot\lambda$ for a simple reflection associated to some affine root $\beta + m\hbar \in \Phi^\bullet \oplus \mathbb{Z}\,\hbar$. We again have

$$K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_{\leqslant\mu};\mathsf{k}) = \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_{\leqslant\mu};\mathsf{k}).$$

The restriction maps from $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_G;\mathsf{k})$ on the left correspond to restriction along $\mathrm{wt}(\mu) \subseteq X_\bullet$ on the right. Note that $\mathcal{G}r_{\leqslant\mu}$ are GKM spaces in the more standard sense – the number of fixed points is finite.

Our aim is to describe this K-theory ring, in other words explicitly solve the system of congruences defining $\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r;\mathsf{k})$. However, it is nontrivial to do so directly, and we need to apply other topological tricks. In any case, our Theorem 20.40 below may be regarded as a solution to this purely combinatorial problem.

20.3. Quasi-diagonals at roots of unity

20.3.1. Power ℓ map. Let k be any coefficient ring. Let G be a split, semisimple, simply connected group over k with maximal torus T and Weyl group W. Consider $T \times T$; we denote respectively $t_1, \dots, t_n$ and $s_1, \dots, s_n$ coordinates on the first and second copy. Denote

$$(-)^\ell : T \to T, \qquad t \mapsto t^\ell$$

the ℓ-th power map, coming from the group structure on T. In coordinates, it sends $t_i \mapsto t_i^\ell$. Since the W-action on T is by group automorphisms, the ℓ-th power map is W-equivariant. It thus induces a map

$$(-)^\ell : T /\!\!/ W \to T /\!\!/ W.$$

The corresponding map $\mathcal{O}(T /\!\!/ W) \to \mathcal{O}(T /\!\!/ W)$ on rings of functions is induced by the substitution $f(t_1,\dots,t_n) \mapsto f(t_1^\ell,\dots,t_n^\ell)$. Similarly for the quotient by any subgroup of W.

20.3.2. Quasi-diagonals at roots of unity. For any $\ell \in \mathbb{N}$, we define the corresponding *quasi-diagonal* by the pullback

$$\begin{array}{ccc} Z_\ell^{T/\!\!/W,T/\!\!/W} & \longrightarrow & T /\!\!/ W \\ \big\downarrow & & \big\downarrow{\scriptstyle (-)^\ell} \\ T /\!\!/ W & \xrightarrow{(-)^\ell} & T /\!\!/ W \end{array}$$

It has trivial derived structure by flatness of $(-)^\ell$. It is a closed subscheme of $T /\!\!/ W \times T /\!\!/ W$ via the pullback square

$$\begin{tikzcd} Z_\ell^{T/\!\!/W, T/\!\!/W} \arrow[r, hook] \arrow[d] & T /\!\!/ W \times T /\!\!/ W \arrow[d, "(-)^\ell \times (-)^\ell"] \\ T /\!\!/ W \arrow[r, hook, "\Delta"] & T /\!\!/ W \times T /\!\!/ W \end{tikzcd}$$

On its functor of points, it parametrizes

$$Z_\ell^{T/\!\!/W, T/\!\!/W} : R \mapsto \{(x, y) \in (T /\!\!/ W \times T /\!\!/ W)(R) \mid x^\ell = y^\ell\}.$$

20.3.3. Variants. More generally, for any two subgroups of $W_1, W_2 \leqslant W$, we define the variant $Z_\ell^{W_1,W_2} = Z_\ell^{T/\!\!/W_1, T/\!\!/W_2}$ by either of the pullback squares

$$\begin{tikzcd} Z_\ell^{T/\!\!/W_1, T/\!\!/W_2} \arrow[r] \arrow[d] & T /\!\!/ W_2 \arrow[d, "(-)^\ell"] \\ T /\!\!/ W_1 \arrow[r, "(-)^\ell"] & T /\!\!/ W \end{tikzcd} \qquad \begin{tikzcd} Z_\ell^{T/\!\!/W_1, T/\!\!/W_2} \arrow[r, hook] \arrow[d] & T /\!\!/ W_1 \times T /\!\!/ W_2 \arrow[d, "(-)^\ell \times (-)^\ell"] \\ T /\!\!/ W \arrow[r, hook, "\Delta"] & T /\!\!/ W \times T /\!\!/ W. \end{tikzcd}$$

It is a closed subscheme of $T /\!\!/ W_1 \times T /\!\!/ W_2$, obtained by the base change of $Z_\ell^{T/\!\!/W, T/\!\!/W}$ along the faithfully flat map $T /\!\!/ W_1 \times T /\!\!/ W_2 \to T /\!\!/ W \times T /\!\!/ W$. On its functor of points, it parametrizes

$$Z_\ell^{T/\!\!/W_1 \times T/\!\!/W_2}(R) = \{(x, y) \in (T /\!\!/ W_1 \times T /\!\!/ W_2)(R) \mid x^\ell W = y^\ell W\}.$$

Notation 20.13. In particular, we will use the following shorter notation for two special cases:

$$\begin{aligned} Z'_\ell &:= Z_\ell^{T,T} \hookrightarrow T \times T, \\ Z_\ell &:= Z_\ell^{T, T/\!\!/W} \hookrightarrow T \times T /\!\!/ W. \end{aligned}$$

Notation 20.14. Furthermore, given $\zeta \in \mu_\infty$ of primitive order ℓ, we will also use the notation $Z_\zeta = Z_\ell \hookrightarrow T \times T /\!\!/ W$. Given any point $x \in T$ in the first copy of the torus, we denote its fiber over x by $Z_{(x,\zeta)} = Z_{(x,\ell)} \hookrightarrow T /\!\!/ W$.

20.3.4. Relationship to specializations of Γ. Assume now k is an algebraically closed field of characteristic zero. Given $\zeta \in \mu_\infty$ of primitive order ℓ, we have just defined the subscheme Z_ζ and its fibers $Z_{(x,\zeta)}$. These are fully described by specializations of the sub-ind scheme $\overline{\Gamma} \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W$ from §18.2.

Lemma 20.15. *Let $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$. Suppose ζ is a root of unity of primitive order ℓ. On the underlying points, we have*

$$|Z'_\ell| = |\Gamma_\zeta|, \qquad |Z'_{(x,\ell)}| = |\Gamma_{(x,\zeta)}|, \qquad |Z_\ell| = |\overline{\Gamma}_\zeta|, \qquad |Z_{(x,\ell)}| = |\overline{\Gamma}_{(x,\zeta)}|.$$

Proof. It is enough to treat the Z'_ℓ-variants, since the Z_ℓ-variant follows by taking quatient by W. Moreover, it is enough to focus on the case of $Z'_{(x,\ell)}$ (which is a fiberwise refinement of Z'_ℓ). Over any (x, ζ), both sets have a transitive action of W_{ext} and base point with the same stabilizer. $\square$

Remark 20.16. More explicitly, we may just compute inside $T \times T$ as follows

$$\begin{aligned}|Z'_\ell| &= \{(x,y) \mid \exists w \in W : x^\ell = w(y^\ell)\}\\ &= \{(x,y) \mid \exists w \in W : (x \cdot w(y)^{-1})^\ell = 1\}\\ &= \{(x,y) \mid \exists \lambda \in X_\bullet, \exists w \in W : x \cdot w(y)^{-1} = \lambda(\zeta)\}\\ &= |\Gamma_\zeta|\end{aligned}$$

The case of $Z'_{(x,\ell)}$ is the same computation with fixed x.

Remark 20.17. The statement of Lemma 20.15 holds on the level of reduced schemes: it identifies $Z_\ell = \overline{\Gamma}_\zeta^{\mathrm{red}}$ and $Z_{(x,\ell)} = \overline{\Gamma}_{(x,\zeta)}^{\mathrm{red}}$.

However, while Z_ℓ is reduced by definition, the fiber $\overline{\Gamma}_\zeta$ will acquire some ind-unreduced structure from the q-direction. In fact, one may heuristically think about the normal cone to Z_ℓ as a counterpart of this ind-unreduced structure.

20.3.5. Generators of the ideal sheaf of Z_ℓ. Let again k be a general coefficient ring. Consider the ring $\mathcal{O}(T /\!\!/ W)$ over k. It naturally identifies with the representation ring $R(G; \mathrm{k})$. Denote $e_1, \dots, e_n \in \mathcal{O}(T /\!\!/ W)$ the elements corresponding to the fundamental representations of G. Since G is semisimple and simply connected, these freely generate $\mathcal{O}(T /\!\!/ W)$.

Lemma 20.18. *The ideal sheaf $\mathcal{I}_{Z_\ell}$ of the closed subscheme $Z_\ell^{T/\!\!/W \times T/\!\!/W} \hookrightarrow T /\!\!/ W \times T /\!\!/ W$ is generated by*

$$f_i = e_i(t_1^\ell, \dots, t_n^\ell) - e_i(s_1^\ell, \dots, s_n^\ell), \qquad i = 1, \dots, n$$

The same works for the ideal sheaf $\mathcal{I}_{Z_\ell}^{T/\!\!/W_1, T/\!\!/W_2}$ of $Z_\ell^{T/\!\!/W_1, T/\!\!/W_2} \hookrightarrow T /\!\!/ W_1 \times T /\!\!/ W_2$ for any $W_1, W_2 \leqslant W$.

Proof. By definition, Z_ℓ fits into the pullback square

$$\begin{tikzcd} Z_\ell \arrow[r, hook] \arrow[d] & T /\!\!/ W \times T /\!\!/ W \arrow[d, "(-)^\ell \times (-)^\ell"] \\ T /\!\!/ W \arrow[r, hook, "\Delta"] & T /\!\!/ W \times T /\!\!/ W \end{tikzcd} \tag{20.4}$$

Note that $T /\!\!/ W$ is smooth (in fact an affine space) and the map $(-)^\ell : T /\!\!/ W \to T /\!\!/ W$ is finite (which is obvious before taking the quotient). The same is true for $(-)^\ell \times (-)^\ell : T /\!\!/ W \times T /\!\!/ W \to T /\!\!/ W \times T /\!\!/ W$, which is thus flat by miracle flatness [Sta, Tag 00R4]. Since it is surjective, it is in fact faithfully flat.

The closed immersions given by the horizontal arrows in (20.4) induce short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_{Z_\ell} \arrow[r] & \mathcal{O}_{T/\!\!/W \times T/\!\!/W} \arrow[r] & \mathcal{O}_{Z_\ell} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{I}_{T/\!\!/W} \arrow[r] \arrow[u] & \mathcal{O}_{T/\!\!/W \times T/\!\!/W} \arrow[r] \arrow[u] & \mathcal{O}_{T/\!\!/W} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Now, the upper sequence is given by base change of the bottom one along the homomorphism $(-)^\ell \times (-)^\ell : \mathcal{O}_{T/\!\!/W \times T/\!\!/W} \hookrightarrow \mathcal{O}_{T/\!\!/W \times T/\!\!/W}$. indeed, this is true for the right-hand term by

(20.4) and tautological for the middle term, hence true for the left-hand term by flatness. In coordinates of $T \times T$, this map is given by $t_i \mapsto t_i^\ell$ and $s_i \mapsto s_i^\ell$ for any $i = 1, \dots, n$.

Now, the ideal sheaf $\mathcal{I}_{T /\!\!/ W}$ of the diagonal in $T /\!\!/ W \times T /\!\!/ W$ is generated by the elements $f_j = e_j(t_1, \dots, t_n) - e_j(s_1, \dots, s_n)$ for $j = 1, \dots, n$. (Indeed, these are just affine spaces with coordinates e_j.) Therefore, the ideal sheaf $\mathcal{I}_{Z_\ell}$ is generated by their images

$$\mathcal{I}_{Z_\ell} = \langle e_j(t_1^\ell, \dots, t_n^\ell) - e_j(s_1^\ell, \dots, s_n^\ell) \mid j = 1, \dots, n \rangle.$$

as an ideal inside $\mathcal{O}_{T /\!\!/ W \times T /\!\!/ W}$.

The general case follows, since $Z_\ell^{T /\!\!/ W_1, T /\!\!/ W_2} \hookrightarrow T /\!\!/ W_1 \times T /\!\!/ W_2$ is given by the base change of $Z_\ell^{T /\!\!/ W, T /\!\!/ W} \hookrightarrow T /\!\!/ W \times T /\!\!/ W$ along the faithfully flat map $T /\!\!/ W_1 \times T /\!\!/ W_2 \to T /\!\!/ W \times T /\!\!/ W$. □

20.4. Deformations of quasi-diagonals to normal cones

20.4.1. Deforming to the normal cone at roots of unity. Recall the general notation for deformations to normal cones and affine blowups from §16.2. We use it in the following case. To simplify the manipulation of roots of unity, assume k is an algebraically closed field of $\operatorname{char} \mathsf{k} = 0$.

Definition 20.19. Take the scheme $T \times T /\!\!/ W \times \mathbb{G}_m^{\mathrm{rot}}$, viewed as a family over $\mathbb{G}_m^{\mathrm{rot}} \hookrightarrow \mathbb{A}^1$. Consider the family of subschemes $\mathcal{Z} = (Z_\zeta \mid \zeta \in \mu_\infty)$ of its fibers over varying $\zeta \in \mu_\infty \hookrightarrow \mathbb{G}_m^{\mathrm{rot}}$. We consider the affine blowup of this family

$$\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W) := \mathrm{Bl}^\circ_{\mathcal{Z}}(T \times T /\!\!/ W \times \mathbb{G}_m^{\mathrm{rot}}).$$

In terms of deformations to normal cones, we can restate the above as follows. Start with the scheme $T \times T /\!\!/ W$. Then $\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)$ is the family over $\mathbb{G}_m^{\mathrm{rot}}$ which is the constant family with fiber $T \times T /\!\!/ W$ away from roots of unity and the deformation to the normal cone of $Z_\ell \hookrightarrow T \times T /\!\!/ W$ on each (formal) neighborhood of each root of unity ζ of primitive order $\ell \in \mathbb{N}$. In other words, we have pullback squares

$$\begin{tikzcd}
C(Z_\zeta, T \times T /\!\!/ W) \arrow[r] \arrow[d] & \mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W) \arrow[d] & T \times T /\!\!/ W \times (\mathbb{G}_m^{\mathrm{rot}} - \mu_\infty) \arrow[l] \arrow[d] \\
\zeta \arrow[r] & \mathbb{G}_m^{\mathrm{rot}} & \mathbb{G}_m^{\mathrm{rot}} - \mu_\infty. \arrow[l]
\end{tikzcd}$$

Our family $\mathcal{N}_{\mu_\infty}(Z, T \times T /\!\!/ W)$ is given the usual Rees constructions for the filtered algebras $(\mathcal{O}_{T \times T /\!\!/ W}, \mathcal{I}^\bullet_{Z_\ell})$ at each ζ_ℓ glued together along $\mathbb{G}_m^{\mathrm{rot}} - \mu_\infty$, where all of them naturally trivialize.

20.4.2. Explicit description of the deformation. Since $T \times T /\!\!/ W$ is affine, the scheme $\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)$ is affine with ring of functions

$$\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) = \mathcal{O}(T \times (T /\!\!/ W))[q^{\pm 1}][\tfrac{f}{q-\zeta} \mid (\zeta, f) \in \Psi]$$

where Ψ is the set of all pairs $(\zeta, f) \in \mathsf{k}^\times \times \mathcal{O}(T \times (T /\!\!/ W))$ with $\zeta^\ell = 1$ and $f \in \mathcal{I}_{Z_\ell}$ for a common $\ell \in \mathbb{N}$. Note that this algebra is *not* finitely generated over k. Alternatively, grouping all ℓ-torsion roots of unity together, we can rewrite it as follows.

Lemma 20.20. *We have*

$$\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) = \mathcal{O}(T \times (T /\!\!/ W))[q^{\pm 1}][\tfrac{f}{1-q^\ell} \mid \ell \in \mathbb{N},\ f \in \mathcal{I}_{Z_\ell}].$$

Proof. Recall $\mathsf{k} = \bar{\mathsf{k}}$ of $\operatorname{char} \mathsf{k} = 0$. Write $(q^\ell - 1) = (q - \zeta_0) \cdots (q - \zeta_{\ell-1})$, $\zeta_i \in \mu_\ell$. Clearly, each $\frac{f}{\zeta - q} = \frac{f}{1-q^\ell} \cdot \prod_{i' \neq i}(q - \zeta_{i'})$ is generated by $\frac{f}{1-q^\ell}$. On the other hand, all the monomials $(q - \zeta_i)$ are pairwise coprime in $\mathsf{k}[q^{\pm 1}]$. By the Chinese remainder theorem, we may express $1 = \sum_{i=0}^{\ell-1} h_i \cdot \prod_{i' \neq i}(q - \zeta_{i'})$ for some $h_i \in \mathsf{k}[q^{\pm 1}]$, hence $1 = \sum_{i=0}^{\ell-1} h_i \cdot (q^\ell - 1)\frac{1}{q - \zeta_i}$. Consequently, $\frac{f}{1-q^\ell} = \sum_{i=0}^{\ell-1} h_i(q^\ell - 1)\frac{f}{\zeta - q}$ is generated by the elements $\frac{f}{\zeta - q}$. □

Remark 20.21. For any $\ell \in \mathbb{N}$, we denote the corresponding cyclotomic polynomial by $\sigma_\ell(q) \in \mathbb{Z}[q^{\pm 1}]$. Since we work over an algebraically closed field k of $\operatorname{char} \mathsf{k} = 0$, we also have

$$\mathcal{O}(T \times (T /\!\!/ W))[q^{\pm 1}][\tfrac{f}{\sigma_\ell(q)} \mid \ell \in \mathbb{N},\ f \in \mathcal{I}_{Z_\ell}]$$

by the same token as the lemma above.

20.4.3. Fundamental Bezrukavnikov classes.

To generate the functions on the deformation, the following distinguished elements suffice. Recall that we are assuming G simply connected.

Definition 20.22. We define the *fundamental Bezrukavnikov classes* as

$$b_{j,\ell} := \tfrac{f_j}{1-q^\ell} := \tfrac{e_j(t_1^\ell, \ldots, t_n^\ell) - e_j(s_1^\ell, \ldots, s_n^\ell)}{1-q^\ell}, \qquad \forall j \in \{1, \ldots, n\}, \forall \ell \in \mathbb{N}\,.$$

Lemma 20.23. *To generate $\mathcal{O}(\mathcal{N}(\mathcal{Z}, T /\!\!/ W \times T /\!\!/ W))$ as an algebra over $\mathcal{O}(T /\!\!/ W \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W)$, it suffices to add the infinite set of ring generators*

$$(b_{j,\ell} \mid \forall j \in \{1, \ldots, n\}, \forall \ell \in \mathbb{N})\,.$$

Proof. By definition, for each $\zeta \in \mu_\infty$ and every $f \in \mathcal{I}_{Z_\zeta}$, we need to add the generator $\frac{f}{q-\zeta}$. For fixed ζ, it is enough to only consider $b_{j,\zeta} := \frac{f_j}{q-\zeta}$ for $f_j = e_j(t_1^\ell, \ldots, t_n^\ell) - e_j(s_1^\ell, \ldots, s_n^\ell)$, because $f_1, \ldots, f_j$ generate $\mathcal{I}_{Z_\ell}$ by Lemma 20.18.

By the same coprimality argument as in Lemma 20.20, we may group all the ℓ-torsion roots of unity together – the elements $b_{j,\zeta}$ and $b_{j,\ell}$ generate the same ring. □

Remark 20.24. Consider the base changed variant $Z_\ell^{T/\!\!/W_1, T/\!\!/W_2} \hookrightarrow T /\!\!/ W_1 \times T /\!\!/ W_2$. The same set $(b_{j,\ell})$ then generates $\mathcal{O}(\mathcal{N}(Z_\ell^{T/\!\!/W_1, T/\!\!/W_2}, T /\!\!/ W_1 \times T /\!\!/ W_2))$ over $\mathcal{O}(T /\!\!/ W_1 \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W_2)$: this follows by base change along the flat map $T /\!\!/ W_1 \times T /\!\!/ W_2 \to T /\!\!/ W \times T /\!\!/ W$.

20.4.4. Completions of the deformation over a root of unity.

The following lemma is the algebraic analogue of a local comparison between equivariant K-theory and cohomology of fixed points.

Lemma 20.25. *Let $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ with $\zeta \in \mu_\infty$. Then*

$$N_{q=\zeta}(Z_\ell, T \times T /\!\!/ W))^\wedge_{(x,\zeta)} = \coprod_{\vartheta \in |Z_{(x,\zeta)}|} N_{h=0}(\mathfrak{t} \times_{\mathfrak{t}/\!\!/W_{[x,\zeta]}} \mathfrak{t} /\!\!/ W_\vartheta, \mathfrak{t} \times \mathfrak{t} /\!\!/ W_\vartheta)^\wedge_{(0,0)}$$

Proof. We first consider the case of $Z'_\zeta \hookrightarrow T \times T$. Recall we have already identified $|Z'_\zeta| = |\overline{\Gamma}_\zeta|$ in Lemma 20.15 and similarly for its fibers $|Z'_{(x,\zeta)}|$.

Decompose $Z'_{(x,\zeta)}$ into its connected components ${Z'}^{\vartheta}_{(x,\zeta)}$ labeled by $\vartheta \in |Z'_{(x,\ell)}|$. We make the following identifications:

$$\begin{aligned} N_{q=\zeta}(Z'_\zeta, T \times T))^{\wedge}_{(x,\zeta)} &= \coprod_{\vartheta \in |Z'_{(x,\zeta)}|} N_{q=\zeta}({Z'}^{\vartheta}_{(x,\zeta)}, T \times T)^{\wedge}_{(x,\zeta)} \\ &= \coprod_{\vartheta \in |Z'_{(x,\zeta)}|} N_{q=1}(T \times_{T /\!\!/ W_{[x,\zeta]}} T, T \times T)^{\wedge}_{(1,1)} \\ &= \coprod_{\vartheta \in |Z'_{(x,\zeta)}|} N_{h=0}(\mathfrak{t} \times_{\mathfrak{t} /\!\!/ W_{[x,\zeta]}} \mathfrak{t}, \mathfrak{t} \times \mathfrak{t})^{\wedge}_{(0,0)}. \end{aligned}$$

The first line holds since the deformation to the normal cone is Zariski local and compatible with connected components, so we only need to study each of ${Z'}^{\vartheta}_{(x,\zeta)} \hookrightarrow T \times T$ separately.

Now, the irreducible components of $|Z'_\ell|$ are labeled by $W_{\mathrm{ext}}/\ell X_\bullet$. Each of these connected components is a copy of T. On the other hand, the irreducible components of its fiber $|Z'_{(x,\zeta)}|$ are labeled by the set $W_{\mathrm{ext}}/\operatorname{Stab}_{(x,\zeta)}$. By (18.6), (18.7) the collisions of the connected components of Z'_ℓ over x are described by the $W_{[x,\zeta]}$-torsor of finite sets

$$W_{[x,\zeta]} \circlearrowright W_{\mathrm{ext}}/\ell X_\bullet \twoheadrightarrow W_{\mathrm{ext}}/\operatorname{Stab}_{(x,\zeta)}.$$

The colliding copies of T at ϑ are labeled by $W_{[x,\zeta]}$, giving the second identification.

The third line is the standard identification of formal neighborhood of the identity in a reductive group with its Lie algebra (also see [BASV22, (2.17)]). This concludes the discussion for Z'_ℓ.

Now, consider the action of W on the left hand side, induced from the tautological action on the second copy of T. It translates to the right-hand side as follows. Firstly, it permutes the connected components: given a component labeled by $\vartheta \in |Z'_{(x,\zeta)}|$, its orbit under $W \circlearrowright |Z'_{(x,\zeta)}|$ identifies with W/W_ϑ. This cuts down the labeling set to $|Z_{(x,\zeta)}|$ as in Corollary 18.6. Furthermore, fixing any such ϑ, we are left with the natural action of W_ϑ on the closed immersion ${Z'}^{\vartheta}_{(x,\zeta)} \hookrightarrow T \times T$.

Altogether, taking the quotient by W on both sides gives the desired equality

$$N_{q=\zeta}(Z_\ell, T \times T /\!\!/ W))^{\wedge}_{(x,\zeta)} = \coprod_{\vartheta \in |Z_{(x,\zeta)}|} N_{h=0}(\mathfrak{t} \times_{\mathfrak{t} /\!\!/ W_{[x,\zeta]}} \mathfrak{t} /\!\!/ W_\vartheta, \mathfrak{t} \times \mathfrak{t} /\!\!/ W_\vartheta)^{\wedge}_{(0,0)}.$$

∎

Remark 20.26. More generally, the final step of the above prove shows that for $(x,\zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ with $\zeta \in \mu_\infty$ and any pair of subgroups $W_1, W_2 \leqslant W$ we have

$$N_{q=\zeta}(Z_\ell^{W_1,W_2}; T /\!\!/ W_1 \times T /\!\!/ W_2))^{\wedge}_{(x,\zeta)} = \coprod_{\vartheta \in |Z^{W_1,W_2}_{(x,\zeta)}|} N_{h=0}(\mathfrak{t} /\!\!/ W_1 \times_{\mathfrak{t} /\!\!/ W_{[x,\zeta]}} \mathfrak{t} /\!\!/ W_{2,\vartheta}, \mathfrak{t} /\!\!/ W_1 \times \mathfrak{t} /\!\!/ W_{2,\vartheta})^{\wedge}_{(0,0)}.$$

20.4.5. Strict transforms of cocharacter graphs. Finally, it is quite useful to keep track of how the cocharacter graphs transform in the affine blowup.

Notation 20.27. Denote

$$\overline{\Upsilon} := \overline{\Gamma}^{\mathrm{st}} \hookrightarrow \mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W) = \mathrm{Bl}^\circ_{\mathcal{Z}}(T \times T /\!\!/ W \times \mathbb{G}_m^{\mathrm{rot}})$$

the strict transform of $\overline{\Gamma} \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W$. This is a closed sub-ind-schemes by §16.2.6. The same construction works with other variants – in particular, we denote $\Upsilon := \Gamma^{\mathrm{st}} \hookrightarrow \mathcal{N}_{\mu_\infty}(\mathcal{Z}', T \times T) = \mathrm{Bl}^\circ_{\mathcal{Z}'}(T \times T \times \mathbb{G}_m^{\mathrm{rot}})$ the strict transform of $\Gamma \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times T$.

We have the following situation

$$\begin{tikzcd}
\mathrm{Bl}^\circ_{\mathcal{Z}}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \arrow[d] & \overline{\Upsilon} \arrow[l, hook'] \arrow[d] & \overline{\Upsilon}_{\mathrm{wt}(\mu)} \arrow[l, hook'] \arrow[d] \\
T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W & \overline{\Gamma} \arrow[l, hook'] & \overline{\Gamma}_{\mathrm{wt}(\mu)}. \arrow[l, hook']
\end{tikzcd} \tag{20.5}$$

The horizontal maps are closed immersions of ind-schemes, the outer terms are actual schemes.

By §16.2.2, each fiber $\overline{\Upsilon}$ is given by the blowup of $\overline{\Gamma}$ at the ind-subscheme $\mathcal{Z}$. In particular, restricting to a formal neighborhood of (x, ζ), we see that

$$\overline{\Upsilon}^\wedge_{(x,\zeta)} = \mathrm{Bl}_{Z^\wedge_\zeta} \overline{\Gamma}^\wedge_{(x,\zeta)}.$$

Remark 20.28. One can trace where $\overline{\Upsilon}$ and Υ go under the identification of Lemma 20.25. Firstly, it decomposes into connected components $\overline{\Upsilon}^\wedge_{(x,\zeta)} = \coprod_{\vartheta \in |Z_{(x,\zeta)}|} \overline{\Upsilon}^{\vartheta,\wedge}_{(x,\zeta)}$. Each of the closed immersions

$$\overline{\Upsilon}^{\vartheta,\wedge}_{(x,\zeta)} \hookrightarrow N_{h=0}(\mathfrak{t} \times_{\mathfrak{t}/\!\!/W_{[x,\zeta]}} \mathfrak{t}/\!\!/W_\vartheta, \mathfrak{t} \times \mathfrak{t}/\!\!/W_\vartheta)^\wedge_{(x,\zeta)}$$

then identifies as a hyperplane arrangement of copies of $(\mathfrak{t} \times \mathbb{G}_a)^\wedge_{(0,0)}$ inside $(\mathfrak{t} \times \mathbb{G}_a \times \mathfrak{t}/\!\!/W_\vartheta)^\wedge_{(0,0)}$ intersecting in the origin, labeled by those elements $\lambda \in X_\bullet$ which lie in the corresponding $\mathrm{Stab}_{(x,\zeta)}$-coset $\vartheta \in X_\bullet / \mathrm{Stab}_{(x,\zeta)}$, see also Corollary 18.6. Depending on the genericity of ζ, the precise arrangement is that of Remarks 20.3 and 20.7, applied to the ambient reductive group $L_{[\vartheta]}$ with the parabolic subgroup P_ϑ.

20.5. A combinatorial map

We construct a comparison map from the completed ring of functions on the above deformation to the GKM ring of the affine Grassmannian. In this section, we work purely combinatorially. The parallel topological meaning is explained in §20.6.

Let k be a coefficient ring. To simplify the manipulation of roots of unity in our deformation, we assume k is an algebraically closed field of $\mathrm{char}\,\mathsf{k} = 0$. However, this assumption is not necessary.

20.5.1. Components of the localization map. We define the localization map

$$\gamma : \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \xrightarrow{\gamma} \prod_{\lambda \in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$$

componentwise as

$$\gamma_\lambda : \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \xrightarrow{\gamma} \prod_{\lambda \in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}}) \xrightarrow{\mathrm{pr}_\lambda} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}}), \tag{20.6}$$

$$\gamma_\lambda : f(x, \zeta, y) \mapsto f(x, \zeta, \lambda(\zeta) \cdot x). \tag{20.7}$$

Consider the closed embedding $\overline{\Gamma}_\lambda \hookrightarrow T \times \mathbb{G}_m^{\mathrm{rot}} \times (T /\!\!/ W)$ and identify $\overline{\Gamma}_\lambda = T \times \mathbb{G}_m^{\mathrm{rot}}$ via the first two factors. The components γ_λ are then geometrically given by restriction to the irreducible components $\overline{\Gamma}_\lambda$ of $\overline{\Gamma}$, namely

$$\gamma_\lambda : \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \twoheadrightarrow \mathcal{O}(\overline{\Gamma}_\lambda). \tag{20.8}$$

We can now combinatorially check that the image of γ satisfies the GKM conditions.

Lemma 20.29. *The image of the map γ from* (20.6) *lands in the subring* $\mathrm{GKM}^0_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_G; \mathsf{k})$.

Proof. We directly check the GKM conditions. To simplify notation, observe that each function from the base $\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ restrict the same to each $\overline{\Gamma}_\lambda$, hence satisfies the GKM conditions trivially. It is thus enough to check on functions $f \in \mathcal{O}(T /\!\!/ W)$ from the second copy. We identify f with the corresponding W-invariant function on T via the embedding $\mathcal{O}(T /\!\!/ W) \hookrightarrow \mathcal{O}(T)$; the restriction to $\overline{\Gamma}_\lambda \hookrightarrow T /\!\!/ W$ now corresponds to restriction to $\Gamma_{\lambda,1} \hookrightarrow T$.

Consider any affine simple reflection $s_{\beta+m\hbar}$. Let $\lambda, \lambda' \in X_\bullet$ be such that $\lambda' = s_{\beta+m\hbar}(\lambda)$, or equivalently

$$\lambda' = s_\beta(\lambda) \cdot (\check{\beta})^m. \tag{20.9}$$

Assume that $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ is in the kernel of the character $\beta + m\hbar$. In other words, this means

$$\beta(x) \cdot \zeta^m = 1, \tag{20.10}$$

or equivalently $\beta(x) = \zeta^{-m}$. Consequently, we have

$$\check{\beta}(\zeta^m) = \check{\beta}(\beta(x^{-1})) = s_\beta(x) \cdot x^{-1}. \tag{20.11}$$

To check the GKM conditions for f, we need to see that if (20.9), (20.10) hold, then $\gamma_\lambda(f) - \gamma_{\lambda'}(f)$ vanishes at (x, ζ). We compute

$$\begin{aligned}
\gamma_\lambda(f(y)) - \gamma_{\lambda'}(f(y)) &= f(\lambda(\zeta) \cdot x) - f(s_\beta(\lambda(\zeta)) \cdot \check{\beta}(\zeta^m) \cdot x) \\
&= f(\lambda(\zeta) \cdot x) - f(s_\beta(\lambda(\zeta)) \cdot s_\beta(x) \cdot x^{-1} \cdot x) \\
&= f(\lambda(\zeta) \cdot x) - f(s_\beta(\lambda(\zeta) \cdot x)) \\
&= f(\lambda(\zeta) \cdot x) - f(\lambda(\zeta) \cdot x) \\
&= 0
\end{aligned}$$

Here, the first line is the definition of γ and (20.9), the second follows by (20.11), the third is clear, and the fourth holds since f is W-invariant. $\square$

20.5.2. Factorization through the deformation. We claim that the localization map γ factors through our deformation.

Lemma 20.30. *There is a factorization*

$$\begin{tikzcd} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \arrow[r, "\gamma"] \arrow[d, hook] & \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r; \mathsf{k}) \arrow[r, hook] & \prod_{\lambda\in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}}) \\ \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \arrow[ur, dashed, "\upsilon"'] & & \end{tikzcd}$$

Proof. We first check that the whole horizontal composition factors through the deformation $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))$. To this end, we need to check that for each root of unity ζ of $\mathrm{ord}(\zeta) = \ell$ and for every $f \in \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W)$ that vanishes on Z_ℓ, the image $\gamma(f)$ inside $\prod_{\lambda\in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ is divisible by $(q - \zeta)$. This can be checked componentwise – we need to see that each $\gamma_\lambda(f) \in \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ is divisible by $(q - \zeta)$, or equivalently that $\gamma_\lambda(f)$ vanishes after substituing $q = \zeta$.

To check this, we may identify f with its image under $\mathcal{O}(T \times T /\!\!/ W) \hookrightarrow \mathcal{O}(T \times T)$. For any (x, ζ), the image of f under the localization map is computed as

$$\gamma_\lambda(f) = f(x, \lambda(\zeta) \cdot x).$$

Specializing to our root of unity ζ, this is the value of $f : T \times T \to \mathbb{A}^1$ on the T-valued point $(x, \lambda(\zeta) \cdot x) : T \to T \times \mathbb{G}_m^{\mathrm{rot}} \times T$. Since ζ is an ℓ-th root of unity, the point $\lambda(\zeta) \in T$ is ℓ-torsion: we have $\lambda(\zeta)^\ell = 1$. Hence

$$x^\ell = (\lambda(\zeta) \cdot x)^\ell.$$

In other words, $(x, \lambda(\zeta) \cdot x)$ is a T-valued point of the closed subscheme $Z'_\ell \hookrightarrow T \times T$. Since $f \in \mathcal{I}_{Z'_\ell}$ by assumption, we deduce the desired vanishing $f(x, \lambda(\zeta) \cdot x) = 0$.

To conclude, we need to see that the image of $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))$ inside $\prod_{\lambda\in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ satisfies the GKM conditions. Given any $f \in \mathcal{I}_{Z'_\ell}$ and $\lambda, \lambda' \in X_\bullet$ with $\lambda' = s_{\beta,m} \cdot \lambda$, we thus need to check that the element

$$\gamma_\lambda(\tfrac{f}{q-\zeta}) - \gamma_{\lambda'}(\tfrac{f}{q-\zeta}) = \tfrac{\gamma_\lambda(f) - \gamma_{\lambda'}(f)}{q-\zeta} \in \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$$

is divisible by $(1 - \beta q^m)$. We already know this is the case for any element in $\mathcal{O}(T \times T /\!\!/ W)$ by Lemma 20.29. Since $(q - \zeta) \lhd \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ is a prime ideal which does not contain $(1 - \beta q^m)$, we deduce that the above ratio is also divisible by $(1 - \beta q^m)$ inside $\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$ as desired. □

20.5.3. Injectivity. The injectivity of the above constructed map is almost obvious.

Lemma 20.31. *The following ring homomorphisms introduced above are injective*

$$\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \to \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \to \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r; \mathsf{k}) \to \prod_{\lambda\in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}}).$$

Proof. All four rings in question are clearly reduced (the leftmost one are functions on a reduced scheme; the middle one is given by adding certain elements from its fraction field; the third one is a subring of the rightmost infinite product of polynomial rings). Since they are moreover algebras over $\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$, they all embed into the localizations at the generic point $\eta \in T \times \mathbb{G}_m^{\mathrm{rot}}$ and it is enough to check the injectivity there. But over η, the first map becomes an isomorphism (by construction); the second map is injective (it is given by restricting functions on a localization of the affine space $\mathbb{A}_\eta^n = \mathrm{Spec}(k(\eta)(T /\!\!/ W))$ over the fraction field $k(\eta)$ to an infinite discrete subset $X_\bullet \subseteq \mathbb{A}_\eta^n$ that spans a full rank sublattice and hence is Zariski dense); the third map becomes an isomorphism (by construction). □

20.5.4. Schubert completion. Consider the closed immersions (20.5). Taking global functions, we get the maps $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \to \mathcal{O}(\overline{\Upsilon}) \to \mathcal{O}(\overline{\Upsilon}_{\mathrm{wt}(\mu)})$. The composites $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \twoheadrightarrow \mathcal{O}(\overline{\Upsilon}_{\mathrm{wt}(\mu)})$ are surjective and compatible as μ varies. Their kernels induce an exhaustive filtration by ideals $(\mathcal{I}_{\leqslant\mu})_{\mu\in X_\bullet}$ on $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))$. For convenience, we also denote the quotients by $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))_{\leqslant\mu} := \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))/\mathcal{I}_{\leqslant\mu}$. We obtain the diagram

$$\begin{tikzcd}
\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \arrow[r, hook] \arrow[d, two heads] & \mathcal{O}(\overline{\Upsilon}) \arrow[d, two heads] \\
\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))_{\leqslant\mu} \arrow[r, equal] & \mathcal{O}(\overline{\Upsilon}_{\mathrm{wt}(\mu)}).
\end{tikzcd} \tag{20.12}$$

Here, the upper horizontal map is injective: restricting to irreducible components of $\overline{\Upsilon}$, the ring $\mathcal{O}(\overline{\Upsilon})$ embeds into $\prod_{\lambda\in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$; the induced map from $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))$ is then is precisely the injection from by Lemma 20.31. The filtration $(\mathcal{I}_{\leqslant\mu})_{\mu\in X_\bullet}$ is thus separated.

Definition 20.32. The *Schubert completion* of $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))$ is the completion with respect to the above separated exhaustive filtration. It is given by

$$\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) := \lim_\mu \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))_{\leqslant\mu}.$$

Since the filtration was separated, we have a natural injection

$$\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \hookrightarrow \widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)).$$

With the above definitions, the following statement is clear.

Lemma 20.33. *We have a ring isomorphism*

$$\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) = \mathcal{O}(\overline{\Upsilon}).$$

Proof. Apply $\lim_\mu(-)$ to the bottom row in (20.12) varying in μ. By definition of the Schubert completion, $\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) = \lim_\mu \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))_{\leqslant\mu}$. Computing functions from an ind-presentation of $\overline{\Upsilon}$, also $\mathcal{O}(\overline{\Upsilon}) = \lim_\mu \mathcal{O}(\overline{\Upsilon}_{\mathrm{wt}(\mu)})$. □

We also easily deduce the following.

Lemma 20.34. *The above induces an injective ring homomorphism*

$$\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \hookrightarrow \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}; \mathsf{k}). \tag{20.13}$$

Proof. To construct the map, observe that $\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}; \mathsf{k})$ is complete with respect to the filtration given by kernels of $\mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}; \mathsf{k}) \to \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu}; \mathsf{k})$. Under the injective homomorphism $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \hookrightarrow \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}; \mathsf{k})$ from Lemma 20.31, this matches the Schubert filtration on the source. Consequently, it induces a map (20.13). This map is injective as both sided embed compatibly into $\prod_{\lambda\in X_\bullet} \mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$. □

20.6. The topological meaning

We now explain the topological meaning of the map from §20.5. This ties to the previous section by localization to fixed points. Ignoring this relationship, the topological arguments are completely independent. In particular, we interpret the map γ and fundamental Bezrukavnikov classes in purely topological terms. This interpretation makes sense over any coefficient ring k, for example over $\mathbb{Z}$.

20.6.1. The map from functions on tori. To allows for topological arguments, the geometry of $\mathcal{G}\mathrm{r}$ will take place over $k = \mathbb{C}$. However, the coefficient ring k may be taken arbitrary.

There is a natural map

$$K^{\mathrm{top}}_{T\times\mathbb{G}_m^{\mathrm{rot}}\times G}(\mathrm{pt}) \to K^{\mathrm{top}}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r})$$

defined as the equivariant structure map under the vertical identifications

$$\begin{tikzcd}
K^{\mathrm{top}}_{T\times\mathbb{G}_m^{\mathrm{rot}}\times G}(\mathrm{pt}) \arrow[rr] \arrow[dd, equal] & & K^{\mathrm{top}}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}) \arrow[d, equal] \\
& & K^{\mathrm{top}}((T\times\mathbb{G}_m^{\mathrm{rot}})\backslash \mathrm{L}G/\mathrm{L}^+G) \arrow[d, equal] \\
K^{\mathrm{top}}((T\times\mathbb{G}_m^{\mathrm{rot}})\backslash \mathrm{pt}/G) \arrow[rr] & & K^{\mathrm{top}}((T\times\mathbb{G}_m^{\mathrm{rot}})\backslash \mathrm{L}G/G).
\end{tikzcd}$$

Remark 20.35. More generally, given two parahoric subgroups $\mathcal{P}$ and $\mathcal{P}'$ of $\mathrm{L}G$ with associated Levi quotients L and L', we get the map

$$K^{\mathrm{top}}_{L\times\mathbb{G}_m^{\mathrm{rot}}\times L'}(\mathrm{pt}) \to K^{\mathrm{top}}_{\mathcal{P}\rtimes\mathbb{G}_m^{\mathrm{rot}}}(\mathrm{L}G/\mathcal{P}').$$

20.6.2. Localization map. We can now identify

$$\begin{tikzcd}
K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}\times G}(\mathrm{pt};\mathsf{k}) \arrow[r] \arrow[d, equal] & K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathsf{k}) \arrow[r, hook] \arrow[d, equal] & K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}^{T\times\mathbb{G}_m^{\mathrm{rot}}};\mathsf{k}) \arrow[d, equal] \\
\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}\times T/\!\!/W) \arrow[r] & \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathsf{k}) \arrow[r, hook] & \prod_{\lambda\in X_\bullet}\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}).
\end{tikzcd} \tag{20.14}$$

In particular, this gives a canonical map

$$\mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}\times T/\!\!/W) \to K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}\times G}(\mathcal{G}\mathrm{r}_G;\mathsf{k}) \tag{20.15}$$

Under the identification from (20.14), the left-right composition is the localization map γ from (20.6).

20.6.3. Tautological bundles on $\mathcal{G}\mathrm{r}$. Let us first take $G = GL_n$. Let $\Lambda_0 = k[\![t]\!]^{\oplus n} \subseteq k(\!(t)\!)^{\oplus n}$ be the standard lattice. Fix some $\mu \in X_\bullet$ and let $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ be the corresponding affine Schubert variety. It parametrizes certain sublattices $\Lambda \subseteq k(\!(t)\!)^{\oplus n}$. Depending on μ, we choose a sufficiently big positive integer $m \geqslant 0$ such that for each $\Lambda \in \mathcal{G}\mathrm{r}_{\leqslant\mu}$ it holds that $t^m\Lambda_0 \subseteq \Lambda \subseteq t^{-m}\Lambda_0$.

After fixing this normalization, we have the tautological vector bundle $\mathcal{E}_{\leqslant\mu}$ on $\mathcal{G}\mathrm{r}_{\leqslant\mu}$, associating to an R-valued points Λ of $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ the algebraic vector bundle $t^{-m}\Lambda_0/\Lambda$ on R:

$$\mathcal{E}_{\leqslant\mu} : \Lambda \mapsto (\operatorname{coker}\Lambda \to t^{-m}\Lambda_0).$$

This is indeed a vector bundle, see [Zhu17b, proof of Theorem 1.1.3]. Similarly, we have the trivial rank mn vector bundle $\mathcal{G}_{\leqslant\mu}$ given by $\operatorname{coker}(\Lambda_0 \to t^{-m}\Lambda_0)$. Both of these are naturally equivariant with respect to the groups $T\times\mathbb{G}_m^{\mathrm{rot}} \leqslant \mathrm{L}^+GL_n \rtimes \mathbb{G}_m^{\mathrm{rot}}$.

20.6.4. Bezrukavnikov classes for GL_n from tautological bundles. Consider the equivariant K-theory class given by the difference of the above vector bundles

$$[\mathcal{E}_{\leqslant\mu}] - [\mathcal{G}_{\leqslant\mu}] = [t^{-m}\Lambda_0/\Lambda] - [t^{-m}\Lambda_0/\Lambda_0] \in K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathsf{k}) \tag{20.16}$$

This can be rewritten in several ways. Firstly, it is given by the perfect complex

$$[\Lambda \to t^{-m}\Lambda_0] + [\Lambda_0 \to t^{-m}\Lambda_0][1] \tag{20.17}$$

on $\mathcal{G}\mathrm{r}_{\leqslant\mu}$; here the $t^{-m}\Lambda_0$ sits in degree zero and $[1]$ is a homological shift. Alternatively, it can be also rewritten as a difference of classes of equivariant coherent sheaves which lie in $\mathrm{Perf}(\mathcal{G}\mathrm{r}_{\leqslant\mu})$, namely

$$[\Lambda_0/\Lambda_0 \cap \Lambda] - [\Lambda/\Lambda_0 \cap \Lambda]. \tag{20.18}$$

Informally, it may be thought of as the difference $[\Lambda_0] - [\Lambda]$, but this does not quite make sense since neither of these projective modules is of finite rank.

In any case, the class (20.16) is clearly independent of the choice of m, and compatible with varying μ. (This is obvious from (20.18); it alternatively follows by $[t^{-m}\Lambda_0/\Lambda]-[t^{-m}\Lambda_0/\Lambda_0] = [t^{-m}\Lambda_0/\Lambda] + [t^{-m'}\Lambda_0/t^{-m}\Lambda_0] - [t^{-m'}\Lambda_0/t^{-m}\Lambda_0] - [t^{-m}\Lambda_0/\Lambda_0] = [t^{-m'}\Lambda_0/\Lambda] - [t^{-m'}\Lambda_0/\Lambda_0]$ for any $m' \geqslant m$.) It thus gives a well defined equivariant K-theory class on the whole of $\mathcal{G}\mathrm{r}_G$. We then have the following.

Lemma 20.36. *Inside $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathsf{k})$, we have $b_{1,1} = [t^{-m}\Lambda_0/\Lambda] - [t^{-m}\Lambda_0/\Lambda_0]$.*

Proof. Given any $\lambda \in X_\bullet$, it is easy to compute the localizations of both sides to the corresponding fixed-point. Denote $\lambda = (t_1^{d_1}, \ldots, t_n^{d_n})$ for $d_1, \ldots, d_n \in \mathbb{Z}$. For any $d \in \mathbb{Z}$, note that

$$[d]_q := \frac{1-q^d}{1-q} = \begin{cases} 1+q+\cdots+q^{d-1} & \text{if } d>0,\\ 0 & \text{if } d=0,\\ -q^{-1}(1+q^{-1}+\cdots+q^{d+1}) & \text{if } d<0. \end{cases} \tag{20.19}$$

We now compute

$$\begin{aligned}
\gamma_\lambda([t^{-m}\Lambda_0/\Lambda] - [t^{-m}\Lambda_0/\Lambda_0]) &= \gamma_\lambda([\Lambda_0/\Lambda_0\cap\Lambda] - [\Lambda/\Lambda_0\cap\Lambda])\\
&= \sum_{\{i|d_i>0\}} (1+q+\cdots+q^{d_i-1})t_i - \sum_{\{i|d_i<0\}} (q^{-1}+\cdots+q^{d_i+1})t_i\\
&= \sum_{i=1}^n \frac{1-q^{d_1}}{1-q}\cdot t_i\\
&= \frac{\sum_{i=1}^n t_i - \sum_{i=1}^n q^{d_i}\cdot t_i}{1-q}\\
&= \gamma_\lambda\left(\frac{e_1(t_1,\ldots,t_n) - e_1(s_1,\ldots,s_n)}{1-q}\right)\\
&= \gamma_\lambda(b_{1,1})
\end{aligned}$$

Here, the first line is (20.18). The second line is by concrete knowledge of the fibers of $t^{-m}\Lambda_0/\Lambda$ and $t^{-m}\Lambda_0/\Lambda$ at the fixed point λ as $T\times\mathbb{G}_m^{\mathrm{rot}}$ representations. The third line is by (20.19). The fourth line is clear, the fifth line follows by the description of γ_λ from (20.6), and the sixth line is by definition of $b_{1,1}$.

Since the above holds for any $\lambda \in X_\bullet$, the lemma follows. ☐

20.6.5. Bezrukavnikov classes from vector bundles for general G. Let G be any reductive group. The affine Grassmannian is functorial in G, as is our description of its equivariant K-theory. Consequently, we can define the fundamental Bezrukavnikov classes from the standard one on $\mathcal{G}\mathrm{r}_{GL_n}$.

Namely, consider any representation $\rho : G \to GL_n$. This induces a map $\mathcal{G}\mathrm{r}_G \to \mathcal{G}\mathrm{r}_{GL_n}$, equivariant with respect to $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}} \to \mathrm{L}GL_n \rtimes \mathbb{G}_m^{\mathrm{rot}}$ and compatible with the Schubert stratifications. We thus have a pullback map $\rho^* : K^{\mathrm{top},0}_{T_{GL_n} \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{GL_n}; \mathsf{k}) \to K^{\mathrm{top},0}_{T_G \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathsf{k})$. Using the double-quotient description §16.4.10 and the construction §20.6.1, it follows that ρ^* induces the obvious map $\rho^* : T_{GL_n} \times \mathbb{G}_m^{\mathrm{rot}} \times T_{GL_n} \to T_G \times \mathbb{G}_m^{\mathrm{rot}} \times T_G$, and hence on the deformation constructions.

In particular, for any $f \in \mathcal{I}_{Z_\ell} \subseteq \mathcal{O}(T_{GL_n} \times T_{GL_n} /\!\!/ W)$, this map sends $\frac{f}{1-q^\ell} \mapsto \frac{\rho^*(f)}{1-q^\ell}$. Specializing further to the case when $\rho = \omega_j$ is the j-th fundamental representation of G and $f = e_1(t_1, \ldots, t_n) - e_1(s_1, \ldots, s_n)$, we deduce it sends $b^{GL_n}_{1,1} \mapsto b^G_{j,1}$. In other words, $b^G_{j,1}$ is the equivariant K-theory class $\omega_j^*([\Lambda_0/\Lambda_0 \cap \Lambda] - [\Lambda/\Lambda_0 \cap \Lambda])$ on $\mathcal{G}\mathrm{r}_G$. We have proved the following.

Lemma 20.37. *We have*

$$b^G_{j,1} = \omega_j^*(b^{GL_n}_{1,1}).$$

20.6.6. Adams operations. The fundamental Bezrukavnikov classes $b_{j,\ell} = \frac{e_j(t_1^\ell, \ldots, t_n^\ell) - e_j(s_1^\ell, \ldots, s_n^\ell)}{1-q^\ell}$ are related to each other through Adams operations in equivariant K-theory.

Lemma 20.38. *Let $1 \leqslant j \leqslant n$. For each $\ell, \ell' \in \mathbb{N}$, we have*

$$\psi^\ell(b_{j,\ell'}) = b_{j,\ell\ell'}. \tag{20.20}$$

Proof. The Adams operation ψ^ℓ is a functorial ring endomorphism of $K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(-)$, which exponentiate classes of line bundles to the ℓ-th power. By injectivity of localization to fixed-points (valid in our setup), it is thus enough to check the equality (20.20) after restriction to each fixed point, in other words after applying γ_λ.

After this restriction, each of the generators $t_i, s_i, q, 1$ is a K-theory class of an equivariant line bundle (a character of $T \times \mathbb{G}_m^{\mathrm{rot}}$). Therefore, ψ^ℓ acts by $(-)^\ell$ on each of them. Since ψ^ℓ is a ring homomorphism, the desired equality follows. □

20.6.7. Topological argument. With the above topological meaning at hand, we have the following clear construction.

Lemma 20.39. *We have ring homomorphisms*

$$\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W) \to \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \to \widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W)) \to K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}; \mathsf{k}).$$

Proof. The map from the leftmost term is given by the equivariant structure maps §20.6.1. To see that it factors through the second term, we need to prove that the fundamental Bezrukavnikov classes $b_{j,\ell}$ give well-defined classes in K-theory. However, we have already interpreted them as Adams operations on the fundamental vector bundles in Lemmata 20.37, 20.38.

Finally, $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathsf{k})$ is complete with respect to the filtration given by kernels of restriction maps to affine Schubert varieties by Proposition 20.8. Equipping $\mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T\times T /\!\!/ W))$ with the Schubert filtration of Definition 20.32 makes the comparison map filtered. Hence it factors through the Schubert completion $\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T\times T /\!\!/ W))$ as desired. □

20.7. Bezrukavnikov–Finkelberg description of equivariant K-theory

We now conclude the proof of our main result for simply–connected G with $\mathbb{C}$-coefficients. To this end, we will assume $k = \mathbb{C} = \mathsf{k}$. We further record the topological Adams generation. We find this case particularly illustrative; technical generalizations are then discusses in §20.8.

20.7.1. The isomorphism. We now prove the surjectivity by a topological argument.

Theorem 20.40. *Assume G is semisimple and simply-connected group over $\mathbb{C}$. Then the above constructed map induces a ring isomorphism*

$$\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T\times T /\!\!/ W)) \xrightarrow{\cong} K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathbb{C}).$$

Proof. Regard both sides as quasi-coherent (so in particular fpqc) sheaves on $T\times\mathbb{G}_m^{\mathrm{rot}}$. To see that the discussed map is an isomorphism, it is thus enough to check it is an isomorphism after formal completion at each closed point $(x,\zeta)\in T\times\mathbb{G}_m^{\mathrm{rot}}$. (Alternatively, it is enough to check the isomorphism at actual fibers by Remark 20.41.) We have thus reduced to checking that for each (x,ζ), the bottom horizontal map in the following diagram is an isomorphism.

$$\begin{tikzcd}
\widehat{\mathcal{O}}(\mathcal{N}_{q=\zeta}(\mathcal{Z}, T\times T /\!\!/ W)) \arrow[r] \arrow[d] & K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathbb{C}) \arrow[d] \\
\widehat{\mathcal{O}}(\mathcal{N}_{q=\zeta}(\mathcal{Z}, T\times T /\!\!/ W))^{\wedge}_{(x,\zeta)} \arrow[r] & K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathbb{C})^{\wedge}_{(x,\zeta)}
\end{tikzcd}$$

To this end, fix a closed point $(x,\zeta)\in T\times\mathbb{G}_m^{\mathrm{rot}}$. To check the isomorphism between the completions, we distinguish cases depending on the genericity of ζ.

Assume first that ζ is a primitive root of unity of order ℓ. Then consider the following diagram.

$$\begin{tikzcd}
\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T\times T /\!\!/ W))^{\wedge}_{(x,\zeta)} \arrow[rr] \arrow[d, equal] & & K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{C})^{\wedge}_{(x,\zeta)} \arrow[d, equal] \\
\widehat{\mathcal{O}}(N_{q=\zeta}(Z_\ell, T\times T /\!\!/ W))^{\wedge}_{(x,\zeta)} \arrow[d, equal] & & \widehat{H}^\bullet_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r});\mathbb{C})^{\wedge}_{(0,0)} \arrow[d, equal] \\
\prod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|}\widehat{\mathcal{O}}(N_{q=\zeta}(Z_\ell^\vartheta, T\times T /\!\!/ W))^{\wedge}_{(x,\zeta)} \arrow[d, equal] & & \widehat{H}^\bullet_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\coprod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|}\mathcal{F}\mathrm{l}^\vartheta_{\ell,L_{[x,\zeta]}};\mathbb{C})^{\wedge}_{(0,0)} \arrow[d, equal] \\
\prod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|}\widehat{\mathcal{O}}(N_{h=0}(\mathfrak{t}\times_{\mathfrak{t}/\!\!/W_{[x,\zeta]}}\mathfrak{t}/\!\!/W_\vartheta, \mathfrak{t}\times\mathfrak{t}/\!\!/W_\vartheta))^{\wedge}_{(0,0)} \arrow[rr, equal] & & \prod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|}\widehat{H}^\bullet_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_{\ell,L_{[x,\zeta]}};\mathbb{C})^{\wedge}_{(0,0)}
\end{tikzcd}$$

The identifications in the left-hand column work as follows. First, since we are completing at (x,ζ), only the deformation to the normal cone of Z_ℓ centered at ζ matters. The next two identifications are covered by Lemma 20.25.

The identifications in the right-hand column go a follows. First use the localization from Corollary 17.30, Theorem 17.20 and Remark 17.22 to pass from the completion of equivariant K-theory over (x,ζ) to the completion of equivariant cohomology of the fixed-points of (x,ζ) at the origin (alternatively see Remark 20.41). Then use the computation of the fixed-points on the affine Grassmannian from Lemma 19.14. Finally, we may commute the disjoint union through cohomology.

The bottom identification is by taking the Schubert completion of the isomorphism from Proposition 20.6. (Note that the completion of equivariant cohomology is given by the Schubert completion.) This finishes the discussion of the case when ζ is a root of unity.

Now, assume on the other hand that $\zeta \in \mathbb{G}_m^{\mathrm{rot}} \backslash \mu_\infty$. We then make the analogous identifications.

$$
\begin{array}{ccc}
\widehat{\mathcal{O}}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times T /\!\!/ W))^\wedge_{(x,\zeta)} & \longrightarrow & K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_G;\mathbb{C})^\wedge_{(x,\zeta)} \\
\| & & \| \\
\widehat{\mathcal{O}}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W)^\wedge_{(x,\zeta)} & & \widehat{H}^\bullet_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}r_G);\mathbb{C})^\wedge_{(0,0)} \\
\| & & \| \\
\prod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|} \mathcal{O}(\mathbb{G}_m^{\mathrm{rot}} \times T \times_{T/\!\!/W_{[x,\zeta]}} T /\!\!/ W_\vartheta)^\wedge_{(x,\zeta)} & & H^\bullet_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\coprod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|} L_{[x,\zeta]}/P_\vartheta;\mathbb{C})^\wedge_{(0,0)} \\
\| & & \| \\
\prod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|} \mathcal{O}(\mathbb{G}_a \times \mathfrak{t} \times_{\mathfrak{t}/\!\!/W_{[x,\zeta]}} \mathfrak{t} /\!\!/ W_\vartheta)^\wedge_{(0,0)} & = & \prod_{\vartheta\in|\overline{\Gamma}_{(x,\zeta)}|} H^\bullet_{T\times\mathbb{G}_m^{\mathrm{rot}}}(L_{[x,\zeta]}/P_\vartheta;\mathbb{C})^\wedge_{(0,0)}
\end{array}
$$

The identifications in the left-hand column are as follows. Firstly, since $\zeta \notin \mu_\infty$, after the completion at (x,ζ) the deformations to the normal cones do not play a role. For the second identification, note that the Schubert completion corresponds to a limit over restrictions to finite subsets of the component set $|\Gamma_{(x,\zeta)}|$. Each of the connected components $\Gamma^\vartheta_{(x,\zeta)}$, $\vartheta \in |\Gamma_{(x,\zeta)}|$ is given by $W_{[x,\zeta]}$ copies of T intersection in the natural way; the stabilizer of this subscheme with respect to the W-action on the right copy of T is precisely W_ϑ. The third identification is the usual comparison of the completion of a semisimple, simply-connected group with its Lie algebra.

For the right hand column, the first identification is the localization of equivariant K-theory as in Corollary 17.30, Theorem 17.20 and Remark 17.22. The second isomorphism is the computation of fixed points from Lemma 19.8 (no completion is necessary, since each of the terms has bounded cohomology). The third line is clear.

The bottom horizontal identification is the well-known description of T-equivariant cohomology of partial flag varieties of reductive groups from Lemma 20.1 and Remark 20.2; note that $\mathbb{G}_m^{\mathrm{rot}}$ acts trivially on each of them. $\square$

Remark 20.41. In fact, in the first reduction step, we may reduce further. Firstly, the map in question is injective by §20.5.3, so we only need to check surjectivity.

Secondly, since we are speaking about a map of quasi-coherent sheaves on an affine scheme, it is enough to check that for each closed point (x,ζ), the following map between the fibers is surjective:

$$\widehat{\mathcal{O}}(\mathcal{N}_{q=\zeta}(\mathcal{Z}, T \times T /\!\!/ W))_{(x,\zeta)} \longrightarrow K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r;\mathbb{C})_{(x,\zeta)}.$$

Consequently, all occurrences of the formal completion $(-)^\wedge_{(x,\zeta)}$ may be replaced by the actual fiber $(-)_{(x,\zeta)}$ throughout the proof. This variant of the proof thus needs a slightly weaker technical input from §17.4, which can be covered by more standard results.

Remark 20.42. The horizontal identifications in the proof may be also compared through the GKM presentations, as outlined in Remarks 20.3, 20.7, 20.28. This is another sanity check for the compatibility of all the identifications that we made.

20.7.2. Finite generation as complete topological λ-ring. As we saw, the equivariant K-theory ring of the affine Grassmannian is quite big – it is a topological completion of an already non-noetherian ring of functions on our affine blowup. Nevertheless, it is finitely generated in the following appropriate sense.

Corollary 20.43. *Let G be a semisimple simply connected grou over $\mathbb{C}$. As complete topological λ-algebra over the equivariant base, $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathbb{C})$ is generated by the finitely many elements*

$$(b_{j,1} \mid j = 1, \dots, n).$$

Proof. By Theorem 20.40 and Lemma 20.23, we know that $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r};\mathbb{C})$ is topologically generated by the fundamental classes $b_{j,\ell}$. Under λ-operations, these are in turn generated by $b_{j,1}$ for $j = 1, \dots, n$ by Lemma 20.38. □

20.7.3. Some intuition. One could try to approach the surjectivity in the proof of Theorem 20.40 in another way as follows. By Lemma 20.33 and Proposition 20.10 we want to prove that the following map is an isomorphism

$$\mathcal{O}(\overline{\Upsilon}) \to \mathrm{GKM}^0_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G;\mathsf{k}).$$

Both sides compatibly embed into

$$\prod_{\lambda\in X_\bullet} \mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}}). \tag{20.21}$$

and we are left to check that the images agree. For the right hand side, the image is given by the GKM conditions, saying that functions on different copies of $T\times\mathbb{G}_m^{\mathrm{rot}}$ agree over subtori of the form $T_{\beta+m\hbar} = \ker(\beta+m\hbar)$. For the left-hand side, we are considering functions on an arrangement of copies of $T\times\mathbb{G}_m^{\mathrm{rot}}$ inside $T\times\mathbb{G}_m^{\mathrm{rot}}\times T /\!\!/ W$. Their intersections combinatorially match the subvarieties imposing the equalizer conditions in the GKM picture.

From what we already said, one would be tempted to conclude. However, this is not a complete reasoning, as Example 20.44 below shows. To complete the above intuition, one needs to, in one way or another, study precisely how $\overline{\Upsilon}$ sits inside $T\times\mathbb{G}_m^{\mathrm{rot}}\times T /\!\!/ W$ via its defining embedding – this is what the above proof accomplishes.

Example 20.44. Consider the following example. Let $X_3 := \operatorname{Spec}\mathsf{k}[x,y,z]/(xy,yz,zx)$ be three lines in general position inside the three-space $\mathbb{A}^3$ intersecting in the origin. Let $X_2 := \operatorname{Spec}\mathsf{k}[x,y]/xy(x-y)$ be three lines inside the plane $\mathbb{A}^2$ intersection in the origin. There is a natural squishing map $X_3 \to X_2$. This map is a homeomorphism – in fact the same arrangement of lines. However, it is not an isomorphism of schemes: the induced injection on the rings of functions $\mathcal{O}(X_2) \to \mathcal{O}(X_3)$ is not surjective. The problem is related to the notion of *semi-normality* (X_3 is seminormal but X_2 is not), or even more explicitly to that of *embedding dimension* (which is 3 and 2, respectively).

20.8. Variants and generalizations

20.8.1. The general answer. We work with coefficients $\mathrm{k} = \mathbb{C}$. As stated in Remark 15.8, our Theorem 20.40 has several direct generalizations. These may be either proved by a direct modifications of the above argument, or bootstrapped from what we already have. Putting these remarks together, the generalized statement is as follows. Recall that G^{sc} denotes the simply connected cover of the derived group of G from §16.3.8.

Theorem 20.45. *Let G be a reductive group over $\mathbb{C}$. Denote by T^{sc} the maximal torus of G^{sc}. Pick two parahoric subsgroups $\mathcal{P}_1, \mathcal{P}_2 \leqslant \mathrm{L}G$. Consider the affine flag variety $\mathcal{F}\mathrm{l}_G^J := \mathrm{L}G/\mathcal{P}_2$, together with the action of $\mathcal{P}_1 \rtimes \mathbb{G}_m^{\mathrm{rot}} \circlearrowright \mathcal{F}\mathrm{l}_G^J$. Then*

$$\widehat{\mathcal{O}}(\mathrm{Bl}^{\circ}_{\mathcal{Z}}(\mathbb{G}_m^{\mathrm{rot}} \times \tfrac{T^{\mathrm{sc}}/\!\!/ W_1 \times T^{\mathrm{sc}}/\!\!/ W_2}{\pi_1(G)} \times \pi_1(G)) = K^{\mathrm{top},0}_{\mathcal{P}_1 \rtimes \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_G^J, \mathbb{C}).$$

Proof. The setup of theorem has four degrees of freedom – we can independently choose: (i) which parahoric will act on the left; (ii) which partial affine flag variety are we taking; (iii) which isogeny class of G we take for $\mathrm{L}G$, or more precisely which connected components of $\mathcal{G}\mathrm{r}_{G^{\mathrm{ad}}}$ we take into account; (iv) which isogeny class of G will be used for the L^+G-action, or alternatively which torus T in between $T^{\mathrm{sc}} \to T \to T^{\mathrm{ad}}$ will act.

We will treat these independently: (i) and (ii) in Lemma 20.46; (iii) in Lemma 20.47; (iv) in Lemma 20.49. It is quite clear that all this bookkeeping can be done simultaneously; we leave the details to the reader. □

20.8.2. Affine flag varieties and positive loop-group equivariance. Assume $G = G^{\mathrm{sc}}$ is simply connected and consider its torus $T = T^{\mathrm{sc}}$. Take two parahoric subgroups $\mathcal{P}_1, \mathcal{P}_2 \leqslant \mathrm{L}G$. Consider the corresponding affine flag variety $\mathcal{F}\mathrm{l}^J = \mathrm{L}G/\mathcal{P}_2$, together with the action $\mathcal{P}_1 \rtimes \mathbb{G}_m^{\mathrm{rot}} \circlearrowright \mathcal{F}\mathrm{l}_G^J$. The equivariant K-theory of this action is the K-theory of the double quotient (quantum Hecke stack)

$$\mathcal{P}_1 \rtimes \mathbb{G}_m^{\mathrm{rot}} \backslash \mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}} / \mathcal{P}_2 \rtimes \mathbb{G}_m^{\mathrm{rot}}$$

which has an apparent symmetry.

Lemma 20.46. *For G semisimple, simply connected, we have a ring isomorphism*

$$\widehat{\mathcal{O}}(\mathrm{Bl}^{\circ}_{\mathcal{Z}}(T /\!\!/ W_1 \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W_2)) = K^{\mathrm{top},0}_{\mathcal{P}_2 \rtimes \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_G^J; \mathbb{C}).$$

Proof. By homotopy invariance and equivariant formality, we have

$$K^{\mathrm{top},0}_{\mathcal{P}_1 \rtimes \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_G^J; \mathbb{C}) = K^{\mathrm{top},0}_{L_1 \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_G^J; \mathbb{C}) = (K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_G^J; \mathbb{C}))^{W_1}.$$

Here, the W_1-action is on the first copy of the torus.

For the other part, note that $\mathcal{F}\mathrm{l}_G^J \to \mathcal{G}\mathrm{r}_G$ is a fpqc locally trivial fibration with fiber G/P_2. The K-theory of $\mathcal{F}\mathrm{l}_G^J$ is thus given by the following base change:

$$K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{F}\mathrm{l}_G^J; \mathbb{C}) = K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C}) \otimes_{\mathcal{O}(T/\!\!/ W)} \mathcal{O}(T /\!\!/ W_2).$$

Since blowups are compatible with base change, we have reduced the desired statement to Theorem 20.40.(Alternatively, one can directly repeat our computation of $T \times \mathbb{G}_m^{\mathrm{rot}}$-equivariant K-theory of $\mathcal{G}\mathrm{r}$ in the more general case of $\mathcal{F}\mathrm{l}_G^J$.) □

20.8.3. Affine Grassmannian of reductive G. Assume now G to be merely reductive, but let us work equivariantly with respect to the torus T^{sc} of G^{sc}, see §16.3.8.

Lemma 20.47. *Assume G is connected reductive. Then the above constructed map induces a ring isomorphism*

$$\widehat{\mathcal{O}}(\mathrm{Bl}^{\circ}_{\mathcal{Z}}(T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} /\!\!/ W \times \pi_1(G))) = K^{\mathrm{top},0}_{T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_G; \mathbb{C}).$$

Proof. The affine Grassmannian $\mathcal{G}\mathrm{r}_G$ has connected components $\mathcal{G}\mathrm{r}_G^{\tau}$ labeled by elements $\tau \in \pi_1(G)$. The identity component $\mathcal{G}\mathrm{r}_G^{1}$ is $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}}$-equivariantly isomorphic to $\mathcal{G}\mathrm{r}_{G^{\mathrm{sc}}}$, which we have treated already in our main theorem.

In general, we can repeat the proof of the main theorem for all components at once, using the corresponding variant of cocharacter graphs from §18.3. Alternatively, we can follows [BF08] and bootstrap the current case as follows. For each $\tau \in \pi_1(G)$ choose a lift $\underline{\tau} \in X_{\bullet}$. Translation by τ then identifies the identity component $\mathcal{G}\mathrm{r}_G^{1}$ with $\mathcal{G}\mathrm{r}_G^{\tau}$. This isomorphism is twisted-equivariant for the $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}}$-action; the twisting is given by the conjugation with t^{τ}. Under this twisting, we identify the copies of $T^{\mathrm{sc}} \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W$ compatibly with the cocharacter graphs inside them as in §18.3.2. The conclusion follows. □

Remark 20.48. The connected components of $\mathcal{G}\mathrm{r}_G$, labeled by the fundamental group $\pi_1(G)$, are all abstractly isomorphic; such isomorphisms are realized by translating with elements of the loop group $\mathrm{L}^{+}G$, for example by an element corresponding to some $\lambda \in X_{\bullet}$. Each of these components is isomorphic to the affine Grassmannian $\mathcal{G}\mathrm{r}_{G^{\mathrm{sc}}}$ of G^{sc}.

However, we would like to emphasize the following caveat: these components are *not* isomorphic equivariantly, not even with respect to the extended torus $T \times \mathbb{G}_m^{\mathrm{rot}}$. Indeed, given two components $\mathcal{G}\mathrm{r}_G^{\sigma}$, $\mathcal{G}\mathrm{r}_G^{\tau}$ for some σ, $\tau \in \pi_1(G)$, their equivariant K-theory rings are usually different as $\mathcal{O}(T \times \mathbb{G}_m^{\mathrm{rot}})$-algebras.

Indeed, as we explained, the fiber of the K-theory ring over a point $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$ is given by the cohomology of fixed-points. Now, take $(1, \zeta)$ for generic ζ, and consider the affine Grassmannian of PGL_2. The fixed points of $(1, \zeta)$ are given by a union of partial flag varieties. In the even component, the fixed-point scheme contains a unique connected component isomorphic to a point. In the odd component, the fixed-point scheme has only components given by partial flag varieties of positive dimension. The cohomology rings of the fixed-points are not the same.

The same situation occurs for cohomology, see [BF08, §3, proof of Theorem 1.(b)].

20.8.4. Equivariance with respect to actual torus. Above, we have always considered equivariance with respect to the maximal torus T^{sc} of the simply connected G^{sc}. However, one can well ask about a description with respect to the actual maximal torus $T \leqslant G$. To simplify notation, we take only the identity component $\mathcal{G}\mathrm{r}_{G^{\mathrm{sc}}}$ of $\mathcal{G}\mathrm{r}_G$.

Lemma 20.49. *We have*

$$\widehat{\mathcal{O}}(\mathrm{Bl}^{\circ}_{\mathcal{Z}}(T \times \mathbb{G}_m^{\mathrm{rot}} \times T^{\mathrm{sc}} /\!\!/ W)) = \widehat{\mathcal{O}}(\mathrm{Bl}^{\circ}_{\mathcal{Z}}(\tfrac{T^{\mathrm{sc}} \times T^{\mathrm{sc}} /\!\!/ W}{\pi_1(G)} \times \mathbb{G}_m^{\mathrm{rot}})) = K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{G^{\mathrm{sc}}}; \mathbb{C}).$$

Proof. By Iwahori pavings, the equivariant K-theory of the affine Grassmannian is equivariantly formal. Since the action of T^{sc} factors through T, we know that $K^{\mathrm{top},0}_{T^{\mathrm{sc}}}(\mathcal{G}\mathrm{r}_G; \mathsf{k})$ is the pullback of $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r_G;\mathsf{k})$ along the $\pi_1(G)$-torsor $T^{\mathrm{sc}} \to T$. Conversely, the T-equivariant K-theory is given by descending the T^{sc}-equivariant one, in other words by taking the quotient by the $\pi_1(G)$-action.

Now, on the inclusion $\Gamma \hookrightarrow T^{\mathrm{sc}} \times \mathbb{G}_m \times T^{\mathrm{sc}} \times \pi_1(G)$, this action is given by §18.3.4. On this level, the quotient by $\pi_1(G)$ is (18.10). Since the affine blowup construction and its Schubert completion is compatible with quotients by finite groups, we deduce the result. $\square$

Remark 20.50. Note the apparent break of symmetry between the two actions. This is due to our desire to explicitly see the equivariant base T as the first copy. If we wish to keep the symmetry, the appropriate description via $\frac{T^{\mathrm{sc}}\times T^{\mathrm{sc}}}{\pi_1(G)}$ is via (18.9).

21. Classical and quantum categories $\mathcal{O}$

The purpose of this section is to recall various categories $\mathcal{O}$ and linkage principles inside them. In particular, we recall their block decompositions and centers. All this is standard material; we review it in a suitable form.

The main relevance for us dwells in the comparison to fixed-points in the affine Grassmannian and fibers of equivariant K-theory. We will discuss this in its natural generality elsewhere. Here, we focus on the cases which can be directly read-off from the literature.

21.1. Quantum groups and their categories $\mathcal{O}$

21.1.1. Conventions. Let k be a coefficient ring; we will soon take $\mathsf{k} = \mathbb{C}$. We work on the Langlands dual side, starting from a split reductive group $\check{G} = \check{G}_{\mathsf{k}}$ over k with root datum $(\check{X}_\bullet, \check{\Phi}_\bullet, \check{X}^\bullet, \check{\Phi}^\bullet) = (X^\bullet, \Phi^\bullet, X_\bullet, \Phi_\bullet)$. In general, our conventions follow those of [Jan96; Lus10; HL16].

21.1.2. Hybrid quantum groups. Let $U^{\mathrm{hb}}_{\mathbb{Z}[q^{\pm1}]} = U^{\mathrm{hb}}_{\mathbb{Z}[q^{\pm1}]}(\check{G})$ be the hybrid quantum group over $\mathbb{Z}[q^{\pm1}]$ associated to the root datum of $\check{G}$. It is an integral for of the rational quantum group [Dri88; Jim85], given by combining Lustig divided power version of the upper unipotent and De Concini–Processi version on the lower unipotent part. Its construction goes back to the work [Hab07; Hab08; HL16]. In the context of geometric representation theory, a variant was independently rediscovered in [Gai21] and further studied in [BASV22; Sit24; Sit22; Sit23; Los23].

Let k be an algebraically closed field of characteristic zero. Its base change to $\mathsf{k}[q^{\pm1}] = \mathcal{O}(\mathbb{G}_m^{\mathrm{rot}})$ is denoted by

$$U^{\mathrm{hb}}_{\mathsf{k}} = U^{\mathrm{hb}}_{\mathsf{k}[q^{\pm1}]}(\check{G}).$$

Given $\zeta \in \mathbb{G}_m^{\mathrm{rot}}(\mathsf{k})$, let U^{hb}_ζ be the quantum group over k specialized via $\mathbb{Z}[q^{\pm1}] \to \mathsf{k}$, $q \mapsto \zeta$.

21.1.3. Quantum categories $\mathcal{O}$. Take $U^{\mathrm{hb}}_{\mathsf{k}[q^{\pm1}]}$ as above. We denote by $\mathcal{O}^{\mathrm{hb}}_\zeta$ the corresponding hybrid quantum category $\mathcal{O}$ with non-integral weights [Los23, Definition 5.2.1]. For $(x,\zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$. We further denote

$$\mathcal{O}^{\mathrm{hb}}_{[x,\zeta]}$$

the fiber of its deformed hybrid quantum category $\mathcal{O}$, it is a Serre subcategory of $\mathcal{O}_\zeta^{\mathrm{hb}}$, containing precisely those Verma modules whose highest weight is ζ-resonant with x. See [AJS94, §2], [Fie03], [Sit24, §2.3.1], [Los23, Definition 5.2.1] for deformed of categories $\mathcal{O}$.

Presently, we wish to discuss the following two special cases, which has been studied in detail elsewhere. Recall k is an algebraically closed field of characteristic zero; take $\mathsf{k} = \mathbb{C}$ for simplicity.

(1) Suppose $\zeta \in \mathbb{G}_m(\mathsf{k}) - \mu_\infty(\mathsf{k})$ is general. Then $\mathcal{O}_{[1,\zeta]}^{\mathrm{hb}}$ is equivalent to the classical integral category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand [Hum08]; see [AM15, §6], [Sit24, end of §1.1] for details. The whole $\mathcal{O}_\zeta^{\mathrm{hb}}$ is the non-integral classical category $\mathcal{O}$. For any $x \in T(\mathsf{k})$, the subcategory $\mathcal{O}_{[x,\zeta]}^{\mathrm{hb}}$ is spanned by those blocks whose weights are ζ-resonant to x.

(2) Suppose $\zeta \in \mu_\infty(\mathsf{k})$ whose primitive order $\ell = \mathrm{ord}(\zeta)$ satisfies Assumption 21.8 below. Then we consider $\mathcal{O}_{[1,\zeta]}^{\mathrm{hb}}$, its ζ-integral category $\mathcal{O}$ as in [Sit24].

We now recall these two cases, comparing to our picture.

21.2. Blocks in the generic case

Consider the coefficient ring $\mathsf{k} = \mathbb{C}$. Given a generic $\zeta \in \mathbb{G}_m^{\mathrm{rot}} - \mu_\infty$, the category $\mathcal{O}_\zeta^{\mathrm{hb}}$ is equivalent to the classical category $\mathcal{O}$ introduced by Bernstein–Gelfand–Gelfand, which we denote simply by $\mathcal{O}_\zeta$. The purpose of this section is to recall its block decomposition. We use the standard reference [Hum08].

21.2.1. Classical linkage of cocharacters. We recall the linkage of cocharacters in the classical setup following [Hum08, §5.1]. Given $\alpha \in X_\bullet$, recall the α-shifted action $(W, \bullet_\alpha) \circlearrowright X_\bullet(T)$ given by $w \bullet_\alpha \lambda = w(\lambda - \alpha) + \alpha$ from §16.3.7. It induces an action on $T(\mathsf{k})$.

(i) Two points $y, x \in T(\mathsf{k})$ are *linked* if they lie in the same orbit of the dot action $(W, \bullet_\rho) \circlearrowright T(\mathsf{k})$.

(ii) We say that y is *strongly linked* to x if

$$y = y_0 = s_1 \bullet_\rho y_1 \leqslant y_1 = s_2 \bullet_\rho y_2 \leqslant \cdots \leqslant y_{r-1} = s_r \bullet_\rho d_r \leqslant y_r = x$$

for some $r \geqslant 0$, where each $s_i \in W$ is a simple reflection $s_i = s_{\beta_i}$ associated to some positive coroot $\beta_i \in \Phi_\bullet^+ = \check{\Phi}_+^\bullet$.

Strong linkage defines a partial order relation $x \uparrow_\zeta y$. We sometimes abbreviate $\uparrow := \uparrow_\zeta$ when there is no risk of confusion.

21.2.2. Orbits of resonant Weyl groups. Since we are assuming G simply connected, we have $T = T^{\mathrm{sc}}$. Consequently, we can identify $T /\!\!/ W = \mathfrak{t} /\!\!/ W$.

Given any $x \in T(\mathsf{k})$, consider its orbit inside $T(\mathsf{k})$ under the dot action of the associated resonant Weyl group: $W_{[x,\zeta]} \bullet_\rho x \subseteq T(\mathsf{k})$. Equivalently, this amount to a point

$$\vartheta = \vartheta_{(x,\zeta)} \in (T /\!\!/ W)(\mathsf{k}).$$

The set of orbits $T /\!\!/ W$ is a labeling set for the blocks of $\mathcal{O}_\zeta$ as we recall next.

21.2.3. Block decomposition for classical category $\mathcal{O}$. Recall ζ is fixed. Given any $x \in T(\mathsf{k})$, we get the corresponding Verma module M_x. Let E_x be its simple quotient. The category $\mathcal{O}_\zeta$ decomposes into blocks as follows.

Proposition 21.1. *Let $\zeta \in \mathbb{G}_m^{\mathrm{rot}}(\mathsf{k})$ be generic. Then the following holds.*

(i) There is a decomposition

$$\mathcal{O}_\zeta \simeq \bigoplus_{\vartheta \in |T /\!\!/ W|} \mathcal{O}_\zeta^\vartheta$$

into indecomposable blocks labeled by the set of closed points of $T /\!\!/ W$.

(ii) More precisely, given two elements $x, y \in T(\mathsf{k})$, we have $[M(x) : E(y)] \neq 0$ if and only if $y \uparrow x$.

Proof. For (i), see [Hum08, §4.9, Theorem]. For (ii), see [Hum08, §5.1, Theorem]. □

In particular, the ζ-*integral* blocks of $\mathcal{O}_\zeta$ are labeled by the orbits of the dot action of the full Weyl group on the cocharacter lattice $(W, \bullet_\rho) \circlearrowright X_\bullet(T)$. More generally, we have the following.

Remark 21.2. Given a point $(x, \zeta) \in T \times \mathbb{G}_m^{\mathrm{rot}}$, consider the subset $|\overline{\Gamma}_{(x,\zeta)}| \subseteq |T /\!\!/ W|$. The points $\vartheta \in |\overline{\Gamma}_{(x,\zeta)}|$ label the blocks that are ζ-*resonant to* x. This labeling set identifies with $X_\bullet / W_{L_{[x,\zeta]}}$, the $W_{[x,\zeta]}$-orbits in the cocharacter lattice of $L_{[x,\zeta]}$.

Remark 21.3. In (ii) the linkage $\uparrow$ automatically takes place with respect to coroots of the resonant Levi $L = L_{[x,\zeta]}$ only. Indeed, non-resonant points x, y end up in different blocks, hence cannot be linked.

21.2.4. Center of the classical category $\mathcal{O}$. The categorical center of each block has a well-known description, which we discuss next. To this end, recall the notation $L_\vartheta \leqslant L_{[\vartheta]}$, $\vartheta \in |T /\!\!/ W|$ from §18.1.6.

Proposition 21.4. *Let $\zeta \in \mathbb{G}_m^{\mathrm{rot}}$ be generic and let $\vartheta \in |T /\!\!/ W|$. Then the categorical center of the block $\mathcal{O}_\zeta^\vartheta$ is given by*

$$Z(\mathcal{O}_\zeta^\vartheta) \cong H^\bullet(L_{[\vartheta]}/P_\vartheta; \mathsf{k})$$

as a commutative ring.

Proof. See [Hum08, §13.12 and §13.11] for a useful exposition. The claim follows from Soergel's Endomorphismsatz [Soe90, §2.2], which describes the center as

$$\mathcal{Z}(\mathcal{O}_\zeta^\vartheta) = \left(\frac{\mathsf{k}[t_1, \ldots, t_n]}{\mathsf{k}[t_1, \ldots, t_n]^{W_{[\vartheta]}}} \right)^{W_\vartheta}.$$

The cohomological interpretation follows from Remark 20.4 (applied to the resonant Levi) and is well known: it is for example stated (for GL_n) in [Str09, §4.4], [Bru08, Introduction]. □

The center of the whole category $\mathcal{O}$ is then given by the product of all these cohomology rings. The moral is as follows.

Proposition 21.5. *Let (x,ζ) with ζ generic. Consider the full subcategory $\mathcal{O}_{[x,\zeta]}$ of $\mathcal{O}_\zeta$ spanned by all the blocks which are ζ-resonant with x. Then*

$$H^\bullet(\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_G);\mathsf{k}) = \mathcal{Z}(\mathcal{O}_{[x,\zeta]})$$

Proof. Categorical centers turn block decomposition into products. Singular cohomology turns arbitrary coproducts into products by the Eilenberg–Steenrod axioms. We match the blocks with connected components via Proposition 21.1 and Proposition 19.7; we conclude from Proposition 21.4. □

In particular, the center of the integral part $\mathcal{O}_{[1,\zeta]}$ is given as the cohomology of the fixed-points of loop rotation by ζ on the affine Grassmannian:

$$\mathcal{Z}(\mathcal{O}_{[1,\zeta]}) = H^\bullet(\mathrm{Fix}_{(1,\zeta)}(\mathcal{G}\mathrm{r});\mathsf{k}).$$

21.2.5. Evaluation at tensor products. Under the identification of $\mathcal{O}^{\mathrm{hb}}_\zeta$ with the classical category $\mathcal{O}_\zeta$, the Verma modules M_x are the usual ones, and the Weyl module $V^\zeta_\mu = V_\mu$ is the usual irreducible highest weight representation of $\check{G}$. From the above description of the center, we deduce the following corollary.

Corollary 21.6. *For generic ζ and any $x \in T$, we have a ring isomorphism*

$$K^{\mathrm{top},0}_{T\times\mathbb{G}^{\mathrm{rot}}_m}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathsf{k})_{(x,\zeta)} = \mathcal{Z}^{\mathrm{cat}}(\mathrm{End}_{\mathcal{O}_\zeta}(M_x\otimes_{\mathsf{k}} V_\mu)).$$

Proof. On the level of whole $\mathcal{G}\mathrm{r}_G$, we have $K^{\mathrm{top},0}_{T\times\mathbb{G}^{\mathrm{rot}}_m}(\mathcal{G}\mathrm{r}_G;\mathbb{C})_{(x,\zeta)} = H^\bullet(\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r});\mathsf{k}) = \mathcal{Z}^{\mathrm{cat}}(\mathcal{O}_{[x,\zeta]})$ by the topological version of Corollary 17.30 and Proposition 21.5. Both sides have a compatible GKM description.

On the topological side, we restrict this to $\mathcal{G}\mathrm{r}_{\leqslant\mu}$. In terms of the GKM description, this is implemented by restriction to the simultaneous fixed points of $T\times\mathbb{G}^{\mathrm{rot}}_m$ in $\mathrm{Fix}_{(x,\zeta)}(\mathcal{G}\mathrm{r}_G)$; we described these combinatorially in Proposition 19.12. On the representation-theoretical side, this corresponds to restriction to the weights λ labeling non-zero weight spaces in $\mathrm{End}_{\mathcal{O}_\zeta}(M_x\otimes_{\mathsf{k}} V_\mu)$. This tensor product has a Verma flag, whose constituents have explicit highest weights. These two restriction combinatorially match and the result follows. □

21.3. Blocks at a root of unity

Assume the coefficient ring is $\mathsf{k} = \mathbb{C}$. Assume $\zeta \in \mu_\infty(\mathsf{k})$ is a root of unity of primitive order ℓ. The structure of the category $\mathcal{O}^{\mathrm{hb}}_\zeta$ has been only studied recently [BASV22; Sit22; Los23], under assumptions on ℓ. Again, the purpose of this section is to recall its block decomposition and center – at least in the known cases.

21.3.1. Linkage of cocharacters at a root of unity. Considering the ℓ-dilated dot action $\bullet^\ell_\rho$ of the affine Weyl group on $X_\bullet$ from §16.4.3, we get the notion of linkage of weights [Jan03, §II.6].

Definition 21.7. Two weights $\lambda,\mu \in X_\bullet$ are said to be

(i) *linked at* ζ if they lie in the same orbit of $(W_{\mathrm{aff}}, \bullet^{\ell}_{\rho}) \circlearrowright X_\bullet$,
(ii) *strongly linked at* ζ if there is a chain

$$\lambda = \lambda_0 = s_1 \bullet^{\ell}_{\rho} \lambda_1 \leqslant \lambda_1 = s_2 \bullet^{\ell}_{\rho} \lambda_2 \leqslant \cdots \leqslant \lambda_{r-1} = s_r \bullet^{\ell}_{\rho} \lambda_r \leqslant \lambda_r = \mu$$

for some $r \geqslant 0$, where each $s_i \in W_{\mathrm{aff}}$ is an affine root reflection $s_i = s_{\beta_i, m_i}$ with $\beta_i \in \check{\Phi}^{\bullet}_{+} = \check{\Phi}^{+}_{\bullet}$ and $m_i \in \mathbb{Z}$.

Strong linkage at ζ defines a partial order relation $\lambda \uparrow_\zeta \mu$ on $X_\bullet$, refining the usual relation $\lambda \leqslant \mu$. We sometimes write $\uparrow := \uparrow_\zeta$ if there is no risk of confusion. The equivalence relation induced by the components of $\uparrow$ on the set $X_\bullet$ is given by the orbits of $(W_{\mathrm{aff}}, \bullet^{\ell}_{\rho})$. See [Jan03, §II.6.4] and [And03, §3].

21.3.2. Integral linkage principle at roots of unity. Let ζ be a root of unity of primitive order ℓ.

Assumption 21.8. As in [Sit24, §2.1.2], assume that the order $\ell = \mathrm{ord}(\zeta)$ is odd, coprime to 3 if $\check{G}$ contains a component of type G_2, coprime to $|\pi_1(G)|$, and greater or equal to the Coxeter number of $\check{G}$.

Under this technical assumptions, the blocks of $\mathcal{O}^{\mathrm{hb}}_{[1,\zeta]}$ look as follows.

Proposition 21.9. *Assume $\zeta \in \mathbb{G}^{\mathrm{rot}}_m(\mathsf{k})$ is a root of unity, whose primitive order satisfies the Assumption 21.8. Then the following holds.*

(i) There is a decomposition

$$\mathcal{O}^{\mathrm{hb}}_{[1,\zeta]} \simeq \bigoplus_{\vartheta} \mathcal{O}^{\mathrm{hb},\vartheta}_{[1,\zeta]}$$

into indecomposable blocks labeled by the orbits ϑ of $(W_{\mathrm{aff}}, \bullet^{\ell}_{\rho}) \circlearrowright X_\bullet$.
(ii) More precisely, consider two elements $\lambda, \mu \in X_\bullet$. Then $[M(\mu) : E(\lambda)] \neq 0$ if and only if $\lambda \uparrow \mu$.

Proof. See [Sit24, Proposition 3.7] and [Sit24, proof of Proposition 3.7] for indecomposability. □

21.3.3. Categorical center at roots of unity. The categorical center $\mathcal{Z}(\mathcal{O}^{\mathrm{hb},\vartheta}_{[1,\zeta]})$ of each block $\mathcal{O}^{\mathrm{hb},\vartheta}_{[1,\zeta]}$ is understood by the work of [Sit24], [Sit22], [BASV22]. It is given by the cohomology ring of the corresponding affine flag variety $\mathcal{F}\mathrm{l}^{\vartheta}_{\ell} = \mathrm{L}_\ell G^{\mathrm{sc}} / \mathcal{P}^{\vartheta}_{\ell}$.

Proposition 21.10. *Let $\zeta \in \mathbb{G}^{\mathrm{rot}}_m$ be a root of unity of primitive order ℓ whose primitive order satisfies the Assumption 21.8, and $\vartheta \in X_\bullet/(W_{\mathrm{aff}}, \bullet^{\ell}_{\rho})$. Then the categorical center of the block $\mathcal{O}^{\mathrm{hb},\vartheta}_{[1,\zeta]}$ is given by*

$$Z(\mathcal{O}^{\mathrm{hb},\vartheta}_{[1,\zeta]}) \cong \widehat{H}^\bullet(\mathcal{F}\mathrm{l}^{\vartheta}_{\ell,G^{\mathrm{sc}}}; \mathsf{k})$$

as a commutative ring.

Proof. See [Sit22, Theorem 5.6]. Also see [Sit24, Theorem A] for a deformed variant. □

Remark 21.11 (Evaluation at tensor products.)**.** Again, this is compatible with the expected statement after evaluation:

$$K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}_{\leqslant\mu};\mathsf{k})_{(1,\zeta)} = \mathcal{Z}^{\mathrm{cat}}(\mathrm{End}_{\mathcal{O}}(M_1\otimes_{\mathsf{k}} V_\mu^\zeta)).$$

We have the statement on the level of whole $\mathcal{G}\mathrm{r}_G$ by Corollary 17.30 and Proposition 21.10. We then restrict to given $\mathcal{G}\mathrm{r}_{\leqslant\mu}$, this is combinatorially given by Proposition 19.18. Representation theoretically, this will match linkage of highest weights in the tensor product.

22. Some explicit formulas in type A

This supplementary section is dedicated to spelling out some of our formulas for equivariant K-theory of the affine Grassmannian for GL_n. We also illustrate some of our results explicitly in the case of GL_2, SL_2, PGL_2. The purpose of this section is to give some space for sanity check.

22.1. Formulas in GL_n

For the convenience of the reader, we record the description of the K-theory of $\mathcal{G}\mathrm{r}_{GL_n}$ together with its localization formulas explicitly in usual coordinates.

22.1.1. Localization formulas in GL_n. Consider $G = GL_n$. The connected components of $\mathcal{G}\mathrm{r}_{GL_n}$ are labeled by $\pi_1(GL_n) = \mathbb{Z}$. We want to describe the localization map from functions on the corresponding union of affine blowups.

$$\gamma : \prod_{\mathbb{Z}} \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T\times T /\!\!/ W)) \to \prod_{\lambda\in X_\bullet} \mathcal{O}(T\times \mathbb{G}_m^{\mathrm{rot}})$$

Let $\lambda \in X_\bullet$ and denote $d_1,\dots,d_n$ the heights of the columns of its Young diagram. The localization γ_λ to the corresponding fixed point $t^\lambda = (t^{d_1},\dots,t^{d_n})$ is given as follows. First project to the component labeled by $\overline{\lambda} = d_1 + d_2 + \cdots + d_n \in \mathbb{Z}$. On generators of this copy, the localization map reads as

$$\begin{aligned}
\gamma_\lambda : \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T\times T /\!\!/ W)) &\to \mathcal{O}(T\times\mathbb{G}_m^{\mathrm{rot}})\\
t_i &\mapsto t_i\\
q &\mapsto q\\
s_i &\mapsto t_i q^{d_i}\\
e_j(s_1,\dots,s_n) &\mapsto e_j(t_1q^{d_1},\dots,t_nq^{d_n})\\
b_{j,\ell} = \tfrac{e_j(t_1^\ell,\dots,t_n^\ell)-e_j(s_1^\ell,\dots,s_n^\ell)}{1-q^\ell} &\mapsto \tfrac{e_j(t_1^\ell,\dots,t_n^\ell)-e_j(t_1\cdot q^{d_1},\dots,t_n\cdot q^{d_n})}{1-q^\ell}.
\end{aligned}$$

Note that these formulas work uniformly with respect to the connected components of $\mathcal{G}\mathrm{r}_{GL_n}$.

22.1.2. Determinant line bundle $\mathcal{L}_{\det}$. Let us first recall the construction of the determinant line bundle $\mathcal{L}_{\det}$ on $\mathcal{G}\mathrm{r}_{GL_n}$.

Recollection 22.1. To a perfect complex $\mathcal{E} \in \mathrm{Perf}(\mathcal{X})$ one can associate a graded line bundle $\det(\mathcal{L}) \in \mathrm{Pic}(\mathcal{X})^{\mathbb{Z}}$. If we represent $\mathcal{E}$ by an actual bounded complex of vector bundles on $\mathcal{X}$, the determinant line bundle is given by an alternating product of the top wedge powers of the terms of $\mathcal{E}$, while the grading records their alternating sum of their ranks. This descends to a map $K(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})^{\mathbb{Z}}$. See [Sta, Tag 0FJI], [BS17, Proposition 5.3, §5, §12].

In particular, to any class in $K^0(\mathcal{X})$ one can associate the isomorphism class of the corresponding line bundle; similarly in the topological setting.

Construction 22.2. We obtain the *determinant line bundle* $\mathcal{L}_{\det}$ on $\mathcal{G}\mathrm{r}_{GL_n}$ as follows. On each $\mathcal{G}\mathrm{r}_{\leqslant\mu}$, set

$$\mathcal{L}_{\leqslant\mu,\det} := \det([\mathcal{E}_{\leqslant\mu}] - [\mathcal{G}_{\leqslant\mu}]).$$

These are compatible in μ, and hence give a well-defined equivariant line bundle on the whole $\mathcal{G}\mathrm{r}_{GL_n}$. For each $\sigma \in \pi_1(G)$, it is known to be an ample generator of the Picard group $\mathrm{Pic}(\mathcal{G}\mathrm{r}^\sigma) = \mathbb{Z}$.

Also note that $\mathcal{L}_{\det}$ is naturally $\mathrm{L}^+GL_n \rtimes \mathbb{G}_m^{\mathrm{rot}}$-equivariant. It is not equivariant for the whole loop group $\mathrm{L}G \rtimes \mathbb{G}_m^{\mathrm{rot}}$; this is corrected by working with respect to its central extension.

For fixed μ, $[\mathcal{L}_{\det}] = \det(b_{1,1})$ is expressible in terms of λ-operations on $b_{1,1}$. With rational coefficients, the restriction of $[\mathcal{L}_{\det}]$ to any $\mathcal{G}\mathrm{r}_{\leqslant\mu}$ may thus be expressed in terms of $b_{j,\ell}$. However, this expression is not uniform in μ – the number of terms grows. (We will see shortly that the class $[\mathcal{L}_{\det}] \in K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r})$ can be seen only after the Schubert completion of our affine blowup.)

Claim 22.3. *The class of the determinant line bundle $\mathcal{L}_{\det}$ inside the GKM description is*

$$\gamma_\lambda([\mathcal{L}_{\det}]) = \prod_{k=1}^{n} t_k^{d_k} \cdot q^{\binom{d_k}{2}}, \qquad \forall \lambda \in X_\bullet.$$

Proof. Directly follows from the localization formulas for $b_{1,1}$ in §20.6.4. □

Remark 22.4. One can easily make the sanity check that the above class $[\mathcal{L}_{\det}]$ indeed satisfies the GKM conditions. Namely, given λ and $\lambda' = s_{\beta,m} \cdot \lambda$, where β corresponds to the simple reflection switching i and j, we can compute

$$\gamma_\lambda([\mathcal{L}_{\det}]) - \gamma_{\lambda'}([\mathcal{L}_{\det}]) = \left(\prod_{k=1}^{n} t_k^{d_k} \cdot q^{\binom{d_k}{2}}\right) \cdot \left(1 - t_j t_i^{-1} q^{d_j - d_i + 1}\right)$$

which is divisible by $(1 - \beta \cdot q^m) = (1 - t_j t_i^{-1} q^{d_j - d_i + 1})$.

Remark 22.5. For general reductive G, the Picard group of each connected component of $\mathcal{G}\mathrm{r}_G$ is given by $\mathbb{Z}$. Pulling back $\mathcal{L}_{\det}^{GL_n}$ along representations $G \to GL_n$ gives nonnegative powers of its ample generator; these can be made explicit. See [YZ11, Lemma 4.2, Remark 4.3] for a discussion.

22.1.3. Necessity of Schubert completion.

The following observation well illustrates the necessity of Schubert completion.

Lemma 22.6. *The class of the determinant line bundle does not lies in the functions on the uncompleted deformation*

$$[\mathcal{L}_{\det}] \notin \mathcal{O}(\mathcal{N}_{\mu_\infty}(\mathcal{Z}, T \times \mathbb{G}_m^{\mathrm{rot}} \times T /\!\!/ W)).$$

Proof. For the sake of contradiction, suppose otherwise. Then we could express $[\mathcal{L}_{\det}]$ as a polynomial f over k in finitely many elements from

$$\{t_i, q, e_j, b_{j,\ell} \mid i = 1, \ldots, n; j = 1, \ldots, n; \ell \in \mathbb{N}\} \tag{22.1}$$

For any such fixed f, the localization map $\gamma = (\gamma_\lambda \mid \lambda \in \mathbb{Z}^d)$ has the following property: The powers of q in the family of polynomials

$$\gamma_\lambda(f) \in \mathsf{k}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}, q^{\pm 1}]$$

grow linearly in $\lambda = (d_1, \ldots, d_n)$. Indeed, this is true for the elements from (22.1) by §22.1.1 and is preserved under polynomial combinations.

However, the powers of q in the family

$$\gamma_\lambda([\mathcal{L}_{\det}]) \in \mathsf{k}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}, q^{\pm 1}]$$

grow quadratically in $\lambda = (d_1, \ldots, d_n)$ by (22.3).

Therefore, there exists a sufficiently big λ, such that $\gamma_\lambda([\mathcal{L}_{\det}]) \neq \gamma_\lambda(f)$, a contradiction. □

Remark 22.7. The statement of Lemma 22.6 follows for any reductive group G. Indeed all the classes in question are base-changed from the GL_n-case; the statement about linear vs. quadratic growth of q-powers for varying λ still holds – the localizations may be also expressed purely Lie theoretically.

In other words, $[\mathcal{L}_{\det}]$ is not present in the uncompleted ring $\mathcal{O}(\mathcal{N}_{\mu_\infty}(T \times T /\!\!/ W))$. At the same time $[\mathcal{L}_{\det}]$ is as nice a line bundle on $\mathcal{G}\mathrm{r}_G$ as one could hope for. This makes it challenging to give a purely K-theoretical interpretation of the decompleted ring $\mathcal{O}(\mathcal{N}_{\mu_\infty}(T \times T /\!\!/ W))$.

22.1.4. Some remarks.

Remark 22.8. When $G = GL_n$ and the coefficient ring k is a field of characteristic zero, it is possible to do with less topological generators. As complete topological λ-over the equivariant base, $K^{\mathrm{top},0}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}\mathrm{r}; \mathbb{C})$ is generated by the single tautological element $b_{1,1}$.

For GL_n, the representation ring $R(GL_n; \mathsf{k}) = \mathcal{O}(T /\!\!/ W)$ is a quotient of the free λ-ring in one variable. Since we are working over a field of characteristic zero, Newton's formulas imply that we may further express any λ operation (an elementary symmetric polynomial) purely in terms of Adams operations (power sums).

However, the above goes wrong if we either pass to more general G, or if we work with more general coefficients such as $\mathbb{Z}$. In total, we view the above simplification rather as a curiosity then a result.

Remark 22.9. Assume $G = GL_n$. In our presentation of K-theory of $\mathcal{G}\mathrm{r}_{GL_n}$, we have used the tautological class $b_{1,1}$ and ψ-operation (or λ-operations) on it. However, for fixed n, there are relations between these classes.

This is closely related to the following phenomenon. The representation ring $R(GL_n) = \mathcal{O}(T /\!\!/ W)$ is a quotient of the free λ-ring in one variable. The free λ-ring in one variable is given by putting all $R(GL_n)$ together for varying n. To have a uniform presentation for the K-theory, one could consider the ind-scheme

$$\mathcal{G}\mathrm{r}_{GL} := \operatorname*{colim}_n \mathcal{G}\mathrm{r}_{GL_n}$$

equivariant with respect to the extended loop group of the ind-group scheme $GL := \operatorname{colim}_n GL_n$. The equivariant K-theory of this object should be cooked out by taking two copies of the free λ-ring, doing our affine blowup at the union of quasi-diagonals over all roots of unity, and appropriately completing. However, we did not consider this seriously.

22.2. Formulas in type A_1

We also recall some formulas in type A_1. We take this as an opportunity to illustrate the passage between isogenous groups.

22.2.1. Isogenies and coordinates. For $G = GL_2$, we have $G^{\mathrm{sc}} = SL_2$ and $G^{\mathrm{ad}} = PGL_2$. We have the maps

$$SL_2 \longrightarrow GL_2 \longrightarrow PGL_2.$$

These induce corresponding maps $\pi_1(SL_2) \to \pi_1(GL_2) \to \pi_1(PGL_2)$, which label the connected components of $\mathcal{G}\mathrm{r}_{SL_2} \to \mathcal{G}\mathrm{r}_{GL_2} \to \mathcal{G}\mathrm{r}_{PGL_2}$. The induced maps on maximal tori identify as

$$\begin{array}{ccccc} T_{SL_2} & \longrightarrow & T_{GL_2} & \longrightarrow & T_{PGL_2} \\ t & \longmapsto & (t, t^{-1}) & & \\ & & (t_1, t_2) & \longmapsto & t_1 t_2^{-1} \end{array}$$

On rings of functions, this means

$$\begin{array}{ccccc} \mathcal{O}(T_{SL_2}) & \longleftarrow & \mathcal{O}(T_{GL_2}) & \longleftarrow & \mathcal{O}(T_{PGL_2}) \\ t & \longleftarrow\!\shortmid & t_1 & & \\ t^{-1} & \longleftarrow\!\shortmid & t_2 & & \\ & & t_1 t_2^{-1} & \longleftarrow\!\shortmid & z \\ t^2 & \longleftarrow & \longleftarrow & \longleftarrow\!\shortmid & z \end{array}$$

Taking quotients by W, these become the following maps of free polynomials rings

$$\begin{array}{ccccc} \mathcal{O}(T_{SL_2} /\!\!/ W) & \longleftarrow & \mathcal{O}(T_{GL_2} /\!\!/ W) & \longleftarrow & \mathcal{O}(T_{PGL_2} /\!\!/ W) \\ \mathsf{k}[t + t^{-1}] & \longleftarrow & \mathsf{k}[t_1 + t_2, t_1 t_2] & \longleftarrow & \mathsf{k}[z + z^{-1}] \\ t + t^{-1} & \longleftarrow\!\shortmid & t_1 + t_2 & & \\ 1 & \longleftarrow\!\shortmid & t_1 t_2 & & \\ & & t_1 t_2^{-1} + t_1^{-1} t_c & \longleftarrow\!\shortmid & z + z^{-1} \\ t^2 + t^{-2} & \longleftarrow & \longleftarrow & \longleftarrow\!\shortmid & z + z^{-1}, \end{array}$$

in other words, the degree two faithhfully flat map $\mathcal{O}(T_{SL_2} /\!\!/ W) \hookleftarrow \mathcal{O}(T_{PL_2} /\!\!/ W)$ given by the composition can be presented as

$$\mathsf{k}[t + t^{-1}] = \mathsf{k}[z + z^{-1}][x]/(x^2 - (z + z^{-1}) - 2).$$

22.2.2. Localization formulas. Even more explicitly, we write out some explicit formulas in GL_2, PGL_2 and SL_2. Recall the q-quantum integers

$$[d]_q := \tfrac{1-q^d}{1-q} \in \mathcal{O}(\mathbb{G}_m^{\mathrm{rot}}).$$

Example 22.10. For $G = GL_2$, we have $\pi_1(GL_2) = \mathbb{Z}$. On each connected component, the multiplicative generators of $K^{\mathrm{top},0}_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathcal{G}r; \mathsf{k})$ are given, together with their localizations, as

$$\begin{aligned}
\mathcal{O}(\mathcal{N}(\mathcal{Z}, T_{GL_2} \times T_{GL_2} /\!\!/ W)) &\xrightarrow{\gamma_\lambda} \mathcal{O}(T_{GL_2} \times \mathbb{G}_m^{\mathrm{rot}}) \\
t_1 &\mapsto t_1 \\
t_2 &\mapsto t_2 \\
q &\mapsto q \\
s_1 &\mapsto q^{d_1} t_1 \\
s_2 &\mapsto q^{d_2} t_2 \\
\tfrac{(t_1+t_2)-(s_1+s_2)}{(1-q)} &\mapsto t_1[d_1]_q + t_2[d_2]_q \\
\tfrac{t_1t_2 - s_1s_2}{(1-q)} &\mapsto t_1t_2[d_1+d_2]_q \\
&\ldots \\
\tfrac{(t_1^\ell+t_2^\ell)-(s_1^\ell+s_2^\ell)}{(1-q^\ell)} &\mapsto t_1^\ell \cdot \tfrac{1-q^{d_1\ell}}{1-q^\ell} + t_2^\ell \cdot \tfrac{1-q^{d_2\ell}}{1-q^\ell} = t_1^\ell[d_1]_{q^\ell} + t_2^\ell[d_2]_{q^\ell} \\
\tfrac{t_1^\ell t_2^\ell - s_1^\ell s_2^\ell}{(1-q^\ell)} &\mapsto t_1^\ell t_2^\ell \cdot \tfrac{1-q^{(d_1+d_1)\ell}}{1-q^\ell} = t_1^\ell t_2^\ell [d_i + d_j]_{q^\ell} \\
&\ldots
\end{aligned}$$

so we have two extra generators for each $\ell \in \mathbb{N}$ given by certain dilated quantum integers.

For SL_2, we have $\pi_1(SL_2) = 1$. On the unique component, the above formulas specialize along $t_1 \mapsto t, t_2 \mapsto t^{-1}$. For PGL_2, we have $\pi_1(PGL_2) = \mathbb{Z}/2\,\mathbb{Z}$. We denote the two connected components by $\mathcal{G}r_{PGL_2} = \mathcal{G}r^{\mathrm{even}}_{PGL_2} \coprod \mathcal{G}r^{\mathrm{odd}}_{PGL_2}$. The formulas again follow by specializing the above.

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