

RESEARCH ARTICLE

Bulletin of the London
Mathematical Society

A short proof of a central limit theorem for the order of the giant component and k -core

Michael Anastos¹  | Joshua Erde²  | Mihyun Kang³  |
Vincent Pfenninger⁴ 

¹Institute of Science and Technology
Austria (ISTA), Klosterneuburg, Austria

²School of Mathematics, University of
Birmingham, Birmingham, UK

³Institute of Discrete Mathematics, Graz
University of Technology, Graz, Austria

⁴Department of Mathematics, London
School of Economics, London, UK

Correspondence

Joshua Erde, School of Mathematics,
University of Birmingham, Birmingham,
UK.

Email: j.erde@bham.ac.uk

Funding information

Austrian Science Fund, Grant/Award
Numbers: 10.55776/P36131, 10.55776/I6502,
10.55776/ESP3863424; Swiss National
Science Foundation, Grant/Award
Number: P500-2_235474

Abstract

In this note we outline a new and simple approach to proving central limit theorems for various ‘global’ graph parameters that have robust ‘local’ approximations, using the Efron–Stein inequality, which relies on a combinatorial analysis of the stability of these approximations under resampling an edge. As an application, we give short proofs of a central limit theorem for the order of the giant component and of the k -core for sparse random graphs.

MSC 2020

05C80, 60F05 (primary)

1 | INTRODUCTION

In discrete probability, and in particular the theory of random graphs, there has been much interest in establishing *central limit theorems* for various parameters of interest. More precisely, given a sequence $(Z_n)_{n \in \mathbb{N}}$ of random variables defined on the probability spaces $G(n, p)$, we are interested in when

$$X_n := \frac{Z_n - \mathbb{E}[Z_n]}{\sqrt{\mathbb{V}[Z_n]}} \xrightarrow{d} \mathcal{N}(0, 1),$$

© 2026 The Author(s). *The Bulletin of the London Mathematical Society* is copyright © London Mathematical Society. This is an open access article under the terms of the [Creative Commons Attribution](https://creativecommons.org/licenses/by/4.0/) License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

as $n \rightarrow \infty$, that is, when X_n converges in distribution to the standard normal distribution. Here, by $G(n, p)$ we denote the binomial random graph, that is, the random graph on n vertices where each edge appears independently with probability p .

Early work on this topic established such results for various types of *subgraph counts* in random graphs, and many different techniques have been developed to deal with these parameters—the method of moments [18, 19, 25], Stein’s method [4], U -statistics and related methods [16, 21] and martingale methods [3, 10]. Moreover, these methods can also be used to show *multivariate central limit theorems* for finite collections of subgraph counts, establishing *joint normality* of the rescaled random variables. In particular, this implies that, for appropriate ranges of p and excluding some pathological cases, any finite linear combination of these parameters also satisfies a central limit theorem, which in turn implies central limit theorems for a broad class of ‘local’ graph parameters—those which can be determined by looking at bounded-order subgraphs of $G(n, p)$.

More recently, central limit theorems have been proved for more ‘global’ graph parameters. An early example of this comes from the work of Pittel and Wormald on the order of the giant component in a supercritical sparse random graph [23, 24] which used enumerative/analytic methods to establish a central limit theorem. More modern proofs [5, 7] involve the analysis of a graph exploration process using martingale methods. Central limit theorems have also been shown for the order of the k -core [9, 17] and very recently the rank and the matching number [14, 15, 20], both of which analyse certain peeling processes on $G(n, p)$, using martingale methods or the differential equations method, respectively.

Recent work of the authors [2] establishes a central limit theorem for another ‘global’ graph parameter, the *circumference*, which is the length of the longest cycle. Rather than analysing the evolution of some random process, the proof instead employs a *perturbative* argument using more elementary, combinatorial methods based on the Efron–Stein inequality, using the joint normality of certain subgraph counts as a ‘black-box’. However, the application in [2] is still quite complicated and technical, due in part to certain ‘non-monotonicity’ issues that arise within the structural analysis in the particular case of the circumference.

The purpose of this note is to demonstrate the simplicity of the method developed by the authors in [2] by giving some short proofs of known results of central limit theorems—for the order of the giant component and the k -core in sparse random graphs in their respective supercritical regimes.

Theorem 1.1. *Let $c > 1$, let $p = \frac{c}{n}$ and let L be the largest component in $G(n, p)$. Then $|L|$ satisfies a central limit theorem.*

Theorem 1.2. *Let $k \geq 3$, and let $\hat{c}_k$ be the unique real number in (k, ∞) such that*

$$\mathbb{P}[\text{Poisson}(\hat{c}_k) \leq k - 1] = (e\hat{c}_k)^{-1}. \quad (1)$$

Let $c > \hat{c}_k$, let $p = \frac{c}{n}$ and let K be the k -core of $G(n, p)$. Then $|V(K)|$ satisfies a central limit theorem.

It is straightforward to show that $\hat{c}_k$ is well defined and that $\hat{c}_k = k + \sqrt{2k \log k} + O(\sqrt{k})$ as $k \rightarrow \infty$. For comparison, the k -core threshold c_k , that is, the threshold value of c for the existence of a (non-empty) k -core, is shown in [22] to be asymptotically $k + \sqrt{k \log k} + O(\log k)$, and so, whilst $\hat{c}_k$ in Theorem 1.2 is not quite sharp, it is very close to c_k . We also note that with a bit more

care, it is possible to extract quantitative bounds on the variances in Theorems 1.2 and 1.1, as in [2].

The proofs for both Theorems 1.2 and 1.1 follow the same rough idea: We take a sequence of ‘local’ approximations to the order of the giant component (or k -core) by taking, for each vertex $v \in V(G)$, an approximation of the indicator of the event that v lies inside the giant component (or inside the k -core, respectively), based on the ‘local’ structure of $G \sim G(n, p)$ around v , taking into account larger and larger neighbourhoods as our sequence develops.

A relatively standard application of Stein’s method will show that each of these local approximations satisfies a central limit theorem and so, modulo some small technical details, it is sufficient to bound in a sufficiently strong manner the errors of these approximations, which we do using the Efron–Stein inequality to bound the variance of these errors.

Roughly, the Efron–Stein inequality allows us to bound the variance of a graph parameter by bounding the expected effect of *resampling a single edge*. A simple combinatorial analysis will show that, in both cases, the effect of resampling an edge $e = \{u, v\}$ on the error of the approximation is in some way ‘localised’ within the components C_u and C_v containing u and v of the *remainder* (or *mantle*), the graph R remaining after we delete the giant component (or k -core, respectively). However, simple first moment calculations demonstrate that, for sufficiently large c , the component structure of R resembles that of a *subcritical* random graph, and so the expected effect of resampling an edge is ‘small’.

In the case of the giant component, this holds for any $c > 1$, whilst for the k -core we require that c is sufficiently large (i.e., larger than $\hat{c}_k$). In fact, heuristically this should hold already for any c above the k -core threshold c_k , although there does not seem to be a simple proof of this fact, and it is unclear if it follows directly from known results in the literature. However, we note that such a result, stating that the mantle has a subcritical distribution for a certain range of p , is all that would be required to extend the proof of Theorem 1.2 to this range of p . We note also that, whilst the statement and proof of Theorem 1.2 extends without change to the case $k = 2$, in fact similar calculations as that for the giant component show that the threshold for the appearance of a linear 2-core coincides with that for the giant component, and that the ‘mantle’ of the 2-core will have a subcritical distribution for any $c > 1$. It is then possible to show an analogue of Theorem 1.2 for $k = 2$ holds for any $c > 1$ using similar methods, although we omit the details.

The case of the circumference, dealt with in [2], follows broadly the same strategy, but has to deal with significant technical difficulties which do not arise in the case of the giant component or k -core. These difficulties are mostly related to the method of ‘approximating’ the vertex set of a longest cycle C , in itself not an obviously straightforward task. In particular, the fact that this approximation is not monotone under edge deletions, which serendipitously is the case for the applications in this paper, leads to sophisticated combinatorial analysis to determine the effect of resampling an edge. Furthermore, whilst it is relatively straightforward to estimate the variance of $|L|$ and $|V(K)|$, which is necessary to the application of Stein’s method, the corresponding calculation for $|C|$ is far from trivial. Indeed, the resolutions of both of these problems take almost as long, if not longer, than this paper.

Looking forward, we note that, in many cases, it can be shown that multiple parameters of interest simultaneously satisfy a *multidimensional central limit theorem*. For example, Pittel and Wormald [23, 24] showed that the order of the giant component and the order of the 2-core, when appropriately rescaled, are jointly Gaussian. It would be interesting to know if our methods could be extended to the multidimensional case, reducing then the derivation of a multidimensional central limit theorem to the combinatorial analysis of the simultaneous effect of resampling an edge on some collection of parameters.

2 | PRELIMINARIES

In this paper, we consider a *component* to be a maximal set of vertices that induces a connected subgraph. For a graph G and a vertex set $A \subseteq V(G)$, we define $G[A]$ to be the *induced subgraph of G with respect to A* , that is, the graph $G[A]$ with vertex set $V(G[A]) = A$ and edge set $E(G[A]) = \{\{x, y\} \in E(G) : x, y \in A\}$. Moreover, we let $G - A := G[V(G) \setminus A]$. A *rooted graph* is a pair (H, r) where H is a graph and $r \in V(H)$. The *radius* of a rooted graph (H, r) is the maximum distance of a vertex in H to the root r (in particular, if H is disconnected, the radius of (H, r) is ∞). Two rooted graphs (H_1, r_1) and (H_2, r_2) are *isomorphic* (written $(H_1, r_1) \cong (H_2, r_2)$) if there exists a bijection $f : V(H_1) \rightarrow V(H_2)$ such that $f(r_1) = r_2$ and $f(u)f(v) \in E(H_2)$ if and only if $uv \in E(H_1)$ for all $u, v \in V(H_1)$.

The k -core of a graph G is defined to be the maximal subgraph $K \subseteq G$ of minimum degree at least k , which is easily seen to be unique. A useful alternative characterisation of the k -core is given by the following *peeling process*. Starting with G , we recursively delete some vertex of degree less than k . Again it is easy to see that the resulting graph K is the k -core of G . We note that, for any vertex w which lies in a component C_w of $G - V(K)$, there is some k -degenerate ordering of C_w , that is, an ordering of the vertices of C_w such that each vertex is adjacent to fewer than k vertices which appear before it or in K .

2.1 | Probabilistic inequalities

We use the following lemma, proved in [2, Lemma 3.6], which bounds the variance of ‘local’ graph parameters using the Efron–Stein inequality [12] by considering the effect of resampling an edge.

Lemma 2.1. *Let $G \sim G(n, p)$ be a random graph, let $\{\psi_G(v) : v \in V(G)\}$ be a set of (isomorphism-invariant) random variables and let $Z := \sum_{v \in V(G)} \psi_G(v)$. Let $f \in \binom{V(G)}{2}$ be arbitrary and let $G^+ = G + f$ and $G^- = G - f$. Suppose that $M > 0$ is such that $|\psi_G(v)| \leq M$ for each $v \in V(G)$ and let*

$$D := \{v \in V(G) : \psi_{G^+}(v) \neq \psi_{G^-}(v)\}.$$

Then

$$\mathbb{V}[Z] \leq 2M^2 p(1-p)n^2 \mathbb{E}[|D|^2].$$

We will also use a probabilistic form of Minkowski’s inequality, see, for example, [6, Equation (19.4)].

Theorem 2.2 (Minkowski’s inequality for variances). *For jointly distributed random variables Y and Z with $\mathbb{E}[Y^2], \mathbb{E}[Z^2], |\mathbb{E}[Y \cdot Z]| < \infty$,*

$$\mathbb{V}[Y + Z] \leq \left(\sqrt{\mathbb{V}[Y]} + \sqrt{\mathbb{V}[Z]} \right)^2.$$

Furthermore, under the same conditions, Theorem 2.2 also implies[†]

$$|\mathbb{V}[Y] - \mathbb{V}[Z]| \leq \mathbb{V}[Y - Z] + 2(\mathbb{V}[Y - Z])^{\frac{1}{2}} \min\{\mathbb{V}[Y]^{\frac{1}{2}}, \mathbb{V}[Z]^{\frac{1}{2}}\}. \quad (2)$$

When applying (2), we further upper bound the minimum by one of its terms, as convenient. We note that equivalent statements can also be derived from a probabilistic version of the Cauchy–Schwarz inequality, as in [2, Theorem 3.1].

In the proof we will consider sequences of approximations which are themselves defined on the sequence $(G(n, p))_{n \in \mathbb{N}}$ of random graphs. For this reason, when talking about limits, it will be useful to have a version of Slutsky’s theorem [26] for ‘double limits’. It can easily be checked that the proof carries over into this setting.

Given a doubly-indexed sequence $(a_{\ell, n})_{\ell, n \in \mathbb{N}}$ of real numbers and $a \in \mathbb{R}$ we write

$$\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} a_{\ell, n} = a, \quad (3)$$

if there exists a function $N : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $\varepsilon > 0$ there exists ℓ_0 such that for all $(\ell, n) \in \mathbb{N}^2$ with $\ell \geq \ell_0$ and $n \geq N(\ell)$ we have $|a_{\ell, n} - a| < \varepsilon$. Given a doubly-indexed sequence $(X_{\ell, n})_{\ell, n \in \mathbb{N}}$ of real random variables and a real random variable X , we write $X_{\ell, n} \xrightarrow{d} X$ as $\ell, n \rightarrow \infty$ if, for every $x \in \mathbb{R}$ at which $x \mapsto \mathbb{P}[X \leq x]$ is continuous, we have

$$\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} \mathbb{P}[X_{\ell, n} \leq x] = \mathbb{P}[X \leq x].$$

Similarly, we write $X_{\ell, n} \xrightarrow{p} X$ as $\ell, n \rightarrow \infty$ if for all $\delta > 0$,

$$\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} \mathbb{P}[|X_{\ell, n} - X| \geq \delta] = 0.$$

Theorem 2.3 (Slutsky’s theorem [26]). *Let $(X_{\ell, n})_{\ell, n \in \mathbb{N}}$, $(Y_{\ell, n})_{\ell, n \in \mathbb{N}}$, and $(Z_{\ell, n})_{\ell, n \in \mathbb{N}}$ be doubly-indexed sequences of real random variables. Let Y be a real random variable and let $x, z \in \mathbb{R}$. If $X_{\ell, n} \xrightarrow{p} x$, $Y_{\ell, n} \xrightarrow{d} Y$ and $Z_{\ell, n} \xrightarrow{p} z$ as $\ell, n \rightarrow \infty$, then*

$$X_{\ell, n} + Y_{\ell, n} \cdot Z_{\ell, n} \xrightarrow{d} x + zY, \quad \text{as } \ell, n \rightarrow \infty.$$

2.2 | A central limit theorem for weighted neighbourhood sums

We will also need the following general result, proved in [2, Lemmas 5.2 and 7.6], which asserts a central limit theorem for a range of graph parameters which have a representation as a weighted sum of neighbourhood counts.

[†] To prove (2), we may assume without loss of generality that $\mathbb{V}[Z] \leq \mathbb{V}[Y]$, then write $Y = Z + (Y - Z)$ and apply Theorem 2.2.

More precisely, given $(\ell, n) \in \mathbb{N}^2$ let $\mathcal{H}$ be the set of all rooted graphs (H, r) with radius at most ℓ and at most n vertices and let $\mathcal{T}$ be the set of rooted trees $(T, r) \in \mathcal{H}$ with $1 \leq v(T) \leq t(c, \ell)$, for some appropriately large function t of c and ℓ . A *weighted ℓ -neighbourhood count* is a random variable of the form

$$X = \sum_{v \in V(G)} \sum_{(H, r) \in \mathcal{H}} \alpha_{(H, r)} \mathbb{1}_{\{(G[B_G(v, \ell)], v) \cong (H, r)\}}$$

where $\alpha_{(H, r)} \in [0, 1]$ for each $(H, r) \in \mathcal{H}$ and $B_G(v, \ell)$ is the set of vertices in G at distance at most ℓ from v . Moreover, we define the *truncation* of X to be

$$Y = \sum_{v \in V(G)} \sum_{(T, r) \in \mathcal{T}} \alpha_{(T, r)} \mathbb{1}_{\{(G[B_G(v, \ell)], v) \cong (T, r)\}}.$$

Lemma 2.4. *Let $c > 1$ and let $p = \frac{c}{n}$. If X is a weighted ℓ -neighbourhood count and Y is the truncation of X , then the following hold.*

- (i) $\mathbb{V}[X - Y] \leq f_1(n, \ell) \cdot n$, where $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_1(n, \ell) = 0$.
- (ii) If $\mathbb{V}[Y] = \omega\left(n^{\frac{2}{3}}\right)$, then Y satisfies a central limit theorem.

For a proof, see [2, Lemmas 5.2 and 7.6]. We note that this follows the classic approach to convergence via Stein's method, which originated in [4], where it was applied to the problem of counting induced subgraphs and more generally to random variables which can be decomposed as a sum of *semi-induced properties*, such as weighted neighbourhood counts, under some mild technical restrictions (see [4, Theorem 5]). This version of Stein's method is by now a standard tool in the area, and the statement of Lemma 2.4, in particular part (ii), should come as no surprise to experts in the field.

2.3 | Typical properties of the remainder and mantle

We will require the following properties of a supercritical sparse random graph which follow from standard arguments, but for completeness we include a short proof. In this paper, $\log$ refers to the natural logarithm.

Lemma 2.5. *Let $c > 1$ and let $p = \frac{c}{n}$. Let $G \sim G(n, p)$, $e := \{u, v\} \in \binom{V(G)}{2}$, $G^- := G - e$, and $G^+ := G + e$. Let $\mathcal{E}$ be the event that for $* \in \{-, +\}$, we have that G^* has a unique component L^* of linear order and all other components have order at most $\log^4 n$. Then $\mathbb{P}[\mathcal{E}] = 1 - o\left(\frac{1}{n^2}\right)$.*

Furthermore, let $R^- = G^- - L^-$ and for each $v \in V(R^-)$, we let C_v^- be the component of R^- containing v and $C_v^- := \emptyset$ for $v \in L^-$. Then for $W := C_u^- \cup C_v^-$, we have

$$\mathbb{E}[|W|^2 \mathbb{1}_{\{|W| \geq \ell\}}] \leq f_2(n, \ell),$$

where $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_2(n, \ell) = 0$.

Proof. The first part of the lemma is a straightforward modification of a standard argument that appears in any textbook on random graphs (see, e.g., [13]).

Furthermore, by the first part we can assume in the second part that all components of R^- have order at most $\log^4 n$ and then calculate

$$\begin{aligned} \mathbb{E}[|W|^2 \mathbb{1}_{\{|W| \geq \ell\}} \mathbb{1}_{\mathcal{E}}] &\leq \sum_{j=\ell}^{\log^4 n} j^2 \binom{n-2}{j-2} j^{j-2} p^{j-2} (1-p)^{j(n-j)} + o(1) \\ &\leq \sum_{j=\ell}^{\log^4 n} j^2 e^2 \left(c e^{1-c+o(1)} \right)^{j-2} + o(1) := f_2(n, \ell) \end{aligned}$$

and the conclusion follows since $ce^{1-c} < 1$ for $c \neq 1$. $\square$

We will also need similar statements about the properties of the mantle.

Lemma 2.6. *Let $k \geq 3$, let $c > \hat{c}_k$, where $\hat{c}_k$ is defined as in (1), and let $p = \frac{c}{n}$. Let $G \sim G(n, p)$, let $e := \{u, v\} \in \binom{V(G)}{2}$, and let $G^- := G - e$ and $G^+ := G + e$. For each $* \in \{-, +\}$, let K^* be the k -core of G^* and let $R^* := G^* - V(K^*)$. Let $\mathcal{E}$ be the event that for $* \in \{-, +\}$, all components of R^* have order at most $\log^4 n$. Then the following hold.*

- (i) $\mathbb{P}[\mathcal{E}] = 1 - o\left(\frac{1}{n^2}\right)$.
- (ii) We have $\hat{c}_k \geq c_k$ where c_k is the k -core threshold.
- (iii) Furthermore, for each $w \in V(R^-)$, we let C_w^- be the component of R^- containing w and $C_w^- := \emptyset$ for $w \in K^-$. Then for $W := C_u^- \cup C_v^-$, we have

$$\mathbb{E}[|W|^2 \mathbb{1}_{\{|W| \geq \ell\}} \mathbb{1}_{\mathcal{E}}] \leq f_3(n, \ell),$$

where $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_3(n, \ell) = 0$.

Proof. We first note that the function $f(x) = x \cdot \mathbb{P}(\text{Poisson}(x) \leq k-1)$ is strictly decreasing for $x > k$. Indeed, for $y > x > k$, as $f(y) = \sum_{i=1}^k e^{-y} y^i / (i-1)! \leq e^{-(y-x)} (y/x)^k f(x)$ we have that

$$\frac{f(y)}{f(x)} \leq e^{-(y-x)} (y/x)^k = e^{-(y-x)} (1 + (y-x)/x)^k \leq e^{-(y-x) + k(y-x)/x} = e^{-(y-x)(1-k/x)} < 1.$$

By Poisson convergence (see, e.g., [11, Theorem 3.6.1]) and the above, we have that for this choice of c , there exists a constant $\alpha > 1$ such that for any $j \leq \log^4 n$,

$$\mathbb{P}(\text{Bin}(n-j, p) \leq k-1) = (1 + o(1)) \mathbb{P}(\text{Poisson}(c) \leq k-1) = (\alpha ec - o(1))^{-1}.$$

The proof of (i) is similar to the argument appearing in [1, Lemma 7 (b)] or [2, Lemma 4.15]. We show the statement for G^+ , and the statement for G^- follows similarly. We first note that with probability $1 - o\left(\frac{1}{n^2}\right)$, the maximum degree of G^+ is at most $\log^2 n$ in this regime of p . Consider then, the k -core peeling process on G^+ . Since the order of the largest component in the remainder can increase by a multiplicative factor of at most $\log^2 n$ each time a vertex is ‘peeled’, if there is a

component of $R^+ := G^+ - V(K^+)$ of order at least $\log^4 n$ at the end of this process, there is some intermediary stage at which there is a component C of peeled vertices of order between $\log^2 n$ and $\log^4 n$. However, by construction there is a k -degenerate ordering of the vertices of C , and hence each vertex in C has at most $k - 1$ neighbours in $V(G) \setminus C$. A simple union bound then shows that the probability that this occurs is at most

$$\sum_{j=\log^2 n}^{\log^4 n} \binom{n}{j} j^{j-2} p^{j-2} (\mathbb{P}(\text{Bin}(n-j, p) \leq k-1))^j \leq \sum_{j=\log^2 n}^{\log^4 n} \frac{n^2}{j^2} (ec)^{j-2} (\alpha ec - o(1))^{-j} = o(n^{-2}).$$

To prove (ii), suppose for a contradiction that $\hat{c}_k < c_k$ and let c be such that $\hat{c}_k < c < c_k$. Then, since $c < c_k$ by results in [22], we have that $K = \emptyset$ with probability $1 - o(1)$. Together with (i), this implies that with probability $1 - o(1)$, we have that G consists only of components of order at most $\log^4 n$. However, since $c > \hat{c}_k > k > 1$, this contradicts Lemma 2.5, which states that G has a component of linear order with probability $1 - o\left(\frac{1}{n^2}\right)$.

Using (i) we can calculate similarly for (iii),

$$\begin{aligned} \mathbb{E}[|W|^2 \mathbb{1}_{\{|W| \geq \ell\}} \mathbb{1}_{\mathcal{E}}] &\leq \sum_{j=\ell}^{\log^4 n} j^2 \binom{n-2}{j-2} j^{j-2} p^{j-2} (\mathbb{P}(\text{Bin}(n-j, p) \leq k-1))^j \\ &\leq \sum_{j=\ell}^{\log^4 n} j^2 e^2 (ec)^{j-2} (\alpha ec - o(1))^{-j} := f_3(n, \ell), \end{aligned}$$

where it is then clear that $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_3(n, \ell) = 0$. □

As mentioned in the introduction, one should expect a qualitatively similar statement to hold for any p above the k -core threshold. Indeed, the work of [8] suggests that the components of the mantle should resemble *subcritical* (multi-type) Galton–Watson trees, although we were not able to extract an explicit statement that would imply Lemma 2.6 from their work.

2.4 | Variance of the order of the giant and k -core

For our arguments, it will be important that the variance of the order of the giant component (or k -core, respectively) is linear in n , which can be shown by elementary considerations. We include a proof for completeness.

Lemma 2.7. *Let $k \geq 3$ and let $\hat{c}_k$ be as defined in (1). Let $p = \frac{c}{n}$, let L be the largest component of $G(n, p)$, and let K be the k -core of $G(n, p)$.*

(V1) *If $c > 1$, then $\mathbb{V}[|L|] = \Omega(n)$.*

(V2) *If $c > \hat{c}_k$, then $\mathbb{V}[|V(K)|] = \Omega(n)$.*

Proof. We just prove the second statement, the first can be proved with an analogous argument.

We utilise a sprinkling argument, taking p_1 and p_2 such that $(1 - p_1)(1 - p_2) = 1 - \frac{c}{n}$, where $p_1 = \frac{c'}{n}$, for some c' such that $c > c' > \hat{c}_k$ thus $p_2 = \Theta\left(\frac{1}{n}\right)$. Let $G_1 \sim G(n, p_1)$ and $G_2 \sim G(n, p_2)$ be independent random graphs and let $G = G_1 \cup G_2$ so that $G \sim G(n, p)$. By Lemma 2.6(ii), we have that $\hat{c}_k$ is above the k -core threshold c_k . Thus standard results on the k -core threshold (see, e.g., [22]) imply that whp[†] G_1 contains a k -core of linear order. Let us write K_1 for the k -core of G_1 , where whp $|V(K_1)| = \Omega(n)$.

Let A_1 be the set of isolated vertices in G_1 and let $A_2 \subseteq A_1$ be the set of vertices which are isolated in $G - E_{G_2}(A_1, V(K_1))$. Then standard calculations show that whp $|A_2| = \Omega(n)$.

We let K_2 be the k -core in $G - E_{G_2}(A_2, V(K_1))$. Then $K_1 \subseteq K_2 \subseteq K$ and we note that $|V(K)| = |V(K_2)| + Z$, where Z is the number of vertices in A_2 adjacent to at least k vertices of K_2 in G_2 . Furthermore, conditioned on G_1 and $G - E_{G_2}(A_2, V(K_1))$, noting that this determines A_1, A_2, K_1 and K_2 , we have that $Z \sim \text{Bin}(|A_2|, q)$ where $q = \mathbb{P}[\text{Bin}(|V(K_1)|, p_2) \geq k]$ thus, $\mathbb{V}[Z] = q(1 - q)|A_2|$.

Let $\mathcal{E}^*$ be the whp event that $|V(K_1)|, |A_2| = \Omega(n)$, which also implies that $q = \Omega(1)$, then a standard calculation (see, e.g., [2, Lemma 6.4]) shows that

$$\mathbb{V}[|V(K)|] \geq (1 - o(1))\mathbb{V}[Z \mid \mathcal{E}^*] = \Omega(n),$$

where the implicit constant depends on c and k . □

3 | A CENTRAL LIMIT THEOREM FOR THE ORDER OF THE GIANT COMPONENT

Let $G \sim G(n, p)$ where $p = \frac{c}{n}$ and $c > 1$. Let us define a function $\phi : V(G) \rightarrow \{0, 1\}$ by

$$\phi(v) := \begin{cases} 1 & \text{if } v \in V(L) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $Z := |L| = \sum_{v \in V(G)} \phi(v)$. For $\ell \in \mathbb{N}$, we also define a ‘localised’ version of ϕ by

$$\phi_\ell(v) := \begin{cases} 1 & \text{if } \partial_G(v, \ell) \neq \emptyset \text{ or } |B_G(v, \ell)| > \log^4 n \\ 0 & \text{otherwise,} \end{cases}$$

where $\partial_G(v, \ell)$ is the set of vertices at distance ℓ from v . Moreover, let $\tilde{Z}_\ell := \sum_{v \in V(G)} \phi_\ell(v)$. Finally, viewing $\tilde{Z}_\ell$ as a weighted neighbourhood count with weights in $\{0, 1\}$, let $\hat{Z}_\ell$ be the truncation of $\tilde{Z}_\ell$ (as defined in Section 2.2).

Lemma 3.1. $\mathbb{V}[Z - \tilde{Z}_\ell] \leq n \cdot f_4(n, \ell)$, where $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_4(n, \ell) = 0$.

Proof. By the definition of the double limit, we may assume that $\ell \leq \log n$. Let us consider the effect of resampling an edge $e := \{u, v\} \in \binom{V(G)}{2}$ and define as before $G^+ = G + e$ and $G^- = G - e$

[†] For a sequence of events $\mathcal{E}_n$, we say that $\mathcal{E}_n$ occurs with high probability (whp) if $\mathbb{P}[\mathcal{E}_n] \rightarrow 1$ as $n \rightarrow \infty$.

and

$$D := \{w \in V(G) : \phi^+(w) - \phi_\ell^+(w) \neq \phi^-(w) - \phi_\ell^-(w)\}. \quad (4)$$

By Lemma 2.1, $\mathbb{V}[Z - \tilde{Z}_\ell] \leq 2cn\mathbb{E}[|D|^2]$. By Lemma 2.5, we have

$$\mathbb{V}[Z - \tilde{Z}_\ell] \leq 2cn\mathbb{E}[|D|^2] \leq 2cn\mathbb{E}[|D|^2 \mathbb{1}_\mathcal{E}] + o(n), \quad (5)$$

where $\mathcal{E}$ is the event as defined in Lemma 2.5.

For $v \in V(G)$ and $* \in \{+, -\}$, let us define C_v^* as in Lemma 2.5, to be the component in $G^* - L^*$ containing v (and $C_v^* = \emptyset$ if $v \in L^*$). Let $W := C_u^- \cup C_v^-$. We note that in the event $\mathcal{E}$, each of G^+ , G^- has a unique linear order component L^- and L^+ , respectively. Moreover, since $G^- \subseteq G^+$, we have that $L^- \subseteq L^+$.

Claim 3.2. In the event $\mathcal{E}$, we have that $D \subseteq W$.

Proof of Claim. Assume that the event $\mathcal{E}$ occurs. Note that if $w \notin W$ then e does not meet the component of w in $G^- - L^-$, that is, $C_w^- \cap \{u, v\} = \emptyset$. It follows that $G^-[C_w^-] = G^+[C_w^+]$. This, and $L^- \subseteq L^+$ imply that $\phi^+(w) = \phi^-(w)$. Next, we will show that $\phi_\ell^*(w)$ is completely determined by $G^*[C_w^*]$ for each $* \in \{-, +\}$. Indeed if $C_w^* \neq \emptyset$, then $\partial_{G^*}(w, \ell) = \partial_{G^*[C_w^*]}(w, \ell)$ and $B_{G^*}(w, \ell) = B_{G^*[C_w^*]}(w, \ell)$. On the other hand, if $C_w^* = \emptyset$, then $w \in L^*$ and thus we have $\partial_{G^*}(w, \ell) \neq \emptyset$ or $|B_{G^*}(w, \ell)| \geq |L^*| > \log^4 n$ implying that $\phi_\ell^*(w) = 1$. Hence $G^-[C_w^-] = G^+[C_w^+]$ also implies $\phi_\ell^+(w) = \phi_\ell^-(w)$. Thus $\phi^+(w) - \phi_\ell^+(w) = \phi^-(w) - \phi_\ell^-(w)$, which implies $w \notin D$. $\square$

Claim 3.3. In the event $\mathcal{E}$, we have that if $|W| < \ell$, then $D = \emptyset$.

Proof of Claim. Assume that the event $\mathcal{E}$ occurs and $|W| < \ell$. By Claim 3.2, it suffices to show that $w \notin D$ for all $w \in W$. Let $w \in W$. Note that it suffices to show that for each $* \in \{-, +\}$, we have $\phi_\ell^*(w) = \phi^*(w)$ (since then $\phi^+(w) - \phi_\ell^+(w) = 0 = \phi^-(w) - \phi_\ell^-(w)$ which implies that $w \notin D$). Let $* \in \{-, +\}$. If $w \in L^*$, then $\phi^*(w) = 1$ by definition and since $\mathcal{E}$ occurs, we have $\partial_{G^*}(w, \ell) \neq \emptyset$ or $|B_{G^*}(w, \ell)| > \log^4 n$, and therefore $\phi_\ell^*(w) = 1$. Hence we may assume that $w \notin L^*$. Thus, by definition, we have $\phi^*(w) = 0$. Note that, since $W := C_u^- \cup C_v^-$ and $e = uv$, we have that $C_w^- \cup C_w^+ \subseteq W$. Hence $0 < |C_w^*| < \ell$. This implies that $\partial_{G^*}(w, \ell) = \emptyset$ and $|B_{G^*}(w, \ell)| \leq |C_w^*| < \ell \leq \log n$. Hence $\phi_\ell^*(w) = 0 = \phi^*(w)$. $\square$

By (5), the two claims above, and Lemma 2.5 it follows that

$$\begin{aligned} \mathbb{V}[Z - \tilde{Z}_\ell] &\leq 2cn\mathbb{E}[|D|^2 \mathbb{1}_\mathcal{E}] + o(n) \leq 2cn\mathbb{E}[|W|^2 \mathbb{1}_{\{|W| \geq \ell\}} \mathbb{1}_\mathcal{E}] + o(n) \\ &\leq 2n \cdot cf_2(n, \ell) + o(n) =: n \cdot f_4(n, \ell), \end{aligned}$$

where $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_4(n, \ell) = \lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} (2c \cdot f_2(n, \ell) + o(1)) = 0$. $\square$

At this point we are essentially done. Lemma 3.1 implies that $\tilde{Z}_\ell$ is a good approximation to Z , and Lemma 2.4 implies that $\hat{Z}_\ell$ is a good approximation to $\tilde{Z}_\ell$ and that $\hat{Z}_\ell$ satisfies a central limit theorem. The rest of the proof consists of relatively straightforward calculations making the above precise, whilst keeping careful track of the double limit over ℓ and n .

Proof of Theorem 1.1. First, from Minkowski's inequality (Theorem 2.2), we obtain

$$\begin{aligned}\mathbb{V}[Z - \hat{Z}_\ell] &= \mathbb{V}[Z - \tilde{Z}_\ell + \tilde{Z}_\ell - \hat{Z}_\ell] \\ &\leq \left(\sqrt{\mathbb{V}[Z - \tilde{Z}_\ell]} + \sqrt{\mathbb{V}[\tilde{Z}_\ell - \hat{Z}_\ell]} \right)^2 \\ &\leq n \cdot \left(\sqrt{f_4(n, \ell)} + \sqrt{f_1(n, \ell)} \right)^2,\end{aligned}\quad (6)$$

where at the last inequality we used Lemmas 2.4 and 3.1.

Let us write $\sigma = \sqrt{\mathbb{V}[Z]}$ and $\hat{\sigma}_\ell = \sqrt{\mathbb{V}[\hat{Z}_\ell]}$, $Y := Z - \mathbb{E}[Z]$ and $\hat{Y}_\ell = \hat{Z}_\ell - \mathbb{E}[\hat{Z}_\ell]$, and

$$X := \frac{Z - \mathbb{E}[Z]}{\sigma} = \frac{Y}{\sigma}, \quad \hat{X}_\ell := \frac{\hat{Z}_\ell - \mathbb{E}[\hat{Z}_\ell]}{\hat{\sigma}_\ell} = \frac{\hat{Y}_\ell}{\hat{\sigma}_\ell}.$$

Then, by Lemma 2.7, we have $\sigma = \Omega(\sqrt{n})$, and so by (6) together with Theorem 2.2, it follows that $\hat{\sigma}_\ell = \Omega(\sqrt{n})$ as well. Moreover, by (2),

$$\begin{aligned}\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} \left| \frac{(\hat{\sigma}_\ell)^2}{\sigma^2} - 1 \right| &= \lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} \frac{|\mathbb{V}[Z] - \mathbb{V}[\hat{Z}_\ell]|}{\sigma^2} \\ &\leq \lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} \left(\frac{\mathbb{V}[Z - \hat{Z}_\ell]}{\sigma^2} + 2 \left(\frac{\mathbb{V}[Z - \hat{Z}_\ell]}{\sigma^2} \cdot \frac{\min\{\mathbb{V}[Z], \mathbb{V}[\hat{Z}_\ell]\}}{\sigma^2} \right)^{\frac{1}{2}} \right) \\ &\leq \lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} \left(\frac{\mathbb{V}[Z - \hat{Z}_\ell]}{\sigma^2} + 2 \left(\frac{\mathbb{V}[Z - \hat{Z}_\ell]}{\sigma^2} \right)^{\frac{1}{2}} \right) = 0.\end{aligned}\quad (7)$$

Then, since $\sigma = \Omega(\sqrt{n})$, by (6) and Chebyshev's inequality

$$\left| \frac{Y - \hat{Y}_\ell}{\sigma} \right| \xrightarrow{p} 0 \quad \text{as } \ell, n \rightarrow \infty. \quad (8)$$

Furthermore, for any $\ell \in \mathbb{N}$, by Lemma 2.4, we have $\hat{X}_\ell \xrightarrow{d} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$ and so

$$\hat{X}_\ell \xrightarrow{d} \mathcal{N}(0, 1) \quad \text{as } \ell, n \rightarrow \infty. \quad (9)$$

Hence, by Slutsky's theorem (Theorem 2.3), using (7)–(9)

$$X = \frac{Y - \hat{Y}_\ell}{\sigma} + \frac{\hat{\sigma}_\ell}{\sigma} \cdot \hat{X}_\ell \xrightarrow{d} 0 + 1 \cdot \mathcal{N}(0, 1) = \mathcal{N}(0, 1) \quad \text{as } \ell, n \rightarrow \infty. \quad (10)$$

However, since X does not depend on ℓ , it follows that $X \xrightarrow{d} \mathcal{N}(0, 1)$ as $n \rightarrow \infty$, as claimed. $\square$

4 | A CENTRAL LIMIT THEOREM FOR THE ORDER OF THE k -CORE

To emphasise the similarities between the proofs of Theorems 1.1 and 1.2 we will use similar variable names to the previous section (e.g., R , C_w , ϕ and ϕ_ℓ). Nevertheless, the reader should be careful to note that these objects are defined differently in these two sections.

Let $k \geq 3$ be given, let $c > \hat{c}_k$, let $p = \frac{c}{n}$ and let $G \sim G(n, p)$. Let K be the k -core of G and let $R := G - V(K)$ be the mantle. For each $v \in V(R)$, we let C_v be the component of R containing v and for each $v \in V(K)$, we set $C_v = \emptyset$.

We define a function $\phi : V(G) \rightarrow \{0, 1\}$ by

$$\phi(v) := \begin{cases} 1 & \text{if } v \in V(K) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $Y := |V(K)| = \sum_{v \in V(G)} \phi(v)$. We also define a ‘localised’ version as follows. Given $v \in V(G)$ and $\ell \in \mathbb{N}$, we define the ℓ -local k -core $K(v, \ell)$ as the maximal subset of $B_G(v, \ell)$ such that $\partial_G(v, \ell) \subseteq K(v, \ell)$ and for every $w \in K(v, \ell) \setminus \partial_G(v, \ell)$, we have that w has at least k neighbours in $K(v, \ell)$. We note that we could equivalently define $K(v, \ell)$ in terms of a peeling process where, starting with $B_G(v, \ell)$ we recursively delete vertices in $B_G(v, \ell - 1)$ of degree less than k .[†] In particular, if a vertex w lies in a component C_w^ℓ of $G[B_G(v, \ell) \setminus K(v, \ell)]$, then there is some ℓ -local k -degenerate ordering of C_w^ℓ , that is, an ordering of the vertices of C_w^ℓ such that each vertex is adjacent to fewer than k vertices which appear before it or are in $\partial_G(v, \ell)$. We then define

$$\phi_\ell(v) := \begin{cases} 1 & \text{if } v \in K(v, \ell) \\ 0 & \text{otherwise.} \end{cases}$$

If v lies in a component C_v^ℓ of $G[B_G(v, \ell) \setminus K(v, \ell)]$, then there is some ℓ -local k -degenerate ordering Σ of C_v^ℓ . Note that Σ can be extended to a k -degenerate ordering of C_v in G and therefore it does not include any vertices outside C_v . Thus $C_v^\ell \subseteq C_v$ and every ℓ -local k -degenerate ordering of C_v^ℓ has the property that every vertex is adjacent to fewer than k vertices which appear before it or are in $\partial_G(v, \ell)$, or are in $N_G(C_v^\ell)$. Moreover, note that if $C_v = \emptyset$ (which happens if and only if $v \in V(K)$) then $\phi_\ell(v) = 1$. It follows that $G[C_v]$ together with the edge boundary of C_v in G completely determine C_v^ℓ .

Let $\tilde{Y}_\ell := \sum_{v \in V(G)} \phi_\ell(v)$ and viewing $\tilde{Y}_\ell$ as a weighted neighbourhood count with weights in $\{0, 1\}$, let $\hat{Y}_\ell$ be the truncation of $\tilde{Y}_\ell$ (as defined in Section 2.2).

Lemma 4.1. $\mathbb{V}[Y - \tilde{Y}_\ell] \leq n \cdot f_5(n, \ell)$, where $\lim_{\substack{\ell, n \rightarrow \infty \\ n \gg \ell}} f_5(n, \ell) = 0$.

Proof. The proof of Lemma 4.1 follows as the proof of Lemma 3.1 verbatim, with Lemma 2.6 in place of Lemma 2.5 and the following minor alterations. In the proof of Claim 3.2, instead of the assertion that $G^-[C_w^-] = G^+[C_w^+]$ implies that $\phi_\ell^+(w) = \phi_\ell^-(w)$, we now use that since $G^-[C_w^-] = G^+[C_w^+]$ and the edge boundaries of C_w^- and C_w^+ are identical in G^- and G^+ , respectively, we have that $\phi_\ell^+(w) = \phi_\ell^-(w)$. In the proof of Claim 3.3, we can instead use the following. If $w \in V(K^*)$ (i.e., $\phi^*(w) = 1$), then also $\phi_\ell^*(w) = 1$ as observed above. Otherwise, we obtain $0 < |C_w^*| < \ell$ and thus

[†] Here the ball $B_G(v, \ell - 1)$ is not updated dynamically in this process.

$\partial_{G^*}(w, \ell) = \emptyset$. This implies that the k -degenerate ordering of C_w^* is also a ℓ -local k -degenerate ordering and thus $\phi^*(w) = 0 = \phi_\ell^*(w)$. $\square$

Proof of Theorem 1.2. The proof of Theorem 1.2 follows as the proof of Theorem 1.1 verbatim, with Lemma 4.1 in place of Lemma 3.1. $\square$

ACKNOWLEDGEMENTS

This research was funded in whole or in part by the Austrian Science Fund (FWF) [10.55776/P36131 (Joshua Erde), 10.55776/I6502 (Mihyun Kang), 10.55776/ESP3863424 (Michael Anastos)] and by the Swiss National Science Foundation (SNSF) [P500-2_235474] (Vincent Pfenninger). For open access purposes, the authors have applied a CC BY public copyright license to any author accepted manuscript version arising from this submission. The authors thank the reviewers for helpful comments on the paper, and for bringing the particular form of Theorem 2.2 to our attention.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Michael Anastos  <https://orcid.org/0000-0001-5475-6522>

Joshua Erde  <https://orcid.org/0000-0003-1129-4270>

Mihyun Kang  <https://orcid.org/0000-0001-8729-2779>

Vincent Pfenninger  <https://orcid.org/0009-0007-0715-970X>

REFERENCES

1. M. Anastos, *A note on long cycles in sparse random graphs*, Electron. J. Comb. **30** (2023), no. 2, P2.21, 24.
2. M. Anastos, J. Erde, M. Kang, and V. Pfenninger, *The law of the circumference of sparse binomial random graphs*, arXiv:2503.14336 [math.CO], 2025.
3. A. D. Barbour, S. Janson, M. Karoński, and A. Ruciński, *Small cliques in random graphs*, Random Struct. Algorithms **1** (1990), no. 4, 403–434.
4. A. D. Barbour, M. Karoński, and A. Ruciński, *A central limit theorem for decomposable random variables with applications to random graphs*, J. Combin. Theory Ser. B **47** (1989), no. 2, 125–145.
5. M. Behrisch, A. Coja-Oghlan, and M. Kang, *Local limit theorems for the giant component of random hypergraphs*, Comb. Probab. Comput. **23** (2014), no. 3, 331–366.
6. P. Billingsley, *Probability and Measure*, 3rd ed., John Wiley & Sons, New York, 1995.
7. B. Bollobás and O. Riordan, *Asymptotic normality of the size of the giant component via a random walk*, J. Comb. Theory, Ser. B **102** (2012), no. 1, 53–61.
8. A. Coja-Oghlan, O. Cooley, M. Kang, and K. Skubch, *How does the core sit inside the mantle?*, Random Struct. Algorithms **51** (2017), no. 3, 459–482.
9. A. Coja-Oghlan, O. Cooley, M. Kang, and K. Skubch, *Core forging and local limit theorems for the k -core of random graphs*, J. Comb. Theory, Ser. B **137** (2019), 178–231.
10. P. de Jong, *A central limit theorem with applications to random hypergraphs*, Random Struct. Algorithms **8** (1996), no. 2, 105–120.
11. R. Durrett, *Probability—theory and examples*, 5th ed., Cambridge Series in Statistical and Probabilistic Mathematics, vol. 49, Cambridge University Press, Cambridge, 2019.
12. B. Efron and C. Stein, *The jackknife estimate of variance*, Ann. Stat. **9** (1981), 586–596.

13. A. Frieze and M. Karoński, *Introduction to random graphs*, Cambridge University Press, Cambridge, 2016.
14. M. Glasgow, M. Kwan, A. Sah, and M. Sawhney, *The Exact Rank of Sparse Random Graphs*, J. Eur. Math. Soc. (2025), published online first.
15. M. Glasgow, M. Kwan, A. Sah, and M. Sawhney, *A central limit theorem for the matching number of a sparse random graph*, J. Lond. Math. Soc. (2) **111** (2025), no. 4, Paper No. e70101, 38.
16. S. Janson, *Orthogonal decompositions and functional limit theorems for random graph statistics*, Mem. Am. Math. Soc., vol. 534, American Mathematical Society (AMS), Providence, RI, 1994.
17. S. Janson and M. J. Luczak, *Asymptotic normality of the k -core in random graphs*, Ann. Appl. Probab. **18** (2008), no. 3, 1085–1137.
18. M. Karoński, *Balanced subgraphs of large random graphs*, Ser. Mat., Uniw. A. Mickiewicza Poznań., vol. 7, Uniwersytet im. Adama Mickiewicza w Poznaniu, Poznań, 1984.
19. M. Karoński and A. Ruciński, *On the number of strictly balanced subgraphs of a random graph*, Graph theory (Proc. Conf., Łagów/Pol. 1981), Lect. Notes Math., vol. 1018, Springer, Berlin, 1983, pp. 79–83.
20. E. Krecic, *Some problems related to the Karp–Sipser algorithm on random graphs*, PhD thesis, University of Oxford, Oxford, 2017.
21. K. Nowicki and J. C. Wierman, *Subgraph counts in random graphs using incomplete U -statistics methods*, Discrete Math. **72** (1988), no. 1-3, 299–310.
22. B. Pittel, J. Spencer, and N. Wormald, *Sudden emergence of a giant k -core in a random graph*, J. Comb. Theory, Ser. B **67** (1996), no. 1, 111–151.
23. B. Pittel and N. C. Wormald, *Counting connected graphs inside-out*, J. Comb. Theory, Ser. B **93** (2005), no. 2, 127–172.
24. B. Pittel and N. C. Wormald, *Corrigendum to: “Counting connected graphs inside-out”*, J. Comb. Theory, Ser. B **98** (2008), no. 4, 835–837.
25. A. Ruciński, *When are small subgraphs of a random graph normally distributed?*, Probab. Theory Relat. Fields **78** (1988), no. 1, 1–10.
26. E. Slutsky, *Über stochastische Asymptoten und Grenzwerte*, Tôhoku Math. J. **27** (1926), 67–70.