



Average-field approximation for dilute almost-bosonic anyons

François L. A. Visconti¹

Received: 9 September 2025 / Accepted: 25 August 2026
© The Author(s) 2026

Abstract

We study the ground state of a system of N two-dimensional trapped almost-bosonic anyons. This setup can equivalently be viewed as bosons interacting through long-range magnetic potentials generated by magnetic charges carried by each particle. These magnetic charges are assumed to be smeared over discs of radius R —a model known as *extended anyons*. To recover the point-like anyons perspective, we consider the joint limit $R \rightarrow 0$ as $N \rightarrow \infty$. We rigorously justify the *average-field approximation* for any radius R that decays polynomially, and even for certain exponentially decaying R . The average-field approximation asserts that the particles behave like independent, identical bosons interacting through a self-consistent magnetic field. Our result significantly improves upon the best-known estimates from [1, 2], and it in particular covers radii that are much smaller than the mean interparticle distance. The proof strategy builds on a recent work on two-dimensional attractive Bose gases by Junge and the author [3].

Mathematics Subject Classification 35Q55 · 82B10 · 82D40

Contents

1	Introduction	1
1.1	Extended anyons	1
1.2	Average-field approximation	1
1.3	The mean-field point of view	1
1.4	Main result	1
2	Notation, preliminaries and proof strategy	2
2.1	Notation	2
2.2	Estimates on the interaction terms	2
2.3	A priori energy bounds	2
2.4	Strategy of proof	2
3	Mean-field limit: proof of Theorem 1.1	3
4	Proof of the main estimates	4
4.1	High-momenta estimates: proof of Lemma 3.1	4
4.2	Low-occupancy estimates: Proof of Lemma 3.2	4
	References	4

✉ François L. A. Visconti
visconti@math.lmu.de

¹ Department of Mathematics, LMU Munich, Theresienstrasse 39, Munich 80333, Germany

1 Introduction

A system of N free two-dimensional anyons with statistics parameter α can *formally* be described by an N -body wavefunction of the form

$$\tilde{\Psi}(x_1, \dots, x_N) = \prod_{1 \leq j < k \leq N} e^{i\alpha\phi_{jk}} \Psi(x_1, \dots, x_N) \quad \text{with} \quad \phi_{jk} = \arg \frac{x_j - x_k}{|x_j - x_k|},$$

where Ψ is a bosonic wavefunction, which is to say symmetric under particle exchange. We refer to [4–8] for physical reviews. The wavefunction $\tilde{\Psi}$ satisfies

$$\tilde{\Psi}(x_1, \dots, x_j, \dots, x_k, \dots, x_N) = e^{i\alpha\pi} \tilde{\Psi}(x_1, \dots, x_k, \dots, x_j, \dots, x_N),$$

meaning that its behaviour under particle exchange interpolates between bosons ($\alpha = 0$) and fermions ($\alpha = 1$). However, this implies that the wavefunction is in general not single-valued, making this description difficult to use in practice. A way to overcome this issue is to realise that the action of the free Hamiltonian

$$\sum_{j=1}^N -\Delta_j + V(x_j)$$

on the anyonic wavefunction $\tilde{\Psi}$ is *formally* equivalent to the action of an effective, α -dependent Hamiltonian on the bosonic wavefunction Ψ [5, 9, 10]. This is called the magnetic gauge picture of the anyonic problem and consists in working with the Hamiltonian

$$H_N = \sum_{j=1}^N (\mathbf{p}_j + \alpha \mathbf{A}_j)^2 + V(x_j) \quad (1.1)$$

acting on

$$\mathfrak{H}^N = \bigotimes_{\text{sym}}^N \mathfrak{H},$$

the symmetric tensor product of N copies of the one-body Hilbert space $\mathfrak{H} := L^2(\mathbb{R}^2)$. Here, we introduced the momentum operator

$$\mathbf{p}_j = -i\nabla_j,$$

and the statistical gauge vector potential

$$\mathbf{A}_j = \sum_{\substack{k=1 \\ k \neq j}}^N \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2} \quad (1.2)$$

felt by the j -th particle due to the influence of all other particles (denoting $(x, y)^\perp = (-y, x)$). The external trapping potential $V : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is assumed to satisfy

$$V(x) \geq c^{-1}|x|^s - C \quad (1.3)$$

for some constants $s, c, C > 0$. Moreover, suppose that

$$\frac{|\nabla V|^2}{2V} \leq c(-\Delta + V) + C(-\Delta + V)^2 + V^2, \quad (1.4)$$

for some constants $c, C > 0$.

In this work we consider α of the form

$$\alpha = \frac{\beta}{N-1}, \quad (1.5)$$

for some given fixed constant β . This places us in a so-called *almost-bosonic* limit, because $\alpha \rightarrow 0$ as $N \rightarrow \infty$, and it ensures that the anyon statistics has an effect at the leading order [1, 2, 11–14].

1.1 Extended anyons

The Hamiltonian (1.1) is actually too singular to be considered as acting on product states $u^{\otimes N}$, regardless of the regularity of the one-particle wavefunction u . To resolve this, we introduce a length scale R over which the magnetic charges are smeared; this is called an *extended anyons* model. For a detailed discussion, we refer to [2, 15, 16] and references therein. We let $R \rightarrow 0$ when $N \rightarrow \infty$ to recover the point-like anyons perspective.

Let w_R be the two-dimensional Coulomb potential generated by a unit charge smeared over a disc of radius R :

$$w_R = \log |\cdot| * \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}. \quad (1.6)$$

Define also $w_0 = \log |\cdot|$. Considering that

$$\nabla^\perp w_0(x) = \frac{x^\perp}{|x|^2},$$

it is natural to introduce the regularised gauge vector potential $\mathbf{A}_j^R$ given by

$$\mathbf{A}_j^R = \sum_{\substack{k=1 \\ k \neq j}}^N \nabla^\perp w_R(x_j - x_k). \quad (1.7)$$

This leads to the regularised Hamiltonian

$$H_N^R = \sum_{j=1}^N \left(\mathbf{p}_j + \alpha \mathbf{A}_j^R \right)^2 + V(x_j). \quad (1.8)$$

For fixed $R > 0$, this operator is essentially self-adjoint and bounded from below (see [17], [18, Theorem X.17]), which can be expanded as

$$\begin{aligned} H_N^R = & \sum_{j=1}^N -\Delta_j + V(x_j) + 2\alpha \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} \mathbf{p}_j \cdot \nabla^\perp w_R(x_j - x_k) \\ & + \alpha^2 \sum_{\substack{1 \leq j, k, \ell \leq N \\ j \neq k \neq \ell \neq j}} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell) + \alpha^2 \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} \left| \nabla w_R(x_j - x_k) \right|^2, \end{aligned} \quad (1.9)$$

where we used that $\mathbf{p}$ and $\nabla^\perp w_R$ commute. Though this operator does not have a unique limit as $R \rightarrow 0$ and the singular Hamiltonian (1.1) is not essentially self-adjoint [19–23], we obtain a unique nonlinear model when taking the joint limit $R \rightarrow 0$ as $N \rightarrow \infty$ as long as $R \ll e^{-N}$. We refer to [11] for a discussion on the other regimes.

1.2 Average-field approximation

The average-field approximation is obtained by substituting the potentials (1.2) and (1.7) with the average-field potentials $\mathbf{A}[\rho]$ and $\mathbf{A}^R[\rho]$ defined by

$$\mathbf{A}[\rho] = \nabla^\perp w_0 * \rho \quad \text{and} \quad \mathbf{A}^R[\rho] = \nabla^\perp w_R * \rho,$$

where ρ is the *normalised* 1-body density of a given wavefunction Ψ :

$$\rho(x) = \int_{\mathbb{R}^{2(N-1)}} dx_2 \dots dx_N |\Psi(x, x_2, \dots, x_N)|^2.$$

Then, the regularised Hamiltonian (1.8) is approximated by the N -body average-field Hamiltonian

$$H_N^{\text{af}}[\rho] = \sum_{j=1}^N \left(\mathbf{p}_j + \alpha(N-1)\mathbf{A}^R[\rho] \right)^2 + V(x_j).$$

Given a fixed density ρ , this is a non-interacting magnetic Hamiltonian acting on $\mathfrak{H}^N$ and its ground state Ψ is therefore a pure Bose–Einstein condensate, that is of the form

$$\Psi = u^{\otimes N},$$

for some 1-body wavefunction $u \in \mathfrak{H}$. The energy per particle associated to such a state is

$$N^{-1} \left\langle u^{\otimes N}, H_N^{\text{af}}[\rho] u^{\otimes N} \right\rangle = \left\langle u, \left[(\mathbf{p} + \alpha(N-1)\mathbf{A}^R[\rho])^2 + V \right] u \right\rangle.$$

For consistency we set

$$\rho = |u|^2,$$

meaning that the density is that of the state $u^{\otimes N}$. This leads to the *nonlinear average-field energy functional* $\mathcal{E}_R^{\text{af}}$ given by

$$\mathcal{E}_R^{\text{af}}[u] = \int_{\mathbb{R}^2} \left(\left| (\mathbf{p} + \mathbf{i}\beta\mathbf{A}^R[|u|^2])u \right|^2 + V|u|^2 \right), \quad (1.10)$$

where $\alpha(N-1)$ was replaced by β in accordance with (1.5). Furthermore, in the limit $R \rightarrow 0$ this energy functional converges to the *singular average-field energy functional* $\mathcal{E}^{\text{af}}$ given by

$$\mathcal{E}^{\text{af}}[u] = \int_{\mathbb{R}^2} \left(\left| (\mathbf{p} + \mathbf{i}\beta\mathbf{A}[|u|^2])u \right|^2 + V|u|^2 \right) \quad (1.11)$$

(see [1, Proposition A.5], [2, Proposition A.6]).

1.3 The mean-field point of view

In principle, the average-field approximation does not require the Hamiltonian H_N^R to exhibit Bose–Einstein condensation. In fact, it has been widely used in the physics literature to perturb around fermions ($\alpha = 1$) [24–31]. However, when taking $\beta = \alpha(N-1)$ fixed, which is natural for the study of $\mathcal{E}_R^{\text{af}}$, the limit $N \rightarrow \infty$ can be seen as a mean-field-like limit of many-body bosonic systems. Indeed, the two-body terms in (1.9) come with prefactors $1/N$ and $1/N^2$, and the three-body term with a prefactor $1/N^2$. See [32, Section 1.4] for a related discussion.

Furthermore, the functionals $\mathcal{E}_R^{\text{af}}$ and $\mathcal{E}^{\text{af}}$ defined in (1.10) and (1.11) may also be derived from the mean-field picture of H_N^R . To see this, we write

$$\begin{aligned} \langle \Psi, H_N^R \Psi \rangle / N &= \text{Tr}((-\Delta + V)\gamma_\Psi^{(1)}) + 2\beta \text{Tr}(\mathbf{p}_1 \cdot \nabla^\perp w_R(x_1 - x_2)\gamma_\Psi^{(2)}) \\ &\quad + \beta^2 \frac{N-2}{N-1} \text{Tr}(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3)\gamma_\Psi^{(3)}) \\ &\quad + \beta^2 \frac{1}{N-1} \text{Tr}(|\nabla w_R(x_1 - x_2)|^2 \gamma_\Psi^{(2)}), \end{aligned} \quad (1.12)$$

for any $\Psi \in \mathfrak{H}^N$ and where $\gamma_\Psi^{(k)}$ denotes the k -body reduced density matrix of Ψ . Since we are considering a mean-field bosonic Hamiltonian (for fixed R at least), it is natural to use the ansatz

$$\Psi \approx u^{\otimes N}, \quad \gamma_\Psi^{(k)} \approx |u^{\otimes k}\rangle \langle u^{\otimes k}|,$$

for which

$$\text{Tr}(\mathbf{p}_1 \cdot \nabla^\perp w_R(x_1 - x_2)|u^{\otimes 2}\rangle \langle u^{\otimes 2}|) = \int_{\mathbb{R}^2} \mathbf{A}^R[|u|^2] \cdot u \overline{\mathbf{p}} u \quad (1.13)$$

and

$$\text{Tr}(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3)|u^{\otimes 3}\rangle \langle u^{\otimes 3}|) = \int_{\mathbb{R}^2} |u|^2 \left| \mathbf{A}^R[|u|^2] \right|^2. \quad (1.14)$$

Inserting these identities into (1.12) and neglecting the last term because it is heuristically of order $\mathcal{O}(N^{-1})$, we obtain

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \approx \mathcal{E}_R^{\text{af}}[u].$$

Taking the joint limit $R \rightarrow 0$ as $N \rightarrow \infty$ again leads to the singular functional $\mathcal{E}^{\text{af}}$.

The convergence of the many-body ground state energy per particle to that of the functional $\mathcal{E}^{\text{af}}$, as well as Bose–Einstein condensation in its minimisers, were first established in [2] for anyons with radius $R = N^{-\eta}$, under the condition

$$0 < \eta < \frac{1}{4} \left(1 + \frac{1}{s}\right)^{-1}$$

with s the exponent in (1.3). This range was later extended to $0 < \eta < 1/4$ in [1], where an external magnetic field was considered. In the present paper we extend these results to anyons with radius $N^{-\eta}$ for any $\eta > 0$, and even to anyons with radius $R = e^{-N^\kappa}$ for any $0 < \kappa < 1$. In particular, we cover the case $R \ll N^{-1/2}$, meaning that the radius of the magnetic charges is much smaller than the mean interparticle distance.

For anyons with radius $R = e^{-N^\kappa}$ for $\kappa \geq 1$, our main result is no longer expected to hold. In this regime, the energy of the fully uncorrelated trial state $\Psi = u^{\otimes N}$ diverges due to the singular two-body interaction, regardless of how smooth u is. This indicates that correlations should be taken into account in the trial state, leading to a different energy functional than $\mathcal{E}^{\text{af}}$. Recently, an upper bound on the energy—which even covers the singular case $R = 0$ —was derived in [11] using Jastrow–Dyson-type trial states. Deriving a matching lower bound remains an interesting but likely challenging open problem.

1.4 Main result

Define the many-body ground state energy per particle

$$e_N^R = N^{-1} \inf \left\{ \langle \Psi, H_N^R \Psi \rangle : \Psi \in \mathfrak{H}^N, \|\Psi\| = 1 \right\}, \quad (1.15)$$

and the ground state energy of the average-field functional (1.11)

$$e^{\text{af}} = \inf \{ \mathcal{E}^{\text{af}}[u] : u \in \mathfrak{H}, \|u\| = 1 \}. \quad (1.16)$$

We prove the convergence of e_N^R to e^{af} . The convergence of the ground states of H_N to condensates in minimisers of $\mathcal{E}^{\text{af}}$ follows directly from the convergence of the energies, thanks to arguments from [33] (see also [1, 2]).

Theorem 1.1 *Consider N extended anyons of radius R in an external potential V satisfying (1.3). Suppose that R is either of the form*

$$R = N^{-\eta}, \quad (1.17)$$

for some $\eta > 0$, or of the form

$$R = e^{-N^\kappa}, \quad (1.18)$$

for some $0 < \kappa < 1$. Take the statistics parameter to scale as

$$\alpha = \frac{\beta}{N-1} \quad (1.19)$$

for some fixed $\beta \in \mathbb{R}$. Define e_N^R and e^{af} as in (1.15) and (1.16). Then,

$$\lim_{N \rightarrow \infty} e_N^R = e^{\text{af}} > -\infty. \quad (1.20)$$

Moreover, for a sequence $\{\Psi_N\}_N$ of ground states of H_N^R , there exists a Borel probability measure μ supported on the set of minimisers of $\mathcal{E}^{\text{af}}$ such that, along a subsequence,

$$\lim_{N \rightarrow \infty} \text{Tr} \left| \gamma_{\Psi_N}^{(k)} - \int d\mu(u) |u^{\otimes k}\rangle \langle u^{\otimes k}| \right| = 0, \quad (1.21)$$

for any integer k . If $\mathcal{E}^{\text{af}}$ has a unique minimiser u_{af} (up to a phase), then for the whole sequence,

$$\lim_{N \rightarrow \infty} \text{Tr} \left| \gamma_{\Psi_N}^{(k)} - |u_{\text{af}}^{\otimes k}\rangle \langle u_{\text{af}}^{\otimes k}| \right| = 0,$$

for any integer k .

Remark 1.2 Instead of considering free anyons, we can introduce an interaction potential into the Hamiltonian (1.8), as was done in [34]. Specifically, we consider the Hamiltonian

$$H_{N,R}^{\text{int}} = \sum_{j=1}^N \left(\mathbf{p}_j + \alpha \mathbf{A}_j^R \right)^2 + V(x_j) + \frac{1}{N-1} \sum_{1 \leq i < j \leq N} v_N(x_i - x_j)$$

acting on $\mathfrak{H}^N$, where the interaction potential v_N is either of the form

$$v_N = N^{2\nu} v(N^\nu \cdot), \quad (1.22)$$

for $\nu > 0$, or of the form

$$v_N = e^{2N^\nu} v(e^{N^\nu} \cdot),$$

for $0 < \nu < 1$. Using the uncorrelated trial state $\Psi = u^{\otimes N}$ leads to the singular functional

$$\mathcal{E}^{\text{int}}[u] = \int_{\mathbb{R}^2} \left(|(\mathbf{p} + i\beta \mathbf{A}[|u|^2])u|^2 + V|u|^2 + \frac{a}{2}|u|^4 \right), \quad (1.23)$$

with $a = \int_{\mathbb{R}^2} v$. To ensure that the functional (1.23) is stable, one needs to assume that

$$a > -a^*(\beta),$$

where $a^*(\beta)$ is the optimal constant in the magnetic Gagliardo–Nirenberg inequality:

$$\frac{a^*(\beta)}{2} \int_{\mathbb{R}^2} |u|^4 \leq \int_{\mathbb{R}^2} |(\mathbf{p} + \beta \mathbf{A}[|u|^2])u|^2,$$

valid for all $u \in H^1(\mathbb{R}^2)$ with $\|u\| = 1$ (see [34, 35] for more details). Assuming $v \in L^1(\mathbb{R}^2) \cap L^{1+\varepsilon}(\mathbb{R}^2)$ for some $\varepsilon > 0$, and taking R as in Theorem 1.1, a straightforward combination of the proof of Theorem 1.1 and the proof of [3, Theorem 2] yields

$$\lim_{N \rightarrow \infty} e_{N,R}^{\text{int}} = e^{\text{int}}, \quad (1.24)$$

where $e_{N,R}^{\text{int}}$ is the ground state energy per particle of $H_{N,R}^{\text{int}}$ and e^{int} is the ground state energy of $\mathcal{E}^{\text{int}}$. Bose–Einstein condensation (BEC) in minimisers of $\mathcal{E}^{\text{int}}$ follows from the energy convergence. In [34], the convergence (1.24) (and consequently BEC) was shown for potentials of the form $v_N = N^{2\nu} v(N^\nu \cdot)$ and for $R = N^{-\eta}$, under the much more restrictive conditions

$$\nu < \frac{s}{6s+6} \quad \text{and} \quad \eta < \frac{s}{4s+4},$$

where s is the exponent appearing in the assumption (1.3) on the trapping potential.

Organisation of the paper. In Sect. 2, we fix some notation, provide preliminary results, and explain the proof strategy of Theorem 1.1. In Sect. 3, we prove Theorem 1.1 assuming the validity of two key estimates: Lemmas 3.1 and 3.2. Section 4 is entirely devoted to the proof of Lemmas 3.1 and 3.2.

2 Notation, preliminaries and proof strategy

2.1 Notation

Let S_N be the group of permutations of $\{1, \dots, N\}$. For $\psi_1 \in \mathfrak{H}^{N_1}$ and $\psi_2 \in \mathfrak{H}^{N_2}$, define the symmetric tensor product $\psi_1 \otimes_s \psi_2 \in \mathfrak{H}^{N_1+N_2}$ by

$$\psi_1 \otimes_s \psi_2(x_1, \dots, x_{N_1+N_2}) = \frac{1}{\sqrt{N_1!N_2!(N_1+N_2)!}} \sum_{\sigma \in S_{N_1+N_2}} \psi_1(x_{\sigma(1)}, \dots, x_{\sigma(N_1)}) \times \psi_2(x_{\sigma(N_1+1)}, \dots, x_{\sigma(N_1+N_2)}).$$

Denote the i -fold tensor product of a vector $f \in \mathfrak{H}$ by $f^{\otimes i} \in \mathfrak{H}^i$, and the i -fold tensor product of an operator $A : \mathfrak{H} \rightarrow \mathfrak{H}$ by $A^{\otimes i}$. Denote by $\mathfrak{S}^1(\mathfrak{X})$ the space of trace-class operators on a given Hilbert space $\mathfrak{X}$ [36]. Define the set of states on $\mathfrak{X}$ by

$$\mathcal{S}(\mathfrak{X}) = \{\Gamma \in \mathfrak{S}^1(\mathfrak{X}) : \Gamma = \Gamma^* \geq 0, \text{Tr}_{\mathfrak{X}} \Gamma = 1\}.$$

The k -particle reduced density matrix $\Gamma^{(k)}$ of a given state $\Gamma \in \mathcal{S}(\mathfrak{H}^N)$ is obtained by taking the partial trace over all but the first k variables:

$$\Gamma^{(k)} = \text{Tr}_{k+1 \rightarrow N} \Gamma.$$

Moreover, we define the k -particle reduced density matrix $\gamma_{\Psi}^{(k)}$ of a normalised wavefunction $\Psi \in \mathfrak{H}^N$ by

$$\gamma_{\Psi}^{(k)} = \text{Tr}_{k+1 \rightarrow N} |\Psi\rangle\langle\Psi|.$$

Finally, given a multi-index $\underline{J} = (j_1, \dots, j_M) \in \mathbb{N}_0^M$, we define

$$|\underline{J}| = \sum_{i=1}^M j_i.$$

2.2 Estimates on the interaction terms

We recall some estimates on the smeared Coulomb potential w_R , which exploit the regularising effect of the smeared charge.

Lemma 2.1 (Smeared Coulomb potential) *Let w_R be defined as in (1.6). Then, there exists a universal constant $C > 0$ such that*

$$\sup_{B(0,1)} |w_R| \leq C + |\log R|, \quad \sup_{\mathbb{R}^2} |\nabla w_R| \leq CR^{-1} \quad \text{and} \quad \sup_{B(0,1)^c} |\nabla w_R| \leq C, \quad (2.1)$$

for R small enough.

Proof See [2, Lemma 2.1]. □

Lemma 2.2 (Singular two-body term) *There exists a universal constant $C > 0$ such that, for R small enough,*

$$|\nabla w_R(x_1 - x_2)|^2 \leq C |\log R|^2 (1 - \Delta_1) \quad (2.2)$$

in the quadratic form sense on $L^2(\mathbb{R}^4)$.

Proof See [14, Lemma 2.5]. □

Lemma 2.3 (Three-body term) *There exists a universal constant $C > 0$ such that, for R small enough,*

$$0 \leq \sum_{\substack{1 \leq j, k, \ell \leq 3 \\ j \neq k \neq \ell \neq j}} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell) \leq C (1 - \Delta_1 - \Delta_2 - \Delta_3) \quad (2.3)$$

in the quadratic form sense on $L^2(\mathbb{R}^6)$.

Proof See [2, Lemma 2.4]. □

2.3 A priori energy bounds

We provide here some bounds on the ground state energy per particle e_N^R , and on the kinetic energy of ground states of H_N^R .

Proposition 2.4 (Energy bounds on the ground state) *Take R as in Theorem 1.1. Then, the many-body ground state energy per particle e_N^R defined in (1.15) satisfies*

$$\limsup_{N \rightarrow \infty} e_N^R \leq e^{\text{af}}, \quad (2.4)$$

with e^{af} given by (1.16). Moreover, for R of the form (1.17) or (1.18) with $\kappa < 1/2$, the estimate

$$\text{Tr} \left((-\Delta + V) \gamma_{\Psi}^{(1)} \right) \leq C \quad (2.5)$$

holds for any normalised ground state Ψ of H_N^R , and for some constant C that depends only on β and the constant C in (1.3).

Proof For R of the form (1.17), the bound (2.4) was already proved in [2, Section 3.3] using the trial state $(u_R^{\text{af}})^{\otimes N}$, where u_R^{af} is a normalised minimiser of $\mathcal{E}_R^{\text{af}}$. Moreover, this bound easily extends to R of the form (1.18) with $\kappa < 1/2$ using the estimate (2.2). For $\kappa \geq 1/2$, however, this estimate is no longer strong enough to control the singular two-body term. Instead, one can use a Jastrow–Dyson trial state to obtain the correct bound, as was done in [11]. We refer to that work for more detail.

The bound on the kinetic energy (2.5) was proved in [2, Proposition 2.6] for R of the form (1.17). Again, this easily extends to the exponential case (1.18) under the condition $\kappa < 1/2$ using the estimate (2.2). For $\kappa \geq 1/2$, however, it is unclear whether a simple proof of (2.5) exists, and we will need to work much harder to prove it; this is explained in the last paragraph of Sect. 3. $\square$

2.4 Strategy of proof

To prove Theorem 1.1 we wish to compare the many-body ground state energy per particle e_N to that of the average-field functional (1.10)

$$e_R^{\text{af}} = \inf \left\{ \mathcal{E}_R^{\text{af}}[u] : u \in \mathfrak{H}, \|u\| = 1 \right\},$$

and use the convergence

$$\lim_{R \rightarrow 0} e_R^{\text{af}} = e^{\text{af}}.$$

The upper bound $\limsup e_N \leq e^{\text{af}}$ is already given in Proposition 2.4 and is essentially obtained using the trial state $(u_R^{\text{af}})^{\otimes N}$, where u_R^{af} is a minimiser of (1.10). Proving a matching lower bound however requires a lot more work, and we follow the general proof strategy of [3], which itself builds on ideas from [33, 37, 38]. We also use ideas from [1, 2]. The main idea of the proof is to use a localisation in momentum space in order to reduce the infinite-dimensional problem to multiple finite-dimensional ones. Then, for each of those finite-dimensional problems, we use the following quantitative version of the quantum de Finetti theorem [39, 40].

Theorem 2.5 *Given a Hilbert space $\mathfrak{X}$ of dimension d and a symmetric state $\Gamma_K \in \mathcal{S}(\mathfrak{X}^K)$, there exists a probability measure μ on $\mathcal{S}(\mathfrak{X})$ such that*

$$\left| \text{Tr} A \otimes B \otimes C \left(\Gamma_K^{(3)} - \int_{\mathcal{S}(\mathfrak{X})} \gamma^{\otimes 3} d\mu(\gamma) \right) \right| \leq C' \sqrt{\frac{\log d}{K}} \|A\|_{\text{op}} \|B\|_{\text{op}} \|C\|_{\text{op}}$$

for all self-adjoint operators A, B and C on $\mathfrak{X}$, and for some universal constant $C' > 0$.

Proof In [39], the statement was proved with A, B, C replaced by quantum measurement. See [41, Proposition 3.3] and [38, Lemma 3.3] for an adaptation to self-adjoint operators. $\square$

We wish to apply Theorem 2.5 on energy subspaces of the 1-body operator

$$h = -\Delta + V.$$

More precisely, on the subspaces defined by the spectral projections

$$P_1 = \mathbb{1}(h < N^{2\varepsilon}), \quad \text{and} \quad P_i = \mathbb{1}(N^{2(i-1)\varepsilon} \leq h < N^{2i\varepsilon}), \quad 2 \leq i \leq M, \quad (2.6)$$

for some nonnegative parameters ε and M .

Before applying Theorem 2.5, we rewrite the energy of a ground state Ψ of H_N^R in terms of the 3-particle reduced density matrix $\Gamma^{(3)}$ of Ψ and of the 3-body Hamiltonian

$$\tilde{H}_3^R = T + \beta W_2 + \beta^2 W_3,$$

with the operators T , W_2 and W_3 given by

$$T = \frac{1}{3} (h_1 + h_2 + h_3), \quad W_2 = \frac{1}{3} \sum_{\substack{1 \leq j, k \leq 3 \\ j \neq k}} \mathbf{p}_j \cdot \nabla^\perp w_R(x_j - x_k)$$

and

$$W_3 = \frac{1}{6} \sum_{\substack{1 \leq j, k, \ell \leq 3 \\ j \neq k \neq \ell \neq j}} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell).$$

Namely, we will bound the energy of Ψ from below by

$$\langle \Psi, H_N^R \Psi \rangle \gtrsim N \operatorname{Tr} (\tilde{H}_3^R \Gamma^{(3)}).$$

Using that the spectral projections defined in (2.6) form a resolution of the identity we will then roughly get

$$\operatorname{Tr} (\tilde{H}_3^R \Gamma^{(3)}) \gtrsim \sum_{\substack{1 \leq i_1, i_2, i_3 \leq M \\ 1 \leq i'_1, i'_2, i'_3 \leq M}} \operatorname{Tr} (P_{i_1} \otimes P_{i_2} \otimes P_{i_3} \tilde{H}_3^R P_{i'_1} \otimes P_{i'_2} \otimes P_{i'_3} \Gamma^{(3)}).$$

The reason we can discard the projections on the subspace $\mathbb{1}(N^{2M\varepsilon} \leq h) \mathfrak{H}$ is that the parameters ε and M will be chosen so that the energy of the one-body operator h is large enough to easily control that of the interaction terms using the estimates (2.1) and (2.3).

Next, we will decompose the many-body wavefunction according to the occupancy of its energy levels. Namely, we decompose Ψ as

$$\Psi = \sum_{\underline{J}} \Psi_{\underline{J}} \quad \text{with} \quad \Psi_{\underline{J}} = P_1^{\otimes j_1} \otimes_s \cdots \otimes_s P_{M+1}^{\otimes j_{M+1}} \Psi,$$

where the sum is taken over multi-indices $\underline{J} = (j_1, \dots, j_{M+1})$ satisfying $|\underline{J}| = N$. Defining also

$$\Gamma_{\underline{J}, \underline{J}'} = |\Psi_{\underline{J}'}\rangle \langle \Psi_{\underline{J}}|,$$

we find

$$\operatorname{Tr} (\tilde{H}_3^R \Gamma^{(3)}) \gtrsim \sum_{\underline{J}, \underline{J}'} \sum_{\substack{1 \leq i_1, i_2, i_3 \leq M \\ 1 \leq i'_1, i'_2, i'_3 \leq M}} \operatorname{Tr} (P_{i_1} \otimes P_{i_2} \otimes P_{i_3} \tilde{H}_3^R P_{i'_1} \otimes P_{i'_2} \otimes P_{i'_3} \Gamma_{\underline{J}, \underline{J}'}^{(3)}). \quad (2.7)$$

After that, we define

$$i_{\max}(\underline{J}) = \max \{i \in \{1, \dots, M\} : j_i \geq N^{1-\delta\varepsilon}\}$$

for some $\delta > 0$, and distinguish between the terms that satisfy $i_{\max}(\underline{J}) = i_{\max}(\underline{J}')$ and for which $i_1, \dots, i'_3$ are all less than $i_{\max}(\underline{J})$, and the ones that do not. In the latter case, we will

show that their energy is much smaller than the overall kinetic energy of the system and that they can therefore be neglected. Once these terms are neglected, we will be left with

$$\mathrm{Tr}(\tilde{H}_3^R \Gamma^{(3)}) \gtrsim \sum_{\substack{\underline{J}, \underline{J}' \\ i_{\max}(\underline{J})=i_{\max}(\underline{J}')}} \sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^{i_{\max}} \mathrm{Tr}(P_{i_1} \otimes P_{i_2} \otimes P_{i_3} \tilde{H}_3^R P_{i'_1} \otimes P_{i'_2} \otimes P_{i'_3} \Gamma_{\underline{J}, \underline{J}'}^{(3)}). \quad (2.8)$$

Going from (2.7) to (2.8) is one of the main difficulties in adapting the proof strategy of [3] and, in particular, new ideas are required to deal with the momentum operators contained in the potential W_2 . Since the proof of (2.8) is the key technical part of the paper, we discuss it briefly. A naive application of the Cauchy–Schwarz inequality shows that the terms we wish to control can be bounded by

$$\mathrm{Tr}(P_{i_1} \otimes P_{i_2} \otimes P_{i_3} W_3 P_{i_1} \otimes P_{i_2} \otimes P_{i_3} \Gamma_{\underline{J}}^{(3)}),$$

where one the indices, say i_3 , is such that $j_{i_3} \leq N^{1-\delta\epsilon}$ (and likewise for W_2). Then, an application of Lemmas 2.3 and 2.6 roughly implies

$$\mathrm{Tr}(P_{i_1} \otimes P_{i_2} \otimes P_{i_3} W_3 P_{i_1} \otimes P_{i_2} \otimes P_{i_3} \Gamma_{\underline{J}}^{(3)}) \lesssim N^{-\delta\epsilon} \mathrm{Tr}(P_{i_1} \otimes P_{i_2} h_1 P_{i_1} \otimes P_{i_2} \Gamma_{\underline{J}}^{(2)}) \ll \mathrm{Tr}(h \Gamma^{(1)}),$$

meaning that we can hope to show that these terms are controlled by the kinetic energy. Though this turns out to be the case—and is the statement of Lemma 3.2 below—we face a multitude of important technical difficulties when trying to prove it, which is why the proof is postponed to Sect. 4.

Lemma 2.6 *Let P_1, P_2, P_3 and Q be orthogonal projections on $\mathfrak{H}$. Given a state*

$$\Gamma \in \mathcal{S}(P_1^{j_1} \otimes_s P_2^{j_2} \otimes_s P_3^{j_3} \otimes_s Q^{N-j_1-j_2} \mathfrak{H}^N),$$

for some $j_1, j_2, j_3 \geq 1$, we have

$$\mathrm{Tr}(P_1^3 \Gamma^{(3)}) = \frac{j_1(j_1-1)(j_1-2)}{N(N-1)(N-2)}, \quad \mathrm{Tr}(P_1^2 \otimes P_2 \Gamma^{(3)}) = \frac{j_1(j_1-1)j_2}{N(N-1)(N-2)} \quad (2.9)$$

and

$$\mathrm{Tr}(P_1 \otimes P_2 \otimes P_3 \Gamma^{(3)}) = \frac{j_1 j_2 j_3}{N(N-1)(N-2)}. \quad (2.10)$$

Proof of Lemma 2.6 The proof is a straightforward adaptation of [3, Lemma 4].

Fixing $i = i_{\max}(\underline{J})$ in (2.8) and defining the projection $\mathbb{P}_j$ by

$$\mathbb{P}_j = \sum_{i=1}^j P_i,$$

we can rewrite

$$\sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^i \mathrm{Tr}(P_{i_1} \otimes P_{i_2} \otimes P_{i_3} \tilde{H}_3^R P_{i'_1} \otimes P_{i'_2} \otimes P_{i'_3} \Gamma_{\underline{J}, \underline{J}'}^{(3)}) = \mathrm{Tr}(\mathbb{P}_i^{\otimes 3} \tilde{H}_3^R \mathbb{P}_i^{\otimes 3} \Gamma_{\underline{J}, \underline{J}'}^{(3)}).$$

This is almost of the correct form to apply Theorem 2.5 on the space $\mathfrak{X} = \mathbb{P}_i \mathfrak{H}$.

The only remaining issue is that we can only apply Theorem 2.5 to a state belonging to $\mathcal{S}(\mathfrak{X}^K)$, for some K , and we currently have $\Gamma_{\underline{J}, \underline{J}'}$, which is in general not even a state.

Said differently, we would like to know exactly how many particle have momenta in $\mathbb{P}_i$, and discard the information about the rest. This can be done by fixing, in addition to $i = i_{\max}(J)$, the number K of particles that have momenta in $\mathbb{P}_i$, and using Lemma 2.7 to construct a state $\gamma_{i,K} \in \mathcal{S}(\mathbb{P}_i^{\otimes K} \mathfrak{H}^K)$. Having done so, we will obtain

$$\mathrm{Tr}(\tilde{H}_3^R \Gamma^{(3)}) \gtrsim \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} \|\Psi_{i,K}\|^2 \mathrm{Tr}(\tilde{H}_3^R \gamma_{i,K}^{(3)}), \quad (2.11)$$

where $\Psi_{i,K}$ is given by

$$\Psi_{i,K} = \sum_{\substack{J \\ i_{\max}(J)=i \\ j_{\max}=K}} \Psi_J, \quad \text{with } j_{\max} = \sum_{k=1}^{i_{\max}(J)} j_k.$$

For each state $\gamma_{i,K}$, we will apply Theorem 2.5 using the CLR-type estimate (3.9) and Proposition 2.8 to bound the error terms, to obtain

$$\mathrm{Tr}(\tilde{H}_3^R \gamma_{i,K}^{(3)}) \gtrsim \int_{\mathcal{S}(\mathbb{P}_i \mathfrak{H})} d\mu(\gamma) \mathrm{Tr}(\tilde{H}_3^R \gamma^{\otimes 3}) \gtrsim e_R^{\mathrm{af}}. \quad (2.12)$$

Inserting (2.12) into (2.11) and using that e_R^{af} converges to e^{af} will yield the desired result.

Lemma 2.7 *Let P and Q be orthogonal projections on $\mathfrak{H}$. Given a state*

$$\Gamma \in \mathcal{S}(P^{\otimes j} \otimes_s Q^{\otimes(N-j)} \mathfrak{H}^N),$$

for some $j \geq 1$, there exists another state

$$\Gamma_j \in \mathcal{S}(P^{\otimes j} \mathfrak{H}^j)$$

such that

$$P^{\otimes 3} \Gamma^{(3)} P^{\otimes 3} = \binom{N}{3}^{-1} \binom{j}{3} \Gamma_j^{(3)}. \quad (2.13)$$

Proof The proof is a straightforward adaptation of [3, Lemma 5]. $\square$

Proposition 2.8 (Plane wave estimate) *Let V satisfy (1.4) and define $h = -\Delta + V$. Denote by $\mathbb{P}_i$ the spectral projection $\mathbb{1}(h < N^{2i\varepsilon})$, for any $i, \varepsilon > 0$. Let $\mathbf{e}_k$ be the multiplication operator on $\mathfrak{H}$ by either $\cos(k \cdot x)$ or $\sin(k \cdot x)$. Then,*

$$\pm \mathbb{P}_i \mathbf{e}_k \mathbb{P}_i \leq \mathbb{P}_i \min \left\{ 1, C \frac{N^{2i\varepsilon}}{|k|^2} \right\} \mathbb{P}_i, \quad (2.14)$$

for some universal constant $C > 0$.

The proof is the same as that of [3, Proposition 4], and we provide in full detail to correct a minor mistake.

Proof Take $k \in \mathbb{R}^2 \setminus \{0\}$ and $\mathbf{e}_k = \cos(k \cdot x)$ ($\mathbf{e}_k = \sin(k \cdot x)$ being similar). The first bound in (2.14) is just $|\mathbf{e}_k| \leq 1$. To prove the other bound, we take some smooth compactly supported function f and use an integration by parts to deduce

$$\langle P_i f, \cos(k \cdot x) P_i f \rangle = \int_{\mathbb{R}^2} dx |P_i f|^2 \cos(k \cdot x) = \int dx (-\Delta |P_i f|^2) |k|^{-2} \cos(k \cdot x).$$

Here, we used the identity $\cos(k \cdot x) = -\Delta \cos(k \cdot x)/|k|^2$. Hence, an application of the Cauchy–Schwarz inequality implies

$$\langle P_i f, \cos(k \cdot x) P_i f \rangle \leq C|k|^{-2} (\|\nabla P_i f\|^2 + \|P_i f\| \|\Delta P_i f\|).$$

Therefore, the claim follows from the bounds

$$P_i(-\Delta)P_i \leq N^{2i\varepsilon} P_i \quad \text{and} \quad P_i(-\Delta)^2 P_i \leq CN^{4i\varepsilon} P_i \quad (2.15)$$

and a density argument. On the one hand, the first bound is an immediate consequence of the nonnegativity of V . On the other hand, to prove the second bound, we start by noticing that

$$(-\Delta)^2 = (-\Delta + V)^2 - (-\Delta)V - V(-\Delta) - V^2.$$

Moreover,

$$(-\Delta)V + V(-\Delta) = 2V^{1/2} \left(-\Delta - \left| \frac{\nabla V}{2V} \right|^2 \right) V^{1/2}$$

Thus, an application of the estimate (1.4) gives the second bound in (2.15). This concludes the proof of Proposition 2.8. $\square$

3 Mean-field limit: proof of Theorem 1.1

We establish the lower bound

$$e_N^R \geq e_R^{\text{af}} - CN^{-\nu} \quad (3.1)$$

for some $\nu > 0$, and where e_R^{af} denotes the ground state energy of the functional $\mathcal{E}_R^{\text{af}}$ defined in (1.10). Then, the energy convergence (1.20) follows from

$$e_R^{\text{af}} \rightarrow e^{\text{af}}$$

as $N \rightarrow \infty$ (see [1, Appendix A]). The state convergence follows directly using ideas from (1.21) [33] (see also [1, 2]). The upper bound matching (3.1) is given in Proposition 2.4.

Let Ψ be a ground state of H_N^R and denote by Γ the projection $|\Psi\rangle\langle\Psi|$. Define also the kinetic and interaction operators

$$T = \frac{1}{3}(h_1 + h_2 + h_3), \quad W_2 = \frac{1}{3} \sum_{\substack{1 \leq j, k \leq 3 \\ j \neq k}} W_2(j, k) \quad \text{and} \quad W_3 = \frac{1}{6} \sum_{\substack{1 \leq j, k, \ell \leq 3 \\ j \neq k \neq \ell \neq j}} W_3(j, k, \ell), \quad (3.2)$$

with

$$h = -\Delta + V, \quad W_2(j, k) = \mathbf{p}_j \cdot \nabla^\perp w_R(x_j - x_k) \quad (3.3)$$

and

$$W_3(j, k, \ell) = \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_\ell). \quad (3.4)$$

By symmetry of Ψ , the nonnegativity of the fourth term in (1.9), and the assumption (1.19) on α , we have

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \geq \text{Tr}(T\Gamma^{(3)}) + \beta \text{Tr}(W_2\Gamma^{(3)}) + \beta^2 \frac{N-2}{N-1} \text{Tr}(W_3\Gamma^{(3)}).$$

The factor $(N - 2)/(N - 1)$ in the right can be replaced by 1 at the cost of a small amount of the kinetic energy. Specifically, Lemma 2.3 implies

$$\beta^2 \left| 1 - \frac{N-2}{N-1} \right| \left| \text{Tr}(W_3 \Gamma^{(3)}) \right| \leq C N^{-1} \text{Tr}((C + h) \Gamma^{(1)}).$$

Hence, by defining the effective three-body Hamiltonian

$$\tilde{H}_3^R = T + \beta W_2 + \beta^2 W_3,$$

we obtain

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \geq \text{Tr}(\tilde{H}_3^R \Gamma^{(3)}) - C N^{-1} \text{Tr}((C + h) \Gamma^{(1)}). \quad (3.5)$$

Next, we split $\mathfrak{H}$ into $M+1$ annuli in momentum space according to the spectral projections

$$\begin{aligned} P_1 &= \mathbb{1}(h < N^{2\varepsilon}), & P_i &= \mathbb{1}(N^{2(i-1)\varepsilon} \leq h < N^{2i\varepsilon}), \quad 2 \leq i \leq M, \\ P_{M+1} &= \mathbb{1}(N^{2M\varepsilon} \leq h), \end{aligned} \quad (3.6)$$

for some nonnegative parameters ε and M that will be chosen later. In the polynomial case $R = N^{-\eta}$, the parameter ε will be universal, and in the exponential case $R = e^{-N^\kappa}$ it will depend only on κ . The parameter M shall be taken such that

$$M = \mathcal{O}(\log R^{-1} / \log N). \quad (3.7)$$

In particular, M will be constant in the polynomial case. To shorten some notation, we define

$$P_{ij} = P_i \otimes P_j, \quad P_{ijk} = P_i \otimes P_j \otimes P_k, \quad \mathbb{P}_i = \sum_{j=1}^i P_j \quad \text{and} \quad \mathbb{Q}_i = 1 - \mathbb{P}_i. \quad (3.8)$$

Thanks to assumption (1.3) and (3.7), we have the Cwikel–Lieb–Rosenblum (CLR) type estimate

$$\dim(\mathbb{P}_M \mathfrak{H}) \leq C R^{-C}. \quad (3.9)$$

(see [33, Lemma 3.3] and references therein).

We claim that

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \geq \sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^M \text{Tr}(P_{i_1 i_2 i_3} \tilde{H}_3^R P_{i'_1 i'_2 i'_3} \Gamma^{(3)}) - C N^{-2\varepsilon} \text{Tr}((C + h) \Gamma^{(1)}) \quad (3.10)$$

(taking ε small enough in (3.6)). To prove this, we use the resolution of the identity

$$\sum_{i=1}^{M+1} P_i = \mathbb{1} \quad (3.11)$$

to expand

$$\text{Tr}(\tilde{H}_3^R \Gamma^{(3)}) = \sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^{M+1} \text{Tr}(P_{i_1 i_2 i_3} \tilde{H}_3^R P_{i'_1 i'_2 i'_3} \Gamma^{(3)}).$$

After that, we bound the interaction terms containing at least one projection P_{M+1} using Lemma 3.1, and insert the result into (3.5) to deduce (3.10). An example of such a term is

$$\sum_{\substack{i_2, i_3 \\ i'_1, i'_2, i'_3}}^M \text{Tr}(P_{(M+1) i_2 i_3} \tilde{H}_3^R P_{i'_1 i'_2 i'_3} \Gamma^{(3)}) = \text{Tr}(P_{M+1} \otimes \mathbb{P}_M \otimes \mathbb{P}_M \tilde{H}_3^R \mathbb{P}_M \otimes \mathbb{P}_M \otimes \mathbb{P}_M \Gamma^{(3)}),$$

which we rewrote using the projection $\mathbb{P}_M$ defined in (3.8); all the others can be rewritten in a similar way. Since Lemma 3.1 will not be used again in the proof of Theorem 1.1, we postpone its own proof to Sect. 4.

Lemma 3.1 (High-momenta estimates) *Take R as in Theorem 1.1, and M satisfying (3.7). Then, there exists a universal constant $C > 0$ such that, for any state $\Gamma \in \mathcal{S}(\mathfrak{H}^3)$, and any projections $Q_1, \dots, Q_5 \in \{\mathbb{P}_M, P_{M+1}\}$,*

$$|\mathrm{Tr}(P_{M+1} \otimes Q_1 \otimes Q_2 W_2 Q_3 \otimes Q_4 \otimes Q_5 \Gamma)| \leq C N^{-2\varepsilon} \mathrm{Tr}((C+h)\Gamma^{(1)}) \quad (3.12)$$

and

$$|\mathrm{Tr}(P_{M+1} \otimes Q_1 \otimes Q_2 W_3 Q_3 \otimes Q_4 \otimes Q_5 \Gamma)| \leq C N^{-2\varepsilon} \mathrm{Tr}((C+h)\Gamma^{(1)}). \quad (3.13)$$

Next, we decompose Ψ according to the occupation of each energy level P_i . For each configuration $\underline{J} = (j_{i_1}, \dots, j_{i_{M+1}}) \in \mathbb{N}_0^{M+1}$ with $|\underline{J}| = N$, define

$$\Psi_{\underline{J}} = P_1^{\otimes j_1} \otimes_s \dots \otimes_s P_{M+1}^{\otimes j_{M+1}} \Psi,$$

as well as

$$\Gamma_{\underline{J}, \underline{J}'} = |\Psi_{\underline{J}'}\rangle\langle\Psi_{\underline{J}}| \quad \text{and} \quad \Gamma_{\underline{J}} = |\Psi_{\underline{J}}\rangle\langle\Psi_{\underline{J}}|. \quad (3.14)$$

Moreover, for any multi-index $\underline{J}$, define

$$\underline{J}^{(i_1 i_2 i_3)} = (\tilde{j}_1, \dots, \tilde{j}_{M+1}) \quad (3.15)$$

with

$$\tilde{j}_k = j_k + \delta_{i_1 k} + \delta_{i_2 k} + \delta_{i_3 k}$$

for all $k \in \{1, \dots, M+1\}$. Define also

$$\Gamma_{\underline{J}^{(i_1 i_2 i_3); i'_1 i'_2 i'_3}} = \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}.$$

Then, we use the identity

$$P_{i'_1 i'_2 i'_3} \Gamma^{(3)} P_{i_1 i_2 i_3} = \sum_{\underline{J}} P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3); i'_1 i'_2 i'_3}}^{(3)} P_{i_1 i_2 i_3}$$

to deduce

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \geq \sum_{\underline{J}} \sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^M \mathrm{Tr}(P_{i_1 i_2 i_3} \tilde{H}_3^R P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3); i'_1 i'_2 i'_3}}^{(3)}) - C N^{-2\varepsilon} \mathrm{Tr}((C+h)\Gamma^{(1)}), \quad (3.16)$$

where we are summing over all multi-indices $\underline{J}$ of size $M+1$ such that $|\underline{J}| = N-3$.

To bound the first term in the right-hand side of (3.16), we define

$$i_{\max}(\underline{J}) = \max \{i \in \{1, \dots, M\} : j_i \geq N^{1-\delta\varepsilon}\}, \quad (3.17)$$

for some $\delta > 0$ to be chosen later. For readability's sake, we sometimes omit $\underline{J}$ and simply write $i_{\max}$. Also, by convention, we take $i_{\max}(\underline{J}) = 0$ if the set on the right is empty. We make the following important distinction regarding the sum over $i_1, \dots, i'_3$ in (3.16): either all indices are less than $i_{\max}$ - in which case we use Theorem 2.5 - or not. In the latter case, we need to bound

$$\mathrm{Tr}(P_{i_1 i_2 i_3} W_2 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3); i'_1 i'_2 i'_3}}^{(3)})$$

and

$$\mathrm{Tr} \left(P_{i_1 i_2 i_3} W_3 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} \right)$$

when one the indices, say i_1 , is such that $j_{i_1} < N^{1-\delta\varepsilon}$. This is done in Lemma 3.2, which, from the technical point of view, is the main novelty of the present work. Since the proof is rather complicated and intricate, we postpone it to Sect. 4.

Lemma 3.2 (Low-occupancy estimates) *Take R as in Theorem 1.1, and M satisfying (3.7). Let $\delta, \Lambda > 0$. Then,*

$$\begin{aligned} & \left| \sum_{\underline{j}} \sum_{i_1=\max+1}^M \sum_{i_2, i_3} \sum_{i'_1, i'_2, i'_3} \mathrm{Tr} \left(P_{i_1 i_2 i_3} W_2 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \\ & \leq C_{\Lambda} (N^{-(\Lambda-3)\varepsilon/2} + N^{-(\delta/2-2)\varepsilon} + N^{-(1-\kappa)/2+2\varepsilon}) \mathrm{Tr} \left((C+h) \Gamma^{(1)} \right) \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} & \left| \sum_{\underline{j}} \sum_{i_1=\max+1}^M \sum_{i_2, i_3} \sum_{i'_1, i'_2, i'_3} \mathrm{Tr} \left(P_{i_1 i_2 i_3} W_3 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \\ & \leq C_{\Lambda} (N^{-(\Lambda-3)\varepsilon/2} + N^{-(\delta/2-2)\varepsilon} + N^{-(1-\kappa)/2+2\varepsilon}) \mathrm{Tr} \left((C+h) \Gamma^{(1)} \right), \end{aligned} \quad (3.19)$$

for some constant $C_{\Lambda} > 0$ depending only on $\Lambda > 0$, and with $\kappa = 0$ in the polynomial case (1.17). Here, we are summing over all multi-indices $\underline{j}$ of size $M+1$ satisfying $|\underline{j}| = N-3$.

Taking $\delta, \Lambda \geq 5$ and inserting (3.18) and (3.19) into (3.16), we obtain

$$\begin{aligned} N^{-1} \langle \Psi, H_N^R \Psi \rangle & \geq \sum_{\underline{j}} \sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^{i_{\max}} \mathrm{Tr} \left(P_{i_1 i_2 i_3} \tilde{H}_3^R P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} \right) \\ & - C(N^{-\varepsilon} - CN^{-(1-\kappa)/2+2\varepsilon}) \mathrm{Tr} \left((C+h) \Gamma^{(1)} \right), \end{aligned} \quad (3.20)$$

with $\kappa = 0$ in the polynomial case (1.17). Hence, we will take

$$\varepsilon < \frac{1-\kappa}{4} \quad (3.21)$$

to make sure that the last term in (3.20) is small.

Before we apply Theorem 2.5, we do some rewriting. We decompose first over the value of $i_{\max}$:

$$\sum_{\substack{\underline{j} \\ i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^{i_{\max}} P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} P_{i_1 i_2 i_3} = \sum_{i=1}^M \sum_{\substack{i_1, i_2, i_3=1 \\ i'_1, i'_2, i'_3=1}}^i \sum_{\substack{\underline{j} \\ i_{\max}(\underline{j})=i}} P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} P_{i_1 i_2 i_3}.$$

At fixed $i_1, \dots, i'_3$, the last sum in the right can be rewritten as

$$\sum_{\substack{\underline{j} \\ i_{\max}(\underline{j})=i}} P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3; i'_1 i'_2 i'_3)}}^{(3)} P_{i_1 i_2 i_3} = \sum_{\substack{\underline{j}', \underline{j}'' \\ i_{\max}(\underline{j}')=i \\ i_{\max}(\underline{j}'')=i}} P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}', \underline{j}''}^{(3)} P_{i_1 i_2 i_3}, \quad (3.22)$$

where we are summing over $\underline{j}$'s satisfying $|\underline{j}| = N-3$ in the left, and over $\underline{j}', \underline{j}''$ satisfying $|\underline{j}'| = |\underline{j}''| = N$ in the right. To see why this holds, notice first that, due to the presence of

the projections $P_{i_1 i_2 i_3}$ and $P_{i'_1 i'_2 i'_3}$, the multi-index $\underline{J}''$ is entirely determined by the choice of $\underline{J}'$. Said differently, the double sum on the right-hand side is essentially only a single sum. Moreover, for every $\underline{J}'$, there is exactly one $\underline{J}$ such that $\underline{J}^{(i_1 i_2 i_3)} = \underline{J}'$.

Define now

$$j_{\max}(\underline{J}) = \sum_{i=1}^{i_{\max}(\underline{J})} j_i,$$

as well as $\Psi_{i,K}$ and $\Gamma_{i,K}$ by

$$\Psi_{i,K} = \sum_{\substack{\underline{J} \\ i_{\max}(\underline{J})=i \\ j_{\max}(\underline{J})=K}} \Psi_{\underline{J}} \quad \text{and} \quad \Gamma_{i,K} = |\Psi_{i,K}\rangle \langle \Psi_{i,K}|. \quad (3.23)$$

Thus, we can rewrite (3.22) further as

$$\sum_{\substack{\underline{J} \\ i_{\max}(\underline{J})=i}} P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} P_{i_1 i_2 i_3} = \sum_K P_{i'_1 i'_2 i'_3} \Gamma_{i,K}^{(3)} P_{i_1 i_2 i_3}.$$

All in all, we have shown that

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \geq \sum_{i=1}^M \sum_K \text{Tr}(\mathbb{P}_i^{\otimes 3} \tilde{H}_3^R \mathbb{P}_i^{\otimes 3} \Gamma_{i,K}^{(3)}) - C(N^{-2\varepsilon} + N^{-(1-\kappa)/2+2\varepsilon}) \text{Tr}((C+h)\Gamma^{(1)}), \quad (3.24)$$

where we are summing over $N - MN^{1-\delta\varepsilon} \leq K \leq N$. The parameter κ in (3.24) is equal to 0 in the polynomial case (1.17).

Since $\Psi_{i,K}$ is symmetric and belongs to $\mathbb{P}_i^{\otimes K} \otimes_s \mathbb{Q}_i^{\otimes(N-K)} \mathfrak{H}^N$, we can use Lemma 2.7 to find a symmetric state $\gamma_{i,K} \in \mathcal{S}(\mathbb{P}_i^{\otimes K} \mathfrak{H}^K)$ such that

$$\mathbb{P}_i^{\otimes 3} \Gamma_{i,K}^{(3)} \mathbb{P}_i^{\otimes 3} = \binom{N}{3}^{-1} \binom{K}{3} \text{Tr}(\Gamma_{i,K}) \gamma_{i,K}^{(3)}.$$

Hence, we may rewrite (3.24) as

$$N^{-1} \langle \Psi, H_N^R \Psi \rangle \geq \sum_{i=1}^M \sum_K \binom{N}{3} \binom{K}{3}^{-1} \text{Tr}(\Gamma_{i,K}) \text{Tr}(\tilde{H}_3^R \gamma_{i,K}^{(3)}) - C(N^{-2\varepsilon} + N^{-(1-\kappa)/2+2\varepsilon}) \text{Tr}((C+h)\Gamma^{(1)}). \quad (3.25)$$

We now use Theorem 2.5 to bound $\text{Tr}(\tilde{H}_3^R \gamma_{i,K}^{(3)})$ from below. To that end, we define

$$\Delta \gamma_{i,K}^{(3)} = \gamma_{i,K}^{(3)} - \int_{\mathcal{S}(\mathbb{P}_i \mathfrak{H})} \gamma^{\otimes 3} d\mu_{i,K}(\gamma),$$

where $\mu_{i,K}$ is the probability measure from Theorem 2.5 applied to the state $\gamma_{i,K} \in \mathcal{S}(\mathbb{P}_i^{\otimes K} \mathfrak{H}^K)$. Thus,

$$\text{Tr}(\tilde{H}_3^R \gamma_{i,K}^{(3)}) = \int_{\mathcal{S}(\mathbb{P}_i \mathfrak{H})} \text{Tr}(\tilde{H}_3^R \gamma^{\otimes 3}) d\mu_{i,K}(\gamma) + \text{Tr}(\tilde{H}_3^R \Delta \gamma_{i,K}^{(3)}). \quad (3.26)$$

Only the first term contributes to the ground state energy; the second is an error as we now show.

Analysis of T . A direct application of Theorem 2.5 yields

$$\mathrm{Tr} \left(T \Delta \gamma_{i,K}^{(3)} \right) \geq -C \sqrt{\frac{\log R^{-1}}{K}} N^{2i\varepsilon}, \quad (3.27)$$

where we used the CLR-type estimate (3.9) to bound the dimension of $\mathbb{P}_i \mathfrak{H}$.

Analysis of W_2 . In order to apply Theorem 2.5, we need to write W_2 in a tensorised form. By symmetry, it suffices to consider

$$W_2(1, 2) = \mathbf{p}_1 \cdot \nabla^\perp w_R(x_1 - x_2).$$

For readability's sake, we define

$$\chi_R = \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}.$$

Then, using the Fourier transform, we write

$$W_2(1, 2) = -2\pi \mathbf{i} \int dk \hat{\chi}_R(k) \frac{e_{k^\perp}}{|k|} e^{-ik \cdot x_2} \mathbf{p}_1 e^{ik \cdot x_1},$$

where $e_{k^\perp}$ denotes the normalised direction of the vector $k^\perp$. Moreover, using that

$$\mathbf{p}_1 e^{ik \cdot x_1} = e^{ik \cdot x_1} (\mathbf{i}k + \mathbf{p}_1)$$

and that $e_{k^\perp} \cdot k = 0$, we get

$$W_2(1, 2) = -2\pi \mathbf{i} \int dk \frac{\hat{\chi}_R(k)}{|k|} e^{ik \cdot (x_1 - x_2)} e_{k^\perp} \cdot \mathbf{p}_1.$$

Developing $e^{ik \cdot (x_1 - x_2)}$ and using that $k \mapsto \frac{\hat{\chi}_R(k)}{|k|} e_{k^\perp}$ is odd, we further rewrite this as

$$W_2(1, 2) = 2\pi \int dk \frac{\hat{\chi}_R(k)}{|k|} (\sin(k \cdot x_1) \cos(k \cdot x_2) - \cos(k \cdot x_1) \sin(k \cdot x_2)) e_{k^\perp} \cdot \mathbf{p}_1. \quad (3.28)$$

Finally, splitting the integral over k between $|k| \leq N^{i\varepsilon}$ and $|k| > N^{i\varepsilon}$, and using Theorem 2.5 and Proposition 2.8, we obtain

$$\begin{aligned} \mathrm{Tr} \left(W_2 \Delta \gamma_{i,K}^{(3)} \right) &\geq -C \sqrt{\frac{\log R^{-1}}{K}} \int_{|k| \leq N^{i\varepsilon}} dk \|\mathbb{P}_i e_{k^\perp} \cdot \mathbf{p}\|_{\mathrm{op}} \frac{|\hat{\chi}_R(k)|}{|k|} \\ &\quad - C \sqrt{\frac{\log R^{-1}}{K}} N^{2i\varepsilon} \int_{|k| > N^{i\varepsilon}} dk \|\mathbb{P}_i e_{k^\perp} \cdot \mathbf{p}\|_{\mathrm{op}} \frac{|\hat{\chi}_R(k)|}{|k|^3} \\ &\geq -C \sqrt{\frac{\log R^{-1}}{K}} N^{2i\varepsilon}. \end{aligned} \quad (3.29)$$

Here, we used that $\|\hat{\chi}_R\|_{L^\infty} \leq \|\chi_R\|_{L^1} \leq C$.

Analysis of W_3 . By symmetry, we need to deal only with

$$W_3(1, 2, 3) = \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3).$$

Using again the Fourier transform, we have

$$W_3(1, 2, 3) = \int dk \chi_R(k) \frac{e_{k^\perp}}{|k|} e^{ik \cdot (x_1 - x_2)} \int dk' \chi_R(k') \frac{e_{k'^\perp}}{|k'|} e^{ik' \cdot (x_1 - x_3)}.$$

Developing the complex exponentials and using the oddness of $k \mapsto \frac{\hat{\chi}_R(k)}{|k|} e_{k^\perp}$ yields

$$\begin{aligned} W_3(1, 2, 3) = & \iint dk dk' \hat{\chi}_R(k) \hat{\chi}_R(k') \frac{e_{k^\perp}}{|k|} \cdot \frac{e_{k'^\perp}}{|k'|} \left(\cos(k \cdot x_1) \cos(k' \cdot x_1) \sin(k \cdot x_2) \sin(k' \cdot x_3) \right. \\ & + \sin(k \cdot x_1) \sin(k' \cdot x_1) \cos(k \cdot x_2) \cos(k' \cdot x_3) \\ & - \cos(k \cdot x_1) \sin(k' \cdot x_1) \sin(k \cdot x_2) \cos(k' \cdot x_3) \\ & \left. - \sin(k \cdot x_1) \cos(k' \cdot x_1) \cos(k \cdot x_2) \sin(k' \cdot x_3) \right). \end{aligned} \quad (3.30)$$

Using once more Theorem 2.5 and Proposition 2.8, we find

$$\mathrm{Tr} \left(W_3 \Delta \gamma_{i,K}^{(3)} \right) \geq -C \sqrt{\frac{\log R^{-1}}{K}} N^{2i\varepsilon}. \quad (3.31)$$

Here we again split the integral over k between $|k| \leq N^{i\varepsilon}$ and $|k| > N^{i\varepsilon}$, and we did the same for k' .

Inserting (3.27)–(3.31) into (3.26), and using that $\mu_{i,K}$ is a probability measure, we obtain

$$\mathrm{Tr} \left(\tilde{H}_3^R \gamma_{i,K}^{(3)} \right) \geq \int_{\mathcal{S}(\mathbb{P}_i \mathfrak{H})} \mathrm{Tr} \left(\tilde{H}_3^R \gamma^{\otimes 3} \right) d\mu_{i,K}(\gamma) - C \sqrt{\frac{\log R^{-1}}{K}} N^{2i\varepsilon}. \quad (3.32)$$

As a result of (3.25) and (3.32), we have

$$\begin{aligned} N^{-1} \langle \Psi, H_N^R \Psi \rangle & \geq \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} \mathrm{Tr} \Gamma_{i,K} \int_{\mathcal{S}(\mathbb{P}_i \mathfrak{H})} \mathrm{Tr} \left(\tilde{H}_3^R \gamma^{\otimes 3} \right) d\mu_{i,K}(\gamma) \\ & - C \sqrt{\frac{\log R^{-1}}{K}} \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} N^{2i\varepsilon} \mathrm{Tr} \Gamma_{i,K} \\ & - C(N^{-2\varepsilon} + N^{-\delta\varepsilon/2+2\varepsilon}) \mathrm{Tr} \left((C+h) \Gamma^{(1)} \right). \end{aligned} \quad (3.33)$$

On the one hand, using $\binom{N}{3}^{-1} \binom{K}{3} \leq 1$ and that $K \geq N - MN^{1-\delta\varepsilon}$, we can bound the second term in (3.33) by

$$\sqrt{\frac{\log R^{-1}}{K}} \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} N^{2i\varepsilon} \mathrm{Tr} \Gamma_{i,K} \leq C N^{\delta\varepsilon/2-1/2} \sqrt{\log R^{-1}} \sum_{\underline{J}} N^{2i_{\max}(\underline{J})\varepsilon} \mathrm{Tr} \Gamma_{\underline{J}},$$

where we recall that $i_{\max}$ was defined in (3.17). Thanks to $1 \leq N^{\delta\varepsilon} j_{i_{\max}}/N$ and Lemma 2.6, we deduce

$$\begin{aligned} & \sqrt{\frac{\log R^{-1}}{K}} \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} N^{2i\varepsilon} \mathrm{Tr} \Gamma_{i,K} \\ & \leq C N^{(3\delta/2+2)\varepsilon-1/2} \sqrt{\log R^{-1}} \sum_{\underline{J}} N^{2(i_{\max}(\underline{J})-1)\varepsilon} \frac{j_{i_{\max}(\underline{J})}}{N} \mathrm{Tr} \Gamma_{\underline{J}} \\ & \leq C N^{(3\delta/2+2)\varepsilon-1/2} \sqrt{\log R^{-1}} \sum_{\underline{J}} \mathrm{Tr} \left(P_{i_{\max}(\underline{J})}(C+h) P_{i_{\max}(\underline{J})} \Gamma_{\underline{J}}^{(1)} \right) \\ & \leq C N^{(3\delta/2+2)\varepsilon-1/2} \sqrt{\log R^{-1}} \mathrm{Tr} \left((C+h) \Gamma^{(1)} \right). \end{aligned}$$

This means that this term is much smaller than the kinetic energy under the condition

$$(3\delta/2 + 2)\varepsilon < (1 - \kappa)/2 \quad (3.34)$$

(with $\kappa = 0$ in the polynomial case). On the other hand, to simplify the first term in (3.33), we define the Borel measure

$$\mu_N = \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} \text{Tr}(\Gamma_{i,K}) \mu_{i,K}$$

and we extend the definition (1.10) of the average-field functional $\mathcal{E}_R^{\text{af}}$ to mixed-states γ through

$$\mathcal{E}_R^{\text{af}}[\gamma] = \text{Tr}(\tilde{H}_3^R \gamma^{\otimes 3}).$$

Summing up, the inequality (3.33) simplifies to

$$\begin{aligned} N^{-1} \langle \Psi, H_N^R \Psi \rangle &\geq \int_{S(\mathbb{P}_M \mathfrak{H})} \mathcal{E}_R^{\text{af}}[\gamma] d\mu_N(\gamma) \\ &\quad - C(N^{-\varepsilon} + N^{-\delta\varepsilon/2+2\varepsilon} + N^{(3\delta/2+2)\varepsilon-(1-\kappa)/2} \sqrt{\log N}) \text{Tr}((C+h)\Gamma^{(1)}), \end{aligned}$$

with $\kappa = 0$ in the polynomial case (1.17). To conclude the proof of proof of Theorem 1.1, we wish to use

$$\text{Tr}(h\Gamma^{(1)}) \leq C \quad (3.35)$$

and

$$\liminf_{N \rightarrow \infty} \int_{S(\mathbb{P}_M \mathfrak{H})} \mathcal{E}_R^{\text{af}}[\gamma] d\mu_N(\gamma) \geq e^{\text{af}}. \quad (3.36)$$

For R of the form $N^{-\beta}$ or of the form e^{-N^κ} with $0 < \kappa < 1/2$, the bound (3.35) is given in Proposition 2.4. Regarding (3.36), we essentially need to show that the sequence $(\mu_N)_N$ converges to a probability measure supported on pure states $|u\rangle\langle u|$. For this, we show that the measure μ_N satisfies

$$1 - CN^{-2\varepsilon} \leq \int_{S(\mathbb{P}_M \mathfrak{H})} \mu_N(\gamma) \leq 1. \quad (3.37)$$

Granted that (3.35) and (3.37) are true, we can follow [1, Section 3.7.] (see references therein) to prove that $(\mu_N)_N$ indeed converges to a probability measure supported on pure states $|u\rangle\langle u|$, from which (3.36) and thus (1.20) follow. The upper bound in (3.37) is a consequence of the estimate $\binom{N}{3}^{-1} \binom{K}{3} \leq 1$ and the identity

$$\sum_{i=0}^{M+1} \sum_K \text{Tr} \Gamma_{i,K} = 1.$$

For the lower bound, we use

$$\begin{aligned} \int_{S(\mathbb{P}_M \mathfrak{H})} \mu_N(\gamma) &= \sum_{i=1}^M \sum_K \binom{N}{3}^{-1} \binom{K}{3} \text{Tr} \Gamma_{i,K} = \sum_{i=1}^M \sum_K \text{Tr}(\mathbb{P}_i^{\otimes 3} \Gamma_{i,K}^{(3)}) \\ &= \sum_{i=0}^M \sum_K \text{Tr} \Gamma_{i,K} - \sum_K \text{Tr} \Gamma_{0,K} \\ &\quad - \sum_{i=1}^M \sum_K (\text{Tr}(\mathbb{P}_i^{\otimes 2} \otimes_s \mathbb{Q}_i \Gamma_{i,K}^{(3)}) + \text{Tr}(\mathbb{P}_i \otimes_s \mathbb{Q}_i^{\otimes 2} \Gamma_{i,K}^{(3)}) + \text{Tr}(\mathbb{Q}_i^{\otimes 3} \Gamma_{i,K}^{(3)})) \end{aligned}$$

$$\geq 1 - C(N^{-2\varepsilon} + N^{-2M\varepsilon}) \operatorname{Tr}((C + h)\Gamma^{(1)})$$

and (3.35). To bound the error terms, we used

$$\sum_{i=1}^M \sum_K \operatorname{Tr}(\mathbb{Q}_i \Gamma_{i,K}^{(1)}) \leq CN^{-2\varepsilon} \sum_{i=1}^M \sum_K \operatorname{Tr}(\mathbb{Q}_i(C + h)\mathbb{Q}_i \Gamma_{i,K}^{(1)}) \leq CN^{-2\varepsilon} \operatorname{Tr}((C + h)\Gamma^{(1)}),$$

and

$$\begin{aligned} \sum_K \operatorname{Tr} \Gamma_{0,K} &\leq CN^{-2M\varepsilon} \sum_K \frac{N}{K} \operatorname{Tr}(P_{M+1} h P_{M+1} \Gamma_{0,K}^{(1)}) \leq CN^{-2M\varepsilon} \sum_K \operatorname{Tr}(P_{M+1} h P_{M+1} \Gamma_{0,K}^{(1)}) \\ &\leq CN^{-2M\varepsilon} \operatorname{Tr}(h\Gamma^{(1)}). \end{aligned}$$

The last estimate results from the fact that we are summing over $N - MN^{1-\delta\varepsilon} \leq K \leq N$, which follows from the convention that $i_{\max} = 0$ (corresponding to $i = 0$) when $j_k \leq N^{1-\delta\varepsilon}$ for all $k \in \{1, \dots, M\}$. This concludes the proof of Theorem 1.1, except for R of the form $e^{-N\kappa}$ with $1/2 \leq \kappa < 1$.

When R is of the form $e^{-N\kappa}$ with $1/2 \leq \kappa < 1$, the estimate (2.5) of Proposition 2.4 no longer applies, and we need to prove (3.35) in a different way. Namely, we will use the nonnegativity of the singular two-body term $|\nabla w_R(x_1 - x_2)|^2$, which we dropped at the beginning of the proof. Let us be more precise. Recall that we are considering the many-body Hamiltonian H_N^R given by

$$H_N^R = \sum_{j=1}^N -\Delta_j + V(x_j) + 2\alpha \mathbf{p}_j \cdot \mathbf{A}_j^R + \alpha^2 (\mathbf{A}_j^R)^2.$$

As a consequence of the Cauchy–Schwarz inequality, we have

$$2\alpha \mathbf{p}_j \cdot \mathbf{A}_j^R \geq -\delta(-\Delta_j) - \delta^{-1}\alpha^2 (\mathbf{A}_j^R)^2,$$

for all $\delta > 0$. Splitting $\mathbf{p}_j \cdot \mathbf{A}_j^R$ into $(1 - \tau)\mathbf{p}_j \cdot \mathbf{A}_j^R + \tau\mathbf{p}_j \cdot \mathbf{A}_j^R$, we deduce

$$H_N^R \geq \sum_{j=1}^N (1 - \tau\delta)(-\Delta_j) + V(x_j) + 2\alpha(1 - \tau)\mathbf{p}_j \cdot \mathbf{A}_j^R + (1 - \tau\delta^{-1})\alpha^2 (\mathbf{A}_j^R)^2,$$

for all $0 < \tau < 1$. Using the nonnegativity of V and doing some rewriting, we get

$$H_N^R \geq \sum_{j=1}^N (1 - \delta)\tau h_j + (1 - \tau)(h_j + 2\alpha \mathbf{p}_j \cdot \mathbf{A}_j^R) + (1 - \delta^{-1}\tau)\alpha^2 (\mathbf{A}_j^R)^2.$$

Next, we expand $(\mathbf{A}_j^R)^2$ to separate the three-body term from the singular two-body one. Testing on both sides with Ψ and using (2.3), we find

$$\begin{aligned} N^{-1} \langle \Psi, H_N^R \Psi \rangle &\geq (1 - \delta - C(1 - \delta^{-1}))\tau \operatorname{Tr}(h\Gamma^{(1)}) + (1 - \tau) \operatorname{Tr}(\tilde{H}_3^R \Gamma^{(3)}) \\ &\quad + (1 - \delta^{-1}\tau)\alpha^2 N^2 \operatorname{Tr}(|\nabla w_R(x_1 - x_2)|^2 \Gamma^{(2)}). \end{aligned} \quad (3.38)$$

Taking $0 < \tau < \delta < 1$, it is easy to see that

$$1 - \delta - C(1 - \delta^{-1}) \geq 0 \quad \text{and} \quad 1 - \delta^{-1}\tau \geq 0.$$

Hence, the last term in (3.38) is nonnegative and we may drop it. Regarding the second one, we can repeat more or less the same proof as before to show that

$$\mathrm{Tr}(\tilde{H}_N^R \Gamma^{(3)}) \geq \int_{\mathcal{S}(\mathbb{P}_M \mathfrak{H})} \mathcal{E}_R^{\mathrm{af}}[\gamma] d\mu_N(\gamma) - C(\log N)^{-1} \mathrm{Tr}((C+h)\Gamma^{(1)}).$$

Moreover, the average-field function $\mathcal{E}_R^{\mathrm{af}}$ satisfies

$$\mathcal{E}_R^{\mathrm{af}}[\gamma] \geq 0.$$

This is shown using the spectral decomposition of γ ; we refer to [1, Section 3.6] for more detail. Combining this with (3.37) and (3.38), we get

$$\mathrm{Tr}(h\Gamma^{(1)}) \leq C(1 + e_N).$$

Thanks to Proposition 2.4, this shows that we have proved (3.35) for R of the form e^{-N^κ} with $1/2 \leq \kappa < 1$. The rest of the proof is exactly the same as before, and we have therefore finished proving Theorem 1.1.

4 Proof of the main estimates

4.1 High-momenta estimates: proof of Lemma 3.1

We begin with the proof of the estimate (3.13). Thanks to the Cauchy–Schwarz inequality and the estimates (2.1) and (2.3), we have

$$\begin{aligned} & |\mathrm{Tr}(P_{M+1} \otimes Q_1 \otimes Q_2 W_3 Q_3 \otimes Q_4 \otimes Q_5 \Gamma)| \\ & \leq CN^{2\varepsilon} R^{-2} \mathrm{Tr}(P_{M+1} \otimes Q_1 \otimes Q_2 \Gamma) \\ & \quad + CN^{-2\varepsilon} \mathrm{Tr}(Q_3 \otimes Q_4 \otimes Q_5 (1 - \Delta_1 - \Delta_2 - \Delta_3) Q_3 \otimes Q_4 \otimes Q_5 \Gamma) \\ & \leq CN^{-2\varepsilon} \mathrm{Tr}((C+h)\Gamma^{(1)}). \end{aligned}$$

In the second inequality we used

$$N^{2\varepsilon} R^{-2} \leq N^{-2\varepsilon} N^{2M\varepsilon} \quad \text{and} \quad P_{M+1} N^{2M\varepsilon} P_{M+1} \leq P_{M+1} h P_{M+1}.$$

The former follows from the definition of R and assumption (3.7). The latter is an immediate consequence of the definition of P_{M+1} .

We now prove (3.12). We only take care of

$$W_2(1, 2) = \mathbf{p}_1 \cdot \nabla^\perp w_R(x_1 - x_2)$$

as the other terms are analogous. Due to $\nabla_1 \cdot \nabla^\perp w_R(x_1 - x_2) = 0$, we have

$$\mathbf{p}_1 \cdot \nabla^\perp w_R(x_1 - x_2) = \nabla^\perp w_R(x_1 - x_2) \cdot \mathbf{p}_1.$$

Combining this with the Cauchy–Schwarz inequality and the estimate (2.1), we find

$$\begin{aligned} & |\mathrm{Tr}(P_{M+1} \otimes Q_1 \otimes Q_2 W_2(1, 2) Q_3 \otimes Q_4 \otimes Q_5 \Gamma)| \\ & \leq CR^{-2} N^{2\varepsilon} \mathrm{Tr}(P_{M+1} \otimes Q_1 \otimes Q_2 \Gamma) \\ & \quad + CN^{-2\varepsilon} \mathrm{Tr}(Q_3 \otimes Q_4 \otimes Q_5 (-\Delta_1) Q_3 \otimes Q_4 \otimes Q_5 \Gamma) \\ & \leq CN^{-2\varepsilon} \mathrm{Tr}((C+h)\Gamma^{(1)}). \end{aligned}$$

This concludes the proof of Lemma 3.1.

4.2 Low-occupancy estimates: Proof of Lemma 3.2

In this section we provide the proof of Lemma 3.2. Although the proof used in the exponential case covers polynomial scalings as well, we first present a simpler proof for the polynomial case for the reader's convenience. Both proofs rely on the following lemma, for which we define

$$\mathbb{P}_{jk} = \mathbb{P}_j \otimes \mathbb{P}_k \quad \text{and} \quad \mathbb{P}_{jk\ell} = \mathbb{P}_j \otimes \mathbb{P}_k \otimes \mathbb{P}_\ell,$$

with $\mathbb{P}_j = \mathbb{1}(h \leq N^{2j\epsilon})$.

Lemma 4.1 *Let $W_2(1, 2)$ and $W_3(1, 2, 3)$ be as in (3.3) and (3.4), with projections as in (3.6). Then, there exists a universal constant $C > 0$ such that, for any i_1, i_2, i'_1, i'_2 and $f, g \in \mathfrak{H}^2$,*

$$\langle \mathbb{P}_{i_1 i_2} f, W_2(1, 2) \mathbb{P}_{i'_1 i'_2} g \rangle \leq C \min(N^{i_1 \epsilon}, N^{\max(i_2, i'_2) \epsilon}) \|\mathbb{P}_{i_1 i_2} f\| \|\mathbf{p}_1 \mathbb{P}_{i'_1 i'_2} g\|. \quad (4.1)$$

Moreover, for any i_1, i_2, i_3 ,

$$P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i_1 i_2 i_3} \leq C N^{(i_2 + i_3) \epsilon}. \quad (4.2)$$

Proof of Lemma 4.1 We start with (4.2). Define

$$\chi_R = \frac{\mathbb{1}_{B(0, R)}}{\pi R^2}$$

and write

$$\begin{aligned} W_3(1, 2, 3) &= \iint dk \, dk' \hat{\chi}_R(k) \hat{\chi}_R(k') \frac{\mathbf{e}_{k^\perp}}{|k|} \\ &\quad \cdot \frac{\mathbf{e}_{k'^\perp}}{|k'|} \left[\cos(k \cdot x_1) \cos(k' \cdot x_1) \sin(k \cdot x_2) \sin(k' \cdot x_3) \right. \\ &\quad + \sin(k \cdot x_1) \sin(k' \cdot x_1) \cos(k \cdot x_2) \cos(k' \cdot x_3) \\ &\quad - \cos(k \cdot x_1) \sin(k' \cdot x_1) \sin(k \cdot x_2) \cos(k' \cdot x_3) \\ &\quad \left. - \sin(k \cdot x_1) \cos(k' \cdot x_1) \cos(k \cdot x_2) \sin(k' \cdot x_3) \right], \end{aligned}$$

as in the decomposition (3.30). Then, we split the integral over $|k| \leq N^{i_2 \epsilon}$ and $|k| > N^{i_2 \epsilon}$, and likewise for k' . Conjugating with $P_{i_1 i_2 i_3}$ and applying Proposition 2.8 to $P_{i_2} \sin(k \cdot x_2) P_{i_2}$ and $P_{i_3} \sin(k' \cdot x_3) P_{i_3}$, we obtain

$$\begin{aligned} &P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i_1 i_2 i_3} \\ &\leq C \left(\int_{|k| \leq N^{i_2 \epsilon}} \frac{dk}{|k|} + \int_{|k| > N^{i_2 \epsilon}} dk \frac{N^{2i_2 \epsilon}}{|k|^3} \right) \left(\int_{|k'| \leq N^{i_3 \epsilon}} \frac{dk'}{|k'|} + \int_{|k'| > N^{i_3 \epsilon}} dk' \frac{N^{2i_3 \epsilon}}{|k'|^3} \right) \\ &\leq C N^{(i_2 + i_3) \epsilon}. \end{aligned}$$

Here, we used $\|\hat{\chi}_R\|_{L^\infty} \leq \|\chi_R\|_{L^1} \leq C$.

Now, we prove (4.1) for f and g smooth with compact support; the result follows from a density argument. We again write in Fourier

$$\begin{aligned} W_2(1, 2) &= 4\pi \int dk \frac{\hat{\chi}_R(k)}{|k|} (\sin(k \cdot x_1) \cos(k \cdot x_2) \\ &\quad - \cos(k \cdot x_1) \sin(k \cdot x_2)) e_{k^\perp} \cdot \mathbf{p}_1, \end{aligned}$$

as in (3.28). To prove the bound involving $N^{2i_1\varepsilon}$, we split the integral over $|k| \leq N^{i_1\varepsilon}$ and $|k| > N^{i_1\varepsilon}$. On the first sector, we have

$$\left| \int_{|k| \leq N^{i_1\varepsilon}} dk \frac{\hat{\chi}_R(k)}{|k|} \left\langle \mathbb{P}_{i_1 i_2} f(\sin(k \cdot x_1) \cos(k \cdot x_2) - \cos(k \cdot x_1) \sin(k \cdot x_2)) e_{k^\perp} \cdot \mathbf{p}_1 \mathbb{P}_{i'_1 i'_2} g \right\rangle \right| \leq C N^{i_1\varepsilon} \|\mathbb{P}_{i_1 i_2} f\| \|\mathbf{p}_1 \mathbb{P}_{i'_1 i'_2} g\|.$$

On the second sector, we write $\sin(k \cdot x_1) = -k \cdot \nabla \cos(k \cdot x_1)/|k|^2$ (resp. $\cos(k \cdot x_1) = k \cdot \nabla \sin(k \cdot x_1)/|k|^2$) and use two successive integrations by parts on x_1 to obtain

$$\left| \int_{|k| > N^{i_1\varepsilon}} dk \frac{\hat{\chi}_R(k)}{|k|} \left\langle \mathbb{P}_{i_1 i_2} f(\sin(k \cdot x_1) \cos(k \cdot x_2) - \cos(k \cdot x_1) \sin(k \cdot x_2)) e_{k^\perp} \cdot \mathbf{p}_1^A \mathbb{P}_{i'_1 i'_2} g \right\rangle \right| \leq C \int_{|k| > N^{i_1\varepsilon}} \frac{dk}{|k|^3} \|\Delta_1 \mathbb{P}_{i_1 i_2} f\| \|\mathbf{p}_1 \mathbb{P}_{i'_1 i'_2} g\| \leq C N^{i_1\varepsilon} \|\mathbb{P}_{i_1 i_2} f\| \|\mathbf{p}_1 \mathbb{P}_{i'_1 i'_2} g\|.$$

Here, we used the bound (2.15).

The bound with $N^{2\max(i_2, i'_2)\varepsilon}$ is proved similarly: we split between $|k| \leq N^{\max(i_2, i'_2)\varepsilon}$ and $|k| > N^{\max(i_2, i'_2)\varepsilon}$, and we use two integrations by parts on the variable x_2 in the second sector. We omit the details. $\square$

Proof of Lemma 3.2 in the polynomial case We begin by proving (3.19) since it is slightly easier. We provide the details only for

$$W_3(1, 2, 3) = \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3)$$

as the other terms are dealt with analogously. Using the Cauchy–Schwarz inequality and Lemmas 2.6 and 4.1, we expand, for all $\tau > 0$,

$$\left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right) \right| \leq \tau C N^{(i_2 + i_3)\varepsilon} \text{Tr} \left(P_{i_1 i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(3)} \right) + \tau^{-1} C N^{(i'_2 + i'_3)\varepsilon} \text{Tr} \left(P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right). \quad (4.3)$$

On the one hand thanks to Lemma 2.6, the first term can be bounded by

$$\text{Tr} \left(P_{i_1 i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(3)} \right) \leq N^{-3} (j_{i_1} + 1)(j_{i_2} + 1)(j_{i_3} + 1) \text{Tr} \Gamma_{\underline{j}(i_1 i_2 i_3)}.$$

Thus, using $j_{i_1} \leq N^{1-\delta\varepsilon}$, the estimate

$$P_i N^{2(i-1)\varepsilon} P_i \leq C P_i (C + h) P_i \quad (4.4)$$

and Lemma 2.6, we deduce

$$N^{(i_2 + i_3)\varepsilon} \text{Tr} \left(P_{i_1 i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(3)} \right) \leq C N^{-\delta\varepsilon + 2\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(1)} \right).$$

On the other hand, thanks to (4.4), we have

$$N^{(i'_2 + i'_3)\varepsilon} \text{Tr} \left(P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right) \leq N^{2\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{j}(i'_1 i'_2 i'_3)}^{(1)} \right).$$

Injecting the previous two inequalities into (4.3) and optimising over τ , we find

$$\left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right) \right| \leq C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(1)} \right) + C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{j}(i'_1 i'_2 i'_3)}^{(1)} \right).$$

Summing over $\underline{J}$, this becomes

$$\sum_{\underline{J}} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \leq C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma^{(1)} \right).$$

Finally, we sum over $i_1, \dots, i'_3$ and use that M is a constant to obtain (3.19).

We now prove (3.18). Again, we bound only

$$W_2(1, 2) = \mathbf{p}_1 \cdot w_R(x_1 - x_2) + w_R(x_1 - x_2) \cdot \mathbf{p}_1$$

and leave the rest to the reader. Thanks to the Cauchy–Schwarz inequality and Lemmas 2.6 and 4.1, we have

$$\begin{aligned} & \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \\ & \leq C N^{\max(i_2, i'_2)\varepsilon} \tau \text{Tr} \left(P_{i_1 i_2 i_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(3)} \right) \\ & \quad + C N^{\max(i_2, i'_2)\varepsilon} \tau^{-1} \text{Tr} \left(P_{i'_1 i'_2 i'_3} (-\Delta_1) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \\ & \leq C N^{\max(i_2, i'_2)\varepsilon} N^{-3} \tau (j_{i_1} + 1)(j_{i_2} + 1)(j_{i_3} + 1) \text{Tr} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}} \\ & \quad + C N^{\max(i_2, i'_2)\varepsilon} N^{-3} \tau^{-1} N^{2i'_1\varepsilon} (j_{i'_1} + 1)(j_{i'_2} + 1)(j_{i'_3} + 1) \text{Tr} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}, \quad (4.5) \end{aligned}$$

for all $\tau > 0$. On the one hand, when $i_2 \geq i'_2$, we use $j_{i_1} \leq N^{1-\delta\varepsilon}$, the estimate (4.4), Lemma 2.6, and optimise over τ to show that

$$\begin{aligned} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| & \leq C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left(P_{i_2 i_3} (C + h_1) P_{i_2 i_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(2)} \right) \\ & \quad + C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left(P_{i'_1 i'_2 i'_3} (C + h_1) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \\ & \leq C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(1)} \right) \\ & \quad + C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}^{(1)} \right). \end{aligned}$$

On the other hand, when $i_2 < i'_2$, an appropriate choice of τ in (4.5) yields

$$\begin{aligned} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| & \leq C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left(P_{i'_2 i'_3} (C + h_1) P_{i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(2)} \right) \\ & \quad + C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left(P_{i'_1 i'_2 i'_3} (C + h_1) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \\ & \leq C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(1)} \right) \\ & \quad + C N^{-(\delta/2-2)\varepsilon} \text{Tr} \left((C + h) \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}^{(1)} \right). \end{aligned}$$

Gathering the previous estimates and carrying out all the sums gives (3.18). This concludes the proof of Lemma 3.2 for polynomially decreasing radii. $\square$

Proof of Lemma 3.2 in the exponential case *Analysis of W_3 .* We first prove (3.19). We give the proof in full detail for

$$W_3(1, 2, 3) = \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3),$$

and we assume that $i_1 \geq i_2 \geq i_3$, as well as $i'_1 \geq i'_2 \geq i'_3$ and $i_1 \geq i'_1$. The rest is left to the reader. To prove (3.19), we make a distinction between $|i_1 - i_3| \leq \Lambda$ and $|i_1 - i_3| > \Lambda$, and likewise for (i'_1, i'_3) and (i_1, i'_1) . Since we only need to use $j_{i_1} \leq N^{1-\delta\varepsilon}$ when

$|i_1 - i_3|, |i'_1 - i'_3|, |i_1 - i'_1| \leq \Lambda$, we begin by treating the other cases. Namely, we bound first

$$\sum_{\underline{J}} \sum_{\substack{i_1 \geq i_2 \geq i_3 \\ i_1 > i_3 + \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \geq i'_3 \\ i'_1 > i'_3 + \Lambda}} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \quad (4.6)$$

and

$$\sum_{\underline{J}} \sum_{\substack{i_1 \geq i_2 \geq i_3 \\ |i_1 - i_3| \leq \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \geq i'_3 \\ |i'_1 - i'_3| \leq \Lambda \\ i_1 > i'_1 + \Lambda}} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right|. \quad (4.7)$$

In order to bound (4.6), we use the Cauchy–Schwarz inequality and Lemmas 2.6 and 4.1 to write

$$\begin{aligned} & \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \\ & \leq \tau C N^{(i_2 + i_3)\varepsilon} N^{-3} (j_{i_1} + 1)(j_{i_2} + 1)(j_{i_3} + 1) \text{Tr} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}} \\ & \quad + \tau^{-1} C N^{(i'_2 + i'_3)\varepsilon} N^{-3} (j_{i'_1} + 1)(j_{i'_2} + 1)(j_{i'_3} + 1) \text{Tr} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}, \end{aligned}$$

for all $\tau > 0$. Swapping j_{i_2} and $j_{i'_2}$ by an appropriate choice of τ , using the estimate

$$P_i N^{2(i-1)\varepsilon} P_i \leq C P_i (C + h) P_i, \quad (4.8)$$

and Lemma 2.6, we thus obtain

$$\begin{aligned} & \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \\ & \leq C N^{(i_2 - i_1)\varepsilon/2} N^{(i_3 - i_1 + 4)\varepsilon/2} N^{(i'_2 - i'_1)\varepsilon/2} N^{(i'_3 - i'_1)\varepsilon/2} \text{Tr} \left(P_{i_1 i'_2 i_3} (C + h_1) P_{i_1 i'_2 i_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(3)} \right) \\ & \quad + C N^{(i_2 - i_1)\varepsilon/2} N^{(i_3 - i_1 + 4)\varepsilon/2} N^{(i'_2 - i'_1)\varepsilon/2} N^{(i'_3 - i'_1)\varepsilon/2} \text{Tr} \left(P_{i'_1 i_2 i'_3} (C + h_1) P_{i'_1 i_2 i'_3} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right), \end{aligned}$$

whenever j_{i_2} and $j_{i'_2}$ are both nonzero. When either of the two indices is null, the previous inequality is no longer true. In fact, if j_{i_2} and $j_{i'_2}$ are both zero, the right-hand side vanishes, whereas the left-hand side can be positive. It is however easy to see that when $j_{i'_2} = 0$, the previous inequality can be made true by replacing $P_{i'_2}$ by $1/\sqrt{N}$ in the right-hand side (recalling that $P_{i_1 i'_2 i_3} = P_{i_1} \otimes P_{i'_2} \otimes P_{i_3}$). The same can also be done with P_{i_2} when $j_{i_2} = 0$. In the following, it will often be the case that inequalities are proved under conditions such as $j_{i_2}, j_{i'_2} \geq 1$ and can easily be adapted to cover $j_{i_2} = 0$ or $j_{i'_2} = 0$. We do not mention this each time to avoid redundancy. Using that

$$\sum_{\underline{J}} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}}^{(3)} \leq \Gamma^{(3)}, \quad (4.9)$$

we may carry out the $\underline{J}$ sum in (4.6) to obtain

$$\begin{aligned} & \sum_{\underline{J}} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}}^{(3)} \right) \right| \\ & \leq C N^{(i_2 - i_1)\varepsilon/2} N^{(i_3 - i_1 + 4)\varepsilon/2} N^{(i'_2 - i'_1)\varepsilon/2} N^{(i'_3 - i'_1)\varepsilon/2} \text{Tr} \left(P_{i_1 i'_2 i_3} (C + h_1) P_{i_1 i'_2 i_3} \Gamma^{(3)} \right) \\ & \quad + C N^{(i_2 - i_1)\varepsilon/2} N^{(i_3 - i_1 + 4)\varepsilon/2} N^{(i'_2 - i'_1)\varepsilon/2} N^{(i'_3 - i'_1)\varepsilon/2} N^{-1} \text{Tr} \left(P_{i_1 i_3} (C + h_1) P_{i_1 i_3} \Gamma^{(2)} \right) \\ & \quad + C N^{(i_2 - i_1)\varepsilon/2} N^{(i_3 - i_1 + 4)\varepsilon/2} N^{(i'_2 - i'_1)\varepsilon/2} N^{(i'_3 - i'_1)\varepsilon/2} \text{Tr} \left(P_{i'_1 i_2 i'_3} (C + h_1) P_{i'_1 i_2 i'_3} \Gamma^{(3)} \right) \end{aligned}$$

$$+ C N^{(i_2-i_1)\varepsilon/2} N^{(i_3-i_1+4)\varepsilon/2} N^{(i'_2-i'_1)\varepsilon/2} N^{(i'_3-i'_1)\varepsilon/2} N^{-1} \operatorname{Tr} \left(P_{i'_1 i'_3} (C + h_1) P_{i'_1 i'_3} \Gamma^{(2)} \right).$$

The two terms containing the factors N^{-1} correspond to $j_{i_2} = 0$ and $j_{i'_2} = 0$. Lastly, carrying out the remaining sums using the geometric series formula, the resolution of the identity

$$\sum_{i=1}^{M+1} P_i = \mathbb{1} \quad (4.10)$$

and the fact that M satisfies (3.7), we find

$$\begin{aligned} & \sum_{\underline{J}} \sum_{\substack{i_1 \geq i_2 \geq i_3 \\ i_1 > i_3 + \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \geq i'_3 \\ i'_1 > i'_3 + \Lambda}} \left| \operatorname{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}} \right) \right| \\ & \leq C N^{-(\Lambda-3)\varepsilon/2} \operatorname{Tr} \left((C + h) \Gamma^{(1)} \right). \end{aligned} \quad (4.11)$$

Hence, we have bounded (4.6) as desired.

To bound (4.7), we begin by writing

$$\begin{aligned} & \left| \operatorname{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}} \right) \right| \\ & \leq \tau C N^{2i_2\varepsilon} N^{-3} (j_{i_1} + 1) (j_{i_2} + 1) (j_{i_3} + 1) \operatorname{Tr} \Gamma_{\underline{J}^{(i_1 i_2 i_3)}} \\ & \quad + \tau^{-1} C N^{2i'_1\varepsilon} N^{-3} (j_{i'_1} + 1) (j_{i'_2} + 1) (j_{i'_3} + 1) \operatorname{Tr} \Gamma_{\underline{J}^{(i'_1 i'_2 i'_3)}}. \end{aligned} \quad (4.12)$$

An appropriate choice of τ then gives

$$\begin{aligned} & \left| \operatorname{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}} \right) \right| \\ & \leq C N^{(i'_1-i_1+2)\varepsilon} \operatorname{Tr} \left(P_{i'_1 i_2 i_3} (C + h_2) P_{i'_1 i_2 i_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}} \right) \\ & \quad + C N^{(i'_1-i_1+2)\varepsilon} \operatorname{Tr} \left(P_{i_1 i'_2 i'_3} (C + h_1) P_{i_1 i'_2 i'_3} \Gamma^{(3)}_{\underline{J}^{(i'_1 i'_2 i'_3)}} \right), \end{aligned} \quad (4.13)$$

for $j_{i_1}, j_{i'_1} \geq 1$. Although the adaptation for $j_{i'_1} = 0$ is straightforward - namely we just replace $P_{i'_1}$ by $1/\sqrt{N}$ - the case $j_{i_1} = 0$ is trickier. Indeed, we cannot simply replace P_{i_1} by $1/\sqrt{N}$ because we also used the estimate $N^{2i_1\varepsilon} P_{i_1} \leq C P_{i_1} (C + h) P_{i_1}$. However, a different choice of τ in (4.12) yields

$$\begin{aligned} & \left| \operatorname{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}} \right) \right| \\ & \leq C N^{-(1-\kappa)/2+2\varepsilon} \operatorname{Tr} \left(P_{i'_2 i_2 i_3} (C + h_2) P_{i'_2 i_2 i_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}} \right) \\ & \quad + C N^{-(1+\kappa)/2+2\varepsilon} \operatorname{Tr} \left(P_{i'_1 i'_3} (C + h_1) P_{i'_1 i'_3} \Gamma^{(2)}_{\underline{J}^{(i'_1 i'_2 i'_3)}} \right), \end{aligned} \quad (4.14)$$

for $j_{i_1} = 0$. Combining (4.13) and (4.14), we can carry out the sums in (4.7) essentially as we did in (4.11) to obtain

$$\begin{aligned} & \sum_{\underline{J}} \sum_{\substack{i_1 \geq i_2 \geq i_3 \\ |i_1 - i_3| \leq \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \geq i'_3 \\ |i'_1 - i'_3| \leq \Lambda \\ i_1 > i'_1 + \Lambda}} \left| \operatorname{Tr} \left(P_{i_1 i_2 i_3} W_3(1, 2, 3) P_{i'_1 i'_2 i'_3} \Gamma^{(3)}_{\underline{J}^{(i_1 i_2 i_3)}, \underline{J}^{(i'_1 i'_2 i'_3)}} \right) \right| \\ & \leq C \left(N^{-(\Lambda-3)\varepsilon/2} + N^{-(1-\kappa)/2+2\varepsilon} \right) \operatorname{Tr} \left((C + h) \Gamma^{(1)} \right). \end{aligned}$$

This is the desired bound on (4.7).

Having bounded (4.6) and (4.7), all that is left to do to prove (3.19) is to deal with

$$\sum_{\underline{j}} \sum_{\substack{i_1 \geq i_2 \geq i_3 \\ |i_1 - i_3| \leq \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \geq i'_3 \\ |i'_1 - i'_3| \leq \Lambda \\ |i_1 - i'_1| \leq \Lambda}} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right) \right|. \quad (4.15)$$

Here, we are summing over $i_1 \geq i_{\max} + 1$, meaning that we always have $j_{i_1} \leq N^{1-\delta\epsilon}$; we do not write it explicitly to avoid cluttering the notation. Proceeding as before, using additionally $j_{i_1} \leq N^{1-\delta\epsilon}$, we can write

$$\begin{aligned} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right) \right| &\leq C N^{-(\delta/2-2)\epsilon} \sum_{i_3} \text{Tr} \left(P_{i_2 i_3} (C + h_1) P_{i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(2)} \right) \\ &\quad + C N^{-(\delta/2-2)\epsilon} \sum_{i'_2 \geq i'_3} \text{Tr} \left(P_{i'_1 i'_2 i'_3} (C + h_1) P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right). \end{aligned}$$

Thus, we can carry out all the sums in (4.15) to obtain the bound

$$\sum_{\underline{j}} \sum_{\substack{i_1 \geq i_2 \geq i_3 \\ |i_1 - i_3| \leq \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \geq i'_3 \\ |i'_1 - i'_3| \leq \Lambda \\ |i_1 - i'_1| \leq \Lambda}} \left| \text{Tr} \left(P_{i_1 i_2 i_3} W_3 P_{i'_1 i'_2 i'_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i'_3)}^{(3)} \right) \right| \leq C N^{-(\delta/2-2)\epsilon} \text{Tr} \left((C + h) \Gamma^{(1)} \right).$$

Gathering the previous estimates yields (3.19).

Analysis of W_2 . We now prove (3.18). We only provide the details of the proof for

$$W_2(1, 2) = \mathbf{p}_1 \cdot w_R(x_1 - x_2) + w_R(x_1 - x_2) \cdot \mathbf{p}_1$$

since it is the most difficult term to treat and that the others are bounded analogously. For the same reason, we take $i_1 \geq i_2 \geq i'_2$ and $i_1 \geq i'_1 \geq i'_2$. To prove (3.18), we distinguish between $|i'_1 - i_2| \leq \Lambda$ and $|i'_1 - i_2| > \Lambda$, and similarly for i_1, i_2 and i_1, i'_1 . Since we only need to use the hypothesis $j_{i_1} \leq N^{1-\delta\epsilon}$ to deal with the case where the three indices i_1, i_2, i'_1 are close, we first take care of the others. Namely, we bound

$$\sum_{\underline{j}} \sum_{i_1 \geq i_2} \sum_{\substack{i'_1 \geq i'_2 \\ i_2 \geq i'_2 \\ i'_1 > i_2 + \Lambda}} \sum_{i_3} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right), \quad (4.16)$$

$$\sum_{\underline{j}} \sum_{i_1 > i_2 + \Lambda} \sum_{\substack{i'_1 \geq i'_2 \\ i_2 \geq i'_2 \\ |i'_1 - i_2| \leq \Lambda}} \sum_{i_3} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right) \quad (4.17)$$

and

$$\sum_{\underline{j}} \sum_{\substack{i_1 \geq i_2 \\ |i_1 - i_2| \leq \Lambda}} \sum_{\substack{i'_1 \geq i'_2 \\ i_2 \geq i'_2 \\ |i'_1 - i_2| \leq \Lambda \\ i_1 > i'_1 + \Lambda}} \sum_{i_3} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right). \quad (4.18)$$

Note that there is only a single sum over the third coefficient i_3 (rather than a double sum) because of the orthogonality of the projections P_i .

To bound (4.16), we start by writing

$$\begin{aligned} & \sum_{\substack{i_1, i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right) \\ &= \left\langle \sum_{i_1} P_{i_1 i_2 i_3} \otimes \mathbb{1}^{\otimes(N-3)} \Psi_{\underline{j}(i_1 i_2 i_3)}, W_2(1, 2) \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} P_{i'_1 i'_2 i_3} \otimes \mathbb{1}^{\otimes(N-3)} \Psi_{\underline{j}(i'_1 i'_2 i_3)} \right\rangle. \end{aligned}$$

Then, thanks to

$$\sum_{\substack{i'_2 \\ i_2 \geq i'_2}} P_{i'_1 i'_2 i_3} \otimes \mathbb{1}^{\otimes(N-3)} \Psi_{\underline{j}(i'_1 i'_2 i_3)} = \mathbb{P}_M \otimes \mathbb{P}_{i_2} \otimes \mathbb{1}^{\otimes(N-2)} \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} P_{i'_1 i'_2 i_3} \otimes \mathbb{1}^{\otimes(N-3)} \Psi_{\underline{j}(i'_1 i'_2 i_3)},$$

we can apply Lemma 4.1 and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \left| \sum_{\substack{i_1, i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right) \right| &\leq \tau C \sum_{i_1} \text{Tr} \left(P_{i_1 i_2 i_3} (-\Delta_1) P_{i_1 i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(3)} \right) \\ &\quad + \tau^{-1} C N^{2i_2 \varepsilon} \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i'_1 i'_2 i_3)}^{(3)} \right), \end{aligned}$$

for all $\tau > 0$. Taking $\tau = N^{(i_2 - i'_1 + 1)\varepsilon}$ and using the estimate (4.8), we deduce

$$\begin{aligned} & \left| \sum_{\substack{i_1, i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right) \right| \\ &\leq C N^{(i_2 - i'_1 + 1)\varepsilon} \sum_{i_1} \text{Tr} \left(P_{i_1 i_2 i_3} (C + h_1) P_{i_1 i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(3)} \right) \\ &\quad + C N^{(i_2 - i'_1 + 1)\varepsilon} \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i'_1 i'_2 i_3} (C + h_1) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i'_1 i'_2 i_3)}^{(3)} \right). \end{aligned}$$

We may now carry out all the sums in (4.16) using (4.9) first, and then (4.10), and the geometric series formula to obtain

$$\left| \sum_{\underline{j}} \sum_{i_1 \geq i_2} \sum_{\substack{i'_1 \geq i'_2 \\ i_2 \geq i'_2 \\ i'_1 > i_2 + \Lambda}} \sum_{i_3} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right) \right| \leq N^{-\Lambda \varepsilon} \text{Tr} \left((C + h) \Gamma^{(1)} \right). \quad (4.19)$$

Thus, we have the bounded (4.16) as wished.

To bound (4.17), we this time write

$$\begin{aligned} & \left| \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3), \underline{j}(i'_1 i'_2 i_3)}^{(3)} \right) \right| \\ &\leq \tau C N^{2i_2 \varepsilon} \text{Tr} \left(P_{i_1 i_2 i_3} \Gamma_{\underline{j}(i_1 i_2 i_3)}^{(3)} \right) \end{aligned}$$

$$+\tau^{-1}C \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i'_1 i'_2 i_3} (\mathbf{p}_1^{\mathbf{A}})^2 P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right),$$

which, by taking $\tau = N^{(i_1 - i_2 + 1)\varepsilon}$, further implies

$$\begin{aligned} & \left| \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right) \right| \\ & \leq C N^{(i_2 - i_1 + 1)\varepsilon} \text{Tr} \left(P_{i_1 i_2 i_3} (C + h_1) P_{i_1 i_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}}^{(3)} \right) \\ & \quad + C N^{(i_2 - i_1 + 1)\varepsilon} \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i'_1 i'_2 i_3} (C + h_1) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right). \end{aligned}$$

Carrying out all the remaining sums as in (4.19), we obtain

$$\left| \sum_{\underline{j}} \sum_{i_1 > i_2 + \Lambda} \sum_{\substack{i'_1 \geq i'_2 \\ i_2 \geq i'_2 \\ |i'_1 - i_2| \leq \Lambda}} \sum_{i_3} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right) \right| \leq N^{-\Lambda\varepsilon} \text{Tr} \left((C + h) \Gamma^{(1)} \right),$$

which is the desired estimate on (4.17).

To bound (4.18), we use Lemma 2.6 to obtain

$$\begin{aligned} & \left| \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right) \right| \\ & \leq \tau C N^{2i_2\varepsilon} N^{-3} (j_{i_1} + 1)(j_{i_2} + 1)(j_{i_3} + 1) \text{Tr} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}} \\ & \quad + \tau^{-1} C N^{2i'_1\varepsilon} \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} N^{-3} (j_{i'_1} + 1)(j_{i'_2} + 1)(j_{i_3} + 1) \text{Tr} \Gamma_{\underline{j}^{(i'_1 i'_2 i_3)}}, \quad (4.20) \end{aligned}$$

for all $\tau > 0$. Choosing τ such that j_{i_1} and $j_{i'_1}$ are swapped and balancing out the front coefficients, we find

$$\begin{aligned} & \left| \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right) \right| \\ & \leq C N^{(i'_1 - i_1 + 2)\varepsilon} \text{Tr} \left(P_{i'_1 i_2 i_3} (C + h_2) P_{i'_1 i_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}}^{(3)} \right) \\ & \quad + C N^{(i'_1 - i_1 + 2)\varepsilon} \text{Tr} \left(P_{i_1 i'_2 i_3} (C + h_1) P_{i_1 i'_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right). \end{aligned}$$

Here, the adaptation for $j_{i'_1} = 0$ is straightforward, whereas it is more delicate for $j_{i_1} = 0$. More specifically, when $j_{i_1} = 0$, we instead write

$$\begin{aligned} & \left| \sum_{\substack{i'_2 \\ i_2 \geq i'_2}} \text{Tr} \left(P_{i_1 i_2 i_3} W_2(1, 2) P_{i'_1 i'_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i'_1 i'_2 i_3)}}^{(3)} \right) \right| \\ & \leq C N^{-(1-\kappa)/2+2\varepsilon} \text{Tr} \left(P_{i_2 i_3} (C + h_1) P_{i_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}}^{(2)} \right) \end{aligned}$$

$$+CN^{-(1+\kappa)/2+2\varepsilon} \sum_{\substack{i_2' \\ i_2 \geq i_2'}} \text{Tr} (P_{i_1' i_2' i_3} (C + h_1) P_{i_1' i_2' i_3} \Gamma_{\underline{j}^{(i_1' i_2' i_3)}}^{(2)}).$$

Carrying out the sums in (4.18), we obtain

$$\begin{aligned} & \left| \sum_{\underline{j}} \sum_{\substack{i_1 \geq i_2 \\ |i_1 - i_2| \leq \Lambda}} \sum_{\substack{i_1' \geq i_2' \\ i_2 \geq i_2' \\ |i_1' - i_2| \leq \Lambda \\ i_1 > i_1' + \Lambda}} \sum_{i_3} \text{Tr} (P_{i_1 i_2 i_3} W_2(1, 2) P_{i_1' i_2' i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i_1' i_2' i_3)}}^{(3)}) \right| \\ & \leq C(N^{-(\Lambda-3)\varepsilon/2} + N^{-(1-\kappa)/2+2\varepsilon}) \text{Tr} ((C + h) \Gamma^{(1)}). \end{aligned}$$

Having bounded (4.16)–(4.18), we are left with

$$\sum_{\underline{j}} \sum_{\substack{i_1 \geq i_2 \\ |i_1 - i_2| \leq \Lambda}} \sum_{\substack{i_1' \geq i_2' \\ i_2 \geq i_2' \\ |i_1' - i_2| \leq \Lambda \\ |i_1' - i_1| \leq \Lambda}} \sum_{i_3} \text{Tr} (P_{i_1 i_2 i_3} W_2(1, 2) P_{i_1' i_2' i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i_1' i_2' i_3)}}^{(3)}). \quad (4.21)$$

Using (4.20), $j_{i_1} \leq N^{1-\delta\varepsilon}$ and balancing out the front coefficients, we find

$$\begin{aligned} & \left| \sum_{\substack{i_2' \\ i_2 \geq i_2'}} \text{Tr} (P_{i_1 i_2 i_3} W_2(1, 2) P_{i_1' i_2' i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i_1' i_2' i_3)}}^{(3)}) \right| \leq CN^{-\delta\varepsilon/2+\varepsilon} \text{Tr} (P_{i_2 i_3} (C + h_1) P_{i_2 i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}}^{(2)}) \\ & + CN^{-\delta\varepsilon/2+\varepsilon} \sum_{\substack{i_2' \\ i_2 \geq i_2'}} \text{Tr} (P_{i_1' i_2' i_3} (C + h_1) P_{i_1' i_2' i_3} \Gamma_{\underline{j}^{(i_1' i_2' i_3)}}^{(2)}). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} & \left| \sum_{\underline{j}} \sum_{\substack{i_1 \geq i_2 \\ |i_1 - i_2| \leq \Lambda}} \sum_{\substack{i_1' \geq i_2' \\ i_2 \geq i_2' \\ |i_1' - i_2| \leq \Lambda \\ |i_1' - i_1| \leq \Lambda}} \sum_{i_3} \text{Tr} (P_{i_1 i_2 i_3} W_2(1, 2) P_{i_1' i_2' i_3} \Gamma_{\underline{j}^{(i_1 i_2 i_3)}, \underline{j}^{(i_1' i_2' i_3)}}^{(3)}) \right| \\ & \leq CN^{-(\delta/2-2)\varepsilon} \text{Tr} ((C + h) \Gamma^{(1)}). \end{aligned}$$

Gathering the previous estimates yields (3.18), thereby concluding the proof of Lemma 3.2. $\square$

Acknowledgements The author would like to express his sincere gratitude to Nicolas Rougerie for introducing him to the problem, for his hospitality, and for the precious discussions. The author would also like to thank Douglas Lundholm and the anonymous referee for their precious feedback. Partial support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the TRR 352 Project ID 470903074 and by the European Research Council through the ERC CoG RAMBAS Project Nr. 101044249 is acknowledged.

Funding Open access funding provided by Institute of Science and Technology (IST Austria).

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest There is no conflict of interest to be declared.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Girardot, T.: Average field approximation for almost bosonic anyons in a magnetic field. *J. Math. Phys.* **61**(7), 071901 (2020)
2. Lundholm, D., Rougerie, N.: The average field approximation for almost bosonic extended anyons. *J. Stat. Phys.* **161**, 1236–1267 (2015)
3. Junge, L., Visconti, F.L.A.: Derivation of Hartree theory for two-dimensional attractive Bose gases in almost Gross-Pitaeski regime, (2025). [arxiv:2410.23150](https://arxiv.org/abs/2410.23150)
4. Khare, A.: Fractional Statistics and Quantum Theory. World Scientific, Singapore (1998)
5. Lundholm, D.: Properties of 2D anyon gas, pp. 450–484. Elsevier (2024)
6. Myrheim, J.: Anyons. In: Topological Aspects of Low Dimensional Systems: Session LXIX. 7–31 July 1998 vol. 69, pp. 265–413. Springer, ??? (2002)
7. Ouvry, S.: In: Duplantier, B., Raimond, J.-M., Rivasseau, V. (eds.) Anyons and Lowest Landau Level Anyons, vol. 55, pp. 71–103. Birkhäuser Basel, Basel (2009)
8. Wilczek, F.: Fractional Statistics and Anyon Superconductivity. World Scientific, ??? (1990)
9. Lundholm, D., Solovej, J.P.: Hardy and Lieb-Thirring inequalities for anyons. *Comm. Math. Phys.* **322**, 883–908 (2013)
10. Myrheim, J.: Anyons. In: Comtet, A., Jolicœur, T., Ouvry, S., David, F. (eds.) Aspects Topologiques de la Physique en Basse Dimension. Topological Aspects of Low Dimensional Systems, pp. 265–413. Springer, Berlin, Heidelberg (1999)
11. Ataei, A., Lundholm, D., Girardot, T.: Microscopic derivation of the stationary Chern-Simons-Schrödinger equation for almost-bosonic anyons, (2025). [arxiv:2504.17488](https://arxiv.org/abs/2504.17488)
12. Correggi, M., Lundholm, D., Rougerie, N.: Local density approximation for the almost-bosonic anyon gas. *Anal. PDE* **10**, 1169–1200 (2017)
13. Correggi, M., Duboscq, R., Rougerie, N., Lundholm, D.: Vortex patterns in the almost-bosonic anyon gas. *EPL* **126**, 20005 (2019)
14. Girardot, T., Lee, J.: Derivation of the Chern-Simons-Schrödinger equation from the dynamics of an almost-bosonic-anyon gas, (2025). [arxiv:2412.13080](https://arxiv.org/abs/2412.13080)
15. Larson, S., Lundholm, D.: Exclusion bounds for extended anyons. *Arch. Ration. Mech. Anal.* **227**, 309–365 (2017)
16. Lundholm, D.: Many-anyon trial states. *Phys. Rev. A* **96**, (2017)
17. Avron, J., Herbst, I., Simon, B.: Schrödinger operators with magnetic fields. I. general interactions. *Duke Math. J.* **45**(4), (1978)
18. Reed, M.C., Simon, B.: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness, New York (1972)
19. Adami, R., Teta, A.: On the Aharonov-Bohm effect. *Lett. Math. Phys.* **43**, 43–53 (1998)
20. Bourdeau, M., Sorkin, R.: When can identical particles collide? *Phys. Rev. D* **45**, 687–696 (1992)
21. Correggi, M., Oddis, L.: Hamiltonians for two-anyon systems. *Rend. Mat. Appl.* **39**, 277–292 (2018)
22. Dabrowski, L., Stovicek, P.: Aharonov-Bohm effect with δ -type interaction. *J. Math. Phys.* **39**, 47–62 (1998)
23. Lundholm, D., Solovej, J.P.: Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics. *Ann. Henri Poincaré* **15**, 1061–1107 (2014)
24. Fetter, A.L., Hanna, C.B., Laughlin, R.B.: Random-phase approximation in the fractional statistics gas. *Phys. Rev. B* **39**, 9679–9681 (1989)

25. Girardot, T., Rougerie, N.: Semiclassical limit for almost fermionic anyons. *Commun. Math. Phys.* **387**, 427–480 (2021)
26. Girardot, T., Rougerie, N.: A Lieb-Thirring inequality for extended anyons. *Lett. Math. Phys.* **113**(6), (2023)
27. Iengo, R., Lechner, K.: Anyon quantum mechanics and Chern-Simons theory. *Phys. Rep.* **213**, 179–269 (1992)
28. Levitt, A., Lundholm, D., Rougerie, N.: Magnetic Thomas-Fermi theory for 2D abelian anyons, (2026). [arxiv:2504.13481](https://arxiv.org/abs/2504.13481)
29. Trugenberger, C.: The anyon fluid in the Bogoliubov approximation. *Phys. Rev. D* **45**, 3807–3817 (1992)
30. Trugenberger, C.: Ground state and collective excitations of extended anyons. *Phys. Lett. B* **288**, 121–128 (1992)
31. Westerberg, E.: Mean field approximation for anyons in a magnetic field. *Int. J. Mod. Phys. B* **7**, 2177–2199 (1993)
32. Rougerie, N.: Scaling limits of bosonic ground states, from many-body to non-linear Schrödinger. *EMS Surv. Math. Sci.* **7**(2), 253–408 (2020)
33. Lewin, M., Nam, P.T., Rougerie, N.: The mean-field approximation and the non-linear Schrödinger functional for trapped Bose gases. *Trans. Amer. Math. Soc.* **368**, 6131–6157 (2014)
34. Nguyen, D.-T.: 2D attractive almost-bosonic anyon gases, (2024). [arxiv:2409.00409](https://arxiv.org/abs/2409.00409)
35. Ataei, A., Lundholm, D., Nguyen, D.-T.: A generalized Liouville equation and magnetic stability, (2025). [arxiv:2404.09332](https://arxiv.org/abs/2404.09332)
36. Simon, B.: *Trace Ideals and Their Applications*. London Mathematical Society Lecture Note Series, vol. 35. Cambridge University Press, Cambridge (1979)
37. Lewin, M., Nam, P.T., Rougerie, N.: A note on 2D focusing many-boson systems. *Proc. Amer. Math. Soc.* **145**(6), 2441–2454 (2017). <https://doi.org/10.1090/proc/13468>
38. Nam, P.T., Rougerie, N.: Improved stability for 2D attractive Bose gases. *J. Math. Phys.* **61**(2), 021901 (2020)
39. Brandão, F.G., Harrow, A.W.: Quantum de Finetti theorems under local measurements with applications. *Commun. Math. Phys.* **353**, 469–506 (2017)
40. Li, K., Smith, G.: Quantum de Finetti theorem under fully-one-way adaptive measurements. *Phys. Rev. Lett.* **114**, 160503 (2015)
41. Rougerie, N.: Non linear Schrödinger limit of bosonic ground states, again. *Conflu. Math.* **12**(1), 69–91 (2020)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.