On the large-scale behavior of interacting particle systems, composite magnetic materials, and interface motions through random obstacle fields

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Abstract

Precise microscopic models of physical processes are often too complex to be useful for understanding or simulating phenomena on a macroscopic scale. The complexity can be drastically reduced if there is a suitable approximation which still accurately captures the effective large-scale behavior of the underlying model. This thesis contains contributions to the rigorous justification of such large-scale approximations for three different models.

Chapter 2 presents joint work with Federico Cornalba, Julian Fischer, and Claudia Raithel. We consider a large system of weakly interacting particles, driven by independent Brownian motions. The physicists Dean and Kawasaki proposed a strongly singular SPDE to describe fluctuations of the particle density. However, without a suitable regularization, the *Dean–Kawasaki equation* has been shown to be ill-posed in the sense that it does not admit martingale solutions for any continuous initial data. We show that a structure-preserving discretization of the Dean–Kawasaki equation – acting as a suitable regularization – captures the density fluctuations of the particle system up to arbitrary order in the inverse number of particles and a discretization error.

Chapter 3 presents joint work with Elisa Davoli and Lorenza D'Elia. We consider a magnetic body consisting of a composite material with a random microstructure. The oscillating magnetic properties are encoded in the *micromagnetic energy functional* which is restricted to functions taking values on the sphere, governing the observable states of magnetization. We obtain a qualitative stochastic homogenization result in the form of Γ-convergence. The Γ-limit is a micromagnetic energy functional with constant, deterministic coefficients, meaning that the magnetic body can be treated as if consisting of a single material when the microstructure is small enough. On a technical level, we obtain an analogous result replacing the sphere with other sufficiently regular manifolds.

Chapter 4 and Chapter 5 present joint work with Julian Fischer. We consider interface motions driven by curvature and a uniform forcing through a stationary field of possibly impenetrable random obstacles satisfying a finite range of dependence assumption. This process models, e.g., the motion of dislocation lines or magnetic domain boundaries in materials with inhibiting impurities. In Chapter 5, for dimension $d = 2$, we introduce an easy-to-check criterion for general obstacle fields which implies ballistic behavior. More precisely, this criterion yields an *almost guaranteed effective minimum speed on large scales* for the interface motion , with 'effective' meaning that enclosures left behind a main front can be ignored. In Chapter 4, for any dimension $d \geq 2$, assuming such an almost guaranteed effective minimum speed, we obtain a quantitative stochastic homogenization result in any dimension, proving that the main front can be approximated with a constant-speed motion on large scales.

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About the Author

Jonas Ingmanns received a BSc and an MSc in Mathematics at the RWTH Aachen University in 2018 and 2020 respectively, both under the supervision of Prof. Heiko von der Mosel. He joined the Institute of Science and Technology Austria (ISTA) first as an intern in June 2021 before transitioning to the PhD program in September 2021, subsequently becoming a part of the research group of Prof. Julian Fischer. From January to April 2026, he joined the Austrian Financial Market Authority (FMA) for an internship, which led to a paper on the market-consistent valuation of health insurance liabilities. His main research interests lie at the intersection where partial differential equations, calculus of variations and geometric analysis meet probability theory and stochastic analysis. The overarching theme of his current research is the rigorous and quantitative justification of large-scale approximations for stochastic models.

List of Collaborators and Publications

This thesis is based on four works with the following collaborators (in alphabetical order): Federico Cornalba, Elisa Davoli, Lorenza D'Elia, Julian Fischer, and Claudia Raithel.

Listed in order of appearance in the thesis, these publications and preprints are:

- Federico Cornalba, Julian Fischer, Jonas Ingmanns, and Claudia Raithel. "Density fluctuations in weakly interacting particle systems via the Dean–Kawasaki equation." *The Annals of Probability* 54.1 (2026). DOI: 10.1214/25-aop1763

  This work, referred to as [Cor+26] throughout the thesis, is presented in Chapter 2.

- Elisa Davoli, Lorenza D'Elia, and Jonas Ingmanns. "Stochastic Homogenization of Micromagnetic Energies and Emergence of Magnetic Skyrmions." *Journal of Nonlinear Science* 34.2 (2024). DOI: 10.1007/s00332-023-10005-3

  This work, referred to as [DDI24] throughout the thesis, is presented in Chapter 3.

- Julian Fischer and Jonas Ingmanns. *Quantitative homogenization of forced geometric motions through random fields of obstacles*. 2026. arXiv: 2603.12179 [math.AP]

  This work, referred to as [FI26a] throughout the thesis, is presented in Chapter 4.

- Julian Fischer and Jonas Ingmanns. "A sufficient criterion for ballistic behavior of forced geometric motions in random environments." *In preparation* (2026+)

  This work, referred to as [FI26b] throughout the thesis, is presented in Chapter 5.

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CHAPTER 1

Introduction

1.1 Large-scale approximations of stochastic models

Randomness is inherent in and used to model small-scale physical processes such as the formation of composite materials or particle behavior. However, for large-scale systems, the usefulness of such models is limited since the effective large-scale properties are not necessarily apparent in the original formulation and the computational cost becomes increasingly unfeasible due to the small-scale complexity. Providing and rigorously justifying large-scale descriptions which approximate such models and reduce their complexity has consistently motivated research across multiple areas of mathematics.

In the present thesis, we rigorously justify such large-scale descriptions for three models:

- Weakly interacting particle systems (Chapter 2),
- Magnetic behavior of random composite materials (Chapter 3),
- Interfaces driven through random obstacle fields (Chapter 4 & Chapter 5).

For the first model, the scale is given by the number of particles. For the latter models, the scale corresponds to the ratio of the length scale on which the effective behavior is approximated compared to the correlation length of the material or obstacle field respectively.

Results justifying large-scale approximations can roughly be assigned to three categories, corresponding to an increasingly better understanding of the large-scale behavior.

1. *Establishing a large-scale limit:* Obtaining a qualitative limit for the model as the scale increases and identifying the properties of this limit provides an effective description for large-scale behavior.
2. *Quantifying the approximation error:* Convergence rates and explicit bounds guarantee the quality of the effective description at a given scale.
3. *Establishing higher-order approximations:* Adding the correct components depending on the given scale to the effective description can improve the convergence rate and thus the approximative quality.

Let us demonstrate these ideas with a well-known example. Consider a one-dimensional environment with random speed $v_n > 0$ associated to each segment $[n-1, n) \subset \mathbb{R}$ for $n \in \mathbb{Z}$ and some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\cdot]$ denoting the expected value. For a traveler, it takes time v_n^{-1} to cross the segment $[n-1, n)$ of length one. The time T_N to reach $N \in \mathbb{N}$ when starting at 0 is given by

$$T_N \coloneqq \sum_{n=1}^{N} v_n^{-1}.$$

Establishing a large-scale limit. Suppose that $(v_n)_{n\in\mathbb{Z}}$ are independent and identically distributed with expected value $\mathbb{E}\big[v_0^{-1}\big] < \infty$. Then the strong law of large numbers yields

$$\frac{T_N}{N} \to \mathbb{E}\big[v_0^{-1}\big] =: \overline{v}^{-1} \qquad \text{almost surely as } N \to \infty.$$

This means that, in the limit, a homogeneous environment with constant speed in $\mathbb{R}$ given by the harmonic mean $\overline{v}$ describes the same effective behavior as the original random environment. The simplified model should hence be a good large-scale description, reducing the complexity of the original model.

Quantifying the approximation error. However, if we consider a large but fixed scale, without error bounds, this result does not provide a guarantee for the quality of the approximation. But if we assume that the speeds $(v_n)_{n\in\mathbb{Z}}$ satisfy $\operatorname{Var}\big(v_0^{-1}\big) = \sigma^2 < \infty$, then we obtain $\operatorname{Var}(T_N) = N\sigma^2$ and convergence rates for the law of large numbers. In particular, for every $\varphi \in C^{0,1}(\mathbb{R})$ with Lipschitz constant ≤ 1, we obtain

$$\left|\mathbb{E}\left[\varphi\left(\frac{T_N}{N}\right)\right] - \varphi(\overline{v}^{-1})\right| \leq \frac{\sigma}{\sqrt{N}}.$$

Due to the size of the variance, the fluctuations of T_N are on average of size $N^{1/2}$, meaning that we can not obtain an error of order better than $N^{-\frac{1}{2}}$ with a deterministic approximation.

Establishing higher-order approximations. As a consequence, we have to modify the effective model if we want a more precise approximation for a given scale. The central limit theorem provides such a modification. For the rescaled error term, we have the weak convergence

$$\sqrt{N}\left(\frac{T_N}{N} - \overline{v}^{-1}\right) \rightharpoonup \sigma Y \qquad \text{as } N \to \infty$$

for a random variable Y with standard normal distribution. In fact, if in addition we assume finite third moments $\mathbb{E}\big[v_0^{-3}\big] = \rho < \infty$, then a weak form of the Berry-Esséen theorem, see [Tao12, Theorem 2.2.14], yields

$$\left|\mathbb{E}\left[\varphi\left(\frac{T_N}{N}\right)\right] - \mathbb{E}\left[\varphi\left(\overline{v}^{-1} + \frac{\sigma}{\sqrt{N}}Y\right)\right]\right| \leq \frac{C}{N}$$

for every $\varphi \in C^{0,1}(\mathbb{R})$ with Lipschitz constant ≤ 1 and for a constant $C = C(\sigma, \rho)$. At a given scale N, a higher order approximation of the arrival time T_N is hence given by $\overline{v}^{-1}N + \sigma\sqrt{N}Y$.

Contributions of the thesis. The present thesis contains results from all three categories justifying large-scale approximations. In Chapter 2, we obtain a higher-order approximation for the density of weakly-interacting particle systems. That is, we show that the Dean–Kawasaki equation, a highly singular stochastic partial differential equation which is ill-posed for smooth initial data, can be used as a 'recipe' for numerical simulations, yielding an approximation error of arbitrarily high order in the inverse number of particles, up to a numerical discretization error.

In Chapter 3, we establish a large-scale limit for the micromagnetic energy in a random composite material. From this limit, we can read off the effective magnetic properties of the composite material on large scales, where it behaves like a single homogeneous (virtual) material.

In Chapter 4 and Chapter 5, we show that curvature-driven interface motions forced through a sufficiently sparse random obstacle field can be approximated with a constant speed motion and establish quantitative bounds for the approximation error. That is, in Chapter 4, we derive the large-scale limit and quantify the approximation error under the abstract assumption of an *'almost guaranteed effective minimum speed'*, encoding the sparseness of the obstacles compared to the driving force. In Chapter 5, we provide a much more tangible criterion for dimension $d = 2$ which implies the abstract assumption introduced in Chapter 4. In particular, we apply this criterion to obtain the large-scale limit for the prototypical case of obstacles distributed according to a Poisson point process.

The following sections of the introduction contain a description of the respective models and their large-scale approximations as well as a brief account of previously obtained justifications and the underlying philosophy.

1.2 Weakly interacting particle systems and the Dean–Kawasaki equation

In Chapter 2, we consider a particle system on the d-dimensional torus $\mathbb{T}^d = [-\pi, \pi)^d$ for a large number of particles $N \in \mathbb{N}$. The particles are

- diffusing driven by independent Brownian motions $(B_i)_{i=1}^N$ with diffusion coefficient $\sigma > 0$, and
- interacting pairwise with all other particles according to the rescaled potential $\frac{1}{N}V$ for $V \in C^2(\mathbb{T}^d)$.

The particle positions $(X_i^N(t))_{i=1}^N$ at time $t \geq 0$ are given by

$$\mathrm{d}X_i^N(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla V\left(X_i^N(t) - X_j^N(t)\right)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}B_i(t). \tag{1.2.1}$$

We say that the particles are weakly interacting due to the scaling of the interaction potential, ensuring that diffusion and interaction remain comparable as the number of particles increases. From a physical point of view, these equations describe the overdamped limit with the forces not acting as an acceleration but immediately translating to the current speed of the particles. Applications include supercooled liquids, active matter, chemotactic particles, or charged particles and electrolytes, see the recent review [Ill25].

We are interested in the large-scale behavior of the particle density, which is given by the *empirical measure* μ^N with

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^N(t)}. \tag{1.2.2}$$

By testing this empirical measure with a smooth function and applying the Itô rule, one can formally obtain the so-called *mean-field limit* $\overline{\rho}$ for $N \to \infty$, given by a *McKean-Vlasov equation*, as candidate for the large-scale limit,

$$\partial_t \overline{\rho} = \sigma \Delta \overline{\rho} + \nabla \cdot (\overline{\rho} \nabla V * \overline{\rho}). \tag{1.2.3}$$

The classical approach by McKean [McK67] of rigorously justifying this large-scale limit is quantitative in nature and hence immediately provides a bound for the approximation error, see, e.g., [Szn91]. We state this result not in its original form, but use the same type of weak metric as later in Chapter 2; in order to compare the empirical measure to a continuous density, we introduce a set of test functions $\boldsymbol{\varphi} = (\varphi_k)_{k=1}^K \in C^\infty(\mathbb{T}^d, \mathbb{R}^K)$ and test times $\boldsymbol{T} = (T_k)_{k=1}^K$ for some $K \in \mathbb{N}$. The resulting random vectors

$$Y_{\mu^N} := \begin{pmatrix} \int \varphi_1 \,\mathrm{d}\mu_{T_1}^N \\ \vdots \\ \int \varphi_K \,\mathrm{d}\mu_{T_K}^N \end{pmatrix}, \qquad Y_{\overline{\rho}} := \begin{pmatrix} \int \varphi_1 \overline{\rho}_{T_1} \,\mathrm{d}x \\ \vdots \\ \int \varphi_K \overline{\rho}_{T_K} \,\mathrm{d}x \end{pmatrix},$$

can be compared in the Wasserstein-1 metric, with its dual formulation for random variables $Y, Z \in \mathbb{R}^K$ given by

$$\mathrm{W}_1(Y,Z) := \sup_{\psi \in C^1(\mathbb{R}^K),\, \|D\psi\|_{L^\infty} \le 1} \mathbb{E}\big[\psi(Y)\big] - \mathbb{E}\big[\psi(Z)\big],$$

which is a negative Sobolev distance between the law of Y and Z.

Theorem 1.2.1 (Theorem 1.4 in [Szn91]). *Let $V \in C^2(\mathbb{T}^d)$. Let $\overline{\rho}_0$ be a probability measure on $\mathbb{T}^d$. Let $(X_i^N(t))_{i=1}^N$ satisfy (1.2.1) with $(X_i^N(t))_{i=1}^N$ sampled independently from $\overline{\rho}_0$. Let $\overline{\rho}$ be a solution of (1.2.3) with initial data $\overline{\rho}_0$. Then there exists a constant $C = C(\sigma, \max \boldsymbol{T}, \|V\|_{C^2}) > 0$ such that*

$$\mathrm{W}_1\big(Y_{\mu^N}, Y_{\overline{\rho}}\big) \le \frac{C\|\boldsymbol{\varphi}\|_{W^{1,\infty}}}{\sqrt{N}}.$$

Due to the size of the fluctuations, as in the example from Section 1.1, any higher order approximation has to be random itself. Tanaka [Tan84], Sznitman [Szn84], and Fernandez and Méléard [FM97] showed that a leading-order description of the fluctuations is given by a stochastic partial differential equation (SPDE),

$$\partial_t \hat{\rho} = \sigma \Delta \hat{\rho} + \nabla \cdot (\hat{\rho} \nabla V * \hat{\rho}) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma\overline{\rho}}\,\xi), \tag{1.2.4}$$

where ξ denotes space-time vector-valued white noise. The strength of the conservative noise $N^{-1/2}\nabla \cdot (\sqrt{2\sigma\overline{\rho}}\,\xi)$ decreases as the number of particles N increases and depends locally on the deterministic mean-field limit $\overline{\rho}$ from (1.2.3). However, with the noise independent of the current state of the random density, this approximation does not preserve positivity and $N^{\frac{1}{2}}(\hat{\rho} - \overline{\rho})$ does not capture the normalized fluctuations $N^{\frac{1}{2}}\big(\mu^N - \overline{\rho}\big)$ up to an order higher than $N^{-\frac{1}{2}}$.

For a precise SPDE description of the particle density, Dean [Dea96] and Kawasaki [Kaw98] proposed the equation

$$\partial_t \rho = \sigma \Delta \rho + \nabla \cdot (\rho \nabla V * \rho) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma\rho}\, \xi). \tag{1.2.5}$$

with the strength of the noise depending on the random density ρ itself rather than the deterministic mean-field limit $\overline{\rho}$ as in (1.2.4). This equation is highly singular: the roughness introduced via the multiplicative noise outscales the regularizing effect of the diffusion. In the language of regularity structures [Hai14] and paracontrolled distributions [GIP15], the Dean–Kawasaki equation is supercritical and hence can not be treated even using these powerful techniques for singular SPDEs. Therefore, a different approach is necessary.

A range of similar conservative, singular SPDEs formally arises from the *theory of fluctuating hydrodynamics*, which aims to derive closed macroscopic equations for the density of dynamical systems, either based on the inherent randomness of an underlying particle system or by adding a suitable thermal noise to the current of a hydrodynamic system, see, e.g., [LL87; HH77; Spo91; BGN16]. Examples also include particle systems on lattices such as the Zero-Range Process (ZRP) or the Symmetric Simple Exclusion Process (SSEP). Other prominent equations from the theory of fluctuating hydrodynamics suffer from the same issue as the Dean–Kawasaki equation: due to their supercritical singularity, it has been unclear how to approach them rigorously.

However, in recent years, it has been understood that a careful regularization of the noise – for example, by truncating high frequencies or by discretizing – yields well-posedness while preserving strong approximative properties with respect to the original particle system. For example, Fehrman and Gess obtain well-posedness for conservative, singular SPDEs with a noise truncation while recovering the correct large deviations principle from the respective fluctuating hydrodynamics equations for the ZRP and, together with Dirr, for the SSEP, see [FG19; FG23; FG24; DFG26]. These results provide a rigorous link between the theory of fluctuating hydrodynamics and macroscopic fluctuation theory, a framework to describe nonequilibrium diffusive systems which centers around a fundamental formula for a large deviations principle, see, e.g., [Der07; Ber+15; BGN16].

For the Dean–Kawasaki equation, Cornalba, Shardlow, and Zimmer [CSZ19; CSZ20; CSZ21] derived a well-posed regularization by smoothing out the underlying particle system. Djurdjevac, Ji, Kremp, and Perkowski [DKP24; DJP26] show that a noise truncation of (1.2.5) yields a higher-order approximation for the empirical measure from 1.2.2.

In Chapter 2, we consider a regularization of the Dean–Kawasaki equation (1.2.5) in the form of a discretization. Based on a framework by Cornalba and Fischer [CF23] for non-interacting Brownian motions, we show that the discretized Dean–Kawasaki equation approximates the density fluctuations up to arbitrary order in N^{-1} and up to a discretization error when measured in suitable weak metrics. When published, to the best of our knowledge, this was the first result which rigorously justified one of the singular SPDEs from the theory of fluctuating hydrodynamics for a particle system with non-local interactions.

The reason why error bounds of arbitrary order are obtainable is the special relation of the Dean–Kawasaki equation to the particle system (1.2.1): Dean showed in [Dea96] that, in a suitable sense, the empirical measure defined in (1.2.2) is an exact solution to (1.2.5). In fact, Konarovskyi, Lehmann, and von Renesse [KLR19b; KLR19a] showed much later that these empirical measures are the only solutions. For example, when replacing N with a non-integer or when the initial data is given by a continuous density, there are no solutions, confirming the highly singular nature of the Dean–Kawasaki equation.

In the following, we will present a version of Dean's derivation from [Dea96] and then describe how this informs our discretization of the noise.

1.2.1 Dean's derivation of the Dean–Kawasaki equation

Testing the empirical measure μ^N from (1.2.2) with a smooth function $\varphi \in C^\infty(\mathbb{T}^d)$, by applying the Itô rule, we obtain

$$\mathrm{d}\int \varphi \,\mathrm{d}\mu_t^N = \left(\int \left(\sigma\Delta\varphi - \nabla\varphi\cdot\left(\nabla V * \mu_t^N\right)\right)\mathrm{d}\mu_t^N\right)\mathrm{d}t + \frac{\sqrt{2\sigma}}{N}\sum_{i=1}^N \nabla\varphi(X_i)\cdot \mathrm{d}B_i. \tag{1.2.6}$$

This means that $\mathcal{M}_{\varphi,N}$ given by

$$\mathcal{M}_{\varphi,N}(t) = \int \varphi \,\mathrm{d}\mu_t^N - \int \varphi \,\mathrm{d}\mu_0^N - \int_0^t \int \left(\sigma\Delta\varphi - \nabla\varphi\cdot\left(\nabla V * \mu_s^N\right)\right)\mathrm{d}\mu_s^N \,\mathrm{d}s \tag{1.2.7}$$

is a martingale with quadratic variation

$$\left[\mathcal{M}_{\varphi,N}\right]_t = \frac{2\sigma}{N}\int_0^t \int \left|\nabla\varphi\right|^2 \mathrm{d}\mu_s^N \,\mathrm{d}s. \tag{1.2.8}$$

Except for the noise term, (1.2.6) corresponds to the weak formulation of a closed equation for μ^N. Formally treating μ^N as if it was a smooth density ρ, one could write the quadratic variation as

$$\begin{aligned}
\mathrm{d}\left[\mathcal{M}_{\varphi,N}\right]_t &= \frac{2\sigma}{N}\int \left|\nabla\varphi\right|^2 \rho \,\mathrm{d}x\,\mathrm{d}t \\
&= \int \left(\frac{1}{\sqrt{N}}\sqrt{2\sigma\rho}\nabla\varphi\right)\cdot\left(\frac{1}{\sqrt{N}}\sqrt{2\sigma\rho}\nabla\varphi\right)\mathrm{d}x\,\mathrm{d}t \\
&= \sum_{m=1}^\infty \int \left(\frac{1}{\sqrt{N}}\sqrt{2\sigma\rho}\nabla\varphi\cdot f_m\right)\mathrm{d}x \int \left(\frac{1}{\sqrt{N}}\sqrt{2\sigma\rho}\nabla\varphi\cdot f_m\right)\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{1.2.9}$$

for any orthonormal basis $(f_m)_{m\in\mathbb{N}}$ of $L^2(\mathbb{T}^d,\mathbb{R}^d)$. If $\sqrt{\rho}\nabla\varphi$ was sufficiently smooth, one could define the martingale

$$\widetilde{M}(t) = N^{-1/2}\sum_{m\in\mathbb{N}}\int_0^t \left(\int \nabla\varphi\cdot\left(\sqrt{2\sigma\rho}f_m\right)\mathrm{d}x\right)\mathrm{d}W_m(s)$$

with independent one-dimensional Brownian motions $(W_m)_{m=1}^\infty$. In view of (1.2.9), the quadratic variation of $\widetilde{M}$ would satisfy (1.2.8). Dean derived the Dean–Kawasaki equation (1.2.5) by replacing $\mathcal{M}_{\varphi,N}(t)$ in (1.2.7) with $\widetilde{M}(t)$, yielding the weak formulation of a closed equation for the particle density. Noting that the time-derivative of $\sum_{m\in\mathbb{N}} f_m W_m$ is space-time white noise, one can read off the Dean–Kawasaki equation.

1.2.2 Our structure-preserving finite-difference discretization

In fact, in the ill-posedness vs. triviality result from [KLR19b; KLR19a], solutions of the Dean–Kawasaki equation are defined as stochastic processes satisfying (1.2.7) and (1.2.8) for all test functions $\varphi \in C^\infty(\mathbb{T}^d)$. Choosing a discretization of the Dean–Kawasaki equation, we want to preserve this martingale structure, which is key for our arguments in Chapter 2.

In Chapter 2, we work with a finite difference discretization, introducing a grid $G_{h,d}$ with distance h between sites chosen such that

$$G_{h,d} := h\mathbb{Z}^d \cap \mathbb{T}^d = \{-\pi, -\pi + h, \dots, \pi - h\}^d.$$

We consider the spaces $L^2_h(G_{h,d}, \mathbb{R}^\ell) := (\mathbb{R}^\ell)^G_{h,d}$ for $\ell \in \mathbb{N}$ endowed with the L^2_h scalar product $(\cdot,\cdot)_h$ approximating integration over $\mathbb{T}^d$, given by

$$(u_h, v_h)_h := \sum_{x \in G_{h,d}} h^d u_h(x) \cdot v_h(x) \qquad \text{for } u_h, v_h \in L^2_h(G_{h,d}, \mathbb{R}^\ell).$$

We use the interpolation operator $\mathcal{I}_h \colon C(\mathbb{T}^d, \mathbb{R}^\ell) \to L^2_h(G_{h,d}, \mathbb{R}^\ell)$ to map continuous functions onto the discrete space with $\mathcal{I}_h\varphi(x) = \varphi(x)$ for $x \in G_{h,d}$.

For a chosen set of finite difference operators ∇_h, Δ_h approximating the differential operators ∇, Δ and for a suitable discrete convolution operator $*_h$, a solution ρ_h to the discretized Dean–Kawasaki equation should hence satisfy that, for every $\varphi_h \in L^2_h(G_{h,d})$,

$$\begin{aligned}\mathcal{M}_{h,\varphi,N}(t) = (\varphi_h,\, \rho_h(t))_h - (\varphi_h,\, \rho_h(0))_h \\ - \int_0^t \big(\sigma\Delta_h\varphi_h - \nabla_h\varphi_h \cdot (\mathcal{I}_h[\nabla V] *_h \rho_h(s)),\, \rho_h(s)\big)_h \,\mathrm{d}s\end{aligned} \tag{1.2.10}$$

is a martingale with quadratic variation

$$[\mathcal{M}_{\varphi,N}]_t = \frac{2\sigma}{N} \int_0^t \Big(|\nabla\varphi|^2,\, \rho_h(s)\Big)_h \,\mathrm{d}s. \tag{1.2.11}$$

Since on the discrete space any measure has a density and since we do not have to worry about the convergence of a Fourier-type decomposition as in (1.2.9), we do not need to make any assumptions on ρ_h to obtain

$$\frac{2\sigma}{N}\Big(|\nabla\varphi|^2,\, \rho_h^+\Big)_h = \sum_{m=1}^{d|G_{h,d}|} \Big(\frac{1}{\sqrt{N}}\sqrt{2\sigma\rho_h^+}\nabla_h\varphi_h,\, f_{h,m}\Big)_h \Big(\frac{1}{\sqrt{N}}\sqrt{2\sigma\rho_h^+}\nabla_h\varphi_h,\, f_{h,m}\Big)_h$$

for any orthonormal basis $(f_{h,m})_{m=1}^{d|G_{h,d}|}$ of $L^2_h(G_{h,d}, \mathbb{R}^d)$. Here, ρ_h^+ denotes the positive part of ρ_h because a-priori we can not exclude that ρ_h might turn negative due to the nature of finite difference discretizations.

This prompts us to define the discretized Dean–Kawasaki equation as

$$\begin{aligned}\mathrm{d}\rho_h = \big(\sigma\Delta_h^R\rho_h + \nabla_h^R \cdot (\rho_h(\mathcal{I}_h[\nabla V] *_h \rho_h))\big)\,\mathrm{d}t \\ + N^{-1/2} \sum_{m=1}^{d|G_{h,d}|} \nabla_h^R \cdot \Big(\sqrt{2\sigma\rho_h^+} f_{h,m}\Big)\,\mathrm{d}W_m \qquad \text{on } G_{h,d}\end{aligned} \tag{1.2.12}$$

for independent one-dimensional Brownian motions $(W_m)_{m=1}^{d|G_{h,d}|}$ and where Δ_h^R and ∇_h^R are the dual operators of Δ_h and $-\nabla_h$ associated to the discrete integration-by-parts rule. Solutions satisfy (1.2.10) and (1.2.11) as long as ρ_h is non-negative, mirroring the behavior of the empirical measure when computing weak metrics.

Naturally, (1.2.12) can be viewed as regularization of the Dean–Kawasaki equation (1.2.5). The discretization does not just allow us to properly define the noise term but the regularizing effect can also be quantified in the following sense. In particular, while we can

not exclude that ρ_h might become negative, we can show that this happens at most with very small probability as long as the deterministic mean-field limit stays strictly positive. Heuristically, we can only expect a regularization if the average number of particles per grid-cell is large, that is,

$$Nh^d \gg 1,$$

allowing individual particle fluctuations to average out. This is also exactly the regime where using (1.2.12) for a density simulation is more efficient than tracking individual particles. Analogously to the proof of Proposition 2.8.2 in Chapter 2, slightly adjusting one of the stopping times there, we obtain the following result, establishing positive regularity.

Proposition 1.2.2 (Regularity of the discretized Dean–Kawasaki equation)**.** *Let ρ_h be a solution to (1.2.12) and let $\overline{\rho}_h$ solve an analogous discretization of the mean-field equation (1.2.3) for*

- *smooth interaction potential $V \in C^\infty(\mathbb{R}^d)$,*
- *deterministic initial data $\overline{\rho}_h(0) = \mathcal{I}_h[\overline{\rho}^0]$ such that there is a smooth solution $\overline{\rho}$ to the mean field equation (1.2.3) with initial data $\overline{\rho}^0$ which satisfies*

  $$\rho_{min} \leq \overline{\rho}(x,t) \leq \rho_{max} \qquad \text{for all } x \in \mathbb{T}^d \text{ and } t \in [0,T]$$

  for some $0 < \rho_{min} \leq \rho_{max}$ and some $T > 0$,
- *random initial data $\rho_h(0) \geq 0$ with a low probability of large fluctuations, that is, there is $C_0 > 0$ such that*

  $$\mathbb{P}\Big[\big\|\rho_h(0) - \overline{\rho}_h(0)\big\|_\infty \geq N^{-1/2}h^{-d/2}R\Big] \leq C_0 \exp\big(-C_0^{-1}R\big),$$
  $$\mathbb{P}\Big[\big\|\rho_h(0) - \overline{\rho}_h(0)\big\|_{H_h^{-\lfloor d/2+1\rfloor}} \geq N^{-1/2}R\Big] \leq C_0 \exp\big(-C_0^{-1}R\big)$$

  for all $R > 0$ given a suitable discrete analogue of the negative Sobolev-norm,
- *some $s_0 > 0$ such that the scaling relationship between N and h satisfies*

  $$Nh^{d+s_0} \geq 1. \tag{1.2.13}$$

Then, for every $0 < \varepsilon < \frac{s_0}{2(d+s_0)}$, there holds

$$\mathbb{P}\Big[\big\|\rho_h(t) - \overline{\rho}_h(t)\big\|_\infty \geq N^{-1/2+\varepsilon}h^{-d/2} \text{ for some } t \in [0,T]\Big] \leq C\exp\Big(-C^{-1}N^{\varepsilon/2}\Big)$$

for a constant $C = C(d, \|\overline{\rho}\|_{C^\infty}, \rho_{min}, \rho_{max}, T, \nabla_h, \Delta_h, C_0, s_0, \varepsilon) > 0$. In particular, due to the scaling relationship (1.2.13), this implies

$$\mathbb{P}\Big[\big\|\rho_h(t) - \overline{\rho}_h(t)\big\|_\infty \geq h^{s_0/2-\varepsilon(d+s_0)} \text{ for some } t \in [0,T]\Big] \leq C\exp\Big(-C^{-1}N^{\varepsilon/2}\Big).$$

Since, for example, $\|\nabla_h\rho_h - \nabla_h\overline{\rho}_h\|_\infty \leq \frac{C}{h}\|\rho_h - \overline{\rho}_h\|_\infty$ for some $C > 0$ and $\overline{\rho}_h$ is essentially smooth in our setting, this result can be interpreted as ρ_h having regularity of order up to $s_0/2$ with very high probability.

While this is very far from the rough behavior one would associate to the Dean–Kawasaki equation, for our main result in Chapter 2, we will establish that such a discretized equation can indeed capture the density fluctuations of the underlying particle system up to arbitrary order in $N^{-\frac{1}{2}}$.

1.3 The micromagnetic energy and stochastic homogenization

In Chapter 3, we consider the mircromagnetic energy, which governs the magnetic behavior of materials, see, e.g., [LL35; Bro63; BMS09]. For example, observable states of the magnetization in a ferromagnet covering a domain $D \subset \mathbb{R}^3$ can be characterized as the local minimizers of the energy. A key feature of the magnetization $M\colon D \to \mathbb{R}^3$ is the *fundamental constraint of micromagnetic theory*: A ferromagnetic body is always locally saturated, meaning that for each temperature there is a material-specific *saturation magnetization* $M_{\mathrm{sat}} \geq 0$ such that

$$\big|M\big| = M_{\mathrm{sat}} \qquad \text{almost everywhere in } D. \tag{1.3.1}$$

The saturation magnetization vanishes past a material-specific temperature known as the Curie point, but we limit ourselves to the regime of a fixed temperature below the Curie point with $M_{\mathrm{sat}} > 0$. While the strength of the magnetization is fixed, the direction might vary: a possible magnetization can be expressed as $M = M_{\mathrm{sat}} m$ with $m\colon D \to \mathbb{S}^2$ taking values on the sphere.

Further material-specific parameters besides the saturation magnetization are

- the *symmetric exchange coefficient* $a > 0$ corresponding to the strength of very-short range interactions,
- the *antisymmetric exchange coefficient* $\kappa \in \mathbb{R}$ describing the strength of chirality effects favoring canting of the magnetization, and
- the *anisotropic energy density* $\varphi \in C(\mathbb{S}^2)$ encoding the preferred magnetization directions of the material.

These properties determine the micromagnetic energy, which for some magnetization $M = M_{\mathrm{sat}} m$ with $m\colon D \to \mathbb{S}^2$ is given by

$$\begin{aligned}\mathcal{F}(m) = \mathcal{F}_{(a,\kappa,M_{\mathrm{sat}},\varphi(\cdot))}(m) \coloneqq{}& \frac{1}{2}\int_D a\nabla m : \nabla m \,\mathrm{d}x + \int_D \kappa \operatorname{curl} m \cdot m \,\mathrm{d}x \\ &- \frac{\mu_0}{2}\int_D h_d\left[M_{\mathrm{sat}} m \chi_D\right] \cdot M_{\mathrm{sat}} m \,\mathrm{d}x \\ &+ \int_D \varphi\left(m\right)\,\mathrm{d}x - \mu_0 \int_D h_a \cdot M_{\mathrm{sat}} m \,\mathrm{d}x \\ =:{}& \mathcal{E}(m) + \mathcal{K}(m) + \mathcal{W}(m) + \mathcal{A}(m) + \mathcal{Z}(m),\end{aligned}$$

where μ_0 is the vacuum permeability and "$\cdot : \cdot$" the double contraction $A : B = \sum_{i,j} A_{ij} B_{ij}$ for $A, B \in \mathbb{R}^{3\times 3}$. The first term is the *exchange energy* $\mathcal{E}(m)$, the second term is the *Dzyaloshinskii–Moriya interaction (DMI)* or antisymmetric exchange energy $\mathcal{K}(m)$, which can be rewritten as

$$\begin{aligned}\mathcal{K}(m) = \int_D \kappa \operatorname{curl} m \cdot m \,\mathrm{d}x = -\int_D \kappa \boldsymbol{\chi}(m) : \nabla m \,\mathrm{d}x, \\ \text{with} \quad \boldsymbol{\chi}(e) \coloneqq (e_1 \times e, e_2 \times e, e_3 \times e)^{\mathsf{T}} \in \mathbb{R}^{3\times 3}_{\mathrm{skew}} \quad \text{for } e \in \mathbb{S}^2.\end{aligned} \tag{1.3.2}$$

The *stray field energy* $\mathcal{W}(m)$ encodes the interaction with the magnetic field $h_d\left[M_{\mathrm{sat}} m \chi_D\right]$ induced by the magnetization in D, given by the magnetostatic Maxwell equation. The final terms are the *anisotropy energy* $\mathcal{A}(m)$ and the *Zeeman energy* which encodes the interaction with an external applied field $h_a \in L^2(D, \mathbb{R}^3)$. Note that, if $\mathcal{F}(m) < \infty$, then $m \in H^1(D, \mathbb{S}^2)$, which hence is the natural space on which to consider the energy.

For a composite material consisting of grains with different magnetic properties, the parameters vary in space, corresponding to the respective local material. In Chapter 3, we consider a *stationary* and *ergodic* composition, that is, parameters given by random fields $a, \kappa, M_{\mathrm{sat}} \in L^\infty(\mathbb{R}^3)$, $a \geq c_a$ for some $c_a > 0$, and $\varphi \in (L^\infty \mathbb{R}^3, C^{0,1}(\mathbb{S}^2))$, which we assume to satisfy:

- *Stationarity:* $\mathrm{law}(a, \kappa, M_{\mathrm{sat}}, \varphi)(\cdot) = \mathrm{law}(a, \kappa, M_{\mathrm{sat}}, \varphi)(\cdot + x)$ for all $x \in \mathbb{R}^3$, meaning that the statistical properties of the material are shift-invariant.
- *Ergodicity:* Any random variable $X = X(a, \kappa, M_{\mathrm{sat}}, \varphi) \in \mathbb{R}$ which is shift invariant – satisfying $\mathrm{law}\, X = \mathrm{law}\, X((a, \kappa, M_{\mathrm{sat}}, \varphi)(\cdot + x))$ for all $x \in \mathbb{R}^3$ – has to be constant almost surely. This essentially means that random fluctuations decorrelate on sufficiently large scales.

In order to model the behavior at the interface between two grains, there are two common options, see [AFM06]. In the first regime, the direction of the magnetization can jump at the interface between grains but such jumps are penalized with an additional surface energy term. Increasing the strength of the penalization leads to the second regime, a *strong coupling* which prevents jumps at the interfaces. We work in this latter regime. Note that even if the direction m does not jump, there will always be a jump of the overall magnetization $M = M_{\mathrm{sat}} m$ whenever the saturation magnetizations differ.

We are interested in the behavior of the micromagnetic energy $\mathcal{F}_\varepsilon \colon H^1(D, \mathbb{S}^2) \to \mathbb{R}$

$$\mathcal{F}_\varepsilon := \mathcal{F}_{(a(\frac{\cdot}{\varepsilon}), \kappa(\frac{\cdot}{\varepsilon}), M_{\mathrm{sat}}(\frac{\cdot}{\varepsilon}), \varphi(\frac{\cdot}{\varepsilon}, \cdot))} \tag{1.3.3}$$

for small $\varepsilon > 0$. Heuristically, this means that the spatial scale on which the material properties oscillate is small compared to the domain D covered by the magnetic body. However, since ergodicity is a purely qualitative description, there is not necessarily an actual scale associated to the oscillations. For trying to obtain convergence rates, one would have to quantify the ergodicity, which then introduces such a scale, see Section 1.3.1 below.

As the main result of Chapter 3, we show that there is a limit energy $\mathcal{F}_{\mathrm{hom}} \colon H^1(D, \mathbb{S}^2) \to \mathbb{R}$ such that the family

$$(\mathcal{F}_\varepsilon)_{\varepsilon > 0} \text{ is equi-coercive and } \Gamma\text{-converges to } \mathcal{F}_{\mathrm{hom}} \text{ almost surely}$$

with respect to the weak topology of $H^1(D, \mathbb{S}^2)$, see [Dal93] for an introduction to Γ-convergence. This means that, for very small ε, minimizers of $\mathcal{F}_\varepsilon$ should behave like minimizers of $\mathcal{F}_{\mathrm{hom}}$. The result is based on a framework which was established for periodic coefficients by Alouges and Di Fratta [AD15] and extended by Davoli and Di Fratta [DD20] to include the DMI. In addition, for both the periodic and the stochastic setting, we show that the Γ-limit is again a micromagnetic energy, that is,

$$\mathcal{F}_{\mathrm{hom}} = \mathcal{F}_{(a_{\mathrm{eff}}, \kappa_{\mathrm{eff}}, M_{\mathrm{eff}}, \varphi_{\mathrm{eff}}(\cdot)))}$$

with constant, deterministic parameters $a_{\mathrm{eff}}, \kappa_{\mathrm{eff}}, M_{\mathrm{eff}}, \varphi_{\mathrm{eff}}(\cdot)$. These effective coefficients do not just correspond to the expected value of the respective quantity but depend on so-called *correctors* $\Phi_a, \Phi_\kappa, \Phi_M$, which, in a suitable sense, solve

$$-\nabla \cdot (a \nabla \Phi_a + a \,\mathrm{Id}) = 0, \quad -\nabla \cdot (a \nabla \Phi_\kappa + \kappa \,\mathrm{Id}) = 0, \quad -\nabla \cdot (\nabla \Phi_M + M_{\mathrm{sat}} \,\mathrm{Id}) = 0. \tag{1.3.4}$$

The effective material properties are given by

$$\begin{aligned} a_{\mathrm{eff}} &= \mathbb{E}\left[a\,\mathrm{Id} + a\nabla\Phi_a\right], \\ \kappa_{\mathrm{eff}} &= \mathbb{E}\left[\kappa\,\mathrm{Id} + a\nabla\Phi_\kappa\right], \\ M_{\mathrm{eff}} &= \mathbb{E}\left[M_{\mathrm{sat}}\right], \\ \varphi_{\mathrm{eff}} &= \mathbb{E}\left[\varphi\right](m) + \frac{1}{2}\boldsymbol{\chi}(m) : \mathbb{E}\left[\kappa\nabla\Phi_\kappa\right]\boldsymbol{\chi}(m) - \frac{\mu_0}{2} m \cdot \mathbb{E}\left[M_{\mathrm{sat}}\nabla\Phi_M\right] m, \end{aligned} \tag{1.3.5}$$

where the matrix-valued anti-symmetric exchange coefficient is understood in the sense of (1.3.2) and with $\boldsymbol{\chi}(\cdot)$ as defined in (1.3.2). In particular, even if the individual materials are isotropic ($\varphi \equiv 0$), preferred magnetization directions might emerge on large scales (non-constant φ_{eff}).

In summary, on large scales, we can treat the composite material as a homogeneous, pure material – *homogenization* occurs. There is a well-established theory of stochastic homogenization. The novelty of Chapter 3 is the restriction to manifold-valued functions – the restriction to the sphere $\mathbb{S}^2$ is a special case of our main homogenization result which holds for arbitrary orientable C^2-hypersurfaces that admit a tubular neighborhood of uniform thickness. One of our key tools is *stochastic two-scale convergence*, the theory of which we extend to Beppo-Levi spaces, which include the potential describing the stray field induced by a given magnetization.

The concept of two-scale convergence was originally introduced by Nguetseng in [Ngu89] and Allaire in [All92] as a weaker notion of L^2-convergence, which allows for the separation of a "fast" periodic variable in the limit. On a technical level, the main advantage of this concept is the compactness of L^2-bounded sets with respect to the weak two-scale convergence, which is stronger than classical weak L^2-convergence and compatible with the increasingly faster oscillations of coefficients associated to the homogenization of a periodic material. In the stochastic setting, two different corresponding notions have been introduced: two-scale convergence in the mean by Bourgeat, Luckhaus, Mikelić, Wright in [WMB94; BLM96] and pointwise or *quenched* stochastic two-scale convergence by Zhikov and Piatnitski in [ZP06]. We work with the quenched notion, which was also used by Alouges, de Bouard, Merlet, and Nicolas in [Alo+21] for a qualitative homogenization result of the Landau–Lifshitz equation, describing the evolution of magnetization over time when only considering exchange interactions.

Since the publication of the paper [DDI24] (corresponding to Chapter 3), there has been further work in this direction by Bach and Ruf in [BR25], who obtain a qualitative homogenization result for integrands satisfying a general p-growth condition rather than the explicit quadratic structure in our setting while placing less restrictive assumptions on the manifold.

In the following, we will give a brief overview of classical stochastic homogenization theory related to our model up to the manifold restriction. In view of the rough framework from Section 1.1, it is noteworthy that an extensive quantitative theory has been developed in the past two decades.

1.3.1 The theory of elliptic homogenization

In the following, our aim is to point out connections of our setting to the existing theory rather than providing a detailed or complete overview of this field – we refer to [AK22] for a self-contained introduction to elliptic homogenization including detailed historical remarks.

If we consider the exchange energy and Zeeman energy of (1.3.3) for scalar instead of manifold-valued functions with constant saturation magnetization, that is

$$\mathcal{F}(u) = \frac{1}{2}\int_D a\Big(\frac{x}{\varepsilon}\Big)\nabla u(x)\cdot\nabla u(x)\,\mathrm{d}x - \mu_0 M_{\mathrm{sat}}\int_D h_a(x)u(x)\,\mathrm{d}x \qquad \text{for } u\in H^1(D),$$

then there exists a unique minimizer $u^\varepsilon \in g + H^1_0(D)$ for each $g\in H^1(D)$, given by the unique solution in the class $g+H^1_0(D)$ to the Euler–Lagrange equation, which is the uniformly elliptic equation

$$-\nabla\cdot\Big(a\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon(x)\Big) = \mu_0 M_{\mathrm{sat}} h_a(x) \qquad \text{in } D.$$

In contrast, for our setting it is unclear whether such unique minimizers exist because the convexity of the energy is broken by the restriction to manifold-valued solutions.

For a stationary and ergodic coefficient field a as above, Papanicolaou and Varadhan [PV81] and Kozlov [Koz80] showed that u^ε converges to $\overline{u}\in g+H^1(D)$ solving the homogenized equation

$$-\nabla\cdot\big(\overline{a}\nabla\overline{u}(x)\big) = \mu_0 M_{\mathrm{sat}} h_a(x) \qquad \text{in } D, \tag{1.3.6}$$

with the effective elliptic coefficient given by

$$\overline{a} := \mathbb{E}\big[a\big(\mathrm{Id}+\nabla\Phi\big)\big] \qquad \text{with} \qquad -\nabla\cdot(a\nabla\Phi_a + a\,\mathrm{Id}) = 0 \quad \text{in } \mathbb{R}^3.$$

The solution $\nabla\Phi_a$ is called the *first-order corrector*, one of the fundamental quantities in the theory of stochastic homogenization, which already appeared in (1.3.4). In fact, our effective exhange coefficient a_{eff} from (1.3.5) is exactly the effective elliptic coefficient from this classical theory.

In order to move beyond a qualitative approximation result and quantify how well $\overline{u}$ approximates u^ε, it is necessary to quantify the purely qualitative ergodicity assumption. Historically, there have been two main approaches. For the purpose of elliptic homogenization, these assumptions can be unified as a mixing condition corresponding to *concentration for sums*, see [AK22, Section 3].

The first approach is to assume an explicit rate of decorrelation for the coefficient field with increasing distance. Under such a *strong α-mixing* assumption, a non-trivial convergence rate was obtained by Yurinskii in [Yur87]. A special case of this assumption is a *finite range of dependence*, meaning that $a|_U$ is independent from $a|_V$ for any $U,V\subset\mathbb{R}^3$ with $\mathrm{dist}(U,V)\geq C$ for some $C>0$.

The second approach is to assume that the random ensemble generating the coefficient field satisfies a *spectral gap inequality (SG)* or a stronger *logarithmic Sobolev inequality (LSI)*, which are functional inequalities on the probability space with (SG) corresponding to a Poincaré inequality. They allow to bound variance (SG) or quadratic entropy (LSI) in terms of derivatives on the probability space, which can be thought of as the sensitivity of the given quantity with respect to locally resampling the coefficient field. In the context of elliptic homogenization, this approach was first used by Naddaf and Spencer in [NS98] to obtain optimal convergence rates in a discrete setting with small ellipticity ratio.

In the past two decades, the quantitative theory of stochastic homogenization developed rapidly, leading to two robust frameworks: one developed by Gloria, Neukamm, and Otto [GO11; GO17; GNO14; GNO20] originally focused on SG/LSI assumptions and one by Armstrong, Kuusi, Mourrat, and Smart [AS16; AM15; AKM16; AK25] originally focused on

the finite-range-of-dependence assumption. Within both frameworks, optimal convergence rates and a large-scale regularity theory for solutions at a given scale $\varepsilon > 0$ have been established.

However, how to quantify the limit for the micromagnetic energy and to which extent the existing theory can be transferred to our setting remains unclear, in particular due to the lack of convexity and the lack of a unique minimizer caused by the restriction to manifold-valued functions.

1.4 Forced mean curvature flow and first passage percolation

In Chapter 4 and Chapter 5, we consider curvature-driven interface motions through a random forcing field. That is, the normal velocity V of an interface at a point $x \in \mathbb{R}^d$ with normal $\nu \in \mathbb{S}^{d-1}$ and mean curvature H is given by

$$V = aH + F(x, \nu) \tag{1.4.1}$$

for some $a \geq 0$ and a stationary random forcing field $F \in C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ with finite range of dependence. At some points in Chapter 4 and Chapter 5, we will allow for non-homogeneous, ansiotropic curvature coefficients a, but the full result requires a constant coefficient.

From a physical perspective, (1.4.1) is used to model the competition between a preference to minimize the interface energy, a uniform forcing driving the interface forwards and local impurities inhibiting the expansion. In particular, applications include dislocation lines and the evolution of domain boundaries in magnetic materials with impurities, see, for example, [IV87; Kar98; BN04].

The forcing field should hence be thought of as the combination of a positive uniform forcing and a random obstacle field modeled with negative forcing. The prototypical example is given by Poisson-distributed obstacles, that is a forcing field

$$F(x, e) = F_{uni} - F_{obst}(x, e) \qquad \text{for } x \in \mathbb{R}^d,\ e \in \mathbb{S}^{d-1}$$

with a uniform forcing $F_{uni} > 0$ and an obstacle field $F_{obst} \in C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ given by

$$F_{obst}(x, e) = \max_{i \in \mathbb{N}} \varphi(x - X_i, e)$$

for a Poisson point process $\{X_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$ with intensity $\rho > 0$ and a Lipschitz continuous obstacle shape $\varphi \in C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ with support $\operatorname{supp} \varphi \subset \overline{B}_1 \times \mathbb{S}^{d-1}$.

We view (1.4.1) as describing the motion of a set's boundary, e.g., with the set corresponding to a magnetic domain. That is, for a closed set $S \subset \mathbb{R}^d$, we work with the set evolution $(\mathscr{R}_t(S))_{t \geq 0}$ given by

$$\mathscr{R}_t(S) \coloneqq \text{the evolution of } S \text{ by (1.4.1) after time } t.$$

This definition can be made rigorous and is well-posed using the level-set formulation in the framework of viscosity solutions, see Chapter 4. We are interested in the large-scale behavior of (1.4.1), that is, we want to approximate the set evolution $\left(\varepsilon \mathscr{R}_{\varepsilon^{-1} t}(\varepsilon^{-1} S)\right)_{t \geq 0}$ for $S \subset \mathbb{R}^d$ and small $\varepsilon > 0$. This set evolution behaves according to

$$V = \varepsilon a H + F\Big(\frac{x}{\varepsilon}\Big).$$

The effective large-scale behavior can differ drastically depending on the relation between the uniform forcing and the obstacle strength and sparsity. For our prototypical example, if $a > 0$ and if the obstacle shape φ is nontrivial, then Dirr, Dondl, and Scheutzow showed in [DDS11] that there exists a threshold $F_* > 0$ such that *pinning* occurs for all $F_{uni} \in [0, F_*]$. Pinning means that above each hyper-plane in $\mathbb{R}^d$ there is a graph which the evolution of sets initialized below the hyper-plane can not cross. This means that even for very sparse obstacles and a positive, albeit very small, uniform forcing, sets end up stuck – this is due to the curvature propagating the inhibiting effect of the obstacles. In this regime, the effective large-scale behavior can hence be characterized by a zero-speed motion.

However, as the forcing increases, there is the so-called de-pinning threshold $F^* \geq F_*$, such that the evolution is *ballistic* for all $F_{uni} > F^*$, meaning that on average the main front of sufficiently large sets sets expands faster than some positive minimum speed, which for our setting was shown by Courte, Dondl, and Ortiz in [CDO22]. In-between the pinning regime and this ballistic regime, there might be a sub-ballistic regime, where sets are not trapped but the average large-scale speed is still zero, see [DS17].

We are interested in the large-scale behavior in the ballistic regime. Here, it is expected that (1.4.1) can be approximated by a homogeneous first-order motion, where the normal velocity V is given by

$$V = \overline{v}(\nu)$$

for a homogeneous speed $\overline{v} \colon \mathbb{S}^{d-1} \to (0, +\infty)$ depending on the normal direction $\nu \in \mathbb{S}^{d-1}$. Such a large-scale limit has been rigorously established with quantitative error bounds by Armstrong and Cardaliaguet in [AC18] and Feldman [Fel19] for $F(x, e) = F(x)$ under the assumption that $\inf_{x \in \mathbb{R}^d} F > 0$ and

$$\inf_{x \in \mathbb{R}^d} \big(F(x)^2 - (d-1)a \big| D_x F(x) \big| \big) > 0 \quad \text{in [AC18]}, \qquad \text{or} \qquad d = 2 \quad \text{in [Fel19]}. \tag{1.4.2}$$

These assumptions ensure that there is a guaranteed minimum speed of expansion $v_{\min} > 0$ with $h_0 > 0$, such that, respectively,

$$\mathscr{R}_{t+\Delta t}(S) \supset \mathscr{R}_t(S) + v_{\min} \Delta t \overline{B}_1 \qquad \text{or} \qquad \mathscr{R}_{t+\Delta t+v_{\min}^{-1}}(S) \supset \mathscr{R}_t(S) + v_{\min} \Delta t \overline{B}_1$$

holds for all $t, \Delta t \geq 0$ and for all sufficiently 'fat' initial sets $S \subset \mathbb{R}^d$ in the sense that $S = \bigcup \{ \overline{B}_{h_0}(x) : x \in \mathbb{R}^d \text{ with } \overline{B}_{h_0}(x) \subset S \}$. With respect to our prototypical example, these assumptions require an extremely strong uniform forcing independent of the obstacle density, implying that the front washes over the individual obstacles.

In the regime of (1.4.2), homogenization results for periodic environments have previously been obtained by Lions and Souganidis in [LS05b] and Caffarelli and Monneau in [CM14]. The techniques in [AC18; Fel19] are based on the quantitative theory of stochastic homogenization for Hamilton–Jacobi equations, which had been established by Armstrong, Cardaliaguet, and Souganidis in [ACS14; AC14] for increasingly general classes of Hamilton–Jacobi equations. This framework was extended in [AC18] to also include the level-set formulation of (1.4.1), which is a viscous Hamilton–Jacobi equation with degenerate diffusion matrix. The qualitative theory assuming just stationarity and ergodicity as in Section 1.3 goes back to Souganidis [Sou99] and Rezakhanlou and Tarver [RT00] for first-order Hamilton-Jacobi equations and Lions and Souganidis [LS05a] and Kosygina, Rezakhanlou, and Varadhan [KRV06] for the semilinear, viscous case. However, there have been no qualitative homogenization results in our setting assuming just ergodicity. In fact, for some Hamilton–Jacobi equations, it has been shown that assuming ergodicity is not sufficient for homogenization to occur, see [Zil16; FS17; FFZ21].

Our results in Chapter 4 and Chapter 5 extend the range of the results from [AC18; Fel19] much closer to the depinning threshold. In particular, this is not just the first quantitative but also the first qualitative homogenization result for curvature-driven interface motions in the ballistic regime which allows for the physically relevant case of impenetrable obstacles and therefore potential enclosures left behind the main front which will never be reached. Obtaining this result required a range of innovations, both on a conceptual and technical level.

One of our key concepts is an *"almost guaranteed effective minimum speed"* $v_{\min} > 0$, which, roughly speaking, means that on large scales $h \gg 1$ there holds

$$\mathscr{R}_{t+v_{\min}^{-1}h}(S) \supset_h \mathscr{R}_t(S) + h\overline{B}_1 \qquad \text{with high probability} \tag{1.4.3}$$

for all $t \geq 0$ and all sets $S \subset \mathbb{R}^d$ with $S = \bigcup\{\overline{B}_h(x) \subset S\}$. Here, the inclusion "$\subset_h$" ignores holes of width up to h and hence accommodates potential enclosures left behind the main front due to the impenetrable obstacles. In Chapter 5, we derive this abstract property for dimension $d = 2$ from an easy-to-check criterion, which in particular applies to the prototypical example of impenetrable obstacles distributed according to a Poisson point process with low enough intensity.

In Chapter 4, we obtain the homogenization result assuming such an *"almost guaranteed effective minimum speed"* for dimension $d \geq 2$, allowing us to adapt the framework from [AC18] for fluctuation bounds. However, this adaptation is not trivial: For instance, we deal with measurability issues arising because the main front can fall back locally due to the obstacles. We further introduce a martingale concentration inequality which only requires control with high probability rather than full probability. Based on the fluctuation bounds, we obtain the full homogenization result, for which we use an explicit construction tailored to geometric motions rather than the classical perturbed test function machinery for Hamilton–Jacobi equations which for us is inaccessible due to a lack of regularity. For an in-depth comparison to [AC18], see Section 4.2.6.

Note that we only consider a random forcing field with a finite range of dependence rather than one satisfying a spectral gap (SG) or logarithmic Sobolev inequality (LSI), see Section 1.3.1. Our proof crucially requires the finite range of dependence for the fluctuation bounds in Chapter 4, where it is used to resolve a conditional expectation. Working with (SG) or (LSI) would require a different approach for these bounds. In addition, it is unclear if it is possible to sufficiently control the probabilistic derivative – resampling the coefficient field in any small area can have a comparatively large impact on the set evolution.

1.4.1 Connections to first passage percolation

In the following, we introduce the setting for the theory of first passage percolation, which studies the effective speed of expansion in a random environment fully determining the local speed. In dimension $d = 1$, this is exactly the example discussed in Section 1.1. In fact, this kind of evolution corresponds to (1.4.1) without a curvature term ($a = 0$) and the local speed given by $F(x, e) = F(x)$ for all $x \in \mathbb{R}^d, e \in \mathbb{S}^{d-1}$, which is known as Riemannian first passage percolation, see [LW10]. Since the theory of first passage percolation has been extensively studied for more than sixty years, see [ADH17], it provides a reference frame and essentially upper bounds for the strength of our results. The key difference and source of additional difficulty in our setting arises from non-local interactions induced by the curvature: we always have to view the set evolution as a whole rather than evaluating individual paths connecting single points. A discussion of individual results from the

theory of first passage percolation and how they relate to our work can be found in Section 4.2.5 and Section 5.2.3. Here, besides introducing the typical setting, we aim to point out the classical regime corresponding most to our own model.

The typical setting for first passage percolation is discrete, where, instead of $\mathbb{R}^d$, the graph $(\mathbb{Z}^d, \mathcal{E}^d)$ with the edges $\mathcal{E}^d$ given by the set of all nearest neighbors in $\mathbb{Z}^d$ is considered. Independent and identically distributed random speeds $(v_e)_{e\in\mathcal{E}^d} \subset [0, +\infty)$ (or, analogously, passage times) are assigned to each edge. A speed of $v_e = 0$ means that an edge is closed and can be thought of as being blocked by an obstacle. The main object of study are the passage times

$$T(x,y) \coloneqq \inf\left\{\sum_{e\in\gamma} v_e^{-1} \,:\, \gamma \subset \mathcal{E}^d \text{ is a path from } x \text{ to } y\right\}$$

for $x, y \in \mathbb{Z}^d$. The evolution $(\mathscr{R}_t(S))_{t\geq 0}$ of a set $S \subset \mathbb{Z}^d$ according to the respective speed is given by

$$\mathscr{R}_t(S) \coloneqq \left\{x \in \mathbb{Z}^d \,:\, \inf_{y\in S} T(y,x) \leq t\right\}.$$

Classical percolation theory, see [Gri99], studies the connectivity properties of the environment. These can differ drastically, depending on the probability

$$p \coloneqq \mathbb{P}\big[v_0 > 0\big]$$

that an edge is open. The set of grid points connected to a set $S \subset \mathbb{Z}^d$ via open edges is called the percolation cluster associated to S, given by $\bigcup_{t\geq 0} \mathscr{R}_t(S)$. For dimension $d \geq 2$, there is a critical value $p_c = p_c(d) \in (0,1)$ such that

$$\begin{cases} \mathbb{P}\left[\bigcup_{t\geq 0} \mathscr{R}_t(\{0\}) \text{ is bounded}\right] = 1 & \text{for } p < p_c, \\ \mathbb{P}\left[\bigcup_{t\geq 0} \mathscr{R}_t(\{0\}) \text{ is bounded}\right] < 1 & \text{for } p > p_c, \end{cases},$$

corresponding to the so-called subcritical and supercritical phases of the environment. In fact, in the supercritical phase $p > p_c$, there almost surely exists exactly one infinite percolation cluster, see [Gri99, Theorem 8.1].

Drawing a parallel to our model, the pinning regime can be compared to the subcritical phase of the percolation model. The ballistic regime, which we are interested in, can be compared to the supercritical phase: combining a result from Antal and Pisztora [AP96] with an argument from Cerf and Théret in [CT16] implies that for all $p > p_c$ there is an effective minimum speed $v_{\min} > 0$. This effective minimum speed holds in the following sense. Due to the definition of p, there exists $M > 0$ such that $\mathbb{P}[v_0 \geq M^{-1}] > p_c$. Let $\mathcal{C}_M \subset (\mathbb{Z}^d, \mathcal{E}^d)$ be the infinite percolation cluster associated to the environment $\widetilde{v}_e = \chi_{v_e \geq M^{-1}}$. Then, by the result from [AP96], there exists a constant $C > 0$ such that

$$\mathbb{P}\big[x, y \in \mathcal{C}_M \text{ and } T(x,y) > v_{\min}^{-1}|x-y|\big] \leq C\exp\big(C^{-1}|x-y|\big) \qquad \text{for all } x, y \in \mathbb{Z}^d,$$

implying ballistic behavior. In addition, combined with the fact from [CT16, Theorem 8] that there is $C > 0$ such that

$$\mathbb{P}\big[\mathcal{C}_M \cap \overline{B}_h(x) = \emptyset\big] \leq C\exp\big(C^{-1}h\big) \qquad \text{for all } x \in \mathbb{Z}^d,\, h > 0,$$

this yields an analogue to our assumption of an "almost guaranteed effective minimum speed", as roughly described in (1.4.3), for the full supercritical phase. This suggests that (1.4.3) could be satisfied throughout the whole ballistic regime all the way down to the de-pinning threshold in dimension $d = 2$, where there are no infinite holes in the supercritical cluster.

The main goal in the theory of first passage percolation is to approximate passage times on large scales with a direction-dependent effective speed and to control the error, in particular the fluctuations, see [ADH17]. The classical results in the theory of first passage percolation, e.g., [HW65; CD81; Kes93; Ale97; BKS03], have been obtained for $p = 1$, guaranteeing that every site $x \in \mathbb{Z}^d$ can be reached. Results for $p \in (p_c, 1)$, that is, the study of arrival times on the infinite percolation cluster in the supercritical phase, have been more recent and rely on the effective minimum speed from [AP96], see, e.g., [GM04; GM09; Mou11; CT16; CD22; Dem24]. Essentially, the relationship between the works [AC18; Fel19] and our results can be compared to the relationship between results from first passage percolation on the full grid $\mathbb{Z}^d$ and on supercritical percolation clusters.

With respect to a higher-order approximation for first-passage percolation and, by extension, also curvature-driven geometric motions, note in particular that the strongest known fluctuation bounds, which have recently been obtained by Basu, Sidoravicius, and Sly in [BSS26], are still far away from the conjectured convergence rate. Rigorously establishing a higher order approximation in the sense of Section 1.1 therefore currently seems out of reach.

CHAPTER 2

Density fluctuations in weakly interacting particle systems via the Dean–Kawasaki equation

This chapter contains (with minimal corrections) the following publication [Cor+26], which is licensed under a Creative Commons Attribution 4.0 International License (CC BY 4.0):

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Abstract The Dean–Kawasaki equation – one of the most fundamental SPDEs of fluctuating hydrodynamics – has been proposed as a model for density fluctuations in weakly interacting particle systems. In its original form it is highly singular and fails to be renormalizable even by approaches such as regularity structures and paracontrolled distributions, hindering mathematical approaches to its rigorous justification. It has been understood recently that it is natural to introduce a suitable regularization, e. g., by applying a formal spatial discretization or by truncating high-frequency noise: This yields well-posed equations that should still precisely approximate the law of the particle density fluctuations.

In the present work, we prove that a regularization in the form of a formal discretization of the Dean–Kawasaki equation indeed accurately describes density fluctuations in systems of weakly interacting diffusing particles: We show that in suitable weak metrics, the law of fluctuations as predicted by the discretized Dean–Kawasaki SPDE approximates the law of fluctuations of the original particle system, up to an error that is of arbitrarily high order in the inverse particle number and a discretization error. In particular, the Dean–Kawasaki equation provides a means for efficient and accurate simulations of density fluctuations in weakly interacting particle systems.

2.1 Introduction

The theory of *Fluctuating Hydrodynamics* [Spo91] describes the dynamics of large, finite-size particle systems subject to fluctuations. In the framework of this theory, the particle system being investigated is described via a suitable stochastic PDE (SPDE), which

captures the fluctuations of the system on top of its deterministic limiting dynamics. The physics applications of this theory are numerous and diversified, and touch upon several different fields, see for instance [Ber+15; MMT99; God+12; Vel+08; Del+16; DFV14; Lut12; Déj+18; Dur+19; CT15; Tho+11].

This work is concerned with giving a fully quantitative justification to a pivotal SPDE from fluctuating hydrodynamics, the so-called *Dean–Kawasaki equation* [Dea96; Kaw98]

$$\partial_t \rho = \sigma \Delta \rho + \nabla \cdot (\rho \nabla V * \rho) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma\rho}\, \xi). \tag{2.1.1}$$

Here, ρ denotes the density of particles, N is the number of particles, V is an interaction potential, $\sigma > 0$ is the diffusion coefficient, and ξ denotes space-time vector-valued white noise. The model (2.1.1) is proposed as a mesoscopic description for the law of the empirical density

$$\mu^N(\cdot, t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^N(t)}$$

of a system of $N \gg 1$ particles with positions $X_i^N(t) \in \mathbb{T}^d$ driven by i.i.d. Brownian motions $(B_i)_{i=1}^N$ and interacting weakly via a smooth potential V:

$$\mathrm{d}X_i^N(t) = -N^{-1} \sum_{j=1}^{N} \nabla V \left(X_i^N(t) - X_j^N(t) \right) \mathrm{d}t + \sqrt{2\sigma}\, \mathrm{d}B_i(t). \tag{2.1.2}$$

The purpose of the Dean–Kawasaki equation (2.1.1) is to correctly describe the law of particle density fluctuations, going beyond the (deterministic) mean-field description of the particle density

$$\partial_t \bar{\rho} = \sigma \Delta \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla V * \bar{\rho})\,. \tag{2.1.3}$$

As we shall see, the Dean–Kawasaki equation (2.1.1) turns out to be even substantially superior in accuracy as compared to the leading-order description of fluctuations by the process

$$\partial_t \hat{\rho} = \sigma \Delta \hat{\rho} + \nabla \cdot (\hat{\rho} \nabla V * \hat{\rho}) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma\bar{\rho}}\, \xi). \tag{2.1.4}$$

The Dean–Kawasaki equation (2.1.1) itself is a highly singular SPDE; it is not even renormalizable by approaches like regularity structures or paracontrolled distributions. As shown in the seminal work [KLR19a], the singular SPDE (2.1.2) turns out to be in fact a formal – mathematically equivalent – rewriting of the associated microscopic particle system (2.3.1): All of its martingale solutions are precisely given as the empirical density of an interacting particle system of the form (2.1.2), i.e., the only solutions to (2.1.1) are of the form

$$\rho(\cdot, t) \equiv \mu^N(\cdot, t) := N^{-1} \sum_{i=1}^{N} \delta_{X_i^N(t)} \tag{2.1.5}$$

for some X_i^N satisfying the system of SDEs (2.1.2). While this might – at first glance – appear to imply that the representation (2.1.1) brings no additional insight over the associated particle dynamics (2.1.2), it has been observed in recent years that this is in fact not the case: Natural regularizations of the formal Dean–Kawasaki SPDE (2.1.1) are better-behaved objects and may be expected to provide meaningful approximations of

density fluctuations. Note that in any practical use for physics simulations, the Dean–Kawasaki equation (2.1.1) is necessarily subjected to a regularization, either by applying a spatial numerical discretization (which implicitly truncates the high-frequency noise modes) or by introducing an explicit frequency cutoff in the noise term. While regularised models may differ from one another in terms of specific features and applicability, they are all usually much more tractable and versatile versions of (2.1.1), and capable of describing – at least some – features of the underlying microscopic systems up to a quantifiable, small error.

In the present work, we prove that upon regularizing the Dean–Kawasaki equation (2.1.9) by applying a formal spatial discretization, it is capable of approximating the law of density fluctuations of the weakly interacting particle system (2.3.1) up to *arbitrary* precision in N^{-1} (plus numerical errors). Note that any meaningful comparison of the empirical density μ^N – a sum of Dirac measures – to the continuous solution ρ_h of a regularized variant of the Dean–Kawasaki equation (2.1.9) must be formulated in terms of weak spatial norms, i.e., in terms of testing the densities against a test function with sufficient regularity. Furthermore, as μ^N and ρ_h live on different probability spaces, any comparison of ρ_h to μ^N can only be phrased in terms of their law (and not as a pathwise statement). The informal statement of our result is as follows. Here, the spatial order of the discretization scheme denotes its accuracy, e.g. for a scheme of order $p+1$ the discrete gradient satisfies $|\nabla_h \varphi - \nabla \varphi| \lesssim \|\varphi\|_{C^{p+2}} h^{p+1}$.

Main Result (Informal statement for high-order approximation of density fluctuations in the single-species case of Theorem 2.2.1 below)**.** *Let $T > 0$ be a fixed time. Let ρ_h denote the solution to a suitable finite-difference discretisation of (2.1.1) on $\mathbb{T}^d \times [0,T]$ with spatial discretisation parameter h and of spatial order $p+1$. Let μ^N denote the empirical density of the particle system (2.1.2) consisting of N particles. Suppose that the initial density $\rho_h(\cdot,0) \geq \inf_x \rho_h(x,0) > 0$ is strictly positive and approximates (in law) the empirical density $\mu^N(\cdot,0)$. Furthermore, let the overline symbol $(^-)$ denote suitable "mean-field" analogues of the densities. For the scaling regime, assume that*

$$N^{1-\delta_0} h^d \geq 1 \quad \text{for some } \delta_0 > 0. \tag{2.1.6}$$

Then, for every $0 < \epsilon < \delta_0/4$, there exists a stopping time $T_\oslash$ with

$$\mathbb{P}(T_\oslash < T) \lesssim \exp\left(-CN^{\epsilon/2}\right), \tag{2.1.7}$$

where for every $\kappa > 0$ there exists $N_0 \in \mathbb{N}$, such that for every $t \in [0,T]$ and $N \geq N_0$,

$$\begin{aligned} &d_{weak,2j+1}\left[N^{1/2}(\rho_h - \overline{\rho}_h)(t \wedge T_\oslash),\, N^{1/2}(\mu^N - \overline{\rho})(t)\right] \\ &\lesssim N^{2\epsilon+\kappa}\left(h^{p+1} + \exp\left(-CN^{\epsilon/2}\right) + N^{-j(1/2-2\epsilon)}\right) \\ &=: Err_{num} + Err_\oslash + Err_{fluct,rel} \end{aligned} \tag{2.1.8}$$

holds for all $j \in \mathbb{N}$. Here, $d_{weak,2j+1}$ is a negative Sobolev-type distance of order $-2j-1$.

This result is desirable for three reasons:

- The approximate scaling regime $Nh^d \gg 1$ (cfr. (2.1.6)) is extremely relevant, as it corresponds to an – on average – large amount of particles per grid cell. Hence, it is the regime in which the direct particle simulation is more expensive than the numerical SPDE model.

- The estimate (2.1.8) shows that the (discrete) Dean–Kawasaki equation provides an accurate description of the underlying particle system, in the sense that the error due to the finiteness of the number of particles is – in suitable weak norms – of arbitrarily high order in N^{-1} and thus typically dominated by the numerical error $N^{2\epsilon+\kappa}h^{p+1}$.
- By (2.1.7), the stopping time (whose role in the proof is to guarantee that the fluctuations $\rho_h - \overline{\rho}_h$ roughly stay in the regime of classical mean field fluctuations, and that ρ_h stays positive) runs short of the final time $T > 0$ only with exponentially small probability.

The key novelty of our result is the derivation of an error estimate for an *interacting* particle system that is *of arbitrary order* in the inverse particle number N^{-1} (plus numerical error Err_{num} and modelling error $Err_{\oslash}$): In particular, our result is the first to show the Dean–Kawasaki equation is a far more accurate descriptor of fluctuations than the leading-order equation (2.1.4) in the case of weakly interacting particle systems.

In the case of independent Brownian particles, i.e., the case $V \equiv 0$ in (2.1.1), the corresponding result has been proven in [CF23]. Again in the non-interacting particle case, a short proof of a quantitative error bound of the order $O(N^{-1/(d+2)})$ has recently been given in [DKP24]: Notably, the result in [DKP24] does not require a positivity lower bound on the initial density profile. Previously, a low-order error estimate $O(N^{-\beta})$ for some $\beta < \frac{1}{2}$ had been established in [FG23], covering also the case of weakly interacting particles.

Let us mention that, having in mind a future application to cross-diffusion limits, throughout the present work we in fact consider a somewhat more general interacting particle system: We allow for multiple species of diffusing particles that interact with each other via possibly mildly (singularly) rescaled potentials $V^{r_I}_{\alpha\beta}(\cdot) = {r_I}^{-d}V(\cdot/r_I)$ with interaction length scale $r_I = r_I(\log N)$. The associated particle dynamics is given by the system of SDEs (2.3.1) below, while the corresponding analogue of the Dean–Kawasaki equation is given by the system

$$\partial_t \rho^{r_I}_\alpha = \sigma_\alpha \Delta \rho^{r_I}_\alpha + \nabla \cdot \left(\sum_{\beta=1}^{n_S} \rho^{r_I}_\alpha \nabla V^{r_I}_{\alpha\beta} * \rho^{r_I}_\beta \right) + N^{-1/2} \nabla \cdot (\sqrt{2\sigma_\alpha \rho^{r_I}_\alpha}\, \xi_\alpha) \tag{2.1.9}$$

for $\alpha, \beta \in \{1, \cdots, n_S\}$, n_S being the number of species. However, we emphasize that our results are already new and relevant in the single-species case $n_S = 1$ and for $r_I \equiv 1$.

2.1.1 Comparison with the independent Brownian case ($V \equiv 0$) treated in [CF23]

In contrast to [CF23], the distinctive trait of this work is the handling of the nonlinear convolution term $\nabla \cdot (\rho \nabla V * \rho)$ in (2.1.1), which corresponds to the mean-field interaction between particles (in this section we take $n_S = 1$ for notational simplicity). Doing this entails working around several difficulties, which we now list informally for the purpose of this introduction. A more thorough analytical discussions of these difficulties can be found in Subsection 2.4.2, directly after – and in relation to – the informal summary of the proof of our main result (Theorem 2.2.1) in Subsection 2.4.1.

Quantitative convergence to the mean-field limit. The presence of the potential V needs handling when proving the convergence of the empirical density μ^N to the mean-field limit $\overline{\rho}$ in (2.1.3). The same also applies for the convergence of the discretised counterparts $\rho_h - \overline{\rho}_h$, although this is a consequence of the next point.

Linearisation argument for convolutional nonlinearity. The most remarkable feature of (2.1.9) is the linear cross-variation structure of the noise (with respect to the density ρ). More precisely, as we will discuss thoroughly, the Itô differential of the process $\int_{\mathbb{T}^d}(\rho-\overline{\rho})\varphi_1 \int_{\mathbb{T}^d}(\rho-\overline{\rho})\varphi_2$, for suitably chosen, smooth test functions φ_1, φ_2, has martingale component with cross-variation given by

$$N^{-1}\int_{\mathbb{T}^d}\rho\nabla\varphi_1\cdot\nabla\varphi_2. \tag{2.1.10}$$

The linearity of (2.1.10) (w.r.t. the density ρ) is the pivotal ingredient upon which most arguments in [CF23] are based. One of the main challenges of this work is – in a nutshell – to linearise the quadratic convolution term $\nabla\cdot(\rho\nabla V*\rho)$ so as to recover the paradigm of [CF23] as much as possible. This involves, crucially, performing a Fourier expansion of $\nabla\cdot(\rho\nabla V*\rho)$ in twice the space dimension: doing this effectively 'detaches' the two instances of the density ρ and leads to a countable number of tractable linear objects of type (2.1.10).

Estimate for probability of large fluctuations. It is important to provide bounds on $\rho_h-\overline{\rho}_h$, as this – among other things – gives an indication of how often the discretised solution ρ_h becomes negative. While – in the absence of the potential – this could be handled with a stopping time argument exclusively in the L^∞-norm, the presence of the potential requires a lengthy technical argument to also keep track of a suitable negative Sobolev norm (this argument also makes use of the convergence of the continuous Mean-Field Limit). Controlling this second norm effectively tackles the superlinear nature of the convolution term, ultimately allowing for the exponentially decaying bound (2.1.7).

Compatible discrete setting. The challenge of this point is twofold. Firstly, in contrast to [CF23], the initial configurations $\mu^N(\cdot,0)$ and $\rho_h(0)$ are random, as sufficient regularity of their expected value is required for the regularity of the Mean-Field Limit (no Mean-Field limit argument is needed in [CF23]). For this point, in order to obtain suitable initial data which satisfies the required high-order (in h) fluctuation estimates for $\rho_h(0)-\overline{\rho}_h(0)$, we propose a scheme based on polynomial interpolation. Secondly, additional care is needed when performing high-order (in h) approximations of the 'time-dual' of the discrete Mean-Field Limit. The need for these approximations will be discussed in due course.

Remark 2.1.1. In order to obtain the fluctuations bound (2.1.8), we require the potential V to be sufficiently smooth. As will become apparent in the proofs, the smoothness of V allows – among other things – for: i) the application of the linearisation argument mentioned above, and ii) small errors entailed by replacing continuous operators with discretised counterparts (in particular, when replacing the continuous convolution $\nabla V*\rho$ with a discrete version $\nabla V *_h \rho$).

We refer the reader to Subsection 2.1.2 for a broader discussion concerning existing literature and future investigations in the case of singular potentials.

2.1.2 Related literature

The leading order description via the additive noise model (2.1.4)

As a description for the fluctuations of the particle density, (2.1.4) with the purely additive noise term had already been established in the 70s and 80s before Dean and Kawasaki

proposed (2.1.1), usually in the sense of a Central Limit Theorem, viewing the convergence to the mean field limit as a law of large numbers. In [BH77], density fluctuations were characterized for a second order system with random initial conditions but dynamics given just via interaction. It was shown, that the normalized fluctuations $N^{1/2}(\mu^N - \overline{\rho})$ converge to a Gaussian stochastic process in a suitable sense. Recently, for this same setting optimal quantitative bounds for this convergence were obtained in [Due21] as a byproduct.

There is a large amount of results in this direction for other particle systems, but here we focus on our specific setting, that is first order dynamics driven by Brownian motion and interaction via a bounded C^2 interaction potential. In [TH81], the limit of the normalized fluctuations is characterized for the simplified version of a quadratic interaction potential V in dimension one: The limit process in our notation corresponds to $N^{1/2}(\hat{\rho} - \overline{\rho})$, thus suggesting (2.1.4) as a good approximation of the particle system. Later, under general assumptions similar to ours the same has been done in [Szn84], showing convergence of finite dimensional marginals. In [FM97], also for more general initial conditions, it has been shown that the normalized fluctuations converge in law as elements of a path-space with values in a weighted negative Sobolev space.

Dean–Kawasaki model

As discussed around (2.1.5), the Dean–Kawasaki equation in its original form is a rather rigid mathematical object, as it only allows for the empirical particle system as solution: this result is given in [KLR19a] and, also, in the (earlier) analogue version for non-interacting particles [KLR19b]. The papers [KLR19b; KLR19a] are related to a series of works [AR10; RS09; KR24; KR19; Del22; KM23; Mar21] which – among others – shed light on the rigid interplay which deterministic and stochastic components of the Dean–Kawasaki model have to abide to (in the context of a suitable stochastic Wasserstein gradient flow).

The aforementioned rigidity of the Dean–Kawasaki model can be broken once suitable regularisations (such as, for instance, noise smoothing or truncation or a discretization) are introduced: A substantial number of works belong to this ever-growing framework.

In [DFG26], a rigorous justification of the SPDE of fluctuating hydrodynamics for the simple exclusion process is provided (with leading order dynamics, although the noise term in this case is nonlinear and therefore much more challenging), together with convergence results concerning the rate functions for large deviation principles. Rigorous links between Dean–Kawasaki type models and large-deviation principles for zero-range processes (and associated thermodynamic setting) are given in [DSZ16].

In [FG19], the well-posedness theory (in a suitable kinetic formulation) of stochastic porous media and fast diffusion equations driven by nonlinear, conservative noise is provided. The generation of a random dynamical system is also discussed. The latter topic is expanded and enriched with uniqueness of invariant measures and mixing for the associated Markov process in [FGG22]. Similar results are also derived in the case of correlated noise [FG24]. For the same noise, derivation of underlying microscopic dynamics is given in [Din22].

Rates of convergence of the discretized Dean–Kawasaki dynamics towards the particle system in the case of independent Brownian particles are discussed in [CF23]. In the recent paper [DKP24], the authors prove weak error estimates in the non-interacting particle case, but they use a suitable SPDE approximation of the Dean–Kawasaki model rather than a discrete numerical approximation of the same. Their mathematical approach

(which is centered around Laplace duality arguments and Kolmogorov backwards equation techniques) is somewhat complementary to ours, and it is safe to say that the current work and [DKP24] have different points of strength. Unlike the results in this work, those in [DKP24] allow for general initial particle profiles, and provide non-negativity of the solution (in addition to other well-posedness properties, including a comparison principle and entropy estimates): However, the weak error accuracy in [DKP24] in terms of N is capped (in terms of relative error) by $N^{-1/(d/2+1)} \log N$ (which gets worse with the spatial dimension d), while our fluctuation rate (2.1.8) can be arbitrarily high.

Recently, conservative stochastic PDEs sharing strong similarities with Dean–Kawasaki models have been proven to be limit of stochastic interacting particle systems in the mean-field limit (e.g., the case of stochastic gradient descent dynamics in overparametrised, shallow neural networks is covered in [GGK25] with optimal convergence rates provided for both convergence and associated Central Limit Theorem).

For regularised Dean–Kawasaki models of inertial type (i.e., models capturing both density and momentum density), high-probability well-posedness for both independent and weakly interacting particle systems is discussed in [CSZ19; CSZ20; CSZ21].

The Dean–Kawasaki model is becoming more and more widespread in physics applications (see, for instance, [CT15; Tho+11; Lut12; MMT99; Vel+08; God+12; Del+16; Déj+18; DFV14; Dur+19]). Consequently, works devoted to numerical approximations of the Dean–Kawasaki model are on the rise. Among such contributions on the numerical side, we mention structure-preserving finite difference and finite element schemes for high-order fluctuation bounds of non-interacting diffusing particles [CF23], analysis of finite element discretisations in the context of reaction-diffusion (agent-based) systems models [Hel+21; Kim+17], analysis of finite differences discretisations of agent-based models describing co-evolving opinion and social dynamics under the influence of multiplicative noise [DKD22], finite-volume schemes for stochastic gradient flow equations [Rus+21; Don+10], full reconstruction of dissipative operators in gradient flow equations from particle fluctuations [Li+19], convergence of finite element schemes for a weak formulation of suitable smoothed Dean–Kawasaki model [BGV23], convergence analysis of discountinuous Galerkin scheme – and modelling – for the regularised inertial Dean–Kawasaki model [CS23].

The mean-field limit of (2.3.1), and discussion of future investigations for singular potentials and cross-diffusion systems

The analysis of the Dean–Kawasaki model (2.1.9) is naturally built on top of the mean-field dynamics of the particle system (2.3.1). The study of the mean-field dynamics of systems of SDEs goes back to the 80s (see, e.g., the reviews [Gol03; JW17]). Results for a weakly interacting system with smooth potential were established early and are now considered classical, see e.g. [Szn91] for lecture notes from the time.

In the present work, we focus on the case of arbitrarily high regularity of the interaction potential V. In recent years there has been major progress concerning the dynamics of diffusive particle systems with a singular mean-field interaction. In particular, we refer to [BJW23; BJW19; CRS23] concerning the mean-field limit of (2.1.2) with singular interaction potentials V and to [WZZ23] for a leading-order description of fluctuations via an SPDE related to (2.1.4). For the kinetic analogue of (2.1.2) we refer to [BD24]. To which degree our present analysis could be extended to the case of singular interaction potentials remains an interesting open problem. While we anticipate that (2.1.1) should still provide a leading-order description of fluctuations, a finer analysis may be needed to

achieve higher-order accuracy in $N^{-1/2}$; for instance, as already hinted in Remark 2.1.1, the Fourier expansion approach for the interaction kernel may need to be replaced by a more careful argument, and caution would need to be taken in discretising the convolution operator (see Corollary 2.B.5).

In the rest of this section, we give a brief account of works related to cross-diffusion-type mean field dynamics (this is in accordance with our intention to also address fluctuations in cross-diffusion systems in the future). In [Oel89], Oelschläger derived reaction-diffusion systems from moderately interacting particle systems – in fact, the case of cross-diffusion is included, however, with a positive-definiteness assumption on the diffusion coefficients. Quadratic porous-medium-type equations are derived from moderately interacting particle systems in [Oel90]. The methods of Oelschläger were significantly extended by Stevens in [Ste00] to derive a chemotaxis system. In [JM98] the mean-field limit and fluctuations of a moderately interacting particle system with nonlinear diffusion coefficients is studied. Further contributions include, e. g., the derivation of a two-phase Stefan problem as the mean-field limit of a master equation [IRS12; KS13], the 2-species Maxwell-Stefan model for the diffusion of gaseous mixtures as the hydrodynamic limit of two (singularly) interacting Brownian motions [Seo18], nonlocal Lotka-Volterra systems with cross-diffusion as limits of a suitable Markov process [FM15], as well as Shigesada-Kawasaki-Teramoto type cross-diffusion systems [Che+21].

Daus, Chen, and Jüngel have shown in [CDJ19] that the limiting behavior of our SDE system (2.3.4) under the simultaneous limit $N \to \infty$ and $r_I \sim (\log N)^{-\beta} \to 0$ is captured by a cross-diffusion system [CDJ19, (1)]; we refer to [GS14] for an earlier more restrictive result and to [JPZ22] for a recent extension. In the present paper we will only concern ourselves with the first limit $N \to \infty$, yielding a mean-field limit of the form (2.3.4); however, many intermediate results are formulated to be of use also in a future work focusing on this cross-diffusion limit, by being established, e.g., uniformly in r_I for a suitable range $(\log N)^c \lesssim r_I \lesssim 1$.

2.2 Main result

In our main result, we rigorously quantify the distance between (a) the law of the density fluctuations arising in the interacting particle system (2.3.1), and (b) the law of density fluctuations in a suitable numerical discretisation of the Dean–Kawasaki SPDE (2.1.9). In order to compare these laws, we make use of the family of distances between $\mathbb{R}^K$-valued random variables X, Y given by

$$d_{-j}[X,Y] := \sup_{\psi\colon \max_{0\le\tilde{j}\le j}\|D^{\tilde{j}}\psi\|_{L^\infty(\mathbb{R}^K)}\le 1} \big|\mathbb{E}\big[\psi(X)\big] - \mathbb{E}\big[\psi(Y)\big]\big|, \qquad j \in \mathbb{N}. \tag{2.2.1}$$

In other words, the distance d_{-j} is a negative Sobolev distance acting on the probability distributions of the arguments. Note that the distance d_{-1} is the Fortet–Mourier norm, a Wasserstein-1-type metric. Up to the zero-th order bound $\|\psi\|_{L^\infty} \le 1$ this corresponds to the dual formulation of the Wasserstein-1 distance.

While the full notation is given below in Section 2.3, we briefly define the remaining minimal ingredients needed to state our main theorem, namely: $G_{h,d}$ (the uniformly spaced grid on $\mathbb{T}^d$ with grid-size parameter $h > 0$); $(\cdot,\cdot)_h$ (the standard inner product in $L^2(G_{h,d})$); $\langle\cdot,\cdot\rangle$ (the standard measure/function duality), and $\mathcal{I}_h$ (the operator interpolating continuous functions on the grid points of $G_{h,d}$).

Our main theorem reads as follows.

Theorem 2.2.1 (High-order approximation of density fluctuations in weakly interacting particle systems). *Let $T > 0$ and $N \in \mathbb{N}$. On $[0,T]$ let*

- *$\mu^{r_I,N} = (\mu_\alpha^{r_I,N})_{\alpha=1}^{n_S}$ as in (2.1.5) be the empirical measures of the cross-diffusing particle system (2.3.1) satisfying Assumption 2.3.7 (particle system),*
- *$\overline{\rho}^{r_I}$ be the intermediate mean-field limit as given in (2.3.4), and satisfying Assumption 2.3.8 (existence and regularity of continuous mean-field limit),*
- *$\rho_h^{r_I}$ be a solution to the discretised Dean–Kawasaki model (h-DK) below in the context of a standard finite-difference discretisation of order $p+1$ and spatial spacing $h > 0$ satisfying Assumption 2.3.9 (discrete finite-difference operators),*
- *$\overline{\rho}_h^{r_I}$ be the corresponding finite difference mean-field limit, as defined in (h-MFL).*

Let the initial conditions satisfy Assumption 2.3.10 and the parameters r_I, N, h satisfy Assumption 2.3.11 with $0 < \delta_0 < 1$ (scaling regime for N and h).

Let $0 < \epsilon < \delta_0/4$. Then there exists a stopping time $T_\oslash \in [0,T]$ with

$$\mathbb{P}\big[T_\oslash < T\big] \lesssim \exp\big(-CN^{\epsilon/2}\big), \tag{2.2.2}$$

such that the discrete Dean-Kawasaki solutions $\{\rho_{h,\alpha}(t \wedge T_\oslash)\}_{\alpha=1}^{n_S}$ capture the fluctuations of the the empirical measures $\mu_{\alpha,t}^{r_I,N}$ in the following sense.

Let $j \in \mathbb{N}$. Assume that with $s(d,p,j) := p + \frac{3d}{2} + 4 + j$ the continuous mean field limit $\overline{\rho}^{r_I}$ is in $L^\infty\big(0,T; C^{s(d,p,j)}\big)$, and that the interaction potentials satisfy $V \in \big[W^{s_I,1}(\mathbb{T}^d)\big]^{n_S \times n_S}$ with $s_I > 2^{j-1}3j^2 s(d,p,j) + 2d + 1$. Then, abbreviating $data := \{V, \overline{\rho}^{r_I}, \rho_{min}, \rho_{max}, d, T, n_S, p, j\}$, for each $\kappa > 0$, there exists $N_0 = N_0(\kappa, \epsilon, K, data)$ such that if $N > N_0$,

$$\begin{aligned} d_{-(2j+1)}\Bigg[& N^{1/2}\begin{pmatrix} \sum_{\alpha=1}^{n_S}\big(\mathcal{I}_h[\varphi_{1,\alpha}], (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(T_1 \wedge T_\oslash)\big)_h \\ \vdots \\ \sum_{\alpha=1}^{n_S}\big(\mathcal{I}_h[\varphi_{K,\alpha}], (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(T_K \wedge T_\oslash)\big)_h \end{pmatrix}, \\ & N^{1/2}\begin{pmatrix} \sum_{\alpha=1}^{n_S}\big\langle \varphi_{1,\alpha}, (\mu_\alpha^{r_I,N} - \overline{\rho}_\alpha^{r_I})(T_1)\big\rangle \\ \vdots \\ \sum_{\alpha=1}^{n_S}\big\langle \varphi_{K,\alpha}, (\mu_\alpha^{r_I,N} - \overline{\rho}_\alpha^{r_I})(T_K)\big\rangle \end{pmatrix}\Bigg] \end{aligned} \tag{2.2.3}$$

$$\begin{aligned} &\leq C(K, \|\boldsymbol{\varphi}\|_{H^{s(d,p,j)}}, data) N^{2\epsilon+\kappa}\big(h^{p+1} + \exp\big(-CN^{\epsilon/2}\big) + N^{-j(1/2-2\epsilon)}\big) \\ &=: Err_{num} + Err_\oslash + Err_{fluct,rel} \end{aligned} \tag{2.2.4}$$

holds for any $\boldsymbol{\varphi} = (\varphi_{k,\alpha})_{\alpha=1,\dots,n_S}^{k=1,\dots,K} \in \big[H^{s(d,p,j)}(\mathbb{T}^d, \mathbb{R}^{n_S})\big]^K$ and any $(T_1,\dots,T_K) \in [0,T]^K$, where the distance $d_{-j}[X,Y]$ has been introduced in (2.2.1).

Remark 2.2.2. Throughout the paper, the generic expression *data* – which may change from use to use – denotes a relevant subset of parameters $\{V, \overline{\rho}^{r_I}, \rho_{min}, \rho_{max}, d, T, n_S, p, j\}$. More specifically, dependencies on $V, \overline{\rho}^{r_I}$ come in terms of Sobolev norms. Note that r_I is not included. We assume that there is a uniform bound in r_I for the respective Sobolev norms of $\overline{\rho}^{r_I}$. This is a reasonable assumption due to the convergence to the solution of a suitable cross-diffusion system given regular enough initial data for $r_I \to 0$, see e.g. [CDJ19].

Remark 2.2.3. Theorem 2.2.1 captures the relative error in fluctuations, $Err_{fluct,err}$: this is due to the prefactor $N^{1/2}$ in the arguments of the metric d_{-j}, which balances out the natural order of fluctuations $N^{-1/2}$ of the inner products $\sum_{\alpha=1}^{n_S}(\mathcal{I}_h[\varphi_{k,\alpha}], (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(T_k \wedge T_\oslash))_h$ and $\sum_{\alpha=1}^{n_S}\langle \varphi_{k,\alpha}, (\mu_{\alpha,T_k}^{r_I,N} - \overline{\rho}_\alpha^{r_I})(T_k)\rangle$. The term Err_{num} accounts for the intrinsic numerical error of the scheme. The term $Err_\oslash$ accounts for the cases where the stopping time runs

short of T. The definition of the stopping time (see Section 2.8) ensures positivity of $\rho_h^{r_I}$: This is due to fact that $T_\oslash$ incorporates an L^∞-bound for $\rho_h^{r_I} - \overline{\rho}_h^{r_I}$, and the fact that the mean-field limit is $\overline{\rho}_h^{r_I}$ is strictly positive by Assumption 2.3.8 below. Additionally, $T_\oslash$ ensures that the fluctuations $\rho_h^{r_I} - \overline{\rho}_h^{r_I}$ roughly stay within the natural regime $N^{-1/2}$. Hence, control of the stopping time is the discrete equivalent of quantifying the mean-field limit convergence.

The additional factors of N^ϵ appear since we are only able to control the stopping time if we relax the fluctuation bound from $N^{-1/2}$ to $N^{-1/2+\epsilon}$. The factor N^κ stems from the logarithmic scaling of the interaction radius r_I with respect to N, see Assumption 2.3.11. If r_I is constant and does not scale, N^κ can be replaced with a constant depending on *data*.

Remark 2.2.4. We will often abbreviate the $\mathbb{R}^K$-valued random variables in (2.2.3) as

$$N^{1/2}\big\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\big\rangle, \qquad N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_\oslash)\big)_h, \tag{2.2.5}$$

where $\boldsymbol{T} := [T_1, \dots, T_K] \in [0,T]^K$.

2.2.1 Structure of the paper

The details of the weakly interacting particle systems we consider, as well as the relevant discretised Dean–Kawasaki model, are given in Section 2.3. Section 2.4 gives an informal – yet exhaustive – summary of the most important results needed to prove Theorem 2.2.1: In particular, Theorem 2.2.1 is of inductive type, and the small fluctuation error $Err_{fluct,rel} \propto N^{2\epsilon+\kappa-j(1/2-2\epsilon)}$ is obtained after j induction steps. Section 2.5 spells out the structure of one of these induction steps and provides all necessary building blocks (above all, Proposition 2.5.9 indicates how to quantitatively include the convolutional nonlinearity in the iteration, thus resolving the mismatch with the linear noise covariance). The proof of Theorem 2.2.1 (i.e., the quantitative performance of all j steps) is finalised in Section 2.6. The core technical lemmas are deferred to subsequent sections, namely: Quantitative convergence to the mean-field limit (Section 2.7); Exponentially decaying bound for the probability of the stopping time $T_\oslash$ coming short of the final time horizon $T > 0$ (Section 2.8).

Finally, the appendix contains the following: Regularity estimates for the continuous test functions (Appendix 2.A); Regularity estimates for the discretised mean-field limit and the discretised test functions, as well as error bounds with respect to their continuous counterparts (Appendix 2.B); Explicit construction of a set of admissible initial conditions for the discrete mean-field limit (Appendix 2.C).

2.3 Setting, notation, and assumptions

Throughout the paper, we use C to denote a generic constant whose value may change from line to line. Relevant dependencies on specific parameters are highlighted whenever needed. Moreover, for generic functions $f, g\colon \mathbb{T}^d \to \mathbb{R}^m$ ($m \in \mathbb{N}$) and $s\colon \mathbb{T}^d \to \mathbb{R}$, we denote $(f * s)(x) := \big[\int_{\mathbb{T}^d} f_\ell(y) s(x-y)\,\mathrm{d}y\big]_{\ell=1}^m$ and $(f *_c g)(x) := \sum_{\ell=1}^m (f_\ell * g_\ell)(x)$.

We now give specific notation and relevant assumptions for the weakly interacting particle system we consider, and its Dean–Kawasaki approximation.

2.3.1 The continuous setting - the particle system

The weakly interacting particle system we are interested in is given by

$$\begin{cases} \mathrm{d}X^{r_I,N}_{\alpha,i}(t) = -\sum_{\beta=1}^{n_S} N^{-1} \sum_{j=1}^{N} \nabla V^{r_I}_{\alpha\beta}\left(X^{r_I,N}_{\alpha,i}(t) - X^{r_I,N}_{\beta,j}(t)\right) \mathrm{d}t + \sqrt{2\sigma_\alpha}\,\mathrm{d}B_{\alpha,i}(t) \\ X^{r_I,N}_{\alpha,i}(0) = \eta_{\alpha,i}, \qquad \alpha = 1,\dots,n_S, \quad i = 1,\dots,N. \end{cases} \tag{2.3.1}$$

In (2.3.1), $\{X^{r_I,N}_{\alpha,i}\}_{i=1}^N \subset \mathbb{T}^d$ denote positions of particles of species α, $\{\sigma_\alpha\}_{\alpha=1}^{n_S} > 0$ are diffusion constants, $B_{\alpha,i}$ are independent Brownian motions (also independent of $\{X^{r_I,N}_{\alpha,i}(0)\}_{\alpha,i}$), $\{\eta_{\alpha,i}\}_{i=1}^N$ are the particles' initial positions, and the potentials $\{V^{r_I}_{\alpha\beta}\}_{\alpha,\beta=1}^{n_S}$ are defined as standard mass-preserving rescaling of smooth potentials $V_{\alpha\beta}$, namely

$$V^{r_I}_{\alpha\beta}(\cdot) = {r_I}^{-d} V_{\alpha\beta}\left(\cdot/r_I\right). \tag{2.3.2}$$

We refer to Assumption 2.3.7 below for the regularity of the potentials $\{V^{r_I}_{\alpha\beta}\}_{\alpha,\beta=1}^{n_S}$ and the law of the particles' initial positions $\{\eta_{\alpha,i}\}_{i=1}^N$.

For each species α we define the empirical measure

$$\mu^{r_I,N}_{\alpha,t} := N^{-1} \sum_{i=1}^{N} \delta_{X^{r_I,N}_{\alpha,i}(t)}. \tag{2.3.3}$$

In the limit $N \to \infty$, the empirical densities (2.3.3) converge almost surely to the deterministic limit $\overline{\rho}^{r_I}_\alpha$ satisfying the PDE

$$\begin{cases} \partial_t \overline{\rho}^{r_I}_\alpha = \sigma_\alpha \Delta \overline{\rho}^{r_I}_\alpha + \nabla \cdot \left(\sum_{\beta=1}^{n_S} \overline{\rho}^{r_I}_\alpha \nabla V^{r_I}_{\alpha\beta} * \overline{\rho}^{r_I}_\beta \right) \\ \overline{\rho}^{r_I}_\alpha(0) = \overline{\rho}^0_\alpha, \end{cases} \tag{2.3.4}$$

where $\overline{\rho}^0_\alpha$ is a suitable deterministic approximation of the initial particle distribution, see Assumption 2.3.7.

Finally, the fluctuating hydrodynamics Dean–Kawasaki equation capturing the fluctuations of the particle system (2.3.1) on top of the mean-field limit (2.3.4) is precisely (2.1.9), the SPDE of interest for this work.

Remark 2.3.1. As already mentioned, we view the current work in the weakly interacting particle setting (which translates in the mollified potentials $V^{r_I}_{\alpha\beta}$ via the parameter $r_I > 0$) as laying the ground for future applications to the purely local cross-diffusion case (i.e., considering $r_I \to 0$).

2.3.2 Discretisation of the Dean–Kawasaki model

We work with the uniformly spaced grid on the d-dimensional torus $\mathbb{T}^d := [-\pi,\pi)^d$. Specifically, for $L \in 2\mathbb{N}$, we define the spatial discretisation parameter $h := 2\pi/L$, and set $G_{h,d} := h\mathbb{Z}^d \cap \mathbb{T}^d = \{-\pi, -\pi+h, \dots, \pi-h\}^d$. For $m \in \mathbb{N}$, we endow the space $[L^2(G_{h,d})]^m$ with the standard inner product

$$(u_h, v_h)_h := \sum_{x \in G_{h,d}} h^d u_h(x) \cdot v_h(x),$$

and the orthonormal basis $f^m_{x,\ell}(y) := h^{-d/2}\delta_{x,y}e_\ell$ for $(x,\ell) \in (G_{h,d}, \{1,\dots,m\})$, where e_ℓ is the ℓ-th vector of the $\mathbb{R}^d$ canonical basis. If there is no ambiguity, the notation is simplified as $f^m_{x,\ell} \equiv f_x$. Later on, we will implicitly extend this inner product to complex valued functions, in which case the complex conjugate is applied to the second argument.

The natural discrete analogue of the continuous convolution operator $*$ (respectively, $*_c$, see beginning of Section 2.3) with respect to the $L^2(G_{h,d})$-inner product is denoted by $*_h$ (respectively, by $*_{c,h}$). Furthermore, we denote by $\mathcal{I}_h$ the interpolation operator of continuous functions onto $[L^2(G_{h,d})]^m$, meaning that $\mathcal{I}_h f(x) = f(x)$ for every $x \in G_{h,d}$.

As for the discrete differential operators, we use: i) a discrete gradient ∇_h and divergence $\nabla_h\cdot$ based on suitable first-order discrete partial derivatives $[\partial_{h,x_1},\dots,\partial_{h,x_d}]$, and ii) second-order discrete derivates D^2_{h,x_ℓ} satisfying a standard integration by parts rule

$$(D^2_{h,x_\ell}u_h, v_h)_h = -(D_{h,x_\ell}u_h, D_{h,x_\ell}v_h)_h \tag{2.3.5}$$

for some other first-order operators D_{h,x_ℓ}. We denote $\Delta_h := \sum_{\ell=1}^d D^2_{h,x_\ell}$ and $\nabla_{h,D} := [D_{h,x_1},\dots,D_{h,x_d}]$. In general, D_{h,x_ℓ} may differ from ∂_{h,x_ℓ}.

We can now define the discretized Dean–Kawasaki model.

Definition 2.3.2 (Finite difference Dean–Kawasaki model of order $p+1$)**.** We say that the $L^2(G_{h,d})$-valued processes $(\rho^{r_I}_{h,\alpha})_{\alpha=1,\dots n_S}$ solve the finite difference Dean–Kawasaki model if they solve the system of stochastic differential equations

$$\begin{cases} \mathrm{d}(\rho^{r_I}_{h,\alpha}, f_x)_h = \left[\sigma_\alpha(\Delta_h\rho^{r_I}_{h,\alpha}, f_x)_h - \displaystyle\sum_{\beta=1}^{n_S}\left(\rho^{r_I}_{h,\alpha}(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h \rho^{r_I}_{h,\beta}), \nabla_h f_x\right)_h\right]\mathrm{d}t \\ \qquad\qquad - \sqrt{2\sigma_\alpha}N^{-1/2}\displaystyle\sum_{\substack{y\in G_{h,d}\\ l\in\{1,\dots,d\}}}\left(\sqrt{(\rho^{r_I}_{h,\alpha})^+}f^d_{y,l}, \nabla_h f_x\right)_h \mathrm{d}W^\alpha_{(y,l)}, \quad \forall x, \\ \rho^{r_I}_{h,\alpha}(0) = \rho^0_{h,\alpha} \end{cases} \tag{h-DK}$$

on a finite-time horizon $T > 0$, where $(f_x)_{x\in G_{h,d}}$ is the basis from above, and where $\{W^\alpha_{(y,l)}\}^{\alpha=1,\dots,n_S}_{(y,l)\in G_{h,d}\times\{1,\dots,d\}}$ are independent Brownian motions. The assumptions on the random initial datum $\{\rho^0_{h,\alpha}\}_{\alpha=1,\dots,n_S}$ will be given in Assumption 2.3.10 below. Moreover, the families $\{W^\alpha_{(y,l)}\}^{\alpha=1,\dots,n_S}_{(y,l)\in G_{h,d}\times\{1,\dots,d\}}$ and $\{\rho^0_{h,\alpha}\}_{\alpha=1,\dots,n_S}$ are independent.

Analogously to the continuous case, the model (h-DK) captures the fluctuations around the following discretised mean-field limit.

Definition 2.3.3 (Mean-field limit for (h-DK))**.** We say that the $L^2(G_{h,d})$-valued functions $(\overline{\rho}^{r_I}_{h,\alpha})_{\alpha=1,\dots n_S}$ solve the discrete version of the mean-field limit of (2.3.1) if they solve the system of differential equations

$$\begin{cases} \partial_t\overline{\rho}^{r_I}_{h,\alpha} = \sigma_\alpha\Delta_h\overline{\rho}^{r_I}_{h,\alpha} + \nabla_h\cdot\left(\overline{\rho}^{r_I}_{h,\alpha}\displaystyle\sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h \overline{\rho}^{r_I}_{h,\beta}\right) \quad \text{on } G_{h,d}\times(0,T), \\ \overline{\rho}^{r_I}_{h,\alpha}(0) = \overline{\rho}^0_{h,\alpha}, \qquad \alpha = 1,\dots,n_S, \end{cases} \tag{h-MFL}$$

for given deterministic initial datum $\overline{\rho}^0_{h,\alpha}$, see Assumption 2.3.10 and Remark 2.3.13.

2.3.3 Relevant functions spaces

The spaces $\mathcal{L}^q_{pow,r}$ of functions with polynomially growing derivatives

While the definition of the metric d_{-j} takes the supremum over 'generalised moment functions' $\psi \in W^{j,\infty} \cap C^j(\mathbb{R}^K, \mathbb{R})$, the natural spaces for such functions will turn out to be

$$\mathcal{L}^q_{pow,r}(\mathbb{R}^K) \coloneqq \left\{ \psi \in C^q(\mathbb{R}^K);\ \|\psi\|_{\mathcal{L}^q_{pow,r}} \coloneqq \max_{0\le\tilde{q}\le q} \|(1+|\cdot|^2)^{-r/2} D^{\tilde{q}}\psi\|_{L^\infty} < \infty \right\}$$

for $K \in \mathbb{N}$, $q, r \in \mathbb{N}_0$. In particular, it holds $W^{j,\infty} \cap C^j = \mathcal{L}^j_{pow,0}$. The norm $\|\psi\|_{\mathcal{L}^q_{pow,r}}$ and thus the definition of the function space can be extended to complex-valued functions with complex-valued arguments.

The discrete Sobolev norms

We denote the L^2-norm induced by the discrete inner product $(\cdot,\cdot)_h$ on $L^2(G_{h,d})$ by either $\|\cdot\|_{L^2(G_{h,d})}$ or, more succinctly, $\|\cdot\|_{L^2_h}$. Analogously to the continuous setting, we also use the notation $\|g\|_{L^\infty(G_{h,d})} = \|g\|_{L^\infty_h} \coloneqq \max_{x\in G_{h,d}} |g(x)|$.

We will also need a discrete version of (also negative) Sobolev norms: This version uses the first order one-sided finite differences given by

$$\partial^{e_\ell}_{h,1} g(x) = \frac{g(x+he_\ell) - g(x)}{h} \qquad \text{for all } g \in L^2(G_{h,d}), \qquad \ell = 1,\dots,d.$$

Definition 2.3.4 (The discrete Sobolev norms)**.** For $s \in \mathbb{N}_0$ and $g \in L^2(G_{h,d})$ set

$$\|g\|_{H^s(G_{h,d})} = \|g\|_{H^s_h} \coloneqq \sup_{|\nu|\le s} \|\partial^\nu_{h,1} g\|_{L^2(G_{h,d})} \tag{2.3.6}$$

with the supremum over multi-indices $\nu \in \mathbb{N}^d_0$.

Remark 2.3.5. The discrete H^s-norms can be equivalently characterized via the discrete Fourier basis: Let $(\vartheta_m)_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \subset L^2(G_{h,d})$ with $\vartheta_m(x) \coloneqq (2\pi)^{-d/2} e^{im\cdot x}$. Then

$$\|g\|^2_{H^s(G_{h,d})} \approx \sum_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} (1+|m|_2^2)^s \big(g,\vartheta_m\big)_h \overline{\big(g,\vartheta_m\big)_h},$$

where $\overline{z}$ denotes the complex conjugate for $z \in \mathbb{C}$. This stems from the observation that due to the discrete version of the Plancherel theorem

$$\|\partial^{e_\ell}_{h,1} g\|_{L^2(G_{h,d})} = \sum_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \frac{1}{h^2} |e^{im_\ell h} - 1|^2 \big(g,\vartheta_m\big)_h \overline{\big(g,\vartheta_m\big)_h}.$$

This remark leads to a natural definition of the discrete negative Sobolev norms.

Definition 2.3.6 (Discrete negative Sobolev norms)**.** For $\mathbb{Z} \ni s < 0$ and $g \in G_{h,d}$, we set

$$\|g\|^2_{H^s(G_{h,d})} \coloneqq \sum_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} (1+|m|_2^2)^s \big(g,\vartheta_m\big)_h \overline{\big(g,\vartheta_m\big)_h}$$

where $(\vartheta_m)_{m\in\mathbb{Z}^d\cap[-\frac{\pi}{h},\frac{\pi}{h})^d} \subset L^2(G_{h,d})$ with $\vartheta_m(x) \coloneqq e^{im\cdot x}$ is the Fourier basis of $L^2(G_{h,d})$.

2.3.4 Assumptions

Assumption 2.3.7 (Weakly interacting particle system and associated mean-field limit). We consider the weakly interacting particle system $\{X^{r_I,N}_{\alpha,i}\}^{i=1,\dots,N}_{\alpha=1,\dots,n_S}$ as given in (2.3.1), and its associated mean-field limit (2.3.4). In terms of regularity, we assume that

$$V_{\alpha\beta} \in C^{p+3}(\mathbb{T}^d), \qquad \forall \alpha,\beta = 1,\dots,n_S,$$

and, furthermore, that $V_{\alpha\beta}$ is symmetric (i.e., $V_{\alpha\beta}(x) = V_{\alpha\beta}(-x)$). This regularity is passed on to the rescaled potentials $V^{r_I}_{\alpha\beta}$.

We assume a product structure for the underlying probability space

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_B \times \Omega_0, \mathcal{F}_B \otimes \mathcal{F}_0, \mathbb{P}_B \otimes \mathbb{P}_0)$$

with some probability space $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$ for the Brownian motions $(B_{\alpha,i})_{\alpha,i}$ driving the particles independent from some space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ for the initial positions $\left(X^{r_I,N}_{\alpha,i}(0)\right)_{i,\alpha}$. For the initial values of the particles, we assume that either they are i.i.d. according to some probability distribution or that they satisfy a spectral gap inequality in the sense that for any $F \in C^1((\mathbb{R}^d)^N)$ and all $\alpha = 1,\dots,n_S$ there holds

$$\mathbb{E}_0\left[\left|F\left(\left(X^{r_I,N}_{\alpha,i}(0)\right)_i\right) - \mathbb{E}\left[F\left(\left(X^{r_I,N}_{\alpha,i}(0)\right)_i\right)\right]\right|^2\right] \le C\mathbb{E}_0\left[\left|\nabla F\left(\left(X^{r_I,N}_{\alpha,i}(0)\right)_i\right)\right|^2\right]. \tag{2.3.7}$$

For the initial value $\overline{\rho}^{r_I}(0) = \overline{\rho}^0$ of the mean field limit we assume that

$$\left\|\mathbb{E}_0[\mu_0^{r_I,N}] - \overline{\rho}^0\right\|_{H^{-d/2-2}} \le CN^{-1/2}. \tag{2.3.8}$$

Assumption 2.3.8 (Continuous mean field limit). The solution $\{\overline{\rho}^{r_I}_\alpha\}^{n_S}_{\alpha=1}$ to (2.3.4) exists and belongs to $L^\infty(0,T;C^{p+3}(\mathbb{T}^d,\mathbb{R}^{n_S}))$. Furthermore, we assume that there exist $\rho_{min},\rho_{max} \in \mathbb{R}$ such that on $[0,T]$ it holds that $0 < \rho_{min} \le \overline{\rho}^{r_I}_\alpha \le \rho_{max}$ for all $\alpha = 1,\dots,n_S$.

Assumption 2.3.9 (Discrete differential operators). Let $p \in \mathbb{N}$ be fixed. The discrete operators $\partial_{h,x_\ell}, D_{h,x_\ell}, D^2_{h,x_\ell}$ introduced in Subsection 2.3.2 are standard finite difference operators of order $p+1$ (meaning $|\partial_{h,x_\ell}\varphi - \partial_{x_\ell}\varphi| \le \|\varphi\|_{C^{p+2}}h^{p+1}$, and similarly for the other operators). In particular, being of finite difference type, these operators commute. Finally, the two first order differential operators $\partial_{h,x_\ell}, D_{h,x_\ell}$ satisfy the inequality

$$\|\partial^R_{h,x_\ell}g\|^2_{L^2(G_{h,d})} \le \frac{1}{C_D}\|D_{h,x_\ell}g\|^2_{L^2(G_{h,d})}, \qquad \forall g \in L^2(G_{h,d}), \tag{2.3.9}$$

for some $C_D > 0$, where ∂^R_{h,x_ℓ} is the reflected version of ∂_{h,x_ℓ} (i.e., its adjoint) which appears when integrating by parts in the discrete setting.

Assumption 2.3.10 (Discrete initial conditions). The initial conditions $\{\rho^0_{h,\alpha}\}_\alpha$ for (h-DK) and $\{\overline{\rho}^0_{h,\alpha}\}_\alpha$ for (h-MFL) are chosen such that the following properties hold.

- *Initialization via interpolation:* We set $\overline{\rho}^0_h := \mathcal{I}_h[\overline{\rho}^{r_I}(0)] = \mathcal{I}_h[\overline{\rho}^0]$.
- *Positivity and mass restriction:* We assume that the random discrete initial data satisfies $\rho^0_{h,\alpha} \ge 0$ and $\|\rho^0_{h,\alpha}\|_{L^1_h} = h^d\sum_{x\in G_{h,d}}|\rho^0_{h,\alpha}(x)| \le 2$ for all $\alpha = 1,\dots,n_S$.
- *High-order fluctuation bound:* For any $K,q \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $\psi \in \mathcal{L}^q_{pow,r}(\mathbb{R}^K)$, the bound

$$\left|\mathbb{E}\left[\psi\left(N^{1/2}\langle\boldsymbol{\varphi}, \mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\rangle\right)\right] - \mathbb{E}\left[\psi\left(N^{1/2}(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I}))_h(0)\right)\right]\right| \le C\|\psi\|_{\mathcal{L}^q_{pow,r}}h^{p+1}\|\boldsymbol{\varphi}\|^{r+1}_{C^{p+1}} \tag{2.3.10}$$

holds for any $\varphi \in \left[C^{p+1}(\mathbb{T}^d, \mathbb{R}^{n_S})\right]^K$. In (2.3.10), we have used the vectorial notation convention specified in Remark 2.2.4.

- *Exponentially decaying bound for probability of observing 'large fluctuations':* We assume that for any $1 \leq R \leq N^{1/2}h^{d/2}$ it holds that

$$\mathbb{P}\left[\|\rho_h^0 - \overline{\rho}_h^0\|_{L_h^\infty} \geq N^{-1/2}h^{-d/2}R\right] \leq C\exp(-CR), \tag{2.3.11}$$

$$\mathbb{P}\left[\|\rho_h^0 - \overline{\rho}_h^0\|_{H_h^{-\lfloor d/2+1\rfloor}} \geq N^{-1/2}R\right] \leq C\exp(-CR). \tag{2.3.12}$$

Assumption 2.3.11 (Parameter scaling). The parameters (N, h, r_I) scale with respect to the following relations: For a chosen, arbitrarily small $0 < \delta_0 < 1$, we assume that

$$N^{1-\delta_0}h^d \geq 1, \tag{2.3.13}$$

$$N^{\delta_0(T+1)}h \leq 1, \tag{2.3.14}$$

$$r_I^{-2(d+2)} \leq \log N. \tag{2.3.15}$$

The scaling regime (2.3.15) is analogous to that of [Che+21, Theorem 3].

Remark 2.3.12 (Ad Assumptions 2.3.7–2.3.8). The regularity of $\{\overline{\rho}_\alpha^{r_I}\}_{\alpha=1}^{n_S}$ prescribed in Assumption 2.3.8 can be met on any time interval $[0, T]$, for instance, if the initial value for the mean field limit $\overline{\rho}_\alpha^0$ belongs to H^s, $s > \max\{d/2+1; p+3+(d/2+1)\}$ and satisfies a smallness assumption $\|\overline{\rho}_\alpha^0\|_{H^s} \leq \{\min_\alpha \sigma_\alpha\}/[C(s,d)\sum_{\alpha,\beta}\|V_{\alpha\beta}\|_{L^1}]$ [CDJ19]. See also [Che+21] for an alternative setting.

Remark 2.3.13 (Ad Assumption 2.3.10). Since $\overline{\rho}_h^{r_I}(0) = \mathcal{I}_h[\overline{\rho}^{r_I}(0)]$, for order $p+1$ finite difference operators $(\partial_h^{e_\ell})_{\ell=1,\dots,d}$ we have

$$\sup_{|\nu|\leq s} \|\mathcal{I}_h[\partial^\nu \overline{\rho}^{r_I}(0)] - \partial_h^\nu \overline{\rho}_h^{r_I}(0)\|_{L_h^2} \leq C\|\overline{\rho}^{r_I}(0)\|_{C^{s+p+2}}h^{p+1}, \tag{2.3.16}$$

where the supremum is over multiindices $\nu \in \mathbb{N}_0^d$ provided $\overline{\rho}_h^{r_I}(0)$ is regular enough.

Remark 2.3.14 (Positivity and Boundedness of the discrete mean field limit). Analogously to the continuous Sobolev inequality, in the discrete setting there exists $C_S > 0$ such that $\|g\|_{L_h^\infty} \leq C_S\|g\|_{H^s}$ for all $s > d/2$, $g \in L^2(G_{h,d})$. In particular, in light of Proposition 2.B.7, as long as $\overline{\rho}^{r_I} \in L^\infty(0, T; C^{p+3+\lfloor d/2\rfloor+1})$, there exist $\rho_{min,h}, \rho_{max,h}$ such that for all $N > N_0$ and $h < h_0$ it holds that on $[0, T]$

$$0 < \rho_{min,h} \leq \overline{\rho}_{h,\alpha}^{r_I} \leq \rho_{max,h} \quad \text{for all } \alpha = 1, \dots, n_S,$$

for some $N_0 = N_0(p, data)$, where $h_0 = h_0(\|\overline{\rho}^{r_I}\|_{C^{\lfloor d/2\rfloor+2}}, \rho_{min})$.

Remark 2.3.15 (Constructing the discrete initial data). The assumptions for the initial random distribution ρ_h^0 (Assumption 2.3.10) may seem extensive at first glance, but such discrete initial data can be rather naturally constructed from the continuous initial data. The most straightforward example is a pathwise approach based on an interpolation scheme – each realisation of ρ_h^0 is derived from the initial particle realisation $\mu_0^{r_I,N}$. Essentially, the mass of a particle starting at $X_{\alpha,i}^{r_I,N}(0)$ is split across surrounding grid points according to interpolation weights. These weights stem from an interpolation scheme of sufficiently high order that is used to approximate functions at $X_{\alpha,i}^{r_I,N}(0)$ based on the function values at the grid points.

2.4 Strategy of the proof of Theorem 2.2.1: an informal view

In this section, we spell out the main ideas behind the proof of Theorem 2.2.1 in an informal way. All arguments will be made precise and rigorous later on.

2.4.1 The induction step

The proof of Theorem 2.2.1 is of inductive type, as we now detail.

In order the compare the fluctuations of the discrete Dean–Kawasaki solution ρ_h (see (h-DK)) and of the particle empirical density $\mu^{r_I,N}$ (see (2.3.3)), we choose a set of times $\boldsymbol{T} = (T_k)_{k=1}^K$, regular enough test functions $\boldsymbol{\varphi} = (\varphi_{k,\alpha})_{k=1,\dots,K}^{\alpha=1,\dots,n_S}$ and set

$$\begin{aligned} \boldsymbol{\zeta}^T &:= N^{1/2}\big\langle \boldsymbol{\varphi}, \mu^{r_I,N}_{\boldsymbol{T}} - \overline{\rho}^{r_I}(\boldsymbol{T})\big\rangle \in \mathbb{R}^K, \\ \boldsymbol{\zeta}^T_h &:= N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(\boldsymbol{T}\wedge T_\oslash)\big)_h \in \mathbb{R}^K, \end{aligned} \tag{2.4.1}$$

which is a shorthand vectorial notation (over the indexes $k = 1,\dots,K$ and $\alpha = 1,\dots,n_S$) for the random variables in (2.2.3), as anticipated in Remark 2.2.4. The role and definition of the stopping time $T_\oslash$ will be discussed in due course.

We measure the distance between $\boldsymbol{\zeta}^T$ and $\boldsymbol{\zeta}^T_h$ using the Wasserstein-type metric d_{-j}-metric (2.2.1), namely

$$d_{-j}[\boldsymbol{\zeta}^T, \boldsymbol{\zeta}^T_h] := \sup_{\psi:\ \max_{1\le\tilde{\jmath}\le j}\|D^{\tilde{\jmath}}\psi\|_{L^\infty}\le 1} \big|\mathbb{E}\big[\psi(\boldsymbol{\zeta}^T)\big] - \mathbb{E}\big[\psi(\boldsymbol{\zeta}^T_h)\big]\big|.$$

Furthermore, we define the shorthand notation[1]

$$\mathcal{M}(\psi,\boldsymbol{\varphi}) := \mathbb{E}\big[\psi(\boldsymbol{\zeta}^T)\big] - \mathbb{E}\big[\psi(\boldsymbol{\zeta}^T_h)\big]. \tag{2.4.2}$$

The proof of Theorem 2.2.1 crucially revolves around essentially obtaining the following relation via the Itô calculus

$$\mathbb{E}\big[\,\mathrm{d}\mathcal{M}(\psi,\boldsymbol{\varphi})\big] \propto N^{-1/2+2\epsilon}\cdot\mathbb{E}\big[\mathcal{M}(\tilde{\psi},\tilde{\boldsymbol{\varphi}})\big]\,\mathrm{d}t + \mathrm{dErr}_\oslash + \mathrm{dErr}_{num}, \tag{2.4.3}$$

where the parameter $\epsilon > 0$ is small (so that $2\epsilon < 1/2$). The bound (2.4.3) has an iterative component, as the right-hand-side contains – yet – another object of kind $\mathcal{M}$, with modified test functions $(\tilde{\psi},\tilde{\boldsymbol{\varphi}})$ derived from the original pair $(\psi,\boldsymbol{\varphi})$. Additionally, there are a numerical error Err_{num}, which will show to be of the type

$$\mathrm{Err}_{num} \propto N^{2\epsilon+\kappa}h^{p+1}, \tag{2.4.4}$$

and a modelling error

$$\mathrm{Err}_\oslash \propto N^{2\epsilon+\kappa}\exp\big(-CN^{\epsilon/2}\big), \tag{2.4.5}$$

where, again, the parameter $\kappa > 0$ is small enough. The role of the small parameters ϵ,κ will be clarified throughout the proofs, and we need not discuss it in this summary.

By iterating (2.4.3) over $\mathcal{M}$, one cumulates as many (small) prefactors $N^{-1/2+2\epsilon}$ as steps performed, together with numerical and modelling errors (2.4.4)–(2.4.5). The number of iterative steps one performs is only capped by the regularity of the initial test functions $\psi,\boldsymbol{\varphi}$ (such regularity usually deteriorates from step to step). Once the regularity of the test functions is exhausted and (2.4.3) is inapplicable, one closes off the argument by obtaining the – final – fluctuation contribution in (2.2.4). The precise details concerning (2.4.3)–(2.4.5) are given in Theorem 2.5.1, which, in turn, relies on several other ingredients which we now list.

[1] we use a slight abuse of notation, as $\psi(\boldsymbol{\zeta}^T)$ and $\psi(\boldsymbol{\zeta}^T_h)$ live in different probability spaces.

Choice of dynamics for test functions φ and $\mathcal{I}_h\varphi$

The particle system (2.3.1) satisfies a crucial property: as the particles are driven by independent Brownian motions, it is easy to see that the cross variation (denoted by square brackets) of the quantities $\langle\varphi_1, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle$ and $\langle\varphi_2, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle$ (these are the 'building blocks' for ζ^T in (2.4.1)) is, for sufficiently regular φ_1, φ_2, given by

$$\begin{aligned}&\big[\langle\varphi_1, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle, \langle\varphi_2, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle\big]\\ &\quad= N^{-1}\langle\nabla\varphi_1\cdot\nabla\varphi_2, \mu^{r_I,N}\rangle\\ &\quad= N^{-1}\langle\nabla\varphi_1\cdot\nabla\varphi_2, \mu^{r_I,N} - \overline{\rho}^{r_I}\rangle + N^{-1}\langle\nabla\varphi_1\cdot\nabla\varphi_2, \overline{\rho}^{r_I}\rangle =: P_1 + P_2.\end{aligned} \tag{2.4.6}$$

With the exception of term P_2 (which we will deal with at a later stage), (2.4.6) amounts to saying that the cross-variation preserves linear functionals of $\mu^{r_I,N} - \overline{\rho}^{r_I}$: this fact is crucial, as it plays directly into the iterative structure of $\mathcal{M}$ in (2.4.2).

In light of (2.4.6), it is convenient to define time-dependent test functions ϕ which:

- coincide with the original test functions φ at the evaluation times $\boldsymbol{T}$ (i.e., $\phi^{\boldsymbol{T}} = \varphi$), and
- reduce the deterministic drift of the Itô differential for ζ^T in (2.4.1) 'as much as possible', as such deterministic drift cannot easily be treated using (2.4.6).

The latter requirement leads to the derivation of the backward evolution for ϕ in Lemma 2.5.2. Noticeably, the only deterministic drift term which survives this cancellation effort is

$$Q_t^{r_I,N}(\phi^t) := \sum_{\alpha,\beta=1}^{n_S}\big\langle\nabla\phi_\alpha^t\cdot\big(\nabla V_{\alpha\beta}^{r_I} * \big(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\big)\big), \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\big\rangle. \tag{2.4.7}$$

The structure of $Q^{r_I,N}$, which appears as a compensation term after linearising the convolution nonlinearity, is – at least not yet – compatible with the structure of $\mathcal{M}$, and will be dealt with in Subsection 2.4.1 below.

A totally analogous discussion also applies for suitable discretisations of (2.1.9): Being statistically equivalent to the particle system (2.3.1), the Dean–Kawasaki model (2.1.9) enjoys the 'mesoscopic analogue' of (2.4.6). Namely, testing (2.1.9) with smooth enough φ_1, φ_2 and integrating by parts, one finds the noise cross-variation to be (2.1.10), which we recall here:

$$\Big[N^{-1/2}\int_{\mathbb{T}^d}\sqrt{\rho_\alpha}\xi\cdot\nabla\varphi_1, N^{-1/2}\int_{\mathbb{T}^d}\sqrt{\rho_\alpha}\xi\cdot\nabla\varphi_2\Big] = N^{-1}\int_{\mathbb{T}^d}\rho_\alpha\nabla\varphi_1\cdot\nabla\varphi_2.$$

Our chosen discretisation for the Dean–Kawasaki model (h-DK) preserves (2.1.10) on the discrete level: Namely, for any $\alpha = 1, \cdots, n_S$ and $\varphi_{1,h}, \varphi_{2,h} \in L^2(G_{h,d})$, we have

$$\begin{aligned}&\Big[\big(\varphi_{1,h}, \rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I}\big)_h, \big(\varphi_{2,h}, \rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I}\big)_h\Big]\\ &= \Bigg[N^{-1/2}\sum_{y\in G_{h,d},\, l\in\{1,\dots,d\}}\Big(\sqrt{(\rho_{h,\alpha}^{r_I})^+}f_{y,l}^d, \nabla_h\varphi_{1,h}\Big)_h\,\mathrm{d}W_{(y,l)}^\alpha,\\ &\qquad N^{-1/2}\sum_{y\in G_{h,d},\, l\in\{1,\dots,d\}}\Big(\sqrt{(\rho_{h,\alpha}^{r_I})^+}f_{y,l}^d, \nabla_h\varphi_{2,h}\Big)_h\,\mathrm{d}W_{(y,l)}^\alpha\Bigg]\\ &\quad= N^{-1}(\nabla_h\varphi_{1,h}\cdot\nabla_h\varphi_{2,h}, \rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})_h + N^{-1}(\nabla_h\varphi_{1,h}\cdot\nabla_h\varphi_{2,h}, \overline{\rho}_{h,\alpha}^{r_I})_h\\ &\qquad + N^{-1}(\nabla_h\varphi_{1,h}\cdot\nabla_h\varphi_{2,h}, (\rho_{h,\alpha}^{r_I})^-)_h =: M_1 + M_2 + M_3.\end{aligned} \tag{2.4.8}$$

With the exception of the terms M_2 and M_3 (which will be treated in Subsections 2.4.1 and 2.4.1), (2.4.8) shows that the cross-variation preserves linear functionals of $\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha}$. In the same way as in the continuous setting, we define a suitable backwards evolution ϕ_h for $\phi_h^{\boldsymbol{T}} = \mathcal{I}_h\boldsymbol{\varphi}$, see Lemma 2.5.4. Taking all species and test functions into account in the definition of ζ_h^T in (2.4.1), the only deterministic drift term which survives is given by

$$Q^{r_I,N}_{h,t}(\phi^t_h) := \sum_{\alpha,\beta=1}^{n_S} \left(\nabla_h \phi^t_{h,\alpha} \cdot (\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t)), (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t)\right)_h. \tag{2.4.9}$$

This term, which is the discrete analogous of $Q^{r_I,N}_t(\phi^t)$ above, and which also does not yet fit the structure of $\mathcal{M}$, is treated in Subsection 2.4.1 below.

The generalised moment structure ψ

The Itô analysis conducted in Subsection 2.4.1 shows that – aside from the mean-field contributions P_2, M_2 and the mismatch $\rho^+_{h,\alpha} \leftrightarrow \rho_{h,\alpha}$ in M_3 – the iterative structure in $\mathcal{M}$ is preserved at the level of noise cross-variations. In order to include the mean-field contributions P_2, M_2, we also need to define a suitable backwards equation for the generalised moment function ψ: This analysis (see Lemma 2.5.5) is somewhat complementary to the derivation of the dynamics for the test functions ϕ discussed earlier. The discussion is tailored to the continuous case associated with the particle system, within the adjustments needed for the discrete case treated in Lemma 2.5.6.

Comparing the generalised moments, the iterative structure

In this step we take the difference of the contributions of the Itô differential on microscopic and SPDE level (discussed in the Subsections 2.4.1 and 2.4.1), and we derive the bound

$$\begin{aligned}\mathbb{E}\left[d\mathcal{M}(\psi^t, \boldsymbol{\phi}^t)\right] \propto N^{-1/2+2\epsilon} \cdot \mathbb{E}\left[\mathcal{M}(\widetilde{\psi}^t, \widetilde{\boldsymbol{\phi}}^t)\right] dt \\ + \mathrm{dErr}_{num} + \mathrm{dErr}_{\oslash} + \mathrm{d}\left(\mathrm{Err}_{lin,a} - \mathrm{Err}_{lin,b}\right).\end{aligned} \tag{2.4.10}$$

The terms $Err_{lin,a}, Err_{lin,b}$ compensate for linearising the convolutional nonlinearity. Note however, that in this form they are not yet suitable for the purposes of iterating. Estimating them without further treatment would yield an order $N^{-1/2}$ error: They are roughly given by

$$\begin{aligned}\mathrm{dErr}_{lin,a} &\propto N^{1/2}\mathbb{E}\left[Q^{r_I,N}_t(\boldsymbol{\phi}^t)\right] \mathrm{d}t, \\ \mathrm{dErr}_{lin,b} &\propto N^{1/2}\mathbb{E}\left[Q^{r_I,N}_{h,t\wedge T_\oslash}(\boldsymbol{\phi}^t_h)\right] \mathrm{d}t,\end{aligned}$$

where $Q^{r_I,N}_t$ and $Q^{r_I,N}_{h,t}$ are the surviving drift terms from Subsection 2.4.1 as defined in (2.4.7) and (2.4.9) respectively. Crucially, in (2.4.10), the mismatch $\rho_h \leftrightarrow \rho^+_h$ (cfr. M_3 in (2.4.8)) has been resolved using the stopping time $T_\oslash$: the definition of such a stopping time and Remark 2.3.14 entail the non-negativity of $\rho_h(t)$ for all $t \le T_\oslash$. A suitable estimate concerning the smallness of $P[T_\oslash < T]$ (i.e., (2.2.2)) is proved separately in Proposition 2.8.2, and justifies the bound (2.4.5). Additionally, the error term Err_{num} in (2.4.10) keeps track of several numerical approximations, including:

- difference of continuous test functions ϕ and discrete counterparts ϕ_h,
- difference of continuous mean-field limit $\overline{\rho}^{r_I}$ and discrete counterpart $\overline{\rho}^{r_I}_h$, and

- difference of initial conditions between the particle system (2.3.1) and the discrete Dean–Kawasaki model (h-DK), see Assumption 2.3.10 and Appendixes 2.A–2.C.

The argument for this subsection is spelled out in Proposition 2.5.7.

Linearising the convolution contributions

Here, we linearise the terms $Err_{lin,a}$, $Err_{lin,b}$ introduced above in order to make them suitable for the iteration. The linearisation is carried our using a $2d$-Fourier argument, which is proved in a separate result (Proposition 2.5.9), and discussed in more detail in Subsection 2.4.2 below.

Closing the estimate

All arguments carried out so far are used to cumulate as many iteration steps as possible (depending on the initial regularity of the initial test functions φ, ψ). When the iteration can no longer be performed, the remaining terms (other than Err_{num}) are bounded using quantitative convergence bounds to the mean-field limit (proven in Section 2.7), and the properties of the stopping time $T_{\oslash}$ (proven in Section 2.8).

2.4.2 Technical challenges

We highlight the four main technical challenges which we address in order to achieve the proof's building blocks sketched in Subsections 2.4.1–2.4.1.

Quantitative convergence to the mean-field limit

In several points of the argument, we need a quantitative bound on the convergence to the mean-field limit in both continuous and discrete setting. Specifically, in Proposition 2.7.1 we prove bounds of the type

$$\begin{aligned}&\big\|\mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\big\|_{H^{-d/2-2}}\\ &\quad\leq C(T+1)\exp\Big(CTr_I^{-(d+2)}\Big)\Big(\big\|\mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\big\|_{H^{-d/2-2}} + \mathcal{C}N^{-1/2}\Big)\end{aligned}$$

for a random variable $\mathcal{C}$ with Gaussian moments $\mathbb{E}\big[\exp(C\mathcal{C}^2)\big] \leq 3$. For the proof, we follow the classical strategy of introducing a coupled auxiliary particle system, where the interaction is calculated with respect to the mean-field limit, see e.g. [Szn91, Theorem 1.4]. The corresponding estimate on the discrete level is a by-product of Section 2.8 below.

Linearisation argument for convolutional nonlinearity

Linearising the quadratic terms $Q_t^{r_I,N}(\phi^t), Q_{h,t}^{r_I,N}(\phi_h^t)$ (given in (2.4.7) and (2.4.9)) essentially revolves around rewriting them into an infinite sum of suitable objects. If the interaction potential is regular enough, the Fourier expansion allows to rewrite the function $(x,y)\mapsto \partial_\ell V_{\alpha\beta}^{r_I}(x-y)\colon \mathbb{T}^{2d}\to\mathbb{R}$ as $\partial_\ell V_{\alpha\beta}^{r_I}(x-y) = \sum_{m,n\in\mathbb{Z}^d}\hat{F}_{k,\alpha\beta}^{\ell}[m,n]e^{im\cdot x}e^{in\cdot y}$, effectively separating

the variables x, y in each addend of the sum. We use this to rewrite $Q_t^{r_I,N}(\phi^t)$ as

$$Q_t^{r_I,N}(\phi^t) \propto \sum_{\alpha,\beta=1}^{n_S} \sum_{\ell=1}^{d} \sum_{n,m\in\mathbb{Z}^d} \hat{F}^{\ell}_{k,\alpha\beta}[m,n] \left(N^{1/2} \sum_{\gamma=1}^{n_S} \left\langle \delta^{\beta\gamma} \vartheta_m, \mu_{\gamma,t}^{r_I,N} - \overline{\rho}_{\gamma}^{r_I}(t) \right\rangle \right) \times \left(N^{1/2} \sum_{\gamma=1}^{n_S} \left\langle \delta^{\alpha\gamma} \partial_\ell \phi_\alpha^t \vartheta_n, \mu_{\gamma,t}^{r_I,N} - \overline{\rho}_{\gamma}^{r_I}(t) \right\rangle \right),$$

where we have set $\vartheta_n(x) := e^{in\cdot x}$, and where $\delta^{\cdot,\cdot}$ is the usual Kronecker delta. The above expression is now compatible with the structure of $\mathcal{M}$, as the nonlinearity has been 'split' – at the expense of having an infinite sum – in two suitable stand-alone contributions (in the round brackets). Analogous discussions are applicable for $Q_{h,t\wedge T_\oslash}^{r_I,N}(\phi_h^t)$.

Estimate for probability of large fluctuations

The exponentially decaying estimate (2.2.2) for $P[T_\oslash < T]$ is centered around proving Gaussian type moment bounds for the quantities $\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I}$. For this purpose, two different stopping times arguments are needed: i) an L^∞-type stopping time, which controls the size of the stochastic noise, and ii) a $H_h^{-2\lfloor d/2\rfloor-2}$-type stopping time (for which the estimates in Subsection 2.4.2 are needed), which allows to linearise the contribution of the quadratic convolutional nonlinearity.

Compatible discrete setting

A number of auxiliary results are needed in order to substantiate our numerical approximations. The two main difficulties here are related to:

- quantifying the difference of relevant continuous functions (namely, the nonlinear mean-field limit and test functions) and bounding their discretised counterparts in higher-order Sobolev norms, see Appendixes 2.A–2.B, and;
- constructing a compatible initial profile of fluctuations in the case of high order operators (i.e., $p > 1$): we use arguments from polynomial interpolation theory, see Appendix 2.C.

2.5 The key step - setting up the iterative structure

In this section we formalize the arguments outlined in Subsection 2.4.1. That is, we compute and compare the Itô formulas (given by Itô's rule) for

$$\mathbb{E}[\psi(\boldsymbol{\zeta}^T)] = \mathbb{E}\big[\psi\big(N^{1/2}\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle\big)\big],$$
$$\mathbb{E}[\psi(\boldsymbol{\zeta}_h^T)] = \mathbb{E}\big[\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\big)\big].$$

Theorem 2.5.1 below is the integrated, more precise version of (2.4.3), that also provides bounds on $\widetilde{\psi}$, $\widetilde{\phi}$, Err_{num}, and Err_{neg}. On the first read of this section, the reader may wish to simply focus on the structure of the proofs and the comparison of the various blocks therein, and skip the quantitative estimates (such estimates rely on the results from the subsequent Section 2.7, Section 2.8, Appendix 2.A and Appendix 2.B).

Theorem 2.5.1 (Iterative Structure). *Let $0 < T$. On $[0,T]$ let*

- $\mu^{r_I,N} = (\mu_\alpha^{r_I,N})_{\alpha=1}^{n_S}$ *as in* (2.3.3) *be the empirical measures of the particle system* (2.3.1) *satisfying Assumption* 2.3.7,
- $\overline{\rho}^{r_I}$ *as in* (2.3.4) *be the intermediate mean-field limit satisfying Assumption* 2.3.8,
- $\rho_h^{r_I}$ *solve the discretised Dean–Kawasaki model* (h-DK) *satisfying Assumption* 2.3.9,
- $\overline{\rho}_h^{r_I}$ *be the finite difference mean-field limit given in* (h-MFL)

Let Assumption 2.3.10 *(concerning the initial conditions) hold. Let the parameters* r_I, N, h *obey the scaling in Assumption* 2.3.11. *Now let* $K \in \mathbb{N}$, $q, r \in \mathbb{N}_0$, $q \geq 3$, *and* $s \in \mathbb{N}$, *satisfying*

$$s > p + 3d/2 + 5. \tag{2.5.1}$$

Let $\psi \in \mathcal{L}^q_{pow,r}(\mathbb{R}^K)$, $\boldsymbol{\varphi} = (\varphi_{k,\alpha})_{k=1,\dots,K}^{\alpha=1,\dots,n_S} \in [H^s(\mathbb{T}^d, \mathbb{R}^{n_S})]^K$, $\boldsymbol{T} = (T_1, \dots, T_K) \in [0,T]^K$. *Additionally, assume that* $\overline{\rho}^{r_I} \in L^\infty\left(0, T; [C^s(\mathbb{T}^d)]^{n_S}\right)$ *and* $V \in W^{s_I,1}(\mathbb{T}^d)$ *for* $s_I \in \mathbb{N}$ *with*

$$s_I > 2d + 1 + (r+2)\lceil d/2 + 1 \rceil. \tag{2.5.2}$$

Let $0 < \epsilon < \delta_0/4$, *and let* $T_\oslash = T_\oslash(\epsilon/2, \epsilon)$ *be the associated stopping time as defined in* (2.8.2) *at the start of Section* 2.8 *for* $\delta = \epsilon/2$.

Then the following facts hold. First, for all $t \in (0,T)$, $k, \tilde{k} \in \{1, \dots, K\}$, $m, n \in \mathbb{Z}^d$, *and* $\alpha, \beta \in \{1, \dots, n_S\}$ *as well as* $\ell \in \{1, \dots, d\}$ *there exist generalised moment functions*

$$\psi^0 \in \mathcal{L}^q_{pow,r}(\mathbb{R}^K), \quad \widetilde{\psi}^t_{k\tilde{k}} \in \mathcal{L}^{q-2}_{pow,r+1}(\mathbb{R}^{K+1}), \quad \breve{\psi}^t_k \in \mathcal{L}^{q-1}_{pow,r+2}(\mathbb{R}^K \times \mathbb{C}^2),$$

and coefficients $\hat{F}^\ell_{k,\alpha\beta}[m,n] \in \mathbb{C}$ *with sets of test functions*

$$\boldsymbol{\phi}^0 \in [H^s(\mathbb{T}^d, \mathbb{R}^{n_S})]^K, \quad , \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}} \in [H^{s-1}(\mathbb{T}^d, \mathbb{R}^{n_S})]^{K+1},$$
$$\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell} \in [H^s(\mathbb{T}^d, \mathbb{R}^{n_S})]^K \times [H^s(\mathbb{T}^d, \mathbb{C}^{n_S})]^2,$$

and test times $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$, $\breve{\boldsymbol{T}}_k \in [0,T]^{K+2}$ *such that*

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(N^{1/2}\langle \boldsymbol{\varphi}, \mu^{r_I,N}_{\boldsymbol{T}} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle\right)\right] \\
&\quad = \mathbb{E}\left[\psi^0\left(N^{1/2}\langle \boldsymbol{\phi}^0, \mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\rangle\right)\right] \\
&\qquad + N^{-1/2}\sum_{k,\tilde{k}=1}^{K}\int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\left[\widetilde{\psi}^t_{k\tilde{k}}\left(N^{1/2}\langle \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, \mu^{r_I,N}_{t\wedge \tilde{\boldsymbol{T}}_{k\tilde{k}}} - \overline{\rho}^{r_I}(t \wedge \tilde{\boldsymbol{T}}_{k\tilde{k}})\rangle\right)\right] \mathrm{d}t \\
&\qquad + N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^\ell_{k,\alpha\beta}[m,n] \\
&\qquad\quad \times \mathbb{E}\left[\breve{\psi}^t_k\left(N^{1/2}\langle \breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}, \mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k} - \overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\rangle\right)\right] \mathrm{d}t,
\end{aligned} \tag{2.5.3}$$

as well as

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\right)\right] \\
&\quad=\mathbb{E}\left[\psi^0\left(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^0],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(0)\big)_h\right)\right] \\
&\quad+N^{-\frac{1}{2}}\sum_{k,\tilde{k}=1}^{K}\int_0^{T_k\wedge T_{\tilde{k}}}\mathbb{E}\Big[\widetilde{\psi}^t_{k\tilde{k}}\big(N^{1/2}(\mathcal{I}_h\big[\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}\big],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\tilde{\boldsymbol{T}}_{k\tilde{k}}\wedge T_\oslash))_h\big)\Big]\,\mathrm{d}t \\
&\quad+N^{-\frac{1}{2}}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n] \\
&\qquad\times\mathbb{E}\left[\breve{\psi}^t_k\left(N^{1/2}\big(\mathcal{I}_h[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\big)_h\right)\right]\mathrm{d}t \\
&\quad+\mathrm{Err}_{neg}+\mathrm{Err}_{num}+\mathrm{Err}_\oslash.
\end{aligned}\tag{2.5.4}$$

Second, for any $\kappa>0$ there exists $N_0=N_0(\kappa,\epsilon,r,s,K,data)$, such that for all $N>N_0$ the following estimates hold. The stopping time satisfies

$$\mathbb{P}\left(T_\oslash<T\right)\leq C\exp\big(-CN^{\epsilon/2}\big).\tag{2.5.5}$$

For $t\in(0,T)$, $k,\tilde{k}\in\{1,\dots,K\}$, $m,n\in\mathbb{Z}^d$, the bounds

$$\|\psi^0\|_{\mathcal{L}^q_{pow,r}}\leq CN^\kappa\left(1+\|\boldsymbol{\varphi}\|^r_{H^s}\right)\|\psi\|_{\mathcal{L}^q_{pow,r}},\tag{2.5.6}$$

$$\|\widetilde{\psi}^t_{k\tilde{k}}\|_{\mathcal{L}^{q-2}_{pow,r+1}}\leq CN^\kappa\left(1+\|\boldsymbol{\varphi}\|^r_{H^s}\right)\|\psi\|_{\mathcal{L}^q_{pow,r}},\tag{2.5.7}$$

$$\|\breve{\psi}^t_k\|_{\mathcal{L}^{q-1}_{pow,r+2}}\leq CN^\kappa\left(1+\|\boldsymbol{\varphi}\|^r_{H^s}\right)\|\psi\|_{\mathcal{L}^q_{pow,r}}\tag{2.5.8}$$

hold. For all $t\in(0,T)$, $k\in\{1,\dots,K\}$, $m,n\in\mathbb{Z}^d$, $\alpha,\beta\in\{1,\dots,n_S\}$ and $\ell\in\{1,\dots,d\}$ the coefficients $\hat{F}^t_{k,\alpha\beta}[m,n]$ are subject to

$$|\hat{F}^t_{k,\alpha\beta}[m,n]|\leq CN^\kappa(|m|_2+|n|_2)^{-s_I+1}.\tag{2.5.9}$$

The test functions are subject to

$$\|\boldsymbol{\phi}^0\|_{H^s}\leq CN^\kappa\|\boldsymbol{\varphi}\|_{H^s},\tag{2.5.10}$$

$$\|\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}\|_{H^{s-1}}\leq CN^\kappa\left(1+\|\boldsymbol{\varphi}\|_{H^s}\right)\|\boldsymbol{\varphi}\|_{H^s},\tag{2.5.11}$$

$$\|\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}\|_{H^{s-1}}\leq CN^\kappa\left(1+\|\boldsymbol{\varphi}\|_{H^s}\right)\left(1+|m|_\infty^{s-1}\vee|n|_\infty^{s-1}\right).\tag{2.5.12}$$

Furthermore, $\mathrm{Err}_\oslash$, Err_{neg} and Err_{num} can be estimated via

$$|\mathrm{Err}_\oslash|\leq C\|\psi\|_{\mathcal{L}^2_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|^{2r+2}_{H^s}\big)N^{r\epsilon+\kappa}\exp\left(-CN^{\epsilon/2}\right),\tag{2.5.13}$$

$$|\mathrm{Err}_{neg}|=0,\tag{2.5.14}$$

$$|\mathrm{Err}_{num}|\leq C\|\psi\|_{\mathcal{L}^3_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|^{2r+3}_{H^s}\big)N^{(r+1)\epsilon+\kappa}h^{p+1}.\tag{2.5.15}$$

Proof. The result will follow from Proposition 2.5.7 and Proposition 2.5.9. □

2.5.1 The backwards evolution equations for the test functions

We first look at the dynamics of the empirical measure and then the solution of the discretized Dean–Kawasaki equation. This will lead to a choice for the backwards evolution for the test functions in both the continuous and discrete setting, summarized in Lemma 2.5.2 and Lemma 2.5.4 respectively.

Dynamics of the empirical measure

Taking a set of generic time-dependent test functions $\eta_\alpha\colon [0,T]\times\mathbb{T}^d\to\mathbb{R}$, writing $\eta_\alpha^t = \eta_\alpha(t,\cdot)$, we apply the Itô rule and calculate

$$\begin{aligned}\mathrm{d}\langle \eta_\alpha^t, \mu_{\alpha,t}^{r_I,N}\rangle &= \left\langle \partial_t\eta_\alpha^t + \sigma_\alpha\Delta\eta_\alpha^t - \nabla\eta_\alpha^t\cdot\left(\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \mu_{\beta,t}^{r_I,N}\right), \mu_{\alpha,t}^{r_I,N}\right\rangle \mathrm{d}t \\ &\quad + \frac{\sqrt{2\sigma_\alpha}}{N}\sum_{i=1}^{N}\nabla\eta_\alpha^t(X_{\alpha,i}^{r_I,N}(t))\,\mathrm{d}B_{\alpha,i}(t).\end{aligned}$$

We linearise the interaction induced term around the mean field limit, that is

$$\begin{aligned}&\left\langle \nabla\eta_\alpha^t\cdot\left(\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \mu_{\beta,t}^{r_I,N}\right), \mu_{\alpha,t}^{r_I,N}\right\rangle \\ &\quad= \left\langle \nabla\eta_\alpha^t\cdot\left(\sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}(t)\right), \mu_{\alpha,t}^{r_I,N}\right\rangle \\ &\qquad+ \sum_{\beta=1}^{n_S}\left\langle \nabla\eta_\alpha^t\cdot\left(\nabla V_{\alpha\beta}^{r_I} * \left(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\right)\right), \overline{\rho}_\alpha^{r_I}(t)\right\rangle + Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t),\end{aligned}$$

where $Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t)$ is the quadratic linearisation compensation given by

$$Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t) \coloneqq \sum_{\beta=1}^{n_S}\left\langle \nabla\eta_\alpha^t\cdot\left(\nabla V_{\alpha\beta}^{r_I} * \left(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\right)\right), \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\right\rangle.$$

Introducing the notation $U_\alpha^{r_I}(t) \coloneqq \sum_{\beta=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I}(t)$, and by subtracting the mean field limit equation (2.3.4), we obtain

$$\begin{aligned}\mathrm{d}\left\langle \eta_\alpha^t, \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\right\rangle &= \left\langle \partial_t\eta_\alpha^t + \sigma_\alpha\Delta\eta_\alpha^t - U_\alpha^{r_I}(t)\cdot\nabla\eta_\alpha^t, \mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\right\rangle \mathrm{d}t \\ &\quad - \sum_{\beta=1}^{n_S}\left\langle \nabla\eta_\alpha^t\cdot\left(\nabla V_{\alpha\beta}^{r_I} * \left(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\right)\right), \overline{\rho}_\alpha^{r_I}(t)\right\rangle \mathrm{d}t \\ &\quad - Q_{\alpha,t}^{r_I,N}(\eta_\alpha^t)\,\mathrm{d}t + \frac{\sqrt{2\sigma_\alpha}}{N}\sum_{i=1}^{N}\nabla\eta_\alpha^t(X_{\alpha,i}^{r_I,N}(t))\,\mathrm{d}B_{\alpha,i}(t).\end{aligned} \tag{2.5.16}$$

Recalling the definition of $*_c$ (cfr. Section 2.3) and summing over all species, we obtain

$$\begin{aligned}&\sum_{\alpha=1}^{n_S}\sum_{\beta=1}^{n_S}\left\langle \nabla\eta_\alpha^t\cdot\left(\nabla V_{\alpha\beta}^{r_I} * \left(\mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\right)\right), \overline{\rho}_\alpha^{r_I}(t)\right\rangle \\ &\quad= -\sum_{\beta=1}^{n_S}\left\langle \sum_{\alpha=1}^{n_S}\nabla V_{\alpha\beta}^{r_I} *_c \left(\overline{\rho}_\alpha^{r_I}(t)\nabla\eta_\alpha^t\right), \mu_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}(t)\right\rangle,\end{aligned}$$

where we used $V^{r_I}_{\alpha\beta}(-x) = V^{r_I}_{\alpha\beta}(x)$ for $x \in \mathbb{T}^d$ (cfr. Assumption 2.3.7). Switching the species indices for this term and setting $Q^{r_I,N}_t(\eta^t) = \sum_{\alpha=1}^{n_S} Q^{r_I,N}_{\alpha,t}(\eta^t_\alpha)$ we have

$$\begin{aligned}
&\mathrm{d}\Big(\sum_{\alpha=1}^{n_S} \langle \eta^t_\alpha, \mu^{r_I,N}_{\alpha,t} - \overline{\rho}^{r_I}_\alpha(t)\rangle\Big) \\
&\quad = \sum_{\alpha=1}^{n_S} \Big\langle \partial_t \eta^t_\alpha + \sigma_\alpha \Delta \eta^t_\alpha - U^{r_I}_\alpha(t)\cdot\nabla\eta^t_\alpha + \sum_{\beta=1}^{n_S} \nabla V^{r_I}_{\beta\alpha} *_c \left(\overline{\rho}^{r_I}_\beta(t)\nabla\eta^t_\beta\right), \mu^{r_I,N}_{\alpha,t} - \overline{\rho}^{r_I}_\alpha(t)\Big\rangle \,\mathrm{d}t \\
&\qquad\qquad - Q^{r_I,N}_t(\eta^t)\,\mathrm{d}t + \sum_{\alpha=1}^{n_S} \frac{\sqrt{2\sigma_\alpha}}{N} \sum_{i=1}^{N} \nabla\eta^t_\alpha(X^{r_I,N}_{\alpha,i}(t))\,\mathrm{d}B_{\alpha,i}(t). \qquad (2.5.17)
\end{aligned}$$

Thus, we can read off the choice for the backwards evolution, which kills as much of the deterministic drift as possible, and obtain Lemma 2.5.2 below.

Lemma 2.5.2 (The backwards evolution equation for the test functions)**.** *Under the assumptions of Theorem 2.5.1, let $\varphi = (\varphi_\alpha)_\alpha \in H^s(\mathbb{T}^d, \mathbb{R}^{n_S})$, $s > 2 + d/2$. Then there exists a unique $\phi \in C^1(0,T; H^s(\mathbb{T}^d, \mathbb{R}^{n_S}))$ solving*

$$\begin{cases} -\partial_t \phi^t_\alpha = \sigma_\alpha \Delta \phi^t_\alpha - U^{r_I}_\alpha(t)\cdot\nabla\phi^t_\alpha + \sum_{\beta=1}^{n_S} \nabla V^{r_I}_{\beta\alpha} *_c \left(\overline{\rho}^{r_I}_\beta(t)\nabla\phi^t_\beta\right) \quad \text{on } \mathbb{T}^d \times (0,T), \\ \phi^T_\alpha = \varphi_\alpha, \qquad \alpha = 1,\dots,n_S, \end{cases} \qquad (2.5.18)$$

where $\phi^t = \phi(t,\cdot)$ and for $\alpha = 1,\dots,n_S$, $t \in [0,T]$,

$$U^{r_I}_\alpha(t) := \sum_{\beta=1}^{n_S} \nabla V^{r_I}_{\alpha\beta} * \overline{\rho}^{r_I}_\beta(t). \qquad (2.5.19)$$

Furthermore, it holds that

$$\begin{aligned}
&\mathrm{d}\Big(\sum_{\alpha=1}^{n_S} \langle \phi^t_\alpha, \mu^{r_I,N}_{\alpha,t} - \overline{\rho}^{r_I}_\alpha(t)\rangle\Big) \\
&\qquad = -Q^{r_I,N}_t(\phi^t)\,\mathrm{d}t + \sum_{\alpha=1}^{n_S} \frac{\sqrt{2\sigma_\alpha}}{N} \sum_{i=1}^{N} \nabla\phi^t_\alpha(X^{r_I,N}_{\alpha,i}(t))\,\mathrm{d}B_{\alpha,i}(t), \qquad (2.5.20)
\end{aligned}$$

where

$$Q^{r_I,N}_t(\phi^t) := \sum_{\alpha,\beta=1}^{n_S} \langle \nabla\phi^t_\alpha \cdot \left(\nabla V^{r_I}_{\alpha\beta} * \left(\mu^{r_I,N}_{\beta,t} - \overline{\rho}^{r_I}_\beta(t)\right)\right), \mu^{r_I,N}_{\alpha,t} - \overline{\rho}^{r_I}_\alpha(t)\rangle. \qquad (2.5.21)$$

Proof. For existence, uniqueness and regularity, see Lemma 2.A.1. Then (2.5.20) immediately follows from (2.5.17), since the regularity is sufficient for the calculations to hold. □

Remark 2.5.3. Note that the reason, for which we are not able to kill all of the deterministic drift by evolving the test functions backwards, is the underlying non-linearity of the system. Specifically, the linearization compensation $Q^{r_I,N}_t(\cdot)$ associated with the nonlinear interaction terms of the cross-diffusion system is the only part of the drift that survives.

Dynamics of the solutions to the discretized Dean–Kawasaki model

The following is the discrete version of Lemma 2.5.2 (recall the definitions of $*_h, *_{h,c}$ from Section 2.3).

Lemma 2.5.4 (The backwards evolution equation for the discrete test functions)**.** *Under the assumptions of Theorem 2.5.1, let $\varphi_h = (\varphi_{h,\alpha})_\alpha \in L^2(G_{h,d}, \mathbb{R}^{n_S})$. Then there exists a unique $\phi_h \in C^1(0,T; L^2(G_{h,d}, \mathbb{R}^{n_S}))$ solving*

$$\begin{cases} -\partial_t \phi^t_{h,\alpha} = \sigma_\alpha \Delta_h \phi^t_{h,\alpha} - U^{r_I}_{h,\alpha} \cdot \nabla_h \phi^t_{h,\alpha} + \sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\overline{\rho}^{r_I}_{h,\beta} \nabla_h \phi^t_{h,\beta}), \\ \phi^T_{h,\alpha} = \varphi_{h,\alpha}, \qquad \alpha = 1, \dots, n_S, \end{cases} \tag{2.5.22}$$

where for $\alpha = 1, \dots, n_S$, $t \in [0,T]$, we have set

$$U^{r_I}_{h,\alpha}(t) := \sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h \overline{\rho}^{r_I}_{h,\beta}. \tag{2.5.23}$$

Furthermore, it holds for $t \in [0,T]$ that

$$\begin{aligned} &\mathrm{d}\Big(\sum_{\alpha=1}^{n_S} (\phi^t_{h,\alpha}, (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t))_h \Big) \\ &\quad = -Q^{r_I,N}_h(\phi^t_h)\,\mathrm{d}t \\ &\qquad - \sum_{\alpha=1}^{n_S} \sqrt{2\sigma_\alpha} N^{-1/2} \sum_{\substack{y \in G_{h,d} \\ l \in \{1,\dots,d\}}} \Big(\sqrt{(\rho^{r_I}_{h,\alpha}(t))^+} f^d_{y,l}, \nabla_h \phi^t_{h,\alpha} \Big)_h \mathrm{d}W^\alpha_{(y,l)}, \end{aligned} \tag{2.5.24}$$

where $\{f^d_{y,l}\}$ is the basis of $L^2(G_{h,d}, \mathbb{R}^d)$ defined in Subsection 2.3.2 and we have set

$$Q^{r_I,N}_{h,t}(\phi^t_h) := \sum_{\alpha,\beta=1}^{n_S} \big(\nabla_h \phi^t_{h,\alpha} \cdot (\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t)), (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t) \big)_h. \tag{2.5.25}$$

Proof. We first take a set of generic discrete test functions $\eta_{h,\alpha} \colon [0,T] \times G_{h,d} \to \mathbb{R}$ which are differentiable in time. Since $\{f_x\}_{x \in G_{h,d}}$ as defined in Section 2.3.2 is an orthonormal basis of $L^2(G_{h,d})$, we expand $\eta_{h,\alpha} = \sum_{x \in G_{h,d}} (\eta_{h,\alpha}, f_x)_h f_x$. Applying the Itô rule to

$$(\eta_{h,\alpha}, \rho^{r_I}_{h,\alpha})_h(t) = \sum_{x \in G_{h,d}} (\eta_{h,\alpha}(t), f_x)_h (\rho^{r_I}_{h,\alpha}(t), f_x)_h$$

using (h-DK), and the expansion $\nabla_h \eta_{h,\alpha} = \sum_{x \in G_{h,d}} (\eta_{h,\alpha}, f_x)_h \nabla_h f_x$, we obtain

$$\begin{aligned} \mathrm{d}(\eta_{h,\alpha}, \rho^{r_I}_{h,\alpha})_h = &\Big[(\partial_t \eta_{h,\alpha}, \rho^{r_I}_{h,\alpha})_h + \sigma_\alpha (\eta_{h,\alpha}, \Delta_h \rho^{r_I}_{h,\alpha})_h \\ &\quad - \sum_{\beta=1}^{n_S} \big(\nabla_h \eta_{h,\alpha}, \rho^{r_I}_{h,\alpha} (\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h \rho^{r_I}_{h,\beta}) \big)_h \Big] \mathrm{d}t \\ &\quad - \sqrt{2\sigma_\alpha} N^{-1/2} \sum_{\substack{y \in G_{h,d} \\ l \in \{1,\dots,d\}}} \Big(\sqrt{(\rho^{r_I}_{h,\alpha})^+} f^d_{y,l}, \nabla_h \eta_{h,\alpha} \Big)_h \mathrm{d}W^\alpha_{(y,l)}. \end{aligned}$$

Following the same steps as in Lemma 2.5.2, we get the discrete analogue to (2.5.17). Inserting the test functions evolving according to the discrete backwards evolution (2.5.22), we obtain (2.5.24). The existence and uniqueness of the solution to (2.5.22) follows from linearity. □

2.5.2 The generalised moment structure

In 2.5.2 we analyze the generalised moment structure in the continuous case, highlighting how we choose the backwards evolution for the generalised moment functions. In 2.5.2 we calculate the resulting generalised moment structure for the discrete setting. Finally, in 2.5.2 we will compare the continuous and discrete setting, collecting and estimating the error terms from adjusting the discrete terms to fit the iterative structure. By defining as well as estimating $\widetilde{\psi}^t_{k\tilde{k}}$, $\widetilde{\phi}^t_{k\tilde{k}}$, and $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$ we complete the first big step towards proving Theorem 2.5.1.

Deriving the generalised moment structure in the continuous setting

Lemma 2.5.5. *Let the assumptions of Theorem 2.5.1 hold. Then there exists a unique set of functions $\boldsymbol{\phi} = (\phi_1, \dots, \phi_K) \in \left[C^1(0,T; H^s(\mathbb{T}^d, \mathbb{R}^{n_S}))\right]^K$ such that*

$$\begin{cases} -\partial_t \phi^t_{k,\alpha} = \sigma_\alpha \Delta \phi^t_{k,\alpha} - U^{r_I}_\alpha(t) \cdot \nabla \phi^t_{k,\alpha} + \sum_{\beta=1}^{n_S} \nabla V^{r_I}_{\beta\alpha} *_c \left(\overline{\rho}^{r_I}_\beta(t) \nabla \phi^t_{k,\beta}\right), & t < T_k, \\ \phi^t_{k,\alpha} = \varphi_{k,\alpha}, \quad t \in [T_k, T] \end{cases} \tag{2.5.26}$$

for all $\alpha = 1, \dots, n_S$ and each $k = 1, \dots, K$. Further, there exists a unique backwards evolution of ψ in $C^1(0,T; \mathcal{L}^q_{pow,r}(\mathbb{R}^K))$ satisfying

$$\begin{cases} -\partial_t \psi^t = \sum_{k,\tilde{k}=1}^{K} \chi_{t \le T_k \wedge T_{\tilde{k}}} \sum_{\alpha=1}^{n_S} \sigma_\alpha \left\langle \nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha}, \overline{\rho}^{r_I}_\alpha(t) \right\rangle \partial_k \partial_{\tilde{k}} \psi^t \quad \text{on } \mathbb{R}^K \times (0,T), \\ \psi^T = \psi. \end{cases} \tag{2.5.27}$$

Then, it holds that

$$\begin{aligned} &\mathbb{E}\left[\psi\left(N^{1/2}\left\langle \boldsymbol{\varphi}, \mu^{r_I,N}_{\boldsymbol{T}} - \overline{\rho}^{r_I}(\boldsymbol{T})\right\rangle\right)\right] \\ &= \mathbb{E}\left[\psi^0\left(N^{1/2}\left\langle \boldsymbol{\phi}^0, \mu^{r_I,N}_0 - \overline{\rho}^{r_I}(0)\right\rangle\right)\right] \\ &+ N^{-1/2} \sum_{k,\tilde{k}=1}^{K} \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\Bigg[\partial_k \partial_{\tilde{k}} \psi^t \left(N^{1/2}\left\langle \boldsymbol{\phi}^t, \mu^{r_I,N}_{t\wedge \boldsymbol{T}} - \overline{\rho}^{r_I}(t \wedge \boldsymbol{T})\right\rangle\right) \\ &\qquad \times N^{1/2} \sum_{\alpha=1}^{n_S} \left\langle \sigma_\alpha \nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha}, \mu^{r_I,N}_{\alpha,t} - \overline{\rho}^{r_I}_\alpha(t)\right\rangle\Bigg] \mathrm{d}t \\ &- N^{1/2} \sum_{k=1}^{K} \int_0^{T_k} \mathbb{E}\left[\partial_k \psi^t \left(N^{1/2}\left\langle \boldsymbol{\phi}^t, \mu^{r_I,N}_{t\wedge \boldsymbol{T}} - \overline{\rho}^{r_I}(t \wedge \boldsymbol{T})\right\rangle\right) Q^{r_I,N}_t(\phi^t_k)\right] \mathrm{d}t. \end{aligned} \tag{2.5.28}$$

Additionally, there exists $N_0 = N_0(\kappa, r, K, data)$ such that for all $0 \le \tilde{q} \le q$ and $t \in [0,T]$

$$\|\psi^t\|_{\mathcal{L}^{\tilde{q}}_{pow,r}} \le C N^\kappa \left(1 + \|\varphi\|^r_{H^s}\right) \|\psi\|_{\mathcal{L}^{\tilde{q}}_{pow,r}}. \tag{2.5.29}$$

Proof. The existence, uniqueness and regularity of ϕ is settled as in Lemma 2.5.2. For the existence, uniqueness and regularity of ψ^t see [CF23, Proof of Proposition 6, Step 1]. These arguments applied for every $0 \le \tilde{q} \le q$ yield

$$\begin{aligned} \|\psi^t\|_{\mathcal{L}^{\tilde{q}}_{pow,r}} &\le C\left(1 + \|\boldsymbol{\phi}\|^r_{L^\infty(0,T;W^{1,\infty}(\mathbb{T}^d))}\right) \|\psi\|_{\mathcal{L}^{\tilde{q}}_{pow,r}} \\ &\le C\left(1 + \|\boldsymbol{\phi}\|^r_{L^\infty(0,T;H^{\lceil (d+3)/2 \rceil}(\mathbb{T}^d))}\right) \|\psi\|_{\mathcal{L}^{\tilde{q}}_{pow,r}}. \end{aligned}$$

Now (2.5.29) follows from $s \geq \lceil (d+3)/2 \rceil$ and Lemma 2.A.1. We now turn to poving (2.5.28). In what follows, the regularity of ϕ and ψ^t is sufficient for the calculations to hold.

For the convenience of notation we set

$$\zeta^t := N^{1/2} \big\langle \boldsymbol{\phi}^t, \mu^{r_I,N}_{t\wedge \boldsymbol{T}} - \overline{\rho}^{r_I}(t \wedge \boldsymbol{T}) \big\rangle.$$

Using this notation, we rewrite (2.5.20) from Lemma 2.5.2 for $k = 1, \dots, K$ as

$$\mathrm{d}\zeta^t_k = -\chi_{t\leq T_k} N^{1/2} Q^{r_I,N}_t(\phi^t_k)\,\mathrm{d}t + \mathrm{d}M^t_k$$

with martingale part M^t_k given by

$$\mathrm{d}M^t_k := \chi_{t\leq T_k} \sum_{\alpha=1}^{n_S} \frac{\sqrt{2\sigma_\alpha}}{N^{1/2}} \sum_{i=1}^{N} \nabla \phi^t_{k,\alpha}(X^{r_I,N}_{\alpha,i}(t))\,\mathrm{d}B_{\alpha,i}(t).$$

Thus, as outlined in 2.4.6, we observe the linearity preserving property of the cross-variation (see Subsection 2.4.1, specifically (2.4.6)), which yields for $k, \tilde{k} \in \{1, \dots, K\}$ that

$$\begin{aligned} \mathrm{d}\left[M^t_k, M^t_{\tilde{k}}\right] &= \chi_{t\leq T_k}\chi_{t\leq T_{\tilde{k}}} \sum_{\alpha=1}^{n_S} \frac{2\sigma_\alpha}{N} \sum_{i=1}^{N} \nabla\phi^t_{k,\alpha}(X^{r_I,N}_{\alpha,i}(t)) \cdot \nabla\phi^t_{\tilde{k},\alpha}(X^{r_I,N}_{\alpha,i}(t))\,\mathrm{d}t \\ &= \chi_{t\leq T_k\wedge T_{\tilde{k}}} \sum_{\alpha=1}^{n_S} 2\sigma_\alpha \big\langle \nabla\phi^t_{k,\alpha} \cdot \nabla\phi^t_{\tilde{k},\alpha}, \mu^{r_I,N}_{\alpha,t} \big\rangle\,\mathrm{d}t. \end{aligned} \tag{2.5.30}$$

Combining these observations with the Itô rule and taking the expected value yields

$$\begin{aligned} \mathrm{d}\mathbb{E}\big[\psi^t(\zeta^t)\big] = \mathbb{E}\big[\partial_t\psi^t(\zeta^t)\big]\,\mathrm{d}t - \mathbb{E}\Bigg[\sum_{k=1}^{K} \partial_k\psi^t(\zeta^t)\chi_{t\leq T_k} N^{1/2} Q^{r_I,N}_t(\phi^t_k)\Bigg]\,\mathrm{d}t \\ + \mathbb{E}\Bigg[\sum_{k,\tilde{k}=1}^{K} \partial_k\partial_{\tilde{k}}\psi^t(\zeta^t)\chi_{t\leq T_k\wedge T_{\tilde{k}}} \sum_{\alpha=1}^{n_S} \sigma_\alpha \big\langle \nabla\phi^t_{k,\alpha} \cdot \nabla\phi^t_{\tilde{k},\alpha}, \mu^{r_I,N}_{\alpha,t} \big\rangle\Bigg]\,\mathrm{d}t. \end{aligned}$$

Thinking of the term with $Q^{r_I,N}_t(\phi^t_k)$ as a linearisation error, we see that (2.5.27) is the clear choice for the backwards evolution of ψ^t if we want the first two terms to fit the iterative structure, that is combining linear functionals of $\mu^{r_I,N} - \overline{\rho}^{r_I}$. Plugging (2.5.27) in and integrating the equation in time yields (2.5.28). □

Deriving the generalised moment structure in the discrete setting

Analogously to Lemma 2.5.5 we obtain the following result in the discrete case. However, since the backwards evolution for ψ^t is tailored for the continuous case, the generalised moment structure does not immediately show the iterative structure. In Subsection 2.5.2 we will estimate the cost of the adjustments necessary in order to obtain this structure.

Lemma 2.5.6. *Let the assumptions of Theorem 2.5.1 hold. Then there exists a unique set of functions $\phi_h = (\phi_{h,1}, \dots, \phi_{h,K}) \in [C^1(0,T; L^2(G_{h,d}, \mathbb{R}^{n_S}))]^K$ solving the respective discrete backwards evolution equation for test functions, that is*

$$\begin{cases} -\partial_t\phi^t_{h,k,\alpha} = \sigma_\alpha\Delta_h\phi^t_{h,k,\alpha} - U^{r_I}_{h,\alpha} \cdot \nabla_h\phi^t_{h,k,\alpha} + \displaystyle\sum_{\beta=1}^{n_S} \mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\overline{\rho}^{r_I}_{h,\beta}\nabla_h\phi_{h,k,\beta}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t < T_k, \\ \phi^t_{h,k,\alpha} = \mathcal{I}_h[\varphi_{k,\alpha}] \quad \text{for } t \in [T_k, T] \end{cases} \tag{2.5.31}$$

for all $\alpha = 1, \dots, n_S$ and each $k = 1, \dots, K$. Further, let $t \to \psi^t$ with $\psi^T = \psi$ be the solution of (2.5.27) from Lemma 2.5.5. As assumed in Theorem 2.5.1, let $T_\oslash = T_\oslash(\epsilon/2, \epsilon)$ be the stopping time as defined in (2.8.2). Then

$$\begin{aligned}
&\mathbb{E}\big[\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T} \wedge T_\oslash)\big)_h\big)\big] \\
&= \mathbb{E}\bigg[\psi^0\bigg(N^{1/2}\big(\boldsymbol{\phi}_h^0, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\big)_h\bigg)\bigg] \\
&+ \sum_{k,\tilde{k}=1}^{K} \sum_{\alpha=1}^{n_S} \sigma_\alpha \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E}\bigg[\partial_k \partial_{\tilde{k}} \psi^t \bigg(N^{1/2}\big(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash)\big)_h\bigg) \\
&\qquad \times \Big(\big(\nabla_h \phi_{h,k,\alpha}^t \cdot \nabla_h \phi_{h,\tilde{k},\alpha}^t, (\rho_{h,\alpha}^{r_I})^+(t \wedge T_\oslash)\big)_h - \big\langle \nabla \phi_{k,\alpha}^t \cdot \nabla \phi_{\tilde{k},\alpha}^t, \overline{\rho}_\alpha^{r_I}(t \wedge T_\oslash)\big\rangle\Big)\bigg] \mathrm{d}t \\
&- N^{1/2} \sum_{k=1}^{K} \int_0^{T_k} \mathbb{E}\bigg[\partial_k \psi^t \bigg(N^{1/2}\big(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash)\big)_h\bigg) Q_{h,t\wedge T_\oslash}^{r_I,N}(\phi_{h,k}^t)\bigg] \mathrm{d}t. + \mathrm{Err}_\oslash.
\end{aligned} \tag{2.5.32}$$

For any $\kappa > 0$, there exists $N_0 = N_0(\kappa, \epsilon, r, s, K, \text{data})$, such that $\mathrm{Err}_\oslash$ is subject to (2.5.13) for all $N > N_0$.

Proof. The existence, uniqueness and regularity for ϕ_h is as in Lemma 2.5.4. The regularity is sufficient for the following calculations to hold.

Step 1: (2.5.32) and defining the error term. We denote

$$\zeta_h^t := N^{1/2}\big(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \boldsymbol{T} \wedge T_\oslash)\big)_h.$$

Rewriting (2.5.24) in Lemma 2.5.4 with this notation yields

$$\mathsf{d}\zeta_{h,k}^{t\wedge T_\oslash} = -\chi_{t \le T_k \wedge T_\oslash} N^{1/2} Q_{h,t}^{r_I,N}(\phi_{h,k}^t)\,\mathrm{d}t + \mathsf{d}M_{h,k}^{t\wedge T_\oslash}$$

with martingale part $M_{h,k}$ given by

$$\mathsf{d}M_{h,k}^{t\wedge T_\oslash} := -\chi_{t \le T_k \wedge T_\oslash} \sum_{\alpha=1}^{n_S} \sqrt{2\sigma_\alpha} \sum_{\substack{y \in G_{h,d} \\ l \in \{1,\dots,d\}}} \Big(\sqrt{(\rho_{h,\alpha}^{r_I}(t))^+}\, f_{y,l}^d, \nabla_h \phi_{h,k,\alpha}^t\Big)_h \mathrm{d}W_{(y,l)}^\alpha(t).$$

The cross variation terms with $k, \tilde{k} \in \{1, \dots, K\}$, as detailed in Subsection 2.4.1 (specifically (2.4.8)), are given by

$$\mathsf{d}\left[M_{h,k}^{t\wedge T_\oslash}, M_{h,\tilde{k}}^{t\wedge T_\oslash}\right] = \chi_{t \le T_k \wedge T_{\tilde{k}} \wedge T_\oslash} \sum_{\alpha=1}^{n_S} 2\sigma_\alpha \big(\nabla_h \phi_{h,k,\alpha}^t \cdot \nabla_h \phi_{h,\tilde{k},\alpha}^t, (\rho_{h,\alpha}^{r_I}(t))^+\big)_h.$$

Now as in the continuous case we have

$$\begin{aligned}
\mathsf{d}\psi^t(\zeta_h^{t\wedge T_\oslash}) &= \partial_t \psi^t(\zeta_h^{t\wedge T_\oslash})\,\mathrm{d}t - \sum_{k=1}^{K} \partial_k \psi^t(\zeta_h^{t\wedge T_\oslash}) \chi_{t \le T_k \wedge T_\oslash} N^{1/2} Q_h^{r_I,N}(\phi_{h,k}^t)\,\mathrm{d}t \\
&+ \sum_{k,\tilde{k}=1}^{K} \partial_k \partial_{\tilde{k}} \psi^t(\zeta_h^{t\wedge T_\oslash}) \chi_{t \le T_k \wedge T_{\tilde{k}} \wedge T_\oslash} \sum_{\alpha=1}^{n_S} \sigma_\alpha \big(\nabla_h \phi_{h,k,\alpha}^t \cdot \nabla_h \phi_{h,\tilde{k},\alpha}^t, (\rho_{h,\alpha}^{r_I}(t))^+\big)_h \,\mathrm{d}t \\
&\qquad + \sum_{k=1}^{K} \partial_k \psi^t(\zeta_h^{t\wedge T_\oslash})\,\mathrm{d}M_{h,k}^{t\wedge T_\oslash}.
\end{aligned} \tag{2.5.33}$$

Integrating in time from 0 to $T_\oslash$, plugging in (2.5.27) and then taking the expected value we obtain (2.5.32) where the error term is given by $\mathrm{Err}_\oslash = \mathrm{Err}_{\oslash,1} + \mathrm{Err}_{\oslash,2}$. The first error term $\mathrm{Err}_{\oslash,1}$ stems from the correction $(\psi^{T_\oslash}, \boldsymbol{\phi}_h^{T_\oslash}) \to (\psi^T, \boldsymbol{\phi}_h^T) = (\psi, \boldsymbol{\varphi})$ given by

$$\mathrm{Err}_{\oslash,1} \coloneqq \mathbb{E}\Big[\chi_{T_\oslash<T}\Big(\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\big) \\ - \psi^{T_\oslash}\big(N^{1/2}\big(\boldsymbol{\phi}_h^{T_\oslash}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\big)\Big)\Big],$$

while $\mathrm{Err}_{\oslash,2}$ comes from extending the integration to T, i.e. $\chi_{t\le T_\oslash} \to 1$, that is

$$\begin{aligned}\mathrm{Err}_{\oslash,2} \coloneqq &-\sum_{k,\tilde{k}=1}^{K}\sum_{\alpha=1}^{n_S}\sigma_\alpha\int_0^{T_k\wedge T_{\tilde{k}}}\mathbb{E}\Big[\chi_{T_\oslash<t\le T}\partial_k\partial_{\tilde{k}}\psi^t(\boldsymbol{\zeta}_h^t)\\ &\times\Big(\big(\nabla_h\phi_{h,k,\alpha}^t\cdot\nabla_h\phi_{h,\tilde{k},\alpha}^t, (\rho_{h,\alpha}^{r_I})^+(t\wedge T_\oslash)\big)_h - \big\langle\nabla\phi_{k,\alpha}^t\cdot\nabla\phi_{\tilde{k},\alpha}^t, \overline{\rho}_\alpha^{r_I}(t\wedge T_\oslash)\big\rangle\Big)\Big]\,\mathrm{d}t\\ &+N^{1/2}\sum_{k=1}^{K}\int_0^{T_k}\mathbb{E}\Big[\chi_{T_\oslash<t\le T}\partial_k\psi^t(\boldsymbol{\zeta}_h^t)Q_{h,t\wedge T_\oslash}^{r_I,N}(\phi_{h,k}^t)\Big]\,\mathrm{d}t.\end{aligned}$$

Step 2: Estimating the error terms. For the following estimates we assume N to be large enough to use the suitable auxiliary results. Additionally, we will rely on the definition of the stopping time $T_\oslash$, see (2.8.2) for $0<\epsilon<\delta_0/4$ and $\delta=\epsilon/2$ as fixed in the assumptions of Theorem 2.5.1. Due to the definition of the $\mathcal{L}^q_{pow,r}$-spaces we have

$$\begin{aligned}&\big|\psi\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\big)\big|\\ &\quad\le\|\psi\|_{\mathcal{L}^0_{pow,r}}\big(1+\big|N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\big|^2\big)^{r/2}\\ &\quad\le\|\psi\|_{\mathcal{L}^0_{pow,r}}\big(1+\|\mathcal{I}_h[\boldsymbol{\varphi}]\|^2_{H_h^{d+2}}N\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\|^2_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big)^{r/2}\\ &\quad\le\|\psi\|_{\mathcal{L}^0_{pow,r}}\big(1+C\|\boldsymbol{\varphi}\|^2_{C^{d+2}}\big(N^{2\epsilon}+N\chi_{T_\oslash=0}\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|^2_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big)\big)^{r/2}\end{aligned}$$

where the last step follows from the definition of the stopping time $T_\oslash$. Additionally plugging in (2.B.11) from Lemma 2.B.8 to estimate $\|\boldsymbol{\phi}_h^{T_\oslash}\|_{H^{d+2}(G_{h,d})}$ and (2.5.29) from Lemma 2.5.5 to bound $\|\psi^{T_\oslash}\|_{\mathcal{L}^q_{pow,r}}$, we obtain that – as long as N is large enough – for any small $\kappa>0$

$$\begin{aligned}&\psi^{T_\oslash}\big(N^{1/2}\big(\boldsymbol{\phi}_h^{T_\oslash}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\big)\\ &\quad\le C\|\psi\|_{\mathcal{L}^0_{pow,r}}N^{\kappa}\big(1+\|\boldsymbol{\varphi}\|^r_{H^s(\mathbb{T}^d)}\big)\\ &\qquad\times\Big(1+N^{\kappa}\|\boldsymbol{\varphi}\|^r_{C^{d+2}(\mathbb{T}^d)}\big(N^{r\epsilon}+\chi_{T_\oslash=0}N^{r/2}\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|^r_{H_h^{-2(\lfloor d/2\rfloor+1)}}\big)\Big).\end{aligned}$$

Since Assumption (2.5.1) implies $s>3d/2+2$, we have $\|\boldsymbol{\varphi}\|_{C^{d+3}}\le C\|\boldsymbol{\varphi}\|_{H^s}$ via Sobolev embedding and thus get

$$\begin{aligned}\mathrm{Err}_{\oslash,1}\le{}&\mathbb{P}(T_\oslash<T)C\|\psi\|_{\mathcal{L}^0_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)N^{r\epsilon+\kappa}\\ &+\mathbb{E}\Big[\chi_{T_\oslash=0}\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|^r_{H_h^{-2(\lfloor d/2\rfloor+1)}}\Big]C\|\psi\|_{\mathcal{L}^0_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)N^{r/2}.\end{aligned}\tag{2.5.34}$$

For now ignoring the case $T_\oslash=0$, plugging in the bound for $\mathbb{P}(T_\oslash<T)$ from Proposition 2.8.2, we obtain (2.5.13).

With respect to the case $T_\oslash = 0$, due to Assumption 2.3.10, the mass restriction and positivity of $\rho_h^{r_I}(0)$, we obtain the very rough estimate

$$\|\rho_h^{r_I}(0)\|_{H_h^{-2(\lfloor d/2\rfloor+1)}} \leq \|\rho_h^{r_I}(0)\|_{L_h^2} \leq \|\rho_h^{r_I}(0)\|_{L_h^\infty} \leq h^{-d}\|\rho_h^{r_I}(0)\|_{L_h^1} \leq Ch^{-d}.$$

Furthermore, since $\overline{\rho}_h^{r_I}(0) = \mathcal{I}_h[\overline{\rho}^{r_I}(0)]$, we also get

$$\|\overline{\rho}_h^{r_I}(0)\|_{H_h^{-2(\lfloor d/2\rfloor+1)}} \leq \|\overline{\rho}_h^{r_I}(0)\|_{L_h^2} \leq C\|\overline{\rho}^{r_I}(0)\|_{C^0}.$$

Both are controlled by the bound for $\mathbb{P}[T_\oslash = 0] \leq \mathbb{P}[T_\oslash < 0]$: In particular

$$\mathbb{P}\big[T_\oslash = 0\big] \leq C\exp\big(-CN^{\epsilon/2}\big) \leq CN^{-r/2}\left(h^{-d} + \|\overline{\rho}^{r_I}(0)\|_{C^1}^{-r}\right)^{-r}\exp\big(-C/2N^{\epsilon/2}\big),$$

where the second inequality is due to the scaling regime, see Assumption 2.3.11, for N large enough (depending on $\|\overline{\rho}^{r_I}(0)\|_{C^1}$). Plugging these estimates into (2.5.34) we obtain the bound given in (2.5.13).

Concerning $\mathrm{Err}_{\oslash,2}$, with the integration we can ignore the case $T_\oslash = 0$. For $0 < t < T_\oslash$ via Remark 2.3.14 we have

$$\|(\rho_{h,\alpha}^{r_I})^+(t \wedge T_\oslash)\|_{L_h^\infty} \leq \|(\overline{\rho}_h^{r_I})\|_{L_h^\infty} + N^{-\epsilon} \leq C\rho_{max,h}, \tag{2.5.35}$$

and using Corollary 2.B.2, Lemma 2.B.8, and that $\|\nabla_h f\|_{L_h^2} \leq C\|f\|_{H_h^1}$ for $f \in L^2(G_{h,d})$,

$$\|\nabla_h\phi_{h,k,\alpha}^t \cdot \nabla_h\phi_{h,\tilde{k},\alpha}^t\|_{L_h^1} \leq C\|\boldsymbol{\phi}_h^t\|_{H_h^1}^2 \leq CN^\kappa\|\mathcal{I}_h[\boldsymbol{\varphi}]\|_{H_h^1}^2 \leq CN^\kappa\|\boldsymbol{\varphi}\|_{H^s}^2. \tag{2.5.36}$$

For the other term, note that

$$\begin{aligned}\left|N^{1/2}Q_{h,t\wedge T_\oslash}^{r_I,N}(\phi_{h,k}^t)\right| &\leq CN^{\epsilon+\kappa}\|\nabla_h\boldsymbol{\phi}_h^t\|_{L_h^1}\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \\ &\leq C\|\boldsymbol{\varphi}\|_{H^s}N^\kappa,\end{aligned} \tag{2.5.37}$$

where for the first inequality we estimated the convolution pointwise via the definition of $T_\oslash$. Together with Assumption 2.3.8 we obtain

$$\mathrm{Err}_{\oslash,2} \leq C\mathbb{P}(T_\oslash < T)\|\psi\|_{\mathcal{L}_{pow,r}^2}(1 + \|\boldsymbol{\varphi}\|_{H^s}^r)\|\boldsymbol{\varphi}\|_{H^s}^r N^{r\epsilon+\kappa}(\|\boldsymbol{\varphi}\|_{H^s}^2 + \|\boldsymbol{\varphi}\|_{H^s}).$$

Combining this with the bound for $\mathbb{P}(T_\oslash < T)$, the scaling from Assumption 2.3.11 and the estimate for $\mathrm{Err}_{\oslash,1}$ we obtain (2.5.13). □

Comparing the generalised moments, the iterative structure

Now, comparing (2.5.28) and (2.5.32), up to some small error terms we obtain the iterative structure, i.e. the same type of terms we started with but with an additional small prefactor of $N^{-1/2}$.

Proposition 2.5.7. *Under the assumptions of Theorem 2.5.1, for all $t \in (0,T)$, as well as $k, \tilde{k} \in \{1,\dots,K\}$, $\alpha,\beta \in \{1,\dots,n_S\}$ there exist functions*

$$\psi^0 \in \mathcal{L}_{pow,r}^q(\mathbb{R}^K), \quad \widetilde{\psi}_{k\tilde{k}}^t \in \mathcal{L}_{pow,r+1}^{q-2}(\mathbb{R}^{K+1}),$$

sets of test functions

$$\boldsymbol{\phi}^0 \in [H^s(\mathbb{T}^d,\mathbb{R}^{n_S})]^K, \quad \widetilde{\boldsymbol{\phi}}_{k\tilde{k}}^t \in [H^{s-1}(\mathbb{T}^d,\mathbb{R}^{n_S})]^{K+1},$$

and test times $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$ *such that*

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(N^{1/2}\langle \boldsymbol{\varphi}, \mu_{\boldsymbol{T}}^{r_I,N} - \overline{\rho}^{r_I}(\boldsymbol{T})\rangle\right)\right] \qquad (2.5.38)\\
&\quad = \mathbb{E}\left[\psi^0\left(N^{1/2}\langle \boldsymbol{\phi}^0, \mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\rangle\right)\right]\\
&\qquad + \frac{1}{N^{1/2}}\sum_{k,\tilde{k}=1}^{K}\int_0^{T_k\wedge T_{\tilde{k}}} \mathbb{E}\left[\widetilde{\psi}^t_{k\tilde{k}}\left(N^{1/2}\langle \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, \mu^{r_I,N}_{t\wedge \tilde{\boldsymbol{T}}_{k\tilde{k}}} - \overline{\rho}^{r_I}(t\wedge \tilde{\boldsymbol{T}}_{k\tilde{k}})\rangle\right)\right]\mathrm{d}t\\
&\qquad + \mathrm{Err}_{lin,a}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\psi\left(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\right)\right] \qquad (2.5.39)\\
&\quad = \mathbb{E}\left[\psi^0\left(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^0], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\big)_h\right)\right]\\
&\qquad + N^{-1/2}\sum_{k,\tilde{k}=1}^{K}\int_0^{T_k\wedge T_{\tilde{k}}} \mathbb{E}\left[\widetilde{\psi}^t_{k\tilde{k}}\left(N^{1/2}\big(\mathcal{I}_h[\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}], (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t\wedge \tilde{\boldsymbol{T}}_{k\tilde{k}}\wedge T_\oslash)\big)_h\right)\right]\mathrm{d}t\\
&\qquad + \mathrm{Err}_\oslash + \mathrm{Err}_{neg} + \mathrm{Err}_{num} + \mathrm{Err}_{lin,b}. \qquad (2.5.40)
\end{aligned}$$

The linearization errors are constructed using ϕ *as in* (2.5.26), ψ^t *as in* (2.5.27), ϕ_h *as in* (2.5.31), $Q^{r_I,N}$ *as in* (2.5.21), *and* $Q_h^{r_I,N}$ *as in* (2.5.25): *explicitly, they read*

$$Err_{lin,a} := -N^{1/2}\sum_{k=1}^{K}\int_0^{T_k}\mathbb{E}\big[\partial_k\psi^t(N^{1/2}\langle \boldsymbol{\phi}^t, \mu^{r_I,N}_{t\wedge\boldsymbol{T}} - \overline{\rho}^{r_I}(t\wedge\boldsymbol{T})\rangle)Q_t^{r_I,N}(\phi_k^t)\big]\,\mathrm{d}t, \qquad (2.5.41)$$

$$\begin{aligned}
Err_{lin,b} := -N^{1/2}\sum_{k=1}^{K}\int_0^{T_k}\mathbb{E}\big[\partial_k\psi^t\big(N^{1/2}\big(\boldsymbol{\phi}_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t\wedge\boldsymbol{T}\wedge T_\oslash)\big)_h\big) \qquad (2.5.42)\\
\times Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi_{h,k}(t))\big]\,\mathrm{d}t.
\end{aligned}$$

For any $\kappa > 0$ *there exists* $N_0 = N_0(\kappa, \epsilon, r, s, K, \text{data})$, *such that for all* $N > N_0$ *the following estimates hold. The stopping time satisfies* (2.5.13), *the new generalised moment functions* ψ^0 *and* $\{\widetilde{\psi}^t_{k\tilde{k}}\}_{t,k,\tilde{k},m,n}$ *are subject to* (2.5.6) *and* (2.5.7) *respectively. The test function* ϕ^0 *satisfy* (2.5.10), *and the test functions* $\{\widetilde{\phi}^t_{k\tilde{k}}\}_{t,k,\tilde{k}}$ *satisfy* (2.5.11). *Finally, for the error terms the estimates* (2.5.13), (2.5.14), *and* (2.5.15) *hold respectively.*

Remark 2.5.8. The terms $\mathrm{Err}_{lin,a}$ and $\mathrm{Err}_{lin,b}$ do not show the itarative structure yet due to the convolutional structure. In Subsection 2.5.3 below we will follow 2.4.2 to separate the two contributions of the quadratic non-linearity.

Proof. In Steps 1 and 2 we will introduce the definitions which yield (2.5.38) and (2.5.39) from Lemma 2.5.5 and Lemma 2.5.6 respectively. In Steps 3, 4, and 5 we will prove the estimates.

Step 1: Defining $\widetilde{\psi}^t_{k\tilde{k}}$, $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}$,$\tilde{\boldsymbol{T}}_{k\tilde{k}}$ **to obtain** (2.5.38)**.** We introduce $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}(x)\colon \mathbb{T}^d \to \mathbb{R}^{K+1}$ and $\tilde{\boldsymbol{T}}_{k\tilde{k}} \in [0,T]^{K+1}$ with

$$\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}(x) = \begin{pmatrix} \big(\phi^t_{1,\alpha}(x)\big)_{\alpha=1,\dots,n_S}\\ \vdots \\ \big(\phi^t_{K,\alpha}(x)\big)_{\alpha=1,\dots,n_S}\\ \big(\sigma_\alpha\nabla\phi^t_{k,\alpha}(x)\cdot\nabla\phi^t_{\tilde{k},\alpha}(x)\big)_{\alpha=1,\dots,n_S}\end{pmatrix}, \qquad \tilde{\boldsymbol{T}}_{k\tilde{k}} = \begin{pmatrix} T_1\\ \vdots\\ T_K\\ T_k\wedge T_{\tilde{k}}\end{pmatrix} \qquad (2.5.43)$$

where ϕ^t and thus particularly ϕ^0 come from (2.5.26). Further defining $\widetilde{\psi}^t_{k\tilde{k}} \colon \mathbb{R}^{K+1} \to \mathbb{R}$,

$$\widetilde{\psi}^t_{k\tilde{k}}(\boldsymbol{z}) = \chi_{t \le T_k \wedge T_{\tilde{k}}} \partial_k \partial_{\tilde{k}} \psi^t(z_1, \dots, z_K) z_{K+1}$$

where ψ^t, and thus also ψ^0 are given by (2.5.27), we immediately obtain (2.5.38) from (2.5.28).

Step 2: (2.5.39) and defining the remaining error terms. From Lemma 2.5.6 we have (2.5.32). Comparing this with (2.5.39), there is a series of slight adjustments to obtain the appropriate form, for which we have to pay with the respective error term. Recall that

$$\boldsymbol{\zeta}^t_h \coloneqq N^{1/2} \big(\boldsymbol{\phi}^t_h, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t \wedge \boldsymbol{T} \wedge T_\oslash) \big)_h .$$

First, in order to adjust $(\rho^{r_I}_{h,\alpha})^+ \longrightarrow \rho^{r_I}_{h,\alpha}$, we have to pay

$$\mathrm{Err}_{neg} \coloneqq \sum_{k,\tilde{k}=1}^{K} \sum_{\alpha=1}^{n_S} \sigma_\alpha \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E} \big[\partial_k \partial_{\tilde{k}} \psi^t(\boldsymbol{\zeta}^t_h) \big(\nabla_h \phi^t_{h,k,\alpha} \cdot \nabla_h \phi^t_{h,\tilde{k},\alpha}, (\rho^{r_I}_{h,\alpha})^- (t \wedge T_\oslash) \big)_h \big] \, \mathrm{d}t.$$

Second, to adjust $\nabla_h \phi^t_{h,k,\alpha} \cdot \nabla_h \phi^t_{h,\tilde{k},\alpha} \longrightarrow \mathcal{I}_h \big[\nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha} \big]$, we pay

$$\begin{aligned} \mathrm{Err}_{num,1} \coloneqq \sum_{k,\tilde{k}=1}^{K} \sum_{\alpha=1}^{n_S} \sigma_\alpha \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E} \Big[& \partial_k \partial_{\tilde{k}} \psi^t(\boldsymbol{\zeta}^t_h) \\ & \times \Big(\big(\nabla_h \phi^t_{h,k,\alpha} \cdot \nabla_h \phi^t_{h,\tilde{k},\alpha} - \mathcal{I}_h [\nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha}], \rho^{r_I}_{h,\alpha} (t \wedge T_\oslash) \big)_h \Big) \Big] \, \mathrm{d}t. \end{aligned}$$

Third, for the adjustment $\overline{\rho}^{r_I}_\alpha \longrightarrow \overline{\rho}^{r_I}_{h,\alpha}$ we have to pay

$$\begin{aligned} \mathrm{Err}_{num,2} \coloneqq \sum_{k,\tilde{k}=1}^{K} \sum_{\alpha=1}^{n_S} \sigma_\alpha \int_0^{T_k \wedge T_{\tilde{k}}} & \mathbb{E} \big[\partial_k \partial_{\tilde{k}} \psi^t(\boldsymbol{\zeta}^t_h) \big] \\ & \times \Big(\big(\mathcal{I}_h [\nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha}], \overline{\rho}^{r_I}_{h,\alpha} (t \wedge T_\oslash) \big)_h - \big\langle \nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha}, \overline{\rho}^{r_I}_\alpha (t \wedge T_\oslash) \big\rangle \Big) \, \mathrm{d}t. \end{aligned}$$

For adjusting $\boldsymbol{\zeta}^t_h \longrightarrow N^{1/2} \big(\mathcal{I}_h [\boldsymbol{\phi}^t], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t \wedge \boldsymbol{T}) \big)_h$ as in the argument of ψ^t, we pay

$$\begin{aligned} \mathrm{Err}_{num,3} \coloneqq \sum_{k,\tilde{k}=1}^{K} \sum_{\alpha=1}^{n_S} \sigma_\alpha \int_0^{T_k \wedge T_{\tilde{k}}} \mathbb{E} \Big[\Big(& \partial_k \partial_{\tilde{k}} \psi^t \big(N^{1/2} \big(\boldsymbol{\phi}^t_h, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t \wedge \boldsymbol{T} \wedge T_\oslash) \big)_h \big) \\ & - \partial_k \partial_{\tilde{k}} \psi^t \big(N^{1/2} \big(\mathcal{I}_h [\boldsymbol{\phi}^t], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t \wedge \boldsymbol{T} \wedge T_\oslash) \big)_h \big) \Big) \\ & \times \big(\mathcal{I}_h [\nabla \phi^t_{k,\alpha} \cdot \nabla \phi^t_{\tilde{k},\alpha}], (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t \wedge T_\oslash) \big)_h \Big] \, \mathrm{d}t. \end{aligned}$$

Finally, in order to adjust $\boldsymbol{\phi}^0_h \to \mathcal{I}_h [\boldsymbol{\phi}^0]$, we pay

$$\mathrm{Err}_{num,4} \coloneqq \mathbb{E} \Big[\psi^0 \Big(N^{1/2} \big(\mathcal{I}_h [\boldsymbol{\phi}^0], (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(0) \big)_h \Big) - \psi^0 \Big(N^{1/2} \big(\boldsymbol{\phi}^0_h, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(0) \big)_h \Big) \Big].$$

Setting

$$\mathrm{Err}_{num} \coloneqq \mathrm{Err}_{num,1} + \mathrm{Err}_{num,2} + \mathrm{Err}_{num,3} + \mathrm{Err}_{num,4},$$

we obtain (2.5.39).

Step 3: Estimates for ψ^0, $\widetilde{\psi}^t_{k\tilde{k}}$, ϕ^0, $\widetilde{\phi}^t_{k\tilde{k}}$. Estimates for ϕ^0 and ψ^0 immediately follow from Lemma 2.5.2 and Lemma 2.5.5 respectively. With respect to $\widetilde{\psi}^t_{k\tilde{k}}$ we observe that

$$\|\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}\|_{H^{s-1}}\leq 2C(s)\sum_{l=0}^{\lfloor(s-1)/2\rfloor}\|D^{l+1}\boldsymbol{\phi}^t\|_{L^\infty}\|D^{s-l}\boldsymbol{\phi}^t\|_{L^2}\leq C(s)\|\boldsymbol{\phi}^t\|^2_{H^s}$$

where we use $s>d+2$ for the Sobolev embedding $H^s\subset\subset C^{\lfloor(s+1)/2\rfloor}$. With the definition of $\widetilde{\psi}^t_{k\tilde{k}}$ and Lemma 2.A.1 we thus obtain (2.5.11). We obtain (2.5.7) for $\widetilde{\psi}^t_{k\tilde{k}}$ by using its definition and (2.5.29), thus getting

$$\|\widetilde{\psi}^t_{k\tilde{k}}\|_{\mathcal{L}^{q-2}_{pow,r+1}}\leq\|\psi^t\|_{\mathcal{L}^q_{pow,r}}\leq CN^\kappa\left(1+\|\boldsymbol{\varphi}\|^r_{H^s}\right)\|\psi\|_{\mathcal{L}^q_{pow,r}}.$$

Step 4: Estimates for the negativity and numerical errors. Estimate (2.5.13) for $\mathrm{Err}_\oslash$ corresponds to the estimates in the proof of Lemma 2.5.6. With respect to (2.5.14) we have $\mathrm{Err}_{neg}=0$ since for $T_\oslash>0$, all $\alpha=1,\dots,n_S$

$$\rho^{r_I}_{h,\alpha}(t\wedge T_\oslash)\geq\overline{\rho}^{r_I}_{h,\alpha}(t\wedge T_\oslash)-N^{-\epsilon}\geq\rho_{\min,h}-N^{-\epsilon}\geq 0 \tag{2.5.44}$$

due to the definition of $T_\oslash$ and Remark 2.3.14 for N large enough.

For the rest of this step, to prove the bound on Err_{num}, take an arbitrary $\kappa>0$. The following estimates hold for N large enough with the κ-depending bound corresponding to the respectively used results. To simplify the notation, we allow for the specific value of κ to change from line to line here. Note that for $T_\oslash>0$

$$|\boldsymbol{\zeta}^t_h|\leq N^{1/2}\|\boldsymbol{\phi}^t_h\|_{H^{d+2}_h}\|(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T}\wedge T_\oslash)\|_{H^{-2\lfloor d/2\rfloor+2}_h}\leq N^{\epsilon+\kappa}\|\boldsymbol{\varphi}\|_{H^s} \tag{2.5.45}$$

due to (2.B.11) from Lemma 2.B.8 paired with $\|\mathcal{I}_h[\cdot]\|_{H^{d+2}_h}\lesssim\|\cdot\|_{C^{d+2}}\lesssim\|\cdot\|_{H^s}$, as well as the definition of $T_\oslash$.

With respect to $\mathrm{Err}_{num,1}$, if $T_\oslash>0$ we have

$$\begin{aligned}\|\rho^{r_I}_h(t\wedge T_\oslash)\|_{L^2_h}&\leq\|\overline{\rho}^{r_I}_h\|_{L^\infty(L^2_h)}+C\|(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge T_\oslash)\|_{L^\infty_h}\\&\leq C\|\overline{\rho}^{r_I}\|_{L^\infty(C^0)}+\|\overline{\rho}^{r_I}_h-\mathcal{I}_h[\overline{\rho}^{r_I}]\|_{L^\infty(L^2_h)}+N^{-\epsilon}\leq C\end{aligned}$$

by the definition of $T_\oslash$, Proposition 2.B.6 and the scaling from Assumption 2.3.11. Thus, with (2.5.29) to bound $\|\psi^t\|_{\mathcal{L}^2_{pow,r}}$ and (2.B.17) from Lemma 2.B.9 we obtain

$$|\mathrm{Err}_{num,1}|\leq C(\|\overline{\rho}^{r_I}\|_{L^\infty})\|\psi\|_{\mathcal{L}^2_{pow,r}}\left(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\right)\|\boldsymbol{\varphi}\|^2_{H^s}N^{r\varepsilon+\kappa}h^{p+1}. \tag{2.5.46}$$

With respect to $\mathrm{Err}_{num,2}$, applying (2.B.1) from Proposition 2.B.6 as well as the Euler-Maclaurin formula – see Lemma 2.B.4 – we have

$$\begin{aligned}&\left|\left(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}],\overline{\rho}^{r_I}_{h,\alpha}(t)\right)_h-\left\langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha},\overline{\rho}^{r_I}_\alpha(t)\right\rangle\right|\\&\quad\leq\left|\left(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}],(\overline{\rho}^{r_I}_{h,\alpha}-\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha])(t)\right)_h\right|\\&\qquad\qquad+\left|\left(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}],\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha](t)\right)_h-\left\langle\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha},\overline{\rho}^{r_I}_\alpha(t)\right\rangle\right|\\&\quad\leq C\|\boldsymbol{\phi}^t\|^2_{C^2}N^\kappa h^{p+1}+C\|\boldsymbol{\phi}^t\|^2_{C^{p+3}}\|\overline{\rho}^{r_I}(t)\|_{C^{p+2}}h^{p+1}.\end{aligned}$$

Since (2.5.1) implies $s > p+3+d/2$ and therefore $\|\cdot\|_{C^{p+3}} \lesssim \|\cdot\|_{H^s}$, with (2.B.11) it follows that

$$|\mathrm{Err}_{num,2}| \leq C(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+2})})\|\psi\|_{\mathcal{L}^2_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r}\big)\|\boldsymbol{\varphi}\|_{H^s}^2 N^{r\epsilon+\kappa}h^{p+1}. \tag{2.5.47}$$

Concerning $\mathrm{Err}_{num,3}$, for $T_\oslash > 0$ by the definition of $T_\oslash$ we have

$$\big|\big(\mathcal{I}_h[\nabla\phi^t_{k,\alpha}\cdot\nabla\phi^t_{\tilde{k},\alpha}],(\rho^{r_I}_{h,\alpha}-\overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash)\big)_h\big| \leq \|\boldsymbol{\phi}^t\|^2_{C^{d+3}}N^{-1/2+\epsilon}, \tag{2.5.48}$$

and, using (2.B.15), we get

$$\begin{aligned}\big|\boldsymbol{\zeta}^t_h - N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^t],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T})\big)_h\big| &\leq C\|\mathcal{I}_h[\boldsymbol{\phi}^t]-\boldsymbol{\phi}^t_h\|_{H^{d+2}_h}N^\epsilon \\ &\leq \|\boldsymbol{\varphi}\|_{H^s}N^\kappa h^{p+1}N^\epsilon,\end{aligned} \tag{2.5.49}$$

where we also used $\|\cdot\|_{C^{d+2+p+3}} \lesssim \|\cdot\|_{H^s}$ since $s > p+3d/2+5$. In particular, for the connecting segment $I := \big[\boldsymbol{\zeta}^t_h, N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^t],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\boldsymbol{T})\big)_h\big] \subset \mathbb{R}^K$ it holds that $\sup_{u\in I}|u| \leq CN^{\epsilon+\kappa}\|\boldsymbol{\varphi}\|_{H^s}$. Thus, by mean value theorem and the definition of $\|\cdot\|_{\mathcal{L}^q_{pow,r}}$,

$$|\mathrm{Err}_{num,3}| \leq C\|\psi\|_{\mathcal{L}^3_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r}\big)\|\boldsymbol{\varphi}\|^3_{H^s}N^{-1/2+(r+2)\epsilon+\kappa}h^{p+1}. \tag{2.5.50}$$

As for $\mathrm{Err}_{num,4}$, including the case $T_\oslash = 0$ in (2.5.49) and dealing with it as in the estimate of $\mathrm{Err}_\oslash$ in the proof of Lemma 2.5.6, via the mean value theorem we obtain

$$|\mathrm{Err}_{num,4}| \leq C\|\psi\|_{\mathcal{L}^1_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r}\big)\|\boldsymbol{\varphi}\|_{H^s}N^{(r+1)\epsilon+\kappa}h^{p+1}. \tag{2.5.51}$$

Combining (2.5.46), (2.5.47), (2.5.50), and (2.5.51) yields (2.5.15) for Err_{num}. □

2.5.3 Including the linearisation errors in the iterative structure

Proposition 2.5.7 leaves us with the task to fit $\mathrm{Err}_{lin,a}$ and $\mathrm{Err}_{lin,b}$ to the iterative structure. These are the terms induced by the linearisation compensations $Q^{r_I,N}_t(\cdot)$ and $Q^{r_I,N}_{h,t}(\cdot)$, which stem from linearising the respective interaction terms in the continuous and discrete setting;

$$\begin{aligned}Q^{r_I,N}_t(\phi^t_k) &= \sum_{\alpha,\beta=1}^{n_S}\big\langle\nabla\phi^t_{k,\alpha}\cdot\big(\nabla V^{r_I}_{\alpha\beta}*\big(\mu^{r_I,N}_{\beta,t}-\overline{\rho}^{r_I}_\beta(t)\big)\big),\mu^{r_I,N}_{\alpha,t}-\overline{\rho}^{r_I}_\alpha(t)\big\rangle,\\ Q^{r_I,N}_{h,t}(\phi^t_{h,k}) &= \sum_{\alpha,\beta=1}^{n_S}\big(\nabla_h\phi^t_{h,k,\alpha}\cdot(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}]*_h(\rho^{r_I}_{h,\beta}-\overline{\rho}^{r_I}_{h,\beta})(t)),(\rho^{r_I}_{h,\alpha}-\overline{\rho}^{r_I}_{h,\alpha})(t)\big)_h.\end{aligned}$$

We now rewrite $Q^{r_I,N}_t(\phi^t_k)$ and $Q^{r_I,N}_{h,t}(\phi^t_{h,k})$ (and thus $\mathrm{Err}_{lin,a}$ and $\mathrm{Err}_{lin,b}$) to conform them to the iterative structure as outlined in Subsection 2.4.2.

Proposition 2.5.9 (Iterative structure for the linearisation compensations)**.** *Under the assumptions of Theorem 2.5.1, let* $\mathrm{Err}_{lin,a}$ *and* $\mathrm{Err}_{lin,b}$ *be defined as in Proposition 2.5.7,* (2.5.41) *and* (2.5.42)*. Then for all* $t\in(0,T)$*,* $k\in\{1,\dots,K\}$*,* $m,n\in\mathbb{Z}^d$*,* $\alpha,\beta\in\{1,\dots,n_S\}$ *and* $\ell\in\{1,\dots,d\}$ *there exist generalised moment functions* $\breve{\psi}^t_k \in \mathcal{L}^{q-1}_{pow,r+2}(\mathbb{R}^K\times\mathbb{C}^2)$*, coefficients* $\hat{F}^\ell_{k,\alpha\beta}[m,n]\in\mathbb{C}$ *with sets of test functions*

$$\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell} \in [H^s(\mathbb{T}^d,\mathbb{R}^{n_S})]^K\times[H^s(\mathbb{T}^d,\mathbb{C}^{n_S})]^2$$

and test times $\breve{\boldsymbol{T}}_k \in [0,T]^{K+2}$ *such that*

$$\mathrm{Err}_{lin,a} = N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n] \\ \times \mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\big\langle\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell},\mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k}-\overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\big\rangle\big)\big]\,\mathrm{d}t$$

and

$$\mathrm{Err}_{lin,b} = N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n] \\ \times \mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\big(\mathcal{I}_h[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}],(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\big)_h\big)\big]\,\mathrm{d}t \\ + \mathrm{Err}_{num,lin}.$$

For any $\kappa>0$ *there exists* $N_0=N_0(\kappa,\epsilon,r,s,K,\text{data})$*, such that for* $N>N_0$ *the following estimates hold. The generalised moment functions* $\{\breve{\psi}^t_k\}_{t,k}$ *are subject to* (2.5.8)*, the coefficients* $\{\hat{F}_{\alpha\beta}[m,n]\}_{\alpha,\beta,m,n}$ *to* (2.5.9)*. The test functions* $\{\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}\}$ *satisfy* (2.5.12)*. For the error term the estimate* (2.5.15) *holds.*

Proof. **Step 1: Rewriting** $Q^{r_I,N}_t(\phi^t_k)$**.** We have $Q^{r_I,N}_t(\phi^t_k)=\sum_{\alpha,\beta=1}^{n_S}I_{\alpha,\beta}$ with

$$I_{\alpha,\beta}=\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\nabla\phi^t_{k,\alpha}(x)\cdot\nabla V^{r_I}_{\alpha\beta}(x-y)\,\mathrm{d}\big(\mu^{r_I,N}_{\beta,t}-\overline{\rho}^{r_I}_\beta(t)\big)(y)\,\mathrm{d}\big(\mu^{r_I,N}_{\alpha,t}-\overline{\rho}^{r_I}_\alpha(t)\big)(x).$$

For $\ell=1,\dots,d$ we use the $2d$–dimensional Fourier series representation

$$\partial_\ell V^{r_I}_{\alpha\beta}(x-y)=\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]e^{im\cdot y}e^{in\cdot x}, \tag{2.5.52}$$

with

$$\hat{F}^{\ell}_{k,\alpha\beta}[m,n]:=\frac{1}{(2\pi)^{2d}}\int_{\mathbb{T}^d\times\mathbb{T}^d}\partial_\ell V^{r_I}_{\alpha\beta}(x-y)e^{im\cdot y}e^{in\cdot x}\,\mathrm{d}x\,\mathrm{d}y\in\mathbb{C}.$$

Since $(x,y)\to\partial_\ell V^{r_I}_{\alpha\beta}(x-y)$ is in $W^{1,s_I-1}(\mathbb{T}^d\times\mathbb{T}^d)$, we have pointwise convergence of the Fourier series (e.g., see [Gra08, Theorem 3.3.9 and Corollaries 3.4.9/3.4.10]) with coefficients estimated by

$$|\hat{F}^{\ell}_{k,\alpha\beta}[m,n]|\le C(s_I)\frac{\|\nabla V^{r_I}_{\alpha\beta}\|_{W^{s_I-1,1}(\mathbb{T}^d)}}{(|m|_2+|n|_2)^{s_I-1}},$$

which yields (2.5.9) after applying the scaling relation (2.3.15) from Assumption 2.3.11 to $\|\nabla V^{r_I}_{\alpha\beta}\|_{W^{s_I-1,1}(\mathbb{T}^d)}\le C{r_I}^{-s_I}$. Thus, since $s_I>2d+1$, we also have uniform convergence of the Fourier series and therefore, abbreviating $\vartheta_n(x):=e^{in\cdot x}$ for $n\in\mathbb{Z}^d$, we obtain

$$I_{\alpha,\beta}=\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\big\langle\vartheta_m,\mu^{r_I,N}_{\beta,t}-\overline{\rho}^{r_I}_\beta(t)\big\rangle\big\langle\partial_\ell\phi^t_{k,\alpha}\vartheta_n,\mu^{r_I,N}_{\alpha,t}-\overline{\rho}^{r_I}_\alpha(t)\big\rangle.$$

Adjusting for N–prefactors, this leads to

$$NQ^{r_I,N}_t(\phi^t_k)=\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\Big(N^{1/2}\sum_{\gamma=1}^{n_S}\big\langle\delta^{\beta\gamma}\vartheta_m,\mu^{r_I,N}_{\gamma,t}-\overline{\rho}^{r_I}_\gamma(t)\big\rangle\Big) \\ \times\Big(N^{1/2}\sum_{\gamma=1}^{n_S}\big\langle\delta^{\alpha\gamma}\partial_\ell\phi^t_{k,\alpha}\vartheta_n,\mu^{r_I,N}_{\gamma,t}-\overline{\rho}^{r_I}_\gamma(t)\big\rangle\Big),$$

where $\delta^{\beta\gamma}$ is the Kronecker-Delta.

Step 2: Rewriting $Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi^t_{h,k})$. First, we get an error for switching from $\nabla_h\phi^t_{h,k,\alpha}$ to $\mathcal{I}_h[\nabla\phi^t_{k,\alpha}]$. That is, we have

$$Q^{r_I,N}_{h,t\wedge T_\oslash}(\phi^t_{h,k}) = \sum_{\alpha,\beta=1}^{n_S} I^h_{\alpha,\beta} + \overline{\mathrm{Err}}^{k,t}_{num,1}$$

with

$$\overline{\mathrm{Err}}^{k,t}_{num,1} \coloneqq \sum_{\alpha,\beta=1}^{n_S} \Big(\big(\nabla_h\phi^t_{h,k,\alpha} - \mathcal{I}_h[\nabla\phi^t_{k,\alpha}]\big)\cdot\big(\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}] *_h (\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t\wedge T_\oslash)\big), \\ (\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash)\Big)_h, \tag{2.5.53}$$

$$I^h_{\alpha,\beta} \coloneqq h^{2d} \sum_{x,y\in G_{h,d}} \nabla\phi^t_{k,\alpha}(x)\cdot\nabla V^{r_I}_{\alpha\beta}(x-y)(\rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta})(t\wedge T_\oslash, y)(\rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha})(t\wedge T_\oslash, x).$$

Plugging in (2.5.52) we obtain

$$I^h_{\alpha,\beta} = \sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d} \hat{F}^\ell_{k,\alpha\beta}[m,n]\big(\mathcal{I}_h[\vartheta_m], \rho^{r_I}_{h,\beta} - \overline{\rho}^{r_I}_{h,\beta}\big)_h\big(\mathcal{I}_h[\partial_\ell\phi^t_{k,\alpha}\vartheta_n], \rho^{r_I}_{h,\alpha} - \overline{\rho}^{r_I}_{h,\alpha}\big)_h(t\wedge T_\oslash)$$

and therefore

$$NQ^{r_I,N}_{h,t}(\phi^t_{h,k}) = \sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\sum_{n,m\in\mathbb{Z}^d} \hat{F}^\ell_{k,\alpha\beta}[m,n]\Big(N^{1/2}\sum_{\gamma=1}^{n_S}\big(\mathcal{I}_h[\delta^{\beta\gamma}\vartheta_m], \rho^{r_I}_{h,\gamma} - \overline{\rho}^{r_I}_{h,\gamma}\big)_h\Big) \\ \times\Big(N^{1/2}\sum_{\gamma=1}^{n_S}\big(\mathcal{I}_h[\delta^{\alpha\gamma}\partial_\ell\phi^t_{k,\alpha}\vartheta_n], \rho^{r_I}_{h,\gamma} - \overline{\rho}^{r_I}_{h,\gamma}\big)_h\Big)(t\wedge T_\oslash) + N\overline{\mathrm{Err}}^{k,t}_{num,1}.$$

Step 3: Defining the iterated functions. Defining $\breve{\psi}^t_k\colon \mathbb{R}^K\times\mathbb{C}^2\to\mathbb{C}$ as

$$\breve{\psi}^t_k(\boldsymbol{z}) \coloneqq -\chi_{t\le T_k}\partial_k\psi^t(z_1,\dots,z_K)z_{K+1}z_{K+2},$$

we have via the dominated convergence theorem (see Step 5 for a full justification)

$$\mathrm{Err}_{lin,a} = N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^\ell_{k,\alpha\beta}[m,n] \\ \times\mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\big\langle\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}, \mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k} - \overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\big\rangle\big)\big]\,\mathrm{d}t \tag{2.5.54}$$

with

$$\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}(x) = \begin{pmatrix} \big(\phi^t_{1,\gamma}(x)\big)_{\gamma=1,\dots,n_S} \\ \vdots \\ \big(\phi^t_{K,\gamma}(x)\big)_{\gamma=1,\dots,n_S} \\ \big(\delta^{\beta\gamma}\vartheta_m(x)\big)_{\gamma=1,\dots,n_S} \\ \big(\delta^{\alpha\gamma}\partial_\ell\phi^t_{k,\alpha}(x)\vartheta_n(x)\big)_{\gamma=1,\dots,n_S}\end{pmatrix}, \qquad \breve{\boldsymbol{T}}_k = \begin{pmatrix} T_1\\ \vdots\\ T_K\\ T_k\\ T_k\end{pmatrix}. \tag{2.5.55}$$

On the discrete side, we analogously have again by dominated convergence (see Step 5)

$$\begin{aligned}\mathrm{Err}_{lin,b} = N^{-1/2}\sum_{k=1}^{K}\sum_{\alpha,\beta=1}^{n_S}\sum_{\ell=1}^{d}\int_0^{T_k}\sum_{n,m\in\mathbb{Z}^d}\hat{F}^{\ell}_{k,\alpha\beta}[m,n]\\ \times\mathbb{E}\big[\breve{\psi}^t_k\big(N^{1/2}\big(\mathcal{I}_h[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\breve{\boldsymbol{T}}_k\wedge T_{\oslash})\big)_h\big)\big]\,\mathrm{d}t\\ +\mathrm{Err}_{num,5}+\mathrm{Err}_{num,6}\end{aligned}\tag{2.5.56}$$

where $\mathrm{Err}_{num,5}$ stems from the errors $\overline{\mathrm{Err}}^{k,t}_{num,1}$ defined in (2.5.53) of Step 2 and is given by

$$\mathrm{Err}_{num,5}\coloneqq-\sum_{k=1}^{K}\int_0^{T_k}\mathbb{E}\big[\partial_k\psi^t\big(N^{1/2}\big(\boldsymbol{\phi}^t_h,(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\boldsymbol{T}\wedge T_{\oslash})\big)_h\big)N^{1/2}\overline{\mathrm{Err}}^{k,t}_{num,1}\big]\,\mathrm{d}t,\tag{2.5.57}$$

while $\mathrm{Err}_{num,6}$ is obtained from replacing $\boldsymbol{\zeta}^t_h\longrightarrow N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^t],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\breve{\boldsymbol{T}}_k\wedge T_{\oslash})\big)_h$, leading to

$$\begin{aligned}\mathrm{Err}_{num,6}\coloneqq-\sum_{k=1}^{K}\int_0^{T_k}\mathbb{E}\Big[\Big(\partial_k\psi^t\big(N^{1/2}\big(\boldsymbol{\phi}^t_h,(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\boldsymbol{T}\wedge T_{\oslash})\big)_h\big)\\ -\partial_k\psi^t\big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\phi}^t],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\boldsymbol{T}\wedge T_{\oslash})\big)_h\big)\Big)N^{1/2}Q^{r_I,N}_{h,t\wedge T_{\oslash}}(\phi^t_{h,k})\Big]\,\mathrm{d}t.\end{aligned}\tag{2.5.58}$$

Step 4: Collecting the estimates. With respect to the test functions we recall that $\vartheta_n(x)\coloneqq e^{in\cdot x}$. Thus, for $n\in\mathbb{Z}^d$, $\tilde{s}\in\mathbb{N}_0$ we have

$$\|\vartheta_n\|_{H^{\tilde{s}}(\mathbb{T}^d)}\leq C|n|^{\tilde{s}}_{\infty}.\tag{2.5.59}$$

This, based on (2.5.55) together with Lemma 2.A.1 and the same argument from Step 3 in the proof of Proposition 2.5.7, yields (2.5.12). Using the definition of the new generalised moment function and (2.5.29) we obtain

$$\|\breve{\psi}^t_k\|_{\mathcal{L}^{q-1}_{pow,r+2}}\leq\|\psi^t\|_{\tilde{\mathcal{L}}^{q}_{pow,r}}\leq CN^{\kappa}\left(1+\|\varphi\|^{r}_{H^s}\right)\|\psi\|_{\mathcal{L}^{q}_{pow,r}}.$$

For estimating the error terms, let $\kappa>0$ be an arbitrary, small exponent. The following estimates hold for N large enough with the κ-depending bound corresponding to the respectively used results. To simplify the notation, we allow for the value of κ to change from line to line. For any $k=1,\dots,K$, if $T_{\oslash}>t>0$ the discrete Hölder inequality (Corollary 2.B.2) entails

$$\big|N^{1/2}\overline{\mathrm{Err}}^{k,t}_{num,1}\big|\leq CN^{\epsilon+\kappa}\|\nabla_h\boldsymbol{\phi}^t_h-\mathcal{I}_h[\nabla\boldsymbol{\phi}^t]\|_{L^1_h}\|(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t)\|_{L^{\infty}_h}\leq C\|\boldsymbol{\varphi}\|_{H^s}N^{\kappa}h^{p+1},$$

where for the second inequality we used the definition of $T_{\oslash}$, that $\|\cdot\|_{L^1_h}\lesssim\|\cdot\|_{L^2_h}$ and (2.B.16); for the first inequality we used that, for $T_{\oslash}>t>0$, it holds that

$$\begin{aligned}&N^{1/2}\|\mathcal{I}_h[\nabla V^{r_I}_{\alpha\beta}]*_h(\rho^{r_I}_{h,\beta}-\overline{\rho}^{r_I}_{h,\beta})(t)\|_{L^{\infty}_h}\\ &\quad\leq N^{1/2}\|\mathcal{I}_h[\nabla V^{r_I}]\|_{H^{2\lfloor d/2\rfloor+2}}\|\rho_h^{r_I}-\overline{\rho}_h^{r_I}\|_{H^{-2\lfloor d/2\rfloor-2}}(t)\\ &\quad\leq N^{\epsilon}\|\nabla V^{r_I}\|_{C^{d+3}}\leq N^{\epsilon}{r_I}^{-(2d+4)}\leq N^{\epsilon+\kappa}.\end{aligned}$$

Here, the last two inequalities are due to the definition of V^{r_I}, given in (2.3.2), and Assumption 2.3.11. Combining this estimate with $|\boldsymbol{\zeta}^t_h|$-estimate (2.5.45) from the proof of Proposition 2.5.7 and (2.5.29) from Lemma 2.5.6 yields

$$|\mathrm{Err}_{num,5}|\leq C\|\psi\|_{\mathcal{L}^1_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|^{2r}_{H^s}\big)\|\boldsymbol{\varphi}\|_{H^s}N^{r\epsilon+\kappa}h^{p+1}.\tag{2.5.60}$$

With respect to $\mathrm{Err}_{num,6}$ using (2.5.37) for $0 < t < T_\oslash$ from the proof of Lemma 2.5.6, analogously to the $|\mathrm{Err}_{num,3}|$-estimate (2.5.50) from the proof of Proposition 2.5.7 via the mean value theorem we obtain

$$C\|\psi\|_{\mathcal{L}^2_{pow,r}}\big(1+\|\boldsymbol{\varphi}\|_{H^s}^{2r}\big)\|\boldsymbol{\varphi}\|_{H^s}^2 N^{(r+1)\epsilon+\kappa}h^{p+1}. \tag{2.5.61}$$

Combining (2.5.60) and (2.5.61) yields (2.5.15) for $\mathrm{Err}_{num,5}+\mathrm{Err}_{num,6}$.

Step 5: Commuting infinite sum and expected value for (2.5.54) and (2.5.56). We have with (2.5.9) and (2.5.12) (or more precisely (2.5.59)) that

$$\begin{aligned}
&\Big|\hat{F}^t_{k,\alpha\beta}[m,n]\breve{\psi}^t_k\big(N^{1/2}\langle\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell},\mu^{r_I,N}_{t\wedge\breve{\boldsymbol{T}}_k}-\overline{\rho}^{r_I}(t\wedge\breve{\boldsymbol{T}}_k)\rangle\big)\Big|\\
&\quad\le CN^\kappa(|m|_2+|n|_2)^{-s_I+1}\|\breve{\psi}^t_k\|_{\mathcal{L}^0_{pow,r+2}}N^{(r+2)/2}\|\breve{\boldsymbol{\phi}}^t_{k,mn,\ell}\|^{r+2}_{H^{\lceil d/2+1\rceil}}\|\mu^{r_I,N}-\overline{\rho}^{r_I}\|^{r+2}_{H^{-\lceil d/2+1\rceil}}\\
&\quad\le CN^\kappa(1+\|\boldsymbol{\varphi}\|^{r+2}_{H^s})(|m|_2+|n|_2)^{-s_I+1}(|m|_\infty+|n|_\infty)^{(r+2)\lceil d/2+1\rceil}\\
&\qquad\qquad\times\|\breve{\psi}^t_k\|_{\mathcal{L}^0_{pow,r+2}}N^{(r+2)/2}\|\mu^{r_I,N}-\overline{\rho}^{r_I}\|^{r+2}_{H^{-\lceil d/2+1\rceil}},
\end{aligned}$$

which is a converging sum in n,m since $s_I>2d+1+(r+2)\lceil d/2+1\rceil$ with (2.5.2), while the expected value is controlled due to Proposition 2.7.1. Thus, the application of the dominated convergence theorem in (2.5.54) is justified. In the discrete case, with $\|\mathcal{I}_h[\cdot]\|_{L^1_h}\lesssim\|\cdot\|_{C^0}$,

$$\begin{aligned}
&\Big|\hat{F}^\ell_{k,\alpha\beta}[m,n]\breve{\psi}^t_k\big(N^{1/2}\big(\mathcal{I}_h[\breve{\boldsymbol{\phi}}^t_{k,mn,\alpha\beta,\ell}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\big)_h\big)\Big|\\
&\quad\le CN^\kappa\|\boldsymbol{\varphi}\|_{H^s}(|m|_2+|n|_2)^{-s_I+1}\|\breve{\psi}^t_k\|_{\mathcal{L}^0_{pow,r+2}}\\
&\qquad\qquad\times N^{(r+2)/2}\|\breve{\boldsymbol{\phi}}^t_{k,mn,\ell}\|^{r+2}_{C^0}\|(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t\wedge\breve{\boldsymbol{T}}_k\wedge T_\oslash)\|^{r+2}_{L^\infty_h}\\
&\quad\le CN^\kappa(1+\|\boldsymbol{\varphi}\|^{r+3}_{H^s})(|m|_2+|n|_2)^{-s+1}N^{(r+2)/2-\epsilon(r+2)}
\end{aligned}$$

due to the definition of the stopping time $T_\oslash$ in (2.8.2) and that $\|\breve{\boldsymbol{\phi}}^t_{k,mn,\ell}\|_{C^0}\le 1+\|\boldsymbol{\phi}^t\|_{H^s}$. Since $s_I>2d+1$, the infinite sum other these terms converges and (2.5.56) is valid. □

2.6 Proof of Theorem 2.2.1

Our main result follows from the iterative application of Theorem 2.5.1, where Proposition 2.7.1, Proposition 2.8.2 and the closeness of the initial fluctuations provided by Assumption 2.3.10 are used to close the estimate after a sufficient number of iterations.

Proof. We choose the stopping time $T_\oslash=T_\oslash(\epsilon/2,\epsilon)$ as defined in (2.8.2). With this choice, Proposition 2.8.2 directly yields (2.2.2). We further assume that in the following N is always large enough for the used auxiliary results to hold. The precise dependencies are summarized in the result. For this proof we allow for the specific value of $\kappa>0$ to change from line to line, since all estimates hold for arbitrary small κ, given N is large enough (this can be easily adjusted down the line).

Step 1: The case $j=1$. Since some of the following considerations will be needed for the case $j>1$, we proceed under slightly more general assumptions than necessary for just the case $j=1$. That is, let $\psi\in\mathcal{L}^q_{pow,r}$, $\boldsymbol{\varphi}\in\big[H^s(\mathbb{T}^d,\mathbb{R}^{n_S})\big]^K$ and $V\in\big[W^{s_I,1}(\mathbb{T}^d)\big]^{n_S\times n_S}$ with $q,r,s,s_I\in\mathbb{N}_0$. As in (2.4.2) we set

$$\begin{aligned}
\mathcal{M}(\psi,\boldsymbol{\varphi},\boldsymbol{T}):=\mathbb{E}\Big[\psi\Big(N^{1/2}\langle\boldsymbol{\varphi},\mu^{r_I,N}_{\boldsymbol{T}}-\overline{\rho}^{r_I}(\boldsymbol{T})\rangle\Big)\Big]\\
-\mathbb{E}\Big[\psi\Big(N^{1/2}\big(\mathcal{I}_h[\boldsymbol{\varphi}],(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\boldsymbol{T}\wedge T_\oslash)\big)_h\Big)\Big].
\end{aligned}$$

Now, if $q \geq 3$, $s > p + 3d/2 + 5$ and $s_I > (d+2)(r+2) + 2d + 1$, Theorem 2.5.1 yields

$$\begin{aligned}\mathcal{M}(\psi, \boldsymbol{\varphi}, \boldsymbol{T}) = N^{-1/2} \sum_{k,\tilde{k}}^{K} \int_0^{T_k \wedge T_{\tilde{k}}} \mathcal{M}(\widetilde{\psi}^t_{k\tilde{k}=1}, \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, t \wedge \tilde{\boldsymbol{T}}_{k\tilde{k}}) \\ + N^{-1/2} \sum_{k=1}^{K} \sum_{\alpha,\beta=1}^{n_S} \sum_{\ell=1}^{d} \int_0^{T_k} \sum_{m \in \mathbb{Z}^{2d}} \hat{F}^{\ell}_{k,\alpha\beta}[m] \\ \times \mathcal{M}(\breve{\psi}^t_k, \breve{\boldsymbol{\phi}}^t_{k,m,\alpha\beta,\ell}, t \wedge \breve{\boldsymbol{T}}_k) + \mathrm{Err}(\psi, \boldsymbol{\varphi}, \boldsymbol{T}),\end{aligned} \tag{2.6.1}$$

with

$$\mathrm{Err}(\psi, \boldsymbol{\varphi}, \boldsymbol{T}) \coloneqq \mathcal{M}(\psi^0, \boldsymbol{\phi}^0, 0) + \mathrm{Err}_{num} + \mathrm{Err}_{neg} + \mathrm{Err}_{\oslash} \tag{2.6.2}$$

for new $\psi^0, \widetilde{\psi}^t_{k\tilde{k}}, \breve{\psi}^t_k$, coefficients $\hat{F}^{\ell}_{k,\alpha\beta}[m]$, test functions $\boldsymbol{\phi}^0, \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, \breve{\boldsymbol{\phi}}^t_{k,m,\alpha\beta,\ell}$ as well as test times $\tilde{\boldsymbol{T}}_{k\tilde{k}}, \breve{\boldsymbol{T}}_k$ and errors $\mathrm{Err}_{num}, \mathrm{Err}_{neg}, \mathrm{Err}_{\oslash}$ as described in Theorem 2.5.1.

With (2.3.10) from Assumption 2.3.10, (2.5.6), (2.5.10) and $\|\cdot\|_{C^{p+1}} \lesssim \|\cdot\|_{H^{(d/2+p+1)+}}$ we have

$$|\mathcal{M}(\psi^0, \boldsymbol{\phi}^0, 0)| \leq C\|\psi\|_{\mathcal{L}^1_{pow,r}} \left(1 + \|\boldsymbol{\varphi}\|^{r+1}_{H^s}\right) h^{p+1}$$

and thus with (2.5.15), (2.5.14) and (2.5.13)

$$|\mathrm{Err}(\psi, \boldsymbol{\varphi}, \boldsymbol{T})| \leq C\|\psi\|_{\mathcal{L}^3_{pow,r}} \left(1 + \|\boldsymbol{\varphi}\|^{2r+3}_{H^s}\right) N^{(r+1)\epsilon+\kappa} \left(h^{p+1} + \exp\left(-CN^{\epsilon/2}\right)\right). \tag{2.6.3}$$

Due to Proposition 2.7.1, definition (2.8.2), and $\|\mathcal{I}_h(\cdot)\|_{H^{d+2}_h} \lesssim \|\cdot\|_{C^{d+2}}$, we obtain the estimate

$$|\mathcal{M}(\psi, \boldsymbol{\varphi}, \boldsymbol{T})| \leq C\|\psi\|_{\mathcal{L}^0_{pow,r}} \|\boldsymbol{\varphi}\|^r_{H^{d/2+2}} N^{\kappa} + C\|\psi\|_{\mathcal{L}^0_{pow,r}} \|\boldsymbol{\varphi}\|^r_{C^{d+2}} N^{r\epsilon}.$$

Recalling the definition (2.5.7) for $\widetilde{\psi}^t_{k\tilde{k}}$ and (2.5.11) for $\widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}$, and using $\|\cdot\|_{C^{d+2}} \lesssim \|\cdot\|_{H^s}$,

$$|\mathcal{M}(\widetilde{\psi}^t_{k\tilde{k}}, \widetilde{\boldsymbol{\phi}}^t_{k\tilde{k}}, t \wedge \tilde{\boldsymbol{T}}_{k\tilde{k}})| \leq C\|\psi\|_{\mathcal{L}^2_{pow,r}} \left(1 + \|\boldsymbol{\varphi}\|^{3r+2}_{H^s}\right) N^{(r+1)\epsilon+\kappa}. \tag{2.6.4}$$

Furthermore, with (2.5.8) and $\|\breve{\boldsymbol{\phi}}^t_m\|_{C^{d+2}} \leq (1 + |m|^{d+2})(1 + N^{\kappa}\|\boldsymbol{\phi}\|_{H^s})$ due to (2.5.55), we get an inequality concerning the terms from the linearization compensation, namely

$$\begin{aligned}|\mathcal{M}(\breve{\psi}^t_k, \breve{\boldsymbol{\phi}}^t_{k,m,\alpha\beta,\ell}, t \wedge \breve{\boldsymbol{T}}_k)| \\ \leq C\|\psi\|_{\mathcal{L}^1_{pow,r}} \left(1 + \|\boldsymbol{\varphi}\|^r_{H^s}\right) \left(1 + |m|^{(r+2)(d+2)}_{\infty}\right) \left(1 + \|\boldsymbol{\varphi}\|^{r+2}_{H^s}\right) N^{(r+2)\epsilon+\kappa}.\end{aligned} \tag{2.6.5}$$

Plugging these back into (2.6.1) and applying (2.5.9) to $\hat{F}^{\ell}_{k,\alpha\beta}[m]$ we obtain

$$\begin{aligned}|\mathcal{M}(\psi, \boldsymbol{\varphi}, \boldsymbol{T})| \\ \leq C\|\psi\|_{\mathcal{L}^3_{pow,r}} \left(1 + \|\boldsymbol{\varphi}\|^{3r+3}_{H^s}\right) \left(N^{-1/2} + h^{p+1} + \exp\left(-CN^{\epsilon/2}\right)\right) N^{(r+2)\epsilon+\kappa},\end{aligned} \tag{2.6.6}$$

where the infinite sum converges since we assumed $s_I > (d+2)(r+2) + 2d + 1$.

These considerations, together with the definition of $s(d,p,j)$ and the assumption on s_I as given in Theorem 2.2.1 yield (2.2.3) (specifically, the assumptions for $j = 1$ correspond to $s = s(d,p,1) = p + 3d/2 + 5$ and $s_I > 3(p + 3d/2 + 5) + 2d + 1$ while the use of the norm d_{-3} corresponds to $q = 3$ and $r = 0$).

Step 2: The case $j > 1$. We iteratively apply Theorem 2.5.1 j-times starting with arbitrary $\boldsymbol{\varphi} \in \left[H^{s_0}(\mathbb{T}^d, \mathbb{R}^{n_S})\right]^K$, $s_0 = s(d,p,j)$, and arbitrary $\psi \in \mathcal{L}^{q_0}_{pow,r_0}$ corresponding to the metric

$d_{-(2j+1)}$, i.e. $q_0 = 2j+1$, $r_0 = 0$. Note that after the first application, some of the test functions and generalized moment functions coming from the Fourier decomposition will be complex-valued. However, Theorem 2.5.1 remains valid when adapting the assumptions on the values of the test functions and generalized moment functions accordingly. The proof does not rely on having real-valued quantities.

This iterative application results in a tree structure: Starting from (ψ, ϕ) the first application of Theorem 2.5.1 spawns three successors: The error term $\mathrm{Err}(\psi, \phi)$ as defined in (2.6.2), the *regular* successor $(\widetilde{\psi}, \widetilde{\phi})$ and the *compensation* successor $(\breve{\psi}, \breve{\phi}_{m,\alpha\beta,\ell})_{m\in\mathbb{Z}^{2d}}$ – for simplicity we do not explicitly keep track of indices $t, k, \tilde{k}, \alpha, \beta$ and the test times; all estimates are independent of these anyways. The edge to the regular successor is labeled with $C_{K,T}N^{-1/2}$, the edge to the compensation successor with $C_{K,T}N^{-1/2}\sum_{m\in\mathbb{Z}^{2d}}\hat{F}[m]$. The error term stays untouched and forms a leaf node of the tree. Then the same process is repeated for the regular and compensation successor. We iterate until we reach depth j.

Each path from the root node to a leaf node corresponds to one distinct group of terms resulting from the iterations. If we add up the bounds for these groups, we obtain a bound for $\mathcal{M}(\psi, \phi)$. Naturally, each path either ends in an error term or is of length j. Thus, with the considerations for the case $j = 1$, the corresponding terms are bounded by either $C(\|\psi\|_{C^{2j+1}}, \|\varphi\|_{s_0})h^{p+1}N^{-\breve{j}/2+\breve{r}\epsilon+\kappa}$ or $C(\|\psi\|_{C^{2j+1}}, \|\varphi\|_{s_0})N^{-j/2+\tilde{r}\epsilon+\kappa}$ for suitable $\breve{j}, \breve{r}, \tilde{r} \in \mathbb{N}_0$. We only need to show three things:

- the remaining regularity of the test functions and generalized moment functions at the second to last node of all paths is sufficient to close the estimates as in Step 1,
- all appearing combinations of $\breve{j}, \breve{r}, \tilde{r}$ fit the scaling in (2.2.4),
- s_I is chosen large enough so that along all paths each infinite sum converges.

Each path is represented by a sequence $(\psi_{0,\vec{m}_0}, \varphi_{0,\vec{m}_0})_{\vec{m}_0}, \dots, (\psi_{\tilde{j},\vec{m}_{\tilde{j}}}, \varphi_{\tilde{j},\vec{m}_{\tilde{j}}})_{\vec{m}_{\tilde{j}}}, (\mathrm{leaf}_{\vec{m}_{\tilde{j}}})_{\vec{m}_{\tilde{j}}}$ with $\tilde{j} \le j-1$, $(\psi_{0,\vec{m}_0}, \varphi_{0,\vec{m}_0})_{\vec{m}_0} = (\psi, \varphi)$, and $\vec{m}_i$ for $0 \le i \le \tilde{j}$ being the vector of the collected $\mathbb{Z}^{2d}$-indices for all compensation nodes up until i.

The necessary regularity is given by sequences $(q_i)_{i=1,\dots,\tilde{j}}, (r_i)_{i=1,\dots,\tilde{j}}, (s_i)_{i=1,\dots,\tilde{j}} \subset \mathbb{N}_0$ such that we control $\|\psi_{i,\vec{m}_i}\|_{\mathcal{L}^{q_i}_{pow,r_i}}$ and $\|\varphi_{i,\vec{m}_i}\|_{H^{s_i}}$ in terms of their predecessors. With respect to $(q_i)_i$, from Step 1 we know that we need $q_{\tilde{j}} \ge 3$. For $i = 1, \dots, \tilde{j}$ we have

$$q_i = \begin{cases} q_{i-1} - 2 & \text{if } (\psi_{i,\vec{m}_i}, \varphi_{i,\vec{m}_i})_{\vec{m}_i} \text{ regular successor, corresponding to (2.5.7),} \\ q_{i-1} - 1 & \text{if } (\psi_{i,\vec{m}_i}, \varphi_{i,\vec{m}_i})_{\vec{m}_i} \text{ compensation successor, see (2.5.8).} \end{cases}$$

Since by definition $q_0 = 2j+1$, we satisfy $q_{\tilde{j}} \ge 3$. For $(r_i)_i$, by definition $r_0 = 0$ and

$$r_i = \begin{cases} r_{i-1} + 1 & \text{if } (\psi_{i,\vec{m}_i}, \varphi_{i,\vec{m}_i})_{\vec{m}_i} \text{ regular successor, corresponding to (2.5.7),} \\ r_{i-1} + 2 & \text{if } (\psi_{i,\vec{m}_i}, \varphi_{i,\vec{m}_i})_{\vec{m}_i} \text{ compensation successor, see (2.5.8).} \end{cases}$$

Concerning $(s_i)_i$, with (2.5.11) and (2.5.12), for both types of successors we have $s_i = s_{i-1} - 1$. Since $s(d, p, j) = p + 3d/2 + 4 + j$ we guarantee $s_{\tilde{j}} > p + 3d/2 + 5$ as assumed in Step 1: In summary, for all i we have

$$r_i \le 2i, \qquad s_i = p + \frac{3d}{2} + 4 + j - i. \tag{2.6.7}$$

It now remains to check that N^ϵ only appears as given in (2.2.4) and that s_I is chosen large enough to guarantee the convergence of the infinite sums. Let $\left((\psi_{i_\ell,\vec{m}_{i_\ell}}, \varphi_{i_\ell,\vec{m}_{i_\ell}})_{\vec{m}_{i_\ell}}\right)_{\ell=1,\dots,l}$

with $(i_\ell)_\ell \subset \{1,\dots,\tilde{j}\}$ be the compensation nodes on the path. Let $\mathcal{G}$ be the group of terms corresponding to this path. Respectively, (2.6.3), (2.6.4), and (2.6.5) yield an estimate for the leaf term as in (2.6.6). Adding the edge labels we obtain

$$|\mathcal{G}| \leq C_{K,T} \sum_{\vec{m}_{\tilde{j}}} N^{-\tilde{j}/2} \Big(\prod_{\ell=1}^{l} \hat{F}[m_{i_\ell}] \Big) \|\psi_{\tilde{j},\vec{m}_{\tilde{j}}}\|_{\mathcal{L}^{q_{\tilde{j}}}_{pow,r_{\tilde{j}}}} \\ \times \Big(1 + \|\boldsymbol{\varphi}_{\tilde{j}}\|^{3r_{\tilde{j}}+3}_{H^{s_{\tilde{j}}}}\Big) \Big(N^{-(j-\tilde{j})/2} + h^{p+1} + \exp\Big(-CN^{\epsilon/2}\Big)\Big) N^{(r_{\tilde{j}}+2)\epsilon+\kappa}.$$

With respect to the N^ϵ-scaling, note that the term $N^{-(j-\tilde{j})/2}$ actually only appears if the leaf does not correspond to an error term, in particular if $\tilde{j} = j-1$. Plugging in $r_{\tilde{j}} \leq 2\tilde{j}$ and using that $N^{-\tilde{j}/2} N^{(2\tilde{j}+2)\epsilon} < N^{2\epsilon}$ since $\epsilon < 1/4$, we obtain the scaling from (2.2.4).

With respect to the convergence of the sums, we iteratively replace $((\psi_{\tilde{j},\vec{m}_{\tilde{j}}}, \boldsymbol{\varphi}_{\tilde{j},\vec{m}_{\tilde{j}}})_{\vec{m}_{\tilde{j}}})$ with their predecessors via (2.5.7), (2.5.8), (2.5.11), and (2.5.12). Assuming convergence for the previous steps, we obtain a sequence $(\lambda_i)_{i=1,\dots,\tilde{j}} \subset \mathbb{N}$ such that

$$|\mathcal{G}| \leq C_{K,T} \sum_{\vec{m}_i} \Big(\prod_{i_\ell \leq i} \hat{F}[m_{i_\ell}] \Big) \|\psi_{i,\vec{m}_i}\|_{\mathcal{L}^{q_i}_{pow,r_i}} \\ \times \big(1 + \|\boldsymbol{\varphi}_i\|^{\lambda_i}_{H^{s_i}}\big) \Big(N^{-(j-\tilde{j})/2} + h^{p+1} + \exp\Big(-CN^{\epsilon/2}\Big)\Big) N^{-\tilde{j}/2+(2\tilde{j}+2)\epsilon+\kappa}. \quad (2.6.8)$$

This sequence is given by $\lambda_{\tilde{j}} := 3r_{\tilde{j}} + 3$ and from (2.5.7)/(2.5.8) for $\|\psi_{i,\vec{m}_i}\|_{\mathcal{L}^{q_i}_{pow,r_i}}$ and (2.5.11)/(2.5.12) for $\|\boldsymbol{\varphi}_i\|$ we can read off that

$$\lambda_{i-1} = \begin{cases} 2\lambda_i + r_{i-1} & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ regular successor,} \\ \lambda_i + r_{i-1} & \text{if } (\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i} \text{ compensation successor.} \end{cases}$$

Plugging in (2.5.9) for $\hat{F}[m_i]$ (assuming that $(\psi_{i,\vec{m}_i}, \boldsymbol{\varphi}_{i,\vec{m}_i})_{\vec{m}_i}$ is a compensation successor) and (2.5.12) for $\|\boldsymbol{\varphi}_i\|_{H^{s_i}}$ into (2.6.8), we see that the convergence of the sum over m_i can be guaranteed if for each $\ell = 1,\dots,l$ we have $s_I > s_{i_\ell}\lambda_{i_\ell} + 2d + 1$. With (2.6.7) we roughly estimate $\lambda_i \leq (j-i)2^{j-i}3j$. Hence, due to our choice of s_I we indeed have for $i \geq 1$ that

$$s_I > 2^{j-1}3j^2 s(d,p,j) + 2d + 1 \geq \lambda_i s_i + 2d + 1.$$

Thus, all infinite sums along the path converge. This completes the proof. □

2.7 Quantitative convergence to the mean-field limit

Proposition 2.7.1. *Let $\{X^{r_I,N}_{\alpha,i}\}^{\alpha=1,\dots,n_S}_{i=1,\dots,N}$ be a solution to the interacting particle system (2.3.1) satisfying Assumption 2.3.7 with mean field limit $\overline{\rho}^{r_I}$ satisfying Assumption 2.3.8. Let $\{\tilde{X}^{r_I,N}_{\alpha,i}\}$ be a solution to the diffusing particle system with mean-field forces*

$$\begin{cases} \mathrm{d}\tilde{X}^{r_I}_{\alpha,i}(t) = -\sum_{\beta=1}^{n_S} (\nabla V^{r_I}_{\alpha\beta} * \overline{\rho}^{r_I}_\beta)(\tilde{X}^{r_I,N}_{\alpha,i}(t), t)\,\mathrm{d}t, + \sqrt{2\sigma_\alpha}\,\mathrm{d}B_{\alpha,i}(t) \\ \tilde{X}^{r_I}_{\alpha,i}(0) = X^{r_I}_{\alpha,i}(0) \end{cases} \quad (2.7.1)$$

(cfr. [CDJ19, System (3)]). Then, for all $\alpha \in \{1,\dots,n_S\}$ and all $i \in \{1,\dots,N\}$, we obtain

$$\sup_{t\in[0,T]} \big|X^{r_I,N}_{\alpha,i} - \tilde{X}^{r_I,N}_{\alpha,i}\big|(t) \quad (2.7.2)$$
$$\leq C(T+1)\exp\Big(CTr_I^{-(d+2)}\Big)\Big(\big\|\mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\big\|_{H^{-d/2-2}} + \mathcal{C}N^{-1/2}\Big),$$

where $\mathcal{C}$ is a random variable with Gaussian moments $\mathbb{E}[\exp \frac{1}{C}\mathcal{C}^2] \leq 3$. Furthermore, for any $\kappa > 0$, also using (2.3.15) from Assumption 2.3.11, there exists $N_0 = N_0(\kappa, data)$ such that for all $N \geq N_0$ we have the estimate

$$\begin{aligned}
&\left\|\mu_{\alpha,t}^{r_I,N} - \overline{\rho}_\alpha^{r_I}(t)\right\|_{H^{-d/2-2}} \\
&\quad \leq C(T+1)\exp\left(CTr_I^{-(d+2)}\right)\left(\left\|\mu_0^{r_I,N} - \overline{\rho}^{r_I}(0)\right\|_{H^{-d/2-2}} + \mathcal{C}N^{-1/2}\right) \qquad (2.7.3)\\
&\quad \leq \mathcal{C}N^{\kappa T}N^{-1/2}. \qquad (2.7.4)
\end{aligned}$$

To prove Proposition 2.7.1, we first establish the following bound.

Lemma 2.7.2. *Suppose that the assumptions of Proposition 2.7.1 hold. Additionally, set $\tilde{\mu}_{\alpha,t}^{r_I,N} := \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{X}_{\alpha,i}^{r_I}(t)}$. Then the estimate*

$$\sup_{t\in[0,T]}\left|X_{\alpha,i}^{r_I,N} - \tilde{X}_{\alpha,i}^{r_I,N}\right|(t) \leq C(T+1)\exp\left(CTr_I^{-(d+2)}\right)\int_0^T \left\|\tilde{\mu}_t^{r_I,N} - \overline{\rho}^{r_I}(t)\right\|_{H^{-d/2-2}}\mathrm{d}t$$

holds for all all $\alpha \in \{1,\dots,n_S\}$ and all $i \in \{1,\dots,N\}$.

Proof. By subtracting (2.3.1) and (2.7.1), we obtain

$$\begin{aligned}
&\mathrm{d}\big(X_{\alpha,i}^{r_I,N} - \tilde{X}_{\alpha,i}^{r_I,N}\big)(t)\\
&= -\sum_{\beta=1}^{n_S} N^{-1}\sum_{j=1}^N \left(\nabla V_{\alpha\beta}^{r_I}\left(X_{\alpha,i}^{r_I,N}(t) - X_{\beta,j}^{r_I,N}(t)\right) - (\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I})(\tilde{X}_{\alpha,i}^{r_I,N}(t))\right)\mathrm{d}t\\
&= -\sum_{\beta=1}^{n_S} N^{-1}\sum_{j=1}^N \left(\nabla V_{\alpha\beta}^{r_I}\left(X_{\alpha,i}^{r_I,N}(t) - X_{\beta,j}^{r_I,N}(t)\right) - \nabla V_{\alpha\beta}^{r_I}\left(X_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\right)\right)\mathrm{d}t\\
&\quad -\sum_{\beta=1}^{n_S} N^{-1}\sum_{j=1}^N \left(\nabla V_{\alpha\beta}^{r_I}\left(X_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\right) - \nabla V_{\alpha\beta}^{r_I}\left(\tilde{X}_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\right)\right)\mathrm{d}t\\
&\quad -\sum_{\beta=1}^{n_S}\left(N^{-1}\sum_{j=1}^N \nabla V_{\alpha\beta}^{r_I}\left(\tilde{X}_{\alpha,i}^{r_I,N}(t) - \tilde{X}_{\beta,j}^{r_I,N}(t)\right) - (\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I})(\tilde{X}_{\alpha,i}^{r_I,N}(t))\right)\mathrm{d}t
\end{aligned}$$

which yields

$$\begin{aligned}
\left|X_{\alpha,i}^{r_I,N} - \tilde{X}_{\alpha,i}^{r_I,N}\right|(T) &\leq C\left\|\nabla^2 V^{r_I}\right\|_{L^\infty}\int_0^T \sum_{\beta=1}^{n_S}\frac{1}{N}\sum_{j=1}^N \left|X_{\beta,j}^{r_I,N} - \tilde{X}_{\beta,j}^{r_I,N}\right|(t)\,\mathrm{d}t\\
&\quad + C\int_0^T \sum_{\beta=1}^{n_S}\|\nabla V^{r_I}\|_{H^{d/2+2}}\left\|\tilde{\mu}_{\beta,t}^{r_I,N} - \overline{\rho}_\beta^{r_I}\right\|_{H^{-d/2-2}}\mathrm{d}t.
\end{aligned}$$

The Gronwall inequality implies an estimate of the desired order for the averaged and time-integrated error $\int_0^T \sum_{\beta=1}^{n_S}\frac{1}{N}\sum_{j=1}^N \left|X_{\beta,j}^{r_I,N} - \tilde{X}_{\beta,j}^{r_I,N}\right|(t)\,\mathrm{d}t$; plugging this back into the previous inequality, we arrive at our claim using $\|\nabla^2 V^{r_I}\|_{L^\infty} \leq C{r_I}^{-(d+2)}$ and absorbing ${r_I}^{-1}$ factors (coming from the potential V^{r_I}) into the exponential. □

Remark 2.7.3 (Initial distribution of the particles). We only prove Proposition 2.7.1 for the case that the particle positions initially satisfy the spectral gap inequality 2.3.7. The proof for the case of i.i.d. positions is analogous, replacing the spectral gap inequality with Hoeffdings inequality, and using that $\{\tilde{X}_{\alpha,i}^{r_I,N}(t)\}_{i=1}^N$ stay independent also for $t > 0$.

Proof of Proposition 2.7.1. In view of the bound from Lemma 2.7.2, we need to bound the term $\int_0^T \|\tilde{\mu}_t^{r_I,N} - \overline{\rho}^{r_I}\|_{H^{-d/2-2}} \,\mathrm{d}t$. Note that $w_\alpha := \mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}] - \overline{\rho}_\alpha^{r_I}$ solve the PDEs

$$\partial_t w_\alpha = \sigma_\alpha \Delta w_\alpha + \nabla \cdot \left(\sum_{\beta=1}^{n_S} w_\alpha \nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I} \right).$$

Testing this PDE with w with respect to the $H^{-d/2-2}$-inner product, with a standard energy estimate, and using the smoothness of V^{r_I} and $\overline{\rho}^{r_I}$ (cfr. Assumptions 2.3.7 and 2.3.8), we get

$$\left\|\mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}] - \overline{\rho}_\alpha^{r_I}\right\|_{H^{-\frac{d}{2}-2}} \lesssim \exp\left(C r_I^{-(d+1)} t\right) \left\|\mathbb{E}[\tilde{\mu}_{\alpha,0}^{r_I,N}] - \overline{\rho}_\alpha^{r_I}(0)\right\|_{H^{-\frac{d}{2}-2}}. \tag{2.7.5}$$

With (2.3.8) from Assumption 2.3.7, it only remains to bound the density fluctuations $\|\tilde{\mu}_{\alpha,t}^{r_I,N} - \mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}]\|_{H^{-d/2-2}}$ for the empirical measure of the auxiliary particle system. Using the product structure of the probability space from Assumption 2.3.7, we write

$$\tilde{\mu}_{\alpha,t}^{r_I,N} - \mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}] = \tilde{\mu}_{\alpha,t}^{r_I,N} - \mathbb{E}_0[\tilde{\mu}_{\alpha,t}^{r_I,N}] + \mathbb{E}_0\left[\tilde{\mu}_{\alpha,t}^{r_I,N} - \mathbb{E}_B[\tilde{\mu}_{\alpha,t}^{r_I,N}]\right],$$

where $\mathbb{E}_0$ and $\mathbb{E}_B$ are the expected values with respect to the probability measures $\mathbb{P}_0$ and $\mathbb{P}_B$ associated to the initial positions and Brownian motions respectively. This decompositions allows us to treat the impact of the initial positions on the fluctuations separately from the impact of the Brownian motions.

We first consider the impact of the random initial positions. Testing $\tilde{\mu}_{\alpha,t}^{r_I,N}$ with elements of the Fourier basis $\exp(i\xi \cdot x)$ with $\xi \in \mathbb{Z}^d$, we have

$$\int_{\mathbb{T}^d} \exp(i\xi \cdot x) \,\mathrm{d}\tilde{\mu}_{\alpha,t}^{r_I,N}(x) = N^{-1} \sum_{j=1}^{N} \exp(i\xi \cdot \tilde{X}_{\alpha,j}^{r_I,N}(t)) =: F_t = F_t\left((X_{\alpha,j}^{r_I,N}(0))_j\right). \tag{2.7.6}$$

Note that by (2.7.1) we have

$$\partial \tilde{X}_{\alpha,j}^{r_I,N}(t) / \partial X_{\alpha,j}^{r_I,N}(0) \leq C {r_I}^{-(d+1)} \qquad \mathbb{P}_B\text{-almost surely for all } t \in [0,T]$$

due to the cancellation of the martingale parts and the Lipschitz continuity of ∇V^{r_I}, where $\mathbb{P}_B$ is the probability measure associated to the Brownian motions. Applying the spectral gap inequality (2.3.7) (with the usual extension to general exponents ≥ 2) we obtain for any $p \geq 1$ that $\mathbb{P}_B$-almost surely

$$\begin{aligned}
&\mathbb{E}_0\left[\left| \int_{\mathbb{T}^d} (\tilde{\mu}_{\alpha,t}^{r_I,N} - \mathbb{E}_0[\tilde{\mu}_{\alpha,t}^{r_I,N}]) \exp(i\xi \cdot x) \,dx \right|^{2p}\right]^{1/p} \\
&\leq Cp\mathbb{E}_0\left[\left(\sum_{j=1}^{N} \left| \frac{\partial F_t}{\partial X_{\alpha,j}^{r_I,N}(0)} \right|^2 \right)^p\right]^{1/p} \leq Cp|\xi|^2 {r_I}^{-2(d+1)} N^{-1},
\end{aligned}$$

where F_t is defined in (2.7.6) and $\mathbb{E}_0$ is the expected value with respect to $\mathbb{P}_0$, the probability measure associated to the initial particle positions. By Chebyshev's inequality and optimizing in p we obtain that $\mathbb{P}_B$-almost surely

$$\mathbb{P}_0\left[\left| \int_{\mathbb{T}^d} (\tilde{\mu}_{\alpha,t}^{r_I,N} - \mathbb{E}_0[\tilde{\mu}_{\alpha,t}^{r_I,N}]) \exp(i\xi \cdot x) \,\mathrm{d}x \right| \geq C \frac{|\xi|\lambda}{{r_I}^{d+1} N^{1/2}}\right] \leq C \exp\left(-\lambda^2\right), \tag{2.7.7}$$

controlling the fluctuations associated to the random initial positions.

Regarding the impact of the Brownian motions, note that – for fixed initial positions – the trajectories $\left(\tilde{X}_{\alpha,j}^{r_I,N}\right)_{\alpha,j}$ of the auxiliary particle system (2.7.1) are independent with respect to $\mathbb{P}_B$ since the Brownian motions are i.i.d. and the interaction effects are calculated with respect to the mean field limit rather than the other particle positions. By Hoeffdings' inequality, we hence have

$$\mathbb{P}_B\left[\left|\int_{\mathbb{T}^d}(\tilde{\mu}_{\alpha,t}^{r_I,N}-\mathbb{E}_B[\tilde{\mu}_{\alpha,t}^{r_I,N}])\exp(i\xi\cdot x)\,\mathrm{d}x\right|\geq C\frac{\lambda}{N^{1/2}}\right]\leq C\exp\left(-\lambda^2\right)$$

for all $\left(X_{\alpha,j}^{r_I,N}(0)\right)_{\alpha,j}\subset\mathbb{T}^d$, and thus

$$\mathbb{P}_B\left[\left|\int_{\mathbb{T}^d}(\mathbb{E}_0[\tilde{\mu}_{\alpha,t}^{r_I,N}]-\mathbb{E}_0[\mathbb{E}_B[\tilde{\mu}_{\alpha,t}^{r_I,N}]])\exp(i\xi\cdot x)\,\mathrm{d}x\right|\geq C\frac{\lambda}{N^{1/2}}\right]\leq C\exp\left(-\lambda^2\right),\tag{2.7.8}$$

controlling the fluctuations associated to the Brownian motions.

Combining (2.7.7) and (2.7.8), using that $\mathbb{P}=\mathbb{P}_0\otimes\mathbb{P}_B$, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\left|\int_{\mathbb{T}^d}(\tilde{\mu}_{\alpha,t}^{r_I,N}-\mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}])\exp(i\xi\cdot x)\,\mathrm{d}x\right|\geq 2C\frac{|\xi|\lambda}{{r_I}^{(d+1)}N^{1/2}}\right]\\
&\quad\leq\mathbb{P}\left[\left|\int_{\mathbb{T}^d}(\tilde{\mu}_{\alpha,t}^{r_I,N}-\mathbb{E}_0[\tilde{\mu}_{\alpha,t}^{r_I,N}])\exp(i\xi\cdot x)\,\mathrm{d}x\right|\geq C\frac{|\xi|\lambda}{{r_I}^{(d+1)}N^{1/2}}\right]\\
&\qquad+\mathbb{P}\left[\left|\int_{\mathbb{T}^d}(\mathbb{E}_0[\tilde{\mu}_{\alpha,t}^{r_I,N}]-\mathbb{E}_0[\mathbb{E}_B[\tilde{\mu}_{\alpha,t}^{r_I,N}]])\exp(i\xi\cdot x)\,\mathrm{d}x\right|\geq C\frac{\lambda}{N^{1/2}}\right]\\
&\quad\leq C\exp\left(-\lambda^2\right).
\end{aligned}$$

This entails

$$\left|\int_{\mathbb{T}^d}(\tilde{\mu}_{\alpha,t}^{r_I,N}-\mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}])\exp(i\xi\cdot x)\,\mathrm{d}x\right|\leq\mathcal{C}|\xi|{r_I}^{-(d+1)}N^{-1/2}$$

for a random variable $\mathcal{C}$ subject to a Gaussian moment bound. We thus deduce

$$\begin{aligned}
&\left\|\tilde{\mu}_{\alpha,t}^{r_I,N}-\mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}]\right\|_{H^{-d/2-2}}^2\\
&\quad=\sum_{\xi\in\mathbb{Z}^d}(1+|\xi|^2)^{-d/2-2}\left|\int_{\mathbb{T}^d}(\bar{\mu}_{\alpha,t}^{r_I,N}-\mathbb{E}[\tilde{\mu}_{\alpha,t}^{r_I,N}])\exp(i\xi\cdot x)\,\mathrm{d}x\right|^2\leq\mathcal{C}{r_I}^{-2(d+1)}N^{-1}.
\end{aligned}\tag{2.7.9}$$

Plugging this estimate as well as (2.7.5) into Lemma 2.7.2, we arrive at (2.7.2). Combining (2.7.9) with (2.7.5) and using the fact that the left-hand side of (2.7.2) controls the 1-Wasserstein distance between $\mu^{r_I,N}$ and $\tilde{\mu}^{r_I,N}$ – which is precisely the $W^{-1,\infty}$ distance – and a Sobolev embedding, we arrive at (2.7.3). Then (2.7.4) follows via another application of the spectral gap inequality, and the bound on $\mathbb{E}\left[\mu_0^{r_I,N}\right]-\overline{\rho}^0$ (cfr. Assumption 2.3.7). □

2.8 Exponentially decaying estimate for the probability that $T<T_\oslash$

The stopping time $T_\oslash$ acts as a discrete analogue to the quantitative convergence of the empirical measure to the mean field limit proven in Section 2.7. That is, it guarantees that also in the discrete case the fluctuations are roughly of order $N^{-1/2}$. Additionally, together with Remark 2.3.14 it ensures that the solution $\rho_h^{r_I}$ of the discretized Dean–Kawasaki

equation (h-DK) stays positive. The stopping time is defined as follows. First, for this section set

$$l := \lfloor d/2 \rfloor + 1. \tag{2.8.1}$$

Given $\rho_h^{r_I}$, the discrete mean field limit $\overline{\rho}_h^{r_I}$, and small parameters $\epsilon, \delta > 0$, we define

$$T_\oslash = T_\oslash(\delta, \epsilon) := T_\oslash^\infty(\delta, \epsilon) \wedge T_\oslash^l(\delta, \epsilon) \wedge T, \tag{2.8.2}$$

where, with discrete L^p-norms as fixed in Definition 2.B.1 and negative Sobolev norms as fixed in Definition 2.3.6,

$$T_\oslash^\infty(\delta, \epsilon) := \begin{cases} \inf\{0 < t \leq T : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \geq N^{-\epsilon}\}, \\ \qquad\qquad \text{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{L_h^\infty} \leq N^{-\epsilon-\delta}, \\ 0, \qquad\qquad \text{otherwise}, \end{cases}$$

$$T_\oslash^l(\delta, \epsilon) := \begin{cases} \inf\{0 < t \leq T : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2\ell}} \geq N^{-1/2+\epsilon}\}, \\ \qquad\qquad \text{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{H_h^{-\ell}} \leq N^{-1/2+\epsilon-\delta}, \\ 0, \qquad\qquad \text{otherwise}. \end{cases}$$

However, for the main result and hence in Sections 2.5 and 2.6, for given $\epsilon > 0$ we always choose $\delta = \epsilon/2$. In any way, this stopping time is only meaningful if we can bound the probability of the stopping time triggering, that is $T_\oslash < T$, which results from the corresponding bounds being broken. Such control of $\mathbb{P}(T_\oslash < T)$ via stretched exponential bounds is established in the following proposition.

Remark 2.8.1. 1.) Note that the conditional definition of the stopping time ensures that if $T_\oslash > 0$, the size of the fluctuations at $t = 0$ is guaranteed to be smaller than the stopping condition by a factor of $N^{-\delta}$ (in a stronger norm too for $T_\oslash^l(\delta, \epsilon)$). If r_I does not scale but remains constant, then replacing $N^{-\delta}$ with a small enough constant would be sufficient.

2.) While the bounds satisfy $N^{-\epsilon} N^{-1/2+\epsilon} = N^{-1/2}$, the parameters for the two auxiliary stopping times are synced just for convenience: The arguments below can be adapted for $T_\oslash := T_\oslash^\infty(\delta, \epsilon) \wedge T_\oslash^l(\tilde{\delta}, \tilde{\epsilon})$. However, we can not hope to get good probability estimates if we choose stricter bounds in the definition of $T_\oslash$. The range for which we get good probability bounds is limited by Assumption 2.3.10 for the size of the fluctuations at $t = 0$. This assumption is justified by the dicussion in Section 2.C.

3.) With our method, we can not obtain probability estimates for the auxiliary stopping times separately due to the nonlinearity. In the proof, the nonlinearity appears in form of the linearization compensation $Q_{h,t}^{r_I,N}(\phi_h^t)$, for which to effectively bound we need an L^∞-bound as well as an order $\backsim N^{-1/2}$ bound.

Proposition 2.8.2. *Let $\rho_h^{r_I}$ solve (h-DK) and $\overline{\rho}_h^{r_I}$ solve (h-MFL). Suppose Assumption 2.3.7 (regularity of the potential), Assumption 2.3.8 (regularity of the continuous mean field limit), Assumption 2.3.9 (discrete differential operators), Assumption 2.3.10 (discrete initial conditions), and Assumption 2.3.11 (scaling regime) hold.*

Let the stopping time $T_\oslash$ be defined as in (2.8.2) with $\epsilon, \delta \in (0, \delta_0/4)$. Then there exists $N_0 = N_0(\delta, \epsilon, data)$ such that for all $N > N_0$ there holds

$$\mathbb{P}\left[T_\oslash < T\right] \leq C \exp\big(-CN^{\epsilon-\delta}\big). \tag{2.8.3}$$

Remark 2.8.3. Despite the definition of $T_\oslash^\infty$, larger ϵ yields better bounds. This is since contributions from $T_\oslash^\infty$ are given and – with Assumption 2.3.11 and $\epsilon < \delta_0/4$ – bounded by

$$\exp\left(-CN^{1/2-\epsilon-\delta} h^{d/2}\right) \leq \exp\big(-CN^{\delta_0/2-\epsilon-\delta}\big) \leq \exp\big(-CN^{\epsilon-\delta}\big).$$

Remark 2.8.4 (Strategy for the proof of Proposition 2.8.2 in Subsection 2.8.2). We will show that, with high probability, for all $0 < t \leq T_\oslash$ we continuously have stricter bounds than guaranteed just by $t \leq T_\oslash$. However, by definition this can only be the case if $T_\oslash = T$. Hence, as long as $T_\oslash > 0$, with high probability we obtain $T_\oslash = T$. Since Assumption 2.3.10 yields $T_\oslash > 0$ with high probability, this proves the proposition (here, 'with high probability' means that we obtain stretched exponential bounds as in (2.8.3) for the complement).
More precisely, we will proceed in seven steps as follows:

- For a fixed time, $\|\cdot\|_{L_h^\infty}$ and $\|\cdot\|_{H_h^{-2l}}$ can be represented via suitable test functions. Hence, we can calculate them with the evolutional scheme from Section 2.5.1 (Steps 1 and 2).
- Via moment bounds for tested fluctuations (see Lemma 2.8.5 below) we obtain stricter bounds at a fixed time T_0 with high probability assuming $T_0 \leq T_\oslash$ (Steps 3 and 4).
- Given a discretization $0 \leq t_1 < t_2 < \ldots \leq T$ of $[0,T]$ with step size $\backsim h^\theta$ for arbitrary $\theta > 0$, we have stricter bounds for all $t_k \leq T_\oslash$ with high probability (Step 5).
- If θ is large enough, then with high probability we have strict bounds on the difference between the fluctuations at any time $t < T_\oslash$ and at the closest previous time step (Step 6).
- We now cover all $t \leq T_\oslash$ and conclude thanks to the bound on $\mathbb{P}[T_\oslash = 0]$ (Step 7).

2.8.1 Moment bounds for the discrete fluctuations

Here, given a stopping time with the same structure as $T_\oslash$, we provide a moment bound crucial for the proof of Proposition 2.8.2.

Lemma 2.8.5 (Moment bounds for the discrete fluctuations). *Let $\rho_h^{r_I}$ solve (h-DK) and $\overline{\rho}_h^{r_I}$ solve (h-MFL). Suppose Assumptions 2.3.7, 2.3.8, and 2.3.9 hold. Let $\overline{\rho}_h^{r_I}(0) = \mathcal{I}_h[\overline{\rho}^{r_I}(0)]$ and with respect to the scaling regime of r_I assume that (2.3.15) from Assumption 2.3.11 holds. Define the stopping time $\widetilde{T}_\oslash := \widetilde{T}_\oslash^\infty \wedge \widetilde{T}_\oslash^l \wedge T$ for some $B_\infty, B_l, B_\infty^0, B_l^0 \in \mathbb{R}$ as*

$$\widetilde{T}_\oslash^\infty := \begin{cases} \inf\{t > 0 : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \geq B_\infty\}, & \text{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{L_h^\infty} \leq B_\infty^0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\widetilde{T}_\oslash^l := \begin{cases} \inf\{t > 0 : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2l}} \geq B_l\}, & \text{if } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(0)\|_{H_h^{-l}} \leq B_l^0, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\varphi_h \in L^2(G_{h,d}, \mathbb{R}^{n_S})$ with backwards evolution $\phi_h^t \in C^1(L^2(G_{h,d}, \mathbb{R}^{n_S}))$ as in (2.5.22).

Then for each $\kappa > 0$ there exists $N_0 = N_0(data, \kappa) \in \mathbb{N}$ independent of j such that for all $N \geq N_0$ and each even $j \in \mathbb{N}$ there holds

$$\mathbb{E}\left[\chi(\widetilde{T}_\oslash > 0) \sup_{0 \leq t \leq \widetilde{T}_\oslash} \left(\phi_h^t, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\right)_h^j\right] \leq j^j \Big(\max\Big\{ CN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2},\ j^{-1}CN^\kappa \min\{\|\varphi_h\|_{L_h^1} B_\infty^0, \|\varphi_h\|_{H_h^l} B_l^0\},$$
$$j^{-1}CN^\kappa \min\{h^{-1}\|\varphi_h\|_{L_h^1}, \|\varphi_h\|_{L_h^2}\} B_\infty B_l \Big\}\Big)^j. \quad (2.8.4)$$

Proof. Lemma 2.5.4 implies that

$$\mathcal{M}_t := \chi(\widetilde{T}_\oslash > 0)\left(\phi_h^{t\wedge\widetilde{T}_\oslash}, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t \wedge \widetilde{T}_\oslash)\right)_h - \int_0^{t\wedge\widetilde{T}_\oslash} Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s$$

is a martingale with Dean–Kawasaki-type noise as in (2.5.24). Hence, Doob's inequality yields

$$\mathbb{E}\left[\chi(\widetilde{T}_{\oslash}>0)\sup_{0\le t\le\widetilde{T}_{\oslash}}\left(\left(\phi_h^t,(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t)\right)_h-\int_0^t Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s\right)^j\right]$$
$$\le\left(\frac{j}{j-1}\right)^j\mathbb{E}\left[\chi(\widetilde{T}_{\oslash}>0)\left(\left(\phi_h^{\widetilde{T}_{\oslash}},(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(\widetilde{T}_{\oslash})\right)_h-\int_0^{\widetilde{T}_{\oslash}}Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s\right)^j\right]. \quad (2.8.5)$$

Note that $(j/(j-1))^j\le 4$. For the right hand side, up to this constant, via the Itô rule with (2.5.24) we obtain for large enough N that

$$\begin{aligned}\mathbb{E}\left[(\mathcal{M}_{\widetilde{T}_{\oslash}})^j\right]&=\mathbb{E}\left[\chi(\widetilde{T}_{\oslash}>0)\left(\phi_h^0,(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(0)\right)_h^j\right]\\&\quad+\mathbb{E}\left[\int_0^{\widetilde{T}_{\oslash}}j(j-1)\,(\mathcal{M}_t)^{j-2}\sum_{\alpha=1}^{n_S}N^{-1}\sigma_\alpha\left(|\nabla_h\phi_{h,\alpha}^t|^2,(\rho_{h,\alpha}^{r_I})^+(t)\right)_h\mathrm{d}t\right].\\&\le\min\left\{\|\phi_h^t\|_{L_h^1}B_\infty^0,\|\phi_h^t\|_{H_h^l}B_l^0\right\}^j\\&\quad+j^2N^{-1}|\sigma|\rho_{max,h}\mathbb{E}\left[\sup_{0\le t\le\widetilde{T}_{\oslash}}(\mathcal{M}_t)^{j-2}\right]\int_0^T\|\nabla_h\phi_h^t\|_{L_h^2}^2\,\mathrm{d}t\\&\le\left(CN^\kappa\min\left\{\|\varphi_h\|_{L_h^1}B_\infty^0,\|\varphi_h\|_{H_h^l}B_l^0\right\}\right)^j\\&\quad+\left(CjN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}\right)^2\mathbb{E}\left[\sup_{0\le t\le\widetilde{T}_{\oslash}}(\mathcal{M}_t)^j\right]^{(j-2)/j}.\end{aligned}$$

For the first inequality, we applied Hölder's inequality (Corollary 2.B.2), used the definitions of $\|\cdot\|_{H_h^s}$ and the stopping time, and that we can bound $\|(\rho^{r_I}(t\wedge\widetilde{T}_{\oslash}))^+\|_{L_h^\infty}\le\rho_{max,h}$ as in Remark 2.3.14. The second inequality follows from bounds for the backwards evolution of test functions in Lemma 2.B.8 and Lemma 2.B.10, as well as an application of Jensen's inequality.

Plugging this inequality back into (2.8.5), and applying Young's inequality to absorb the moment-term (this is necessary only for $j>2$), we obtain

$$\mathbb{E}\left[\chi(\widetilde{T}_{\oslash}>0)\sup_{0\le t\le\widetilde{T}_{\oslash}}\left(\left(\phi_h^t,(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t)\right)_h-\int_0^t Q_{h,s}^{r_I,N}(\phi_h^s)\,\mathrm{d}s\right)^j\right]$$
$$\le\left(CN^\kappa\min\left\{\|\varphi_h\|_{L_h^1}B_\infty^0,\|\varphi_h\|_{H_h^l}B_l^0\right\}\right)^j+j^j\left(CN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}\right)^j.$$

Taking the j-th root, applying the reverse triangle inequality on the left, moving the Q-term to the other side, taking the j-th power, and applying $(a+b)^j\le 2^ja^j+2^jb^j$ yields

$$\mathbb{E}\left[\chi(\widetilde{T}_{\oslash}>0)\sup_{0\le t\le\widetilde{T}_{\oslash}}\left(\phi_h^t,(\rho_h^{r_I}-\overline{\rho}_h^{r_I})(t)\right)_h^j\right]\le j^j\left(CN^{-1/2+\kappa}\|\varphi_h\|_{L_h^2}\right)^j$$
$$+\left(CN^\kappa\min\left\{\|\varphi_h\|_{L_h^1}B_\infty^0,\|\varphi_h\|_{H_h^l}B_l^0\right\}\right)^j+2^j\mathbb{E}\left[\left(\int_0^{\widetilde{T}_{\oslash}}\left|Q_{h,t}^{r_I,N}(\phi_h^t)\right|\,\mathrm{d}t\right)^j\right].$$

Hence, to finish the proof it is enough to show that

$$\mathbb{I}_Q := \int_0^{\widetilde{T}_\oslash} \left| Q_{h,t}^{r_I,N}(\phi_h^t) \right| \mathrm{d}t \leq CN^\kappa \min\{\|\varphi_h\|_{L_h^2}, h^{-1}\|\varphi_h\|_{L_h^1}\} B_\infty B_l. \tag{2.8.6}$$

This follows immediately from the two following estimates by applying the test function estimates from Lemma 2.B.8 and Lemma 2.B.10, the definition of the stopping time, and the observation $\|\mathcal{I}_h[\nabla V^{r_I}]\|_{H_h^{2l}} \lesssim \|V^{r_I}\|_{C^{2l+1}} \lesssim {r_I}^{-(d+2l+1)}$: Hölder's inequality yields

$$\begin{aligned}
\mathbb{I}_Q &= \int_0^{\widetilde{T}_\oslash} \left| \sum_{\alpha,\beta=1}^{n_S} \left(\nabla_h \phi_{h,\alpha}^t \cdot (\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h (\rho_{h,\beta}^{r_I} - \overline{\rho}_{h,\beta}^{r_I})(t)), (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(t) \right)_h \right| \mathrm{d}t \\
&\leq \begin{cases} \left(\int_0^T \|\nabla_h \phi_h^t\|_{L_h^2}^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^{\widetilde{T}_\oslash} \|\mathcal{I}_h[\nabla V^{r_I}]\|_{H_h^{2l}}^2 \|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{H_h^{-2l}}^2 \|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{L_h^\infty}^2 \,\mathrm{d}t \right)^{\frac{1}{2}}, \\ h^{-1} \int_0^{\widetilde{T}_\oslash} \|\phi_h^t\|_{L_h^1} \|\mathcal{I}_h[\nabla V^{r_I}]\|_{H_h^{2l}} \|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{H_h^{-2l}} \|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{L_h^\infty} \,\mathrm{d}t. \end{cases}
\end{aligned}$$

Here, for the first option we used that $\|\cdot\|_{L_h^2} \leq C\|\cdot\|_{L_h^\infty}$ before applying Young's convolution inequality (Corollary 2.B.3). For the second option, we used that $\|\nabla_h f\|_{L_h^1} \leq Ch^{-1}\|f\|_{L_h^1}$ for any $f \in L^2(G_{h,d})$. □

2.8.2 Proof of the bound for $\mathbb{P}(T_\oslash < T)$

Proof of Proposition 2.8.2. The proof consists of seven steps, outlined in Remark 2.8.4.

Step 1: $\|\cdot\|_{L_h^\infty}$ via test functions. We define $\varphi_{h,x_0}^\gamma \in [L^2(G_{h,d})]^{n_S}$ for any $\gamma \in \{1,\dots,n_S\}$ and $x_0 \in G_{h,d}$ as $\varphi_{h,x_0,\alpha}^\gamma := h^{-d/2}\delta_{\gamma,\alpha} f_{x_0}$ where $\delta_{\gamma,\alpha}$ is the Dirac delta and f_{x_0} as defined in Section 2.3.2. Hence, for any $\eta_h \in [L^2(G_{h,d})]^{n_S}$ we have $\left(\varphi_{h,x_0}^\gamma, \eta_h\right)_h = \eta_{h,\gamma}(x_0)$ and therefore

$$\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} = \max_{x_0 \in G_{h,d}} \max_{\gamma=1,\dots,n_S} \left| \left(\varphi_{h,x_0}^\gamma, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\right)_h \right|. \tag{2.8.7}$$

Note that by the definition of φ_{h,x_0}^γ we have

$$\|\varphi_{h,x_0}^\gamma\|_{L_h^1} = 1, \qquad \|\varphi_{h,x_0}^\gamma\|_{L_h^2} = h^{-d/2}. \tag{2.8.8}$$

Step 2: $\|\cdot\|_{H_h^{-2l}}$ via test functions. We define $\varphi_{h,m}^\gamma \in [L^2(G_{h,d})]^{n_S}$ for any $\gamma \in \{1,\dots,n_S\}$ and $m \in \mathbb{Z}^d \cap \left[-\frac{\pi}{h}, \frac{\pi}{h}\right)^d$ as $\varphi_{h,m,\alpha}^\gamma := \delta_{\gamma,\alpha}\vartheta_m$ where $(\vartheta_m)_m$ is the discrete Fourier basis as introduced in Remark 2.3.5. Hence, with Definition 2.3.6 we have

$$\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2l}}^2 = \sum_{\gamma=1}^{n_S} \sum_m \frac{1}{(1+|m|^2)^{2l}} \left(\varphi_{h,m}^\gamma, (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\right)_h^2, \tag{2.8.9}$$

where the sum is over $m \in \mathbb{Z}^d \cap \left[-\frac{\pi}{h}, \frac{\pi}{h}\right)^d$. For $\varphi_{h,m}^\gamma$ we have

$$\|\varphi_{h,m}^\gamma\|_{L_h^1} = (2\pi)^d, \qquad \|\varphi_{h,m}^\gamma\|_{L_h^2} = (2\pi)^{d/2}, \qquad \|\varphi_{h,m}^\gamma\|_{H_h^l} \leq (1+|m|^2)^{l/2}, \tag{2.8.10}$$

where the last equation is an immediate consequence of Remark 2.3.5.

Step 3: Stricter L_h^∞-bounds for a single time point $0 < T_0 \leq T_\oslash$. Let $0 < \kappa < \delta$. In this step, we will show that, if N is large enough, for any fixed $T_0 \in [0,T]$ we have

$$\mathbb{P}\left[T_0 \leq T_\oslash \text{ and } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(T_0)\|_{L_h^\infty} \geq \frac{N^{-\epsilon}}{4} \right] \leq n_S h^{-d} \exp\left(-CN^{1/2-\epsilon-\kappa} h^{d/2}\right) \tag{2.8.11}$$

In order to achieve this, we will apply Lemma 2.8.5 to the backwards evolution of the test functions φ^γ_{h,x_0} from Step 1 with $j = 2\left\lfloor \frac{1}{4}C_0^{-1}\frac{N^{-\epsilon}}{4}N^{1/2-\kappa}\|\varphi^\gamma_{h,x_0}\|^{-1}_{L^2_h}\right\rfloor$ where C_0 is the constant in Lemma 2.8.5. The reasoning for this choice will be explained at the end of this step. Note that we have $j \in 2\mathbb{N}$ for large enough N since with (2.8.8) and (2.3.13) from Assumption 2.3.11

$$N^{1/2-\kappa-\epsilon}\|\varphi^\gamma_{h,x_0}\|^{-1}_{L^2_h} = N^{1/2-\kappa-\epsilon}h^{d/2} \geq N^{\delta_0/2-\kappa-\epsilon}.$$

We will now show that in our setting the maximum on the right hand side of (2.8.4) is dominated by $C_0N^{-1/2+\kappa}\|\varphi^\gamma_{h,x_0}\|_{L^2_h}$. Note that for our choice of j

$$N^{-\epsilon}/16 \leq j\big(C_0N^{-1/2+\kappa}\|\varphi^\gamma_{h,x_0}\|_{L^2_h}\big) \leq N^{-\epsilon}/8. \tag{2.8.12}$$

Regarding the middle term on the right hand side of (2.8.4), with (2.8.8) and $\kappa < \delta$, we have

$$C_0N^\kappa\|\varphi^\gamma_{h,x_0}\|_{L^1_h}N^{-\epsilon-\delta} = C_0N^{-\epsilon}N^{-(\delta-\kappa)} \leq N^{-\epsilon}/16$$

for large enough N. With respect to the last term, with (2.3.13) from Assumption 2.3.11 and large enough N

$$C_0N^\kappa\|\varphi^\gamma_{h,x_0}\|_{L^2_h}N^{-1/2} = C_0h^{-d/2}N^{-1/2}N^\kappa \leq C_0N^{-(\delta_0/2-\kappa)} \leq N^{-\epsilon}/16.$$

Now we apply Lemma 2.8.5. Denoting with $\phi^{\gamma,t}_{h,x_0}$ the backwards evolution according to (2.5.22) with $\phi^{\gamma,t}_{h,x_0} = \varphi^\gamma_{h,x_0}$ for $t \geq T_0$, we obtain

$$\mathbb{E}\left[\chi(T_\oslash > 0)\sup_{0\leq t\leq T_\oslash\wedge T_0}\left(\phi^{\gamma,t}_{h,x_0}, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t)\right)^j_h\right] \leq j^j\left(C_0N^{-1/2+\kappa}\|\varphi^\gamma_{h,x_0}\|_{L^2_h}\right)^j.$$

With Chebyshev's inequality, then plugging in the choice of j and applying (2.8.12), this yields

$$\begin{aligned}
&\mathbb{P}\left[T_\oslash > 0 \text{ and } T_0 \leq T_\oslash \text{ and } \left|\left(\varphi^\gamma_{h,x_0}, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(T_0)\right)_h\right| \geq \frac{N^{-\epsilon}}{4}\right]\\
&\quad\leq \mathbb{P}\left[\chi(T_\oslash > 0)\sup_{0\leq t\leq T_\oslash\wedge T_0}\left|\left(\phi^{\gamma,t}_{h,x_0}, (\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(t)\right)_h\right| \geq \frac{N^{-\epsilon}}{4}\right]\\
&\quad\leq \exp\left(j\left[\log(j) + \log\left(4N^\epsilon C_0N^{-1/2+\kappa}\|\varphi^\gamma_{h,x_0}\|_{L^2_h}\right)\right]\right)\\
&\quad\leq \exp\left(-CN^{1/2-\epsilon-\kappa}h^{d/2}\right).
\end{aligned}$$

This last inequality dictates the choice of j: Up to constants, it is the largest j such that the logarithms combine to a negative constant. Via (2.8.7) from Step 1, repeating the same argument for all different $\gamma = 1, \dots, n_S$ and the h^{-d} grid points $x_0 \in G_{h,d}$ yields (2.8.11).

Step 4: Stricter H_h^{-2l}-bounds for a single time point $T_0 \leq T_\oslash$. Again, let $0 < \kappa < \delta$. In this step, we show that for any fixed $0 < T_0 \leq T$, if N is large enough, it holds that

$$\mathbb{P}\left[T_0 \leq T_\oslash \text{ and } \|(\rho^{r_I}_h - \overline{\rho}^{r_I}_h)(T_0)\|_{H_h^{-2l}} \geq \frac{N^{-1/2+\epsilon}}{4}\right] \leq n_S\exp\left(-CN^{\epsilon-\kappa}\right). \tag{2.8.13}$$

As in Step 3, the main part of the argument is an application of Lemma 2.8.5, here to the functions $\varphi^\gamma_{h,m}$ from Step 2. We choose $j = 2\left\lfloor\frac{1}{4}C_d^{-1}C_0^{-1}N^{\epsilon-\kappa}\right\rfloor$ independent of m, where

C_0 denotes the constant in Lemma 2.8.5 and C_d is a constant, which we will fix later in this step. Clearly, $j \in 2\mathbb{N}$ for N large enough.

With these choices, the maximum in (2.8.4) is dominated by $C_0(1+|m|^2)^{l/2}N^{-1/2+\kappa}$ for large enough N. Indeed, with our choice of j and (2.8.9) we have

$$j^{-1}C_0N^{\kappa}\|\varphi^{\gamma}_{h,m}\|_{L^2_h}N^{-1/2} \leq C_0N^{-1/2+\kappa}\|\varphi^{\gamma}_{h,m}\|_{L^2_h} = C_0N^{-1/2+\kappa},$$
$$j^{-1}C_0N^{\kappa}\|\varphi^{\gamma}_{h,m}\|_{H^l_h}N^{-1/2+\epsilon-\delta} \leq 4C_dC_0^2(1+|m|^2)^{l/2}N^{-1/2+\kappa}N^{-(\delta-\kappa)}.$$

Thus, applying Lemma 2.8.5 with $\phi^{\gamma,t}_{h,m}$ denoting the backwards evolution according to (2.5.22) with $\phi^{\gamma,t}_{h,m} = \varphi^{\gamma}_{h,m}$ for $t \geq T_0$, we obtain

$$\mathbb{E}\left[\chi(T_\oslash > 0)\sup_{0\leq t\leq T_\oslash\wedge T_0}\left(\phi^{\gamma,t}_{h,m},(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t)\right)^j_h\right] \leq j^j\left(C_0(1+|m|^2)^{l/2}N^{-1/2+\kappa}\right)^j.$$

Thus, already (2.8.9) for $\|\cdot\|_{H^{-2l}_h}$ in mind, after applying the triangle inequality we obtain

$$\begin{aligned}
&\mathbb{E}\left[\chi(T_\oslash > 0)\sup_{0\leq t\leq T_\oslash\wedge T_0}\left(\sum_m\frac{1}{(1+|m|^2)^{2l}}\left(\phi^{\gamma,t}_{h,m},(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t)\right)^2_h\right)^{j/2}\right]^{2/j}\\
&\quad\leq\sum_m\frac{C}{(1+|m|^2)^{2l}}\mathbb{E}\left[\chi(T_\oslash > 0)\sup_{0\leq t\leq T_\oslash\wedge T_0}\left(\phi^{\gamma,t}_{h,m},(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t)\right)^j_h\right]^{2/j}\\
&\quad\leq\sum_m\frac{1}{(1+|m|^2)^{l}}\left(jC_0N^{-1/2+\kappa}\right)^2.
\end{aligned}$$

With Chebyshev's inequality, choosing C_d only depending on the dimension and n_S such that $\sum_m\frac{n_S}{(1+|m|^2)^l}\leq C_d^2$ (recall that $l := \lfloor d/2\rfloor+1$), with the above choice of j we obtain

$$\begin{aligned}
&\mathbb{P}\left[T_0\leq T_\oslash \text{ and } \sum_m\frac{1}{(1+|m|^2)^{2l}}\left(\varphi^{\gamma}_{h,m},(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(t)\right)^2_h\geq\frac{(N^{-1/2+\epsilon})^2}{n_S4^2}\right]\\
&\qquad\leq\exp\left(j\left[\log(j)+\log\left(4C_dC_0N^{-\epsilon+\kappa}\right)\right]\right)\leq\exp\left(-CN^{\epsilon-\kappa}\right)
\end{aligned}$$

as in Step 3. Iterating through all species yields (2.8.13) via (2.8.9).

Step 5: Extension to finitely many time points in $[0,T_\oslash]$ with step size h^θ. We discretize the time interval $[0,T]$ with step size h^θ for some large $\theta>1$, which we will determine in the next step. Adding up the bounds in (2.8.11) from Step 3 as well as (2.8.13) from Step 4 for $T_0 = h^\theta, 2h^\theta,\dots,\lfloor T/h^\theta\rfloor h^\theta$, for any $0<\kappa<\delta$ and large enough N we obtain

$$\begin{aligned}
&\mathbb{P}\left[T_\oslash>0 \text{ and } \exists i\in\mathbb{N},\, ih^\theta\leq T_\oslash\colon \|(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(ih^\theta)\|_{L^\infty_h}\geq\frac{N^{-\epsilon}}{4}\right] \qquad (2.8.14)\\
&\quad\leq Ch^{-d-\theta}\exp\left(-CN^{1/2-\epsilon-\kappa}h^{d/2}\right),\\
&\mathbb{P}\left[T_\oslash>0 \text{ and } \exists i\in\mathbb{N},\, ih^\theta\leq T_\oslash\colon \|(\rho^{r_I}_h-\overline{\rho}^{r_I}_h)(ih^\theta)\|_{H^{-2l}_h}\geq\frac{N^{-1/2+\epsilon}}{4}\right] \qquad (2.8.15)\\
&\quad\leq Ch^{-\theta}\exp\left(-CN^{\epsilon-\kappa}\right).
\end{aligned}$$

Step 6: Strict bounds for changes of $\rho^{r_I}_h-\overline{\rho}^{r_I}_h$ in small time increments. In this step, we show that, given any $\kappa>0$ and $j>1$, for large enough N and any $i=1,\dots,\lfloor T/h^\theta\rfloor$ we

have

$$\mathbb{E}\left[\chi(T_\oslash > 0) \sup_{t\in[ih^\theta,(i+1)h^\theta\wedge T_\oslash]} \left\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t) - (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\right\|_\infty^j\right]^{1/j} \leq jCh^{\theta/2-5d/2-4}N^{-1/2+\epsilon+\kappa}. \tag{2.8.16}$$

This implies (via Chebyshev's inequality and optimization in j as in Steps 3 and 4) that

$$\mathbb{P}\left[T_\oslash > 0 \text{ and } \sup_{t\in[ih^\theta,(i+1)h^\theta\wedge T_\oslash]} \left\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t) - (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\right\|_\infty \geq \frac{N^{-\frac{1}{2}+\epsilon}}{4}\right] \leq \exp\left(-Ch^{-\theta/2+5d/2+4}N^{-\kappa}\right).$$

With (2.3.14) from Assumption 2.3.11, choosing $\kappa = \delta_0$,

$$h^{-\theta/2+5d/2-4}N^{-\kappa} \geq h^{-\theta/2+5d/2+5}$$

and hence, choosing $\theta = 5d + 12$ and adding up the bounds for all i,

$$\mathbb{P}\Big[T_\oslash > 0 \text{ and } \exists i \in \mathbb{N},\, ih^\theta \leq T_\oslash\colon \sup_{t\in[ih^\theta,(i+1)h^\theta\wedge T_\oslash]} \left\|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t) - (\rho_h^{r_I} - \overline{\rho}_h^{r_I})(ih^\theta)\right\|_\infty \geq \frac{N^{-1/2+\epsilon}}{4}\Big] \leq Ch^{-\theta}\exp\left(-Ch^{-1}\right). \tag{2.8.17}$$

The exact estimate to (2.8.17) holds when replacing L_h^∞ with H_h^{-2l}, since $\|\cdot\|_{H_h^{-2l}} \leq C\|\cdot\|_{L_h^\infty}$ for a constant C independent of h (for L_h^∞ we can replace $N^{-1/2+\epsilon}$ with $N^{-\epsilon}$ since $\epsilon < 1/2$).

We now prove (2.8.16) by bounding the increments for each species $\gamma = 1, \dots, n_S$ and $x_0 \in G_{h,d}$. With the basis functions f_{x_0} as in Section 2.3.3, following the calculations in Section 2.5.1, the discrete version of (2.5.16) yields

$$\begin{aligned}
&(\rho_{h,\gamma}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(t,x_0) - (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(ih^\theta,x_0) \\
&= h^{-d/2}\left(\left(f_{x_0},(\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(t)\right)_h - \left(f_{x_0},(\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(ih^\theta)\right)_h\right) \\
&= h^{-d/2}\int_{ih^\theta}^t \left(\sigma_\alpha\Delta_h f_{x_0} - U_{h,\alpha}^{r_I}(s)\cdot\nabla_h f_{x_0}, (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(s)\right)\mathrm{d}s \\
&\quad - h^{-d/2}\sum_{\beta=1}^{n_S}\int_{ih^\theta}^t \left(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_{h,c} \left(\overline{\rho}_{h,\alpha}^{r_I}(s)\nabla_h f_{x_0}\right), (\rho_{h,\beta}^{r_I} - \overline{\rho}_{h,\beta}^{r_I})(s)\right)\mathrm{d}s \\
&\quad + h^{-d/2}\int_{ih^\theta}^t \mathrm{d}\mathcal{W}_h((\rho_{h,\alpha}^{r_I})^+, f_{x_0})(s),
\end{aligned} \tag{2.8.18}$$

where $U_{h,\alpha}^{r_I}$ is defined as in (2.5.23) and where $\mathcal{W}_h((\rho_{h,\alpha}^{r_I})^+, f_{x_0})$ is a real-valued martingale corresponding to the Dean–Kawasaki noise with quadratic variation satisfying

$$\mathrm{d}\left[\mathcal{W}((\rho_{h,\alpha}^{r_I})^+, f_{x_0}), \mathcal{W}((\rho_{h,\alpha}^{r_I})^+, f_{x_0})\right](t) = 2\sigma_\alpha N^{-1}\left((\rho_{h,\alpha}^{r_I})^+(t), \nabla_h f_{x_0}\cdot\nabla_h f_{x_0}\right)_h.$$

Applying the Burkholder-Davis-Gundy inequality (for $j > 1$) then gives

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[ih^\theta,(i+1)h^\theta\wedge T_\oslash]}\left|\int_{ih^\theta}^{t} \mathrm{d}\mathcal{W}_h((\rho_{h,\alpha}^{r_I})^+, f_{x_0})(s)\right|^j\right]^{1/j} \qquad (2.8.19)\\
&\leq Cj\mathbb{E}\left[\left(\int_{ih^\theta}^{(i+1)h^\theta\wedge T_\oslash} 2\sigma_\alpha N^{-1}\left((\rho_{h,\alpha}^{r_I})^+(t), \nabla_h f_{x_0}\cdot\nabla_h f_{x_0}\right)_h \mathrm{d}t\right)^{j/2}\right]^{1/j}\\
&\leq Cjh^{\theta/2}N^{-1/2}\sqrt{\rho_{max,h}}h^{-1},
\end{aligned}$$

where we used that $\|(\rho_{h,\alpha}^{r_I})^+(t)\|_{L_h^\infty} \leq \rho_{max,h}$ as in Remark 2.3.14 and $\|\nabla_h f_{x_0}\|_{L_h^2} \leq Ch^{-1}$.

With respect to the first term in (2.8.18), with $\|\cdot\|_{H_h^{2l}} \leq Ch^{-2l}\|\cdot\|_{L_h^2}$ we have

$$\begin{aligned}
&\left|\int_{ih^\theta}^{t}\left(\sigma_\alpha\Delta_h f_{x_0} - U_{h,\alpha}^{r_I}(s)\cdot\nabla_h f_{x_0}, (\rho_{h,\alpha}^{r_I} - \overline{\rho}_{h,\alpha}^{r_I})(s)\right)\mathrm{d}s\right| \qquad (2.8.20)\\
&\leq Ch^{-2l}\int_{ih^\theta}^{(i+1)h^\theta\wedge T_\oslash}\left(\|\Delta_h f_{x_0}\|_{L_h^2} + \|U_h^{r_I}\|_{L_h^\infty}\|\nabla_h f_{x_0}\|_{L_h^2}\right)\|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{H_h^{-2l}}\,\mathrm{d}s\\
&\leq Ch^{-2l+\theta}\left(h^{-2} + {r_I}^{-d-1}h^{-1}\right)N^{-1/2+\epsilon}
\end{aligned}$$

where we used the definition of $T_\oslash$ and that $\|U^{r_I}\|_{L_h^\infty} \leq \|\mathcal{I}_h[V^{r_I}]\|_{L^\infty}\|\overline{\rho}^{r_I}\|_{L_h^1} \leq C{r_I}^{-d-1}$ due to Young's inequality, see Corollary 2.B.3. Note that if $\overline{\rho}^{r_I}(t) \geq 0$, in particular for $t \leq T_\oslash$ with Remark 2.3.14, it holds that $\|\overline{\rho}^{r_I}(t)\|_{L_h^1} = \|\overline{\rho}^{r_I}(0)\|_{L_h^1}$.

For the second term in (2.8.18), an analogous argument yields

$$\begin{aligned}
&\left|\int_{ih^\theta}^{t}\left(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_{h,c}\left(\overline{\rho}_{h,\alpha}^{r_I}(s)\nabla_h f_{x_0}\right), (\rho_{h,\beta}^{r_I} - \overline{\rho}_{h,\beta}^{r_I})(s)\right)\mathrm{d}s\right| \qquad (2.8.21)\\
&\leq Ch^{-2l}\int_{ih^\theta}^{(i+1)h^\theta\wedge T_\oslash}\|\mathcal{I}_h[V^{r_I}]\|_{L_h^2}\|\overline{\rho}_h^{r_I}\|_{L_h^2}\|\nabla_h f_{x_0}\|_{L_h^2}\|\rho_h^{r_I} - \overline{\rho}_h^{r_I}\|_{H_h^{-2l}}\,\mathrm{d}s\\
&\leq Ch^{-2l+\theta}{r_I}^{-d-1}\rho_{max,h}h^{-1}N^{-1/2+\epsilon}.
\end{aligned}$$

Plugging the sum of (2.8.18) over all $\gamma = 1,\dots,n_S$ and the h^{-d} different $x_0 \in G_{h,d}$ into the left hand side of (2.8.16), applying the triangle inequality and then (2.8.20), (2.8.21), and (2.8.19), we obtain the right hand side of (2.8.16) with l chosen as in (2.8.1) for large enough N with (2.3.15) from Assumption 2.3.11.

Step 7: Conclusion from Steps 5 and 6. Combining (2.8.14) and (2.8.15) for the discrete time points with (2.8.17) for the increments inbetween with respect to $\|\cdot\|_{L_h^\infty}$ and $\|\cdot\|_{H_h^{-2l}}$ yields

$$\begin{aligned}
\mathbb{P}\left[T_\oslash = T\right] &\geq \mathbb{P}\Bigg[T_\oslash > 0 \text{ and } \forall 0 \leq t \leq T_\oslash : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} \leq \frac{N^{-\epsilon}}{2}\\
&\qquad\qquad \text{and } \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2\ell}} \leq \frac{N^{-1/2+\epsilon}}{2}\Bigg]\\
&\geq 1 - \mathbb{P}\left[T_\oslash = 0\right] - \mathbb{P}\left[T_\oslash > 0 \text{ and } \exists 0 \leq t \leq T_\oslash : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{L_h^\infty} > \frac{N^{-\epsilon}}{2}\right]\\
&\quad - \mathbb{P}\left[T_\oslash > 0 \text{ and } \exists 0 \leq t \leq T_\oslash : \|(\rho_h^{r_I} - \overline{\rho}_h^{r_I})(t)\|_{H_h^{-2\ell}} > \frac{N^{-1/2+\epsilon}}{2}\right]\\
&\geq 1 - C\exp\left(-CN^{1/2-\epsilon-\delta}h^{d/2}\right) - C\exp\left(-CN^{\epsilon-\delta}\right)\\
&\quad - Ch^{-\theta}\left(h^{-d}\exp\left(-CN^{1/2-\epsilon-\kappa}h^{d/2}\right) + \exp\left(-CN^{\epsilon-\kappa}\right) + \exp\left(-Ch^{-1}\right)\right),
\end{aligned}$$

where we used (2.3.11) and (2.3.12) from Assumption 2.3.10 to bound $\mathbb{P}[T_{\oslash} = 0]$. After absorbing the h^{-1} prefactors and applying Assumption 2.3.11 (e.g., as in Remark 2.8.3), we obtain (2.8.3). □

2.A Regularity of continuous test functions

Lemma 2.A.1 (Bounds on Sobolev norms of continuous test functions). *Let $\varphi_\alpha \in H^s(\mathbb{T}^d)$ for all $\alpha \in 1, \dots, n_S$, for some $s \in \mathbb{N}_0$. Assume $\overline{\rho}^{r_I} \in C^s$. Then there exists a unique H^s-valued strong solution ϕ^t_α to the backwards evolution equation (2.5.18). Additionally, given an arbitrary $\kappa > 0$, there exists $N_0 = N_0(\kappa, data)$, such that for all $N > N_0$ we have the bound*

$$\|\phi^t\|_{H^s} \leq N^{\kappa(T-t)} \|\varphi\|_{H^s}. \tag{2.A.1}$$

The proof of Lemma 2.A.1 is straightforward, and we refer the reader to the arXiv version of this paper (https://arxiv.org/abs/2303.00429).

2.B Relevant estimates in the discrete setting

We prove several auxiliary results relating to the discretised setting, namely: error bounds for the difference of continuous mean-field limit $\overline{\rho}^{r_I}$ and discrete mean-field limit $\overline{\rho}^{r_I}_h$ (Subsection 2.B.1); error bounds for the difference of continuous test functions (2.5.18) and discrete test functions (2.5.22), and L^1-bound for the discrete test functions (2.5.22) (Subsection 2.B.2).

In what follows, we will use a discrete version of Hölder's inequality and Young's convolution inequality. To state these, we need a discrete notion of L^p-norms.

Definition 2.B.1 (Discrete L^p-norms). For all $p \in [1, \infty)$ and $f \in L^2(G_{h,d})$ we write

$$\|f\|_{L^p(G_{h,d})} = \|f\|_{L^p_h} = \Big(\sum_{x \in G_{h,d}} h^d |f(x)|^p \Big)^{1/p}, \qquad \|f\|_{L^\infty(G_{h,d})} = \max_{x \in G_{h,d}} |f(x)|.$$

In particular, this is consistent with the previous definition of $\|\cdot\|_{L^2(G_{h,d})}$.

Corollary 2.B.2 (Discrete Hölder inequality). *Let $q, r \in [1, \infty]$ with $\frac{1}{q} + \frac{1}{r} = 1$. Then for all $f, g \in L^2(G_{h,d})$ it holds that*

$$(f, g)_h \leq \|f\|_{L^q(G_{h,d})} \|g\|_{L^r(G_{h,d})}.$$

With the validity of a Hölder inequality, Young's convolution inequality automatically holds in the same framework.

Corollary 2.B.3 (Discrete Young's convolution inequality). *Assume $q, \widetilde{q}, r \in [1, \infty]$ satisfy $\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{1}{r} + 1$ and $f, g \in L^2(G_{h,d})$. Then*

$$\|f *_h g\|_{L^r(G_{h,d})} \leq \|f\|_{L^q(G_{h,d})} \|g\|_{L^{\widetilde{q}}(G_{h,d})}.$$

We obtain a quantitative error bound for comparing continuous and discrete convolutions based on the classical Euler-Maclaurin formula for numerical integration.

Lemma 2.B.4 (A multidimensional Euler-Maclaurin formula). *Let $s \in \mathbb{N}_0$. Then there exists $C_s > 0$, such that for all $f \in C^{2s+2}(\mathbb{T}^d)$*

$$\left| \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x - \sum_{x \in G_{h,d}} h^d f(x) \right| \leq C_s h^{2s+2} \|f\|_{C^{2s+2}(\mathbb{T}^d)}.$$

Proof. For $d = 1$ see, for instance, [Che07, Chapter 1]. The lemma then follows with Fubini's theorem by induction. □

These facts imply the following approximation property of the discrete convolution.

Corollary 2.B.5 (Approximation order of convolutions on periodic grids). *Let $s \in \mathbb{N}_0$. Then there exists $C_s > 0$, such that for all $f, g \in C^{2s+2}(\mathbb{T}^d)$*

$$\left\| \mathcal{I}_h[f * g] - \mathcal{I}_h[f] *_h \mathcal{I}_h[g] \right\|_{L^\infty(G_{h,d})} \leq C_s h^{2s+2} \|f\|_{C^{2s+2}(\mathbb{T}^d)} \|g\|_{C^{2s+2}(\mathbb{T}^d)}.$$

2.B.1 Error bounds for difference of continuous and discrete mean field limit

In this section we compare the continuous and discretised mean field limits. First, we obtain an L^2-type bound of order h^{p+1} via an energy estimate. Since we need to assume the difference to be small to deal with the quadratic nonlinearity, the estimate only holds up to a finite positive time. Second, building upon the L^2-estimate we will derive higher order estimates difference.

Proposition 2.B.6 (L^2-Estimate for the continuous-discrete mean field limit difference). *Let Assumption 2.3.7 hold. Let $\overline{\rho}^{r_I}$ be the continuous mean-field limit as in (2.3.4) satisfying Assumption 2.3.8 with discrete counterpart $\overline{\rho}_h^{r_I}$ as in (h-MFL). Assume that the initial conditions satisfy Assumption 2.3.10 while the parameters r_I, N, h are subject to Assumption 2.3.11. Set*

$$\overline{T}_h := \inf\left\{ t \in [0,\infty);\ \|\mathcal{I}_h[\overline{\rho}^{r_I}](t) - \overline{\rho}_h^{r_I}(t)\|_{L^2(G_{h,d})} > 1 \right\} \overset{\text{Assumption 2.3.10}}{>} 0.$$

Then for all $\kappa > 0$ there exists $N_0 = N_0(p, \kappa, \text{data})$ such that for all $t \in [0, \overline{T}_h]$ and $N > N_0$

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}(t) \leq C h^{2(p+1)} N^{\kappa t} \tag{2.B.1}$$

In particular, $\overline{T}_h \geq T$, since for small $\kappa > 0$ and $h > 0$ with Assumption 2.3.11 it holds that

$$C h^{2(p+1)} N^{\kappa t} \leq C h^{2(p+1)} N^{\frac{\kappa}{\delta_0}\delta_0 T} \leq C h^{2(p+1)} h^{-\kappa/\delta_0} \leq 1.$$

Proof. Testing (h-MFL), the definition of the discrete mean field limit, with $\eta_h \in L^2(G_{h,d})$ and applying integration by parts for ∇_h we obtain its adjoint/reflected version ∇_h^R

$$\left(\partial_t \overline{\rho}_{h,\alpha}^{r_I}, \eta_h\right)_h = \sigma_\alpha \left(\Delta_h \overline{\rho}_{h,\alpha}^{r_I}, \eta_h\right)_h - \sum_{\beta=1}^{n_S} \left(\overline{\rho}_{h,\alpha}^{r_I} (\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \overline{\rho}_{h,\beta}^{r_I}), \nabla_h^R \eta_h\right)_h$$

for all species $\alpha = 1, \dots, n_S$. After applying $\mathcal{I}_h$ to (2.3.4) and testing with η_h, for the continuous mean field limit we have

$$\left(\mathcal{I}_h[\partial_t \overline{\rho}_\alpha^{r_I}], \eta_h\right)_h = \sigma_\alpha \left(\mathcal{I}_h[\Delta \overline{\rho}_\alpha^{r_I}], \eta_h\right)_h + \sum_{\beta=1}^{n_S} \left(\mathcal{I}_h\left[\nabla \cdot \left(\overline{\rho}_\alpha^{r_I} (\nabla V_{\alpha\beta}^{r_I} * \overline{\rho}_\beta^{r_I})\right)\right], \eta_h\right)_h. \tag{2.B.2}$$

In order to make both evolutions comparable, we want to rephrase (2.B.2) in terms of discrete differential operators. For the diffusion term, with Assumption 2.3.9 – the order of the differential operators – and the discrete Hölder inequality we have

$$\left|\left(\mathcal{I}_h[\Delta\overline{\rho}_\alpha^{r_I}],\eta_h\right)_h-\left(\Delta_h\overline{\rho}_\alpha^{r_I},\eta_h\right)_h\right|\leq C\|\eta_h\|_{L^1(G_{h,d})}\|\overline{\rho}_\alpha^{r_I}\|_{C^{p+3}(\mathbb{T}^d)}h^{p+1}$$

Using integration by parts, we rewrite (for all $l=1,\dots,d$ and $\alpha,\beta=1,\dots,n_S$)

$$\begin{aligned}
&\left(\mathcal{I}_h\left[\partial_{x_\ell}\left(\overline{\rho}_\alpha^{r_I}(\partial_{x_\ell}V_{\alpha\beta}^{r_I}*\overline{\rho}_\beta^{r_I})\right)\right],\eta_h\right)_h\\
&\quad=\left(\partial_{h,x_\ell}\mathcal{I}_h\left[\overline{\rho}_\alpha^{r_I}(\partial_{x_\ell}V_{\alpha\beta}^{r_I}*\overline{\rho}_\beta^{r_I})\right],\eta_h\right)_h+\widetilde{R}_\nabla\\
&\quad=-\left(\mathcal{I}_h\left[\overline{\rho}_\alpha^{r_I}(\partial_{x_\ell}V_{\alpha\beta}^{r_I}*\overline{\rho}_\beta^{r_I})\right],\partial_{h,x_\ell}^R\eta_h\right)_h+\widetilde{R}_\nabla\\
&\quad=-\left(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]\left(\mathcal{I}_h[\partial_{x_\ell}V_{\alpha\beta}^{r_I}]*_h\mathcal{I}_h[\overline{\rho}_\beta^{r_I}]\right),\partial_{h,x_\ell}^R\eta_h\right)_h+\widetilde{R}_\nabla+\widetilde{R}_*,
\end{aligned}$$

where for $\nabla\to\nabla_h$ with Assumption 2.3.9 and the discrete Hölder inequality we pay

$$\begin{aligned}
|\widetilde{R}_\nabla|&\leq C\|\eta_h\|_{L^1(G_{h,d})}\|\overline{\rho}_\alpha^{r_I}(\partial_{x_\ell}V_{\alpha\beta}^{r_I}*\overline{\rho}_\beta^{r_I})\|_{C^{p+2}(\mathbb{T}^d)}h^{p+1}\\
&\leq C\|\eta_h\|_{L^1(G_{h,d})}\|\overline{\rho}^{r_I}\|_{C^{p+2}(\mathbb{T}^d)}^2\|\nabla V_{\alpha\beta}^{r_I}\|_{L^1(\mathbb{T}^d)}h^{p+1},
\end{aligned}$$

while for $*\to *_h$ due to Lemma 2.B.5 with the discrete Hölder inequality we have

$$|\widetilde{R}_*|\leq C\|D_\ell^h\eta_h\|_{L^1(G_{h,d})}\|\overline{\rho}_\alpha^{r_I}\|_{L^\infty}\|\nabla V_{\alpha\beta}^{r_I}\|_{C^{p+2}}\|\overline{\rho}_\beta^{r_I}\|_{C^{p+2}}h^{p+1}.$$

Thus, combining these two observations we obtain

$$\begin{aligned}
&\left(\partial_t(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}),\eta_h\right)_h\\
&\quad=\sigma_\alpha\left(\Delta_h(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}),\eta_h\right)_h\\
&\qquad-\sum_{\beta=1}^{n_S}\left((\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\left(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}]*_h\mathcal{I}_h[\overline{\rho}_\beta^{r_I}]\right),\nabla_h^R\eta_h\right)_h\\
&\qquad-\sum_{\beta=1}^{n_S}\left(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]\left(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}]*_h(\mathcal{I}_h[\overline{\rho}_\beta^{r_I}]-\overline{\rho}_{h,\beta}^{r_I})\right),\nabla_h^R\eta_h\right)_h\\
&\qquad+\sum_{\beta=1}^{n_S}\left((\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\left(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}]*_h(\mathcal{I}_h[\overline{\rho}_\beta^{r_I}]-\overline{\rho}_{h,\beta}^{r_I})\right),\nabla_h^R\eta_h\right)_h\\
&\qquad+R_{\Delta,\nabla}\|\eta_h\|_{L^1(G_{h,d})}+R_*\|\nabla_h^R\eta_h\|_{L^1(G_{h,d})},
\end{aligned}\tag{2.B.3}$$

with

$$\begin{aligned}
|R_{\Delta,\nabla}|&\leq C\left(\|\overline{\rho}^{r_I}\|_{C^{p+3}}+\|\overline{\rho}^{r_I}\|_{C^{p+2}}^2\|\nabla V^{r_I}\|_{L^1}\right)h^{p+1},\\
|R_*|&\leq C\|\overline{\rho}^{r_I}\|_{L^\infty}\|\overline{\rho}^{r_I}\|_{C^{p+2}}\|\nabla V^{r_I}\|_{C^{p+2}}h^{p+1}.
\end{aligned}$$

The remainder of the proof will follow the steps of a standard energy estimate. Thus, we will choose $\eta_h=\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}$ as a test function. Note that with Hölder

$$\begin{aligned}
\|\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}\|_{L^1(G_{h,d})}&\leq|\mathbb{T}^d|^{1/2}\|\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}\|_{L^2(G_{h,d})},\\
\|\nabla_h^R(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\|_{L^1(G_{h,d})}&\leq|\mathbb{T}^d|^{1/2}\|\nabla_h^R(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\|_{L^2(G_{h,d})},
\end{aligned}$$

Applying (2.3.5), comparing the first-order operators via (2.3.9) from Assumption 2.3.9, and integrating by parts, we obtain

$$\begin{aligned}\sigma_\alpha\big(\Delta_h(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}),\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}\big)_h &= -\sigma_\alpha\|\nabla_{D,h}(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\|^2_{L^2(G_{h,d})}\\ &\leq -\sigma_\alpha C_D\|\nabla_h^R(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\|^2_{L^2(G_{h,d})}.\end{aligned}$$

Now, applying a combination of the discrete Hölder inequality, Young's discrete convolution inequality and Young's inequality for products to (2.B.3), we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\|\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I}\|^2_{L^2(G_{h,d})}+\frac{C_D\underline{\sigma}}{5}\|\nabla_h^R(\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}]-\overline{\rho}_{h,\alpha}^{r_I})\|^2_{L^2(G_{h,d})}\\ &\quad\leq 2\frac{5}{C_D\underline{\sigma}}\|\mathcal{I}_h[\nabla V]\|^2_{L^2(G_{h,d})}\|\mathcal{I}_h[\overline{\rho}^{r_I}]\|^2_{L^2(G_{h,d})}\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}\\ &\qquad+\frac{5}{C_D\underline{\sigma}}\|\mathcal{I}_h[\nabla V]\|^2_{L^2(G_{h,d})}\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^4_{L^2(G_{h,d})}\\ &\qquad+\frac{5}{C_D\underline{\sigma}}|\mathbb{T}^d||R_*|^2+|\mathbb{T}^d||R_{\Delta,\nabla}|^2+\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})},\end{aligned}$$

where $\underline{\sigma}:=\min_\alpha\sigma_\alpha$. For $t\in[0,\overline{T}_h]$, which implies

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^4_{L^2(G_{h,d})}(t)\leq\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}(t),$$

summing over all species results in

$$\begin{aligned}&\partial_t\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}+\|\nabla_h^R(\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I})\|^2_{L^2(G_{h,d})}\\ &\quad\leq C\left(1+\big(1+\|\mathcal{I}_h[\overline{\rho}^{r_I}]\|^2_{L^2(G_{h,d})}\big)\|\mathcal{I}_h[\nabla V^{r_I}]\|^2_{L^2(G_{h,d})}\right)\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}+|R|^2,\end{aligned}$$

with

$$|R|\leq C\big(1+\|\nabla V^{r_I}\|_{C^{p+2}}\|\overline{\rho}^{r_I}\|_{C^{p+2}}\big)\|\overline{\rho}^{r_I}\|_{C^{p+3}}h^{p+1}.$$

Now, Assumption 2.3.10 yields

$$\|\mathcal{I}_h[\overline{\rho}^0]-\overline{\rho}_h^{r_I}(0)\|_{L^2(G_{h,d})}\leq Ch^{p+1}.$$

Further using $\|\mathcal{I}_h[f]\|_{L^2(G_{h,d})}\leq C\|f\|_{C(\mathbb{T}^d)}$ and that $\|\nabla V^{r_I}\|_C\leq C{r_I}^{-(d+1)}$ as well as $\|\nabla V^{r_I}\|_{C^{p+2}}\leq C{r_I}^{-(d+p+3)}$, the Gronwall lemma then yields

$$\begin{aligned}\|\mathcal{I}_h[\overline{\rho}^{r_I}]-\overline{\rho}_h^{r_I}\|^2_{L^2(G_{h,d})}(t)\leq&\left(Ch^{2(p+1)}+tC\big(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})}\big)h^{2(p+1)}{r_I}^{-2(d+p+3)}\right)\\ &\times\exp\left(tC(\rho_{\max}){r_I}^{-2(d+1)}\right).\end{aligned}\tag{2.B.4}$$

In order to simplify the structure, we estimate

$$tC(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})}){r_I}^{-2(d+p+3)}\leq\exp\left(tC\big(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})},p\big){r_I}^{-2(d+1)}\right).$$

With the scaling from Assumption 2.3.11 we have, for any chosen $\kappa>0$

$${r_I}^{-2(d+1)}\leq\log(N)^{\frac{2d+2}{2d+4}}\leq\kappa\left(C\big(\|\overline{\rho}^{r_I}\|_{L^\infty(C^{p+3})},p\big)+C(\rho_{\max})\right)^{-1}\log(N)$$

for N large enough. Combining these observations with (2.B.4) yields (2.B.1). □

Higher Order bounds for the mean field limit difference

We will obtain higher order bounds for the mean field limit difference via induction. Opposed to Proposition 2.B.6 in this proof we do not need additional smallness assumptions to deal with the quadratic non-linearity because – when taking derivatives – at most one factor is of the highest order. The other one is controlled by the induction assumption.

The first order one-sided finite differences given by

$$\partial_{h,1}^{e_\ell} f(x) = \frac{f(x + he_\ell) - f(x)}{h} \tag{2.B.5}$$

for $\ell = 1, \dots, d$, satisfy the product rule

$$\partial_{h,1}^{e_\ell}(fg) = (\tau_{he_\ell} f)(\partial_{h,1}^{e_\ell} g) + (\partial_{h,1}^{e_\ell} f)g \tag{2.B.6}$$

for any $f, g \in L^2(G_{h,d})$ with the shift operators $\tau_{h\nu}$, $\nu \in \mathbb{Z}^d$, given by $\tau_{h\nu} f(x) = f(x + h\nu)$.

Proposition 2.B.7 (Higher order bound for the discrete mean field limit)**.** *Let $s \in \mathbb{N}$. In the setting of Proposition 2.B.6 with the additional assumption $\overline{\rho}^{r_I} \in L^\infty(0, T; C^{s+p+3}(\mathbb{T}^d))$, for every $\kappa > 0$ there exists $N_0 = N_0(\kappa, s, \text{data})$ such that for all $N > N_0$*

$$\|\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}_h^{r_I}\|^2_{H^s(G_{h,d})}(t) \le Ch^{2(p+1)} N^{\kappa t}, \qquad \forall t \in [0, T]. \tag{2.B.7}$$

Proof. We use induction over $s \in \mathbb{N}_0$. The base case is settled by Proposition 2.B.6. For the induction step assume that (2.B.7) holds for $s - 1$, $s \ge 1$, and large enough N. Let $\nu \in \mathbb{N}_0^d$ with $|\nu| = s$. With (2.B.6), similarly to the proof of Proposition 2.B.6, for $\eta_h \in L^2(G_{h,d})$, we get

$$\begin{aligned}\big(\partial_t(\partial_{h,1}^\nu \overline{\rho}_{h,\alpha}^{r_I}), \eta_h\big)_h &= -\sigma_\alpha\big(\nabla_{D,h}\partial_{h,1}^\nu \overline{\rho}_{h,\alpha}^{r_I}, \nabla_{D,h}\eta_h\big)_h \\ &\quad - \sum_{\gamma \le \nu}\sum_{\beta=1}^{n_S} b_{\nu,\gamma}\big(\tau_{h\gamma}\partial_{h,1}^{\nu-\gamma}\overline{\rho}_{h,\alpha}^{r_I}(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \partial_{h,1}^\gamma \overline{\rho}_{h,\beta}^{r_I}), \nabla_h^R \eta_h\big)_h,\end{aligned} \tag{2.B.8}$$

where ∇_h^R is the adjoint/reflected version of ∇_h while $(b_{\nu,\gamma})_\gamma$ are the typical product rule prefactors (hence binomial coefficients), and

$$\begin{aligned}\big(\partial_t\partial_{h,1}^\nu \mathcal{I}_h[\overline{\rho}_\alpha^{r_I}], \eta_h\big)_h &= -\sigma_\alpha\big(\nabla_{D,h}\partial_{h,1}^\nu \mathcal{I}_h[\overline{\rho}_\alpha^{r_I}], \nabla_{D,h}\eta_h\big)_h \\ &\quad - \sum_{\gamma \le \nu}\sum_{\beta=1}^{n_S} b_{\nu,\gamma}\big(\tau_{h\gamma}\partial_{h,1}^{\nu-\gamma}\mathcal{I}_h[\overline{\rho}_\alpha^{r_I}](\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \partial_{h,1}^\gamma \mathcal{I}_h[\overline{\rho}_\beta^{r_I}]), \nabla_h^R \eta_h\big)_h \\ &\quad + R_{\Delta,\nabla}\|\eta_h\|_{L^2(G_{h,d})} + R_*\|\nabla_h^R \eta_h\|_{L^2(G_{h,d})}\end{aligned} \tag{2.B.9}$$

for all species $\alpha = 1, \dots, n_S$ with consistency errors

$$|R_{\Delta,\nabla}|, |R_*| \le C\left(1 + \|\nabla V^{r_I}\|_{C^{p+2}}\|\overline{\rho}^{r_I}\|_{C^{s+p+2}}\right)\|\overline{\rho}^{r_I}\|_{C^{s+p+3}} h^{p+1}. \tag{2.B.10}$$

To obtain this bound for $R_{\Delta,\nabla}$ we use that

$$\|\partial_{h,1}^\nu(\Delta_h - \Delta)\overline{\rho}_\alpha^{r_I}\|_{L^\infty} \le Ch^{p+1}\|\partial_{h,1}^\nu \overline{\rho}_\alpha^{r_I}\|_{C^{p+3}} \le Ch^{p+1}\|\overline{\rho}_\alpha^{r_I}\|_{C^{p+s+3}}$$

and the analogous estimate for ∇_h, which hold since $\partial_{h,1}^\nu$ is an order $|\nu|$ finite difference operator and with the mean value theorem. To bound R_* note that with Lemma 2.B.5

$$\begin{aligned}&\|\partial_{h,1}^\nu(\overline{\rho}_\alpha^{r_I}(\nabla V^{r_I} * \overline{\rho}_\beta^{r_I})) - \partial_{h,1}^\nu(\overline{\rho}_\alpha^{r_I}(\mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \overline{\rho}_\beta^{r_I}))\|_{L^\infty} \\ &\quad = \Big\|\sum_{\gamma \le \nu} b_{\nu,\gamma}\tau_{h\gamma}\partial_{h,1}^{\nu-\gamma}\overline{\rho}_\alpha^{r_I}(\nabla V_{\alpha\beta}^{r_I} * \partial_{h,1}^\gamma \overline{\rho}_\beta^{r_I} - \mathcal{I}_h[\nabla V_{\alpha\beta}^{r_I}] *_h \partial_{h,1}^\gamma \overline{\rho}_\beta^{r_I})\Big\|_{L^\infty} \\ &\quad \le Ch^{p+1}\sum_{\gamma \le \nu}\|\partial_{h,1}^{\nu-\gamma}\overline{\rho}^{r_I}\|_{L^\infty}\|\nabla V^{r_I}\|_{C^{p+2}}\|\partial_{h,1}^\gamma \overline{\rho}^{r_I}\|_{C^{p+2}}.\end{aligned}$$

Taking the difference of (2.B.8) and (2.B.9), we obtain a structure corresponding to (2.B.3). Further proceeding as in the previous proof, we choose $\eta_h = \partial^\nu_{h,1}\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \partial^\nu_{h,1}\overline{\rho}^{r_I}_{h,\alpha}$, apply a combination of Young's discrete convolution inequality, the discrete Hölder inequality, as well as Young's inequality for products, and then absorb the $\|\nabla^R_h(\mathcal{I}_h[\partial^\nu\overline{\rho}^{r_I}] - \partial^\nu_{h,1}\overline{\rho}^{r_I}_h)\|_{L^2}$-terms. Note that in the interaction terms always at least one of the differences only has derivatives of order at most $s-1$. Evoking the induction assumption, we thus obtain

$$\begin{aligned}\partial_t\|\partial^\nu_{h,1}\mathcal{I}_h[\overline{\rho}^{r_I}_\alpha] - \partial^\nu_{h,1}\overline{\rho}^{r_I}_{h,\alpha}\|_{L^2} \le C\|\mathcal{I}_h[\nabla V^{r_I}]\|^2_{L^2_h}\Big(\sup_{|\gamma|\le s}\|\partial^\gamma_{h,1}\mathcal{I}_h[\overline{\rho}^{r_I}]\|^2_{L^2} + h^{2(p+1)}N^{\kappa t}\Big)\\ \times\Big(\|\partial^\nu_{h,1}\mathcal{I}_h[\overline{\rho}^{r_I}] - \partial^\nu_{h,1}\overline{\rho}^{r_I}_h)\|^2_{L^2} + h^{2(p+1)}N^{\kappa t}\Big)\\ + \|\partial^\nu_{h,1}\mathcal{I}_h[\overline{\rho}^{r_I}] - \partial^\nu_{h,1}\overline{\rho}^{r_I}_h\|^2_{L^2} + |R_{\Delta,\nabla}|^2 + C|R_*|^2.\end{aligned}$$

Assumption 2.3.11 implies $h^{2(p+1)}N^{\kappa T} \le 1$ for $\kappa \le \delta_0$, while as for the $R_{\Delta,\nabla}$-bound

$$\sup_{|\gamma|\le s}\|\partial^\gamma_{h,1}\mathcal{I}_h[\overline{\rho}^{r_I}]\|^2_{L^2} \le C\|\overline{\rho}^{r_I}\|^2_{L^\infty(0,T;C^s(\mathbb{T}^d))}.$$

Thus, after summing over all species and replacing the V^{r_I}-norms with the respective r_I-scaling, since $\partial^\nu_{h,1}(\mathcal{I}_h[\overline{\rho}^{r_I}] - \overline{\rho}^{r_I}_h)(0) = 0$ via the Gronwall inequality we obtain

$$\begin{aligned}\|\mathcal{I}_h[\partial^\nu\overline{\rho}^{r_I}] - \partial^\nu_{h,1}\overline{\rho}^{r_I}_h\|^2_{L^2(G_{h,d})}(t) \le Ct\big(1+\|\overline{\rho}^{r_I}\|^4_{L^\infty(C^{s+p+3})}\big)h^{2(p+1)}N^{\kappa t}{r_I}^{-2(d+p+2)}\\ \times\exp\Big(Ct(1+\|\overline{\rho}^{r_I}\|^2_{L^\infty(C^s)}){r_I}^{-2(d+1)}\Big).\end{aligned}$$

Now we plug in the scaling of r_I and simplify as we did at the end of the proof for Proposition 2.B.6. Looping over all $|\nu| = s$ leads to (2.B.7) for N large enough (and a slightly larger κ than chosen for the induction assumption), thus finishing the proof by induction. □

2.B.2 Regularity of discrete test functions

Lemma 2.B.8 (Bounds on Sobolev norms of discrete test functions)**.** *Let ϕ^t_h be the solution to (2.5.22), and let the assumptions of Proposition 2.B.7 be satisfied. Let $\kappa > 0$. Then, for every $s \in \mathbb{N}_0$, for every $t \in [0,T]$, and for N large enough, we have the bound*

$$\|\phi^t_h\|^2_{H^s_h} + \int_t^T \|\nabla_h\phi^\tau_h\|^2_{H^s_h}\,\mathrm{d}\tau \le N^{\kappa(T-t)}\|\mathcal{I}_h\varphi\|^2_{H^s_h} \le N^{\kappa(T-t)}\|\varphi\|^2_{C^s}. \tag{2.B.11}$$

We omit the proof, a straightforward adaption of Lemma 2.A.1 combined with (2.B.7).

Lemma 2.B.9 (High-order bounds on Sobolev norms for difference of continuous and discrete test functions)**.** *Let ϕ^t_α be the solution to (2.5.18) with final datum $\phi^T = \varphi$. Let $\phi^t_{h,\alpha}$ be the solution to (2.5.22). Furthermore, let all the assumptions in Proposition 2.B.6 be satisfied. Let $\kappa > 0$. Then the following estimates hold for large enough N.*

- ***Part I: bound for L^2-difference (analogue of Lemma 2.B.6).*** *If the regularity requirements*

$$V_{\beta\alpha} \in C^{p+3}(\mathbb{T}^d), \qquad \phi_\alpha \in C^{p+3}(\mathbb{T}^d), \qquad \overline{\rho}^{r_I} \in C^{p+3} \tag{2.B.12}$$

are satisfied, we have the bound

$$\|\phi^t_{h,\alpha} - \mathcal{I}_h\phi^t_\alpha\|^2_h \le Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|_{C^{p+3}}. \tag{2.B.13}$$

- ***Part II: bound for discrete Sobolev norm of difference (analogue of Proposition 2.B.7).*** *If the following stricter regularity requirement*

$$V_{\beta\alpha} \in C^{s+(p+3)}(\mathbb{T}^d), \qquad \phi_\alpha \in C^{s+(p+3)}(\mathbb{T}^d), \qquad \overline{\rho}^{r_I} \in C^{s+(p+3)} \tag{2.B.14}$$

is satisfied, we have the bound

$$\|\phi^t_{h,\alpha} - \mathcal{I}_h\phi^t_\alpha\|^2_{H^s(G_{h,d})}(t) \leq Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|_{C^{s+p+3}}. \tag{2.B.15}$$

- ***Part III: bound for gradient difference.*** *The assumptions of Part II) being satisfied for $s = 1$, we have the bound*

$$\|\nabla_h\phi^t_{h,\alpha} - \mathcal{I}_h\nabla\phi^t_\alpha\|_h \leq Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|_{C^{p+4}}. \tag{2.B.16}$$

- ***Part IV: bound for difference of product of gradients.*** *Assuming the validity of the hypotheses in Parts I), II), III), we obtain, as a special case, that*

$$\|\nabla_h\phi^t_{h,\alpha_1}\cdot\nabla_h\phi^t_{h,\alpha_2} - \mathcal{I}_h[\nabla\phi^t_{\alpha_1}\cdot\nabla\phi^t_{\alpha_2}]\|^2_h \leq Ch^{2(p+1)}N^{\kappa(T-t)}\|\varphi\|^2_{C^{\lceil d/2\rceil+p+5}}. \tag{2.B.17}$$

The proof of Lemma 2.B.9 closely follows the proofs of Lemma 2.B.6, Proposition 2.B.7, and makes use of Lemma 2.A.1, and Lemma 2.B.8. Such a proof – which we omit here – can be found in the arXiv version of this paper (https://arxiv.org/abs/2303.00429).

Lemma 2.B.10 (L^1 bound for discrete test functions)**.** *Let ϕ^t_h be the solution to (2.5.22). Let $\kappa > 0$. Then, for large enough N, ϕ^t_h satisfies the L^1 bound*

$$\sup_{t\in[0,T]}\|\phi^t_h\|_{L^1(G_{h,d})} \leq N^{\kappa T}\|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})}. \tag{2.B.18}$$

Proof. We expand the solution ϕ_h using the discrete Green's function $\mathcal{G}_h$ associated with the discrete Laplace Δ_h, obtaining

$$\begin{aligned}\phi^{T-t}_{h,\alpha}(x) = (\mathcal{G}^t_h *_h \mathcal{I}_h\varphi)(x) &+ \int_{T-t}^T \left(\mathcal{G}^{T-s}_h(x-\cdot), \left[-U^{r_I}_{h,\alpha}(s)\cdot\nabla_h\phi^s_{h,\alpha}\right]\right)_h \mathrm{d}s \\ &+ \int_{T-t}^T \left(\mathcal{G}^{T-s}_h(x-\cdot), \sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\overline{\rho}^{r_I}_{h,\beta}\nabla_h\phi^s_{h,\beta})\right)_h \mathrm{d}s.\end{aligned}$$

Using integration by parts so as to remove all first derivatives from ϕ_h, we get

$$\begin{aligned}\phi^{T-t}_{h,\alpha}(x) = (\mathcal{G}^t_h *_h \mathcal{I}_h\varphi)(x) &+ \int_{T-t}^T \left(\phi^s_{h,\alpha}, \nabla_h\cdot\left[\mathcal{G}^{T-s}_h(x-\cdot)U^{r_I}_{h,\alpha}(s)\right]\right)_h \mathrm{d}s \\ &- \int_{T-t}^T \left(\nabla_h\mathcal{G}^{T-s}_h(x-\cdot), \sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\overline{\rho}^{r_I}_{h,\beta}\phi^s_{h,\beta})\right)_h \mathrm{d}s \\ &- \int_{T-t}^T \left(\mathcal{G}^{T-s}_h(x-\cdot), \sum_{\beta=1}^{n_S}\mathcal{I}_h[\nabla V^{r_I}_{\beta\alpha}] *_{h,c} (\nabla_h[\overline{\rho}^{r_I}_{h,\beta}]\phi^s_{h,\beta})\right)_h \mathrm{d}s.\end{aligned}$$

Using the inequality

$$\|\nabla_h\mathcal{G}^{T-s}_h(\cdot - y)\|_{L^1(G_{h,d})} \lesssim (T-s)^{-1/2},$$

the bound $\|U^{r_I}_{h,\alpha}\|_{W^{1,\infty}} \le \|V^{r_I}_{\beta\alpha}\|_{H^2}\|\overline{\rho}^{r_I}_h\|_{W^{1,\infty}(G_{h,d})}$, the discrete Young's convolution inequality given in Corollary 2.B.3), and summing up over all species, we obtain

$$\sum_{\alpha=1}^{n_S}\|\phi^{T-t}_{h,\alpha}\|_{L^1(G_{h,d})} \le \|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})} + \|V^{r_I}_{\beta\alpha}\|_{H^2}\|\overline{\rho}^{r_I}_h\|_{W^{1,\infty}(G_{h,d})} \times \int_{T-t}^{T}(1+(T-s)^{-1/2})\sum_{\alpha=1}^{n_S}\|\phi^{s}_{h,\alpha}\|_{L^1(G_{h,d})}\,\mathrm{d}s.$$

Taking the supremum for $t \in [0,\tilde{t}]$, we deduce

$$\sup_{t\in[0,\tilde{t}]}\sum_{\alpha=1}^{n_S}\|\phi^{T-t}_{h,\alpha}\|_{L^1(G_{h,d})} \le \|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})} + \|V^{r_I}_{\beta\alpha}\|_{H^2}\|\overline{\rho}^{r_I}_h\|_{W^{1,\infty}(G_{h,d})} \times \left[\int_{T-\tilde{t}}^{T}(1+(T-s)^{-\frac{1}{2}})\,\mathrm{d}s\right]\sup_{t\in[0,\tilde{t}]}\sum_{\alpha=1}^{n_S}\|\phi^{T-t}_{h,\alpha}\|_{L^1(G_{h,d})}. \quad (2.B.19)$$

Choosing $\tilde{t} \le C\|V^{r_I}_{\beta\alpha}\|^{-2}_{H^2}\|\overline{\rho}^{r_I}_h\|^{-2}_{W^{1,\infty}(G_{h,d})}$ with small enough C allows to perform an absorbtion argument in (2.B.19), leading to

$$\sup_{t\in[0,\tilde{t}]}\sum_{\alpha=1}^{n_S}\|\phi^{T-t}_{h,\alpha}\|_{L^1(G_{h,d})} \le 2\|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})}.$$

Repeating the analysis over all time intervals $[j\tilde{t},(j+1)\tilde{t}]$ up to saturation of $[0,T]$ (there are $\lceil T/\tilde{t}\rceil \lesssim \|V^{r_I}_{\beta\alpha}\|^2_{H^2}\|\overline{\rho}^{r_I}_h\|^2_{W^{1,\infty}(G_{h,d})}T$ of such intervals) we get

$$\sup_{t\in[0,T]}\|\phi_h(t)\|_{L^1(G_{h,d})} \le 2^{T\|V^{r_I}_{\beta\alpha}\|^2_{H^2}\|\overline{\rho}^{r_I}_h\|^2_{W^{1,\infty}(G_{h,d})}}\|\mathcal{I}_h\varphi\|_{L^1(G_{h,d})},$$

and the proof is now concluded bounding $\|\overline{\rho}^{r_I}_{h,\beta}\|_{W^{1,\infty}}$ using (2.B.7), and bounding $\|V^{r_I}_{\beta\alpha}\|^2_{H^2}$ using the scaling (2.3.15). □

2.C Constructing discrete initial data via interpolation

We substantiate Remark 2.3.15: specifically, we show how to construct discrete initial data satisfying Assumption 2.3.10 from given continuous initial data (i.i.d.) by redistributing mass to an appropriate set of surrounding grid points according to interpolation weights.

2.C.1 The multivariate interpolation scheme

We briefly discuss the polynomial multivariate interpolation scheme of order $p+1$. First, for each $y \in \mathbb{T}^d$, we have to fix the set of surrounding grid points $B^p_h(y)$ onto which to distribute mass from the continuous initial data. For schemes of order $p+1$ (interpolating polynomials of order at most p exactly) we need $\binom{p+d}{d}$ points in a suitable formation. As a subset of our rectangular grid we choose, e.g., the "lower left triangular cone" defined as

$$B^p_h := \{x \in G_{h,d};\ \forall = 1,\dots,d : 0 \le x_i,\ x_1+\dots+x_d \le ph\},$$

see [GS00, Section 2.2]. Thus, for $y \in [0,h)^d$ we choose $B^p_h(y) = B^p_h$. In case y is in another grid section, we appropriately shift the set of interpolation points: $B^p_h(y) = B^p_h + h\lfloor y/h\rfloor$. With this choice, for $f \in C^{p+1}$ and $y \in [0,h)^d$ we have the error estimate

$$f(y) = \sum_{x\in B^p_h} f(x)\Lambda_x(y) + \mathcal{O}\big(\|\varphi\|_{C^{p+1}}h^{p+1}\big), \quad (2.C.1)$$

where the Λ_x are the Lagrange fundamental polynomials with $\Lambda_x(\tilde{x}) = \delta_{x\tilde{x}}$ for $\tilde{x} \in B_h^p$, see [GS00, Section 4.1]. Note though that we can rephrase $\Lambda_x(y)$ as samples $\Lambda(x-y)$ from a continuously defined convolution kernel

$$\Lambda\colon \mathbb{T}^d \to \mathbb{R}, \quad \Lambda(z) \coloneqq \begin{cases} \Lambda_x(x-z), & \text{for } x - z \in [0,h)^d,\ x \in B_h^p, \\ 0, & \text{otherwise.} \end{cases}$$

Note that in the interior of each grid section the kernel Λ is a polynomial and thus continuous. Due to the definition of the Lagrange polynomials, Λ is further continuous at all grid points, but in general discontinuous on the remaining contact areas between grid sections.

With this definition we can extend (2.C.1) to all $y \in \mathbb{T}^d$ as

$$f(y) = \sum_{x \in G_{h,d}} f(x)\Lambda(x-y) + \mathcal{O}\big(\|\varphi\|_{C^{p+1}} h^{p+1}\big). \tag{2.C.2}$$

2.C.2 Constructing the discrete initial data

To construct discrete initial data from the continuous one, for $\alpha = 1, \dots, n_S$ we would like to straight-up distribute the mass of a particle starting at $X_{\alpha,i}^{r_I,N}(0)$ according to the interpolation weights, that is for $x \in G_{h,d}$

$$\rho_{h,\alpha}^{0,*}(x) \coloneqq h^{-d} \int_{\mathbb{T}^d} \Lambda(x-y)\, \mathrm{d}\mu_{0,\alpha}^{r_I,N}(y) = h^{-d}\big(\Lambda * \mu_{0,\alpha}^{r_I,N}\big)(x),$$

where the volume factor h^{-d} comes from the interpretation as a density. In order to abide by the wanted properties of the fluctuations (Assumption 2.3.10), we have to adjust for the fact that the discrete mean field limit is initialized as $\overline{\rho}_h^0 = \mathcal{I}_h[\overline{\rho}^0]$. Thus, for $x \in G_{h,d}$ we set

$$\rho_{h,\alpha}^0(x) \coloneqq h^{-d}\big(\Lambda * \mu_{0,\alpha}^{r_I,N}\big)(x) - h^{-d}\big(\Lambda * \overline{\rho}_\alpha^0\big)(x) + \mathcal{I}_h[\overline{\rho}_\alpha^0](x).$$

From (2.C.2) it follows that this definition satisfies (2.3.10).

With respect to (2.3.11) and (2.3.12), note that since the particles are initially i.i.d., Hoeffding's inequality implies for any $m \in \mathbb{Z}^d \cap \left[-\frac{h}{\pi}, \frac{h}{\pi}\right)^d$

$$\left| h^d \sum_{x \in G_{h,d}} (\rho_h^0 - \overline{\rho}_h^0)(x) \exp(im \cdot x) \right|^2 \leq \mathcal{C}^2 N^{-1}$$

where $\mathcal{C}$ is subject to a Gaussian moment bound. Multiplying with $(1+|m|)^{-d-1}$ and summing in m, we deduce $\|\rho_h^0 - \overline{\rho}_h^0\|_{H_h^{-(\lfloor d/2 \rfloor + 1)}} \leq \mathcal{C} N^{-1/2}$, which immediately implies that (2.3.12) is satisfied. Similarly, by Bernstein's inequality we obtain

$$\mathbb{P}\left[|\rho_h^0(x) - \overline{\rho}_h^0(x)| \geq \sqrt{\frac{CN \operatorname{Var} \rho_h^0(x)}{N}} R + \frac{C|N \max_\Omega \rho_h^0(x)| R^2}{N} \right] \leq \exp(-R^2),$$

for every $x \in G_{h,d}$ and any $R \geq 1$, and thus by bounding the local mollifier $h^{-d}\Lambda$

$$\mathbb{P}\Big[|\rho_h^0(x) - \overline{\rho}_h^0(x)| \geq CN^{-1/2} h^{-d/2} R + Ch^{-d} N^{-1} R^2 \Big] \leq \exp(-R^2).$$

This implies that this set of initial conditions satisfies (2.3.11). The only parts of Assumption 2.3.10 left are positivity and the mass restriction. Due to $\overline{\rho}^0 \geq \rho_{min}$ and (2.3.11), the

probability of negative initial data decays exponentially – hence, for large enough N, we can restrict to positive data at the cost of only an additional insignificant error in (2.3.10). Since the interpolation weights preserve mass, for positive initial data we have $\|\rho^0_{h,\alpha}\|_{L^1_h} = \|\mathcal{I}_h[\overline{\rho}^0_\alpha]\|_{L^1_h} \approx 1$ for h small enough. Thus, restricting to $\rho^0_h \geq 0$, this construction satisfies Assumption 2.3.10.

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CHAPTER 3

Stochastic homogenization of micromagnetic energies

This chapter contains the following publication [DDI24], reproduced with permission from Springer Nature:

Elisa Davoli, Lorenza D'Elia, and Jonas Ingmanns. "Stochastic Homogenization of Micromagnetic Energies and Emergence of Magnetic Skyrmions." *Journal of Nonlinear Science* 34.2 (2024). DOI: 10.1007/s00332-023-10005-3

Abstract We perform a stochastic-homogenization analysis for composite materials exhibiting a random microstructure. Under the assumptions of stationarity and ergodicity, we characterize the Γ-limit of a micromagnetic energy functional defined on magnetizations taking value in the unit sphere, and including both symmetric and antisymmetric exchange contributions. This Γ-limit corresponds to a micromagnetic energy functional with homogeneous coefficients. We provide explicit formulas for the effective magnetic properties of the composite material in terms of homogenization correctors. Additionally, the variational analysis of the two exchange energy terms is performed in the more general setting of functionals defined on manifold-valued maps with Sobolev regularity, in the case in which the target manifold is a bounded, orientable smooth surface with tubular neighborhood of uniform thickness. Eventually, we present an explicit characterization of minimizers of the effective exchange in the case of magnetic multilayers, providing quantitative evidence of Dzyaloshinskii's predictions on the emergence of helical structures in composite ferromagnetic materials with stochastic microstructure.

3.1 Introduction

Many key properties and applications of magnetic materials are strongly intertwined with the spatial distribution of magnetic moments inside the corresponding specimens. In addition to classical magnetic structures such as Weiss domains, where the magnetization is almost collinear and slowly-varying, and Bloch walls, forming thin transition layers and allowing the magnetization to rotate coherently from one magnetic structure to the other, magnetic skyrmions [Bod+07; BH99; FRC17] have recently emerged as an extremely active field of research due to their potential as building blocks for innovative functional devices. Since skyrmions can be manipulated (written and deleted)

individually on a magnetic stripe, in fact, these quasiparticles are regarded as possible carriers of information for future storage devices [FCS13]. Named after the physicist T. Skyrme, skyrmions have been first predicted in magnetic crystals in [Dzy58]. They were then experimentally identified both in thin and ultrathin films as well as in multilayers.

The specific chirality of magnetic skyrmions, namely their lack of inversion symmetry [FRC17], is determined by the antisymmetric Dzyaloshinskii-Moriya interaction (DMI, also called antisymmetric exchange)[Dzy58; Mor60]. For an open bounded domain $D \subset \mathbb{R}^3$ representing a specimen of a single-crystal chiral magnet, assuming that m satisfies a normalized saturation constraint $|m(x)| = 1$, the micro-magnetic energy is given by the functional (see [Bro63])

$$\begin{aligned}\mathcal{F}(m) &:= \frac{1}{2}\int_D a|\nabla m(x)|^2\,\mathrm{d}x + \int_D \kappa \operatorname{curl} m(x)\cdot m(x)\,\mathrm{d}x - \frac{\mu_0}{2}\int_D h_d\left[M_{\mathrm{sat}} m\chi_D\right](x)\cdot M_{\mathrm{sat}} m(x)\,\mathrm{d}x \\ &\quad + \int_D \varphi\left(m(x)\right)\,\mathrm{d}x - \mu_0\int_D h_a\cdot M_{\mathrm{sat}} m(x)\,\mathrm{d}x \\ &=: \mathcal{E}(m) + \mathcal{K}(m) + \mathcal{W}(m) + \mathcal{A}(m) + \mathcal{Z}(m) \end{aligned} \tag{3.1.1}$$

In the expression above, $\mathcal{E}(m)$ and $\mathcal{K}(m)$ represent the exchange energy and the bulk DMI, encoding energetic contributions due to spatial changes of the magnetization and to asymmetries in the crystalline structure of the material, respectively. The parameters $a > 0$ and $\kappa \in \mathbb{R}$ are material length-scales tuning the strength of very short-range interactions, and of chirality effects. The term $\mathcal{W}(m)$ in (3.1.1) is the stray-field energy. It is a nonlocal energy contribution encoding the effects of the magnetic field $h_d[m\chi_D]$ induced by the magnetization m, and favoring configurations involving solenoidal magnetic fields. Note that $m\chi_D$ denotes an extension of m to the whole $\mathbb{R}^3$ by setting it equal to zero outside of the set D. These two latter quantities are related through the magnetostatic Maxwell equation

$$\operatorname{div}(-\mu_0 h_d[m\chi_D] + m\chi_D) = 0 \quad \text{in } \mathbb{R}^3,$$

where μ_0 is the vacuum permeability. The contribution $\mathcal{A}(m)$ is the magnetocrystalline anisotropic, whose density $\varphi : \mathbb{S}^2 \to [0, +\infty)$ encodes the presence of easy axes in the micromagnetic specimen, namely, of preferred magnetization directions (here $\mathbb{S}^2$ denotes, as usual, the unit sphere in $\mathbb{R}^3$). Eventually, $\mathcal{Z}(m)$ is the Zeeman energy, modeling the tendency of the magnetization to align with the externally applied magnetic field h_a. A mathematical analysis of micromagnetics models incorporating DMI terms can be found in [LM18; Mel14; MS17]. We also refer to the works [CRS16; CS15; CFO19] for the discrete-to-continuum setting, as well as to [GZ22] for a study of associated energy scaling laws, and to [Hrk+19; Dav+22] for related numerical and dimension-reduction results.

The goal of this paper is to advance the mathematical modeling of magnetic skyrmions by analyzing the interplay of stochastic microstructures and chirality. In a recent work [DD20], jointly with G. Di Fratta, one of the authors has undertaken a periodic homogenization analysis for a micromagnetic energy functional involving a DMI bulk energy term. In particular, the results in [DD20] provided a first quantitative counterpart to the theoretical predictions in the seminal work by Dzyaloshinskii on the existence of helicoidal textures as the result of possible instabilities of ferromagnetic structures under small relativistic spin-lattice or spin-spin interactions [Dzy58; Dzy65a; Dzy65b; BJ80].

In the present manuscript we move radically beyond the even distribution of material heterogeneities assumed in [DD20] to identify effective theories for composite chiral magnetic materials with a microstructure encompassing random effects. The novelty of our contribution is threefold. First, we provide a stochastic homogenization analysis in the general setting of manifold-valued Sobolev spaces, having as a corollary the aforementioned application in micromagnetics. To the Authors' knowledge, this is the first mathematical study in this direction. Second, we delineate the framework for stochastic two-scale convergence in Beppo Levi spaces. Third, we present an explicit characterization of minimizers of the limiting exchange energies in the case of stochastic multilayers providing a further quantitative evidence, this time in an aperiodic setting, to Dzyaloshinskii predictions and to the experimental observations in [Yu+10; FRC17; Jia+17], as well as a stochastic counterpart to [DD20].

Before describing our findings in detail, we briefly review the mathematical literature on homogenization in micromagnetics. Among the many contributions, we refer the Reader to the following papers and to the references therein: the setting of ferromagnetic laminates has been studied in [HJ03], whereas that of perforated domains in [San07]. An analysis relying on the notion of $\mathcal{A}$-quasiconvexity has been carried out in [Pis10]. We refer to [Alo+21; AD15] for two extensive homogenization results in the periodic and stochastic case. Periodic homogenization of chiral magnetic materials was tackled in the aforementioned [DD20].

When considering a composite ferromagnetic body, it is important to keep track of the local interactions of grains with different magnetic properties at their interface [AFM06]. In particular, under the assumption of strong coupling conditions, meaning that the direction m of the magnetization does not jump through an interface and only the magnitude is allowed to be discontinuous, homogenization problems are formulated by considering point- and microstructure-dependent material parameters, and for magnetizations m taking value in the unit sphere. Assuming that D is a specimen of a multi-crystal ferromagnet with random microstructure, we thus consider the family of energy functionals $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$ with $\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon^{\widetilde{\omega}} \colon H^1(D;\mathbb{S}^2) \to \mathbb{R}$ given by

$$\begin{aligned}\mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) \coloneqq{}& \frac{1}{2}\int_D a\left(\frac{x}{\varepsilon},\widetilde{\omega}\right)|\nabla m(x)|^2\,\mathrm{d}x + \int_D \kappa\left(\frac{x}{\varepsilon},\widetilde{\omega}\right)\operatorname{curl} m(x)\cdot m(x)\,\mathrm{d}x\\ &- \frac{\mu_0}{2}\int_D h_d\left[M_{\mathrm{sat}}\left(\frac{x}{\varepsilon},\widetilde{\omega}\right)m\chi_D\right](x)\cdot M_{\mathrm{sat}}\left(\frac{x}{\varepsilon},\widetilde{\omega}\right)m(x)\,\mathrm{d}x\\ &+ \int_D \varphi\left(\frac{x}{\varepsilon},m(x),\widetilde{\omega}\right)\mathrm{d}x - \mu_0\int_D h_a\cdot M_{\mathrm{sat}}\left(\frac{x}{\varepsilon},\widetilde{\omega}\right)m(x)\,\mathrm{d}x\\ ={}& \mathcal{E}_\varepsilon^{\widetilde{\omega}}(m) + \mathcal{K}_\varepsilon^{\widetilde{\omega}}(m) + \mathcal{W}_\varepsilon^{\widetilde{\omega}}(m) + \mathcal{A}_\varepsilon^{\widetilde{\omega}}(m) + \mathcal{Z}_\varepsilon^{\widetilde{\omega}}(m)\end{aligned} \tag{3.1.2}$$

for a set of stationary and ergodic random parameters $a,\kappa,M_{\mathrm{sat}}\colon\mathbb{R}^3\to\mathbb{R}$ and $\varphi\colon\mathbb{R}^3\to C^{0,1}(\mathbb{S}^2)$ on an appropriate probability space $(\Omega,\Sigma,\mathbb{P})$ with $\widetilde{\omega}\in\Omega$. The specific stochastic framework and assumptions on the parameters will be described in Section 3.2.1. We only briefly mention here that requiring stationarity amounts, roughly speaking, to imposing a sort of stochastic periodicity (or periodicity in law, cf [DM86a]) guaranteeing that any given microstructure and all its translations occur with equal probability. Ergodicity, instead, enforces that the behavior of the effective material is independent from the specific stochastic realization, see [DM86b].

Note that the DMI term can be rewritten as

$$\mathcal{K}_\varepsilon^{\widetilde{\omega}}(m) = \int_D \kappa\left(\frac{x}{\varepsilon}, \widetilde{\omega}\right) \operatorname{curl} m(x) \cdot m(x)\,\mathrm{d}x = -\int_D \kappa\left(\frac{x}{\varepsilon}, \widetilde{\omega}\right) \boldsymbol{\chi}(m(x)) : \nabla m(x)\,\mathrm{d}x, \tag{3.1.3}$$

where $A : B = \sum_{i,j} A_{ij}B_{ij}$ for $A, B \in \mathbb{R}^{3\times 3}$ and the map $\boldsymbol{\chi}$ is defined by

$$\boldsymbol{\chi}(s) \coloneqq (e_1 \times s, e_2 \times s, e_3 \times s)^\intercal \in \mathbb{R}^{3\times 3}_{\mathrm{skew}}, \quad \text{for any } s \in \mathbb{R}^3, \tag{3.1.4}$$

where $\{e_i\}_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$. Let Id denote the identity matrix in $\mathbb{R}^{3\times 3}$, and for a random variable f, let $\mathbb{E}[f] \coloneqq \int_\Omega f(\omega)\,\mathrm{d}\mathbb{P}$ be its expected value. We present below a slightly simplified statement of our first main result. We refer to Theorem 3.2.3 for its precise formulation.

Theorem 3.1.1. *Let $D \subset \mathbb{R}^3$ be an open, bounded set with Lipschitz boundary. Then, under the assumption of stochasticity and ergodicity of all material parameters, for almost every $\widetilde{\omega} \in \Omega$ the family $(\mathcal{F}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$ is equi-coercive and Γ-converges with respect to the weak topology on $H^1(D;\mathbb{S}^2)$ to the energy functional $\mathcal{F}_{\mathrm{hom}}$, which corresponds to the micromagnetic energy of a homogeneous material with a-priori anisotropic exchange parameter and DMI constant,*

$$\mathcal{F}_{\mathrm{hom}} = \mathcal{E}_{\mathrm{eff}} + \mathcal{K}_{\mathrm{eff}} + \mathcal{W}_{\mathrm{eff}} + \mathcal{A}_{\mathrm{eff}} + \mathcal{Z}_{\mathrm{eff}},$$

where, in the sense of Remark 3.2.4, denoting by $\nabla\Phi_a, \nabla\Phi_\kappa, \nabla\Phi_M$ the solutions in $L^2(\Omega, L^2_{\mathrm{loc}}(\mathbb{R}^3))$ of the corrector-type equations

$$\nabla \cdot (a\nabla\Phi_a + a\,\mathrm{Id}) = 0, \qquad \nabla \cdot (a\nabla\Phi_\kappa + \kappa\,\mathrm{Id}) = 0, \qquad \nabla \cdot (\nabla\Phi_M + M_{\mathrm{sat}}\,\mathrm{Id}) = 0, \tag{3.1.5}$$

there holds

$$\begin{aligned}
\mathcal{E}_{\mathrm{eff}} &= \frac{1}{2}\int_D \nabla m : \mathbb{E}\left[a\,\mathrm{Id} - (\nabla\Phi_a)^\intercal a \nabla\Phi_a\right]\nabla m\,\mathrm{d}x,\\
\mathcal{K}_{\mathrm{eff}} &= -\int_D \nabla m : \mathbb{E}\left[\kappa\,\mathrm{Id} - (\nabla\Phi_a)^\intercal a \nabla\Phi_\kappa\right]\boldsymbol{\chi}(m)\,\mathrm{d}x,\\
\mathcal{W}_{\mathrm{eff}} &= -\frac{\mu_0}{2}\int_D h_d\left[\mathbb{E}\left[M_{\mathrm{sat}}\right] m\chi_D\right] \cdot \mathbb{E}\left[M_{\mathrm{sat}}\right] m\,\mathrm{d}x,\\
\mathcal{A}_{\mathrm{eff}} &= \int_D \left(\mathbb{E}\left[\varphi\right](m) - \frac{1}{2}\boldsymbol{\chi}(m) : \mathbb{E}\left[(\nabla\Phi_\kappa)^\intercal a \nabla\Phi_\kappa\right]\boldsymbol{\chi}(m) + \frac{\mu_0}{2} m \cdot \mathbb{E}\left[(\nabla\Phi_M)^\intercal \nabla\Phi_M\right] m\right)\mathrm{d}x\\
\mathcal{Z}_{\mathrm{eff}} &= -\mu_0 \int_D h_a \cdot \mathbb{E}\left[M_{\mathrm{sat}}\right] m\,\mathrm{d}x.
\end{aligned}$$

We stress that the effective material identified via our homogenization procedure reduces, in the periodic case, to the one found in [DD20]. In this latter setting, the corrector equations are replaced by corresponding cell problems and the expected values of the limiting random variables by the averages of the corresponding periodic quantities in their periodicity cell. We also point out that the different energy contributions denoted with the pedix "eff" in Theorem 3.1.1 are not the Γ-limit of the corresponding ε-energies (which we will denote by "hom" instead) but have rather been rearranged in order to highlight the role of correctors in the limiting problem, as well as the meaning of the various energy terms.

The proof of Theorem 3.2.3 is based on the notion of quenched stochastic two-scale convergence. Whereas the notion of two-scale convergence in the periodic case [LNW02; All92; Ngu89] is uniquely defined, in the stochastic setting two alternative notions have

been introduced, corresponding to two different choices for the topology in which the limiting description is identified: when working with stochastic two-scale convergence in the mean [PV81; AW98; BLM96; WMB94], the relevant fields are integrated with respect to the probability space (see also [NV18; HNV21] for a corresponding notion of stochastic unfolding), whereas quenched stochastic limits [Alo+21; ZP06] are taken pointwise for almost every realization. We refer to [HNV21] for a comparison between the two convergences. We will argue here with the quenched variant, henceforth simply referred to, for shortness, as stochastic two-scale convergence.

The key ingredient for showing Theorem 3.1.1 consists in proving the Γ-convergence of $\mathcal{G}_\varepsilon^{\widetilde{\omega}} := \mathcal{E}_\varepsilon^{\widetilde{\omega}} + \mathcal{K}_\varepsilon^{\widetilde{\omega}}$, for the remaining energy contributions can be treated as continuous perturbations. Studying the asymptotic behavior of $\mathcal{G}_\varepsilon^{\widetilde{\omega}}$ is of independent mathematical interest, because it represents a first stochastic homogenization result for Dirichlet-type energies in manifold-valued Sobolev spaces. We therefore prove a slightly more general result for functionals defined on maps taking value in a general bounded, orientable, C^2 hypersurface of $\mathbb{R}^3$ admitting a tubular neighborhood of uniform thickness, cf. Theorem 3.2.6. We refer to [AD15; DD20; Dac+99; BM09] for periodic homogenization problems in manifold-valued Sobolev spaces, as well as to [BSS17] and the references therein for a overview on stochastic homogenization in classical Sobolev spaces for convex and non-convex integral functionals.

The Γ-convergence of the magnetostatic self-energy is characterized in Proposition 3.2.8, whereas anisotropic energy and Zeeman contributions are studied in Proposition 3.2.9. In particular, as a by-product of our analysis, we provide an extension of the theory of two-scale convergence in Beppo-Levi spaces in [AD15] to the stochastic setting, cf. Subsection 3.5.1.

Eventually, we specify our analysis to the case in which our micromagnetic specimen is a multilayer with random microstructure. In this latter framework, we provide an explicit characterization of the minimizers of $\mathcal{G}_{\mathrm{hom}} := \Gamma\text{-}\lim_{\varepsilon\to 0}(\mathcal{E}_\varepsilon^{\widetilde{\omega}} + \mathcal{K}_\varepsilon^{\widetilde{\omega}})$ and show the emergence of chiral structures. A simplified statement of our second main result reads as follows. We refer to Proposition 3.6.2 and Lemma 3.6.4 for its precise formulation.

Theorem 3.1.2. *If $\mathbb{E}[\kappa] = 0$, the only minimizers of $\mathcal{G}_{\mathrm{hom}}$ are given by*

$$m_*(x) := \cos(\theta(x\cdot e_3))e_1 + \sin(\theta(x\cdot e_3))e_2, \quad \theta(t) := \theta_0 + \mathbb{E}\left[\frac{\kappa}{a}\right] t,$$

for every $t \in \mathbb{R}$, with $\theta_0 \in \mathbb{R}$ arbitrary.

Note that the assumption $\mathbb{E}[\kappa] = 0$ is only needed to obtain an explicit formulation of minimizer of $\mathcal{G}_{\mathrm{hom}}$. To the authors' knowledge, removing such an assumption makes the computation very challenging.

The paper is organized as follows. In Section 3.2 we describe the stochastic framework and specify the assumptions on the random parameters as well as the anisotropic energy density. We conclude this section providing the precise statements of our main results. In Section 3.3 we recall the concept of stochastic two-scale convergence and show the existence of the corrector quantities given in (3.2.1). Section 3.3.3 is devoted to the characterization of the two-scale limits of $H^1(D;\mathcal{M})$-maps. In Section 3.4, we obtain the Γ-convergence of $\mathcal{G}_\varepsilon^{\widetilde{\omega}}$ (Theorem 3.2.6). In particular, the limiting density is characterized in Subsection 3.4.1, whereas the liminf and limsup inequalities are proven in Subsections 3.4.2 and 3.4.3, respectively. Section 3.5 contains the rest of the proof of Theorem 3.1.1 (for the precise assumptions, Theorem 3.2.3): adapting the strategy in [AD15] to the

stochastic setting, we prove the convergence of $\mathcal{W}_\varepsilon^{\widetilde{\omega}}$ (Proposition 3.2.8) in Subsection 3.5.1 and the convergence of $\mathcal{A}_\varepsilon^{\widetilde{\omega}}$ and $\mathcal{Z}_\varepsilon^{\widetilde{\omega}}$ (Proposition 3.2.9) in Subsection 3.5.2. The equi-coerciveness of the micromagnetic functionals is shown in Subsection 3.5.3. Finally, Section 3.6 is devoted to obtain Theorem 3.1.2: we calculate the effective material properties if the microstructure is given by laminates and thus obtain a characterization of the minimizers.

3.2 Setting of the problem and main results

In this section we collect our main assumptions and state our main results. For simplicity, we will formulate all results in $\mathbb{R}^3$. We stress, however, that they hold more generally in $\mathbb{R}^d$.

Notation In what follows, we denote by Id the identity matrix in $\mathbb{R}^{3\times 3}$. For $A \in \mathbb{R}^{3\times 3}$, we indicate by $A^\intercal$ its transpose. Given a random variable f, the expected value of f is denoted by $\mathbb{E}[f] := \int_\Omega f(\omega)\,\mathrm{d}\mathbb{P}$. We will use standard notation for Lebesgue and Sobolev spaces. We will sometimes omit the target space whenever this is immediately clear from the context. We will often denote by $C > 0$ a generic constant, whose value in a formula might change from line to line. Diagonal matrices in $\mathbb{R}^{3\times 3}$ with entries d_1, d_2, d_3 will be denoted by $\mathrm{diag}(d_1, d_2, d_3)$.

3.2.1 Stochastic framework and parameter assumptions

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space endowed with a *dynamical system* $(T_x)_{x\in\mathbb{R}^3}$, i.e., a family of measurable bijective mappings $T_x : \Omega \to \Omega$ satisfying the following properties

(T1) $T_0 = \mathrm{Id}$ and $T_{x+y} = T_x \circ T_y$, for every $x, y \in \mathbb{R}^3$;

(T2) $\mathbb{P}(T_x^{-1}(B)) = \mathbb{P}(B)$ for all $x \in \mathbb{R}^3$ and $B \in \Sigma$;

(T3) the map $(x, \omega) \in \mathbb{R}^3 \times \Omega \mapsto T_x\omega \in \Omega$ is measurable.

We also assume that Ω is a compact metric space, Σ is a complete σ-algebra on Ω, and $(x, \omega) \mapsto T_x\omega$ as a map from $\mathbb{R}^3 \times \Omega$ to Ω is continuous. We will work under the assumption that the dynamical system T_x is *ergodic*, namely:

(T4) The only measurable sets $A \in \Sigma$ which are translation invariant, i.e., are such that (up to a null subset) $T_x A = A$ for every $x \in \mathbb{R}^d$, satisfy $\mathbb{P}(A) \in \{0, 1\}$.

Remark 3.2.1. The ergodicity property can be equivalently formulated by requiring that the only measurable functions $f : \Omega \to \mathbb{R}$ such that $f(\omega) = f(T_x\omega)$ for every $x \in \mathbb{R}^3$ and $\mathbb{P}$-a.e. $\omega \in \Omega$ are those coinciding $\mathbb{P}$-almost surely with a constant.

Remark 3.2.2. A range of settings and examples, where the above assumptions on the probability space and the dynamical system are satisfied, can be found in [ZP06, Section 1.1].

For $1 \le p \le \infty$ we denote by $L^p(\Omega)$ the L^p-space on $(\Omega, \Sigma, \mathbb{P})$, and, similarly, we indicate by $L^p(\mathbb{R}^3 \times \Omega)$ the L^p-space on the product measure space.

A random field $(x, \omega) \in \mathbb{R}^3 \times \Omega \mapsto \tilde{f}(x, \omega)$ is said to be *stationary* if there exists a measurable function f on Ω such that $\tilde{f}(x, \omega) = f(T_x\omega)$. Hence, a random variable $f : \mathbb{R}^3 \times \Omega \to \mathbb{R}$ is *stationary ergodic* if f is stationary and the underlying dynamical system T_x is ergodic.

We assume that all the material-dependent parameters and the anisotropic energy density are stationary ergodic random variables. With a slight abuse of notation, we will use the same letter for each of these random fields as well as for their corresponding measurable functions. To be precise,

(P1) the exchange coefficient is given by $a(x,\omega) = a(T_x\omega)$ with $a \in L^\infty(\Omega)$ bounded from below and above by two positive constants $c_{\mathrm{ex}}, C_{\mathrm{ex}}$, i.e., $0 < c_{\mathrm{ex}} \leq a(\omega) \leq C_{\mathrm{ex}}$ for all $\omega \in \Omega$;

(P2) the DMI constant is given by $\kappa(x,\omega) = \kappa(T_x\omega)$, for some $\kappa \in L^\infty(\Omega)$ bounded by a positive constant C_{DMI}, i.e. $|\kappa(\omega)| \leq C_{\mathrm{DMI}}$ for all $\omega \in \Omega$;

(P3) the local saturation magnetization is given by $M_{\mathrm{sat}}(x,\omega) = M_{\mathrm{sat}}(T_x\omega)$ with $M_{\mathrm{sat}} \in L^\infty(\Omega)$ positive and bounded from above, i.e., $0 \leq M_{\mathrm{sat}}(\omega) \leq C_{\mathrm{sat}}$ for some $C_{\mathrm{sat}} > 0$ and all $\omega \in \Omega$;

(P4) the anisotropic energy density $\varphi\colon \mathbb{R}^3 \times \Omega \times \mathbb{S}^2 \to \mathbb{R}^+$ is given by $\varphi(x,s,\omega) = \varphi(T_x\omega, s)$, where $\varphi(\cdot, s) : \Omega \to \mathbb{R}^+ \in L^\infty(\Omega)$ for every $s \in \mathbb{S}^2$, and there exists a positive constant $L_{\mathrm{an}} > 0$ such that

$$\operatorname*{ess\text{-}sup}_{\omega\in\Omega} |\varphi(\omega, s_1) - \varphi(\omega, s_2)| \leq L_{\mathrm{an}}|s_1 - s_2|, \quad \forall\, s_1, s_2 \in \mathbb{S}^2.$$

3.2.2 Statement of the main results

We are now in a position to provide the precise statement of our main result. Recall (3.1.4) and (3.1.2). In order to phrase our stochastic framework in terms of an ergodic dynamical system, we need to introduce two suitable function spaces: the space $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ of gradients functions and $L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3)$ of divergence-free functions. The space $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ is defined as the closure of the set $\{\nabla_\omega u : u \in C^1(\Omega;\mathbb{R}^3)\}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. We set $L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3) := (L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3))^\perp$ with respect to the standard scalar product in $L^2(\Omega;\mathbb{R}^3)$, see Section 3.3.1 for a more detailed introduction to these spaces.

Theorem 3.2.3. *Let D be an open, bounded domain with Lipschitz boundary. Assume that, with respect to a probability space $(\Omega, \Sigma, \mathbb{P})$ as described in Section 3.2.1, the random parameters $a_\varepsilon, \kappa_\varepsilon, M_\varepsilon$ and the random anisotropic density φ_ε satisfy (P1)–(P4).*
Then, for almost every $\widetilde{\omega} \in \Omega$ the family $(\mathcal{F}^{\widetilde{\omega}}_\varepsilon)_{\varepsilon>0}$ is equi-coercive and Γ-converges with respect to the weak topology on $H^1(D;\mathbb{S}^2)$ to the energy functional $\mathcal{F}_{\mathrm{hom}}$, which corresponds to the micromagnetic energy of a homogeneous material with a-priori anisotropic exchange parameter and DMI constant,

$$\mathcal{F}_{\mathrm{hom}} = \mathcal{E}_{\mathrm{eff}} + \mathcal{K}_{\mathrm{eff}} + \mathcal{W}_{\mathrm{eff}} + \mathcal{A}_{\mathrm{eff}} + \mathcal{Z}_{\mathrm{eff}},$$

where, denoting by $\Theta_a, \Theta_\kappa, \Theta_M \in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^{3\times 3})$ the solutions of

$$(a\Theta_a + a\,\mathrm{Id})\,,\ (a\Theta_\kappa + \kappa\,\mathrm{Id})\,,\ (\Theta_M + M_{\mathrm{sat}}\,\mathrm{Id}) \in L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3), \tag{3.2.1}$$

the effective energy components are given by

$$\begin{aligned}
\mathcal{E}_{\mathrm{eff}} &= \frac{1}{2}\int_D \nabla m : \mathbb{E}\left[a\,\mathrm{Id} - \Theta_a^{\intercal} a \Theta_a\right]\nabla m\,\mathrm{d}x,\\
\mathcal{K}_{\mathrm{eff}} &= -\int_D \nabla m : \mathbb{E}\left[\kappa\,\mathrm{Id} - \Theta_a^{\intercal} a \Theta_\kappa\right]\boldsymbol{\chi}(m)\,\mathrm{d}x,\\
\mathcal{W}_{\mathrm{eff}} &= -\frac{\mu_0}{2}\int_D h_d\left[\mathbb{E}\left[M_{\mathrm{sat}}\right]m\right]\cdot\mathbb{E}\left[M_{\mathrm{sat}}\right]m\,\mathrm{d}x,\\
\mathcal{A}_{\mathrm{eff}} &= \int_D \left(\mathbb{E}\left[\varphi\right](m) - \frac{1}{2}\boldsymbol{\chi}(m) : \mathbb{E}\left[\Theta_\kappa^{\intercal} a \Theta_\kappa\right]\boldsymbol{\chi}(m) + \frac{\mu_0}{2} m\cdot\mathbb{E}\left[\Theta_M^{\intercal}\Theta_M\right]m\right)\,\mathrm{d}x\\
\mathcal{Z}_{\mathrm{eff}} &= -\mu_0\int_D h_a\cdot\mathbb{E}\left[M_{\mathrm{sat}}\right]m\,\mathrm{d}x.
\end{aligned}$$

Remark 3.2.4 (On the homogenization correctors). Our correctors Θ_a, Θ_κ, and Θ_M are equivalently characterized as those maps for which $\nabla\Phi_a, \nabla\Phi_\kappa, \nabla\Phi_M$ are the solutions in $L^2(\Omega, L^2_{\mathrm{loc}}(\mathbb{R}^3))$ of the corrector-type equations

$$\nabla\cdot(a\nabla\Phi_a + a\,\mathrm{Id}) = 0, \qquad \nabla\cdot(a\nabla\Phi_\kappa + \kappa\,\mathrm{Id}) = 0, \qquad \nabla\cdot(\nabla\Phi_M + M_{\mathrm{sat}}\,\mathrm{Id}) = 0.$$

We will address the existence and uniqueness of the corrector quantities given via (3.2.1) later in Subsection 3.3.2 and Lemma 3.3.21.

Remark 3.2.5 (Typical trajectories). The set $\widetilde{\Omega}\subset\Omega$ with $\mathbb{P}(\widetilde{\Omega}) = 1$, on which we prove the main theorem, is given by $\widetilde{\Omega} = \widetilde{\Omega}_{\mathrm{d}\mathbb{P}}\cap\widetilde{\Omega}_{M_{\mathrm{sat}}\,\mathrm{d}\mathbb{P}}\cap\widetilde{\Omega}_{M_{\mathrm{sat}}^2\,\mathrm{d}\mathbb{P}}\cap\widetilde{\Omega}_{a\,\mathrm{d}\mathbb{P}}\cap\widetilde{\Omega}_{\kappa\,\mathrm{d}\mathbb{P}}\cap\widetilde{\Omega}_{\kappa^2/a\,\mathrm{d}\mathbb{P}}$ (see Theorem 3.4.3). In other words, $\widetilde{\Omega}$ is the intersection of the sets of typical trajectories with respect to the measures $\mathrm{d}\mathbb{P}$, $a(\cdot)\,\mathrm{d}\mathbb{P}(\cdot)$, $\kappa(\cdot)\,\mathrm{d}\mathbb{P}(\cdot)$, $\kappa^2/a(\cdot)\,\mathrm{d}\mathbb{P}(\cdot)$, $M_{\mathrm{sat}}(\cdot)\,\mathrm{d}\mathbb{P}(\cdot)$, and $M_{\mathrm{sat}}(\cdot)^2\,\mathrm{d}\mathbb{P}(\cdot)$, respectively, see Definition 3.3.3, Proposition 3.3.4, and Remark 3.3.6 below.

The key step for the proof of Theorem 3.2.3 is the analysis of the asymptotic behavior via Γ-convergence of the two exchange contributions. Since this latter result is of independent interest, we prove it in a broader setting. In what follows, $\mathcal{M}$ is a bounded, C^2 orientable hypersurface of $\mathbb{R}^3$ that admits a tubular neighborhood of uniform thickness (in the micromagnetic setting, usually $\mathcal{M}$ is the unit sphere $\mathbb{S}^2$). The tangent space at $s\in\mathcal{M}$ is denoted by $\mathcal{T}_s\mathcal{M}$. The vector bundle $\boldsymbol{\mathcal{T}}\mathcal{M}$ is given by $\boldsymbol{\mathcal{T}}\mathcal{M} := \bigcup_{s\in\mathcal{M}}\{s\}\times\boldsymbol{\mathcal{T}}_s\mathcal{M}$, with $\boldsymbol{\mathcal{T}}_s\mathcal{M} := (\mathcal{T}_s\mathcal{M})^3$. We will indicate by $\xi^{\intercal} := (\xi_1^{\intercal}, \xi_2^{\intercal}, \xi_3^{\intercal})$ a generic element of $\boldsymbol{\mathcal{T}}_s\mathcal{M}$. This notation is motivated by the fact that if $\xi := \nabla m$, with $m\in H^1(D,\mathcal{M})$ and ∇m is the transpose of the Jacobian matrix of m, the columns $(\xi_1^{\intercal}, \xi_2^{\intercal}, \xi_3^{\intercal}) := (\partial_1 m(x), \partial_2 m(x), \partial_3 m(x))$ of $(\nabla m(x))^{\intercal}$ belong to $\mathcal{T}_{m(x)}\mathcal{M}$ for a.e. $x\in D$. Recalling (3.1.2) and (3.1.4), our result is the following.

Theorem 3.2.6 (Γ-convergence of exchange energy and Dzyaloshinskii–Moriya interaction)**.** *Let $\mathcal{M}$ be a bounded, orientable, C^2 hypersurface of $\mathbb{R}^3$ admitting a tubular neighborhood of uniform thickness. Under the assumptions of Theorem 3.2.3, for almost every $\widetilde{\omega}\in\Omega$ the family $(\mathcal{G}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0} := (\mathcal{E}_\varepsilon^{\widetilde{\omega}} + \mathcal{K}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$ Γ-converges with respect to the weak topology on $H^1(D;\mathcal{M})$ to the energy functional*

$$\begin{aligned}
\mathcal{G}_{\mathrm{hom}}(m) := \frac{1}{2}\int_D \nabla m : \mathbb{E}\left[a\,\mathrm{Id} - (\nabla\Phi_a)^{\intercal} a \nabla\Phi_a\right]\nabla m\,\mathrm{d}x\\
- \int_D \nabla m : \mathbb{E}\left[\kappa\,\mathrm{Id} - (\nabla\Phi_a)^{\intercal} a\nabla\Phi_\kappa\right]\boldsymbol{\chi}(m)\,\mathrm{d}x\\
- \frac{1}{2}\int_D \boldsymbol{\chi}(m) : \mathbb{E}\left[(\nabla\Phi_\kappa)^{\intercal} a\nabla\Phi_\kappa\right]\boldsymbol{\chi}(m)\pi_{\mathcal{T}_m\mathcal{M}}\,\mathrm{d}x. \qquad (3.2.2)
\end{aligned}$$

with $\nabla\Phi_a$, $\nabla\Phi_\kappa$ as in (3.1.5) and where $\pi_{\mathcal{T}_s\mathcal{M}}\in\mathbb{R}^{3\times 3}$ is the orthogonal projection onto the tangent space $\mathcal{T}_s\mathcal{M}$ at $s\in\mathcal{M}$. Note that $\boldsymbol{\chi}(m)\pi_{\mathcal{T}_m\mathcal{M}} = \boldsymbol{\chi}(m)$ for $\mathcal{M} = \mathbb{S}^2$.

Remark 3.2.7 (Anisotropic exchange parameter and DMI constant). We stress that Theorem 3.2.6 also holds (with the same proof) when the exchange parameter is given by a symmetric, uniformly elliptic coefficient field with $a(x,\omega) \in \mathbb{R}^{3\times 3}$ and the DMI constant by a bounded coefficient field with $\kappa(x,\omega) \in \mathbb{R}^{3\times 3}$ (interpreted as in (3.1.3)). However, to keep the notation simpler, we assume scalar parameters for the remainder of the paper (so that, e.g., we can write $a|\nabla m|^2$ instead of $\nabla m : (a\nabla m)$).

The remaining energy components can be treated individually since they converge Γ-continuously.

Proposition 3.2.8 (Γ-continuous convergence of the magnetostatic self-energy)**.** *Under the assumptions of Theorem 3.2.3, for almost every $\widetilde{\omega} \in \Omega$ the family $(\mathcal{W}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$ continuously Γ-converges with respect to the strong $L^2(D;\mathbb{S}^2)$-topology and hence also with respect to the weak topology on $H^1(D;\mathbb{S}^2)$ to the energy functional*

$$\mathcal{W}_{\mathrm{hom}}(m) = -\frac{\mu_0}{2}\int_D h_d\left[\mathbb{E}\left[M_{\mathrm{sat}}\right] m\right] \cdot \mathbb{E}\left[M_{\mathrm{sat}}\right] m \,\mathrm{d}x + \frac{\mu_0}{2}\int_D m \cdot \mathbb{E}\left[(\nabla\Phi_M)^{\mathsf{T}}\nabla\Phi_M\right] m \,\mathrm{d}x \tag{3.2.3}$$

with $\nabla\Phi_M$ as in (3.2.1).

Proposition 3.2.9 (Γ-continuous convergence of the anisotropic and Zeeman energy)**.** *Under the assumptions of Theorem 3.2.3, for almost every $\widetilde{\omega} \in \Omega$ the families $(\mathcal{A}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$ and $(\mathcal{Z}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$ continuously Γ-converge with respect to the strong $L^2(D;\mathbb{S}^2)$-topology and hence also with respect to the weak topology on $H^1(D;\mathbb{S}^2)$ to the energy functionals*

$$\mathcal{A}_{\mathrm{hom}}(m) = \int_D \mathbb{E}\left[\varphi\right](m)\,\mathrm{d}x, \tag{3.2.4}$$

$$\mathcal{Z}_{\mathrm{hom}}(m) = -\mu_0\int_D h_a \cdot \mathbb{E}\left[M_{\mathrm{sat}}\right] m \,\mathrm{d}x. \tag{3.2.5}$$

3.3 Stochastic two-scale convergence and homogenization correctors

We recall here the basic properties of two-scale convergence and introduce the homogenization correctors. For most of the proofs we rely on the Birkhoff Ergodic Theorem and on the concept of stochastic two-scale convergence (see, e.g., [Alo+21; ZP06] for a more extensive overview on this topic). For convenience of the reader, in this section we follow [Alo+21, Section 2.2], recalling the definitions and main results which we will apply in the forthcoming sections. In what follows, recall that D is a bounded, open domain in $\mathbb{R}^3$, and that $(\Omega,\Sigma,\mathbb{P})$ is a probability space endowed with an ergodic dynamical system $(T_x)_{x\in\mathbb{R}^3}$.

Theorem 3.3.1 (Birkhoff Ergodic Theorem)**.** *Let $f \in L^1(\Omega)$. Then, for $\mathbb{P}$-almost every $\widetilde{\omega} \in \Omega$ and for every bounded Borel set $A \subset \mathbb{R}^3$, there holds*

$$\lim_{t\to\infty} \frac{1}{t^3|A|}\int_{tA} f(T_x\widetilde{\omega})\,\mathrm{d}x = \int_\Omega f(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[f]. \tag{3.3.1}$$

Remark 3.3.2 (Equivalent formulation of the Birkhoff Ergodic Theorem). The previous result can be equivalently formulated in terms of weak* convergence of oscillating test functions. We refer the interested reader, e.g., to [Zep16, Theorem 2.9] and [Kre85].

Note that the set of $\widetilde{\omega} \in \Omega$ for which (3.3.1) holds depends on the function f. Hence, the Birkhoff Ergodic Theorem is not sufficient to obtain results valid almost surely for all functions f. This inconvenience can be avoided by restricting the study to the set of typical trajectories.

Definition 3.3.3. Let $\widetilde{w} \in \Omega$. We say that $\widetilde{\omega}$ is a *typical trajectory* if

$$\lim_{t\to\infty} \frac{1}{t^3|A|} \int_{tA} g(T_x\widetilde{\omega})\,\mathrm{d}x = \int_\Omega g(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[g],$$

for every bounded Borelian $A \subset \mathbb{R}^3$ with $|A| > 0$ and for every $g \in C(\Omega)$. The set of typical trajectories $\widetilde{\omega}$ will be denoted by $\widetilde{\Omega}_{\mathrm{d}\mathbb{P}}$.

Due to the separability of the space $C(\Omega)$ of continuous random variables, this restriction to the set of typical trajectories actually still includes almost every $\omega \in \Omega$.

Proposition 3.3.4 ([Alo+21, Proposition 2.2.1]). *Let $\widetilde{\Omega}_{\mathrm{d}\mathbb{P}}$ be the set of typical trajectories. Then $\mathbb{P}(\widetilde{\Omega}_{\mathrm{d}\mathbb{P}}) = 1$.*

For our purposes in this paper, rather than the original formulation of Birkhoff's Theorem, we will use the following extension on the set of typical trajectories obtained by rescaling and approximating compactly supported continuous via simple functions.

Proposition 3.3.5 (Mean-value property, [Alo+21, Proposition 2.2.2]). *Let $g \in C(\Omega)$. Then, for every $\varphi \in C_{\mathrm{c}}(\mathbb{R}^3)$ and for every $\widetilde{\omega} \in \widetilde{\Omega}_{\mathrm{d}\mathbb{P}}$, there holds*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \varphi(x) g(T_{x/\varepsilon}\widetilde{\omega})\,\mathrm{d}x = \mathbb{E}[g] \int_{\mathbb{R}^3} \varphi(x)\,\mathrm{d}x.$$

Remark 3.3.6 (Typical trajectories and extension of the mean-value property). While we can often rely on the mean-value property as stated in Proposition 3.3.5, we will mostly need a more general result due to the low regularity of the coefficients. Let ρ be a random variable, with $\rho \in L^1(\Omega)$. The measures $\mathrm{d}x$ and $\mathrm{d}\mathbb{P}(\omega)$ in (3.3.1) can be replaced with the *random stationary measure* $\mathrm{d}\mu_\omega = \rho(T_x\omega)\,\mathrm{d}x$ on $\mathbb{R}^3$ and the respective *Palm measure* $\mathrm{d}\boldsymbol{\mu}(\omega) = \rho(\omega)\,\mathrm{d}\mathbb{P}(\omega)$ on Ω (see [ZP06, Section 1 and Theorem 1.1] for the general definitions and the generalized ergodic theorem). Hence, $\widetilde{\omega} \in \Omega$ is a $\rho\,\mathrm{d}\mathbb{P}$-typical trajectory if

$$\lim_{t\to\infty} \frac{1}{t^3|A|} \int_{tA} f(T_x\widetilde{\omega})\,\mathrm{d}\mu_{\widetilde{\omega}}(x) = \int_\Omega f(\omega)\,\mathrm{d}\boldsymbol{\mu}(\omega),$$

for every bounded Borelian $A \subset \mathbb{R}^3$ with $|A| > 0$ and for every $f \in C(\Omega)$. We denote with $\widetilde{\Omega}_{\rho\,\mathrm{d}\mathbb{P}}$ the set of $\rho\,\mathrm{d}\mathbb{P}$-typical trajectories. Slightly adopting the arguments presented in [Alo+21], we obtain that $\mathbb{P}(\widetilde{\Omega}_{\rho\,\mathrm{d}\mathbb{P}}) = 1$ and an analogous version of the mean-value property holds in $\widetilde{\Omega}_{\rho\,\mathrm{d}\mathbb{P}}$.

We recall below the notion of stochastic two-scale convergence and its main properties.

Definition 3.3.7. Let $\widetilde{\omega} \in \widetilde{\Omega}_{\mathrm{d}\mathbb{P}}$ be fixed, let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of elements of $L^2(D)$ and let $v \in L^2(D \times \Omega)$. We say that the subsequence $\{v_\varepsilon\}$ weakly stochastically two-scale converges to v if, for every $\psi \in C_{\mathrm{c}}^\infty(D)$ and every $b \in C(\Omega)$,

$$\lim_{\varepsilon\to 0} \int_D v_\varepsilon(x)\psi(x)b(T_{x/\varepsilon}\widetilde{\omega})\,\mathrm{d}x = \int_D \int_\Omega v(x,\omega)\psi(x)b(\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.$$

We write

$$v_\varepsilon \in L^2(D) \xrightarrow{\text{stoch 2-s}} v \in L^2(D \times \Omega).$$

Note that the definition of two-scale convergence may depend on the choice of the typical trajectory $\widetilde{\omega} \in \widetilde{\Omega}_{\mathrm{d}\mathbb{P}}$ and accordingly the limit may depend on $\widetilde{\omega}$. If not further specified, we refer to two-scale convergence with respect to the element $\widetilde{\omega} \in \widetilde{\Omega}$ in the definition of $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$, see also Remark 3.2.5.
An important property of this convergence is the following compactness result.

Theorem 3.3.8 ([Alo+21, Theorem 2.2.3]). *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a bounded family in $L^2(D)$. Then, there exist a subsequence $\{\varepsilon_k\}_{k>0}$ going to zero and a function $v_0 \in L^2(\mathbb{R}^3 \times \Omega)$ such that v_{ε_k} weakly two-scale-converges to v_0.*

We stress that, by Definition 3.3.7, any $\widetilde{\omega}$ might in principle give rise to a different two-scale limit. In other words, the specific v_0 identified in Theorem 3.3.8 will be dependent on the choice of $\widetilde{\omega}$ in Definition 3.3.7.

An explicit example for a class of weakly stochastically two-scale converging sequences is provided by a combination of a density argument and of the mean value property.

Example 3.3.9. Let $\varphi \in L^2(\mathbb{R}^3) = \overline{C_c(\mathbb{R}^3)}^{\|\cdot\|_{L^2}}$, $g \in C(\Omega)$. Then due to Proposition 3.3.5 the sequence $\{v_\varepsilon\}_{\varepsilon>0}$ given by $v_\varepsilon(x) = \varphi(x)g(T_{x/\varepsilon}\widetilde{\omega})$ weakly two-scale converges to $v \in L^2(D \times \Omega)$ with $v(x,\omega) = \varphi(x)g(\omega)$.

Before providing the stochastic two-scale version of the weak-strong convergence principle, we introduce the notion of strong two-scale convergence.

Definition 3.3.10. A sequence $\{v_\varepsilon\}_{\varepsilon>0} \in L^2(D)$ is said to strongly two-scale converge to a function $v_0 \in L^2(D \times \Omega)$ if v_ε weakly two-scale converges to v_0 and if

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^2(D)} = \|v_0\|_{L^2(D\times\Omega)}.$$

We write that $v_\varepsilon \in L^2(D) \xrightarrow{\text{stoch 2-s}} v_0 \in L^2(D \times \Omega)$.

Proposition 3.3.11 (Weak-strong convergence principle, [Alo+21, Proposition 2.2.4]). *Let $\{v_\varepsilon\}_{\varepsilon>0}$ and $\{u_\varepsilon\}_{\varepsilon>0}$ be two families of functions in $L^2(D)$. If $\{v_\varepsilon\}_{\varepsilon>0}$ strongly two-scale converges to v_0 and $\{u_\varepsilon\}_{\varepsilon>0}$ weakly two-scale converges to u_0, for every $\psi \in C_c^\infty(D)$ and $b \in C(\Omega)$,*

$$\lim_{\varepsilon\to 0} \int_D u_\varepsilon(x)v_\varepsilon(x)\psi(x)b(T_{x/\varepsilon}\widetilde{\omega})\,\mathrm{d}x = \int_D \int_\Omega u_0(x,\omega)v_0(x,\omega)\psi(x)b(\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.$$

Remark 3.3.12 (L^2-convergence vs. two-scale convergence). As in the classical periodic case, on the one hand, strong L^2-convergence implies strong two-scale convergence. Namely, if $u_\varepsilon \to u \in L^2(\mathbb{R}^3)$ strongly in $L^2(\mathbb{R}^3)$ as $\varepsilon \to 0$ for $\{u_\varepsilon\} \subset L^2(\mathbb{R}^3)$, then $u_\varepsilon \xrightarrow{\text{stoch 2-s}} \widetilde{u} \in L^2(\mathbb{R}^3 \times \Omega)$ with $\widetilde{u}(x,\omega) = u(x)$.
On the other hand, weak two-scale convergence implies weak L^2-convergence. If $u_\varepsilon \xrightarrow{\text{stoch 2-s}} u \in L^2(\mathbb{R}^3 \times \Omega)$, then $u_\varepsilon \rightharpoonup \mathbb{E}[u]$ weakly in $L^2(\mathbb{R}^3)$.

3.3.1 The space $H^1(\Omega)$

The goal of this subsection is to state a compactness result, similar to Theorem 3.3.8, for sequences of uniformly bounded gradients. The language to describe the emerging additional structure is provided via the spaces $H^1(\Omega)$, $L^2_{\text{pot}}(\Omega)$, and $L^2_{\text{sol}}(\Omega)$. We briefly recall the definitions and main results.

First, we define a notion of derivatives on Ω that is compatible with the dynamical system $(T_x)_{x\in\mathbb{R}^3}$.

Definition 3.3.13. A function $u \in L^2(\Omega)$ is said to be differentiable at $\omega \in \Omega$ if the limits

$$\lim_{\delta\to 0} \frac{u(T_{\delta e_i}\omega) - u(\omega)}{\delta} =: (D_i u)(\omega)$$

exist for every $i \in \{1,2,3\}$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. In this case, we write $D_\omega u = (D_1 u, D_2 u, D_3 u)$, $\nabla_\omega u = (D_\omega u)^\intercal$, and adapt the notation accordingly if u is vector-valued.

This allows us to define a space $C^1(\Omega)$ of continuously differentiable functions on the probability space $(\Omega, \Sigma, \mathbb{P})$. Indeed, $C^1(\Omega)$ is the set of functions $u \in C(\Omega)$ that are differentiable, in the sense of Definition 3.3.13, at any $\omega \in \Omega$ and such that $\nabla_\omega u$ is continuous on Ω. It turns out that $C^1(\Omega)$ is a dense subspace in $L^2(\Omega)$ (see [Alo+21, Lemma 2.3.2]). Due to the ergodicity assumption, in particular via the mean-value property, we obtain an integration-by-parts result for these functions.

Lemma 3.3.14 ([Alo+21, Lemma 2.3.3]). *Let $w \in C^1(\Omega)$. Then $\mathbb{E}[D_i w] = 0$ for $i \in \{1,2,3\}$. In particular, for $u, v \in C^1(\Omega)$ there holds*

$$\int_\Omega (D_i u) v \,\mathrm{d}\mathbb{P} + \int_\Omega u (D_i v) \,\mathrm{d}\mathbb{P} = 0.$$

This leads to the definition of weak derivatives and thus to a Sobolev-type space $H^1(\Omega)$ on the probability space $(\Omega, \Sigma, \mathbb{P})$.

Definition 3.3.15. Let $u \in L^2(\Omega)$, $i \in \{1,2,3\}$ and let $w_i \in L^2(\Omega)$. We say that u admits w_i as a (unique) weak derivative in the i-th direction if

$$-\int_\Omega u D_i v \,\mathrm{d}\mathbb{P} = \int_\Omega w_i v \,\mathrm{d}\mathbb{P} \quad \text{for all } v \in C^1(\Omega).$$

In this case, we denote by $D_i u = w_i$ the weak derivative of u in the i-th direction.

Note that both definitions of D_i coincide for a C^1-random variable. The space $H^1(\Omega)$ is the space of all functions $u \in L^2(\Omega)$ that admit weak derivatives in $L^2(\Omega)$ in all directions and we denote $D_\omega u = (D_1 u, D_2 u, D_3 u)$, $\nabla_\omega u = (D_\omega u)^\intercal$. Clearly, $H^1(\Omega)$ is a vector space endowed with the inner product

$$(u, v)_{H^1(\Omega)} \coloneqq (u, v)_{L^2(\Omega)} + (\nabla_\omega u, \nabla_\omega v)_{L^2(\Omega;\mathbb{R}^3)}.$$

Proposition 3.3.16 ([Alo+21, Proposition 2.3.4]). *The space $H^1(\Omega)$ is a separable Hilbert space and the subspace $C^1(\Omega)$ is dense in it.*

Remark 3.3.17. In [ZP06, Section 2], the authors provide another definition of the Sobolev space $H^1(\Omega)$, which is similar to the one of $H^1(\mathbb{R}, \mu)$ (see, e.g., [BBS97; BF01; Zhi00]). For completeness, we recall the definition of $H^1(\Omega, \mu)$ below. Let μ be a Borel measure on Ω. A function $u \in L^2(\Omega, \mu)$ belongs to $H^1(\Omega, \mu)$ and $z \in (L^2(\Omega, \mu))^3$ is the gradient of u if there exists a sequence $\{u_k\}_{k>0}$ in $C^1(\Omega)$ such that $u_k \to u$ in $L^2(\Omega, \mu)$ and $D_i u_k \to z_i$ in $L^2(\Omega, \mu)$, for $i = 1, 2, 3$, as $k \to \infty$. The equivalence of the two definitions is shown in [ZP06, Section 4].

The ergodicity of the dynamical system implies that, if the weak derivatives of a Sobolev map vanish, then the original function is constant.

Proposition 3.3.18 ([Alo+21, Proposition 2.3.6]). *For any $u \in H^1(\Omega)$, we have*

$$\nabla_\omega u \equiv 0 \quad \Longrightarrow \quad u \text{ is constant } \mathbb{P}\text{-a.s.}$$

This property is also referred to as the measure $\mathbb{P}$ being ergodic with respect to translations.

We proceed by introducing the spaces of gradients and divergence-free functions in $L^2(\Omega;\mathbb{R}^3)$, which are used to define the homogenization correctors.

Definition 3.3.19. We denote by $L^2_{\mathrm{pot}}(\Omega)$ the set of all functions of $L^2(\Omega;\mathbb{R}^3)$ which can be viewed as limits of a sequence of gradients. In other words,

$$L^2_{\mathrm{pot}}(\Omega) := \overline{\{\nabla_\omega u : u \in C^1(\Omega))\}}^{\|\cdot\|_{L^2(\Omega;\mathbb{R}^3)}}.$$

We denote by $L^2_{\mathrm{sol}}(\Omega)$ the orthogonal complement of $L^2_{\mathrm{pot}}(\Omega)$.

Finally, we state the H^1-extension of the L^2-compactness result from Theorem 3.3.8.

Theorem 3.3.20 ([Alo+21, Theorem 2.4.2])**.** *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $H^1(D)$. Then, there exists v_0 in $H^1(D)$ and $\xi \in L^2(D, L^2_{\mathrm{pot}}(\Omega))$ such that, up to the extraction of a non-relabelled subsequence, there holds*

$$v_\varepsilon \xrightarrow{\text{stoch 2-s}} v_0(x) \quad \text{strongly in } L^2(D\times\Omega;\mathbb{R}^3),$$
$$\nabla v_\varepsilon \xrightarrow{\text{stoch 2-s}} \nabla v_0(x) + \xi(x,\omega) \quad \text{weakly in } L^2(D\times\Omega;\mathbb{R}^{3\times 3}).$$

Moreover, if $v_\varepsilon \in H^1_0(\Omega)$ for every $\varepsilon>0$, then the limit v_0 is in $H^1_0(\Omega)$.

3.3.2 The correctors Θ_a, Θ_κ, and Θ_M

We generalize the spaces introduced in Definition 3.3.19 to vector-valued functions. The space $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ is defined as the closure of the set $\{\nabla_\omega u : u \in C^1(\Omega;\mathbb{R}^3)\}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Analogously to the scalar case, we set $L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3) := (L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3))^\perp$ with respect to the $L^2(\Omega;\mathbb{R}^{3\times 3})$ scalar product

$$(A,B)_{L^2(\Omega;\mathbb{R}^{3\times 3})} = \int_\Omega A(\omega) : B(\omega)\,\mathrm{d}\mathbb{P}(\omega).$$

Lemma 3.3.21 (Existence and uniqueness of the homogenization correctors)**.** *There exist unique solutions $\Theta_a, \Theta_\kappa, \Theta_M \in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ to the problems*

$$(a\Theta_a + a\,\mathrm{Id}) \in L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3), \tag{3.3.2}$$
$$(a\Theta_\kappa + \kappa\,\mathrm{Id}) \in L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3), \tag{3.3.3}$$
$$(\Theta_M + M_{\mathrm{sat}}\,\mathrm{Id}) \in L^2_{\mathrm{sol}}(\Omega;\mathbb{R}^3). \tag{3.3.4}$$

Proof. As a closed subspace of $L^2(\Omega;\mathbb{R}^{3\times 3})$, the set $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ is a Hilbert space. Condition (3.3.2) is equivalent to

$$\int_\Omega a(\omega)\Theta_a(\omega) : B(\omega)\,\mathrm{d}\mathbb{P}(\omega) = -(a\,\mathrm{Id}, B)_{L^2(\Omega;\mathbb{R}^{3\times 3})} \quad \text{for all } B \in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3).$$

Hence, due to (P1) – the uniform positivity and boundedness of a – there exists a unique solution due to the Lax–Milgram theorem. The same follows analogously for (3.3.3) and (3.3.4) owing to (P2) and (P3). □

Remark 3.3.22 (The correctors via orthogonal projections). Note that $(\cdot,\cdot)_a := (\cdot, a\,\cdot)_{L^2}$ is a scalar product on $L^2(\Omega;\mathbb{R}^3)$ or $L^2(\Omega;\mathbb{R}^{3\times 3})$, respectively, due to (P1), and that such a scalar product induces the same topology on the respective spaces. From this perspective, $\Theta_a e_i$ and $\Theta_\kappa e_i$ are the orthogonal projections with respect to $(\cdot,\cdot)_a$ of $\omega \mapsto -e_i$ and $\omega \mapsto -\frac{\kappa(\omega)}{a(\omega)}e_i$ onto $L^2_{\mathrm{pot}}(\Omega)$ for $i=1,2,3$. Similarly, Θ_M is the orthogonal projection with respect to $(\cdot,\cdot)_{L^2}$ of $\omega \mapsto -M_{\mathrm{sat}}e_i$ for $i=1,2,3$.

3.3.3 Two-scale limits of fields in $H^1(D;\mathcal{M})$

We conclude this section with a characterization of the stochastic two-scale limit of manifold-valued sequences in H^1. This result will be instrumental for the analysis in Section 3.4. Throughout this subsection, $\mathcal{M}$ will be a C^2 orientable hypersurface in $\mathbb{R}^3$. In this setting, two-scale limits will be the sum of standard L^2 gradients with elements of the subspace $L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M}) \subset L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ with $s \in \mathcal{M}$, which is defined as the closure of the set $\{\nabla_\omega u : u \in C^1(\Omega, \mathcal{T}_s\mathcal{M})\}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$.

Proposition 3.3.23. *Let $\mathcal{M}$ be a C^2 orientable hypersurface in $\mathbb{R}^3$ admitting a tubular neighborhood of uniform thickness, and let $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence of functions in $H^1(D;\mathcal{M})$. Let $u_0 \in H^1(D;\mathbb{R}^3)$ and $\Xi \in L^2(D; L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3))$ be such that*

$$u_\varepsilon \to u_0 \qquad \text{strongly in } L^2(D;\mathbb{R}^3), \tag{3.3.5}$$

$$\nabla u_\varepsilon \xrightarrow{\text{stoch 2-s}} \nabla u_0(x) + \Xi(x,\omega) \qquad \text{weakly in } L^2(D\times\Omega;\mathbb{R}^{3\times 3}). \tag{3.3.6}$$

Then, $u_0 \in H^1(D;\mathcal{M})$ and $\Xi \in L^2(D; L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_{u_0(x)}\mathcal{M}))$.

Proof. The assumptions on $\mathcal{M}$ ensure the existence of an open tubular neighborhood $U \in \mathbb{R}^3$ of $\mathcal{M}$ and a C^2 function $\gamma : U \to \mathbb{R}$ which has 0 as a regular value and with $\mathcal{M} = \gamma^{-1}(\{0\})$. Thanks to (3.3.5) combined with the continuity of γ we deduce that, up to the extraction of a non-relabelled subsequence, $0 = \gamma(u_\varepsilon(x)) \to \gamma(u_0(x)) = 0$ for almost every $x \in D$. This implies that $u_0(x) \in \mathcal{M}$ for almost every $x \in D$, and hence $u_0 \in H^1(D,\mathcal{M})$.

Next, we show that $\Xi(x,\cdot) \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_{u_0(x)}\mathcal{M})$ for almost every $x \in D$. For $x \in D$ fixed, we need to prove that there exists a sequence $\{\widetilde{v}_n\}_{n\in\mathbb{N}} \subset C^1(\Omega,\mathcal{T}_{u_0(x)}\mathcal{M})$, possibly dependent on x, and such that

$$\|\nabla_\omega \widetilde{v}_n - \Xi(x,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \to 0 \qquad \text{as } n\to\infty. \tag{3.3.7}$$

First, note that since $\Xi(x,\cdot) \in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$, there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset C^1(\Omega;\mathbb{R}^3)$ such that

$$\|\nabla_\omega v_n - \Xi(x,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \to 0 \qquad \text{as } n\to\infty. \tag{3.3.8}$$

We define the maps $\widetilde{v}_n \in C^1(\Omega,\mathcal{T}_{u_0(x)}\mathcal{M})$ as the nearest point projections of v_n onto $\mathcal{T}_{u_0(x)}\mathcal{M}$. Namely,

$$\widetilde{v}_n(\omega) \coloneqq v_n(\omega) - \left(v_n(\omega)\cdot\frac{\nabla\gamma(u_0(x))}{|\nabla\gamma(u_0(x))|}\right)\frac{\nabla\gamma(u_0(x))}{|\nabla\gamma(u_0(x))|}.$$

Now,

$$\|\nabla_\omega\widetilde{v}_n - \Xi(x,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le \|\nabla_\omega\widetilde{v}_n - \nabla_\omega v_n\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\nabla_\omega v_n - \Xi(x,\cdot)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}.$$

To prove (3.3.7), thanks to (3.3.8), it is enough to show that $\|\nabla_\omega\widetilde{v}_n - \nabla_\omega v_n\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \to 0$ as $n\to\infty$. To this end, note that for any $\varepsilon > 0$, there holds

$$0 = \nabla(\gamma\circ u_\varepsilon) = \nabla u_\varepsilon \nabla\gamma(u_\varepsilon).$$

From (3.3.5), it follows that $\nabla\gamma(u_\varepsilon) \to \nabla\gamma(u_0)$ strongly in $L^2(D;\mathbb{R}^3)$. Thus, in view of (3.3.6) we obtain that, for every $\varphi \in C^\infty_{\mathrm{c}}(D)$ and for every $b \in C(\Omega;\mathbb{R}^3)$,

$$0 = \int_D \nabla u_\varepsilon(x)\nabla\gamma(u_\varepsilon)\cdot b(T_{x/\varepsilon}\widetilde{\omega})\varphi(x)\,\mathrm{d}x \to \int_D\int_\Omega (\nabla u_0(x)+\Xi(x,\omega))\nabla\gamma(u_0)\cdot b(\omega)\varphi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x$$

as $\varepsilon \to 0$. Since $\nabla\gamma(u_0)\nabla u_0 = \nabla(\gamma \circ u_0) = 0$, we conclude that for every $\varphi \in C_c^\infty(D)$ and for every $b \in C(\Omega;\mathbb{R}^3)$

$$\int_D \int_\Omega \Xi(x,\omega)\nabla\gamma(u_0)\cdot b(\omega)\varphi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x = 0.$$

It follows that, almost everywhere in D, and for every $b \in C(\Omega;\mathbb{R}^3)$,

$$\int_\Omega \Xi(x,\omega)\nabla\gamma(u_0)\cdot b(\omega)\,\mathrm{d}\mathbb{P}(\omega) = 0. \tag{3.3.9}$$

Therefore, for almost every $x \in D$, we deduce that

$$\begin{aligned}\|\nabla_\omega(v_n - \widetilde{v}_n)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} &= |\nabla\gamma(u_0(x))|^{-4}\|\nabla\gamma(u_0(x))\otimes\nabla_\omega v_n\nabla\gamma(u_0(x))\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\\ &\le 2|\nabla\gamma(u_0(x))|^{-2}\Big(\|(\nabla_\omega v_n - \Xi(x,\cdot))\nabla\gamma(u_0(x))\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Xi(x,\cdot)\nabla\gamma(u_0(x))\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big)\\ &\le 2\|\nabla_\omega v_n - \Xi(x,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\end{aligned}$$

where in the last inequality we have used (3.3.9) by approximating $\Xi(x,\cdot)\nabla\gamma(u_0(x))$ in $C(\Omega,\mathbb{R}^3)$. This along with (3.3.8) implies (3.3.7) and concludes the proof.

□

3.4 The limit behavior of the exchange and DMI energies

In this section we prove Theorem 3.2.6. In what follows, $\mathcal{M}$ is a bounded, C^2 orientable hypersurface of $\mathbb{R}^3$ that admits a tubular neighborhood of uniform thickness. The tangent space at $s \in \mathcal{M}$ is denoted by $\mathcal{T}_s\mathcal{M}$. The vector bundle $\boldsymbol{\mathcal{T}}\mathcal{M}$ is given by $\boldsymbol{\mathcal{T}}\mathcal{M} := \bigcup_{s\in\mathcal{M}}\{s\}\times \boldsymbol{\mathcal{T}}_s\mathcal{M}$, with $\boldsymbol{\mathcal{T}}_s\mathcal{M} := (\mathcal{T}_s\mathcal{M})^3$. We will indicate by $\xi^\intercal := (\xi_1^\intercal,\xi_2^\intercal,\xi_3^\intercal)$ a generic element of $\boldsymbol{\mathcal{T}}_s\mathcal{M}$. As a corollary of Theorem 3.2.6, in the special case $\mathcal{M} = \mathbb{S}^2$ we will infer a characterization of the asymptotic behavior of the sum of exchange energy and DMI term. Recall that

$$\begin{aligned}\mathcal{G}^{\widetilde{\omega}}_\varepsilon(m) &:= \mathcal{E}^{\widetilde{\omega}}_\varepsilon(m) + \mathcal{K}^{\widetilde{\omega}}_\varepsilon(m)\\ &= \frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega})\nabla m(x) : \nabla m(x)\,\mathrm{d}x - \int_D \kappa(T_{x/\varepsilon}\widetilde{\omega})\boldsymbol{\chi}(m(x)) : \nabla m(x)\,\mathrm{d}x\\ &= \frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega})\left|\nabla m(x) - \frac{\kappa(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}\boldsymbol{\chi}(m(x))\right|^2\mathrm{d}x - \int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m(x)|^2\,\mathrm{d}x,\end{aligned} \tag{3.4.1}$$

for every $m \in H^1(D;\mathcal{M})$, where $\boldsymbol{\chi}$ is the map defined in (3.1.4) and $\widetilde{\omega} \in \widetilde{\Omega}$ as in Remark 3.2.5.

This section is organized as follows. First, in Subsection 3.4.1 we provide an alternative formulation of the homogenized energy $\mathcal{G}_{\mathrm{hom}}$ defined in (3.2.2) via a pointwise minimization problem on $L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$. In Subsection 3.4.2, we prove a compactness result as well as the liminf inequality. Finally, Subsection 3.4.3 is devoted to the proof of the optimality of the lower bound identified in Subsection 3.4.2.

3.4.1 Description of the tangentially homogenized energy density

Proposition 3.4.1 (Characterization of the homogenized energy density)**.** *Under the assumptions of Theorem 3.2.6, there holds*

$$\mathcal{G}_{\mathrm{hom}}(m) = \int_D T_{\mathrm{hom}}(m(x),\nabla m(x))\,\mathrm{d}x,$$

for every $m \in H^1(D;\mathcal{M})$, where, for every $s \in \mathcal{M}$ and $A^\intercal \in \mathcal{T}_s\mathcal{M} \subset \mathbb{R}^{3\times 3}$, the tangentially homogenized energy density is defined as

$$T_{\mathrm{hom}}(s,A) \coloneqq \min\Bigg\{ \int_\Omega \left(\frac{1}{2} a(\omega)|A + \Xi(\omega)|^2 - \kappa(\omega)\boldsymbol{\chi}(s) : (A + \Xi(\omega)) \right) \mathrm{d}\mathbb{P}(\omega) \\ \text{where } \Xi \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M}) \Bigg\}. \tag{3.4.2}$$

Additionally, the minimum problem above admits a unique solution, given by

$$\Xi[s,A] = \Theta_a A - \Theta_\kappa \boldsymbol{\chi}(s)\pi_{\mathcal{T}_s\mathcal{M}}, \tag{3.4.3}$$

where $\pi_{\mathcal{T}_s\mathcal{M}}$ is the orthogonal projection onto the tangent space $\mathcal{T}_s\mathcal{M}$ at $s \in \mathcal{M}$.

Proof. We will prove the proposition in 2 steps. First, we will show the characterization (3.4.3) of the unique solution to the minimization problem (3.4.2). Second, plugging this minimizer back into (3.4.2), we will obtain the density of $\mathcal{G}_{\mathrm{hom}}$ as defined in (3.2.2).

Step 1: Finding the minimizer for (3.4.2). We set

$$\Xi[s,A] \coloneqq \operatorname*{arg\,min}_{\Xi \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})} \left\{ \int_\Omega \left(\frac{1}{2} a(\omega)|A + \Xi(\omega)|^2 - \kappa(\omega)\boldsymbol{\chi}(s) : (A + \Xi(\omega)) \right) \mathrm{d}\mathbb{P}(\omega) \right\}$$

and via the quadratic structure, omitting the constant terms $-\frac{1}{2}\int_\Omega \frac{\kappa^2(\omega)}{a(\omega)}|\boldsymbol{\chi}(s)|^2 \,\mathrm{d}\mathbb{P}(\omega)$, we obtain

$$\begin{aligned} &= \operatorname*{arg\,min}_{\Xi \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})} \left\{ \frac{1}{2} \int_\Omega a(\omega) \left| A + \Xi(\omega) - \frac{\kappa(\omega)}{a(\omega)} \boldsymbol{\chi}(s) \right|^2 \mathrm{d}\mathbb{P}(\omega) \right\} \\ &= \operatorname*{arg\,min}_{\Xi \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})} \left\{ \left(\Xi + A - \kappa a^{-1}\boldsymbol{\chi}(s),\, \Xi + A - \kappa a^{-1}\boldsymbol{\chi}(s)\right)_a \right\} \end{aligned}$$

where $(\cdot,\cdot)_a$ is the scalar product on $L^2(\Omega;\mathbb{R}^{3\times 3})$ introduced in Remark 3.3.22. Hence, $\Xi[s,A]$ is given as the orthogonal projection with respect to $(\cdot,\cdot)_a$ of $-A + \kappa a^{-1}\boldsymbol{\chi}(s)$ onto $L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$. On the one hand, due to Remark 3.3.22, the projection onto $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ is given by $\Theta_a A - \Theta_\kappa \boldsymbol{\chi}(s)$. On the other hand, the projection from $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ onto $L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$ is given by right-multiplication with $\pi_{\mathcal{T}_s\mathcal{M}} \in \mathbb{R}^{3\times 3}$ (the representation of the projection from $\mathbb{R}^3$ onto $\mathcal{T}_s\mathcal{M}$). Hence,

$$\Xi[s,A] = \Theta_a A \pi_{\mathcal{T}_s\mathcal{M}} - \Theta_\kappa \boldsymbol{\chi}(s)\pi_{\mathcal{T}_s\mathcal{M}} = \Theta_a A - \Theta_\kappa \boldsymbol{\chi}(s)\pi_{\mathcal{T}_s\mathcal{M}}, \tag{3.4.4}$$

where we used that $A\pi_{\mathcal{T}_s\mathcal{M}} = (\pi_{\mathcal{T}_s\mathcal{M}} A^\intercal)^\intercal = A$ with the first equation following from $\pi_{\mathcal{T}_s\mathcal{M}} = (\pi_{\mathcal{T}_s\mathcal{M}})^\intercal$ and the second from $A^\intercal \in \mathcal{T}_s\mathcal{M}$.

Step 2: Plugging $\Xi[s,A]$ back into (3.4.2). Due to the properties of orthogonal projections, we have that for all $\Psi \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$

$$\left(\Xi[s,A] + A - \kappa a^{-1}\boldsymbol{\chi}(s),\, \Psi\right)_a = 0.$$

Since in particular $\Xi[s,A] \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$, it follows that

$$\int_\Omega \kappa\boldsymbol{\chi}(s) : \Xi[s,A] \,\mathrm{d}\mathbb{P} = \int_\Omega a\left(\Xi[s,A] + A\right) : \Xi[s,A] \,\mathrm{d}\mathbb{P}.$$

Plugging this into (3.4.2), resolving the square, and then applying (3.4.4), we obtain

$$\begin{aligned} T_{\mathrm{hom}}(s,A) &= \int_\Omega \left(\frac{a}{2}|A|^2 - \frac{a}{2}|\Xi[s,A]|^2 - \kappa\boldsymbol{\chi}(s) : A\right) \mathrm{d}\mathbb{P} \\ &= \int_\Omega \left(\frac{a}{2}A : A - \frac{a}{2}A : \Theta_a^\intercal\Theta_a A + aA : \Theta_a^\intercal\Theta_\kappa\boldsymbol{\chi}(s) - \frac{a}{2}\boldsymbol{\chi}(s) : \Theta_\kappa^\intercal\Theta_\kappa\boldsymbol{\chi}(s)\pi_{\mathcal{T}_s\mathcal{M}} - \kappa A : \boldsymbol{\chi}(s)\right) \mathrm{d}\mathbb{P} \end{aligned} \tag{3.4.5}$$

where for the last step we additionally used that $X : (ZY^\intercal) = (XY) : Z = Y : (X^\intercal Z)$ for any $X, Y, Z \in \mathbb{R}^{3\times 3}$, which also yields the absorption of $\pi_{\mathcal{T}_s\mathcal{M}}$ in the mixed term as at the end of Step 1. Also note that $\pi_{\mathcal{T}_s\mathcal{M}}(\pi_{\mathcal{T}_s\mathcal{M}})^\intercal = (\pi_{\mathcal{T}_s\mathcal{M}})^2 = \pi_{\mathcal{T}_s\mathcal{M}}$. Reordering the energy terms yields the density in (3.2.2) and thus completes the proof. Note that the measurability of $x \to T_{\mathrm{hom}}(m(x), \nabla m(x))$ follows directly by the explicit expression in (3.4.5). □

Remark 3.4.2 (The micromagnetic case). For $\mathcal{M} = \mathbb{S}^2$ we observe $\boldsymbol{\chi}(s)\pi_{\mathcal{T}_s\mathcal{M}} = \boldsymbol{\chi}(s)$ and – following the proof above – we could therefore omit the projection from $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ onto $L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$. This means, that the minimizer of (3.4.2) within $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ is automatically in $L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_s\mathcal{M})$:

$$T_{\mathrm{hom}}(s,A) = \min_{\Xi\in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)} \left\{ \int_\Omega \left(\frac{1}{2}a(\omega)|A+\Xi(\omega)|^2 - \kappa(\omega)\boldsymbol{\chi}(s) : (A+\Xi(\omega))\right) \mathrm{d}\mathbb{P}(\omega)\right\}.$$

On a geometric level, this is the case because s is the normal vector of $\mathcal{T}_s\mathbb{S}^2$. An alternative generalization from $\mathbb{S}^2$ to arbitrary manifolds is given by replacing (3.1.4) with

$$\boldsymbol{\chi}\colon s \in \mathcal{M} \mapsto (e_1 \times \nu_s, e_2 \times \nu_s, e_3 \times \nu_s)^\intercal \in \mathbb{R}^{3\times 3}_{\mathrm{skew-symmetric}},$$

where ν_s is the normal vector of $\mathcal{T}_s\mathcal{M}$. This choice would allow to relax to the minimum over all $\Xi \in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ also in the case of general manifolds, making the additional projection in (3.2.2) unnecessary.

3.4.2 The liminf inequality

In this subsection, we provide a compactness result, as well as a lower bound for the asymptotic behavior of the energy $\mathcal{G}_\varepsilon^{\widetilde{\omega}}$ using the characterization of $\mathcal{G}_{\mathrm{hom}}$ from Proposition 3.4.1.

Theorem 3.4.3. *Let $\{m_\varepsilon\}_{\varepsilon>0} \subset H^1(D;\mathcal{M})$ be such that $\sup_{\varepsilon>0}\mathcal{G}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) < \infty$, and let $\widetilde{\omega} \in \widetilde{\Omega}$, cf. Remark 3.2.5. Then, there exists $m_0 \in H^1(D;\mathcal{M})$ and $M_1 \in L^2(D; L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_{m_0}\mathcal{M}))$ such that*

$$\begin{aligned} m_\varepsilon &\rightharpoonup m_0 \quad \textit{weakly in } H^1(\Omega;\mathcal{M}) \\ \nabla m_\varepsilon &\xrightarrow{\text{stoch 2-s}} \nabla m_0(x) + M_1(x,\omega) \quad \textit{weakly in } L^2(D\times\Omega;\mathbb{R}^{3\times 3}). \end{aligned}$$

Moreover,

$$\mathcal{G}_{\mathrm{hom}}(m_0) \le \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon), \tag{3.4.6}$$

where $\mathcal{G}_{\mathrm{hom}}$ is given by (3.2.2).

Proof. The compactness result is a direct consequence of Proposition 3.3.23. To prove (3.4.6), we write $\mathcal{G}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon)$ as the sum of two functionals $\mathcal{H}_\varepsilon^{\widetilde{\omega}}$ and $\mathcal{I}_\varepsilon^{\widetilde{\omega}}$ given by

$$\mathcal{H}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) := \frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega}) \left|\nabla m_\varepsilon(x) - \frac{\kappa(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}\boldsymbol{\chi}(m_\varepsilon(x))\right|^2 \mathrm{d}x,$$

and

$$\mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) := -\int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m_\varepsilon(x)|^2\,\mathrm{d}x.$$

Let $\{\psi_{0,n}\}_{n\in\mathbb{N}} \subset C^\infty(D;\mathbb{R}^3)$ and let $\{\psi_{1,n}\}_{n\in\mathbb{N}}$ be a sequence of elements in the linear span of admissible test functions in $C_c^\infty(D)\times C(\Omega;\mathbb{R}^{3\times 3})$ for stochastic two-scale convergence. We define

$$\Psi_n(x,\omega) := \nabla\psi_{0,n}(x) + \psi_{1,n}(x,\omega) - \frac{\kappa(\omega)}{a(\omega)}\boldsymbol{\chi}(\psi_{0,n}(x)).$$

For every $\varepsilon > 0$ and by the positivity of a, we find

$$\frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega})\left|\nabla m_\varepsilon(x) - \frac{\kappa(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}\chi(m_\varepsilon(x)) - \Psi_n(x,T_{x/\varepsilon}\widetilde{\omega})\right|^2\,\mathrm{d}x > 0.$$

Hence,

$$\begin{aligned}\mathcal{H}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) \geq -\frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega})|\Psi_n(x,T_{x/\varepsilon}\widetilde{\omega})|^2\,\mathrm{d}x\\ + \int_D \big(a(T_{x/\varepsilon}\widetilde{\omega})\nabla m_\varepsilon(x) - \kappa(T_{x/\varepsilon}\widetilde{\omega})\boldsymbol{\chi}(m_\varepsilon(x))\big) : \Psi_n(x,T_{x/\varepsilon}\widetilde{\omega})\,\mathrm{d}x.\end{aligned}$$

Since $\psi_{1,n}$ belongs to the linear span of admissible test functions and since $\widetilde{\omega}$ is a typical trajectory with respect to $a\,\mathrm{d}\mathbb{P}, \kappa\,\mathrm{d}\mathbb{P}, \kappa^2/a\,\mathrm{d}\mathbb{P}$, see Remark 3.2.5, we can pass to the stochastic two-scale convergence, obtaining

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathcal{H}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) \geq -\frac{1}{2}\int_D\int_\Omega a(\omega)|\Psi_n(x,\omega)|^2\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x\\ + \int_D\int_\Omega a(\omega)\left[\nabla m_0(x) + M_1(x,\omega) - \frac{\kappa(\omega)}{a(\omega)}\boldsymbol{\chi}(m_0(x))\right] : \Psi_n(x,\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.\end{aligned} \tag{3.4.7}$$

Choosing $\psi_{0,n}$ and $\psi_{1,n}$ such that $\psi_{0,n}\to m_0$ in $H^1(D;\mathbb{R}^3)$ and $\psi_{1,n}\to M_1$ in $L^2(D\times\Omega;\mathbb{R}^{3\times 3})$, and taking the limit as $n\to\infty$ in (3.4.7), we deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\mathcal{H}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) &\geq \frac{1}{2}\int_D\int_\Omega a(\omega)\left|\nabla m_0(x) + M_1(x,\omega) - \frac{\kappa(\omega)}{a(\omega)}\boldsymbol{\chi}(m_0(x))\right|^2\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x\\ &\geq \frac{1}{2}\int_D \inf_{\Xi\in L^2(\Omega;\mathcal{T}_{m_0(x)}\mathcal{M})}\int_\Omega a(\omega)\left|\nabla m_0(x) + \Xi(\omega) - \frac{\kappa(\omega)}{a(\omega)}\boldsymbol{\chi}(m_0(x))\right|^2\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x\\ &\geq \int_D \inf_{\Xi\in L^2(\Omega;\mathcal{T}_{m_0(x)}\mathcal{M})}\int_\Omega \frac{1}{2}a(\omega)|\nabla m_0(x)+\Xi(\omega)|^2 - \kappa(\omega)\boldsymbol{\chi}(m_0(x)) : (\nabla m_0(x)+\Xi(\omega))\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x\\ &\quad + \frac{1}{2}\mathbb{E}\left[\frac{\kappa^2}{a}\right]\int_D |m_0(x)|^2\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.\end{aligned} \tag{3.4.8}$$

Concerning $\mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon)$ we have

$$\lim_{\varepsilon\to 0}\mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) = -\frac{1}{2}\mathbb{E}\left[\frac{\kappa^2}{a}\right]\int_D |m_0(x)|^2\,\mathrm{d}x =: \mathcal{I}_0(m_0), \tag{3.4.9}$$

since

$$|\mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{I}_0(m_0)| \leq |\mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_0)| + |\mathcal{I}_\varepsilon^{\widetilde{\omega}}(m_0) - \mathcal{I}_0(m_0)| \to 0$$

as $\varepsilon\to 0$ and $m_\varepsilon\to m_0$ strongly in $L^2(D;\mathcal{M})$. The convergence of the first term is due to the bounds on a,κ from (P1) and (P2). The convergence of the second term holds due to the boundedness of $\mathcal{M}$ and since $\widetilde{\omega}$ is a typical trajectory with respect to $\kappa^2/a(\cdot)\,\mathrm{d}\mathbb{P}(\cdot)$, see Remark 3.3.6. Combining (3.4.8) and (3.4.9), via Proposition 3.4.1 we obtain (3.4.6). □

3.4.3 The limsup inequality

In this subsection, we provide an upper bound for the asymptotic behavior of the family $(\mathcal{G}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$. To be precise, we show that the lower bound identified in Theorem 3.4.3 is optimal.

Theorem 3.4.4. *Let $\mathcal{M}$ be a C^2 orientable hypersurface of $\mathbb{R}^3$ such that $\mathcal{M}$ has a tubular neighborhood of uniform thickness $\delta > 0$. Let $m_0 \in H^1(D;\mathcal{M})$. Then, there exists a sequence $\{m_\varepsilon\}_{\varepsilon>0} \subset H^1(D;\mathcal{M})$, such that, as $\varepsilon \to 0$,*

$$m_\varepsilon \rightharpoonup m_0 \quad \textit{weakly in } H^1(D;\mathcal{M}),$$

and

$$\mathcal{G}_{\mathrm{hom}}(m_0) \geq \limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon),$$

where the functional $\mathcal{G}_{\mathrm{hom}}$ is given by (3.2.2).

Proof. Let U_δ be a tubular neighborhood of size δ around $\mathcal{M}$. Denote by $\pi_{\mathcal{M}}\colon U_\delta \to \mathcal{M}$ the pointwise projection onto the manifold. Since $\mathcal{M}$ is C^2, we have $\pi_{\mathcal{M}} \in C^1(U_\delta;\mathcal{M})$. In particular, we choose $\delta > 0$ small enough such that $\pi_{\mathcal{M}} \in C^1(\overline{U}_\delta;\mathcal{M})$. We split the proof into two steps.

Step 1: Recovery sequence for smooth 2-scale perturbations of m_0. For $i = 1,\dots,n$, with $n \in \mathbb{N}$, consider $\varphi_i \in C_c^\infty(D;\mathbb{R}^{3\times 3})$, $b_i \in C^1(\Omega;\mathbb{R}^3)$ for $i = 1,\dots,n$. For every $\varepsilon > 0$ and for almost every $x \in D$, we set

$$\widetilde{m}_\varepsilon(x) \coloneqq m_0(x) + \varepsilon \sum_{i=1}^n \varphi_i(x) b_i(T_{x/\varepsilon}\widetilde{\omega}),$$
$$m_\varepsilon \coloneqq \pi_{\mathcal{M}} \circ \widetilde{m}_\varepsilon \in H^1(D;\mathcal{M}).$$

For $\varepsilon > 0$ sufficiently small, we find $\widetilde{m}_\varepsilon(x) \in U_\delta$ for almost every $x \in D$. Additionally, as $\varepsilon \to 0$,

$$\widetilde{m}_\varepsilon, m_\varepsilon \to m_0 \quad \text{strongly in } L^2(D;\mathbb{R}^3). \tag{3.4.10}$$

Thanks to the regularity of $\pi_{\mathcal{M}}$, there exists a positive constant $C_{\mathcal{M}}$ depending only on $\mathcal{M}$ such that

$$|\nabla m_\varepsilon| \leq C_{\mathcal{M}} |\nabla \widetilde{m}_\varepsilon| \quad \text{a.e. in } D.$$

To be precise, a direct computation shows that

$$(\nabla \widetilde{m}_\varepsilon(x))_{jk} = (\nabla m_0(x))_{jk} + \varepsilon \sum_{i=1}^n \sum_{\ell=1}^3 b_{i,\ell}^{\intercal}(T_{x/\varepsilon}\widetilde{\omega}) \partial_{x_j} \varphi_{i,k\ell}(x) + \sum_{i=1}^n (\nabla_\omega b_i(T_{x/\varepsilon}\widetilde{\omega}) \varphi_i^{\intercal}(x))_{jk},$$
$$\nabla m_\varepsilon(x) = \nabla \widetilde{m}_\varepsilon(x) \nabla \pi_{\mathcal{M}}[\widetilde{m}_\varepsilon(x)].$$

Since $\{\widetilde{m}_\varepsilon\}_\varepsilon$ is bounded in $H^1(D;\mathbb{R}^{3\times 3})$, the sequence $\{m_\varepsilon\}_\varepsilon$ is bounded in $H^1(D;\mathbb{R}^{3\times 3})$ as well. This, combined with (3.4.10) implies that, up to the extraction of a non-relabelled subsequence,

$$\widetilde{m}_\varepsilon, m_\varepsilon \rightharpoonup m_0 \quad \text{weakly in } H^1(D;\mathbb{R}^3).$$

Since m_ε converges uniformly to m_0 and $\nabla\pi_{\mathcal{M}}$ is uniformly continuous as a continuous function on the compact set $\overline{U}_\delta$, we also infer

$$\nabla\pi_{\mathcal{M}}[\widetilde{m}_\varepsilon] \to \nabla\pi_{\mathcal{M}}[m_0] \quad \text{strongly in } L^\infty(D;\mathbb{R}^{3\times 3}).$$

On the one hand, we deduce that, for all $j,k,\ell = 1,\dots,3$, $i = 1,\dots,n$,

$$\varepsilon\big(b_{i,\ell}^\intercal(T_{x/\varepsilon}\widetilde{\omega})\partial_{x_j}\varphi_{i,k\ell}(x)\big)_{jk} \to 0 \quad \text{strongly in } L^\infty(D).$$

On the other hand, since $\widetilde{\omega}$ is a typical trajectory, we find

$$\nabla_\omega b_i(T_{x/\varepsilon}\widetilde{\omega})\varphi_i^\intercal(x) \xrightarrow{\text{stoch 2-s}} \nabla_\omega b_i(\omega)\varphi_i^\intercal(x) \quad \text{strongly in } L^2(D\times\Omega;\mathbb{R}^{3\times 3}).$$

where we have applied the mean value property (see Proposition 3.3.5) to $\nabla_\omega b_i(T_{x/\varepsilon}\widetilde{\omega})\varphi_i^\intercal(x)$ for the weak and to $|\nabla_\omega b_i(T_{x/\varepsilon}\widetilde{\omega})|^2|\varphi_i(x)|^2$ for the strong stochastic 2-scale convergence. Hence, since $\nabla m_0 \nabla\pi_{\mathcal{M}}[m_0] = \nabla m_0$ since $\pi_{\mathcal{M}}[m_0] = m_0$, we conclude that

$$\nabla m_\varepsilon(x) \xrightarrow{\text{stoch 2-s}} \nabla m_0(x) + \sum_{i=1}^n \nabla_\omega b_i(\omega)\varphi_i^\intercal(x)\nabla\pi_{\mathcal{M}}[m_0(x)] \quad \text{strongly in } L^2(D\times\Omega;\mathbb{R}^{3\times 3}).$$

Applying Proposition 3.3.11 and Remark 3.2.5 yields

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathcal{G}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) &= \lim_{\varepsilon\to 0}\left(\frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega})|\nabla m_\varepsilon(x)|^2\,\mathrm{d}x - \int_D \kappa(T_{x/\varepsilon}\widetilde{\omega})\boldsymbol{\chi}(m_\varepsilon(x))\colon\nabla m_\varepsilon(x)\,\mathrm{d}x\right)\\
&= \frac{1}{2}\int_{D\times\Omega} a(\omega)\left|\nabla m_0(x) + \sum_{i=1}^n \nabla_\omega b_i(\omega)\varphi_i^\intercal(x)\nabla\pi_{\mathcal{M}}[m_0(x)]\right|^2 \mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x\\
&\quad - \int_{D\times\Omega}\kappa(\omega)\boldsymbol{\chi}(m_0(x))\colon\left(\nabla m_0(x) + \sum_{i=1}^n \nabla_\omega b_i(\omega)\varphi_i^\intercal(x)\nabla\pi_{\mathcal{M}}[m_0(x)]\right)\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x. \qquad (3.4.11)
\end{aligned}$$

Step 2: Recovery sequence for perturbations of m_0 with the correctors. Let $m_0 \in H^1(D;\mathcal{M})$. Based on Proposition 3.4.1 and (3.4.3), we want the perturbations to result in the gradient correction

$$\Xi(x,\omega) := \Theta_a(\omega)\nabla m_0(x) - \Theta_\kappa(\omega)\boldsymbol{\chi}(m_0(x))\pi_{\mathcal{T}_s\mathcal{M}} \qquad (3.4.12)$$

with $\Xi(x,\cdot) \in L^2_{\mathrm{pot}}(\Omega;\mathcal{T}_{m_0(x)}\mathcal{M})$ for almost every $x\in D$. Let $\{\varphi_{a,\delta}\}_{\delta>0},\{\varphi_{\kappa,\delta}\}_{\delta>0} \subset C_c^\infty(D;\mathbb{R}^{3\times 3})$ be such that

$$\begin{aligned}
\varphi_{a,\delta}^\intercal &\xrightarrow{\delta\to 0} \nabla m_0 \quad \text{strongly in } L^2(D;\mathbb{R}^{3\times 3}),\\
\varphi_{\kappa,\delta}^\intercal &\xrightarrow{\delta\to 0} \boldsymbol{\chi}(m_0(x))\pi_{\mathcal{T}_s\mathcal{M}} \quad \text{strongly in } L^2(D;\mathbb{R}^{3\times 3}).
\end{aligned}$$

By the definition of $L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$ there exist sequences $\{b_{a,\delta}\}_{\delta>0},\{b_{\kappa,\delta}\}_{\delta>0}\subset C^1(\Omega;\mathbb{R}^3)$ with

$$\begin{aligned}
\nabla_\omega b_{a,\delta} &\xrightarrow{\delta\to 0} \Theta_a \quad \text{strongly in } L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3),\\
\nabla_\omega b_{\kappa,\delta} &\xrightarrow{\delta\to 0} \Theta_\kappa \quad \text{strongly in } L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3).
\end{aligned}$$

To summarize,

$$\Psi_\delta := \nabla_\omega b_{a,\delta}\varphi_{a,\delta}^\intercal + \nabla_\omega b_{\kappa,\delta}\varphi_{\kappa,\delta}^\intercal \xrightarrow{\delta\to 0} \Xi(x,\omega) \quad \text{strongly in } L^2(D;L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)).$$

In the spirit of Step 1, for fixed $\delta>0$, we define the sequence $\{m_\varepsilon^\delta\}_{\varepsilon>0}$ by

$$m_\varepsilon^\delta := \pi_{\mathcal{M}}\big[m_0 + \varepsilon(\varphi_{a,\delta}b_{a,\delta} + \varphi_{\kappa,\delta}b_{\kappa,\delta})\big].$$

According to (3.4.11), we have that

$$\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon^\delta) = F(m_0, \Psi_\delta),$$

where $F(m_0, \Psi)$ for $\Psi \in L^2(D; L^2_{\mathrm{pot}}(\Omega; \mathbb{R}^3))$ is given by

$$F(m_0,\Psi) := \frac{1}{2}\int_{D\times\Omega} a(\omega)\Big|\nabla m_0(x) + \Psi(x,\omega)\nabla\pi_{\mathcal{M}}[m_0(x)]\Big|^2 \,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x \\ - \int_{D\times\Omega} \kappa(\omega)\boldsymbol{\chi}(m_0(x)) : \Big(\nabla m_0(x) + \Psi(x,\omega)\nabla\pi_{\mathcal{M}}[m_0(x)]\Big)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.$$

Additionally, from Proposition 3.4.1 and by the definition of Ξ in (3.4.12) we have

$$F(m_0, \Xi) = \mathcal{G}_{\mathrm{hom}}(m_0).$$

This equality is a consequence of the fact that $\Xi(x,\omega)\nabla\pi_{\mathcal{M}}[m_0(x)] = \Xi(x,\omega)$, which we prove in Lemma 3.4.5 below. Now, due to the strong L^2-convergence of $(\Psi_\delta)_{\delta>0}$, there holds

$$\lim_{\delta\to 0} F(m_0, \Psi_\delta) = F(m_0, \Xi) = \mathcal{G}_{\mathrm{hom}}(m_0).$$

The thesis follows then by Step 1. □

Lemma 3.4.5. *Let $\mathcal{M}$ be a C^2 orientable hypersurface of $\mathbb{R}^3$ with a tubular neighborhood U_δ of size δ. Let $\pi_{\mathcal{M}} \in C^1(U_\delta; \mathcal{M})$ be the pointwise projection onto the manifold. Then for every $s \in \mathcal{M}$ and every $\tau \in \mathcal{T}_s\mathcal{M}$,*

$$\tau^{\intercal}\nabla\pi_{\mathcal{M}}[s] = \tau^{\intercal}.$$

Proof. Let $s \in \mathcal{M}$ and $\tau \in \mathcal{T}_s\mathcal{M}$. By the definition of the tangent space and in view of the regularity of $\mathcal{M}$, we have that

$$\pi_{\mathcal{M}}(s + h\tau) = s + h\tau + \mathcal{O}(h^2) \quad \text{as } h \to 0.$$

Now, the claim follows from the following equality

$$\tau^{\intercal}\nabla\pi_{\mathcal{M}}[s] = \frac{\mathrm{d}}{\mathrm{d}h}\Big(\pi_{\mathcal{M}}(s+h\tau)\Big|_{h=0}\Big)^{\intercal} = \tau^{\intercal},$$

which concludes the proof. □

3.5 Proof of the main result

This section is devoted to completing the proof of Theorem 3.2.3. With Theorem 3.2.6 proven in Section 3.4, in this section it remains to prove Proposition 3.2.8, i.e. the characterization of the limiting behaviour of the magnetostatic self-energy functionals $\mathcal{W}_\varepsilon^{\widetilde{\omega}}$ (Subsection 3.5.1), Proposition 3.2.9, i.e. the study of the asymptotic behavior of the anisotropic and Zeeman energy (Subsection 3.5.2), and the equi-coerciveness of the overall family $(\mathcal{F}_\varepsilon^{\widetilde{\omega}})_{\varepsilon>0}$.

3.5.1 Homogenization of the demagnetizing field

In this section, we show that the family of magnetostatic self-energy functionals $\mathcal{W}_\varepsilon^{\widetilde{\omega}}$ continuously converges to the energy $\mathcal{W}_{\mathrm{hom}}$ defined in (3.2.3). To this end, we recall the definition of Beppo-Levi spaces. Set $\theta(x) := (1+|x|^2)^{-1/2}$ and let $L^2_\theta(\mathbb{R}^3)$ be the weighted Lebesgue space $L^2_\theta(\mathbb{R}^3) := \{u \in \mathcal{D}'(\mathbb{R}^3) : u\theta \in L^2(\mathbb{R}^3)\}$. The Beppo-Levi space $\mathrm{BL}^1(\mathbb{R}^3)$ is defined as

$$\mathrm{BL}^1(\mathbb{R}^3) := \left\{u \in \mathcal{D}'(\mathbb{R}^3) : u \in L^2_\theta(\mathbb{R}^3) \quad \text{and} \quad \nabla u \in L^2(\mathbb{R}^3;\mathbb{R}^3)\right\}.$$

The space $\mathrm{BL}^1(\mathbb{R}^3)$ endowed with $(u,v)_{\mathrm{BL}^1(\mathbb{R}^3)} := (\nabla u, \nabla v)_{L^2(\mathbb{R}^3)}$ is a Hilbert space due to the Hardy inequality.
Recall that a given magnetization $m \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ generates the stray field $h_d[m] = \nabla u_m$, where the potential u_m solves

$$\Delta u_m = -\operatorname{div}(m) \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{3.5.1}$$

By Lax-Milgram's theorem, there exists a unique solution to the variational formulation associated to (3.5.1): namely, a potential $u_m \in \mathrm{BL}^1(\mathbb{R}^3)$ such that, for all $\varphi \in \mathrm{BL}^1(\mathbb{R}^3)$,

$$(u_m,\varphi)_{\mathrm{BL}^1(\mathbb{R}^3)} := \int_{\mathbb{R}^3} \nabla u_m \cdot \nabla\varphi \,\mathrm{d}x = -\int_{\mathbb{R}^3} m\cdot\nabla\varphi \,\mathrm{d}x. \tag{3.5.2}$$

Additionally, using both Cauchy-Schwarz and Hölder inequality, we get

$$\|\nabla u_m\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} = -\int_{\mathbb{R}^3} m\cdot\nabla u_m \,\mathrm{d}x \le \int_{\mathbb{R}^3} |m\cdot\nabla u_m| \,\mathrm{d}x \le \|m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|\nabla u_m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)},$$

which implies

$$\|u_m\|_{\mathrm{BL}^1(\mathbb{R}^3)} = \|\nabla u_m\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \le \|m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}. \tag{3.5.3}$$

The family of magnetostatic self-energy functionals $\mathcal{W}_\varepsilon^{\widetilde{\omega}} : L^2(D;\mathcal{M}) \to \mathbb{R}$ is given by

$$\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m) := -\frac{\mu_0}{2}\left(h_d\left[M_{\mathrm{sat}}\left(T_{\cdot/\varepsilon}\widetilde{\omega}\right) m\chi_D\right], M_{\mathrm{sat}}\left(T_{\cdot/\varepsilon}\widetilde{\omega}\right) m\right)_{L^2(D)}$$

for $\widetilde{\omega} \in \widetilde{\Omega}$ as in Remark 3.2.5, where $m\chi_D$ denotes the extension of m to $\mathbb{R}^3$ vanishing outside D. We will prove the Γ-continuous convergence of $\mathcal{W}_\varepsilon^{\widetilde{\omega}}$ via an application of Proposition 3.3.11, the weak-strong convergence principle for stochastic two-scale convergence. However, we will first need to transfer some basic stochastic two-scale convergence results to the weighted spaces L^2_θ and BL^1. In particular, we provide compactness results for bounded sequences in these spaces, which are an adaption of [AD15, Propositions 4.1 and 4.2] to the stochastic setting.

Proposition 3.5.1 (Weighted stochastic 2-scale compactness in L^2)**.** *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L^2_\theta(\mathbb{R}^3)$. Let $\widetilde{\omega} \in \widetilde{\Omega}$. Then, there exists a function $u \in L^2_{\mathrm{loc}}(\mathbb{R}^3\times\Omega)$ such that $\mathbb{E}[u] \in L^2_\theta(\mathbb{R}^3)$ and, up to subsequences,*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} u_\varepsilon(x)\varphi(x)b(T_{x/\varepsilon}\widetilde{\omega})\,\mathrm{d}x = \int_{\mathbb{R}^3\times\Omega} u(x,\omega)\varphi(x)b(\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x,$$

for all $\varphi \in C_c^\infty(\mathbb{R}^3)$ and $b \in C(\Omega)$.

We say that a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ as in Proposition 3.5.1 stochastically L^2_θ-two-scale converges to u.

Proof. The boundedness of $\{u_\varepsilon\}_\varepsilon$ in $L^2_\theta(\mathbb{R}^3)$ implies the existence of $u_\infty \in L^2_\theta(\mathbb{R}^3)$ and of a subsequence $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ such that

$$u_{\varepsilon_n} \rightharpoonup u_\infty \quad \text{weakly in } L^2_\theta(\mathbb{R}^3).$$

In particular, for every bounded domain $D \in \mathbb{R}^3$, we have that $u_{\varepsilon_n} \rightharpoonup u_\infty$ weakly in $L^2(D)$. Let $(D_i)_{i\in\mathbb{N}}$ be a sequence of bounded domains covering $\mathbb{R}^3$. An application of Theorem 3.3.8 to domain D_1 yields to the existence of a subsequence $\{u_{\varepsilon_{n(k_1)}}\}$ of $\{u_{\varepsilon_n}\}$ and $u_1 \in L^2(D_1 \times \Omega)$ such that

$$u_{\varepsilon_{n(k_1)}} \xrightarrow{\text{stoch 2-s}} u_1 \quad \text{in } L^2(D_1 \times \Omega).$$

This subsequence $\{u_{\varepsilon_{n(k_1)}}\}$ is again bounded in $L^2(D_2)$, hence there exists a subsequence $\{u_{\varepsilon_{n(k_2)}}\}$ such that $u_{\varepsilon_{n(k_2)}}$ stochastic two-scale converges to some $u_2 \in L^2(D_2 \times \Omega)$. In addition, the uniqueness of the two-scale limits implies that u_1 and u_2 coincide on $(D_1 \cap D_2) \times \Omega$. Repeating the same arguments for any D_i, with $i \in \mathbb{N}$, we conclude the existence of a subsequence $\{u_{\varepsilon_{n(k_i)}}\} \subset \{u_{\varepsilon_{n(k_{i-1})}}\}$ such that $u_{\varepsilon_{n(k_i)}}$ stochastic two-scale converges to some $u_i \in L^2(D_i \times \Omega)$. This allows us to define a diagonal sequence of indices

$$n(k_\infty(1)) = n(k_1(1)),\ n(k_\infty(2)) = n(k_2(2)), \dots, n(k_\infty(i)) = n(k_i(i)), \dots. \tag{3.5.4}$$

Since for any $i \in \mathbb{N}$, up to the first $(i-1)$ terms, the sequence of indices $n(k_\infty)$ is included in $n(k_i)$, we deduce that $u_{\varepsilon_{n(k_\infty)}}$ stochastic two-scale converges to u_i in $L^2(D_i \times \Omega)$ for all $i \in \mathbb{N}$. Once again, the uniqueness of the two-scale limits implies that $u_i \equiv u_j$ on $(D_i \cap D_j) \times \Omega$ whenever $D_i \cap D_j \neq \emptyset$. Therefore, thanks to the *principle du recollement des morceaux* (see, e.g., [Sch66]), we deduce that there exists a unique distribution $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times \Omega)$ such that u coincides with u_i on $D_i \times \Omega$ and, for $\varphi \in C^\infty_c(D)$ and $b \in C(\Omega)$,

$$\lim_{k_\infty \to \infty} \int_{\mathbb{R}^3} u_{\varepsilon_{n(k_\infty)}}(x)\varphi(x)b(T_{x/\varepsilon_{n(k_\infty)}}\omega)\,\mathrm{d}x = \int_{\mathbb{R}^3\times\Omega} u(x,\omega)\varphi(x)b(\omega)\,\mathrm{d}x\,\mathrm{d}\mathbb{P}(\omega).$$

Additionally, $\{u_{\varepsilon_{n(k_\infty)}}\}$ weakly converges to $\mathbb{E}[u]$ in $L^2(D_i)$ for every $i \in \mathbb{N}$, see Remark 3.3.12. Hence, since $\{u_{\varepsilon_n}\}$ and thus $\{u_{\varepsilon_{n(k_\infty)}}\}$ weakly converges in $L^2(D)$ to u_∞ for every bounded domain $D \in \mathbb{R}^3$, we deduce that $\mathbb{E}[u] \equiv u_\infty \in L^2_\theta(\mathbb{R}^3)$. This concludes the proof. □

Proposition 3.5.2 (Stochastic 2-scale compactness in BL^1)**.** *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $\mathrm{BL}^1(\mathbb{R}^3)$ which weakly converges to u_∞. Then, u_ε stochastically L^2_θ-two-scale converges to $u_\infty \in \mathrm{BL}^1(\mathbb{R}^3)$ and there exists $\xi \in L^2(\mathbb{R}^3; L^2_{\text{pot}}(\Omega))$ such that, up to a subsequence,*

$$\nabla u_\varepsilon \xrightarrow{\text{stoch 2-s}} \nabla u_\infty(x) + \xi(x,\omega) \quad \textit{weakly in } L^2(\mathbb{R}^3 \times \Omega; \mathbb{R}^3). \tag{3.5.5}$$

Proof. First, note that the weak convergence of u_ε in $BL^1(\mathbb{R}^3)$ implies the weak convergence of u_ε in $L^2_\theta(\mathbb{R}^3)$ to u_∞. Hence, an application of Proposition 3.5.1 yields the existence of a function $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times \Omega)$ such that, up to a subsequence, u_ε stochastically L^2_θ-two scale converges to $u \in L^2_{\text{loc}}(\mathbb{R}^3 \times \Omega)$ and $\mathbb{E}[u] \in L^2_\theta(\mathbb{R}^3)$.

Repeating similar arguments as those in Proposition 3.5.1, we find that, for every $i \in \mathbb{N}$, there exists a subsequence $\{u_{\varepsilon_{n(k_i)}}\}$ such that $n(k_i) \subset n(k_{i-1})$ and

$$u_{\varepsilon_{n(k_i)}} \xrightarrow{\text{stoch 2-s}} u(x,\omega) \quad \text{in } L^2(D_i \times \Omega).$$

Here, $\{D_i\}_{i\in\mathbb{N}}$ is a covering of $\mathbb{R}^3$ consisting of bounded domains. Defining the diagonal sequence of indices as in (3.5.4), for every $i \in \mathbb{N}$, up to the first $(i-1)$ terms, the sequence of indices $n(k_\infty)$ is included in $n(k_i)$ and

$$\begin{cases} u_{\varepsilon_{n(k_\infty)}} \xrightarrow{\text{stoch 2-s}} u(x,\omega) & \text{in } L^2(D_i \times \Omega), \\ u_{\varepsilon_{n(k_\infty)}} \rightharpoonup u_\infty(x) = \mathbb{E}[u] & \text{in } L^2(D_i). \end{cases}$$

Let $b \in C^1(\Omega;\mathbb{R}^3)$ and $\varphi \in C_c^1(D_i)$. For $\varepsilon > 0$, we have

$$\int_{D_i} u_{\varepsilon_{n(k_\infty)}}(x)\varphi(x)\,\mathrm{div}_\omega\, b(T_{x/\varepsilon_{n(k_\infty)}}\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x \\ = -\varepsilon \int_{D_i} \nabla_x \left(u_{\varepsilon_{n(k_\infty)}}(x)\varphi(x)\right) b(T_{x/\varepsilon_{n(k_\infty)}}\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.$$

Passing to the limit as ε goes to 0, we deduce that

$$\int_{D_i} \varphi(x) \left[\int_\Omega u(x,\omega)\,\mathrm{div}_\omega\, b(\omega)\,\mathrm{d}\mathbb{P}(\omega)\right] \mathrm{d}x = 0.$$

Therefore, for almost every $x \in D_i$ and any $b \in C^1(\Omega;\mathbb{R}^3)$,

$$\int_\Omega u(x,\omega)\,\mathrm{div}_\omega\, b(\omega)\,\mathrm{d}\mathbb{P}(\omega) = 0.$$

For such x, $u(x,\cdot) \in H^1(\Omega)$ with $\nabla_\omega u(x,\cdot) = 0$. Thanks to Proposition 3.3.18, it follows that $u(x,\omega)$ does not depend on ω up to a $\mathbb{P}$-negligible set. Setting $\hat{u}(x) := u(x,\omega)$, we conclude that $u_\infty \equiv \mathbb{E}[u] \equiv \hat{u} \in L^2_\theta(\mathbb{R}^3)$ in $\mathbb{R}^3$.

To conclude the proof, it remains to show (3.5.5). For simplicity, with a slight abuse of notation, from now on we omit the index $n(k_\infty)$ and simply denote the above extracted subsequence by ε. Once again, the weak convergence of u_ε in $BL^1(\mathbb{R}^3)$ to u_∞ implies that ∇u_ε weakly converges to ∇u_∞ in $L^2(\mathbb{R}^3)$, and that ∇u_ε is bounded in $L^2(\mathbb{R}^3)$. Due to Theorem 3.3.8, there exists $\lambda_\infty \in L^2(\mathbb{R}^3 \times \Omega;\mathbb{R}^3)$ such that, up to subsequences, ∇u_ε stochastically two-scale converges to λ_∞. Let $\varphi \in C_c^\infty(\mathbb{R}^3)$. Integrating by parts, we deduce that

$$\int_{\mathbb{R}^3} \nabla u_\varepsilon(x)\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^3} u_\varepsilon(x)\nabla\varphi(x)\,\mathrm{d}x.$$

Passing to the two-scale limit, we get

$$\int_{\mathbb{R}^3} \varphi(x)\left[\int_\Omega \lambda_\infty(x,\omega)\,\mathrm{d}\mathbb{P}(\omega)\right]\mathrm{d}x = -\int_{\mathbb{R}^3} u_\infty(x)\nabla\varphi(x)\,\mathrm{d}x.$$

Therefore, we define the distribution ∇u_∞ by

$$\nabla u_\infty := \int_\Omega \lambda_\infty(\cdot,\omega)\,\mathrm{d}\mathbb{P}(\omega) \in L^2(\mathbb{R}^3).$$

Now, let $\sigma \in C(\Omega;\mathbb{R}^3) \cap L^2_{\mathrm{sol}}(\Omega)$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$. Hence,

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \sigma(T_{x/\varepsilon}\omega)\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^3} \nabla[u_\varepsilon\varphi(x)] \cdot \sigma(T_{x/\varepsilon}\omega)\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} u_\varepsilon(x)\nabla\varphi(x)\cdot\sigma(T_{x/\varepsilon}\omega)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} u_\varepsilon(x)\nabla\varphi(x)\cdot\sigma(T_{x/\varepsilon}\omega)\,\mathrm{d}x. \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ and integrating by parts, we find

$$\int_{\mathbb{R}^3\times\Omega} \lambda_\infty(x,\omega)\cdot\sigma(\omega)\varphi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x = -\int_{\mathbb{R}^3\times\Omega} u_\infty(x)\nabla\varphi(x)\cdot\sigma(\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^3\times\Omega} \nabla u_\infty(x)\cdot\sigma(\omega)\varphi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.$$

The last equality shows that $\xi(x,\omega) \coloneqq \lambda_\infty(x,\omega) - \nabla u_\infty(x) \in (C(\Omega;\mathbb{R}^3)\cap L^2_{\mathrm{sol}}(\Omega))^\perp$ for almost every $x\in\mathbb{R}^3$. By the density of $C(\Omega)$ in $L^2(\Omega)$, we conclude that $\xi \in L^2(\mathbb{R}^3;(L^2_{\mathrm{sol}}(\Omega))^\perp) = L^2(\mathbb{R}^3;L^2_{\mathrm{pot}}(\Omega))$. □

With this result at hand, we are now in a position to characterize the behaviour of the stray fields $\{h_d[m_\varepsilon]\}_{\varepsilon>0}$ defined in (3.5.1). Applying the characterization to the sequence $M_{\mathrm{sat}}(T_{\cdot/\varepsilon}\widetilde{\omega})m\chi_D$ for $m\in L^2(D;\mathbb{S}^2)$ as in the definition of $\mathcal{W}_\varepsilon$, will then allow us to prove Proposition 3.2.8.

Proposition 3.5.3 (Stochastic 2-scale convergence and the demagnetizing field)**.** *Let $\{m_\varepsilon\}_{\varepsilon>0}$ be a bounded family in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ which stochastically 2-scale converges to $m = m(x,\omega)\in L^2(\mathbb{R}^3\times\Omega;\mathbb{R}^3)$. Then, the stochastic 2-scale limit of $\{h_d[m_\varepsilon]\}_{\varepsilon>0}\subset L^2(\mathbb{R}^3;\mathbb{R}^3)$ exists and is given by*

$$h_d(x,\omega) = h_d\big[\mathbb{E}[m]\big](x) + \xi_m(x,\omega),$$

where, for a.e. $x\in\mathbb{R}^3$, the map $\xi_m(x,\cdot)$ is the unique solution in $L^2_{\mathrm{pot}}(\Omega)$ to

$$m(x,\cdot) + \xi_m(x,\cdot) \in L^2_{\mathrm{sol}}(\Omega). \tag{3.5.6}$$

Remark 3.5.4. The variational formulation of (3.5.6) reads

$$\int_\Omega \xi_m(x,\omega)\cdot\Psi(\omega)\,\mathrm{d}\mathbb{P}(\omega) = -\int_\Omega m(x,\omega)\cdot\Psi(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

for any $\Psi\in L^2_{\mathrm{pot}}(\Omega)$.

Proof. Combining the stability estimate (3.5.3) with the boundedness in $L^2(\mathbb{R}^3)$ of m_ε, the sequence of magnetostatic potentials $\{u_{m_\varepsilon}\}_{\varepsilon>0}$ is uniformly bounded in $BL^1(\mathbb{R}^3)$. Thus, there exists $u_m\in BL^1(\mathbb{R}^3)$ such that, up to the extraction of a non-relabelled subsequence, u_{m_ε} weakly converges to u_m in $BL^1(\mathbb{R}^3)$. In view of Proposition 3.5.2, there exist functions $u_m\in BL^1(\mathbb{R}^3)$ and $\xi_m\in L^2(\mathbb{R}^3;L^2_{\mathrm{pot}}(\Omega))$ such that

$$u_{m_\varepsilon} \xrightarrow{\text{stoch 2-s}} u_m \quad \text{in } L^2_\theta \quad \text{and} \quad \nabla u_{m_\varepsilon} \xrightarrow{\text{stoch 2-s}} \nabla u_m + \xi_m \quad \text{in } L^2(\mathbb{R}^3\times\Omega;\mathbb{R}^3).$$

Testing (3.5.2) with $\varphi(x)+\varepsilon\psi(x)b(T_{x/\varepsilon}\omega)$, for $\varphi,\psi\in C^\infty_c(\mathbb{R}^3)$ and $b\in C^1(\Omega)$, we infer

$$\int_{\mathbb{R}^3} \nabla u_{m_\varepsilon}\cdot(\nabla\varphi(x) + \psi(x)\nabla_\omega b(T_{x/\varepsilon}\omega) + \varepsilon b(T_{x/\varepsilon}\omega)\nabla\psi(x))\,\mathrm{d}x$$
$$= -\int_{\mathbb{R}^3} m_\varepsilon\cdot(\nabla\varphi(x) + \psi(x)\nabla_\omega b(T_{x/\varepsilon}\omega) + \varepsilon b(T_{x/\varepsilon}\omega)\nabla\psi(x))\,\mathrm{d}x.$$

Passing to the stochastic two-scale limit yields

$$\int_{\mathbb{R}^3\times\Omega} (\nabla u_m(x) + \xi_m(x,\omega))\cdot(\nabla\varphi(x) + \psi(x)\nabla_\omega b(\omega))\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x$$
$$= -\int_{\mathbb{R}^3\times\Omega} m(x,\omega)\cdot(\nabla\varphi(x) + \psi(x)\nabla_\omega b(\omega))\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x. \tag{3.5.7}$$

In particular, choosing $\psi \equiv 0$ we have

$$\begin{aligned} -\int_{\mathbb{R}^3} \mathbb{E}[m(x,\cdot)] \cdot \nabla\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^3\times\Omega} (\nabla u_m(x) + \xi_m(x,\omega)) \cdot \nabla\varphi(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \nabla u_m(x) \cdot \nabla\varphi(x)\,\mathrm{d}x, \end{aligned}$$

where the last equality is a consequence of Lemma 3.3.14. In other words, $\nabla u_m = h_d\,[\mathbb{E}[m]]$, since $u_m \in \mathrm{BL}^1(\mathbb{R}^3)$ is a solution to

$$-\operatorname{div}(\nabla u_m + \mathbb{E}[m]) = 0.$$

Now, choosing $\varphi \equiv 0$, (3.5.7) yields

$$\int_{\mathbb{R}^3\times\Omega} (\nabla u_m(x) + \xi_m(x,\omega)) \cdot \nabla_\omega b(\omega)\psi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x = -\int_{\mathbb{R}^3\times\Omega} m(x,\omega) \cdot \nabla_\omega b(\omega)\psi(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}x.$$

This implies that for almost every $x \in \mathbb{R}^3$

$$\begin{aligned} -\int_\Omega m(x,\omega) \cdot \nabla_\omega b(\omega)\,\mathrm{d}\mathbb{P}(\omega) &= \int_\Omega (\nabla u_m(x) + \xi_m(x,\omega)) \cdot \nabla_\omega b(\omega)\,\mathrm{d}\mathbb{P}(\omega) \\ &= \int_\Omega \xi_m(x,\omega) \cdot \nabla b(\omega)\,\mathrm{d}\mathbb{P}(\omega), \end{aligned} \tag{3.5.8}$$

where in the last equality we used again Lemma 3.3.14. The equality (3.5.8) is the variational formulation of

$$\operatorname{div}_\omega \xi_m(x,\omega) = -\operatorname{div}_\omega m(x,\omega) \quad \text{in } L^2_{\mathrm{pot}}(\Omega).$$

The well-posedness of this problem is a consequence of Lax-Milgram's theorem in $L^2_{\mathrm{pot}}(\Omega)$. This concludes the proof. □

We conclude this subsection with the proof of Proposition 3.2.8.

Proof of Proposition 3.2.8. Let $m \in L^2(D;\mathbb{S}^2) \subset L^2(\mathbb{R}^3;\mathbb{R}^3)$ and fix $\widetilde{\omega} \in \widetilde{\Omega}$, see Remark 3.3.6. Define $M_\varepsilon^{\widetilde{\omega}} \in L^\infty(\mathbb{R}^3)$ by $M_\varepsilon^{\widetilde{\omega}}(x) := M_{\mathrm{sat}}(T_{x/\varepsilon}\widetilde{\omega})$. Then, the sequence $\{M_\varepsilon^{\widetilde{\omega}} m\}$ weakly stochastically two-scale converges to $M_{\mathrm{sat}}(\omega)m(x)$ componentwise. Since $\widetilde{\omega}$ is a typical trajectory with respect to $M_{\mathrm{sat}}^2\,\mathrm{d}\mathbb{P}$, and $m(x) \in \mathbb{S}^2$ for every $x \in D$,

$$\|M_\varepsilon^{\widetilde{\omega}} m\|^2_{L^2(D)} = \int_D M_{\mathrm{sat}}(T_{x/\varepsilon}\widetilde{\omega})^2\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} \|M_{\mathrm{sat}} m\|^2_{L^2(D\times\Omega)}.$$

Thus, $\{M_\varepsilon^{\widetilde{\omega}} m\}$ strongly stochastically two-scale converges to $M_{\mathrm{sat}} m$. Hence, we can apply Proposition 3.5.3 for the corresponding stray fields. Note that by the definition of $\Theta_M \in L^2_{\mathrm{pot}}(\Omega;\mathbb{R}^3)$, cf. (3.3.4), for almost every $x \in D$ we have

$$M_{\mathrm{sat}} m(x) + \Theta_M m(x) \in L^2_{\mathrm{sol}}(\Omega) \tag{3.5.9}$$

and therefore

$$h_d\left[M_\varepsilon^{\widetilde{\omega}} m \chi_D\right] \xrightarrow{\text{stoch 2-s}} h_d\,[\mathbb{E}[M_{\mathrm{sat}}]m] + \Theta_M m \quad \text{in } L^2(\mathbb{R}^3\times\Omega;\mathbb{R}^3).$$

Thus, the weak-strong convergence principle in Proposition 3.3.11 yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m) &= -\lim_{\varepsilon\to 0}\frac{\mu_0}{2}(h_d[M_\varepsilon^{\widetilde{\omega}}m], M_\varepsilon^{\widetilde{\omega}}m)_{L^2(D)}\\ &= -\frac{\mu_0}{2}\mathbb{E}\left[(h_d\left[\mathbb{E}[M_{\mathrm{sat}}]m\right], M_{\mathrm{sat}}m)_{L^2(D)} + (\Theta_M m, M_{\mathrm{sat}}m)_{L^2(D)}\right]\\ &= -\frac{\mu_0}{2}\left(h_d\left[\mathbb{E}[M_{\mathrm{sat}}]m\right], \mathbb{E}[M_{\mathrm{sat}}]m\right)_{L^2(D)} - \frac{\mu_0}{2}\mathbb{E}\left[(\Theta_M m, \Theta_M m)_{L^2(D)}\right]\\ &= \mathcal{W}_{\mathrm{hom}}(m)\end{aligned}$$

where we used (3.5.9) with $\Theta_M m(x) \in L^2_{\mathrm{pot}}(\Omega)$ for almost every $x \in D$. To conclude the proof, it remains to show that the family $\mathcal{W}_\varepsilon^{\widetilde{\omega}}$ continuously converges to $\mathcal{W}_{\mathrm{hom}}$ in L^2, i.e. that for $m_\varepsilon \to m_0$ strongly in $L^2(D;\mathbb{S}^2)$ there holds

$$\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) \to \mathcal{W}_{\mathrm{hom}}(m_0) \quad \text{as } \varepsilon \to 0. \tag{3.5.10}$$

To this end, we write

$$|\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{W}_{\mathrm{hom}}(m_0)| \le |\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_0)| + |\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_0) - \mathcal{W}_{\mathrm{hom}}(m_0)|.$$

Due to Proposition 3.5.3, it follows that $|\mathcal{W}_\varepsilon(m_0) - \mathcal{W}_{\mathrm{hom}}(m_0)| \to 0$ as $\varepsilon \to 0$. Since $\{m_\varepsilon\}_{\varepsilon>0} \subset L^2(D;\mathbb{S}^2)$, by (3.5.3) we infer

$$\begin{aligned}|\mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{W}_\varepsilon^{\widetilde{\omega}}(m_0)| &\le |(h_d[M_\varepsilon^{\widetilde{\omega}}m_\varepsilon], M_\varepsilon^{\widetilde{\omega}}(m_\varepsilon - m_0))_{L^2(D)} + (h_d[M_\varepsilon^{\widetilde{\omega}}(m_\varepsilon - m_0)], M_\varepsilon^{\widetilde{\omega}}m_0)_{L^2(D)}|\\ &\le 2\|M_{\mathrm{sat}}\|^2_{L^\infty(D)}\|m_\varepsilon - m_0\|_{L^2(D;\mathbb{R}^3)},\end{aligned}$$

which in turn yields (3.5.10). □

3.5.2 The homogenized anisotropic and the interaction energies

In this subsection we prove Proposition 3.2.9, i.e. we focus on the continuous convergence of the family of anisotropic energy functionals $\{\mathcal{A}_\varepsilon^{\widetilde{\omega}}\}$ and of the family of Zeeman energy functionals $\{\mathcal{Z}_\varepsilon^{\widetilde{\omega}}\}$.

Proof of Proposition 3.2.9. Step 1: The convergence of $\{\mathcal{A}_\varepsilon^{\widetilde{\omega}}\}$. We prove that for every $m_0 \in L^2(D;\mathbb{S}^2)$ and every sequence $\{m_\varepsilon\} \subset L^2(D;\mathbb{S}^2)$ with $m_\varepsilon \to m_0$ strongly in $L^2(D;\mathbb{S}^2)$ there holds

$$\mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) \to \mathcal{A}_{\mathrm{hom}}(m_0) \quad \text{as } \varepsilon \to 0. \tag{3.5.11}$$

In fact, we have

$$|\mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{A}_{\mathrm{hom}}(m_0)| \le |\mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_0)| + |\mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_0) - \mathcal{A}_{\mathrm{hom}}(m_0)|. \tag{3.5.12}$$

Now, we estimate the two terms on the right-hand side of (3.5.12). An application of Birkhoff's ergodic theorem combined with an approximation argument yields $|\mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_0) - \mathcal{A}_{\mathrm{hom}}(m_0)| \to 0$, as $\varepsilon \to 0$ (see also Remark 3.3.2). On the other hand, using the global Lipschitz continuity of φ combined with Hölder inequality, we deduce

$$\begin{aligned}|\mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{A}_\varepsilon^{\widetilde{\omega}}(m_0)| &\le \int_D \left|\varphi\left(T_{x/\varepsilon}\omega, m_\varepsilon\right) - \varphi\left(T_{x/\varepsilon}\omega, m_0\right)\right| \mathrm{d}x\\ &\le L_{\mathrm{an}}\int_D |m_\varepsilon(x) - m_0(x)|\,\mathrm{d}x\\ &\le L_{\mathrm{an}}|D|^{1/2}\|m_\varepsilon - m_0\|_{L^2(D;\mathbb{R}^3)},\end{aligned}$$

which in turn implies (3.5.11).

Step 2: The convergence of $\{\mathcal{Z}_\varepsilon^{\widetilde{\omega}}\}$. The convergence of the Zeeman energies follows directly via an analogous argument to Step 1. On the one hand, since $\{M_{\mathrm{sat}}(T_{\cdot/\varepsilon}\widetilde{\omega})m_0\}$ weakly converges to $\mathbb{E}[M_{\mathrm{sat}}]m_0$ in $L^2(D)$,

$$\mathcal{Z}_\varepsilon^{\widetilde{\omega}}(m_0) = -\mu_0 \int_D h_a \cdot M_{\mathrm{sat}}(T_{x/\varepsilon}\widetilde{\omega})m_0(x)\,\mathrm{d}x \overset{\varepsilon\to 0}{\longrightarrow} -\mu_0 \int_D h_a \cdot \mathbb{E}[M_{\mathrm{sat}}]m_0(x)dx = \mathcal{Z}_{\mathrm{hom}}(m_0).$$

On the other hand,

$$\left|\mathcal{Z}_\varepsilon^{\widetilde{\omega}}(m_\varepsilon) - \mathcal{Z}_\varepsilon^{\widetilde{\omega}}(m_0)\right| \le \mu_0 C_{\mathrm{sat}} \|h_a\|_{L^2(D;\mathbb{R}^3)} \|m_\varepsilon - m_0\|_{L^2(D;\mathbb{R}^3)}.$$

□

3.5.3 Equi-coerciveness of the micromagnetic functional

The main part left for the proof of Theorem 3.2.3 is to show that the family $\{\mathcal{F}_\varepsilon^{\widetilde{\omega}}\}$ is equi-coercive in the weak topology of $H^1(D,\mathbb{S}^2)$. This then guarantees the validity of the fundamental theorem of Γ-convergence concerning the variational convergence of minimum problems (see [BD98; Dal93]).

Proposition 3.5.5. *The family of functionals $\mathcal{F}_\varepsilon^{\widetilde{\omega}}$ is equi-coercive in the weak topology of $H^1(D;\mathbb{S}^2)$. In other words, there exists a weakly compact set $K \subset H^1(D;\mathbb{S}^2)$ such that*

$$\inf_{m\in H^1(D;\mathbb{S}^2)} \mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) = \inf_{m\in K} \mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) \qquad \text{for every } \varepsilon > 0.$$

Proof. Rewriting $\mathcal{G}_\varepsilon^{\widetilde{\omega}} = \mathcal{F}_\varepsilon^{\widetilde{\omega}} + \mathcal{K}_\varepsilon^{\widetilde{\omega}}$ as in (3.4.1) and applying the inequality $(a-b)^2 \ge \frac{1}{2}a^2 - b^2$, along with assumption (P1), yields

$$\begin{aligned}
\mathcal{G}_\varepsilon^{\widetilde{\omega}}(m) &= \frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega}) \left|\nabla m(x) - \frac{\kappa(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}\boldsymbol{\chi}(m(x))\right|^2 \mathrm{d}x - \frac{1}{2}\int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m(x)|^2\,\mathrm{d}x. \\
&\ge \frac{1}{4}\int_D a(T_{x/\varepsilon}\widetilde{\omega})|\nabla m(x)|^2\,\mathrm{d}x - \int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m(x)|^2\,\mathrm{d}x \\
&\ge \frac{1}{4}c_{\mathrm{ex}}\int_D |\nabla m(x)|^2\,\mathrm{d}x - \int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m(x)|^2\,\mathrm{d}x. \qquad (3.5.13)
\end{aligned}$$

In view of assumptions (P1) and (P2) combined with an application of the Cauchy-Schwarz inequality, the last integral in (3.5.13) is bounded by

$$\left|\int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m(x)|^2\,\mathrm{d}x\right| \le \frac{C_{\mathrm{DMI}}^2}{c_{\mathrm{ex}}}\|m\|_{L^2(D;\mathbb{S}^2)} \le C\frac{C_{\mathrm{DMI}}^2}{c_{\mathrm{ex}}}|D|,$$

where we have used the fact that $\|m\|_{L^2(D;\mathbb{S}^2)} \le C|D|$. Therefore,

$$\mathcal{G}_\varepsilon^{\widetilde{\omega}}(m) \ge \frac{1}{4}c_{\mathrm{ex}}\int_D |\nabla m(x)|^2\,\mathrm{d}x - C\frac{C_{\mathrm{DMI}}^2}{c_{\mathrm{ex}}}|D|. \qquad (3.5.14)$$

The interaction energy $\mathcal{Z}_\varepsilon^{\widetilde{\omega}}$ is estimated by

$$\left|\mu_0\int_D h_a \cdot M_\varepsilon m(x)\,\mathrm{d}x\right| \le \mu_0\|h_a\|_{L^2(D)}\|M_\varepsilon m\|_{L^2(D)} \le C\mu_0|D||h_a|. \qquad (3.5.15)$$

Since the magnetostatic self-energy $\mathcal{W}_\varepsilon^{\widetilde{\omega}}$, as well as the anisotropic energy $\mathcal{A}_\varepsilon^{\widetilde{\omega}}$ are non-negative terms, and owing to (3.5.14) and (3.5.15), we conclude that

$$\mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) \geq \frac{1}{4} c_{\mathrm{ex}} \int_D |\nabla m(x)|^2 \,\mathrm{d}x - C \frac{C_{\mathrm{DMI}}^2}{c_{\mathrm{ex}}} |D| - C\mu_0 |D||h_a|. \tag{3.5.16}$$

Now, set

$$\widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m) \coloneqq \mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) + C\left(\frac{C_{\mathrm{DMI}}^2}{c_{\mathrm{ex}}} + \mu_0 |h_a|\right)|D|,$$

so that, from inequality (3.5.16), it follows that

$$C\|m\|_{H^1(D;\mathbb{S}^2)} \leq \widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m), \tag{3.5.17}$$

for every magnetization $m \in H^1(D;\mathbb{S}^2)$. On the other hand, by assumptions (P1), (P2), and (P4), we deduce that

$$\begin{aligned}
\mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) &\leq \frac{1}{2}\int_D a(T_{x/\varepsilon}\widetilde{\omega})|\nabla m(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_D \frac{\kappa^2(T_{x/\varepsilon}\widetilde{\omega})}{a(T_{x/\varepsilon}\widetilde{\omega})}|m(x)|^2 \,\mathrm{d}x + \frac{\mu_0}{2}\int_D |h_a[M_\varepsilon(x)m(x)]|^2 \,\mathrm{d}x \\
&\quad + C_{\mathrm{an}}|D| + C\mu_0|D||h_a|C_s \\
&\leq \frac{C_{\mathrm{ex}}}{2}\int_D |\nabla m(x)|^2 \,\mathrm{d}x + \frac{C_{\mathrm{DMI}}^2}{2c_{\mathrm{ex}}}\|m\|_{L^2(D;\mathcal{M})}^2 + C,
\end{aligned}$$

where we used (3.5.3) for the second inequality. In particular, there exists a positive constant $\widetilde{C}$ such that $\widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m) \leq \widetilde{C}\|m\|_{H^1(D;\mathbb{S}^2)}^2$ Therefore, for every magnetization $m \in H^1(D;\mathbb{S}^2)$, there holds $\widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m) \leq C|D|$. Let $K(D;\mathbb{S}^2)$ be the set defined by

$$K(D;\mathbb{S}^2) \coloneqq \{m \in H^1(D;\mathbb{S}^2) : \widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m) \leq C|D|\}.$$

Note that in view of (3.5.17), if the magnetization $m \in K(D;\mathbb{S}^2)$, then $\|\nabla m\|_{L^2(D;\mathbb{S}^2)} \leq \widetilde{C}|D|$. Hence, $K(D;\mathbb{S}^2)$ is contained in a ball $B_{\mathbb{S}^2}$ of $H^1(D;\mathbb{R}^3)$. Setting $K \coloneqq K(D;\mathbb{S}^2) \coloneqq B_{\mathbb{S}^2} \cap H^1(D;\mathbb{S}^2)$, we conclude that

$$\inf_{m\in H^1(D;\mathbb{S}^2)} \widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m) = \inf_{m\in K} \widetilde{\mathcal{F}}_\varepsilon^{\widetilde{\omega}}(m),$$

where K is weakly compact being the intersection of the weakly closed set $H^1(D;\mathbb{S}^2)$ and the weakly compact set $B_{\mathbb{S}^2}$. This concludes the proof. □

3.5.4 Limiting behavior of the micromagnetic functionals - the proof of Theorem 3.2.3

We are now in a position to prove Theorem 3.2.3. First, the equi-coerciveness of the family $\mathcal{F}_\varepsilon^{\widetilde{\omega}}$ with respect to the weak topology $H^1(D;\mathbb{S}^2)$ is proven in Section 3.5.3. In view of the stability property of Γ-limits under the sum with a continuously convergent family of functionals (see [Dal93, Proposition 6.20]), we deduce that, for $m \in L^2(D;\mathbb{S}^2)$,

$$\begin{aligned}
F_{\mathrm{hom}}(m) &= \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon^{\widetilde{\omega}}(m) = \Gamma\text{-}\lim_{\varepsilon\to 0} (\mathcal{E}_\varepsilon^{\widetilde{\omega}} + \mathcal{K}_\varepsilon^{\widetilde{\omega}})(m) + \mathcal{W}_{\mathrm{hom}}(m) + \mathcal{A}_{\mathrm{hom}}(m) + \mathcal{Z}_{\mathrm{hom}}(m) \\
&= \mathcal{G}_{\mathrm{hom}}(m) + \mathcal{W}_{\mathrm{hom}}(m) + \mathcal{A}_{\mathrm{hom}}(m) + \mathcal{Z}_{\mathrm{hom}}(m),
\end{aligned}$$

where $\mathcal{G}_{\mathrm{hom}}$ is as in (3.2.2), $\mathcal{W}_{\mathrm{hom}}$ is defined by (3.2.3), $\mathcal{A}_{\mathrm{hom}}$ and $\mathcal{Z}_{\mathrm{hom}}$ are given, respectively, by (3.2.4) and (3.2.5). This completes the proof of Theorem 3.2.3.

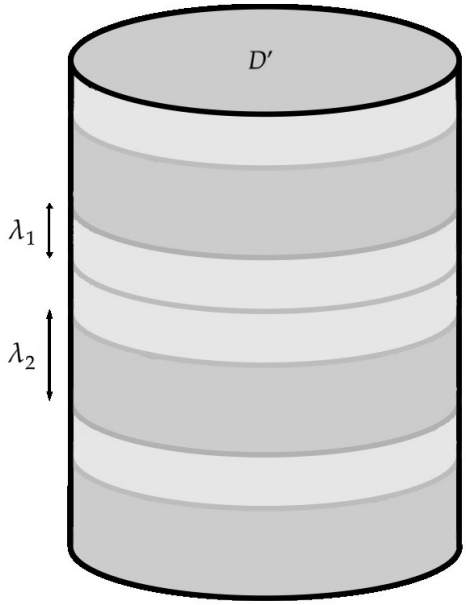


Figure 3.6.1: We consider a chiral multilayer having a random laminated structure. E.g. we assume here that there are two possible layer ratios $\lambda_1, \lambda_2 \in (0,1)$ both occurring with a certain probability.

3.6 Application to multilayers

In this section, we specify the effective energy $\mathcal{G}_{\mathrm{hom}}$ from Theorem 3.2.6 in the micromagnetic setting of chiral multilayers. In such a case, $\mathcal{M} = \mathbb{S}^2$ and $D \subset \mathbb{R}^3$ has a laminated structure. More specifically, we assume that $D := D' \times I_\lambda$, where D' is a bounded open subset of $\mathbb{R}^2$ with $|D'| = 1$ and $I_\lambda = (0, \lambda)$, for some $\lambda > 0$ (see Figure 3.6.1). In addition to the usual assumptions, we let the dynamical system satisfy

$$T_{(x_1,x_2,x_3)} = T_{(0,0,x_3)} = T_{x_3 e_3} \quad \text{for all } x \in \mathbb{R}^3. \tag{3.6.1}$$

In other words, the ergodic action described by T does not depend on the variable $(x_1, x_2) \in D'$.

Remark 3.6.1 (Differentiability in the multilayer setting). In particular, any function $u \in L^2(\Omega)$ is everywhere differentiable in the directions e_1, e_2 with $D_1 u = D_2 u = 0$ in the sense of Definition 3.3.13. Therefore, the first two rows of every function in $L^2_{\mathrm{pot}}(\Omega; \mathbb{R}^3)$ are 0 almost everywhere. This holds, in particular, for $\Theta_a, \Theta_\kappa \in L^2_{\mathrm{pot}}(\Omega; \mathbb{R}^3)$ as defined in Lemma 3.3.21.

Proposition 3.6.2 (Characterization of $\mathcal{G}_{\mathrm{hom}}$ given a laminated structure)**.** *Under the assumptions of Theorem 3.2.3, and assuming additionally that the dynamical system satisfies* (3.6.1)*, the homogenized energy functional $\mathcal{G}_{\mathrm{hom}}$ as defined in* (3.2.2) *is given by*

$$\mathcal{G}_{\mathrm{hom}}(m) = \frac{1}{2}\int_D \nabla m : a_{\mathrm{eff}} \nabla m \,\mathrm{d}x - \int_D \nabla m : \kappa_{\mathrm{eff}} \boldsymbol{\chi}(m) \,\mathrm{d}x + \int_D \varphi_{\mathrm{eff}}(m) \,\mathrm{d}x. \tag{3.6.2}$$

for $m \in H^1(D; \mathbb{S}^2)$ with constants

$$a_{\mathrm{eff}} := \mathrm{diag}\left(\mathbb{E}[a], \mathbb{E}[a], \mathbb{E}\left[a^{-1}\right]^{-1}\right) \in \mathbb{R}^{3\times 3},$$
$$\kappa_{\mathrm{eff}} := \mathrm{diag}\left(\mathbb{E}[\kappa], \mathbb{E}[\kappa], \mathbb{E}\left[\frac{\kappa}{a}\right] \mathbb{E}\left[a^{-1}\right]^{-1}\right) \in \mathbb{R}^{3\times 3},$$

and effective anisotropic density $\varphi_{\mathrm{eff}} \in C^1(\mathbb{S}^2, \mathbb{R})$ *defined as*

$$s \mapsto \varphi_{\mathrm{eff}}(s) := -\frac{1}{2}\left(\mathbb{E}\left[\frac{\kappa^2}{a}\right] - \mathbb{E}\left[\frac{\kappa}{a}\right]^2 \mathbb{E}[a^{-1}]^{-1}\right)\left(s_1^2 + s_2^2\right).$$

Remark 3.6.3 (Comparison to the periodic case). It is easy to see that this is the same energy which Di Fratta and the first author obtained for periodic multilayers [DD20, Theorem 3.1], only due to the setting the expected values are replaced with cell averages.

Proof. We provide explicit characterizations of the correctors $\Theta_a = (\Theta_{a.ij})_{i,j=1,2,3}$ $\Theta_\kappa = (\Theta_{\kappa.ij})_{i,j=1,2,3}$ in this setting. We then plug these characterizations into the definition of $\mathcal{G}_{\mathrm{hom}}$, cf (3.2.2), to obtain the result.

Step 1: The characterization of Θ_a. With Remark 3.6.1, it remains to identify $\Theta_{a,3i}$ for $i = 1, 2, 3$, which are defined by

$$a\Theta_{a,3i}e_3 + ae_i \in L^2_{\mathrm{sol}}(\Omega). \tag{3.6.3}$$

Again from Remark 3.6.1, it immediately follows that $ae_1, ae_2 \in L^2_{\mathrm{sol}}(\Omega)$ and hence $\Theta_{a,31} = \Theta_{a,32} = 0$. For $i = 3$ we rewrite (3.6.3) as

$$\int_\Omega (a(\omega)\Theta_{a,33}(\omega) + a(\omega))D_3 b(\omega)\,\mathrm{d}\mathbb{P}(\omega) = 0 \quad \text{for all } b \in C^1(\Omega).$$

By Definition 3.3.15, we thus have $D_3(a\Theta_{a,33} + a) = 0$ and hence $a\Theta_{a,33} + a \in H^1(\Omega)$ with $\nabla_\omega(a\Theta_{a,33} + a) = 0$ due to Remark 3.6.1. Thus, by Proposition 3.3.18 there exists $C_a \in \mathbb{R}$ such that $\mathbb{P}$-almost surely

$$a\Theta_{a,33} + a = C_a.$$

We extend Lemma 3.3.14 to components of $L^2_{\mathrm{pot}}(\Omega)$-functions via approximation, and have

$$0 = \mathbb{E}[\Theta_{a,33}] = \mathbb{E}\left[a^{-1}\right] C_a - 1$$

Hence, we obtain $C_a = \mathbb{E}[a^{-1}]^{-1}$ and therefore

$$\Theta_{a,33} = \frac{\mathbb{E}[a^{-1}]^{-1}}{a} - 1. \tag{3.6.4}$$

Step 2: The characterization of Θ_κ. As in Step 1, it holds that $\Theta_{\kappa,ij} = 0$ for $(i,j) \neq (3,3)$ and there exists a constant $C_\kappa \in \mathbb{R}$, such that $\mathbb{P}$-almost surely

$$a\Theta_{\kappa,33} + \kappa = C_\kappa.$$

Analogously, we obtain $C_\kappa = \mathbb{E}[\kappa/a]\mathbb{E}[a^{-1}]^{-1}$ and therefore

$$\Theta_{\kappa,33} = \mathbb{E}\left[\frac{\kappa}{a}\right]\frac{\mathbb{E}[a^{-1}]^{-1}}{a} - \frac{\kappa}{a}. \tag{3.6.5}$$

Step 3: Plugging Θ_a, Θ_κ into (3.2.2). With (3.6.4) we have

$$\mathbb{E}\left[a\Theta_a^{\intercal}\Theta_a\right] = \mathbb{E}\left[a^{-1}\left(\mathbb{E}[a^{-1}]^{-1} - a\right)^2\right] e_3 \otimes e_3 = \left(\mathbb{E}[a] - \mathbb{E}[a^{-1}]^{-1}\right) e_3 \otimes e_3.$$

Additionally considering (3.6.5), for the mixed term we obtain

$$\begin{aligned}\mathbb{E}\left[a\Theta_a^{\intercal}\Theta_\kappa\right] &= \mathbb{E}\left[\left(\mathbb{E}[a^{-1}]^{-1} - a\right)\left(\mathbb{E}\left[\frac{\kappa}{a}\right]\frac{\mathbb{E}[a^{-1}]^{-1}}{a} - \frac{\kappa}{a}\right)\right] e_3 \otimes e_3 \\ &= \left(\mathbb{E}[\kappa] - \mathbb{E}\left[\frac{\kappa}{a}\right]\mathbb{E}[a^{-1}]^{-1}\right) e_3 \otimes e_3.\end{aligned}$$

For the effective anisotropic energy term, (3.6.5) yields

$$\begin{aligned}\mathbb{E}\left[a\Theta_\kappa^\intercal\Theta_\kappa\right] &= \mathbb{E}\left[a^{-1}\left(\mathbb{E}\left[\frac{\kappa}{a}\right]\mathbb{E}[a^{-1}]^{-1}-\kappa\right)^2\right]e_3\otimes e_3\\ &= \left(\mathbb{E}\left[\frac{\kappa^2}{a}\right]-\mathbb{E}\left[\frac{\kappa}{a}\right]^2\mathbb{E}[a^{-1}]^{-1}\right)e_3\otimes e_3.\end{aligned}$$

Additionally, note that via direct calculation

$$\boldsymbol{\chi}(m):(e_3\otimes e_3)\boldsymbol{\chi}(m)=|e_3\times m|^2=m_1^2+m_2^2.$$

Plugging these identities into (3.2.2) yields (3.6.2). □

Due to Remark 3.6.3, via a result from [DD20] we immediately obtain a characterization for the minimizers of $\mathcal{G}_{\mathrm{hom}}$ given $\mathbb{E}[\kappa]=0$.

Lemma 3.6.4 (Characterization of minimizers, [DD20, Theorem 3.1]). *In the setting of Proposition 3.6.2, additionally assuming that $\mathbb{E}[\kappa]=0$, the only minimizers in $H^1(D;\mathbb{S}^2)$ of $\mathcal{G}_{\mathrm{hom}}$ given by (3.6.2) are helical textures m_* of the form*

$$\begin{aligned}m_*(x) &:= \cos(\theta(x\cdot e_3))e_1+\sin(\theta(x\cdot e_3))e_2,\\ \theta(t) &:= \theta_0+\mathbb{E}\left[\frac{\kappa}{a}\right]t, \quad \text{for } t\in\mathbb{R},\end{aligned}$$

with arbitrary $\theta_0\in\mathbb{R}$. The minimum value of the energy is given by

$$\mathcal{G}_{\mathrm{hom}}(m_*)=-\frac{\lambda}{2}\mathbb{E}\left[\frac{\kappa^2}{a}\right].$$

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CHAPTER 4

Quantitative homogenization of forced geometric motions through random fields of obstacles

This chapter contains (with minimal modifications) the following preprint [FI26a]:

Julian Fischer and Jonas Ingmanns. *Quantitative homogenization of forced geometric motions through random fields of obstacles*. 2026. arXiv: 2603.12179 [math.AP]

Abstract We establish a quantitative homogenization result for an interface moving through a field of sufficiently sparse but possibly impenetrable random obstacles. From a physical viewpoint, such problems arise e. g. in the context of the motion of dislocations or magnetic domain walls in a material with impurities.

More precisely, given an interface moving by forced mean curvature flow – with a positive global driving force plus a spatially fluctuating (negative) driving force modeling the obstacles – , we prove that the effective large-scale behavior of the forward front is governed by a constant-speed effective motion. For typical values of the global forcing, on large scales of the order ε^{-1} we obtain a (relative) error estimate for arrival times of the front of the order $\varepsilon^{1/9-}$.

Previous stochastic homogenization results for forced mean curvature motion in the literature have required a positive pointwise lower bound on the combined forcing, which implies the absence of any actual obstacles capable of locally blocking the interface motion. In contrast, our results are valid even in the presence of islands with locally negative forcing, potentially allowing for locally pinned interfaces and eventually enclosures left behind the main front. Thus, our homogenization result applies to settings closer to (but still strictly away from) the pinning-depinning transition.

4.1 Introduction

In the physics of materials, lower dimensional features in the material – like dislocation lines, magnetic domain walls, or grain boundaries – are central to the quantitative understanding of many phenomena, like e. g. plasticity, ferromagnetism, or fracture. The motion of lines and interfaces in materials is often governed by the combination of a

global force, driving the line or interface in a certain direction, with the general preference of minimization of the interface energy. In the simplest setting, the corresponding evolution is thus given by a forced mean curvature motion [IV87; Kar98; BN04]. In a material with impurities, these impurities may act as obstacles to the evolution of the line or interface, giving rise to a complex range of expected phenomena, such as pinning [BT14; CDO22; DY06; DDS11], de-pinning [BT14; CDL10; CDO22; DY06; DS12], and possibly homogenization [AC18; CM14; DKY08; DY06; Fel19; LS05b].

In the present work, we consider the motion of an interface driven by curvature and a uniform forcing through a field of sufficiently sparse but possibly impenetrable random obstacles. We prove that this motion can be approximated on large scales by a homogeneous first-order motion, i. e., in the case of isotropic media, by a constant speed motion; see Figure 4.1.1 for an illustration.

More precisely, we treat the case of an interface motion with the normal velocity V at $x \in \mathbb{R}^d$ given in terms of the normal $\nu \in \mathbb{S}^{d-1}$ and the mean curvature H by

$$V = H + F\big(x, \nu\big), \tag{4.1.1}$$

with the random forcing field $F(x,\nu) = F_{uni}(\nu) - F_{obst}(x,\nu)$ combining the uniform positive forcing and the obstacles. On large scales, that is zooming out by ε^{-1} and accelerating time by ε^{-1} for some small $\varepsilon > 0$, the interface motion is given by

$$V = \varepsilon H + F\Big(\frac{x}{\varepsilon}, \nu\Big). \tag{4.1.2}$$

For small $\varepsilon > 0$ and under suitable conditions on the obstacle distribution, one expects that this motion can be approximated by a first-order evolution with a homogeneous speed, that is with

$$V = \overline{v}(\nu) \tag{4.1.3}$$

for a suitable function $\overline{v} \colon \mathbb{S}^1 \to \mathbb{R}_{>0}$ capturing the effective speed in each direction. For our result, the effective speed $\overline{v}$ will be independent of the direction since we assume isotropy of the random obstacle distribution, that is that the law of $F(\cdot)$ is invariant with respect to rotations. Under suitable further assumptions on the probability distribution of F – assumptions that in particular cover the case of sufficiently sparse but possibly impenetrable random obstacles F_{obst} and generic positive forcings F_{uni} – , we provide a quantitative error estimate for this approximation of the order of $\varepsilon^{1/9-}$.

Previous stochastic homogenization results in the literature for forced mean curvature flow (4.1.2) and similar PDEs had been limited to strictly positive forcings $F > 0$: The first homogenization result for (4.1.2) in the random setting was obtained by Armstrong and Cardaliaguet [AC18], who required that $\inf_x F^2 - (d-1)|\nabla F| > 0$. In $d = 2$ dimensions, Feldman [Fel19] weakened this positivity assumption to $\inf_x F > 0$. Analogous results in the periodic setting had previously been derived by Lions and Souganidis [LS05b] and by Caffarelli and Monneau [CM14] respectively. In all of these cases, a consequence of the uniform lower bound on the forcing is the existence of a minimum speed for the expansion of sufficiently regular sets, and thus the absence of any impenetrable obstacles.

In the present paper, we allow for potentially negative forcing and hence actual impenetrable obstacles. This means that locally there is a non-zero probability that the interface gets stuck. In particular, some locations might never be reached. Hence, a pointwise comparison between the evolutions (4.1.2) and (4.1.3) is not possible. Instead, we compare the interface motions (4.1.2) and (4.1.3) with respect to arrival times of interfaces at some $x_0 \in \mathbb{R}^d$. In fact, for (4.1.2) we consider the arrival time in a neighborhood $\overline{B}_{h_\varepsilon}(x_0)$ to ensure

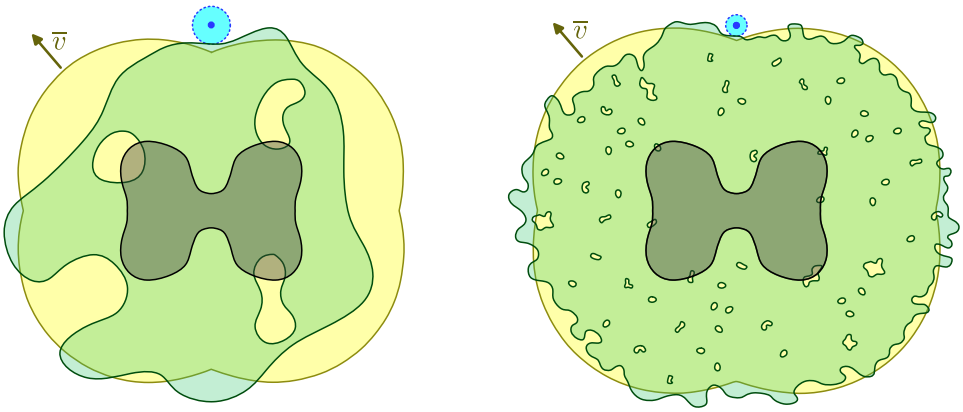


Figure 4.1.1: A comparison of the evolution by forced mean curvature flow in a field of random obstacles with the evolution according to a homogeneous first-order motion. The initial set is depicted in grey, the evolution by forced mean curvature flow in green, and the evolution by the first-order motion in yellow. The target area for which the arrival times are compared is depicted in blue. Left: The length scale of the initial set is of the same order as the range of dependence. Right: The length scale of the initial set is much larger than the range of dependence, and homogenization is observed

that the probability of the arrival time being infinity converges to 0 as $\varepsilon \to 0$. In addition, we truncate the arrival time to have a well defined expected value. The quantitative error estimate then consists of two parts:

- For the fluctuations of the truncated arrival time for (4.1.2) in $\overline{B}_{h_\varepsilon}(x_0)$ we obtain a bound of order $\varepsilon^{\frac{1}{2}-}$ with essentially exponential tail bounds.
- For the bias, the difference between the expected value and the arrival time for the homogenized evolution (4.1.3), we obtain a bound of order $\varepsilon^{\frac{1}{9}-}$.

Briefly summarizing our setting and assumptions, the coefficient field F is sampled from a stationary, isotropic probability measure $\mathbb{P}$ satisfying a finite range of dependence assumption. Regarding the regularity of the coefficient field, $F \in C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ is assumed to be uniformly bounded and uniformly Lipschitz.

Moreover, there are two less straightforward assumptions: First, we assume that actual obstacles are sufficiently sparse. That is, we require that for any sufficiently large and regular set with overwhelming probability there exists an interior approximation, which does not shrink under (4.1.1) and expands at an 'effective minimum speed' $v_{\min} > 0$ on large scales. Here, expanding at an 'effective minimum speed' $v_{\min} > 0$ on a fixed scale means that at any time $t \geq 0$ the evolved set grows at speed $v_{\min}$ up to left-behind holes of size below this scale, see Figure 4.3.2. In fact, this is a generalization of the assumptions on the forcing from [Fel19] and [AC18], since their assumptions guarantee that the original sets themselves are non-shrinking and expand with an actual minimum speed.

Second, we require that our candidate for the homogenized speed $\overline{v}$ is Hölder continuous at $\Delta F = 0$ with respect to changing the forcing by a constant from F to $F + \Delta F$. We will show that this assumption holds generically in the sense that $\Delta F \mapsto \overline{v}_{\Delta F}$ is increasing and hence Lipschitz continuous at almost every ΔF.

In order to illustrate our result, we first state it in the special case of sufficiently sparse Poisson-distributed obstacles of bounded diameter in dimension $d = 2$ as in [DDS11; CDO22]. In Chapter 5 we will rigorously prove that this setting satisfies the assumptions for the present chapter's main result. We further expect that the same holds in arbitrary dimension $d \geq 3$ and that these arguments should also be applicable to essentially

Gaussian obstacle fields, where F_{obst} arises from suitable local modifications of a Gaussian random field with finite range of dependence and where $F_{uni} > 0$ is sufficiently large.

4.1.1 An application of the main result: Poisson-distributed obstacles

Before we discuss the informal version of our main result and the technical notions necessary to state it, we present this application for the special case of potentially impenetrable obstacles distributed according to a Poisson point process in dimension $d = 2$ and a uniform forcing which is sufficiently large compared to the intensity of the point process. In Chapter 5, we will show that the requirements for the application of the results in the present chapter are satisfied. We expect that an analogous result holds in dimension $d \geq 3$ up to a different scaling of the lower bound for the uniform forcing.

Theorem 4.1.1 (Homogenization for Poisson-distributed obstacles)**.** *Consider Poisson-distributed obstacles in dimension $d = 2$ represented by*

$$F_{obst} \colon \mathbb{R}^2 \to \mathbb{R}, \quad F_{obst}(x) \coloneqq \max_{i \in \mathbb{N}} \varphi(|x - X_i|)$$

for

- *a Poisson point process $\{X_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$ with intensity $\rho > 0$, and*
- *a Lipschitz continuous obstacle shape $\varphi \in C^{0,1}(\mathbb{R})$ with support $\operatorname{supp} \varphi \subset \overline{B}_1$.*

There exist $\rho_0 > 0$ and $C_0 > 0$ independent of the obstacle shape φ such that – given any intensity $\rho \in (0, \rho_0]$ – for almost every $F_{uni} \geq C_0\sqrt{\rho}$ the interface motion given by

$$\text{Normal velocity} = \text{mean curvature} + F_{uni} - F_{obst}(x) \qquad \text{at } x \in \mathbb{R}^2 \tag{4.1.4}$$

is approximated by a first-order motion with constant positive speed $\overline{v} = \overline{v}(\rho, \varphi, F_{uni}) > 0$ in the following sense.

Let $x_0 \in \mathbb{R}^2$ and let $S \subset \mathbb{R}^2$ be a union of balls with radius $r > 0$. For $\varepsilon > 0$, let $(\mathscr{R}_t(\varepsilon^{-1}S))_{t \geq 0}$ be the set evolution of $\varepsilon^{-1}S$ corresponding to the interface motion (4.1.4), see Definition 4.1.2 below. Let $\vartheta \in (0, 1)$. Then the arrival time of the set evolution in the target area $\overline{B}_{\varepsilon^{-\vartheta}}(\varepsilon^{-1}x_0)$, given by

$$m_{\varepsilon^{-\vartheta}}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big) \coloneqq \min\big\{t \geq 0 \,:\, \overline{B}_{\varepsilon^{-\vartheta}}(\varepsilon^{-1}x_0) \cap \mathscr{R}_t\big(\varepsilon^{-1}S\big) \neq \emptyset\big\},$$

is described well by a constant-speed motion in the sense that

$$\mathbb{P}\Big[\big|\varepsilon m_{\varepsilon^{-\vartheta}}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big) - \overline{v}^{-1}\operatorname{dist}(x_0, S)\big| \geq C\varepsilon^{\frac{1-\vartheta}{9}}\Big] \leq C\exp\big(-C^{-1}\varepsilon^{-\alpha\vartheta}\big)$$

for the probability measure $\mathbb{P}$ associated to the Poisson point process and for some $\alpha > 0$ and some $C = C(\rho, \varphi, F_{uni}, r, \operatorname{dist}(x_0, S), \vartheta) > 0$.

Since ρ_0 and C_0 are independent of the obstacle shape, the shape can in particular be chosen such that the obstacles are impenetrable for a large range of uniform forcings $F_{uni} \geq C_0\sqrt{\rho}$. The scaling $\sqrt{\rho}$ of the lower bound for the uniform forcing corresponds to the Taylor scaling of the pinning to depinning transition (see Subsection 4.2.4) established by Courte, Dondl and Ortiz in [CDO22]. Note however that the assumption of an 'effective minimum speed' in our main result below is stronger than the existence of a ballistic evolution proven in [CDO22]. For a brief discussion of the Chapter 5 and its application to Poisson distributed obstacles see Example 4.3.12 and Example 4.3.13.

4.1.2 Set evolutions and arrival times

Next, we want to state the informal version of our main result, comparing the arrival times for the heterogeneous and homogenized geometric motions. In order to state this informal version, we first define these arrival times at points $x_0 \in \mathbb{R}^d$ or in target areas $\overline{B}_h(x_0) \in \mathbb{R}^d$.

We view the geometric motions as set evolutions with (4.1.1), (4.1.2) and (4.1.3) describing the speed at which the interface moves. To obtain well-defined set evolutions, we work with level-set formulations in the framework of viscosity solutions, see [Gig06] for a self-contained account of the theory and Appendix 4.A for a summary of the definitions and results used by us. In this framework, a set evolves according to for example (4.1.2) if it corresponds to the sublevel set of a function u^ε which in the case of (4.1.2) is a viscosity solution to

$$\partial_t u^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\right) D^2 u^\varepsilon\right) + F\left(\frac{x}{\varepsilon}, \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|}\right)|\nabla u^\varepsilon| = 0 \tag{4.1.5}$$

in $\mathbb{R}^d \times (0,\infty)$ with $A(e) = \mathrm{Id} - e \otimes e$ for $x \in \mathbb{R}^d$, $e \in \mathbb{S}^{d-1}$. More precisely, we work with the following notion of set evolutions.

For some partial results, we allow $A(\cdot,\cdot)$ to take a more general form and in particular to be space dependent, see Remark 4.1.8 below. This is why we state the definition of set evolutions for a larger class of coefficient fields corresponding to geometric motions.

Definition 4.1.2 (The set evolution in the heterogeneous environment)**.** Let $(A,F) \in \Omega$, with Ω denoting the admissible coefficient fields as specified in Section 4.3.1. Given a closed set $S \subset \mathbb{R}^d$, we define the evolution $(\mathscr{R}_t(S))_{t\geq 0}$ of S with respect to (A,F) as

$$\mathscr{R}_t(S) = \mathscr{R}_t^{(A,F)}(S) := \{x \in \mathbb{R}^d \,:\, u_S(x,t) \leq 0\},$$

where u_S with continuous initial data $u_S(\cdot,0)$ satisfying $S = \{x \in \mathbb{R}^d \,:\, u_S(x,0) \leq 0\}$ is a (viscosity) solution of

$$\partial_t u + G_{(A,F)}(x, \nabla u, D^2 u) = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty) \tag{4.1.6}$$

$$\text{with} \quad G_{(A,F)}(x,\xi,X) := -\operatorname{tr}\left(A\left(x, \frac{\xi}{|\xi|}\right)X\right) + F\left(x, \frac{\xi}{|\xi|}\right)|\xi| \tag{4.1.7}$$

for $(x,\xi,X) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Such a solution always exists and $\mathscr{R}_t(S)$ is independent of the specific initial data $u_S(\cdot,0)$. Further, we can assume that u_S is uniformly continuous on $\mathbb{R}^d \times (0,T)$ for any $T > 0$, see Theorem 4.A.14, Remark 4.A.15, Theorem 4.A.17.

Remark 4.1.3 (Hyperbolic scaling regime)*.* The scaling of $\varepsilon \to 0$ in (4.1.2) and (4.1.5) corresponds to zooming out and accelerating time by a factor of ε^{-1}. The evolution of a set S with respect to these equations is hence given by $t \mapsto \varepsilon \mathscr{R}^{(A,F)}_{\varepsilon^{-1}t}(\varepsilon^{-1}S)$.

Remark 4.1.4 (Selection criterion for fattening)*.* Instead of a set evolution, one might hope to directly obtain an interface evolution from (4.1.6) with the initial hypersurface given by $\{x \in \mathbb{R}^d \,:\, u_S(x,0) = 0\}$ with Hausdorff-dimension $d-1$. However, the level set $\{x \in \mathbb{R}^d \,:\, u_S(x,t) = 0\}$ might at some point develop a non-empty interior, a phenomenon which is called fattening [BSS93]. In this case, there can be multiple interfaces within the level set for which we could say that they evolve by (4.1.1). Our choice for $\mathscr{R}_t(S)$ implies that the interface $\partial\mathscr{R}_t(S)$ is the fastest or outermost of the interfaces within the level set.

Definition 4.1.5 (The homogenized set evolution)**.** Let $v \in C(\mathbb{S}^{d-1}, \mathbb{R}_{>0})$. Given a closed set $S \subset \mathbb{R}^d$, we define

$$\mathscr{R}_t^{hom,v}(S) := \{x \in \mathbb{R}^d \,:\, u_S^{hom}(x,t) \leq 0\}$$

where u_S^{hom} with $S = \{x \in \mathbb{R}^d : u_S^{hom}(x,0) \leq 0\}$ is a (viscosity) solution of

$$\partial_t u^{hom} + v\left(\frac{\nabla u^{hom}}{|\nabla u^{hom}|}\right)|\nabla u^{hom}| = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty). \tag{4.1.8}$$

As stated above, we compare the set evolutions $(\mathscr{R}_t^{(A,F)}(S))_{t\geq 0}$ and $(\mathscr{R}_t^{hom,\overline{v}}(S))_{t\geq 0}$ in terms of their arrival times at appropriately scaled target areas.

Definition 4.1.6 (Arrival times)**.** For admissible coefficient fields (A,F), $x_0 \in \mathbb{R}^d$ and a closed set $S \subset \mathbb{R}^d$ we define the (actual) *arrival time*

$$m(x_0,S) = m^{(A,F)}(x_0,S) := \min\Big\{t \geq 0 : x_0 \in \mathscr{R}_t^{(A,F)}(S)\Big\}.$$

We further denote the *truncated arrival time in the target area* $\overline{B}_h(x_0)$ for $h > 0$ with

$$m_h(x_0,S) = m_h^{(A,F)}(x_0,S) := \min\Big\{\min_{y\in\overline{B}_h(x_0)} m^{(A,F)}(y,S),\, T_h(x_0,S)\Big\}.$$

Here, $T_h(x_0,S)$ is the time for which with high probability we will have a guarantee that the set evolution has reached the target area $\overline{B}_h(x_0)$, given by

$$T_h(x_0,S) := v_{\min}^{-1}\operatorname{dist}(x_0,S) + C(\text{data})h.$$

The guarantee is due to the assumption on the existence of the effective minimum speed $v_{\min}$ from Assumption '(P⊙)' below, see Lemma 4.5.7.

Considering the homogenized set evolutions, for $v \in C(\mathbb{S}^{d-1},\mathbb{R}_{>0})$ we set

$$m^{hom,v}(x_0,S) := \min\Big\{t \geq 0 : x_0 \in \mathscr{R}_t^{hom,v}(S)\Big\}.$$

4.1.3 The informal main result

Before we state the informal version of our main theorem, we provide an informal version of the key assumption, which effectively replaces and relaxes the deterministic, uniform bounds on the forcing required in [AC18] or [Fel19]. For the rigorous formulation see Assumption (P⊙) in Subsection 4.3.2 and Subsection 4.3.3, where the respective notions are made precise.

'(P⊙)' **Almost guaranteed effective minimum speed**. We assume that there exist $v_{\min}, C > 0$, $h_0 \geq 1$ and a rate $\alpha_\odot > 0$ such that for all $h \geq h_0$ and $M \subset \mathbb{R}^d$

$$\begin{aligned}&\mathbb{P}\Big[\big\{\text{In } M\text{, on the scale } h\text{, every 'fat' set has a 'stable approximation',}\\ &\qquad\qquad\text{which propagates with an 'effective minimum speed' of } v_{\min}\big\}\Big]\\ &\geq 1 - C\big(1+\operatorname{diam}(M)\big)^d \exp\big(-C^{-1}h^{\alpha_\odot}\big),\end{aligned}$$

where 'fat' sets $S \subset \mathbb{R}^d$ are unions of sufficiently large balls and 'stable approximations' are non-shrinking interior approximations $S_h \subset S$ (see Figure 4.3.3). An 'effective minimum speed' on the scale h means that for any $t \geq 0$ the evolved set $\mathscr{R}_t(S_h)$ after another time step of $v_{\min}^{-1}h$ has advanced by at least h in all directions while allowing for enclosures of width smaller than $2h$ to be left behind the main front (see Figure 4.3.2).

Up to replacing this assumption with its rigorous formulation stated in Assumption (P⊙), the following informal version of our main result is a precise statement.

Theorem 4.1.7 (Informal main result)**.** *Let $F_{obst} = F_{obst}(x,e)$ be a stationary random field with a finite range of dependence, which is uniformly bounded and Lipschitz. Let $F_{uni} \in \mathbb{R}$ be a uniform forcing such that $F(x,e) = F_{uni} - F_{obst}(x,e)$ satisfies Assumption '(P⊙)'. Let $x_0 \in \mathbb{R}^d$ and let $S \subset \mathbb{R}^d$ be a fat set with bounded boundary.*

Then the truncated arrival time in the target area $\overline{B}_{h_\varepsilon}(\varepsilon^{-1}x_0)$ for the set evolution $(\mathscr{R}_t(\varepsilon^{-1}S))_{t\geq 0}$ corresponding to (4.1.1) with $h_\varepsilon = \varepsilon^{-\vartheta}$ for arbitrary $0 < \vartheta < 1$ has fluctuations of order at most $\varepsilon^{\frac{1-\vartheta}{2}}$ with essentially exponential tail bounds, that is

$$\mathbb{P}\Big[\big|\varepsilon m_{h_\varepsilon}\big(\varepsilon^{-1}x_0,\varepsilon^{-1}S\big) - \mathbb{E}\big[\varepsilon m_{h_\varepsilon}\big(\varepsilon^{-1}x_0,\varepsilon^{-1}S\big)\big]\big| \geq C\varepsilon^{\frac{1-\vartheta}{2}}\lambda\Big] \leq \max\Big\{C\exp\big(-\lambda\big), C\exp\Big(-\frac{\varepsilon^{-\vartheta\alpha_\odot}}{C\log(\varepsilon^{-1})}\Big)\Big\} \tag{4.1.9}$$

for $\alpha_\odot$ from Assumption '(P⊙)' and some constant $C > 0$ independent of ε.

Moreover, if F_{obst} is sampled from an isotropic probability distribution, then for Lebesgue-almost every $F_{uni} \in \mathbb{R}$ with $F(x,e) = F_{uni} - F_{obst}(x,e)$ satisfying Assumption '(P⊙)' there is a homogenized speed $\overline{v} \geq v_\infty$ such that

$$\big|\mathbb{E}\big[\varepsilon m_{h_\varepsilon}\big(\varepsilon^{-1}x_0,\varepsilon^{-1}S\big)\big] - m^{hom,\overline{v}}\big(x_0,S\big)\big| \leq C\varepsilon^{\frac{1-\vartheta}{9}}.$$

Remark 4.1.8. In fact, we obtain the fluctuation bounds in (4.1.9) also for space-dependent, anisotropic curvature terms, that is for interface motions with the normal velocity V at $x \in \mathbb{R}^d$ given in terms of the normal $\nu \in \mathbb{S}^{d-1}$ by

$$V = \operatorname{tr}\big(a(x,\nu)\nabla\nu\big) + F\big(x,\nu\big)$$

for a positive semi-definite coefficient field $a(x,e) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. In terms of the level-set formulation, this corresponds to (4.1.6) with $A(x,e) = (\mathrm{Id} - e\otimes e)a(x,e)(\mathrm{Id} - e\otimes e)$.

4.1.4 Structure of the paper

In Section 4.2, we first collect related literature and then compare our result and the strategy of our proofs to [AC18]. In Section 4.3, we lay out our assumptions and rigorously introduce the notions used in Assumption '(P⊙)'. In Section 4.4, we state our main results and then describe the strategy of the overall proof in detail. In Section 4.5, we collect frequently used basic properties of the set evolutions and stable sets, as well as provide a measurability criterion, with their proofs given in Appendix 4.B. The major parts of the proof for the main results are presented in Section 4.6, Section 4.7, Section 4.8, Section 4.9, and Appendix 4.C. A more detailed description of these Sections is given at the start of Subsection 4.4.1.

4.2 Related literature

In this section we will discuss some of the literature which is related to our setting. In Subsection 4.2.1, we provide motivation to study curvature-driven interface motions in random environments based on applications in physics. In Subsection 4.2.2, we review homogenization results for geometric motions in periodic environments. In Subsection 4.2.3, we review the known homogenization results for geometric motions in random

environments, including some of the work on Hamilton–Jacobi equations which these results are based on. In Subsection 4.2.4, we discuss results on pinning and de-pinning phenomena for geometric motions in random obstacle fields. In Subsection 4.2.5, we discuss some results from first passage percolation, which can be viewed as an edge case of the type of problem we study. Finally, in Subsection 4.2.6 we compare the methods of our proof to those used in the existing literature on the stochastic homogenization of interface motions, in particular in [AC18].

4.2.1 Physical motivation

Equation (4.1.1) describes the motion of an interface under the combined effects of surface tension – represented by the mean curvature – and a forcing field. The forcing field can be thought of as the combination of a uniform external driving field and the effects of the random spatially heterogeneous environment including obstacles. Models of type (4.1.1) are used to describe the evolution of dislocation lines in a slip plane [IV87; BN04] ($d = 2$), and the evolution of domain boundaries in magnetic materials with impurities [IV87; Kar98; BN04], in the random-field Ising model [KL85; Kar98], or in a simplified version of immiscible-fluid displacement in porous media [KL85; Kar98] ($d = 2, 3$).

In addition, there are phenomenologically related models for charge density waves [FL78; IV87; Kar98; BN04], contact lines [Gen85; Kar98], Wigner crystals [NS00; BN04], and vertex (or flux) lines in type-II superconductors [Kar98; NS00; BN04].

4.2.2 Results for geometric motions in periodic environments

A detailed survey on the homogenization of surfaces in periodic media up until 2013 can be found in [Caf13]. In the periodic (and almost periodic) setting, Lions and Souganidis [LS05b] have provided a qualitative homogenization result for a class of second-order degenerate PDE in $\mathbb{R}^d$, which includes the level set formulation (4.1.5). The coefficient fields are required to satisfy a technical assumption which in our setting corresponds to

$$\inf_{x\in\mathbb{R}^d} \left(F(x)^2 - (d-1)|A||DF(x)|\right) > 0 \tag{4.2.1}$$

and implies that for sufficiently large and regular sets the arrival times are uniformly Lipschitz continuous, which corresponds to a uniform minimum speed for the set evolutions. Caffarelli and Monneau [CM14] have shown that for $d = 2$ this assumption can be relaxed to $\inf_x F(x) > 0$, which yields a minimum speed on large scales. In addition, a counterexample was provided showing that homogenization might fail in $d \geq 3$ for the same relaxation. Gao and Kim [GK19] have shown that for $d \geq 3$, under the assumption of a strictly positive forcing with $\inf_x F(x) > 0$, for each direction there is a head and a tail speed for the expansion of sets. For the counterexample from [CM14], head and tail speed differ, which is the reason why there can be no homogenization.

For signed, but very small forcing compared to the strength of the curvature, it was shown by Dirr, Karali, and Yip [DKY08] that surfaces initialized as a plane stay within a fixed distance of a plane which moves with a fixed velocity. For signed forcing, pinning and de-pinning phenomena occur when adding a constant F_{uni} to the forcing, see Subsection 4.2.4 for a brief introduction. In the work by Dirr and Yip [DY06] it was shown for approximations of forced mean curvature flow in the graph setting that homogenization occurs and further that the homogenized speed scales like $(F_{uni} - F_*)^{\frac{1}{2}}$ close to the de-pinning threshold F_*.

4.2.3 Results for Hamilton–Jacobi equations in random environments with geometric motions as a special case

The level set formulation (4.1.6) of the set evolution is a viscous Hamilton–Jacobi equation with a geometric and hence degenerate diffusion matrix and a 1-homogeneous, non-coercive Hamiltonian. Our setting corresponds to a hyperbolic scaling regime with the strength of the curvature proportional to the average distance between obstacles. We limit our summary to results on stochastic homogenization for Hamilton–Jacobi equations which can be interpreted as the level set formulation for geometric motions and an overview of the works which led to these results.

For first order Hamilton–Jacobi equations, the first qualitative homogenization results were obtained in [RT00; Sou99] for convex Hamiltonians assuming just ergodicity. Qualitative results for viscous, that is semilinear second-order equations were first proven by Lions and Souganidis [LS05a] and Kosygina, Rezakhanlou, and Varadhan [KRV06]. Extensions to larger classes of Hamilton–Jacobi equations and new proofs followed [LS10; Sch09; AS12a; AS12b] with many more since.

However, there have been no results for Hamilton–Jacobi equations corresponding to curvature-driven geometric motions with general ergodicity assumptions for the random environment. The first stochastic homogenization result for geometric motions was obtained for random environments with a finite range of dependence: It is included as a special case in the last paper of a series of works by Armstrong, Cardaliaguet, and Souganidis [ACS14] and by Armstrong and Cardaliaguet [AC14; AC18] on the quantitative homogenization of increasingly larger classes of Hamilton–Jacobi equations. As in the work [LS05b] on periodic environments, the coefficient fields are assumed to satisfy a technical assumption, which in our setting corresponds to (4.2.1), implying a guaranteed minimum speed of propagation. Feldman [Fel19] has shown for forced mean curvature flow in $d = 2$ that this technical assumption can be relaxed to $\inf_x F(x) > 0$, which again implies a minimum speed on large scales as in [CM14] for the periodic case. Our arguments also cover the setting from [Fel19] and the case of geometric motions from [AC18] with the additional assumption of statistical isotropy, substantially improving the convergence rates. However, the main novel feature of our work is that we allow for actual obstacles, which might locally pin the interface and thus cause enclosures left behind a main front. Compared to the existing homogenization results in the literature, one can thus view our setting to be much closer to the de-pinning threshold, see Subsection 4.2.4 below.

4.2.4 Pinning and de-pinning phenomena in random environments

For curvature driven geometric motions through random obstacle fields, there has been a lot of interest in pinning and de-pinning phenomena. The forcing field is often written as $F(\cdot) = F_{uni} - F_{obst}(\cdot)$ with a uniform forcing $F_{uni} \geq 0$ and a random field $F_{obst}(\cdot) \geq 0$ encoding the environment and obstacles. For a fixed obstacle field F_{obst}, the choice of F_{uni} can lead to qualitatively very different behavior. First, there is the pinning regime $0 \leq F_{uni} \leq F_*$, where F_* is the maximal forcing for which there exists an interface above a given hyper-plane which corresponds to a stationary subsolution (in the sign-convention we use for our level-set formulation). By comparison principle, this interface is a barrier for the evolution of any interface initialized below. Second, there is the ballistic regime $F_{uni} > F^*$, where F^* is the infimum over all forcings such that the hyper-plane expands according to some minimum speed, that is that the expected height of the main front above each point of the hyperplane grows linear in time. Due to ergodicity, these critical

forcings are deterministic. While $F_* \leq F^*$, in general it is not clear that $F_* = F^*$: There might be a so-called sub-ballistic regime inbetween, where the interfaces advance only at a sub-linear pace. Clearly our Assumption (P⊙) can hold only in the ballistic regime, but it is unclear whether it is satisfied for all $F_{uni} > F^*$.

In the prototypical case, $F_{obst}(\cdot)$ corresponds to obstacles of sufficient strength which are distributed according to a Poisson point process. Alternatively, the obstacles can be placed on a grid and only the obstacle strength is random. Many results on pinning and de-pinning have been derived for the quenched Edwards–Wilkinson model, which is a linearized version of forced mean curvature flow in a graph setting and hence does not develop topological singularities. For this model or (semi-)discrete versions, under varying assumptions on the obstacle distribution it has been shown that it is possible to have $F_* > 0$ [DDS11; DJS21], $F_* < +\infty$ [CDL10], $F^* < +\infty$ [DS12; BT14; DS17], $F_* = F^*$ [BT14], $F_* < F^*$ [DS17], or $F_* = +\infty$ [DJS22].

For forced mean curvature flow as in (4.1.1) with obstacles distributed according to a Poisson point process, it was shown by Dirr, Dondl, and Scheutzow [DDS11] for $d \geq 2$ that $F_* > 0$. The proof relies on a percolation result proven in [Dir+10]. In the recent work by Courte, Dondl, and Ortiz [CDO22], it was shown in $d = 2$ also for more general interface motions that the estimate $c\sqrt{\rho} \leq F_* \leq F^* \leq C\sqrt{\rho}$ holds, where $c, C > 0$ are constants and where $\rho > 0$ denotes the intensity of the Poisson point process. Note that $\sqrt{\rho}$ is proportional to the average obstacle distance.

4.2.5 Related results from first passage percolation

Our setting can be seen as a generalized version of first passage percolation. The theory of first passage percolation studies the large-scale behavior of the passage time $T(x, y)$ between two points in a random environment, that is the optimal cost among all paths connecting the two points. For a detailed survey of the theory up until 2015 see [ADH17]. The most common choice for the underlying space is the lattice $\mathbb{Z}^d$ with a random cost or time associated to the traversal of each edge.

In our setting, for $A = 0$ and direction independent forcing $F(x, \xi) = F(x)$, the arrival times from Definition 4.1.6 correspond to the passage times in Riemannian first passage percolation on the continuum as introduced in [LW10]. For a description of this connection in the discrete setting see [ADH17, Section 2.2] . In particular, the scaling of the fluctuation bounds from Theorem 4.4.1 and the convergence of the mean in Theorem 4.4.2 in this general setting can not be better than the best possible bounds for first passage percolation.

It has been predicted that the fluctuations of the passage times $T(0, \varepsilon^{-1}e)$ scale like $\sim \varepsilon^{-\frac{1}{3}}$ for $d = 2$ and that they decrease further for higher dimensions. Proving this without some stronger, unproven additional hypothesis is still open. Very recently, the best upper bound has been improved to $\lesssim \varepsilon^{-\frac{1}{2}+\delta}$ for $d = 2$ in a statistically isotropic environment by Basu, Sidoravicius, and Sly [BSS26]. For general dimension $d \geq 2$, the best upper bound is $\lesssim \frac{\varepsilon^{-\frac{1}{2}}}{\log(\varepsilon^{-1})}$ for $d \geq 2$ which has first been achieved for Bernoulli distributed edge costs by Benjamini, Kalai, and Schramm [BKS03] and was later extended to more general settings. The best prior bound was $\lesssim \varepsilon^{-\frac{1}{2}}$ by Kesten [Kes93], for the proof of which the martingale method of bounded increments was used. This proof together with the martingale structure used in [Zha10] inspired the fluctuation estimates for the metric problem in the stochastic homogenization of Hamilton–Jacobi equations [ACS14; AC14; AC18; Fel19] and thus in turn our proof of Theorem 4.4.1.

Regarding the rate of convergence to the mean, Alexander [Ale97] showed that

$$\left|\mathbb{E}\left[T(0,\varepsilon^{-1}e)\right]-\varepsilon^{-1}\mu(e)\right|\lesssim\left(\varepsilon^{-1}\log\left(\varepsilon^{-1}\right)\right)^{\frac{1}{2}}$$

for all $e\in\mathbb{S}^{d-1}$. The existence of the limit $\mu(e)$ had already been established beforehand in the form of shape theorems, see [ADH17] for an overview.

For first passage percolation on a graph, our setting with potentially impenetrable areas corresponds to potentially closed edges and therefore first passage percolation on an infinite percolation cluster. Shape theorems [GM04; Mou11; CT16] and fluctuation bounds [GM07; GM09; Dem24] for the passage times, also called 'chemical distance' in this setting, rely on an upper bound for passage times which holds on large scales with exponentially increasing probability [AP96]. Together with control of the size of holes in the infinite cluster, this lower bound corresponds to the 'effective minimum speed' which we require in Assumption (P$\circledcirc$). Note that these fluctuation bounds are proven with techniques based on Poincaré or log-Sobolev inequalities respectively which require to bound the impact of locally re-sampling the coefficient field and rely on localization properties which are not available in our setting, such as, e.g., that an increase of the passage time is very unlikely since it can only occur if an area crossed by the minimizing path is re-sampled. In [CD22], it has been shown that the limit speed $\mu(e)^{-1}$ is Lipschitz continuous in the probability that an individual edge is open, which can be compared to our Assumption (Pη).

4.2.6 Comparison to [AC18]

Even though the work by Armstrong and Cardaliaguet [AC18] also treats geometric motions, we use quite a different language. This is due to both regularity issues in our setting and the fact that they are treating the level-set equation for forced mean curvature flow as a special case of more general Hamilton–Jacobi equations rather than just those corresponding to geometric motions. The basic building block in [AC18] is the metric problem, whose solutions for geometric motions would be exactly our arrival times. To be more precise, for $\mu>0$ and a set $S\subset\mathbb{R}^d$ the metric problem is given by

$$\begin{cases}-\operatorname{tr}\left(A\left(x,\frac{\nabla\widetilde{m}_\mu}{|\nabla\widetilde{m}_\mu|}\right)D^2\widetilde{m}_\mu\right)+F\left(x,\frac{\nabla\widetilde{m}_\mu}{|\nabla\widetilde{m}_\mu|}\right)|\nabla\widetilde{m}_\mu|=\mu & \text{in } \mathbb{R}^d\setminus S,\\ \widetilde{m}_\mu=0 & \text{on } S.\end{cases}\tag{4.2.2}$$

Here, since the underlying Hamilton–Jacobi equation corresponds to a geometric motion, we have $\widetilde{m}:=\widetilde{m}_1=\mu^{-1}\widetilde{m}_\mu$ for all $\mu>0$. If a solution $\widetilde{m}$ to the metric problem existed, then $(x,t)\mapsto\max\{0,\,-t+\widetilde{m}(x)\}$ would be a solution to the level-set equation (4.1.6) with the initial 0-sublevel set given by S. In particular, our arrival time for S at some $x_0\in\mathbb{R}^d$ would be given by $m(x_0,S)=\widetilde{m}(x_0)$. However, in our setting the metric problem is ill-posed because our arrival times might attain infinity as a value since large areas might never be reached by the set evolution.

Translating the properties of the metric problem from [AC18], one of their central assumptions is that all realizations of F satisfy an inequality which implies the Lipschitz continuity of the metric problem for sufficiently 'fat' sets S. In our language this corresponds to a guarantee that these sets expand at a general minimum speed and hence the absence of any actual obstacles. They heavily use this property both on large scales for the general concept of the proofs and on small scales for specific regularity issues. With our potentially negative forcing, there is no guaranteed minimum speed. In addition, it might happen that the front is not strictly expanding but might shrink locally. Both of

these possibilities potentially damage the effective independence between the 'past' and 'future' of the evolution, which is central in [AC18]. In order to salvage this property, we introduce a suitable notion of 'stability', preventing shrinkage on large scales, as well as the notion of an 'effective minimum speed of propagation' on large scales, ignoring enclosures behind the main front. While the 'stability' property and the 'effective minimum speed' are not guaranteed to be applicable, we can however reasonably assume that on large scales they are very likely to be applicable.

Applied on large enough scales, we use these two notions to obtain the fluctuation bounds for the truncated arrival times in a similar fashion to the fluctuation bounds for the metric problem in [AC18]: We also use the method of bounded martingale differences, writing the truncated arrival time as the endpoint of a suitably defined martingale. However, in addition to significant technical adaptions due to the lack of an actual minimum speed, we obtain bounds for the martingale differences not almost surely, but just with high probability since our 'effective minimum speed of propagation' is only applicable with high probability. Despite this, we are still able to obtain good fluctuation bounds by introducing a suitable replacement for Azuma's inequality.

On small scales we have much worse regularity properties assuming an effective instead of an actual minimum speed. This has two major consequences: First, we only obtain an 'approximate finite speed of propagation of perturbations' property corresponding to [AC18, Proposition 4.2] at the cost of increasing the forcing by a small constant, that is replacing (A, F) with $(A, F + \Delta F)$ for some small $\Delta F > 0$. To obtain the overall quantitative homogenization result, we hence have to introduce an assumption on the Hölder continuity of the homogenized speed with respect to these forcing modifications. Second, since the metric problem is ill-posed in our setting, we are not able to reproduce the classical homogenization argument for these kind of problems, which would be to use the perturbed test function method to lift the homogenization bounds from the metric problem first to an approximate corrector equation and then to the level-set equation. Instead, we use a more constructive approach specific to geometric motions, which is not applicable for the general viscous Hamilton–Jacobi equations treated in [AC18].

4.3 Setting and assumptions

For convenience, we denote dependencies on the parameters introduced with the assumptions below via

$$\text{data} := \big(d, C_{1A}, C_{1F}, v_{\min}, h_0, \alpha_\circledcirc, c_\circledcirc, C_\circledcirc, c_{\mathcal{F}}, c_s, \eta, C^*\big).$$

4.3.1 Admissible coefficient fields

We say that (A, F) is an admissible coefficient field if the following conditions (4.3.1), (4.3.2) and (4.3.3) are satisfied. A is assumed to be continuously differentiable, degenerate elliptic and uniformly bounded and Lipschitz, that is there exist $\sigma \in C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1}; \mathbb{R}^{d\times d}_{\text{sym}})$ and $C_{1A} > 0$ such that

$$A = \sigma\sigma^\intercal \quad \text{with } \big\|\sigma\big\|_{C^{0,1}(\mathbb{R}^d\times\mathbb{S}^{d-1};\mathbb{R}^{d\times d}_{\text{sym}})}, \big\|A\big\|_{C^{0,1}(\mathbb{R}^d\times\mathbb{S}^{d-1};\mathbb{R}^{d\times d}_{\text{sym}})} \le C_{1A}. \tag{4.3.1}$$

To guarantee that (4.1.5) is the level-set formulation of a surface motion, we require

$$A(x,e)\big(\mathrm{Id}_{d\times d} - e\otimes e\big) = A(x,e) \text{ for all } e \in \mathbb{S}^{d-1},\ x \in \mathbb{R}^d. \tag{4.3.2}$$

F is also assumed to be uniformly bounded and Lipschitz, that is there exists $C_{1F} > 0$ such that

$$F \in C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1};\, \mathbb{R}) \text{ with } \|F\|_{C^{0,1}(\mathbb{R}^d \times \mathbb{S}^{d-1};\, \mathbb{R})} \leq C_{1F}. \tag{4.3.3}$$

These conditions on (A, F) guarantee that the theory for surface evolution equations is applicable to (4.1.6), see Appendix 4.A for a summary of the results relevant to us. In addition, the uniform bounds will be used in some later sections. To simplify the notation we use the 0-homogeneous extension of A and F to $\mathbb{R}^d \times \big(\mathbb{R}^d \setminus \{0\}\big)$, for A given by $A(x, \xi) := A\big(x, |\xi|^{-1}\xi\big)$.

4.3.2 Assumptions on the probability distribution

In short, we assume that the coefficient field (A, F) in (4.1.6) is a random variable sampled from a stationary probability measure with finite range of dependence. Some of the later results we are only able to prove under two additional assumptions: The assumption of A being deterministic and the assumption of isotropy of the probability measure. Finally, we introduce two non-standard assumptions: First, one of our key assumptions is that for 'fat' sets with overwhelming probability there are 'stable approximations', for which an 'effective minimum speed' is applicable. Second, we require that our candidate for the homogenized speed, which is well-defined due to the previous assumptions, is Hölder continuous with respect to changing the forcing F by a constant.

More explicitly, the probability space is defined as the set of admissible coefficient fields

$$\Omega := \big\{(A, F) \,:\, A \text{ satisfies (4.3.1) and (4.3.2)},\ F \text{ satisfies (4.3.3)}\big\}. \tag{4.3.4}$$

On Ω we define the σ-algebras $\{\mathcal{F}(U)\}$ for all Borel sets $U \subset \mathbb{R}^d$,

$$\mathcal{F}(U) := \text{the } \sigma\text{-algebra generated by the family of maps}$$
$$\Omega \to \mathbb{R}^{d\times d} \times \mathbb{R},\ (A, F) \mapsto \big(A(x, e), F(x, e)\big) \quad \text{with } x \in U,\ e \in \mathbb{S}^{d-1}. \tag{4.3.5}$$

The σ-algebras $\mathcal{F}(U)$ can be seen as containing all of the knowledge which can be gained by observing the coefficient field on $U \subset \mathbb{R}^d$.

We assume that the randomness of the coefficient field is described by a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(\mathbb{R}^d))$, which satisfies the following conditions. For some of our results, only a subset of these assumptions will be required. First, we have two classical key assumptions from the theory of stochastic homogenization:

(P1) **Stationarity.** $\mathbb{P}$ is invariant under translations of the coefficient field:

$$\mathbb{P} = \mathbb{P} \circ \tau_x, \quad \text{for any } x \in \mathbb{R}^d,$$

where $\tau_x(\widetilde{\Omega}) = \{(A(\cdot + x, \cdot), F(\cdot + x, \cdot)) \,:\, (A, F) \in \widetilde{\Omega}\}$ for $\widetilde{\Omega} \in \mathcal{F}(\mathbb{R}^d)$.

(P2) **Finite range of dependence.** For all Borel sets $U, V \subset \mathbb{R}^d$ with $\operatorname{dist}(U, V) \geq 1$

$$\mathcal{F}(U) \text{ and } \mathcal{F}(V) \text{ are } \mathbb{P}\text{-independent.}$$

Regarding the two additional assumptions, we require that the following holds:

(P3) **Spatially constant and thus deterministic second order-term.** There holds

$$\mathbb{P}\big[(A, F) \in \Omega \,:\, A(x, e) = A(0, e) \text{ for all } x \in \mathbb{R}^d, e \in \mathbb{S}^{d-1}\big] = 1.$$

(P4) **Isotropy.** $\mathbb{P}$ is invariant under rotations, that is there exists a non-negative constant $a \geq 0$ such that

$$\mathbb{P}\big[(A,F) \in \Omega \,:\, A(x,e) = a\big(\mathrm{Id}_{d\times d} - e\otimes e\big) \text{ for all } x \in \mathbb{R}^d, e \in \mathbb{S}^{d-1}\big] = 1,$$

and $\mathrm{Law}(F(\cdot,\cdot)) = \mathrm{Law}\, F(R\cdot, R\cdot)$ for all $R \in \mathrm{SO}(d)$.

Regarding our first non-standard assumption, we require that on large scales there is an almost guaranteed effective minimum speed for the evolution of sets.

(P$\circledcirc$) **Almost guaranteed effective minimum speed.** There is $v_{\min} > 0$, $h_0 \geq 1$ and a rate $\alpha_\circledcirc > 0$ with constants $c_\circledcirc, C_\circledcirc, c_{\mathcal{F}}, c_s > 0$, such that for each $h \geq h_0$ there is a family of events $(E_h^\circledcirc(M))_{M\subset\mathbb{R}^d}$ with

$$\mathbb{P}[E_h^\circledcirc(M)] \geq 1 - C_\circledcirc\left(1 + \frac{\mathrm{diam}(M)}{h}\right)^d \exp(-c_\circledcirc h^{\alpha_\circledcirc}), \tag{4.3.6}$$

which satisfies

(∗i) $E_h^\circledcirc(M)$ is measurable with respect to $\mathcal{F}(M + c_{\mathcal{F}}\overline{B}_h)$,

(∗ii) for all $M_1 \subset M_2$ there holds $E_h^\circledcirc(M_1) \supset E_h^\circledcirc(M_2)$,

(∗iii) for all $(A,F) \in E_h^\circledcirc(M)$, every h-fat set S with $\partial S \subset M$ has a stable (h, c_s)-approximation, for which (A,F) admits $v_{\min} > 0$ as an effective minimum speed of propagation on the scale h in $M \subset \mathbb{R}^d$, see Definition 4.3.8 below for the precise introduction of these notions.

Remark 4.3.1. We do not explicitly define $E_h^\circledcirc(M)$ directly as the set such that (∗iii) holds, because then in general we would not have measurability as in (∗i).

Regarding our second non-standard assumption, we require Hölder continuity of the homogenized speed with respect to changing the forcing by a constant $\Delta F \in \mathbb{R}$. Note that generically this assumption is satisfied, see Remark 4.3.2 below. We want to choose this homogenized speed as the large scale limit of the average speed corresponding to truncated arrival times for half-spaces. Since we are not able to guarantee the existence of this limit under just the previous assumptions, we approximate the half-spaces with bounded sets as in Figure 4.8.1 and define the homogenized speed for the forcing $F + \Delta F$ as

$$\Delta F \mapsto \overline{v}_{\Delta F,\beta}(e) := \left(\lim_{r\to\infty} \frac{1}{r}\mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big]\right)^{-1},$$

where $r \mapsto S^-_{e,\beta,h(r)}(r)$ are disks of radius $\sim r^\beta$ with $\beta \geq \frac{3}{2}$ and thickness $\sim h(r) \ll r$ approximating the boundary of the half space $\mathcal{H}_e^- = \{x \in \mathbb{R}^d \,:\, x\cdot e \leq 0\}$ for $e \in \mathbb{S}^{d-1}$, see Figure 4.8.1 and Corollary 4.8.1 for the precise definition. In Section 4.8 we will prove the existence of these limits for all β and all $\Delta F \in \mathbb{R}$ if Assumptions (P1), (P2), (P3) and (P$\circledcirc$) hold for the random coefficient field $(A, F + \Delta F)$. Assumption (P4) further guarantees that these quantities are the same for all $e \in \mathbb{S}^{d-1}$. We assume that the following holds.

(Pη) **Hölder-continuity of the homogenized speed with respect to changing the forcing by a constant.** There is $0 < \eta \leq 1$ and a width scaling $\beta^* > \frac{3}{2}$ for the half-space approximations $\left(S^-_{e,\beta^*,h(r)}(r)\right)_{e\in\mathbb{S}^{d-1},r>0}$ defined in Corollary 4.8.1, such that for all $e \in \mathbb{S}^{d-1}$ the function

$$\Delta F \mapsto \overline{v}_{\Delta F}(e) := \overline{v}_{\Delta F,\beta^*}(e) = \left(\lim_{r\to\infty} \frac{1}{r}\mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta^*,h(r)}(r)\Big)\Big]\right)^{-1}$$

is Hölder-continuous at $\Delta F = 0$ with Hölder coefficient η. That is, there exists $C^* > 0$ such that

$$\left|\overline{v}_{\Delta F}(e) - \overline{v}_0(e)\right| \leq C^* \left|\Delta F\right|^\eta \text{ for all } \Delta F \in \left[-\frac{1}{C^*}, \frac{1}{C^*}\right] \text{ for which } \overline{v}_{\Delta F}(e) \text{ exists.}$$

For the homogenized speed given the unchanged forcing we write $\overline{v} = \overline{v}_0$.

Remark 4.3.2 (On Assumption (Pη)). 1. We will show in Lemma 4.8.7 that the width scaling β^* of the half-space approximations is arbitrary: If all the previous assumptions hold, then Assumption (Pη) implies that the function $\Delta F \mapsto \overline{v}_{\Delta F,\beta}(e)$ is η-Hölder continuous at $\Delta F = 0$ with the same constant C^* and $\overline{v}_{0,\beta}(e) = \overline{v}_{0,\beta^*}(e) = \overline{v}$ for all $\beta > \frac{3}{2}$.

2. Assumption (Pη) can be expected to hold generically in the following sense: Since by the comparison principle $\Delta F \mapsto \overline{v}_{\Delta F}$ is a nondecreasing function, it is Lipschitz-continuous at Lebesgue-almost every ΔF.

3. Assumption (Pη) is satisfied if there is an actual minimum speed as for example in the setting of [AC18] for forced mean curvature flow. This can be seen by manipulating the metric problem, whose solutions in this case are well posed, Lipschitz continuous and correspond to our arrival times.

4. Heuristic arguments suggest that $\Delta F \mapsto \overline{v}_{\Delta F}$ is always at least $\frac{1}{2}$-Hölder continuous when all the previous assumptions hold. One reason for the failure of Lipschitz continuity can be the sufficiently frequent occurrence of local de-pinning when increasing F by ΔF, thus unlocking 'shortcuts'. In this case, morally the effective speed can be expected to increase at most like $\sim \Delta F^{\frac{1}{2}}$, see the discussion on de-pinning transitions in the introduction of [DY06]. However, further investigating this is out of scope for this paper.

Remark 4.3.3 (Core assumptions versus additional requirements). While (P2) and (P$\circledcirc$) are central assumptions for this paper, (P1), (P3), (P4) and (Pη) are only required in the latter half of the paper for the deterministic error estimate, see the beginning of Subsection 4.4.1 for a description of where the assumptions enter the overall proof. Note that in particular the isotropy assumption (P4) is only required for the very last part of the proof and can be replaced by a less restrictive but more technical assumption on the homogenized speed $\overline{v}\colon \mathbb{S}^{d-1} \mapsto \mathbb{R}_{>0}$, see Remark 4.9.3.

Remark 4.3.4 (On Assumption (P4)). Strictly speaking, Assumption (P4) is stronger than just assuming that the probability distribution is isotropic in the context of the level-set formulation in (4.1.6), which would correspond to the following more general assumption.

(P4)' **Isotropy with respect to** (4.1.6)**.** $\mathbb{P}$ is invariant under rotations:

$$\mathbb{P} = \mathbb{P} \circ \mathrm{rot}_R, \quad \text{for any } R \in \mathrm{SO}(d),$$

where $\mathrm{rot}_R(\widetilde{\Omega})$ for any set of coefficients $\widetilde{\Omega} \in \mathcal{F}(\mathbb{R}^d)$ is given by

$$\mathrm{rot}_R(\widetilde{\Omega}) = \{\big((x,e) \mapsto R^\intercal A(Rx, Re)R, F(Rx, Re)\big) \,:\, (A,F) \in \widetilde{\Omega}\}.$$

However, taken together with the other assumptions, this indeed requires that the second-order term corresponds to mean curvature up to a constant factor and hence (P4) holds, as we state in the following lemma, which we prove in Appendix 4.B.

Lemma 4.3.5. *In view of* (4.3.1) *and* (4.3.2)*, if Assumption (P1), Assumption (P2), Assumption (P3), and Assumption (P4)' hold, then there exists a constant* $a \geq 0$ *such that*

$$\mathbb{P}\big[(A,F) \in \Omega \,:\, A(x,e) = a\big(\mathrm{Id}_{d\times d} - e\otimes e\big) \text{ for all } x \in \mathbb{R}^d, e \in \mathbb{S}^{d-1}\big] = 1.$$

For $d = 2$, Assumptions (P1), (P2), (P3), (P4), and (P$\circledcirc$) are satisfied for mean curvature flow with a uniform, positive forcing except for Poisson-distributed symmetric obstacles of diameter smaller than one, as long as the intensity of the underlying Poisson process is low enough. Assumption (Pη) holds for Lebesgue-almost every choice of the uniform forcing as stated in Remark 4.3.2. For more details see Theorem 4.1.1 above and Example 4.3.13 below.

4.3.3 Stability and the notion of effective minimum speeds

In this subsection, we will properly introduce the notions used to state Assumption (P$\circledcirc$) from above and its informal version '(P$\circledcirc$)' from the introduction. That is, we will make precise what we mean with 'fat' sets, 'stable approximations' and an 'effective minimum speed'. In particular, we first need to introduce stable sets.

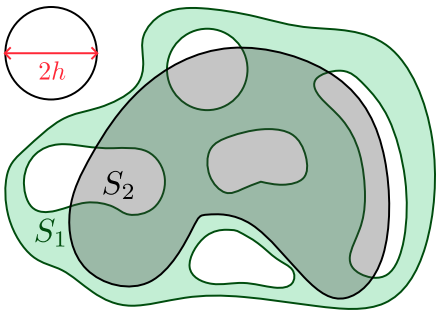


Figure 4.3.1: An illustration of our notion of h-envelopment $S_2 \subset_h S_1$. A set $S_1 \subset \mathbb{R}^d$ (depicted in green) which h-envelops a set $S_2 \subset \mathbb{R}^d$ (depicted in grey)

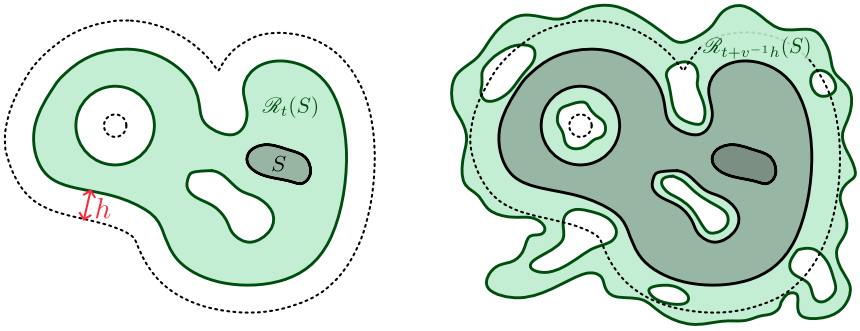


Figure 4.3.2: An illustration of our notion of effective minimum speed. Left: An initial set S (depicted in grey) and its evolution $\mathscr{R}_t(S)$ after some time t (depicted green). The dotted line corresponds to a further expansion of $\mathscr{R}_t(S)$ by $\overline{B}_h$. Right: If an effective minimum speed v holds for S, then after an additional time step $v^{-1}h$ the evolved set $\mathscr{R}_{t+v^{-1}h}(S)$ (depicted in green) must h-envelop the dotted area

Introduction of Stable Sets

Throughout this paper, sets which do not shrink under the interface evolution will be central to many arguments. We call these sets or the corresponding interfaces *stable*. We will discuss properties of these sets in Section 4.5.

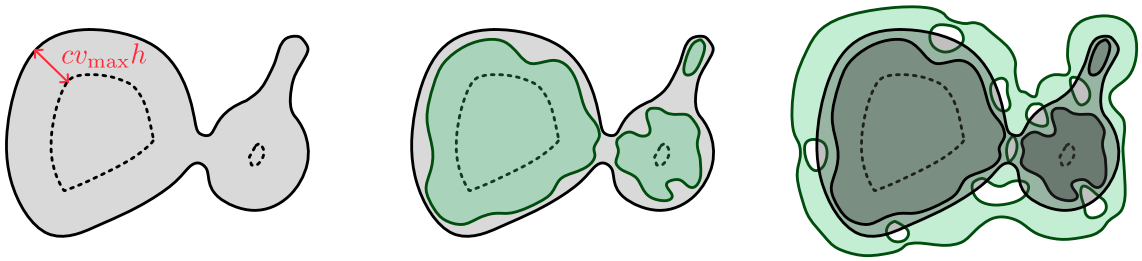


Figure 4.3.3: An illustration of our notion of a stable (h, c)-approximation S_h of a set S. Left: The set $S \subset \mathbb{R}^d$ to be approximated is depicted in grey. The subset of S that must be included in S_h is depicted with a dotted boundary. Center: In addition to the aforementioned sets, the approximation S_h is depicted in green. Right: After a time step ch, the evolved set $\mathscr{R}_{ch}(S_h)$ (depicted in green) must h-envelop the set S

Definition 4.3.6 (Stable Sets)**.** A closed set $S \subset \mathbb{R}^d$ is called stable (with respect to $(A, F) \in \Omega$) if $u := 1 - \chi_S$ is a supersolution of

$$-\operatorname{tr}\left(A\left(x, \frac{\nabla u}{|\nabla u|}\right) D^2 u\right) + F\left(x, \frac{\nabla u}{|\nabla u|}\right)|\nabla u| = 0 \quad \text{in } \mathbb{R}^d. \tag{4.3.7}$$

Remark 4.3.7. Due to the forcing possibly being negative, there is no condition that would guarantee that any 'flat enough' set is stable with respect to all coefficient fields $(A, F) \in \Omega$. This is different from [AC18], where admissible coefficient fields have to satisfy stricter conditions. There, essentially the "ball condition" $S = \bigcup\{\overline{B}_1(x) : \overline{B}_1(x) \subset S\}$ guarantees that $S \subset \mathbb{R}^d$ is stable with respect to all admissible coefficient fields.

Approximate stability and effective minimum speeds

We now make precise what we mean in Assumption (P⊙) with an 'effective minimum speed' of propagation, 'approximate stability' and 'fat' sets. We first give a brief intuitive description of these notions and then provide the formal definition.

Loosely speaking, a coefficient field (A, F) admits $v > 0$ as an *effective minimum speed of propagation* on the scale h in an area $M \subset \mathbb{R}^d$ for some set $S \subset \mathbb{R}^d$, if at any time $t \geq 0$ the evolved set $\mathscr{R}_t(S)$ within a time step $v^{-1}h$ further expands by h in each direction, see Figure 4.3.2. However, for this expansion we ignore enclosed holes of width smaller than $2h$. Morally we can view such thin enclosed areas as being part of the set since we look at arrival times in target areas of diameter $2h$, see Definition 4.1.6.

A set S is (h, c)-*approximately stable* if it has a stable subset S_h, which approximates S in two ways: First, it covers the part of the interior of S, which is further than $cv_{\max}h$ from the boundary. Second, after evolving for ch, S_h envelops all of S except for potential holes of width smaller than $2h$, see Figure 4.3.3.

A set S is h-*fat*, if it is same as the union of all included balls with radius h - that is, if there are no parts on a scale smaller than h protruding from the set.

Definition 4.3.8. Let $(A, F) \in \Omega$. For a set $S \subset \mathbb{R}^d$, let $t \mapsto \mathscr{R}_t^{(A,F)}(S)$ denote the set evolution from Definition 4.1.2. Let $h > 0$, $c > 0$.

(i) We say that a set $S_1 \subset \mathbb{R}^d$ h-*envelops* another set $S_2 \subset \mathbb{R}^d$, denoted by $S_2 \subset_h S_1$, if for each connected component H of $(S_1)^c$ either $H \cap S_2 = \emptyset$ or $H \subset S_1 + \overline{B}_h$, see Figure 4.3.1.

(ii) We say that (A, F) admits $v > 0$ as an *effective minimum speed of propagation* on the scale h in $M \subset \mathbb{R}^d$ for a set $S \subset \mathbb{R}^d$ if for any $t \geq 0$ there holds

$$\left(\mathscr{R}_t^{(A,F)}(S) \cap M\right) + \overline{B}_h \subset_h \mathscr{R}_{t+v^{-1}h}^{(A,F)}(S). \tag{4.3.8}$$

(iii) We say that a set $S \subset \mathbb{R}^d$ is (h, c)-*approximately stable* with respect to (A, F) if there exists a set $S_h \subset S$ such that

$$S_h \text{ is stable}, \qquad S \subset_h \mathscr{R}_{ch}^{(A,F)}(S_h), \qquad S_h \supset \mathbb{R}^d \setminus \left(S^c + cv_{\max}B_h\right),$$

where $v_{\max} = v_{\max}(\text{data})$ corresponds to the maximum speed with which the interface propagates on larger scales, see Lemma 4.5.3. In this case, S_h is called a *stable* (h, c)-*approximation* of S.

(iv) We say that a set $S \subset \mathbb{R}^d$ is h-*fat* if

$$S = \bigcup \{\overline{B}_h(x) \,:\, x \in \mathbb{R}^d \text{ with } \overline{B}_h(x) \subset S\}.$$

We say that S is *fat*, if there exists some $h_0 > 0$ such that S is h_0-fat.

Note that the notion of h-envelopes is chosen to be compatible with the truncated arrival times from Definition 4.1.6, as we will show in the following lemma. For the proof see Appendix 4.B.

Lemma 4.3.9 (Compatibility of the arrival times $m_h(\cdot,\cdot)$ and "$\subset_h$"). *Let $(A, F) \in \Omega$. Let $h > 0$, $x_0 \in \mathbb{R}^d$ and $S_1, S_2 \subset \mathbb{R}^d$ be closed sets with*

$$\mathscr{R}_t^{(A,F)}(S_1) \cap \overline{B}_{\max\{h,\operatorname{dist}(x_0,S_2)\}}(x_0) \subset_h \mathscr{R}_t^{(A,F)}(S_2) \qquad \text{for all } t \leq T_h(x_0, S_2) \tag{4.3.9}$$

with $T_h(x_0, S_2) = v_{\min}^{-1} \operatorname{dist}(x_0, S_2) + Ch$ from Definition 4.1.6. Then there holds

$$m_h^{(A,F)}(x_0, S_1) \geq m_h^{(A,F)}(x_0, S_2).$$

4.3.4 Examples

In this subsection, we list a range of settings for which the assumptions from Subsection 4.3.1 are applicable.

Example 4.3.10 (In the absence of actual obstacles as in [AC18]). Let $\mathbb{P}$ satisfy Assumptions (P1) and (P2). In addition, for almost every $(A, F) \in \Omega$ let

$$A(x,\xi) = \mathrm{Id} - \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}, \qquad F(x,\xi) = F(x),$$

which corresponds to mean curvature flow with an additional forcing of $F(\cdot)$. As shown in [AC18], if we have $\delta > 0$ such that

$$\inf_{x \in \mathbb{R}^d} \left(F(x)^2 - (d-1)\left|DF(x)\right|\right) \geq \delta > 0 \quad \mathbb{P}\text{-almost surely},$$

then there exists $R_0 > 0$ and a guaranteed minimum speed $v_{\min} > 0$, which applies to any R_0-fat set $S \subset \mathbb{R}^d$, that is

$$\mathscr{R}_{t+s}(S) \supset \mathscr{R}_t(S) + v_{\min} s \overline{B}_1 \quad \mathbb{P}\text{-almost surely for all } t, s \geq 0.$$

Hence, Assumption (P◎) is satisfied with the stable approximations given by the sets themselves. Moreover, Assumption (P4) also is satisfied, see Remark 4.3.2.

In this special case and additionally requiring the isotropy from Assumption (P4), our result hence improves the error rate for the bias from [AC18]. Of course, [AC18] covers a much more general setting and there had been no attempt to optimize this error rate. We do not believe that our error rate is optimal either.

Example 4.3.11 (First passage percolation in a continuous environment)**.** For $A = 0$ and direction independent forcing $F(x, \xi) = F(x)$, our setting corresponds to first passage percolation on continuous space, that is Riemannian first passage percolation as introduced in [LW10]. In this case, the arrival time $m(x_0, S)$ can be seen as the fastest time at which x_0 can be reached via an arbitrary path from a closed set $S \subset \mathbb{R}^d$, with the speed along the path at some point $x \in \mathbb{R}^d$ given by $F(x)$, see [LS05a; ADH17]. Areas with $F \leq 0$ would be impassable in this case.

Hence, the best possible bounds on fluctuations and bias in our general setting can not be better than the best possible bounds for first passage percolation.

Example 4.3.12 (A box criterion for $d = 2$)**.** In Chapter 5, we will provide a practical, constructive criterion which we prove to be sufficient for Assumption (P◎) in dimension $d = 2$ if Assumptions (P1) and (P2) hold. In order to check the criterion, it suffices to sample the coefficient field in a thin, long box $Q^{\uparrow}_{r,w} = [0, w] \times [-w, r]$ with width $w > 0$ and height $r > 1$ not necessarily much larger than the range of dependence. If we did not have the isotropy from Assumption (P4), then we would also have to sample the coefficient field in a 90° rotation of the box.

The criterion is satisfied if it is likely enough that there is a stable set in the bottom of the box, which at some point reaches the top of the box when evolving with respect to (A, F) restricted to the box, see Figure 4.3.4.

To be more precise, we define the event

$$E^{\uparrow}_{r,w} := \Big\{ (A,F) \in \Omega \,:\, \text{there is a w.r.t. } (A,F) \text{ stable set } S \subset [0,w] \times [-w,0] \\ \text{with } \mathscr{R}_t^{(A,F)_{Q^{\uparrow}_{r,w}}}(S) \cap [0,w] \times \{r\} \neq \emptyset \text{ for some } t > 0 \Big\},$$

where $(A, F)_{Q^{\uparrow}_{r,w}}$ is the restriction of (A, F) to the box $Q^{\uparrow}_{r,w}$ as in Definition 4.5.8. The criterion is satisfied if for some $p_{r,w} \geq 0$

$$\mathbb{P}\big[E^{\uparrow}_{r,w}\big] \geq 1 - p_{r,w}, \tag{4.3.10}$$

where $p_{r,w} > 0$ for sufficiently long and thin boxes.

Example 4.3.13 (Poisson-distributed obstacles)**.** An explicit setting in which our results are applicable is given by forced mean curvature flow through a field of obstacles distributed according to a Poisson point process with small enough intensity.

To be more precise, we assume that our coefficient fields are given by

$$A(x,\xi) = \mathrm{Id} - \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}, \qquad\qquad F(x,\xi) = F_{uni} - F_{obst}(x),$$

where $F_{uni} > 0$ represents a uniform forcing and where F_{obst} representing the obstacles is the maximum of compactly supported, symmetric, smooth but steep bumps with their centers chosen according to the Poisson point process, see Theorem 4.1.1.

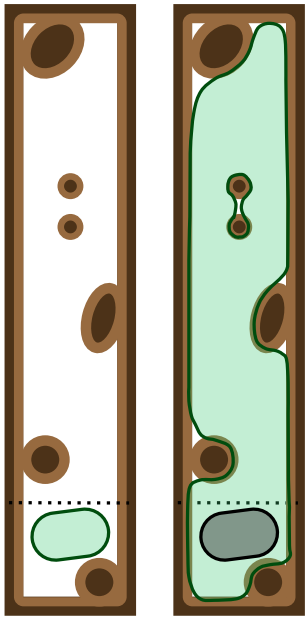


Figure 4.3.4: An illustration of our box criterion (Example 4.3.12) sufficient for Assumption (P◎) in dimension $d = 2$. Left: A rectangular box with a sample of the obstacles in the interior (depicted in brown) and the artificial obstacle from the restriction surrounding the boundary (also depicted in brown) with an initial stable set S in the bottom of the box (depicted in green). Right: After some time t, we require the evolution $\mathscr{R}_t(S)$ (depicted in green) of the set S to have reached the top of the box

Assumptions (P1), (P2), (P3), and (P4) are satisfied for all $F_{uni} > 0$, with the finite range of dependence given by the diameter of the obstacles. If Assumption (P◎) is satisfied for some $F^*_{uni} > 0$, then with Remark 4.3.2 Assumption (Pη) will be satisfied for Lebesgue-almost every $F_{uni} > F^*_{uni}$.

For $d = 2$, we can check that these coefficient fields satisfy the box criterion from Example 4.3.12 and hence Assumption (P◎) if F_{uni} is large enough compared to the intensity or average density ρ of the Poisson point process. That is, we can show that (4.3.10) holds for sufficiently large boxes if $F_{uni} \geq C\sqrt{\rho}$ for some $C > 0$ with the support of the obstacles small enough, where C is independent of the individual strength of the obstacles. In particular, (4.3.10) holds even if the obstacles are chosen to individually be impenetrable for a given uniform forcing, see Chapter 5.

For $d \geq 3$, we also expect that Assumption (P◎) is satisfied if F_{uni} is large enough compared to the intensity of the Poisson point process, based on the properties of sufficiently wide obstacle free areas. Exploring this specific setting is left for future research.

4.4 Main result and the strategy for the proof

4.4.1 The main results

Our main result can be split into two parts: First the fluctuation bounds, for which we need much less assumptions and the arguments of which are applicable to a much broader range of set evolutions, as long as they satisfy a comparison principle. Second the homogenization error, for which we need the remaining more specific requirements to make up for the lack of regularity from not having a guaranteed minimum speed of propagation.

Theorem 4.4.1 (Main result for fluctuation bounds)**.** *Let $(A, F) \in \Omega$ be a random coefficient field with a respective probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(\mathbb{R}^d))$, which*

- *has a finite range of dependence (Assumption (P2)), and*
- *admits an effective minimum speed of propagation $v_{\min}$ with overwhelming probability (Assumption (P⊙)).*

Choose $0 < \vartheta < 1$ for the scaling of the target areas and set $h(r) := r^{\vartheta}$. Let $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ have bounded boundary and be a fat set, as in Definition 4.3.8. Set $R = \operatorname{dist}(x_0, S)$. Then there exists $\varepsilon_0 = \varepsilon_0(\mathrm{data}, \vartheta, R, \operatorname{diam}(\partial S)) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the fluctuations of the truncated arrival times on the scale ε^{-1} can be bounded by

$$\mathbb{P}\left[\left|\varepsilon m^{(A,F)}_{h(\varepsilon^{-1}R)}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big) - \mathbb{E}\left[\varepsilon m^{(A,F)}_{h(\varepsilon^{-1}R)}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big)\right]\right| \geq CR\Big(\frac{\varepsilon}{R}\Big)^{\frac{1-\vartheta}{2}}\lambda\right] \leq \max\left\{C\exp\big(-\lambda\big), C\exp\left(-\frac{R^{\vartheta\alpha_\odot}\varepsilon^{-\vartheta\alpha_\odot}}{C\log\Big(\big(\frac{R}{\varepsilon}\big)^{1-\vartheta}\Big)}\right)\right\}$$

for any $\lambda > 0$ and for some $C = C(\mathrm{data}) > 0$. Here, $\alpha_\odot > 0$ is the exponential rate for the probability bounds from Assumption (P⊙).

This result follows from Proposition 4.6.1, which treats general large scale fluctuations, by carefully plugging in the ε-scaling.

Theorem 4.4.2 (Main result for homogenization)**.** *Let $(A, F) \in \Omega$ be a random coefficient field with a respective probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(\mathbb{R}^d))$, which*

- *is stationary, that is invariant with respect to translations (Assumption (P1)),*
- *has a finite range of dependence (Assumption (P2)),*
- *has a constant, deterministic second-order coefficient A (Assumption (P3)),*
- *is isotropic, that is invariant with respect to rotations (Assumption (P4)),*
- *admits an effective minimum speed of propagation $v_{\min}$ with overwhelming probability for all forcings $F + \Delta F$ with ΔF in some interval around 0 (Assumption (P⊙) for all $(A, F + \Delta F)$ with $\Delta F \in [-\delta_F, \delta_F]$ for some $\delta_F > 0$), and*
- *generates a candidate $\overline{v}$ for the homogenized speed that is η-Hölder continuous with respect to changing the forcing by a constant (Assumption (Pη)).*

Choose $0 < \vartheta < 1$ for the scaling of the target areas and set $h(r) := r^{\vartheta}$. Let $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ be a fat set, as in Definition 4.3.8. Set $R = \operatorname{dist}(x_0, S)$. Then for all $\varepsilon > 0$ the bias of the truncated arrival times on the scale ε^{-1} compared to the arrival times for the homogeneous evolution can be bounded by

$$\left|\mathbb{E}\left[\varepsilon m^{(A,F)}_{h(\varepsilon^{-1}R)}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big)\right] - m^{hom,\overline{v}}\big(x_0, S\big)\right| \leq CR\Big(\frac{\varepsilon}{R}\Big)^{\alpha}\log(\max\{R, 2\})^{\frac{3-\vartheta}{2}}\log(\varepsilon^{-1})^{\frac{3-\vartheta}{2}}$$

for $C = C(\mathrm{data}, \delta_F, \vartheta) > 0$ with $\alpha := \Big(1 + 4\Big(\frac{1}{1-\vartheta} + \frac{1}{\eta}\Big)\Big)^{-1}$, where η is the Hölder coefficient from Assumption (Pη). For $\eta = 1$ we can hence obtain any rate $0 < \alpha < \frac{1}{9}$ if we choose $\vartheta > 0$ small enough.

This result will be proven in Section 4.9, using the fluctuation bounds from Section 4.6, the control on the propagation of influence via increasing the forcing from Section 4.7 and the quantitative convergence of the arrival times for the half-space approximations from Section 4.8, which are used to define the homogenized speed $\overline{v} > 0$.

4.4.2 Overall strategy of the proof

We give a rough overview of the overall proof and then go into more detail regarding the major sections in the following paragraphs.

In Section 4.6, we will prove Theorem 4.4.1, controlling the fluctuations of truncated arrival times for sets with bounded boundary. This estimate will be heavily used for the overall homogenization proof. For this section we only need the finite range of dependence from Assumption (P2) and the very likely guaranteed minimum speed and stability from Assumption (P$\circledcirc$). The proof is based on the martingale method of bounded differences, for which a quantity is written as the endpoint of a martingale with bounded increments and Azuma's inequality – a martingale concentration inequality – is applied. We provide a detailed description of this method for set evolutions in Subsection 4.4.3. However, in our setting we only obtain *likely* bounded martingale differences. We exploit these bounds with an alternative martingale concentration inequality, which we introduce and prove in Appendix 4.C.

In Section 4.7 we introduce the other major tool for the overall homogenization proof. We provide a comparison principle that allows us to ignore the propagation of far-away influences at the cost of increasing the forcing by a small constant for one of the sets. This is the only major part of the proof for which we have to work on the level of the specific level-set equation (4.1.6). Since without an actual minimum speed of propagation we lack regularity on small scales, we are not able to deal with space-dependent second-order coefficients, such that from here on we have to use Assumption (P3), that is that $A(\cdot,\cdot)$ is constant in space and hence deterministic.

Next, we introduce the homogenized speed in Section 4.8. In the already established frameworks, this could be translated as the average speed corresponding to large-scale arrival times for half spaces. In our case, we have to replace the half spaces with bounded half-space approximations in order to apply our fluctuation bounds. Combining these fluctuation bounds from Section 4.6 with the comparison principle from Section 4.7 and using the stationarity from Assumption (P1), we show that the large scale arrival times behave approximately linearly and hence obtain a quantitative convergence result for the averages. However, as in Section 4.7 we are only able to do so at the cost of slightly changing the forcing. To properly apply this intermediate result, we will hence from here on use Assumption (Pη), quantifying the continuity of the homogenized speed with respect to changing the forcing by a constant. Hence, at the end of this section we prove Remark 4.3.2 (1), that is that the choice of half-space approximations is arbitrary with respect to Assumption (Pη).

Finally, we look at general fat sets in Section 4.9 to prove Theorem 4.4.2. On a mesoscopic level, we align the half-space approximations from Section 4.8 with the interface of the homogeneous first-order evolution in order to compare it to the heterogeneous second-order evolution. We use the isotropy from Assumption (P4) to ensure the smoothness of the homogeneous evolution, allowing us to properly align the half-space approximations. With this method we can lift the quantitative convergence result from Section 4.8 to general sets fat sets and obtain Theorem 4.4.2.

Fluctuation bounds for large-scale arrival times

In Section 4.6 we will provide fluctuation bounds for large sets with bounded boundaries. Let $h \geq h_0$ and $S \subset \mathbb{R}^d$ be an h-fat set with $x_0 \in \mathbb{R}^d$ such that $\operatorname{diam}(\partial S)$ and $\operatorname{dist}(x_0, S)$ are not exponentially larger than h. Then for the truncated arrival time for S in the target

area $\overline{B}_h(x_0)$ we obtain

$$\mathbb{P}\big[\big|m_h(x_0,S)-\mathbb{E}\big[m_h(x_0,S)\big]\big|\geq\lambda\big]\leq C\exp\left(-\frac{\lambda}{C\sqrt{h\,\mathrm{dist}(x_0,S)}}\right)\quad\text{for }\lambda\leq\lambda_0$$

with $\lambda_0=h^{\alpha_\circledcirc}\frac{\sqrt{h\,\mathrm{dist}(x_0,S)}}{\log\left(\frac{\mathrm{dist}(x_0,S)}{h}\right)}$, where $\alpha_\circledcirc>0$ is the exponential rate from Assumption (P◎). That is, we have exponential tail bounds which are capped essentially by the probability bounds from Assumption (P◎). This cap is natural since we do not expect any control for coefficient fields in areas with no effective minimum speed of propagation on the scale h.

The proof for these bounds is based on the martingale method of bounded differences, using the framework developed in [ACS14; AC14; AC18] as orientation. The truncated arrival time can be viewed as the endpoint of a martingale, whose increments are bounded to then apply a concentration result. This martingale is the conditional expectation of the truncated arrival time with respect to the environment explored by the set evolution at different times. We will describe this established framework in more detail in Subsection 4.4.3. The novelty of our result comes from applying this framework in a setting with no guaranteed minimum speed of propagation. This requires two types of innovation: First, we use our concepts of an effective minimum speed and stable approximations to cover all the relevant use cases of an actual minimum speed on larger scales with some technical adjustments. Second, we have no full guarantee that there actually are stable approximations and that the effective minimum speed is applicable, only a very high probability. This yields bounds for the martingale increments which do not hold almost surely as usual, but only on a set of very high probability. We hence need a replacement for Azuma's inequality, the concentration result via which the fluctuation bounds are usually obtained. We provide a suitable replacement in Appendix 4.C, via which we obtain the result.

Note that this method is applicable to much more general set evolutions, there are only four prerequisites:

- The evolution admits a comparison principle.
- The range of dependence for the coefficient field is finite (Assumption (P2)).
- With overwhelming probability, stable approximations exist and an effective minimum speed of propagation is applicable (Assumption (P◎)).
- The set evolution only depends on the values of the coefficient field on the interface or in a small neighborhood of the overall interface.

Bounding the propagation of influence at the cost of increasing the forcing

Our building blocks for homogenizing the evolution of general large-scale sets will be a range of standardized disks. Later in Section 4.8, we will obtain good control on the evolution close to the center of these disks. For the final homogenization argument, we will align these disks with the boundary of much larger sets in order to extend the control on their evolution to arbitrary sets. Therefore, we need a comparison principle for the evolution close to the center of the disk, which allows us to ignore the relation between the disk and the general set far away from this area.

In Section 4.7 we will show that for such a local comparison we can indeed ignore far away influences but at the cost of slightly increasing the forcing. More precisely, we

consider a stable set S_1, that is 'ahead' of a second set S_2 within a large ball of radius $R > 0$ – outside of this ball, S_2 might be much larger than S_1, see Figure 4.7.1. We then show that within in a much smaller ball of radius $r > 0$ and for a limited time $T > 0$, the evolution of S_1 stays ahead. However, this only works at the cost of increasing the forcing in the evolution of S_1. More precisely, if for some $h \geq 1$

$$S_1 \supset \left(S_2 + \overline{B}_h\right) \cap \overline{B}_R,$$

then

$$\mathscr{R}_t^{(A,F+\Delta F)}(S_1) \supset \mathscr{R}_t^{(A,F)}(S_2) \cap \overline{B}_r \qquad \text{for all } 0 \leq t \leq T$$

if the change to the forcing satisfies

$$\Delta F \geq C \frac{T^{\frac{3}{4}}}{(R-h-r)^{\frac{1}{2}}}.$$

Since we are interested in truncated arrival times on the scale r, due to the truncation we only need to compare the sets up to time $T \approx r$. If we choose $R \approx r^\beta >> r, h$ for some $\beta \geq 1$, then we need $\Delta F \gtrsim r^{\frac{3}{4}-\frac{\beta}{2}}$. For only a small change of the forcing to be sufficient, we hence require $\beta > \frac{3}{2}$ for the scaling of the larger radius.

The same result holds true if we replace the balls $\overline{B}_r$ and $\overline{B}_R$ with the half-balls $\overline{B}_r \cap \mathcal{H}_e^+$ and $\overline{B}_R \cap \mathcal{H}_e^+$ with $\mathcal{H}_e^+ := \{x \in \mathbb{R}^d : x \cdot e \geq 0\}$ for $e \in \mathbb{S}^{d-1}$. In this case we have to assume in addition that S_1 covers the boundary corresponding to the half-space, that is

$$S_1 \supset \left(\partial\mathcal{H}_e^+ + \overline{B}_h\right) \cap \overline{B}_R.$$

The proof for these results is an adaptation of the proof for the comparison principle. For this, it is crucial that the locally large set is stable, which for the solution of the level-set equation implies increasing sub-level sets. Thus, the solution is decreasing and we obtain a one-sided bound for the time derivative. This is the only way for us to impose any control on the time derivatives of the level set evolution. If we knew that there was a general minimum speed of propagation as in the setting from [AC18], then we would obtain a decreasing, Lipschitz-continuous solution. Since we lack this regularity, we can not deal with space dependent second order terms and additionally need to increase the forcing for the evolution of the stable set to compensate. Hence from here on we require Assumption (P3), that is that the second-order coefficient A is constant in space and thus deterministic due to the stationarity.

Introduction of the homogenized speed and quantitative convergence bounds.

Next, in Section 4.8 we introduce our candidate for the homogenized speed $\overline{v}$, for which we need both the fluctuation bounds from Section 4.6 and the bounds on the propagation of influence from Section 4.7 corresponding to the Subsections 4.4.2 and 4.4.2. We would have liked to straight-up define $\overline{v}$ as the average large scale speed for the evolution of the half-spaces, that is

$$\overline{v} = \overline{v}(e) = \left(\lim_{r\to\infty} \frac{1}{r}\mathbb{E}\left[m_{h(r)}\left(re, \mathcal{H}_e^-\right)\right]\right)^{-1} \tag{4.4.1}$$

with the half-spaces $\mathcal{H}_e^- = \{x \in \mathbb{R}^d : x \cdot e \leq 0\}$ for $e \in \mathbb{S}^{d-1}$. The invariance with respect to $e \in \mathbb{S}^{d-1}$ is due to the isotropy from Assumption (P4), although technically we will

only use this property later in Section 4.9. If we were able to control the fluctuations of arrival times for sets with unbounded boundary, then we would be able to show the approximate sub-linearity of the truncated arrival times $r \mapsto m_{h(r)}(re, \mathcal{H}_e^-)$ for half spaces $\mathcal{H}_e^- = \{x \in \mathbb{R}^d : x \cdot e \leq 0\}$ with $h(r) \ll r$, which would yield the existence of the limit in (4.4.1). If in addition we had approximate super-linearity, we would obtain quantitative bounds for the convergence.

However, we only control the fluctuations for sets with bounded boundary. Thus, we approximate the boundary of the half space with disks $r \mapsto S^-_{e,\beta,h(r)}(r)$ of radius $\sim r^\beta$ for $\beta \geq \frac{3}{2}$ and thickness $\sim h(r) \ll r$, see Figure 4.8.1. In order to justify the existence of the limit and quantify the convergence in (4.4.1), we will replace $\mathcal{H}_e^-$ with $S^-_{e,\beta,h(r)}(r)$. We then show that the arrival times are approximately linear in r, which yields the convergence of the averages and quantitative error bounds.

First, just using the fluctuation bounds from Section 4.6, we obtain approximate super-linearity for the arrival times

$$r \mapsto m_{h(r)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)$$

in the sense that

$$\begin{aligned}\mathbb{E}\Big[m_{h(r_1)}\Big(r_1 e, S^-_{e,\beta,h_1(r_1)}(r_1)\Big)\Big] + \mathbb{E}\Big[m_{h(r_2)}\Big(r_2 e, S^-_{e,\beta,h_2(r_2)}(r_2)\Big)\Big] \\ \geq \mathbb{E}\Big[m_{h(r_1+r_2)}\Big((r_1+r_2)e, S^-_{e,\beta,h(r_1+r_2)}(r_1+r_2)\Big)\Big] - C\sqrt{h(r_2)r_2}\log(r_1). \quad (4.4.2)\end{aligned}$$

The main idea for proving this is to place translations of the small disk $S^-_{e,\beta,h_2(r_2)}(r_2)$ in $S^-_{e,\beta,h(r_1+r_2)}(r_1+r_2)$ such that the translated target areas $\overline{B}_{h_2(r_2)}(r_2 e)$ cover the smaller disk $r_2 e + S^-_{e,\beta,h_1(r_1)}(r_1)$. For the first step, we wait until all of these disks have reached the corresponding target area r_2 above them, see Figures 4.8.2 and 4.8.3. For the second step, now that the upper, smaller disk is completely covered, we take the time until this disk reaches the target area $\overline{B}_{h(r_1)}(r_1+r_2)$. By comparison principle, adding step 1 and step 2 takes longer than the time which the larger disk needs to reach the target area $\overline{B}_{h(r_1+r_2)}(r_1+r_2)$. Taking the expected value, we can resolve the maximum for the arrival times over the distance r_2 from the first step at the cost of the typical-fluctuation-sized error $\sim C\sqrt{h_2 r_2}\log(r_1)$ because due to the exponential tail bounds from Section 4.6 it is unlikely that any of the single arrival times is much larger.

We then show the approximate superlinearity. The idea of this proof is similar to the one of (4.4.2): Instead of waiting until the intermediate disk is fully covered and then restarting with that disk, we now restart with that disk as soon as a larger intermediate disk is reached for the first time by one of the lower disks, see Figure 4.8.4. This yields a faster time than the original arrival time for the larger disk. However, in this case, in addition to the fluctuation bounds from Section 4.6 we need the comparison principle from Section 4.7: This comparison principle allows us to favorably compare the smaller disks $S^-_{e,\beta,h_1(r_1)}(r_1)$ and $S^-_{e,\beta,h_2(r_2)}(r_2)$ to the large disk at the cost of increasing the forcing for their evolution. Thus, we obtain

$$\begin{aligned}\mathbb{E}\Big[m^{(A,F+f(r_1))}_{h(r_1)}\Big(r_1 e, S^-_{e,\beta,h_1(r_1)}(r_1)\Big)\Big] + \mathbb{E}\Big[m^{(A,F+f(r_2))}_{h(r_2)}\Big(r_2 e, S^-_{e,\beta,h_2(r_2)}(r_2)\Big)\Big] \\ \leq \mathbb{E}\Big[m^{(A,F)}_{h(r_1+r_2)}\Big((r_1+r_2)e, S^-_{e,\beta,h(r_1+r_2)}(r_1+r_2)\Big)\Big] + C\sqrt{h(r_1+r_2)r_2}\log(r_1), \quad (4.4.3)\end{aligned}$$

where the forcing is changed by the constant $f(r) = r^{-\frac{\beta}{2}+\frac{3}{4}}$, depending on the scaling of the width of the disks.

We now look at the limit of the averaged arrival times, having established the approximate linearity. Already with just the sub-linearity, we obtain the existence of the limits

$$\Delta F \mapsto \overline{v}_{\Delta F,\beta} := \left(\lim_{r\to\infty} \frac{1}{r} \mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big]\right)^{-1},$$

where we use that if the underlying assumptions hold for the random coefficient field $(A, F-\delta_F)$, then they hold for all fields $(A, F+\Delta F)$ with $\Delta F \geq -\delta_F$. The reason that we need to consider these limits after changing the forcing by different constants ΔF is that the quantitative convergence bounds, which we obtain from (4.4.2) and (4.4.3), are given by

$$\begin{aligned}\overline{v}^{-1}_{\Delta F,\beta} r - C\sqrt{h(r)r}\log(r) \leq \mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big] \\ \leq \overline{v}^{-1}_{\Delta F - Cf(r),\beta} r + C\sqrt{h(r)r}\log(r). \quad (4.4.4)\end{aligned}$$

We would hence get proper quantitative bounds for the convergence if we knew that $\Delta F \mapsto \overline{v}_{\Delta F,\beta}$ is Hölder continuous, which Assumption (Pη) yields for some $\beta = \beta^* > \frac{3}{2}$. We justified this assumption in Remark 4.3.2. At the end of Section 4.8, we will show that if Assumption (Pη) holds for $\beta = \beta^*$, then it holds for all $\beta > \frac{3}{2}$ via the result from Section 4.7 bounding the propagation of far-away influence. In fact, if Assumption (Pη) holds, then the limit in (4.4.1) for the full half spaces exists and is the same as $\overline{v}_{0,\beta}$ for all $\beta > \frac{3}{2}$.

Homogenization for general sets using half-space approximations as building blocks

The final argument for the homogenization result from Theorem 4.4.2 is based on using the half-space approximations from Section 4.8 and Subsection 4.4.2 as building blocks. The first step is to reduce the case of a general fat set $S \subset \mathbb{R}^d$ to two much simpler sets: Given a target point $x_0 \in \mathbb{R}^d$ with $R = \operatorname{dist}(x_0, S)$, we sandwich $\varepsilon^{-1}S$ between

$$\overline{B}_{h(\varepsilon^{-1}R)}(y_\varepsilon) \subset \varepsilon^{-1}S \subset \mathbb{R}^d \setminus B_{\varepsilon^{-1}R}(\varepsilon^{-1}x_0)$$

with y_ε chosen such that $\operatorname{dist}(\varepsilon^{-1}x_0, \overline{B}_{h(\varepsilon^{-1}R)}(y_\varepsilon)) = \varepsilon^{-1}\operatorname{dist}(x_0, S)$, see Figure 4.9.1. This is possible if we choose ε small enough such that S is $h(\varepsilon^{-1}R)$-fat, see Definition 4.3.8. The reason why the ball and the round hole are much simpler to treat is because they have self-similar first-order evolutions:

$$\mathscr{R}_t^{hom,\overline{v}}\big(\overline{B}_{h(\varepsilon^{-1}R)}\big) = \big(h(\varepsilon^{-1}R) + \overline{v}t\big)\overline{B}_1, \quad (4.4.5)$$

$$\mathscr{R}_t^{hom,\overline{v}}\big(\mathbb{R}^d \setminus B_{\varepsilon^{-1}R}\big) = \mathbb{R}^d \setminus \big(\varepsilon^{-1}R - \overline{v}t\big)B_1. \quad (4.4.6)$$

In particular, the arrival times at $\varepsilon^{-1}x_0$ with respect to the homogenized first-order evolution are the same for the small ball, the original set $\varepsilon^{-1}S$, and the large hole. Thus, in order to obtain Theorem 4.4.2, it is sufficient to show two things:

1. On large scales the expansion of a ball with respect to the heterogeneous second-order evolution is not much slower than (4.4.5).
2. On large scales the shrinking of a hole with respect to the heterogeneous second-order evolution is not much faster than (4.4.6).

We obtain this by comparing the evolutions for discrete time-steps. For the ball, starting with radius $R_0 = h(\varepsilon^{-1}R)$, we incrementally choose expansions $(r_n)_n$ until the radius

$R_N = R_0 + \sum_{n=1}^N r_n$ covers the distance between $\varepsilon^{-1}S$ and $\varepsilon^{-1}x_0$. We show that the expected value for the time t_{n+1} until the ball $\overline{B}_{R_n}$ fully envelops the next larger ball $\overline{B}_{R_{n+1}}$ is less than $\overline{v}^{-1}(R_{n+1} - R_n) + \tau_{n+1,ball}$ for a controlled error $\tau_{n+1,ball}$. For the hole, starting with radius $R_0 = \varepsilon^{-1}R$, we incrementally choose expansions $(r_n)_n$ until $R_N = R_0 - \sum_{n=1}^N r_n$ corresponds to the target area of radius $h(\varepsilon^{-1}R)$. We show that the expected value for the time t_{n+1} until $\mathbb{R}^d \setminus B_{R_n}$ would first leave $\mathbb{R}^d \setminus B_{R_{n+1}}$ is more than $\overline{v}^{-1}(R_n - R_{n+1}) - \tau_{n+1,hole}$ for an error term $\tau_{n+1,hole}$ similar to $\tau_{n+1,ball}$. By comparison principle, the overall homogenization error can thus be bounded by adding up these error terms.

Since the shapes are self-similar, each step of this iteration is essentially the same. For first-order motions with an arbitrary continuous speed $e \mapsto v(e)$, there does always exist such a self-similar, convex grower K given by $\mathscr{R}_t^{hom,v}(\{0\}) = tK$, which in our isotropic case is the ball. There are two potential issues in the anisotropic case though: First, if K is convex but not smooth, then it is not possible to properly align the half-space approximations. Second, in general it is not true that the mirror image $-K$ yields a self-similar shrinking hole as in the isotropic case.

For the error terms $(\tau_{n,ball})_n$ and $(\tau_{n,hole})_n$, there are three contributions, which we will exemplify for the shrinking hole; the proof for the growing ball is very similar. We will choose the discrete step sizes $r_n = R_n - R_{n-1}$ to optimally balance these three error contributions. In order to relate the time t_n until $\mathbb{R}^d \setminus B_{R_{n-1}}$ first leaves $\mathbb{R}^d \setminus B_{R_n}$ to $\overline{v}^{-1}r_n$, we use half-space approximations $S^-_{e,\beta,h(r_n)}(r_n)$ with target area $\overline{B}_{h(r_n)}(r_n e)$ from Subsection 4.4.2, which were used to define the homogenized speed. We align translations of these half-space approximations with the boundary of $\mathbb{R}^d \setminus B_{R_{n-1}}$, but place them slightly in front, see Figure 4.9.5. The size ρ_n of the resulting gap due to the curvature is given by

$$\big(R_{n-1} - \rho_n\big)^2 + \big(C_\beta r_n^\beta\big)^2 = R_{n-1}^2, \quad \text{implying} \quad \rho_n \le C(\beta, \text{data})\frac{r_n^{2\beta}}{R_{n-1}},$$

The combined target areas for these half-space approximations approximate ∂B_{R_n} up to an error corresponding to ρ_n. Making up for the propagation of the far away influences with the half-space approximations locally ahead of the boundary of $\mathbb{R}^d \setminus B_{R_{n-1}}$ as described in Subsection 4.4.2, it takes longer for the hole $\mathbb{R}^d \setminus B_{R_{n-1}}$ to reach these target areas than for the respective half-space approximation as long as for their evolution we slightly increase the forcing by $f(r_n) = f_\beta(r_n)$. The time t_n is hence larger than the minimum of these arrival times for half-space approximations up to an error of approximately $v_{\min}^{-1}\rho_n$. Taking the expected value, we can resolve this minimum at the cost of a typical-fluctuation-sized error since the essentially exponential tail bounds for the fluctuations make it unlikely that any of the single arrival times is much smaller. Plugging in the convergence rate of the arrival times for half-space approximations from (4.4.4) and $\rho_n \approx \frac{r_n^{2\beta}}{R_{n-1}}$, we obtain

$$\begin{aligned}\mathbb{E}\big[t_n\big] &\ge \overline{v}^{-1}r_n - \Big(\overline{v}^{-1} - \overline{v}^{-1}_{f_\beta(r_n)}\Big)r_n - C\sqrt{h(r_n)r_n}\log\big(R_{n-1}\big) - C\frac{r_n^{2\beta}}{R_{n-1}} \\ &=: \overline{v}^{-1}r_n - \tau_n.\end{aligned}$$

The first error term comes from comparing the homogenized speed for $(A, F + \Delta F)$ with $\Delta F = f_\beta(r_n)$ and $\Delta F = 0$. The second error term comes from using the fluctuation bounds and the convergence rate to the homogenized speed. The third error term comes from the gap between the half-space approximations and the boundary of $\mathbb{R}^d \setminus B_{R_{n-1}}$.

We choose $r_n = r_n(R_{n-1})$ and $\beta > \frac{3}{2}$ in order to optimally balance these three error contributions and obtain

$$\tau_n \lesssim \sqrt{r_n^{1+\vartheta}}\log(R_{n-1}) \qquad \text{for} \qquad r_n \approx R_{n-1}^{\frac{2}{1-\vartheta+4\left(\frac{1-\vartheta}{\eta}+1\right)}},$$

yielding a recursive scheme for the choice of the enlargements, where η is the Hölder coefficient from Assumption (Pη). Adding up these errors for each step until $R_N = h(\varepsilon^{-1}R)$ and hence the original target area $\overline{B}_{h(\varepsilon^{-1}R)}$ is reached yields the overall error. Similar considerations yield an analogous result for the growing hole, such that in summary we obtain Theorem 4.4.2 after multiplying with ε.

4.4.3 The proof of the fluctuation bounds

We will illustrate the main ideas behind the proof for the fluctuation bounds of the arrival times in Section 4.6 in more detail. For this, we prove an analogous result under the much more restrictive assumption that there exists a guaranteed minimum speed of propagation $v_\infty > 0$:

$$\mathscr{R}_t^{(A,F)}(S) \supset \mathscr{R}_s^{(A,F)}(S) + v_\infty(t-s)\overline{B}_1 \text{ for all } t \geq s,\ S = \bigcup\{\overline{B}_1(x) \subset S\}. \tag{4.4.7}$$

We know from [AC18] that this is satisfied for example if

$$A(x,\xi) = a\left(\mathrm{Id} - \frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right), \qquad F(x,\xi) = F(x) \text{ with } F^2 - (d-1)a|DF| \geq c,$$

for some $c > 0$ large enough since (4.4.7) corresponds to Lipschitz continuity of the metric problem for S. This is because for geometric motions the metric problem corresponds to the arrival times, if it is well-posed – which this assumption implies. Hence, this case has been covered by the fluctuation bounds for the metric problem in [AC18].

The ideas presented here originate from first passage percolation theory [Kes93; Zha10]. More recent approaches for supercritical percolation clusters [GM09; Dem24] – corresponding to our setting with $A \equiv 0$ – use concentration results based on Poincaré or log-Sobolev inequalities but rely on localization properties which do not seem accessible in our case. That is, due to the curvature component we have to view the set evolving as a whole while in first passage percolation the arrival times can be obtained by individually computing the traversal times for all paths connecting the target area to the initial set.

The ideas from [Kes93; Zha10] have been extended and adapted to increasingly general Hamilton–Jacobi equations in the series [ACS14; AC14; AC18], finally including this case in [AC18]. However, the proof of [AC18, Proposition 3.1] does not just cover interface motions but also general viscous Hamilton–Jacobi equations and therefore comes with significant additional technical baggage. Therefore, we think that it is worthwhile to present a concise proof in order to illustrate the already established main components and provide orientation for readers unfamiliar with these kind of arguments. For this, we use our language of arrival times and set evolutions instead of working on the level of the equation describing the metric problem.

The technical innovations allowing us to adapt the ideas presented here to our setting later in Section 4.6 are fourfold:

- Usage of arrival times in a target area instead of solutions to the metric problem,

- Usage of stable approximations and an *effective* minimum speed of propagation instead of a strict minimum speed of propagation,
- Applying the arguments on a set of high instead of full probability,
- An alternative to Azuma's inequality requiring good martingale bounds with just high probability (instead of full probability).

The simpler result which we will prove to demonstrate the main ideas is the following:

Proposition 4.4.3 (Fluctuation bounds for guaranteed minimum speed of propagation)**.** *Assume $\mathbb{P}$ is a probability measure on $(\widetilde{\Omega}, \mathcal{F}(\mathbb{R}^d))$ with $\widetilde{\Omega} = \{(A,F) \in \Omega : \text{(4.4.7) holds}\}$ with a finite range of dependence as in Assumption (P2). Then for the arrival times from Definition 4.1.6 there exists $C = C(\mathrm{data}, v_\infty) > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ with $S = \bigcup\{\overline{B}_1(x) \subset S\}$ as well as ∂S bounded, for any $\lambda > 0$ there holds*

$$\mathbb{P}\big[\big|m\big(x_0,S\big) - \mathbb{E}\big[m\big(x_0,S\big)\big]\big| \geq \lambda\big] \leq C\exp\bigg(-\frac{\lambda^2}{C(1+\operatorname{dist}(x_0,S))}\bigg).$$

Fluctuation bounds via Azuma's inequality

We obtain this inequality by viewing $m(x_0,S) - \mathbb{E}[m(x_0,S)]$ as the endpoint of a martingale, for which we apply a standard concentration result: We set $(\widetilde{\mathcal{M}}_n)_{n\in\mathbb{N}_0} \coloneqq (\mathcal{M}_{n\Delta t})_n$ with $\Delta t > 0$ to be chosen later and

$$\mathcal{M}_t \coloneqq \mathbb{E}\big[m\big(x_0,S\big)\,\big|\,\mathcal{G}_t\big] - \mathbb{E}\big[m\big(x_0,S\big)\big],$$

with

$$\begin{aligned}&\mathcal{G}_0 \coloneqq \{\emptyset,\Omega\},\\ &\mathcal{G}_t \coloneqq \text{“}\sigma\text{-algebra corresponding to information on the state of } \mathscr{R}_s(S)\\ &\qquad\qquad\qquad

If we could show that there was $C > 0$ such that for all $0 \leq s \leq t$ there holds

$$\left|\mathcal{M}_t - \mathcal{M}_s\right| \leq C(t-s) + C, \tag{4.4.9}$$

then we could apply Azuma's inequality to obtain

$$\mathbb{P}\left[\left|\mathcal{M}_N - \mathcal{M}_0\right| \geq \lambda\right] \leq 2\exp\left(-\frac{\lambda^2}{2C^2N(\Delta t+1)^2}\right).$$

Choosing for example $\Delta t = 1$ then yields Proposition 4.4.3. Essentially, we need to show that 'information' gained while evolving for Δt impacts the current best guess for the arrival time at most linearly in Δt. Due to the finite range of dependence, this will come down to comparing the slowest to the fastest possible evolution for this time step.

Measurability and the application of the finite range of dependence assumption

In order to point out some of the key considerations, we first need to make the definition of $\mathcal{G}_t$ from (4.4.8) more precise: Due to measurability issues, we need to approximate the possible states of $\mathscr{R}_t(S)$ with countable many sets. Fixing $h > 0$, we define the discrete approximation of a set $M \subset \mathbb{R}^d$ as

$$K_h(M) \coloneqq \bigcup\left\{x + [0,h]^d \,:\, x \in h\mathbb{Z}^d,\, M \cap \left(x + [0,h]^d\right) \neq \emptyset\right\}.$$

Clearly, $\left\{K_h(M) \,:\, M \subset \mathbb{R}^d\right\}$ is not countable and hence too large for the discrete state space approximating the set evolutions. However, due to the minimum speed from (4.4.7) and the maximum speed of propagation (see Lemma 4.5.3), for any $t \geq 0$ there holds

$$\mathscr{R}_t\big(S\big) \in \left\{M \subset \mathbb{R}^d \,:\, S \subset M \subset S + B_R \text{ for some } R > 0\right\},$$

and since ∂S is bounded, there are countable many sets $(K_i)_{i\in\mathbb{N}}$ such that

$$\left\{K_i \,:\, i \in \mathbb{N}\right\} = \left\{K_h(M) \,:\, M \subset \mathbb{R}^d \text{ with } S \subset M \subset S + B_R \text{ for some } R > 0\right\}.$$

Based on these sets we define the events $\{E_i(t)\}_{i\in\mathbb{N}}$ capturing the approximate state of the set evolution at time $t \geq 0$ as

$$E_i(t) \coloneqq \left\{(A,F) \in \widetilde{\Omega} \,:\, K_h\Big(\mathscr{R}_s^{(A,F)}\big(S\big)\Big) = K_i\right\}.$$

Note that $\bigcup_i E_i(t) = \widetilde{\Omega}$ for all $t \geq 0$. Finally, this allow us to properly define $\mathcal{G}_t$ as

$$\mathcal{G}_t \coloneqq \sigma\text{-algebra generated by events } E_i(s) \cap F,\ 0 \leq s \leq t,\ F \in \mathcal{F}\big(K_i\big),$$

where $\mathcal{F}(K_i)$ was the σ-algebra containing the information of the coefficient field in K_i as introduced in (4.3.5). Essentially, $\mathcal{G}_t$ contains the information of the approximate states of the set evolution up to time t and on the coefficient field in the area covered by these approximate states. In order to work with the conditional expectations of arrival times with respect to $\mathcal{G}_t$ from the definition of the martingale, we need the following measurability properties.

Lemma 4.4.5 (Measurability of arrival times)**.** *In the setting of Proposition of 4.4.3 and with $\mathcal{G}_t$ and $\mathcal{M}_t$ defined as above, the following holds with the σ-algebras $\{\mathcal{F}(U)\})$ from (4.3.5) containing the information on the coefficient field in $U \subset \mathbb{R}^d$ respectively.*

(a) $m(x_0, S)$ is $\mathcal{F}(\overline{S^c})$-measurable,

(b) $E_i(t)$ *is* $\mathcal{F}(K_i)$*-measurable for all* $i \in \mathbb{N}$ *and* $t \geq 0$*,*

(c) $\{(A,F) : m^{(A,F)}(x_0,S) \leq t\}$ *and* $m(x_0,S)\chi_{\{(A,F):m^{(A,F)}(x_0,S)\leq t\}}$ *are* $\mathcal{G}_t$*-measurable,*

(d) $\mathbb{E}\big[m\big(x_0, K_h(\mathscr{R}_t(S)) + \overline{B}_1\big) \,\big|\, \mathcal{G}_t\big] = \sum_{i=1}^{\infty} \mathbb{E}\big[m\big(x_0, K_i + \overline{B}_1\big)\big]\chi_{E_i(t)}$ *for all* $t \geq 0$*.*

The proof of (d) is the one and only point where we use the finite range of dependence from Assumption (P2). Resolving the conditional expectation will be key when we bound the martingale increments below – the combination of (c) and (d) allows us to do so. Further note that (c) implies that indeed $\widetilde{M}_N = m(x_0,S) - \mathbb{E}[m(x_0,S)]$ since due to the minimum speed we have $m(x_0,S) \leq T(x_0,S)$ and thus the $\mathcal{G}_{N\Delta t}$-measurability of $m(x_0,S)$.

Sketch proof of Lemma 4.4.5. In order to show measurability with respect to information about the coefficient field on only a certain region $U \subset \mathbb{R}^d$, we use some localization $(A,F)_U \in \widetilde{\Omega}$ essentially depending only on $A|_U$ and $F|_U$ with $(A,F)_U = (A,F)$ on U. The precise definition of this localization is flexible, see [AC18, Section 3.2] or our restricted coefficient fields from Definition 4.3.6. For example, one could choose $(A,F)_U = 0$ on $\mathbb{R}^d \setminus \big(U + \overline{B}_1\big)$ with a smooth transition between U and $\mathbb{R}^d \setminus \big(U + \overline{B}_1\big)$.

Regarding (a), due to the guaranteed growth from (4.4.7) we have $\partial\mathscr{R}_t^{(A,F)}(S) \subset \overline{S^c}$ for all $t \geq 0$ and hence the evolution depends only on the coefficient field on $\overline{S^c}$, yielding $\mathscr{R}_t^{(A,F)}(S) = \mathscr{R}_t^{(A,F)_{\overline{S^c}}}(S)$ for all $t \geq 0$. From here it is straightforward to show (a), see our main measurability criterion in Lemma 4.5.12.

Analogously, regarding (b), $E_i(t)$ is $\mathcal{F}(K_i)$-measurable because we have

$$E_i(t) = \Big\{(A,F) \in \widetilde{\Omega} : K_h\Big(\mathscr{R}_t^{(A,F)_{K_i}}(S)\Big) = K_i\Big\}.$$

The inclusion "$\subset$" holds since if $K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) = K_i$, then $\mathscr{R}_s^{(A,F)}(S) \subset K_i$ for all $0 \leq s \leq t$ due to the minimum speed from (4.4.7) and hence $\mathscr{R}_s^{(A,F)}(S) = \mathscr{R}_s^{(A,F)_{K_i}}(S)$ because the evolution only depends on the coefficient field in K_i for all $0 \leq s \leq t$. The reverse inclusion "$\supset$" follows in the same way.

Regarding (c), with $\mathscr{R}_s^{(A,F)}(S) = \mathscr{R}_s^{(A,F)_{K_i}}(S)$ for $s \leq t$ and $(A,F) \in E_i(t)$ we have

$$\Big\{(A,F) : m^{(A,F)}\big(x_0,S\big) \leq t\Big\} = \bigcup_{i\in\mathbb{N}}\Big(\Big\{(A,F) : x_0 \in \mathscr{R}_t^{(A,F)_{K_i}}\big(S\big)\Big\} \cap E_i(t)\Big)$$

which is $\mathcal{G}_t$-measurable due to the definition of $\mathcal{G}_t$. For the same reason, we obtain

$$m^{(A,F)}\big(x_0,S\big) = \sum_{i\in\mathbb{N}} m^{(A,F)_{K_i}}\big(x_0,S\big)\chi_{E_i(t)}$$

for $(A,F) \in \{(A',F') : m^{(A',F')}(x_0,S) \leq t\}$ and thus $\mathcal{G}_t$-measurability.

Finally, (d) follows immediately by summing over $i \in \mathbb{N}$ if we can show

$$\mathbb{E}\big[m\big(x_0, K_i + \overline{B}_1\big)\chi_{E_i(t)} \,\big|\, \mathcal{G}_t\big] = \mathbb{E}\big[m\big(x_0, K_i + \overline{B}_1\big)\big]\chi_{E_i(t)},$$

which follows if for any $E \in \mathcal{G}_t$ we can show that

$$\mathbb{E}\big[m\big(x_0, K_i + \overline{B}_1\big)\chi_{E_i(t)}\chi_E\big] = \mathbb{E}\big[m\big(x_0, K_i + \overline{B}_1\big)\big]\mathbb{P}\big[E_i(t) \cap E\big].$$

This is the case due to the finite range of dependence from Assumption (P2): From (a) we know that $m(x_0, K_i + \overline{B}_1)$ is $\mathcal{F}(\overline{(K_i + \overline{B}_1)^c})$-measurable, while $E \cap E_i(t) \in \mathcal{F}(K_i)$: It is sufficient to prove this for $E = E_j(s) \cap F$ with $s \leq t$ and $F \in \mathcal{F}(K_j)$ due to the definition of $\mathcal{G}_t$. We know from (b) that $E_j(s) \in \mathcal{F}(K_j)$ and hence $E \in \mathcal{F}(K_j)$. If $E_i(t) \cap E_j(s) = \emptyset$, then we are done. However, if there is $(A, F) \in E_i(t) \cap E_j(s)$, then due to the guaranteed minimum speed from (4.4.7) there holds

$$K_j = K_h\Big(\mathscr{R}_s^{(A,F)}(S)\Big) \subset K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) = K_i$$

and hence $E_j(s) \cap F \cap E_i(t) \in \mathcal{F}(K_i)$. □

Bounds for the martingale increments

With the measurability properties from Lemma 4.4.5 established, we can now obtain bounds for the martingale increments as in (4.4.9) and thus prove Proposition 4.4.3.

Sketch proof of Proposition 4.4.3. It remains to find $C > 0$ such that (4.4.9) holds for all $0 \leq s \leq t$. For $\mathbb{E}[m(x_0, S) \,|\, \mathcal{G}_s]$, that is the 'guess' for the arrival time given the information from $\mathcal{G}_s$, we consider two cases: The case where x_0 has already been reached at time s, and the case where x_0 is still outside of the already covered area. We have

$$\begin{aligned}
\mathcal{M}_s &= \mathbb{E}\big[m\big(x_0, S\big) \,\big|\, \mathcal{G}_s\big] - \mathbb{E}\big[m\big(x_0, S\big)\big] \\
&= m\big(x_0, S\big)\chi_{\{m(x_0,S)\leq s\}} + \mathbb{E}\big[\big(s + m\big(x_0, \mathscr{R}_s(S)\big)\big)\chi_{\{m(x_0,S)>s\}} \,\big|\, \mathcal{G}_s\big] \\
&\quad - \mathbb{E}\big[m\big(x_0, S\big)\big] \\
&= m\big(x_0, S\big)\chi_{\{m(x_0,S)\leq s\}} + s\chi_{\{m(x_0,S)>s\}} \\
&\quad + \mathbb{E}\big[m\big(x_0, \mathscr{R}_s(S)\big) \,\big|\, \mathcal{G}_s\big]\chi_{\{m(x_0,S)>s\}} - \mathbb{E}\big[m\big(x_0, S\big)\big],
\end{aligned}$$

where we used Lemma 4.4.5(c) and the definition of the arrival times. We can match this for t by writing

$$\begin{aligned}
\mathcal{M}_t &= m\big(x_0, S\big)\big(\chi_{\{m(x_0,S)\leq s\}} + \chi_{\{m(x_0,S)\in(s,t]\}}\big) + t\chi_{\{m(x_0,S)>t\}} \\
&\quad + \mathbb{E}\big[m\big(x_0, \mathscr{R}_t(S)\big)\chi_{\{m(x_0,S)>t\}} \,\big|\, \mathcal{G}_t\big] - \mathbb{E}\big[m\big(x_0, S\big)\big] \\
&= m\big(x_0, S\big)\chi_{\{m^{(A,F)}(x_0,S)\leq s\}} + t\chi_{\{m(x_0,S)>s\}} \\
&\quad + \mathbb{E}\big[m\big(x_0, \mathscr{R}_t(S)\big) \,\big|\, \mathcal{G}_t\big]\chi_{\{m(x_0,S)>s\}} - \mathbb{E}\big[m\big(x_0, S\big)\big] + R_t,
\end{aligned}$$

introducing a small error term R_t for changing arrival times in $(s, t]$ to t, given by

$$\begin{aligned}
R_t = \big(m\big(x_0, S\big) - t\big)\chi_{\{m(x_0,S)\in(s,t]\}} - \mathbb{E}\big[m\big(x_0, \mathscr{R}_t(S)\big)\chi_{\{m(x_0,S)\in(s,t]\}} \,\big|\, \mathcal{G}_t\big] \\
= \big(m\big(x_0, S\big) - t\big)\chi_{\{m(x_0,S)\in(s,t]\}} \in [s - t, 0].
\end{aligned}$$

Taking the difference, we obtain

$$\begin{aligned}
\mathcal{M}_t - \mathcal{M}_s = \big(\mathbb{E}\big[m\big(x_0, \mathscr{R}_t(S)\big) \,\big|\, \mathcal{G}_t\big] - \mathbb{E}\big[m\big(x_0, \mathscr{R}_s(S)\big) \,\big|\, \mathcal{G}_s\big]\big)\chi_{\{m(x_0,S)>s\}} \\
+ (t - s)\chi_{\{m(x_0,S)>s\}} + R_t. \qquad (4.4.10)
\end{aligned}$$

Due to the finite range of dependence, the future evolution should essentially not depend on the already explored environment, but just on the current state of the set. Accordingly, we want to resolve the conditional expectation and replace it with just an expected value, which corresponds to Lemma 4.4.5(d). In order to use the finite range of dependence in the form of Lemma 4.4.5(d), we need to slightly enlarge the set. Note that due to

the minimum speed from (4.4.7) we have $\mathscr{R}_t(S) \subset K_h(\mathscr{R}_t(S)) + \overline{B}_1 \subset \mathscr{R}_{t+v_\infty^{-1}(\sqrt{d}h+1)}(S)$ and hence there holds

$$m\big(x_0, \mathscr{R}_t(S)\big) \geq m\big(x_0, K_h\big(\mathscr{R}_t(S)\big) + \overline{B}_1\big) \geq m\big(x_0, \mathscr{R}_t(S)\big) - v_\infty^{-1}\Big(\sqrt{d}h + 1\Big).$$

Thus, enlarging $\mathscr{R}_t(S)$ and $\mathscr{R}_s(S)$ in (4.4.10) and then using the independence to resolve the conditional expectation via Lemma 4.4.5(d) and that $\sum_i \chi_{E_i(t)} = \sum_j \chi_{E_j(s)} = 1$ we obtain

$$\begin{aligned}\mathcal{M}_t - \mathcal{M}_s &= (t-s)\chi_{\{m(x_0,S)>s\}} + R_t + R_K \\ &\quad + \sum_{i,j=1}^{\infty} \big(\mathbb{E}\big[m\big(x_0, K_i + \overline{B}_1\big)\big] - \mathbb{E}\big[m\big(x_0, K_j + \overline{B}_1\big)\big]\big)\chi_{E_i(t)\cap E_j(s)}\chi_{\{m(x_0,S)>s\}},\end{aligned} \tag{4.4.11}$$

with $|R_K| \leq v_\infty^{-1}\big(\sqrt{d}h + 1\big)$. In order to bound this term, we compare the expected arrival times from $K_i + \overline{B}_1$ and $K_j + \overline{B}_1$ if $E_i(t) \cap E_j(s) \neq \emptyset$, which can be done by comparing the fastest and slowest evolution during the time step $t - s$: Due to the minimum speed from (4.4.7) and the maximum speed from Lemma 4.5.3 there holds

$$\mathscr{R}_s\big(S\big) + v_\infty(t-s)\overline{B}_1 \subset \mathscr{R}_t\big(S\big) \subset \mathscr{R}_s\big(S\big) + \big(1 + v_{\max}(t-s)\big)\overline{B}_1.$$

Since $K_i = K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big)$ and $K_j = K_h\Big(\mathscr{R}_s^{(A,F)}(S)\Big)$ if there is $(A,F) \in E_i(t) \cap E_j(s)$, this implies

$$K_j + \Big(v_\infty(t-s) - \sqrt{d}h\Big)\overline{B}_1 \subset K_i \subset K_j + \Big(1 + \sqrt{d}h + v_{\max}(t-s)\Big)\overline{B}_1.$$

On the one hand, the maximum speed implies that

$$\mathscr{R}_\tau\big(K_j + \overline{B}_1\big) \subset K_i + \overline{B}_1 \quad \text{for all } \tau \leq \frac{v_\infty}{v_{\max}}(t-s) - \frac{1+\sqrt{d}h}{v_{\max}},$$

while on the other hand, the minimum speed implies that

$$\mathscr{R}_\tau\big(K_j + \overline{B}_1\big) \supset K_i + \overline{B}_1 \quad \text{for } \tau = \frac{v_{\max}}{v_\infty}(t-s) + \frac{1+\sqrt{d}h}{v_\infty}.$$

Hence, by the definition of the arrival times, for $E_i(t) \cap E_j(s) \neq \emptyset$ there holds

$$m\big(x_0, K_j + \overline{B}_1\big) - m\big(x_0, K_i + \overline{B}_1\big) \begin{cases} \geq \dfrac{v_\infty}{v_{\max}}(t-s) - \dfrac{1+\sqrt{d}h}{v_{\max}}, & \text{if } x_0 \notin K_i + \overline{B}_1 \\ \geq 0, & \text{if } x_0 \in K_i + \overline{B}_1 \\ \leq \dfrac{v_{\max}}{v_\infty}(t-s) + \dfrac{1+\sqrt{d}h}{v_\infty}. \end{cases}$$

Plugging these bounds into (4.4.11) and using that $1 - \frac{v_\infty}{v_{\max}} \leq \frac{v_{\max}}{v_\infty} - 1$, we obtain

$$\big|\mathcal{M}_t - \mathcal{M}_s\big| \leq \left(\frac{v_{\max}}{v_\infty} - 1\right)(t-s) + C(\text{data}, h) + \big|R_t\big| + \big|R_x\big|$$

with

$$\begin{aligned}\big|R_t\big| &\leq (t-s)\chi_{\{m(x_0,S)\in(s,t]\}}, \\ \big|R_x\big| &\leq \max\left\{1 - \left(\frac{v_{\max}}{v_\infty} - 1\right), 0\right\}(t-s)\sum_{i\in\mathbb{N}} \chi_{E_i(t)\cap\{x_0\in K_i+\overline{B}_1\}\cap\{m(x_0,S)>s\}}\end{aligned}$$

which yields (4.4.9) once we fix an arbitrary small $h > 0$. In summary, during a time step Δt the best guess for the arrival time changes at most linearly in Δt with rate at most $\frac{v_{\max}}{v_\infty} - 1$ up to error terms. With a more careful approach, the error terms R_t and R_x can be mitigated so that they do not influence this scaling. □

4.5 Preliminaries

Throughout this Section, if not specified further, we always assume $(A,F) \in \Omega$ or $(A,F) = (\widetilde{A},\widetilde{F})_Q$, that is that (A,F) is the restriction of some admissible coefficient field $(\widetilde{A},\widetilde{F}) \in \Omega$ to a closed set $Q \subset \mathbb{R}^d$, see Definition 4.5.8 below. We first state some basic properties of the set evolutions defined in Definition 4.1.2. Then we will present some more specific properties related to stable sets and restricted coefficient fields. Finally, we will provide a tool to check measurability with respect to $\mathcal{F}(U)$. The proofs of these basic properties can be found in Appendix 4.B.

Remark 4.5.1 (Semigroup property of $\mathscr{R}_t(\cdot)$). Letting closed sets $S \subset \mathbb{R}^d$ evolve with respect to a fixed coefficient field (A,F) for different times can be seen as a semigroup, that is

$$\mathscr{R}_{t+s}^{(A,F)}(S) = \mathscr{R}_t^{(A,F)}\Big(\mathscr{R}_s^{(A,F)}(S)\Big) \qquad \text{for any } t,s \geq 0,$$

see e.g. [Gig06, Theorem 4.5.1]. We will often implicitly use this property.

Throughout this paper, we will often use that the (negative) characteristic function of the evolution of a set is a supersolution to the level-set equation (4.1.6).

Lemma 4.5.2. *Let $S \subset \mathbb{R}^d$ be a closed set. The function $u_S^{\mathscr{R}} = u_{S,(A,F)}^{\mathscr{R}}$ given by*

$$u_{S,(A,F)}^{\mathscr{R}}(x,t) \coloneqq 1 - \chi_{\mathscr{R}_t^{(A,F)}(S)}(x)$$

is a supersolution of (4.1.6) with the coefficient field (A,F) in $\mathbb{R}^d \times (0,\infty)$.

A key property of the set evolutions is that – while for example around closing holes or small gaps the interface might move at arbitrary high speeds as typical for the standard mean curvature flow – on larger scales, there is a maximum speed which is essentially dominated by the first order term.

Lemma 4.5.3 (Maximum speed of front propagation). *Let $S \subset \mathbb{R}^d$ be a closed set. Then for any $\delta > 0$ there holds $\mathscr{R}_T^{(A,F)}(S) \subset S + (\delta + T v_{\max,\delta})\overline{B}_1$ for all $T > 0$, where*

$$v_{\max,\delta} \coloneqq \max\{C_{1F},\, 6C_{1A}\} + C_{1A}\delta^{-1}.$$

We write $v_{\max} \coloneqq v_{\max,1}$.

4.5.1 Stable sets

One of the key concepts in this work is the guarantee that a set will not shrink with respect to the evolution according to a given coefficient field (A,F). As stated in Definition 4.3.6, we call a set *stable* with respect to (A,F) if its negative characteristic function is a stationary supersolution to the level-set equation (4.1.6). The first lemma of this subsection states that these stable sets indeed do not shrink under the evolution with respect to (A,F) and in fact the sets preserve the stability under the evolution.

Lemma 4.5.4 (Preservation of Stability). *Let $S \subset \mathbb{R}^d$ be a closed set. If S is stable with respect to (A,F), then also $\mathscr{R}_t^{(A,F)}(S)$ is stable and $\mathscr{R}_s^{(A,F)}(S) \subset \mathscr{R}_t^{(A,F)}(S)$ for any $0 \leq s \leq t$.*

Next, we translate the comparison principle for viscosity solutions to our set evolutions: When a set envelops another set, at any time the evolution of the first set will envelop that of the other. As long as the larger set is stable, this stays true if we replace the full envelopment with h-envelopment from Definition 4.3.8, that is when ignoring holes of width smaller than $2h$.

Lemma 4.5.5 (The comparison principle for set evolutions)**.**

(a) Let $S_1 \subset S_2$. Then $\mathscr{R}_t(S_1) \subset \mathscr{R}_t(S_2)$ holds for all $t \geq 0$.

(b) Let $S_1 \subset_h S_2$ for some $h > 0$. If in addition S_2 is stable, then $\mathscr{R}_t(S_1) \subset_h \mathscr{R}_t(S_2)$ holds for all $t \geq 0$.

We use the above comparison principle to obtain the following lemma for the iterative application of the effective minimum speed from Definition 4.3.8.

Lemma 4.5.6 (Iterative application of the effective minimum speed)**.** *Let $S \subset \mathbb{R}^d$ be a stable set with respect to some coefficient field (A, F). Assume that (A, F) admits $v > 0$ as an effective minimum speed of propagation on a scale $h > 0$ in some set $U \subset \mathbb{R}^d$ as in Definition 4.3.8. Then for any $n \in \mathbb{N}$ with $U \supset \big(S + n\overline{B}_h\big) \cap \big(\partial S \cup S^c\big)$ there holds*

$$S + n\overline{B}_h \subset_h \mathscr{R}^{(A,F)}_{nv^{-1}}\big(S\big).$$

Assumption (P◎) provides an effective minimum speed $v_{\min}$ on scales $h \geq h_0$, which applies for an area $U \subset \mathbb{R}^d$ if the event $E^{\circledcirc}_h(U)$ holds. For bounded U and large enough $h \geq h_0$, this event has high probability. We now use Lemma 4.5.6 to show that this implies that the arrival time of an h-fat set in a target area $\overline{B}_h(x_0)$ is very likely smaller than the time $T_h(x_0, S)$, which essentially corresponds to the arrival time of the same set expanding continuously at that minimum speed.

Lemma 4.5.7 (Guaranteed upper bound for arrival times in target areas)**.** *Let $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ be an h-fat set with bounded boundary. Then with $v_{\min} > 0$, $c_s \geq 0$ from Assumption (P◎) there holds*

$$\min\big\{m\big(y, S\big) \,:\, y \in \overline{B}_h(x_0)\big\}\chi_{E^{\circledcirc}_h(U)} \leq T_h\big(x_0, S\big) \coloneqq v_{\min}^{-1}\operatorname{dist}(x_0, S) + c_s h$$

for any $U \subset \mathbb{R}^d$ with

$$U \supset \big(S + \operatorname{dist}(x_0, S)\overline{B}_1\big) \cap \big(\partial S \cup S^c\big).$$

This means that the artificial upper bound in the definition of the truncated arrival times $m^{(A,F)}_h(x_0, S)$ in the target area $\overline{B}_h(x_0)$ only kicks in for $(A, F) \notin E^{\circledcirc}_h(U)$ from Assumption (P◎), which by this assumption is very unlikely, see (4.3.6).

4.5.2 Restricted solutions

In order to apply the finite range of dependence condition (P2), we need to guarantee that some properties of solutions depend on the coefficient field only in a fixed area, as discussed for example in the proof of Lemma 4.4.5. In Chapter 5, we further want to ensure that a set evolves within or along a fixed area. For this purpose, we introduce suitably restricted coefficient fields.

Introduction of restricted coefficient fields

Given an admissible coefficient field (A, F), we restrict it to some set $Q \subset \mathbb{R}^d$ by leaving it untouched on Q and then continuously transition to constant coefficients within $Q + \overline{B}_1 \setminus Q$. In fact, instead of 1 we could choose an arbitrary small scale for the transition, but this would only impact the constants in our results. While in principle we could transition to arbitrary coefficients, we choose a very negative forcing, such that sets starting within Q are guaranteed to stay trapped within a small neighborhood of Q.

Definition 4.5.8 (Restricted coefficient fields)**.** Let $Q \subset \mathbb{R}^d$ be a Borel set. Given $(A, F) \in \Omega$ with $A = \sigma\sigma^\intercal$ we define $(A, F)_Q := (A_Q, F_Q)$ with $A_Q = \sigma_Q \sigma_Q^\intercal$, the coefficient field restricted to Q, as

$$\sigma_Q(x,e) := \begin{cases} \sigma(x,e) & \text{for } \operatorname{dist}(x,Q) \le \frac{2}{3}, \\ 9\big(1-\operatorname{dist}(x,Q)\big)^2\sigma(x,e) & \text{for } \operatorname{dist}(x,Q) \in \left[\frac{2}{3},1\right], \\ 0 & \text{for } \operatorname{dist}(x,Q) \ge 1, \end{cases} \tag{4.5.1}$$

$$F_Q(x,e) := \begin{cases} F(x,e) & \text{for } \operatorname{dist}(x,Q) \le 0, \\ 3\left(\frac{1}{3}-\operatorname{dist}(x,Q)\right)F(x,e) - 3\operatorname{dist}(x,Q)\max\{C_{1F},\,6C_{1A}\} & \\ & \text{for } \operatorname{dist}(x,Q) \in \left[0,\frac{1}{3}\right], \\ -\max\{C_{1F},\,6C_{1A}\} & \text{for } \operatorname{dist}(x,Q) \ge \frac{1}{3}. \end{cases} \tag{4.5.2}$$

By definition, the evolution of a set with respect to $(A, F)_Q$ only depends on (A, F) restricted to $Q + B_1$ and hence should be $\mathcal{F}(Q + B_1)$ measurable, see Lemma 4.5.12. As a consequence of this definition, any $S \subset Q$ stays stuck in $Q + B_{\frac{2}{3}}$ when evolving by (4.1.6) with coefficients $(A, F)_Q$. In addition, in this case the evolution of a set with respect to $(A, F)_Q$ provides a lower bound to the evolution of the same set with respect to the unmodified coefficient field (A, F). For the rigorous statement see Lemma 4.5.10 below.

Remark 4.5.9 (Well-posedness of evolution for restricted coefficient fields). Existence, uniqueness and uniform continuity for solutions of the level-set equation as well as invariance for the evolution of the level sets as stated in Definition 4.1.2 for $(A, F) \in \Omega$ also hold if $(A, F) = (\widetilde{A}, \widetilde{F})_Q$ for some $(\widetilde{A}, \widetilde{F}) \in \Omega$ and $Q \subset \mathbb{R}^d$.

Properties of restricted coefficient fields

As already stated above, the choice for the constant coefficients in the definition of restricted coefficient fields $(A, F)_Q$ means that sets stay trapped in a neighborhood of Q, which we make precise in the following lemma.

Lemma 4.5.10 (Properties of restricted coefficient fields)**.** *Let u be a supersolution to (4.1.6) with coefficients $(A, F)_Q$ for $(A, F) \in \Omega$ and some closed set $Q \subset \mathbb{R}^d$, which satisfies*

$$u(\cdot,0) \le C \quad \text{in } \mathbb{R}^d, \qquad \{x : u(\cdot,0) < C\} \subset \big(Q^\oslash\big)^c,$$

where, as in Figure 4.5.1,

$$Q^\oslash := \bigcup \left\{ B_{\frac{1}{3}}(x) : \operatorname{dist}(x,Q) \ge \frac{2}{3} \right\}, \quad \text{see Figure 4.5.1.} \tag{4.5.3}$$

Then for any $t \ge 0$ there holds

$$u(\cdot,t) \le C \quad \text{in } \mathbb{R}^d, \qquad \{x : u(\cdot,t) < C\} \subset \big(Q^\oslash\big)^c.$$

In addition, u is a supersolution also to (4.1.6) with coefficients $(A, F)_{\widetilde{Q}}$ for any $\widetilde{Q} \supset Q$.

In particular, for any $S \subset (Q^\oslash)^c$ and $\widetilde{Q} \supset Q$ there holds

(a) $\mathscr{R}_t^{(A,F)_Q}(S) \subset (Q^\oslash)^c \subset Q + \overline{B}_{\frac{2}{3}}$ *for any* $t \ge 0$.

(b) If S is stable with respect to $(A,F)_Q$, then the set is stable with respect to $(A,F)_{\widetilde{Q}}$.

(c) $u^{\mathscr{R}}_{S,(A,F)_Q} = 1 - \chi_{\mathscr{R}_\cdot^{(A,F)_Q}(S)}$ as in Lemma 4.5.2 is a supersolution to (4.1.6) with coefficients $(A,F)_{\widetilde{Q}}$.

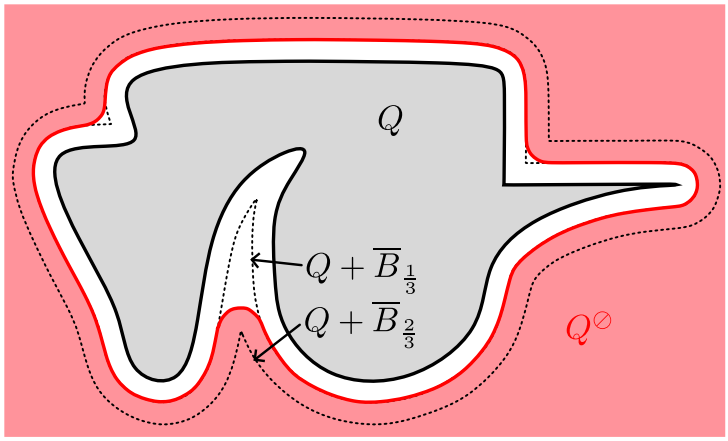


Figure 4.5.1: An illustration of our notion of restricted coefficient fields. The set Q to which the coefficient field is restricted is depicted in grey. The set $Q^{\oslash}$, which sets initialized in $(Q^{\oslash})^c$ will not reach, is depicted in red

Having chosen the constant coefficients such that sets stay trapped in a neighborhood of Q makes it easier to compare the evolution with respect to the original coefficient field to the evolution with respect to the restricted field.

Lemma 4.5.11 (A condition for the equivalence of restricted and unrestricted solutions)**.** *Let $(A,F) \in \Omega$ and $S \subset \mathbb{R}^d$ be a closed set. Suppose $t_0 > 0$ and $Q \subset \widetilde{Q} \subset \mathbb{R}^d$. If $S \subset (Q^{\oslash})^c$ as in Lemma 4.5.10, then*

$$\mathscr{R}_t^{(A,F)_Q}(S) \subset \mathscr{R}_t^{(A,F)_{\widetilde{Q}}}(S) \quad \text{for all } 0 \le t.$$

If $\mathscr{R}_t^{(A,F)_Q}(S) + B_\varepsilon \subset Q$ for all $0 \le t \le t_0$ given some $\varepsilon, t_0 > 0$, then there holds

$$\mathscr{R}_t^{(A,F)_Q}(S) = \mathscr{R}_t^{(A,F)_{\widetilde{Q}}}(S) \quad \text{for all } 0 \le t \le t_0.$$

4.5.3 Measurability of events with respect to $\mathcal{F}(U)$

To prove the measurability of events with respect to the σ-algebras $\mathcal{F}(U)$ defined in (4.3.5), our main strategy is the application of the following lemma, which states that it is sufficient to check if the event is closed with respect to uniform convergence of the coefficient fields in U.

Lemma 4.5.12. *Let $U \subset \mathbb{R}^d$ be a closed set. Let $E \subset \Omega \subset C(\mathbb{R}^d \times \mathbb{S}^{d-1}, \mathbb{R}^{d\times d} \times \mathbb{R})$ be an event which is closed in Ω with respect to the norm*

$$\|(A,F)\|_{C_w(U\times\mathbb{S}^{d-1},\mathbb{R}^{d\times d}\times\mathbb{R})} := \sup_{x\in U,\, e\in\mathbb{S}^{d-1}} \frac{1}{1+|x|^2}\big|(A(x,e),F(x,e))\big|,$$

which corresponds to weighted uniform convergence and is well defined on Ω due to the uniform bounds from (4.3.1) and (4.3.3). Then E is measurable with respect to $\mathcal{F}(U)$.

4.6 Fluctuation bounds for the truncated arrival times in target areas

In this section, we aim to bound fluctuations of the arrival time at a point $x_0 \in \mathbb{R}^d$ around its expectation given a fixed starting set $S \subset \mathbb{R}^d$. We will follow the approach as laid out in Subsection 4.4.3. However, compared to Subsection 4.4.3 there are multiple main issues:

1.) Since we have no guaranteed minimum speed, x_0 might never be reached, that is potentially $\mathbb{P}[m(x_0, S) = \infty] > 0$.

2.) Since in general sets might shrink, the independence of their evolution from the values of the coefficient fields in their interior breaks down.

3.) Since locally the effective minimum speed might not hold, we will not get uniform bounds for the martingale increments. Hence, Azuma's inequality is not applicable.

To deal with these issues, we innovate as follows:

1.) We do not work with the actual arrival time at $x_0 \in \mathbb{R}^d$, but the arrival time in the surrounding target area $\overline{B}_h(x_0)$, ignoring holes of width up to $2h$ such that the effective minimum speed guaranteed by Assumption (P◎) applies. Since with low probability this might still yield an infinite arrival time, we introduce a truncation, which however is increasingly unlikely to kick in on larger scales, see Definition 4.1.6 and Lemma 4.5.7.

2.) We use stable approximations from Assumption (P◎) to guarantee that sets can not shrink by too much and hence the independence between the coefficients in the interior of the set and its future evolution stays intact. We keep track of the error from the unlikely case that the approximations do not exist.

3.) We introduce an alternative to of Azuma's inequality in Appendix 4.C, so that it suffices to obtain uniform bounds on the martingale increments only with overwhelming and not necessarily full probability.

To be precise, let us briefly recall the precise definition of the arrival times from Definition 4.1.6 for $S \subset \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$ and $h \geq h_0$. We write

$$\begin{aligned} m(x_0, S) = m^{(A,F)}(x_0, S) &:= \min\Big\{t \geq 0 \,:\, x_0 \in \mathscr{R}_t^{(A,F)}(S)\Big\}, \\ m_h(x_0, S) = m_h^{(A,F)}(x_0, S) &:= \min\Big\{\min_{y \in \overline{B}_h(x_0)} m(y, S),\, T_h(x_0, S)\Big\} \\ &\qquad \text{with } T_h(x_0, S) := v_{\min}^{-1} \operatorname{dist}(x_0, S) + c_s h, \end{aligned}$$

where $t \mapsto \mathscr{R}_t^{(A,F)}(S)$ is the evolution of the set S with respect to the coefficient field (A, F) from Definition 4.1.2 and $c_s > 0$ is a parameter from Assumption (P◎) for the stable approximations.

Note that these are well defined since if x_0 or $\overline{B}_h(x_0)$ are reached at some point, there exists a minimum due to the maximum speed of propagation from Lemma 4.5.3. Further, these are random variables with respect to $(\Omega, \mathcal{F}(\mathbb{R}^d), \mathbb{P})$, see Lemma 4.6.4 below for the measurability. We now state the main result from this section, showing that the fluctuations of these truncated arrival times in the target area $\overline{B}_h(x_0)$ are of order $\sqrt{h \operatorname{dist}(x_0, S)}$.

Proposition 4.6.1 (Fluctuation bounds for truncated arrival times)**.** *Assume that (P2) and (P⊙) hold. There are* $c, C_{\mathrm{fl}}, c_{\mathrm{fl}}, C_{\mathrm{co}} = c(\mathrm{data}), C_{\mathrm{fl}}(\mathrm{data}), c_{\mathrm{fl}}(\mathrm{data}), C_{\mathrm{co}}(\mathrm{data}) > 0$ *such that for all* $x_0 \in \mathbb{R}^d$, $h \geq h_0$ *and* h*-fat sets* $S \subset \mathbb{R}^d$ *with bounded boundary as well as*

$$\frac{\operatorname{diam}(\partial S) + \operatorname{dist}(x_0, S)}{h} \leq \exp\bigl(c h^{\alpha_\odot}\bigr) \tag{4.6.1}$$

there holds

$$\begin{aligned}
&\mathbb{P}\bigl[\bigl|m_h(x_0,S) - \mathbb{E}\bigl[m_h(x_0,S)\bigr]\bigr| \geq \lambda\bigr] \\
&\quad\leq \begin{cases}
C_{\mathrm{fl}} \exp\left(-\dfrac{c_{\mathrm{fl}}\lambda}{\sqrt{h \operatorname{dist}(x_0,S)}}\right), & \text{if } \lambda \leq C_{\mathrm{co}} \dfrac{h^{\alpha_\odot}\sqrt{h \operatorname{dist}(x_0,S)}}{\log\left(\frac{\operatorname{dist}(x_0,S)}{h}\right)}, \\
C_{\mathrm{fl}} \exp\left(-\dfrac{c_{\mathrm{fl}} h^{\alpha_\odot}}{\log\left(\frac{\operatorname{dist}(x_0,S)}{h}\right)}\right), & \text{if } \lambda > C_{\mathrm{co}} \dfrac{h^{\alpha_\odot}\sqrt{h \operatorname{dist}(x_0,S)}}{\log\left(\frac{\operatorname{dist}(x_0,S)}{h}\right)}, \\
0, & \text{if } \lambda > T_h(x_0,S),
\end{cases}
\end{aligned} \tag{4.6.2}$$

where $\alpha_\odot > 0$ *is the exponential rate from the probability bounds in Assumption (P⊙) for the existence of stable approximation and the applicability of the effective minimum speed of propagation.*

Analogously to Subsection 4.4.3, we now define the martingale

$$\mathcal{M}_t := \mathbb{E}\bigl[m_h(x_0,S) \,\big|\, \mathcal{G}_t\bigr] - \mathbb{E}\bigl[m_h(x_0,S)\bigr], \tag{4.6.3}$$

where $(\mathcal{G}_t)_{t\geq 0}$ is a filtration with the σ-algebra $\mathcal{G}_t \subset \mathcal{F}(\mathbb{R}^d)$ containing the information on the approximate state of the set evolution at times $s \leq t$ and on the coefficient field around the area explored up to time t.

Let us first make precise the definition of $\mathcal{G}_t$. As in Subsection 4.4.3, for any set $M \subset \mathbb{R}^d$ we define its discrete approximation as

$$K_h(M) := \bigcup \bigl(y + [0,h]^d \,:\, y \in h\mathbb{Z}^d \text{ and } y + [0,h]^d \cap M \neq \emptyset\bigr).$$

Since $\bigl\{K_h(M) \,:\, M \subset \mathbb{R}^d\bigr\}$ is not countable, we need a smaller state space for the discretized set evolutions. Assuming that there is a stable (h, c_s)-approximation $S_h \subset S$ of the set $S \subset \mathbb{R}^d$ as in Definition 4.3.8, we have

$$\mathbb{R}^d \setminus \bigl(S^c + c_s v_{\max} B_h\bigr) \subset S_h \subset \mathscr{R}_t(S_h) \subset \mathscr{R}_t(S) \subset S + (1 + v_{\max} t)\overline{B}_1$$

with the first inclusion due to the definition of stable (h, c_s)-approximations, the second inclusion due to the stability of S_h and Lemma 4.5.4, the third inclusion due to the comparison principle as in Lemma 4.5.5 and the final inclusion due to the maximum speed of the evolution as in Lemma 4.5.3. Putting these limitations on the state space and since ∂S is bounded, we obtain a countable family $\{K_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\bigl\{K_i\bigr\}_{i\in\mathbb{N}} = \bigl\{K_h(M) \,:\, \mathbb{R}^d \setminus \bigl(S^c + c_s v_{\max} B_h\bigr) \subset M \subset S + B_R \text{ for some } R > 0\bigr\}.$$

Based on this state space we define the events $\{E_i(t)\}_{i\in\mathbb{N}}$ capturing the approximate state of the set evolution at time $t \geq 0$ as

$$E_i(s) := \Bigl\{(A,F) \in \Omega \,:\, K_h\Bigl(\mathscr{R}_s^{(A,F)}(S)\Bigr) = K_i\Bigr\} \cap E_h^{\odot}(\partial S), \tag{4.6.4}$$

where the restriction to $E_h^{\circledcirc}(\partial S)$ from Assumption (P⊙) guarantees the existence of a stable approximation. Finally, we can now define the appropriate filtration $\{\mathcal{G}_t\}_{t\geq 0}$ capturing the information on the current state of the set evolution and the coefficient field in a neighborhood of the explored area as

$$\begin{aligned}\mathcal{G}_0 &\coloneqq \{\emptyset, \Omega\},\\ \mathcal{G}_t &\coloneqq \sigma\text{-algebra generated by the events } E_i(s)\cap F\\ &\qquad\qquad \text{for any } \ 0\leq s\leq t \text{ and } F\in\mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)\end{aligned} \tag{4.6.5}$$

with $\{\mathcal{F}(U)\}_U$ from (4.3.5) denoting the σ-algebras containing the information on the coefficient field in $U\subset\mathbb{R}^d$ and $C_{\mathcal{G}} = C_{\mathcal{G}}(\text{data}) > 0$ chosen large enough. Essentially, we choose $C_{\mathcal{G}}$ such that expanding K_i by $C_{\mathcal{G}}\overline{B}_h$ covers the additional area which the evolution of a set might have covered before falling back to within K_i, at least assuming that the set has a stable (h, c_s)-approximation. For the precise requirements on $C_{\mathcal{G}}$, see Lemma 4.6.5, Lemma 4.6.6 and Lemma 4.6.7 as well as their proofs.

In order to prove Proposition 4.6.1, we largely follow the approach laid out in Subsection 4.4.3. In Subsection 4.6.1, we will deal with the necessary measurability issues corresponding to Lemma 4.4.5, essentially using the finite range of dependence to show that the future evolution of a set is independent from the coefficient field explored while reaching its current state. In Subsection 4.6.2 we will use these observations to bound the martingale increments given an event of overwhelming probability and then apply the alternative to Azuma's inequality from Appendix 4.C to obtain Proposition 4.6.1.

But first, since in our setting S is not fully guaranteed to be stable, two obvious properties from the minimum speed case treated in Subsection 4.4.3 – $\bigcup_{i\in\mathbb{N}} E_i(t) = \Omega$ and $S\subset K_i$ for $i\in\mathbb{N}$ – do not hold and we need replacements.

Lemma 4.6.2. *Let $S\subset\mathbb{R}^d$ be an h-fat set with bounded boundary. Let the events $(E_i(t))_i$ for some $t\geq 0$ be defined as in (4.6.4). The sets $(E_i(t))_i$ are disjoint and there holds*

$$\bigcup_{i\in\mathbb{N}} E_i(t) = E_h^{\circledcirc}\big(\partial S\big).$$

Proof. By the definition (4.6.4) it is clear that '$\subset$' holds. It remains to show '$\supset$'. Assume that $(A,F)\in E_h^{\circledcirc}(\partial S)$. Then with (∗iii) from Assumption (P⊙) there exists a stable (h,c_s)-approximation S_h of S. In particular, S_h is stable with respect to (A,F), satisfies $S_h\subset S$ and $S_h\supset\mathbb{R}^d\setminus(S^c + c_s v_{\max}B_h)$. Therefore, with Lemma 4.5.4 and 4.5.5 we obtain

$$S_h\subset\mathscr{R}_t\big(S_h\big)\subset\mathscr{R}_t\big(S\big)\quad\text{ and }\quad\mathscr{R}_t\big(S\big)\supset\mathbb{R}^d\setminus\big(S^c + c_s v_{\max}B_h\big).$$

Due to the maximum speed of propagation from Lemma 4.5.3, we have

$$\mathscr{R}_t\big(S\big)\subset S + \big(1+v_{\max}t\big)\overline{B}_1.$$

Thus there exists $i_0\in\mathbb{N}$ with $K_h(\mathscr{R}_t(S)) = K_{i_0}$. Hence, $E_{i_0}(t)$ holds and since the events $\{E_i(t)\}_i$ are clearly disjoint, we are done. □

Lemma 4.6.3. *Let $S\subset\mathbb{R}^d$ be an h-fat set with bounded boundary. Consider the events $(E_i(t))_i$ for some $t\geq 0$ as defined in (4.6.4). Then for any $i\in\mathbb{N}$ with $E_i(t)\neq\emptyset$ there holds*

$$S\subset K_i + \big(1+\big(1+v_{\max}c_s\big)h\big)\overline{B}_1.$$

Proof. Let $(A,F) \in E_i(t)$, in particular $(A,F) \in E_h^{\circledcirc}(\partial S)$. Then with (∗iii) from Assumption (P◎) there exists a stable (h, c_s)-approximation S_h of S with respect to (A,F). In particular, we have $S_h \subset S \subset_h \mathscr{R}_{c_s h}^{(A,F)}(S_h)$. On the one hand, with Lemma 4.5.4 and the stability of S_h as well as the comparison principle from Lemma 4.5.5 we obtain

$$S_h \subset \mathscr{R}_t^{(A,F)}\big(S_h\big) \subset \mathscr{R}_t^{(A,F)}\big(S\big) \subset K_i$$

since $K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) = K_i$. On the other hand, with $v_{\max}$ from Lemma 4.5.3 we have

$$S \subset_h \mathscr{R}_{c_s h}^{(A,F)}\big(S_h\big) \subset S_h + (1 + v_{\max} c_s h)\overline{B}_1.$$

Combining both, we obtain

$$S \subset K_i + (1 + v_{\max} c_s h)\overline{B}_1 + \overline{B}_h.$$

□

4.6.1 Measurability issues

In this subsection we will provide replacements for the measurability properties in Lemma 4.4.5, which relied on the guaranteed minimum speed. We first obtain an analogous result to Lemma 4.4.5(a): Arrival times are indeed measurable and further independent from the coefficient field in the interior of the starting set – as long as a stable approximation for this starting set exists, which is guaranteed by the event $E_h^{\circledcirc}(\partial S)$ introduced in Assumption (P◎). If $E_h^{\circledcirc}(\partial S)$ does not hold, then the interface might partially collapse into the interior of the starting set, breaking the independence Hence, we multiply the truncated arrival times with $\chi_{E_h^{\circledcirc}(\partial S)}$ to only consider the case that $E_h^{\circledcirc}(\partial S)$ holds.

Lemma 4.6.4 (Measurability of arrival times)**.** *Let $S \subset \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$. Then*

$$m\big(x_0, S\big) \text{ and } m_h\big(x_0, S\big) \text{ are measurable with respect to } \mathcal{F}(\mathbb{R}^d). \tag{4.6.6}$$

Assume that in addition S is an h-fat set. Then for the event $E_h^{\circledcirc}(\partial S)$ from Assumption (P◎) there holds

$$m\big(x_0, S\big)\chi_{E_h^{\circledcirc}(\partial S)} \text{ and } m_h\big(x_0, S\big)\chi_{E_h^{\circledcirc}(\partial S)} \text{ are measurable with respect to } \mathcal{F}\big(\overline{S^c} + \max\{c_s v_{\max}, c_{\mathcal{F}}\}\overline{B}_h\big). \tag{4.6.7}$$

Proof. We only prove the measurability of the truncated arrival times $m_h(x_0, S)$ in the target area $\overline{B}_h(x_0)$, since the proofs for the actual arrival times $m(x_0, S)$ are analogous. Regarding (4.6.6), it is sufficient to show for any $T \geq 0$ that

$$\Big\{(A,F) \in \Omega \,:\, m_h^{(A,F)}\big(x_0, S\big) \leq T\Big\} \in \mathcal{F}(\mathbb{R}^d).$$

This is clear for $T \geq T_h(x_0, S)$. For $T < T_h(x_0, S)$ we will apply Lemma 4.5.12, for which we need to check that the set is closed with respect to weighted uniform convergence of the coefficient fields.

Let $(A_k, F_k) \in \Omega$ with $T_k := m_h^{(A_k,F_k)}(x_0, S) \leq T$ and $(A_k, F_k) \to (A, F) \in \Omega$ with respect to the norm $\|\cdot\|_{C_w(\mathbb{R}^d \times \mathbb{S}^{d-1}, \mathbb{R}^{d\times d} \times \mathbb{R})}$ from Lemma 4.5.12. By definition, for $k \in \mathbb{N}$ there exists

$y_k \in \overline{B}_h(x_0) \cap \mathscr{R}_{T_k}^{(A_k,F_k)}(S)$. Then there exists a (non-relabeled) subsequence such that $T_k \to T_0 \leq T$ and $y_k \to y_0 \in \overline{B}_h(x_0)$. We take the lower relaxed limit as defined in (4.A.5),

$$v := \liminf_{k\to\infty}{}_* \, u^{\mathscr{R}}_{S,(A_k,F_k)},$$

where $u^{\mathscr{R}}_{S,(A_k,F_k)}(x,t) = 1 - \chi_{\mathscr{R}_t^{(A_k,F_k)}(S)}(x)$ are the supersolutions from Lemma 4.5.2. Due to the stability of viscosity solutions from Theorem 4.A.7 and the weighted uniform convergence of (A_k, F_k), we obtain that v is a supersolution of the level-set equation (4.1.6) with the coefficient field (A, F). Since $S \subset \{x \in \mathbb{R}^d : v(x,0) \leq 0\}$, with the comparison principle from Theorem 4.A.9 and Remark 4.A.10, we obtain $u_{S,(A,F)} \leq v$ for a continuous solution $u_{S,(A,F)}$ from Definition 4.1.2 and thus in particular

$$\mathscr{R}_{T_0}^{(A,F)}(S) = \{x \in \mathbb{R}^d : u_{S,(A,F)}(x,T_0) \leq 0\} \supset \{x \in \mathbb{R}^d : v(x,T_0) \leq 0\} \supset \{y_0\},$$

because $v(y_0,T_0) \leq \liminf_{k\to 0} u^{\mathscr{R}}_{S,(A_k,F_k)}(y_k,T_k) = 0$ due to the definition of the lower relaxed limit. Since $y_0 \in \overline{B}_h(x_0)$, this implies $m_h^{(A,F)}(x_0,S) \leq T_0 \leq T$ and hence Lemma 4.5.12 yields the measurability of $m_h(x_0,S)$ with respect to $\mathcal{F}(\mathbb{R}^d)$.

Regarding (4.6.7), we write

$$E := \{(A,F) \in \Omega : \text{there exists } S_h \subset S \text{ stable w.r.t. } (A,F) \text{ and with } S_h \supset \mathbb{R}^d \setminus \big(S^c + c_s v_{\max} B_h\big)\}$$

for the event that there exists a stable set approximating S from inside. Clearly, Assumption (P◎) implies $E_h^{\circledcirc}(\partial S) \subset E$, since a stable (h,c_s)-approximation of S satisfies these properties. Therefore we have

$$m_h(x_0,S)\chi_{E_h^{\circledcirc}(\partial S)} = m_h(x_0,S)\chi_E \chi_{E_h^{\circledcirc}(\partial S)},$$

Since $E_h^{\circledcirc}(\partial S) \in \mathcal{F}\big(\partial S + c_{\mathcal{F}}\overline{B}_h\big)$ by assumption, it is hence sufficient to show for any $T > 0$ that

$$O_0 := \Big\{(A,F) \in \Omega : m_h^{(A,F)}(x_0,S)\chi_E = 0\Big\} \in \mathcal{F}\big(\overline{S^c} + c_s v_{\max}\overline{B}_h\big)$$

$$\text{and} \quad O_T := \Big\{(A,F) \in \Omega : 0 < m_h^{(A,F)}(x_0,S)\chi_E \leq T\Big\} \in \mathcal{F}\big(\overline{S^c} + c_s v_{\max}\overline{B}_h\big).$$

Regarding O_0, if $\big(x_0 + \overline{B}_h\big) \cap S = \emptyset$, then $m_h^{(A,F)}(x_0,S) > 0$ due to the bounds on the speed of expansion from Lemma 4.5.3 and we have $O_0 = E^c$. If $\big(x_0 + \overline{B}_h\big) \cap S \neq \emptyset$, then we clearly have $m_h^{(A,F)}(x_0,S) = 0$ and hence $O_0 = \Omega$. Hence, to show the measurability of O_0, it is enough to show that $E \in \mathcal{F}\big(\overline{S^c} + c_s v_{\max}\overline{B}_h\big)$. Regarding O_T, note that there has to be $\tau > 0$ with $m_h^{(A,F)}(x_0,S) \geq \tau$ for all $(A,F) \in O_T$ if $O_T \neq \emptyset$, because otherwise we would obtain $\big(x_0 + \overline{B}_h\big) \cap S \neq \emptyset$ again due to the bounds on the speed of expansion from Lemma 4.5.3. This would imply $O_0 = \Omega$ and hence $O_T = \emptyset$, yielding a contradiction. With these considerations and taking the proof of (4.6.6) from above as a blueprint, both measurability statements are straightforward to show with Lemma 4.5.12 using the stability of viscosity solutions, i.e. Theorem 4.A.7. The key insight is that due to the existence of the stable set $S_h \subset S$ from E for any $t \geq 0$ we have

$$\mathbb{R}^d \setminus \big(S^c + c_s v_{\max} B_h\big) \subset S_h \subset \mathscr{R}_t(S_h) \subset \mathscr{R}_t(S)$$

where we used that with Lemma 4.5.4 the stable set S_h does not shrink and then applied the comparison principle from Lemma 4.5.5. Thus, the interface never leaves $\overline{S^c} + c_s v_{\max}\overline{B}_h$, because of which the evolution is not impacted by the values of the coefficient field outside of this area. □

In order to obtain replacements for the remaining parts of Lemma 4.4.5, we need the following lemma. It describes one of the key implications of the existence of a stable (h, c_s)-approximation of a set $S \subset \mathbb{R}^d$: The area explored before the evolution of S reaches a given state can not be much larger than that state. In particular, in order to check if the set evolution ends up in that state, it is sufficient to use the evolution with respect to the coefficient field restricted to a slightly larger area than the state for which we check. This will imply that the events E_i defined in (4.6.4) are measurable with respect to information from a set only slightly larger than K_i.

Lemma 4.6.5. *Let $h \geq h_0$ and $S \subset \mathbb{R}^d$ be an h-fat set. Let $(A, F) \in E_h^{\circledcirc}(\partial S)$. Let $U \subset \mathbb{R}^d$ be such that*

$$S + (2 + v_{\max} c_s h)\overline{B}_1 \subset \hat{U} := U + \overline{B}_R \quad \text{for some } R \geq 2 + \big(1 + v_{\max} c_s\big)h. \tag{4.6.8}$$

Let $(A, F)_{\hat{U}}$ be the restriction of the coefficient field to $\hat{U}$ as in Definition 4.5.8. If there holds $\mathscr{R}_{t_0}^{(A,F)_{\hat{U}}}(S) \subset U$ for some $t_0 \geq 0$, then $\mathscr{R}_t^{(A,F)}(S) \subset \hat{U}$ for all $0 \leq t \leq t_0$ and

$$\mathscr{R}_t^{(A,F)_{\hat{U}}}\big(S\big) = \mathscr{R}_t^{(A,F)}\big(S\big) \quad \text{for all } 0 \leq t \leq t_0.$$

Proof. Once we obtain that $\mathscr{R}_t^{(A,F)_{\hat{U}}}(S) + B_1 \subset \hat{U}$ for all $0 \leq t \leq t_0$, Lemma 4.5.11 yields $\mathscr{R}_t^{(A,F)_{\hat{U}}}(S) = \mathscr{R}_t^{(A,F)}(S)$ for all $0 \leq t \leq t_0$ and the result follows immediately. The idea for proving that $\mathscr{R}_t^{(A,F)_{\hat{U}}}(S)$ can not be too large is the following:

- S is sandwiched between a stable set S_h from the inside and from the outside by the same S_h but evolved for a short additional time span $c_s h$.
- S_h has to stay within U at all times, otherwise it and thus S can not drop back into U.
- This also means that the set bounding S from the outside can not be farther from U than evolving U with the maximum speed from Lemma 4.5.3 for the short time span $c_s h$.

We will now formalize this argument, also considering the restriction of the coefficient field to $\hat{U}$ as in Definition 4.5.8. If $(A, F) \in E_h^{\circledcirc}(\partial S)$ holds, then with $(*\text{iii})$ from Assumption (P$\circledcirc$) there exists a stable (h, c_s)-approximation S_h of S. In particular, S_h is stable with respect to (A, F) and satisfies $S_h \subset S \subset_h \mathscr{R}_{c_s h}^{(A,F)}(S_h)$. Due to the maximum speed of propagation described in Lemma 4.5.3, for all $0 \leq t \leq c_s h$ there holds

$$\mathscr{R}_t^{(A,F)_{\hat{U}}}\big(S_h\big) + B_1 \subset S_h + (2 + v_{\max} c_s h)\overline{B}_1 \subset S + (2 + v_{\max} c_s h)\overline{B}_1 \subset \hat{U}.$$

Hence, with Lemma 4.5.11 we obtain $\mathscr{R}_{c_s h}^{(A,F)}(S_h) = \mathscr{R}_{c_s h}^{(A,F)_{\hat{U}}}(S_h)$ and thus

$$S_h \subset S \subset_h \mathscr{R}_{c_s h}^{(A,F)_{\hat{U}}}\big(S_h\big), \quad \text{with } S_h \text{ stable with respect to } (A, F)_{\hat{U}},$$

where the stability with respect to $(A, F)_{\hat{U}}$ follows since $(A, F) = (A, F)_{\hat{U}}$ on $\hat{U} \supset S_h$.

With the comparison principle from Lemma 4.5.5, which states that for stable sets not only "$\subset$" but also "$\subset_h$" is preserved under the evolution, and the stability of S_h via Lemma 4.5.4, for all $t \geq s \geq 0$ we obtain

$$\mathscr{R}_s^{(A,F)_{\hat{U}}}\big(S_h\big) \subset \mathscr{R}_t^{(A,F)_{\hat{U}}}\big(S_h\big) \subset \mathscr{R}_t^{(A,F)_{\hat{U}}}\big(S\big) \subset_h \mathscr{R}_{t+c_s h}^{(A,F)_{\hat{U}}}\big(S_h\big). \tag{4.6.9}$$

In particular, the assumption that $\mathscr{R}_{t_0}^{(A,F)_{\hat{U}}}(S) \subset U$ implies that $\mathscr{R}_{t}^{(A,F)_{\hat{U}}}(S_h) \subset U$ for all $0 \leq t \leq t_0$. Again due to the maximum speed of propagation from Lemma 4.5.3, we have

$$\mathscr{R}_{t+c_s h}^{(A,F)_{\hat{U}}}\big(S_h\big) \subset \mathscr{R}_{t}^{(A,F)_{\hat{U}}}\big(S_h\big) + (1 + v_{\max} c_s h)\overline{B}_1 \subset U + (1 + v_{\max} c_s h)\overline{B}_1$$

for all $0 \leq t \leq t_0$. Together with (4.6.9), and since $S_1 \subset_h S_2$ implies $S_1 \subset S_2 + \overline{B}_h$ for any $S_1, S_2 \subset \mathbb{R}^d$ by Definition 4.3.8, this yields

$$\mathscr{R}_{t}^{(A,F)_{\hat{U}}}\big(S\big) \subset U + (1 + (1 + v_{\max} c_s h))\overline{B}_1 \qquad \text{for all } 0 \leq t \leq t_0$$

and hence

$$\mathscr{R}_{t}^{(A,F)_{\hat{U}}}\big(S\big) + B_1 \subset U + (2 + (1 + v_{\max} c_s)h)\overline{B}_1 \subset \hat{U} \qquad \text{for all } 0 \leq t \leq t_0,$$

which concludes the proof as described at the start. □

With Lemma 4.6.5 established, for the event $E_i(t)$, which checks if $\mathscr{R}_t(S) \approx K_i$, we are now able to prove the measurability with respect to the σ-algebra $\mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$ containing the information of the coefficient field on a set only slightly larger than K_i.

Lemma 4.6.6 (Measurability of E_i)**.** *Let $S \subset \mathbb{R}^d$ be an h-fat set with bounded boundary. For $i \in \mathbb{N}$ and $t \geq 0$, the event $E_i(t)$ as defined in (4.6.4) is measurable with respect to $\mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$ if $C_{\mathcal{G}} = C_{\mathcal{G}}(\mathrm{data}) > 0$ has been chosen large enough.*

Proof. Assume that $E_i(t) \neq \emptyset$, otherwise we are done. We claim that with Lemma 4.6.5 we obtain

$$E_i(t) = \left\{(A,F) \in \Omega \,:\, K_h\left(\mathscr{R}_t^{(A,F)_{K_i + (C_{\mathcal{G}}-1)\overline{B}_h}}\big(S\big)\right) = K_i\right\} \cap E_h^{\circledcirc}(\partial S), \tag{4.6.10}$$

that is we use $\mathscr{R}_t^{(A,F)_{K_i + (C_{\mathcal{G}}-1)\overline{B}_h}}(S)$ instead of $\mathscr{R}_t^{(A,F)}(S)$ as in the definition, replacing the unmodified coefficient field with the restriction to an area only slightly larger than the set, for which we are checking if the final state of the evolution corresponds to it. Having established (4.6.10), we quickly obtain that $E_i(t)$ is measurable with respect to $\mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$: It is straightforward to show that the event $\left\{(A,F) \in \Omega \,:\, K_h\left(\mathscr{R}_t^{(A,F)_{K_i + (C_{\mathcal{G}}-1)\overline{B}_h}}\big(S\big)\right) = K_i\right\}$ is $\mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$-measurable with Lemma 4.5.12 as for example in the proof of Lemma 4.6.4. With (∗i) from Assumption (P⊙) we have $E_h^{\circledcirc}(\partial S) \in \mathcal{F}\big(\partial S + c_{\mathcal{F}}\overline{B}_h\big)$. Since $E_i(t) \neq \emptyset$, with Lemma 4.6.3 for $C_{\mathcal{G}}$ large enough we have

$$\partial S + c_{\mathcal{F}}\overline{B}_h \subset K_i + \big(1 + \big(1 + v_{\max} c_s\big)h\big)\overline{B}_1 + c_{\mathcal{F}}\overline{B}_h \subset K_i + C_{\mathcal{G}}\overline{B}_h$$

and since $\mathcal{F}(M_1) \subset \mathcal{F}(M_2)$ for $M_1 \subset M_2$, we obtain that $E_h^{\circledcirc}(\partial S)$ is measurable with respect to $\mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$. Thus, $E_i(t) \in \mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$ due to (4.6.10).

It remains to show that (4.6.10) holds. In order to apply Lemma 4.6.5, we need to check that (4.6.8) holds for $U = K_i$, $\hat{U} = K_i + (C_{\mathcal{G}} - 1)\overline{B}_h$ and S. Since $E_i(t) \neq \emptyset$, we know from Lemma 4.6.3 that

$$S \subset K_i + \big(1 + \big(1 + v_{\max} c_s\big)h\big)\overline{B}_1.$$

In particular, for $C_{\mathcal{G}}$ large enough we have

$$S + \big(2 + v_{\max} c_s h\big)\overline{B_1} \subset K_i + \big(3 + \big(1 + 2v_{\max} c_s\big)h\big)\overline{B}_1 \subset K_i + \big(C_{\mathcal{G}} - 1\big)\overline{B}_h,$$

satisfying condition (4.6.8) for Lemma 4.6.5.

To show that "$\supset$" in (4.6.10) holds, note that if $K_h\Big(\mathscr{R}_t^{(A,F)_{K_i+(C_\mathcal{G}-1)\overline{B}_h}}(S)\Big) = K_i$ and $(A,F) \in E_h^\circledcirc(\partial S)$, Lemma 4.6.5 yields $\mathscr{R}_t^{(A,F)_{K_i+(C_\mathcal{G}-1)\overline{B}_h}}(S) = \mathscr{R}_t^{(A,F)}(S)$ and hence $K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) = K_i$.

Regarding "$\subset$" in (4.6.10), if $K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) = K_i$ and $(A,F) \in E_h^\circledcirc(\partial S)$, then the evolution with respect to the restricted coefficient field also ends up in K_i since by Lemma 4.5.11 the evolution with respect to restricted coefficient fields is lower than the one with respect to the original coefficient field. Now, Lemma 4.6.5 again yields that the evolutions are the same up to time t. Thus, we have shown that (4.6.10) holds. □

Using a similar approach, we can leverage the bounds on potential shrinkage from Lemma 4.6.5 to obtain that arrival times smaller than t are fully determined with the information from $\mathcal{G}_t$, corresponding to Lemma 4.4.5(c). However, again this only works when a stable approximation exists as for the event $E_h^\circledcirc(\partial S)$ from Assumption (P⊙).

Lemma 4.6.7 (Measurability of arrival times within the already explored environment)**.** *Let $t \geq 0$, $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ be an h-fat set with bounded boundary. Then*

$$(A,F) \mapsto m_h(x_0,S)\chi_{\left\{(A',F'):\, m_h^{(A',F')}(x_0,S)\leq t\right\}}((A,F))\chi_{E_h^\circledcirc(\partial S)}((A,F)) \tag{4.6.11}$$

is measurable with respect to $\mathcal{G}_t$ as defined in (4.6.5) with $\mathcal{G}_0 = \{\emptyset,\Omega\}$. In addition,

$$\begin{aligned}\Big\{(A,F)\,:\, m_h^{(A,F)}(x_0,S) \leq t\Big\} \cap E_h^\circledcirc(\partial S) \quad &\text{is } \mathcal{G}_t\text{-measurable for } t>0,\\ \Big\{(A,F)\,:\, m_h^{(A,F)}(x_0,S) = 0\Big\} \quad &\text{is } \mathcal{G}_0\text{-measurable.}\end{aligned}$$

In particular, for the truncated arrival time in the target area $\overline{B}_h(x_0)$ there holds

$$\mathbb{E}\Big[m_h(x_0,S)\chi_{E_h^\circledcirc(\partial S)}\,\Big|\,\mathcal{G}_{T_h(x_0,S)}\Big] = m_h(x_0,S)\chi_{E_h^\circledcirc(\partial S)}. \tag{4.6.12}$$

Proof. We only show (4.6.11), the measurability of the sets follows accordingly and (4.6.12) is implied by (4.6.11) since

$$m_h(x_0,S)\chi_{\left\{(A',F'):\, m_h^{(A',F')}(x_0,S)\leq T_h(x_0,S)\right\}} = m_h(x_0,S).$$

Regarding $t=0$ in (4.6.11), we clearly have

$$m_h(x_0,S)\chi_{\left\{(A',F'):\, m_h^{(A',F')}(x_0,S)\leq t\right\}} = 0.$$

Therefore, for the rest of the proof let $t>0$. Due to Lemma 4.6.5 we obtain $\mathscr{R}_s^{(A,F)}(S) = \mathscr{R}_s^{(A,F)_{K_i+(C_\mathcal{G}-1)\overline{B}_h}}(S)$ for $s \leq t$ and $(A,F) \in E_i(t)$, as long as $C_\mathcal{G} = C_\mathcal{G}(\mathrm{data}) > 0$ from the definition of $\mathcal{G}_t$ in (4.6.5) is chosen large enough. Hence, with $E_h^\circledcirc(\partial S) = \bigcup_{i\in\mathbb{N}} E_i(t)$ from

Lemma 4.6.2 for the disjoint events $(E_i(t))_i$, for $(A,F) \in \Omega$ there holds

$$\begin{aligned} m_h^{(A,F)}(x_0,S)\chi_{\left\{(A',F'):m_h^{(A',F')}(x_0,S)\le t\right\}}((A,F))\chi_{E_h^\circledcirc(\partial S)}((A,F)) \\ = \sum_{i\in\mathbb{N}} m_h^{(A,F)}(x_0,S)\chi_{\left\{(A',F'):m_h^{(A',F')}(x_0,S)\le t\right\}\cap E_i(t)}((A,F)) \\ = \sum_{i\in\mathbb{N}} m_h^{(A,F)_{K_i+(C_{\mathcal{G}}-1)\overline{B}_h}}(x_0,S) \\ \times \chi_{\left\{(A',F'):m_h^{(A',F')_{K_i+(C_{\mathcal{G}}-1)\overline{B}_h}}(x_0,S)\le t\right\}\cap E_i(t)}((A,F)). \end{aligned}$$

By the definition of $\mathcal{G}_t$, it only remains to show that the term

$$\sum_{i\in\mathbb{N}} m_h^{(A,F)_{K_i+(C_{\mathcal{G}}-1)\overline{B}_h}}(x_0,S)\chi_{\left\{(A',F'):m_h^{(A',F')_{K_i+(C_{\mathcal{G}}-1)\overline{B}_h}}(x_0,S)\le t\right\}}$$

is $\mathcal{F}(K_i + C_{\mathcal{G}}\overline{B}_h)$-measurable. This follows from a straightforward application of Lemma 4.5.12 with the proof of Lemma 4.6.4 as a blueprint. □

Finally, we are able to resolve the conditional expectation for arrival times which have not yet been reached, corresponding to the last part of Lemma 4.4.5. As in Subsection 4.4.3, this is the only place where the finite range of dependence from Assumption (P2) enters: Essentially, it provides the independence of the future development of the set from the coefficients in the already explored environment.

Lemma 4.6.8 (Resolving the conditional expectation)**.** *There exists $C_K = C_K(C_{\mathcal{G}}, \mathrm{data})$ depending on the choice of the constant $C_{\mathcal{G}}$ in the definition of $\mathcal{G}_t$ from (4.6.5), such that the following holds. Let $h \ge h_0$. Let $S \subset \mathbb{R}^d$ be an h-fat set with bounded boundary. Then for any $i \in \mathbb{N}$ and $t \ge 0$ there holds*

$$\mathbb{E}\Big[m_h(x_0, K_i + C_K\overline{B}_h)\chi_{E_{h,i}^\circledcirc}\chi_{E_i(t)}\,\Big|\,\mathcal{G}_t\Big] = \mathbb{E}\Big[m_h(x_0, K_i + C_K\overline{B}_h)\chi_{E_{h,i}^\circledcirc}\Big]\chi_{E_i(t)} \tag{4.6.13}$$

with

$$E_{h,i}^\circledcirc := E_h^\circledcirc\big(\partial(K_i + C_K\overline{B}_h)\big).$$

In particular, we obtain

$$\begin{aligned} \mathbb{E}\Big[m_h(x_0, K_h(\mathscr{R}_t(S)) + C_K\overline{B}_h)\chi_{E_h^\circledcirc(t)}\,\Big|\,\mathcal{G}_t\Big] \\ = \sum_{i\in\mathbb{N}} \mathbb{E}\Big[m_h(x_0, K_i + C_K\overline{B}_h)\chi_{E_{h,i}^\circledcirc}\Big]\chi_{E_i(t)}, \end{aligned} \tag{4.6.14}$$

with the union $E_h^\circledcirc(t) := \bigcup_{i\in\mathbb{N}}\Big(E_{h,i}^\circledcirc \cap E_i(t)\Big)$ of disjoint events. For all $0 \le t \le T_h(x_0,S)$ there holds

$$E_h^\circledcirc(t) \supset E_h^\circledcirc\big(M_h(x_0,S)\big), \tag{4.6.15}$$

where the set $M_h(x_0,S)$ is given by

$$M_h(x_0,S) := \big(\overline{S^c} + c_s v_{\max} B_h\big) \cap \Big(S + \big(1 + v_{\max}T_h(x_0,S)\big)\overline{B}_1 + \Big(C_K + \sqrt{d}\Big)\overline{B}_h\Big), \tag{4.6.16}$$

covering all of the area through which – assuming that there is a stable (h,c_s)-approximation – the interface of S could travel until time $T_h(x_0,S)$, plus the additional area due to the discretization and expansion by $C_K\overline{B}_h$.

Proof. We first prove (4.6.15) by showing that $E_i(t) \neq \emptyset$ implies $\partial(K_i + C_K\overline{B}_h) \subset M_h(x_0, S)$ and hence $E_h^{\circledcirc}\big(\partial\big(K_i + C_K\overline{B}_h\big)\big) \subset E_h^{\circledcirc}(M_h(x_0, S))$ due to the monotonicity of the events $(E_h^{\circledcirc}(M))_M$ from Assumption (P$\circledcirc$). Let $E_i(t) \neq \emptyset$. By the definition of $E_i(t)$ from (4.6.4) there exists $(A, F) \in E_h^{\circledcirc}(\partial S)$ with $K_i = K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big)$. In particular, S has a stable (h, c_s)-approximation S_h with respect to (A, F), which hence satisfies $\mathbb{R}^d \setminus (S^c + c_s v_{\max} B_h) \subset S_h \subset S$. On the one hand, due to the comparison principle and the stability of S_h with Lemma 4.5.4 we hence obtain

$$K_i = K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) \supset \mathscr{R}_t^{(A,F)}(S) \supset \mathscr{R}_t^{(A,F)}(S_h) \supset \mathbb{R}^d \setminus \big(S^c + c_s v_{\max} B_h\big).$$

On the other hand, due the maximum speed of propagation from Lemma 4.5.3 we have

$$K_i = K_h\Big(\mathscr{R}_t^{(A,F)}(S)\Big) \subset \mathscr{R}_t^{(A,F)}(S) + \big[-h, h\big]^d \subset S + \big(1 + t v_{\max}\big)\overline{B}_1 + \big[-h, h\big]^d.$$

Combining these two observations yields $\partial(K_i + C_K\overline{B}_h) \subset M_h(x_0, S)$ and hence (4.6.15).

Regarding (4.6.13), we need to show for any $E \in \mathcal{G}_t$ that

$$\mathbb{E}\Big[m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_{h,i}^{\circledcirc}}\chi_{E_i(t)}\chi_E\Big] = \mathbb{E}\Big[m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_{h,i}^{\circledcirc}}\Big]\mathbb{P}\big[E_i(t) \cap E\big].$$

This follows because

1.) $m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_{h,i}^{\circledcirc}}$ is $\mathcal{F}\Big(\overline{\big(K_i + C_K\overline{B}_h\big)^c} + \max\{c_s v_{\max}, c_{\mathcal{F}}\}\overline{B}_h\Big)$-measurable,

2.) $E_i(t) \cap E \in \mathcal{F}\big(K_i + \big(C_{\mathcal{G}} + 3 + \sqrt{d} + v_{\max}c_s\big)\overline{B}_h\big)$,

3.) for $C_K = C_K(C_{\mathcal{G}}, \text{data})$ large enough, $\mathcal{F}\Big(\overline{\big(K_i + C_K\overline{B}_h\big)^c} + \max\{c_s v_{\max}, c_{\mathcal{F}}\}\overline{B}_h\Big)$ and $\mathcal{F}\big(K_i + \big(C_{\mathcal{G}} + 3 + \sqrt{d} + v_{\max}c_s\big)\overline{B}_h\big)$ are $\mathbb{P}$-independent.

Here, 1.) holds due to (4.6.7) from Lemma 4.6.4 since $K_i + C_K\overline{B}_h$ is h-fat by definition. We obtain 3.) from the finite range of dependence in Assumption (P2) because

$$\begin{aligned}
&\operatorname{dist}\Big(\overline{\big(K_i + C_K\overline{B}_h\big)^c} + \max\big\{c_s v_{\max}, c_{\mathcal{F}}\big\}\overline{B}_h,\ K_i + \Big(C_{\mathcal{G}} + 3 + \sqrt{d} + v_{\max}c_s\Big)\overline{B}_h\Big) \\
&\quad \geq \operatorname{dist}\big(\big(K_i + C_K\overline{B}_h\big)^c,\ K_i\big) - \Big(C_{\mathcal{G}} + 3 + \sqrt{d} + v_{\max}c_s + \max\big\{c_s v_{\max}, c_{\mathcal{F}}\big\}\Big)h \\
&\quad \geq C_K h - \Big(C_{\mathcal{G}} + 3 + \sqrt{d} + v_{\max}c_s + \max\big\{c_s v_{\max}, c_{\mathcal{F}}\big\}\Big)h.
\end{aligned}$$

It only remains to show 2.). By definition of $\mathcal{G}_t$ from (4.6.5), it is sufficient to deal with the case $E = E_j(s) \cap F$, where $j \in \mathbb{N}$, $s \leq t$ and $F \in \mathcal{F}\big(K_j + C_{\mathcal{G}}\overline{B}_h\big)$. From Lemma 4.6.6 we know that $E \in \mathcal{F}\big(K_j + C_{\mathcal{G}}\overline{B}_h\big)$.

If $E_i(t) \cap E_j(s) = \emptyset$, then $E_i(t) \cap E = \emptyset$ and we are done. Hence, assume that there is $(A, F) \in E_i(t) \cap E_j(s)$. Since $\mathscr{R}_t^{(A,F)}(S) \subset K_i$ and $(A, F) \in E_h^{\circledcirc}(\partial S)$ by the definition of $E_i(t)$ in (4.6.4), the set evolution can not have reached much further than K_i before. In fact, with Lemma 4.6.5 we obtain $\mathscr{R}_s^{(A,F)}(S) \subset K_i + (2 + (1 + v_{\max}c_s)h)\overline{B}_1$ and thus

$$K_j = K_h\Big(\mathscr{R}_s^{(A,F)}(S)\Big) \subset K_i + \big(2 + \big(1 + v_{\max}c_s\big)h\big)\overline{B}_1 + [-h, h]^d.$$

In summary, we obtain

$$E \in \mathcal{F}\big(K_j + C_{\mathcal{G}}\overline{B}_h\big) \subset \mathcal{F}\Big(K_i + C_{\mathcal{G}}\overline{B}_h + \Big(2 + \Big(1 + \sqrt{d} + v_{\max}c_s\Big)h\Big)\overline{B}_1\Big)$$

and thus have shown 2.) since $E_i(s) \in \mathcal{F}\big(K_i + C_{\mathcal{G}}\overline{B}_h\big)$ with Lemma 4.6.6.

Regarding (4.6.14), plugging in the definition of $E_h^{\circledcirc}(t)$ and using the definition of $\{E_i(t)\}_{i\in\mathbb{N}}$ from (4.6.4) we have

$$\begin{aligned}
&\mathbb{E}\Big[m_h\big(x_0, K_h\big(\mathscr{R}_t(S)\big) + C_K\overline{B}_h\big)\chi_{E_h^{\circledcirc}(t)} \,\Big|\, \mathcal{G}_t\Big] \\
&\quad= \sum_{i\in\mathbb{N}} \mathbb{E}\Big[m_h\big(x_0, K_h\big(\mathscr{R}_t(S)\big) + C_K\overline{B}_h\big)\chi_{E_{h,i}^{\circledcirc}\cap E_i(t)} \,\Big|\, \mathcal{G}_t\Big] \\
&\quad= \sum_{i\in\mathbb{N}} \mathbb{E}\Big[m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_{h,i}^{\circledcirc}\cap E_i(t)} \,\Big|\, \mathcal{G}_t\Big] \\
&\quad= \sum_{i\in\mathbb{N}} \mathbb{E}\Big[m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_{h,i}^{\circledcirc}}\Big]\chi_{E_i(t)},
\end{aligned}$$

where in the final step we applied (4.6.13). □

4.6.2 Bounding the martingale increments

In this subsection, we will bound the increments of the martingale $\mathcal{M}_t = \mathbb{E}[m_h(x_0, S)\,|\,\mathcal{G}_t] - \mathbb{E}[m_h(x_0, S)]$ as introduced at the start of the section with the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ defined in (4.6.5). Once we obtain these bounds, we will apply our alternative to and our generalization of Azuma's inequality from Appendix 4.C to prove Proposition 4.6.1. In order to obtain the bounds for the martingale increments, we will follow the approach laid out in Subsection 4.4.3. However, to use the effective minimum speed from Assumption (P◎), we will restrict the arrival times to the event $E_h^{\circledcirc}(M_h(x_0, S))$ with $M_h(x_0, S)$ as defined in (4.6.16) essentially covering the maximum range of the interface. Since this breaks the independence used in Lemma 4.6.8 to resolve the conditional expectation, we will have to relax this restriction at the appropriate occasion. The error terms from these operations show up in the final bounds, but they are controlled with high probability, since Assumption (P◎) guarantees that $E_h^{\circledcirc}(M_h(x_0, S))$ holds with overwhelming probability. Naturally, the probability bounds for very large fluctuations in Proposition 4.6.1 are limited by this bound from Assumption (P◎).

Proposition 4.6.9 (Bounds for the martingale increments)**.** *Let $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ be an h-fat set with bounded boundary. Then there exists a constant $C = C(\mathrm{data}) > 0$ such that the martingale $(\mathcal{M}_t)_{t\geq 0}$ as defined in (4.6.3) for all $0 \leq s \leq t \leq T_h(x_0, S)$ satisfies*

$$\begin{aligned}
\big|\mathcal{M}_t - \mathcal{M}_s\big| \leq\ & C(t-s) + Ch \\
&+ \big(\delta_{t=0} + \delta_{s=0}\big)T_h\big(x_0, S\big)\Big(1 - \chi_{E_h^{\circledcirc}(M_h(x_0,S))}\Big) \\
&+ 3T_h\big(x_0, S\big)\Big(\mathbb{E}\Big[1 - \chi_{E_h^{\circledcirc}(M_h(x_0,S))} \,\Big|\, \mathcal{G}_t\Big] + \mathbb{E}\Big[1 - \chi_{E_h^{\circledcirc}(M_h(x_0,S))} \,\Big|\, \mathcal{G}_s\Big]\Big) \\
&+ T_h\big(x_0, S\big)\big(1 - \mathbb{P}\big[E_h^{\circledcirc}\big(M_h\big(x_0, S\big)\big)\big]\big),
\end{aligned} \tag{4.6.17}$$

with $\delta_{t=0}, \delta_{s=0}$ denoting Dirac-deltas and with $M_h(x_0, S)$ as introduced in Lemma 4.6.8 covering the maximum range of the interface until time $T_h(x_0, S)$ and given by

$$M_h\big(x_0, S\big) := \big(\overline{S^c} + c_s v_{\max} B_h\big) \cap \Big(S + \big(1 + v_{\max}T_h\big(x_0, S\big)\big)\overline{B}_1 + \Big(C_K + \sqrt{d}\Big)\overline{B}_h\Big). \tag{4.6.16}$$

Proof of Proposition 4.6.9. We will follow the same strategy as in Subsection 4.4.3 to obtain good bounds for the increments uniformly on Ω. However, as described above we will not obtain uniform bounds but might have to restrict ourselves to events such as for

example $E_h^{\circledcirc}(M_h(x_0,S))$ from Assumption (P$\circledcirc$), which guarantees the existence of stable (h,c_s)-approximations and the applicability of the effective minimum speed. For some shorthand notation, we write $M \coloneqq M_h(x_0,S)$ denoting the area covering the potential range of the interface and its modifications until time $T_h(x_0,S)$ as defined in (4.6.16).

We first restrict the conditional expectation in $\mathcal{M}_t$ to $E_h^{\circledcirc}(M)$ in order to be able to use the effective minimum speed from Assumption (P$\circledcirc$). That is, at the cost of an error term R_1^t we obtain

$$\begin{aligned}\mathcal{M}_t &= \mathbb{E}\big[m_h(x_0,S)\,\big|\,\mathcal{G}_t\big] - \mathbb{E}\big[m_h(x_0,S)\big]\\ &= \mathbb{E}\Big[m_h(x_0,S)\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big] - \mathbb{E}\big[m_h(x_0,S)\big] + R_1^t\end{aligned}$$

with

$$0 \le R_1^t \le T_h(x_0,S)\mathbb{E}\Big[1-\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big].$$

Next, we separate the case that $m_h(x_0,S)\le s$, that is that x_0 is within the environment already explored at time $t\ge s$, which makes the arrival time measurable with respect to $\mathcal{G}_t$, see Lemma 4.6.7. In the remaining case that x_0 has not been reached at time s, that is $m_h(x_0,S)>s$, we can write $m_h(x_0,S)=s+m_h(x_0,\mathscr{R}_s(S))$ since Lemma 4.5.7 guarantees that the truncation of the arrival times does not kick in because of the restriction to $E_h^{\circledcirc}(M)$. Writing $\{m_h(x_0,S)\le s\} = \Big\{(\widetilde{A},\widetilde{F}) : m_h^{(\widetilde{A},\widetilde{F})}(x_0,S)\le s\Big\}$ and introducing the error term R_0^t, we obtain

$$\begin{aligned}\mathcal{M}_t &= \mathbb{E}\Big[m_h(x_0,S)\chi_{E_h^{\circledcirc}(\partial S)}\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big] - \mathbb{E}\big[m_h(x_0,S)\big] + R_1^t\\ &= \mathbb{E}\Big[\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big]m_h(x_0,S)\chi_{\{m_h(x_0,S)\le s\}}\chi_{E_h^{\circledcirc}(\partial S)}\\ &\quad + \mathbb{E}\Big[\big(s+m_h(x_0,\mathscr{R}_s(S))\big)\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big]\chi_{\{m_h(x_0,S)>s\}}\chi_{E_h^{\circledcirc}(\partial S)}\\ &\quad - \mathbb{E}\big[m_h(x_0,S)\big] + R_1^t + R_0^t,\end{aligned}$$

where we used that $E_h^{\circledcirc}(M)\subset E_h^{\circledcirc}(\partial S)$ and then applied Lemma 4.6.7. The error R_0^t captures that for $t=0$ only $\{m_h(x_0,S)>0\}\in\{\emptyset,\Omega\}$ is $\mathcal{G}_0$-measurable, but not necessarily $\{m_h(x_0,S)>0\}\cap E_h^{\circledcirc}(\partial S)\in\mathcal{G}_0$:

$$\begin{aligned}0\le R_0^t &= \delta_{t=0}\mathbb{E}\Big[m_h(x_0,S)\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_0\Big]\chi_{\{m_h(x_0,S)>0\}}\Big(1-\chi_{E_h^{\circledcirc}(\partial S)}\Big)\\ &\le \delta_{t=0}T_h(x_0,S)\Big(1-\chi_{E_h^{\circledcirc}(\partial S)}\Big).\end{aligned}$$

Since these considerations analogously hold for $\mathcal{M}_s$, we obtain

$$\begin{aligned}\big|\mathcal{M}_t-\mathcal{M}_s\big| \le{}& \Big|\mathbb{E}\Big[\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big] - \mathbb{E}\Big[\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_s\Big]\Big|\\ &\qquad\times m_h(x_0,S)\chi_{\{m_h(x_0,S)\le s\}}\chi_{E_h^{\circledcirc}(\partial S)}\\ &+\Big|\mathbb{E}\Big[\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big] - \mathbb{E}\Big[\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_s\Big]\Big|\\ &\qquad\times s\chi_{\{m_h(x_0,S)>s\}}\chi_{E_h^{\circledcirc}(\partial S)}\\ &+\Big|\mathbb{E}\Big[m_h(x_0,\mathscr{R}_s(S))\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_t\Big] - \mathbb{E}\Big[m_h(x_0,\mathscr{R}_s(S))\chi_{E_h^{\circledcirc}(M)}\,\Big|\,\mathcal{G}_s\Big]\Big|\\ &\qquad\times \chi_{\{m_h(x_0,S)>s\}}\chi_{E_h^{\circledcirc}(\partial S)}\\ &+\big|R_1^t-R_1^s+R_0^t-R_0^s\big|.\end{aligned}$$

Viewing the first two contributions as another error term R_2, we have

$$\begin{aligned}|\mathcal{M}_t - \mathcal{M}_s| \leq \Big|\mathbb{E}\Big[m_h\big(x_0, \mathscr{R}_s(S)\big)\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_t\Big] - \mathbb{E}\Big[m_h\big(x_0, \mathscr{R}_s(S)\big)\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_s\Big]\Big| \\ + R_2 + \big|R_1^t - R_1^s + R_0^t - R_0^s\big|,\end{aligned} \tag{4.6.18}$$

where with $m_h(x_0, S)\chi_{\{m_h(x_0,S)\leq s\}} \leq s\chi_{\{m_h(x_0,S)\leq s\}}$ we have

$$\begin{aligned}R_2 &\leq s\Big|\mathbb{E}\Big[\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_t\Big] - \mathbb{E}\Big[\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_s\Big]\Big| \\ &= s\Big|\mathbb{E}\Big[1 - \chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_t\Big] - \mathbb{E}\Big[1 - \chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_s\Big]\Big|.\end{aligned}$$

Ignoring the error terms in (4.6.18), we essentially almost eliminated dependence on the information of the already explored area by time s, encoded in $\mathcal{G}_s$ – at least if we knew that $\mathscr{R}_s(S)$ was strictly expanding.

Regarding the conditional expectation with respect to $\mathcal{G}_t$, we want to replace $\mathscr{R}_s(S)$ with $\mathscr{R}_t(S)$. For $(A, F) \in E_h^\circledcirc(M)$, let $S_h = S_h^{(A,F)}$ be the stable (h, c_s)-approximation from (∗iii) in Assumption (P⊙), which exists since $\partial S \subset M$ by definition 4.6.16. Then we have $\mathscr{R}_{\tilde{t}}(S_h) \subset \mathscr{R}_{\tilde{t}}(S) \subset_h \mathscr{R}_{\tilde{t}+c_s h}(S_h)$ for all $\tilde{t} \geq 0$ on $E_h^\circledcirc(M)$ due to the comparison principle as in Lemma 4.5.5. On the one hand, note that

$$\begin{aligned}m_h\big(x_0, \mathscr{R}_s(S)\big)\chi_{E_h^\circledcirc(M)} &\geq m_h\big(x_0, \mathscr{R}_{s+c_s h}(S_h)\big)\chi_{E_h^\circledcirc(M)} \\ &\geq m_h\big(x_0, \mathscr{R}_{t+c_s h}(S_h)\big)\chi_{E_h^\circledcirc(M)} \\ &\geq m_h\big(x_0, \mathscr{R}_t(S_h)\big)\chi_{E_h^\circledcirc(M)} - c_s h\chi_{E_h^\circledcirc(M)} \\ &\geq m_h\big(x_0, \mathscr{R}_t(S)\big)\chi_{E_h^\circledcirc(M)} - c_s h\chi_{E_h^\circledcirc(M)}\end{aligned}$$

where we first used that if $S_1 \subset_h S_2$ and $x \in S_1 + \overline{B}_h$, then $x \in S_2 + \overline{B}_h$. The second inequality is due to S_h being stable, the third and fourth inequality follow from the definition of the arrival times. On the other hand, we have

$$m_h\big(x_0, \mathscr{R}_s(S)\big)\chi_{E_h^\circledcirc(M)} \leq m_h\big(x_0, \mathscr{R}_t(S)\big)\chi_{E_h^\circledcirc(M)} + (t-s)\chi_{E_h^\circledcirc(M)}.$$

In summary, we obtain

$$\big|m_h\big(x_0, \mathscr{R}_s(S)\big) - m_h\big(x_0, \mathscr{R}_t(S)\big)\big|\chi_{E_h^\circledcirc(M)} \leq \max\{c_s h,\, t-s\}\chi_{E_h^\circledcirc(M)}. \tag{4.6.19}$$

Now that we can replace $\mathscr{R}_s(S)$ with $\mathscr{R}_t(S)$, we want to resolve the conditional expectation in $\mathbb{E}\Big[m_h(x_0, \mathscr{R}_t(S))\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_t\Big]$ (and in the corresponding term with s instead of t) via Lemma 4.6.8. However, in order to apply Lemma 4.6.8, we need to replace $\mathscr{R}_t(S)$ with a slightly larger set, for which the evolution is independent of the coefficient field in the already explored area. For $C_K = C_K(C_{\mathcal{G}}, \text{data}) > 0$ from Lemma 4.6.8, on $E_h^\circledcirc(M)$ we have

$$\mathscr{R}_t(S) \subset K_h\big(\mathscr{R}_t(S)\big) + C_K\overline{B}_h \subset_h \mathscr{R}_{t+c_s h + C_K v_{\min}^{-1} h}(S_h).$$

The last inclusion '$\subset_h$' above comes from as in the proof of Lemma 4.5.7 iteratively applying the effective minimum speed of propagation guaranteed by $E_h^\circledcirc(M)$ since

$$\big(\mathscr{R}_t(S) + C_K(\text{data})\overline{B}_h\big) \cap \overline{(S_h)^c} \subset M$$

for $t \le T_h(x_0, S)$ by the definition of M in (4.6.16). We hence have

$$\begin{aligned} m_h\big(x_0, \mathscr{R}_t(S)\big)\chi_{E_h^\circledcirc(M)} &\le m_h\big(x_0, K_h\big(\mathscr{R}_t(S)\big) + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(M)} \\ &\le \big(m_h\big(x_0, \mathscr{R}_t(S)\big) + c_s h + C_K v_{\min}^{-1} h\big)\chi_{E_h^\circledcirc(M)}. \end{aligned} \tag{4.6.20}$$

Replacing $\mathscr{R}_s(S)$ with $\mathscr{R}_t(S)$, then $\mathscr{R}_t(S)$ with $K_h(\mathscr{R}_t(S))+C_K\overline{B}_h$ and $\mathscr{R}_s(S)$ with $K_h(\mathscr{R}_s(S))+C_K\overline{B}_h$ in (4.6.18), we obtain

$$\begin{aligned} \big|\mathcal{M}_t - \mathcal{M}_s\big| \le \Big|&\mathbb{E}\Big[m_h\big(x_0, K_h\big(\mathscr{R}_t(S)\big) + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_t\Big] \\ &- \mathbb{E}\Big[m_h\big(x_0, K_h\big(\mathscr{R}_s(S)\big) + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_s\Big]\Big| \\ &+ R_3 + R_2 + \big|R_1^t - R_1^s + R_0^t - R_0^s\big|, \end{aligned}$$

where due to (4.6.19) and (4.6.20) the corresponding error term R_3 is bound by

$$\big|R_3\big| \le \max\big\{c_s h,\, t-s\big\} + c_s h + C_K v_{\min}^{-1} h.$$

In order to apply Lemma 4.6.8, we now only need to get rid of the restriction to $E_h^\circledcirc(M)$. Because $S \subset K_h(\mathscr{R}_t(S)) + C_K\overline{B}_h$ on $E_h^\circledcirc(\partial S)$ with Lemma 4.6.3 and since C_K was chosen accordingly, we have $T_h(x_0, S) \ge T_h\big(x_0, K_h(\mathscr{R}_t(S)) + C_K\overline{B}_h\big)$ and hence obtain

$$\begin{aligned} 0 \le m_h\big(x_0, K_h\big(\mathscr{R}_t(S)\big) + C_K\overline{B}_h\big)&\Big(\chi_{E_h^\circledcirc(t)} - \chi_{E_h^\circledcirc(M)}\Big) \\ &\le T_h\big(x_0, S\big)\Big(1 - \chi_{E_h^\circledcirc(M)}\Big), \end{aligned} \tag{4.6.21}$$

where $E_h^\circledcirc(t) \supset E_h^\circledcirc(M)$ is the less restrictive event defined in Lemma 4.6.8, which also is independent from the coefficient field in the already explored area. Thus, applying Lemma 4.6.8 after replacing the restriction to $E_h^\circledcirc(M)$ with one to $E_h^\circledcirc(t)$ at the cost of the error term R_4, we obtain

$$\begin{aligned} \big|\mathcal{M}_t - \mathcal{M}_s\big| \le \Bigg|&\sum_{i\in\mathbb{N}} \mathbb{E}\Big[m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(t)}\Big]\chi_{E_i(t)} \\ &- \sum_{j\in\mathbb{N}} \mathbb{E}\Big[m_h\big(x_0, K_j + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(s)}\Big]\chi_{E_j(s)}\Bigg| \\ &+ R_4 + R_3 + R_2 + \big|R_1^t - R_1^s + R_0^t - R_0^s\big|, \end{aligned}$$

with

$$R_4 \le T_h\big(x_0, S\big)\Big(\mathbb{E}\Big[1 - \chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_t\Big] + \mathbb{E}\Big[1 - \chi_{E_h^\circledcirc(M)} \,\Big|\, \mathcal{G}_s\Big]\Big).$$

With Lemma 4.6.2 we have $\bigcup E_i(t) = E_h^\circledcirc(\partial S) \supset E_j(s)$ for any $j \in \mathbb{N}$ and therefore

$$\begin{aligned} \big|\mathcal{M}_t - \mathcal{M}_s\big| \le \sum_{i,j\in\mathbb{N}} \chi_{E_i(t)\cap E_j(s)}\Big|&\mathbb{E}\Big[m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(t)}\Big] \\ &- \mathbb{E}\Big[m_h\big(x_0, K_j + C_K\overline{B}_h\big)\chi_{E_h^\circledcirc(s)}\Big]\Big| \\ &+ R_4 + R_3 + R_2 + \big|R_1^t - R_1^s + R_0^t - R_0^s\big|. \end{aligned} \tag{4.6.22}$$

Now assume that there exists $(\widetilde{A}, \widetilde{F}) \in E_i(t) \cap E_j(s) \subset E_h^\circledcirc(\partial S)$ for $i, j \in \mathbb{N}$. Let S_h be the stable (h, c_s)-approximation of S with respect to $(\widetilde{A}, \widetilde{F})$. On the one hand, we have

$$\begin{aligned} \mathscr{R}_s^{(\widetilde{A},\widetilde{F})}(S) \subset_h \mathscr{R}_{c_s h+s}^{(\widetilde{A},\widetilde{F})}(S_h) &\subset \mathscr{R}_s^{(\widetilde{A},\widetilde{F})}(S_h) + \big(1 + v_{\max} c_s h\big)\overline{B}_1 \\ &\subset \mathscr{R}_t^{(\widetilde{A},\widetilde{F})}(S_h) + \big(1 + v_{\max} c_s h\big)\overline{B}_1 \subset \mathscr{R}_t^{(\widetilde{A},\widetilde{F})}(S) + \big(1 + v_{\max} c_s h\big)\overline{B}_1 \end{aligned}$$

due to the comparison principle from Lemma 4.5.5, the maximum speed of propagation from Lemma 4.5.3, and the stability of S_h. On the other hand, again due to the maximum speed of propagation we have

$$\mathscr{R}_t^{(\widetilde{A},\widetilde{F})}(S) \subset \mathscr{R}_s^{(\widetilde{A},\widetilde{F})}(S) + \big(1 + v_{\max}(t-s)\big)\overline{B}_1.$$

On the level of the discretization of the states, this implies that

$$\begin{aligned} K_j = K_h\Big(\mathscr{R}_s^{(\widetilde{A},\widetilde{F})}(S)\Big) &\subset K_i + \Big(1 + \Big(\sqrt{d} + v_{\max}c_s\Big)h\Big)\overline{B}_1, \\ K_i = K_h\Big(\mathscr{R}_t^{(\widetilde{A},\widetilde{F})}(S)\Big) &\subset K_j + \Big(1 + v_{\max}(t-s) + \sqrt{d}h\Big)\overline{B}_1. \end{aligned}$$

In particular, with the effective minimum speed of propagation, analogously to the proof of Lemma 4.5.7, for $(A,F) \in E_h^{\circledcirc}(M)$ and S_k denoting the stable (h,c_s)-approximation of $K_k + C_K\overline{B}_h$ for $k = i,j$ we obtain

$$\begin{aligned} K_j + C_K\overline{B}_h &\subset_h \mathscr{R}^{(A,F)}_{c_sh + v_{\min}^{-1}(1+(1+\sqrt{d}+v_{\max}c_s)h)}(S_i), \\ K_i + C_K\overline{B}_h &\subset_h \mathscr{R}^{(A,F)}_{c_sh + v_{\min}^{-1}(1+v_{\max}(t-s)+(1+\sqrt{d})h)}(S_j). \end{aligned}$$

For the arrival times, with the comparison principle as in Lemma 4.5.5 this yields

$$\begin{aligned} &m_h\big(x_0, K_i + C_K\overline{B}_h\big)\chi_{E_h^{\circledcirc}(M)} \\ &\quad \geq \Big(m_h\big(x_0, K_j + C_K\overline{B}_h\big) - c_sh - v_{\min}^{-1}\Big(1 + \Big(1 + \sqrt{d} + v_{\max}c_s\Big)h\Big)\Big)\chi_{E_h^{\circledcirc}(M)}, \\ &m_h\big(x_0, K_j + C_K\overline{B}_h\big)\chi_{E_h^{\circledcirc}(M)} \\ &\quad \geq \Big(m_h\big(x_0, K_i + C_K\overline{B}_h\big) - c_sh - v_{\min}^{-1}\Big(1 + v_{\max}(t-s) + \Big(1 + \sqrt{d}\Big)h\Big)\Big)\chi_{E_h^{\circledcirc}(M)}. \end{aligned}$$

In order to apply these observations to (4.6.22), we move back to the restriction on $E_h^{\circledcirc}(M)$,

$$\begin{aligned} \big|\mathcal{M}_t - \mathcal{M}_s\big| \leq \sum_{i,j\in\mathbb{N}} \chi_{E_i(t)\cap E_j(s)}\Big|\mathbb{E}\Big[\Big(&m_h\big(x_0, K_i + C_K\overline{B}_h\big) \\ &- m_h\big(x_0, K_j + C_K\overline{B}_h\big)\Big)\chi_{E_h^{\circledcirc}(M)}\Big]\Big| \\ + R_5 + R_4 + R_3 + R_2 + \big|R_1^t - R_1^s + R_0^t - R_0^s\big|, \end{aligned}$$

where we again bound the resulting error via (4.6.21) with

$$\begin{aligned} R_5 &\leq T_h\big(x_0, S\big)\Big|\mathbb{E}\Big[\Big(1 - \chi_{E_h^{\circledcirc}(M)}\Big)\Big(\chi_{E_h^{\circledcirc}(t)} - \chi_{E_h^{\circledcirc}(s)}\Big)\Big]\Big| \\ &\leq T_h\big(x_0, S\big)\big(1 - \mathbb{P}\big[E_h^{\circledcirc}(M)\big]\big). \end{aligned}$$

Now plugging in the inequalities for $E_i(t) \cap E_j(s) \neq \emptyset$, we obtain

$$\begin{aligned} \big|\mathcal{M}_t - \mathcal{M}_s\big| \leq c_sh + v_{\min}^{-1}\Big(1 + \Big(1 + \sqrt{d}\Big)h + v_{\max}\max\big\{c_sh, (t-s)\big\}\Big) \\ + R_5 + R_4 + R_3 + R_2 + \big|R_1^t - R_1^s + R_0^t - R_0^s\big|. \end{aligned}$$

Collecting the bounds for the error terms R_0^t, R_1^t, R_2, R_3, R_4, and R_5 and using that $1 \leq h$ thus yields (4.6.17). □

Now that with Proposition 4.6.9 we have established the 'almost' uniform bounds (see Lemma 4.C.1) for the martingale increments, we are able to prove Proposition 4.6.1. The first two inequalities in (4.6.2) will follow from an application of Proposition 4.C.2, our alternative to Azuma's inequality. The final inequality trivially follows from the truncation imposed on the arrival times.

Proof of Proposition 4.6.1. Without loss of generality, let $\operatorname{dist}(x_0, S) > h$, since otherwise $m_h(x_0, S) = \mathbb{E}[m_h(x_0, S)] = 0$. We will now apply Proposition 4.C.2, our replacement for Azuma's inequality to the martingale $(\widetilde{\mathcal{M}}_n)_{n\in\mathbb{N}_0}$ defined by

$$\widetilde{\mathcal{M}}_n := \begin{cases} \mathcal{M}_{n\Delta t} & \text{for } n\Delta t < T_h(x_0, S), \\ \mathcal{M}_{T_h(x_0,S)} & \text{for } n\Delta t \geq T_h(x_0, S), \end{cases} \qquad \text{with } \Delta t := h,$$

with $(\mathcal{M}_t)_t$ defined in (4.6.3) and Δt chosen in view of (4.6.17) from Proposition 4.6.9. We set

$$N := \left\lceil \frac{T_h(x_0, S_0)}{\Delta t} \right\rceil \leq C \frac{\operatorname{dist}(x_0, S)}{h}, \tag{4.6.23}$$

with the inequality satisfied for some $C = C(\text{data}) > 0$ since $\operatorname{dist}(x_0, S) > h$.

Since $\mathcal{G}_0 = \{\emptyset, \Omega\}$ there holds $\widetilde{\mathcal{M}}_0 = 0$. With (4.6.12) from Lemma 4.6.7 we have

$$\begin{aligned} \widetilde{M}_N &= \mathbb{E}\big[m_h(x_0, S) \,\big|\, \mathcal{G}_{T_h(x_0,S)}\big] - \mathbb{E}\big[m_h(x_0, S)\big] \\ &= m_h(x_0, S)\chi_{E_h^{\circledcirc}(\partial S)} - \mathbb{E}\big[m_h(x_0, S)\big] + R_1 \\ &= m_h(x_0, S) - \mathbb{E}\big[m_h(x_0, S)\big] + R_1 + R_2 \end{aligned} \tag{4.6.24}$$

with

$$\begin{aligned} |R_1| &\leq T_h(x_0, S)\mathbb{E}\Big[1 - \chi_{E_h^{\circledcirc}(\partial S)} \,\Big|\, \mathcal{G}_{T_h(x_0,S)}\Big], \\ |R_2| &\leq T_h(x_0, S)\Big(1 - \chi_{E_h^{\circledcirc}(\partial S)}\Big). \end{aligned}$$

In order to apply our replacement for Azuma's inequality, we will first show that there is an event $E_{\mathcal{M}} \in \mathcal{F}(\mathbb{R}^d)$ such that

$$\Big|\widetilde{\mathcal{M}}_{n+1} - \widetilde{\mathcal{M}}_n\Big|\chi_{E_{\mathcal{M}}} \leq Ch \quad \text{for all } n \in \mathbb{N} \tag{4.6.25}$$

for some $C = C(\text{data}) > 0$, which holds with overwhelming probability, that is

$$1 - \mathbb{P}\big[E_{\mathcal{M}}\big] \leq C \exp\Big(-\frac{c_{\circledcirc}}{2} h^{\alpha_{\circledcirc}}\Big).$$

From Assumption (P$\circledcirc$) we have

$$\begin{aligned} 1 - \mathbb{P}\big[E_h^{\circledcirc}\big(M_h(x_0, S)\big)\big] &\leq C_{\circledcirc}\left(1 + \frac{\operatorname{diam}(M_h(x_0, S))}{h}\right)^d \exp\big(-c_{\circledcirc} h^{\alpha_{\circledcirc}}\big) \\ &\leq C\left(1 + \frac{\operatorname{diam}(\partial S) + \operatorname{dist}(x_0, S)}{h}\right)^d \exp\big(-c_{\circledcirc} h^{\alpha_{\circledcirc}}\big) \end{aligned}$$

for some constant $C = C(\text{data})$. Regarding the last term in the bounds for the martingale increments from (4.6.17), we hence have

$$\begin{aligned} &T_h(x_0, S)\big(1 - \mathbb{P}\big[E_h^{\circledcirc}\big(M_h(x_0, S)\big)\big]\big) \\ &\qquad \leq C\left(1 + \frac{\operatorname{diam}(\partial S) + \operatorname{dist}(x_0, S)}{h}\right)^{d+1} \exp\big(-c_{\circledcirc} h^{\alpha_{\circledcirc}}\big) \leq C \exp\Big(-\frac{c_{\circledcirc}}{2} h^{\alpha_{\circledcirc}}\Big) \end{aligned} \tag{4.6.26}$$

since we assumed in (4.6.1) that ∂S and $\operatorname{dist}(x_0, S)$ are not exponentially larger than h.

Regarding the terms with conditional expectations in (4.6.17), Lemma 4.C.1 yields that

$$\begin{aligned} E_{\mathcal{M}} \coloneqq{}& E_h^{\circledcirc}\big(M_h(x_0,S)\big) \\ &\cap \Big\{(A,F)\in\Omega : \mathbb{E}\Big[1-\chi_{E_h^{\circledcirc}(M_h(x_0,S))}\,\Big|\,\mathcal{G}_{n\Delta t\wedge T_h(x_0,S)}\Big]((A,F)) \\ &\qquad\qquad \le \frac{h}{T_h(x_0,S)} \quad \text{for all } n\in\mathbb{N}\Big\} \end{aligned}$$

as in (4.6.26) for some $C = C(\mathrm{data}) > 0$ satisfies

$$1-\mathbb{P}\big[E_{\mathcal{M}}\big] \le \left(1+\frac{T_h(x_0,S)}{h}\right)\big(1-\mathbb{P}\big[E_h^{\circledcirc}\big(M_h(x_0,S)\big)\big]\big) \le C\exp\Big(-\frac{c_{\circledcirc}}{2}h^{\alpha_{\circledcirc}}\Big). \tag{4.6.27}$$

Plugging these definitions and (4.6.26) into the bounds for the martingale increments from (4.6.17) in Proposition 4.6.9 yields for all $n\in\mathbb{N}$ that

$$\big|\widetilde{\mathcal{M}}_{n+1}-\widetilde{\mathcal{M}}_n\big|\chi_{E_{\mathcal{M}}} \le Ch+Ch+0+6h+C\exp\Big(-\frac{c_{\circledcirc}}{2}h^{\alpha_{\circledcirc}}\Big)$$

and hence we obtain that this definition of $E_{\mathcal{M}}$ satisfies (4.6.25).

Now that we have the bounds from (4.6.25) for the increments on the set $E_{\mathcal{M}}$ satisfying (4.6.27), we apply our replacement for Azuma's inequality. Since further $|\widetilde{\mathcal{M}}_n| \le T_h(x_0,S)$ for all $n\in\mathbb{N}_0$, via Proposition 4.C.2 we obtain

$$\mathbb{E}\Big[\widetilde{M}_N^{2k}\chi_{E_{\mathcal{M}}}\Big] \le (2k)!(Ch)^{2k}N^k + C(2k)!N^kT_h\big(x_0,S\big)^{2k}\big(1-\mathbb{P}\big[E_{\mathcal{M}}\big]\big)$$

for all $k\in\mathbb{N}$. Based on (4.6.24) with the errors R_1, R_2, we know that for the endpoint of the martingale we have

$$\widetilde{M}_N\chi_{E_{\mathcal{M}}} = \big(m_h(x_0,S)-\mathbb{E}\big[m_h(x_0,S)\big]\big)\chi_{E_{\mathcal{M}}} + R_1\chi_{E_{\mathcal{M}}}$$

where due to the definition of $E_{\mathcal{M}}$

$$\big|R_1\big|\chi_{E_{\mathcal{M}}} \le h.$$

Therefore, with the value of $C = C(\mathrm{data}) > 0$ changing from line to line, we have

$$\begin{aligned} &\mathbb{E}\Big[\big(m_h(x_0,S)-\mathbb{E}\big[m_h(x_0,S)\big]\big)^{2k}\Big] \\ &\qquad\le \mathbb{E}\Big[\big(\widetilde{M}_N-R_1\big)^{2k}\chi_{E_{\mathcal{M}}}\Big] + CT_h\big(x_0,S\big)^{2k}\big(1-\mathbb{P}\big[E_{\mathcal{M}}\big]\big) \\ &\qquad\le \Big(\mathbb{E}\Big[\widetilde{M}_N^{2k}\chi_{E_{\mathcal{M}}}\Big]^{\frac{1}{2k}}+h\Big)^{2k} + CT_h\big(x_0,S\big)^{2k}\big(1-\mathbb{P}\big[E_{\mathcal{M}}\big]\big) \\ &\qquad\le (2k)!(Ch)^{2k}N^k + C(2k)!N^kT_h\big(x_0,S\big)^{2k}\big(1-\mathbb{P}\big[E_{\mathcal{M}}\big]\big). \end{aligned}$$

Normalizing this in view of Proposition 4.C.2, we have

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{m_h(x_0,S)-\mathbb{E}[m_h(x_0,S)]}{Ch}\right)^{2k}\right] \\ &\qquad\le (2k)!N^k + C(2k)!N^k\left(\frac{T_h(x_0,S)}{h}\right)^{2k}\big(1-\mathbb{P}\big[E_{\mathcal{M}}\big]\big). \end{aligned}$$

As in Proposition 4.C.2, from these moment bounds we obtain fluctuation bounds corresponding to (4.C.3), which with $\log(1-\mathbb{P}[E_{\mathcal{M}}]) \approx -h^{\alpha_\circledcirc}$ as in (4.6.27) for $\widetilde{\lambda} > 0$ yields

$$\mathbb{P}\left[\left|\frac{m_h(x_0,S)-\mathbb{E}[m_h(x_0,S)]}{Ch}\right| \geq \widetilde{\lambda}\right] \leq \begin{cases} C\exp\left(-\dfrac{\widetilde{\lambda}}{2\sqrt{N}}\right) & \text{if } \widetilde{\lambda} \leq \dfrac{ch^{\alpha_\circledcirc}\sqrt{N}}{\log\left(\frac{T_h(x_0,S)}{h}\right)}, \\ C\exp\left(-\dfrac{ch^{\alpha_\circledcirc}}{\log\left(\frac{T_h(x_0,S)}{h}\right)}\right) & \text{if } \widetilde{\lambda} \geq \dfrac{ch^{\alpha_\circledcirc}\sqrt{N}}{\log\left(\frac{T_h(x_0,S)}{h}\right)}. \end{cases}$$

Plugging in $T_h(x_0,S) \approx \operatorname{dist}(x_0,S)$ and $N \approx \frac{\operatorname{dist}(x_0,S)}{h}$ from (4.6.23) as well as substituting $\lambda = Ch\widetilde{\lambda}$ yields the two central inequalities in (4.6.2). □

4.7 Compensating the propagation of influence by increasing the forcing

In this section we will show that we can compensate far-away influences on the evolution of sets by adding a small constant to the forcing. More precisely, if a stable set is initially ahead of a second set within a large ball, we can guarantee that the set stays ahead near the center of this large ball for a finite time period at the cost of increasing the forcing by a constant – no matter how much bigger the second set is outside of the large ball, see Figure 4.7.1. The larger the radius of the large ball, the less the forcing has to be increased.

However, with our approach we are only able to obtain this result if we assume that the diffusion matrix A for the second-order term is constant in space, the reasons for which we will discuss in Remark 4.7.3 below.

Proposition 4.7.1 (Compensating far-away influence with higher forcing)**.** *Let $(A,F) \in \Omega$ as defined in (4.3.4) with the additional assumption that A is constant in space in the sense that $A(x,e) = A(0,e)$ for any $x \in \mathbb{R}^d$ and $e \in \mathbb{S}^{d-1}$. Let $S_1, S_2 \subset \mathbb{R}^d$ be closed sets. Assume that S_1 is stable with respect to (A,F) and that for some $h, R > 0$*

$$\big(S_2 + \overline{B}_h\big) \cap \overline{B}_R \subset S_1.$$

Then when increasing the forcing for S_1 with a constant $\Delta F > 0$ there holds

$$\mathscr{R}_t^{(A,F)}\big(S_2\big) \cap \overline{B}_r \subset \mathscr{R}_t^{(A,F+\Delta F)}\big(S_1\big) \qquad \text{for all } 0 \leq t \leq T$$

for all $T \geq 0$ with

$$T \leq c\min\Big\{\big(R-1-h-r\big),\ \big(R-1-h-r\big)^{\frac{2}{3}}h^{\frac{4}{3}},\ \big(R-1-h-r\big)^{\frac{2}{3}}\Delta F^{\frac{4}{3}}\Big\}, \tag{4.7.1}$$

where $c = c(\text{data}) > 0$.

In order to prove this proposition, we first need an auxiliary result, which provides a decreasing (sub-)solution with initial 0-sublevel set S_1 for coefficients $(A, F+\Delta F)$.

Lemma 4.7.2. *Let $(A,F) \in \Omega$ with the additional assumption that A is constant in space in the sense that $A(x,e) = A(0,e)$ for any $x \in \mathbb{R}^d$ and $e \in \mathbb{S}^{d-1}$. Let $S \subset \mathbb{R}^d$ be stable with respect to (A,F). Then for any $\Delta F \in \mathbb{R}_+$, the set $S + \overline{B}_\delta$ is stable with respect to $(A, F+\Delta F)$ for all $0 \leq \delta \leq C_{1F}^{-1}\Delta F$. In particular, there exists a solution u_S to (4.1.6) for coefficients $(A, F+\Delta F)$ with initial data satisfying $S = \{x : u_S(x,0) \leq 0\}$, which is decreasing.*

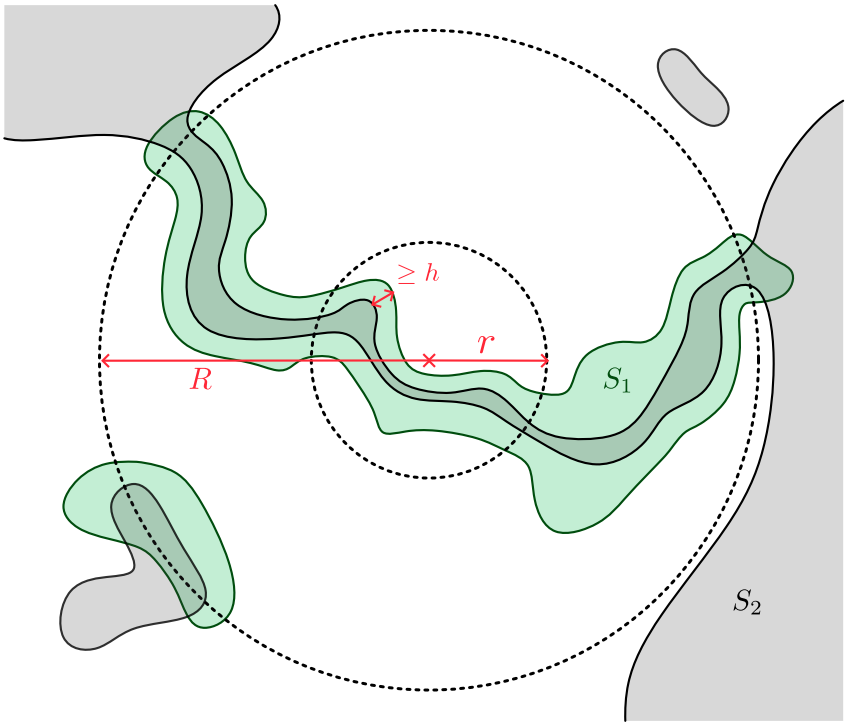


Figure 4.7.1: An illustration of the fact that far-away influences on the evolution of sets may be compensated for by increasing the forcing. Within the large ball B_R, the set S_1 (depicted in green) is distance h ahead of the set S_2 (depicted in grey). Within the small ball B_r, S_1 evolving by $(A, F + \Delta F)$ will stay ahead of S_2 evolving by (A, F) for a time depending on $R - r, h, \Delta F > 0$

Proof of Lemma 4.7.2. In order to prove that $S + \overline{B}_\delta$ is stable for some $\delta > 0$, we need to show that $u_\delta = 1 - \chi_{S+\overline{B}_\delta}$ is a supersolution to $G_{(A,F+\Delta F)}(x, \nabla u, D^2 u) = 0$. Thus, let $(\xi, X) \in \mathcal{J}^{2,-} u_\delta(x_0)$ for $x_0 \in \mathbb{R}^d$. If $x_0 \notin \big(S + \overline{B}_\delta\big)$, then $\xi = 0$ with $X \leq 0$ and hence $\big(G_{(A,F+\Delta F)}\big)^*(x_0, \xi, X) \geq 0$. If $x_0 \in \big(S + \overline{B}_\delta\big)$, then there is $y_0 \in S$ with $|x_0 - y_0| \leq \delta$. Lemma 4.A.2 yields $(\xi, X) \in \mathcal{J}^{2,-} u_0(x_0)$ with $u_0 = 1 - \chi_S$. Since S is stable, u_0 is a supersolution and hence we obtain $\big(G_{(A,F)}\big)^*(y_0, \xi, X) \geq 0$. For $\xi = 0$, by the definition of the upper semi-continuous envelope and since the second order term is independent of x_0 and y_0, we have

$$\big(G_{(A,F+\Delta F)}\big)^*(x_0, \xi, X) = \big(G_{(A,F+\Delta F)}\big)^*(x_0, 0, X) = \big(G_{(A,F)}\big)^*(y_0, 0, X) \geq 0.$$

For $\xi \neq 0$, due to the Lipschitz-continuity of F from Assumption (4.3.3) we obtain

$$\begin{aligned}
\big(G_{(A,F+\Delta F)}\big)^*(x_0, \xi, X) &= -\operatorname{tr}\big(A(\xi)X\big) + \big(F(x_0, \xi) + \Delta F\big)|\xi| \\
&\geq -\operatorname{tr}\big(A(\xi)X\big) + \big(F(y_0, \xi) - C_{1F}|x_0 - y_0| + \Delta F\big)|\xi| \\
&= \big(G_{(A,F)}\big)^*(y_0, \xi, X) + \big(\Delta F - C_{1F}\delta\big)|\xi| \\
&\geq 0.
\end{aligned}$$

Thus, we have shown that u_δ is a supersolution to $G_{(A,F+\Delta F)}(x, \nabla u, D^2 u) = 0$ and hence $S + \overline{B}_\delta$ is stable with respect to $(A, F + \Delta F)$ if $\delta \leq C_{1F}^{-1} \Delta F$.

To obtain the decreasing solution, we choose

$$u_0(x) = \begin{cases} 0 & \text{if } x \in S, \\ 1 & \text{if } x \notin S + B_{C_{1F}^{-1}\Delta F}, \\ \dfrac{C_{1F}}{\Delta F}\delta & \text{if } x \in \partial\big(S + \overline{B}_\delta\big) \text{ for } \delta \in (0, C_{1F}^{-1}\Delta F). \end{cases}$$

Clearly, u_0 is uniformly continuous with $S = \{x : u_0(x) \leq 0\}$. Hence, due to Theorem 4.A.14 there exists a unique solution u_S to the level-set equation (4.1.6) for coefficients $(A, F + \Delta F)$ with $u_S(\cdot, 0) = u_0$. Since the initial data has been chosen such that all of the initial sublevel sets are stable with respect to $(A, F + \Delta F)$, all sub-level sets stay stable and do not shrink over time due to Lemma 4.5.4. This implies that u_S is decreasing. □

Equipped with Lemma 4.7.2, we can now prove Proposition 4.7.1.

Proof of Proposition 4.7.1. Let $\Delta F \in \mathbb{R}_+$ and $0 < r < R$. We will obtain the result by comparing

- $0 \leq u_1 \leq 1$ with $\mathscr{R}_t^{(A,F+\Delta F)}(S_1) = \{x : u_1(x,t) \leq 0\}$, the decreasing solution to the level-set equation (4.1.6) with coefficients $(A, F + \Delta F)$ from Lemma 4.7.2, and
- $u_2^{\mathscr{R}}$ given by $u_2^{\mathscr{R}}(x,t) = 1 - \chi_{\mathscr{R}_t^{(A,F)}(S_2)}(x)$, which is a supersolution to (4.1.6) with coefficients (A, F) by Lemma 4.5.2

after applying suitable modifications: Instead of u_1, we will actually use

$$(x,t) \mapsto u_1(x,t) - \zeta t - \varphi_\varepsilon(x)$$

for small $\zeta, \varepsilon > 0$, morally corresponding to slightly speeding up the evolution via ζ and gently extending S_1 when moving away from the origin with $\varphi_\varepsilon \geq 0$ and

$$\varphi_\varepsilon = 0 \quad \text{on } \overline{B}_r, \qquad \varphi_\varepsilon(x) \geq \varepsilon\big(|x| - r - 1\big) \quad \text{for all } x \in \mathbb{R}^d, \qquad |\nabla\varphi_\varepsilon|, |D^2\varphi_\varepsilon| \leq \varepsilon.$$

Since we have no control of the gradients, we use the standard approach of doubling the variables to introduce some regularization, to then directly obtain a comparison. Introducing a regularization parameter $\delta > 0$ we write

$$\Psi(x,y,t) \coloneqq u_1(x,t) - u_2(y,t) - \zeta t - \varphi_\varepsilon(x) - \frac{|x-y|^4}{4\delta}.$$

Since Ψ is upper semicontinuous, upper bounded and decaying for $t \to \infty$, it attains its maximum in $\overline{B}_R \times \overline{B}_R \times [0,\infty)$, say at (x_0, y_0, t_0). We obtain

$$-1 \leq \Psi(0,0,0) \leq \Psi(x_0,y_0,t_0) \leq 1 - \frac{|x_0-y_0|^4}{4\delta}$$

and hence

$$|x_0 - y_0| \leq (8\delta)^{\frac{1}{4}}. \tag{4.7.2}$$

The rest of the proof will consist out of these three steps:

1.) We will show that the maximum is attained on the boundary of $\overline{B}_R \times \overline{B}_R \times [0,\infty)$ if the choices of the parameters $\zeta, \varepsilon, \delta$ are suitably restricted.

2.) We will show that $\Psi(x_0, y_0, t_0) \leq 0$ for a suitable choice of the parameters.

3.) Combining the restrictions on the parameters and using that the argument is invariant with respect to reparametrizations of u_1 we will obtain the result.

Step 1: No maximum in the interior. Assume that (x_0, y_0, t_0) is in the interior of $\overline{B}_R \times \overline{B}_R \times [0,\infty)$. We will show that this leads to a contradiction under suitable restrictions on the parameters. With Theorem 4.A.11, known as Ishii's Lemma or a parabolic maximum principle for semicontinuous functions, fixing $\gamma > 0$ there exist $b_1, b_2 \in \mathbb{R}$ and $X, Y \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ such that

$$\begin{cases} \big(b_1, \xi_0 + \nabla\varphi_\varepsilon(x_0), X\big) \in \overline{\mathcal{P}}^{2,+} u_1(x_0,t_0) \\ \big(b_2, \xi_0, Y\big) \in \overline{\mathcal{P}}^{2,-} u_2^{\mathscr{R}}(y_0,t_0) \end{cases}, \tag{4.7.3}$$

$$b_1 - b_2 = \zeta, \tag{4.7.4}$$

$$-\left(\frac{1}{\gamma} + |U|\right)\mathrm{Id}_{2d\times 2d} \le \begin{pmatrix} X - D^2\varphi_\varepsilon(x_0) & 0 \\ 0 & -Y \end{pmatrix} \le U + \gamma U^2 \tag{4.7.5}$$

with the spatial derivatives of the smooth 'test function' given by

$$\xi_0 := \frac{|x_0 - y_0|^2}{\delta}(x_0 - y_0)$$

and

$$U := \frac{1}{\delta}\begin{pmatrix} N & -N \\ -N & N \end{pmatrix} \qquad \text{with } N := |x_0 - y_0|^2\,\mathrm{Id}_{d\times d} + 2(x_0 - y_0)\otimes(x_0 - y_0).$$

If $x_0 \neq y_0$ we choose $\gamma = \delta|x_0 - y_0|^{-2}$ and hence obtain from (4.7.5) that

$$-\frac{C}{\delta}|x_0 - y_0|^2\,\mathrm{Id} \le \begin{pmatrix} X - D^2\varphi_\varepsilon(x_0) & 0 \\ 0 & -Y \end{pmatrix} \le \frac{C}{\delta}|x_0 - y_0|^2 \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}. \tag{4.7.6}$$

In fact, we next show that always $x_0 \neq y_0$ and hence this estimate holds. More precisely, we claim that there is $c = c(\mathrm{data}) > 0$ such that

$$|x_0 - y_0| \ge c(\delta\zeta)^{\frac{1}{2}} \qquad \text{if } \zeta\delta \le c. \tag{4.7.7}$$

On the one hand, note that since $u_2^{\mathscr{R}}$ is a supersolution to (4.1.6), due to the inclusion in the subjet from (4.7.3) we have

$$\big(G_{(A,F)}\big)_*(y_0, \xi_0, Y) \ge -b_2 = \zeta - b_1 \ge \zeta$$

where in the last step we used (4.7.4) that $b_1 \le 0$ since u_1 is decreasing due to the choice of the initial data from Lemma 4.7.2. On the other hand, choosing γ as above for $x_0 \neq y_0$ and arbitrarily for $x_0 = y_0$, from (4.7.5) we obtain

$$-Y \le C\delta^{-1}|x_0 - y_0|^2\,\mathrm{Id}$$

and hence

$$\big(G_{(A,F)}\big)_*(y_0, \xi_0, Y) \le CC_{1A}\delta^{-1}|x_0 - y_0|^2 + C_{1F}|\xi_0|.$$

Combining both equations and plugging in the definition of ξ_0, we obtain

$$\zeta\delta \le C|x_0 - y_0|^2 + C_{1F}|x_0 - y_0|^3,$$

which yields (4.7.7) and hence shows that the claim holds.

Next, we set up the contradiction. As above, with (4.7.3) and using that u_1 is a sub- and u_2 a supersolution we obtain

$$\begin{aligned} b_1 - \mathrm{tr}\big(A(\xi_0 + \nabla\varphi_\varepsilon(x_0))X\big) + \big(F(\xi_0 + \nabla\varphi_\varepsilon(x_0), x_0) + \Delta F\big)|\xi_0 + \nabla\varphi_\varepsilon(x_0)| &\le 0 \\ b_2 - \mathrm{tr}\big(A(\xi_0)Y\big) + F(\xi_0, y_0)|\xi_0| &\ge 0 \end{aligned}$$

knowing that $\xi_0 \neq 0$ and for now assuming that $\xi_0 + \nabla\varphi_\varepsilon(x_0) \neq 0$. Combining both and using (4.7.4) yields

$$\zeta \leq \operatorname{tr}\big(A(\xi_0 + \nabla\varphi_\varepsilon(x_0))X\big) - \operatorname{tr}\big(A(\xi_0)Y\big) + F\big(\xi_0, y_0\big)|\xi_0| - \big(F\big(\xi_0 + \nabla\varphi_\varepsilon(x_0), x_0\big) + \Delta F\big)|\xi_0 + \nabla\varphi_\varepsilon(x_0)|. \tag{4.7.8}$$

We will derive suitable restrictions on the parameters for which the right hand side is smaller than ζ, which will yield the contradiction. First, to ensure that $\xi_0 + \nabla\varphi_\varepsilon(x_0) \neq 0$, besides $\delta\zeta \leq c$ as in (4.7.7) we require that

$$\varepsilon < c^3\delta^{\frac{1}{2}}\zeta^{\frac{3}{2}} \qquad \text{such that } \big|\nabla\varphi_\varepsilon(x_0)\big| \leq \epsilon < \frac{\Big(c(\delta\zeta)^{\frac{1}{2}}\Big)^3}{\delta} \leq \big|\xi_0\big|$$

with the constant $c = c(\text{data}) > 0$ originating in (4.7.7). Further, note that

$$\left|\frac{\xi_0}{|\xi_0|} - \frac{\xi_0 + \nabla\varphi_\varepsilon(x_0)}{|\xi_0 + \nabla\varphi_\varepsilon(x_0)|}\right| \leq 2\frac{|\nabla\varphi_\varepsilon(x_0)|}{|\xi_0|} \leq \frac{2\varepsilon\delta}{|x_0 - y_0|^3}. \tag{4.7.9}$$

Regarding the first order terms, we claim that there is some $C = C(\text{data}) > 0$ such that

$$F\big(\xi_0, y_0\big)|\xi_0| - \big(F\big(\xi_0 + \nabla\varphi_\varepsilon(x_0), x_0\big) + \Delta F\big)|\xi_0 + \nabla\varphi_\varepsilon(x_0)| \leq C\varepsilon \quad \text{if} \quad \Delta F \geq C\delta^{\frac{1}{4}}. \tag{4.7.10}$$

In order to compare the two different locations, using the Lipschitz continuity of F on $\mathbb{S}^{d-1}$ from (4.3.3) and (4.7.9) from above, we first bound the error for neglecting $|\nabla\varphi_\varepsilon(x_0)| \leq \varepsilon$,

$$\begin{aligned}&\left|\left(F\left(\frac{\xi_0 + \nabla\varphi_\varepsilon(x_0)}{|\xi_0 + \nabla\varphi_\varepsilon(x_0)|}, x_0\right) + \Delta F\right)|\xi_0 + \nabla\varphi_\varepsilon(x_0)| - \left(F\left(\frac{\xi_0}{|\xi_0|}, x_0\right) + \Delta F\right)|\xi_0|\right| \\ &\quad \leq \big(C_{1F} + \Delta F\big)\varepsilon + C_{1F}\frac{2\varepsilon}{|\xi_0|}|\xi_0|.\end{aligned}$$

To bound the error from switching location, we use the Lipschitz continuity of F in $\mathbb{R}^d$ from (4.3.3) and obtain

$$F\big(\xi_0, y_0\big)|\xi_0| - \left(F\left(\frac{\xi_0}{|\xi_0|}, x_0\right) + \Delta F\right)|\xi_0| \leq \big(C_{1F}|x_0 - y_0| - \Delta F\big)|\xi_0| \leq 0$$

where the last inequality is guaranteed to hold via (4.7.2) if $\Delta F \geq C_{1F}(8\delta)^{\frac{1}{4}}$.

Regarding the second order terms, we claim that there is $C = C(\text{data}) > 0$ such that

$$\operatorname{tr}\big(A(\xi_0 + \nabla\varphi_\varepsilon(x_0))X\big) - \operatorname{tr}\big(A(\xi_0)Y\big) \leq C\varepsilon + C\delta^{-1}\zeta^{-2}\varepsilon^2. \tag{4.7.11}$$

In order to use (4.7.6), we first get an error for inserting $D^2\varphi_\varepsilon(x_0)$,

$$\big|\operatorname{tr}\big(A(\xi_0 + \nabla\varphi_\varepsilon(x_0))X\big) - \operatorname{tr}\big(A(\xi_0 + \nabla\varphi_\varepsilon(x_0))\big(X - D^2\varphi_\varepsilon(x_0)\big)\big)\big| \leq C_{1A}\varepsilon.$$

Since with Assumption (4.3.1) we have $A = \sigma\sigma^\intercal$, we can write

$$\begin{aligned}&\operatorname{tr}\big(A(\xi_0 + \nabla\varphi_\varepsilon(x_0))\big(X - D^2\varphi_\varepsilon(x_0)\big)\big) - \operatorname{tr}\big(A(\xi_0)Y\big) \\ &\quad = \operatorname{tr}\big(\sigma^\intercal(\xi_0 + \nabla\varphi_\varepsilon(x_0))\big(X - D^2\varphi_\varepsilon(x_0)\big)\sigma(\xi_0 + \nabla\varphi_\varepsilon(x_0)) - \sigma^\intercal(\xi_0)Y\sigma(\xi_0)\big) \\ &\quad \leq \frac{C}{\delta}\big|x_0 - y_0\big|^2\left|\sigma\left(\frac{\xi_0 + \nabla\varphi_\varepsilon(x_0)}{|\xi_0 + \nabla\varphi_\varepsilon(x_0)|}\right) - \sigma\left(\frac{\xi_0}{|\xi_0|}\right)\right|^2 \\ &\quad \leq \frac{C}{\delta}\big|x_0 - y_0\big|^2 C_{1A}^2\frac{\varepsilon^2}{|\xi_0|^2} \\ &\quad = C\delta\big|x_0 - y_0\big|^{-4}\varepsilon^2 \\ &\quad \leq C\delta^{-1}\zeta^{-2}\varepsilon^2\end{aligned}$$

where the first inequality is an application of (4.7.6), the second follows from the Lipschitz continuity of σ and (4.7.9), and the last inequality follows from (4.7.7). Combining the two estimates above yields (4.7.11).

To obtain the contradiction, we now plug the inequalities (4.7.10) and (4.7.11) into (4.7.8) and collect the restrictions on the parameters to obtain

$$\zeta \leq C\varepsilon + C\delta^{-1}\zeta^{-2}\varepsilon^2 < \zeta \qquad \text{if} \quad \begin{cases} \zeta\delta \leq c_1(\text{data}), \\ \varepsilon < c_2(\text{data})\delta^{\frac{1}{2}}\zeta^{\frac{3}{2}}, \\ \Delta F \geq C(\text{data})\delta^{\frac{1}{4}}, \end{cases} \tag{4.7.12}$$

where the contradiction holds if we choose $c_2 = c_2(\text{data}, c_1) > 0$ small enough.

Step 2: Bounding the maximum. Because the maximum is not attained in the interior, (x_0, y_0, t_0) has to be on the boundary of $\overline{B}_R \times \overline{B}_R \times [0,\infty)$. Thus, there holds $x_0 \in \partial B_R$, $y_0 \in \partial B_R$ or $t_0 = 0$. If $x_0 \in \partial B_R$, then with $|x_0| = R$ and the definition of φ_ε note that

$$\Psi(x_0, y_0, t_0) \leq 1 - 0 - 0 - \varepsilon(R - r - 1) - 0.$$

Similarly, if $y_0 \in \partial B_R$ then due to (4.7.2) also $|x_0| \geq R - (8\delta)^{\frac{1}{4}}$ and hence

$$\Psi(x_0, y_0, t_0) \leq 1 - 0 - 0 - \varepsilon\Big(R - (8\delta)^{\frac{1}{4}} - r - 1\Big) - 0.$$

If $t_0 = 0$ and $x_0 \in S_1$, then $u_1(x_0, t_0) \leq 0$ and hence $\Psi(x_0, y_0, t_0) \leq 0$. If however $x_0 \notin S_1$, then because $S_2 + \overline{B}_h \cap \overline{B}_R \subset S_1$ we have $y_0 \notin S_2$ if $|x_0 - y_0| \leq h$ and thus due to (4.7.2) we have

$$\Psi(x_0, y_0, t_0) \leq 1 - 1 - 0 - 0 - 0 = 0 \qquad \text{if} \quad (8\delta)^{\frac{1}{4}} \leq h.$$

In summary, we obtain

$$\Psi(x_0, y_0, t_0) \leq 0 \qquad \text{if next to (4.7.12) also} \quad \begin{cases} \varepsilon \geq \dfrac{1}{R - 1 - h - r}, \\ \delta \leq 8h^4. \end{cases} \tag{4.7.13}$$

Step 3: Choosing the parameters and comparing u_1 and $u_2^{\mathscr{R}}$. For $x \in \overline{B}_r$ and $t \geq 0$, having $\max_{(x,y,t)\in\overline{B}_R\times\overline{B}_R\times[0,\infty)} \Psi(x, y, t) \leq 0$ implies

$$u_1(x, t) - u_2^{\mathscr{R}}(x, t) - \zeta t = \Psi(x, x, t) \leq 0.$$

As an intermediate step, we want to obtain

$$\mathscr{R}_t^{(A,F)}\big(S_2\big) \cap \overline{B}_r \subset \left\{ x \,:\, u_1(x, t) \leq \frac{1}{2} \right\} \qquad \text{for all } 0 \leq t \leq T. \tag{4.7.14}$$

Thus, we choose $\zeta = \frac{1}{2T}$. If we can choose the rest of the parameters such that the requirements from (4.7.12) and (4.7.13) are satisfied, then we obtain (4.7.14). Thus, we choose ε and δ as small as possible, that is

$$\varepsilon = \frac{1}{R - 1 - h - r}, \qquad \delta = C(\text{data})\frac{\varepsilon^2}{\zeta^3} = C(\text{data})\frac{T^3}{(R - 1 - h - r)^2},$$

where the constant for δ is chosen such that the corresponding inequality in (4.7.12) is satisfied. For (4.7.13) to be satisfied, we hence require

$$\big(R - 1 - h - r\big) \geq C(\text{data})\frac{T^{\frac{3}{2}}}{h^2}.$$

For (4.7.12) to be satisfied, when plugging in the choice of δ we see that we need

$$\big(R-1-h-r\big) \geq C(\text{data})T, \qquad \Delta F \geq C(\text{data})\frac{T^{\frac{3}{4}}}{(R-1-h-r)^{\frac{1}{2}}}.$$

Reformulating these requirements in terms of T yields (4.7.1). To conclude the proof, note that all of the bounds above are invariant with respect to reparametrizing the initial data of u_1. Repeating the same arguments with $\widetilde{u}_1 \coloneqq \max\{su_1, 1\}$ for $s \geq 1$, we also arrive at (4.7.14), now corresponding to

$$\mathscr{R}_t^{(A,F)}\big(S_2\big) \cap \overline{B}_r \subset \left\{x \,:\, \widetilde{u}_1(x,t) \leq \frac{1}{2}\right\} = \left\{x \,:\, u_1(x,t) \leq \frac{1}{2s}\right\} \qquad \text{for } 0 \leq t \leq T.$$

Since $s \geq 1$ is arbitrary, we get $\mathscr{R}_t^{(A,F)}(S_2) \cap \overline{B}_r \subset \{x \,:\, u_1(x,t) \leq 0\} = \mathscr{R}_t^{(A,F+\Delta F)}(S_1)$. □

Remark 4.7.3. The main reason why we increase the forcing for the comparison is not to obtain Lemma 4.7.2 – this could also be solved using a slightly different concept of stability – but to control the difference of the first-order terms in (4.7.10) of Step 1 in the proof of Proposition 4.7.1. The reason why we have to rely on this makeshift approach is that we lack a guaranteed minimum speed of propagation for the interfaces. This minimum speed would translate to Lipschitz continuity of the level-set solutions, in terms of the notation from the proof allowing us to bound ξ_0 and thus obtain better control of $|x_0 - y_0|$ and smallness of $\delta^{-1}|x_0 - y_0|^4$, which would make increasing the forcing obsolete.

Similarly, we are not able to treat inhomogeneous curvature terms because due to the lack of control on the gradients also the bounds for X and Y obtained via the regularization explode too fast.

In the following, instead of Proposition 4.7.1 we will use this variation for the half-ball instead of the full ball, see Figure 4.7.2.

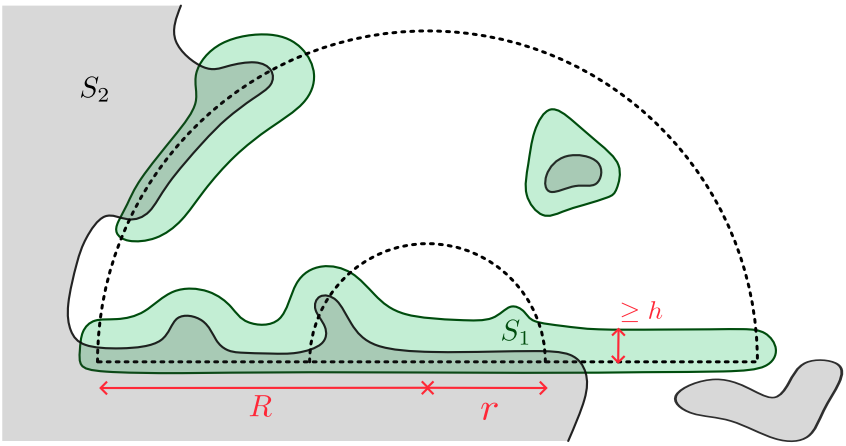


Figure 4.7.2: An illustration of the fact that far-away influences on the evolution of sets within the half-ball may be compensated for by increasing the forcing. In the upper half-ball $\overline{B}_R \cap \mathcal{H}_e^+$, the set S_1 (depicted in green) is distance h ahead of the set S_2 (depicted in grey) as well as h ahead of the boundary of the half-space. Within the small half-ball $\overline{B}_r \cap \mathcal{H}_e^+$, S_1 evolving by $(A, F + \Delta F)$ will stay ahead of S_2 evolving by (A, F) for a time depending on $R - r, h, \Delta F > 0$

Proposition 4.7.4 (Compensating far-away influence with higher forcing in the half-ball)**.** *Let $(A, F) \in \Omega$ with the additional assumption that $A(x, e) = A(0, e)$ for $x \in \mathbb{R}^d$, $e \in \mathbb{S}^{d-1}$ is constant in space. Let $S_1, S_2 \subset \mathbb{R}^d$ be closed sets. Assume that S_1 is stable with respect to (A, F) and that for some $h, R > 0$ in the upper half-space $\mathcal{H}_e^+ = \{x \in \mathbb{R}^d \,:\, x \cdot e \geq 0\}$ in direction $e \in \mathbb{S}^{d-1}$ there holds*

$$\big(S_2 + \overline{B}_h\big) \cap \overline{B}_R \cap \mathcal{H}_e^+ \subset S_1 \qquad \textit{and} \qquad \big(\partial\mathcal{H}_e^+ + \overline{B}_h\big) \cap \overline{B}_R \cap \mathcal{H}_e^+ \subset S_1.$$

Then when increasing the forcing for S_1 with a constant $\Delta F \in \mathbb{R}_+$ there holds

$$\mathscr{R}_t^{(A,F)}(S_2) \cap \overline{B}_r \cap \mathcal{H}_e^+ \subset \mathscr{R}_t^{(A,F+\Delta F)}(S_1) \qquad \text{for all } 0 \le t \le T$$

for all T with

$$T \le c \min\Big\{ (R-1-h-r),\ (R-1-h-r)^{\frac{2}{3}} h^{\frac{4}{3}},\ (R-1-h-r)^{\frac{2}{3}} \Delta F^{\frac{4}{3}} \Big\}, \tag{4.7.1}$$

where $c = c(\mathrm{data}) > 0$.

Proof. The proof is analogous to the proof of Proposition 4.7.1. The only difference is that in Step 2 we might also have $x_0 \in \partial\mathcal{H}_e^+ \cap \overline{B}_R$ or $y_0 \in \partial\mathcal{H}_e^+$. Either way, since with the same choice of parameters we have $|x_0 - y_0| \le h$ and hence $x_0 \in S_1$ by assumption. Because S_1 is stable, this implies $x_0 \in \mathscr{R}_t^{(A,F+\Delta F)}(S_1)$ for all $t \ge 0$, thus $u_1(x_0, t_0) = 0$ and hence also $\Psi(x_0, y_0, t_0) \le 0$ as in all of the other cases. □

4.8 The homogenized speed via arrival times for half-space approximations

For a direction $e \in \mathbb{S}^{d-1}$ we denote the corresponding upper and lower half-space with

$$\mathcal{H}_e^+ = \{x \in \mathbb{R}^d : x \cdot e \ge 0\}, \qquad \mathcal{H}_e^- = \{x \in \mathbb{R}^d : x \cdot e \le 0\}.$$

Translating the results from [AC18] to our setting, we would like to choose the homogenized speed as the average speed with which the half space reaches an 'infinitely far away' target area, that is

$$\overline{v}(e) := \left(\lim_{r\to\infty} \frac{1}{r} \mathbb{E}\Big[m_{h(r)}\big(re, \mathcal{H}_e^-\big) \Big] \right)^{-1} \tag{4.8.1}$$

for suitable chosen $r \mapsto h(r)r$. However, we can not guarantee that this limit exists. Since the fluctuation bounds from Proposition 4.6.1 and some of the following arguments require a set with bounded boundary, we will approximate the boundary of the half-space with a family of bounded disks

$$S^-_{e,\beta,h(r)}(r) \approx \mathcal{H}_e^- \qquad \text{on the scale } r \ge 0 \text{ for a fixed parameter } \beta,$$

see Figure 4.8.1. In Corollary 4.8.1, we will define these sets in a way such that Proposition 4.7.4 guarantees good approximative properties with respect to the half-space.

The main task of this section is to show that the expected values of the corresponding arrival times,

$$r \mapsto \mathbb{E}\Big[m_{h(r)}\Big(re, S^-_{e,\beta,h(r)}(r) \Big) \Big],$$

behave almost linearly, see Proposition 4.8.2 for the approximate sub- and Proposition 4.8.4 for the approximate superlinearity. Already the approximate sub-linearity then implies the existence of a limit for the averaged arrival times as in (4.8.1). The approximate linearity further provides quantitative bounds for the convergence, see Corollary 4.8.3 and Corollary 4.8.5. We obtain the approximate sub-linearity adapting arguments from [AC18, Section 4.2] to our setting. However, we only obtain the approximate superlinearity at the cost of slightly changing the forcing due to applications of Proposition 4.7.4. As a result, the quality of the quantitative bounds depends on the continuity of arrival times with respect to changing the forcing by a constant, which is why we will later need Assumption (Pη). At the end of this section, we will define the homogenized speed and discuss the implications of Assumption (Pη).

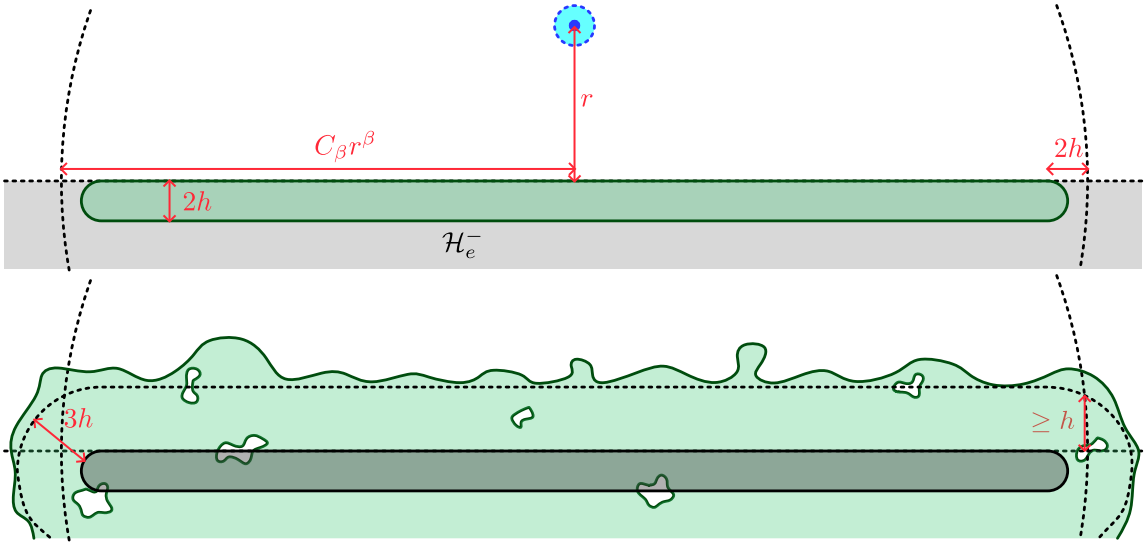


Figure 4.8.1: An illustration of the half-space approximations and the fact that they are quickly h ahead of the half-space in the corresponding half-ball. Top: The half-space approximation $S^-_{e,\beta,h}(r)$ is depicted in green, the target area $\overline{B}_h(re)$ in blue. Bottom: The evolution $\mathscr{R}_{(c_s+3v_{\min}^{-1})h}(S_h)$ (depicted in green) of a stable (h,c_s)-approximation S_h of $S^-_{e,\beta,h}(r)$ after time $(c_s+3v_{\min}^{-1})h$ fully h-envelopes the set $(\partial\mathcal{H}_e^+ + \overline{B}_h)\cap\overline{B}_R\cap\mathcal{H}_e^+$

4.8.1 The half-space approximations

We start by introducing the disks $S^-_{e,\beta,h(r)}(r)$ approximating the respective half-space $\mathcal{H}_e^-$ on the scale $r\geq h_0$, see Figure 4.8.1. To obtain the approximative properties, we compensate the far-away influence by slightly increasing the forcing as in Proposition 4.7.4: We only need to compare the evolution of the approximation to the original half-space (or other large sets) up to the time $T_h(re,\mathcal{H}_e^-)\approx r$ at which we truncate the arrival times. Plugging this into (4.7.1) from Proposition 4.7.4 and choosing a scaling for the width of the disk yields the scaling of the forcing increase.

Corollary 4.8.1 (Justifying half-space approximations with Proposition 4.7.4)**.** *Let $e\in\mathbb{S}^{d-1}$ and $\beta\geq\frac{3}{2}$. For all $r,h\geq h_0$ we approximate $\mathcal{H}_e^-$ with the fattened disks*

$$S^-_{e,\beta,h}(r) := \Big\{x\in\overline{B}_{C_\beta r^\beta-2h}(-he) \text{ with } (x+he)\cdot e=0\Big\}+\overline{B}_h\subset\overline{B}_{C_\beta r^\beta},$$

where $C_\beta=C_\beta(\mathrm{data})>0$ is chosen large enough such that (4.7.1) from Proposition 4.7.1 and Proposition 4.7.4 is satisfied for all $r\geq h\geq h_0$ with $(T(r,h),2r,h,R(r),\Delta F(r))$ given by

$$T(r,h)=T_h\big(re,\mathcal{H}_e^-\big)\leq C(\mathrm{data})r \quad \text{as in Definition 4.1.6}$$

and

$$R(r)=C_\beta r^\beta,\qquad\qquad \Delta F(r)=r^{-\frac{\beta}{2}+\frac{3}{4}}.$$

Let $(A,F)\in E_h^{\circledcirc}\big(\overline{B}_{C_\beta r^\beta}\big)$ from Assumption (P⊙) with the additional assumption that $A(x,e)=A(0,e)$ for $x\in\mathbb{R}^d$, $e\in\mathbb{S}^{d-1}$ is constant in space. Then

$$\begin{aligned} m_h^{(A,F)}\Big(re,S^-_{e,\beta,h}(r)\Big)&\geq m_h^{(A,F)}\big(re,\mathcal{H}_e^-\big)\\ &\geq m_h^{\left(A,F+r^{-\frac{\beta}{2}+\frac{3}{4}}\right)}\Big(re,S^-_{e,\beta,h}(r)\Big)-(c_s+3v_{\min}^{-1})h. \end{aligned}\tag{4.8.2}$$

The same inequalities hold when replacing $\mathcal{H}_e^-$ with any other closed set $S \subset \mathbb{R}^d$ which satisfies $S_{e,\beta,h}^-(r) \subset S \subset \mathcal{H}_e^- \cup \left(\mathbb{R}^d \setminus \overline{B}_{C_\beta r^\beta}\right)$.

Proof. We only show the proof for $S = \mathcal{H}_e^-$, it is the same for any other closed set S with $S_{e,\beta,h}^-(r) \subset S \subset \mathcal{H}_e^- \cup \left(\mathbb{R}^d \setminus \overline{B}_{C_\beta r^\beta}\right)$.

Via the comparison principle from Lemma 4.5.5 we know that for all $t \geq 0$ there holds $\mathscr{R}_t^{(A,F)}\left(S_{e,\beta,h}^-(r)\right) \subset \mathscr{R}_t^{(A,F)}(\mathcal{H}_e^-)$. The first inequality in (4.8.2) hence follows via Lemma 4.3.9. For the second inequality, due to Lemma 4.3.9 it is sufficient if we can show that

$$\mathscr{R}_t^{(A,F)}(\mathcal{H}_e^-) \cap \overline{B}_{2r} \cap \mathcal{H}_e^+ \subset_h \mathscr{R}_{t+(c_s+3v_{\min}^{-1})h}^{\left(A,F+r^{-\frac{\beta}{2}+\frac{3}{4}}\right)}\left(S_{e,\beta,h}^-(r)\right) \quad \text{for all } t \leq T_h(re, \mathcal{H}_e^-). \tag{4.8.3}$$

Since $S_{e,\beta,h}^-(r)$ is h-fat and $(A,F) \in E_h^{\circledcirc}\left(\overline{B}_{C_\beta r^\beta}\right)$, Assumption (P$\circledcirc$) guarantees that there is a stable (h, c_s)-approximation S_h of $S_{e,\beta,h}^-(r)$ with respect to (A,F), for which the minimum effective speed applies. On the one hand, since $S_h \subset S_{e,\beta,h}^-(r)$, by the comparison principle from Lemma 4.5.5 there holds

$$\mathscr{R}_t^{\left(A,F+r^{-\frac{\beta}{2}+\frac{3}{4}}\right)}(S_h) \subset \mathscr{R}_t^{\left(A,F+r^{-\frac{\beta}{2}+\frac{3}{4}}\right)}\left(S_{e,\beta,h}^-(r)\right) \qquad \text{for all } t \geq 0.$$

On the other hand, due to the definition of $S_{e,\beta,h}^-(r)$, with $S_{e,\beta,h}^-(r) \subset_h \mathscr{R}_{c_s h}^{(A,F)}(S_h)$ and with the minimum effective speed as in Lemma 4.5.6 we have

$$(\partial\mathcal{H}_e^+ + \overline{B}_h) \cap \overline{B}_{C_\beta r^\beta} \cap \mathcal{H}_e^+ \subset \left(S_{e,\beta,h}^-(r) + 3\overline{B}_h\right) \cap \overline{B}_{C_\beta r^\beta} \subset_h \mathscr{R}_{(c_s+3v_{\min}^{-1})h}^{\left(A,F+r^{-\frac{\beta}{2}+\frac{3}{4}}\right)}(S_h).$$

Filling potential holes of width smaller than $2h$ in $\mathscr{R}_{(c_s+3v_{\min}^{-1})h}^{\left(A,F+r^{-\frac{\beta}{2}+\frac{3}{4}}\right)}(S_h)$ as in the proof of Lemma 4.5.5, we obtain a stable set which satisfies the conditions of Proposition 4.7.4 as S_1 with $S_2 = \mathcal{H}_e^-$. Proposition 4.7.4 with the parameters $(T(r,h), \widetilde{r}(r), h, R(r), \Delta F(r))$ chosen as above in the assumptions hence yields (4.8.3), which concludes the proof. □

4.8.2 Approximate sub-linearity, convergence of the averages and lower bounds

We will now show that the expected values of the truncated arrival times for the half-space approximations are sub-linear up to a vanishing relative error as $r \to \infty$.

Proposition 4.8.2 (Approximate sub-linearity of the truncated arrival times)**.** *Assume that (P1), (P2), (P3) and (P$\circledcirc$) hold. Let $\beta \geq \frac{3}{2}$ with $C_\beta > 0$ and $S_{e,\beta,h}^-(r)$ for $e \in \mathbb{S}^{d-1}$ and $r, h \geq h_0$ as defined in Corollary 4.8.1.*

Let $r_2 \geq r_1 \geq h_0$ and $h_3 \geq h_2 \geq h_1 \geq h_0$ with $\log(h_2) \geq c\log(r_2)$ for some $c > 0$. Then there exists $C = C(c, \beta, \text{data}) > 0$ such that

$$\begin{aligned}\mathbb{E}\Big[m_{h_3}\Big((r_1+r_2)e, S_{e,\beta,h_3}^-(r_1+r_2)\Big)\Big] \\ \leq \mathbb{E}\Big[m_{h_1}\Big(r_1 e, S_{e,\beta,h_1}^-(r_1)\Big)\Big] + \mathbb{E}\Big[m_{h_2}\Big(r_2 e, S_{e,\beta,h_2}^-(r_2)\Big)\Big] + C\sqrt{h_2 r_2}\log(r_1).\end{aligned} \tag{4.8.4}$$

Proof. We set $r_3 = r_1 + r_2$. The time until the set $S^-_{e,\beta,h_3}(r_3)$ effectively reaches $\overline{B}_{h_3}(r_3 e)$ is clearly less than adding the time it takes for $S^-_{e,\beta,h_3}(r_3)$ to fully envelop $r_2 e + S^-_{e,\beta,h_1}(r_1)$ to the time it then takes until the 'intermediate set' $r_2 e + S^-_{e,\beta,h_1}(r_1)$ reaches $\overline{B}_{h_1}(r_3 e)$, see Figure 4.8.2.

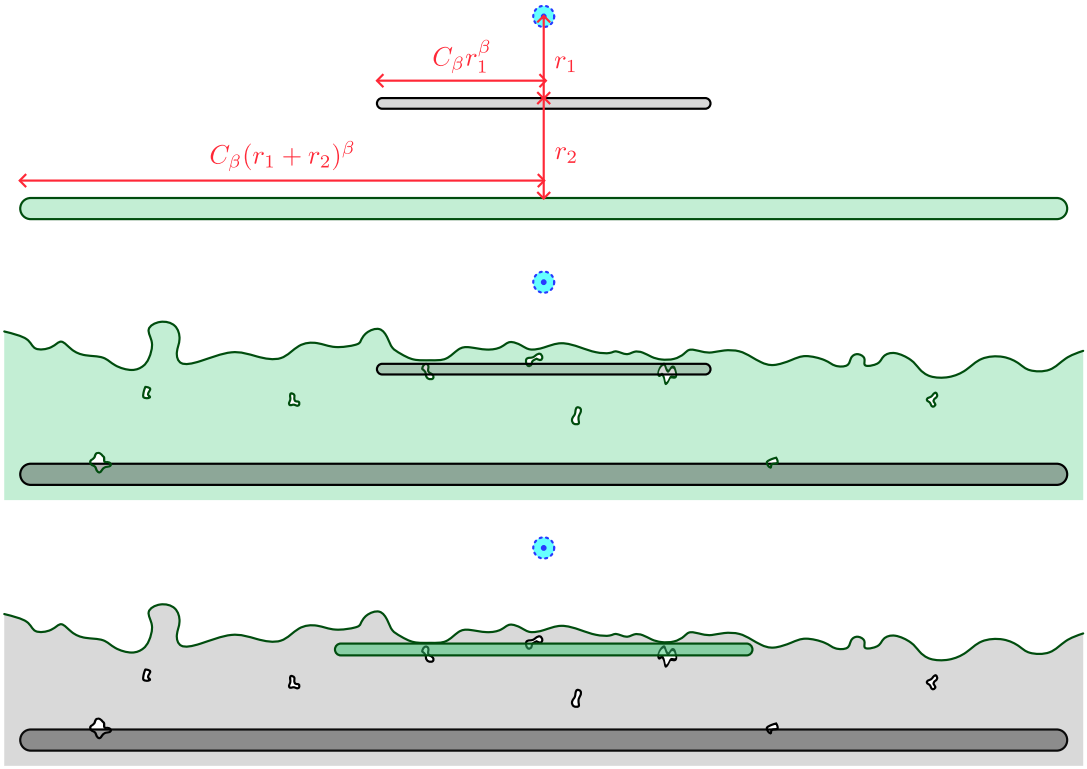


Figure 4.8.2: An illustration of the argument for the sub-linearity of the arrival times. Top: The large disk is depicted in green with its target area in blue, and the smaller 'intermediate' disk is depicted in grey. Center: The evolution of the large disk at time T_{full} (depicted in green) h_2-envelopes the smaller disk. Bottom: At time T_{full}, we compare the evolved set with the smaller intermediate disk (depicted in green)

More precisely, we denote the time by which $r_2 e + S^-_{e,\beta,h_1}(r_1)$ is fully h_2-enveloped with

$$T_{\text{full}} := \inf\Big\{ t \geq 0 \,:\, r_2 e + S^-_{e,\beta,h_1}(r_1) \subset_{h_2} \mathscr{R}_t\Big(S^-_{e,\beta,h_3}(r_3)\Big)\Big\}.$$

Then for $(A,F) \in E^{\circledcirc}_{h_2}\Big(\overline{B}_{C_\beta r_3^\beta}\Big)$ there holds

$$m^{(A,F)}_{h_3}\Big(r_3 e, S^-_{e,\beta,h_3}(r_3)\Big) \leq T^{(A,F)}_{\text{full}} + c_s h_2 + m^{(A,F)}_{h_1}\Big(r_3 e, r_2 e + S^-_{e,\beta,h_1}(r_1)\Big) \tag{4.8.5}$$

because with S_{h_2} denoting the stable (h_2, c_s)-approximation of $S^-_{e,\beta,h_3}(r_3)$ from (P$\circledcirc$) and with the comparison principle as in Lemma 4.5.5, for all $t \geq 0$ we have

$$\mathscr{R}^{(A,F)}_t\Big(r_2 e + S^-_{e,\beta,h_1}(r_1)\Big) \subset_{h_2} \mathscr{R}^{(A,F)}_{t+T^{(A,F)}_{\text{full}}+c_s h_2}(S_{h_2}) \subset \mathscr{R}^{(A,F)}_{t+T^{(A,F)}_{\text{full}}+c_s h_2}\Big(S^-_{e,\beta,h_3}(r_3)\Big).$$

In order to obtain (4.8.4) from (4.8.5), two steps remain:

1.) We will show that the expected value of T_{full} on $E^{\circledcirc}_{h_2}\Big(\overline{B}_{C_\beta r_3^\beta}\Big)$ can be bounded by the expected value of $m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)$ up to an error essentially corresponding to the fluctuation bounds from Proposition 4.6.1.

2.) We take the expected value of (4.8.5), use the stationarity from Assumption (P1) and manage the error on the complement of $E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)$.

Step 1: Bounding T_{full}. For $(A,F) \in E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)$, due to the minimum effective speed we can essentially reduce T_{full} to the maximum of a finite amount of truncated arrival times in target areas of radius h_2: Instead of every point in the upper disk $r_2 e + S^-_{e,\beta,h_1}(r_1)$, in a first step we only care about reaching every point on a grid. Whenever the evolution of the large disk $S^-_{e,\beta,h_3}(r_3)$ reaches the target area around one of these points, its stable (h_2, c_s)-approximation from Assumption (P$\circledcirc$) will reach the area just $c_s h_2$ later. Since (A,F) admits $v_{\min}$ as an effective minimum speed for the stable approximation, $\{x\}$ will be h_2-enveloped $v_{\min}^{-1} h_2$ later. Once all gridpoints are h_2-enveloped, the whole upper disk will be h_2-enveloped after $v_{\min}^{-1} h_2$ and another time step proportional to the gridsize by the evolution of the stable approximation and thus due to the comparison principle also by the evolution of the large disk. Hence, for $\lambda > 0$ and $\mathbb{Z}^{d-1}_e$ denoting $\mathbb{Z}^{d-1}$ placed in the hyperplane $\{x \in \mathbb{R}^d \,:\, x \cdot e = 0\}$, we obtain

$$T^{(A,F)}_{\text{full}} \le \sup\left\{ m^{(A,F)}_{h_2}\left(x + r_2 e, S^-_{e,\beta,h_3}(r_3)\right) \,:\, x \in \frac{v_{\min}}{\sqrt{d}} \lambda \mathbb{Z}^{d-1}_e \cap S^-_{e,\beta,h_1}(r_1) \right\} + \left(c_s + 2v_{\min}^{-1}\right) h_2 + \lambda.$$

For each target point $x + r_2 e$, we can further replace the large disk with the smaller disks $x + S^-_{e,\beta,h_2}(r_2) \subset S^-_{e,\beta,h_3}(r_3)$, see Figure 4.8.3.

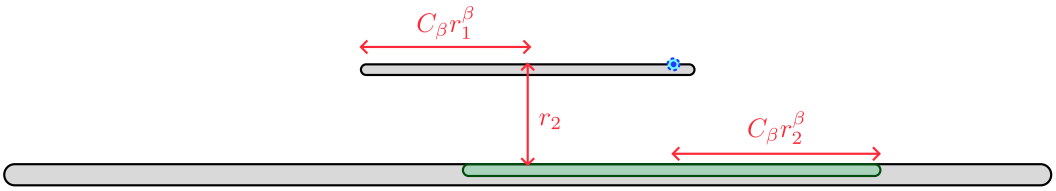


Figure 4.8.3: An illustration of the comparison argument from Step 1 of the proof for the sub-linearity of the arrival times. A small disk (depicted in green) is placed within the large disk $S^-_{e,\beta,h_3}(r_3)$. Its target area (depicted in blue) covers part of the intermediate disk

The comparison principle as in Lemma 4.5.5 yields

$$T^{(A,F)}_{\text{full}} \le \sup\left\{ m^{(A,F)}_{h_2}\left(x + r_2 e, x + S^-_{e,\beta,h_2}(r_2)\right) \,:\, x \in \frac{v_{\min}}{\sqrt{d}} \lambda \mathbb{Z}^{d-1}_e \cap S^-_{e,\beta,h_1}(r_1) \right\} + \left(c_s + 2v_{\min}^{-1}\right) h_2 + \lambda.$$

We set

$$T_2 = \mathbb{E}\left[m_{h_2}\left(r_2 e, S^-_{e,\beta,h_2}(r_2)\right)\right].$$

Using this bound on $T^{(A,F)}_{\text{full}}$, for $\lambda > \left(c_s + 2v_{\min}^{-1}\right) h_2$ and with a union bound and the station-

arity of the probability distribution from Assumption (P1), we obtain

$$\begin{aligned}
&\mathbb{P}\Big[T_{\text{full}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)} - T_2 \geq 3\lambda\Big] \\
&\leq \mathbb{P}\Bigg[\sup_{x\in\frac{v_{\min}\lambda}{\sqrt{d}}\mathbb{Z}_e^{d-1}\cap S^-_{e,\beta,h_1}(r_1)} m_{h_1}\Big(x + r_2 e, x + S^-_{e,\beta,h_2}(r_2)\Big) - T_2 \geq \lambda\Bigg] \\
&\leq \sum_{x\in\frac{v_{\min}\lambda}{\sqrt{d}}\mathbb{Z}_e^{d-1}\cap S^-_{e,\beta,h_1}(r_1)} \mathbb{P}\Big[m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big) - T_2 \geq \lambda\Big] \\
&\leq C\frac{r_1^{\beta(d-1)}}{\lambda^{d-1}}\mathbb{P}\Big[m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big) - \mathbb{E}\Big[m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] \geq \lambda\Big]
\end{aligned}$$

with $C = C(\beta, \text{data}) > 0$. Inserting the fluctuation bounds from Proposition 4.6.1, for $\Lambda \geq (c_s + 2v_{\min}^{-1})h_2$ we now obtain

$$\begin{aligned}
&\mathbb{E}\Big[T_{\text{full}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)}\Big] - \mathbb{E}\Big[m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] \\
&= \int_0^\infty 3\mathbb{P}\Big[T_{\text{full}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)} - \mathbb{E}\Big[m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] \geq 3\lambda\Big]\,\mathrm{d}\lambda \\
&\leq 3\Lambda + \int_\Lambda^\infty C\frac{r_1^{\beta(d-1)}}{\lambda^{d-1}}\exp\Big(-\frac{c_{\text{fl}}\lambda}{4\sqrt{h_2 r_2}}\Big)\,\mathrm{d}\lambda \\
&\qquad + \int_{h_2^{\alpha_\circledcirc}\frac{\sqrt{h_2 r_2}}{\log(r_2)}}^{Cr_2} C\frac{r_1^{\beta(d-1)}}{\lambda^{d-1}}\exp\Big(-C^{-1}\frac{h_2^{\alpha_\circledcirc}}{\log(r_2)}\Big)\,\mathrm{d}\lambda \\
&\leq 3\Lambda + C\frac{r_1^{\beta(d-1)}}{\Lambda^{d-1}}\sqrt{h_2 r_2}\int_{\frac{c_{\text{fl}}\Lambda}{4\sqrt{h_2 r_2}}}^\infty \exp(-s)\,\mathrm{d}s \\
&\qquad + C\frac{r_1^{\beta(d-1)}\log(r_2)^{d-1}}{h_2^{(\alpha_\circledcirc+\frac{1}{2})(d-1)}r_2^{\frac{d-1}{2}-1}}\exp\Big(-C^{-1}\frac{h_2^{\alpha_\circledcirc}}{\log(r_2)}\Big) \\
&\leq 3\Lambda + C\frac{r_1^{\beta(d-1)}}{\Lambda^{d-1}}\sqrt{h_2 r_2}\exp\Big(-\frac{c_{\text{fl}}\Lambda}{4\sqrt{h_2 r_2}}\Big) + C\exp\Big(-C^{-1}\frac{h_2^{\alpha_\circledcirc}}{\log(r_2)}\Big),
\end{aligned}$$

where in the last step we used that $\log(h_2) \geq c\log(r_2)$ to absorb the polynomial terms in front. Now choosing $\Lambda = \frac{4\sqrt{h_2 r_2}}{c_{\text{fl}}}\log\big(r_1^{\beta(d-1)}\big)$, we have

$$\begin{aligned}
&\mathbb{E}\Big[T_{\text{full}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)}\Big] - \mathbb{E}\Big[m_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] \\
&\qquad\qquad \leq C\sqrt{h_2 r_2}\log(r_1) + C\exp\Big(-C^{-1}\frac{h_2^{\alpha_\circledcirc}}{\log(r_2)}\Big)
\end{aligned}$$

for some $C = C(c, \beta, \text{data}) > 0$.

Step 2: Managing the error on the complement of $E^{\circledcirc}_h$. Restricting to $E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)$ at the cost of an error term, applying the estimate (4.8.5), and using the stationarity of the probability distribution, after absorbing polynomial prefactors since $\log(h_2) \geq c\log(r_2) \geq$

$c\log\left(\frac{r_3}{2}\right)$ we obtain

$$\begin{aligned}
&\mathbb{E}\Big[m_{h_3}\Big(r_3 e, S^-_{e,\beta,h_3}(r_3)\Big)\Big] \\
&\quad\le \mathbb{E}\left[m_{h_3}\Big(r_3 e, S^-_{e,\beta,h_3}(r_3)\Big)\chi_{E^{\circledcirc}_{h_1}\left(\overline{B}_{C_\beta r_3^\beta}\right)}\right] + Cr_3\Big(1-\mathbb{P}\Big[E^{\circledcirc}_{h_2}\Big(\overline{B}_{C_\beta r_3^\beta}\Big)\Big]\Big) \\
&\quad\le \mathbb{E}\left[T_{\text{full}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)}\right] + Ch_2 + \mathbb{E}\Big[m_{h_1}\Big(r_1 e, S^-_{e,\beta,h_1}(r_1)\Big)\Big] + C\exp\big(-C^{-1}h_2^{\alpha_{\circledcirc}}\big).
\end{aligned}$$

Plugging in the final inequality for $\mathbb{E}\left[T_{\text{full}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{C_\beta r_3^\beta}\right)}\right]$ from Step 1 and absorbing all smaller terms into $C\sqrt{h_2 r_2}\log(r_1)$ yields the result. □

Having obtained the approximate sub-linearity, we are now able to prove the existence of the limit for the averaged arrival times and provide a rate such that the limit can not be approached any slower from below.

Corollary 4.8.3. *Assume that (P1), (P2), and (P3) hold. Let $\beta \ge \frac{3}{2}$ and $0<\vartheta<1$. Let $S^-_{e,\beta,h}(r)$ be the approximation of the half-space from Corollary 4.8.1 with $e\in\mathbb{S}^{d-1}$ and $r,h\ge h_0$. For $r\ge h_0^{-\vartheta}$ we set $h(r):=r^\vartheta$.*

Then for all $\Delta F\in\mathbb{R}$ with the coefficient field $(A,F+\Delta F)$ satisfying (P⊙) the limit

$$\overline{m}_{\Delta F,\beta}(e) := \lim_{r\to\infty}\frac{1}{r}\mathbb{E}\Big[m^{(A,F+\Delta F)}_{h(r)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big]$$

exists, is independent of ϑ and there exists $C=C(\Delta F,\beta,\vartheta,\text{data})>0$ such that for all $r\ge h_0^{-\vartheta}$ there holds

$$\overline{m}_{\Delta F,\beta}(e) \le \frac{1}{r}\mathbb{E}\Big[m^{(A,F+\Delta F)}_{h(r)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big] + C\frac{\log(r)}{\sqrt{r^{1-\vartheta}}}. \tag{4.8.6}$$

The dependence of C on ΔF is due to replacing C_{1F} from (4.3.3) by $C_{1F}+|\Delta F|$ in data, and hence the same constant can be used for all $\Delta F'$ with $|\Delta F'|\le|\Delta F|$.

Proof. If Assumptions (P1), (P2), and (P3) hold for the random variable (A,F), then they clearly also hold for $(A,F+\Delta F)$.

We first comment on the independence of the limit $\overline{m}_{\Delta F,\beta}(e)$ with respect to ϑ. The existence of the limit will be proven below. Let $0<\vartheta_1<\vartheta_2<1$. Let $\overline{m}^i_{\Delta F,\beta}(e)$ denote the respective limit with $h(r)$ replaced with $h_i(r)=r^{\vartheta_i}$ for $i=1,2$. Note that for $E(r):=E^{\circledcirc}_{h_1(r)}\left(\overline{B}_{C_\beta r^\beta}\right)$ with the probability bounds from Assumption (P⊙) we have $\lim_{r\to\infty}\mathbb{P}E(r)=1$ and hence due to the truncation of the arrival times $i=1,2$ we obtain

$$\overline{m}^i_{\Delta F,\beta}(e) = \lim_{r\to\infty}\frac{1}{r}\mathbb{E}\Big[m^{(A,F+\Delta F)}_{h_i(r)}\Big(re, S^-_{e,\beta,h_i(r)}(r)\Big)\chi_{E(r)}((A,F))\Big]. \tag{4.8.7}$$

For coefficients $(A,F)\in E(r)$, Assumption (P⊙) yields stable $(h_1(r),c_s)$-approximations for the disks $S^-_{e,\beta,h_i(r)}(r)$ with $i=1,2$, for which (A,F) and hence $(A,F+\Delta F)$ admit an effective minimum speed $v_{\min}$ in $\overline{B}_{C_\beta r^\beta}$ on the scale $h_i(r)$. Due to this effective minimum speed, the evolution of the stable approximation for a disk $S^-_{e,\beta,h_i(r)}(r)$ will h_1-envelop

the other disk after a time step proportional to $h_1(r) + h_2(r)$. The comparison principle from Lemma 4.5.5 and the compatibility of the arrival times with "$\subset_{h_1}$" from Lemma 4.3.9 yields

$$\left| m_{h_1(r)}^{(A,F+\Delta F)}\left(re, S^-_{e,\beta,h_1(r)}(r)\right) - m_{h_1(r)}^{(A,F+\Delta F)}\left(re, S^-_{e,\beta,h_2(r)}(r)\right)\right| \leq C\big(h_1(r) + h_2(r)\big)$$

for $C = C(\mathrm{data}) > 0$. We further replace the target area $re + \overline{B}_{h_1(r)}$ for the second arrival time with $re + \overline{B}_{h_2(r)}$, which yields an additional error proportional to $h_1(r) + h_2(r)$: Clearly, the larger target area is reached first, but since the stable approximation expands with effective minimum speed $v_{\min}$ on the scale $h_1(r)$, the smaller target area will be reached after an additional time step proportional to $h_1(r) + h_2(r)$. Combining these bounds on the difference with (4.8.7), we obtain

$$\left|\overline{m}^1_{\Delta F,\beta}(e) - \overline{m}^2_{\Delta F,\beta}(e)\right| \leq \lim_{r\to\infty} C\frac{h_1(r) + h_2(r)}{r} = 0,$$

which yields the independence of the limit $\overline{m}_{\Delta F,\beta}(e)$ with respect to ϑ.

We now prove the existence of the limit as well as (4.8.6). For $r \geq h_0^{-\vartheta}$ we abbreviate

$$G(r) := \frac{1}{r}\mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big].$$

From Proposition 4.8.2 we have $C = C(\Delta F, \beta, \vartheta, \mathrm{data}) > 0$, such that for $r \geq s$ we obtain

$$\begin{aligned} G(r) - G(s) &= \frac{1}{r}\big(rG(r) - sG(s) - (r-s)G(r-s)\big) - \frac{r-s}{r}\big(G(s) - G(r-s)\big) \\ &\leq C\frac{\log(r)}{\sqrt{r^{1-\vartheta}}} + \frac{r-s}{r}\big(G(r-s) - G(s)\big). \end{aligned} \tag{4.8.8}$$

The proof consists of 3 steps, only using the boundedness of $G(r)$ and (4.8.8).

1.) For any $s \geq h_0^{-\vartheta}$ the sequence $(G(ns))_{n\in\mathbb{N}}$ is essentially bounded from above by $G(s)$ in the sense that there is some $C = C(\Delta F, \beta, \vartheta, \mathrm{data}) > 0$ such that

$$G(ns) \leq G(s) + C\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} \qquad \text{for all } n \in \mathbb{N}. \tag{4.8.9}$$

2.) We extend the bounds to all $r \geq s$ with the additional error decreasing for $r \to \infty$ with some $C = C(\Delta F, \beta, \vartheta, \mathrm{data}) > 0$ as

$$G(r) \leq G(s) + C\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} + C\frac{s}{r} \qquad \text{for all } r \geq s. \tag{4.8.10}$$

3.) The first step implies that the sequence $\big(G(2^k s)\big)_{k\in\mathbb{N}}$ is essentially decreasing and hence converges. In combination with the second step, this implies the existence of $\lim_{r\to\infty} G(r)$.

Finally, (4.8.6) then follows from sending $n \to \infty$ in (4.8.9).

Step 1: $G(ns)$ is essentially bounded from above by $G(s)$. We will show (4.8.9) by induction, keeping track of the constant to show that it stays bounded. Assume that for some $k \in \mathbb{N}_0$ there is $C_k > 0$ such that

$$G(ns) \leq G(s) + C_k\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} \qquad \text{for all } n \in \{1, \dots, 2^k\}. \tag{4.8.11}$$

This clearly holds for $k = 0$ with $C_k = 0$. For any even $n \in \{2^k + 1, \dots, 2^{k+1}\}$, with (4.8.8) and this assumption we obtain

$$\begin{aligned} G(ns) - G(s) &= G(ns) - G\Big(\frac{n}{2}s\Big) + G\Big(\frac{n}{2}s\Big) - G(s) \\ &\le C\frac{\log(ns)}{\sqrt{n^{1-\vartheta}}\sqrt{s^{1-\vartheta}}} + C_k\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} \\ &\le \widetilde{C}_{k+1}\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} \end{aligned}$$

with $\widetilde{C}_{k+1} = C_k + C\frac{\max\{k,1\}}{2^{\frac{1-\vartheta}{2}k}}$.

For any odd $n \in \{2^k + 1, \dots, 2^{k+1}\}$, with (4.8.8) and since $n - 1 \le 2^{k+1}$ is even, we hence obtain

$$\begin{aligned} G(ns) - G(s) &\le C\frac{\log(ns)}{\sqrt{n^{1-\vartheta}}\sqrt{s^{1-\vartheta}}} + \frac{n-1}{n}\big(G((n-1)s) - G(s)\big) \\ &\le C\frac{\log(ns)}{\sqrt{n^{1-\vartheta}}\sqrt{s^{1-\vartheta}}} + \widetilde{C}_{k+1}\frac{\log(s)}{\sqrt{s^{1-\vartheta}}}, \end{aligned}$$

which yields (4.8.11) for $k+1$ with $C_{k+1} = C_k + C\frac{\max\{k,1\}}{2^{\frac{1-\vartheta}{2}k}}$. Since the increasing sequence $(C_k)_{k\in\mathbb{N}}$ converges, we hence obtain (4.8.9) with the constant given by $\lim_{k\to\infty} C_k$.

Step 2: Bounding $G(r)$ for all $r \ge s$. For $r \ge s$, let $n \in \mathbb{N}$ such that $ns \le r < (n+1)s$. Then with (4.8.8) and (4.8.9) from Step 1 we have

$$\begin{aligned} G(r) - G(s) &= G(r) - G(ns) + G(ns) - G(s) \\ &\le C\frac{\log(r)}{\sqrt{r^{1-\vartheta}}} + \frac{r - ns}{r}\big(G(r - ns) - G(ns)\big) + C\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} \\ &\le C\frac{\log(s)}{\sqrt{s^{1-\vartheta}}} + C\frac{s}{r}, \end{aligned}$$

where in the last step we used $r - ns \le s$ by the choice of n and that $0 \le G(\cdot) \le C(\mathrm{data})$.

Step 3: Existence of $\lim_{r\to\infty} G(r)$. Applying Step 1 for each $r_k := 2^k s$, for the sequence $(G(r_k))_{k\in\mathbb{N}}$ we obtain that

$$G(r_\ell) \le G(r_k) + C\frac{\log(r_k)}{\sqrt{r_k^{1-\vartheta}}} \qquad \text{for all } \ell > k.$$

In particular, the sequence $b_n := G(r_k) - \sum_{\ell=1}^{k-1} C\frac{\log(r_\ell)}{\sqrt{r_\ell^{1-\vartheta}}}$ is decreasing. Since it is bounded from below, it converges and thus also $(G(r_k))_{k\in\mathbb{N}}$.

It remains to prove that also away from this sequence $G(r)$ converges to the same limit for $r \to \infty$. Let $\overline{G} = \lim_{k\to\infty} G(r_k)$. Let $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that

$$\big|G(r_k) - \overline{G}\big| \le \frac{\epsilon}{3}, \qquad C\frac{\log(r_k)}{r_k} \le \frac{\epsilon}{3} \qquad \text{for all } k \ge k_0,$$

with the constant C as in (4.8.10). Let $r \ge \max\{r_{k_0}, 3C\varepsilon^{-1}r_{k_0}\}$ with C as in (4.8.10). On the one hand, (4.8.10) yields

$$G(r) \le G(r_{k_0}) + C\frac{\log(r_{k_0})}{r_{k_0}} + C\frac{r_{k_0}}{r} \le \overline{G} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le \overline{G} + \varepsilon.$$

On the other hand, there is $k \geq k_0$ such that $r_k \geq \max\{r, 3C\varepsilon^{-1}r\}$, for which (4.8.10) yields

$$G(r) \geq G(r_k) - C\frac{\log(r)}{r} - C\frac{r}{r_k} \geq \overline{G} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \geq \overline{G} - \varepsilon.$$

We have thus shown that $G(r)$ converges to $\overline{G}$ for $r \to \infty$, concluding the last step of the proof. □

4.8.3 Approximate super-linearity and upper bounds for the convergence

Next, we will obtain the super-linearity of the truncated arrival times, albeit at the cost of changing the forcing by a small constant due to applying Corollary 4.8.1.

Proposition 4.8.4 (Approximate super-linearity of the truncated arrival times)**.** *Assume that (P1), (P2), (P3) and (P⊙) hold. Let $\beta \geq \frac{3}{2}$ with $C_\beta > 0$ and $S^-_{e,\beta,h}(r)$ for $e \in \mathbb{S}^{d-1}$ and $r, h \geq h_0$ as in Corollary 4.8.1.*

Let $r_2 \geq r_1 \geq h_0$ and $h_3 \geq h_2 \geq h_1 \geq h_0$ with $\frac{\log(h_2)}{\log(r_2)}, \frac{\log(h_1)}{\log(r_1)} \geq c$ and for some $c > 0$. Then there exists $C = C(c, \beta, \mathrm{data}) > 0$ such that for $f(r) = r^{-\frac{\beta}{2}+\frac{3}{4}}$ there holds

$$\begin{aligned}
&\mathbb{E}\Big[m^{(A,F)}_{h_3}\Big((r_1+r_2)e, S^-_{e,\beta,h_3}(r_1+r_2)\Big)\Big] \\
&\quad \geq \mathbb{E}\Big[m^{(A,F+f(r_1))}_{h_1}\Big(r_1 e, S^-_{e,\beta,h_1}(r_1)\Big)\Big] + \mathbb{E}\Big[m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] \\
&\qquad - C\sqrt{h_3 r_2}\log(r_2).
\end{aligned} \tag{4.8.12}$$

Proof. We set $r_3 = r_1 + r_2$. We claim that the time until $S^-_{e,\beta,h_3}(r_3)$ reaches $\overline{B}_{h_3}(r_3 e)$ is larger than adding the time it takes until $S^-_{e,\beta,h_3}(r_3)$ first reaches any point of $r_2 e + S^-_{e,\beta,h_2}(r_3)$ to the time it takes until the smaller 'intermediate set' $r_2 e + S^-_{e,\beta,h_1}(r_1)$ reaches $\overline{B}_{h_1}(r_3 e)$ with a slightly increased forcing, see Figure 4.8.4.

More precisely, we denote the time when the disk $r_2 e + S^-_{e,\beta,h_2}(r_3)$ is first reached with

$$T_{\mathrm{first}} := \inf\Big\{t \geq 0 \,:\, \Big(r_2 e + S^-_{e,\beta,h_2}(r_3)\Big) \cap \mathscr{R}_t\Big(S^-_{e,\beta,h_3}(r_3)\Big) \neq \emptyset\Big\}.$$

We will show that for $(A,F) \in E^{\odot}_{h_2}\Big(r_2 e + \overline{B}_{C_\beta r_3^\beta}\Big) \cap E^{\odot}_{h_1}\Big(r_2 e + \overline{B}_{C_\beta r_1^\beta}\Big)$ there holds

$$\begin{aligned}
&m^{(A,F)}_{h_3}\Big(r_3 e, S^-_{e,\beta,h_3}(r_3)\Big) \\
&\qquad \geq T^{(A,F)}_{\mathrm{first}} + m^{(A,F+f(r_1))}_{h_1}\Big(r_3 e, r_2 e + S^-_{e,\beta,h_1}(r_1)\Big) - 2(c_s + 2v^{-1}_{\min})h_3.
\end{aligned} \tag{4.8.13}$$

In order to obtain (4.8.12), we will follow these three steps:

1.) We will prove (4.8.13).

2.) We will show that the expected value of T_{first} restricted to $E^{\odot}_{h_2}\Big(\overline{B}_{2C_\beta r_3^\beta}\Big)$ can be bounded from below by the expected value of $m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)$ up to an error corresponding to the fluctuation bounds from Proposition 4.6.1.

3.) We take the expected value of (4.8.13), use the stationarity from Assumption (P1) and manage the error on the complement of $E^{\odot}_{h_2}\Big(\overline{B}_{C_\beta r_3^\beta}\Big)$.

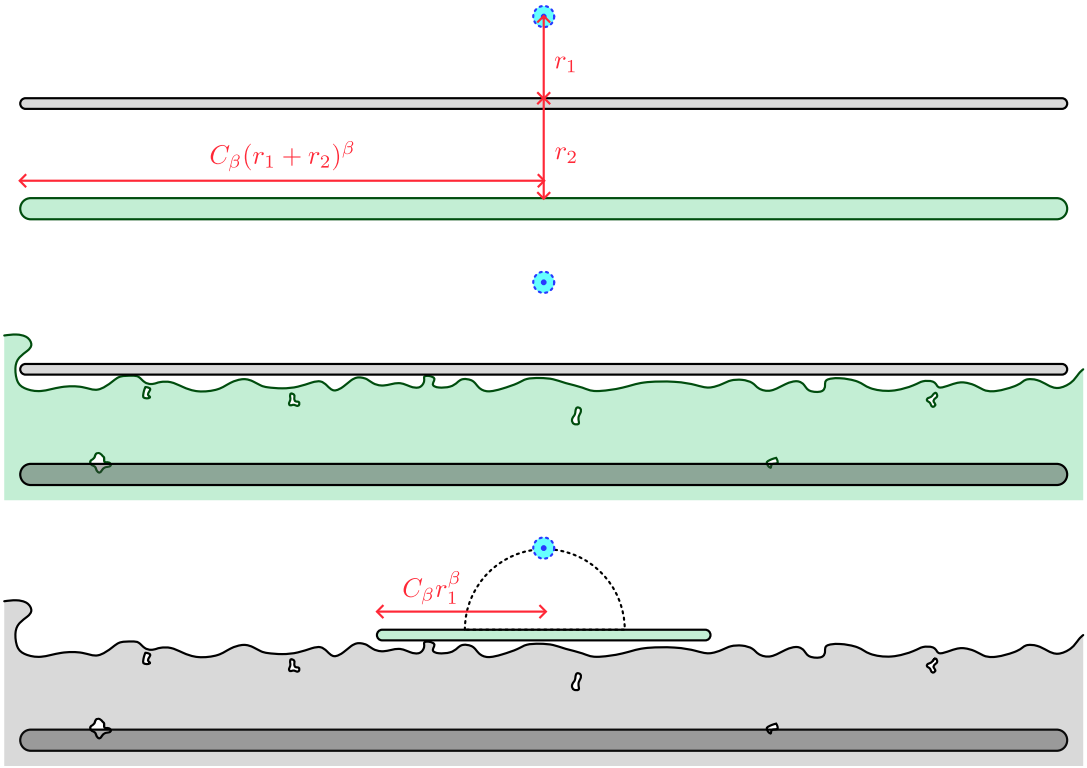


Figure 4.8.4: An illustration of the argument for the super-linearity of arrival times. Top: The large disk is depicted in green with its target area in blue, and the 'intermediate' translation of the large disk is depicted in grey. Center: The evolution of the large disk at time T_{first} (depicted in green) reaches the intermediate disk. Bottom: At time T_{first}, we replace the evolved set with the smaller intermediate disk (depicted in green), which stays ahead the evolution of the large disk within the dotted half-ball when increasing the forcing by a constant as in Proposition 4.7.4

Step 1: Proving (4.8.13). We claim that for $(A,F) \in E^{\circledcirc}_{h_2}\left(r_2 e + \overline{B}_{C_\beta r_3^\beta}\right)$ and for $r_3 \geq C = C(\beta, \text{data}) > 0$ large enough there holds

$$\left(r_2 e + \left(\overline{B}_{C_\beta r_1^\beta} \cap \mathcal{H}^+_e\right)\right) \cap \mathscr{R}^{(A,F)}_t\left(S^-_{e,\beta,h_3}(r_3)\right) = \emptyset \qquad \text{for all } 0 \leq t \leq T^{(A,F)}_{\text{first}}. \tag{4.8.14}$$

Assuming that this claim holds and if in addition $(A,F) \in E^{\circledcirc}_{h_1}\left(r_2 e + \overline{B}_{C_\beta r_1^\beta}\right)$, similar to the proof of Corollary 4.8.1 we apply Proposition 4.7.4 in $r_2 e + \overline{B}_{C_\beta r_1^\beta}$: For the set S_1 we choose the evolution of the stable (h_1, c_s)-approximation of $r_2 e + S^-_{e,\beta,h_1}(r_1)$ after time $\left(c_s + 3v_{\min}^{-1}\right)h_1$ with the holes of width smaller than $2h_1$ filled as in the proof of the comparison principle from Lemma 4.5.5. This comparison principle further implies

$$S_1 \subset \mathscr{R}^{(A,F+f(r_1))}_{(c_s+3v_{\min}^{-1})h_1}\left(r_2 e + S^-_{e,\beta,h_1}(r_1)\right).$$

For the set S_2 we choose

$$S_2 = \mathscr{R}^{(A,F)}_{T^{(A,F)}_{\text{first}}}\left(S^-_{e,\beta,h_3}(r_3)\right).$$

The parameters are given by

$$r = r_1, \qquad R = C_\beta r_1^\beta, \qquad \Delta F = f(r_1) = r_1^{-\frac{\beta}{2}+\frac{3}{4}}, \qquad T = T_{h_1}\left(r_3 e, r_2 e + S^-_{e,\beta,h_1}(r_1)\right).$$

The choice of C_β from Corollary 4.8.1 guarantees that (4.7.1) holds for these parameters, which together with the claim (4.8.14) allows us to apply Proposition 4.7.4. We hence

obtain

$$\left(r_2 e + \left(\overline{B}_{C_\beta r_1^\beta} \cap \mathcal{H}_e^+\right)\right) \cap \mathscr{R}^{(A,F)}_{t+T^{(A,F)}_{\text{first}}}\left(S^-_{e,\beta,h_3}(r_3)\right) \subset \mathscr{R}^{(A,F+f(r_1))}_{t+(c_s+3v_{\min}^{-1})h_1}\left(r_2 e + S^-_{e,\beta,h_1}(r_1)\right)$$

for $0 \leq t \leq T_{h_1}\left(r_3 e, r_2 e + S^-_{e,\beta,h_1}(r_1)\right)$, the time by which the set is guaranteed to reach $\overline{B}_{h_1}(r_3 e)$. This implies (4.8.13) when considering that if the set $r_2 e + S^-_{e,\beta,h_1}(r_1)$ reaches $\overline{B}_{h_3}(r_3 e)$, it will reach $\overline{B}_{h_1}(r_3 e)$ at most $c_s h_1 + v_{\min}^{-1} h_3$ later due to the effective minimum speed of the stable approximation from Assumption (P◎).

To obtain (4.8.14) we compare – within the domain $r_2 e + \left(\overline{B}_{C_\beta r_3^\beta} \cap \mathcal{H}_e^+\right)$ – the evolution of $S^-_{e,\beta,h_3}(r_3)$ to a stable approximation of the upper arc of the boundary domain given by $r_2 e + \left(\partial B_{C_\beta r_3^\beta} \cap \mathcal{H}_e^+\right)$, which exists since $(A,F) \in E^\circledcirc_{h_2}\left(r_2 e + \overline{B}_{C_\beta r_3^\beta}\right)$, see Figure 4.8.5. To

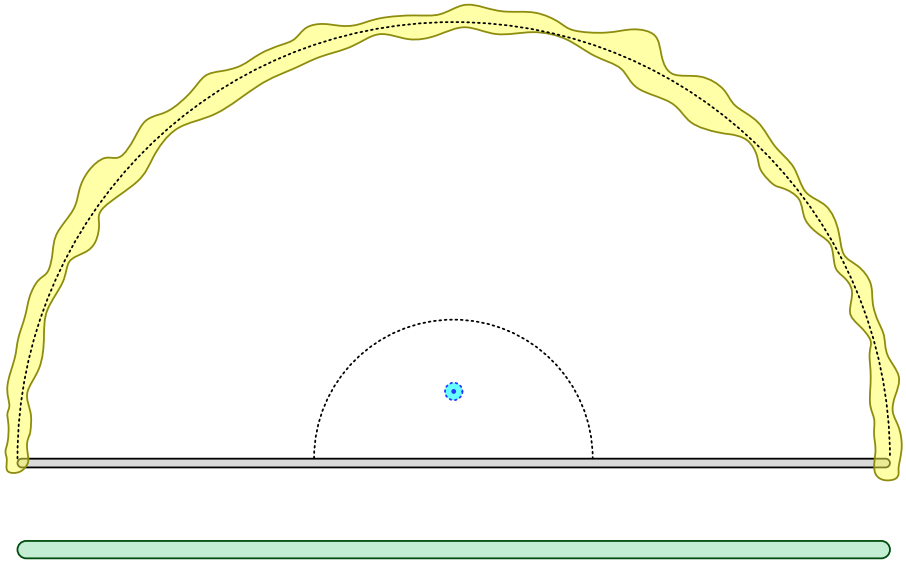


Figure 4.8.5: An illustration of the comparison argument from Step 1 of the proof for the super-linearity of arrival times. The stable approximation of the upper arc of the half-ball is depicted in yellow. In this half-ball, we compare its evolution to the evolution of the large disk (depicted in green). Until time T_{first}, by definition this evolution does not reach the intermediate disk (depicted in grey), which makes up the straight part of the half-ball's boundary

be precise, we apply the comparison principle as in Theorem 4.A.8 on the bounded domain $r_2 e + \left(\overline{B}_{C_\beta r_3^\beta} \cap \mathcal{H}_e^+\right)$ to the supersolution $u^{\mathscr{R}}_{S^-_{e,\beta,h_3}(r_3)}$ from Lemma 4.5.2 and a solution $\widetilde{u} \leq 1$ to (4.1.6) as in Definition 4.1.6 with the stable approximation of the arc corresponding to the initial 0-sublevel set. Clearly $u^{\mathscr{R}}_{S^-_{e,\beta,h_3}(r_3)} = 1 \geq \widetilde{u}$ for $t = 0$ on the domain. Regarding the boundary, on the flat part we have $u^{\mathscr{R}}_{S^-_{e,\beta,h_3}(r_3)} = 1 \geq \widetilde{u}$ for $0 \leq t \leq T_{\text{first}}$ due to the definition of T_{first}. On the arc, due to the stability of the approximation we have $\widetilde{u} = 0 \leq u^{\mathscr{R}}_{S^-_{e,\beta,h_3}(r_3)}$. Taking into account the maximum speed of propagation from Lemma 4.5.3, the comparison principle yields

$$\mathscr{R}^{(A,F)}_t\left(S^-_{e,\beta,h_3}(r_3)\right) \cap \left(r_2 e + \left(\overline{B}_{C_\beta r_3^\beta} \cap \mathcal{H}_e^+\right)\right) \subset r_2 e + \mathbb{R}^d \setminus \left(C_\beta r_3^\beta - C h_2 - v_{\max} t\right)\overline{B}_1$$

for $0 \leq t \leq T_{\text{first}}$. Because $T^{(A,F)}_{\text{first}} \leq C r_2 \leq C r_3 \ll C_\beta r_3^\beta$ for $(A,F) \in E^\circledcirc_{h_3}\left(r_2 e + \overline{B}_{C_\beta r_3^\beta}\right)$ and r_3 large enough, this yields (4.8.14) and hence (4.8.13).

Step 2: Bounding T_{first}. For $(A,F) \in E^\circledcirc_{h_2}\left(\overline{B}_{2C_\beta r_3^\beta}\right)$, due to the minimum effective speed we can replace T_{first} with the minimum of a finite amount of truncated arrival times

at the cost of a small error. Considering only points on a grid within the upper disk, whenever any point of the disk is reached, the closest grid point will be reached just shortly afterwards: The first point of contact will be reached $c_s h_2$ later by the stable approximation of $S^-_{e,\beta,h_3}(r_3)$, which will then reach the closest grid point at the rate of the effective minimum speed. That is, for $\lambda > 0$ and $\mathbb{Z}^{d-1}_e$ denoting $\mathbb{Z}^{d-1}$ placed in the hyperplane $\{x \in \mathbb{R}^d : x \cdot e = 0\}$, we have

$$\begin{aligned} T^{(A,F)}_{\text{first}} &\geq \inf\Big\{ m^{(A,F)}_{h_2}\Big(x + r_2 e, S^-_{e,\beta,h_3}(r_3)\Big) : x \in \frac{v_{\min}}{\sqrt{d}}\lambda\mathbb{Z}^{d-1}_e \cap S^-_{e,\beta,h_2}(r_3)\Big\} \\ &\quad - \big(c_s + v^{-1}_{\min}\big)h_2 - \lambda \\ &\geq \inf\Big\{ m^{(A,F+f(r_2))}_{h_2}\Big(x + r_2 e, x + S^-_{e,\beta,h_2}(r_2)\Big) : x \in \frac{v_{\min}}{\sqrt{d}}\lambda\mathbb{Z}^{d-1}_e \cap S^-_{e,\beta,h_2}(r_3)\Big\} \\ &\quad - 2\big(c_s + 2v^{-1}_{\min}\big)h_2 - \lambda, \end{aligned}$$

where for the second inequality Corollary 4.8.1 allows us to replace the large disk with the smaller disks $x + S^-_{e,\beta,h_2}(r_2)$ placed as in Figure 4.8.3, only with a larger upper disk. Abbreviating

$$T_2 = \mathbb{E}\Big[m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big],$$

for $\lambda \geq 2\big(c_s + 2v^{-1}_{\min}\big)h_2$ we thus obtain

$$\begin{aligned} &\mathbb{P}\Big[T_{\text{first}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{2C_\beta r_3^\beta}\right)} - T_2 \leq -3\lambda\Big] \\ &\leq \mathbb{P}\Big[\inf_{x \in \frac{v_{\min}\lambda}{\sqrt{d}}\mathbb{Z}^{d-1}_e \cap S^-_{e,\beta,h_2}(r_3)} m^{(A,F+f(r_2))}_{h_2}\Big(x + r_2 e, x + S^-_{e,\beta,h_2}(r_2)\Big)\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{2C_\beta r_3^\beta}\right)} \leq T_2 - \lambda\Big] \\ &\leq C\frac{r_3^{\beta(d-1)}}{\lambda^{d-1}}\mathbb{P}\Big[m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big) - T_2 \leq -\lambda\Big] \\ &\quad + 1 - \mathbb{P}\Big[E^{\circledcirc}_{h_2}\Big(\overline{B}_{2C_\beta r_3^\beta}\Big)\Big] \end{aligned}$$

with $C = C(\beta, \text{data}) > 0$, where for the second inequality we used a union bound, the stationarity of the probability distribution from Assumption (P1) and obtained an error term for loosing the restriction to $E^{\circledcirc}_{h_2}\Big(\overline{B}_{2C_\beta r_3^\beta}\Big)$. Applying the fluctuation bounds from Proposition 4.6.1 and with calculations analogous to Step 1 in the proof of Proposition 4.8.2 while absorbing the additional term $Cr_2\Big(1 - \mathbb{P}\Big[E^{\circledcirc}_{h_2}\Big(\overline{B}_{2C_\beta r_3^\beta}\Big)\Big]\Big)$, we obtain

$$\begin{aligned} &\mathbb{E}\Big[T_{\text{first}}\chi_{E^{\circledcirc}_{h_2}\left(\overline{B}_{2C_\beta r_3^\beta}\right)}\Big] - \mathbb{E}\Big[m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] \\ &\qquad\qquad \geq -C\sqrt{h_2 r_2}\log(r_2) - C\exp\Big(-C^{-1}\frac{h_2^{\alpha_\circledcirc}}{\log(r_2)}\Big) \end{aligned}$$

for some $C = C(c, \beta, \text{data}) > 0$.

Step 3: Managing the error on the complement of $E^{\circledcirc}_h$. Taking the expected value of the inequality (4.8.13), using the stationarity of the probability distribution from Assumption

(P1), applying Step 2 and writing $E = E^{\circledcirc}_{h_2}\left(r_2 e + \overline{B}_{C_\beta r_3^\beta}\right) \cap E^{\circledcirc}_{h_1}\left(r_2 e + \overline{B}_{C_\beta r_1^\beta}\right)$ we obtain

$$\begin{aligned}
&\mathbb{E}\Big[m^{(A,F)}_{h_3}\Big(r_3 e, S^-_{e,\beta,h_3}(r_3)\Big)\Big] \\
&\quad \geq \mathbb{E}\Big[m^{(A,F)}_{h_3}\Big(r_3 e, S^-_{e,\beta,h_3}(r_3)\Big)\chi_E\Big] \\
&\quad \geq \mathbb{E}\Big[m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] - C\sqrt{h_3 r_2}\log(r_2) \\
&\qquad + \mathbb{E}\Big[m^{(A,F+f(r_1))}_{h_1}\Big(r_1 e, S^-_{e,\beta,h_1}(r_1)\Big)\chi_E\Big] \\
&\quad \geq \mathbb{E}\Big[m^{(A,F+f(r_2))}_{h_2}\Big(r_2 e, S^-_{e,\beta,h_2}(r_2)\Big)\Big] + \mathbb{E}\Big[m^{(A,F+f(r_1))}_{h_1}\Big(r_1 e, S^-_{e,\beta,h_1}(r_1)\Big)\Big] \\
&\qquad - C\sqrt{h_3 r_2}\log(r_2)
\end{aligned}$$

for some $C = C(c, \beta, \mathrm{data})$. For the second inequality, we also used that $Ch_3 \leq C\sqrt{h_3 r_2}$ since $h_3 \leq r_3 \leq 2r_2$. For the last inequality, we loosened the restriction to E and used that $\frac{\log(h_1)}{\log(r_1)} \geq c$ to absorb the resulting error term, which is controlled by the bounds on $1 - \mathbb{P}\Big[E^{\circledcirc}_{h_2}\Big(r_2 e + \overline{B}_{C_\beta r_3^\beta}\Big)\Big]$ and $1 - \mathbb{P}\Big[E^{\circledcirc}_{h_1}\Big(r_2 e + \overline{B}_{C_\beta r_1^\beta}\Big)\Big]$ from Assumption (P⊙). □

Now that we have the approximate superlinearity, we can use this for the convergence of the averaged arrival times: We obtain a rate such that the limit can not be approached any slower from above, albeit this only works if we slightly increase the forcing for the arrival times compared to the ones used for the limit process.

Corollary 4.8.5. *Assume that (P1), (P2), and (P3) hold. Let $S^-_{e,\beta,h}(r)$ be the approximation of the half-space from Corollary 4.8.1 with $e \in \mathbb{S}^{d-1}$ and $r, h \geq h_0$. Let $\beta > \frac{3}{2}$, $0 < \vartheta < 1$, $h(r) := r^\vartheta$ and $\overline{m}_{\Delta F,\beta}(e)$ as in Corollary 4.8.3 as well as $f(r) = r^{-\frac{\beta}{2}+\frac{3}{4}}$.*

Let $\Delta F \in \mathbb{R}$ and suppose that Assumption (P⊙) is satisfied for all coefficient fields $(A, F + \Delta F + c)$ with $0 \leq c \leq C_1 f(r)$, where $C_1 = C_1(\beta) > 0$ is determined in the proof below. Then

$$\overline{m}_{\Delta F,\beta}(e) \geq \frac{1}{r}\mathbb{E}\Big[m^{(A,F+\Delta F+C_1 f(r))}_{h(r)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big] - C_2\frac{\log(r)}{\sqrt{r^{1-\vartheta}}}$$

with $C_2 = C_2(\Delta F, \beta, \vartheta, \mathrm{data}) > 0$. The dependence of C_2 on ΔF is due to replacing C_{1F} from (4.3.3) by $C_{1F} + |\Delta F|$ in data, and hence the same constant can also be used for all $\Delta F'$ with $|\Delta F'| \leq |\Delta F|$.

Proof. For $r \geq h_0^{-\vartheta}$ and $\Delta F \geq 0$ we abbreviate

$$G_{\Delta F}(r) := \frac{1}{r}\mathbb{E}\Big[m^{(A,F+\Delta F)}_{h(r)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big].$$

With Proposition 4.8.4, for $k \in \mathbb{N}_0$ we obtain

$$\begin{aligned}
&G_{\Delta F}(2^{k+1}r) - G_{\Delta F + f(2^k r)}(2^k r) \\
&\quad = \frac{1}{2^{k+1}r}\big(2^{k+1}r G_{\Delta F}(2^{k+1}r) - 2^k r G_{\Delta F + f(2^k r)}(2^k r) - 2^k r G_{\Delta F + f(2^k r)}(2^k r)\big) \\
&\quad \geq -C\frac{\log(2^{k+1}r)}{\sqrt{(2^{k+1}r)^{1-\vartheta}}} \\
&\quad \geq -C\frac{\max\{k,1\}}{2^{\frac{1-\vartheta}{2}k}}\frac{\log(r)}{\sqrt{r^{1-\vartheta}}}.
\end{aligned}$$

Chaining this inequality, in particular increasing the forcing at every step, we obtain

$$G_{\Delta F}(2^{k+1}r) - G_{\Delta F+\sum_{i=0}^{k} f(2^k r)}(r) \geq -C\left(\sum_{i=0}^{k} \frac{\max\{k,1\}}{2^{\frac{1-\vartheta}{2}k}}\right)\frac{\log(r)}{\sqrt{r^{1-\vartheta}}}.$$

Note that the sum of the forcing increments converges and scales like $f(r)$: Because we chose $\beta > \frac{3}{2}$, there is $C_1 = C_1(\beta) > 0$ such that

$$\sum_{i=0}^{k} f(2^k r) = r^{-\frac{\beta}{2}+\frac{3}{4}} \sum_{i=0}^{k} 2^{-\left(\frac{\beta}{2}-\frac{3}{4}\right)k} \leq C_1 r^{-\frac{\beta}{2}+\frac{3}{4}} = C_1 f(r).$$

Since the arrival times decrease when increasing the forcing, we have $G_{\Delta F}(r) \geq G_{\Delta F'}(r)$ for any $0 \leq \Delta F \leq \Delta F'$ and hence obtain

$$G_{\Delta F}(2^{k+1}r) \geq G_{\Delta F + C_1 f(r)}(r) - C\frac{\log(r)}{\sqrt{r^{1-\vartheta}}}.$$

From Corollary 4.8.5 we know that $\overline{m}_{\Delta F,\beta}(e) = \lim_{s\to\infty} G_{\Delta F}(s)$ exists. Taking the limit for $k \to \infty$ in the inequality above thus concludes the proof. □

4.8.4 Definition of the homogenized speed and comments on Assumption (Pη)

Corollary 4.8.3 guarantees that the limit of the averaged arrival times for the half-space approximations exist. This allows us to define our candidates for the homogenized speed for different disk width scalings.

Definition 4.8.6 (The candidates for the homogenized speed)**.** Assume that (P1), (P2), and (P3) hold. Then for any $\Delta F \in \mathbb{R}$ with $(A, F + \Delta F)$ satisfying (P⊙), $\beta > \frac{3}{2}$, and $e \in \mathbb{S}^{d-1}$ we set

$$\overline{v}_{\Delta F,\beta}(e) := \left(\lim_{r\to\infty} \frac{1}{r}\mathbb{E}\left[m_{h(r)}^{(A,F+\Delta F)}\left(re, S^-_{e,\beta,h(r)}(r)\right)\right]\right)^{-1},$$

where $S^-_{e,\beta,h(r)}(r)$ are the half-space approximations with $h(r) := r^\vartheta$ for some $0 < \vartheta < 1$. The existence of the limits and their independence with respect to ϑ follows from Corollary 4.8.3. It is straightforward to see that

$$v_{\min} \leq \overline{v}_{\Delta F,\beta}(e) \leq v_{\max}(\Delta F) \qquad \text{for all } \beta > \frac{3}{2}, \text{ and } e \in \mathbb{S}^{d-1}$$

with $v_{\max}(\Delta F)$ for $(A, F + \Delta F)$ corresponding to the maximum speed $v_{\max}$ for (A, F) from Lemma 4.5.3. The lower bound follows from Assumption (P⊙) via Lemma 4.5.7.

In terms of the homogenized speed, the quantitative bounds from Corollary 4.8.3 and Corollary 4.8.5 yield

$$\mathbb{E}\left[m_{h(r)}^{(A,F+\Delta F)}\left(re, S^-_{e,\beta,h(r)}(r)\right)\right] \geq \overline{v}^{-1}_{\Delta F,\beta}(e)r - C\log(r)\sqrt{r^{1+\vartheta}} \tag{4.8.15}$$

and, if all coefficient fields $(A, F + \Delta F + c)$ with $0 \leq c \leq C_1 f(r)$ satisfy (P⊙),

$$\mathbb{E}\left[m_{h(r)}^{(A,F+\Delta F)}\left(re, S^-_{e,\beta,h(r)}(r)\right)\right] \leq \overline{v}^{-1}_{\Delta F - C_1 f(r),\beta}(e)r + C\log(r)\sqrt{r^{1+\vartheta}}. \tag{4.8.16}$$

Note that based on these inequalities we can fully quantify the convergence only if the function $\Delta F \mapsto \overline{v}_{\Delta F,\beta}$ is Hölder continuous. Assumption (Pη), which we argued to be plausible in Remark 4.3.2, provides this Hölder continuity for some width scaling $\beta^* > \frac{3}{2}$.

However, morally the Hölder continuity should not be dependent on a specific width scaling β^*. In fact, in the following lemma we show that the Hölder continuity for β^* from Assumption (Pη) implies Hölder continuity with the same constant and coefficient for any $\beta > \frac{3}{2}$ and that all candidates from Definition 4.8.6 yield the same homogenized speed independent of β.

Lemma 4.8.7. *[The homogenized speed is independent of the width scaling] Assume that the random coefficient field $(A, F) \in \Omega$ with probability distribution $\mathbb{P}$ satisfies Assumptions (P1), (P2), (P3) and (Pη). Assume that there is $\delta_F \in (0, (C^*)^{-1}]$ such that all coefficient fields $(A, F + \Delta F)$ with $\Delta F \in [-\delta_F, \delta_F]$ satisfy Assumption (P$\circledcirc$).*

Then for any $\beta > \frac{3}{2}$, $\Delta F > -\delta_F$ and $e \in \mathbb{S}^{d-1}$ the candidates for the homogenized speed from Definition 4.8.6 satisfy

$$\left|\overline{v}_{\Delta F,\beta}(e) - \overline{v}_{0,\beta}(e)\right| \leq C^* \left|\Delta F\right|^\eta \qquad \text{for all } \Delta F \in [-\delta_F, \delta_F]$$

with $C^* > 0$ and $0 < \eta \leq 1$ as in Assumption (Pη). In addition, there holds

$$\overline{v}(e) \coloneqq \overline{v}_{0,\beta^*}(e) = \overline{v}_{0,\beta}(e) \qquad \text{for all } \beta > \frac{3}{2}. \tag{4.8.17}$$

Proof. Let $\beta_1 \geq \beta_2 > \frac{3}{2}$ be two different scalings for the width of the half-space approximations. We claim that

$$\overline{v}_{\Delta F,\beta_1}(e) \geq \overline{v}_{\Delta F,\beta_2}(e) \geq \overline{v}_{\widetilde{\Delta F},\beta_1}(e) \quad \text{for any } \Delta F > \widetilde{\Delta F} \geq -\delta_F,\ e \in \mathbb{S}^{d-1}. \tag{4.8.18}$$

If this claim holds, we immediately obtain that $\overline{v}_{0,\beta} = \overline{v}_{0,\beta^*}$ for all $\beta > \frac{3}{2}$ since due to Assumption (Pη) we have $\overline{v}_{0,\beta^*}(e) = \lim_{\Delta F \to 0} \overline{v}_{\Delta F,\beta^*}(e)$. We further obtain the Hölder continuity for arbitrary $\beta > \frac{3}{2}$ as follows. For $\beta > \beta^*$ and $\Delta F \geq 0$, let $\widetilde{\Delta F} > \Delta F$. Then (4.8.18) yields

$$\left|\overline{v}_{\Delta F,\beta}(e) - \overline{v}_{0,\beta}(e)\right| = \overline{v}_{\Delta F,\beta}(e) - \overline{v}_{0,\beta}(e) \leq \overline{v}_{\widetilde{\Delta F},\beta^*}(e) - \overline{v}_{0,\beta^*}(e) \leq C^* \left|\widetilde{\Delta F}\right|^\eta,$$

where we used the Hölder continuity of $\widetilde{\Delta F} \mapsto \overline{v}_{\widetilde{\Delta F},\beta^*}(e)$ from Assumption (Pη). Taking the limit $\widetilde{\Delta F} \searrow \Delta F$ yields the desired inequality. The cases of $\Delta F \leq 0$ as well as $\beta^* > \beta > \frac{3}{2}$ with $\Delta F \geq 0$ and $\Delta F \leq 0$ respectively are dealt with analogously.

It remains to prove (4.8.18). Let $\Delta F > -\delta_F$. For width scalings $\beta_1 \geq \beta_2$, due to the definition of the half-space approximations from Corollary 4.8.1 we have the inclusion $S^-_{e,\beta_1,h(r)}(r) \supset S^-_{e,\beta_2,h(r)}(r)$ for all $r > h_0$ large enough. Hence, the comparison principle from Lemma 4.5.5 and the definition of the truncated arrival times yield

$$\frac{1}{r} m^{(A,F+\Delta F)}_{h(r)}\left(re, S^-_{e,\beta_1,h(r)}(r)\right) \leq \frac{1}{r} m^{(A,F+\Delta F)}_{h(r)}\left(re, S^-_{e,\beta_2,h(r)}(r)\right)$$

for all $(A, F) \in \Omega$ if $r > h_0$ is large enough. Taking the expected value and the limit $r \to \infty$ we thus obtain $\overline{v}_{\Delta F,\beta_1}(e) \geq \overline{v}_{\Delta F,\beta_2}(e)$.

Regarding the second inequality in (4.8.18), we apply Corollary 4.8.1, that is we compensate for the propagation of influence from far away by increasing the forcing by an appropriate constant to obtain

$$m^{(A,F+\Delta F)}_{h(r)}\left(re, S^-_{e,\beta_2,h(r)}(r)\right) \leq m^{(A,F+\Delta F-f(r))}_{h(r)}\left(re, S^-_{e,\beta_1,h(r)}(r)\right) + Ch(r)$$

with $f(r) = r^{-\frac{\beta_2}{2}+\frac{3}{4}}$. Since $\beta_2 > \frac{3}{2}$, for any $\widetilde{\Delta F} < \Delta F$ we have $\Delta F - f(r) \geq \widetilde{\Delta F}$ for all $r \geq h_0$ large enough and we hence obtain

$$m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta_2,h(r)}(r)\Big) \leq m_{h(r)}^{(A,F+\widetilde{\Delta F})}\Big(re, S^-_{e,\beta_1,h(r)}(r)\Big) + Ch(r).$$

Multiplying with $\frac{1}{r}$, taking the expected value and the limit $r \to \infty$ we thus obtain $\overline{v}_{\Delta F,\beta_2}(e) \geq \overline{v}_{\widetilde{\Delta F},\beta_1}(e)$, using that $\frac{h(r)}{r} = r^{\vartheta-1} \to 0$ as $r \to \infty$. We have thus shown (4.8.18), concluding the proof. □

If Assumption (Pη) holds, then the bounds (4.8.15) and (4.8.16) for the arrival times of the half-space approximations based on Corollary 4.8.3 and Corollary 4.8.5 improve as follows, where we also use Lemma 4.8.7.

Corollary 4.8.8 (Bounds for the truncated arrival times of half-space approximations)**.** *Assume that the random coefficient field $(A, F) \in \Omega$ with probability distribution $\mathbb{P}$ satisfies Assumptions (P1), (P2), (P3), and (Pη). Assume that there is $\delta_F \in (0, 1]$ such that all coefficient fields $(A, F + \Delta F)$ with $\Delta F \in [-\delta_F, \delta_F]$ satisfy Assumption (P$\circledcirc$). Let $h(r) = r^\vartheta$ for some $0 < \vartheta < 1$ and $e \mapsto \overline{v}(e)$ be the homogenized speed as defined in Lemma 4.8.7. Then there exist $C = C(\mathrm{data}) > 0$ and $r = r_0(\mathrm{data}, \delta_F)$ such that for the truncated arrival times of the half-space approximations $S^-_{e,\beta,h(r)}(r)$ with $e \in \mathbb{S}^{d-1}$, width scaling $\beta > \frac{3}{2}$ and $r \geq r_0$ there holds*

$$\mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big] \geq \overline{v}^{-1}(e)r - C\log(r)\sqrt{r^{1+\vartheta}} - Cr\big|\Delta F\big|^\eta$$

for any $\Delta F \in [-\delta_F, \delta_F]$ with the Hölder coefficient η from Assumption (Pη) and further

$$\mathbb{E}\Big[m_{h(r)}^{(A,F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big] \leq \overline{v}^{-1}(e)r + C\log(r)\sqrt{r^{1+\vartheta}} + Cr^{1-\left(\frac{\beta}{2}-\frac{3}{4}\right)\eta}.$$

4.9 Final homogenization argument and the proof of Theorem 4.4.2

In this section we will prove the overall homogenization result for arrival times, as stated in Theorem 4.4.2. In the first subsection, we will list the used assumptions and recall the definition of the homogenized speed. In the second subsection, we will show that in order to prove the result we only need to look at two types of sets: Balls and round holes. In the third subsection, we will prove bounds for the homogenization error of the arrival times given these two sets and comment on the necessity of the isotropy assumption.

4.9.1 Assumptions and the homogenized speed

We assume that the random coefficient field (A, F) with the probability distribution $\mathbb{P}$ satisfies Assumptions (P1), (P2), (P3), (P4) and (Pη). We require that there is $\delta_F \in (0, 1]$ such that all coefficient fields $(A, F + \Delta F)$ with $\Delta F \in [-\delta_F, \delta_F]$ satisfy Assumption (P$\circledcirc$).

Due to Assumptions (P1), (P2), (P3) and (P$\circledcirc$) we know from Corollary 4.8.3 that the limits

$$\overline{v}_{\Delta F,\beta}(e) := \left(\lim_{r\to\infty} \frac{1}{r}\mathbb{E}\Big[m_{h(r)}^{(A,F+\Delta F)}\Big(re, S^-_{e,\beta,h(r)}(r)\Big)\Big]\right)^{-1}$$

exist for all $\Delta F \in [-\delta_F, \delta_F]$, $e \in \mathbb{S}^{d-1}$, and width scalings $\beta > \frac{3}{2}$, where $S^-_{e,\beta,h(r)}(r)$ are the half-space approximations from Corollary 4.8.1. Since Assumption (Pη) holds, we know

from Lemma 4.8.7 that for $\Delta F = 0$ these limits are independent of β. Due to the isotropy of the probability distribution from Assumption (P4) – which we use here for the first time – all these limits are the same for each direction $e \in \mathbb{S}^{d-1}$. For the homogenized speed we hence set

$$\overline{v} := \overline{v}_{0,\beta}(e) \qquad \text{for arbitrary } e \in \mathbb{S}^{d-1},\ \beta > \frac{3}{2}. \tag{4.9.1}$$

In order to relate truncated arrival times of arbitrary sets to the homogenized speed, we will align half-space approximations with the boundary of these sets and use the quantitative error bounds from Corollary 4.8.8.

4.9.2 Reduction from arbitrary sets to a self-similar grower and shrinker

In order to prove Theorem 4.4.2, we want to compare the arrival time for a fat set $S \subset \mathbb{R}^d$ at some target $x_0 \in \mathbb{R}^d$ with respect to the homogenized first-order evolution from Definition 4.1.5 to the heterogeneous second-order evolution on large scales. To be more precise, for $0 < \varepsilon \ll 1$ we normalize the arrival time of $\varepsilon^{-1}S$ in the target area $\varepsilon^{-1}x_0 + \overline{B}_{h(R_\varepsilon)}$, where

$$\begin{aligned} R_\varepsilon &:= \varepsilon^{-1} \operatorname{dist}\big(x_0, S\big), \\ h(r) &:= r^\vartheta \qquad\qquad \text{for some fixed } 0 < \vartheta < 1, \end{aligned}$$

to ultimately compare

$$m^{hom,\overline{v}}\big(x_0, S\big) = \overline{v}^{-1} \operatorname{dist}\big(x_0, S\big) \qquad \text{to} \qquad \varepsilon m^{(A,F)}_{h(R_\varepsilon)}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big).$$

However, instead of dealing with arbitrary sets S, we want to reduce this comparison to two special sets.

Since S is fat, there exists $\varepsilon_0 = \varepsilon_0(S) > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ the set $\varepsilon^{-1}S$ is $h(R_\varepsilon)$-fat. In particular, there exists $y_\varepsilon \in \mathbb{R}^d$ such that

$$\overline{B}_{h(R_\varepsilon)}(y_\varepsilon) \subset \varepsilon^{-1}S \qquad \text{and} \qquad \operatorname{dist}\big(\varepsilon^{-1}x_0, \overline{B}_{h(R_\varepsilon)}(y_\varepsilon)\big) = R_\varepsilon.$$

Thus, as shown in Figure 4.9.1 we can sandwich $\varepsilon^{-1}S$ between

$$\overline{B}_{h(R_\varepsilon)}(y_\varepsilon) \subset \varepsilon^{-1}S \subset \mathbb{R}^d \setminus B_{R_\varepsilon}(\varepsilon^{-1}x_0). \tag{4.9.2}$$

and therefore obtain

$$\begin{aligned} \varepsilon m_{h(R_\varepsilon)}\big(\varepsilon^{-1}x_0, \overline{B}_{h(R_\varepsilon)}(y_\varepsilon)\big) &\ge \varepsilon m_{h(R_\varepsilon)}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big) \\ &\ge \varepsilon m_{h(R_\varepsilon)}\big(\varepsilon^{-1}x_0, \mathbb{R}^d \setminus B_{R_\varepsilon}(\varepsilon^{-1}x_0)\big). \end{aligned} \tag{4.9.3}$$

Recall from Definition 4.1.5 and Definition 4.1.6 that for $\widetilde{S} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ the homogenized arrival times are given by

$$m^{hom,\overline{v}_{\Delta F}}\Big(x, \widetilde{S}\Big) := \min\Big\{t \ge 0 \,:\, x \in \mathscr{R}^{hom,\overline{v}_{\Delta F}}_t\Big(\widetilde{S}\Big)\Big\},$$

where $t \mapsto \mathscr{R}^{hom,\overline{v}_{\Delta F}}_t\Big(\widetilde{S}\Big)$ is the evolution of the set $\widetilde{S}$ according to (4.1.8) with homogeneous coefficient field $\overline{v}_{\Delta F}$. The key consequence of an isotropic homogenized speed resulting from Assumption (P4) is that

$$\mathscr{R}^{hom,\overline{v}_{\Delta F}}_t\Big(\widetilde{S}\Big) = \widetilde{S} + t\overline{v}_{\Delta F}\overline{B}_1 \qquad \text{for all } t > 0$$

and thus in particular from (4.9.2) we obtain

$$\begin{aligned}\varepsilon m^{hom,\overline{v}}\big(\varepsilon^{-1}x_0, \overline{B}_{h(R_\varepsilon)}(y_\varepsilon)\big) = m^{hom,\overline{v}}\big(x_0, S\big) &= \overline{v}^{-1}\operatorname{dist}\big(x_0, S\big)\\ &= \varepsilon m^{hom,\overline{v}}\big(\varepsilon^{-1}x_0, \mathbb{R}^d \setminus B_{R_\varepsilon}(\varepsilon^{-1}x_0)\big). \quad (4.9.4)\end{aligned}$$

To obtain a homogenization result, it is therefore sufficient to only treat two shapes: A ball and a round hole. In the view of (4.9.3), we want to show that the ball expands faster with respect to the heterogeneous evolution than with respect to the homogeneous evolution, that is

$$\varepsilon\mathbb{E}\Big[m_{h(R_\varepsilon)}\big(\varepsilon^{-1}x_0, \overline{B}_{h(R_\varepsilon)}(y_\varepsilon)\big)\Big] \le \varepsilon m^{hom,\overline{v}}\big(\varepsilon^{-1}x_0, \overline{B}_{h(R_\varepsilon)}(y_\varepsilon)\big) + \mathrm{Err}_{hom}$$

for a suitable error term. Analogously, we want to show that the round hole collapses slower with respect to the heterogeneous evolution than with respect to the homogeneous one, that is

$$\varepsilon\mathbb{E}\Big[m_{h(R_\varepsilon)}\big(\varepsilon^{-1}x_0, \mathbb{R}^d \setminus B_{R_\varepsilon}(\varepsilon^{-1}x_0)\big)\Big] \ge \varepsilon m^{hom,\overline{v}}\big(\varepsilon^{-1}x_0, \mathbb{R}^d \setminus B_{R_\varepsilon}(\varepsilon^{-1}x_0)\big) - \mathrm{Err}_{hom}.$$

If we are able to show both, then for the original set (4.9.3) and (4.9.4) yield

$$\Big|\varepsilon\mathbb{E}\Big[m_{h(\varepsilon^{-1}\operatorname{dist}(x_0,S))}\big(\varepsilon^{-1}x_0, \varepsilon^{-1}S\big)\Big] - m^{hom,\overline{v}}\big(x_0, S\big)\Big| \le \mathrm{Err}_{hom}$$

and thus the kind of homogenization result we aim for. In the following we will provide such a comparison for the arrival times of the ball (Proposition 4.9.1) and for the arrival times of the round hole (Proposition 4.9.2).

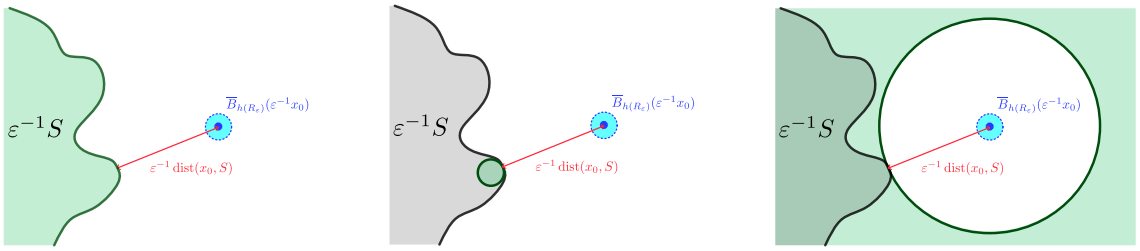


Figure 4.9.1: An illustration of the comparison argument used to reduce the homogenization problem to expanding balls and shrinking holes. Left: The original set $\varepsilon^{-1}S \subset \mathbb{R}^d$ is depicted in green. The target area $\overline{B}_{h(R_\varepsilon)}(\varepsilon^{-1}x_0)$ is depicted in blue. Center: The ball $\overline{B}_{h(R_\varepsilon)}(y_\varepsilon)$ (depicted in green) is placed at the point in $\varepsilon^{-1}S$ closest to the target area. Right: The set $\mathbb{R}^d \setminus B_{R_\varepsilon}(\varepsilon^{-1}x_0)$ (depicted in green) separates $\varepsilon^{-1}S$ and the target area

4.9.3 Bounding the homogenization error for the self-similar sets

Proposition 4.9.1 (Upper bound for arrival times)**.** *Assume that the random coefficient field $(A, F) \in \Omega$ with probability distribution $\mathbb{P}$ satisfies Assumptions (P1), (P2), (P3), (P4), and (Pη). Assume that there is $\delta_F \in (0, 1]$ such that all coefficient fields $(A, F + \Delta F)$ with $\Delta F \in [-\delta_F, 0]$ satisfy Assumption (P$\circledcirc$). Let $\overline{v} > 0$ be the homogenized speed defined in (4.9.1) and $h(r) = r^\vartheta$ for $0 < \vartheta < 1$.*

Then there exists a constant $C = C(\delta_F, \vartheta, \mathrm{data}) > 0$ such that for any $e \in \mathbb{S}^{d-1}$, $R > 0$ and $\varepsilon > 0$ with $R_\varepsilon := \varepsilon^{-1}R$ there holds

$$\begin{aligned}\varepsilon\mathbb{E}\Big[m^{(A,F)}_{h(R_\varepsilon)}\big((R_\varepsilon + h(R_\varepsilon))e, \overline{B}_{h(R_\varepsilon)}\big)\Big] - \overline{v}^{-1}R&\\ \le CR^{1-\alpha}\log(\max\{R, 2\})^{\frac{3-\vartheta}{2}}\varepsilon^\alpha\log(\varepsilon^{-1})^{\frac{3-\vartheta}{2}} \quad (4.9.5)&\end{aligned}$$

with $\alpha := \Big(1 + 4\Big(\frac{1}{1-\vartheta} + \frac{1}{\eta}\Big)\Big)^{-1}$, where η is the Hölder coefficient from Assumption (Pη).

Proof. The idea is to show that we can expect the evolution of the ball according to the heterogeneous coefficient field to expand not much slower than the first order evolution with respect to the homogenized speed, that is

$$\mathscr{R}_{t+\mathrm{Err}_t}\big(\overline{B}_{h(R_\varepsilon)}\big) \supset_{h(R_\varepsilon)} \overline{B}_{h(R_\varepsilon)} + t\overline{v}\overline{B}_1 = \mathscr{R}_t^{hom,\overline{v}}\big(\overline{B}_{h(R_\varepsilon)}\big)$$

for all $0 \le t \le \overline{v}^{-1}R$ with a suitably controlled random error term $t \mapsto \mathrm{Err}_t$.

More precisely, we will iteratively choose a suitable sequence of ball enlargements $0 < r_1, \dots, r_N$ and show that the times until one ball fully envelops the next largest one correspond to expansion not much slower than at speed $\overline{v}$. That is, with the enlargements adding up to the radii $R_n := h(R_\varepsilon) + \sum_{k=1}^n r_k$ with $R_N = R_\varepsilon + h(R_\varepsilon)$, for $n \in \{0, \dots, N\}$ we define the times

$$t_n = t_n^{(A,F)} := \sup\Big\{ m_{h(r_n)}^{(A,F)}\big(y, \overline{B}_{R_{n-1}}\big) \,:\, y \in \partial\overline{B}_{R_n} \Big\} \qquad \text{for } n \in \{1, \dots, N\}, \tag{4.9.6}$$

see Figure 4.9.2. We will show that for a suitable choice of the enlargements $(r_n)_n$ there

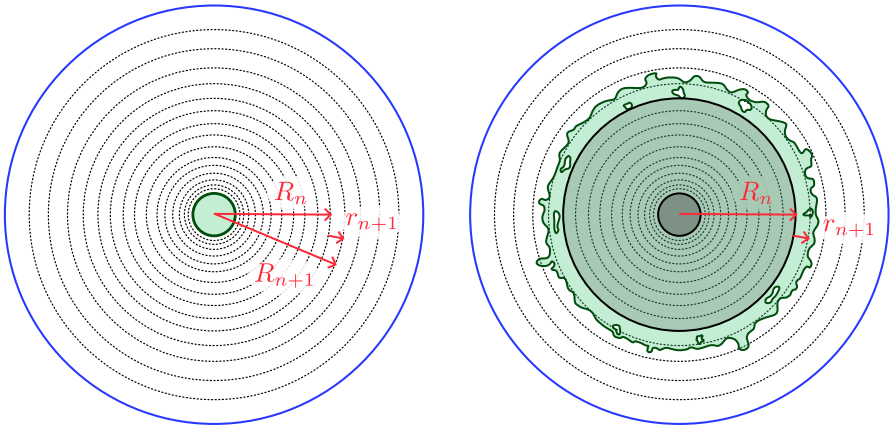


Figure 4.9.2: An illustration of the argument for bounding the expansion speed of balls from below. Left: The initial ball with radius $R_0 = h(R_\varepsilon)$ is depicted in green. The boundary of the ball with radius R_N corresponding to the target distance is depicted in blue. In-between, dotted lines denote the boundaries of intermediate balls with radii $(R_n)_n$ corresponding to the consecutive enlargements $(r_n)_n$. Right: For $n \in \{0, \dots, N-1\}$, after time t_{n+1} the evolution $\mathscr{R}_{t_{n+1}}(\overline{B}_{R_n})$ (depicted in green) of the ball $\overline{B}(R_n)$ (depicted in light grey) fully envelops the boundary of the next larger ball $\overline{B}_{R_{n+1}}$

are relatively small errors $(\tau_n)_n$ such that

$$\mathbb{E}\big[t_n\big] \le \overline{v}^{-1} r_n + \tau_n \qquad \text{for all } n \in \big\{1, \dots, N\big\}. \tag{4.9.7}$$

We claim that the overall arrival time is smaller than the sum of the times for all full ball expansions, that is that

$$m_{h(R_\varepsilon)}^{(A,F)}\big((R_\varepsilon + h(R_\varepsilon))e, \overline{B}_{h(R_\varepsilon)}\big) \le \sum_{n=1}^N \Big(t_n^{(A,F)} + Ch(r_n) \Big) \tag{4.9.8}$$

for $(A,F) \in \bigcap_{n=1}^N E^{\circledcirc}_{h(r_n)}(\overline{B}_{R_n})$. Taking the expected value of (4.9.8) and applying (4.9.7) will yield (4.9.5) with the right hand side given by

$$\varepsilon \sum_{n=1}^N \big(\tau_n + Ch(r_n)\big) + CR \sum_{n=1}^N \Big(1 - \mathbb{P}\Big[E^{\circledcirc}_{h(r_n)}(\overline{B}_{R_n})\Big]\Big). \tag{4.9.9}$$

Our goal is to choose the enlargements $(r_n)_n$ such that this overall error is as small as possible. In the first part of the proof, we will establish the claim in (4.9.10). In the second

part of the proof, we will estimate $\mathbb{E}[t_n]$ for $n \in \mathbb{N}$ and the different contributions to the error τ_n for a given enlargement choice r_n in order to deduce a recursive formula for $(r_n)_n$ which optimizes the scaling of $(\tau_n)_n$ and the overall error. In the third part of the proof, we will analyze the resulting sequence and calculate the overall error.

Step 1: Establishing (4.9.8). The claim (4.9.8) follows directly from

$$\partial B_{R_n} \subset_{h(r_n)} \mathscr{R}^{(A,F)}_{\sum_{k=1}^n \left(t_k^{(A,F)} + Ch(r_k)\right)}\left(\overline{B}_{h(R_\varepsilon)}\right), \qquad \text{for all } n = 1, \dots, N \tag{4.9.10}$$

for $(A,F) \in \bigcap_{n=1}^N E^{\circledcirc}_{h(r_n)}(\overline{B}_{R_n})$, which we obtain via induction.

For $n = 0$ with $R_0 = h(R_\varepsilon)$ we have $\partial B_{R_0} \subset \overline{B}_{h(R_\varepsilon)}$. Let $n \geq 1$ with

$$\partial B_{R_{n-1}} \subset_{h(r_{n-1})} \mathscr{R}^{(A,F)}_{T_{n-1}}\left(\overline{B}_{h(R_\varepsilon)}\right) \qquad \text{for } T_{n-1} := \sum_{k=1}^{n-1}\left(t_k^{(A,F)} + Ch(r_k)\right).$$

Since $(A,F) \in E^{\circledcirc}_{h(r_n)}(\overline{B}_{R_n})$, due to Assumption (P$\circledcirc$) there is a stable $(h(r_n), c_s)$-approximation S_h of $\overline{B}_{h(R_\varepsilon)}$, for which the comparison principle from Lemma 4.5.5 yields

$$\partial B_{R_{n-1}} \subset_{h(r_{n-1})} \mathscr{R}^{(A,F)}_{T_{n-1}}\left(\overline{B}_{h(R_\varepsilon)}\right) \subset_{h(r_n)} \mathscr{R}^{(A,F)}_{T_{n-1}+Ch(r_n)}(S_h).$$

Since by Assumption (P$\circledcirc$) (A,F) admits an effective minimum speed on the scale $h(r_n)$ for S_h, an additional time step of size $Ch(r_n)$ yields $\partial B_{R_{n-1}} \subset_{h(r_n)} \mathscr{R}^{(A,F)}_{T_{n-1}+Ch(r_n)}(S_h)$. Using the stability of $\mathscr{R}^{(A,F)}_{T_{n-1}+Ch(r_n)}(S_h)$ inherited from S_h via Lemma 4.5.4, we can fill any of its holes which intersect $\overline{B}_{R_{n-1}}$ as in the proof of the comparison principle from Lemma 4.5.5(b) without changing any arrival times outside of $\overline{B}_{R_{n-1}}$. Due to the comparison principle, the definition of t_n in (4.9.6) and the stability we thus obtain

$$x + \overline{B}_{h(r_n)} \cap \mathscr{R}^{(A,F)}_{T_{n-1}+Ch(r_n)+t_n^{(A,F)}}(S_h) \neq \emptyset \qquad \text{for all } x \in \partial B_{R_n}.$$

With the effective minimum speed, after another time step of size $Ch(r_n)$ we hence obtain

$$\partial B_{R_n} \subset_{h(r_n)} \mathscr{R}^{(A,F)}_{T_{n-1}+t_n^{(A,F)}+Ch(r_n)}(S_h) \subset \mathscr{R}^{(A,F)}_{T_{n-1}+t_n^{(A,F)}+Ch(r_n)}\left(\overline{B}_{h(R_\varepsilon)}\right),$$

where we used that $S_h \subset \overline{B}_{h(R_\varepsilon)}$ and the comparison principle. This yields (4.9.10) by induction and thus establishes (4.9.8).

Step 2: Estimate on $\mathbb{E}[t_n]$. Assume that $r_1, \dots, r_n$ and thus $R_n = h(R_\varepsilon) + \sum_{k=1}^n r_k$ are already chosen. In order to relate the arrival times in the definition of t_{n+1} from (4.9.6) to the homogenized speed $\overline{v}$, we need to find proper placements for half-space approximations $S^-_{e,\beta,h(r_{n+1})}(r_{n+1})$, which were used to define the homogenized speed. For each $y = (R_n + r_{n+1})e$ with $e \in \mathbb{S}^{d-1}$, we place a half-space approximation $S^-_{e,\beta,h(r_{n+1})}(r_{n+1})$ inside $\overline{B}_{R_n}$, but as close to y as possible, see Figure 4.9.3. Due to the curvature of the ball, as noted in Figure 4.9.3, there is a gap of size $\rho_{n+1} > 0$ between the boundary of the ball and the center of the half-space approximation given by

$$\left(R_n - \rho_{n+1}\right)^2 + \left(C_\beta r_{n+1}^\beta\right)^2 = R_n^2, \quad \text{implying} \quad \rho_{n+1} \leq C\frac{r_{n+1}^{2\beta}}{R_n}, \tag{4.9.11}$$

where $C_\beta r_{n+1}^\beta$ is the width of $S^-_{e,\beta,h(r_{n+1})}(r_{n+1})$ as defined in Corollary 4.8.1. We hence bound

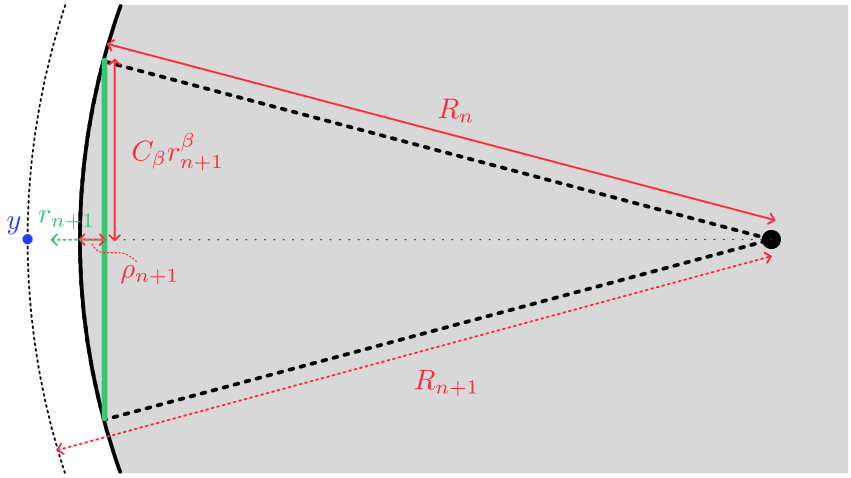


Figure 4.9.3: A sketch of the argument used to estimate the gap between the boundary of balls and the aligned half-space approximations in the proof of the lower bounds for the expansion speed of balls. The ball $\overline{B}_{R_n}$ is depicted in grey with a target point $y \in \partial\overline{B}_{R_{n+1}}$ depicted in blue. A half-space approximation (depicted in green) is aligned inside $\overline{B}_{R_n}$, which results in a gap of size $\rho_{n+1} > 0$ between the center of the half-space approximation and the boundary of the ball

the arrival time of the ball $\overline{B}_{R_n}$ at the point $y = (R_n + r_{n+1})e$ – as in the definition of t_{n+1} from (4.9.6) – by the arrival time from the half-space approximation $S^-_{e,\beta,h(r_{n+1})}(r_{n+1}) + (R_n - \rho_{n+1})e$ at the point $y - \rho_{n+1}e$ plus an error term corresponding to bridging the remaining gap at minimum effective speed, that is

$$\begin{aligned} &m^{(A,F)}_{h(r_{n+1})}\big(y, \overline{B}_{R_n}\big) \\ &\quad \le m^{(A,F)}_{h(r_{n+1})}\Big(r_{n+1}e + \big(R_n - \rho_{n+1}\big)e, S^-_{e,\beta,h(r_{n+1})}\big(r_{n+1}\big) + \big(R_n - \rho_{n+1}\big)e\Big) \\ &\qquad + C\big(\rho_{n+1} + h(r_{n+1})\big) \end{aligned}$$

for $(A,F) \in E^{\circledcirc}_{h(r_{n+1})}\big(\overline{B}_{R_{n+1}}\big)$. We have thus reduced the expansion of the ball to the expansion of infinitely many half-space approximations at the cost of the error term $C(\rho_{n+1} + h(r_{n+1}))$, obtaining

$$\begin{aligned} &t_{n+1}\chi_{E^{\circledcirc}_{h(r_{n+1})}\left(\overline{B}_{R_{n+1}}\right)} - C\big(\rho_{n+1} + h(r_{n+1})\big) \\ &\qquad \le \sup_{e\in\mathbb{S}^{d-1}} m_{h(r_{n+1})}\Big(r_{n+1}e + \big(R_n - \rho_{n+1}\big)e, S^-_{e,\beta,h(r_{n+1})}\big(r_{n+1}\big) + \big(R_n - \rho_{n+1}\big)e\Big). \end{aligned}$$

Analogously to the proof of Proposition 4.8.2, for $(A,F) \in E^{\circledcirc}_{h(r_{n+1})}\big(\overline{B}(R_{n+1})\big)$ we can reduce the supremum over infinitely many arrival times to one over finitely many, control the fluctuations with a union bound and the fluctuation bounds from Proposition 4.6.1, and finally use the stationarity and isotropy from Assumptions (P1) and (P4) to obtain

$$\begin{aligned} \mathbb{E}\big[t_{n+1}\big] \le \mathbb{E}\Big[m_{h(r_{n+1})}\Big(r_{n+1}e, S^-_{e,\beta,h(r_{n+1})}\big(r_{n+1}\big)\Big)\Big] \\ + C\sqrt{h(r_{n+1})r_{n+1}}\log\big(R_{n+1}\big) + C\big(\rho_{n+1} + h(r_{n+1})\big) \end{aligned}$$

for some $e \in \mathbb{S}^{d-1}$ and $C = C(\vartheta, \mathrm{data})$. Now, assuming without loss of generality that $r_{n+1} \ge C = C(\mathrm{data}, \delta_F)$ is large enough to apply Corollary 4.8.8 and plugging in the bound for ρ_{n+1} from (4.9.11) yields

$$\begin{aligned} \mathbb{E}\big[t_{n+1}\big] &\le \overline{v}^{-1}r_{n+1} + Cr_{n+1}^{1-\left(\frac{\beta}{2}-\frac{3}{4}\right)\eta} + C\sqrt{h(r_{n+1})r_{n+1}}\log\big(R_{n+1}\big) + C\frac{r_{n+1}^{2\beta}}{R_n} \\ &=: \overline{v}^{-1}r_{n+1} + \tau_{n+1,force} + \tau_{n+1,fluct} + \tau_{n+1,curve}. \end{aligned} \tag{4.9.12}$$

In order to balance the three error contributions $\tau_{n+1,force}, \tau_{n+1,fluct}, \tau_{n+1,curve}$ and to optimize the overall error, we make two choices: First, for the width scaling of the half-space approximations we choose $\beta = \frac{3}{2} + \frac{1-\vartheta}{\eta}$, balancing the error $\tau_{n+1,force}$ due to the forcing change and the Hölder continuity from Assumption (Pη) with the error $\tau_{n+1,fluct}$ from the fluctuation bounds. Second, we choose the enlargement r_{n+1} to balance these errors with the error $\tau_{n+1,curve}$ due to the gap.

In summary, we have

$$\mathbb{E}\big[t_{n+1}\big] \leq \overline{v}^{-1} r_{n+1} + C\log(R_{n+1})\sqrt{r_{n+1}^{1+\vartheta}} \qquad \text{for} \quad r_{n+1}^{2\left(\frac{3}{2}+\frac{1-\vartheta}{\eta}\right)-\frac{1+\vartheta}{2}} \approx R_n, \tag{4.9.13}$$

establishing the scaling for our recursive choice of r_{n+1} based on $r_1, \dots, r_n$.

Step 3: Analyzing the recursive formula and calculating the overall error. We choose $r_{n+1} = R_{n+1} - R_n$ as in (4.9.13), that is

$$\begin{aligned} R_{n+1} &:= R_n + c_{n+1} R_n^{\gamma} = \big(1 + c_{n+1} R_n^{\gamma-1}\big) R_n, \\ R_0 &:= h(R_\varepsilon) = R_\varepsilon^{\vartheta}, \end{aligned}$$

with

$$\gamma := \left(2\left(\frac{3}{2} + \frac{1-\vartheta}{\eta}\right) - \frac{1+\vartheta}{2}\right)^{-1} = \frac{2}{1 - \vartheta + 4\left(\frac{1-\vartheta}{\eta} + 1\right)} \in (0,1), \tag{4.9.14}$$

and with the coefficients $(c_n)_n \subset \left(\frac{1}{2}, 1\right)$ chosen such that there is $N \in \mathbb{N}$ with $R_N = R_\varepsilon$. Since the radii are increasing, for $1 \leq n \leq N$ with $R_0 \leq R_n \leq R_N = R_\varepsilon + h(R_\varepsilon) \leq 2R_\varepsilon$ we have

$$R_\varepsilon \geq R_0 \prod_{n=1}^{N} \left(1 + \frac{1}{2} R_n^{\gamma-1}\right) \geq \left(1 + \frac{1}{4} R_\varepsilon^{\gamma-1}\right)^N R_0$$

and thus

$$N \leq \frac{\log\left(\frac{R_\varepsilon}{R_0}\right)}{\log\left(1 + \frac{1}{4} R_\varepsilon^{\gamma-1}\right)} \leq C R_\varepsilon^{1-\gamma} \log\big(R_\varepsilon\big).$$

Based on (4.9.8), (4.9.9), (4.9.12) and (4.9.13) due to the choice of the incremental enlargements from Step 1, we obtain

$$\begin{aligned} \mathbb{E}\Big[m_{h(R_\varepsilon)}^{(A,F)}\big((R_\varepsilon + h(R_\varepsilon))e, \overline{B}_{h(R_\varepsilon)}\big)\Big] &\leq \sum_{k=1}^{N} \mathbb{E}\big[t_n\big] + Ch(R_\varepsilon) \\ &\leq \overline{v}^{-1} R_\varepsilon + Ch(R_\varepsilon) + C\log\big(R_\varepsilon\big) \sum_{n=1}^{N} \sqrt{r_n^{1+\vartheta}}, \end{aligned}$$

where the term $CR\sum_{n=1}^{N}\Big(1 - \mathbb{P}\Big[E^{\circledcirc}_{h(r_n)}(\overline{B}_{R_n})\Big]\Big)$ from (4.9.9) can be bound by a constant due to the probability bound from Assumption (P$\circledcirc$), which allows us to absorb all polynomial prefactors since $h(r_n) \geq h(r_1) \geq \frac{1}{2} h(R_\varepsilon)^{\gamma}$. Regarding the main error term, applying Jensen's inequality we obtain

$$\begin{aligned} \sum_{n=1}^{N} r_n^{\frac{1+\vartheta}{2}} = N \frac{1}{N} \sum_{n=1}^{N} r_n^{\frac{1+\vartheta}{2}} \leq N \left(\frac{1}{N} \sum_{n=1}^{N} r_n\right)^{\frac{1+\vartheta}{2}} &= N^{\frac{1-\vartheta}{2}} R_\varepsilon^{\frac{1+\vartheta}{2}} \\ &\leq C R_\varepsilon^{(1-\gamma)\frac{1-\vartheta}{2} + \frac{1+\vartheta}{2}} \big(\log\big(R_\varepsilon\big)\big)^{\frac{1-\vartheta}{2}}. \end{aligned}$$

Multiplying with ε and plugging in $R_\varepsilon = \varepsilon^{-1}R$ as well as the definition of $\gamma \in (0,1)$ from (4.9.14) we hence obtain (4.9.5). □

Proposition 4.9.2 (Lower bound for arrival times)**.** *Assume that the random coefficient field $(A,F) \in \Omega$ with probability distribution $\mathbb{P}$ satisfies Assumptions (P1), (P2), (P3), (P4), and (Pη). Assume that there is $\delta_F \in (0,1]$ such that all coefficient fields $(A, F+\Delta F)$ with $\Delta F \in [0,\delta_F]$ satisfy Assumption (P$\circledcirc$). Let $\overline{v} > 0$ be the homogenized speed defined in (4.9.1) and $h(r) = r^\vartheta$ for $0 < \vartheta < 1$.*

Then there exists a constant $C = C(\vartheta, \mathrm{data}) > 0$ such that for any $R > 0$ and $\varepsilon > 0$ with $R_\varepsilon := \varepsilon^{-1}R$ there holds

$$\varepsilon \mathbb{E}\Big[m_{h(R_\varepsilon)}\big(0, \mathbb{R}^d \setminus B_{R_\varepsilon}\big)\Big] - \overline{v}^{-1}R \geq -CR^{1-\alpha}\log\big(\max\{R,2\}\big)^{\frac{3-\vartheta}{2}}\varepsilon^\alpha \log(\varepsilon^{-1})^{\frac{3-\vartheta}{2}} \tag{4.9.15}$$

with $\alpha := \Big(1 + 4\Big(\frac{1}{1-\vartheta} + \frac{1}{\eta}\Big)\Big)^{-1}$, where η is the Hölder coefficient from Assumption (Pη).

Proof. Up to some smaller details, this proof is analogous to the proof of Proposition 4.9.1, only that the geometric setup is mirrored. The idea here is to show that we can expect the round hole to shrink not much faster when evolving according to the heterogeneous coefficient field compared to the homogenized speed, that is

$$\mathscr{R}_{t-\mathrm{Err}_t}\big(\mathbb{R}^d \setminus B_{R_\varepsilon}\big) \subset_{h(R_\varepsilon)} \mathbb{R}^d \setminus \big(R_\varepsilon - t\overline{v}\big)B_1 = \mathscr{R}_t^{hom,\overline{v}}\big(\mathbb{R}^d \setminus B_{R_\varepsilon}\big)$$

for all $0 \leq t \leq \overline{v}^{-1}R$ with a suitably controlled random error term $t \mapsto \mathrm{Err}_t$.

To be more precise, we again iteratively choose a suitable sequence of enlargements $0 < r_1, \dots, r_N$ resulting in shrinking radii $R_n := R_\varepsilon - \sum_{k=1} r_k$ for $n \in \{1, \dots, N\}$. We will show that the time until the evolution $\mathbb{R}^d \setminus B_{R_{n-1}}$ first reaches the boundary of the next smaller hole $\mathbb{R}^d \setminus B_{R_n}$ is not much faster than $\overline{v}^{-1}r_n$. That is, for $n \in \{1, \dots, N\}$ we define the times

$$t_n = t_n^{(A,F)} := \inf\Big\{m_{h(r_n)}^{(A,F)}\big(y, \mathbb{R}^d \setminus B_{R_{n-1}}\big) \,:\, y \in \partial B_{R_n}\Big\}, \tag{4.9.16}$$

see Figure 4.9.4. Clearly, following the actual evolution to the center of the hole is slower

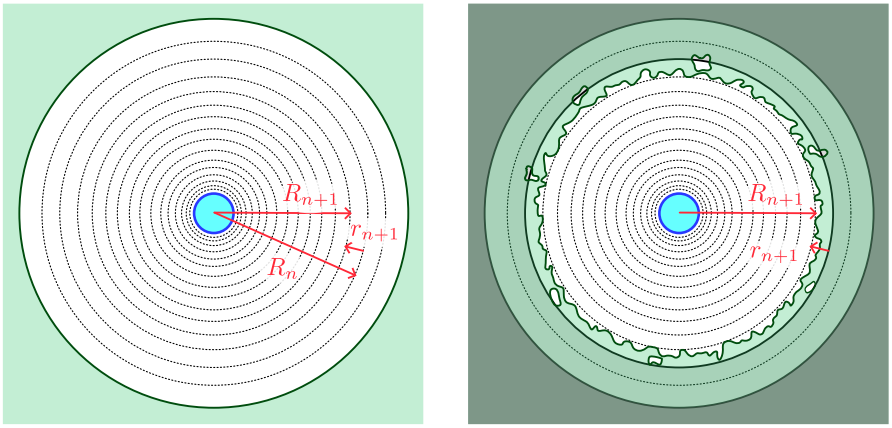


Figure 4.9.4: An illustration of the argument for bounding the shrinking speed of holes from above. Left: The exterior of the initial hole with radius $R_0 = R_\varepsilon$ is depicted in green. The target area $\overline{B}_{R_N}$ with radius $R_N = h(R_\varepsilon)$ is depicted in blue. In-between, dotted lines denote the boundaries of intermediate holes with radii $(R_n)_n$ corresponding to the consecutive enlargements $(r_n)_n$. Right: For $n \in \{0, \dots, N-1\}$, after time t_{n+1} the evolution $\mathscr{R}_{t_{n+1}}\big(\mathbb{R}^d \setminus B_{R_n}\big)$ (depicted in green) of $\mathbb{R}^d \setminus B_{R_n}$ (depicted in light grey) first reaches $\partial B_{R_{n+1}}$

than fully filling up the complement every time the boundary of the next smaller hole is reached and then continuing the evolution. That is, with $R_N = h(R_\varepsilon)$ we have

$$m^{(A,F)}_{h(R_\varepsilon)}\big(0, \mathbb{R}^d \setminus B_{R_\varepsilon}\big) \geq \sum_{n=1}^{N} t_n^{(A,F)} \tag{4.9.17}$$

for $(A,F) \in E^{\circledcirc}_{h(R_\varepsilon)}\big(\overline{B}_{R_\varepsilon}\big)$, because using the definition of the $(t_n)_n$ and iteratively applying the comparison principle yields $\mathscr{R}_{\sum_{k=1}^n t_n}\big(\mathbb{R}^d \setminus B_{R_\varepsilon}\big) \subset \mathbb{R}^d \setminus B_{R_n}$ for all n. Mirroring the proof of Proposition 4.9.1, we will choose the enlargements $(r_n)_n$ such that there are relatively small errors $(\tau_n)_n$ satisfying

$$\mathbb{E}\big[t_n\big] \geq \overline{v}^{-1} r_n - \tau_n \qquad \text{for all } n \in \{1, \dots, N\}, \tag{4.9.18}$$

which will yield (4.9.15) with the error term given by $\varepsilon \sum_{k=1}^N \tau_n$.

For the first part of the proof, we will estimate $\mathbb{E}[t_n]$ for $n \in \mathbb{N}$ and the different contributions to the error τ_n for a given enlargement choice r_n in order to deduce a recursive formula for $(r_n)_n$ which optimizes the scaling of $(\tau_n)_n$ and the overall error. Since the resulting scaling will be essentially the same as in the proof of Proposition 4.9.1, in the second part of the proof we will then briefly summarize the analysis of the resulting sequence and the calculation of the overall error.

Step 1: Estimate on $\mathbb{E}[t_n]$. Assume that $r_1, \dots, r_n$ and thus $R_n = R_\varepsilon - \sum_{k=1}^n r_k$ are already chosen. Again, in order to relate the arrival times in the definition of t_{n+1} from (4.9.16) to the homogenized speed $\overline{v}$, we need to properly place half-space approximations $S^-_{e,\beta,h(r)}(r)$, ideally tangentially to the hole. However, since we now want to bound the arrival times from below, we need to place the disk approximating the half-space 'ahead' of the hole's curve, such that the boundary of the disk is aligned with the boundary of the hole, see Figure 4.9.5. The gap $\rho_{n+1} > 0$ between the center of the disk and the boundary

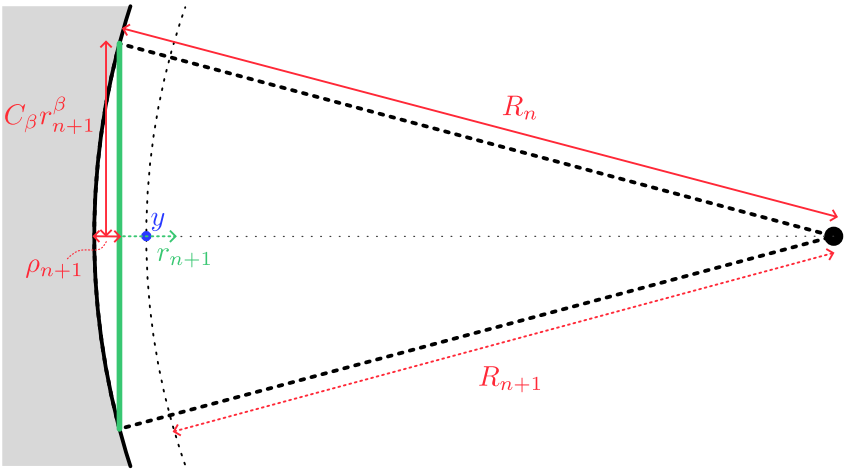


Figure 4.9.5: A sketch of the argument used to estimate the gap between the boundary of holes and the aligned half-space approximations in the proof of the upper bounds for the shrinking speed of holes. The set $\mathbb{R}^d \setminus B_{R_n}$ is depicted in grey with a target point $y \in \partial B_{R_{n+1}}$ depicted in blue. A half-space approximation (depicted in green) is aligned ahead of $\mathbb{R}^d \setminus B_{R_n}$, which results in a gap of size $\rho_{n+1} > 0$ between the center of the half-space approximation and the boundary of the hole

of the hole is the same as in the mirrored situation from the proof of Proposition 4.9.1, given by

$$\big(R_n - \rho_{n+1}\big)^2 + \Big(C_\beta r_{n+1}^\beta\Big)^2 = R_n^2, \quad \text{implying} \quad \rho_{n+1} \leq C(\beta, \text{data}) \frac{r_{n+1}^{2\beta}}{R_n},$$

where $C_\beta r_{n+1}^\beta$ is the width of $S^-_{e,\beta,h(r_{n+1})}(r_{n+1})$ as defined in Corollary 4.8.1.

Consider the definition of t_{n+1} from (4.9.6). For a target point $y = -(R_n - r_{n+1})e$ on the boundary of the smaller hole, we now apply Corollary 4.8.1 to compare the half-space approximation $S^-_{e,\beta,h(r_{n+1})}(r_{n+1}) - (R_n - \rho_{n+1})e$ to the hole $\mathbb{R}^d \setminus B_{R_n}$. For $f(r) = r^{-\frac{\beta}{2}+\frac{3}{4}}$ from Corollary 4.8.1, and $(A,F) \in E^\circledcirc_{h(r_{n+1})}(\overline{B}_{R_n})$, using the thus guaranteed minimum effective speed we obtain

$$\begin{aligned}
&m^{(A,F)}_{h(r_{n+1})}\big(y, \mathbb{R}^d \setminus B_{R_n}\big)\\
&\geq m^{(A,F)}_{h(r_{n+1})}\big(r_{n+1}e - \big(R_n - \rho_{n+1}\big)e, \mathbb{R}^d \setminus B_{R_n}\big) - C\big(\rho_{n+1} + h(r_{n+1})\big)\\
&\geq m^{(A,F+f(r_{n+1}))}_{h(r_{n+1})}\Big(r_{n+1}e - \big(R_n - \rho_{n+1}\big)e, S^-_{e,\beta,h(r_{n+1})}(r_{n+1}) - \big(R_n - \rho_{n+1}\big)e\Big)\\
&\quad - C\big(\rho_{n+1} + h(r_{n+1})\big).
\end{aligned}$$

We have thus reduced the shrinking of the hole to the expansion of infinitely many half-space approximations at the cost of the error term $C(\rho_{n+1} + h(r_{n+1}))$ and changing the forcing by a constant, obtaining

$$\begin{aligned}
&t^{(A,F)}_{n+1} + C\big(\rho_{n+1} + h(r_{n+1})\big)\\
&\qquad \geq \inf_{e\in\mathbb{S}^{d-1}} m^{(A,F+f(r_{n+1}))}_{h(r_{n+1})}\Big(r_{n+1}e - \big(R_n - \rho_{n+1}\big)e, S^-_{e,\beta,h(r_{n+1})}(r_{n+1}) - \big(R_n - \rho_{n+1}\big)e\Big).
\end{aligned}$$

for $(A,F) \in E^\circledcirc_{h(r_{n+1})}(\overline{B}_{R_{n+1}})$. Analogously to the proof of Proposition 4.8.4, we can reduce the infimum to one over finitely many arrival times, control the fluctuations with the bounds from Proposition 4.6.1 and use the stationarity and isotropy from Assumptions (P1) and (P4) to obtain

$$\begin{aligned}
\mathbb{E}\big[t_{n+1}\big] \geq \mathbb{E}\Big[m^{(A,F+f(r_{n+1}))}_{h(r_{n+1})}\Big(r_{n+1}e, S^-_{e,\beta,h(r_{n+1})}\big(r_{n+1}\big)\Big)\Big]\\
- C\sqrt{h(r_{n+1})r_{n+1}}\log\big(R_n\big) - C\big(\rho_{n+1} + h(r_{n+1})\big)
\end{aligned}$$

for some $e \in \mathbb{S}^{d-1}$ and $C = C(\beta, \vartheta, \mathrm{data}) > 0$. The expected value of the arrival times from the half-space approximations now provides the link to the homogenized speed, with the change in the forcing resulting in an error depending on the Hölder coefficient η from Assumption (Pη). That is, we apply Corollary 4.8.8 to obtain

$$\begin{aligned}
\mathbb{E}\big[t_{n+1}\big] &\geq \overline{v}^{-1} r_{n+1} - C r_{n+1} f(r_{n+1})^\eta - C\sqrt{r_{n+1}^{1+\vartheta}}\log\big(R_n\big) - C\rho_{n+1}\\
&=: \overline{v}^{-1} r_{n+1} - \tau_{n+1,force} - \tau_{n+1,fluct} - \tau_{n+1,curve}.
\end{aligned}$$

Balancing these errors just as in the proof of Proposition 4.9.1, we choose $\beta = \frac{3}{2} + \frac{1-\vartheta}{\eta}$ and obtain

$$\mathbb{E}\big[t_{n+1}\big] \geq \overline{v}^{-1} r_{n+1} - C\log(R_n)\sqrt{r_{n+1}^{1+\vartheta}} \qquad \text{for} \quad r_{n+1}^{2\left(\frac{3}{2}+\frac{1-\vartheta}{\eta}\right)-\frac{1+\vartheta}{2}} \approx R_n, \tag{4.9.19}$$

resulting in the same error bounds and sequence of radii as above but in reverse order.

Step 2: Analyzing the recursive formula and calculating the overall error. Again, we look at the sequence of radii for the hole instead of the single enlargements. Instead of increasing as in the proof of Proposition 4.9.1, these radii are decreasing with $R_{n+1} = R_n - r_{n+1}$. We set

$$\begin{aligned}
R_{n+1} &:= R_n - c_{n+1}R_n^\gamma = \big(1 - c_{n+1}R_n^{\gamma-1}\big)R_n,\\
R_0 &:= R_\varepsilon
\end{aligned}$$

with

$$\gamma := \left(2\left(\frac{3}{2} + \frac{1-\vartheta}{\eta}\right) - \frac{1+\vartheta}{2}\right)^{-1} = \frac{2}{1-\vartheta+4\left(\frac{1-\vartheta}{\eta}+1\right)} \in (0,1).$$

and with the coefficients $(c_n)_n \subset \left(\frac{1}{2}, 1\right)$ chosen such that there is $N \in \mathbb{N}$ with $R_N = h(R_\varepsilon)$. To bound this number of steps, note that since we have $h(R_\varepsilon) = R_N \leq R_n \leq R_\varepsilon$ for all $1 \leq n \leq N$, there holds

$$h(R_\varepsilon) \leq \left(1 - \frac{1}{2}R_\varepsilon^{\gamma-1}\right)^N R_\varepsilon$$

and thus

$$N \leq \frac{\log\left(\frac{R_\varepsilon}{h(R_\varepsilon)}\right)}{\log\left(\frac{1}{1-\frac{1}{2}R_\varepsilon^{\gamma-1}}\right)} \leq C R_\varepsilon^{1-\gamma}\log(R_\varepsilon).$$

Plugging (4.9.19) into (4.9.18) and the resulting bound for $\mathbb{E}[t_n]$ into (4.9.17), we obtain

$$\begin{aligned}\mathbb{E}\left[m_{h(R_\varepsilon)}\left(0, \mathbb{R}^d \setminus B_{R_\varepsilon}\right)\right] &\geq \sum_{n=1}^{N} \mathbb{E}\left[t_n\right] \\ &\geq \overline{v}^{-1} R_\varepsilon - C\log\left(R_\varepsilon\right) \sum_{n=1}^{N} r_n^{\frac{1+\vartheta}{2}} \\ &\geq \overline{v}^{-1} R_\varepsilon - C\log\left(R_\varepsilon\right) N^{\frac{1-\vartheta}{2}} R_\varepsilon^{\frac{1+\vartheta}{2}},\end{aligned}$$

where for the last inequality we used Jensen's inequality as at the end of the proof of Proposition 4.9.1. In the same way, plugging in the bound for N and multiplying with ε we obtain (4.9.15). □

In summary, combining Proposition 4.9.1 and Proposition 4.9.2 with the arguments from Subsection 4.9.2, which allowed us to replace arbitrary sets with balls and holes, proves Theorem 4.4.2.

Remark 4.9.3 (The anisotropic case). In principle, Assumption (P4) is not necessary for our argument to work. Its role is to guarantees that as in (4.9.4) we can reduce the analysis of arrival times for arbitrary sets to obtaining bounds for the arrival times of just the ball and its inverted shape, a round hole. This kind of reduction is still valid for an anisotropic homogeneous speed $e \mapsto \overline{v}(e)$ if the convex shape $K \subset \mathbb{R}^d$ given by

$$tK = \mathscr{R}_t^{hom,\overline{v}}\left(\{0\}\right) \qquad \text{for all } t > 0$$

has bounded curvature and also satisfies

$$\mathscr{R}_t^{hom,\overline{v}}\left(\overline{\mathbb{R}^d \setminus (-K)}\right) = (1-t)\overline{\mathbb{R}^d \setminus (-K)} \qquad \text{for all } 0 < t < 1.$$

In the isotropic case, we have $K = \overline{v}\overline{B}_1$ and hence clearly both conditions are satisfied. Both conditions do not hold in general and depend on the properties of $e \mapsto \overline{v}(e)$. For example, the second condition is satisfied if the homogenized evolution satisfies a dynamic programming principle, that is that the evolution of a set $S \subset \mathbb{R}^d$ is given by the union over the separate evolution of all included points:

$$\mathscr{R}_t^{hom,v}(S) = \bigcup_{y \in S} \mathscr{R}_t^{hom,v}(\{y\}) \qquad \text{for all } t \geq 0.$$

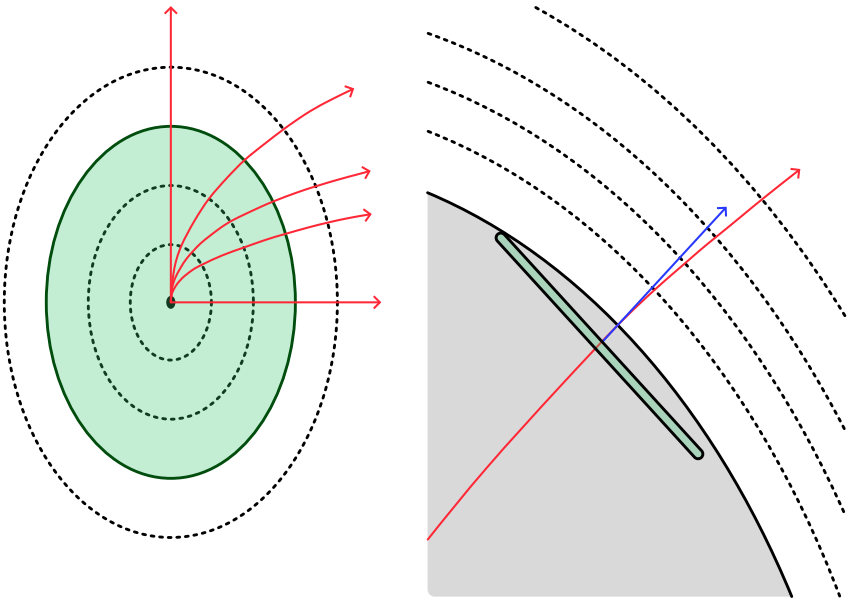


Figure 4.9.6: A depiction of the evolution of a self-similar grower with respect to a homogeneous first-order motion in the non-isotropic case. Left: Some normal trajectories (depicted in red) for an invariant shape K (depicted in green). Right: A half-space approximation (depicted in green) is tangentially aligned inside the invariant shape K (depicted in grey). The normal of the half-space approximation (depicted in blue) is compared to the normal trajectory of the set evolution

A sufficient criterion for the dynamic programming principle to hold is level-set convexity of the Hamiltonian given by the 1-homogeneous extension of the speed, that is the convexity of the shape given by $\mathbb{S}^{d-1} \ni e \mapsto \overline{v}(e)^{-1}e$, see for example [ACS14].

The only major adjustment for the part of the proof presented here in Subsection 4.9.3 is that we would have to more carefully keep track of the homogenized motions' 'trajectories': These are the paths obtained from following the outwards normal of the invariant shape K scaled up to the current point of the trajectory, see Figure 4.9.6. In the isotropic case these trajectories of course are straight and the velocity of the interface stays constant. In the anisotropic case, these trajectories might be curved and hence the velocity might vary along the trajectory, but the curvature is controlled by the curvature of the invariant shape K. In the proof, for a suitably chosen time step we need to show that the evolution of a tangentially fitted half-space approximation reaches the endpoint of the corresponding trajectory in a comparable time, see Figure 4.9.6. For the half-space approximation, Corollary 4.8.8 up to an error guarantees the evolution along a straight trajectory with the average speed corresponding to the homogenized velocity for that direction. Choosing the time steps small enough, the trajectory for the homogenized motion will be close to that straight line. The difference in velocities can be controlled by showing that $e \mapsto \overline{v}^{-1}(e)$ is Hölder continuous, adapting [AC18, Lemma 4.7] to our setting. This leads to an additional error term in the first step of the proof of Proposition 4.9.1 and Proposition 4.9.2 respectively, which can be handled analogously.

4.A Results from the theory of viscosity solutions

We consider the level-set evolution equation (4.1.5) in the framework of viscosity solutions. The notion of viscosity solutions was developed in the 1980s and provides a framework which yields well-posedness and comparison principles for a wide range of scalar fully nonlinear partial differential equations of second order. For a general introduction see [CIL92] with a detailed history of the development in the notes at the end of sections.

Here, we mainly follow the presentation in [Gig06], which summarizes and provides results for surface evolution equations.

The results presented in this section apply to evolution equations of the form

$$\partial_t u + G(x, \nabla u, D^2 u) = 0 \tag{4.A.1}$$

where $G\colon \mathbb{R}^d \times \big(\mathbb{R}^d \backslash \{0\}\big) \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ satisfies

(G1) (Continuity) $G\colon \mathbb{R}^d \times \big(\mathbb{R}^d \backslash \{0\}\big) \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ is continuous.

(G2) (Degenerate ellipticity) For any $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d \setminus \{0\}$ there holds

$$G(x,\xi,X) \le G(x,\xi,Y) \quad \text{for } X \ge Y,\ X, Y \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

(G3) (Mild singularity) Given the semicontinuous envelopes G^* and G_* as defined below in (4.A.2), for any $x \in \mathbb{R}^d$ there holds $-\infty < G_*(x,0,0) = G^*(x,0,0) < \infty$.

(G4) (Geometricity) G satisfies for all $\lambda > 0$ and $\sigma \in \mathbb{R}$ that

$$\begin{aligned} G(x,\lambda\xi,\lambda X) &= \lambda G(x,\xi,X) \quad \text{for all } x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d \setminus \{0\},\ X \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\\ G(x,\xi,X + \sigma \xi\otimes\xi) &= G(x,\xi,X) \quad \text{for all } x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d \setminus \{0\},\ X \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \end{aligned}$$

In our case, recall that for $(A,F) \in \Omega$ in (4.1.7) we set

$$G_{(A,F)}\colon \mathbb{R}^d \times \big(\mathbb{R}^d \backslash \{0\}\big) \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}, \quad (x,\xi,X) \mapsto -\operatorname{tr}\Big(A\Big(x,\frac{\xi}{|\xi|}\Big)X\Big) + F\Big(x,\frac{\xi}{|\xi|}\Big)|\xi|,$$

which due to the definition of Ω satisfies (G1), (G2), and (G4).

Regarding (G3), given a metric space M with $U \subset M$, for $g\colon U \to \mathbb{R}\cup\{\pm\infty\}$ the upper and lower semicontinuous envelopes are given by

$$\begin{aligned} g^*\colon \overline{U} \to \mathbb{R}\cup\{\pm\infty\}, \quad g^*(x) &\coloneqq \limsup_{r\searrow 0}\big\{g(y) \,:\, y \in U \cap B_r(x)\big\},\\ g_*\colon \overline{U} \to \mathbb{R}\cup\{\pm\infty\}, \quad g_*(x) &\coloneqq \liminf_{r\searrow 0}\big\{g(y) \,:\, y \in U \cap B_r(x)\big\}. \end{aligned} \tag{4.A.2}$$

For example, $G_{(A,F)} = G^*_{(A,F)} = \big(G_{(A,F)}\big)_*$ on $\mathbb{R}^d \times \big(\mathbb{R}^d\backslash\{0\}\big) \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and

$$\begin{aligned} C_{1A}\min\{0,\lambda_1(X)\} \le \big(G_{(A,F)}\big)_*(x,0,X)& \\ \le G^*_{(A,F)}(x,0,X) \le C_{1A}\max\{0,\lambda_d(X)\}& \end{aligned} \tag{4.A.3}$$

for $x \in \mathbb{R}^d$ and $X \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, where $\lambda_1(X) \le \lambda_d(X)$ are the smallest and largest eigenvalue of X respectively. In particular, $G_{(A,F)}$ satisfies (G3).

With respect to the stability of sub- and supersolutions, the natural limit procedure is taking the *upper* and *lower relaxed limits*. For a family of functions $(g_\varepsilon)_{\varepsilon>0}$ on $U \subset M$ with values in $\mathbb{R}\cup\{\pm\infty\}$, these are respectively defined on $\overline{U}$ by

$$\Big(\limsup_{\varepsilon\to 0}{}^* g_\varepsilon\Big)(x) \coloneqq \limsup_{\varepsilon\to 0}\big\{g_\delta(y) \,:\, y \in U\cap B_\varepsilon(x) \text{ and } 0<\delta<\varepsilon\big\}, \tag{4.A.4}$$

$$\Big(\liminf_{\varepsilon\to 0}{}_* g_\varepsilon\Big)(x) \coloneqq \liminf_{\varepsilon\to 0}\big\{g_\delta(y) \,:\, y \in U\cap B_\varepsilon(x) \text{ and } 0<\delta<\varepsilon\big\}. \tag{4.A.5}$$

In order to obtain values for the derivatives necessary to evaluate (4.A.1) given just semi-continuous functions, semijets are introduced. These semijets effectively approximate the behavior of the function around a point with smooth functions from above and from below.

Definition 4.A.1 (Semijets)**.** For an upper semicontinuous function $u\colon U \to \mathbb{R} \cup \{-\infty\}$ with $U \subset \mathbb{R}^d$ locally compact, an element $(\xi, X) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is called a *superjet* of u at some $x_0 \in U$ if $u(x_0)$ is finite and

$$u(x) - u(x_0) \leq \xi \cdot (x - x_0) + \frac{1}{2}(x - x_0)^\intercal X (x - x_0) + o\Big(\big|x - x_0\big|^2\Big)$$

for $x \in U$ and $x \to x_0$. As usual, $o(s)$ denotes a function such that $o(s)/s \to 0$ as $h \to 0$. For the set of all superjets of u at $x_0 \in U$ we write $\mathcal{J}^{2,+}_U u(x_0)$.

For a lower semicontinuous function $u\colon U \to \mathbb{R} \cup \{+\infty\}$ an element $(\xi, X) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a *subjet* of u at $x_0 \in U$ if $(-\xi, -X) \in -\mathcal{J}^{2,+}_U(-u)(x_0)$. For the set of all subjets of u at $x_0 \in U$ we write $\mathcal{J}^{2,-}_U u(x_0)$.

For the proof of Lemma 4.7.2 we use the following property of semijets with respect to characteristic functions.

Lemma 4.A.2. *Let $S, B \subset \mathbb{R}^d$ be closed sets. Let $v_S = 1 - \chi_S$ and $v_{S+B} = 1 - \chi_{S+B}$. Then for any $x_0 \in S$, $y_0 \in \{x_0\} + B$ there holds*

$$\mathcal{J}^{2,-}_{\mathbb{R}^d} v_{S+B}(y_0) \subset \mathcal{J}^{2,-}_{\mathbb{R}^d} v_S(x_0).$$

Proof. We will show that $\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \setminus \mathcal{J}^{2,-}_{\mathbb{R}^d} v_{S+B}(y_0) \supset \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \setminus \mathcal{J}^{2,-}_{\mathbb{R}^d} v_S(x_0)$. Let $(\xi, X) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \setminus \mathcal{J}^{2,-}_{\mathbb{R}^d} v_S(x_0)$. Then there exists a sequence $(x_n)_n \subset S$ and a constant $c > 0$ such that $x_0 = \lim_{n\to\infty} x_n$ but

$$\frac{0 - 0 + \xi \cdot (x_n - x_0) + \frac{1}{2}(x_n - x_0)^\intercal X (x_n - x_0)}{|x_n - x_0|^2} \geq c \quad \text{for all } n \in \mathbb{N}.$$

Note that $y_0 - x_0 \in B$. Hence, we have $y_n \coloneqq x_n + (y_0 - x_0) \in S + B$. The sequence $(y_n)_n$ satisfies $y_0 = \lim_{n\to\infty} y_n$ and

$$\frac{v_{S+B}(y_n) - v_{S+B}(y_0) + \xi \cdot (y_n - y_0) + \frac{1}{2}(y_n - y_0)^\intercal X (y_n - y_0)}{|y_n - y_0|^2} \geq c \quad \text{for all } n \in \mathbb{N}.$$

This implies $(\xi, X) \notin \mathcal{J}^{2,-}_{\mathbb{R}^d} v_{S+B}(y_0)$ and concludes the proof. □

With the semijets providing a surrogate for the derivatives, we can define semicontinuous sub- and supersolutions.

Definition 4.A.3. Let $U \subset \mathbb{R}^d$ be an open set.

(i) $u\colon U \to \mathbb{R} \cup \{-\infty\}$ is a subsolution of $G(x, \nabla u, D^2 u) = 0$ in U if $u^*(x) < \infty$ for $x \in U$ and

$$G_*(x, \xi, X) \leq 0 \quad \text{for all } x \in U \text{ and } (\xi, X) \in \mathcal{J}^{2,+}_U u^*(x).$$

(ii) We call $u\colon U \to \mathbb{R} \cup \{-\infty\}$ a supersolution of $G(x, \nabla u, D^2 u) = 0$ in U if $u_*(x) > -\infty$ for $x \in U$ and

$$G^*(x, \xi, X) \geq 0 \quad \text{for all } x \in U \text{ and } (\xi, X) \in \mathcal{J}^{2,-}_{\mathcal{O}} u_*(x).$$

For equations such as (4.A.1) it is convenient to separate the spatial and time derivatives, using parabolic semijets instead of the standard semijets.

Definition 4.A.4 (Parabolic semijets)**.** Fixing a locally compact set $\mathcal{O} \subset \mathbb{R}^d \times \mathbb{R}$, let $u\colon \mathcal{O} \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function.

(i) An element $(a, \xi, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}}$ is called a *parabolic superjet* of u at some $(x_0, t_0) \in \mathcal{O}$ if

$$u(x,t) - u(x_0,t_0) \le a(t-t_0) + \xi \cdot (x-x_0) + \frac{1}{2}(x-x_0)^\intercal X (x-x_0) + o\Big(|t-t_0| + |x-x_0|^2\Big) \quad \text{for } (x,t) \in \mathcal{O} \text{ as } (x,t) \to (x_0,t_0).$$

The set of all parabolic superjets of u at (x_0, t_0) is denoted with $\mathcal{P}^{2,+}_{\mathcal{O}} u(x_0, t_0)$.

(ii) The closure of the superjets at $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$ is restricted to sequences with a continuous approach of $u(x_0, t_0)$:

$$\overline{\mathcal{P}}^{2,+}_{\mathcal{O}} u(t_0,x_0) := \Big\{(a,\xi,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}} : \text{ there is } (t_j,x_j,a_j,\xi_j,X_j)_{j\in\mathbb{N}} \text{ with } (x_j,t_j,u(x_j,t_j),a_j,\xi_j,X_j) \overset{j\to\infty}{\to} (x_0,t_0,u(x_0,t_0),a_0,\xi_0,X_0) \text{ and } (a_j,\xi_j,X_j) \in \mathcal{P}^{2,+}_{\mathcal{O}} u(x_j,t_j)\Big\}.$$

(iii) We call $(a, \xi, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}}$ a *relaxed parabolic superjet* of u at $(x_0, t_0) \in \mathcal{O}$ if the inequality only holds in the past:

$$u(x,t) - u(x_0,t_0) \le a(t-t_0) + \xi \cdot (x-x_0) + \frac{1}{2}(x-x_0)^\intercal X (x-x_0) + o\Big(|t-t_0| + |x-x_0|^2\Big) \quad \text{for } (x,t) \in \mathcal{O},\ t \le t_0 \text{ as } (x,t) \to (x_0,t_0).$$

The set of all relaxed parabolic superjets of u at (x_0, t_0) is denoted with $\widetilde{\mathcal{P}}^{2,+}_{\mathcal{O}} u(x_0, t_0)$.

The sets of parabolic subjets $\mathcal{P}^{2,-}_{\mathcal{O}} u(x_0,t_0)$, $\overline{\mathcal{P}}^{2,-}_{\mathcal{O}} u(x_0,t_0)$, $\widetilde{\mathcal{P}}^{2,-}_{\mathcal{O}} u(x_0,t_0)$ for lower semicontinuous functions are defined analogously. Additionally, if $\mathcal{O}$ is an open neighborhood of (x_0, t_0), then the semijets are the same as for any other open neighborhood and we might omit $\mathcal{O}$ in the notation.

Since we are only interested in either problems on $\mathbb{R}^d$ or with Dirichlet boundary conditions, we only define sub- and supersolutions in open sets. For compatibility with more complicated boundary conditions see e.g. [Gig06, Definition 2.3.1].

Definition 4.A.5 (Parabolic (Viscosity) Sub- and Supersolutions)**.** Let $T > 0$ and $\mathcal{O} \subset \mathbb{R}^d \times [0, T]$ be an open set.

(i) $u\colon \mathcal{O} \to \mathbb{R} \cup \{-\infty\}$ is a subsolution of (4.A.1) in $\mathcal{O}$ if $u^*(x,t) < \infty$ for $(x,t) \in \mathcal{O}$ and

$$a + G_*(x,\xi,X) \le 0 \quad \text{for all } (x,t) \in \mathcal{O} \text{ and } (a,\xi,X) \in \mathcal{P}^{2,+}_{\mathcal{O}} u^*(x,t).$$

(ii) We call $u\colon \mathcal{O} \to \mathbb{R} \cup \{-\infty\}$ a supersolution of (4.A.1) in $\mathcal{O}$ if $u_*(x,t) > -\infty$ for $(x,t) \in \mathcal{O}$ and

$$a + G^*(x,\xi,X) \ge 0 \quad \text{for all } (x,t) \in \mathcal{O} \text{ and } (a,\xi,X) \in \mathcal{P}^{2,-}_{\mathcal{O}} u_*(x,t).$$

Lemma 4.A.6 (Localization in space). *Let $T > 0$ and $U \subset \mathbb{R}^d$ be open. If u is a subsolution (resp. supersolution) of (4.A.1) in $U \times (0,T)$, then u is a subsolution (resp. supersolution) of (4.A.1) in $\mathcal{O}$ for any open set $\mathcal{O} \subset U \times (0,T)$.*

Proof. The lemma follows immediately via the independence of $\mathcal{P}^{2,+}_{\mathcal{O}} u^*(t,x)$ from $\mathcal{O}$ as long as $\mathcal{O}$ is a neighborhood of (t,x). □

Theorem 4.A.7 (Stability, see Theorem 2.2.1 in [Gig06]). *Let $T > 0$ and $\mathcal{O} \subset \mathbb{R}^d \times (0,T)$ be open. Let $(G_\varepsilon)_{\varepsilon>0}$ and G satisfy (G1) and (G3) with*

$$G_* \leq \liminf_{\varepsilon \to 0}{}_* G_\varepsilon \qquad \Big(\text{resp. } G^* \geq \limsup_{\varepsilon \to 0}{}^* G_\varepsilon\Big) \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

If $(u_\varepsilon)_{\varepsilon>0}$ are subsolutions (resp. supersolutions) in $\mathcal{O}$ of

$$\partial_t u_\varepsilon + G_\varepsilon\big(x, \nabla u, D^2 u\big) = 0,$$

Then $\overline{u} := \limsup_{\varepsilon\to 0}{}^* u_\varepsilon$ (resp. $\underline{u} := \liminf_{\varepsilon\to 0}{}_* u_\varepsilon$) is a subsolution (resp. supersolution) in $\mathcal{O}$ of (4.A.1), as long as $\overline{u} < \infty$ (resp. $\underline{u} > -\infty$) in $\mathcal{O}$.

In general, a comparison principle does not hold for parabolic equations with spatial inhomogeneities in the second order term. We do not state the precise general assumptions, since this would take up too much space. However, in our case a comparison principle holds due to the structure of (4.1.6) and the assumptions for admissible coefficient fields.

Theorem 4.A.8 (Comparison principle for bounded domains, see Corollary 3.6.6 in [Gig06]). *Let $(A,F) \in \Omega$ or $(A,F) = (\widetilde{A},\widetilde{F})_Q$ for some $(\widetilde{A},\widetilde{F}) \in \Omega$ and a closed set $Q \subset \mathbb{R}^d$. Let $T > 0$ and $U \subset \mathbb{R}^d$ be an open and bounded set. Let u be a sub- and v be a supersolution of (4.1.6) in $U \times (0,T)$. If $-\infty < u^* \leq v_* < \infty$ on the parabolic boundary $\big(\overline{U} \times \{0\}\big) \cup (\partial U \times [0,T])$, then $u^* \leq v_*$ in $U \times (0,T)$.*

For unbounded domains, the requirements are more complicated to check.

Theorem 4.A.9 (Comparison principle for unbounded domains). *Let $(A,F) \in \Omega$ or $(A,F) = (\widetilde{A},\widetilde{F})_Q$ for some $(\widetilde{A},\widetilde{F}) \in \Omega$ and a closed set $Q \subset \mathbb{R}^d$. Let $T > 0$ and $U \subset \mathbb{R}^d$ be open. Let u be a sub- and v be a supersolution of (4.1.6) in $U \times (0,T)$. Assume that*

(i) there is some $C > 0$ independent of $(x,t) \in U \times (0,T)$ such that

$$u(x,t) \leq C(|x|+1) \quad \text{and} \quad v(x,t) \geq -C(|x|+1),$$

(ii) there is a modulus m such that

$$u^*(x,t) - v_*(y,t) \leq m(|x-y|) \quad \text{for all } (x,y,t) \in \partial_p\big((U\times U)\times(0,T)\big),$$

where $\partial_p((U\times U)\times(0,T))$ denotes the parabolic boundary of $(U\times U)\times(0,T)$.

(iii) there is some $C > 0$ independent of $(x,y,t) \in \partial_p((U\times U)\times(0,T))$, such that

$$u^*(x,t) - v_*(y,t) \leq C(|x-y|+1) \quad \text{on } \partial_p\big((U\times U)\times(0,T)\big).$$

Then $u^* \leq v_*$ in $U \times (0,T)$.

Proof. See [Gig+91, Theorem 4.2]. The comment in [Gig+91] right below Theorem 4.2 implies that $G_{(A,F)}$ satisfies the conditions for applying Theorem 4.2 when ignoring the first order term. However, it is easy to see that adding the first order term does not break any of the conditions. □

Remark 4.A.10. Sufficient conditions for Assumptions (i), (ii) and (iii) in Theorem 4.A.8 to be satisfied are as follows:

a) If u and v are uniformly bounded, then (i) and (iii) hold.

b) If $U = \mathbb{R}^d$ and $u(\cdot, 0)$ or $v(\cdot, 0)$ is uniformly continuous with $u^* \leq v_*$ on $U \times \{0\}$, then (ii) holds choosing m as the modulus of continuity.

c) If u is uniformly continuous with $u \leq v_*$ on $U \times \{0\}$ and $u \leq \inf_{U\times(0,T)} v$ on $\partial U \times (0, T)$, then (ii) holds choosing m as the modulus of continuity for u.

d) If v is uniformly continuous with $u^* \leq v$ on $U \times \{0\}$ and $v \geq \sup_{U\times(0,T)} u$ on $\partial U \times (0, T)$, then (ii) holds choosing m as the modulus of continuity for v.

A classical ingredient to proving comparison principles is a maximum principle for semicontinuous functions, also known as Ishii's lemma. In particular, this principle is used whenever the comparison principle itself is not applicable, but the machinery behind it still leads to results. Here we use a parabolic version.

Theorem 4.A.11 (Theorem 8.3 in [CIL92]). *Let $n \in \mathbb{N}$. Let $T > 0$, let $d_j \in \mathbb{N}$ and $U_j \subset \mathbb{R}^{d_j}$ be open sets for $j = 1, \ldots, n$, and let $u_j \colon U_j \times (0, T)$ be upper semicontinuous functions. Let $\varphi \colon U_1 \times \ldots \times U_n \times (0, T)$ be once continuously differentiable in t and twice continuously differentiable in $(x_1, \ldots, x_n) \in U_1 \times \ldots \times U_n$.*

Assume that $(t, x_1, \ldots, x_n) \mapsto u_1(x_1, t) + \ldots + u_n(x_n, t) - \varphi(x_1, \ldots, x_n, t)$ has a maximum in $(\widetilde{x}_{,1}, \ldots, \widetilde{x}_n, \widetilde{t})$ and that

$$\begin{aligned} &\text{there exists } r > 0 \text{ such that for all } L > 0 \text{ there is a } C > 0 \text{ such that} \\ &\text{whenever } (a_j, \xi_j, X_j) \in \mathcal{J}^{2,+} u_j(x_j, t) \text{ for } |x_j - \widetilde{x}_j| + |t - \widetilde{t}| \leq r \\ &\text{with } |u_j(x_j, t)| + |\xi_j| + |X_j| \leq L, \text{ then } a_j \leq C. \end{aligned} \tag{4.A.6}$$

Then for each $\gamma > 0$ there are $b_1, \ldots, b_n \in \mathbb{R}$ and $X_j \in \mathbb{R}^{d_j \times d_j}_{\mathrm{sym}}$, $j = 1, \ldots, n$, such that with $Z = D^2_x \varphi(\widetilde{x}_{,1}, \ldots, \widetilde{x}_n, \widetilde{t})$ there hold

$$\begin{aligned} &(i) \quad (b_j, \nabla_{x_j} \varphi(\widetilde{x}_{,1}, \ldots, \widetilde{x}_n, \widetilde{t}), X_j) \in \overline{\mathcal{P}}^{2,+} u(\widetilde{x}_j, \widetilde{t}), \\ &(ii) \quad -\left(\frac{1}{\gamma} + |Z|\right) \mathrm{Id} \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_n \end{pmatrix} \leq Z + \gamma Z^2, \\ &(iii) \quad b_1 + \ldots + b_n = \partial_t \varphi(\widetilde{x}_{,1}, \ldots, \widetilde{x}_n, \widetilde{t}). \end{aligned}$$

Remark 4.A.12. (4.A.6) holds for example whenever u_j is a subsolution or $-u_j$ is a supersolution of a parabolic equation.

If G satisfies a parabolic comparison principle, then replacing the parabolic semijets with the relaxed parabolic semijets from Definition 4.A.4 yields an equivalent definition.

Proposition 4.A.13 (Localization in time). *Let $T > 0$ and $U \subset \mathbb{R}^d$ be an open set. Let $(A, F) \in \Omega$ or $(A, F) = (\widetilde{A}, \widetilde{F})_Q$ for some $(\widetilde{A}, \widetilde{F}) \in \Omega$ and a closed set $Q \subset \mathbb{R}^d$. Then u is a subsolution (resp. supersolution) of (4.1.6) in $U \times (0, T)$ if and only if*

$$\begin{aligned} a + \big(G_{(A,F)}\big)_*(x, \xi, X) \leq 0 \quad &\text{for all } (x, t) \in \mathcal{O} \text{ and } (a, \xi, X) \in \widetilde{\mathcal{P}}^{2,+}_{\mathcal{O}} u^*(x, t) \\ \Big(a + \big(G_{(A,F)}\big)^*(x, \xi, X) \geq 0 \quad &\text{for all } (x, t) \in \mathcal{O} \text{ and } (a, \xi, X) \in \widetilde{\mathcal{P}}^{2,-}_{\mathcal{O}} u_*(x, t)\Big). \end{aligned}$$

Proof. See [Juu01, Theorem 1], which requires only degenerate ellipticity and that a parabolic comparison principle as in Theorem 4.A.8 holds. While the result in [Juu01] is stated for a continuous functional without singularities, the proof works in the same way given a singularity using the semi-continuous envelopes from (4.A.2). While the relaxed parabolic semijets in [Juu01] are defined via test functions, it is easy to see that – as for the usual parabolic semijets – this is equivalent to our formulation, see e.g. [Gig06, Lemma 3.2.7]. □

For sufficiently nice initial data and some further regularity assumptions on G, there exists a unique solution to (4.A.1). We state the result only for $G_{(A,F)}$ to avoid listing all of the extra assumptions. The appropriate space given a metric space M is defined by

$$UC_*(M) \coloneqq \big\{g \in C(M) : (g \wedge L) \vee (-L) \text{ is uniformly continuous for all } L > 0\big\}.$$

Theorem 4.A.14 (Existence and uniqueness of solutions). *Let $T > 0$ and $(A, F) \in \Omega$ or $(A, F) = (\widetilde{A}, \widetilde{F})_Q$ for some $(\widetilde{A}, \widetilde{F}) \in \Omega$ and a closed set $Q \subset \mathbb{R}^d$. Then for every $u_0 \in UC_*(\mathbb{R}^d)$ there exists a unique solution $u \in UC_*(\mathbb{R}^d \times [0, T))$ of (4.1.6) in $\mathbb{R}^d \times (0, T)$ with $u(\cdot, 0) = u_0$.*

Proof. With Theorem 4.A.9, a comparison principle for unbounded domains holds for (4.A.1). With this, the assumptions for [Gig06, Theorem 4.3.5] hold, which yields the unique existence result. □

Remark 4.A.15 (Suitable initial data given a set as initial data). Given a closed set S, a suitable function $u_0 \in UC_*(\mathbb{R}^d)$ with $S = \big\{x \in \mathbb{R}^d : u_0(x) \leq 0\big\}$ is given for example by

$$u_0(x) \coloneqq \operatorname{dist}(x, S).$$

While the existence result requires uniformly continuous initial data, the evolution of a fixed level set is independent of the specific initial data, as long as the level set is preserved.

Theorem 4.A.16 (Invariance, see Theorem 4.2.1 in [Gig06]). *Let $T > 0$. Assume that G satisfies (G1) and (G4). Let $\theta \colon \mathbb{R} \to \mathbb{R}$ be nondecreasing and upper semicontinuous (resp. lower semicontinuous). Let u be a subsolution (resp. supersolution) of (4.A.1) in $\mathbb{R}^d \times (0, T)$. Then $\theta \circ u$ also is a subsolution (resp. supersolution) of (4.A.1) in $\mathbb{R}^d \times (0, T)$.*

Theorem 4.A.17 (Uniqueness of set evolutions, see Theorem 4.2.11 in [Gig06]). *Let $T > 0$ and $(A, F) \in \Omega$ or $(A, F) = (\widetilde{A}, \widetilde{F})_Q$ for some $(\widetilde{A}, \widetilde{F}) \in \Omega$ and a closed set $Q \subset \mathbb{R}^d$. Let u and $\widetilde{u}$ be solutions of (4.1.6) in $\mathbb{R}^d \times (0, T)$ where*

$$\big\{x \in \mathbb{R}^d : u(x, 0) \leq 0\big\} = \big\{x \in \mathbb{R}^d : \widetilde{u}(x, 0) \leq 0\big\}.$$

Then for every $t \in (0, T)$ there holds

$$\big\{x \in \mathbb{R}^d : u(x, t) \leq 0\big\} = \big\{x \in \mathbb{R}^d : \widetilde{u}(x, t) \leq 0\big\}.$$

4.B Proofs for the preliminaries

In this section we collect our proofs for the preliminary properties of the set evolutions, stable sets, restricted coefficient fields and measurability from Section 4.3 and Section 4.5.

4.B.1 Proofs for Section 4.3

Proof of Lemma 4.3.5. Due to Assumption (P1), Assumption (P2), and Assumption (P3) there exists $\widetilde{A} \in C^{0,1}(\mathbb{S}^{d-1}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfying (4.3.1) and (4.3.2) such that

$$\mathbb{P}\Big[(A,F) \in \Omega \,:\, A(x,e) = \widetilde{A}(e) \text{ for all } x \in \mathbb{R}^d, e \in \mathbb{S}^{d-1}\Big] = 1.$$

Assumption (P4)' from Remark 4.3.4 further yields that

$$\widetilde{A}(e) = R^{\intercal}\widetilde{A}(Re)R \qquad \text{for all } R \in \mathrm{SO}(d) \text{ and } e \in \mathbb{S}^{d-1}. \tag{4.B.1}$$

We claim that for any $e \in \mathbb{S}^{d-1}$ there exists $a(e) \geq 0$ such that

$$\widetilde{A}(e) = a(e)\big(Id - e \otimes e\big). \tag{4.B.2}$$

Then (4.B.1) would immediately imply that a is constant, concluding the proof.

It remains to show (4.B.2). Let $e_1 \in \mathbb{S}^{d-1}$. First note that $\widetilde{A}(e_1)$ is positive semi-definite and symmetric due to (4.3.1) and hence has an eigendecomposition with only non-negative eigenvalues. With (4.3.1) we obtain that $\widetilde{A}(e_1)e_1 = 0$ and hence that e_1 is an eigenvector with eigenvalue 0. For $d = 2$, this already implies 4.B.2. For $d \geq 3$, we choose eigenvectors $e_2, \dots, e_d \in \mathbb{S}^{d-1}$ with eigenvalues $\lambda_2, \dots, \lambda_d \geq 0$ such that $e_1, \dots, e_d$ is an orthonormal basis of $\mathbb{R}^d$. For $i \in \{3, \dots, d\}$, let $R \in \mathrm{SO}(d)$ be the rotation given by $Re_j = e_j$ for $j \notin \{2, i\}$, $Re_2 = e_i$ and $Re_i = -e_2$. Then with (4.B.1) we obtain

$$\lambda_2 e_2 = A(e_1)e_2 = R^{\intercal}\widetilde{A}(Re_1)Re_2 = R^{\intercal}\widetilde{A}(e_1)e_i = \lambda_i R^{\intercal}e_i = \lambda_i e_2.$$

Since $i \in \{3, \dots, d\}$ was arbitrary, this implies $\widetilde{A}(e_1) = \lambda_2(Id - e_1 \otimes e_1)$ and thus establishes the claim (4.B.2). □

Proof of Lemma 4.3.9. Let $S_3, S_4 \subset \mathbb{R}^d$. We claim that

$$\text{if } S_3 \subset_h S_4, \text{ then } S_3 + \overline{B}_r \subset S_4 + \overline{B}_{\max\{r,h\}} \text{ for all } r > 0. \tag{4.B.3}$$

Note that for closed sets $S \subset \mathbb{R}^d$ we can rewrite the arrival time from Definition 4.1.6 as

$$m_h^{(A,F)}(x_0, S) = \min\Big\{\min\Big\{t \geq 0 \,:\, x_0 \in \mathscr{R}_t^{(A,F)}(S) + \overline{B}_h\Big\}, T_h(x_0, S)\Big\}.$$

For any $0 \leq t \leq T_h(x_0, S_2)$ with $x_0 \in \mathscr{R}_t^{(A,F)}(S_1) + \overline{B}_h$, we in particular have $x_0 \in \Big(\mathscr{R}_t^{(A,F)}(S_1) \cap \overline{B}_h(x_0)\Big) + \overline{B}_h$ and thus $x_0 \in \mathscr{R}_t^{(A,F)}(S_2) + \overline{B}_h$ due to (4.3.9) and (4.B.3). This implies

$$\begin{aligned} &m_h^{(A,F)}(x_0, S_1) \\ &\qquad \geq \min\Big\{\min\Big\{t \geq 0 \,:\, x_0 \in \mathscr{R}_t^{(A,F)}(S_2) + \overline{B}_h\Big\}, T_h(x_0, S_2), T_h(x_0, S_1)\Big\}. \end{aligned} \tag{4.B.4}$$

Note that for closed sets we can rewrite the distance as

$$\operatorname{dist}(x_0, S) = \min\big\{r \geq 0 \,:\, x_0 \in S + \overline{B}_r\big\}.$$

Thus, if $\operatorname{dist}(x_0, S_1) \leq h$, then (4.B.3) implies that also $\operatorname{dist}(x_0, S_2) \leq h$ and hence $m_h^{(A,F)}(x_0, S_2) = 0$, which yields $m_h^{(A,F)}(x_0, S_1) \geq m_h^{(A,F)}(x_0, S_2)$. Alternatively, if $\operatorname{dist}(x_0, S_1) > h$, then the above characterization of distance and (4.3.9) in combination with (4.B.3) imply that $\operatorname{dist}(x_0, S_1) \geq \operatorname{dist}(x_0, S_2)$ and hence $T_h(x_0, S_1) \geq T_h(x_0, S_2)$. Plugging this into (4.B.4) yields $m_h^{(A,F)}(x_0, S_1) \geq m_h^{(A,F)}(x_0, S_2)$.

It remains to show that (4.B.3) holds. Let $x \in S_3 + \overline{B}_r$ with $y \in S_3$ such that $x - y \in \overline{B}_r$. Without loss of generality assume that $x \in (S_4)^c$. If x is in the same connected component of $(S_4)^c$ as y, then $x \in S_4 + \overline{B}_h$ by Definition 4.3.8 since $S_3 \subset_h S_4$. If x is not in the same connected component, then there exists $z \in S_4$ on the line segment connecting x and y. In particular, there holds $x - z \in \overline{B}_r$ and hence $x \in S_4 + \overline{B}_r$. This establishes (4.B.3) and thus concludes the proof. □

4.B.2 General properties of the set evolutions

Proof of Lemma 4.5.2. Let u_S be as in Definition 4.1.2. Then writing $u^{\mathscr{R}}_{S,(A,F)}$ as

$$u^{\mathscr{R}}_{S,(A,F)}(x,t) = 1 - \chi_{\mathscr{R}_t(S)}(x) = \big(\chi_{\{r\in\mathbb{R}\,:\,r>0\}} \circ u_S\big)(x,t)$$

yields the result by Theorem 4.A.16. □

Proof of Lemma 4.5.3. Let $x_0 \in \mathbb{R}^d$ with $\operatorname{dist}(x_0, S) > \delta + \overline{T} v_{\max,\delta}$, for some $\overline{T} > T$. We will prove the lemma by showing that $x_0 \notin \mathscr{R}_T^{(A,F)}(S)$. Let $u^{\mathscr{R}}_S = 1 - \chi_{\mathscr{R}_\cdot(S)}$ be the supersolution to (4.1.6) from Lemma 4.5.2. A comparable subsolution v of (4.1.6) in $\mathbb{R}^d \times (0, \overline{T})$ is given by a closing hole moving according to the maximum forcing and with a curvature contribution corresponding to the hole of radius δ reached after time $\overline{T}$, see Figure 4.B.1.

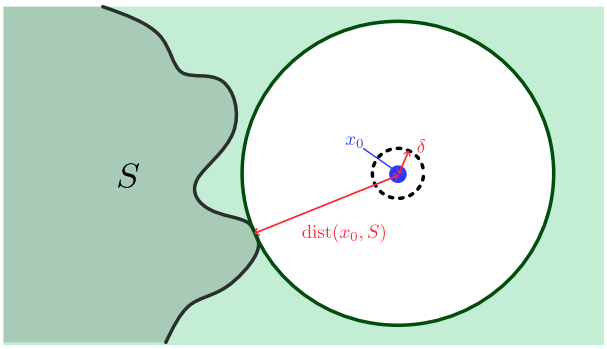


Figure 4.B.1: An illustration of the argument bounding the maximum speed of front propagation. The evolution of a set $S \subset \mathbb{R}^d$ (depicted in grey) is compared to the evolution of a hole (its exterior is depicted in green) around the target point. For the evolution of the hole, the forcing is set to its maximum value

That is, v is given by

$$v(x,t) \coloneqq \chi_{(\delta + (\overline{T}-t)v_{\max,\delta})B_1(x_0)}(x),$$

where we use that on the one hand $0 \leq A \leq C_{1A}$ and $|F| \leq \max\{C_{1F}, 6C_{1A}\}$, while on the other hand the interface has 'curvature' larger or equal to δ^{-1} for $t \leq \overline{T}$. Relaxing the estimate $|F| \leq C_{1F}$ to $|F| \leq \max\{C_{1F}, 6C_{1A}\}$ covers the case that (A, F) is the restriction of an admissible coefficient field, see Definition 4.5.8. Note that $u^{\mathscr{R}}_S(\cdot, 0) \geq v(\cdot, 0)$ since $\operatorname{dist}(x_0, S) > \delta + \overline{T} v_{\max,\delta}$. Now applying the comparison principle (see Theorem 4.A.8) to the supersolution $u^{\mathscr{R}}_S$ and the subsolution v on the ball $(\delta + \overline{T} v_{\max,\delta})B_1(x_0)$ yields $\mathscr{R}_t(S) \subset \big((\delta + (\overline{T} - t)v_{\max,\delta})B_1(x_0)\big)^c$ for all $t < \overline{T}$. □

4.B.3 Properties of stable sets

Proof of Lemma 4.5.4. Let $T > 0$. We first show the non-shrinking property and then the preservation of stability for any $t < T$. Since T is arbitrary, this yields the properties for all $t > 0$. The idea of the proof is to take the union of all sets up to a certain time. We show that this is a supersolution, which due to the comparison principle means that original evolution of the set has to be at least as large and thus can not shrink.

Step 1: Non-shrinking. Let u_S be as in Definition 4.1.2, without loss of generality assume $u_S = \min\{1, u_S\}$. Let $u_S^{\mathscr{R}} = 1 - \chi_{\mathscr{R}_\cdot(S)}$ be the supersolution to (4.1.6) from Lemma 4.5.2. We claim that $\widetilde{u}$ given by $\widetilde{u}(x,t) \coloneqq \inf_{s\in[0,t]} u_S^{\mathscr{R}}(x,s)$ is also a supersolution of (4.1.6). If this was the case, then we can apply the comparison principle (see Theorem 4.A.9) to $\widetilde{u}$ and u_S on $\mathbb{R}^d \times (0,T)$ with Remark 4.A.10b). Since by definition $\widetilde{u}(\cdot,0) \geq u_S(\cdot,0)$, this yields $\widetilde{u} \geq u_S$. Hence,

$$\mathscr{R}_t(S) = \{x : u_S(x,t) \leq 0\} \supset \{x : \widetilde{u}(x,t) \leq 0\} \supset \{x : u_S^{\mathscr{R}}(x,t) \leq 0\} = \mathscr{R}_t(S)$$

and therefore $\mathscr{R}_t(S) = \{x : \widetilde{u}(x,t) \leq 0\}$, in particular $\mathscr{R}_s^{(A,F)}(S) \subset \mathscr{R}_t^{(A,F)}(S)$ for any $0 \leq s \leq t < T$ by the definition of $\widetilde{u}$.

It remains to show that indeed $\widetilde{u}$ is a supersolution. Let $(a,\xi,X) \in \mathcal{P}^{2,-}\widetilde{u}(x_0,t_0)$ for some $(x_0,t_0) \in \mathbb{R}^d \times (0,T)$. There exists $0 \leq t_1 \leq t_0$ such that $\widetilde{u}(x_0,t_0) = u_S^{\mathscr{R}}(x_0,t_1)$.
Case 1: $t_1 > 0$. Since $\widetilde{u}$ is non-increasing and by the definition of $\mathcal{P}^{2,-}\widetilde{u}(x_0,t_0)$,

$$\begin{aligned} u_S^{\mathscr{R}}(x_0+x,t_1+t) &\geq \widetilde{u}(x_0+x,t_1+t) \geq \widetilde{u}(x_0+x,t_0+t) \\ &\geq u_S^{\mathscr{R}}(x_0,t_1) + at + \xi\cdot x + \frac{1}{2}x^\intercal X x + o\Big(|t| + |x|^2\Big) \end{aligned}$$

and therefore $(a,\xi,X) \in \mathcal{P}^{2,-}u_S^{\mathscr{R}}(x_0,t_1)$. Because $u_S^{\mathscr{R}}$ is a supersolution, we obtain

$$a + G^*_{(A,F)}(x_0,\xi,X) \geq 0.$$

Case 2: $t_1 = 0$. By the definition of $\widetilde{u}$, we have $u_S^{\mathscr{R}}(x_0,0) = \widetilde{u}(x_0,t)$ for all $t \leq t_0$. On the one hand, by the definition of the parabolic subjet this implies $a \geq 0$. On the other hand, since $\widetilde{u}$ is non-increasing,

$$u_S^{\mathscr{R}}(x_0+x,0) \geq \widetilde{u}(x_0+x,t_0) \geq u_S^{\mathscr{R}}(x_0,0) + \xi\cdot x + \frac{1}{2}x^\intercal X x + o\Big(|x|^2\Big),$$

and thus $(\xi,X) \in \mathcal{J}^{2,-}\big(u_S^{\mathscr{R}}(\cdot,0)\big)(x_0)$. Since S is stable, by definition $u_S^{\mathscr{R}}(\cdot,0)$ is a supersolution of (4.3.7), which yields

$$0 \leq G^*_{(A,F)}(x_0,\xi,X) \leq a + G^*_{(A,F)}(x_0,\xi,X).$$

Step 2: Preservation of stability. We need to show that $u_S^{\mathscr{R}}(\cdot,t_0)$ is a supersolution of (4.3.7). Let $(\xi,X) \in \mathcal{J}^{2,-}(u(\cdot,t_0))(x_0)$ for some $x_0 \in \mathbb{R}^d$. Then $(0,\xi,X) \in \widetilde{\mathcal{P}}^{2,-}u(x_0,t_0)$, where $\widetilde{\mathcal{P}}^{2,-}u(x_0,t_0)$ is the relaxed parabolic superjet introduced in Definition 4.A.4: Since $t \mapsto u_S^{\mathscr{R}}(x,t)$ is non-increasing by Step 1, for $t \leq 0$ we have

$$u_S^{\mathscr{R}}(x_0+x,t_0+t) \geq u_S^{\mathscr{R}}(x_0+x,t_0) \geq u_S^{\mathscr{R}}(x_0,t_0) + \xi\cdot x + \frac{1}{2}x^\intercal X x + o\Big(|x|^2\Big).$$

Since $u_S^{\mathscr{R}}$ is a supersolution (by Proposition 4.A.13 also with relaxed superjets), we obtain

$$0 + G^*_{(A,F)}(x_0,\xi,X) \geq 0.$$

Hence, $u_S^{\mathscr{R}}(\cdot,t_0) = 1 - \chi_{\mathscr{R}_{t_0}(S)}$ is a supersolution to (4.3.7) and thus $\mathscr{R}_{t_0}(S)$ is stable. □

Proof of Lemma 4.5.5. We first prove (a). Let $T > 0$. Let $u^{\mathscr{R}}_{S_1}(x,t) = 1 - \chi_{\mathscr{R}_t(S_1)}(x)$ be the supersolution of (4.1.6) from Lemma 4.5.2. Let $u_{S_2} \colon \mathbb{R}^d \times [0,\infty) \to [0,1]$ be a uniformly continuous solution of (4.1.6) on $\mathbb{R}^d \times (0,T)$ with $S_2 = \{x \in \mathbb{R}^d \,:\, u_{S_2}(x,0) \leq 0\}$ as in Definition 4.1.2 and hence $\mathscr{R}_t(S_2) = \{x \in \mathbb{R}^d \,:\, u_{S_2}(x,t) \leq 0\}$. We now apply Theorem 4.A.9, the comparison principle for unbounded domains, with Remark 4.A.10a) and b) to $v = u^{\mathscr{R}}_{S_1}$ and $u = u_{S_2}$: Since $S_1 \subset S_2$, we have $u_{S_2} \leq u^{\mathscr{R}}_{S_1}$ on $\mathbb{R}^d \times \{0\}$ and hence the comparison yields $u^{\mathscr{R}}_{S_1} \geq u_{S_2}$ on $\mathbb{R}^d \times (0,T)$. Since $T > 0$ was arbitrary, comparing the 0-sublevel sets implies $\mathscr{R}_t(S_1) \subset \mathscr{R}_t(S_2)$ for all $t \geq 0$.

Regarding (b), let $\widetilde{S}_2$ be S_2 with all holes of width smaller than $2h$ removed:

$$\widetilde{S}_2 \coloneqq S_2 \cup \bigcup \{H \,:\, H \text{ is a connected component of } (S_2)^c \text{ with } H \subset S_2 + \overline{B}_h\}.$$

By the definition of '$\subset_h$', we have $\widetilde{S}_2 \subset_h S_2 \subset \mathscr{R}_t(S_2)$ for all $t \geq 0$ using Lemma 4.5.4 since S_2 is stable. If we can show that $\mathscr{R}_t(S_1) \cap (\widetilde{S}_2)^c \subset \mathscr{R}_t(S_2)$ for all $t \geq 0$, then we are done. This follows by applying Theorem 4.A.9 to $u^{\mathscr{R}}_{S_1}$ and u_{S_2} as in the first part of the proof, only now not on $\mathbb{R}^d \times (0,T)$ but on $(\widetilde{S}_2)^c \times (0,T)$. Since S_2 is stable, Lemma 4.5.4 yields

$$u_{S_2} = 0 \leq \inf_{\mathbb{R}^d \times (0,\infty)} u^{\mathscr{R}}_{S_1} \quad \text{on} \quad \partial S_2 \times [0,T] \supset \partial(\widetilde{S}_2)^c \times [0,T].$$

Hence with Remark 4.A.10a) and c), the conditions for Theorem 4.A.9 hold and we obtain $u^{\mathscr{R}}_{S_1} \geq u_{S_2}$ on $(\widetilde{S}_2)^c \times (0,T)$ and thus $\mathscr{R}_t(S_1) \cap (\widetilde{S}_2)^c \subset \mathscr{R}_t(S_2)$ for all $t \geq 0$, which concludes the proof. □

Proof of Lemma 4.5.6. We prove the Lemma by induction, iteratively applying the effective minimum speed of propagation from Definition 4.3.8 in combination with the comparison principle from Lemma 4.5.5. For $n = 0$ we clearly have $S + n\overline{B}_h \subset_h S_n \coloneqq \mathscr{R}^{(A,F)}_{nv^{-1}_{\min}h}(S)$. Assume that $S + (n-1)\overline{B}_h \subset_h S_{n-1}$ for some $n \in \mathbb{N}$. Due to the definition of "$\subset_h$", see Definition 4.3.8, for each $x \in S + n\overline{B}_h$ we either already have $\{x\} \subset_h S_{n-1}$ or there exists $y \in S_{n-1} \cap \left(S + n\overline{B}_h\right)$ on the shortest path connecting x to $S + (n-1)\overline{B}_h$, see Figure 4.B.2. Either due to the stability of S or due to the effective minimum speed and $x \in \overline{B}_h(y) \subset (S_{n-1} \cap U) + \overline{B}_h$ we hence obtain $\{x\} \subset_h \mathscr{R}^{(A,F)}_{c_s h + nv^{-1}_{\min}h}(S)$ and thus $S + n\overline{B}_h \subset_h S_n$, concluding the proof. □

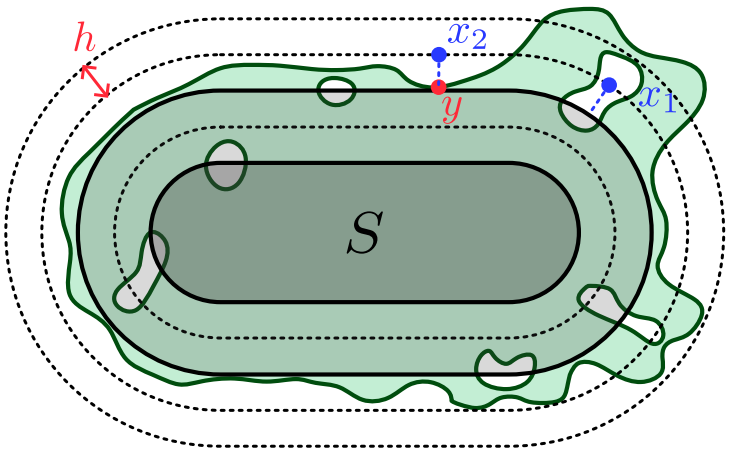


Figure 4.B.2: An illustration of a setup for the iterative application of the effective minimum speed. The set S_{n-1} (depicted in green) h-envelopes $S + (n-1)\overline{B}_h$ (depicted in light grey). For each point $x \in S + n\overline{B}_h$, either $\{x\} \subset_h S_{n-1}$ or there exists $y \in S_{n-1}$ on the shortest path connecting x to $S + (n-1)\overline{B}_h$, exemplified for x_1 and x_2 (both depicted in blue)

Proof of Lemma 4.5.7. The proof is a rather straightforward application of Lemma 4.5.6 using the properties which Assumption (P⊙) guarantees for $(A,F) \in E_h^{\odot}(U)$. Since by assumption $\partial S \subset U$, for $(A,F) \in E_h^{\odot}(U)$ there exists a stable (h,c_s)-approximation S_h with $S_h \subset S \subset_h \mathscr{R}^{(A,F)}_{c_s h}(S_h)$, for which (A,F) admits $v_{\min}$ as an effective minimum speed of propagation on the scale h in U. Since $S_1 \subset_h S_2$ implies $S_1 + \overline{B}_h \subset S_2 + \overline{B}_h$ as in (4.B.3), with Lemma 4.5.6 we obtain

$$S + n\overline{B}_h \subset \mathscr{R}^{(A,F)}_{c_s h}(S_h) + n\overline{B}_h \subset_h \mathscr{R}^{(A,F)}_{c_s h + n v_{\min}^{-1} h}(S_h)$$

for $n \in \mathbb{N}$ as long as $(S + n\overline{B}_h) \cap (\partial S \cup S^c) \subset U \cup S$. In particular, due to the assumption on U and using that the evolution of the stable set S_h is increasing as in Lemma 4.5.4 to not round down we have

$$S + \left\lfloor \frac{\operatorname{dist}(x_0,S)}{h} \right\rfloor \overline{B}_h \subset_h \mathscr{R}^{(A,F)}_{c_s h + \operatorname{dist}(x_0,S) v_{\min}^{-1}}(S_h) \subset \mathscr{R}^{(A,F)}_{c_s h + \operatorname{dist}(x_0,S) v_{\min}^{-1}}(S),$$

where the last inclusion is due to $S_h \subset S$ and the comparison principle from Lemma 4.5.5. This implies

$$x_0 \in S + \frac{\operatorname{dist}(x_0,S)}{h}\overline{B}_h \subset_h \mathscr{R}^{(A,F)}_{c_s h + \operatorname{dist}(x_0,S) v_{\min}^{-1}}(S) + \overline{B}_h$$

and hence $m_h^{(A,F)}(x_0,S) \le c_s h + \operatorname{dist}(x_0,S) v_{\min}^{-1} = T_h(x_0,S)$. □

4.B.4 Properties of restricted coefficient fields

Proof of Lemma 4.5.10. Without loss of generality, assume that $C = 1$ and $u \ge 0$. The result for general u then follows since $e^{u-C} \colon \mathbb{R}^d \times [0,\infty) \to [0,1]$ has the same level sets and is a supersolution due to Theorem 4.A.16. For $0 < \varepsilon < \frac{1}{2}$ we denote

$$Q_\varepsilon^{\oslash} := \bigcup \left\{ \overline{B}_{\frac{1-\varepsilon}{3}}(x) \,:\, \operatorname{dist}(x,Q) \ge \frac{2}{3} \right\}.$$

Note that $Q^{\oslash} = \bigcup_{0<\varepsilon<1} Q_\varepsilon^{\oslash}$. We claim that for all $0 < \varepsilon < \frac{1}{2}$

$$v_\varepsilon(\cdot,t) := \chi_{Q_\varepsilon^{\oslash}} \text{ is a stationary subsolution of (4.1.6) with coefficients } (A,F)_Q. \tag{4.B.5}$$

Assuming that (4.B.5) holds, we apply Theorem 4.A.9, the comparison principle for unbounded domains, to u and v_ε on $\mathbb{R}^d \times (0,T)$ for any $T > 0$. The conditions (i) and (iii) from Theorem 4.A.9 are satisfied since u and v_ε are bounded. Regarding condition (ii), note that whenever $v_\varepsilon(x,0) = 1$, then $v_\varepsilon(x,0) = 1 \le u(y,0)$ for all $y \in B_{\frac{\varepsilon}{6}}(x)$, because

$$\operatorname{dist}\big(\{x \,:\, v_\varepsilon(x,0) = 1\}, \{x \,:\, u(x,0) \ne 1\}\big) \ge \operatorname{dist}\big(Q_\varepsilon^{\oslash}, (Q^{\oslash})^c\big) \ge \frac{\varepsilon}{3}.$$

Hence, Theorem 4.A.9 yields $v_\varepsilon \le u$ on $\mathbb{R}^d \times [0,\infty)$ for any $0 < \varepsilon < \frac{1}{2}$ and therefore

$$\emptyset = \{x \,:\, u(\cdot,t) < 1\} \cap \bigcup_{0<\varepsilon<1} Q_\varepsilon^{\oslash} = \{x \,:\, u(\cdot,t) < C\} \cap Q^{\oslash}.$$

Now, with respect to u being a supersolution to (4.1.6) with coefficients $(A,F)_{\widetilde{Q}}$, by the definition of the restricted coefficient fields we have $G_{(A,F)_Q} \le G_{(A,F)_{\widetilde{Q}}}$ on $Q + \overline{B}_{\frac{2}{3}}$. Hence, for $t_0 > 0$ and $x_0 \in Q + \overline{B}_{\frac{2}{3}}$ and $(a,\xi,X) \in \mathcal{J}^{2,-} u^{\mathscr{R}}_{S,(A,F)_Q}(x_0,t_0)$, we obtain

$$a + G^*_{(A,F)_{\widetilde{Q}}}(x_0,\xi,X) \ge a + G^*_{(A,F)_Q}(x_0,\xi,X) \ge 0$$

because u is a supersolution to (4.1.6) with coefficients $(A,F)_Q$. For $t_0>0$ and $x_0\notin Q+\overline{B}_{\frac{2}{3}}$, we know that $u=C$ in a neighborhood of (x_0,t_0) since $\{x:u(\cdot,t)<C\}\subset (Q^{\oslash})^c\subset Q+\overline{B}_{\frac{2}{3}}$ for all $t>0$. Hence, if $(a,\xi,X)\in\mathcal{J}^{2,-}u^{\mathscr{R}}_{S,(A,F)_Q}(x_0,t_0)$, then $(a,\xi,X)=(0,0,X)$ with $X\le 0$ and thus $a+G^*_{(A,F)_{\widetilde{Q}}}(x_0,\xi,X)\ge 0$.

Thus, we have shown that u is a supersolution to (4.1.6) with coefficients $(A,F)_{\widetilde{Q}}$. We obtain (a) and (c) by choosing $u=u^{\mathscr{R}}_{S,(A,F)_Q}$ as in Lemma 4.5.2. We obtain (b) by plugging in the stationary supersolution $u(x,t)=1-\chi_S(x)$.

It remains to prove Claim (4.B.5): For $(x_0,t_0)\notin\partial Q_\varepsilon^{\oslash}\times(0,\infty)$, there is a neighborhood of (x_0,t_0) on which v_ε is constant. Hence, if $(a,\xi,X)\in\mathcal{P}^{2,+}v_\varepsilon(x_0,t_0)$, then we have $(a,\xi,X)=(0,0,X)$ with $X\ge 0$ and thus $a+\big(G_{(A,F)_Q}\big)_*(x_0,\xi,X)\le 0$.

For $(x_0,t_0)\in\partial Q_\varepsilon^{\oslash}\times(0,\infty)$, we know that $v_\varepsilon(x_0,t)=1$, $\frac{1}{3}\le\operatorname{dist}(x_0,Q)\le\frac{2}{3}$ and that there exists x_1 such that $x_0\in\partial B_{\frac{1-\varepsilon}{3}}(x_1)$ and $v_\varepsilon=1$ on $\overline{B}_{\frac{1-\varepsilon}{3}}(x_1)$. Hence, a superjet $(a,\xi,X)\in\mathcal{P}^{2,+}v_\varepsilon(x_0,t_0)$ has to satisfy $a=0$. If $\xi=0$, then by approaching x_0 from within $B_{\frac{1}{3}}(x_1)$ we obtain $X\ge 0$ and thus $a+\big(G_{(A,F)_Q}\big)_*(x_0,\xi,X)\le 0$, exactly as for $(x_0,t_0)\notin\partial Q^{\oslash}\times(0,\infty)$.

If $\xi\neq 0$, we know that $\xi\cdot\nu\ge 0$, where $\nu:=\frac{3}{1-\varepsilon}(x_1-x_0)$. Let $\nu^\perp$ be a normal orthogonal component of ν. We approach x_0 along $\partial B_{\frac{1-\varepsilon}{3}}(x_1)$ with

$$s\mapsto x_0+s\nu^\perp+f(s)\nu,\qquad\text{where}\qquad f(s):=\frac{1-\varepsilon}{3}\left(1-\sqrt{1-\left(\frac{3}{1-\varepsilon}s\right)^2}\right).$$

Since v_ε is constant on the ball, by the definition of $\mathcal{P}^{2,+}v_\varepsilon(x_0,t_0)$, we have

$$0\le\lim_{s\to 0}\frac{1}{s^2}\left(\xi\cdot\big(s\nu^\perp+f(s)\nu\big)+\frac{1}{2}\big(s\nu^\perp+f(s)\nu\big)^{\intercal}X\big(s\nu^\perp+f(s)\nu\big)\right).$$

Note that $f(s)=\frac{3}{2(1-\varepsilon)}s^2+\mathcal{O}(|s|^3)$ for $s\to 0$. Comparing the coefficients for the different powers of s, this yields $\xi\cdot\nu^\perp=0$ (thus $\xi=|\xi|\nu$) and further

$$0\le\frac{3}{1-\varepsilon}\xi\cdot\nu+\nu^\perp\cdot X\nu^\perp=\frac{3}{1-\varepsilon}|\xi|+\nu^\perp\cdot X\nu^\perp.$$

Since $\nu^\perp$ was arbitrary and because $\frac{\xi}{|\xi|}=\nu$, we obtain

$$\left(\mathrm{Id}_{d\times d}-\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)X\left(\mathrm{Id}_{d\times d}-\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)\ge-\frac{3}{1-\varepsilon}|\xi|.\tag{4.B.6}$$

Using that $\xi\neq 0$, that $G_{(A,F)_{\widetilde{Q}}}$ is strongly geometric due to (4.3.2) and [Gig06, Theorem 1.6.12 and Lemma 1.6.2], we obtain

$$\begin{aligned}G^*_{(A,F)_{\widetilde{Q}}}(x_0,\xi,X)&=G_{(A,F)_{\widetilde{Q}}}\left(x_0,\xi,\left(\mathrm{Id}-\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)X\left(\mathrm{Id}-\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)\right)\\&=-\operatorname{tr}\left(A(x_0,\nu)\left(\mathrm{Id}-\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)X\left(\mathrm{Id}-\frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}\right)\right)+F_Q(x_0,\nu)|\xi|\\&\le\frac{3}{1-\varepsilon}C_{1A}|\xi|-\max\{C_{1F},\,6C_{1A}\}|\xi|\le 0\end{aligned}$$

where in the last step we used (4.B.6) and that A is nonnegative definite due to (4.3.1), the definition of F_Q from (4.5.2) and that $\operatorname{dist}(x_0,Q)\in\left[\frac{1}{3},\frac{2}{3}\right]$ as well as $0<\varepsilon<\frac{1}{2}$. □

Proof of Lemma 4.5.11. $\mathscr{R}_t^{(A,F)_Q}(S) \subset \mathscr{R}_t^{(A,F)_{\widetilde{Q}}}(S)$: Let $\widetilde{u}_S\colon \mathbb{R}^d\times[0,\infty)\to[0,1]$ be a uniformly continuous solution of (4.1.6) for $(A,F)_{\widetilde{Q}}$, with $S=\{x\in\mathbb{R}^d : \widetilde{u}_S(x,0)\le 0\}$ as in Definition 4.1.2. From Lemma 4.5.10 we know that $u_S^{\mathscr{R}}(x,t)=1-\chi_{\mathscr{R}_t^{(A,F)_Q}(S)}(x)$ is a supersolution to (4.1.6) for coefficients $(A,F)_{\widetilde{Q}}$. Since $u_S^{\mathscr{R}}\ge\widetilde{u}_S$ on $\mathbb{R}^d\times\{0\}$, with Remark 4.A.10a) and b) we can apply the comparison principle Theorem 4.A.9 and obtain $u_S^{\mathscr{R}}\ge\widetilde{u}_S$ on $\mathbb{R}^d\times(0,\infty)$. Comparing the 0-sublevel sets, we obtain $\mathscr{R}_t^{(A,F)_Q}(S)\subset\mathscr{R}_t^{(A,F)_{\widetilde{Q}}}(S)$.

$\mathscr{R}_t^{(A,F)_Q}(S)\supset\mathscr{R}_t^{(A,F)_{\widetilde{Q}}}(S)$: Let u_S be be a uniformly continuous solution of (4.1.6) for coefficients $(A,F)_Q$ satisfying $S=\{x\in\mathbb{R}^d : u_S(x,0)\le 0\}$ as in Definition 4.1.2. Since u_S is uniformly continuous and for all $0\le t\le t_0$ there holds

$$Q\supset\mathscr{R}_t^{(A,F)_Q}\big(S\big)+B_\varepsilon=\big\{x\in\mathbb{R}^d : u_S(x,t)\le 0\big\}+B_\varepsilon,$$

there exists $C>0$, such that $\{x\in\mathbb{R}^d : u_S(x,t)\le C\}\subset Q$ for all $0\le t\le t_0$. Without loss of generality, assume that $C=1$ and $0\le u_S\le 1$ (otherwise use Theorem 4.A.16 to continue with $(0\vee C^{-1}u_S)\wedge 1$ instead).

We claim that u_S is a subsolution to (4.1.6) not just for the coefficients $(A,F)_Q$, but also $(A,F)_{\widetilde{Q}}$: For $(x_0,t_0)\in Q\times(0,\infty)$, if $(a,\xi,X)\in\mathcal{P}^{2,+}u_S(x_0,t_0)$, we have

$$a+\big(G_{(A,F)_{\widetilde{Q}}}\big)_*(x_0,\xi,X)=a+\big(G_{(A,F)_Q}\big)_*(x_0,\xi,X)\le 0$$

since u_S is a solution of (4.1.6) for coefficients $(A,F)_Q$ and $(A,F)_{\widetilde{Q}}=(A,F)_Q$ on $\mathbb{S}^{d-1}\times Q$. For $(x_0,t_0)\in Q^c\times(0,\infty)$, there is a neighborhood of (x_0,t_0) on which $u_S=1$. Hence, if $(a,\xi,X)\in\mathcal{P}^{2,+}u_S(x_0,t_0)$, then we have $(a,\xi,X)=(0,0,X)$ with $X\ge 0$ and thus in this case we also have $a+\big(G_{(A,F)_{\widetilde{Q}}}\big)_*(x_0,\xi,X)\le 0$.

Let $\tilde{u}_S^{\mathcal{R}}=1-\chi_{\mathscr{R}_t^{(A,F)_{\widetilde{Q}}}(S)}(x)$ be the supersolution of (4.1.6) for coefficients $(A,F)_{\widetilde{Q}}$ from Lemma 4.5.2. Since $\tilde{u}_S^{\mathcal{R}}\ge u_S$ on $\mathbb{R}^d\times\{0\}$, via the comparison principle – Theorem 4.A.9 and Remark 4.A.10a) and b) – we obtain $\tilde{u}_S^{\mathcal{R}}\ge u_S$ on $\mathbb{R}^d\times(0,\infty)$ and hence finish the proof by comparing the 0-sublevel sets. □

4.B.5 Measurability criterion

Proof of Lemma 4.5.12. Note that Ω with the topology induced by $\|\cdot\|_{C_w(U\times\mathbb{S}^{d-1},\mathbb{R}^{d\times d}\times\mathbb{R})}$ is separable due to the uniform Lipschitz bounds from (4.3.1) and (4.3.3). It is straightforward to show that any subset of a separable metric space is again separable: For each element of the original dense set take a sequence inside the subset with the distance to the point converging to the distance between the point and the subset, which combined yields a dense countable set in the subset. In particular, E is separable with respect to the topology induced by $\|\cdot\|_{C_w(U\times\mathbb{S}^{d-1},\mathbb{R}^{d\times d}\times\mathbb{R})}$. For brevity, in this proof we write $C_w(U\times\mathbb{S}^{d-1})$ for $C_w(U\times\mathbb{S}^{d-1},\mathbb{R}^{d\times d}\times\mathbb{R})$. Since $E\subset\Omega$ is closed with respect to $\|\cdot\|_{C_w(U\times\mathbb{S}^1)}$, there exist $((A_k,F_k))_k\in\mathbb{N}\subset E$ such that

$$E=\overline{\bigcup_{k\in\mathbb{N}}\{(A_k,F_k)\}}^{C_w(U\times\mathbb{S}^1)}\cap\Omega=\bigcap_{\ell\in\mathbb{N}}\left(\bigcup_{k\in\mathbb{N}}\overline{B}_{\frac{1}{\ell}}^{C_w(U\times\mathbb{S}^1)}\big((A_k,F_k)\big)\cap\Omega\right).$$

This shows that E is measurable with respect to $\mathcal{F}(U)$, since $\overline{B}_\delta^{C_w(U\times\mathbb{S}^1)}\big((\widetilde{A},\widetilde{F})\big)\cap\Omega$ is $\mathcal{F}(U)$-measurable for any $\delta>0$ and $(\widetilde{A},\widetilde{F})\in\Omega$, which can be seen as follows. Let

$((x_k, e_k))_{k\in\mathbb{N}}$ be dense in $U \times \mathbb{S}^1$, again using that subsets of separable space are separable. Using continuity, we have

$$\begin{aligned}
&\overline{B}_\delta^{C_w(U\times\mathbb{S}^1)}\Big((\widetilde{A},\widetilde{F})\Big) \cap \Omega \\
&\quad = \Big\{(A,F)\in\Omega \,:\, \Big|(A,F)(x,e) - (\widetilde{A},\widetilde{F})(x,e)\Big| \le \delta\Big(1+|x|^2\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } (x,e)\in U\times\mathbb{S}^1\Big\} \\
&\quad = \bigcap_{k\in\mathbb{N}}\Big\{(A,F)\in\Omega \,:\, \Big|(A,F)(x_k,e_k) - (\widetilde{A},\widetilde{F})(x_k,e_k)\Big| \le \delta\Big(1+|x_k|^2\Big)\Big\},
\end{aligned}$$

which clearly is measurable with the definition of $\mathcal{F}(U)$ in (4.3.5). □

4.C Azuma's inequality – a generalization and an alternative

In Subsection 4.C.1, we introduce and prove our alternative for Azuma's inequality, which we cited in Proposition 4.4.4 and which provides fluctuation bounds for martingales if the increments based on uniform bounds for the increments. In contrast, our results only require uniform bounds on a set of very large probability. Compared to Azuma's inequality, our alternative only yields exponential instead of Gaussian tail bounds.

While it is possible to generalize Azuma's inequality to martingales with only likely bounded increments, this does not provide meaningful Gaussian fluctuation bounds in our setting – at least if it was possible to obtain such bounds, a different strategy would be required. In order to illustrate this, we provide a concentration inequality with Gaussian tail bounds in Subsection 4.C.2, generalizing Azuma's inequality. We further provide the probability bounds which we would obtain in the setting of Proposition 4.6.1. However, in this setting the probability of the uniform bounds holding is not strong enough to recover good fluctuation bounds in the same regime as with the alternative from Subsection 4.C.1.

4.C.1 An alternative to Azuma's inequality

Lemma 4.C.1. *Let $(\Omega,\mathcal{G},\mathbb{P})$ be a probability space. Let $(\mathcal{G}_n)_{n\in\mathbb{N}_0}\subset\mathcal{G}$ be a filtration. Then for any $E\in\mathcal{G}$ and $\delta>0$ there holds*

$$\mathbb{P}\big[\{\omega\in\Omega \,:\, \exists n\in\mathbb{N}_0 \,:\, \mathbb{E}[\chi_E\,|\,\mathcal{G}_n](\omega)\ge\delta\}\big] \le \delta^{-1}\mathbb{P}\big[E\big].$$

Proof. For $n\in\mathbb{N}_0$ we set

$$E_n := \big\{\omega\in\Omega \,:\, \mathbb{E}[\chi_E\,|\,\mathcal{G}_n](\omega)\ge\delta\big\}\in\mathcal{G}_n, \qquad \widetilde{E}_n := E_n \setminus \bigcup_{i=0}^{n-1} E_i \in \mathcal{G}_n.$$

Due to the measurability of these events, we obtain

$$\mathbb{P}\big[E\big] = \mathbb{E}\big[\chi_E\big] \ge \sum_n \mathbb{E}\Big[\chi_E\chi_{\widetilde{E}_n}\Big] = \sum_n \mathbb{E}\Big[\mathbb{E}[\chi_E\,|\,\mathcal{G}_n]\chi_{\widetilde{E}_n}\Big] \ge \sum_n \delta\mathbb{P}\Big[\widetilde{E}_n\Big] = \delta\mathbb{P}\Big[\bigcup_n \widetilde{E}_n\Big].$$

Since $\mathbb{P}[\{\omega\in\Omega \,:\, \exists n\in\mathbb{N}_0 \,:\, \mathbb{E}[\chi_E\,|\,\mathcal{G}_n](\omega)\ge\delta\}] = \mathbb{P}[\bigcup_n E_n]$, this concludes the proof. □

We now introduce our alternative to Azuma's inequality. Assuming that we would have uniform bounds on the martingale increments this result would be strictly weaker than the original cited in Proposition 4.4.4: While we obtain the same typical size for the

fluctuations, the stochastic integrability is worse since we only have exponential instead of Gaussian tail bounds. However, we can afford the probability of the set on which we do not have uniform bounds to be much larger compared to the straightforward generalization of Azuma's inequality, which we present below in Proposition 4.C.4.

Proposition 4.C.2 (An alternative to Azuma's inequality). *Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to a filtration $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ on a probability space $(\Omega,\mathcal{G},\mathbb{P})$. Suppose that $X_0=0$, $|X_n|\le T$ for some $T\ge 1$ and all $n\in\mathbb{N}_0$. Set $Z_n := X_n - X_{n-1}$ for $n\in\mathbb{N}$ and let $E\in\mathcal{G}$ be a set such that $|Z_n|(\omega)\le 1$ for all $\omega\in E$. Then, for all $k\in\mathbb{N}_0$ and $N\in\mathbb{N}_0$ there holds*

$$\mathbb{E}\big[\chi_E X_N^{2k}\big] \le (2k)!N^k\mathbb{P}\big[E\big] + C(T)(2k)!N^kT^{2k}\mathbb{P}\big[E^c\big]. \tag{4.C.1}$$

with $C(T) := \frac{2T^2}{T^2+2-e}$ and, as a consequence,

$$\mathbb{E}\big[X_N^{2k}\big] \le (2k)!N^k\mathbb{P}\big[E\big] + \big(1+C(T)(2k)!N^k\big)T^{2k}\mathbb{P}\big[E^c\big]. \tag{4.C.2}$$

In particular, there exists $C>0$ such that for all $\lambda>0$ this implies

$$\mathbb{P}\big[|X_N|\ge\lambda\big] \le \begin{cases} C\exp\left(-\dfrac{\lambda}{2\sqrt{N}}\right) & \text{if } \lambda \le \dfrac{-\log(\mathbb{P}[E^c])}{2\log(T)}\sqrt{N}, \\ C\mathbb{P}\big[E^c\big]^{\frac{1}{2\log(T)}} & \text{if } \lambda \ge \dfrac{-\log(\mathbb{P}[E^c])}{2\log(T)}\sqrt{N}. \end{cases} \tag{4.C.3}$$

Proof. We prove the proposition by induction. For both $k=0$ with arbitrary $N\in\mathbb{N}_0$ and $N\in\{0,1\}$ with arbitrary $k\in\mathbb{N}_0$ clearly (4.C.1) holds. Now fix $(k,N)\in\mathbb{N}\times\mathbb{N}$, $N\ge 2$ and assume that (4.C.1) holds for all $(\ell,n)\in\mathbb{N}_0\times\mathbb{N}_0$ with either $\ell<k$, $n\le N$ or $n<N$, $\ell\le k$. Showing that then (4.C.1) holds for (k,N) yields the result.

Using that $\mathbb{E}\big[Z_N X_{N-1}^{2k-1}\big] = \mathbb{E}\big[\mathbb{E}\big[Z_N X_{N-1}^{2k-1}\,\big|\,\mathcal{G}_{N-1}\big]\big] = \mathbb{E}\big[\mathbb{E}[Z_N\,|\,\mathcal{G}_{N-1}]X_{N-1}^{2k-1}\big]$ and $\mathbb{E}[Z_N\,|\,\mathcal{G}_{N-1}] = 0$ hold since X_n is a martingale , we obtain

$$\begin{aligned}
\mathbb{E}\big[\chi_E X_N^{2k}\big] &= \mathbb{E}\Big[\chi_E\big(X_{N-1}+Z_N\big)^{2k}\Big] \\
&= \mathbb{E}\big[\chi_E X_{N-1}^{2k}\big] - 2k\mathbb{E}\big[\chi_{E^c}Z_N X_{N-1}^{2k-1}\big] + \sum_{\ell=0}^{2k-2}\binom{2k}{\ell}\mathbb{E}\big[\chi_E Z_N^{2k-\ell}X_{N-1}^{\ell}\big] \\
&\le \mathbb{E}\big[\chi_E X_{N-1}^{2k}\big] + 4kT^{2k}\mathbb{P}\big[E^c\big] + \sum_{\ell=0}^{k-1}\binom{2k}{2\ell}\mathbb{E}\big[\chi_E X_{N-1}^{2\ell}\big] \\
&\quad + \sum_{\ell=0}^{k-2}\binom{2k}{2\ell+1}\mathbb{E}\big[\chi_E(N-1)X_{N-1}^{2\ell}\big],
\end{aligned}$$

where in the last step we used $|Z_N|\le 2T$, $\chi_E|Z_N|\le 1$ and $\chi_E|X_{N-1}|\le N-1$. Applying

(4.C.1), which holds due to the induction assumption, now yields

$$\begin{aligned}\mathbb{E}\big[\chi_E X_N^{2k}\big] \le (2k)!N^k\mathbb{P}[E]\bigg(\bigg(\frac{N-1}{N}\bigg)^k + \frac{1}{N}\bigg(\sum_{\ell=0}^{k-1}\frac{1}{(2k-2l)!}\frac{(N-1)^\ell}{N^{k-1}} \\ + \sum_{\ell=0}^{k-2}\frac{1}{(2k-2l-1)!(2l+1)}\frac{(N-1)^{\ell+1}}{N^{k-1}}\bigg)\bigg) \\ + C(T)(2k)!N^kT^{2k}\mathbb{P}\big[E^c\big]\bigg(\bigg(\frac{N-1}{N}\bigg)^k + \frac{1}{N}\bigg(\frac{4k}{C(T)(2k)!N^{k-1}} \\ + \sum_{\ell=0}^{k-1}\frac{1}{(2k-2l)!}\frac{(N-1)^\ell}{N^{k-1}}T^{2(\ell-k)} \\ + \sum_{\ell=0}^{k-2}\frac{1}{(2k-2l-1)!(2l+1)}\frac{(N-1)^{\ell+1}}{N^{k-1}}T^{2(\ell-k)}\bigg)\bigg)\end{aligned}$$

Factoring out and using that $\frac{(N-1)^\ell}{N^{k-1}} \le \big(\frac{N-1}{N}\big)^{k-1}$ since $N \ge 2$, we have

$$\begin{aligned}\mathbb{E}\big[\chi_E X_N^{2k}\big] \le (2k)!N^k\mathbb{P}[E]\bigg(\frac{N-1}{N}\bigg)^{k-1}\bigg(1 - \frac{1}{N} + \frac{1}{N}\big(\mathrm{I}_k + \mathrm{II}_k\big)\bigg) \\ + C(T)(2k)!N^kT^{2k}\mathbb{P}\big[E^c\big]\bigg(\frac{N-1}{N}\bigg)^{k-1} \\ \times\bigg(1 - \frac{1}{N} + \frac{1}{N}\big(\mathrm{III}_k + \mathrm{I}_{k,T} + \mathrm{II}_{k,T}\big)\bigg),\end{aligned}$$

where, adjusting the indices of the sums,

$$\begin{aligned}\mathrm{I}_k &\coloneqq \sum_{\ell=1}^{k}\frac{1}{(2\ell)!}, & \mathrm{II}_k &\coloneqq \sum_{\ell=2}^{k}\frac{1}{(2\ell-1)!(2k-2\ell+1)}, \\ \mathrm{I}_{k,T} &\coloneqq \sum_{\ell=1}^{k}\frac{1}{(2\ell)!T^{2\ell}} \le \frac{1}{T^2}\mathrm{I}_k, & \mathrm{II}_{k,T} &\coloneqq \sum_{\ell=2}^{k}\frac{1}{(2\ell-1)!(2k-2\ell+1)T^{2\ell}} \le \frac{1}{T^4}\mathrm{II}_k, \\ \mathrm{III}_k &\coloneqq \frac{4k}{C(T)(2k)!} \le \frac{2}{C(T)}.\end{aligned}$$

Note that $\mathrm{I}_k + \mathrm{II}_k \le e - 2 < 1$ and hence $\mathrm{III}_k + \mathrm{I}_{k,T} + \mathrm{II}_{k,T} \le 1$ by definition of $C(T)$. Plugging this back into the estimate for $\mathbb{E}\big[\chi_E X_N^{2k}\big]$ yields (4.C.1) and thus finishes the proof by induction.

Regarding (4.C.3), with (4.C.2) we obtain for any $k \in \mathbb{N}$ that

$$\begin{aligned}\mathbb{P}\big[\big|X_N\big| \ge \lambda\big] &\le \lambda^{-2k}\mathbb{E}\big[X_N^{2k}\big] \\ &\le \lambda^{-2k}(2k)!N^k\big(1 + CT^{2k}\mathbb{P}\big[E^c\big]\big).\end{aligned}$$

Using that $(2k)! \le C\sqrt{k}\big(\frac{2k}{e}\big)^{2k} \le C\Big(\frac{2k}{\sqrt{e}}\Big)^{2k}$ for some $C > 0$, we obtain

$$\mathbb{P}\big[\big|X_N\big| \ge \lambda\big] \le C\exp\bigg(-2k\bigg(\frac{1}{2} + \log\bigg(\frac{\lambda}{\sqrt{N}}\bigg) - \log(2k)\bigg)\bigg)\big(1 + CT^{2k}\mathbb{P}\big[E^c\big]\big).$$

Just focusing on the argument of the exponential, we want to choose k as large as possible while the overall sign stays negative, that is $k_\lambda \coloneqq \big\lfloor\frac{\lambda}{2\sqrt{N}}\big\rfloor$. However, this only

makes sense while this is not counteracted by the last factor stays under control, that is for $T^{2k}\mathbb{P}[E^c] \leq 1$, which is guaranteed if $k \leq k_0 := \left\lfloor \frac{-\log(\mathbb{P}[E^c])}{2\log(T)} \right\rfloor$. We hence obtain (4.C.3) by choosing $k = k_\lambda$ if $k_\lambda \leq k_0$ and $k = k_0$ if $k_\lambda \geq k_0$. □

4.C.2 A generalization of Azuma's inequality

For our generalization of Azuma's inequality, we will use the following lemma.

Lemma 4.C.3 (Hoeffding's lemma for conditional expectations). *Let $X \in \mathbb{R}$ be a random variable with respect to some probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and $\widetilde{\mathcal{G}} \subset \mathcal{G}$ be a coarser σ-algebra. Suppose there are $\widetilde{\mathcal{G}}$-measurable random-variables $a, b \in \mathbb{R}$ such that $a \leq X \leq b$. Then for all $s \in \mathbb{R}$ there holds*

$$\mathbb{E}\Big[\exp(sX) \,\Big|\, \widetilde{\mathcal{G}}\Big] \leq \exp\bigg(s\mathbb{E}\Big[X \,\Big|\, \widetilde{\mathcal{G}}\Big] + \frac{s^2(b-a)^2}{8}\bigg).$$

Proof. Analogous to the proof of Hoeffding's lemma for the expected value. □

For the following generalization of Azuma's inequality, note that the limit up to which we obtain the tail bounds is much stricter than in Proposition 4.C.2. In particular, it is not even necessarily better than the typical size of the fluctuations.

Proposition 4.C.4 (A generalization of Azuma's inequality). *Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to a filtration $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Suppose that $|X_n| \leq T$ for some $T > 0$ and all $n \in \mathbb{N}_0$. Set $Z_n := X_n - X_{n-1}$ for $n \in \mathbb{N}$ and let $E \in \mathcal{G}$ be a set such that $|Z_n|(\omega) \leq c_n$ for all $\omega \in E$, $n \in \mathbb{N}$ and some $(c_n)_n \subset \mathbb{R}_+$. Then, for all $N \in \mathbb{N}$ and $s > 0$ there holds*

$$\mathbb{E}\big[\exp\big(s(X_N - X_0)\big)\big] \leq \exp\bigg(s^2\sum_{n=1}^N c_n^2\bigg)\bigg(1 + N\bigg(1 + \frac{4TN}{s\sum_{n=1}^N c_n^2}\bigg)\mathbb{P}\big[E^c\big]e^{2sT}\bigg)$$

and the same for $\mathbb{E}[\exp(s(X_0 - X_N))]$. In particular, there holds

$$\mathbb{P}\big[|X_N - X_0| \geq \lambda\big] \leq 4\exp\bigg(-\frac{\lambda^2}{4\sum_{n=1}^N c_n^2}\bigg) \qquad \text{for all } 0 < \lambda \leq \lambda_0, \tag{4.C.4}$$

where $\lambda_0 = -\log\bigg(\bigg(N + \frac{8TN^2}{\sqrt{\sum_{n=1}^N c_n^2}}\bigg)\mathbb{P}[E^c]\bigg)\frac{\sum_{n=1}^N c_n^2}{T}$.

Proof. We provide bounds for the exponential moments of $X_N - X_0$, the bounds for $X_0 - X_N$ follow analogously. For any $s \geq 0$, using that X_n is $\mathcal{G}_{N-1}$-measurable for all $n \leq N-1$, for any $E_N \in \mathcal{G}_{N-1}$ we obtain

$$\begin{aligned}\mathbb{E}\big[\exp\big(s(X_N - X_0)\big)\big] &= \mathbb{E}\bigg[(\chi_{E^c} + \chi_E)(\chi_{(E_N)^c} + \chi_{E_N})\prod_{n=1}^N \exp\big(sZ_n\big)\bigg] \\ &\leq \mathbb{P}\big[E^c\big]e^{2sT} + \mathbb{P}\big[(E_N)^c\big]e^{2sT} + \mathbb{E}\bigg[\chi_{E_N}\prod_{n=1}^{N-1}\exp\big(sZ_n\big)\mathbb{E}\big[\chi_E\exp\big(sZ_N\big)\,\big|\,\mathcal{G}_{N-1}\big]\bigg].\end{aligned}$$

We now want to apply Hoeffding's Lemma in the form of Lemma 4.C.3. Note that by assumption $|\chi_E Z_N| \leq c_N$ and due to the martingale property

$$\mathbb{E}[\chi_E Z_N \,|\, \mathcal{G}_{N-1}] = \mathbb{E}[Z_N \,|\, \mathcal{G}_{N-1}] - \mathbb{E}[\chi_{E^c} Z_N \,|\, \mathcal{G}_{N-1}] \leq 2T\mathbb{E}[\chi_{E^c} \,|\, \mathcal{G}_{N-1}]. \tag{4.C.5}$$

This kind of leftover term is small with probability proportional to $\mathbb{P}[E^c]$. To control this leftover term, we define suitable sets $E_n \in \mathcal{G}_{n-1}$, namely

$$E_n := \left\{ \omega \in \Omega \text{ such that } 2T\mathbb{E}[\chi_{E^c} \,|\, \mathcal{G}_{n-1}](\omega) \leq \frac{s \sum_{k=1}^N c_k^2}{2N} \right\},$$

for whose probability as in the proof of Lemma 4.C.1 we have the bound

$$\mathbb{P}\big[(E_n)^c\big] \leq \frac{4TN}{s \sum_{n=1}^N c_n^2} \mathbb{P}\big[E^c\big].$$

Now, first applying Hoeffding's Lemma in the form of Lemma 4.C.3 with (4.C.5) and then using the definition of E_N yields

$$\begin{aligned}
\mathbb{E}\big[\exp\big(s\big(X_N - X_0\big)\big)\big] &\leq \big(\mathbb{P}\big[E^c\big] + \mathbb{P}\big[(E_N)^c\big]\big) e^{2sT} \\
&\quad + \mathbb{E}\left[\chi_{E_N} \prod_{n=1}^{N-1} \exp\big(sZ_n\big) \exp\big(s2T\mathbb{E}[\chi_{E^c} \,|\, \mathcal{G}_{N-1}]\big) \right] e^{\frac{s^2 c_N^2}{2}} \\
&\leq \left(1 + \frac{4TN}{s \sum_{n=1}^N c_n^2}\right) \mathbb{P}\big[E^c\big] e^{2sT} \\
&\quad + \mathbb{E}\left[\prod_{n=1}^{N-1} \exp\big(sZ_n\big) \right] e^{\frac{s^2 c_N^2}{2} + \frac{s^2 \sum_{n=1}^N c_n^2}{2N}}.
\end{aligned}$$

Repeating this procedure with $\mathbb{E}[\prod_{n=1}^m \exp(sZ_n)]$ for all $m \in \{1, \dots, N-1\}$, we obtain

$$\mathbb{E}\big[\exp\big(s\big(X_N - X_0\big)\big)\big] \leq N\left(1 + \frac{4TN}{s \sum_{n=1}^N c_n^2}\right) \exp\left(s^2 \sum_{n=2}^N c_n^2\right) \mathbb{P}\big[E^c\big] e^{2sT} + \exp\left(s^2 \sum_{n=1}^N c_n^2\right),$$

which yields the exponential moment bounds. Regarding (4.C.4), we have

$$\begin{aligned}
\mathbb{P}\big[X_N - X_0 \geq \lambda\big] &\leq e^{-s\lambda} \mathbb{E}\big[\exp\big(s\big(X_N - X_0\big)\big)\big] \\
&\leq \exp\left(-s\lambda + s^2 \sum_{n=1}^N c_n^2\right) \left(1 + N\left(1 + \frac{4TN}{s \sum_{n=1}^N c_n^2}\right) \mathbb{P}\big[E^c\big] e^{2sT}\right).
\end{aligned}$$

Just focusing on the argument of the first exponential, to minimize it we would like to choose $s = s_\lambda := \frac{\lambda}{2 \sum_{n=1}^N c_n^2}$. However, we still need to make sure that the second term stays under control. Since the probability bounds for (4.C.4) are only meaningful if they are smaller than 1 and we are only considering half of the total contribution, we only need to consider $\lambda \geq \sqrt{\sum_{n=1}^N c_n^2}$ because otherwise just the first term breaks these bounds. In particular, we would like to choose at least $s \geq s_0 := \frac{1}{2\sqrt{\sum_{n=1}^N c_n^2}}$. With this restriction, we obtain for the second term that

$$N\left(1 + \frac{4TN}{s \sum_{n=1}^N c_n^2}\right) \mathbb{P}\big[E^c\big] e^{2sT} \leq N\left(1 + \frac{8TN}{\sqrt{\sum_{n=1}^N c_n^2}}\right) \mathbb{P}\big[E^c\big] e^{2sT} \leq 1$$

for all

$$s_0 \leq s \leq s_1 := \frac{-\log\left(\left(N + \frac{8TN^2}{\sqrt{\sum_{n=1}^N c_n^2}}\right) \mathbb{P}[E^c]\right)}{2T}.$$

For $s_\lambda \leq s_1$, we choose $s = s_\lambda$. This is the case for all

$$\lambda \leq \lambda_0 := -\log\left(\left(N + \frac{8TN^2}{\sqrt{\sum_{n=1}^N c_n^2}}\right)\mathbb{P}\big[E^c\big]\right)\frac{\sum_{n=1}^N c_n^2}{T}.$$

Note that we can ignore the case that $s_{\lambda_0} = s_1 < s_0$ because then the resulting probability bounds are anyways larger than 1. For $\lambda > \lambda_0$, we would want to choose $s = s_1$, but this is not necessary since in the overall context of this paper the resulting bounds are worse than the ones provided by Proposition 4.C.2. Repeating the same argument for $\mathbb{P}[X_N - X_0 \leq -\lambda]$ yields (4.C.4). □

If in the proof of Proposition 4.6.1 we apply this generalization of Azuma's inequality instead of Proposition 4.C.2, we obtain Gaussian instead of exponential tail bounds for the same fluctuation size up to some lower bound, see the lemma below. However, for this lower bound to actually be small we would have to require that $h^{1+2\alpha_\circledcirc} \gg \operatorname{dist}(x_0, S)$. This is different to the lower bound resulting from Proposition 4.C.2, where we only need that h is large.

Lemma 4.C.5 (Fluctuation bounds with improved stochastic integrability)**.** *Assume that (P2) and (P⊙) hold. There are $c, C_{\mathrm{fl}}, c_{\mathrm{fl}}, C_{\mathrm{co}} = c(\mathrm{data}), C_{\mathrm{fl}}(\mathrm{data}), c_{\mathrm{fl}}(\mathrm{data}), C_{\mathrm{co}}(\mathrm{data}) > 0$ such that for all $x_0 \in \mathbb{R}^d$, $h \geq h_0$ and h-fat sets $S \subset \mathbb{R}^d$ with bounded boundary as well as*

$$\frac{\operatorname{diam}(\partial S) + \operatorname{dist}(x_0, S)}{h} \leq \exp\big(ch^{\alpha_\circledcirc}\big)$$

there holds

$$\mathbb{P}\big[\big|m_h(x_0, S) - \mathbb{E}\big[m_h(x_0, S)\big]\big| \geq \lambda\big] \leq C_{\mathrm{fl}} \exp\left(-\frac{c_{\mathrm{fl}}^2\lambda^2}{h \operatorname{dist}(x_0, S)}\right) \quad \text{for all } 0 < \lambda \leq C_{\mathrm{co}}h^{1+\alpha_\circledcirc},$$

where $\alpha_\circledcirc > 0$ is the exponential rate from the probability bounds in Assumption (P⊙) for the existence of stable approximation and the applicability of the effective minimum speed of propagation.

Proof. To obtain these fluctuation bounds, we repeat the argument from the proof of Proposition 4.6.1 with Proposition 4.C.4 instead of Proposition 4.C.2. That is, we apply our generalization of Azuma's inequality to the martingale $(\widetilde{\mathcal{M}}_n)_n$ from the proof of Proposition 4.6.1 with

$$\widetilde{\mathcal{M}}_N = m_h(x_0, S) - \mathbb{E}\big[m_h(x_0, S)\big] + R_1 + R_2 \tag{4.6.24}$$

for $N \leq Ch^{-1} \operatorname{dist}(x_0, S)$ and

$$\begin{aligned}
\big|R_1\big| &\leq T_h(x_0, S)\mathbb{E}\Big[1 - \chi_{E_h^\circledcirc(\partial S)}\,\Big|\,\mathcal{G}_{T_h(x_0,S)}\Big], \\
\big|R_2\big| &\leq T_h(x_0, S)\Big(1 - \chi_{E_h^\circledcirc(\partial S)}\Big).
\end{aligned}$$

From the proof of Proposition 4.6.1 we further have the event $E_{\mathcal{M}} \in \mathcal{F}(\mathbb{R}^d)$ such that

$$\Big|\widetilde{\mathcal{M}}_{n+1} - \widetilde{\mathcal{M}}_n\Big|\chi_{E_{\mathcal{M}}} \leq Ch \quad \text{for all } n \in \mathbb{N}.$$

For the failure of this event we have the stretched exponential bounds

$$1 - \mathbb{P}\big[E_{\mathcal{M}}\big] \leq C \exp\Big(-\frac{c_{\circledcirc}}{2} h^{\alpha_{\circledcirc}}\Big).$$

Using (4.6.24) and the associated error bounds right below to then apply Proposition 4.C.4 yields

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(s\big(m_h(x_0,S) - \mathbb{E}\big[m_h(x_0,S)\big]\big)\big)\big] \\
&\qquad \leq \mathbb{E}\Big[\exp\Big(s\Big(\widetilde{M}_N - R_1\Big)\Big)\chi_{E_{\mathcal{M}}}\Big] + \exp\big(sT_h(x_0,S)\big)\big(1 - \mathbb{P}\big[E_{\mathcal{M}}\big]\big) \\
&\qquad \leq \exp(sh)\mathbb{E}\Big[\exp\Big(s\Big(\widetilde{M}_N\Big)\Big)\chi_{E_{\mathcal{M}}}\Big] + \exp\big(sT_h(x_0,S)\big)\big(1 - \mathbb{P}\big[E_{\mathcal{M}}\big]\big) \\
&\qquad \leq \exp\big(Cs^2Nh^2 + sh\big) \\
&\qquad\qquad \times \left(1 + C\left(N + \frac{T_h(x_0,S)N}{sh^2}\right)\big(1 - \mathbb{P}\big[E_{\mathcal{M}}\big]\big)e^{2sT_h(x_0,S)}\right).
\end{aligned}$$

This is essentially the same moment bound as in Proposition 4.C.4. Following the proof of (4.C.4) from Proposition 4.C.4 – except for using that $e^{-s\lambda+sh} \leq e^{-\frac{1}{2}s\lambda}$ for $\lambda \geq 2h$ to deal with the additional term e^{sh} – we obtain

$$\mathbb{P}\big[\big|m_h(x_0,S) - \mathbb{E}\big[m_h(x_0,S)\big]\big| \geq \lambda\big] \leq 4\exp\left(-\frac{\lambda^2}{CNh^2}\right) \qquad \text{for all } 0 < \lambda \leq \lambda_0,$$

with $\lambda_0 = -\log\Big(\frac{CTN\sqrt{N}}{h}(1 - \mathbb{P}[E_{\mathcal{M}}])\Big)\frac{Nh^2}{T}$. Plugging in $N \approx \frac{\operatorname{dist}(x_0,S)}{h}$ and $T_h(x_0,S) \approx \operatorname{dist}(x_0,S)$ from (4.6.23) as well as using Assumption (4.6.1) with (4.6.27) to obtain

$$\log\left(\frac{CTN\sqrt{N}}{h}\big(1 - \mathbb{P}\big[E_{\mathcal{M}}\big]\big)\right) \approx -h^{\alpha_{\circledcirc}}$$

yields the first inequality in (4.6.2). □

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CHAPTER 5

A sufficient criterion for ballistic behavior of forced geometric motions in random environments

This chapter contains the following yet unpublished preprint [FI26b]:

Julian Fischer and Jonas Ingmanns. "A sufficient criterion for ballistic behavior of forced geometric motions in random environments." *In preparation* (2026+)

Abstract We introduce a sufficient criterion for ballistic motion of interface evolutions in random environments. More precisely, we consider interfaces moving by forced mean curvature flow in dimension $d = 2$ with a spatially fluctuating (possibly negative) driving force. Such models arise e.g. from the motions of dislocation lines or magnetic domain walls in a material with impurities. We show that the evolution admits an 'effective minimum speed on large scales' if for a sample of the random environment in some long and thin box there is a sufficiently large probability that an interface initialized in the bottom of the box evolves to reach its top. The concept of an 'effective minimum speed on large scales' was introduced and is required for the quantitative stochastic homogenization result in Chapter 4. We show that our criterion and hence the requirements for the homogenization result in Chapter 4 are satisfied for compactly supported obstacle shapes distributed according to a Poisson point process with sufficiently low intensity.

5.1 Introduction

We consider the question of when ballistic motion occurs for interface evolutions in random environments, that is, whether on large time scales the interface propagates with positive velocity. In the present work, we introduce a sufficient criterion for ballistic motion in dimension $d = 2$ for arbitrary random environments with a finite range of dependence. While we expect that our criterion and methods can be extended to general interface evolutions with a comparison principle, we focus on interfaces evolving by forced mean curvature flow through a random environment: The normal velocity V of an interface at $x \in \mathbb{R}^2$ is given in terms of the normal $\nu \in \mathbb{S}^1$ and the (mean) curvature H by

$$V = a(x,\nu)H + F(x,\nu), \tag{5.1.1}$$

with the curvature coefficient field $a = \sigma^2 \geq 0$ and the forcing field

$$F(x, \nu) = F_{uni}(\nu) - F_{obst}(x, \nu).$$

Here, F_{uni} can be thought of as a deterministic, external uniform forcing while F_{obst} encodes local obstacles. We assume that $\sigma, F \in C^{0,1}(\mathbb{R}^2 \times \mathbb{S}^1)$ are sampled from a stationary probability measure $\mathbb{P}$ with finite range of dependence, see Section 5.3 for a detailed description of the assumptions.

For a given random environment in the form of (σ, F_{obst}), qualitatively very different behavior might emerge depending on the strength of the uniform forcing F_{uni}. For (5.1.1) and related interface motions such as the quenched Edwards-Wilkinson model, the occurrence of pinning and depinning phenomena has been established both for periodic obstacle fields [DY06] and for certain stationary random obstacle fields with a finite range of dependence [CDL10; DS12; BT14; DS17; DJS21; DJS22]. In short, for prototypical obstacle fields, there are critical forcings $0 < F_* \leq F^*$ such that

- for $F_{uni} < F_*$, any interface gets pinned, meaning that its evolution is bounded by a stationary surface (pinning regime),
- for $F_{uni} > F^*$, for increasingly large initial interfaces the probability that their evolution is ballistic converges to one, meaning that the main front covers a strictly positive average distance per time unit (ballistic regime).

Both $F_* = F^*$ [BT14] or $F_* < F^*$ [DS17] are possible, depending on the specific model. In the case of $F_* < F^*$, there is an intermediate sub-ballistic regime in which the interface propagates indefinitely but the limit velocity is zero.

Pinning and depinning can be observed for a range of physical phenomena in material science such as dislocation lines, magnetic domain walls, or grain boundaries [IV87; Kar98; BN04]. The motions of these lines or interfaces is driven by the competition between three effects: A preference for the minimization of the interface area, an external, uniform forcing pushing the interface forwards, and material impurities which may act as local obstacles to the interface motion. Forced mean curvature flow as in (5.1.1) is commonly used to model these phenomena with the strength of these effects encoded in a, F_{uni}, and F_{obst} respectively.

Most results for random environments have been established for explicitly modeled obstacle fields, usually obstacles distributed according to a Poisson point process or a discrete analogue. We introduce a criterion applicable to arbitrary random environments which implies ballistic behavior of the interface motion. In order to show that this criterion is satisfied, it is sufficient to check statistical properties of a sample from the random fields in a box not necessarily much larger than the range of dependence. For a discrete version of the quenched Edwards-Wilkinson model, [BT14] also provides a criterion for ballistic behavior given arbitrary random environments while allowing for slowly decaying long-range correlations rather than the finite range of dependence which we require. However, their criterion already requires control on all scales for the probability that there is a pinned solution. Moreover, our criterion implies not just ballistic behavior but the existence of an almost guaranteed 'effective minimum speed of propagation', a notion that has been introduced and is required for the homogenization result established in Chapter 4.

As an application, we show that our criterion is satisfied for Poisson distributed obstacles with low enough intensity compared to the uniform forcing. In particular, having

thus established the existence of an almost guaranteed 'effective minimum speed of propagation', we obtain the homogenization result Theorem 4.1.1 announced at the start of Chapter 4. Ballistic motion in this setting has already been established in [CDO22]: Starting with a flat interface, the average height of the main front above the original line grows with positive speed. The notion of an almost guaranteed 'effective minimum speed of propagation' is stronger: On large scales, we show that with high probability any sufficiently regular interface expands at least according to some minimum speed in any direction – up to relatively small enclosures left behind the main front – and that this holds not just for the initial interface but at any stage of the evolution even when the initial regularity assumption might not be satisfied anymore.

We now state an informal version of our *box criterion*, which roughly speaking requires that it is sufficiently likely for an interface starting in the bottom of a long but thin box sample of the environment to reach the top of the box. The threshold for this probability depends on the dimensions of the box and can be arbitrarily small, see Assumption (P□) in Section 5.3 for the rigorous formulation of the criterion. Using this criterion, we will later obtain improved probability bounds for the same event in larger boxes with the width $w(r)$ of the boxes scaling sublinearly compared to the length r. Since this sublinear scaling is central for obtaining a positive speed in the limit, we incorporate it into the criterion to streamline the notation.

'(P□)' **Box criterion.** We assume that there exist a velocity $v_0 > 0$, a width scaling exponent $q > 1$, and a box length $r_0 > C(q)$ such that within the thin boxes $Q_r^\uparrow = [0, w(r)] \times [-w(r), r]$ with width $w(r) \sim r^{\frac{1}{q}}$, as in Figure 5.1.1, it holds that

$$\begin{aligned}\mathbb{P}\Big[\big\{&\text{There is a 'stable' set in the bottom } [0,w(r)]\times[-w(r),0] \text{ of } Q_r^\uparrow,\\ &\quad\text{which evolving 'within } Q_r^\uparrow\text{' by (5.1.1)}\\ &\quad\text{reaches the top } [0,w(r)]\times\{r\} \text{ by the time } t = v_0^{-1}r\big\}\Big]\\ &\geq 1 - p_0,\end{aligned} \tag{5.1.2}$$

for all $r \in [r_0, r_1)$ with the acceptable probability of failure $p_0 = p_0(r_0, w(\cdot)) \geq 0$ depending on the dimensions of the boxes. The upper end r_1 of the range for the box lengths is chosen such that a rotation of $Q_{r_0}^\uparrow$ fits into $Q_{r_1}^\uparrow$, see Figure 5.3.1, meaning that $w(r_1) \approx r_0$. Here, 'stable' means that the set is non-shrinking and 'within $Q_r^\uparrow$' means that the coefficient fields are modified to restrict the evolution to the box. In addition, we assume that the same is true for the accordingly rotated boxes $Q_r^\rightarrow, Q_r^\downarrow, Q_r^\leftarrow$.

The following informal version of our main result is a precise statement up to replacing '(P□)' with the rigorous formulation stated in Assumption (P□) and the informal version '(P◎)' of the guarantee for an 'effective minimum speed of propagation' from Chapter 4, which is stated below the result, with its rigorous formulation (P◎) given in Section 5.3.5.

Theorem 5.1.1 (Informal main result)**.** *Let the coefficient fields $a = \sigma^2$ and F be sampled from a stationary probability measure $\mathbb{P}$ with range of dependence one and let $\sigma, F \in C^{0,1}(\mathbb{R}^2 \times \mathbb{S}^1)$ be uniformly bounded and Lipschitz. Assume that $\mathbb{P}$ satisfies the box criterion '(P□)'. Then the evolution (5.1.1) with the coefficient field (a, F) admits an almost guaranteed 'effective minimum speed' $v_{\min} = v_{\min}(r_0, w(\cdot), v_0) > 0$ in the sense of '(P◎)' from Chapter 4, see below. In particular, this evolution is in the ballistic regime.*

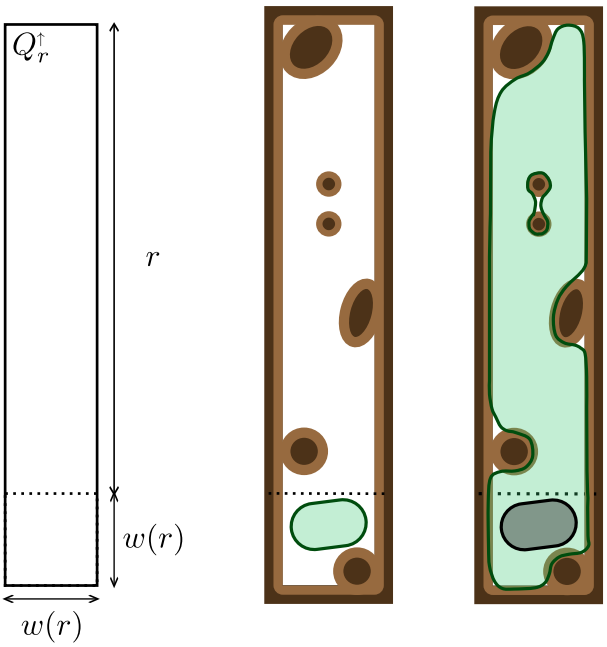


Figure 5.1.1: An illustration of the event $E_r^{\uparrow,v}$ introduced for our box criterion. Left: The dimensions of the box $Q_r^\uparrow$ and its 'starting area'. Center: A sample of the obstacle field (depicted in brown) inside the box with the artificial obstacle surrounding the boundary (also depicted in brown) since the coefficient field is restricted to the box. For the event $E_r^{\uparrow,v}$ to hold, there has to be a stable set S in the starting area of the box (depicted in green). Right: After time $t = v^{-1}r$, the evolution $\mathscr{R}_t(S)$ (depicted in green) of the set S is required to have reached the top of the box.

We now provide the aforementioned informal version of the almost guarantee for an 'effective minimum speed'. We will state the rigorous formulation and the required definitions of the respective notions from Chapter 4 later in Section 5.3.5.

'(P⊙)' **Almost guaranteed effective minimum speed**. There exist $v_{\min}, C > 0$ and a rate $\alpha_\odot > 0$ such that for all $h \geq h_0$ and $M \subset \mathbb{R}^2$

$$\mathbb{P}\Big[\big\{\text{In } M\text{, on the scale } h\text{, every 'fat' set has a 'stable approximation',}\\ \text{which propagates with an 'effective minimum speed' of } v_{\min}\big\}\Big]\\ \geq 1 - C\big(1+\operatorname{diam}(M)\big)^2 \exp\big(-C^{-1}h^{\alpha_\odot}\big),$$

where 'fat' sets $S \subset \mathbb{R}^2$ are unions of sufficiently large balls and 'stable approximations' are non-shrinking interior approximations $S_h \subset S$. An 'effective minimum speed' on the scale h means that for any $t \geq 0$ the evolved set $\mathscr{R}_t(S_h)$ after another time step of $v_{\min}^{-1}h$ has advanced by at least h in all directions while allowing for enclosures of width smaller than $2h$ to be left behind the main front.

Such an almost guaranteed effective minimum speed allows us to apply the homogenization results from the companion paper under the additional assumptions that the probability measure is isotropic and that the curvature coefficient field is constant.

Corollary 5.1.2 (Application of Theorem 4.4.1 and Theorem 4.4.2)**.** *Let the coefficient field F be sampled from a stationary probability measure $\mathbb{P}$ with range of dependence one and $F \in C^{0,1}(\mathbb{R}^2 \times \mathbb{S}^1)$ uniformly bounded and Lipschitz. Let $\mathbb{P}$ be isotropic, that is $\operatorname{law} F(\cdot,\cdot) = \operatorname{law} F(R\cdot, R\cdot)$ for all rotations $R \in \mathrm{SO}(2)$. Assume that $\mathbb{P}$ satisfies the box criterion '(P□)' for (5.1.1) with the coefficient field $(a, F_0 + F)$ for some constants $a \geq 0$ and $F_0 \in \mathbb{R}$. Then for Lebesgue-almost every $F_{uni} > F_0$ there exists $\overline{v} = \overline{v}(a, \mathbb{P}, F_{uni}) > 0$ such*

that on large scales the evolution (5.1.1) with $(a, F_{uni} + F(x))$ can be approximated by a constant-speed motion

$$V = \overline{v}$$

in terms of arrival times in target areas which are small with respect to the large scale but large compared to the range of dependence, see Chapter 4.

As an application of the main result, we show that '(P□)' and hence '(P◎)' is satisfied for a compactly supported obstacle shape distributed according to a Poisson point process with sufficiently low intensity compared to the uniform forcing. We obtain the same scaling relationship ("Taylor scaling") between the intensity and F^*, the uniform forcing marking the transition to the ballistic regime, as in [CDO22]. Note however that '(P□)' is stronger than the ballistic property shown in [CDO22].

Theorem 5.1.3 (Almost guaranteed effective minimum speed for Poisson obstacles)**.** *Consider Poisson-distributed obstacles in dimension $d = 2$ represented by*

$$F_{obst} \colon \mathbb{R}^2 \to \mathbb{R}, \quad F_{obst}(x) := \max_{i \in \mathbb{N}} \varphi(|x - X_i|)$$

for

- *a Poisson point process $\{X_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$ with intensity $\rho > 0$, and*
- *a Lipschitz continuous obstacle shape $\varphi \in C^{0,1}(\mathbb{R})$ with support $\operatorname{supp} \varphi \subset \overline{B}_1$.*

There exist $\rho_0 > 0$ and $C_0 > 0$ independent of the obstacle shape φ such that – given any intensity $\rho \in (0, \rho_0]$ – for every $F_{uni} \geq C_0 \sqrt{\rho}$ the interface motion given by

$$\text{Normal velocity} = \text{curvature} + F_{uni} - F_{obst}(x) \qquad \text{at } x \in \mathbb{R}^2$$

satisfies the box criterion (P□) for some $v_0 = v_0(F_{uni}) > 0$, $r_0 = r_0(F_{uni}) > 1$, and $w(r) = \left(\frac{r}{c_w}\right)^{\frac{1}{q}}$ with $q > 1$ and $c_w = c_w(F_{uni}) > 0$.

Combining this result with Theorem 5.1.1 allows the application of Theorem 4.4.1 and Theorem 4.4.2 to sufficiently sparse Poisson obstacles and hence yields the quantitative homogenization result stated in Theorem 4.1.1.

5.1.1 Structure of the paper

In Section 5.2, we discuss the related literature, in particular with respect to results on pinning-depinning phenomena. In Section 5.3, we state our assumptions including the box criterion and give precise definitions of the notions used throughout the present work such as set evolutions, restricted coefficient fields, stable sets, stable approximations and an effective minimum speed of propagation. In Section 5.4, we provide the formal version of our main result and describe the strategy of the proof. In Section 5.5, we gather preliminary results on the basic properties of the set evolutions which will be used in the main part of the proof. In Section 5.6 and Section 5.7, the main part of the proof is presented, see Section 5.4.2 for a more detailed description. In Section 5.8, we will apply our box criterion for obstacles distributed according to a Poisson point process, establishing Theorem 5.1.3.

5.2 Related literature

One of the major motivations of the present work is to provide an easy-to-check, tangible criterion in dimension $d = 2$ for the quite general assumption of an 'almost guaranteed effective minimum speed' introduced in Chapter 4. For a more detailed review of related literature on first passage percolation [Kes93; Ale97; BKS03; LW10; Zha10; BSS26], which corresponds to the situation without surface tension ($a \equiv 0$), and homogenization in a periodic setting [LS05b; DKY08; CM14; GK19] as well as a random setting [AC18; Fel19] we refer to Section 4.2. Here, we focus on results for pinning and depinning phenomena and some more related results from percolation theory.

5.2.1 Physical motivation

The competition between some form of surface tension, a uniform driving force and local obstacles or impurities as represented by (5.1.1) governs a range of physical phenomena such as the random field Ising model [KL85; Kar98], the evolution of domain boundaries in magnetic materials [IV87; Kar98; BN04] or the evolution of dislocation lines in a slip plane [IV87; BN04]. In addition, (5.1.1) is used as a simplification of more complex models, for example immiscible-fluid displacement in porous media [KL85; Kar98]. A key phenomenon observed in these settings is a pinning-depinning transition when increasing the uniform driving force: The interface remains pinned even for a small positive driving force. Only when increasing the forcing past a critical threshold will the interface propagate at a positive effective speed [Kar98; BN04]. Such behavior is also investigated with respect to charge density waves [FL78; IV87; Kar98; BN04], contact lines [Gen85; Kar98], Wigner crystals [NS00; BN04] and vertex (or flux) lines in type-II superconductors [Kar98; NS00; BN04].

5.2.2 Pinning and depinning

Pinning and depinning effects have been extensively studied for the *quenched Edwards-Wilkinson (QEW) model*, which heuristically approximates the evolution by forced mean curvature flow as in (5.1.1) if the interface can be represented as a graph with small slope. For a constant, isotropic curvature coefficient ($a \equiv 1$ in (5.1.1)) and a forcing $F(x) = F_{uni} - F_{obst}(x)$, the height h of the approximating interface in the QEW model is given by

$$\partial_t h(y,t) = \Delta_y h(y,t) + F_{uni} - F(y, h(y)) \qquad \text{on } \mathbb{R}^{d-1}. \tag{5.2.1}$$

However, there are some fundamental differences compared to forced mean curvature flow: This model enforces the graph structure while an interface evolving by forced mean curvature flow can "wrap around" obstacles, leaving enclosures behind the main front.

The main objects of study in the related mathematical literature on pinning and depinning phenomena are the critical thresholds F_*, F^* for the uniform forcing F_{uni} introduced at the start of Section 5.1, below and above which the behavior of the interface evolution changes qualitatively. For the QEW model, these can be defined as follows.

- F_* is the supremum of all $F \geq 0$ such that the height h of the interface remains pointwise bounded for all $F_{uni} \leq F$, that is, there exists a stationary supersolution to (5.2.1) for all $F_{uni} \leq F$ (pinning regime).
- F^* is the infimum of all $F \geq 0$ such that $\liminf_{t\to\infty} \frac{h(y,t)}{t} > 0$ for all $y \in \mathbb{R}^{d-1}$ and for all $F_{uni} > F$ (ballistic regime).

One might assume that there is a dichotomy between these two behaviors ($F_* = F^*$), but in general there might be a sub-ballistic regime between them, in which no pinning occurs but in which the effective speed vanishes, $\limsup_{t\to\infty} \frac{h(y,t)}{t} = 0$. Note that for (5.1.1) without a curvature component and for compactly supported, separated obstacle shapes we would have $F_* = F^* = 0$ since the interface would propagates with velocity F_{uni} along the obstacle-free areas. However, with a curvature component – corresponding to surface tension –, the interface might remain pinned in this setting even for $F_{uni} > 0$ with the obstacles acting as 'anchors', see the results cited below.

Pinning and depinning for periodic media

For the QEW model in arbitrary dimension $d \in \mathbb{N}$ and a periodic obstacle field F_{obst}, it was shown in [DY06] that $0 < F_* < +\infty$ and, under a non-degeneracy assumption on the obstacle field, that $F^* = F_*$ with the height advancing at an effective speed $\overline{v} \sim (F_{uni} - F^*)^{\frac{1}{2}}$ for $0 < F_{uni} - F^* \ll 1$. For the QEW model in dimension $d = 2$ and periodically spaced, soft, and compactly supported obstacles, a scaling relation between the strength of the obstacles and the critical forcings F_*, F^* was established in [CBD20].

Pinning and depinning for random media

For the QEW model in dimension $d = 2$ and periodically spaced, compactly supported obstacles with random strength independently sampled from an exponential distribution, it was shown in [CDL10] that $F_* < +\infty$. In the same setting, $F^* < +\infty$ was established in [DS12] and extended to arbitrary dimension $d \in \mathbb{N}$ for a spatial semi-discretization in [DS17]. For a full time-space-discretization of this model, $F_* = F^*$ was obtained in [BT14]. However, in [DS17] it was shown that $F_* = F^*$ does not hold in general, using parallel planar obstacles with random sparse holes as counterexample with a sub-ballistic regime.

For general discrete interface motions in the graph setting, in [BT14] it was shown that the evolution is ballistic assuming that the probability for the evolution getting blocked in a box decays polynomially as the box size increases, using a general mixing condition rather than the strict independence required for most of the other results in the literature.

For compactly supported obstacles distributed according to a Poisson point process in arbitrary dimension $d \in \mathbb{N}$, $0 < F_* < +\infty$ was established in [DDS11] for both the QEW model and forced mean curvature flow using a percolation result from [Dir+10]. For the QEW model in $d = 2$, the same has been shown while allowing for obstacles with different signs in [DJS21]. If in addition the strength of the individual obstacles is sampled from a probability distribution with a sufficiently thick tail, then for the QEW model it holds that $F_* = +\infty$, that is, pinning occurs no matter how strong the uniform forcing is, see [DJS22]. In contrast, an interface evolving by mean curvature flow with a sufficiently strong uniform forcing would just "wrap around" the obstacles, no matter how strong they are. In fact, given impenetrable obstacles distributed according to a Poisson point process with intensity ρ in dimension $d = 2$, it was shown in [CDO22] that $c\sqrt{\rho} \le F_* \le F^* \le C\sqrt{\rho}$ holds with suitable constants $c, C > 0$ for forced mean curvature flow and related geometric motions. An alternative proof for $F^* \le C\sqrt{\rho}$ will be given in Section 5.8 by applying our main result to this setting.

5.2.3 Percolation theory

Without its curvature component ($a \equiv 0$) and forcing and obstacles independent of the direction ($F(x, e) = F(x)$), the evolution (5.1.1) would correspond to Riemannian first

passage percolation as introduced in [LW10] with the speed at $x \in \mathbb{R}^2$ given by $F(x)$. The domain is hence split into an 'open' area $\{x : F(x) > 0\}$ and a 'closed' area $\{x : F(x) \leq 0\}$. Rather than having to consider the global evolution of a set in order to determine if or when a point is reached, it is sufficient to check for paths along open areas connecting the point of the initial set and compute the passage time. In classical (first passage) percolation theory [Gri99; ADH17], typically a discrete grid is considered instead of continuum space. In this setting, our box criterion breaks down into two questions. First, in a given box, is there path in the open area connecting opposite sides of the box? Second, is there such a path along which the travel time is faster than $v_0^{-1} r$ for some fixed speed $v_0 > 0$?

In percolation theory [Gri99], i.e., the study of the connectivity properties, the first question plays a prominent role. The probability that the answer is yes is called the crossing probability. For example, RSW theory allows the transfer of probability bounds for crossing probabilities between rectangles with different aspect ratios, e.g., lower bounds for crossing the short side of a box imply lower bounds for crossing parallel to the long side of a thin box. This theory originated with the seminal works of Russo [Rus78] and Seymour and Welsh [SW78] on Bernoulli percolation.

Regarding the behavior on large scales, in the supercritical phase (where an infinite open cluster exists), the crossing probability converges to one exponentially fast as the box size increases, see [DP96; DT20]. This is similar to our Corollary 5.6.2. We however only obtain stretched exponential bounds, but note that the aspect ratio of our boxes gets worse as the size increases. Our argument in Section 5.6 corresponds to a renormalization or coarse graining scheme which in spirit is similar to such approaches in percolation theory, e.g., [AP96; DT20].

In first passage percolation on infinite percolation clusters [GM04; GM07; GM09; Mou11; CT16; Dem24], i.e. the study of passage times or the so-called 'chemical distance', a key tool has been an almost guaranteed upper bound for passage times on large scales established in [AP96], which together with control of the size of holes in the cluster essentially corresponds to our 'effective minimum speed'.

5.3 Setting and Assumptions

For convenience, we denote dependencies on the parameters introduced with the assumptions below via

$$\text{data} := (C_{1A}, C_{1F}, q, c_w, v_0, p_0, r_0).$$

5.3.1 Admissible coefficient fields

We say that (A, F) is an admissible coefficient field if the following conditions are satisfied. Here, F takes the role of the forcing in (5.1.1) while A corresponds to the (degenerate) diffusion matrix for the level-set formulation – see Section 5.3.2 below – derived from a curvature coefficient field $a = \sigma^2$ as in (5.1.1) by combining it with a projection onto the tangent space. We use this diffusion matrix rather than the curvature coefficient field in order to keep the notation aligned with Chapter 4.

We assume that A is uniformly Lipschitz continuous and bounded, given by

$$A(x,e) = \sigma^2(x,e)\big(\mathrm{Id}_{2\times 2} - e \otimes e\big) \quad \text{with } \big\|\sigma\big\|_{C^{0,1}(\mathbb{R}^2\times\mathbb{S}^1)}, \big\|A\big\|_{C^{0,1}(\mathbb{R}^2\times\mathbb{S}^1;\,\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \leq C_{1A} \tag{5.3.1}$$

for some $\sigma \in C^{0,1}(\mathbb{R}^2 \times \mathbb{S}^1)$ and a uniform constant C_{1A}. We further assume that F is also uniformly bounded and Lipschitz, that is there exists $C_{1F} > 0$ such that

$$F \in C^{0,1}(\mathbb{R}^2 \times \mathbb{S}^1;\, \mathbb{R}) \text{ with } \|F\|_{C^{0,1}(\mathbb{R}^2 \times \mathbb{S}^1;\, \mathbb{R})} \leq C_{1F}. \tag{5.3.2}$$

To simplify the notation we use the 0-homogeneous extension of A and F to $(\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\})$, for A given by $A(x,\xi) \coloneqq A\big(x, |\xi|^{-1}\xi\big)$. For these admissible coefficient fields, we will now introduce the level-set formulation of (5.1.1), which is well-posed in the viscosity solution framework.

5.3.2 Set evolutions and the level-set formulation

In general, interface evolutions such as (5.1.1) can lead to topological changes of the evolving interfaces, allowing for different interpretations of (5.1.1). In order to give rigorous meaning to the evolution, a variety of frameworks have been introduced, which for some cases are equivalent but differ in others. As in Chapter 4, we work with viscosity solutions of the level-set formulation. For a collection of results from the theory of viscosity solutions relevant for our setting, which have been gathered from [CIL92; Gig06; Gig+91; Juu01], see Section 4.A.

Rather than as an interface evolution, we view

$$V = a\big(x,\nu\big)H + F\big(x,\nu\big) \tag{5.1.1}$$

as a set evolution with V denoting the outwards normal speed of the boundary, ν its normal direction, and H the curvature of the boundary. In our chosen framework, these set evolutions are given as the sublevel sets of a function $u\colon \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ satisfying the level-set equation, a viscous Hamilton–Jacobi equation derived from (5.1.1) with degenerate diffusion matrix $A(x,e) = a(x,e)(\mathrm{Id}_{2\times 2} - e \otimes e)$ for $x \in \mathbb{R}^2$, $e \in \mathbb{S}^1$. While it is also possible to view the level sets $t \mapsto \{x : u(x,t) = 0\}$ as an interface evolution, 'fattening' might occur [BSS93], meaning that the level set develops non-empty interior.

Definition 5.3.1 (The set evolution in the heterogeneous environment)**.** Let (A,F) be an admissible coefficient field satisfying (5.3.1) and (5.3.2) respectively. Given a closed set $S \subset \mathbb{R}^2$, we define the evolution $(\mathscr{R}_t(S))_{t\geq 0}$ of S with respect to (A,F) as

$$\mathscr{R}_t\big(S\big) = \mathscr{R}_t^{(A,F)}\big(S\big) \coloneqq \big\{x \in \mathbb{R}^2 \,:\, u_S(x,t) \leq 0\big\},$$

where u_S with continuous initial data $u_S(\cdot,0)$ satisfying $S = \{x \in \mathbb{R}^2 \,:\, u_S(x,0) \leq 0\}$ is the (viscosity) solution of

$$\partial_t u + G_{(A,F)}(x, \nabla u, D^2 u) = 0 \quad \text{in } \mathbb{R}^2 \times (0,\infty) \tag{5.3.3}$$

$$\text{with} \quad G_{(A,F)}(x,\xi,X) \coloneqq -\operatorname{tr}\left(A\left(x, \frac{\xi}{|\xi|}\right)X\right) + F\left(x, \frac{\xi}{|\xi|}\right)|\xi| \tag{5.3.4}$$

for $(x,\xi,X) \in \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and the singularity at $\xi = 0$ treated via semi-continuous envelopes in the framework of viscosity solutions. Such a solution always exists and $\mathscr{R}_t(S)$ is unique and independent of the specific initial data $u_S(\cdot,0)$. We can assume that u_S is uniformly continuous on $\mathbb{R}^2 \times (0,T)$ for any $T > 0$, see Theorem 4.A.14, Remark 4.A.15, and Theorem 4.A.17.

5.3.3 Assumptions on the probability distribution

In short, we assume that the coefficient field (A, F) in (5.3.3) is a random variable sampled from a stationary probability measure with finite range of dependence, which satisfies the box criterion as informally described in Assumption '(P□)'.

More explicitly, the probability space is defined as the set of admissible coefficient fields

$$\Omega := \big\{(A,F) \,:\, A \text{ satisfies (5.3.1)}, \, F \text{ satisfies (5.3.2)}\big\}. \tag{5.3.5}$$

On Ω we define the σ-algebras $\{\mathcal{F}(U)\}$ for all Borel sets $U \subset \mathbb{R}^2$,

$$\begin{aligned}\mathcal{F}(U) := \text{ the } \sigma\text{-algebra generated by the family of maps}\\ \Omega \to \mathbb{R}^{2\times 2} \times \mathbb{R},\ (A,F) \mapsto \big(A(x,e), F(x,e)\big) \quad \text{with } x \in U,\ e \in \mathbb{S}^1.\end{aligned} \tag{5.3.6}$$

The σ-algebras $\mathcal{F}(U)$ can be seen as containing all of the information derived from observing the coefficient field in $U \subset \mathbb{R}^2$.

We assume that the randomness of the coefficient field is described by a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(\mathbb{R}^2))$, which satisfies the following conditions. The first two assumptions are standard in the theory of stochastic homogenization:

(P1) **Stationarity.** $\mathbb{P}$ is invariant under translations of the coefficient field:

$$\mathbb{P} = \mathbb{P} \circ \tau_x, \quad \text{for any } x \in \mathbb{R}^2,$$

where $\tau_x(\widetilde{\Omega}) = \{(A(\cdot + x, \cdot), F(\cdot + x, \cdot)) \,:\, (A,F) \in \widetilde{\Omega}\}$ for $\widetilde{\Omega} \in \mathcal{F}(\mathbb{R}^2)$.

(P2) **Finite range of dependence.** For all Borel sets $U, V \subset \mathbb{R}^2$ with $\operatorname{dist}(U,V) \geq 1$,

$$\mathcal{F}(U) \text{ and } \mathcal{F}(V) \text{ are } \mathbb{P}\text{-independent.}$$

The final assumption is our box criterion, which is illustrated in Figure 5.1.1 and the informal version of which was stated in the introduction as '(P□)'. For sampling the coefficient field we use boxes scaling sub-linearly in width, that is, given a length scale $r > 0$ we consider the box

$$Q_r^{\uparrow} := [0, w(r)] \times [-w(r), r] \qquad \text{with} \quad w(r) := \left(\frac{r}{c_w}\right)^{\frac{1}{q}}. \tag{5.3.7}$$

for some fixed $q > 1$ and $c_w > 0$. For such a box $x + Q_r^{\uparrow}$ of size $r > 0$ anchored at $x \in \mathbb{R}^2$, for a prescribed average speed $v > 0$ we consider the event $E_{r,x}^{\uparrow,v}$ that there is a non-shrinking set in the bottom of the box which within the box evolves to reach the top of the box with an average speed of at least v. Formally, we define this event as

$$\begin{aligned} E_{r,x}^{\uparrow,v} = \Big\{(A,F) \in \Omega \,:\, &\text{there is a closed set } S \subset x + [0, w(r)] \times [-w(r), 0] && (5.3.8)\\ &\text{such that } S \text{ is stable with respect to } (A,F)_{x+Q_r^{\uparrow}} && (5.3.9)\\ &\text{and } \mathscr{R}^{(A,F)_{x+Q_r^{\uparrow}}}_{v^{-1}r}\big(S\big) \cap \big(x + [0, w(r)] \times \{r\}\big) \neq \emptyset\Big\}. && (5.3.10)\end{aligned}$$

where *'stable'* sets and the *'restricted coefficient field'* $(A,F)_{x+Q_r^{\uparrow}}$ are defined below in Section 5.3.4. The evolution of stable sets is non-shrinking, ensuring that the evolved set stays connected to the *'starting area'* $x + [0, w(r)] \times [-w(r), 0]$, see Lemma 5.5.11. The

restriction of the coefficient field confines the evolution to a small neighborhood of the box and ensures the measurability of $E^{\uparrow,v}_{r,x}$ with respect to $\mathcal{F}(x+Q^{\uparrow}_r+B_1)$, see Lemma 5.5.16.

We write

$$E^{\uparrow,v}_r := E^{\uparrow,v}_{r,0} = \tau_x\big(E^{\uparrow,v}_{r,x}\big),$$

where τ_x is a translation as defined in Assumption (P1). Due to the stationarity from Assumption (P1), we have $\mathbb{P}\big[E^{\uparrow,v}_r\big] = \mathbb{P}\big[E^{\uparrow,v}_{r,x}\big]$ for all $x \in \mathbb{R}^2$. We require that there is an initial length scale $[r_0, r_1)$ and a speed $v_0 > 0$ such that this event and its counterparts $E^{\rightarrow,v_0}_r, E^{\downarrow,v_0}_r, E^{\leftarrow,v_0}_r$ for the other directions are sufficiently likely, with the threshold depending on the length scale and the width-scaling:

(P□) **Box criterion for $d = 2$.** There exist a velocity $v_0 > 0$, a width scaling exponent $q > 1$ and a width scaling factor $c_w > 0$ such that

$$\min\big\{\mathbb{P}\big[E^{\uparrow,v_0}_r\big], \mathbb{P}\big[E^{\rightarrow,v_0}_r\big], \mathbb{P}\big[E^{\downarrow,v_0}_r\big], \mathbb{P}\big[E^{\leftarrow,v_0}_r\big]\big\} \geq 1 - p_0, \quad \text{for all } r \in [r_0, r_1), \tag{5.3.11}$$

with $r_1 := w^{-1}(r_0 + 2w(r_0))$, see Figure 5.3.1, for some initial box length $r_0 > 0$ and an acceptable probability of failure $p_0 \geq 0$ satisfying

$$r_0 \geq \max\Big\{3^q c_w,\ \big(24q\big)^{\frac{q}{q-1}} c_w^{-\frac{1}{q-1}}\Big\}, \tag{5.3.12}$$

$$p_0 < \min\Big\{\exp\Big(-\frac{\log(6) + q\log(k_0)}{k_0 - 1}\Big), \exp\Big(-\frac{2q}{k_0}\Big)\Big\},$$

$$\text{where} \quad k_0 := \Big\lfloor \frac{1}{12q} c_w^{\frac{1}{q}} r_0^{\frac{q-1}{q}} \Big\rfloor \geq 2. \tag{5.3.13}$$

Here, the events $E^{\rightarrow,v}_{r,x}, E^{\downarrow,v}_{r,x}, E^{\leftarrow,v}_{r,x}$ above are defined analogously to $E^{\uparrow,v}_{r,x}$, replacing the vertical box $Q^{\uparrow}_r$ with

$$\begin{aligned} Q^{\rightarrow}_r &:= [-w(r), r] \times [-w(r), 0] && \text{with starting area } [-w(r), 0]^2, \\ Q^{\downarrow}_r &:= [-w(r), 0] \times [-r, w(r)] && \text{with starting area } [-w(r), 0] \times [0, w(r)], \\ Q^{\leftarrow}_r &:= [-r, w(r)] \times [0, w(r)] && \text{with starting area } [0, w(r)]^2. \end{aligned}$$

Note that the threshold for p_0 in (5.3.13) can be arbitrarily close to one if the initial scale is sufficiently large (and the corresponding boxes sufficiently thin).

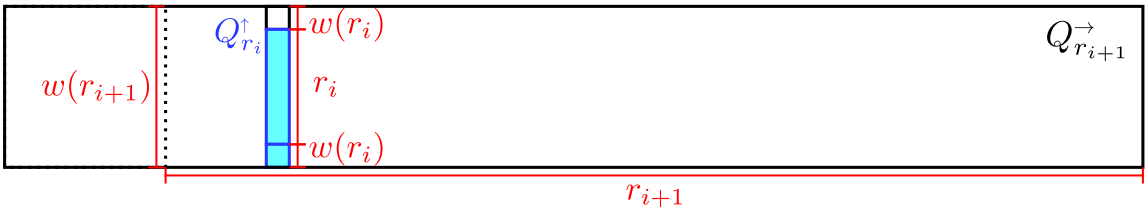


Figure 5.3.1: An illustration of how r_0 and r_1 relate. The length scales are chosen such that translations of $Q^{\uparrow}_{r_0}$ and $Q^{\downarrow}_{r_0}$ fit into $Q^{\rightarrow}_{r_1}$ without their starting areas overlapping.

Remark 5.3.2. Note that for (P□) it can be sufficient to check a single box size rather than every size in a given length scale, albeit with harsher requirements on the probability bound. Probability bounds for a continuous range of sizes and thus Assumption (P□) can then be obtained using a construction similar to the one in Section 5.6.

5.3.4 Stable sets and restricted coefficient fields

We now make precise the notions of 'stable' sets and 'restricted coefficient fields' used to define the event $E_{r,x}^{\uparrow,v}$ in (5.3.8) for the box criterion (P□).

Introduction of Stable Sets

Throughout this paper, sets which do not shrink under the interface evolution will be central to many arguments. We call these sets or the corresponding interfaces *stable*. We will discuss properties of these sets in Section 5.5.1.

Definition 5.3.3 (Stable Sets)**.** A closed set $S \subset \mathbb{R}^2$ is called stable (with respect to $(A, F) \in \Omega$) if $u := 1 - \chi_S$ is a supersolution of

$$-\operatorname{tr}\left(A\left(x, \frac{\nabla u}{|\nabla u|}\right) D^2 u\right) + F\left(x, \frac{\nabla u}{|\nabla u|}\right)|\nabla u| = 0 \quad \text{in } \mathbb{R}^2. \tag{5.3.14}$$

Introduction of restricted coefficient fields

For the box criterion, we want to modify the forcing to ensure that the initial set evolves and stays within a small neighborhood of the given box. This should also allow us to apply the finite range of dependence from Assumption (P2) for boxes which are sufficiently far apart. For this purpose, we introduce suitably restricted coefficient fields.

Definition 5.3.4 (Restricted coefficient fields)**.** Let $Q \subset \mathbb{R}^2$ be a Borel Set. Given $(A, F) \in \Omega$ we define $(A, F)_Q := (A_Q, F_Q)$, the coefficient fields restricted to Q, as

$$A_Q(x,e) := \begin{cases} A(x,e) & \text{for } \operatorname{dist}(x,Q) \le \frac{2}{3}, \\ 9\big(1-\operatorname{dist}(x,Q)\big)^2 A(x,e) & \text{for } \operatorname{dist}(x,Q) \in \left[\frac{2}{3}, 1\right], \\ 0 & \text{for } \operatorname{dist}(x,Q) \ge 1, \end{cases} \tag{5.3.15}$$

$$F_Q(x,e) := \begin{cases} F(x,e) & \text{for } \operatorname{dist}(x,Q) \le 0, \\ 3\left(\frac{1}{3} - \operatorname{dist}(x,Q)\right) F(x,e) - 3\operatorname{dist}(x,Q)\max\{C_{1F},\, 6C_{1A}\} & \\ & \text{for } \operatorname{dist}(x,Q) \in \left[0, \frac{1}{3}\right], \\ -\max\{C_{1F},\, 6C_{1A}\} & \text{for } \operatorname{dist}(x,Q) \ge \frac{1}{3}. \end{cases} \tag{5.3.16}$$

By definition, the evolution of a set with respect to $(A, F)_Q$ only depends on (A, F) restricted to $Q + B_1$ and hence should be $\mathcal{F}(Q + B_1)$ measurable, see e.g. Lemma 5.5.16. As a consequence of this definition, any $S \subset Q$ stays stuck in $Q + B_{\frac{2}{3}}$ when evolving by (5.3.3) with coefficients $(A, F)_Q$. In addition, in this case the evolution of a set with respect to $(A, F)_Q$ provides a lower bound to the evolution of the same set with respect to the unmodified coefficient field (A, F). For the rigorous statement see Lemma 5.5.8 in Section 5.5.2.

Remark 5.3.5 (Well-posedness of evolution for restricted coefficient fields). Existence, uniqueness and uniform continuity for solutions of the level-set equation as well as invariance for the evolution of the level sets as stated in Definition 5.3.1 for $(A, F) \in \Omega$ also hold if $(A, F) = (\widetilde{A}, \widetilde{F})_Q$ for some $(\widetilde{A}, \widetilde{F}) \in \Omega$ and $Q \subset \mathbb{R}^2$.

Remark 5.3.6 (Choice of (5.3.15) and (5.3.16)). Instead of transitioning from $(A, F)(\cdot)$ on Q to $(0, \overline{F})$ with $\overline{F} = -\max\{C_{1F}, 6C_{1A}\}$ via interpolation on a scale of 1, we could choose an arbitrarily small scale $s > 0$ – as long as we adapt $\overline{F}$ accordingly (see the proof of Lemma 4.5.10, adaptation only necessary for non-convex Q). Then the evolution of a set with respect to $(A, F)_Q$ would be $\mathcal{F}(Q + B_s)$ measurable. This and being more precise in the proof of Proposition 5.6.1 would for example allow us to weaken the assumption on r_0 in (5.3.12) and (5.3.13) from (P□). The arguments in Section 5.6 stay the same, except that we place boxes with a distance of $1 + 2s$ instead of 3 for independence. Therefore, we just choose a scale of 1 for a more stringent notation.

5.3.5 The effective minimum speed assumption from Chapter 4

We now precisely state the effective minimum speed assumption (P⊙) including the notions of an 'effective minimum speed' of propagation, 'approximate stability' and 'fat' sets as introduced in Chapter 4. For detailed descriptions and illustrations of these notions see Section 4.3.3.

Intuitively, a set evolves with an 'effective minimum speed' on a scale $h > 0$ if – at any step of the evolution – the set is guaranteed to expand by at least h in every direction within a time proportional to h, up to ignoring enclosures of size up to h left behind the main front. To formalize ignoring such enclosures, we introduce the notion of h-envelopes. A set is 'approximately stable' on a scale $h > 0$ if there is a good interior approximation which is stable, i.e., non-shrinking, and h-envelops the original set after a time proportional to h. A set is h-'fat' if it can be written as the union of balls with radius h.

Definition 5.3.7. Let $(A, F) \in \Omega$. For a set $S \subset \mathbb{R}^2$, let $t \mapsto \mathscr{R}_t^{(A,F)}(S)$ denote the set evolution from Definition 5.3.1. Let $h > 0$, $c > 0$.

(i) We say that a set $S_1 \subset \mathbb{R}^2$ *h-envelops* another set $S_2 \subset \mathbb{R}^2$, denoted by $S_2 \subset_h S_1$, if for each connected component H of $(S_1)^c$ either $H \cap S_2 = \emptyset$ or $H \subset S_1 + \overline{B}_h$.

(ii) We say that (A, F) admits $v > 0$ as an *effective minimum speed of propagation* on the scale h in $M \subset \mathbb{R}^2$ for a set $S \subset \mathbb{R}^2$ if for any $t \geq 0$ it holds that

$$\left(\mathscr{R}_t^{(A,F)}(S) \cap M\right) + \overline{B}_h \subset_h \mathscr{R}_{t+v^{-1}h}^{(A,F)}(S). \tag{5.3.17}$$

(iii) We say that a set $S \subset \mathbb{R}^2$ is *(h, c)-approximately stable* with respect to (A, F) if there exists a set $S_h \subset S$ such that

$$S_h \text{ is stable}, \qquad S \subset_h \mathscr{R}_{ch}^{(A,F)}(S_h), \qquad S_h \supset \mathbb{R}^2 \setminus \left(S^c + cv_{\max}B_h\right),$$

where $v_{\max} = v_{\max}(\text{data})$ corresponds to the maximum speed with which the interface propagates on larger scales, see Lemma 5.5.4. In this case, S_h is called a *stable (h, c)-approximation* of S.

(iv) We say that a set $S \subset \mathbb{R}^2$ is *h-fat* if

$$S = \bigcup \{\overline{B}_h(x) : x \in \mathbb{R}^2 \text{ with } \overline{B}_h(x) \subset S\}.$$

Using these notions, we now state the rigorous version of the effective minimum speed assumption (P⊙) from Chapter 4, with the informal version '(P⊙)' stated in the introduction. In short, for coefficient fields satisfying (P⊙), there is a velocity $v_{\min} > 0$ such that – as

the scale increases – it becomes increasingly likely that all sufficiently fat sets in an area have stable approximations for which $v_{\min}$ is admitted as an effective minimum speed of propagation. In order to ensure measurability with respect to a small neighborhood of the area under consideration, we require a family of events guaranteeing the effective minimum speed rather than defining these events directly.

In the present work, our goal is to establish (P◎) for suitable parameters based on Assumptions (P1), (P2), and (P□) from Section 5.3.3.

(P◎) **Almost guaranteed effective minimum speed**. There is $v_{\min} > 0$, $h_0 \geq 1$ and a rate $\alpha_{◎} > 0$ with constants $c_{◎}, C_{◎}, c_{\mathcal{F}}, c_s > 0$, such that for each $h \geq h_0$ there is a family of events $(E_h^{◎}(M))_{M \subset \mathbb{R}^2}$ with

$$\mathbb{P}[E_h^{◎}(M)] \geq 1 - C_{◎}\left(1 + \frac{\operatorname{diam}(M)}{h}\right)^2 \exp(-c_{◎} h^{\alpha_{◎}}), \tag{5.3.18}$$

which satisfies

(∗i) $E_h^{◎}(M)$ is measurable with respect to $\mathcal{F}(M + c_{\mathcal{F}}\overline{B}_h)$,

(∗ii) for all $M_1 \subset M_2$ it holds that $E_h^{◎}(M_1) \supset E_h^{◎}(M_2)$,

(∗iii) for all $(A, F) \in E_h^{◎}(M)$, every h–fat set S with $\partial S \subset M$ has a stable (h, c_s)–approximation, for which (A, F) admits $v_{\min} > 0$ as an effective minimum speed of propagation on the scale h in $M \subset \mathbb{R}^2$, see Definition 5.3.7 above for the precise introduction of these notions.

In particular, (P◎) implies ballistic motion on large scales. However, ballistic motion does not necessarily imply the existence of an (almost guaranteed) effective minimum speed as introduced here, at least not in general dimension. For ballistic motion it is sufficient to compare the main front to the regular initial set. But for an effective minimum speed, the main front has to expand compared to the state of the set evolution at any time, which is potentially much less regular than the initial set.

For an illustration, consider the counterexample from [CM14] to periodic homogenization of forced mean curvature flow despite a uniformly positive combined forcing in dimension $d = 3$: Thin, infinitely long, parallel cylinders are placed periodically in space with the forcing inside the cylinders much stronger than outside with a transition layer in-between. Due to the mean curvature, sets initialized inside the cylinders are not able to pass the transition layer. The cylinders can hence be thought of as pipes along which sets expand much faster but to which these sets remain constrained. Choosing the half space orthogonal to the pipes as the initial set, the main front will expand in the direction parallel to the pipes at different speeds: With a fast head speed inside the pipes and with a much slower but still positive tail speed in the remaining space, lagging increasingly far behind the evolution inside the pipes. We hence observe a ballistic motion, but the evolution does not admit an effective minimum speed. For the points reached early within a pipe, the set will not expand into the close-by area between the pipes: This area remains connected to the complement of the set evolution until it will be covered much later by the part of the front which advances with the slower tail speed.

5.4 Main Result and the Strategy for the Proof

5.4.1 The main result

Our main result establishes an effective minimum speed and thus ballistic behavior for sets evolving by (5.1.1) for random coefficient fields sampled from a stationary coefficient field with a finite range of dependence and which satisfies our box criterion.

Theorem 5.4.1 (Main result). *Let $(A, F) \in \Omega$ be a random coefficient field sampled from a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(\mathbb{R}^2))$ which*

- *is stationary, that is invariant with respect to translations (Assumption (P1)),*
- *has finite range of dependence (Assumption (P2)), and*
- *satisfies our box criterion (Assumption (P□)) with width-scaling $q > 1$).*

Then the coefficient field admits an almost guaranteed effective minimum speed: the coefficient field satisfies (P◎) for every $\alpha_\circledcirc \in \left(0, \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\right)$ with coefficients $v_{\min} = v_{\min}(\alpha_\circledcirc, \mathrm{data}) > 0$, $h_0 = r_0$, $c_\circledcirc = c_\circledcirc(\alpha_\circledcirc, \mathrm{data}) > 0$, $C_\circledcirc > 0$, $c_{\mathcal{F}} > 0$, and $c_s = c_s(\mathrm{data}) > 0$.

As a consequence, we obtain Theorem 5.1.3 by showing that the assumptions of Theorem 5.4.1 are satisfied for obstacles distributed according to a Poisson point process and a sufficiently large uniform forcing, see Section 5.8. Theorem 5.4.1 then yields an almost guaranteed effective minimum speed, which allows the application of the homogenization results from Chapter 4 to Poisson distributed obstacles and yields Theorem 4.1.1.

5.4.2 Strategy of the proof

In Section 5.5, we establish some preliminary results about the properties of stable sets and set evolutions with respect to restricted coefficient fields. The proof of Theorem 5.4.1 then consists of two major steps:

1. The box criterion guarantees lower bounds for the probability that the interface propagates along boxes of length $r \in [r_0, r_1)$ with average speed v_0. The key step of our argument in Section 5.6 is to iteratively extend these probability bounds to larger box lengths $r \in [r_i, r_{i+1})$ with average speed v_i for $i \in \mathbb{N}$, while at the same time improving the probability bounds with each step and sufficiently controlling the average speed such that $\liminf_{i\to\infty} v_i > 0$.
2. In Section 5.7, we show that a suitable network of such boxes along which the interface propagates guarantees an effective minimum speed in the sense of (P◎). Due to the results from Section 5.6, the existence of such a network is sufficiently likely for the probability bounds in (P◎) to be satisfied.

A key tool in the present work is Lemma 5.5.9, our 'stitching lemma'. Combining our notion of stable sets from Definition 5.3.3 with the properties of the restricted coefficient fields from Definition 5.3.4 allows us to chain the guaranteed propagation for consecutive boxes crossing each other, see Figure 5.4.1. The main method for our proofs is to construct such overlapping box configurations and derive the targeted property on the larger scale. The finite range of dependence from Assumption (P2) is applied for subgroups of suitably spaced boxes to control or improve the probability.

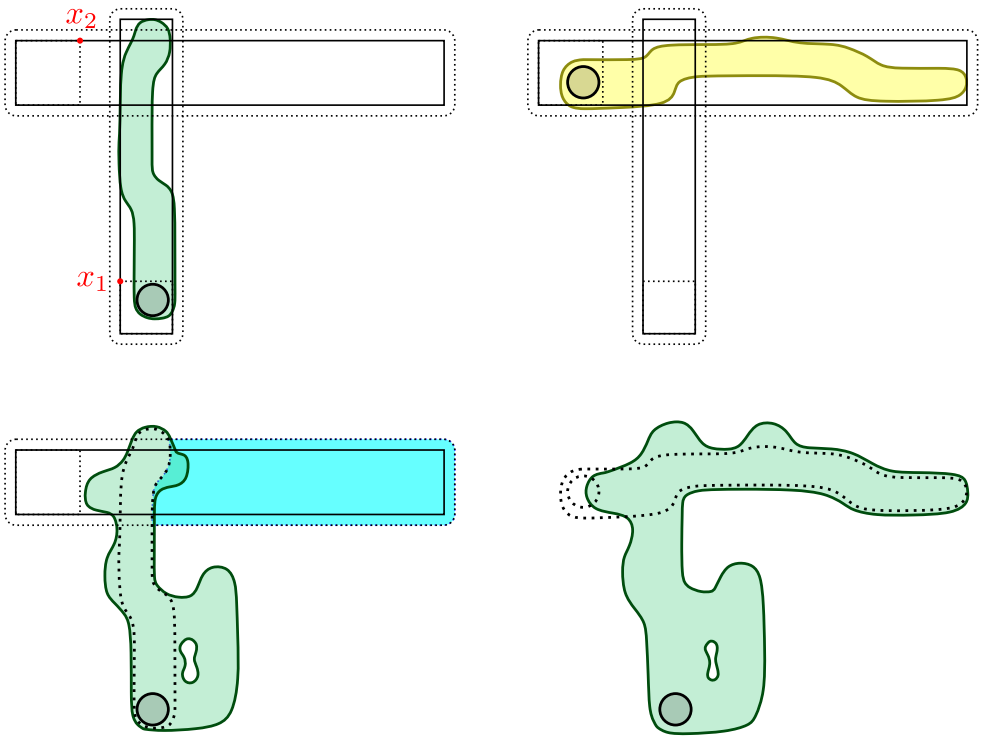


Figure 5.4.1: An illustration of our 'stitching lemma' applied to chain the propagation along two consecutive boxes. Assume that for two intersecting boxes the respective events $E^{\uparrow,v}_{r_1,x_1}$ and $E^{\rightarrow,v}_{r_2,x_2}$ hold for the coefficient field (A, F). Top left: Due to $E^{\uparrow,v}_{r_1,x_1}$, there is a stable set S_1 (depicted in grey) which reaches the top of the first box after evolving for $t_1 = v^{-1}r_1$ with respect to the coefficient field restricted to the first box (depicted in green). Top right: Due to $E^{\rightarrow,v}_{r_2,x_2}$, there is a stable set S_2 (depicted in grey) which reaches the right end of the second box after evolving for $t_2 = v^{-1}r_2$ with respect to the coefficient field restricted to the second box (depicted in yellow). Bottom left: By comparison, the evolution of S_1 with respect to the unrestricted coefficient field at t_1 (depicted in green) separates S_2 from the right side of the second box (depicted in blue). Bottom right: Since S_1 and thus its evolution are stable, the stitching lemma allows us to compare the evolution of S_1 and S_2 in that area and we obtain that the evolution of S_1 reaches the right end of the second box at $t_1 + t_2$ (depicted in green)

Lifting the box criterion to larger scales while improving the probability bound

The box criterion (P□) guarantees that $\mathbb{P}\big[E_r^{\uparrow,v_0}\big] \geq 1 - p_0$ for $r \in [r_0, r_1)$ and the same for the other directions. The first goal is to iteratively lift these to larger scales while improving the probability bound. That is, for all $i \in \mathbb{N}$ we want to show that

$$\min\big\{\mathbb{P}\big[E_r^{\uparrow,v_i}\big], \mathbb{P}\big[E_r^{\rightarrow,v_i}\big], \mathbb{P}\big[E_r^{\downarrow,v_i}\big], \mathbb{P}\big[E_r^{\leftarrow,v_i}\big]\big\} \geq 1 - p_i \qquad \text{for all } r \in [r_i, r_{i+1}) \tag{5.4.1}$$

for $v_i > 0$ and $p_i \in [0, 1)$ while ensuring that $v_i \geq v_\infty$ for some $v_\infty > 0$ and $p_i \to 0$ for $i \to \infty$ as fast as possible. As r_1 in (P□), the sequence $(r_i)_i$ is given by $w(r_{i+1}) = r_i + 2w(r_i)$, meaning that the box-width of the next scale corresponds to the box-length (and an additional starting area) of the current scale, see Figure 5.3.1.

Given a large box $Q_r^\uparrow$, the idea is to place layers of smaller boxes within and show that if the interface propagates along suitable subsets of the smaller boxes then the interface propagates from the starting area of the large box to its top. To be more precise, starting from the top of the starting area, we place multiple layers of smaller boxes in $Q_r^\uparrow$, see Figure 5.6.1. Each layer consists of three subgroups of size $k_\parallel, k_= \in \mathbb{N}$ to be fixed later:

1. $k_\parallel$ parallel translations of $Q^\uparrow_{r_\parallel}$ with $r_\parallel = r_\parallel(r, k_\parallel) \ll r$,

2. $k_=$ parallel translations of $Q^{\rightarrow}_{r_=}$ with $r_= + 2w(r_=) = w(r)$ all cutting through the top of the first subgroup from left to right, potentially enabling the application of the stitching lemma,

3. $k_=$ parallel translations of $Q^{\leftarrow}_{r_=}$ all cutting through the top of the first subgroup from right to left.

Within each subgroup, the boxes are spaced with distance 3 such that the respective restricted coefficient fields are independent. The next layer is then placed such that the horizontal boxes of the previous layer cut through the bottom of the new vertical boxes, until the top of $Q^{\uparrow}_r$ is reached. For the number of layers we write $N = N(r, k_\parallel, k_=, w(\cdot))$. Note that the boxes are placed in such a way that, if for each subgroup there is at least one box with a stable set in its starting area propagating to the opposite end, then the stable set from the first subgroup is in the starting area of $Q^{\uparrow}_r$ and – due to the stitching lemma – propagates layer by layer until it reaches the top of $Q^{\uparrow}_r$, see Figure 5.6.2.

Suppose that

$$\min\Big\{\mathbb{P}\Big[E^{\uparrow,v}_{r_\parallel}\Big], \mathbb{P}\Big[E^{\rightarrow,v}_{r_=}\Big], \mathbb{P}\Big[E^{\leftarrow,v}_{r_=}\Big]\Big\} \geq 1 - p$$

for some $v > 0$ and $p \in [0, 1)$. Then due to the stationarity from Assumption (P1) and due to the finite range of dependence from Assumption (P2), with the argument from above we obtain

$$\begin{aligned}\mathbb{P}\big[E^{\uparrow,\widetilde{v}}_r\big] &\geq 1 - N\Big(1 - \mathbb{P}\Big[E^{\uparrow,v}_{r_\parallel}\Big]\Big)^{k_\parallel} - N\big(1 - \mathbb{P}\big[E^{\rightarrow,v}_{r_=}\big]\big)^{k_=} - N\big(1 - \mathbb{P}\big[E^{\leftarrow,v}_{r_=}\big]\big)^{k_=} \\ &\geq 1 - Np^{k_\parallel} - 2Np^{k_=} \end{aligned} \tag{5.4.2}$$

for some average speed $\widetilde{v} = \widetilde{v}(v, r, k_\parallel, k_=, w(\cdot))$.

While the average speed along the smaller boxes is v, this average speed $\widetilde{v}$ for reaching the top of the large box is slower due to the extra distance covered. For each of the N layers, the horizontal distance $r_=$ is traversed at average speed v. In addition, between each of the layers there is an overlap of the vertical subgroups corresponding to the height $k_=(w(r_=) + 3)$ of the horizontal subgroups. Finally, the last layer might have to be pushed down by up to $r_\parallel$ to stay within the larger box $Q^{\uparrow}_r$. In summary, we obtain

$$\widetilde{v} \geq v\frac{r}{r + Nr_= + Nk_=(w(r_=) + 3) + r_\parallel}. \tag{5.4.3}$$

Iteratively repeating this argument for all directions yields (5.4.1) for all $i \in \mathbb{N}$ with (5.4.2) governing the behavior of $(p_i)_{i\in\mathbb{N}}$ and with (5.4.3) governing the behavior $(v_i)_{i\in\mathbb{N}}$. Due to the sublinear scaling of the box-width, an increasingly smaller percentage of the minimum average speed is shaved off in (5.4.3) while at the same time we are still able to choose increasingly large $k_=, k_\parallel$ as $r \to \infty$. This leads to exponentially improving probability bounds provided that relative to each other r_0 is large enough and p_0 is small enough, corresponding to the bounds in Assumption (P□).

As a consequence, we do indeed obtain $v_\infty > 0$ such that $v_i \geq v_\infty$ for all $i \in \mathbb{N}$ and a positive rate $\alpha_\uparrow > 0$ such that

$$\min\big\{\mathbb{P}\big[E^{\uparrow,v_\infty}_r\big], \mathbb{P}\big[E^{\rightarrow,v_\infty}_r\big], \mathbb{P}\big[E^{\downarrow,v_\infty}_r\big], \mathbb{P}\big[E^{\leftarrow,v_\infty}_r\big]\big\} \geq 1 - \exp\big(-cr^{\alpha_\uparrow}\big) \tag{5.4.4}$$

for all $r \geq r_0$ and some $c > 0$, which is the main result from Section 5.6, see Corollary 5.6.2.

Effective minimum speed

In order to show that there is an almost guaranteed effective minimum speed $v_{\min} > 0$ in the sense of (P◎), for a given set $M \subset \mathbb{R}^2$ and a scale $h \gg 1$ we need to define an event $E_h^{\circledcirc}(M)$ which implies that all h-fat sets with boundary in M have a stable approximation which within M propagates with effective minimum speed $v_{\min}$.

In Section 5.7, we choose $E_h^{\circledcirc}(M)$ to be the event that there is a net of boxes along which the interface propagates with mesh width $\sim h$. More precisely, we set

$$E_{2h}^{\circledcirc}(M) := \bigcap_{x \in h\mathbb{Z}^2 \,:\, (x+[-2h,3h]^2)\cap M \neq \emptyset} E_{h,x}^{\square, v_\infty},$$

where $E_{h,x}^{\square,v_\infty}$ is the event that there are boxes $y^{\uparrow} + Q_{2h}^{\uparrow}$, $y^{\rightarrow} + Q_{2h}^{\rightarrow}$, $y^{\downarrow} + Q_{2h}^{\downarrow}$, and $y^{\leftarrow} + Q_{2h}^{\leftarrow}$ which first cut through the grid cell $x + [0,h]^2$ and then the respective neighboring cell and for which the events $E_{2h,y^{\uparrow}}^{\uparrow,v_\infty}$, $E_{2h,y^{\rightarrow}}^{\rightarrow,v_\infty}$, $E_{2h,y^{\downarrow}}^{\downarrow,v_\infty}$, and $E_{2h,y^{\leftarrow}}^{\leftarrow,v_\infty}$ hold, see Figure 5.7.1. Probability bounds as required in (P◎) follow from (5.4.4) in a straightforward fashion.

For $(A,F) \in E_{2h}^{\circledcirc}(M)$, we obtain the stable approximation of a $2h$-fat set S with $\partial S \subset M$ by combining stable sets from grid cells included in S and close to the boundary. More precisely, for each boundary point $y \in \partial S$ there is $y' \in \mathbb{R}^2$ such that $y \in \partial B_{2h}(y')$ and $\overline{B}_{2h}(y') \subset S$. Hence, the grid cell $x + [0,h]^2$ with $x \in h\mathbb{Z}^2$ which includes y' intersects $M + [-2h, 2h]^2$ and its neighborhood $x + [0,h]^2 + \overline{B}_{\frac{h}{2}}$ is included in $\overline{B}_{2h}(y') \subset S$. Using that the evolution of the stable sets along the boxes associated to these grid cells preserves the stability, see Lemma 5.5.6, and that the union of stable sets is again stable, see Lemma 5.5.5, we obtain a stable set which roughly tracks the boundary of S from the inside, see Figure 5.7.3. Adding the interior of S enclosed by this set removes the inwards facing part of the interface – therefore preserves the stability – and yields our candidate for a stable approximation of S. Iteratively applying the stitching lemma guarantees that S_h propagates along the boxes associated to the surrounding grid cells with average speed at least v_∞. Since these boxes form a net with mesh width smaller than $2h$, we obtain $S \subset_h \mathscr{R}_{Ch}^{(A,F)}(S_h)$ for C sufficiently large, see Figure 5.7.4 and Figure 5.7.5, and thus have shown that S_h is a stable approximation of S.

It remains to show that (A,F) admits an effective minimum speed $v_{\min} > 0$ in M for the stable approximation S_h. Suppose that $y \in \mathscr{R}_t^{(A,F)}(S_h) \cap M$ for some $t > 0$. We need to show that $\overline{B}_h(y) \subset_h \mathscr{R}_{t+v_{\min}^{-1}h}^{(A,F)}(S_h)$. Consider the grid cell $x + [0,h]^2$ with $x \in h\mathbb{Z}^2$ which includes y and the ring of eight grid cells surrounding it. Without loss of generality let $\operatorname{dist}(y, S_h) \geq 3h$, meaning that the initial set S_h does not intersect these cells. Since S_h is stable, $\mathscr{R}_t^{(A,F)}(S_h)$ has to cut through the ring of surrounding grid cells, otherwise y could not have been reached. In particular, $\mathscr{R}_t^{(A,F)}(S_h)$ has to cut side to side through a single cell or two adjacent cells and hence also through the respective boxes associated to the cells, see Figure 5.7.6. This allows the iterative application of the stitching lemma, guaranteeing the propagation along the net of boxes and hence that the set evolution h-envelops $\overline{B}_h(x)$ after a fixed amount of time steps $\Delta t = 2v_\infty^{-1}h$, see Figure 5.7.7 and Figure 5.7.8. This yields the desired result if $v_{\min}$ is chosen accordingly.

Note that for this last step it is critical that we work in dimension $d = 2$. In principle, we expect that it is possible to translate all of the previous arguments to higher dimensions, for example by replacing the boxes with thin sheets in dimension $d = 3$ and requiring that the evolution of the stable set does not just reach the top but also separates the front and back of the sheet. However, a stable set for which we want to apply the stitching

lemma would have to cut through the whole sheet, splitting it in half. The connection between any reached point and the original set is sufficient to split a box in dimension $d = 2$ but of course not a sheet in dimension $d = 3$.

5.5 Preliminaries

Throughout this Section, if not specified further, we always assume $(A, F) \in \Omega$ or $(A, F) = (\widetilde{A}, \widetilde{F})_Q$, that is that (A, F) is the restriction of some admissible coefficient field $(\widetilde{A}, \widetilde{F}) \in \Omega$ to a closed set $Q \subset \mathbb{R}^2$ as in Definition 5.3.4. We first state some basic properties of the set evolutions from Definition 5.3.1. Then we will present some more specific properties related to stable sets and restricted coefficient fields. While some of these have already been established in Chapter 4, we also introduce one of the key tools for the present work in form of Lemma 5.5.9, our 'stitching lemma', and apply it to stitch together set evolutions along overlapping boxes. Finally, we will recall a tool to check measurability with respect to $\mathcal{F}(U)$ from Chapter 4 and use it to show the measurability of the event $E_r^{\uparrow, v_0}$ defined in (5.3.8).

Remark 5.5.1 (Semigroup property of $\mathscr{R}_t(\cdot)$). Letting closed sets $S \subset \mathbb{R}^2$ evolve with respect to a fixed coefficient field (A, F) for different times can be seen as a semigroup, that is

$$\mathscr{R}_{t+s}^{(A,F)}(S) = \mathscr{R}_t^{(A,F)}\Big(\mathscr{R}_s^{(A,F)}(S)\Big) \qquad \text{for any } t, s \geq 0,$$

see e.g. [Gig06, Theorem 4.5.1]. We will often implicitly use this property.

When working with the level-set formulation based on properties of a set evolution, we will often use the following lemma to obtain a supersolution.

Lemma 5.5.2 (see Lemma 4.5.2). *Let $S \subset \mathbb{R}^2$ be a closed set. The function $u_S^{\mathscr{R}} = u_{S,(A,F)}^{\mathscr{R}}$ given by*

$$u_{S,(A,F)}^{\mathscr{R}}(x, t) \coloneqq 1 - \chi_{\mathscr{R}_t^{(A,F)}(S)}(x)$$

is a supersolution of (5.3.3) with the coefficient field (A, F) in $\mathbb{R}^2 \times (0, \infty)$.

The comparison principle for the level-set formulation naturally translates to a comparison principle for the set evolutions.

Lemma 5.5.3 (The comparison principle for set evolutions, see Lemma 4.5.5(a)). *Let $S_1 \subset S_2$. Then $\mathscr{R}_t(S_1) \subset \mathscr{R}_t(S_2)$ holds for all $t \geq 0$.*

For long distances, there is a maximum speed at which the set evolutions can propagate. For shorter distances, the speed might be arbitrarily large due to curvature effects.

Lemma 5.5.4 (Maximum speed of front propagation, see Lemma 4.5.3). *Let $S \subset \mathbb{R}^2$ be a closed set. Then for any $\delta > 0$ it holds that $\mathscr{R}_T^{(A,F)}(S) \subset S + (\delta + T v_{\max,\delta})\overline{B}_1$ for all $T > 0$, where*

$$v_{\max,\delta} \coloneqq \max\{C_{1F},\, 6C_{1A}\} + C_{1A}\delta^{-1}.$$

We write $v_{\max} \coloneqq v_{\max,1}$.

5.5.1 Properties of stable sets

We now state some properties for stable sets and their evolutions with respect to (5.1.1).

Lemma 5.5.5 (Unions of stable sets)**.** *Any union of stable sets which is closed is again stable.*

Proof. The result follows immediately from the fact that supersolutions are closed under the infimum: The infimum over a set of subsolutions still is a subsolution, see [Gig06, Lemma 2.4.1]. □

A key property of stable sets is that they do not shrink while evolving according to (5.1.1) and that the stability is preserved.

Lemma 5.5.6 (Preservation of Stability, see Lemma 4.5.4)**.** *Let $S \subset \mathbb{R}^2$ be a closed set. If S is stable with respect to (A, F), then also $\mathscr{R}_t^{(A,F)}(S)$ is stable and $\mathscr{R}_s^{(A,F)}(S) \subset \mathscr{R}_t^{(A,F)}(S)$ for any $0 \le s \le t$.*

The evolution of a stable set is continuous in time with respect to the Hausdorff distance due to the maximum speed of propagation from Lemma 5.5.4. For sets which are not stable, this is not the case since a connected component might fully disappear in finite time.

Corollary 5.5.7. *Let $S \subset \mathbb{R}^2$ be a closed set. If S is stable, then $t \mapsto \mathscr{R}_t^{(A,F)}(S)$ is continuous with respect to the Hausdorff distance.*

Proof of Corollary 5.5.7. Using the monotonicity from Lemma 5.5.6, the semigroup property introduced in Remark 5.5.1, and $v_{\max,\delta}$ as in Lemma 5.5.4, we have for $t > s$

$$\mathscr{R}_s(S) \subset \mathscr{R}_t(S) = \mathscr{R}_{t-s}(\mathscr{R}_s(S)) \subset \bigcap_{\delta>0} (\mathscr{R}_s(S) + (\delta + (t-s)v_{\max,\delta})\overline{B}_1).$$

□

5.5.2 Properties of restricted solutions

The reasoning behind restricting coefficient fields to a set Q as in Definition 5.3.4 is to ensure that the set evolution remains in a neighborhood of Q and provides a 'lower bound' for the evolution with respect to the unrestricted coefficient field.

Lemma 5.5.8 (Properties of restricted coefficient fields, see Lemma 4.5.10)**.** *Let u be a supersolution to (5.3.3) with coefficients $(A, F)_Q$ for $(A, F) \in \Omega$ and some closed set $Q \subset \mathbb{R}^2$, which satisfies*

$$u(\cdot, 0) \le C \quad \text{in } \mathbb{R}^2, \qquad \{x : u(\cdot, 0) < C\} \subset \left(Q^{\oslash}\right)^c,$$

where

$$Q^{\oslash} := \bigcup \left\{ B_{\frac{1}{3}}(x) : x \in \mathbb{R}^2 \text{ with } \operatorname{dist}(x, Q) \ge \frac{2}{3} \right\}. \tag{5.5.1}$$

Then for any $t \ge 0$ it holds that

$$u(\cdot, t) \le C \quad \text{in } \mathbb{R}^2, \qquad \{x : u(\cdot, t) < C\} \subset \left(Q^{\oslash}\right)^c.$$

In addition, u is a supersolution also to (5.3.3) with coefficients $(A, F)_{\widetilde{Q}}$ for any $\widetilde{Q} \supset Q$.

In particular, for any $S \subset (Q^{\oslash})^c$ and $\widetilde{Q} \supset Q$ it holds that

(a) $\mathscr{R}_t^{(A,F)_Q}(S) \subset (Q^\oslash)^c \subset Q + \overline{B}_{\frac{2}{3}}$ *for any* $t \geq 0$.

(b) If S *is stable with respect to* $(A,F)_Q$*, then the set is stable with respect to* $(A,F)_{\widetilde{Q}}$.

(c) $u^{\mathscr{R}}_{S,(A,F)_Q} = 1 - \chi_{\mathscr{R}_\cdot^{(A,F)_Q}(S)}$ *as in Lemma 5.5.2 is a supersolution to (5.3.3) with coefficients* $(A,F)_{\widetilde{Q}}$.

Combining the properties of stable sets and our restricted coefficient fields yields a 'stitching lemma', one of the key tools in the present work. It allows us to guarantee the expansion of stable set by comparison to a set evolving with respect to a restricted coefficient field, even if there is no overlap between the sets. It is sufficient if the stable set cuts through the area to which the coefficient field is restricted, separating the second set from a part of that area. The stitching lemma allows us to compare the set evolutions in that part of the area, see Figure 5.5.1.

Lemma 5.5.9 (Stitching lemma)**.** *Let* $(A,F) \in \Omega$ *and* Q *be a closed and bounded set, let* $\widetilde{Q} \supset Q$ *be a closed set. Let* $\widetilde{S}$ *be a stable set with respect to* $(A,F)_{\widetilde{Q}}$*. Let* $Q_1 \subset (Q^\oslash)^c$ *be such that* $\partial Q_1 \subset \widetilde{S} \cup \partial Q^\oslash$*, where* $Q^\oslash$ *has been defined in (5.5.1). Then, for any closed set* $S \subset \widetilde{S} \cup (Q^\oslash)^c \setminus Q_1$ *and* $t \geq 0$ *it holds that*

$$\mathscr{R}_t^{(A,F)_Q}(S) \cap Q_1 \subset \mathscr{R}_t^{(A,F)_{\widetilde{Q}}}\left(\widetilde{S}\right).$$

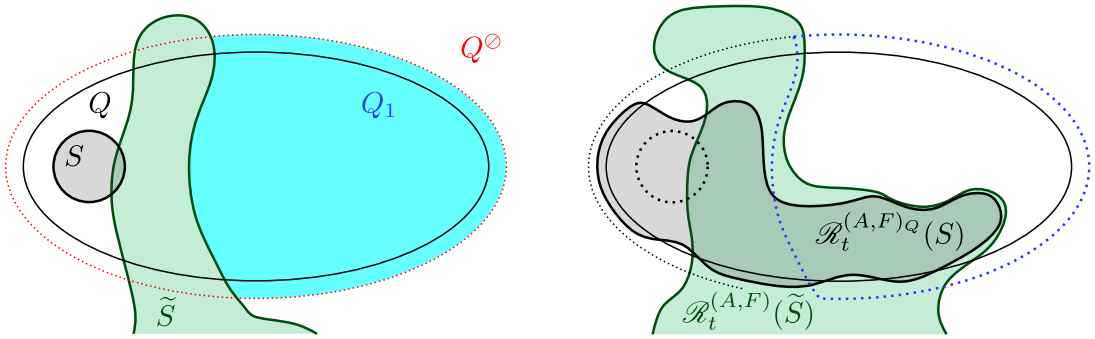


Figure 5.5.1: An illustration of Lemma 5.5.9, our 'stitching lemma'. Left: The stable set $\widetilde{S}$ (depicted in green) cuts through $(Q^\oslash)^c$ (depicted with red dotted line) separating the open area Q_1 (depicted in blue) from S (depicted in grey). Right: The set evolution $\mathscr{R}_t^{(A,F)_{\widetilde{Q}}}\left(\widetilde{S}\right)$ (depicted in green) with respect to the less restricted coefficient field stays ahead of $\mathscr{R}_t^{(A,F)_Q}(S)$ (depicted in grey) in Q_1 (depicted with blue dotted line)

Proof. Let $v_{\widetilde{S}} \coloneqq \min\{1, u_{\widetilde{S}}\}$, where $u_{\widetilde{S}}$ as in Definition 5.3.1 is a solution of (5.3.3) for the coefficient fields $(A,F)_{\widetilde{Q}}$ with $\widetilde{S} = \{x : u_{\widetilde{S}}(x,0) \leq 0\}$. In particular, by Theorem 4.A.16, $v_{\widetilde{S}}$ also is a solution. By Lemma 5.5.8, $u^{\mathscr{R}}_{S,(A,F)_Q} = 1 - \chi_{\mathscr{R}_\cdot^{(A,F)_Q}(S)}$ is a supersolution to (5.3.3) for the coefficient fields $(A,F)_{\widetilde{Q}}$ and $u^{\mathscr{R}}_{S,(A,F)_Q} = 1$ on $\partial Q^\oslash$.

We apply Theorem 4.A.8, the comparison principle, to $u^{\mathscr{R}}_{S,(A,F)_Q}$ and $v_{\widetilde{S}}$ in Q_1. For this, we first need to compare the two functions on the parabolic boundary. Because $S \subset \widetilde{S} \cup (Q_1)^c$, we have $u^{\mathscr{R}}_{S,(A,F)_Q}(\cdot,0) \geq 1 - \chi_{\widetilde{S}} \geq v_{\widetilde{S}}(\cdot,0)$ in Q_1. Since $\widetilde{S}$ is stable, Lemma 5.5.6 yields $v_{\widetilde{S}}(\cdot,t) \leq 0 \leq u^{\mathscr{R}}_{S,(A,F)_Q}(\cdot,t)$ on $\widetilde{S}$ for $t \geq 0$. As mentioned above, on $\partial Q^\oslash \times [0,\infty)$ we have $u^{\mathscr{R}}_{S,(A,F)_Q} = 1 \geq v_{\widetilde{S}}$. Hence, $v_{\widetilde{S}} \leq u^{\mathscr{R}}_{S,(A,F)_Q}$ on $\partial Q_1 \times [0,\infty)$. Now, by comparison we obtain

$v_{\widetilde{S}} \leq u^{\mathscr{R}}_{S,(A,F)_Q}$ in $Q_1 \times [0,\infty)$ and therefore in particular

$$\mathscr{R}_t^{(A,F)_Q}(S) \cap Q_1 = \left\{x \in Q_1 \,:\, u^{\mathscr{R}}_{S,(A,F)_Q}(x,t) = 0\right\} \subset \left\{x \,:\, v_{\widetilde{S}}(x,t) \leq 0\right\} = \mathscr{R}_t^{(A,F)_{\widetilde{Q}}}\left(\widetilde{S}\right).$$

□

5.5.3 Stitching together set evolutions along overlapping boxes

Recall that the events $(E^{\uparrow,v}_{r,x})_{r,x}$, $(E^{\rightarrow,v}_{r,x})_{r,x}$, $(E^{\downarrow,v}_{r,x})_{r,x}$, $(E^{\leftarrow,v}_{r,x})_{r,x}$ as introduced in Section 5.3.3 guarantee the existence of a stable set in the starting area of the respective associated box which will propagate along that box at some average speed at least $v > 0$. We want to apply the stitching lemma to guarantee the propagation of such a stable set not just within the initial box but along networks of suitably placed overlapping boxes for which the respective events hold.

For this purpose, we introduce the notion of a set cutting vertically or horizontally through a box, which means that the set separates the respective opposite sides of that box. If such a set is stable, this separation will allow the application of the stitching lemma in the respective halves of the box.

Definition 5.5.10. Let $S \subset \mathbb{R}^2$ be a closed set and $x \in \mathbb{R}^2$ with $R_1, R_2 > 0$. We say that S *cuts vertically (horizontally) through the box* $Q = x + [0,R_1] \times [0,R_2]$ if there is no path in $Q \setminus S$ connecting the opposing sides $x + \{0\} \times [0,R_2]$ and $x + \{R_1\} \times [0,R_2]$ ($x + [0,R_1] \times \{0\}$ and $x + [0,R_1] \times \{R_2\}$).

In order for a stable set in the starting area of a box associated to an event $E^{\uparrow,v}_{r,x}$ to reach the opposite side, its evolution will have to cut along the main body of that box. Due to the maximum speed of front propagation from Lemma 5.5.4 we expect that this implies that the set evolution will contain a path connecting the starting area to the top. However, we did not easily find a proof and hence chose to fall back to the weaker notion of cuts as introduced here, which is sufficient for our purposes.

Lemma 5.5.11 (Characterization of $E^{\uparrow,v}_{r,x}$)**.** *For $x \in \mathbb{R}^2$, $v > 0$ assume that $(A,F) \in E^{\uparrow,v}_{r,x}$ as defined in (5.3.8). By definition, there exists a stable set $S \subset x + [0,w(r)] \times [-w(r),0]$ in the starting area of the box $x + Q^{\uparrow}_r$ associated to the event $E^{\uparrow,v}_{r,x}$ with*

$$\mathscr{R}^{(A,F)_{x+Q^{\uparrow}_r}}_{v^{-1}r}(S) \cap \left(x + [0,w(r)] \times \{r\}\right) \neq \emptyset.$$

Then $\mathscr{R}^{(A,F)_{x+Q^{\uparrow}_r}}_{v^{-1}r}(S)$ cuts vertically through the main body $Q = x + [-1,w(r)+1] \times [0,r]$ of the box $x + Q^{\uparrow}_r$.

Proof. Without loss of generality let $x = 0$. We prove the result by contrapositive, showing that if the set evolution does not cut through the box, then due to the comparison principle the top of the box can not have been reached. Suppose that $\mathscr{R}^{(A,F)_{Q^{\uparrow}_r}}_{v^{-1}r}(S)$ does not cut vertically through the main body Q. Then there exists a path

$$\gamma \in C\left([0,1], Q \setminus \mathscr{R}^{(A,F)_{Q^{\uparrow}_r}}_{v^{-1}r}(S)\right) \quad \text{with } \gamma(0) \in \{-1\} \times [0,r],\ \gamma(1) \in \{w(r)+1\} \times [0,r].$$

Without loss of generality let $[0,w(r)] \times \{r\} \not\subseteq \gamma([0,1])$, otherwise we are done. Consider the part of the thickened box $Q^{\uparrow}_r + B_1$ above γ, that is, the connected component U of

the top of the box $[0, w(r)] \times \{r\}$ in $(Q_r^\uparrow + B_1) \setminus \gamma([0,1])$. Note that $U \cap S = \emptyset$ because this would imply that the top and bottom side of the main body Q are connected in $Q \setminus \gamma$ which is not possible due to the Jordan curve theorem. In addition, $\partial U \subset \gamma([0,1]) \cup \partial(Q_r^\uparrow + B_1)$ and hence

$$\partial U \cap \mathscr{R}_t^{(A,F)_{Q_r^\uparrow}}(S) = \emptyset \qquad \text{for all } t \in [0, v^{-1}r]$$

because $\partial(Q_r^\uparrow + B_1)$ cannot be reached with the coefficient field restricted to $x + Q_r^\uparrow$ by Lemma 5.5.8 and if $\gamma([0,1])$ had been reached at any time $t < v^{-1}r$, then those parts would still be covered at time $t = v^{-1}r$ by Lemma 5.5.6 due to the stability of S, contradicting the definition of γ.

We can thus compare

$$u(y,t) = 1 - \chi_{\mathscr{R}_t^{(A,F)_{Q_r^\uparrow}}(S)}(y) \qquad \text{and} \qquad v(y,t) = 1$$

on $U \times (0, v^{-1}r)$ with u a supersolution by Lemma 5.5.2 and v a trivial solution: On the parabolic boundary of $U \times (0, v^{-1}r)$ we have $u = 1 = v$. The comparison principle for bounded domains as in Theorem 4.A.8 and the continuity of the set evolution from Lemma 5.5.7 yield $u \geq 1$ on $\overline{U} \times [0, v^{-1}r]$ and therefore in particular the contrapositive

$$\mathscr{R}_{v^{-1}r}^{(A,F)_{Q_r^\uparrow}}(S) \cap \left([0, w(r)] \times \{r\}\right) = \emptyset,$$

concluding the proof. □

In order to iteratively apply the stitching lemma along a network of overlapping boxes, we alternate horizontal and vertical boxes. If, for example, $E_r^{\uparrow,v}$ holds, we obtain a stable set which evolves to cut vertically along the associated box as in Lemma 5.5.11. For horizontal boxes crossed by this vertical box, we can hence apply the stitching lemma, see Figure 5.4.1. Lemma 5.5.13 below states that, given two such horizontal boxes oriented in opposing directions along which the set evolution propagates, the original stable set will evolve to cut horizontally across the area spanned by the these two boxes, see Figure 5.5.2. This cut can be used as a base for an extension of the original vertical cut: Lemma 5.5.14 states that, given an additional vertical box crossed by the horizontal cut and along which the set evolution propagates, the original stable set will further evolve to cut vertically across the area spanned by both vertical boxes, see Figure 5.5.3.

Remark 5.5.12. Lemma 5.5.13 and Lemma 5.5.14 below remain valid for translations and rotations of the overall box configurations and when replacing the coefficient field for the set evolution with the restriction to an area including the configuration and with respect to which S remains stable. To simplify the notation, we state these two lemmas for a box configuration anchored at $x = 0$ and with respect to the unrestricted coefficient field.

Lemma 5.5.13 (Cutting through two complementing boxes to form a base)**.** *Consider a configuration of boxes $Q_0 = [0, R_1] \times [0, R_2]$, and $x_1 + Q_r^\rightarrow$, $x_2 + Q_r^\leftarrow$ as introduced in Section 5.3.3 where the main parts of the latter two cross Q_0, see Figure 5.5.2. That is, let*

$$r > 0, \qquad R_1 \in (0, r], \qquad R_2 \geq w(r) + 2, \qquad \begin{aligned} x_1 &\in \{0\} \times [w(r) + 1, R_2 - 1], \\ x_2 &\in \{R_1\} \times [1, R_2 - w(r) - 1], \end{aligned}$$

where $w(r)$ is the width of the boxes $Q_r^\rightarrow$, $Q_r^\leftarrow$ as defined in (5.3.7).

Let $S \subset \mathbb{R}^2$ be a stable set with respect to some coefficient field $(A, F) \in \Omega$ which cuts vertically through Q_0. If $(A, F) \in E_{r,x_1}^{\rightarrow,v} \cap E_{r,x_2}^{\leftarrow,v}$ for some $v > 0$, then the evolved set $\mathscr{R}_{v^{-1}r}^{(A,F)}(S)$ cuts horizontally through the box $\widetilde{Q}_0 = [-(r - R_1), r] \times [0, R_2]$ spanning the main parts of $x_1 + Q_r^\rightarrow$ and $x_2 + Q_r^\leftarrow$.

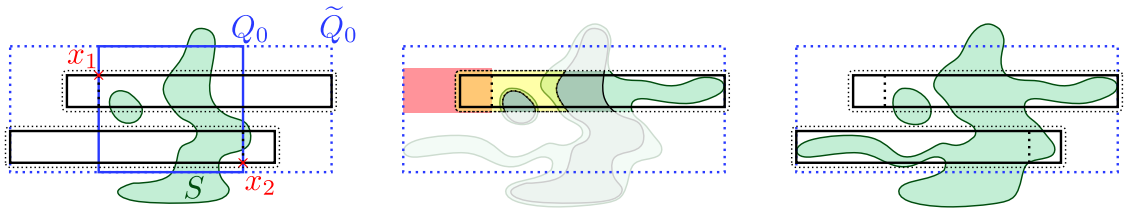


Figure 5.5.2: An illustration of the box configuration in Lemma 5.5.13 and its proof. Left: The set S (depicted in green) cuts vertically through the box Q_0 (depicted in blue), which is crossed by the main parts of $x_1 + Q_r^{\rightarrow}$ and $x_2 + Q_r^{\leftarrow}$ (depicted in black). Together with Q_0, they span the box $\widetilde{Q}_0$ (depicted with blue dotted lines). Center: The layer $\widetilde{Q}_1 \subset \widetilde{Q}_0$ corresponding to $x_1 + Q_r^{\rightarrow}$ is cut horizontally by the union of the extension Q_1' (depicted in red), the connected component $U_1 \subset \mathbb{R}^2 \setminus S$ of the starting area (depicted in yellow), and the set evolution $\mathscr{R}_{v^{-1}r}^{(A,F)}(S)$ (depicted in green). Right: The set evolution $\mathscr{R}_{v^{-1}r}^{(A,F)}(S)$ (depicted in green) cuts horizontally through $\widetilde{Q}_0$

Proof. We first consider the two horizontal layers $\widetilde{Q}_1, \widetilde{Q}_2 \subset \widetilde{Q}_0$ corresponding to the thin boxes $x_1 + Q_r^{\rightarrow}$ and $x_2 + Q_r^{\leftarrow}$:

$$\widetilde{Q}_1 \coloneqq x_1 + [-(r - R_1), r] \times [-w(r) - 1, 1],$$
$$\widetilde{Q}_2 \coloneqq x_2 + [-r, r - R_1] \times [-1, w(r) + 1].$$

Let U_1 be the union of all connected components in $\left(x_1 + Q_r^{\rightarrow} + B_{\frac{2}{3}}\right) \setminus S$ which intersect the starting area $x_1 + [-w(r), 0]^2$. Let U_2 be defined analogously for the box $x_2 + Q_r^{\leftarrow}$. Let Q_1' be the extension of the main body of $x_1 + Q_r^{\rightarrow}$ to cover the whole layer $\widetilde{Q}_1$ with Q_2' chosen analogously for $\widetilde{Q}_2$, that is,

$$Q_1' \coloneqq x_1 + [-(r - R_1), 0] \times [-w(r) - 1, 1],$$
$$Q_2' \coloneqq x_2 + [0, r - R_1] \times [-1, w(r) + 1].$$

We claim that

$$\mathscr{R}_{v^{-1}r}^{(A,F)}(S) \cup U_i \cup Q_i' \text{ cuts horizontally through } \widetilde{Q}_i \text{ for } i = 1, 2. \tag{5.5.2}$$

We only show the claim for $i = 1$ since it follows analogously for $i = 2$. First note that it is sufficient to show that $\mathscr{R}_{v^{-1}r}^{(A,F)}(S) \cup U_1$ cuts horizontally through the main body $x_1 + [0, r] \times [1, -w(r) - 1]$ of $x_1 + Q_r^{\rightarrow}$ since the extension Q_1' fully covers the complement in $\widetilde{Q}_1$. Let $S_1 \subset (x_1 + [-w(r), 0]^2)$ be the stable set propagating along $x_1 + Q_r^{\rightarrow}$ whose existence is guaranteed with $(A, F) \in E_{r,x_1}^{\rightarrow,v}$. By Lemma 5.5.11, the evolved set $\mathscr{R}_{v^{-1}r}^{(A,F)_{x_1+Q_r^{\rightarrow}}}(S_1)$ cuts horizontally through the main body of $x_1 + Q_r^{\rightarrow}$. We hence only need to show that $\mathscr{R}_{v^{-1}r}^{(A,F)_{x_1+Q_r^{\rightarrow}}}(S_1) \subset \mathscr{R}_{v^{-1}r}^{(A,F)}(S) \cup U_1$. This follows from an application of Lemma 5.5.9, our stitching lemma, which yields that

$$\mathscr{R}_{v^{-1}r}^{(A,F)_{x_1+Q_r^{\rightarrow}}}(S_1) \cap \left(\left(x_1 + Q_r^{\rightarrow} + B_{\frac{2}{3}}\right) \setminus U_1\right) \subset \mathscr{R}_{v^{-1}r}^{(A,F)}(S),$$

where we used that S is stable, that $\partial\left(\left(x_1 + Q_r^{\rightarrow} + B_{\frac{2}{3}}\right) \setminus U_1\right) \subset S \cup \partial\left(x_1 + Q_r^{\rightarrow} + B_{\frac{2}{3}}\right)$, and that, as a subset of the starting area, $S_1 \subset U_1 \cup S$ by the definition of U_1. After noting that $\mathscr{R}_{v^{-1}r}^{(A,F)_{x_1+Q_r^{\rightarrow}}}(S_1) \subset x_1 + Q_r^{\rightarrow} + B_{\frac{2}{3}}$ by Lemma 5.5.8 due to the restriction of the evolution, this establishes (5.5.2).

Now, suppose that $\mathscr{R}_{v^{-1}r}^{(A,F)}(S)$ does not cut horizontally through $\widetilde{Q}_0$. Then there exists a path γ connecting the top and bottom of $\widetilde{Q}_0$ in $\widetilde{Q}_0 \setminus \mathscr{R}_{v^{-1}r}^{(A,F)}(S)$ and hence also in $\widetilde{Q}_0 \setminus S$ due to the stability of S and Lemma 5.5.6. In particular, there exist segments of γ which

connect the top and bottom of $\widetilde{Q}_1$ and $\widetilde{Q}_2$ inside these layers respectively. However, due to (5.5.2), this means that γ has to pass through U_i or Q_i' for each $i = 1, 2$. Since each connected component in U_i is open, every point in U_i is path-connected in $U_i \subset \widetilde{Q}_0 \setminus S$ to the respective starting area of the associated box and hence Q_i'. In summary, we obtain either a segment of γ or an extension of such a segment along U_1 and/or U_2 which connects Q_1' on the left side of Q_0 with Q_2' on the right of Q_0 in $\widetilde{Q}_0 \setminus S$. This is a contradiction since we assumed that S cuts vertically through Q_0. Thus, we have shown that $\mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ cuts horizontally through $\widetilde{Q}_0$. □

Lemma 5.5.14 (Extending a cut through a base)**.** *Consider a configuration of boxes given by $Q_0 = [0, R_1] \times [0, R_2]$, a smaller box $\widetilde{Q}_0 = [0, R_1] \times [\widetilde{R}_2, R_2]$ covering its top, and a thin box $x + Q_r^{\uparrow}$ as in (5.3.7) with its starting area aligned below $\widetilde{Q}_0$, see Figure 5.5.3. That is, let*

$$r > 0, \qquad R_1 \geq w(r) + 2, \qquad R_2 > 0, \qquad \widetilde{R}_2 \in (R_2 - r, R_2), \quad x \in [1, R_1 - w(r) - 1] \times \{\widetilde{R}_2\},$$

where $w(r)$ is the width of the box $Q_r^{\uparrow}$ as defined in (5.3.7).

Let $S \subset \mathbb{R}^2$ be a stable set with respect to some coefficient field $(A, F) \in \Omega$ which cuts vertically through Q_0 and horizontally through $\widetilde{Q}_0$. If $(A, F) \in E^{\uparrow, v}_{r,x}$ for some $v > 0$, then the evolved set $\mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ cuts vertically through the box $Q_1 = [0, R_1] \times [0, \widetilde{R}_2 + r]$, extending Q_0 by the height added via $x + Q_r^{\uparrow}$.

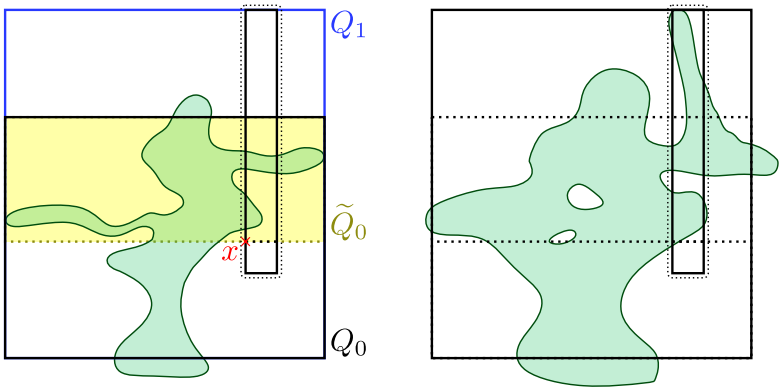


Figure 5.5.3: An illustration of the box configuration in Lemma 5.5.14. Left: The set S (depicted in green) cuts vertically through the box Q_0 (depicted in black) and horizontally through its top horizontal layer $\widetilde{Q}_0$ (depicted in yellow), which is crossed by the main part of $x + Q_r^{\uparrow}$ (depicted in black). The box Q_1 (depicted in blue) extends Q_0 to the top of $x + Q_r^{\uparrow}$. Right: The set evolution $\mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ (depicted in green) cuts vertically through $Q_1 \supset Q_0$

Proof. Consider the main part $Q_2 = x + [-1, w(r) + 1] \times [0, r]$ of the box $x + Q_r^{\uparrow}$. Let U be the union of all connected components in $\left(x + Q_r^{\uparrow} + B_{\frac{2}{3}}\right) \setminus S$ which intersect the starting area $x + [0, w(r)] \times [-w(r), 0]$. Analogously to (5.5.2) in the Proof of Lemma 5.5.13, we obtain that

$$\mathscr{R}^{(A,F)}_{v^{-1}r}(S) \cup U \text{ cuts vertically through } Q_2. \tag{5.5.3}$$

Now, suppose that $\mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ does not cut vertically through Q_1. Then there exists a path γ connecting the left and right side of Q_1 in $Q_1 \setminus \mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ and hence also in $Q_1 \setminus S$ due to the stability of S and Lemma 5.5.6. Since S and hence by stability and Lemma 5.5.6 also $\mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ cut vertically through Q_0, at some point γ has to visit the vertical extension

$Q_1 \setminus Q_0$. This implies that γ cannot visit the area $Q_0 \setminus \widetilde{Q}_0$ below the horizontal layer $\widetilde{Q}_0$ since otherwise there would have to be a segment connecting the top and bottom of $\widetilde{Q}_0$, which would contradict the assumption that S cuts horizontally through $\widetilde{Q}_0$. With γ thus staying inside the box $Q_1 \setminus \left(Q_0 \setminus \widetilde{Q}_0\right)$ and connecting its left and right side, there has to be a segment of γ which connects the left and right side of the vertical layer given by the main part Q_2 of the box $x + Q_r^{\uparrow}$. Due to (5.5.3), the path γ hence has to visit U. Since each connected component in U is open, every point in U is path-connected in $U \subset Q_1 \setminus S$ to the starting area of $x + Q_r^{\uparrow}$. With γ visiting the vertical extension $Q_1 \setminus Q_0$, we hence obtain a path in $Q_1 \setminus S$ connecting $Q_1 \setminus Q_0$ to the starting area of $x + Q_r^{\uparrow}$ below the horizontal layer $\widetilde{Q}_0$. This path thus connects the top to the bottom of $\widetilde{Q}_0$, which is a contradiction since we assumed that S cuts horizontally through $\widetilde{Q}_0$. Thus, we have shown that $\mathscr{R}^{(A,F)}_{v^{-1}r}(S)$ cuts vertically through Q_1. □

5.5.4 Measurability of events with respect to $\mathcal{F}(U)$

We recall the measurability criterion from Chapter 4 for the σ-algebras $\{\mathcal{F}(U)\}_U$ defined in (5.3.6).

Lemma 5.5.15 (see Lemma 4.5.12). *Let $U \subset \mathbb{R}^2$ be a closed set. Let $E \subset \Omega \subset C(\mathbb{R}^2 \times \mathbb{S}^1, \mathbb{R}^{2\times 2} \times \mathbb{R})$ be an event which is closed in Ω with respect to the norm*

$$\|(A,F)\|_{C_w(U\times\mathbb{S}^1,\mathbb{R}^{2\times 2}\times\mathbb{R})} := \sup_{x\in U,\, e\in\mathbb{S}^1} \frac{1}{1+|x|^2}\big|(A(x,e),F(x,e))\big|,$$

which corresponds to weighted uniform convergence and is well defined on Ω due to the uniform bounds from (5.3.1) and (5.3.2). Then E is measurable with respect to $\mathcal{F}(U)$.

In order to show the well-posedness of the box criterion (P□), we use Lemma 5.5.15 to show the measurability of the event that there is a stable set in the starting area of a given box which reaches the opposite side before a prescribed time.

Lemma 5.5.16 (Measurability of 'propagation in a box'). *For any $r, v > 0$ and $x \in \mathbb{R}^2$, the event $E^{\uparrow,v}_{r,x} \subset \Omega$ defined in (5.3.8) is measurable with respect to $\mathcal{F}(x + Q_r^{\uparrow} + B_1)$, where $Q_r^{\uparrow} = [0, w(r)] \times [-w(r), r]$ as in (5.3.7). Analogous statements hold for $E^{\rightarrow,v}_{r,x}$, $E^{\downarrow,v}_{r,x}$, and $E^{\leftarrow,v}_{r,x}$.*

Proof. By Lemma 5.5.15, it is enough to show that the event $E^{\uparrow,v}_{r,x} \subset \Omega$ is closed with respect to $C_w((x + Q_r^{\uparrow} + B_1) \times \mathbb{S}^1, \mathbb{R}^{2\times 2} \times \mathbb{R})$. That is, assuming there are $(A_k, F_k) \in E^{\uparrow,v}_{r,x}$ for $k \in \mathbb{N}$ with $(A_k, F_k) \to (A, F)$ uniformly on $(x + Q_r^{\uparrow} + B_1) \times \mathbb{S}^1$ as $k \to \infty$ for some $(A, F) \in \Omega$, we only need to show that $(A, F) \in E^{\uparrow,v}_{r,x}$. By the definition of restricted coefficient fields in (5.3.15) and (5.3.16), we immediately have that $(A_k, F_k)_{x+Q_r^{\uparrow}} \to (A, F)_{x+Q_r^{\uparrow}}$ uniformly on $\mathbb{R}^2 \times \mathbb{S}^1$ as $k \to \infty$.

Recalling the definition of $E^{\uparrow,v}_{r,x}$, for $k \in \mathbb{N}$ let $S_k \subset [0, w(r)] \times [-w(r), 0]$ satisfy (5.3.9) and (5.3.10) with respect to (A_k, F_k). Let $u_k = u^{\mathscr{R}}_{S^0_k,(A_k,F_k)} = 1 - \chi_{\mathscr{R}^{(A_k,F_k)}_{\cdot}(S^0_k)}$ be the supersolutions of (5.3.3) from Lemma 5.5.2.

The non-empty compact subsets of a compact set are a compact metric space when endowed with the Hausdorff metric. Hence, there exist (without loss of generality non-relabeled) subsequences and $S^0, \widetilde{S}^{v^{-1}r} \neq 0$ such that

$$S_k \to S^0 \subset x + \big[0, w(r)\big] \times \big[-w(r), 0\big],$$
$$\mathscr{R}^{(A_k,F_k)_{x+Q_r^{\uparrow}}}_{v^{-1}r}(S_k) \cap \big(x + \big[0, w(r)\big] \times \{r\}\big) \to \widetilde{S}^{v^{-1}r} \subset \big(x + \big[0, w(r)\big] \times \{r\}\big).$$

We define $u := \liminf_* u_k$, with $\liminf_* u_k$ denoting the lower relaxed limit as in (4.A.5), which does not just take the infimum pointwise but also with respect to approaches from the surrounding neighborhood. By Corollary 5.5.7, the evolutions of the starting sets are uniformly continuous with respect to the Hausdorff metric and hence $u(\cdot,t) = \liminf_*\Big(1 - \chi_{\mathscr{R}_t^{(A_k,F_k)}(S_k)}\Big)$. Therefore, $u(\cdot,0) = 1 - \chi_{S^0}$ and $\{y : u(y, v^{-1}r) \le 0\} \supset \widetilde{S}^{v^{-1}r}$. Due to the stability of viscosity solutions (see Theorem 4.A.7) and that $(A_k,F_k)_{x+Q_r^\uparrow} \to (A,F)_{x+Q_r^\uparrow}$ uniformly, we further know that S_0 is stable and that u is a supersolution to (5.3.3) with respect to $(A,F)_{x+Q_r^\uparrow}$. Applying the comparison principle from Theorem 4.A.8 to u and u_{S^0} from Definition 5.3.1, we in particular obtain

$$\mathscr{R}_{v^{-1}r}^{(A,F)_{x+Q_r^\uparrow}}(S^0) = \{y : u_{S^0}(y, v^{-1}r) \le 0\} \supset \{y : u(y, v^{-1}r) \le 0\} \supset \widetilde{S}^{v^{-1}r}.$$

Thus, $(A,F) \in E_{r,x}^{\uparrow,v}$ with S^0 satisfying (5.3.9) and (5.3.10), showing that $E_{r,x}^{\uparrow,v}$ is closed and therefore measurable. □

5.6 Lifting the box criterion to larger scales

In this section, we aim to lift the probability bound (5.3.11) from the box criterion (P□) to arbitrary scales. That is, for arbitrary $r \ge r_0$ we aim to obtain probability bounds for the events that the interface propagates along the boxes $Q_r^\uparrow$, $Q_r^\rightarrow$, $Q_r^\downarrow$, and $Q_r^\leftarrow$ respectively, see Figure 5.1.1. We are looking to find speeds $v_i > 0$ and probability bounds $p_i \in [0,1)$ such that for all $i \in \mathbb{N}$

$$\min\big\{\mathbb{P}\big[E_r^{\uparrow,v_i}\big], \mathbb{P}\big[E_r^{\rightarrow,v_i}\big], \mathbb{P}\big[E_r^{\downarrow,v_i}\big], \mathbb{P}\big[E_r^{\leftarrow,v_i}\big]\big\} \ge 1 - p_i \qquad \text{for all } r \in [r_i, r_{i+1}), \tag{5.6.1}$$

where the sequence $(r_i)_{i\in\mathbb{N}}$ is recursively defined by

$$r_i := w^{-1}\big(r_{i-1} + 2w\big(r_{i-1}\big)\big). \tag{5.6.2}$$

This means that the width of the box $Q_{r_i}^\uparrow$ from Section 5.3.4 is given by $r_{i-1} + 2w(r_{i-1})$. Therefore, translations of $Q_{r_{i-1}}^\rightarrow$ and $Q_{r_{i-1}}^\leftarrow$ fit inside $Q_{r_i}^\uparrow$ without their starting areas overlapping, see Figure 5.3.1.

This setup already suggests that we will obtain (5.6.1) iteratively by arranging smaller boxes, for which we will already have a probability bound, inside a given box and exploiting the random fields' stationarity and finite range of dependence.

Proposition 5.6.1 (The box criterion for large scales)**.** *Assume that (P1), (P2), and (P□) hold. Let $(r_i)_{i\in\mathbb{N}}$ be recursively defined box lengths as in (5.6.2). Let $0 < \delta < \frac{q-1}{q}$. Then there are sequences $(p_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ such that (5.6.1) holds for all $i \in \mathbb{N}$ with the sequences satisfying*

$$p_{i+1} \le \min\left\{p_0, p_i^{\left(\frac{r_i}{r_0}\right)^{\frac{q-1}{q}-\delta}-1}\right\}, \qquad v_{i+1} \ge \left(\frac{1}{1 + C_v r_{i+1}^{-\min\left\{\delta,\, \frac{q-1}{q}-\delta\right\}}}\right)^{\ell} v_i, \tag{5.6.3}$$

where $C_v = C_v(q, c_w, r_0, \delta)$ and $\ell = \ell(q, c_w, r_0, \delta)$ depend only on q, c_w, r_0, δ.

Once we obtain Proposition 5.6.1, relating the probability bounds to the scale of the boxes yields stretched exponential bounds for the probability that propagation along the boxes fails.

Corollary 5.6.2. *Assume that (P1), (P2), and (P□) hold. For* $\alpha_\uparrow \in \left(0, \frac{q-1}{q^3}\right)$*, there exist* $C_\uparrow = C_\uparrow(v_0, q, c_w, r_0, p_0, \alpha_\uparrow) > 0$ *and a velocity* $v_\infty = v_\infty(q, c_w, r_0, v_0, \alpha_\uparrow)$ *such that*

$$\min\left\{\mathbb{P}\left[E_r^{\uparrow, v_\infty}\right], \mathbb{P}\left[E_r^{\rightarrow, v_\infty}\right], \mathbb{P}\left[E_r^{\downarrow, v_\infty}\right], \mathbb{P}\left[E_r^{\leftarrow, v_\infty}\right]\right\} \geq 1 - \exp\left(-C_\uparrow r^{\alpha_\uparrow}\right) \tag{5.6.4}$$

for all $r \geq r_0$.

Proof of Corollary 5.6.2. Let $\alpha_\uparrow \in \left(0, \frac{q-1}{q^3}\right)$. Choose $\delta \in \left(0, \frac{q-1}{q}\right)$ such that $\alpha_\uparrow = \frac{q-1}{q^3} - \frac{\delta}{q^2}$. Given δ, let $(p_i)_i$, $(v_i)_i$ be as in Proposition 5.6.1. Using (5.6.3), we will first show that $v_\infty = v_\infty(q, c_w, r_0, v_0, \delta) := \liminf_{i\to\infty} v_i > 0$ and then provide the probability bound in (5.6.4) based on (5.6.1) and the bounds for $(p_i)_i$ in (5.6.3).

Regarding v_∞, we set $\beta := \min\left\{\delta, \frac{q-1}{q} - \delta\right\} > 0$. From (5.6.3), for any $i \in \mathbb{N}$, we obtain

$$v_i \geq \left(\frac{1}{1 + C_v r_i^{-\beta}}\right)^\ell v_{i-1} \geq \left(\prod_{j=1}^{i} \frac{1}{1 + C_v r_j^{-\beta}}\right)^\ell v_0 \geq \exp\left(-C_v \ell \sum_{j=1}^{i} r_j^{-\beta}\right) v_0,$$

where we used that $1 + x \leq \exp(x)$. Note that, using the formula for geometric sums,

$$r_i = w^{-1}\left(r_{i-1} + 2w(r_{i-1})\right) \geq w^{-1}\left(r_{i-1}\right) = c_w r_{i-1}^q \geq c_w^{\sum_{j=0}^{i-1} q^j} r_0^{q^i} = c_w^{-\frac{1}{q-1}} \left(c_w^{\frac{1}{q-1}} r_0\right)^{q^i}.$$

Plugging this into the bound for v_i above yields

$$v_i \geq \exp\left(-C_v \ell c_w^{\frac{\beta}{q-1}} \sum_{j=1}^{i} \left(c_w^{\frac{1}{q-1}} r_0\right)^{-\beta q^j}\right) v_0.$$

For $v_\infty = \liminf_{i\to\infty} v_i$, we hence obtain

$$v_\infty \geq \exp\left(-C_v \ell c_w^{\frac{\beta}{q-1}} \sum_{j=1}^{\infty} \left(c_w^{\frac{1}{q-1}} r_0\right)^{-\beta q^j}\right) v_0 > 0,$$

where we used that $q > 1$, $\beta > 0$ and $c_w^{\frac{1}{q-1}} r_0 = \left(c_w^{\frac{1}{q}} r_0^{\frac{q-1}{q}}\right)^{\frac{q}{q-1}} \geq 24^{\frac{q}{q-1}} > 1$ due to (5.3.12) from Assumption (P□).

Regarding the probability bound, let $r \in [r_{i+1}, r_{i+2})$, $i \geq 1$. Then, since $v_\infty \leq v_i$ for all $i \in \mathbb{N}_0$ by definition, (5.6.3) yields the bound

$$\begin{aligned}
1 - \mathbb{P}\left[E_r^{\uparrow, v_\infty}\right] \leq p_{i+1} &\leq p_0^{\left(\frac{r_i}{r_0}\right)^{\frac{q-1}{q} - \delta} - 1} \\
&\leq p_0^{\left(1 - \left(\frac{r_0}{r_1}\right)^{\frac{q-1}{q} - \delta}\right) \frac{r_i}{r_0}^{\frac{q-1}{q} - \delta}} \\
&\leq \exp\left(-\log\left(p_0^{-1}\right)\left(r_0^{\delta - \frac{q-1}{q}} - r_1^{\delta - \frac{q-1}{q}}\right) r_i^{\frac{q-1}{q} - \delta}\right).
\end{aligned}$$

where we used that $r_i \geq 2r_0$ for $i \geq 1$. Since we have $r_j \geq \frac{1}{2} w(r_{j+1})$ by the definition of the sequence $(r_j)_j$ from (5.6.2), we have

$$r_i \geq \frac{1}{2} w\left(\frac{1}{2} w\left(r_{i+2}\right)\right) \geq \frac{1}{2} w\left(\frac{1}{2} w\left(r\right)\right) = \left(\frac{1}{2 c_w^{\frac{1}{q}}}\right)^{\frac{q+1}{q}} r^{\frac{1}{q^2}}$$

and we hence obtain a constant $C(q, c_w, r_0, p_0, \delta) > 0$ such that

$$1 - \mathbb{P}\big[E_r^{\uparrow, v_\infty}\big] \leq \exp\Big(-C(q, c_w, r_0, p_0, \delta) r^{\frac{q-1}{q^3} - \frac{\delta}{q^2}}\Big) =: \exp\big(-C_\uparrow r^{\alpha_\uparrow}\big).$$

□

Proof of Proposition 5.6.1. We obtain the result by induction. The box criterion (P□) yields (5.6.1) for $i = 0$. Assuming that (5.6.1) holds for some $i \in \mathbb{N}_0$ with $v_i > 0$ and $p_i > 0$ small enough, we will show that (5.6.1) also holds for $i + 1$ with some p_{i+1}, v_{i+1} satisfying (5.6.3). The requirements on r_0 from (5.3.12) and p_0 from (5.3.13) in (P□) are chosen to ensure that p_0 is small enough and via (5.6.3) also p_i for all $i \in \mathbb{N}$.

To be more precise, we will proceed in seven steps as follows:

1.) Given $r \geq r_1$, we introduce a configuration of smaller boxes inside the box $Q_r^\uparrow$ consisting of N layers. Each layer consists of one subgroup of $k_\parallel$ parallel vertical boxes of length $r_\parallel = r_\parallel(r, k_\parallel)$ and two subgroups of $k_=$ parallel horizontal boxes of length $r_= = r_=(r)$.

2.) Assuming that for each subgroup there is at least one box along which the interface propagates with minimum average speed v, using the stitching lemma we show that $E_r^{\uparrow, \widetilde{v}}$ holds for suitable $\widetilde{v} = \widetilde{v}(r, k_\parallel, k_=)$.

3.) From the previous step, with the stationarity from Assumption (P1) and the finite range of dependence from Assumption (P2), we deduce a probability bound for $\mathbb{P}\Big[E_r^{\uparrow, \widetilde{v}}\Big]$ depending on $\mathbb{P}\big[E_{r_\parallel}^{\uparrow, v}\big]$, $\mathbb{P}[E_{r_=}^{\rightarrow, v}]$, and $\mathbb{P}[E_{r_=}^{\leftarrow, v}]$, as well as $k_\parallel$ and $k_=$.

4.) We discuss the requirements on the number of boxes per subgroup and fix a choice for $k_\parallel = k_\parallel(r)$ and $k_= = k_=(r)$.

5.) Having fixed $k_\parallel$ and $k_=$, we quantify the loss of speed from v to $\widetilde{v}$ from Step 2.

6.) In the same vein, we ensure that the probability bound from Step 3 is an improvement compared to the smaller boxes, explaining the bounds for r_0 in (5.3.12) and p_0 in (5.3.13) from Assumption (P□).

7.) Repeatedly applying this argument yields the result. Note that, assuming (5.6.1) holds for some $i \in \mathbb{N}$, multiple intermediate steps are necessary in order to expand the probability bounds from r_{i+1} to r_{i+2}: while $r_=(r) \in [r_i, r_{i+1})$ for $r \in [r_{i+1}, r_{i+2})$, for r closer to r_{i+2} we have $r_\parallel(r) > r_{i+1}$.

Repeating the procedure for $Q_r^\rightarrow$, $Q_r^\downarrow$, and $Q_r^\leftarrow$ yields the result.

Step 1: Setting up smaller boxes in $Q_r^\uparrow$. Given $r \geq r_1$, we set up multiple layers of smaller boxes within the large vertical box $Q_r^\uparrow$, see Figure 5.6.1. Except for the first layer, each consists of three groups of boxes:

- Two groups of horizontal boxes form the 'base' of the layer. The first group consists of $k_= \in \mathbb{N}$ translations of $Q_{r_=}^\rightarrow$, where $r_= > 0$ is chosen such that $r_= + 2w(r_=) = w(r)$. These boxes are stacked on top of each other with distance 3 between them, their starting areas aligning with the left side of $Q_r^\uparrow$.

- The second group respectively consists of $k_= \in \mathbb{N}$ translations of $Q^{\leftarrow}_{r_=}$. These are stacked overlapping with the first group, but such that their starting areas align with the right side of $Q^{\uparrow}_r$. Hence, each box from the second group overlaps one from the first, except for their starting areas to the right/left respectively.
- The third group consists of $k_\parallel \in \mathbb{N}$ translations of the vertical box $Q^{\uparrow}_{r_\parallel}$, which are stacked next to each other with distance 3 between them. Here, $r_\parallel$ is chosen such that the overall width of the group is less than $r_= - 2$. This group of vertical boxes is placed such that the top of their starting areas is 1 below the lowest of the horizontal boxes.

For the first layer, we scrap the horizontal base, leaving only the vertical boxes. This first layer of vertical boxes is placed such that the top of their starting areas is aligned with the top of the starting area $[0, w(r)] \times [-w(r), 0] \subset Q^{\uparrow}_r$. Each consecutive layer is then placed such that the base is 1 below the top of the vertical boxes from the previous layer. For the last layer, if necessary we adjust the position of the vertical boxes downwards until their top aligns with the top of $Q^{\uparrow}_r$.

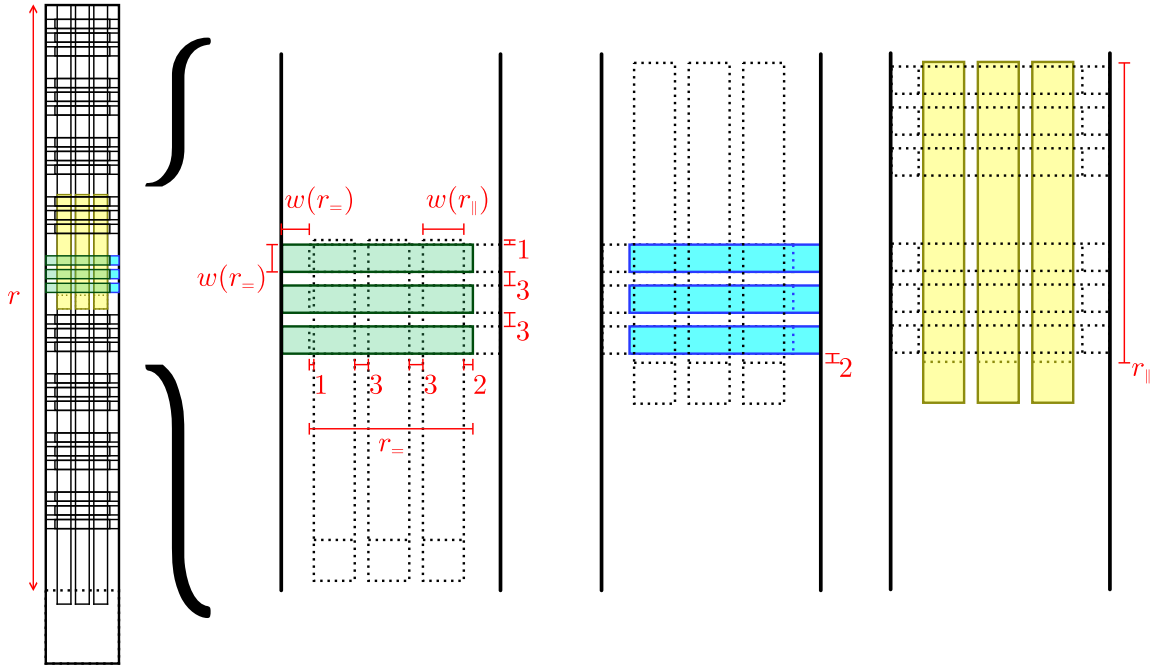


Figure 5.6.1: An illustration of the box configuration introduced in Step 1 of the proof of Proposition 5.6.1. Left: The box $Q^{\uparrow}_r$ with layers each consisting of three subgroups of smaller boxes stacked inside. Center left: For a single layer, the first subgroup consists of $k_=$ parallel boxes pointing to the right (depicted in green), cutting through the top of the vertical subgroup from the previous layer (depicted with dotted lines). Center right: The second subgroup (depicted in blue) mirrors the first subgroup. Right: The third subgroup consists of $k_\parallel$ parallel vertical boxes (depicted in yellow) spaced with distance 3 between them and with their starting areas placed below the previous subgroups

To be precise, given $r \geq r_1$ and $k_=, k_\parallel \in \mathbb{N}$, since $w\colon [0,\infty) \to [0,\infty)$ is onto and strictly increasing, there exist $r_= \geq r_0$ and – as along as r is large enough compared to $k_\parallel$ – further $r_\parallel \geq r_0$ such that

$$w(r) = r_= + 2w(r_=), \qquad r_= = k_\parallel w(r_\parallel) + 3k_\parallel. \tag{5.6.5}$$

The number of layers needed to reach the top of $Q^{\uparrow}_r$ is

$$N := \left\lceil \frac{r - k_= w(r_=) - 3k_=}{r_\parallel - k_= w(r_=) - 3k_=} \right\rceil \tag{5.6.6}$$

since for the height 'gained' per layer (after the first one) we have to subtract the base of the layer, i.e. the horizontally stacked boxes. The n-th layer, $n \in \{0, \dots, N-1\}$, consists of the boxes

$$\begin{aligned} &x^{\rightarrow}_{n,j} + Q^{\rightarrow}_{r_=}, && \text{for } j \in \{0, \dots, k_= - 1\}, && \text{(only for } n \neq 0) \\ &x^{\leftarrow}_{n,j} + Q^{\leftarrow}_{r_=}, && \text{for } j \in \{0, \dots, k_= - 1\}, && \text{(only for } n \neq 0) \\ &x^{\uparrow}_{n,j} + Q^{\uparrow}_{r_\parallel}, && \text{for } j \in \{0, \dots, k_\parallel - 1\}, \end{aligned}$$

where, see Figure 5.6.1,

$$\begin{aligned} x^{\rightarrow}_{n,j} &:= \begin{pmatrix} w(r_=) \\ w(r_=) + 2 \end{pmatrix} + j \begin{pmatrix} 0 \\ w(r_=) + 3 \end{pmatrix} + n \begin{pmatrix} 0 \\ r_\parallel - k_=(w(r_=) + 3) \end{pmatrix}, \\ x^{\leftarrow}_{n,j} &:= \begin{pmatrix} w(r) - w(r_=) \\ 2 \end{pmatrix} + j \begin{pmatrix} 0 \\ w(r_=) + 3 \end{pmatrix} + n \begin{pmatrix} 0 \\ r_\parallel - k_=(w(r_=) + 3) \end{pmatrix}, \\ x^{\uparrow}_{n,j} &:= \begin{cases} \begin{pmatrix} w(r_=) + 1 \\ 0 \end{pmatrix} + j \begin{pmatrix} w(r_\parallel) + 3 \\ 0 \end{pmatrix} + n \begin{pmatrix} 0 \\ r_\parallel - k_=(w(r_=) + 3) \end{pmatrix} & \text{for } n < N-1, \\ \begin{pmatrix} w(r_=) + 1 \\ r - r_\parallel \end{pmatrix} + j \begin{pmatrix} w(r_\parallel) + 3 \\ 0 \end{pmatrix} & \text{for } n = N-1. \end{cases} \end{aligned}$$

Step 2: $E^{\uparrow,\widetilde{v}}_r$ holds for some $\widetilde{v}$, if enough of the smaller boxes allow propagation. Still assuming that $k_=, k_\parallel \in \mathbb{N}$ are already chosen, we require that

- r is large enough such that $r_=, r_\parallel$ are well defined,
- $0 < N < \infty$, that is $r_\parallel - k_= w(r_=) - 3k_= > 0$,

In this step, we will show that if for each layer and for each subgroup there is at least one box along which the interface propagates with average speed $v > 0$, then $E^{\uparrow,\widetilde{v}}_r$ holds for some $\widetilde{v} = \widetilde{v}(v, k_\parallel, k_=, r) > 0$:

$$E^{\uparrow,\widetilde{v}}_r \supset \bigcap_{n=0}^{N-1} \left(\bigcup_{j=0}^{k_\parallel - 1} E^{\uparrow,v}_{r_\parallel, x^{\uparrow}_{n,j}} \right) \cap \bigcap_{n=1}^{N-1} \left(\bigcup_{j=0}^{k_= - 1} E^{\rightarrow,v}_{r_=, x^{\rightarrow}_{n,j}} \right) \cap \bigcap_{n=1}^{N-1} \left(\bigcup_{j=0}^{k_= - 1} E^{\leftarrow,v}_{r_=, x^{\leftarrow}_{n,j}} \right). \tag{5.6.7}$$

Hence, we will show that if for each $n \in \{0, \dots, N-1\}$ there exist $j_{n\uparrow} \in \{0, \dots, k_\parallel - 1\}$, and $j_{n\rightarrow}, j_{n\leftarrow} \in \{0, \dots, k_= - 1\}$ (for $n > 0$), such that $E^{\uparrow,v}_{r_\parallel, x^{\uparrow}_{n,j_{n\uparrow}}}$, $E^{\rightarrow,v}_{r_=, x^{\rightarrow}_{n,j_{n\rightarrow}}}$, and $E^{\leftarrow,v}_{r_=, x^{\leftarrow}_{n,j_{n\leftarrow}}}$ hold, then $E^{\uparrow,v}_r$ holds with $\widetilde{v}$ defined below in (5.6.9).

Since $E^{\uparrow,v}_{r_\parallel, x^{\uparrow}_{0,j_{0\uparrow}}}$ holds, there exists a closed set

$$S_0 \subset \left(x^{\uparrow}_{0,j_{0\uparrow}} + \left[0, w(r_\parallel)\right] \times \left[-w(r_\parallel), 0\right] \right) \subset \left[0, w(r)\right] \times \left[-w(r), 0\right]$$

which is stable with respect to $(A, F)_{x^{\uparrow}_{0,j_{0\uparrow}} + Q^{\uparrow}_{r_\parallel}}$ and therefore by Lemma 5.5.8 also stable with respect to the less restricted coefficient field $(A, F)_{Q^{\uparrow}_r}$. Thus, S_0 satisfies (5.3.8) and (5.3.9) for $E^{\uparrow,\widetilde{v}}_r$.

In order to show that S_0 satisfies (5.3.10), we use Lemma 5.5.13 and Lemma 5.5.11 to iteratively "stitch" together the guaranteed set evolutions along the the various small boxes,

see Figure 5.6.2. We will show that $\mathscr{R}_t^{(A,F)_{Q_r^\uparrow}}(S_0)$ passes the n-th layer, $n \in \{0,\dots,N-1\}$, at the latest after time

$$t_n := v^{-1}r_\parallel + nv^{-1}\big(r_= + r_\parallel\big).$$

More precisely, we will show by induction that

$\mathscr{R}_{t_n}^{(A,F)_{Q_r^\uparrow}}(S_0)$ cuts vertically through the box

$$Q_n := [w(r_=), w(r) - w(r_=)] \times [0, \min\big\{r, (n+1)r_\parallel - nk_=\big(w(r_=)+3\big)\big\}] \quad (5.6.8)$$

in the sense of Definition 5.5.10 for $n \in \{0,\dots,N-1\}$. For $n = N-1$, this implies that $\mathscr{R}_{t_{N-1}}^{(A,F)_{Q_r^\uparrow}}(S_0)$ must have reached the top of $Q_r^\uparrow$. Hence, S_0 satisfies (5.3.10) with

$$\widetilde{v} := \frac{r}{t_{N-1}} = v\frac{r}{(N-1)(r_\parallel + r_=) + r_\parallel}. \quad (5.6.9)$$

Thus, $E_r^{\uparrow,\widetilde{v}}$ and therefore $E_r^{\uparrow,\hat{v}}$ for any $\hat{v} \le \widetilde{v}$ hold.

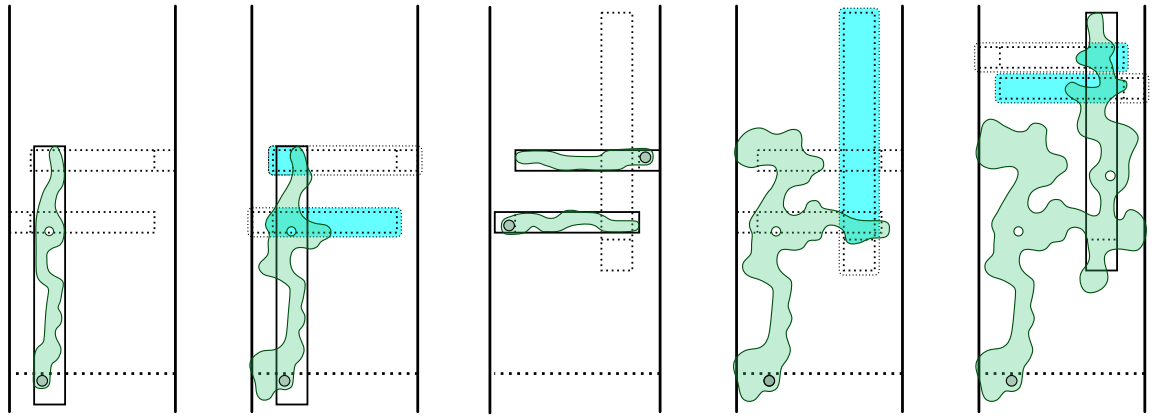


Figure 5.6.2: An illustration of the stitching argument from Step 2 of the proof of Proposition 5.6.1. Left: After time $v^{-1}r_\parallel$, the restricted evolution (depicted in green) of the set S_0 (depicted in grey) associated to one of the vertical boxes in the first layer cuts through the horizontal boxes from the second layer. Center left: This evolution is contained in the less restricted evolution $\mathscr{R}_{v^{-1}r_\parallel}^{(A,F)_{Q_r^\uparrow}}(S_0)$ (depicted in green), which hence cuts vertically through the horizontal boxes separating their end (depicted in blue) from their starting area. Center: The stable sets associated to the horizontal boxes (depicted in grey) after evolving with respect to the restriction for time $v^{-1}r_=$ (depicted in green) cut horizontally through their boxes. Center right: Due to the stitching lemma, $\mathscr{R}_{v^{-1}r_\parallel + v^{-1}r_=}^{(A,F)_{Q_r^\uparrow}}(S_0)$ (depicted in green) cuts horizontally through the vertical box of the second layer along which the interface propagates, separating its top (depicted in blue) from its starting area. Right: Another application of the stitching lemma yields that $\mathscr{R}_{2v^{-1}r_\parallel + v^{-1}r_=}^{(A,F)_{Q_r^\uparrow}}(S_0)$ (depicted in green) extends the vertical cut through the horizontal boxes of the third layer

It remains to prove (5.6.8). For $n = 0$, due to the characterization of $E_{r_\parallel}^{\uparrow,v}$ from Lemma 5.5.11, the set evolution of S_0 restricted to $x_{0,j_{0\uparrow}}^\uparrow + Q_{r_\parallel}^\uparrow$ cuts vertically through the main part of that box after time $t_0 = v^{-1}r_\parallel$. By Lemma 5.5.8, the same is hence true for $\mathscr{R}_{t_0}^{(A,F)_{Q_r^\uparrow}}(S_0)$ after evolving in a less restricted environment and for the wider box Q_0, for which the main part of $x_{0,j_{0\uparrow}}^\uparrow + Q_{r_\parallel}^\uparrow$ is a vertical layer.

Assuming that (5.6.8) holds for some $n \in \{0,\dots,N-2\}$, we know that $\mathscr{R}_{t_n}^{(A,F)_{Q_r^\uparrow}}(S_0)$ cuts vertically through the box

$$\widetilde{Q}_n = \begin{pmatrix} w(r_=) \\ (n+1)(r_\parallel - k_=(w(r_=)+3)) \end{pmatrix} + [0, w(r) - 2w(r_=)] \times [0, k_=\big(w(r_=)+3\big)]$$

which includes the main bodies of $x^{\rightarrow}_{n+1,j_{(n+1)\rightarrow}} + Q^{\rightarrow}_{r_=}$ and $x^{\leftarrow}_{n+1,j_{(n+1)\leftarrow}} + Q^{\leftarrow}_{r_=}$, the horizontal boxes of the $(n+1)$-th layer. An application of Lemma 5.5.13 hence yields that $\mathscr{R}^{(A,F)_{Q^{\uparrow}_r}}_{t_n+v^{-1}r_=}(S_0)$ cuts horizontally through $\widetilde{Q}_n$. Since the bottom side of $\widetilde{Q}_n$ is aligned with the starting area of $x^{\uparrow}_{n+1,j_{(n+1)\uparrow}} + Q^{\uparrow}_{r_\parallel}$, the vertical box of the $(n+1)$-th layer, we can apply Lemma 5.5.14. With Q_{n+1} corresponding to the extension of Q_n by the height of this vertical box, we obtain that $\mathscr{R}^{(A,F)_{Q^{\uparrow}_r}}_{t_n+v^{-1}r_=+v^{-1}r_\parallel}(S_0)$ cuts vertically through Q_{n+1}. By induction, we have thus shown that (5.6.8) holds for all $n \in \{0,\dots,N-1\}$.

Step 3: A probability bound. In the previous step we showed (5.6.7), that is, that $E^{\uparrow,\widetilde{v}}_r$ holds if for each of the N groups of vertical boxes and $2(N-1)$ groups of horizontal boxes there is at least one box such that the corresponding event holds for speed v. Using that $\mathbb{P}[E_1 \cap E_2] \geq 1 - \mathbb{P}[(E_1)^c] - \mathbb{P}[(E_2)^c]$ and (P1), the stationarity of the probability distribution, (5.6.7) yields

$$\mathbb{P}\big[E^{\uparrow,\widetilde{v}}_r\big] \geq 1 - N\,\mathbb{P}\left[\bigcap_{j=0}^{k_\parallel-1}\left(E^{\uparrow,v}_{r_\parallel,x^{\uparrow}_{0,j}}\right)^c\right] - (N-1)\mathbb{P}\left[\bigcap_{j=0}^{k_=-1}\left(E^{\rightarrow,v}_{r_=,x^{\rightarrow}_{1,j}}\right)^c\right] - (N-1)\mathbb{P}\left[\bigcap_{j=0}^{k_=-1}\left(E^{\leftarrow,v}_{r_=,x^{\leftarrow}_{1,j}}\right)^c\right].$$

Since the distance between any two boxes in one group is larger than 3, and since by Lemma 5.5.16 whether the interface propagates along the box only depends on the coefficient field in a 1-neighborhood, the corresponding events are independent due to the finite range of dependence from Assumption (P2). Hence, again invoking the stationarity of the probability distribution, we obtain

$$\begin{aligned}\mathbb{P}\big[E^{\uparrow,\widetilde{v}}_r\big] &\geq 1 - N\mathbb{P}\Big[\big(E^{\uparrow,v}_{r_\parallel}\big)^c\Big]^{k_\parallel} - N\mathbb{P}\big[(E^{\rightarrow,v}_{r_=})^c\big]^{k_=} - N\mathbb{P}\big[(E^{\leftarrow,v}_{r_=})^c\big]^{k_=} \\ &= 1 - N\Big(1 - \mathbb{P}\Big[E^{\uparrow,v}_{r_\parallel}\Big]\Big)^{k_\parallel} - N\big(1 - \mathbb{P}\big[E^{\rightarrow,v}_{r_=}\big]\big)^{k_=} - N\big(1 - \mathbb{P}\big[E^{\leftarrow,v}_{r_=}\big]\big)^{k_=}. \qquad (5.6.10)\end{aligned}$$

Step 4: Choosing the parameters $k_\parallel$ and $k_=$. Given $r \in [r_i, r_{i+1})$ for some $i \in \mathbb{N}$, in particular we have $r \geq r_1$. Hence, for the length $r_= = r_=(r)$ of the horizontal boxes as defined in (5.6.5) we automatically obtain

(C1) $r_= = r_=(r) \in [r_{i-1}, r_i)$ and $r_= = r_=(r) \geq r_0$ by the definition of $(r_i)_i$ in (5.6.2).

Considering (5.6.10), we set $k_= = k_\parallel = k$ with the intent to choose $k = k(r) \geq 2$ as large as possible. For $r_=$ and the length $r_\parallel = r_\parallel(r)$ of the vertical boxes as defined in (5.6.5) we still want to guarantee though that

(C2) $r_\parallel = r_\parallel(r)$ exists with $r_= \leq r_\parallel \leq k^{-q} r$ (the last inequality holds by definition),

(C3) $w(r_=) \geq 3$,

(C4) $N \in [k^q, 2k^q]$ (by definition $N \geq k^q$, the constant 2 is chosen arbitrarily for an initial bound on the loss of speed[1]).

[1] In order to obtain a positive limit speed with this iteration, we need $\frac{\widetilde{v}}{v} \to 1$ as $r \to \infty$. This requires $\frac{Nr_=}{r}$, the additional time corresponding to traversing the horizontal boxes relative to r, to disappear. Since $N \geq k^q$, this implies $\frac{k^q}{r_=^{q-1}} \to 0$ and thus plugging this into (5.6.6) also $\frac{N}{k^q} \to 1$ as $r \to \infty$. Requiring that $\frac{k^q}{r_=^{q-1}} \to 0$ also explains why below we choose $k \sim r_=^{\frac{q-1}{q}-\delta}$ and not $k \sim r_=^{\frac{q-1}{q}}$.

First, (C3) is equivalent to

$$r_= \geq 3^q c_w. \tag{5.6.11}$$

With (5.6.5), the definition of $r_\parallel$, existence is guaranteed for $r_= > 3k$. Using (C3), we obtain that the inequality $r_= \leq r_\parallel$ in (C2) is implied via

$$\begin{aligned} r_\parallel = w^{-1}\Big(\frac{r_=}{k} - 3\Big) \geq r_= \qquad &\Leftarrow \qquad r_= \geq k\big(w(r_=) + 3\big) \\ \Leftarrow \qquad r_= \geq 2kw(r_=) \qquad &\Leftarrow \qquad r_=^{q-1} \geq 2^q k^q c_w^{-1}. \end{aligned} \tag{5.6.12}$$

The second inequality, $r_\parallel \leq k^{-q} r$, follows from (5.6.5), the definition of $r_=$, via

$$r_\parallel = w^{-1}\Big(\frac{r_=}{k} - 3\Big) \leq w^{-1}\Big(\frac{w(r)}{k}\Big) = \frac{r}{k^q}.$$

With respect to (C4), using that $s \mapsto \frac{C_1 - s}{C_2 - s}$ increases for $C_1 > C_2 > s$ we have

$$N = \left\lceil \frac{r - kw(r_=) - 3k}{r_\parallel - kw(r_=) - 3k} \right\rceil \geq \frac{r}{r_\parallel} \geq \frac{w^{-1}(r_=)}{w^{-1}\big(\frac{r_=}{k}\big)} = k^q,$$

where for the last inequality we used that $r \geq w^{-1}(r_=)$ and that $r_\parallel \leq w^{-1}\big(\frac{r_=}{k}\big)$ due to the definition of $r_=, r_\parallel$ in (5.6.5) and the monotonicity of $w(\cdot)$. On the other hand, using that $r = w^{-1}(r_= + 2w(r_=))$ and $r_\parallel = w^{-1}\big(\frac{r_=}{k} - 3\big)$, as well as that $w(r_=) \geq 3$ from (C3), we obtain

$$\begin{aligned} N &\leq \frac{r}{r_\parallel - kw(r_=) - 3k} + 1 \\ &= \left(\frac{c_w\big(\frac{r_=}{k} - 3\big)^q}{c_w(r_= + 2w(r_=))^q} - \frac{k(w(r_=) + 3)}{c_w(r_= + 2w(r_=))^q} \right)^{-1} + 1 \\ &= k^q \left(\left(1 - \frac{3k + 2w(r_=)}{r_= + 2w(r_=)}\right)^q - \frac{k^{q+1}(w(r_=) + 3)}{c_w(r_= + 2w(r_=))^q} \right)^{-1} + 1 \\ &\leq k^q \left(\left(1 - \frac{(k+2)w(r_=)}{r_=}\right)^q - \frac{2k^{q+1} w(r_=)}{c_w r_=^q} \right)^{-1} + 1 \\ &\leq k^q \left(1 - q\frac{(k+2)w(r_=)}{r_=} - \frac{2k^{q+1} w(r_=)}{c_w r_=^q} \right)^{-1} + \frac{k^q}{2} \end{aligned}$$

where for the last inequality we used that $k \geq 2$, $q > 1$. In particular, $N \leq 2k^q$ holds if

$$\begin{aligned} \frac{(k+2)w(r_=)}{r_=} \leq \frac{1}{6q}, \qquad &\text{which is implied by} \qquad r_=^{\frac{q-1}{q}} \geq 6q(k+2)c_w^{-\frac{1}{q}}, \\ \frac{2k^{q+1} w(r_=)}{c_w r_=^q} \leq \frac{1}{6}, \qquad &\text{which is implied by} \qquad r_=^{\frac{q^2-1}{q}} \geq 12k^{q+1} c_w^{-\frac{q+1}{q}}. \end{aligned}$$

Combining this with (5.6.11) and (5.6.12), we obtain that (C2), (C3), and (C4) hold if

$$r_= \geq \max\left\{ 3^q c_w, \left(12qkc_w^{-\frac{1}{q}}\right)^{\frac{q}{q-1}} \right\}. \tag{5.6.13}$$

This prompts us to fix $0 < \delta < \frac{q-1}{q}$, $k_0 \in \mathbb{N}$ and choose

$$k = k(r) := \left\lfloor k_0 \left(\frac{r_=(r)}{r_0} \right)^{\frac{q-1}{q} - \delta} \right\rfloor, \qquad k_0 := \left\lfloor \frac{1}{12q} c_w^{\frac{1}{q}} r_0^{\frac{q-1}{q}} \right\rfloor, \tag{5.6.14}$$

where k_0 is the largest $k \in \mathbb{N}$ such that (5.6.13) holds for $r_=(r_1) = r_0$ and hence $k(r_1) = k_0$. The bound (5.3.12) for r_0 from Assumption (P□) guarantees $k_0 \geq 2$ and hence $k(r) \geq 2$ for all $r \geq r_1$. Note that (5.3.12) also guarantees $r_=(r) \geq r_0 \geq 3^q c_w$ for $r \geq r_1$.

Hence, choosing $k(r)$ and k_0 as in (5.6.14) yields (5.6.13) and thus (C2), (C3), and (C4) for all $r \geq r_1$. In addition, using that $k(r) \geq 2$ for all $r \geq r_1$, we obtain the bound

$$k(r) \geq k_0 \left(\frac{r_=(r)}{r_0}\right)^{\frac{q-1}{q}-\delta} - 1 \tag{5.6.15}$$

$$\geq \frac{k_0}{2}\left(\frac{r_=(r)}{r_=(r_1)}\right)^{\frac{q-1}{q}-\delta} \geq \frac{k_0}{2}\left(\frac{w(r)}{w(r_1)}\right)^{\frac{q-1}{q}-\delta} = \frac{k_0}{2}\left(\frac{r}{r_1}\right)^{\frac{q-1}{q^2}-\frac{\delta}{q}}, \tag{5.6.16}$$

where we used that, with the definition of $r_=(r)$ from (5.6.5) and due to $r_= \mapsto \frac{r_=}{r_=+2w(r_=)}$ increasing, we have

$$\frac{r_=(r)}{r_=(r_1)} = \frac{w(r)}{w(r_1)} \frac{\frac{r_=(r)}{r_=(r)+2w(r_=(r))}}{\frac{r_=(r_1)}{r_=(r_1)+2w(r_=(r_1))}} \geq \frac{w(r)}{w(r_1)}.$$

Step 5: Quantifying the loss of speed per step. In Step 2, given $r \geq r_1$ and $v > 0$ for the speed associated with the smaller boxes inside $Q_r^\uparrow$, for the minimum average speed in $Q_r^\uparrow$ we obtained

$$\widetilde{v} = \widetilde{v}(v, r) \coloneqq v \frac{r}{(N-1)(r_\parallel + r_=) + r_\parallel}. \tag{5.6.9}$$

We bound the loss factor by bounding the excess distance from following the path along the smaller boxes. With (5.6.6), the definition of N, we have

$$\begin{aligned}
&(N-1)\big(r_\parallel + r_=\big) + r_\parallel \\
&\quad = (N-1)\big(r_\parallel - k\big(w(r_=)+3\big) + k\big(w(r_=)+3\big)\big) + (N-1)r_= + r_\parallel \\
&\quad \leq r + (N-1)k\big(w(r_=)+3\big) + (N-1)r_= + r_\parallel.
\end{aligned} \tag{5.6.17}$$

The three additional terms correspond to the overlap of the layers, the horizontally traversed distance, and the potential overshoot in the last layer. For the overlap, using $r_= \leq r_\parallel$ from (C2), the monotonicity of $w(\cdot)$, and the definition of $r_\parallel$ from (5.6.5), we obtain a bound corresponding to the horizontal distance,

$$(N-1)k\big(w(r_=)+3\big) \leq (N-1)k\big(w(r_\parallel)+3\big) = (N-1)r_=. \tag{5.6.18}$$

For the horizontal distance, with (C4), the choice of k, and $r_= \leq w(r)$, we have

$$(N-1)r_= \leq 2k^q r_= \leq 2k_0^q \left(\frac{r_=}{r_0}\right)^{q-1-\delta q} r_= \leq r\, 2k_0^q r_0^{\delta q - q + 1} c_w^{-(1-\delta)} r^{-\delta}. \tag{5.6.19}$$

For the overshoot, first applying (C2) and then the bound on $k(r)$ from(5.6.16), we obtain

$$r_\parallel \leq r k^{-q} \leq r \left(\frac{r_1}{r}\right)^{\frac{q-1}{q}-\delta}. \tag{5.6.20}$$

In summary, plugging (5.6.17) with (5.6.18), (5.6.19), and (5.6.20) into (5.6.9) and then using that $k_0 \leq \frac{1}{12q} c_w^{\frac{1}{q}} r_0^{\frac{q-1}{q}}$ yields

$$\widetilde{v} \geq v \frac{1}{1 + 4k_0^q r_0^{\delta q - (q-1)} c_w^{\delta - 1} r^{-\delta} + \left(\frac{r_1}{r}\right)^{\frac{q-1}{q}-\delta}} \geq v \frac{1}{1 + C_v r^{-\min\left\{\delta, \frac{q-1}{q}-\delta\right\}}} \tag{5.6.21}$$

with $C_v = C_v(q, c_w, r_0, \delta) := \frac{4}{(12q)^q} r_0^{\delta q} c_w^{\delta} + r_1^{\frac{q-1}{q}-\delta}$ where we used that $k_0 \leq \frac{1}{12q} c_w^{\frac{1}{q}} r_0^{\frac{q-1}{q}}$.

Step 6: Ensuring improved probability per step. Referring to the probability bound (5.6.10) from Step 3, for $0 < p < 1$ and $r \geq r_1$ with $r_=(r)$, $N(r)$, and $k(r)$ as above, we set

$$\overline{p} = \overline{p}(p, r) := 3N(r)p^{k(r)}. \tag{5.6.22}$$

In this step, we will provide bounds for $\overline{p}$. In particular, we are interested in sufficient conditions to guarantee that $\overline{p} < p$. Note that due to the bounds on N from (C4) it holds that

$$\overline{p} \leq 6k^q p^k = p \exp\big(\log(6) + q\log(k) - (k-1)\log(p^{-1})\big). \tag{5.6.23}$$

We obtain the desired bound if the exponent is negative. For $p = p_0$ and $r = r_1$ with $k(r_1) = k_0$ as chosen in (5.6.14), we hence have $\overline{p}(p_0, r_1) < p_0$ if

$$\begin{aligned} & & 0 &> \log(6) + q\log(k_0) - (k_0 - 1)\log(p_0^{-1}) \\ \Leftarrow & & \log(p_0^{-1}) &> \frac{\log(6) + q\log(k_0)}{k_0 - 1}, \end{aligned}$$

which is guaranteed by (5.3.13) from Assumption (P□).

Since $r \mapsto k(r)$ is increasing and we only want to apply the iteration for $r \geq r_1$ and $p \leq p_0$, we now only need to check that

$$g\colon (p, k) \mapsto \log(6) + q\log(k) - (k-1)\log(p^{-1})$$

is decreasing in k for $k \geq k_0 = k(r_1)$ and increasing in p for $p \leq p_0$. We have

$$\begin{aligned} &\frac{\partial g}{\partial p}(p, k) = \frac{k-1}{p} > 0, \\ &\frac{\partial g}{\partial k}(p, k) = \frac{q}{k} + \log(p) \leq \frac{1}{2}\log(p) \qquad \Leftarrow \qquad \log(p^{-1}) \geq \frac{2q}{k}, \end{aligned}$$

which holds for $p \leq p_0$ and $k \geq k_0$ due to (5.3.13) from Assumption (P□). In particular, we have $g(p, k) < 0$ for $p \leq p_0$ and $k \geq k_0$ and, due to the mean value theorem, we obtain

$$g(p, k) \leq g(p, k_0) + (k - k_0)\frac{\log(p)}{2}.$$

In summary, plugging these bounds into (5.6.23) yields

$$\overline{p}(p, r) \leq p \exp\big(g(p, k(r))\big) < p, \tag{5.6.24}$$

$$\overline{p}(p, r) \leq p \exp\left(g(p, k_0) + (k(r) - k_0)\frac{\log(p)}{2}\right) \leq p^{\frac{1}{2}(k(r)-k_0)+1} \tag{5.6.25}$$

for all $r \geq r_1$ and $0 < p \leq p_0$.

Step 7: The recursive formulas for v_i and p_i. Assume that (5.6.1) holds for some $i \in \mathbb{N}_0$, that is, assume that there exist $0 \leq p_i \leq p_0$ and $v_i > 0$ such that

$$\min\big\{\mathbb{P}\big[E_r^{\uparrow, v_i}\big], \mathbb{P}\big[E_r^{\rightarrow, v_i}\big], \mathbb{P}\big[E_r^{\downarrow, v_i}\big], \mathbb{P}\big[E_r^{\leftarrow, v_i}\big]\big\} \geq 1 - p_i, \quad \text{for all } r \in [r_i, r_{i+1}).$$

Let $r \in [r_{i+1}, r_{i+2})$. We want to apply the iterative scheme to obtain a probability bound for $\mathbb{P}\big[E_r^{\uparrow, v_{i+1}}\big]$ with some v_{i+1} via (5.6.10). By the definition of $(r_i)_i$, we have $r_=(r) \in [r_i, r_{i+1})$. However, while $r_\parallel(r) \geq r_i$ by (C2), $r_\parallel(r) > r_{i+1}$ for r close enough to r_{i+2}. Hence, we will

first obtain bounds for $r \in [r_{i+1}, r_{i+1}^{(1)})$, with $r_{i+1}^{(1)}$ chosen such that $r_{\shortparallel}(r) < r_{i+1}^{(0)} := r_{i+1}$ for all such r. Having extended the range of the probability bounds, we repeat this step until we have obtained a bound for all $r < r_{i+2}$, see Figure 5.6.3. To be more precise, we set

$$r_{i+1}^{(0)} := r_{i+1}, \qquad r_{i+1}^{(j)} := \inf\Big\{ r \in [r_{i+1}, \infty) \,:\, r_{\shortparallel}(r) \geq r_{i+1}^{(j-1)} \Big\} \quad \text{for } j \in \mathbb{N}.$$

The amount of such steps necessary to reach r_{i+2} we denote with

$$\ell_{i+1} := \min\Big\{ j \in \mathbb{N} \,:\, r_{i+1}^{(j)} \geq r_{i+2} \Big\}. \tag{5.6.26}$$

Now, since $r_{=}(r), r_{\shortparallel}(r) \in [r_i, r_{i+1}^{(j-1)})$ for $r \in [r_{i+1}^{(j-1)}, r_{i+1}^{(j)})$ by (C1), (C2) and the definition of $(r_{i+1}^{(j)})_j$, iteratively applying (5.6.10) yields

$$\begin{aligned} &\mathbb{P}\Big[E_r^{\uparrow, v_i^{(j)}} \Big] \geq 1 - p_{i+1}, \quad \text{for all } r \in [r_i, r_{i+1}^{(j)} \wedge r_{i+2}),\ j \in \{1, \dots, \ell_{i+1}\}, \\ &\text{with} \quad v_i^{(j)} := \inf\Big\{ \widetilde{v}\Big(v_i^{(j-1)}, r\Big) \,:\, r \in \big[r_{i+1}^{(j-1)}, r_{i+1}^{(j)}\big) \Big\}, \quad v_i^{(0)} := v_i, \\ &\qquad\quad p_{i+1} := \sup\big\{ \overline{p}(p_i, r) \,:\, r \in [r_{i+1}, r_{i+2}) \big\}, \end{aligned} \tag{5.6.27}$$

where $\widetilde{v}$ and $\overline{p}$ were defined in (5.6.9) and (5.6.22). We also used that $\widetilde{v}(v, r) \leq v$ and $\overline{p}(p, r) < p$ for $r \geq r_1$ and $p \leq p_0$ by (5.6.24). We obtain the same for $\mathbb{P}\Big[E_r^{\rightarrow, v_i^{(j)}} \Big]$, $\mathbb{P}\Big[E_r^{\downarrow, v_i^{(j)}} \Big]$, and $\mathbb{P}\Big[E_r^{\leftarrow, v_i^{(j)}} \Big]$ by respectively adjusting the procedure in the previous steps of the proof. In summary, this yields (5.6.1) with $(p_i)_i$ defined recursively as above and $(v_i)_i$ defined recursively via $v_{i+1} := v_i^{(\ell_{i+1})}$.

For the probabilities, by (5.6.10), (5.6.22), and (5.6.25), we have

$$p_{i+1} \leq \sup_{r \in [r_{i+1}, r_{i+2})} 3N(r) p_i^{k(r)} \leq p_i^{\frac{1}{2}(k(r_{i+1}) - k_0) + 1} \leq p_i^{\frac{k_0}{2}\left(\left(\frac{r_i}{r_0}\right)^{\frac{q-1}{q} - \delta} - 1 \right)}, \tag{5.6.28}$$

using the lower bound for $k(r)$ from (5.6.15).

It remains to prove the recursive bounds for $(v_i)_i$. To achieve this, we first determine a uniform bound on the amount of steps $i \mapsto \ell_i$ for the respective intervals as defined in (5.6.26). To obtain such a bound we need to show that $j \mapsto r_{i+1}^{(j)}$ grows fast enough. Note that, since $r \mapsto r_{\shortparallel}(r)$ is right-continuous by the definition of $r \mapsto k(r)$, we obtain $r_{\shortparallel}\Big(r_{i+1}^{(j)}\Big) \geq r_{i+1}^{(j-1)}$ with the definition of $r_{i+1}^{(j)}$. By the definition of $r_{=}(r), r_{\shortparallel}(r)$ from (5.6.5), this implies

$$w\Big(r_{i+1}^{(j)}\Big) = k\Big(r_{i+1}^{(j)}\Big)\Big(w\Big(r_{\shortparallel}\Big(r_{i+1}^{(j)}\Big)\Big) + 3\Big) + 2w\Big(r_{=}\Big(r_{i+1}^{(j)}\Big)\Big) \geq k\Big(r_{i+1}^{(j)}\Big) w\Big(r_{i+1}^{(j-1)}\Big).$$

Combining this inequality with the lower bound for $k(r) \geq 2$ from (5.6.16), we obtain

$$r_{i+1}^{(j)} \geq k^q\Big(r_{i+1}^{(j)}\Big) r_{i+1}^{(j-1)} \geq \max\left\{ 2^q, \left(\frac{r_{i+1}^{(j)}}{r_1} \right)^{\frac{q-1}{q} - \delta} \right\} r_{i+1}^{(j-1)},$$

implying that ℓ_{i+1} is finite. From this inequality we deduce that

$$\begin{aligned} r_{i+1}^{(j)} \geq r_1^{1 - \frac{q}{1+\delta q}} \Big(r_{i+1}^{(j-1)}\Big)^{\frac{q}{1+\delta q}} &= \Big(r_{i+1}^{(j-1)}\Big)^{\frac{q}{1+\delta q} - \left(\frac{q}{1+\delta q} - 1\right) \frac{\log(r_1)}{\log\left(r_{i+1}^{(j-1)}\right)}} \\ &\geq \Big(r_{i+1}^{(j-1)}\Big)^{\frac{q}{1+\delta q} - \left(\frac{q}{1+\delta q} - 1\right) \frac{\log(r_1)}{\log(r_{i+1})}}, \end{aligned}$$

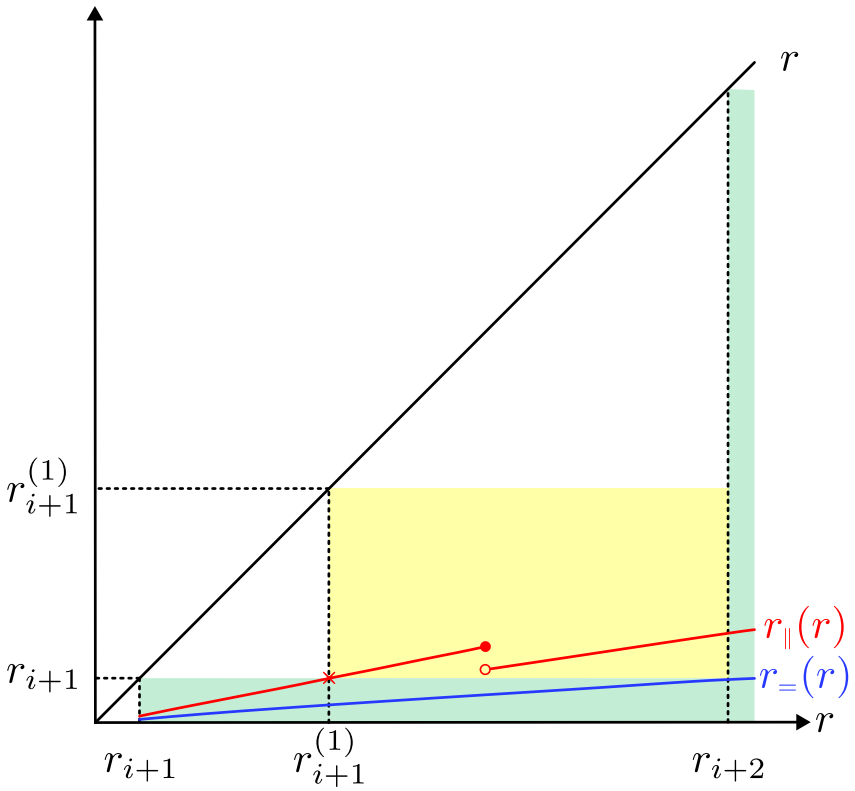


Figure 5.6.3: An illustration of the functions $r \mapsto (r_=(r), r_\parallel(r), r)$ with the thresholds from Step 7 of the proof of Proposition 5.6.1. Assuming we have bounds for boxes in the range $[r_i, r_{i+1})$ (depicted in green), we can use these to obtain bounds for boxes of length $r \geq r_{i+1}$ (depicted in black) until $r_\parallel(r)$ (depicted in red) leaves this range at $r = r_{i+1}^{(1)}$ (marked with a dotted line). In the next step, these bounds can be used until $r_\parallel(r)$ leaves the range $[r_{i+1}, r_{i+1}^{(1)})$ (depicted in yellow) or until $r_=(r)$ (depicted in blue) leaves the original range at $r = r_{i+2}$. This process is repeated until the second case occurs, which is when we can update the propagation probability for all smaller boxes and restart the loop. The discontinuous jumps in $r_\parallel(r)$ correspond to increases of $k(r) \in \mathbb{N}$

where we used that $r_{i+1}^{(j)} \geq r_{i+1}$ for all $j \in \mathbb{N}_0$. Iteratively applying this inequality yields

$$r_{i+1}^{(j)} \geq r_{i+1}^{\left(\frac{q}{1+\delta q} - \left(\frac{q}{1+\delta q} - 1\right)\frac{\log(r_1)}{\log(r_{i+1})}\right)^j}, \tag{5.6.29}$$

which provides a good lower bound for the growth of $j \mapsto r_{i+1}^{(j)}$ in terms of j. In order to bound the amount of steps necessary to reach r_{i+2}, note that with $2w(r_=) \leq 2w(r_\parallel) \leq r_=$ due to (C2) we have

$$r_{i+2} = w^{-1}\big(r_{i+1} + 2w(r_{i+1})\big) \leq w^{-1}\big(2r_{i+1}\big) = 2^q c_w r_{i+1}^q = r_{i+1}^{q + \frac{\log(2^q c_w)}{\log(r_{i+1})}}. \tag{5.6.30}$$

Hence, by the definition of ℓ_{i+1} from (5.6.26) and by comparing the exponents in (5.6.29) and (5.6.30), we know that

$$\left(\frac{q}{1+\delta q} - \left(\frac{q}{1+\delta q} - 1\right)\frac{\log(r_1)}{\log(r_{i+1})}\right)^{\ell_{i+1}-1} < q + \frac{\log(2^q c_w)}{\log(r_{i+1})}$$

and can conclude that for any $i \geq 1$ we obtain the bound

$$\begin{aligned}\ell_{i+1} &\leq \frac{\log\Big(q + \frac{\log(2^q c_w)}{\log(r_{i+1})}\Big)}{\log\Big(\frac{q}{1+\delta q} - \Big(\frac{q}{1+\delta q} - 1\Big)\frac{\log(r_1)}{\log(r_{i+1})}\Big)} + 1 \\ &\leq \frac{\log\Big(q + \max\Big\{0, \frac{\log(2^q c_w)}{\log(r_2)}\Big\}\Big)}{\log\Big(\frac{q}{1+\delta q} - \Big(\frac{q}{1+\delta q} - 1\Big)\frac{\log(r_1)}{\log(r_2)}\Big)} + 1 \\ &=: \overline{\ell} = \overline{\ell}(q, c_w, r_0, \delta).\end{aligned}$$

That is, for all $i \in \mathbb{N}$ we have $\ell_i \leq \ell = \ell(q, c_w, r_0, \delta) := \max\big\{\ell_1, \overline{\ell}(q, c_w, r_0, \delta)\big\}$.

Concluding, we obtain (5.6.1) from (5.6.27) with the iterative bounds on $(p_i)_i$ due to (5.6.28) and the iterative bounds on $(v_i)_i$ due to ℓ applications of (5.6.21), given by

$$\begin{aligned}v_{i+1} &\geq \left(\frac{1}{1 + C_v(q, c_w, r_0, \delta) r_{i+1}^{-\min\left\{\delta,\ \frac{q-1}{q} - \delta\right\}}}\right)^{\ell(q,c_w,r_0,\delta)} v_i, \\ p_{i+1} &\leq p_i^{\left(\frac{r_i}{r_0}\right)^{\frac{q-1}{q}-\delta} - 1}.\end{aligned}$$

□

5.7 Establishing an almost guaranteed effective minimum speed

We will now show that, via Proposition 5.6.2, the box criterion (P□) implies that there is an almost guaranteed effective minimum speed in the sense of (P◎), see Section 5.3.5. That is, we need to find a family of events $(E_h^{\circledcirc}(M))_{h,M}$ for areas $M \subset \mathbb{R}^2$ and scales $h \gg 1$ which are (∗i) measurable with respect to a neighborhood of M, (∗ii) ordered with respect to set inclusions, and, most importantly, (∗iii) for given $M \subset \mathbb{R}^2$ and scales $h \gg 1$ guarantees that all sufficiently h-fat sets with boundary in M have a stable approximation which propagates with some effective minimum speed. These notions have been introduced in Definition 5.3.7. In addition, we need to provide strong probability bounds in the form of (5.3.18).

Our candidate for $E_h^{\circledcirc}(M)$ is the event that there is a net of boxes with mesh width $\sim h$ covering M along which the interface is guaranteed to propagates with some minimum average speed. That is, given $h \geq \frac{r_0}{2}$, $M \subset \mathbb{R}^2$ and $v > 0$, we define the event

$$E_h^{\boxplus, v}(M) := \bigcap_{x \in h\mathbb{Z}^2 \,:\, (x+[0,h]^2) \cap M \neq \emptyset} E_{h,x}^{\square, v}, \tag{5.7.1}$$

where, with the measurability to be established in Lemma 5.7.1,

$$\begin{aligned}E_{h,x}^{\square,v} := &\left(\bigcup_{y \in x + [1, h - w(2h) - 1] \times \{0\}} E_{2h,y}^{\uparrow,v}\right) \cap \left(\bigcup_{y \in x + \{0\} \times [1 + w(2h), h-1]} E_{2h,y}^{\rightarrow,v}\right) \\ &\cap \left(\bigcup_{y \in x + [1 + w(2h), h-1] \times \{h\}} E_{2h,y}^{\downarrow,v}\right) \cap \left(\bigcup_{y \in x + \{h\} \times [1, h - w(2h) - 1]} E_{2h,y}^{\leftarrow,v}\right).\end{aligned}$$

That is, if $E_h^{\boxplus,v}(M)$ holds, then for each cell of the grid $h\mathbb{Z}^2$ touching M and for each direction there exists a box along which the interface propagates crossing both this cell and then the neighboring cell in the corresponding direction, see Figure 5.7.1.

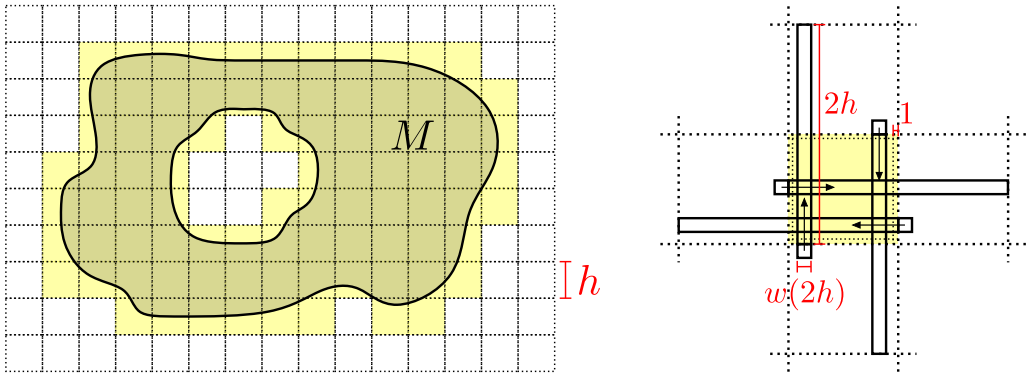


Figure 5.7.1: An illustration of the event $E_h^{\boxplus,v}(M)$ introduced in Section 5.7. Left: For a set $M \subset \mathbb{R}^2$ (depicted in grey), we consider all cells of the grid $h\mathbb{Z}^2$ which intersect M (depicted in yellow). Right: If $E_h^{\boxplus,v}(M)$ holds, then for each of these cells and for each direction there exists a box along which the interface propagates cutting through that cell as well as through the neighboring cell in the respective direction

In this section, we will show that this event indeed guarantees

- an effective minimum speed of propagation on the scale of h, and
- approximate stability on the scale of h for $2h$-fat sets.

Due to Corollary 5.6.2 we will obtain stretched exponential bounds for the complement of $E_h^{\boxplus,v_\infty}(M)$ as required in (5.3.18) for (P⊙). Finally, we will show that choosing $E_h^{\odot}(M) := E_{\frac{h}{2}}^{\boxplus,v}\big(M + 2[-h,h]^2\big)$ satisfies the requirements for (P⊙) and hence we will establish Theorem 5.4.1.

5.7.1 The scaled up version of the box criterion implies (P⊙)

Regarding the measurability for (∗i) in (P⊙), we obtain the following lemma analogously to Lemma 5.5.16 on the measurability of the event that the interface percolates along a single box.

Lemma 5.7.1 (Measurability of $E_h^{\boxplus,v}(M)$)**.** *Let $M \subset \mathbb{R}^2$, $h \geq \frac{r_0}{2}$, $v > 0$. Then $E_h^{\boxplus,v}(M)$ is measurable with respect to $\mathcal{F}(M + 3[-h,h]^2)$.*

Proof. We only need to show that $E_{h,x}^{\square,v} \in \mathcal{F}(x + [-2h,3h]^2)$ for any $x \in h\mathbb{Z}^2$ because if $(x + [0,h]^2) \cap M \neq \emptyset$ then $\mathcal{F}(x + [-2h,3h]^2) \subset \mathcal{F}(M + 3[-h,h]^2)$. As in the proof of Lemma 5.5.16 for the measurability of an event such as $E_{2h}^{\uparrow,v}$, we apply Lemma 5.5.15. That is, we have to show that the event is closed with respect to weighted uniform convergence of the coefficient field. A small modification to the proof for Lemma 5.5.16 suffices, accounting for the fact that the box for which the respective event holds might vary along a given sequence of coefficient fields.

To be precise, let there be a sequence $(A_k, F_k) \to (A, F)$ which converges uniformly on the domain $(x + [-2h,3h]^2) \times \mathbb{S}^1$ with

$$(A_k, F_k) \in E_{2h,y_k}^{\uparrow,v}, \quad y_k \in x + \big[1, h - w(2h) - 1\big] \times \{0\}.$$

There is a (without loss of generality non-relabeled) subsequence with $y_k \to y$ for some $y \in x + [1, h - w(2h) - 1] \times \{0\}$. Thus, $(A_k, F_k)_{y_k + Q^{\uparrow}_{2h}} \to (A, F)_{y + Q^{\uparrow}_{2h}}$ uniformly. Analogously to the proof for Lemma 5.5.16, we obtain $(A, F) \in E^{\uparrow, v}_{2h, y}$. An application of Lemma 5.5.15 hence yields

$$\bigcup_{y \in x + [1, h - w(2h) - 1] \times \{0\}} E^{\uparrow, v}_{2h, y} \subset \mathcal{F}\big(x + [-2h, 3h]^2\big).$$

The other directions in the definition of $E^{\square, v}_{h, x}$ are treated analogously. □

Next, in view of (∗iii) in (P◎), we show that $E^{\boxplus, v}_h(M)$ guarantees the existence of stable approximations as introduced in Definition 5.3.7 for sufficiently fat sets with boundary suitably covered by M.

Lemma 5.7.2 (Approximate stability guaranteed by $E^{\boxplus, v}_h(M)$)**.** *Let $h \geq \frac{r_0}{2}$ and $S \subset \mathbb{R}^2$ be a $\big(\sqrt{2} + \frac{1}{2}\big)h$-fat set. Assume that $(A, F) \in E^{\boxplus, v}_h\big(\partial S + \big(\sqrt{2} + \frac{1}{2}\big)B_h\big)$ for some velocity $0 < v \leq v_{\max}$. Then S is $(h,\, 12v^{-1})$-approximately stable with respect to (A, F).*

Proof. We will construct a stable $(h,\, 12v^{-1})$-approximation S_h for S, which is a stable set with $S_h \supset \mathbb{R}^2 \setminus \big(S^c + 12\frac{v_{\max}}{v} B_h\big)$ and $S \subset_h \mathscr{R}_{12v^{-1}h}(S_h)$.

In order to relate the set to the grid cells used to define the events $\big(E^{\boxplus, v}_h(M)\big)_M$, we select those inner cells which together with a small neighborhood are fully included in S, that is,

$$\operatorname{int}_h(S) \coloneqq \bigcup \Big\{ y + [0, h]^2 \,:\, y \in h\mathbb{Z}^2 \text{ and } y + [0, h]^2 + \frac{1}{2} B_h \subset S \Big\}. \tag{5.7.2}$$

Note that in each ball of radius $\big(\sqrt{2} + \frac{1}{2}\big)h$, the grid cell $y + [0, h]^2$ with $y \in h\mathbb{Z}^2$ which includes the center of the ball is itself included in the ball with $\frac{h}{2}$ to spare and thus an inner cell of the ball, see Figure 5.7.2. In particular, since S is $\big(\sqrt{2} + \frac{1}{2}\big)h$-fat, we obtain

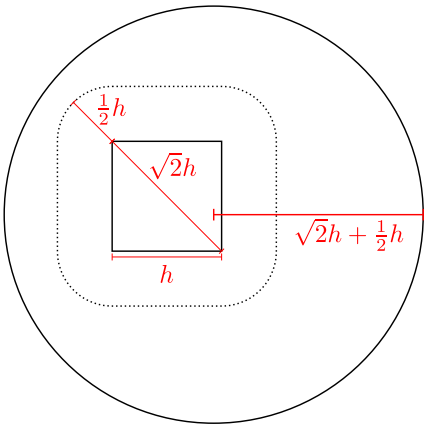


Figure 5.7.2: An illustration of the fact that a sufficiently large ball contains an 'inner cell' as defined in (5.7.2). The $\frac{1}{2}h$-neighborhood of a cell given by $x + [0, h]^2 + \frac{1}{2}B_h$ (depicted with dotted lines) has diameter $\sqrt{2}h + h$. Hence, any ball with radius at least $\sqrt{2}h + \frac{1}{2}h$ and center in the cell fully includes this neighborhood.

$$S \subset \operatorname{int}_h(S) + \left(\sqrt{2} + \frac{1}{2}\right)\overline{B}_h. \tag{5.7.3}$$

The grid cells making up $\operatorname{int}_h(S)$ can be split into two groups, see Figure 5.7.3: The boundary cells are those, where at least one of the eight surrounding cells is not in $\operatorname{int}_h(S)$. Accordingly, the interior cells are those with all surrounding cells also in $\operatorname{int}_h(S)$. Note

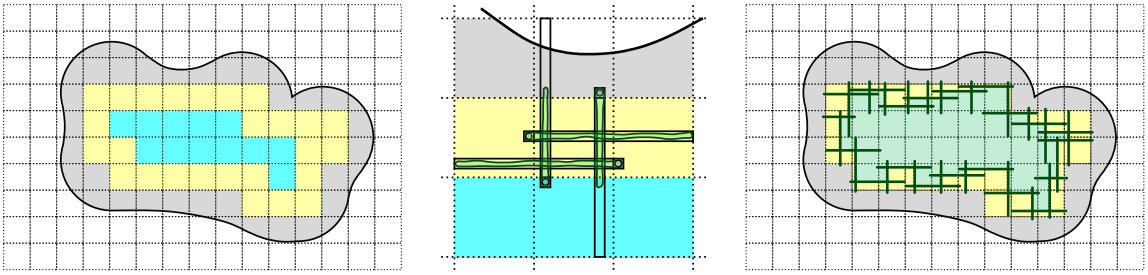


Figure 5.7.3: An illustration of the construction of the stable approximation in the proof of Lemma 5.7.2. Left: For a fat set S, the boundary cells of $\mathrm{int}_h(S)$ (depicted in yellow) approximate ∂S (depicted in black) from the inside, separating the interior cells of $\mathrm{int}_h(S)$ (depicted in blue) from the outside. Center: For each boundary cell and each direction, we take the stable 'seed' guaranteed by $E_h^{\boxplus,v}(\partial S + [-h,h]^2 + \frac{1}{2}B_h)$. Its evolution (depicted in green) with respect to the coefficient field restricted to the respective box remains stable and cuts through the cell and a potential neighboring boundary cell. If the respective neighbor is not a boundary cell, we instead consider the set evolution at the moment it enters that cell. Right: The stable approximation is the union of all these sets with the surrounded interior (depicted in green)

that this means that for $x \in h\mathbb{Z}^2$ we have

$$x + [0,h]^2 \text{ is an interior cell of } \mathrm{int}_h(S) \quad \Leftrightarrow \quad x + [0,h]^2 + [-h,h]^2 + \frac{1}{2}B_h \subset S. \tag{5.7.4}$$

This means that

$$S \subset \bigcup \{x + [0,h]^2 \,:\, x + [0,h]^2 \text{ is an interior cell}\} \cup \left(\partial S + 2[-h,h]^2 + \frac{1}{2}B_h\right)$$

and therefore

$$\bigcup \{x + [0,h]^2 \,:\, x + [0,h]^2 \text{ is an interior cell}\} \supset \mathbb{R}^2 \setminus \left(S^c + \left(2\sqrt{2} + \frac{1}{2}\right)B_h\right). \tag{5.7.5}$$

In fact, since our candidate S_h for the stable approximation will include the union of the interior cells, this will ensure $S_h \supset \mathbb{R}^2 \setminus (S^c + cv^{-1}v_{\max}B_h)$ for $c \geq \left(2\sqrt{2} + \frac{1}{2}\right){v_{\max}}^{-1}$.

To satisfy the other properties of a stable approximation, we now aim to find a stable interface surrounding the union of the interior cells inside S, which will evolve to envelope S. Contrasting (5.7.4), for each boundary cell $x + [0,h]^2 \subset \mathrm{int}_h(S)$ we have

$$\left(x + [0,h]^2\right) \cap \left(\partial S + [-h,h]^2 + \frac{1}{2}B_h\right) \neq \emptyset.$$

Thus, if $E_h^{\boxplus,v}(\partial S + [-h,h]^2 + \frac{1}{2}B_h)$ holds, then $E_{h,x}^{\square,v}$ holds for the boundary cell $x + [0,h]^2$. This means that for each direction there is a box crossing the cell and its neighbor along which the interface propagates. To be more precise, there is a stable set ('seed') which within the respective translation and rotation of $Q_{2h}^{\uparrow} + B_1$ would evolve to cross $x + [0,h]^2$ and then the corresponding neighboring cell. By Lemma 5.5.6 and 5.5.8, these sets remain stable with respect to (A,F) throughout this evolution. For each of these sets, we stop the evolution when it leaves $\bigcup\{y + [0,h]^2 \,:\, y + [0,h]^2 \text{ is a boundary cell}\} + B_{w(2h)+1}$, using that by Corollary 5.5.7 the evolution is continuous. By a straightforward adaption of Lemma 5.5.11 and in the sense of Definition 5.5.10, the resulting sets cut through $x + [0,h]^2$ in the respective direction, as well as through the next neighboring cell if that one is also a boundary cell, see Figure 5.7.3. We take the union of all these evolved sets, denoted with $\widetilde{S}_h$. By Lemma 5.5.5, $\widetilde{S}_h$ is again stable as a locally finite union of stable sets. Furthermore, using arguments as in the proofs of Lemma 5.5.13 and Lemma 5.5.14, we obtain that $\widetilde{S}_h$ separates the interior cells from the complement of $\mathrm{int}_h(S)$ in the sense that there exists no path in $\mathbb{R}^2 \setminus \widetilde{S}_h$ connecting the two.

We now define S_h as the union of $\widetilde{S}_h$ with all the path-connected components of $\left(\widetilde{S}_h\right)^c$ that intersect interior cells. Note that S_h is also stable since $\partial S_h \subset \partial \widetilde{S}_h$ with $S_h \supset \widetilde{S}_h$ and hence $\mathcal{J}^{2,-}(1-\chi_{S_h})(\cdot) \subset \mathcal{J}^{2,-}\left(1-\chi_{\widetilde{S}_h}\right)(\cdot)$ for the subjets in $\mathbb{R}^2$, see Definition 4.A.1. Moreover, due to (5.7.5), we have $S_h \supset \mathbb{R}^2 \setminus \left(S^c + \left(2\sqrt{2}+\frac{1}{2}\right)B_h\right)$.

It remains to show that $S \subset_h \mathscr{R}_{4\lceil c+1\rceil v^{-1}h}(S_h)$ for suitable $c > 0$. We obtain this result by consecutively applying the stitching lemma in the form of Lemma 5.5.13 to S_h with time steps of size $2hv^{-1}$. We know that for each of these time steps, S_h evolving with respect to (A, F) will propagate along the boxes associated to the surrounding cells which have been cut in the sense of Definition 5.5.10 in the previous step, see Figure 5.7.4 and 5.7.5.

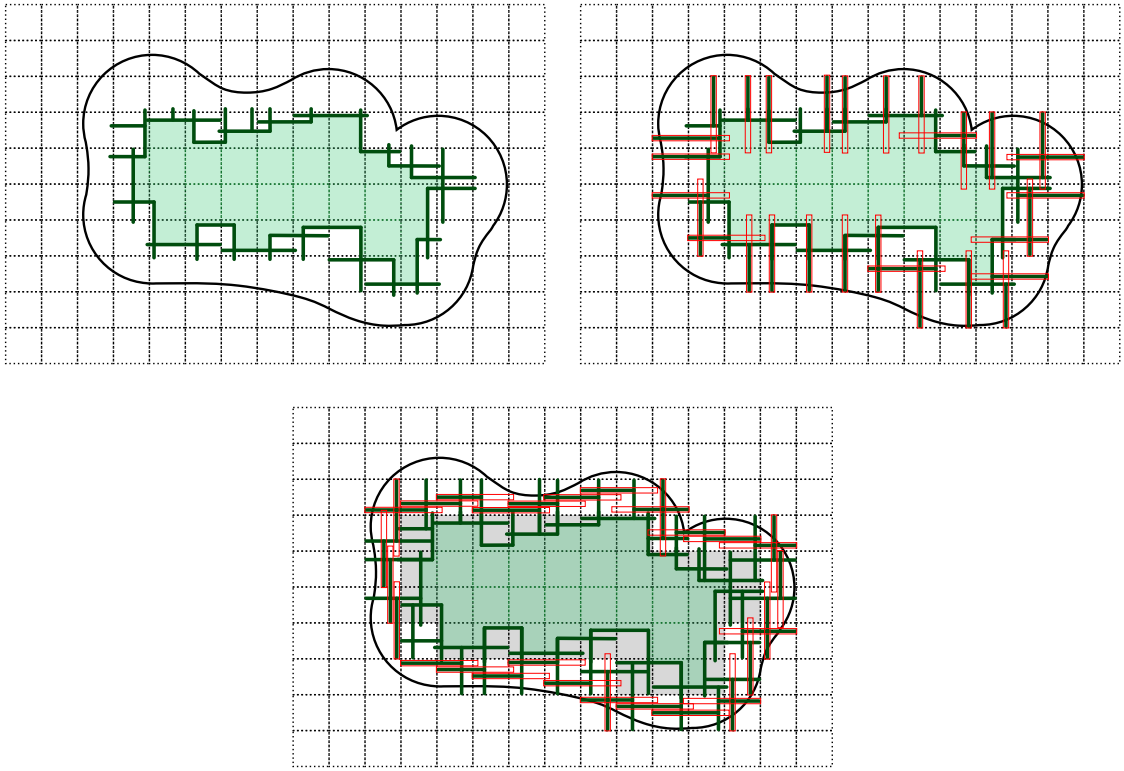


Figure 5.7.4: An illustration of the first steps of the guaranteed expansion of the stable approximation in the proof of Lemma 5.7.2. Top left: The constructed stable approximation S_h (depicted in green) of the set S (depicted with black boundary). Top right: After time steps $\Delta t = 2v^{-1}h$, due to the stitching lemma, the stable approximation S_h expands at least according to the evolution of the stable seeds along previously crossed boxes associated to the surrounding cells (depicted in red). Bottom: After two such steps, all inner cells (depicted in grey) are h-enveloped. After every two additional cells, all horizontally, diagonally and vertically neighboring cells are h-enveloped. In particular, the original set S is h-enveloped after six steps

More precisely, let $K_{h,0} = \operatorname{int}_h(S)$ be the union of all interior and boundary cells. For $i \in \mathbb{N}$, we define $K_{h,i+1}$ by adding to $K_{h,i}$ all cells $x + [0,h]^2$, which are neighboring $K_{h,i}$ horizontally, vertically, or diagonally, and for whose eight surrounding cells and itself $E_h^{\square,v}$ holds. Using Lemma 5.5.13, $\mathscr{R}_{2v^{-1}h}(S_h)$ will horizontally (or vertically) cut through all cells that horizontally (or vertically) border $K_{h,0}$. Next, due to another application of Lemma 5.5.13, $\mathscr{R}_{4v^{-1}h}(S_h)$ will orthogonally cut across those cells and their orthogonal neighbors, which includes all cells diagonally bordering $K_{h,0}$. In summary,

$$\mathscr{R}_{4v^{-1}h}(S_h) \text{ cuts horizontally and vertically} \\ \text{through any two neighboring cells in } K_{h,0} + [-h,h]^2. \quad (5.7.6)$$

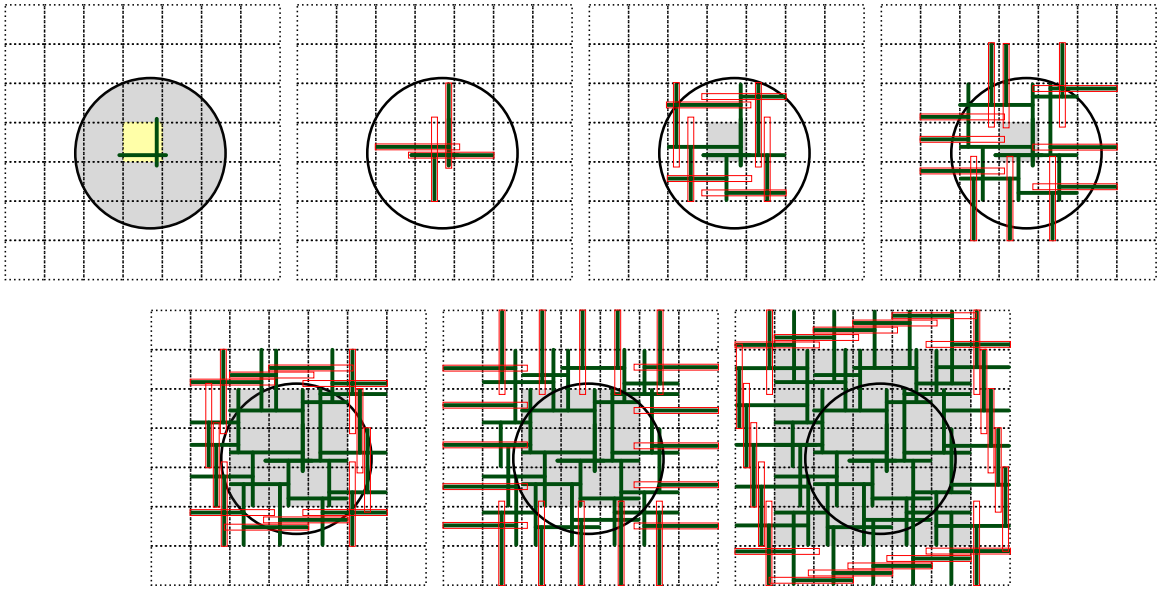


Figure 5.7.5: An illustration of the guaranteed expansion of the stable approximation in the proof of Lemma 5.7.2 for a single inner cell. Top left: The stable approximation S_h (depicted in green) constructed for a $(\sqrt{2}+\frac{1}{2})h$-fat set S (depicted in grey) with $\operatorname{int}_h(S)$ consisting of a single inner cell (depicted in yellow). Center: After every time step $\Delta t = 2v^{-1}h$, due to the stitching lemma, the stable approximation S_h expands at least according to the evolution of the stable seeds along boxes associated to the surrounding cells which have been crossed in the previous step (depicted in red). After two steps, $\operatorname{int}_h(S)$ is h-enveloped. Every two further steps, the set of cells which are guaranteed to be h-enveloped (depicted in grey) expands by all surrounding cells. Bottom right: After six steps, the evolution of S_h is guaranteed to h-envelop S (depicted with black boundary)

We claim that this implies $K_{h,0} \subset_h \mathscr{R}_{4v^{-1}h}(S_h)$. Consider a cell in $K_{h,0}$ and its eight surrounding cells. The nine cells are included in $\mathscr{R}_{4v^{-1}h}(S_h) + \overline{B}_h$ since otherwise there would be a straight segment in $\mathbb{R}^2 \setminus \mathscr{R}_{4v^{-1}h}(S_h)$ crossing one of the boxes, contradicting (5.7.6). Hence, for a point in the center cell to not be h-enveloped, it would have to be connected to the exterior of the nine cells in $\mathbb{R}^2 \setminus \mathscr{R}_{4v^{-1}h}(S_h)$. Assuming that this was the case, since $\mathbb{R}^2 \setminus \mathscr{R}_{4v^{-1}h}(S_h)$ is open, there exists a path in $\mathbb{R}^2 \setminus \mathscr{R}_{4v^{-1}h}(S_h)$ connecting the center cell to the boundary of the nine cells. Without loss of generality, assume that the path ends in the bottom side of the nine cells, meaning that there exists a path segment connecting the top and bottom side of the bottom row of cells, see Figure 5.7.6. Now there are two cases: First, if all three of the bottom cells are visited, there has to be a segment connecting the left and right side within the center cell, contradicting (5.7.6). Second, the path segment connects the top and bottom side of two adjacent cells, contradicting (5.7.6). Thus, the center cell has to be h-enveloped, which yields $K_{h,0} \subset_h \mathscr{R}_{4v^{-1}h}(S_h)$. Repeating this argument, we obtain for $i \in \mathbb{N}$ that $K_{h,i-1} \subset_h \mathscr{R}_{4iv^{-1}h}(S_h)$.

To finish the proof, we show that $S \subset K_{h,2}$ and hence $S \subset_h \mathscr{R}_{12v^{-1}h}(S_h)$. Recall that in (5.7.3) we had $S \subset \operatorname{int}_h(S) + \big(\sqrt{2}+\frac{1}{2}\big)\overline{B}_h$ and thus in particular

$$y \in \partial S + \left(\sqrt{2}+\frac{1}{2}\right)\overline{B}_h \qquad \text{for all } y \in S \setminus \operatorname{int}_h(S).$$

Since $E_h^{\boxplus,v}\big(\partial S + \big(\sqrt{2}+\frac{1}{2}\big)\overline{B}_h\big)$ holds, $E_h^{\square,v}$ holds for all of the grid cells that intersect $S \setminus \operatorname{int}_h(S) = S \setminus K_{h,0}$ as well as their respective eight surrounding cells. Since there are at most two cells necessary to connect $y \in S \setminus \operatorname{int}_h(S)$ to $K_{h,0} = \operatorname{int}_h(S)$, we obtain $S \subset K_{h,2}$. □

Continuing the argument from the proof above, we would obtain that the evolution of the constructed stable approximation S_h essentially stays ahead of the original set expanding

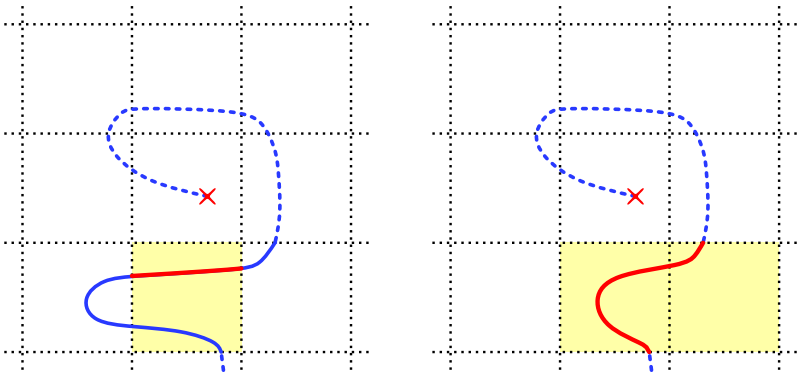


Figure 5.7.6: A path connecting a grid cell to the exterior across the eight surrounding cells at some point has to cut either across a single cell or across two neighboring cells. Left: If the last segment of the path crosses all three cells on the side of its exit, a sub-segment has to cut the center cell (depicted in yellow). Right: Otherwise, the last segment cuts through the union of two neighboring cells (depicted in yellow)

at some fixed speed. However, in order to obtain the effective minimum speed required for (∗iii) in (P⊙), $E_h^{\boxplus,v}(M)$ has to guarantee not just that S_h expands accordingly in M, but also the set evolution $\mathscr{R}_t(S_h)$ at any time $t \geq 0$.

Lemma 5.7.3 (An effective minimum speed of propagation guaranteed by $E_h^{\boxplus,v}(M)$)**.** *Let $M \subset \mathbb{R}^2$, $h \geq \frac{r_0}{2}$, $v > 0$. Let $(A,F) \in E_h^{\boxplus,v}(M + 2[-h,h]^2)$ and $S \subset \mathbb{R}^2$ be a stable set with respect to (A,F). If $x \in \mathscr{R}_t^{(A,F)}(S) \cap M$ for some $t > 0$ with $\operatorname{dist} x, S \geq 3h$, then $x + \overline{B}_h \subset_h \mathscr{R}_{t+14v^{-1}h}^{(A,F)}(S)$.*

Proof. Let $x \in \mathscr{R}_t^{(A,F)}(S) \cap M$ be in the grid cell $y + [0,h]^2$ with $y \in h\mathbb{Z}^2$. First, since $\operatorname{dist} x, S \geq 3h$, the initial set has no intersection with the 3×3 group of cells $y + [-h, 2h]^2$ surrounding $y + [0,h]^2$. We will show that this implies that $\mathscr{R}_t^{(A,F)}(S)$ cuts through the eight cells surrounding $y + [0,h]^2$ in the sense of Definition 5.5.10. Second, since $(A,F) \in E_h^{\boxplus,v}(M + 2[-h,h]^2)$ holds, the event $E_h^{\square,v}$ holds for all cells in the surrounding 5×5 group $y + [-2h, 3h]^2$. Starting with the cut from the first step, we will obtain the h-envelopment due to the guaranteed propagation along the boxes associated to $E_h^{\boxplus,v}(M + 2[-h,h]^2)$ in those cells.

Regarding the first step, we claim that $\mathscr{R}_t^{(A,F)}(S)$ cuts vertically through the top or bottom row of 1×3 cells in $y + [-h, 2h]^2$ or horizontally through the left or right column of 3×1 cells. If this was not the case, then by definition there exist paths in $\mathbb{R}^2 \setminus \mathscr{R}_t^{(A,F)}(S)$ connecting the opposing narrow sides in all four triples. Combining these paths yields a loop which does not intersect $\mathscr{R}_t^{(A,F)}(S)$ and hence neither $\mathscr{R}_s^{(A,F)}(S)$ for any $s \in [0,t]$ due to the stability of S and Lemma 5.5.6. The interior of the loop contains the center cell and is itself included in the surrounding 3×3 group. Therefore, the set evolution of S does not intersect the parabolic boundary of the loop's interior until time t. By the comparison principle – as in the proof of Lemma 5.5.11 – this implies that $\mathscr{R}_t^{(A,F)}(S) \cap (y + [0,h]^2) = \emptyset$, contradicting the existence of $x \in \mathscr{R}_t^{(A,F)}(S) \cap (y + [0,h]^2)$.

Hence, without loss of generality, let $\mathscr{R}_t^{(A,F)}(S)$ cut vertically through the bottom row of 1×3 cells. Now there are two cases, similar to Figure 5.7.6: In the first case, $\mathscr{R}_t^{(A,F)}(S)$ cuts vertically through two adjacent cells, say without loss of generality through $x + [0,2h] \times [-h, 0]$. In the second case, $\mathscr{R}_t^{(A,F)}(S)$ has to horizontally cut through the bottom center cell $x + [0,h] \times [-h, 0]$. If this was not the case, then there would exist paths in $(x + [0,2h] \times [-h,0]) \setminus \mathscr{R}_t^{(A,F)}(S)$ connecting left and right side of the two bottom left cells,

connecting the top and bottom of the bottom center cell, and connecting the left and right side of the two bottom right cells. Combining these three paths we could obtain a path connecting the left and right side of the full bottom row, contradicting the assumed vertical cut.

Regarding the second step, for both of these cases, we now consider the expansion of $\mathscr{R}_t^{(A,F)}(S)$ along the boxes associated to $E_h^{\boxplus,v}(M + 2[-h,h]^2)$. As in the proof of Lemma 5.7.2, we consecutively apply the stitching lemma in the form of Lemma 5.5.13 using the respective translations and rotations of $Q_{2h}^{\uparrow}$ associated to $E_h^{\boxplus,v}(M + 2[-h,h]^2)$, which as observed above includes boxes for all cells in the surrounding 5×5 grid $y + [-2h, 3h]^2$.

In the first case, we obtain $y + [-h, 2h]^2 \subset_h \mathscr{R}_{7 \cdot 2hv^{-1}}(S)$, see Figure 5.7.7. In the second case, we obtain $y + [-h, 2h]^2 \subset_h \mathscr{R}_{6 \cdot 2hv^{-1}}(S)$, see Figure 5.7.8. In summary, using the stability of S, we obtain $x + \overline{B}_h \subset_h \mathscr{R}_{14v^{-1}h}(S)$. □

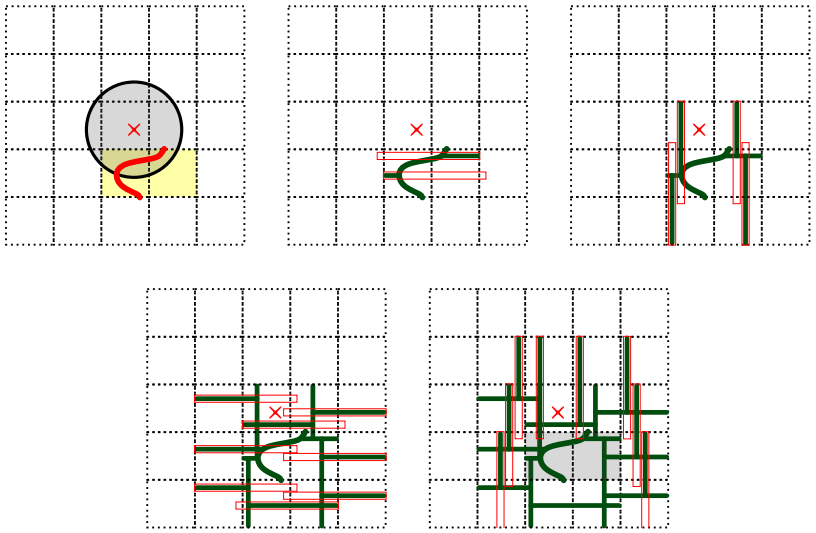


Figure 5.7.7: An illustration of the guaranteed further expansion around a point x reached by the set evolution for Case 1 in the proof of Lemma 5.7.3. Top left: The stable set $\mathscr{R}_t(S)$ cuts vertically through the union of two neighboring cells below x, providing a base for the stitching lemma (depicted in red). Center: Consecutive applications of the stitching lemma to the stable seeds in the boxes associated to the surrounding cells (depicted in red) guarantee the further expansion of the set evolution within time steps $\Delta t = 2v^{-1}h$. Bottom right: After four time steps, the cells with the path segment (depicted in grey) are guaranteed to be h-enveloped. With every further time step, this set of guaranteed h-enveloped cells alternately expands by all vertically neighboring cells and all horizontally neighboring cells. Hence, the ball $\overline{B}_h(y)$ (depicted in black) is guaranteed to be h-enveloped after seven such time steps

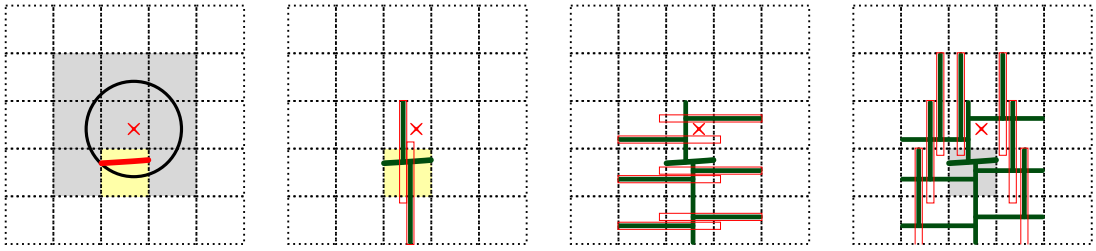


Figure 5.7.8: An illustration of the guaranteed further expansion around a point x reached by the set evolution for Case 2 in the proof of Lemma 5.7.3. Left: The stable set $\mathscr{R}_t(S)$ cuts horizontally through the cell below x, providing a base for the stitching lemma (depicted in red). Center: Consecutive applications of the stitching lemma to the stable seeds from $E_h^{\boxplus,v}(M + 2[-h,h]^2)$ in the boxes associated to the surrounding cells (depicted in red) guarantee the further expansion of the set evolution within time steps $\Delta t = 2v^{-1}h$. Right: After three time steps, the cell with the path segment (depicted in grey) is guaranteed to be h-enveloped. With every further time step, this set of guaranteed h-enveloped cells alternately expands by all vertically neighboring cells and all horizontally neighboring cells. Hence, the ball $\overline{B}_h(y)$ (depicted in black) is guaranteed to be h-enveloped after six such time steps

We obtain probability bounds for $E_h^{\boxplus,v_\infty}(M)$ corresponding to (5.3.18) for (P⊙) based on the results from Section 5.6, in particular Corollary 5.6.2.

Corollary 5.7.4. *Assume that Assumption (P1), Assumption (P2), and the box criterion (P□) hold. Let $h \geq \frac{1}{2}r_0$. For any $\alpha_\uparrow \in \left(0, \frac{q-1}{q^3}\right)$ and $v_\infty = v_\infty(q, c_w, r_0, v_0, \alpha_\uparrow)$ as in Corollary 5.6.2, there exists $C_\square = C_\square(q, c_w, p_0, r_0, \alpha_\uparrow) > 0$ such that for any $M \subset \mathbb{R}^2$*

$$\mathbb{P}\left[E_h^{\boxplus,v_\infty}(M)\right] \geq 1 - 16\left(1 + \frac{\operatorname{diam}(M)}{h}\right)^2 \exp\left(-C_\square h^{\alpha_\uparrow + \frac{q-1}{q}}\right).$$

Proof. Note that

$$\#\left\{x \in h\mathbb{Z}^2 \,:\, \left(x + [0,h]^2\right) \cap M \neq \emptyset\right\} \leq \left(2 + \frac{\operatorname{diam}(M)}{h}\right)^2$$

and therefore by Assumption (P1) (Stationarity), it holds that

$$\mathbb{P}\left[E_h^{\boxplus,v_\infty}(M)\right] \geq 1 - 4\left(1 + \frac{\operatorname{diam}(M)}{h}\right)^2 \left(1 - \mathbb{P}\left[E_{h,0}^{\square,v_\infty}\right]\right). \tag{5.7.7}$$

With respect to bounding the probability of $E_{h,0}^{\square,v}$, for each direction we use only finitely many box positions rather than along the whole interval. However, if the distance between

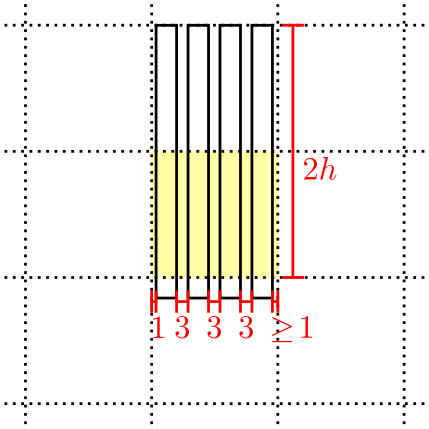


Figure 5.7.9: An illustration of the box positions considered to bound the probability of the event $E_h^{\boxplus,v}(M)$. The distance between the boxes guarantees that the events of propagation along the respective boxes are independent.

them is sufficiently large we can apply the finite range of dependence from Assumption (P2). That is, for boxes as in Figure 5.7.9 we consider the union of the associated events,

$$\bigcup_{y \in [1, h - w(2h) - 1] \times \{0\}} E_{2h,y}^{\uparrow,v} \supset \bigcup_{y \in \{1, 1 + w(2h) + 3, \ldots, 1 + (j-1)(w(2h)+3)\} \times \{0\}} E_{2h,y}^{\uparrow,v}$$

with

$$j := \left\lfloor \frac{h+1}{w(2h)+3} \right\rfloor \geq \frac{h}{4w(2h)} = \frac{c_w^{\frac{1}{q}}}{2^{\frac{1}{q}}4} h^{1-\frac{1}{q}}$$

where we used that $h+1 \geq h \geq 4w(2h) \geq 2(w(2h)+3)$ since $2h \geq r_0$ and $w(r_0) \geq 3$ by (5.3.12) from Assumption (P□). Hence, with the measurability from Lemma 5.5.16, the finite range of dependence from Assumption (P2) and the stationarity from Assumption (P1), using

the bounds from Corollary 5.6.2 we obtain

$$\begin{aligned}\mathbb{P}\Big[E_{h,0}^{\square,v_\infty}\Big] &\geq 1-\big(1-\mathbb{P}\big[E_{2h}^{\uparrow,v_\infty}\big]\big)^j-\big(1-\mathbb{P}\big[E_{2h}^{\rightarrow,v_\infty}\big]\big)^j\\ &\qquad -\big(1-\mathbb{P}\big[E_{2h}^{\downarrow,v_\infty}\big]\big)^j-\big(1-\mathbb{P}\big[E_{2h}^{\leftarrow,v_\infty}\big]\big)^j\\ &\geq 1-4\exp\left(-C_\uparrow 2^{\alpha_\uparrow}c_w^{\frac{1}{q}}\frac{1}{2^{\frac{1}{q}}4}2^{\alpha_\uparrow}h^{\alpha_\uparrow+\frac{q-1}{q}}\right).\end{aligned}$$

Plugging this bound into (5.7.7) yields the result. □

We now obtain our main result by combining the previous considerations of the section.

Theorem 5.7.5 (Explicit version of Theorem 5.4.1)**.** *Assume that Assumption (P1), Assumption (P2), and the box criterion (P□) hold. Let* $\alpha_\uparrow \in \left(0, \frac{q-1}{q^3}\right)$*. Then (P⊙) holds with* $h_0 := r_0$ *and*

$$\begin{aligned}&\alpha_\odot := \frac{q-1}{q}+\alpha_\uparrow, && c_\odot := \frac{C_\square(q,c_w,r_0,p_0,\alpha_\uparrow)}{2^{\frac{q-1}{q}+\alpha_\uparrow}}, && C_\odot := 2500,\\ &c_{\mathcal{F}} := \frac{7}{\sqrt{2}}, && v_{\min} := \frac{1}{16}v_\infty\big(v_0,q,c_w,r_0,p_0,\alpha_\uparrow\big), && c_s := 6v_\infty^{-1},\end{aligned}$$

where $C_\square(q,c_w,p_0,r_0,\alpha_\uparrow)$ *is as in Corollary 5.7.4 and* $v_\infty(v_0,q,c_w,r_0,p_0,\alpha_\uparrow)$ *is as in Corollary 5.6.2.*

Proof. For the family of events $\big(E_{\tilde h}^\odot(M)\big)_{\tilde h,M}$ in (P⊙) which will guarantee the effective minimum speed on the respective scale $\tilde h \geq h_0$ in $M \subset \mathbb{R}^2$, we choose a slightly finer scale $h := \frac{\tilde h}{2}$ and set

$$E_{\tilde h}^\odot(M) = E_h^{\boxplus,v_\infty}\big(M+4[-h,h]^2\big) \qquad \text{for } \tilde h \geq h_0 := r_0.$$

Let $\alpha_\uparrow \in \left(0, \frac{q-1}{q^3}\right)$. Then the probability bound (5.3.18) is satisfied for all $\tilde h \geq h_0$ due to Corollary 5.7.4 with

$$C_\odot \geq 16(8\sqrt{2}+1)^2, \qquad c_\odot := \frac{C_\square(q,c_w,r_0,p_0,\alpha_\uparrow)}{2^{\frac{q-1}{q}+\alpha_\uparrow}}, \qquad \alpha_\odot := \frac{q-1}{q}+\alpha_\uparrow.$$

Due to Lemma 5.7.1, using that $\mathcal{F}(M_1) \subset \mathcal{F}(M_2)$ for $M_1 \subset M_2$ by the definition of $\mathcal{F}(\cdot)$ in (5.3.6), we obtain

$$E_h^\odot(M) \in \mathcal{F}\big(M+7[-h,h]^2\big) \subset \mathcal{F}\big(M+7\sqrt{2}B_h\big) = \mathcal{F}\left(M+\frac{7\sqrt{2}}{2}B_{\tilde h}\right)$$

and hence the measurability in (∗i) is satisfied with $c_{\mathcal{F}} := \frac{7}{\sqrt{2}}$. In addition, the monotonicity of the events in (∗ii) is satisfied since $E_h^{\boxplus,v_\infty}(M_1) \supset E_h^{\boxplus,v_\infty}(M_2)$ for any sets $M_1 \subset M_2$ by the definition in (5.7.1).

Regarding (∗iii), let $S \subset \mathbb{R}^2$ be $\tilde h$-fat and thus $\big(\sqrt{2}+\frac{1}{2}\big)h$-fat with $\partial S \subset M$ and let $(A,F) \in E_{\tilde h}^\odot(M)$. Then, due to Lemma 5.7.2 we know that S is $(h, 12v_\infty^{-1})$-approximately stable and thus $(\tilde h, 6v_\infty^{-1})$-approximately stable. Let S_h be the stable approximation of S as constructed in the proof of Lemma 5.7.2.

It remains to show that (A, F) admits an effective minimum speed on the scale $\tilde{h}$ for S_h in M. Let $x \in \mathscr{R}_t(S_h) \cap M$ for some $t \geq 0$. Now there are two cases, considering the 3×3 block of cells from $h\mathbb{Z}^2$ with x in the central cell.

For the first case, assume that the intersection of this block with S_h is empty. From this point on, following the arguments in the proof of Lemma 5.7.3 we obtain

$$x + \overline{B}_h \subset_h \mathscr{R}_{t+14v_\infty^{-1}h}(S_h).$$

For the second case, assume that the intersection of the block with S_h is non-empty. Due to the construction of S_h, this means that either one of these nine cells or one of the neighboring cells is in $\operatorname{int}_h(S)$ as defined in the proof of Lemma 5.7.2. With $(K_{h,i})_{i\in\mathbb{N}_0}$ denoting the same family of sets as in the proof of Lemma 5.7.2, we hence obtain $x \in K_{h,2}$ and

$$x + \overline{B}_h \subset K_{h,3} \subset_h \mathscr{R}_{4\cdot 4v_\infty^{-1}h}(S_h) \subset \mathscr{R}_{t+16v_\infty^{-1}h}(S_h).$$

Repeating the argument once more, we obtain

$$x + \overline{B}_{\tilde{h}} = x + \overline{B}_h + \overline{B}_h \subset_h \mathscr{R}_{t+16v_\infty^{-1}h+16v_\infty^{-1}h}(S_h) = \mathscr{R}_{t+16v_\infty^{-1}\tilde{h}}(S_h).$$

In summary, (∗iii) is satisfied with

$$c_s := 6v_\infty^{-1}, \qquad v_{\min} := \frac{1}{16}v_\infty.$$

□

5.8 Sparse Poisson obstacles admit an effective minimum speed

In this section, we will prove Theorem 5.1.3, that is, we will show that the box criterion (P□) is satisfied if the intensity ρ of the Poisson point process $\{X_i\}_i$ is small enough compared to the uniform forcing F_{uni}, irrespective of the obstacle shape $\varphi \in C^{0,1}(\mathbb{R})$ with $\operatorname{supp}\varphi \subset \overline{B}_1$.

It is straightforward to see that, for all directions $e \in \mathbb{S}^1$, the characteristic function $1 - \chi_{S_t}$ of the set evolution

$$t \mapsto S_t := \bigcup_{s\in[0,t]} \overline{B}_{\frac{2}{F_{uni}}}\left(\frac{F_{uni}}{2}se\right)$$

is a supersolution to the level-set formulation corresponding to the interface motion

$$V = H + F_{uni}.$$

The curvature H is always at least $-\frac{F_{uni}}{2}$ and the normal velocity V is at most $\frac{F_{uni}}{2}$. In particular, S_0 is a stable set.

This means that if there is no obstacle in the 1-neighborhood of a box with width at least $\frac{4}{F_{uni}}$, then a translation of S_0 is a stable set in the starting area which propagates to the opposite end of the box with average speed at least $v_0 := \frac{F_{uni}}{2}$. That is, for all $r > 0$ with $w(r) \geq \frac{4}{F_{uni}}$, using the definition of a Poisson point process we obtain

$$\mathbb{P}\big[E_r^{\uparrow,v_0}\big] \geq \mathbb{P}\big[\#\big\{i \in \mathbb{N} : X_i \in Q_r^\uparrow + \overline{B}_1\big\} = 0\big] = \exp\big(-\rho\big|Q_r^\uparrow + \overline{B}_1\big|\big) \tag{5.8.1}$$

with $\#$ denoting the number of elements in a set and $|\cdot|$ the area of a set. Analogous bounds hold for $\mathbb{P}[E_r^{\rightarrow,v_0}]$, $\mathbb{P}\big[E_r^{\downarrow,v_0}\big]$, and $\mathbb{P}[E_r^{\leftarrow,v_0}]$.

For a given uniform forcing $F_{uni} > 0$ and arbitrary width-scaling $q > 1$, we choose the width function $w(r) = \left(\frac{r}{c_w}\right)^{\frac{1}{q}}$ such that $w(r_0) = \frac{4}{F_{uni}}$ and such that the ratio of the sides of $Q_{r_0}^{\uparrow}$ is the same for all F_{uni}, that is,

$$c_w := 24q\left(\frac{F_{uni}}{4}\right)^{q-1}, \qquad r_0 := \frac{96q}{F_{uni}}. \tag{5.8.2}$$

The remaining degree of freedom has been used to further guarantee that

$$(24q)^{\frac{q}{q-1}} c_w^{-\frac{1}{q-1}} = (24q)^{\frac{q-1}{q-1}} \frac{4}{F_{uni}} = r_0$$

and hence (5.3.12) from (P□) is automatically satisfied if in addition $w(r_0) \geq 3$, which corresponds to $F_{uni} \leq \frac{4}{3}$.

Due to the scaling invariance of the shape of $Q_{r_0}^{\uparrow}$ with respect to F_{uni}, the required probability bounds (5.3.13) in (P□) for the first range of boxes are independent of F_{uni}, that is,

$$k_0 = \left\lfloor \frac{1}{12q} c_w^{\frac{1}{q}} r_0^{\frac{q-1}{q}} \right\rfloor = 2.$$

It remains to show the probability bound (5.3.11) for some $p_0 < \min\left\{\frac{2^{-q}}{6}, e^{-q}\right\}$. Note that, for the sides of the the largest box $Q_{r_1}^{\uparrow}$ in the relevant range, we have

$$w(r_1) + 2 = r_0 + 2w(r_0) + 2 \leq 2r_0 = \frac{192q}{F_{uni}},$$
$$r_1 + 2 \leq w^{-1}(2r_0) = \frac{4(24q)^{q+1}2^q}{F_{uni}}.$$

For $r \in [r_0, r_1)$, the bound (5.8.1) hence yields

$$\mathbb{P}\big[E_r^{\uparrow,v_0}\big] \geq \exp\big(-\rho\big|Q_{r_1}^{\uparrow} + \overline{B}_1\big|\big) \geq \exp\big(-\rho\big(w(r_1)+2\big)\big(r_1+2\big)\big)$$
$$\geq \exp\left(-2^q 24^{q+1} q^{q+2} 768 \frac{\rho}{F_{uni}^2}\right),$$

where ρ is the intensity of the Poisson point process. With analogous bounds for $\mathbb{P}[E_r^{\rightarrow,v_0}]$, $\mathbb{P}\big[E_r^{\downarrow,v_0}\big]$, and $\mathbb{P}[E_r^{\leftarrow,v_0}]$, we thus obtain the box criterion if

$$\rho < 768^{-1} 2^{-q} 24^{-q-1} q^{-q-2} \min\left\{\frac{2^{-q}}{6}, e^{-q}\right\} F_{uni}^2 =: c_0(q) F_{uni}^2$$

and $F_{uni} \leq \frac{4}{3}$. By the comparison principle, the box criterion is of course also satisfied for any stronger uniform forcings. We have therefore established Theorem 5.1.3 with the constants $C_0 = 2c_0^{-\frac{1}{2}}(q)$ and $\rho_0 = \frac{16}{9} c_0(q)$ for any $q > 1$.

This scaling relationship independent of the obstacle shape is obviously not preserved for arbitrarily large intensities – it breaks at the very least when the critical threshold is reached where the support of the obstacles forms an infinite percolation cluster.

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References

[AC14] Scott N. Armstrong and Pierre Cardaliaguet. “Quantitative Stochastic Homogenization of Viscous Hamilton-Jacobi Equations.” *Communications in Partial Differential Equations* 40.3 (2014), pp. 540–600. DOI: 10.1080/03605302.2014.971372.

[AC18] Scott Armstrong and Pierre Cardaliaguet. “Stochastic homogenization of quasilinear Hamilton–Jacobi equations and geometric motions.” *Journal of the European Mathematical Society* 20.4 (2018), pp. 797–864. DOI: 10.4171/jems/777.

[ACS14] Scott N. Armstrong, Pierre Cardaliaguet, and Panagiotis E. Souganidis. “Error estimates and convergence rates for the stochastic homogenization of Hamilton-Jacobi equations.” *Journal of the American Mathematical Society* 27.2 (2014), pp. 479–540. DOI: 10.1090/s0894-0347-2014-00783-9.

[AD15] François Alouges and Giovanni Di Fratta. “Homogenization of composite ferromagnetic materials.” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 471.2182 (2015), p. 20150365. DOI: 10.1098/rspa.2015.0365.

[ADH17] Antonio Auffinger, Michael Damron, and Jack Hanson. *50 Years of First-Passage Percolation*. American Mathematical Society, 2017. DOI: 10.1090/ulect/068.

[AFM06] Emilio Acerbi, Irene Fonseca, and Giuseppe Mingione. “Existence and regularity for mixtures of micromagnetic materials.” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 462.2072 (2006), pp. 2225–2243. DOI: 10.1098/rspa.2006.1655.

[AK22] Scott Armstrong and Tuomo Kuusi. *Elliptic homogenization from qualitative to quantitative*. 2022. DOI: 10.48550/ARXIV.2210.06488.

[AK25] Scott Armstrong and Tuomo Kuusi. “Renormalization group and elliptic homogenization in high contrast.” *Inventiones mathematicae* 242.3 (2025), pp. 895–1086. DOI: 10.1007/s00222-025-01370-9.

[AKM16] Scott Armstrong, Tuomo Kuusi, and Jean-Christophe Mourrat. “The additive structure of elliptic homogenization.” *Inventiones mathematicae* 208.3 (2016), pp. 999–1154. DOI: 10.1007/s00222-016-0702-4.

[Ale97] Kenneth S. Alexander. “Approximation of subadditive functions and convergence rates in limiting-shape results.” *The Annals of Probability* 25.1 (1997). DOI: 10.1214/aop/1024404277.

[All92] Grégoire Allaire. “Homogenization and Two-Scale Convergence.” *SIAM Journal on Mathematical Analysis* 23.6 (1992), pp. 1482–1518. DOI: 10.1137/0523084.

[Alo+21] François Alouges, Anne de Bouard, Benoît Merlet, and Léa Nicolas. “Stochastic homogenization of the Landau–Lifshitz–Gilbert equation.” *Stochastics and Partial Differential Equations: Analysis and Computations* 9.4 (2021), pp. 789–818. DOI: 10.1007/s40072-020-00185-4.

[AM15] Scott N. Armstrong and Jean-Christophe Mourrat. “Lipschitz Regularity for Elliptic Equations with Random Coefficients.” *Archive for Rational Mechanics and Analysis* 219.1 (2015), pp. 255–348. DOI: 10.1007/s00205-015-0908-4.

[AP96] Peter Antal and Agoston Pisztora. “On the chemical distance for supercritical Bernoulli percolation.” *The Annals of Probability* 24.2 (1996). DOI: 10.1214/aop/1039639377.

[AR10] Sebastian Andres and Max-K. von Renesse. “Particle approximation of the Wasserstein diffusion.” *J. Funct. Anal.* 258.11 (2010), pp. 3879–3905. DOI: 10.1016/j.jfa.2009.10.029.

[AS12a] Scott N. Armstrong and Panagiotis E. Souganidis. “Stochastic homogenization of Hamilton–Jacobi and degenerate Bellman equations in unbounded environments.” *Journal de Mathématiques Pures et Appliquées* 97.5 (2012), pp. 460–504. DOI: 10.1016/j.matpur.2011.09.009.

[AS12b] Scott N. Armstrong and Panagiotis E. Souganidis. “Stochastic Homogenization of Level-Set Convex Hamilton–Jacobi Equations.” *International Mathematics Research Notices* 2013.15 (2012), pp. 3420–3449. DOI: 10.1093/imrn/rns155.

[AS16] N. Scott Armstrong and Charles K. Smart. “Quantitative stochastic homogenization of convex integral functionals.” *Annales scientifiques de l’École normale supérieure* 49.2 (2016), pp. 423–481. DOI: 10.24033/asens.2287.

[AW98] Kevin T. Andrews and Steve Wright. “Stochastic homogenization of elliptic boundary-value problems with Lp-data.” *Asymptotic Analysis* 17.3 (1998), pp. 165–184. DOI: 10.3233/asy-1998-300.

[Azu67] Kazuoki Azuma. “Weighted sums of certain dependent random variables.” *Tohoku Mathematical Journal* 19.3 (1967). DOI: 10.2748/tmj/1178243286.

[BBS97] Guy Bouchitte, Giuseppe Buttazzo, and Pierre Seppecher. “Energies with respect to a measure and applications to low dimensional structures.” *Calculus of Variations and Partial Differential Equations* 5.1 (1997), pp. 37–54. DOI: 10.1007/s005260050058.

[BD24] Armand Bernou and Mitia Duerinckx. “Uniform-in-time estimates on the size of chaos for interacting Brownian particles.” *arXiv preprint arXiv:2405.19306* (2024). DOI: https://doi.org/10.48550/arXiv.2405.19306.

[BD98] Andrea Braides and Anneliese Defranceschi. *Homogenization of Multiple Integrals*. Oxford University PressOxford, 1998. DOI: 10.1093/oso/9780198502463.001.0001.

[Ber+15] Lorenzo Bertini, Alberto De Sole, Davide Gabrielli, Giovanni Jona-Lasinio, and Claudio Landim. “Macroscopic fluctuation theory.” *Reviews of Modern Physics* 87.2 (2015), pp. 593–636. DOI: 10.1103/revmodphys.87.593.

[BF01] Guy Bouchitté and Ilaria Fragalà. “Homogenization of Thin Structures by Two-Scale Method with Respect to Measures.” *SIAM Journal on Mathematical Analysis* 32.6 (2001), pp. 1198–1226. DOI: 10.1137/s0036141000370260.

[BGN16] Freddy Bouchet, Krzysztof Gawedzki, and Cesare Nardini. “Perturbative Calculation of Quasi-Potential in Non-equilibrium Diffusions: A Mean-Field Example.” *Journal of Statistical Physics* 163.5 (2016), pp. 1157–1210. DOI: 10.1007/s10955-016-1503-2.

[BGV23] L’ubomír Baňas, Benjamin Gess, and Christian Vieth. “Numerical approximation of singular-degenerate parabolic stochastic PDEs.” *IMA Journal of Numerical Analysis* 44 (2023). DOI: 10.1093/imanum/drad061.

[BH77] W. Braun and K. Hepp. “The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles.” *Communications in Mathematical Physics* 56.2 (1977), pp. 101–113. DOI: 10.1007/BF01611497.

[BH99] A. Bogdanov and A. Hubert. “The stability of vortex-like structures in uniaxial ferromagnets.” *Journal of Magnetism and Magnetic Materials* 195.1 (1999), pp. 182–192. DOI: 10.1016/s0304-8853(98)01038-5.

[BJ80] P Bak and M H Jensen. “Theory of helical magnetic structures and phase transitions in MnSi and FeGe.” *Journal of Physics C: Solid State Physics* 13.31 (1980), pp. L881–L885. DOI: 10.1088/0022-3719/13/31/002.

[BJW19] Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. “On mean-field limits and quantitative estimates with a large class of singular kernels: application to the Patlak-Keller-Segel model.” *C. R. Math. Acad. Sci. Paris* 357.9 (2019), pp. 708–720. DOI: 10.1016/j.crma.2019.09.007.

[BJW23] Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. “Mean field limit and quantitative estimates with singular attractive kernels.” *Duke Mathematical Journal* 172.13 (2023), pp. 2591–2641. DOI: 10.1215/00127094-2022-0088.

[BKS03] Itai Benjamini, Gil Kalai, and Oded Schramm. “First passage percolation has sublinear distance variance.” *The Annals of Probability* 31.4 (2003). DOI: 10.1214/aop/1068646373.

[BLM96] Alain Bourgeat, Stephan Luckhaus, and Andro Mikelić. “Convergence of the Homogenization Process for a Double-Porosity Model of Immiscible Two-Phase Flow.” *SIAM Journal on Mathematical Analysis* 27.6 (1996), pp. 1520–1543. DOI: 10.1137/s0036141094276457.

[BM09] Jean-François Babadjian and Vincent Millot. “Homogenization of variational problems in manifold valued Sobolev spaces.” *ESAIM: Control, Optimisation and Calculus of Variations* 16.4 (2009), pp. 833–855. DOI: 10.1051/cocv/2009025.

[BMS09] Giorgio Bertotti, Isaak D. Mayergoyz, and Claudio Serpico. *Nonlinear Magnetization Dynamics in Nanosystems*. Elsevier Series in Electromagnetism. Elsevier, 2009. DOI: 10.1016/b978-0-08-044316-4.x0001-1.

[BN04] Serguei Brazovskii and Thomas Nattermann. “Pinning and sliding of driven elastic systems: from domain walls to charge density waves.” *Advances in Physics* 53.2 (2004), pp. 177–252. DOI: 10.1080/00018730410001684197.

[Bod+07] M. Bode, M. Heide, K. von Bergmann, P. Ferriani, S. Heinze, G. Bihlmayer, A. Kubetzka, O. Pietzsch, S. Blügel, and R. Wiesendanger. “Chiral magnetic order at surfaces driven by inversion asymmetry.” *Nature* 447.7141 (2007), pp. 190–193. DOI: 10.1038/nature05802.

[BR25] Annika Bach and Matthias Ruf. *Stochastic homogenization of integral functionals defined on manifold-valued Sobolev maps*. cvgmt preprint. 2025.

[Bro63] W.F. Brown. *Micromagnetics*. Interscience tracts on physics and astronomy. Interscience Publishers, 1963.

[BSS17] Leonid Berlyand, Etienne Sandier, and Sylvia Serfaty. "A two scale Gamma-convergence approach for random non-convex homogenization." *Calculus of Variations and Partial Differential Equations* 56.6 (2017). DOI: 10.1007/s00526-017-1249-y.

[BSS26] Riddhipratim Basu, Vladas Sidoravicius, and Allan Sly. *Rotationally invariant first passage percolation: Breaking the $n/\log n$ variance barrier*. 2026. arXiv: 2604.01214 [math.PR].

[BSS93] G. Barles, H. M. Soner, and P. E. Souganidis. "Front Propagation and Phase Field Theory." *SIAM Journal on Control and Optimization* 31.2 (1993), pp. 439–469. DOI: 10.1137/0331021.

[BT14] T. Bodineau and A. Teixeira. "Interface Motion in Random Media." *Communications in Mathematical Physics* 334.2 (2014), pp. 843–865. DOI: 10.1007/s00220-014-2152-4.

[Caf13] Luis Caffarelli. "The homogenization of surfaces and boundaries." *Bulletin of the Brazilian Mathematical Society, New Series* 44.4 (2013), pp. 755–775. DOI: 10.1007/s00574-013-0033-7.

[CBD20] Luca Courte, Kaushik Bhattacharya, and Patrick Dondl. "Bounds on precipitate hardening of line and surface defects in solids." *Zeitschrift für angewandte Mathematik und Physik* 71.3 (2020). DOI: 10.1007/s00033-020-01327-3.

[CD22] Raphaël Cerf and Barbara Dembin. "The time constant for Bernoulli percolation is Lipschitz continuous strictly above pc." *The Annals of Probability* 50.5 (2022). DOI: 10.1214/22-aop1565.

[CD81] J. Theodore Cox and Richard Durrett. "Some Limit Theorems for Percolation Processes with Necessary and Sufficient Conditions." *The Annals of Probability* 9.4 (1981). DOI: 10.1214/aop/1176994364.

[CDJ19] Li Chen, Esther S Daus, and Ansgar Jüngel. "Rigorous mean-field limit and cross-diffusion." *Zeitschrift für angewandte Mathematik und Physik* 70.4 (2019), pp. 1–21. DOI: 10.1007/s00033-019-1170-7.

[CDL10] Jérôme Coville, Nicolas Dirr, and Stephan Luckhaus. "Non-existence of positive stationary solutions for a class of semi-linear PDEs with random coefficients." *Networks and Heterogeneous Media* 5.4 (2010), pp. 745–763. DOI: 10.3934/nhm.2010.5.745.

[CDO22] Luca Courte, Patrick Dondl, and Michael Ortiz. "A Proof of Taylor Scaling for Curvature-Driven Dislocation Motion Through Random Arrays of Obstacles." *Archive for Rational Mechanics and Analysis* 244.2 (2022), pp. 317–341. DOI: 10.1007/s00205-022-01765-5.

[CF23] Federico Cornalba and Julian Fischer. "The Dean–Kawasaki Equation and the Structure of Density Fluctuations in Systems of Diffusing Particles." *Archive for Rational Mechanics and Analysis* 247.5 (2023). DOI: 10.1007/s00205-023-01903-7.

[CFO19] Marco Cicalese, Marwin Forster, and Gianluca Orlando. "Variational Analysis of a Two-Dimensional Frustrated Spin System: Emergence and Rigidity of Chirality Transitions." *SIAM Journal on Mathematical Analysis* 51.6 (2019), pp. 4848–4893. DOI: 10.1137/19m1257305.

[Che+21] Li Chen, Esther S. Daus, Alexandra Holzinger, and Ansgar Jüngel. “Rigorous derivation of population cross-diffusion systems from moderately interacting particle systems.” *J. Nonlinear Sci.* 31.6 (2021), Paper No. 94, 38. DOI: 10.1007/s00332-021-09747-9.

[Che07] Hung Cheng. *Advanced analytic methods in applied mathematics, science, and engineering*. LuBan Press, 2007.

[CIL92] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. “User’s guide to viscosity solutions of second order partial differential equations.” *Bulletin of the American Mathematical Society* 27.1 (1992), pp. 1–67. DOI: 10.1090/s0273-0979-1992-00266-5.

[CM14] L. A. Caffarelli and R. Monneau. “Counter-Example in Three Dimension and Homogenization of Geometric Motions in Two Dimension.” *Archive for Rational Mechanics and Analysis* 212.2 (2014), pp. 503–574. DOI: 10.1007/s00205-013-0712-y.

[Cor+26] Federico Cornalba, Julian Fischer, Jonas Ingmanns, and Claudia Raithel. “Density fluctuations in weakly interacting particle systems via the Dean–Kawasaki equation.” *The Annals of Probability* 54.1 (2026). DOI: 10.1214/25-aop1763.

[CRS16] Marco Cicalese, Matthias Ruf, and Francesco Solombrino. “Chirality transitions in frustrated S2-valued spin systems.” *Mathematical Models and Methods in Applied Sciences* 26.08 (2016), pp. 1481–1529. DOI: 10.1142/s0218202516500366.

[CRS23] Antonin Chodron de Courcel, Matthew Rosenzweig, and Sylvia Serfaty. “Sharp uniform-in-time mean-field convergence for singular periodic Riesz flows.” *arXiv preprint arXiv:2304.05315* (2023). DOI: https://doi.org/10.48550/arXiv.2304.05315.

[CS15] Marco Cicalese and Francesco Solombrino. “Frustrated Ferromagnetic Spin Chains: A Variational Approach to Chirality Transitions.” *Journal of Nonlinear Science* 25.2 (2015), pp. 291–313. DOI: 10.1007/s00332-015-9230-4.

[CS23] Federico Cornalba and Tony Shardlow. “The regularised inertial Dean–Kawasaki equation: discontinuous Galerkin approximation and modelling for low-density regime.” *ESAIM Math. Model. Numer. Anal.* 57.5 (2023), pp. 3061–3090. DOI: 10.1051/m2an/2023077.

[CSZ19] Federico Cornalba, Tony Shardlow, and Johannes Zimmer. “A Regularized Dean–Kawasaki Model: Derivation and Analysis.” *SIAM Journal on Mathematical Analysis* 51.2 (2019), pp. 1137–1187. DOI: 10.1137/18m1172697.

[CSZ20] Federico Cornalba, Tony Shardlow, and Johannes Zimmer. “From weakly interacting particles to a regularised Dean–Kawasaki model.” *Nonlinearity* 33.2 (2020), pp. 864–891. DOI: 10.1088/1361-6544/ab5174.

[CSZ21] Federico Cornalba, Tony Shardlow, and Johannes Zimmer. “Well-posedness for a regularised inertial Dean–Kawasaki model for slender particles in several space dimensions.” *Journal of Differential Equations* 284 (2021), pp. 253–283. DOI: 10.1016/j.jde.2021.02.048.

[CT15] Michael E Cates and Julien Tailleur. “Motility-induced phase separation.” *Annu. Rev. Condens. Matter Phys.* 6.1 (2015), pp. 219–244. DOI: https://doi.org/10.1146/annurev-conmatphys-031214-014710.

[CT16] Raphaël Cerf and Marie Théret. “Weak shape theorem in first passage percolation with infinite passage times.” *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques* 52.3 (2016). DOI: 10.1214/15-aihp686.

[Dac+99] B. Dacorogna, I. Fonseca, J. Malý, and K. Trivisa. “Manifold constrained variational problems.” *Calculus of Variations and Partial Differential Equations* 9.3 (1999), pp. 185–206. DOI: 10.1007/s005260050137.

[Dal93] Gianni Dal Maso. *Introduction to Gamma-convergence*. Birkhäuser, 1993, p. 340.

[Dav+22] Elisa Davoli, Giovanni Di Fratta, Dirk Praetorius, and Michele Ruggeri. “Micromagnetics of thin films in the presence of Dzyaloshinskii–Moriya interaction.” *Mathematical Models and Methods in Applied Sciences* 32.05 (2022), pp. 911–939. DOI: 10.1142/s0218202522500208.

[DD20] Elisa Davoli and Giovanni Di Fratta. “Homogenization of Chiral Magnetic Materials: A Mathematical Evidence of Dzyaloshinskii’s Predictions on Helical Structures.” *Journal of Nonlinear Science* 30.3 (2020), pp. 1229–1262. DOI: 10.1007/s00332-019-09606-8.

[DDI24] Elisa Davoli, Lorenza D’Elia, and Jonas Ingmanns. “Stochastic Homogenization of Micromagnetic Energies and Emergence of Magnetic Skyrmions.” *Journal of Nonlinear Science* 34.2 (2024). DOI: 10.1007/s00332-023-10005-3.

[DDS11] Nicolas Dirr, Patrick Dondl, and Michael Scheutzow. “Pinning of interfaces in random media.” *Interfaces and Free Boundaries* (2011), pp. 411–421. DOI: 10.4171/ifb/265.

[Dea96] David S Dean. “Langevin equation for the density of a system of interacting Langevin processes.” *Journal of Physics A: Mathematical and General* 29.24 (1996), pp. L613–L617. DOI: 10.1088/0305-4470/29/24/001.

[Déj+18] PM Déjardin, Y Cornaton, P Ghesquière, Cyril Caliot, and R Brouzet. “Calculation of the orientational linear and nonlinear correlation factors of polar liquids from the rotational Dean-Kawasaki equation.” *J. Chem. Phys.* 148.4 (2018), p. 044504. DOI: 10.1063/1.5010295.

[Del+16] Jean-Baptiste Delfau, Hélène Ollivier, Cristóbal López, Bernd Blasius, and Emilio Hernández-García. “Pattern formation with repulsive soft-core interactions: Discrete particle dynamics and Dean-Kawasaki equation.” *Phys. Rev. E* 94.4 (2016), p. 042120. DOI: 10.1103/PhysRevE.94.042120.

[Del22] Lorenzo Dello Schiavo. “The Dirichlet–Ferguson diffusion on the space of probability measures over a closed Riemannian manifold.” *Ann. Probab.* 50.2 (2022), pp. 591–648. DOI: 10.1214/21-aop1541.

[Dem24] Barbara Dembin. “The variance of the graph distance in the infinite cluster of percolation is sublinear.” *Latin American Journal of Probability and Mathematical Statistics* 21.1 (2024), p. 307. DOI: 10.30757/alea.v21-13.

[Der07] Bernard Derrida. “Non-equilibrium steady states: fluctuations and large deviations of the density and of the current.” *Journal of Statistical Mechanics: Theory and Experiment* 2007.07 (2007), P07023–P07023. DOI: 10.1088/1742-5468/2007/07/p07023.

[DFG26] Nicolas Dirr, Benjamin Fehrman, and Benjamin Gess. “Conservative Stochastic PDE and Fluctuations of the Symmetric Simple Exclusion Process.” *Communications in Mathematical Physics* 407.4 (2026). DOI: 10.1007/s00220-026-05587-4.

[DFV14] Aleksandar Donev, Thomas G Fai, and Eric Vanden-Eijnden. “A reversible mesoscopic model of diffusion in liquids: from giant fluctuations to Fick’s law.” *Journal of Statistical Mechanics: Theory and Experiment* 2014.4 (2014), P04004. DOI: 10.1088/1742-5468/2014/04/p04004.

[Din22] Hao Ding. *A new particle approximation to the diffusive Dean–Kawasaki equation with colored noise*. 2022. DOI: 10.48550/arXiv.2204.11309. arXiv: 2204.11309.

[Dir+10] Nicolas Dirr, Patrick Dondl, Geoffrey Grimmett, Alexander Holroyd, and Michael Scheutzow. “Lipschitz percolation.” *Electronic Communications in Probability* 15.none (2010). DOI: 10.1214/ecp.v15-1521.

[DJP26] Ana Djurdjevac, Xiaohao Ji, and Nicolas Perkowski. “Weak error of Dean–Kawasaki equation with smooth mean-field interactions.” *Journal de Mathématiques Pures et Appliquées* 213 (2026), p. 103937. DOI: 10.1016/j.matpur.2026.103937.

[DJS21] Patrick W. Dondl, Martin Jesenko, and Michael Scheutzow. “Pinning of interfaces in a random medium with zero mean.” *Interfaces and Free Boundaries, Mathematical Analysis, Computation and Applications* 23.3 (2021), pp. 305–321. DOI: 10.4171/ifb/455.

[DJS22] Patrick Dondl, Martin Jesenko, and Michael Scheutzow. “Infinite pinning.” *Bulletin of the London Mathematical Society* 54.2 (2022), pp. 760–771. DOI: 10.1112/blms.12599.

[DKD22] Nataša Djurdjevac Conrad, Jonas Köppl, and Ana Djurdjevac. “Feedback Loops in Opinion Dynamics of Agent-Based Models with Multiplicative Noise.” *Entropy* 24.10 (2022), p. 1352. DOI: 10.3390/e24101352.

[DKP24] Ana Djurdjevac, Helena Kremp, and Nicolas Perkowski. “Weak error analysis for a nonlinear SPDE approximation of the Dean–Kawasaki equation.” *Stochastics and Partial Differential Equations: Analysis and Computations* 12.4 (2024), pp. 2330–2355. DOI: 10.1007/s40072-024-00324-1.

[DKY08] N. Dirr, G. Karali, and N. K. Yip. “Pulsating wave for mean curvature flow in inhomogeneous medium.” *European Journal of Applied Mathematics* 19.6 (2008), pp. 661–699. DOI: 10.1017/s095679250800764x.

[DM86a] Gianni Dal Maso and Luciano Modica. “Nonlinear Stochastic Homogenization.” *Annali di Matematica Pura ed Applicata* 144.1 (1986), pp. 347–389. DOI: 10.1007/bf01760826.

[DM86b] Gianni Dal Maso and Luciano Modica. “Nonlinear stochastic homogenization and ergodic theory.” *Journal für die reine und angewandte Mathematik* 368 (1986), pp. 28–42.

[Don+10] Aleksandar Donev, Eric Vanden-Eijnden, Alejandro Garcia, and John Bell. “On the accuracy of finite-volume schemes for fluctuating hydrodynamics.” *Communications in Applied Mathematics and Computational Science* 5.2 (2010), pp. 149–197. DOI: 10.2140/camcos.2010.5.149.

[DP96] Jean-Dominique Deuschel and Agoston Pisztora. “Surface order large deviations for high-density percolation.” *Probability Theory and Related Fields* 104.4 (1996), pp. 467–482. DOI: 10.1007/bf01198162.

[DS12] Patrick W. Dondl and Michael Scheutzow. “Positive speed of propagation in a semilinear parabolic interface model with unbounded random coefficients.” *Networks and Heterogeneous Media* 7.1 (2012), pp. 137–150. DOI: 10.3934/nhm.2012.7.137.

[DS17] Patrick W. Dondl and Michael Scheutzow. “Ballistic and sub-ballistic motion of interfaces in a field of random obstacles.” *The Annals of Applied Probability* 27.5 (2017). DOI: 10.1214/17-aap1279.

[DSZ16] Nicolas Dirr, Marios Stamatakis, and Johannes Zimmer. “Entropic and gradient flow formulations for nonlinear diffusion.” *Journal of Mathematical Physics* 57.8 (2016), p. 081505. DOI: 10.1063/1.4960748.

[DT20] Hugo Duminil-Copin and Vincent Tassion. “Renormalization of Crossing Probabilities in the Planar Random-Cluster Model.” *Moscow Mathematical Journal* 20.4 (2020), pp. 711–740. DOI: 10.17323/1609-4514-2020-20-4-711-740.

[Due21] Mitia Duerinckx. “On the Size of Chaos via Glauber Calculus in the Classical Mean-Field Dynamics.” *Communications in Mathematical Physics* 382.1 (2021), pp. 613–653. DOI: 10.1007/s00220-021-03978-3.

[Dur+19] Miguel A Durán-Olivencia, Rishabh S Gvalani, Serafim Kalliadasis, and Grigorios A Pavliotis. “Instability, rupture and fluctuations in thin liquid films: theory and computations.” *Journal of Statistical Physics* 174.3 (2019), pp. 579–604. DOI: 10.1007/s10955-018-2200-0.

[DY06] N. Dirr and N.K. Yip. “Pinning and de-pinning phenomena in front propagation in heterogeneous media.” *Interfaces and Free Boundaries* (2006), pp. 79–109. DOI: 10.4171/ifb/136.

[Dzy58] I. Dzyaloshinsky. “A thermodynamic theory of “weak” ferromagnetism of antiferromagnetics.” *Journal of Physics and Chemistry of Solids* 4.4 (1958), pp. 241–255. DOI: 10.1016/0022-3697(58)90076-3.

[Dzy65a] Igor Dzyaloshinskii. “Theory of helicoidal structures in antiferromagnets. i. nonmetals.” *Soviet Physics JETP* 19.4 (1965).

[Dzy65b] Igor Dzyaloshinskii. “Theory of helicoidal structures in antiferromagnets. ii. metals.” *Soviet Physics JETP* 20.1 (1965).

[FCS13] Albert Fert, Vincent Cros, and João Sampaio. “Skyrmions on the track.” *Nature Nanotechnology* 8.3 (2013), pp. 152–156. DOI: 10.1038/nnano.2013.29.

[Fel19] William M Feldman. *Mean curvature flow with positive random forcing in 2-d*. 2019. DOI: 10.48550/ARXIV.1911.00488.

[FFZ21] William M. Feldman, Jean-Baptiste Fermanian, and Bruno Ziliotto. “An example of failure of stochastic homogenization for viscous Hamilton-Jacobi equations without convexity.” *Journal of Differential Equations* 280 (2021), pp. 464–476. DOI: https://doi.org/10.1016/j.jde.2021.01.009.

[FG19] Benjamin Fehrman and Benjamin Gess. “Well-Posedness of Nonlinear Diffusion Equations with Nonlinear, Conservative Noise.” *Archive for Rational Mechanics and Analysis* 233.1 (2019), pp. 249–322. DOI: 10.1007/s00205-019-01357-w.

[FG23] Benjamin Fehrman and Benjamin Gess. “Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.” *Inventiones mathematicae* 234.2 (2023), pp. 573–636. DOI: 10.1007/s00222-023-01207-3.

[FG24] Benjamin Fehrman and Benjamin Gess. “Well-Posedness of the Dean–Kawasaki and the Nonlinear Dawson–Watanabe Equation with Correlated Noise.” *Archive for Rational Mechanics and Analysis* 248.2 (2024). DOI: 10.1007/s00205-024-01963-3.

[FGG22] Benjamin Fehrman, Benjamin Gess, and Rishabh S Gvalani. “Ergodicity and random dynamical systems for conservative SPDEs.” *arXiv preprint arXiv:2206.14789* (2022). DOI: https://doi.org/10.48550/arXiv.2206.14789.

[FI26a] Julian Fischer and Jonas Ingmanns. *Quantitative homogenization of forced geometric motions through random fields of obstacles*. 2026. arXiv: 2603.12179 [math.AP].

[FI26b] Julian Fischer and Jonas Ingmanns. “A sufficient criterion for ballistic behavior of forced geometric motions in random environments.” *In preparation* (2026+).

[FL78] H. Fukuyama and P. A. Lee. “Dynamics of the charge-density wave. I. Impurity pinning in a single chain.” *Physical Review B* 17.2 (1978), pp. 535–541. DOI: 10.1103/physrevb.17.535.

[FM15] Joaquin Fontbona and Sylvie Meleard. “Non local Lotka-Volterra system with cross-diffusion in an heterogeneous medium.” *JOURNAL OF MATHEMATICAL BIOLOGY* 70.4 (2015), pp. 829–854. DOI: 10.1007/s00285-014-0781-z.

[FM97] Begoña Fernandez and Sylvie Méléard. “A Hilbertian approach for fluctuations on the McKean-Vlasov model.” *Stochastic Processes and their Applications* 71.1 (1997), pp. 33–53. DOI: 10.1016/s0304-4149(97)00067-7.

[FRC17] Albert Fert, Nicolas Reyren, and Vincent Cros. “Magnetic skyrmions: advances in physics and potential applications.” *Nature Reviews Materials* 2.7 (2017). DOI: 10.1038/natrevmats.2017.31.

[FS17] William M. Feldman and Panagiotis E. Souganidis. “Homogenization and non-homogenization of certain non-convex Hamilton–Jacobi equations.” *Journal de Mathématiques Pures et Appliquées* 108.5 (2017), pp. 751–782. DOI: 10.1016/j.matpur.2017.05.016.

[Gen85] P. G. de Gennes. “Wetting: statics and dynamics.” *Reviews of Modern Physics* 57.3 (1985), pp. 827–863. DOI: 10.1103/revmodphys.57.827.

[GGK25] Benjamin Gess, Rishabh S. Gvalani, and Vitalii Konarovskyi. “Conservative SPDEs as fluctuating mean field limits of stochastic gradient descent.” *Probability Theory and Related Fields* 192.3-4 (2025), pp. 1447–1515. DOI: 10.1007/s00440-024-01353-6.

[Gig+91] Y. Giga, S. Goto, H. Ishii, and M.-H. Sato. “Comparison Principle and Convexity Preserving Properties for Singular Degenerate Parabolic Equations on Unbounded Domains.” *Indiana University Mathematics Journal* 40.2 (1991), pp. 443–470.

[Gig06] Yoshikazu Giga. *Surface Evolution Equations*. Birkhäuser-Verlag, 2006. DOI: 10.1007/3-7643-7391-1.

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. "Paracontrolled distributions and singular PDEs." *Forum of Mathematics, Pi* 3 (2015). DOI: 10.1017/fmp.2015.2.

[GK19] Hongwei Gao and Inwon Kim. "Head and Tail Speeds of Mean Curvature Flow with Forcing." *Archive for Rational Mechanics and Analysis* 235.1 (2019), pp. 287–354. DOI: 10.1007/s00205-019-01423-3.

[GM04] Olivier Garet and Régine Marchand. "Asymptotic shape for the chemical distance and first-passage percolation on the infinite Bernoulli cluster." *ESAIM: Probability and Statistics* 8 (2004), pp. 169–199. DOI: 10.1051/ps:2004009.

[GM07] Olivier Garet and Régine Marchand. "Large deviations for the chemical distance in supercritical Bernoulli percolation." *The Annals of Probability* 35.3 (2007). DOI: 10.1214/009117906000000881.

[GM09] Olivier Garet and Régine Marchand. "Moderate deviations for the chemical distance in Bernoulli percolation." *Latin American journal of probability and mathematical statistics* 7 (2009).

[GNO14] A. Gloria, S. Neukamm, and F. Otto. "Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics." *Inventiones mathematicae* 199.2 (2014), pp. 455–515. DOI: 10.1007/s00222-014-0518-z.

[GNO20] Antoine Gloria, Stefan Neukamm, and Felix Otto. "A Regularity Theory for Random Elliptic Operators." *Milan Journal of Mathematics* 88.1 (2020), pp. 99–170. DOI: 10.1007/s00032-020-00309-4.

[GO11] Antoine Gloria and Felix Otto. "An optimal variance estimate in stochastic homogenization of discrete elliptic equations." *The Annals of Probability* 39.3 (2011). DOI: 10.1214/10-aop571.

[GO17] Antoine Gloria and Felix Otto. "Quantitative results on the corrector equation in stochastic homogenization." *Journal of the European Mathematical Society* 19.11 (2017), pp. 3489–3548. DOI: 10.4171/jems/745.

[God+12] Benjamin D Goddard, Andreas Nold, Nikos Savva, Grigorios A Pavliotis, and Serafim Kalliadasis. "General dynamical density functional theory for classical fluids." *Phys. Rev. Lett.* 109.12 (2012), p. 120603. DOI: 10.1103/PhysRevLett.109.120603.

[Gol03] F. Golse. "The mean-field limit for the dynamics of large particle systems." *Journ. Équ. dériv. partielles* (2003), pp. 1–47. DOI: 10.5802/jedp.623.

[Gra08] Loukas Grafakos. *Classical fourier analysis*. Vol. 2. Springer, 2008. DOI: https://doi.org/10.1007/978-1-4939-1194-3.

[Gri99] Geoffrey Grimmett. *Percolation (Grundlehren der mathematischen Wissenschaften)*. Springer, 1999, p. 444.

[GS00] Mariano Gasca and Thomas Sauer. "Polynomial interpolation in several variables." *Adv. Comput. Math.* 12.4 (2000), pp. 377–410. DOI: 10.1023/A:1018981505752.

[GS14] G. Galiano and V. Selgas. "On a cross-diffusion segregation problem arising from a model of interacting particles." *Nonlin. Anal. Real World Appl.* 18 (2014), pp. 34–49. DOI: 10.1016/J.NONRWA.2014.02.001.

[GZ22] Janusz Ginster and Barbara Zwicknagl. “Energy Scaling Law for a Singularly Perturbed Four-Gradient Problem in Helimagnetism.” *Journal of Nonlinear Science* 33.1 (2022). DOI: 10.1007/s00332-022-09847-0.

[Hai14] Martin Hairer. “A theory of regularity structures.” *Inventiones mathematicae* 198.2 (2014), pp. 269–504. DOI: 10.1007/s00222-014-0505-4.

[Hel+21] Luzie Helfmann, Nataša Djurdjevac Conrad, Ana Djurdjevac, Stefanie Winkelmann, and Christof Schütte. “From interacting agents to density-based modeling with stochastic PDEs.” *Communications in Applied Mathematics and Computational Science* 16.1 (2021), pp. 1–32. DOI: 10.2140/camcos.2021.16.1.

[HH77] P. C. Hohenberg and B. I. Halperin. “Theory of dynamic critical phenomena.” *Reviews of Modern Physics* 49.3 (1977), pp. 435–479. DOI: 10.1103/revmodphys.49.435.

[HJ03] H. Haddar and P. Joly. “Homogenized model for a laminar ferromagnetic medium.” *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 133.3 (2003), pp. 567–598. DOI: 10.1017/s0308210500002547.

[HNV21] Martin Heida, Stefan Neukamm, and Mario Varga. “Stochastic homogenization of Lambda-convex gradient flows.” *Discrete and Continuous Dynamical Systems - S* 14.1 (2021), pp. 427–453. DOI: 10.3934/dcdss.2020328.

[Hrk+19] Gino Hrkac, Carl-Martin Pfeiler, Dirk Praetorius, Michele Ruggeri, Antonio Segatti, and Bernhard Stiftner. “Convergent tangent plane integrators for the simulation of chiral magnetic skyrmion dynamics.” *Advances in Computational Mathematics* 45.3 (2019), pp. 1329–1368. DOI: 10.1007/s10444-019-09667-z.

[HW65] J. M. Hammersley and D. J. A. Welsh. “First-Passage Percolation, Subadditive Processes, Stochastic Networks, and Generalized Renewal Theory.” *Bernoulli 1713 Bayes 1763 Laplace 1813: Anniversary Volume*. Ed. by Jerzy Neyman and Lucien M. Le Cam. Berlin, Heidelberg: Springer Berlin Heidelberg, 1965, pp. 61–110. DOI: 10.1007/978-3-642-99884-3_7.

[Ill25] Pierre Illien. “The Dean-Kawasaki equation and stochastic density functional theory.” *Reports on Progress in Physics* (2025). DOI: 10.1088/1361-6633/adee2e.

[IRS12] K. Ichikawa, M. Rouzimaimaiti, and T. Suzuki. “Reaction diffusion equation with non-local term arises as a mean field limit of the master equation.” *Discrete Cont. Dyn. Sys. Ser.* Ser. S.5 (2012), pp. 115–126. DOI: 10.3934/dcdss.2012.5.115.

[IV87] L B Ioffe and V M Vinokur. “Dynamics of interfaces and dislocations in disordered media.” *Journal of Physics C: Solid State Physics* 20.36 (1987), pp. 6149–6158. DOI: 10.1088/0022-3719/20/36/016.

[Jia+17] Wanjun Jiang, Gong Chen, Kai Liu, Jiadong Zang, Suzanne G.E. te Velthuis, and Axel Hoffmann. “Skyrmions in magnetic multilayers.” *Physics Reports* 704 (2017), pp. 1–49. DOI: 10.1016/j.physrep.2017.08.001.

[JM98] B. Jourdain and S. Méléard. “Propagation of chaos and fluctuations for a moderate model with smooth initial data.” *Ann. inst. Henri Poincare (B) Probab. Stat.* 34.6 (1998), pp. 727–766. DOI: 10.1016/S0246-0203(99)80002-8.

[JPZ22] Ansgar Jüngel, Stefan Portisch, and Antoine Zurek. “Nonlocal cross-diffusion systems for multi-species populations and networks.” *Nonlin. Anal.* 219 (2022), p. 112800. DOI: 10.1016/j.na.2022.112800.

[Juu01] Petri Juutinen. “On the definition of viscosity solutions for parabolic equations.” *Proceedings of the American Mathematical Society* 129.10 (2001), pp. 2907–2911. DOI: 10.1090/s0002-9939-01-05889-0.

[JW17] P.E. Jabin and Z. Wang. “Active Particles.” Vol. 1. Springer, Boston, 2017. Chap. Mean field limit for stochastic particle systems, pp. 379–402. DOI: https://doi.org/10.1007/978-3-319-49996-3.

[Kar98] Mehran Kardar. “Nonequilibrium dynamics of interfaces and lines.” *Physics Reports* 301.1–3 (1998), pp. 85–112. DOI: 10.1016/s0370-1573(98)00007-6.

[Kaw98] Kyozi Kawasaki. “Microscopic Analyses of the Dynamical Density Functional Equation of Dense Fluids.” *Journal of Statistical Physics* 93.3/4 (1998), pp. 527–546. DOI: 10.1023/b:joss.0000033240.66359.6c.

[Kes93] Harry Kesten. “On the Speed of Convergence in First-Passage Percolation.” *The Annals of Applied Probability* 3.2 (1993). DOI: 10.1214/aoap/1177005426.

[Kim+17] Changho Kim, Andy Nonaka, John B. Bell, Alejandro L. Garcia, and Aleksandar Donev. “Stochastic simulation of reaction-diffusion systems: A fluctuating-hydrodynamics approach.” *The Journal of Chemical Physics* 146.12 (2017), p. 124110. DOI: 10.1063/1.4978775.

[KL85] Joel Koplik and Herbert Levine. “Interface moving through a random background.” *Physical Review B* 32.1 (1985), pp. 280–292. DOI: 10.1103/physrevb.32.280.

[KLR19a] Vitalii Konarovskyi, Tobias Lehmann, and Max von Renesse. “On Dean–Kawasaki Dynamics with Smooth Drift Potential.” *Journal of Statistical Physics* 178.3 (2019), pp. 666–681. DOI: 10.1007/s10955-019-02449-3.

[KLR19b] Vitalii Konarovskyi, Tobias Lehmann, and Max-K. von Renesse. “Dean-Kawasaki dynamics: ill-posedness vs. triviality.” *Electronic Communications in Probability* 24.none (2019). DOI: 10.1214/19-ecp208.

[KM23] Vitalii Konarovskyi and Victor Marx. “On conditioning Brownian particles to coalesce.” *J. Theoret. Probab.* 36.4 (2023), pp. 2126–2164. DOI: 10.1007/s10959-023-01267-4.

[Koz80] S M Kozlov. “Averaging of random operators.” *Mathematics of the USSR-Sbornik* 37.2 (1980), pp. 167–180. DOI: 10.1070/sm1980v037n02abeh001948.

[KR19] Vitalii Konarovskyi and Max-K von Renesse. “Modified massive Arratia flow and Wasserstein diffusion.” *Commun. Pure Appl. Math.* 72.4 (2019), pp. 764–800. DOI: 10.1002/cpa.21758.

[KR24] Vitalii Konarovskyi and Max-K. von Renesse. “Reversible coalescing-fragmentating Wasserstein dynamics on the real line.” *J. Funct. Anal.* 286.8 (2024), Paper No. 110342, 60. DOI: 10.1016/j.jfa.2024.110342.

[Kre85] Ulrich Krengel. *Ergodic Theorems*. Berlin, New York: De Gruyter, 1985. DOI: 10.1515/9783110844641.

[KRV06] Elena Kosygina, Fraydoun Rezakhanlou, and S. R. S. Varadhan. “Stochastic homogenization of Hamilton-Jacobi-Bellman equations.” *Communications on Pure and Applied Mathematics* 59.10 (2006), pp. 1489–1521. DOI: 10.1002/cpa.20137.

[KS13] N. Kavallaris and T. Suzuki. “Non-local reaction-diffusion system involved by reaction radius I.” *IMA J. Appl. Math.* 78 (2013), pp. 614–632. DOI: 10.1093/imamat/hxr068.

[Li+19] Xiaoguai Li, Nicolas Dirr, Peter Embacher, Johannes Zimmer, and Celia Reina. “Harnessing fluctuations to discover dissipative evolution equations.” *J. Mech. Phys. Solids* 131 (2019), pp. 240–251. DOI: 10.1016/j.jmps.2019.05.017.

[LL35] Lev Davidovich Landau and E Lifshitz. “On the theory of the dispersion of magnetic permeability in ferromagnetic bodies.” *Phys. Z. Sowjet.* 8 (1935), p. 153.

[LL87] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics, Second Edition. Volume 6 (Course of Theoretical Physics)*. Butterworth-Heinemann, 1987, p. 552.

[LM18] Xinye Li and Christof Melcher. “Stability of axisymmetric chiral skyrmions.” *Journal of Functional Analysis* 275.10 (2018), pp. 2817–2844. DOI: 10.1016/j.jfa.2018.01.019.

[LNW02] Dag Lukkassen, Gabriel Nguetseng, and Peter Wall. “Two-scale convergence.” *International journal of pure and applied mathematics* 2.1 (2002), pp. 35–86.

[LS05a] Pierre-Louis Lions and Panagiotis E. Souganidis. “Homogenization of “Viscous” Hamilton–Jacobi Equations in Stationary Ergodic Media.” *Communications in Partial Differential Equations* 30.3 (2005), pp. 335–375. DOI: 10.1081/pde-200050077.

[LS05b] Pierre-Louis Lions and Panagiotis E. Souganidis. “Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications.” *Annales de l’Institut Henri Poincaré C, Analyse non linéaire* 22.5 (2005), pp. 667–677. DOI: 10.1016/j.anihpc.2004.10.009.

[LS10] Pierre-Louis Lions and Panagiotis E. Souganidis. “Stochastic homogenization of Hamilon-Jacobi and “viscous”-Hamilton-Jacobi equations with convex nonlinearities – Revisited.” *Communications in Mathematical Sciences* 8.2 (2010), pp. 627–637. DOI: cms/1274816896.

[Lut12] James F Lutsko. “A dynamical theory of nucleation for colloids and macromolecules.” *J. Chem. Phys.* 136.3 (2012), p. 034509. DOI: 10.1063/1.3677191.

[LW10] T. LaGatta and J. Wehr. “A shape theorem for Riemannian first-passage percolation.” *Journal of Mathematical Physics* 51.5 (2010). DOI: 10.1063/1.3409344.

[Mar21] Victor Marx. “Infinite-dimensional regularization of McKean–Vlasov equation with a Wasserstein diffusion.” *Ann. inst. Henri Poincare (B) Probab. Stat.* 57.4 (2021), pp. 2315–2353. DOI: 10.1214/20-AIHP1136.

[McK67] Henry P. McKean. “Propagation of chaos for a class of non-linear parabolic equations.” *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)* (1967), pp. 41–57.

[Mel14] Christof Melcher. “Chiral skyrmions in the plane.” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 470.2172 (2014), p. 20140394. DOI: 10.1098/rspa.2014.0394.

[MMT99] Umberto Marconi, Bettolo Marini, and Pedro Tarazona. “Dynamic density functional theory of fluids.” *J. Chem. Phys.* 110.16 (1999), pp. 8032–8044. DOI: 10.1088/0953-8984/12/8A/356.

[Mor60] Tôru Moriya. “Anisotropic Superexchange Interaction and Weak Ferromagnetism.” *Physical Review* 120.1 (1960), pp. 91–98. DOI: 10.1103/physrev.120.91.

[Mou11] Jean-Christophe Mourrat. “Lyapunov exponents, shape theorems and large deviations for the random walk in random potential.” *Alea* 9 (2011).

[MS17] Cyrill B. Muratov and Valeriy V. Slastikov. “Domain structure of ultrathin ferromagnetic elements in the presence of Dzyaloshinskii–Moriya interaction.” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 473.2197 (2017), p. 20160666. DOI: 10.1098/rspa.2016.0666.

[Ngu89] Gabriel Nguetseng. “A General Convergence Result for a Functional Related to the Theory of Homogenization.” *SIAM Journal on Mathematical Analysis* 20.3 (1989), pp. 608–623. DOI: 10.1137/0520043.

[NS00] T. Nattermann and S. Scheidl. “Vortex-glass phases in type-II superconductors.” *Advances in Physics* 49.5 (2000), pp. 607–704. DOI: 10.1080/000187300412257.

[NS98] Ali Naddaf and Thomas J. Spencer. “Estimates On the Variance of Some Homogenization Problems.” 1998.

[NV18] Stefan Neukamm and Mario Varga. “Stochastic Unfolding and Homogenization of Spring Network Models.” *Multiscale Modeling & Simulation* 16.2 (2018), pp. 857–899. DOI: 10.1137/17m1141230.

[Oel89] K. Oelschläger. “On the derivation of reaction-diffusion equations as limit dynamics of systems of moderately increasing stochastic processes.” *Probab. Theory Relat. Fields.* 82.4 (1989), pp. 565–586. DOI: 10.1007/BF00341284.

[Oel90] K. Oelschläger. “Large systems of interacting particles and the porous medium equation.” *J. Differ. Equ.* 88.2 (1990), pp. 294–346. DOI: 10.1016/0022-0396(90)90101-T.

[Pis10] Giovanni Pisante. “Homogenization of micromagnetics large bodies.” eng. *ESAIM: Control, Optimisation and Calculus of Variations* 10.2 (2010), pp. 295–314.

[PV81] G. C. Papanicolaou and S. R. S. Varadhan. “Boundary value problems with rapidly oscillating random coefficients.” *Random fields, Vol. I, II (Esztergom, 1979)*. Vol. 27. Colloq. Math. Soc. János Bolyai. North-Holland, Amsterdam, 1981, pp. 835–873.

[RS09] Max-K von Renesse and Karl-Theodor Sturm. “Entropic measure and Wasserstein diffusion.” *Ann. Probab* 37.3 (2009), pp. 1114–1191. DOI: 10.1214/08-AOP430.

[RT00] Fraydoun Rezakhanlou and James E. Tarver. “Homogenization for Stochastic Hamilton-Jacobi Equations.” *Archive for Rational Mechanics and Analysis* 151.4 (2000), pp. 277–309. DOI: 10.1007/s002050050198.

[Rus+21] Antonio Russo, Sergio P. Perez, Miguel A. Durán-Olivencia, Peter Yatsyshin, José A. Carrillo, and Serafim Kalliadasis. “A finite-volume method for fluctuating dynamical density functional theory.” *Journal of Computational Physics* 428 (2021), p. 109796. DOI: 10.1016/j.jcp.2020.109796.

[Rus78] Lucio Russo. “A note on percolation.” *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 43.1 (1978), pp. 39–48. DOI: 10.1007/bf00535274.

[San07] Kévin Santugini-Repiquet. “Homogenization of the demagnetization field operator in periodically perforated domains.” *Journal of Mathematical Analysis and Applications* 334.1 (2007), pp. 502–516. DOI: 10.1016/j.jmaa.2007.01.001.

[Sch09] Russell W. Schwab. “Stochastic Homogenization of Hamilton-Jacobi Equations in Stationary Ergodic Spatio-Temporal Media.” *Indiana University Mathematics Journal* 58.2 (2009), pp. 537–581.

[Sch66] Laurent Schwartz. *Théorie des distributions*. Vol. 1. Paris: Hermann, 1966.

[Seo18] I. Seo. “Scaling limit of two-component interacting Brownian motions.” *Ann. Probab.* 46 (2018), pp. 2038–2063. DOI: 10.1214/17-AOP1220.

[Sou99] Panagiotis E. Souganidis. “Stochastic homogenization of Hamilton–Jacobi equations and some applications.” *Asymptotic Analysis* 20 (1999), pp. 1–11.

[Spo91] Herbert Spohn. *Large Scale Dynamics of Interacting Particles*. Springer Berlin Heidelberg, 1991. DOI: 10.1007/978-3-642-84371-6.

[Ste00] Angela Stevens. “The Derivation of Chemotaxis Equations as Limit Dynamics of Moderately Interacting Stochastic Many-Particle Systems.” *SIAM Journal on Applied Mathematics* 61.1 (2000), pp. 183–212. DOI: 10.1137/s0036139998342065.

[SW78] P.D. Seymour and D.J.A. Welsh. “Percolation Probabilities on the Square Lattice.” *Advances in Graph Theory*. Elsevier, 1978, pp. 227–245. DOI: 10.1016/s0167-5060(08)70509-0.

[Szn84] Alain-Sol Sznitman. “Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated.” *Journal of Functional Analysis* 56.3 (1984), pp. 311–336. DOI: 10.1016/0022-1236(84)90080-6.

[Szn91] Alain-Sol Sznitman. “Topics in propagation of chaos.” *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, 1991, pp. 165–251. DOI: 10.1007/bfb0085169.

[Tan84] Hiroshi Tanaka. “Limit Theorems for Certain Diffusion Processes with Interaction.” *Stochastic Analysis, Proceedings of the Taniguchi International Symposium on Stochastic Analysis*. Elsevier, 1984, pp. 469–488. DOI: 10.1016/s0924-6509(08)70405-7.

[Tao12] Terence Tao. *Topics in random matrix theory*. Graduate Studies in Mathematics. American Mathematical Society, 2012.

[TH81] Hiroshi Tanaka and Masuyuki Hitsuda. “Central limit theorem for a simple diffusion model of interacting particles.” *Hiroshima Mathematical Journal* 11.2 (1981). DOI: 10.32917/hmj/1206134109.

[Tho+11] AG Thompson, J Tailleur, ME Cates, and RA Blythe. “Lattice models of nonequilibrium bacterial dynamics.” *J. Stat. Mech: Theory Exp. P* 2029 (2011). DOI: 10.1088/1742-5468/2011/02/P02029.

[Vel+08] Andrea Velenich, Claudio Chamon, Leticia F Cugliandolo, and Dirk Kreimer. “On the Brownian gas: a field theory with a Poissonian ground state.” *J. Phys. A Math. Theor.* 41.23 (2008), p. 235002. DOI: 10.1088/1751-8113/41/23/235002.

[WMB94] S. Wright, A. Mikelic, and Alain Bourgeat. “Stochastic two-scale convergence in the mean and applications.” *Journal für die reine und angewandte Mathematik* 1994.456 (1994), pp. 19–52. DOI: doi:10.1515/crll.1994.456.19.

[WZZ23] Zhenfu Wang, Xianliang Zhao, and Rongchan Zhu. “Gaussian fluctuations for interacting particle systems with singular kernels.” *Archive for Rational Mechanics and Analysis* 247.5 (2023), p. 101. DOI: 10.1007/s00205-023-01932-2.

[Yu+10] X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N. Nagaosa, and Y. Tokura. “Real-space observation of a two-dimensional skyrmion crystal.” *Nature* 465.7300 (2010), pp. 901–904. DOI: 10.1038/nature09124.

[Yur87] V. V. Yurinskii. “Averaging of symmetric diffusion in random medium.” *Siberian Mathematical Journal* 27.4 (1987), pp. 603–613. DOI: 10.1007/bf00969174.

[Zep16] Caterina Ida Zeppieri. “Stochastic homogenisation of singularly perturbed integral functionals.” *Annali di Matematica Pura ed Applicata* 195.6 (2016), pp. 2183–2208. DOI: 10.1007/s10231-016-0558-7.

[Zha10] Yu Zhang. “On the concentration and the convergence rate with a moment condition in first passage percolation.” *Stochastic Processes and their Applications* 120.7 (2010), pp. 1317–1341. DOI: https://doi.org/10.1016/j.spa.2010.03.001.

[Zhi00] Vasilii V. Zhikov. “On an extension of the method of two-scale convergence and its applications.” *Sbornik: Mathematics* 191.7 (2000), pp. 973–1014. DOI: 10.1070/sm2000v191n07abeh000491.

[Zil16] Bruno Ziliotto. “Stochastic Homogenization of Nonconvex Hamilton-Jacobi Equations: A Counterexample.” *Communications on Pure and Applied Mathematics* 70.9 (2016), pp. 1798–1809. DOI: 10.1002/cpa.21674.

[ZP06] Vasilii V. Zhikov and Andrey Piatnitski. “Homogenization of random singular structures and random measures.” *Izvestiya: Mathematics* 70.1 (2006), pp. 19–67. DOI: 10.1070/im2006v070n01abeh002302.

Declaration of the use of Generative AI and AI tools

In this thesis, large language models have been used occasionally to search for related literature in addition to classical methods. ChatGPT 5.6 was used in August 2026 by a collaborator as part of the proof-reading process for the unpublished preprint [Fl26b] contained in Chapter 4, finding an error in Section 5.5, the preliminary results. Without further use of generative AI or AI tools, the error has been fixed for the final version of this thesis, leading to Section 5.5.3 and subsequent adjustments in the following sections. Outside of these two use case, no generative AI or AI tools were used in the research or writing process.