Colloquium: Many-body localization, thermalization, and entanglement

Dmitry A. Abanin

Department of Theoretical Physics, University of Geneva, 1211 Geneva, Switzerland

Ehud Altman

Department of Physics, University of California, Berkeley, California 94720, USA

Immanuel Bloch

Fakultät für Physik, Ludwig-Maximilians-Universität München, 80799 Munich, Germany and Max-Planck-Institut für Quantenoptik, 85748 Garching, Germany

Maksym Serbyn

IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria

(published 22 May 2019)

Thermalizing quantum systems are conventionally described by statistical mechanics at equilibrium. However, not all systems fall into this category, with many-body localization providing a generic mechanism for thermalization to fail in strongly disordered systems. Many-body localized (MBL) systems remain perfect insulators at nonzero temperature, which do not thermalize and therefore cannot be described using statistical mechanics. This Colloquium reviews recent theoretical and experimental advances in studies of MBL systems, focusing on the new perspective provided by entanglement and nonequilibrium experimental probes such as quantum quenches. Theoretically, MBL systems exhibit a new kind of robust integrability: an extensive set of quasilocal integrals of motion emerges, which provides an intuitive explanation of the breakdown of thermalization. A description based on quasilocal integrals of motion is used to predict dynamical properties of MBL systems, such as the spreading of quantum entanglement, the behavior of local observables, and the response to external dissipative processes. Furthermore, MBL systems can exhibit eigenstate transitions and quantum orders forbidden in thermodynamic equilibrium. An outline is given of the current theoretical understanding of the quantum-to-classical transition between many-body localized and ergodic phases and anomalous transport in the vicinity of that transition. Experimentally, synthetic quantum systems, which are well isolated from an external thermal reservoir, provide natural platforms for realizing the MBL phase. Recent experiments with ultracold atoms, trapped ions, superconducting qubits, and quantum materials, in which different signatures of many-body localization have been observed, are reviewed. This Colloquium concludes by listing outstanding challenges and promising future research directions.

CONTENTS

I. INTRODUCTION

Dramatic experimental progress of the last few years has enabled the realization of quantum many-body systems that are well isolated from the environment and therefore evolve under their intrinsic quantum dynamics. Examples of systems that offer a large degree of control include ultracold atoms in optical lattices (Bloch, Dalibard, and Zwerger, 2008), trapped ions (Blatt and Roos, 2012), as well as nuclear and electron spins associated with impurity atoms in diamond (Doherty *et al.*, 2013; Schirhagl *et al.*, 2014). The tunability and long coherence times of these systems, along with the ability to prepare highly nonequilibrium states, enable one to probe quantum dynamics and thermalization in closed systems. What are the possible regimes of quantum-coherent many-body dynamics? How does classical hydrodynamic transport, seen at long times when a system thermalizes, emerge from the unitary quantum evolution? Under what conditions does a system fail to thermalize, thus evading the conventional classical fate even at long times? In contrast to a majority of experiments in solid state systems, these questions pertain to highly nonequilibrium states of matter with nonzero energy density that could translate to high and even infinite effective temperature. Can quantum effects survive at long times in many-body systems at such high-energy densities? Answering these basic questions is a necessary step toward understanding a potentially rich variety of new states of matter that can appear in highly nonequilibrium quantum systems.

The most common class of dynamics leads to thermalization in *ergodic*[1] systems, different degrees of freedom exchange energy, and information. At long times, the system effectively reaches thermal equilibrium, even though as a whole it remains in a pure quantum state. Intuitively, in ergodic dynamics the system as a whole acts as a thermal reservoir for its subsystems, provided those are small enough. Stationary states in such systems are described by quantum statistical mechanics (Deutsch, 1991; Srednicki, 1994).

The approach to equilibrium is illustrated in Fig. 1 for a particular setup, known as a quantum quench, in which a system described by the Hamiltonian $\hat{H}$ is prepared in a nonequilibrium state $|\psi(0)\rangle$, e.g., characterized by a nonuniform density of particles. Under unitary evolution $e^{-i\hat{H}t}$, at sufficiently long times, the state $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$ of an ergodic system will have local observables which appear thermal. Information encoded in the initial state is effectively erased in the course of time evolution as it is transferred to highly nonlocal inaccessible correlations. First to be washed away are the quantum correlations in the initial state, while the last to disappear are the inhomogeneities of conserved densities that are transported by slow diffusion modes. Ultimately the local physical observables will be determined just by the values of the few global conserved quantities, total energy, particle number, etc.

---

[1]We note that in the context of quantum many-body systems the term ergodicity is defined somewhat differently compared to classical mechanics. Our use of this term is synonymous with thermalization, as discussed in Sec. II.A.
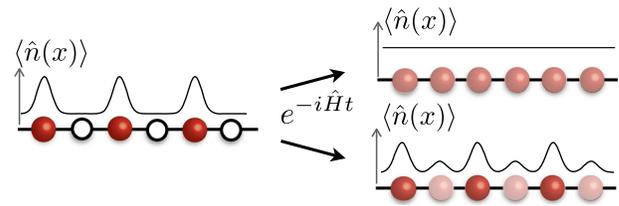

FIG. 1. In a quantum quench, interacting particles on a lattice are, e.g., initially prepared in a state with nonuniform density. Following unitary quantum dynamics, the thermalizing system relaxes toward the state where all lattice sites are equally populated and the density profile is uniform (shown at the top). In contrast, a nonthermalizing many-body localized system retains the memory of initial state even at infinite time (bottom).

The mechanism of thermalization and the approach to thermal equilibrium in different systems are issues of central importance in statistical mechanics. While there exist different regimes of thermalization (e.g., it can be parametrically slow), it is of particular interest to find systems which avoid thermalization. In this case, quantum information encoded in the initial state can persist and govern the dynamics at long times as well as the steady state. Thus, ergodicity-breaking systems can allow for new forms of stable quantum phases and phase transitions that are unique to the nonequilibrium settings. Moreover, understanding ergodicity-breaking mechanisms could provide new insight into the workings of thermalization.

Thermalization requires that different parts of ergodic systems exchange energy and particles, and consequently thermal systems must be conducting. Therefore, a natural way to break ergodicity is to find systems which are insulating. One familiar and well-studied example of insulating behavior is Anderson localization in noninteracting disordered systems (Anderson, 1958; Abrahams, 2010). The essence of Anderson localization is that a disorder potential can completely change the nature of single-particle eigenstates in a crystal: instead of propagating Bloch states, which are similar to plane waves [Fig. 2(a)], wave functions become localized in some region of space and decay exponentially far away from that region [Fig. 2(b)]. The origin of localization can be most easily understood in the limit of strong disorder, in which the variance of the random potential W is much larger than the tunneling between neighboring sites of the lattice t. In that limit, resonant transitions between typical neighboring sites are impossible. The same holds for transitions between sites separated by long distances: indeed, tunneling processes between two sites at a distance of $\sim n$ lattice sites apart typically occur in the nth level in perturbation theory and are therefore suppressed as $t_n \sim (t/W)^n$. In contrast, the typical energy mismatch of two sites δ_n will decay only algebraically with distance n, $\delta_n \sim W/n^d$, where d is the number of spatial dimensions. This simple argument intuitively explains why long-range hopping processes remain off resonant, and the wave functions are truly localized in the strong-disorder limit. Anderson localization thereby leads to the absence of diffusion, suppressing transport.

After almost 60 years following Anderson's original paper (Anderson, 1958), much progress has been achieved
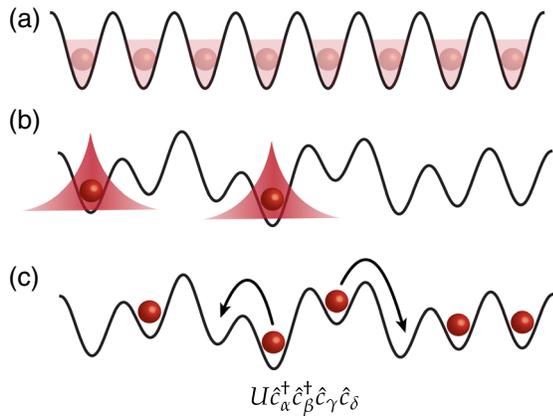
FIG. 2. (a) In a clean crystal, eigenstates are Bloch waves, which extend throughout the sample. (b) The essence of Anderson localization of noninteracting particles is that for sufficiently strong disorder there is a vanishing probability for a particle to make a resonant transition from one site to another one spatially separated from it. This leads to eigenstates which are localized in some region of space, decaying exponentially away from it. (c) Adding interactions to an Anderson localized system. To first order, the effect of interaction is to induce hopping of pairs of particles between the single-particle localized orbitals. One may ask if the localized phase, with vanishing particle and thermal conductivities, is robust to this process.

in understanding single-particle localization. Some prominent developments include the scaling theory of localization (Abrahams *et al.*, 1979), multifractality of the critical wave functions at the metal-insulator transition (Evers and Mirlin, 2008), as well as understanding intricate effects of symmetries (e.g., time reversal) on localization [see, e.g., Abrahams (2010) for a review]. Since an Anderson insulator is noninteracting, it is not clear if it is a true phase of matter, and a key challenge envisioned in a pioneering work of Anderson, which remained open for several decades, was to understand the fate of localization in the presence of interactions between particles. The new, interaction-induced processes which may potentially destroy localization are illustrated in Fig. 2(c).

The interplay of interaction and disorder was addressed in early ground-breaking work: Fleishman and Anderson (1980) provided qualitative arguments in favor of localization in the presence of weak short-ranged interactions. Renormalization group approaches have been used to generalize the notion of the Anderson insulator to describe the quantum ground states of interacting systems. In this context Finkelstein (1983) extended the scaling theory of localization to account for the interplay of interactions and disorder. Giamarchi and Schulz (1988) developed a controlled renormalization group approach to describe zero-temperature properties of disordered, interacting 1D quantum liquids.

More recently, the existence of the localized phase at *nonzero temperatures*, as a dynamical phase of matter, was put on a firm footing. The possibility of localization in an interacting setting was established for a zero-dimensional case of a quantum dot (Altshuler *et al.*, 1997), and in higher-dimensional systems with local interactions (Gornyi, Mirlin,

and Polyakov, 2005; Basko, Aleiner, and Altshuler, 2006). Such a perfect interacting insulator at nonzero temperature is said to be *many-body localized* (MBL). Many-body localization represents a robust dynamical phase of matter because it is stable within a range of interaction and other Hamiltonian parameters.

We emphasize that the question whether a given interacting system is MBL is fundamentally different from the issue of the Anderson localization of its ground state. In order to establish MBL, one has to consider states with a finite density of excitations above the ground state, or, equivalently, states with a finite energy density, and show that they remain localized. In contrast, zero-temperature localization requires only the localization of a finite number of excitations in the whole system, corresponding to a vanishing energy density as the system size is taken to infinity.

From the fundamental theoretical perspective, MBL provides the only known *robust* mechanism to avoid thermalization in a closed system. Other examples of systems that do not thermalize are noninteracting systems and Yang-Baxter integrable quantum models in one spatial dimension, where any multiparticle interaction process can be reduced to two-particle collisions (Sutherland, 2004). Unlike MBL, these are not robust with respect to small perturbations: generally adding even weak interactions or changing the form of the Hamiltonian leads to thermalization [see D'Alessio *et al.* (2016) and references therein]. Thus, such models do not describe stable phases of matter.

Recently, the phenomenon of MBL was investigated extensively, in both theory and experiment. This led to many exciting developments and new research directions. Much of this progress, on the theory side, was fueled by applying quantum information concepts, such as quantum entanglement, to describe the miscroscopic structure of MBL eigenstates and dynamics in those systems. Theoretical advances have largely been guided by the new experimental capabilities, which shifted the focus from traditional condensed matter setups (e.g., linear-response measurements of conductivity) to setups, which are naturally realized in isolated synthetic quantum systems (quantum quench experiments, Fig. 1 being one of the main examples). The goal of this Colloquium is to review the recent progress and current status of MBL in an accessible manner.

We start Sec. II with a brief review of thermalization in quantum models. Afterward, we introduce the notion of a many-body localized phase and survey its early studies. In Sec. II.C we outline the phenomenological theory of the MBL phase. The key insight of this theory is that MBL systems exhibit a new kind of integrability: they are characterized by the emergence of an extensive set of quasilocal integrals of motion (LIOMs). The emergent integrability strongly constrains the system's dynamics and thus provides an intuitive explanation of why it fails to thermalize. We relate the entanglement structure of MBL eigenstates to the emergent integrability. Finally, we discuss the robustness of the emergent integrability of the MBL phase, which distinguishes it from other integrable systems.

The remainder of Sec. II is devoted to exploring properties of the MBL phase. Section II.D discusses dynamical properties of the MBL phase that follow from a LIOM description.

We explain the logarithmic growth of entanglement entropy in time—a property which is taken as one of the key characteristics of the MBL phase. For the local observables, the entanglement spreading implies their equilibration to highly nonthermal values at long times set by initial conditions. Furthermore, we discuss the effect of dissipation on the dynamics of MBL systems. Finally, Sec. II.E discusses new efficient algorithms to obtain highly excited MBL eigenstates that are possible due to the simple entanglement structure of MBL eigenstates.

In Sec. III we discuss the new phenomena made possible by the fact that MBL phases avoid thermalization and are not described by statistical mechanics. This allows for localization-protected quantum orders in eigenstates, e.g., infinite temperature breaking of discrete symmetry in one-dimensional systems, which would be otherwise prohibited by statistical mechanics. In a different direction, we discuss that MBL is possible in systems with parameters that periodically depend on time (periodically driven or Floquet systems), hence preventing energy absorption and equilibration to the infinite temperature states. This makes MBL an essential ingredient that can provide the thermodynamic stability of new driven phases, such as time crystals and anomalous Floquet insulators. Such phases provide examples of new states which are possible only out of equilibrium.

Section IV summarizes the present understanding of the MBL delocalization transition. This is a novel kind of dynamical phase transition between MBL and ergodic phases. From the MBL phase, the transition can be visualized as a proliferation of resonances as one increases interaction strength or decreases disorder. On the other hand, when approaching the transition from the ergodic phase, Griffiths effects which create bottlenecks in the transport become progressively important, especially in one dimension. We discuss the basic setup and predictions of the existing renormalization group approaches.

Section V summarizes recent experimental developments in studies of MBL. To be in the MBL phase at nonvanishing temperature, the system must be isolated from any external heat bath. In disordered solids, unavoidable coupling to a bath of delocalized phonons ultimately destroys the localized state of the electrons, leading to slow transport by variable-range hopping. However, systems of ultracold atoms are phonon free and thus allow for a better control of residual couplings to the environment. Thereby they offer a laboratory to observe and systematically study many-body localization and thermalization phenomena. More recently, trapped ions, superconducting qubits, and spins of NV centers in diamond have also emerged as promising systems where thermalization can be studied, and new nonequilibrium phases of matter can be realized.

Finally, Sec. VI concludes this Colloquium by presenting a broader perspective on the ongoing research efforts aimed to understand the quantum nonergodic behaviors. We outline some open questions and discuss future directions and possible synergies between research on MBL systems and other fields.

II. THE MANY-BODY LOCALIZED PHASE

A. Thermalization in quantum systems

We start by discussing thermalization in isolated quantum systems. In particular, we review the eigenstate thermalization hypothesis (ETH), which explains the microscopic mechanism of thermalization in isolated quantum systems. We further discuss its implications for the entanglement properties of eigenstates. Since the main focus of this review is on MBL, our discussion of thermalization is brief; a more complete overview can be found in the original papers (Deutsch, 1991; Srednicki, 1994, 1999) and reviews (Nandkishore and Huse, 2015; D'Alessio *et al.*, 2016).

First let us recall that thermalization and more generally the statistical mechanics description of classical systems are based on the powerful ergodicity hypothesis, which states that over a long period of time all microstates of the system are accessed with equal probability. Directly translating this definition of ergodicity to quantum systems is problematic, since quantum mechanics operates in Hilbert space where dynamics is unitary and one cannot track a trajectory in the phase space.

To see this, let us consider an isolated quantum many-body system with a Hamiltonian $\hat{H}$. While the discussion applies to general local lattice Hamiltonians (and can be further extended to continuum models), as a concrete example the reader may keep in mind an interacting chain sketched in Fig. 1. The generic initial nonequilibrium state $|\psi(0)\rangle$ can be expanded over the basis of many-body eigenstates $|\alpha\rangle$ as $|\psi(0)\rangle = \sum_\alpha A_\alpha |\alpha\rangle$. Over the course of the quantum evolution, each coefficient A_α acquires a phase factor determined by the corresponding eigenenergy E_α,

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle = \sum_\alpha A_\alpha e^{-iE_\alpha t}|\alpha\rangle. \qquad (1)$$

The probability of finding the system in a given eigenstate $|\alpha\rangle$, $p_\alpha = |A_\alpha|^2$, is set by the choice of the initial state and does not change over time. This is unlike classical systems, which during their evolution explore different states in phase space. Thus, we need to modify the notion of ergodicity in the quantum case.

Intuitively, thermalization in an isolated system means that starting from a physical initial state[2] the system's observables reach values given by the microcanonical (and Gibbs) ensembles at sufficiently long times. The infinite-time average of a physical observable described by an operator $\hat{O}$ (which is typically a linear combination of few-body operators) can be found from Eq. (1):

$$\langle \hat{O} \rangle_\infty = \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \psi(t)|\hat{O}|\psi(t)\rangle dt = \sum_\alpha p_\alpha \langle \alpha|\hat{O}|\alpha\rangle. \qquad (2)$$

---

[2]By physical we mean, e.g., product states, extensive superposition of many eigenstates, or any other states that can be experimentally prepared. In contrast, an individual eigenstate of a generic many-body system is inaccessible, as its preparation requires time which is exponentially long in a system's size.

Thus, $\langle \hat{O} \rangle_\infty$ is encoded in the probabilities p_α along with the expectation values of the observable $\langle \alpha | \hat{O} | \alpha \rangle$, because the terms that involve off-diagonal matrix elements of $\hat{O}$ oscillate at different frequencies and therefore average out. Since p_α are fixed by the initial state, the natural way to ensure that an observable $\hat{O}$ reaches a thermal expectation value at long times for generic initial states is to assume that the expectation values in individual eigenstates $\langle \alpha | \hat{O} | \alpha \rangle$ agree with the microcanonical ensemble.

Such an explanation of thermalization using properties of individual eigenstates, proposed by Deutsch (1991) and Srednicki (1994), is known as the ETH. More precisely, the ETH states that in ergodic systems individual many-body eigenstates have thermal observables, identical to microcanonical ensemble values at energy $E = E_\alpha$, $\langle \alpha | \hat{O} | \alpha \rangle \approx \mathcal{O}_{\text{mc}}(E)$. Thus, even if the entire system is prepared in an eigenstate, its subsystems experience the remainder as an effective heat bath and explore possible configurations, restricted only by global conservation laws (e.g., energy). In this sense the ETH mechanism of thermalization implies ergodicity, so in what follows we use both notions interchangeably. ETH has been extensively tested in numerical simulations of small quantum systems (D'Alessio *et al.*, 2016). While all known examples of thermalizing systems obey ETH, at present it is not clear if ETH is a necessary condition for thermalization.

The ETH, as previously formulated, implies thermalization at infinitely long times. More specifically, since for physical initial states the probabilities p_α are concentrated around a certain energy, from Eq. (2) one can show that $\langle \hat{O} \rangle_\infty \approx \mathcal{O}_{\text{mc}}$. However, in order to describe the approach to the equilibrium values and bound the temporal fluctuations, one needs further information about off-diagonal matrix elements. Srednicki (1999) introduced the following ansatz for both diagonal and off-diagonal matrix elements of local operators $\hat{O}$ in the basis of eigenstates:

$$\langle \alpha | \hat{O} | \beta \rangle = \mathcal{O}_{\text{mc}}(\bar{E}) \delta_{\alpha\beta} + e^{-S_{\text{th}}^{\bar{E}}/2} R_{\alpha\beta} f(\omega, \bar{E}), \qquad (3)$$

where $\bar{E} = (E_\alpha + E_\beta)/2$ denotes the average eigenenergy, and $\omega = E_\alpha - E_\beta$ is the energy difference. $S_{\text{th}}^{\bar{E}}$ is the thermodynamic entropy, and $R_{\alpha\beta}$ is a normal-distributed random number. The expectation value of the local observable and the spectral function, denoted as $\mathcal{O}_{\text{mc}}(\bar{E})$ and $f(\omega, \bar{E})$, respectively, are smooth functions of ω and $\bar{E}$. Srednicki (1999) demonstrated that such an ansatz Eq. (3) is sufficient to ensure thermalization; it remains an open question whether this is also a necessary condition (D'Alessio *et al.*, 2016).

ETH has direct implications for the structure, and, in particular, for the entanglement properties of ergodic eigenstates. For an eigenstate $|\alpha\rangle$ obeying ETH, all observables within a sufficiently small subsystem A will have thermal expectation values. This implies that the reduced density matrix of this subsystem $\rho_A = \text{tr}_B |\alpha\rangle\langle\alpha|$ (here B is the complement of A) is thermal. Therefore, the entanglement entropy of A in state $|\alpha\rangle$, which is defined as the von Neumann entropy of ρ_A, is equal to the thermodynamic entropy

$$S_{\text{ent}}(A) = -\text{tr}\rho_A \log \rho_A = S_{\text{th}}(A). \qquad (4)$$

Since thermodynamic entropy is extensive, this implies that for highly excited eigenstates $|\alpha\rangle$ the entanglement entropy obeys the "volume law," scaling proportionally to the volume of the subsystem $S_{\text{ent}}(A) \propto \text{vol}(A)$. As we will see, entanglement properties of MBL eigenstates are dramatically different.

The matrix element ansatz Eq. (3) also implies the strong sensitivity of ergodic eigenstates to external perturbations of the Hamiltonian. Let us perturb the Hamiltonian by adding a small term $\epsilon \hat{O}$ to it and ask how this modifies the eigenstates. For a typical operator $\hat{O}$, the function $f(\omega, \bar{E})$ exhibits an algebraic decay for $\omega \lesssim J$, where J is the characteristic energy scale of $\hat{H}$ (e.g., hopping and local interaction strength). This decay saturates at the Thouless energy (D'Alessio *et al.*, 2016). Then Eq. (3) implies that the off-diagonal matrix element is exponentially larger than the many-body level spacing, which scales as $\Delta \sim J e^{-S_{\text{th}}^{\bar{E}}}$. Therefore, a small local perturbation of Hamiltonian, $\epsilon \hat{O}$ with $\epsilon \ll 1$, generally has a nonlocal effect in Hilbert space, mixing an exponentially large number of original eigenstates (since $S_{\text{th}}^{\bar{E}}$ is proportional to the system's volume). Thus, the new eigenstates in the presence of a small perturbation are very different from the original ones, reflecting a chaotic character of ergodic quantum systems.

Finally, the sensitivity of ETH eigenstates to external perturbations implies level repulsion. The statistics of many-body level spacings $s_\alpha = E_{\alpha+1} - E_\alpha$, where $\{E_\alpha\}$ is the ordered set of eigenenergies, was previously established to be an indicator of quantum chaos for few-body systems (Wigner, 1951), e.g., stadium billiards [see, e.g., D'Alessio *et al.* (2016) and references therein]. In particular, zero-dimensional quantum systems whose classical counterpart exhibits chaotic dynamics display level repulsion. The level spacings in such systems obey Wigner-Dyson statistics, where the probability density $p(s)$ vanishes as a power law s^β, as $s \to 0$ with $\beta = 1$, 2, and 4 depending on the symmetry class of the model. Wigner-Dyson statistics was also found in thermalizing many-body lattice models (D'Alessio *et al.*, 2016). In contrast, if the system has an extensive number of integrals of motion (as is the case for Bethe-ansatz integrable models), the eigenenergies that belong to different sectors behave as independent random variables. Hence, in such systems the distribution of level spacings is Poisson, and $p(0) = 1$.

As we will demonstrate, in the MBL phase the eigenstates break ETH and display very different properties. In particular, entanglement scaling obeys area law (in contrast to the volume law for ergodic systems), the effect of local perturbations remains local, and the eigenspectrum has Poisson level statistics.

B. Escaping thermalization by disorder

What are the possible routes of escaping thermalization? As discussed in the Introduction, the absence of transport would be sufficient, and therefore, Anderson localization in single-particle disordered systems provides a natural starting point to look for nonthermalizing systems. In fact, Anderson himself pointed out this possibility in 1958, stating that a localized

system provides "an example of a real physical system with an infinite number of degrees of freedom, having no obvious oversimplification, in which the approach to equilibrium is simply impossible" (Anderson, 1958).

For noninteracting particles on a lattice there exist two different possibilities of localization, depending on the dimensionality and form of disorder. In low spatial dimensions ($d = 1, 2$) and for random uncorrelated disorder, all single-particle states can be localized for arbitrarily weak disorder. In spatial dimensions $d \geq 3$, however, systems may exhibit a metal-insulator transition as a function of disorder strength. Such a transition is manifested by the appearance of a single-particle mobility edge in the energy spectrum, which separates localized states at low energy from extended states (Abrahams, 2010). Single-particle mobility edges can also exist in lower spatial dimensions with quasiperiodic rather than random potential (Boers *et al.*, 2007; Li, Li, and Das Sarma, 2017; Lüschen *et al.*, 2018); see Sec. V.

A fully localized Anderson insulator (without the mobility edge) of noninteracting particles is not able to transport energy or charge and has zero conductivity at any temperature. However, in realistic systems, interactions between particles are inevitable, and therefore, to claim victory over thermalization, their effect on localization must be investigated. Interactions may open up new transport channels: in particular, a particle can decay from a high-energy (single-particle) localized state by producing an avalanche of excitations at lower energies, potentially restoring transport. Fleishman and Anderson (1980) related such delocalization to a nonvanishing broadening of single-particle levels by interactions. Within second order perturbation theory they showed that such broadening vanishes when disorder is sufficiently strong, suggesting the stability of localization for short-range interacting systems.

Later, Altshuler *et al.* (1997) examined interaction-induced decay of an excitation in a zero-dimensional quantum dot beyond perturbation theory. In this work, the process where a particle decays into a shower of particle-hole excitations was formulated as a single-particle Anderson localization problem in a Fock space. Utilizing an approximate mapping to localization on a graph without loops, Altshuler *et al.* (1997) showed that the states below a certain energy remain localized. Despite the presence of interactions, these localized states remain close to a noninteracting single-particle excitation with a perturbatively small admixture of a few particle-hole excitations.

Building on these results, Gornyi, Mirlin, and Polyakov (2005) and Basko, Aleiner, and Altshuler (2006) analyzed the stability of the Anderson insulator with respect to short-ranged interactions. As a starting point, they considered a model in which all single-particle states are localized, with a typical localization length ζ. Interactions are characterized by a dimensionless parameter λ, given by the ratio of the two-particle transition matrix element to the level spacing δ_ζ of excitations in a "localization volume" of size ζ. Basko, Aleiner, and Altshuler (2006) calculated the broadening of a single-particle level using the self-consistent Born approximation. This approach captures the subset of decay processes where a particle at each step decays into a maximal possible

number of excitations, hence maximizing the available phase space. They argued that the problem of level broadening is reminiscent of the Anderson localization problem on a tree with connectivity K that depends on the temperature T: $K \sim T/\delta_\zeta$, which, intuitively, stems from the fact that at higher temperatures the available phase space for decay processes grows. Using an analogy with localization on a Cayley tree (Abou-Chacra, Thouless, and Anderson, 1973), Basko, Aleiner, and Altshuler (2006) estimated the critical temperature below which the interacting model is localized as $K_c \approx T_c/\delta_\zeta \approx 1/(\lambda \ln |\lambda|)$. The vanishing probability to have a nonzero decay rate of a single-particle level below a critical temperature $T \leq T_c$ serves as a criterion of the stability of a localized phase for a finite range of interaction strength in arbitrary spatial dimensions. The resulting perfect insulator at nonvanishing temperature is termed a many-body localized phase.

At high-energy densities $T > T_c$, the phase space for allowed transitions increases and the many-body eigenstates in the model of Basko, Aleiner, and Altshuler (2007) become delocalized. The transition between localized and delocalized many-body eigenstates happens at a finite energy density, which was named a "many-body mobility edge" (Basko, Aleiner, and Altshuler, 2007). The perfectly insulating behavior, however, is difficult to observe in conventional solid state systems, since phonons are protected from localization and can act as a heat bath, giving rise to slow, variable-range hopping transport of localized electrons.

An important step, which opened the door to investigating the properties of the MBL phase in numerical simulations, was taken by Oganesyan and Huse (2007). They pointed out that disordered lattice models with a finite dimension of the local Hilbert space can remain in the MBL phase even at infinite temperature. As a specific model, they studied a 1D chain of spinless fermions with an on-site disorder, nearest-neighbor interactions, and hopping between nearest-neighbor and next-nearest-neighbor sites. Subsequent studies concentrated on a simpler model without longer-range hopping,

$$\hat{H} = t \sum_i (\hat{c}_i^\dagger \hat{c}_{i+1} + \text{H.c.}) + V \sum_i \hat{n}_i \hat{n}_{i+1} + \sum_i \epsilon_i \hat{n}_i, \quad (5)$$

where $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$ is the density operator on site i, and $\epsilon_i \in [-W; W]$ is the disorder potential distribution. Using numerical exact diagonalization of finite-size lattice systems, Oganesyan and Huse (2007) demonstrated signatures of an MBL phase in fermionic spin chains.

Furthermore, Znidarič, Prosen, and Prelovsek (2008) and subsequently Pal and Huse (2010) presented extensive numerical studies of a disordered Heisenberg spin chain, defined by the Hamiltonian

$$\hat{H}_{XXZ} = \frac{J_\perp}{2} \sum_{i=1}^{L} (\hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y) + \sum_{i=1}^{L} \left(\frac{J_z}{2} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + h_i^z \hat{\sigma}_i^z \right),$$

$$(6)$$

where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ is a vector of three spin-1/2 Pauli operators, and h_i^z are randomly distributed on-site magnetic
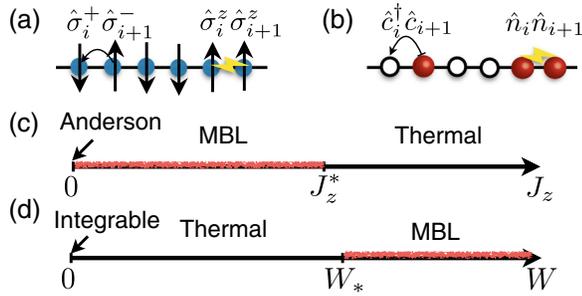
FIG. 3. Sketch of the (a) Heisenberg spin chain and (b) spinless fermions in one dimension, which are used as generic models for the MBL phase. Bottom panels show the phase diagram of the spin chain as a function of (c) interaction and (d) disorder strength.

fields $h_i^z \in [-W, W]$ (thus, W is the disorder strength). As illustrated in Fig. 3, this random-field *XXZ* spin chain can be mapped onto the chain of spinless fermions in Eq. (5), using a Jordan-Wigner transformation (Jordan and Wigner, 1928). In this mapping, $J_\perp$ and J_z terms are transformed into fermion hopping and nearest-neighbor interaction, respectively, while the $h_i^z \hat\sigma_i^z$ term is transformed into random on-site energies.

In the limit $J_z \to 0$, the *XXZ* spin chain is equivalent to free fermions moving in a disorder potential, and therefore, all states are Anderson localized, for arbitrary values of disorder strength W. For a fixed and not too large W (i.e., $W/J_\perp \sim 1$) increasing interactions J_z above some threshold value, $|J_z| > J_z^*(W)$, may lead to delocalization. The corresponding schematic phase diagram is sketched in Fig. 3(c), where we do not illustrate a possible recurrent MBL phase for very strong $J_z \gg J_\perp$. Similarly, at fixed interaction strength there is a critical value W_* above which the many-body states become fully localized; see Fig. 3(d). The numerical results (Luitz, Laflorencie, and Alet, 2015; Serbyn, Papić, and Abanin, 2015) also indicate that for disorder $W < W_*$, eigenstates in the middle of the band become delocalized leading to a many-body mobility edge. Thus, already in one dimension there is a transition between a localized and a thermal phase—a property which distinguishes MBL from single-particle Anderson localization.

In the localized regime Pal and Huse (2010) observed a breakdown of ETH through a number of metrics. In particular, the spin expectation value $\langle \alpha | \hat\sigma_i^z | \alpha \rangle$ in eigenstates was found to fluctuate widely between adjacent many-body eigenstates, in contradiction to the ansatz Eq. (3), requiring local observables to be smooth functions of energy. To locate the critical disorder strength for which the system enters an MBL phase, Oganesyan and Huse (2007) and Pal and Huse (2010) used the average ratio of adjacent level spacings as a diagnostic probe of the level statistics. Pal and Huse (2010) confirmed that for weak disorder there is level repulsion and the level statistics is of Wigner-Dyson form, as expected for a thermalizing system; see Sec. II.A. In contrast, for strong disorder the level repulsion disappears and the level statistics approaches a Poisson distribution, which, as we will explain in the next section, is a consequence of a new form of emergent integrability present in the MBL phase. For $J_z = J_\perp = 1$ the crossover point between the two behaviors $W_* \approx 3.5$ was

identified as the location of the transition between MBL and ergodic phases.

The introduction of such microscopic lattice models enabled investigations of highly nonequilibrium dynamics of the MBL phase, as opposed to traditional computations of Ohmic conductivity. Znidarič, Prosen, and Prelovsek (2008) and Bardarson, Pollmann, and Moore (2012) numerically studied the behavior of a model of Eq. (5) in an MBL phase in a quantum quench experiment, similar to the one shown in Fig. 1. First, they numerically observed that starting from initial product states, while there was no transport, entanglement between two parts of the spin chain kept growing logarithmically in time. Such a growth was absent in the Anderson insulating phase. The second surprising observation was that at very long times, the entanglement entropy saturated at values which were proportional to the system size, albeit smaller than the expected thermal value of the entropy at the same energy density. This result demonstrated that quantum information spreads throughout the entire system and indicated that the MBL phase has qualitatively different properties compared to the noninteracting Anderson insulator.

Next we introduce an effective theory of the MBL phase based on local integrals of motion, which provides a unified description of most known properties of the MBL phase, such as absence of transport, integrability, logarithmic dynamics of entanglement, as well as the breakdown of ETH. In addition, we discuss other predictions of this theory, in particular, the dynamics of local observables in a quantum quench experiment.

C. Emergent integrability of the MBL phase

In this section we introduce a new kind of integrability, which characterizes the MBL phase. The construction builds on the entanglement structure of the eigenstates in this phase. We start with a heuristic argument, which exploits the intuitive definition of MBL (in the absence of a many-body mobility edge), as a phase where local perturbations have only local effects on the eigenstates. This intuition can be used to understand the entanglement structure of eigenstates and to infer the existence of local integrals of motion. In addition, we discuss alternate viewpoints on how the local integrals of motion emerge. Finally, we demonstrate how integrability explains the breakdown of thermalization and contrast MBL systems to other examples of integrable systems.

1. Area-law entanglement in MBL eigenstates

MBL eigenstates display a low amount of entanglement, obeying the so-called area law: that is, the entanglement entropy of a subsystem A in an MBL egeinstate scales proportional to the volume of the boundary ∂A of A, as both the size of the system and the size of A are taken to infinity $S_{\text{ent}}(A) \propto \text{vol}(\partial A)$. Area-law entanglement scaling is typical of ground states in gapped systems (Verstraete, Murg, and Cirac, 2008; Eisert, Cramer, and Plenio, 2010). However, as we explain, in MBL systems even highly excited states obey area-law scaling, in contrast to thermal eigenstates which have volume-law entanglement.

The low entanglement of MBL eigenstates can be intuitively inferred from the following thought experiment (Serbyn,
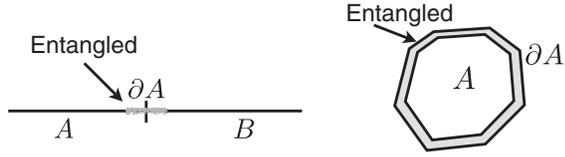
FIG. 4. Illustration of the area-law entanglement entropy in one and two spatial dimensions where only the shaded boundary regions that include $\propto \mathrm{vol}(\partial A)$ degrees of freedom contribute to the entanglement. In contrast, for systems with volume-law entanglement, extensively many degrees of freedom $\propto \mathrm{vol}(A)$ are entangled with the exterior region.

Papić, and Abanin, 2013a). Let us consider an MBL system with a local Hamiltonian $\hat{H}$ and specify a region A (e.g., a block of adjacent spins in a one-dimensional spin chain, in which case ∂A is just two end points of the block). We divide the Hamiltonian into three parts: $\hat{H}_A$, which contains the terms acting only on spins in A, $\hat{H}_B$ acting only on spins in the complement of A, and terms $\hat{V}_{AB}$ which couple spins in A, B along the boundary ∂A. Let us turn off the couplings along the boundary of region A. Then the eigenstates are simply tensor product states of eigenstates $|\alpha\rangle_A$, $|\beta\rangle_B$ of $\hat{H}_A$ and $\hat{H}_B$:

$$|I\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B. \qquad (7)$$

These states have zero entanglement entropy for region A. Now let us turn on the coupling $\hat{V}_{AB}$, which acts locally near the boundary. Since the system is in the MBL phase, introducing a local perturbation will only significantly affect degrees of freedom situated within the localization length ξ from the boundary.[3] Thus, we expect that the new eigenstates can be obtained from the states $|I\rangle_{AB}$ by entangling spins in A and B over a distance $\sim\xi$ away from the boundary ∂A. The effect of introducing a local perturbation on spins far away from the boundary is expected to decay exponentially with the distance leading to an area-law scaling of entanglement entropy $S_{\mathrm{ent}} \propto \mathrm{vol}(\partial A)$ (see Fig. 4). The area-law entanglement scaling of MBL eigenstates, suggested by this argument, was demonstrated numerically by Bauer and Nayak (2013) and Serbyn, Papić, and Abanin (2013a). As discussed in Sec. II.D, despite the area-law entanglement of eigenstates, the dynamics of the MBL phase after a global quench leads to a volume-law saturation value of entanglement (Serbyn, Papić, and Abanin, 2013a).

2. Quasilocal integrals of motion

The low entanglement of MBL eigenstates implies that they can be connected to product states by a sequence of quasilocal unitary transformations (Serbyn, Papić, and Abanin, 2013a) except for the case when MBL eigenstates exhibit topological order (see Sec. III.A). Such unitary transformations diagonalize the Hamiltonian in a given product state basis. Their quasilocal nature can be used to map physical degrees of freedom into quasilocal integrals of motion.
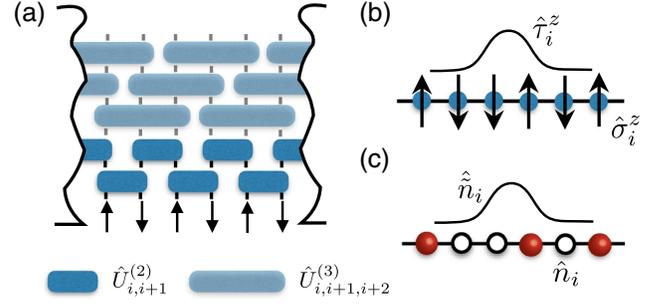

FIG. 5. (a) Rotation of the product states into the exact many-body eigenstates can be achieved by a sequence of quasilocal unitary transformations. (b), (c) The same quasilocal unitary transformation can be used to obtain the quasilocal operators $\hat{\tau}^z$ and $\hat{n}_i$ which commute with the Hamiltonian.

To make this intuition more precise, let us consider the disordered Heisenberg model of Eq. (6). In the limit $J_\perp \to 0$ the Hamiltonian $\hat{H}_0 = \sum h_i \hat{\sigma}_i^z + J_z \sum \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$ commutes with the $\hat{\sigma}_i^z$ operator on every site, and therefore the eigenstates are nonentangled product states, where each spin has a definite z projection:

$$|\{\sigma\}\rangle = |\sigma_1 \sigma_2 \cdots \sigma_N\rangle, \qquad \sigma_i = \uparrow, \downarrow. \qquad (8)$$

In total, we have 2^L eigenstates, where L is the number of spins, labeled by strings $\{\sigma\}$.

Now, let us turn on a weak flip-flop (kinetic) term $J_\perp$, such that the system remains in the MBL phase but the Hamiltonian is no longer diagonal in the $|\{\sigma\}\rangle$ basis. The argument for the area-law entanglement implies that the new eigenstates can be obtained from the product states Eq. (8) by a quasilocal unitary transformation. We say that $\hat{U}$ is quasilocal if it can be factored into a sequence of 2-site, 3-site, 4-site, …, unitary operators as $\hat{U} = \prod_i \cdots \hat{U}^{(3)}_{i,i+1,i+2} \hat{U}^{(2)}_{i,i+1}$ (see Fig. 5 for a schematic illustration). In this expansion, the long-range unitary operators have progressively decreasing rotation angles, so that $\|1 - \hat{U}^{(n)}_{i,i+1,\dots,i+n}\|_F^2 < e^{-n/\xi}$, where $\|\cdot\|_F$ is the Frobenius operator norm. In contrast, if the Hamiltonian describes a thermalizing phase, the operator $\hat{U}$ that diagonalizes it is highly nonlocal since it rotates the product states into states with volume-law entanglement.

The unitary operator $\hat{U}$ transforms the integrals of motion $\hat{\sigma}_i^z$ of $\hat{H}_0$ into the integrals of motion $\hat{\tau}_i^z = \hat{U}\hat{\sigma}_i^z \hat{U}^\dagger$ of $\hat{H}$. Because $\hat{U}$ is quasilocal in an MBL system, the $\hat{\tau}_i^z$ are typically close to the microscopic spin operators $\hat{\sigma}_i^z$ at least at strong disorder. Specifically, $\hat{\tau}_i^z$ can be expanded as

$$\hat{\tau}_i^z = Z\hat{\sigma}_i^z + \sum_{n=1}^{\infty} V_i^{(n)} \hat{O}_i^{(n)}, \qquad (9)$$

where $\hat{O}_i^{(n)}$ contains up to $(2n+1)$-body operators with contributions from sites at distance n from i (i.e., sites $i-n, \dots, i, \dots, i+n$ could contribute) and is normalized to $\|\hat{O}_i^{(n)}\|_F = 1$. Each $\hat{\tau}_i^z = \hat{U}\hat{\sigma}_i^z \hat{U}^\dagger$ has a finite overlap Z with the microscopic spin operator $\hat{\sigma}_i^z$. Moreover, the

---

[3] We will provide a more precise definition of the many-body localization length later.

coefficients of longer-range operators decay as $V_i^{(n)} \sim e^{-n/\xi}$, so the effect of operator $\hat{\tau}_i^z$ on spin j situated far away from site i is exponentially small. This locality is the key property which distinguishes the MBL phase from the thermal phase. The length scale ξ, which controls the locality of $\hat{\tau}_i^z$, or equivalently, the locality of the unitary $\hat{U}$ can be viewed as the localization length in the MBL phase.

The operators $\hat{\tau}_i^z$ form a complete set of independent quasilocal integrals of motion (LIOMs) (in the literature they are also sometimes called localized bits or l-bits). Eigenstates of $\hat{H}$ can be fully specified by labeling them with the eigenvalues of all $\hat{\tau}_i^z$. One can view each $\hat{\tau}_i^z$ operator as an emergent conserved pseudospin-like degree of freedom; it cannot decay during quantum evolution as long as an MBL system is not coupled to an external heat bath. In principle, one could define operators $\hat{\tau}_i^z$ for thermal systems, but in that case they would be highly nonlocal, have vanishing overlap with the microscopic spin operators and thus they would be of little use.

To form a complete basis of operators, we introduce operators $\hat{\tau}_i^{x,y} = \hat{U}\hat{\sigma}_i^{x,y}\hat{U}^\dagger$, which are also quasilocal. The operators $\hat{\tau}_i^{x,y,z}$ and their products form a complete basis in the operator space. Therefore, any physical operator described by $\hat{\sigma}_i^\alpha$ and their products can be decomposed in the τ basis.

The τ representation is particularly useful when analyzing the dynamics in the MBL phase as we will see. This is due to the simple form that the system's Hamiltonian $\hat{H}$ takes in the τ basis. Indeed, since $[\hat{\tau}_i^z, \hat{H}] = 0$, $\hat{H}$ cannot include any $\hat{\tau}_i^{x,y}$ operators. This results in the following general form (Serbyn, Papić, and Abanin, 2013a; Huse, Nandkishore, and Oganesyan, 2014):

$$\hat{H}_{\text{MBL}} = \sum_i \tilde{h}_i \hat{\tau}_i^z + \sum_{i>j} J_{ij} \hat{\tau}_i^z \hat{\tau}_j^z + \sum_{i>j>k} J_{ijk} \hat{\tau}_i^z \hat{\tau}_j^z \hat{\tau}_k^z + \cdots. \quad (10)$$

Here and in what follows we denote by $\hat{H}_{\text{MBL}}$ the Hamiltonian written in terms of LIOMs. Now since this representation results from the action of the quasilocal transformation $\hat{U}$ on the local Hamiltonian $\hat{H}$, the couplings

$$J_{ij} \propto J_0 e^{-|i-j|/\kappa}, \qquad J_{ijk} \propto J_0 e^{-|i-k|/\kappa}, \dots \quad (11)$$

decay exponentially with separation between the LIOMs. The above form of the Hamiltonian equation (10) is often viewed as the universal Hamiltonian of the MBL phase.

Assuming that the operators $\hat{O}_n^{(i)}$ entering Eq. (9) are superposition of Pauli strings with coefficients that follow a narrow distribution, the corresponding length scale κ can be shown to satisfy $\kappa^{-1} \geq (\xi^{-1} + \ln 2)/2$.[4] This implies that κ must remain finite even if ξ diverges at the MBL transition

---

[4] To derive this inequality, we equate $\hat{H}_{\text{MBL}}$ to the original Hamiltonian, e.g., Eq. (6) and substitute the expansion of τ_i^α via σ_i^α operators, Eq. (9). The extra term $\ln 2$ accounts for the exponentially large number of possible couplings that contribute to the operator $\hat{O}_n^{(i)}$ of a given range n, and the overall factor $1/2$ accounts for their random signs.

(see Sec. IV). We note that it would be interesting to test both the distribution of coefficients and the resulting bound, which so far remains a hypothesis.

It is instructive to consider the analogy between the effective MBL Hamiltonian Eq. (10) and Landau's Fermi-liquid theory of interacting fermion systems. Within the equivalent fermion description of the XXZ chain, Eq. (5), the operators $\hat{\sigma}_i^z$ are the site occupation numbers $\hat{n}_i$, while the LIOMs $\hat{\tau}_i^z$ map to "quasiparticle" occupation numbers $\hat{\tilde{n}}_i$. In both cases the effective theory can be written entirely in terms of the commuting integrals of motion. Furthermore, there is a nonvanishing overlap between the bare fermion or spin operators and the dressed operators (Bera *et al.*, 2015). The main difference between the Fermi liquid and the MBL phase is that in Fermi liquids the effective theory is valid only in the low-energy limit, while the MBL Hamiltonian provides an exact description at all energies. In the former case, quasi-particle operators are true integrals of motion only for wave vectors asymptotically close to the Fermi surface, while in the latter case the LIOMs are a complete set of commuting operators that fully specify all eigenstates.

The emergent integrability in the MBL phase naturally explains the Poisson level statistics, characteristic of integrable quantum many-body systems, observed in the early numerical studies (Oganesyan and Huse, 2007; Pal and Huse, 2010). It also explains the breakdown of ergodicity in dynamics because it implies that during its evolution, an MBL system retains the local memory of the initial states, encoded in the initial values of LIOMs. Moreover, in many cases the LIOMs have an overlap with conserved densities such as energy or particle number, which explains the absence of transport in the MBL phase. We note, however, that the existence of global conserved quantities is not essential for the MBL phase—for example, as discussed in Sec. III.B, MBL is possible in periodically driven systems where even energy is not conserved.

So far, we outlined the description of Serbyn, Papić, and Abanin (2013a) and Huse, Nandkishore, and Oganesyan (2014), which introduced the Hamiltonian in Eq. (10) as an effective model of the MBL phase using an intuitive definition of this phase and its entanglement properties. This is reminiscent of Landau's hypothesis of the Fermi-liquid Hamiltonian based on adiabatic continuity. However, in a modern perspective, the integrable Fermi-liquid Hamiltonian can be obtained systematically, as a renormalization group (RG) fixed point (Polchinski, 1992; Shankar, 1994). Similarly, the integrability of the MBL state as well as an approximate form of the LIOMs and the effective Hamiltonian were also obtained through a perturbative RG approach (Vosk and Altman, 2013; Pekker *et al.*, 2014).

The RG transformation can be formulated as a dynamical scheme that captures the time evolution by successively integrating out the fastest modes (Vosk and Altman, 2013). A related scheme, so-called real-space RG for excited states (RSRG-X), uses an approximate succession of local unitary transformations to construct eigenstates of the system at all energies (Pekker *et al.*, 2014). In the renormalization process one successively integrates out local degrees of freedom with the highest frequency scale in analogy to the strong disorder real-space RG scheme of Dasgupta and Ma (1980). A crucial

difference however is that in the standard scheme (Dasgupta and Ma, 1980) the eliminated degrees of freedom are always put into their lowest energy configuration, while in the RSRG-X one chooses either low-energy or the high-energy manifold of the local term. For instance, the spin with the magnetic field that is the largest scale in the Hamiltonian $h_z \hat{\sigma}_i^z$ can be put into either of configurations $\sigma^z = \pm 1$. These degrees are therefore not really eliminated, but rather become the emergent LIOMs $\hat{\tau}_i^z$. By choosing the ground state or excited state of the decimated pair of spins one assigns a given value $\tau_i^z = \pm 1$ to this integral of motion. Hence in this way one is able to obtain the entire spectrum of the many-body Hamiltonian. The universal Hamiltonian Eq. (10) emerges as a fixed point of such an RG flow. It should be noted that the RSRG-X method is only approximate, as it keeps track only of a certain subset of many-body processes and does not capture (rare) long-range resonances that are related to the absence of the adiabatic limit in the MBL phase.

The quasilocal nature of the unitary transformation $\hat{U}$ and therefore the existence of the MBL phase were subsequently proven by Imbrie (2016a, 2016b) for the strongly disordered Ising spin chain with transverse and longitudinal magnetic fields. In essence, the strategy is to perform a sequence of more and more nonlocal unitary transformations which gradually diagonalize the Hamiltonian. The proof relies on a reasonable assumption that limits the attraction between many-body eigenenergies and puts bounds on the probability to have long-range resonances which could potentially destroy the quasilocal structure of the unitary $\hat{U}$. The proof makes explicit use of the one-dimensional nature of the system, thus it does not apply to higher spatial dimensions.

Other perturbative approaches were used to obtain an approximate construction of LIOMs that would be valid also in higher dimensions. Ros, Müller, and Scardicchio (2015) constructed the LIOMs using a perturbative technique similar to the self-consistent Born approximation of Basko, Aleiner, and Altshuler (2006). Localization length and other properties of LIOMs constructed numerically were further studied by Chandran *et al.* (2015), O'Brien *et al.* (2016), Rademaker and Ortuño (2016), Pekker *et al.* (2017), and Thomson and Schiró (2018).

The description of the MBL phase in terms of LIOMs is expected to be valid throughout the phase. In Sec. II.D we explore the physical implications of the universal MBL Hamiltonian for dynamics in the MBL phase. Upon approaching the delocalization transition, the LIOM operators are expected to become increasingly nonlocal due to long-range resonances. Thus perturbative approaches fail to describe the delocalization transition as well as critical phenomena associated with the approach to this transition. We review various aspects of the delocalization transition in Sec. IV.

3. Comparison to other integrable systems

The existence of an extensive set of local conserved operators seemingly places MBL systems in the same category with other integrable models. However, as we discuss, the MBL integrability is conceptually different compared to other previously known kinds of integrability in noninteracting systems and in Yang-Baxter integrable systems (Sutherland, 2004).

First, the integrals of motion in the MBL phase are quasilocal operators, in contrast to the integrals of motion for Yang-Baxter integrable systems, which are extensive sums of local operators (Sutherland, 2004). Second, the emergent integrability of the MBL phase is robust: if an MBL Hamiltonian is perturbed by a weak, but finite perturbation, the system stays in the MBL phase, and therefore a deformed set of LIOMs can be defined. In contrast, if a noninteracting system of (delocalized) fermions, characterized by conserved occupations of single-particle eigenstates, is perturbed by introducing an arbitrarily weak two-body interaction, the integrability is immediately destroyed. A similar scenario is expected to hold for Yang-Baxter integrable systems (D'Alessio *et al.*, 2016). The robustness of LIOMs reflects the fact that MBL is a dynamical phase of matter, while noninteracting systems and Yang-Baxter integrability represents isolated points or lines in the phase space of possible Hamiltonians.

It is also instructive to draw a parallel between MBL and weakly perturbed integrable classical systems. For the latter, the powerful Kolmogorov-Arnold-Moser (KAM) theory establishes that weak integrability-breaking perturbations transform most periodic orbits into quasiperiodic ones (Vogtmann, Weinstein, and Arnol'd, 2013). An important assumption of the KAM theory is the incommensurability of frequencies, which ensures the absence of resonant processes between different degrees of freedom. Similarly, in MBL systems, the incommensurability of frequencies and energies arises naturally due to disorder. The stability of the MBL phase, where weak local perturbations deform but do not destroy LIOMs, may be viewed as the analog of the KAM theorem for quantum many-body systems. Moreover, the MBL phase is the only known example of a KAM-type integrable system that survives in the thermodynamic limit.

D. Dynamical properties of the MBL phase

The emergent integrability underlies ergodicity breaking in the MBL phase and strongly constrains the dynamics therein. In this section, we discuss largely universal dynamical properties that stem from the existence of LIOMs. In particular, we analyze the behavior of an isolated MBL system following a quantum quench and explain the origin of the logarithmic spreading of entanglement. Surprisingly, despite localization, physical observables (such as local magnetization) do equilibrate at long times, albeit to highly nonthermal values, which carry information about the initial state. This and other dynamical signatures, which have been predicted from the theory based on LIOMs, can be understood using the effective Hamiltonian Eq. (10) of the MBL phase.

To illustrate the origin of entanglement generation in a localized system, let us consider the evolution of an MBL spin chain starting from a low-entanglement state. In numerical simulations (Znidarič, Prosen, and Prelovsek, 2008; Bardarson, Pollmann, and Moore, 2012), the initial state was typically taken to be a product state. For simplicity, let us instead focus on a different state: a product state in the basis of LIOMs, in which each τ spin is pointing in some direction on the Bloch sphere:
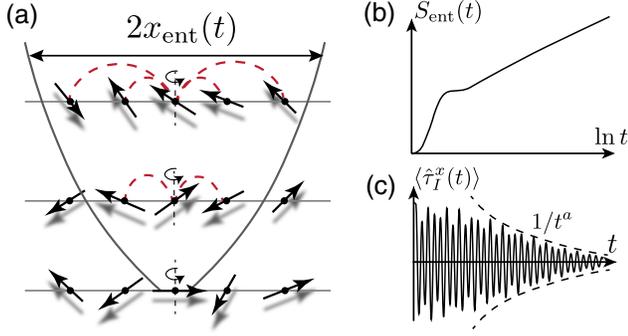
FIG. 6. Mechanism of dynamics in the MBL phase: (a) The central LIOM precesses with time, and interactions with other LIOMs lead to the dependence of its precession frequency on states of neighboring spins. This process is responsible for logarithmic growth of entanglement shown in (b) and also for the relaxation of fluctuations of $\langle \hat{\tau}^x(t) \rangle$ illustrated in (c).

$$|\psi_0\rangle = \otimes_i (A_i|\Uparrow_i\rangle + B_i|\Downarrow_i\rangle), \qquad (12)$$

where we introduced double arrows $\Uparrow_i$, $\Downarrow_i$, which refer to eigenstates of $\hat{\tau}_i^z = \pm 1$, and $|A_i|^2 + |B_i|^2 = 1$. Otherwise, the coefficients A_i and B_i can be arbitrary.[5]

The evolution of this state can be understood using the Hamiltonian Eq. (10): each τ spin precesses in the external magnetic field created by other τ spins. The coefficients A_i and B_i thereby acquire phases. Crucially, these phases depend on the state of other τ spins, which leads to entanglement generation; see Fig. 6(a). This can be illustrated for just two LIOMs prepared in a superposition

$$|\psi_2\rangle = \tfrac{1}{2}(|\Uparrow_1\rangle + |\Downarrow_1\rangle) \otimes (|\Uparrow_2\rangle + |\Downarrow_2\rangle).$$

While initially $|\psi_2\rangle$ is a product state, evolution with the effective Hamiltonian $\hat{H}_{\mathrm{MBL}} = J_{12}\hat{\tau}_1^z\hat{\tau}_2^z$ (we omit single-spin terms as they do not produce entanglement) introduces the phases that depend on the relative state of spins 1 and 2, resulting in the evolution

$$
\begin{aligned}
|\psi_2(t)\rangle &= e^{-i\hat{H}_{\mathrm{MBL}}t}|\psi_2\rangle \\
&= \tfrac{1}{2}e^{-iJ_{12}t}(|\Uparrow\Uparrow\rangle + |\Downarrow\Downarrow\rangle) + \tfrac{1}{2}e^{iJ_{12}t}(|\Uparrow\Downarrow\rangle + |\Downarrow\Uparrow\rangle).
\end{aligned}
$$

This wave function $|\psi_2(t)\rangle$ now has entanglement between two spins that grows up to $\ln 2$ for times when the accumulated phase $J_{12}t = \pi/4$ (Serbyn, Papić, and Abanin, 2013b).

We sketch the generalization of this argument for the case of many spins, following Serbyn, Papić, and Abanin (2013a, 2013b) and Huse, Nandkishore, and Oganesyan (2014). A given spin acquires a phase of the order of one dependent on the state of another spin a distance x away after a time $t(x)$ set by the condition $\tilde{h}_{i,i+x}t(x) \sim 1$. Here we introduced the strength of an effective magnetic field acting on spin at site i

---

[5]One should, however, avoid the situation where all $A_i = 1$ or $B_i = 1$, because this would correspond to an eigenstate, which would take an exponentially long time (of the order of the inverse level spacing) to prepare.

due to spin distance x away. This field depends on the state of other spins between spin i and $i + x$, $\tilde{h}_{i,i+x} = J_{i,i+x} + J_{i,i+1,i+x}\hat{\tau}_{i+1}^z + \cdots$. Using exponential decay of couplings $J_{ij}, \ldots$, see Eq. (11), one can show that the effective magnetic field is also exponentially suppressed with the interaction range $\tilde{h}_{i,i+x} \sim J_0 e^{-x/\xi'}$, where the length scale ξ' is discussed later. This decay of the magnetic field together with the condition $\tilde{h}_{i,i+x}t(x) \sim 1$ yields the logarithmic entanglement "light cone" and logarithmic growth of entanglement,

$$x_{\mathrm{ent}}(t) = \xi' \log(J_0 t), \qquad S_{\mathrm{ent}}(t) \propto \xi' \log(J_0 t). \quad (13)$$

Indeed, at time t, for typical initial states, all spins within volume $x_{\mathrm{ent}}(t)$ acquire phases dependent on the states of other spins, and therefore entanglement spreads over that volume. In a finite-size system, $x_{\mathrm{ent}}(t)$ is bounded by the system size; hence the entanglement entropy in a quantum quench saturates to a value that is proportional to the system size $S_{\mathrm{ent}}(\infty) \propto L$.

The length ξ' which controls entanglement growth is related to κ as $\xi'^{-1} \le \kappa^{-1} + (\ln 2)/2$, where extra contribution comes from the exponentially large number of possible interaction terms within a given range. Thus, ξ' can be shown to satisfy $\xi' \le 2\xi$. Note that we started with a single localization length ξ in Sec. II.C and introduced two additional lengths κ and ξ' which control different physical properties. These length scales may be viewed as phenomenological parameters which could in principle account for the presence of multiple intrinsic length scales within the MBL phase. Establishing whether the three length scales ξ, κ, and ξ' are directly related remains an outstanding challenge.

Two comments are in order. First, while for simplicity we focused on initial product states of τ spins, the logarithmic growth of entanglement holds generally, e.g., for initial product states of physical spins (Serbyn, Papić, and Abanin, 2013b). This is because a generic initial state is an extensive superposition of many-body eigenstates, and each τ spin undergoes the dephasing dynamics. Second, the proportionality coefficient in Eq. (13) depends on the diagonal entropy of the initial state (Polkovnikov, 2011), which, in particular, is influenced by the disorder strength, as discussed by Serbyn, Papić, and Abanin (2013b). For initial states of Eq. (12) this entropy is determined by the probability distribution of coefficients A_i and B_i.

Entanglement spreading in MBL systems can also be described by the dynamical RG approach introduced by Vosk and Altman (2013). This approach gives an entanglement entropy that grows logarithmically in time for usual MBL states in agreement with Eq. (13), while in the vicinity of a transition between two distinct MBL phases (see Sec. III), the growth changes to $S_{\mathrm{ent}}(t) \propto \ln^\phi t$ with $\phi > 1$.

The logarithmic propagation of entanglement in the MBL phase is in a stark contrast to the ballistic entanglement spreading in ergodic systems (Kim and Huse, 2013) and in Yang-Baxter integrable models (Calabrese and Cardy, 2009). The logarithmic spreading of entanglement also distinguishes the MBL phase from an Anderson insulator. It is therefore often viewed as one of the defining features of MBL, especially in numerical simulations.

The growth of entanglement is difficult to measure experimentally in large systems (see, however, Sec. V). It is therefore important to identify alternative manifestations of the dephasing dynamics in the MBL phase that are more directly observable. Serbyn, Papić, and Abanin (2014) showed that the dephasing dynamics in MBL systems leads to equilibration of local observables in a quantum quench setup, with a characteristic, power-law approach to equilibrium values.

To illustrate this, let us again consider the simple initial state (12) with $A_i = B_i = 1/\sqrt{2}$ and study the single-spin observables, described by operators $\hat{\tau}_I^{x,y}$ (operator $\hat{\tau}_I^z$, being conserved, is time independent). At $t = 0$, the τ spin I is pointing in the x direction. Further, it undergoes precession in the x-y plane, and its rotation angle depends on the state of other spins. At time t, spins within distance $x_{ent}(t)$ away, such that $t J_0 e^{-x_{ent}(t)/\xi'} \gtrsim 1$, significantly affect the rotation angle of spin I. The generation of entanglement with those spins leads to dephasing. Accordingly, the off-diagonal elements of the reduced density matrix or, equivalently, observables $\langle \hat{\tau}_I^{x,y}(t) \rangle$ decay. Observing that at time t spin I is entangled with an "environment" that has the Hilbert space dimension $\mathcal{D}(t) \approx 2^{2x_{ent}(t)}$ spins, and using Eq. (13) we conclude that

$$|\langle \hat{\tau}_I^{x,y}(t) \rangle| \sim \frac{1}{\sqrt{\mathcal{D}(t)}} \approx \frac{1}{(J_0 t)^a}, \qquad (14)$$

which describes a power-law decay of x, y spin projections; see Fig. 6(c). We note that the exponent of this power law is not universal, and in general is given by $a = \xi' s_0$, where s_0 is the diagonal entropy (Polkovnikov, 2011) of the initial state per spin (Serbyn, Papić, and Abanin, 2014). For the state in Eq. (12) with all $A_i = B_i = 1/\sqrt{2}$, the diagonal entropy attains its maximal value, $\ln 2$ per spin, leading to an exponent $a = \xi' \ln 2$.

The dynamics of physical spin operators can be analyzed by expressing them via $\hat{\tau}_i^\alpha$, $\alpha = x$, y, z operators and their products. The terms in that expansion which involve only $\hat{\tau}^z$ operators will remain unchanged, while any term which involves at least one $\hat{\tau}_i^{x,y}$ operator will decay in a power-law fashion. Thus, generic local observables approach their long-time equilibrium values in a power-law fashion. We emphasize that equilibrium values of observables retain the memory of the initial state due to the extensive set of LIOMs.

Another interesting implication of the dephasing dynamics is that the standard spin-echo protocol can fully recover the state of a given τ spin (Serbyn *et al.*, 2014; Bahri *et al.*, 2015), implying that the intrinsic T_1 relaxation time remains infinite in the MBL phase. On the other hand, the T_2 time induced by the entanglement dynamics with distant spins increases exponentially with the distance to these spins, reflecting the logarithmic dynamics of entanglement growth (Serbyn *et al.*, 2014). Of course, in practice one would perform the spin-echo protocol on a physical spin, which leads only to an incomplete recovery of the initial state, but the revival probability is large at strong enough disorder, when the physical $\hat{\sigma}_i^z$ operators are close to $\hat{\tau}_i^z$.

Other, closely related experimental signatures of the dephasing dynamics include temporal revivals of local observables Vasseur, Parameswaran, and Moore (2015) and

double-electron resonance, which can be viewed as a modification of the spin-echo protocol, allowing one to probe the dephasing of a given spin by distant spins (Serbyn *et al.*, 2014). In addition, power-law decays of various quantities due to the same mechanism have been identified, including mutual information (De Tomasi *et al.*, 2017), fluctuations of the out-of-time correlation functions (Chen *et al.*, 2017), and fluctuations of the Loschmidt echo (Serbyn and Abanin, 2017).

In this discussion, we addressed the dynamics of isolated MBL systems. In practice, no system is perfectly isolated from the environment, and therefore it is important to understand how dissipation affects the dynamical behavior. In general, coupling to the bath leads to delocalization and restores slow transport. A classic example is the variable-range hopping: electrons in a solid can hop between localized states, with the mismatch energy being provided by phonon absorption or emission (Mott, 1968).

Recent experimental studies of MBL were, however, performed in synthetic quantum systems of, e.g., ultracold atoms or ions, which are free from phonons, since the lattice potential is generated by lasers. Dissipation is still present, with two significant sources being inelastic scattering on lattice lasers and particle loss for atomic and trapped ion experiments. Fischer, Maksymenko, and Altman (2016), Levi *et al.* (2016), and Medvedyeva, Prosen, and Žnidarič (2016) developed a theoretical approach for describing these kinds of dissipation. They formulated the Lindblad equation in terms of LIOMs, and, having reduced it to a classical rate equation, analyzed the resulting dynamics. It was shown that relaxation of an initial density modulation displays a certain degree of universality, following a stretched-exponential law. In a different direction, Nandkishore, Gopalakrishnan, and Huse (2014) demonstrated that the spectral function of an MBL system weakly coupled to a heat bath still carries signatures of localization.

E. New numerical and analytical approaches

As discussed, the theory based on LIOMs provides a natural description of the MBL phase. In the basis of the LIOMs, the eigenstates become product states and the dynamics is reduced to dephasing. Yet, in contrast to the Yang-Baxter-type integrability, where integrals of motion are known exactly, the presence of disorder in the MBL phase precludes an explicit analytic construction of LIOMs. Thus, in order to better understand the properties of LIOMs and the way they become nonlocal in the vicinity of the MBL transition, it is necessary to develop new theoretical and numerical tools for constructing highly excited eigenstates.

The low, area-law entanglement of the highly excited MBL eigenstates in 1D allows for their efficient representation by the matrix product state (MPS) ansatz (Schollwöck, 2005). This opens the door to using the density matrix renormalization group (DMRG), originally developed for the ground states of the one-dimensional systems. (Note, however, that extending DMRG techniques to the excited states is highly nontrivial due to the fact that the level spacing becomes very small at a finite energy density.) Works by Kennes and Karrasch (2015), Khemani, Pollmann, and Sondhi (2015), Lim and Sheng (2015), Yu, Pekker, and Clark (2015), and

Serbyn *et al.* (2016) provided a proof of principle of the applicability of DMRG algorithms by extracting highly excited eigenstates in MBL system and studying their properties. A more ambitious approach proposed by Pollmann *et al.* (2015) and Wahl, Pal, and Simon (2016) uses the matrix product operator ansatz (Schollwöck, 2005) to find the unitary $\hat{U}$, previously introduced, which diagonalizes the full Hamiltonian and allows for an explicit construction of LIOMs.

In a different direction, the generalizations of the real-space renormalization group to highly excited states provided an alternative set of tools to describe the properties of the MBL phase. We already discussed in Sec. II.C the dynamical RG of Vosk and Altman (2013) and the real-space RG for excited states of Pekker *et al.* (2014). While MPS methods are more suitable for area-law entangled states, the RG-based approaches are also capable of capturing the structure of the eigenstates with a logarithmic scaling of entanglement. In particular, works by Pekker *et al.* (2014) and Slagle, You, and Xu (2016) investigated critical properties of the transition between different MBL phases for Ising and *XYZ* chains, while Vasseur, Potter, and Parameswaran (2015) studied random $SU(2)_k$ anyon chains.

The development of new numerical methods in the two directions outlined above is an active subject of current research. In addition, tensor-network-based approaches can allow access to the long time dynamics of large open systems (Fischer, Maksymenko, and Altman, 2016; Znidarič, Scardicchio, and Varma, 2016). Many of the properties of the simplest models of the MBL phase, Eqs. (5) and (6), can be studied using exact diagonalization. At the same time, the new numerical tools are essential for describing 1D systems with local Hilbert space larger than 2 (bosons, spinful fermions, higher spins) and for studies of phase transitions between MBL and thermal phases, as well as between different MBL phases. Finally, the development of tensor-network methods for excited states is necessary for investigating MBL in higher dimensions (Wahl, Pal, and Simon, 2017), which is also the subject of current experimental studies; see Sec. V.

III. MBL-PROTECTED PHASES OF MATTER

Landau's theory of symmetry breaking gives a description of phases of matter in thermodynamic equilibrium. The fact that MBL systems are not able to reach thermodynamic equilibrium calls for reexamining the notion of a quantum phase of matter in the presence of localization. Later we discuss the properties of MBL that arise from the presence of additional symmetries. In particular, in the presence of MBL, discrete Abelian symmetries, such as a Z_2 symmetry, can either be spontaneously broken even in highly excited eigenstates or remain intact, giving rise to distinct MBL phases. In contrast, we show that non-Abelian symmetries are incompatible with MBL, at least in the sense of having area-law entanglement of eigenstates and a complete set of quasilocal integrals of motion. Next we will turn to periodically driven (Floquet) systems. We will discuss that MBL may persist in the presence of periodic driving, resulting in a Floquet-MBL phase. Periodic driving enriches possible kinds of dynamics, and, as a result, MBL can enable Floquet phases of matter with no counterpart in equilibrium systems. Finally,

we will briefly discuss the relation between MBL and spin and Bose glass physics.

A. Symmetries and localization-protected orders

In conventional statistical mechanics, phases of matter can be characterized by the order parameter and its symmetry in thermal (Gibbs) states of the system. The assumption of thermal equilibrium puts constraints on the existence of possible phases and phase transitions. One well-known example is the absence of symmetry breaking at finite temperature in one-dimensional systems with short-range interactions (Mermin and Wagner, 1966). The eigenstates of ergodic systems, obeying ETH, are expected to behave similar to the Gibbs states and to satisfy the same thermodynamic constraints.

The eigenstates of MBL systems, in contrast, violate ETH, and therefore can exhibit richer order compared to thermal ensembles. Intuitively, as described by Huse *et al.* (2013), localization of excitations that would have destroyed the order in a thermal state, "protects" the order in individual eigenstates. To illustrate this, and to see how distinct MBL phases may arise, let us consider a disordered Ising spin chain with Z_2 symmetry:

$$\hat{H}_{\text{Ising}} = \sum_i J_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \sum_i h_i \hat{\sigma}_i^x + \lambda \sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x, \quad (15)$$

where J_i, h_i are independent, positive random couplings with nonzero means $\bar{J}$, $\bar{h}$, and variances δJ^2, δh^2. The global Z_2 symmetry is implemented by the spin-flip operator $\hat{P} = \prod_i \hat{\sigma}_i^x$.

This model can be mapped to a fermionic model via the Jordan-Wigner transformation; the first two terms map onto a free fermion model with disorder, while the third term corresponds to a four-fermion interaction term. In the ground state, two phases are possible: the ordered phase for $J_i \gg h_i$, which breaks the Z_2 symmetry, and the paramagnetic phase for $h_i \gg J_i$, which respects the symmetry. The two phases are separated by the infinite-randomness fixed point described by the strong-disorder real-space renormalization group (Dasgupta and Ma, 1980).

Turning to the properties of eigenstates at finite energy density, it is convenient to start from the noninteracting (in the fermionic language) limit $\lambda = 0$. In the ordered phase, the excitations are domain walls, which are localized by arbitrarily weak disorder (since the model is noninteracting). Likewise, spin-flip excitations which are relevant in the paramagnetic phase are also localized. Upon including sufficiently weak interactions $\lambda \neq 0$, both kinds of excitations may remain MBL (Huse *et al.*, 2013; Kjäll, Bardarson, and Pollmann, 2014; Pekker *et al.*, 2014). Thus, even in the presence of interactions, two distinct phases arise: the ordered phase, characterized by long-range "spin-glass" order $\langle \hat{\sigma}_i^z \hat{\sigma}_j^z \rangle_\alpha \neq 0$ for $|i - j| \to \infty$ (here $\langle \cdots \rangle_\alpha$ denotes expectation value in an eigenstate $|\alpha\rangle$) and the paramagnetic phase.

Huse *et al.* (2013) and Pekker *et al.* (2014) argued that the eigenstate transition between the two MBL phases (spin-glass and paramagnetic) may be of the same infinite-randomness universality class as the ground state transition. Within the approximation used by Pekker *et al.* (2014) this

infinite-randomness fixed point remains stable at finite energy density. At the same time, the presence of other delocalization channels which may change the kind of transition (e.g., giving rise to a thermal phase separating the two MBL phases) has not been ruled out.

These ideas can be extended to other discrete, Abelian symmetries and higher-dimensional systems. Bauer and Nayak (2013) and Huse *et al.* (2013) argued that MBL can protect the Z_2 topological order in $d = 2$ at finite energy density. Further, Chandran *et al.* (2014) and Bahri *et al.* (2015) considered certain models of symmetry-protected topological phases, arguing that they can be MBL. The bulk topology can give rise to a protected qubit at the edge, surprisingly, even when the system is very "hot" and strongly coupled to the degrees of freedom making up the qubit (Bahri *et al.*, 2015).

While Abelian symmetries enrich the variety of possible MBL states as previously discussed, the presence of an unbroken non-Abelian symmetry places strong restrictions on the structure of eigenstates and is incompatible with MBL as defined. More specifically, it is impossible to have the area-law entanglement of excited eigenstates and a complete set of LIOMs (Potter and Vasseur, 2016; Vasseur *et al.*, 2016; Protopopov, Ho, and Abanin, 2017). For instance, Protopopov, Ho, and Abanin (2017) demonstrated that the area-law entanglement is incompatible with the SU(2) symmetry and the eigenstate entanglement entropy must scale at least logarithmically with system size. The interplay of MBL and different discrete non-Abelian symmetries was also addressed by Vasseur *et al.* (2016), Friedman *et al.* (2017), and Prakash *et al.* (2017). Searching for possible nonergodic phases beyond MBL in the presence of non-Abelian symmetries is a promising research direction.

B. Many-body localization in Floquet systems

We proceed by discussing another application of MBL: in periodically driven (Floquet) systems, MBL can prevent heating to an infinite temperature state, opening up the possibility of having new nonequilibrium Floquet-MBL phases.

Subjecting a physical system to an external, time-periodic perturbation, e.g., with lasers, is a powerful experimental tool. Recently, this tool has been used to control and engineer properties of synthethic quantum systems, leading, e.g., to the realization of topological Bloch bands in systems of ultracold atoms (Cooper, Dalibard, and Spielman, 2018). Furthermore, it was shown theoretically that single-particle Floquet systems exhibit a rich variety of novel topological states, which are not possible in equilibrium (Kitagawa *et al.*, 2010). A prominent example of such topological, "Floquet-only" phases is a 2D system with protected edge states, but no bulk bands with nonzero Chern numbers (Rudner *et al.*, 2013).

Real systems are interacting, and intuitively one expects that "shaking" an interacting system would almost inevitably cause heating. Such heating arising from energy absorption in interacting systems will wash out interesting topological features, such as edge states, thus being a central obstacle in the field of Floquet engineering. This intuition was recently made precise by D'Alessio and Rigol (2014), Lazarides, Das, and Moessner (2014), and Ponte, Chandran *et al.* (2015),

which argued that ergodic systems satisfying ETH heat up indefinitely under driving.

However, MBL systems break ETH and therefore may avoid heating (Ponte, Chandran *et al.*, 2015). Lazarides, Das, and Moessner (2015), Ponte, Papić *et al.* (2015), and Abanin, De Roeck, and Huveneers (2016) established the fact that MBL can indeed remain stable in periodically driven systems as long as the frequency of the drive is sufficiently high. At low driving frequency, in contrast, delocalization is inevitable, even for a drive with a small amplitude (Abanin, De Roeck, and Huveneers, 2016). The *Floquet-MBL phase* is characterized by a complete set of LIOMs, area-law entanglement of Floquet eigenstates, but most importantly, MBL prevents heating to an infinite temperature, opening the door to stabilizing Floquet-only phases in isolated systems. Signatures of the Floquet-MBL phase have been observed in a recent experiment with ultracold atoms (Bordia *et al.*, 2017).

One example of an MBL-enabled Floquet-only phase is the discrete time crystal (Else, Bauer, and Nayak, 2016; Khemani *et al.*, 2016), characterized by the breaking of discrete time-translation symmetry of the drive ($t \rightarrow t + nT$). As a result, at long times local observables generally do not relax, instead exhibiting persistent oscillations at multiple integers of the driving period. First experimental signatures have been reported in NV-center spin system in diamond and in trapped ionic systems (Choi *et al.*, 2017; Zhang *et al.*, 2017). However, these systems exhibit long-range interaction, which precludes localization; see discussion in Sec. V.B. Ho *et al.* (2017) demonstrated that the observed transient time-crystal-line signatures stemmed from the parametrically slow relaxation caused by rare two-spin resonances ("critical time crystal").

Another, qualitatively different example of an interacting, Floquet-only phase is the anomalous Floquet insulator. This is a two-dimensional phase of matter which is MBL in the bulk, yet has topologically protected chiral edge states. The stability of this phase was shown by Nathan *et al.* (2017), while Gross *et al.* (2012) and Po *et al.* (2016) discussed the topological invariants which protect this and related phases. Finally, we note that Else and Nayak (2016), Po *et al.* (2016), Potter, Morimoto, and Vishwanath (2016), von Keyserlingk and Sondhi (2016a, 2016b), and Harper and Roy (2017) put forward (partial) classifications of distinct Floquet-MBL phases in the presence of additional symmetries. Theoretical and experimental investigations of new Floquet-MBL phases, their physical properties, and topological invariants remain a subject of active research.

C. Many-body localization, spin, and Bose glasses

There are apparent similarities between the physics of glasses, which has been a subject of intense study for many years, and the phenomenon of many-body localization. In particular, both phenomena involve breaking of ergodicity. The goal of this section is to compare the two phenomena and clarify the essential differences between them. We discuss two kinds of glasses, which can occur in quantum systems: a spin glass, and a zero-temperature phase commonly referred to as a Bose glass.

1. Spin glass

A spin glass is a low-temperature phase found in certain disordered (either classical or quantum) spin models at sufficiently low temperatures. In the ideal case, there is a thermodynamic phase transition at a critical temperature below which the system breaks ergodicity. As discussed next, this ergodicity breaking has a different character compared to the case of MBL, since a glassy system can remain ergodic within separate parts of the phase space. More generally (even if there is no sharp transition), spin glasses are characterized by a wealth of unusual dynamical phenomena, including very slow dynamics of observables (e.g., magnetization) and memory (aging) effects in a quench experiment (Binder and Young, 1986). While MBL is also associated with slow dynamics and breaking of ergodicity, the origin of these effects is very different, allowing one to make a sharp distinction between MBL systems and spin glasses.

The source of the unconventional properties of glasses is frustration. The fact that the interaction terms cannot all be minimized simultaneously results in a large number of low-energy states that are separated by energy barriers that increase with the system size; this leads to the characteristic "rugged" energy landscape in phase space. When coupled to a bath maintained at sufficiently low temperature, the large barriers prevent the system from exploring the entire phase space. The broad distribution of energy barrier heights leads to a broad distribution of relaxation time scales in glasses. Note that this basic fact about glasses is true regardless of whether the microscopic degrees of freedom are classical or quantum (e.g., quantum spins). As long as there is a broad distribution of energy barriers, fluctuations induced by coupling to a (cold) thermal bath generate glassy dynamics. Hence, classical glasses are not only robust with respect to coupling to an external bath, but their dynamics is in fact often generated by such coupling.

In stark contrast, MBL is a fully quantum phenomenon, which does not require frustration, and instead relies on the discreteness of the spectrum, or equivalently, on interference effects. The intuitive criterion for MBL is that the transition rate between two many-body configurations that are very close in energy is much smaller than the many-body level spacing, resulting in the absence of resonances between such configurations. Coupling to an external bath destroys the interference effects that ensure the discrete spectrum and therefore generally also destroys MBL; see Fischer, Maksymenko, and Altman (2016) and Sec. V.E. We can therefore make a distinction between an MBL system and a glass by studying how its respective dynamics is affected by coupling to a low-temperature bath. MBL relies on the system being isolated from the environment, whereas glassiness does not. Of course, the presence of frustration may help MBL, but it is not essential for its existence.

This discussion implies another important difference between the two phases. While certain observables fail to relax (or relax very slowly) in a glassy system, the information retained in these observables is completely classical. In an MBL system, on the other hand, it is possible to recover local quantum information (i.e., local phase information of a q bit) using spin echoes after arbitrary long times (Serbyn *et al.*, 2014; Bahri *et al.*, 2015).

While the basic mechanisms of an MBL and a spin-glass phase are qualitatively different, there is an interesting and largely unexplored question regarding their possible coexistence. More generally, it is desirable to develop an understanding of dynamical properties of isolated quantum systems with glassy classical energy landscape subject to unitary quantum dynamics.

To this end, one may consider the spin-1/2 Edwards-Anderson model with a transverse field, which gives rise to quantum dynamics:

$$\hat{H}_{\text{EA-}q} = -\sum_{ij} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Gamma \sum_i \hat{\sigma}_i^x, \tag{16}$$

where J_{ij} are random couplings. The statistics and dependence of interaction J_{ij} on the distance between spins determine the phase diagram of the corresponding classical model. For instance, infinite-ranged interactions where the distribution of J_{ij} is independent of i, j lead to the Sherrington-Kirkpatrick model (Sherrington and Kirkpatrick, 1975). On the other hand, embedding spins into a d-dimensional lattice with only nearest-neighbor $J_{ij} \neq 0$ describes a short-range spin glass. While the classical ($\Gamma = 0$) phase diagram of the model in Eq. (16) is known in many cases, much less is known about the quantum model. What is the nature of quantum eigenstates? Can this model be in the MBL phase?

Baldwin *et al.* (2017) investigated these issues for mean-field spin-glass models with p-spin interactions (the Sherrington-Kirkpatrick model corresponds to the case $p = 2$). They found a parameter regime in which eigenstates at a given energy density cluster into groups with different values of observables, such that ETH is satisfied within one cluster. While an MBL phase is impossible in such systems due to an infinite range of interactions, the behavior of the model in Eq. (16) with short-range interactions may allow for an MBL phase or nonergodic phase similar to one found by Baldwin *et al.* (2017). Investigating these issues comprises a promising direction for future research.

2. Bose glass

Bose glass (BG) is the term used to describe insulating quantum phases of interacting bosons in a disorder potential at zero temperature. Such phases are found in a broad range of physical systems, including ⁴He in porous media, cold atoms in disordered optical lattices (Fallani *et al.*, 2007; Meldgin *et al.*, 2016), thin superconducting films (Haviland, Liu, and Goldman, 1989; Hebard and Paalanen, 1990), and disordered magnets (Yu *et al.*, 2012). Theoretically, the existence of Bose glass was established by Giamarchi and Schulz (1988), who used a perturbative renormalization group to analyze the transition between BG and fluid phase in one dimension. BG in higher-dimensional systems was studied by Fisher *et al.* (1989), who derived the critical exponents for the BG-superfluid transition.

Because Bose glasses are zero-temperature ground states, while MBL is a property of highly excited states, these two phenomena refer to different parts of the spectrum and are not

directly related. However, it is natural to ask if having a Bose glass ground state necessarily implies many-body localization at low nonvanishing temperatures. This question was investigated for disordered weakly interacting bosons in one and two dimensions by Aleiner, Altshuler, and Shlyapnikov (2010) and Bertoli *et al.* (2017), who argued that the BG phase is smoothly connected to an MBL phase as the temperature is increased. Aleiner, Altshuler, and Shlyapnikov (2010) and Michal *et al.* (2016) analyzed the critical temperature for the MBL-delocalization transition (equivalently, the position of the many-body mobility edge) as a function of the disorder strength and interactions. Furthermore, Michal *et al.* (2016) studied MBL for strongly interacting bosons in 1D, also finding that the zero-temperature BG phase at both weak and strong disorder is smoothly connected to the MBL phase at finite temperature.

It would be interesting to confront these theoretical arguments for the smooth connection between BG and MBL phases with an experimental test. In addition, the connection between the BG and MBL phases when interactions are strong is an interesting open question that remains unexplored.

IV. DELOCALIZATION TRANSITION

The breakdown of many-body localization upon changing the disorder strength or some other control parameter provides an intriguing opportunity to study the emergence of thermalization in a quantum system, possibly with the control afforded by proximity to a critical point.[6] In the MBL phase, quantum information encoded in local observables is protected and affects the dynamics at arbitrarily long times. On the other hand, in a thermalizing system quantum information is lost to nonlocal degrees of freedom and the remaining slow modes are described by classical hydrodynamics. Thus investigating the transition can shed light on the elusive boundary between quantum and classical behavior in interacting systems.

In addition to providing insight into the mechanism of quantum thermalization, the delocalization transition represents a new type of quantum phase transition, which differs in crucial aspects from both thermal and zero-temperature quantum phase transitions. One important distinction is that unlike conventional phase transitions, there need not be any thermodynamic signatures of the delocalization transition, and it can be manifested only in dynamical quantities, such as energy conduction or entanglement propagation. Another unique feature is that the MBL transition must involve a dramatic change in the real-space entanglement structure of many-body eigenstates. At the critical point, the entanglement entropy of energy eigenstates changes from area-law on the localized side to volume-law entanglement entropy, consistent with the thermodynamic entropy, on the thermal side of it. In contrast, the usual quantum critical points mark the transition between two ground states with area-law entanglement. In spite of those differences, important insight into the MBL

---

[6]We assume the direct transition between MBL and thermal phases throughout this section and do not consider the possibility of, e.g., an intervening glassy phase at finite temperature.

transition has been gained by adapting renormalization group ideas.

A. Renormalization group approach to the MBL transition

The MBL phase is stable against thermalization due to the extreme rarity of resonances when the disorder is sufficiently strong. The prevailing view of the delocalization transition is that it is driven by the emergence of resonant clusters. As the strength of randomness is reduced, resonant clusters start to occur more frequently in the system. At the critical disorder strength a critical cluster grows to encompass the entire system. The RG approaches to the MBL transition attempt to describe the fluctuations that give rise to such a critical cluster at multiple scales and predict how they impact properties of the system near the critical point.

Different RG schemes were proposed to describe this process (Potter, Vasseur, and Parameswaran, 2015; Vosk, Huse, and Altman, 2015; Zhang *et al.*, 2016; Dumitrescu, Vasseur, and Potter, 2017; Thiery, Müller, and De Roeck, 2017; Goremykina, Vasseur, and Serbyn, 2019). The RG scheme in Vosk, Huse, and Altman (2015) starts from a phenomenological coarse-grained description of a one-dimensional system as a chain of regions with varying local character: some regions behave locally as insulators and others have a local thermal character. The RG scheme then attempts to describe the competition between the growth of the thermal regions as they hybridize with nearby clusters, on the one hand, and their potential isolation by surrounding MBL regions, on the other hand. The RG scheme of Potter, Vasseur, and Parameswaran (2015) takes a more microscopic starting point, operating on spin-1/2-like degrees of freedom. The scheme attempts to construct the critical cluster from the bottom up: identifying resonant pairs of spins, joining them into resonant miniclusters, then joining those to even larger clusters, and so on. In the MBL phase this process stops with a finite cluster, when no more resonant clusters can be merged, while in the delocalized phase a resonant cluster eventually encompasses the entire system.

In both cases, the key scaling variable that identifies the degree to which a cluster is localized or thermalizing is a ratio $g = \Gamma/\Delta$ between the "decay rate" Γ associated with the rate of information loss across the cluster and the many-body level spacing. If $g \gg 1$ then the cluster is resonant and we say it is locally thermalized, while if $g \ll 1$ then the cluster is said to be localized. This ratio is reminiscent of the Thouless conductance, defined in noninteracting systems by the ratio between a single-particle relaxation rate (the Thouless energy) to the level spacing, which is the central object of the scaling theory of Anderson localization (Abrahams *et al.*, 1979). The ratio g can be viewed as a "many-body Thouless parameter." A microscopic version of the many-body Thouless parameter was introduced by Serbyn, Papić, and Abanin (2015) and used to diagnose the MBL transition and its properties.

An important difference from the single-particle case is that the many-body level spacing is exponentially small with the cluster size l. For a spin-1/2 chain $\Delta \sim e^{-(l/a)\ln 2}$ near the middle of the spectrum (infinite temperature) and more generally $\Delta \sim e^{-sl/a}$, where s is the entropy density and a

the lattice spacing. In the MBL phase the relaxation rate of a cluster is also exponentially small with the cluster size (distance to conducting leads) $\Gamma \sim e^{-l/\ell_*}$, where ℓ_* is the localization length. Comparison between the two exponentially small scales imposes a stringent cutoff for the localization length associated with propagation of information through an MBL system. For the system to remain localized with $g \ll 1$ we must have $\ell_* < a/s$.

From the existence of a finite (noncritical) localization length ℓ_*, it is tempting to infer that the MBL transition must be first order. However, the RG schemes (Potter, Vasseur, and Parameswaran, 2015; Vosk, Huse, and Altman, 2015) do find a critical fixed point with a diverging length scale $\zeta_{\mathrm{RG}} \sim |g - g_c|^{-\nu}$ with $\nu \approx 3.3$. Nonetheless the transition appears as a noncritical jump when viewed through the lens of typical observables, measured on a single sample. Vosk, Huse, and Altman (2015) pointed out that to observe the growing critical cluster on the MBL side of the transition one must measure average values. As seen in the next section, the critical behavior is much more accessible to experiments done on the thermal side of the MBL phase transition.

It is interesting, and encouraging, that in spite of the different philosophies underlying the two RG schemes, they give similar predictions for the critical properties, including the correlation length exponent, dynamical exponents, and more. The critical exponent $\nu \approx 3.3$ found by both RG schemes is consistent with the Harris bound $\nu > 2/d$ (Harris, 1974; Chayes *et al.*, 1986). [Chandran, Laumann, and Oganesyan (2015) argued that the Harris bound holds for the MBL critical point in spite of its unconventional aspects.] By contrast, exact diagonalization results for small spin chains give the appearance of scaling with a critical exponent that violates the bound $\nu \approx 1 < 2/d$ (Kjäll, Bardarson, and Pollmann, 2014; Luitz, Laflorencie, and Alet, 2015). The violation of the Harris criterion by the exponent extracted numerically for *XXZ* spin chains may be due to the fact that systems that can be diagonalized exactly are too short to be in the asymptotic scaling regime.

At this point it is worth noting that the RG scheme has identified two distinct localization lengths ζ_{RG} and ℓ_*, where the former length diverges and the latter stays finite at the transition. The divergent localization length ζ_{RG} is rather hard to observe inside the insulating phase as it is not manifested in typical values [at least of the quantities considered by Vosk, Huse, and Altman (2015)]. Instead one must measure average values which are sensitive to rare events and therefore may require very large sample sizes to converge. While the relation between the phenomenological length scales ζ_{RG}, ℓ_* and the microscopic parameters ξ, κ, ξ' is unclear at the moment, we note that (i) ξ' controls the entanglement spreading, as does ℓ_* and therefore they are likely proportional to each other; and (ii) the localization length ξ (or its average) may diverge at the transition, just like ζ_{RG}.

The entanglement structure of eigenstates is intriguing and helps in developing a theoretical picture of the MBL transition; however, it is not accessible to experimental measurement. In the next section we discuss critical relaxation dynamics and transport properties which can serve as realistic probes of the critical point in experiments.
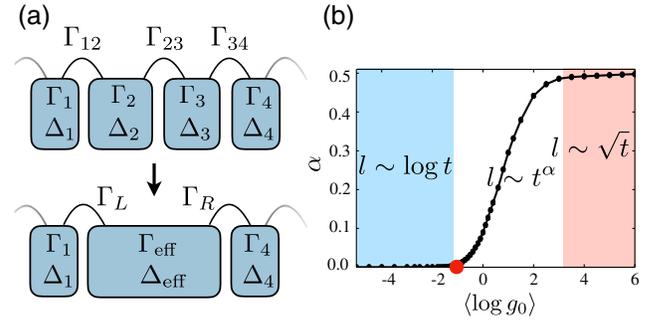

FIG. 7. (a) RG description starts with a collection of blocks each characterized by the hybridization rate and level spacing. Within one RG step two blocks with the strongest coupling rate are merged together resulting in a block with new effective parameters. (b) The inverse dynamical exponent α vanishes in the MBL phase and interpolates continuously between 0 and 1/2 in the thermal phase.

B. Subdiffusion and Griffith regions

The renormalization group approaches (Potter, Vasseur, and Parameswaran, 2015; Vosk, Huse, and Altman, 2015) give explicit predictions for the transport properties near MBL transition. The scheme of Vosk, Huse, and Altman (2015) allows one to plot the energy relaxation time $\tau_{\mathrm{tr}} = (l/l_0)\Gamma^{-1}$ of clusters versus their average size l to obtain the dynamical exponent $1/\alpha$, $l \sim \tau_{\mathrm{tr}}^{\alpha}$. In the MBL phase one finds a logarithmic relation $l \sim \ell_* \ln(\tau_{\mathrm{tr}})$, hence $\alpha = 0$ in Fig. 7(b). While naively one would expect to see the diffusive scaling throughout the thermal phase $l \sim \sqrt{D\tau_{\mathrm{tr}}}$, with a diffusion constant that vanishes at the critical point, the RG scheme observes $\alpha = 1/2$ only far from transition. Closer to the transition, however, both RG schemes (Potter, Vasseur, and Parameswaran, 2015; Vosk, Huse, and Altman, 2015) as well as earlier numerical studies (Agarwal *et al.*, 2015; Bar Lev, Cohen, and Reichman, 2015; Znidarič, Scardicchio, and Varma, 2016) found subdiffusive transport $\tau_{\mathrm{tr}} \sim l^{1/\alpha}$ with the inverse dynamical exponent α varying continuously. The dynamical exponent $z \equiv 1/\alpha$ diverges at the critical point as $z \sim \zeta_{\mathrm{RG}} \sim (g_0 - g_{0c})^{-\nu}$, with $\nu \approx 3.3$, while at criticality the transport shows exponential scaling just as in the insulating phase.

The subdiffusive scaling is understood to be a result of rare critical inclusions in the thermal phase. Singularities due to rare regions were first discussed by Griffiths (1969) and McCoy (1969) in the context of conventional phase transitions of random spin systems and since then are known as Griffiths effects. The key to the contribution of rare regions to the transport near the unconventional MBL critical point is the balance between the low probability of finding such a long region and the long delay it would affect as a bottleneck to transport. The probability to find a long critical cluster of length l falls off exponentially with its length. For a system of length L near the critical point it is $P_L(l) \sim (L/\zeta_{\mathrm{RG}}) \exp(-l/\zeta_{\mathrm{RG}})$, where ζ_{RG} is the correlation length that diverges at the critical point. Thus, the longest critical inclusion we are likely to find in a system of length L (with probability of order 1) is $l_m(L) \sim \zeta_{\mathrm{RG}} \ln(L/\zeta_{\mathrm{RG}})$. If such a rare inclusion serves as

the dominant bottleneck on transport, it leads to a relaxation time $\tau(L) \sim \exp[l_m(L)/\ell_*] \sim L^{\zeta_{RG}/\ell_*}$. From this scaling we can read off the dynamical exponent $1/\alpha \sim \zeta_{RG}/\ell_*$ whose singular behavior at the MBL transition directly reflects the divergence of the correlation length. The RG results shown in Fig. 7(b) agree with the conclusion of these elementary considerations, showing that $\alpha \sim (g_0 - g_{0c})^\nu$.

The dynamical scaling affects not only the transport through the system, but also relaxation of global observables, such as the decay of a charge density wave imposed on the system (Schreiber *et al.*, 2015). In this case the rare regions lead to power-law relaxation instead of the exponential decay that would otherwise be expected from a nonconserved operator.

So far we discussed only the effects of rare regions on transport on the thermal side of the critical point. The implication of such regions to the ac conductivity in the MBL phase were investigated by Gopalakrishnan *et al.* (2015). However, those effects compete with other types of many-body resonances and it is not clear what the relative contribution of the different effects is. Hence relaxational dynamics in the MBL side of the transition is not well understood at this point.

Our discussion of the MBL critical point so far pertained to one-dimensional systems. How do these ideas generalize to higher dimensions? Naively, the thermalization criterion based on the many-body Thouless parameter previously defined always predicts thermalization in a sufficiently large system. Indeed, if the thermalization rate Γ falls off exponentially with the linear size of the system L, the level spacing decreases as $\exp(-sL^2)$ and is therefore always much smaller as $L \to \infty$. De Roeck and Huveneers (2017) turned this insight into a more systematic bootstrap scheme, which suggests that the MBL phase may not be stable to the presence of a sufficiently large but finite thermal inclusion. On the other hand, these arguments rely on the assumption that a metallic region is able to "perfectly thermalize" a nearby insulator, hence the issue of MBL (in)stability in higher dimensions is far from being settled. In Sec. V we return to the discussion of experiments which display signatures of MBL in two-dimensional lattice models.

V. EXPERIMENTAL DEVELOPMENTS

Probing for MBL is challenging in experiments as the system under investigation has to be isolated from any thermal environment. This makes it very hard to observe MBL in standard materials as almost all of them are connected to a thermal reservoir during cooling and experimental cycles. Quantum simulators based on ultracold atoms and ions have therefore been among the first systems in which MBL could be observed, owing to their almost perfect isolation and small couplings to the outside world. There is currently also an active search for MBL in real materials. It requires finding degrees of freedom which are extremely weakly coupled to the standard thermal bath of phonons present in a solid. NV centers of spins in diamond crystals and electrons in disordered superconducting films have shown interesting possible signatures of localization, but research is still ongoing to unravel the complex interplay of phenomena in these systems. One should note that

in any experiment, even the ones based on almost ideally isolated systems of ultracold atoms or ions, a small coupling to the environment is inevitable. The question of how such couplings can affect MBL and the associated phase transition will therefore be discussed in a separate section. In addition, finite-size effects of smaller experimental samples can play an important role in the interpretation of the data.

A. MBL with ultracold atoms

When exploring MBL, we recall that one is trying to identify a new phase and phase transition at high-energy densities, far away from the ground state of an interacting many-body system. Earlier related experiments on the ground state physics of disordered interacting particles had been successful in establishing evidence for the existence of a Bose glass phase [see Sec. III.C.2 and Fallani *et al.* (2007) and Meldgin *et al.* (2016)] and disorder induced localization of a metallic state (Kondov *et al.*, 2015). To study MBL and the associated transition, experiments have mostly resorted to preparing highly out-of-equilibrium initial states and probing their subsequent time evolution. As discussed in Sec. II.D, an MBL system will evolve into a stationary state, in which some local observables will assume nonthermal expectation values, whereas an ergodic thermalizing system would exhibit thermal expectation values for all local observables. The presence of such nonthermal local observables directly indicates a nonergodic evolution of the system and therefore can be used as a way to identify the localized nonthermal phase. Note that it is difficult to show that a system is thermalized, as this would require demonstrating that all local observables are thermal. The opposite, demonstrating localization, can in contrast be rather straightforward: a *single* local observable with a nonthermal value is sufficient to show this.

Initial experiments on MBL with ultracold atoms at high-energy densities were carried out using one-dimensional Fermi-Hubbard chains of interacting spin mixtures of two (hyperfine) spin components (Schreiber *et al.*, 2015). In order to realize a detuning landscape for the atoms, a quasiperiodic potential was applied, giving rise to the following Hamiltonian in 1D:

$$\hat{H} = -J\sum_{j,\sigma}(\hat{c}^\dagger_{j,\sigma}\hat{c}_{j+1,\sigma} + \text{H.c.})$$
$$+ \Delta\sum_{j,\sigma}\cos(2\pi\beta j + \phi)\hat{n}_{j,\sigma} + U\sum_j \hat{n}_{j,\uparrow}\hat{n}_{j,\downarrow}. \quad (17)$$

Here $\hat{c}^\dagger_{j,\sigma}$ ($\hat{c}_{j,\sigma}$) denote the fermionic creation (annihilation) operators for a particle in spin state $\sigma = \uparrow, \downarrow$ on lattice site i, U is the on-site interaction strength, and Δ denotes the strength of the quasiperiodic detuning potential. A Feshbach resonance between the atomic spin states allows one to tune the interaction between the particles U, enabling one to directly compare the evolution of repulsively, attractively, and noninteracting systems starting from the *same initial state*.

For the noninteracting case, $U = 0$, the model realizes the celebrated Aubry-André (AA) transition, exhibiting Anderson localization for $\Delta/J > 2$, which can serve as a well-understood reference point for the problem. Furthermore, in

1D and for infinitely strong interactions $U \to \infty$, the system also maps back onto the noninteracting AA problem, if initially no doubly occupied sites are present in the system. This work, together with theoretical results (Iyer *et al.*, 2013; Michal, Altshuler, and Shlyapnikov, 2014), indicates that MBL can also occur in the system with quasiperiodic detuning potentials. The question of how the transition in this case is different from the generic MBL transition is a subject of current research (Li *et al.*, 2015; Khemani, Sheng, and Huse, 2017; Li, Li, and Das Sarma, 2017; Setiawan, Deng, and Pixley, 2017).

In experiment (Schreiber *et al.*, 2015), the system is initially prepared in a density-wave state, with particles predominantly occupying even sites. The subsequent time evolution of the state is monitored (see Fig. 8), keeping track of the remnant density wave. This is quantified through the imbalance $I = \langle (N_e - N_o)/(N_e + N_o) \rangle$ (N_e and N_o being the even an odd site populations of the system), in analogy to the visibility of an interference pattern in optics. For a weak quasiperiodic detuning potential, the imbalance relaxes rapidly—compatible with a thermalized state of the system. However, above a critical detuning strength, larger than the one in the noninteracting system, the imbalance saturates to a nonvanishing value; see Fig. 8. This is incompatible with thermalization and indicates a localized phase, since a thermal phase occupies even and odd sites with equal probability. Subsequently, the delocalizing effect of coupling many 1D systems subject to identical quasiperiodic potential was experimentally studied by Bordia *et al.* (2016). In a different direction, Lukin *et al.* (2018) recently observed logarithmic spreading of entanglement in small Bose-Hubbard chains subject to a quasiperiodic potential, consistent with the theoretical picture described in Sec. II.D.

However, first experiments in a system of two-dimensional interacting bosons exposed to a *two-dimensional disorder pattern* seem to also indicate the presence of a localized phase that is reached above a critical disorder strength (Choi *et al.*, 2016). Here again the nonequilibrium dynamics of the system was used to probe for a nonthermal evolution, by monitoring the time dynamics of a domain wall in the density of the system (see Fig. 9).
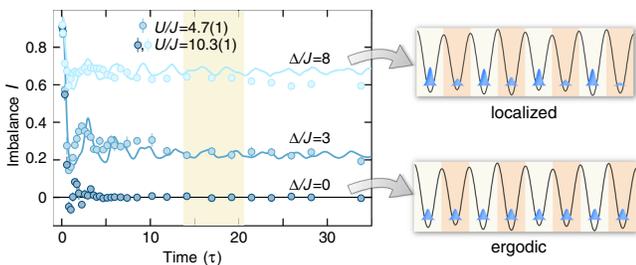

FIG. 8. Nonthermalizing out-of-equilibrium evolution of an initial density wave in the presence of a quasiperiodic detuning potential in the interacting Aubry-André model [see Eq. (17)]. Time traces of the imbalance I for various strengths of the detuning potential Δ. Points are experimental measurements, averaged over six different phases ϕ of the quasiperiodic detuning lattice. Lines denote DMRG simulations that take into account the trapping potential and the averaging over neighboring tubes, which are present in the experiment. From Schreiber *et al.*, 2015.
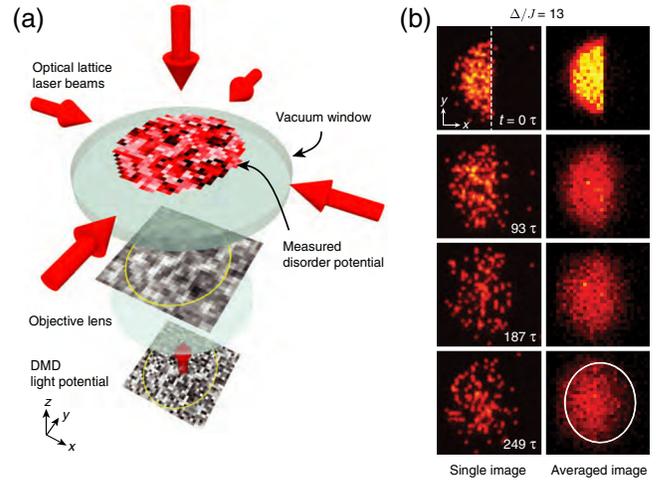

FIG. 9. Probing many-body localization in two dimensions. (a) Almost arbitrary disorder potentials of light are projected onto an ultracold bosonic atom cloud. The subsequent quantum evolution of an initial nonequilibrium state can then be tracked in the experiment. (b) In the experiment an initial domain wall of a bosonic Mott insulator is prepared ("half circle" in images). Even for long evolution times of $\simeq 250$ tunneling times, the system fails to thermalize, indicated by the remnant domain wall still visible in the experiment. In contrast, a thermalized state would not carry any information about the initial state of the system. From Choi *et al.*, 2016.

B. MBL with ultracold ions

Experiments using one-dimensional strings of ten ultracold ions were used to implement the disordered transverse field Ising model with long-range interactions:

$$H_{\text{Ising}} = \sum_{i<j} J_{i,j}\hat{\sigma}_i^x \hat{\sigma}_j^x + \frac{1}{2}\sum_i h_i \sigma_i^z + \frac{1}{2}B\sum_i \sigma_i^z. \quad (18)$$

A specialty of the experiment was the long-range interactions between spins, which decay algebraically with distance $J_{i,j} = J_{\text{max}}/|i-j|^\alpha$ and exhibit a tunable decay exponent $\alpha = 0.85\text{--}1.81$. Random on-site disorder h_i was generated by spin dependent ac Stark shifts of a laser beam and sampled from a uniform distribution $h_i \in [-\Delta, \Delta]$. Starting from an initial Néel state, the system was evolved in time and exhibited a stationary magnetization above a critical disorder strength, evidencing the presence of a localized phase (Smith *et al.*, 2016).

An interesting additional feature of the experiment was the measurement of entanglement in the system through the quantum Fisher information (Braunstein and Caves, 1994). Whereas the nondisordered system showed an initial rapid increase of entanglement, but no subsequent growth, the quantum Fisher information of the interacting system exhibited an increase even for intermediate evolution times. Such an intermediate time increase of the quantum Fisher information is consistent with the logarithmic growth of entanglement entropy due to the dephasing between LIOMs (see Sec. II.D), whereas a simple noninteracting Anderson insulator does not exhibit such a continued growth of entanglement as a function of time.

Theoretically, sufficiently long-range hopping and interactions are known to destroy single-particle (Anderson, 1958; Levitov, 1990) and many-body localization (Burin, 2006, 2015; Yao *et al.*, 2014; Gutman *et al.*, 2016). In the single-particle problem in d dimensions with on-site disorder and power-law hopping $t_{ij} \propto 1/r_{ij}^\alpha$, localization is destroyed for $\alpha \leq d$ due to inevitable resonant transitions, which occur between remote sites in the limit of strong on-site disorder (Anderson, 1958; Levitov, 1990). Burin (2006) considered a many-body problem with on-site disorder and power-law hopping and interactions decaying with the same exponent α and showed that hierarchical resonances are inevitable and prohibit many-body localization provided $\alpha < 2d$. Further, Yao *et al.* (2014), Burin (2015), and Gutman *et al.* (2016) investigated fermionic and spin models in which hopping (or, equivalently, flip-flop processes in spin models) and interactions decay as power laws with different exponents α, β, and identified the regimes in which hierarchical resonances destroy MBL.

Interestingly, Burin (2015) showed that the model of Eq. (18) realized in the trapped ion experiments delocalizes for $\alpha < 3d/2$, which is in an apparent contradiction with the experimental observation of localization in the range $0.85 < \alpha < 1.81$. We note, however, that the hierarchical resonances occur at large length scales, exceeding the size of the realized ion chains. Thus, the fact that delocalization was not observed at $\alpha < 3/2$ in the experiment likely stems from the pronounced finite-size effects. Indeed, numerical simulations of the experiment (Wu and Das Sarma, 2016) confirmed the absence of relaxation in small ion chains.

We note that Nandkishore and Sondhi (2017) recently pointed out an intriguing possibility that MBL can persist at low temperatures in systems where long-range interactions induce charge confinement (e.g., in one-dimensional systems where interactions grow proportional to the distance between particles).

C. MBL with superconducting circuits

Superconducting qubits have emerged as another powerful platform for tunable and isolated quantum many-body systems (Houck, Türeci, and Koch, 2012). Recently localization signatures for two interacting photons on a nine-site large lattice modeled by a disordered Bose-Hubbard model were obtained through a novel many-body spectroscopy technique. This technique has enabled one to retrieve the many-body eigenenergies of the system and thereby obtain information on the level statistics of the underlying Hamiltonian (Roushan *et al.*, 2017). Already at the level of two particles it was found that the level statistics parameter characterizing the average ratio of adjacent level spacings has markedly different properties in the disordered versus the nondisordered case. Whereas the nondisordered system exhibited a distribution compatible with the Wigner-Dyson Gaussian orthogonal ensemble, for increasing disorder a more Poissonian shaped distribution was found; see Fig. 10. This is indeed expected deep in the MBL phase, where the localized nature of eigenstates leads to a vanishing level repulsion between the system in the limit of large system sizes (Pal and Huse, 2010).
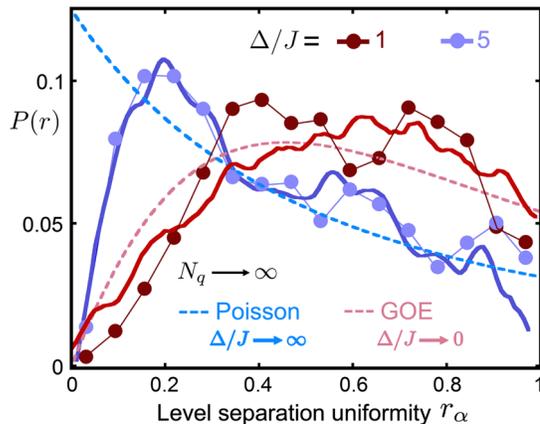

FIG. 10. Level spacing statistics obtained in a system of two interacting photons on nine lattice sites and a local disorder potential using many-body spectroscopy. For weak disorder the system exhibits a distribution resembling the one of a Wigner-Dyson Gaussian orthogonal ensemble, whereas for stronger disorder strengths a change toward the one of a Poissonian distribution is observed. Deviations at very small level spacings r are attributed to the finite size and small number of particles in the system. From Roushan *et al.*, 2017.

In a different setting, an all-to-all coupled system of ten superconducting qubits, characterized by a disordered XY-spin Hamiltonian, was probed (Xu *et al.*, 2018). An initial Néel state, in close connection to the density wave prepared in cold atom experiments, was used to monitor the dynamical evolution of the system for different disorder strengths. The system exhibited evolution into a steady state, with finite staggered magnetization (imbalance) after a few interaction times, indicating a localized phase of spins. Using quantum state tomography, it was also possible to map out the entanglement entropy of a five-qubit subsystem, which exhibited signs of logarithmic entanglement entropy growth (see Sec. II.D).

D. MBL in real materials

In real materials the coupling between phonons and electrons renders the observation of MBL difficult. However, promising signatures of localization were reported in thin films of a : InO, which at high magnetic fields undergo a superconductor to insulator transition. For temperatures below 100 mK and magnetic fields in the range $0.5 < B < 2$ T the system exhibited a dramatic drop in conductivity as the temperature was lowered. The conductivity data were found to be incompatible with a simple exponential activation but could, instead, be fitted with a functional form that indicated a critical temperature below which the conductivity vanishes (Ovadia *et al.*, 2015). Additional results seem to indicate that the origin of this vanishing conductivity is intimately connected to a decoupling between electron and phonon temperatures in the system. Both these observations are compatible with theoretical descriptions of an MBL scenario; however, the microscopic origins of the indirectly measured decoupling of electron and phonon temperatures remain unclear so far. While more research seems needed to clarify these questions, the striking results are the most promising ones so far for the existence of MBL in real materials.

Separate experiments using nuclear magnetic resonance probes of nuclear spin chains showed evidence for growing correlations in an interacting localized system as a function of evolution time (Wei, Ramanathan, and Cappellaro, 2018). The results thereby provided additional support for the entangling evolution in MBL systems (see Sec. II.D) in strong contrast to the absence of such an evolution in a noninteracting Anderson insulator.

Yet in another direction, Silevitch, Aeppli, and Rosenbaum (2017) studied a disordered magnet $LiHo_\alpha Y_{1-\alpha} F_4$ using pump-probe techniques. They observed that low-energy excitations, with energies much smaller than the microscopic spin-spin interaction scale, were very long lived, suggesting a dramatic slowdown of thermalization of those excitations. While the precise nature of these (most likely collective) excitations and the origin of their slow decay remain to be understood, it is evident that pump-probe experiments with disordered magnets provide a promising setting for exploring the breakdown of thermalization and MBL.

E. Residual coupling to the outside world

In an ideal MBL scenario, the system is completely isolated from the outside world preventing thermalization with such an infinite size bath. All experiments are, however, to a varying degree, coupled to the external world. How does this affect the observability of different ergodic and localized phases in the system? Let us imagine that the system exhibits a finite coupling strength γ to the outside world. Even if γ is small, we expect the system to eventually thermalize with the environment for very long-time evolutions. For intermediate time scales, however, long compared to any relaxation times in the system and short compared to the coupling to the outside world, we can nevertheless expect the system to exhibit genuine MBL properties. In general, it is therefore suggestive that the situation is rather similar to the case of a
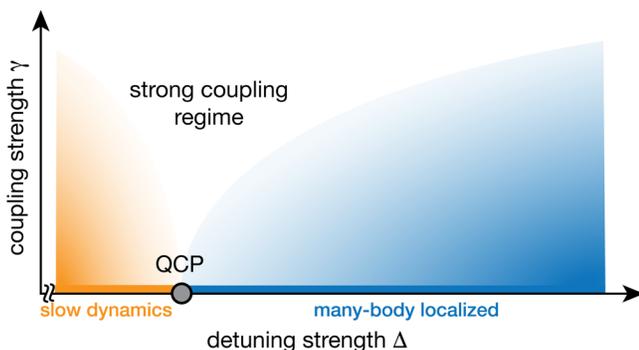

FIG. 11. Schematic phase diagram of an open MBL system: Coupling an MBL system to a bath destroys the signatures of the system on a time scale that depends on the coupling strength. Signatures might, e.g., be a persistent imbalance in the MBL phase or a power-law decay of the imbalance in the thermalizing phase. The yellow and blue shaded regions indicate the regimes where those quantities are still accessible. In the white regime of strong couplings, the bath dominates the dynamics of the system. The ideal phases and an actual transition exist only at $\gamma = 0$. From Lüschen, 2018.

finite-temperature quantum phase transition, with the temperature being replaced by the coupling rate γ; see Fig. 11.

In contrast to glassy systems, we expect couplings to thermal reservoirs to have a much stronger influence on the MBL phase, which thereby also may act as an experimental signature that can distinguish the two from each other. While a classical glass can remain in a glassy phase even when coupled to a reservoir, the MBL phase is expected to be destroyed (becoming either a thermalizing phase or a glassy phase, in case of a frustrated system at low-energy density). Measuring the susceptibility of any MBL phase to external couplings might therefore be a useful probe to distinguish between the two cases (Lüschen *et al.*, 2017). In a different direction, Lenarčič, Altman, and Rosch (2018) considered a setup where the system is coupled to phonons and simultaneously subject to a white noise drive, showing that measuring local temperature fluctuations in such a setting provides a way to distinguish MBL from a thermal and potentially a glassy phase.

VI. OUTLOOK

Finally, we close this Colloquium with a discussion of experimental and theoretical challenges related to MBL and to ergodicity breaking in a broader context.

While there has been significant progress in our ability to describe many aspects of the MBL phase theoretically, many important questions remain open. The transition region remains especially challenging for both theory and experiment. For experiment it is a challenging regime mainly due to the very long time, and possibly also long length scales, on which the critical behavior develops. To capture these long times the experiments must become even better isolated from the environment. It would also be interesting to design experiments to measure the evolution of the entanglement entropy (Islam *et al.*, 2015), or to monitor the evolution of fluctuations in the system as an alternative measure of many-body entanglement (Serbyn, Papić, and Abanin, 2014).

Another question concerns the existence of MBL in higher dimensions $d > 1$. It has been argued that in this case small thermal inclusions can trigger avalanches in the system that destroy the localized phase (De Roeck and Huveneers, 2017; Luitz, Huveneers, and De Roeck, 2017; Ponte *et al.*, 2017). The essence of the argument is that a small inclusion thermalizes its immediate neighborhood, thereby becoming a larger bath. If the number of spatial dimensions is larger than 1, the incipient bath continues to grow and ultimately thermalizes the entire system. These arguments rest on the applicability of random matrix theory at every stage of the avalanche, which is a subject of debate. Moreover, it is not clear how to reconcile this argument with the approach of Basko, Aleiner, and Altshuler (2006) which predicts the existence of an MBL phase in arbitrary spatial dimension. Experiments can help to shed light on this fundamental question by using structured disorder patterns, where the disorder is interrupted by small nondisordered, thermalizing regions whose density and size can be tuned at will; see Fig. 12.

Along similar lines, one can ask how stable MBL is when coupled to an ergodic system of approximately the same size (Li *et al.*, 2015; Nandkishore, 2015; Hyatt *et al.*, 2017). Does
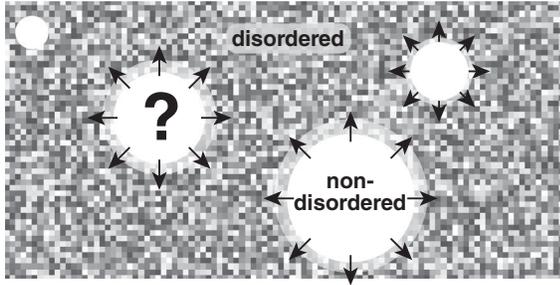
FIG. 12. Engineered disorder pattern. Different techniques allow one to engineer disorder potentials in highly controllable ways. It is, for example, feasible to include nondisordered regions in the disorder pattern, where the system can locally thermalize. The stability of the overall MBL upon different densities and sizes of such thermalizing inclusions could thereby be investigated. This would enable one to probe the stability of MBL in fundamentally new ways.

the disordered system localize the ergodic one (which plays the role of a bath) or does the combined system become fully ergodic? Could both phases even coexist when the system and the bath are strongly coupled to each other? A natural setting in which localized and delocalized states coexist is a noninteracting system with a single-particle mobility edge (Li *et al.*, 2015; Li, Li, and Das Sarma, 2017). Such a system was recently realized in experiment (Lüschen *et al.*, 2018). Very weak interactions are expected to immediately drive such a system into an ergodic phase. In contrast, numerical results at relatively strong interactions show indications of localization in this setting (Li *et al.*, 2015). While it is conceivable that strong interactions may localize the initially delocalized single-particle states above the single-particle mobility edge, more detailed theoretical studies are needed. Experiments can again help to shed light on the competition between localization and thermalization in regimes which are inaccessible for numerics.

In a broader context, an exciting challenge for both theory and experiment is to establish whether MBL provides the only robust mechanism of ergodicity breaking in quantum systems. MBL systems with their description in terms of LIOMs and a simple area-law entanglement structure of eigenstates provide a useful starting point for addressing this question. In particular, is it natural to ask if there could exist a system which exhibits only a partial (rather than a complete as in MBL phase) set of quasilocal integrals of motion with a number of LIOMs scaling as some fraction of the total number of physical spins? A possible example of such partial integrability could be provided by systems with many-body mobility edges. It is highly desirable to extend the theory based on LIOMs to such systems (Geraedts, Bhatt, and Nandkishore, 2017).

Turning to the area-law entanglement structure of MBL eigenstates, one can ask if the breakdown of thermalization can occur for eigenstates which violate area-law entanglement scaling. As discussed in Sec. III.A, disordered systems with continuous non-Abelian symmetries, such as SU(2) symmetry, must have logarithmic scaling of entanglement and thus may exhibit new kinds of ergodicity-breaking phases with only partial integrability.

To make progress in the challenges previously outlined, new theoretical and numerical methods are needed. In particular, tensor-network-based methods for studying the dynamics and eigenstates have to be significantly improved to allow studies of large quantum systems. On the analytical side, extending real-space RG methods to include multispin processes that are typically neglected appears to be a promising direction.

To conclude, theoretical and experimental advances in many-body localization have revealed a new universality class of quantum dynamics, opening up a new frontier in nonequilibrium physics. As discussed, much of the theory progress was inspired by applying concepts from quantum information theory to nonequilibrium phenomena. On the experimental side, studies of MBL in quantum systems were made possible by the remarkable technological progress in realizing synthetic quantum systems. Looking forward, a close collaboration between experiment and different branches of theory will almost certainly lead to the discovery of new nonequilibrium states with unexpected and potentially useful properties.

REFERENCES

Abanin, Dmitry A., Wojciech De Roeck, and François Huveneers, 2016, "Theory of many-body localization in periodically driven systems," Ann. Phys. (Amsterdam) **372**, 1–11.

Abou-Chacra, R., D. J. Thouless, and P. W. Anderson, 1973, "A selfconsistent theory of localization," J. Phys. C **6**, 1734.

Abrahams, E., 2010, Ed., *50 Years of Anderson Localization* (World Scientific, Singapore).

Abrahams, E., P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, 1979, "Scaling theory of localization: Absence of quantum diffusion in two dimensions," Phys. Rev. Lett. **42**, 673–676.

Agarwal, Kartiek, Sarang Gopalakrishnan, Michael Knap, Markus Müller, and Eugene Demler, 2015, "Anomalous diffusion and

Griffiths effects near the many-body localization transition," Phys. Rev. Lett. **114**, 160401.

Aleiner, I. L., B. L. Altshuler, and G. V. Shlyapnikov, 2010, "A finite-temperature phase transition for disordered weakly interacting bosons in one dimension," Nat. Phys. **6**, 900–904.

Altshuler, Boris L., Yuval Gefen, Alex Kamenev, and Leonid S. Levitov, 1997, "Quasiparticle lifetime in a finite system: A non-perturbative approach," Phys. Rev. Lett. **78**, 2803–2806.

Anderson, P. W., 1958, "Absence of diffusion in certain random lattices," Phys. Rev. **109**, 1492–1505.

Bahri, Yasaman, Ronen Vosk, Ehud Altman, and Ashvin Vishwanath, 2015, "Localization and topology protected quantum coherence at the edge of hot matter," Nat. Commun. **6**, 7341.

Baldwin, C. L., C. R. Laumann, A. Pal, and A. Scardicchio, 2017, "Clustering of nonergodic eigenstates in quantum spin glasses," Phys. Rev. Lett. **118**, 127201.

Bardarson, Jens H., Frank Pollmann, and Joel E. Moore, 2012, "Unbounded growth of entanglement in models of many-body localization," Phys. Rev. Lett. **109**, 017202.

Bar Lev, Yevgeny, Guy Cohen, and David R. Reichman, 2015, "Absence of diffusion in an interacting system of spinless fermions on a one-dimensional disordered lattice," Phys. Rev. Lett. **114**, 100601.

Basko, D. M., I. L. Aleiner, and B. L. Altshuler, 2006, "Metal–insulator transition in a weakly interacting many-electron system with localized single-particle states," Ann. Phys. (Amsterdam) **321**, 1126–1205.

Basko, D. M., I. L. Aleiner, and B. L. Altshuler, 2007, "Possible experimental manifestations of the many-body localization," Phys. Rev. B **76**, 052203.

Bauer, Bela, and Chetan Nayak, **2013**, "Area laws in a many-body localized state and its implications for topological order," J. Stat. Mech. 09, P09005.

Bera, Soumya, Henning Schomerus, Fabian Heidrich-Meisner, and Jens H. Bardarson, 2015, "Many-body localization characterized from a one-particle perspective," Phys. Rev. Lett. **115**, 046603.

Bertoli, G., V. P. Michal, B. L. Altshuler, and G. V. Shlyapnikov, 2017, "Finite temperature disordered bosons in two dimensions," arXiv:1708.03628.

Binder, K., and A. P. Young, 1986, "Spin glasses: Experimental facts, theoretical concepts, and open questions," Rev. Mod. Phys. **58**, 801–976.

Blatt, R., and C. F. Roos, 2012, "Quantum simulations with trapped ions," Nat. Phys. **8**, 277–284.

Bloch, Immanuel, Jean Dalibard, and Wilhelm Zwerger, 2008, "Many-body physics with ultracold gases," Rev. Mod. Phys. **80**, 885–964.

Boers, Dave J., Benjamin Goedeke, Dennis Hinrichs, and Martin Holthaus, 2007, "Mobility edges in bichromatic optical lattices," Phys. Rev. A **75**, 063404.

Bordia, Pranjal, Henrik Lüschen, Ulrich Schneider, Michael Knap, and Immanuel Bloch, 2017, "Periodically driving a many-body localized quantum system," Nat. Phys. **13**, 460–464.

Bordia, Pranjal, Henrik P. Lüschen, Sean S. Hodgman, Michael Schreiber, Immanuel Bloch, and Ulrich Schneider, 2016, "Coupling identical one-dimensional many-body localized systems," Phys. Rev. Lett. **116**, 140401.

Braunstein, Samuel L., and Carlton M. Caves, 1994, "Statistical distance and the geometry of quantum states," Phys. Rev. Lett. **72**, 3439–3443.

Burin, A. L., 2006, "Energy delocalization in strongly disordered systems induced by the long-range many-body interaction," arXiv:cond-mat/0611387.

Burin, A. L., 2015, "Localization in a random xy model with long-range interactions: Intermediate case between single-particle and many-body problems," Phys. Rev. B **92**, 104428.

Calabrese, Pasquale, and John Cardy, 2009, "Entanglement entropy and conformal field theory," J. Phys. A **42**, 504005.

Chandran, A., C. R. Laumann, and V. Oganesyan, 2015, "Finite size scaling bounds on many-body localized phase transitions," arXiv:1509.04285.

Chandran, Anushya, Vedika Khemani, C. R. Laumann, and S. L. Sondhi, 2014, "Many-body localization and symmetry-protected topological order," Phys. Rev. B **89**, 144201.

Chandran, Anushya, Isaac H. Kim, Guifre Vidal, and Dmitry A. Abanin, 2015, "Constructing local integrals of motion in the many-body localized phase," Phys. Rev. B **91**, 085425.

Chayes, J. T., L. Chayes, Daniel S. Fisher, and T. Spencer, 1986, "Finite-size scaling and correlation lengths for disordered systems," Phys. Rev. Lett. **57**, 2999–3002.

Chen, Xiao, Tianci Zhou, David A. Huse, and Eduardo Fradkin, 2017, Ann. Phys. (Berlin) **529**, 1600332.

Choi, Jae-yoon, Sebastian Hild, Johannes Zeiher, Peter Schauß, Antonio Rubio-Abadal, Tarik Yefsah, Vedika Khemani, David A. Huse, Immanuel Bloch, and Christian Gross, 2016, "Exploring the many-body localization transition in two dimensions," Science **352**, 1547–1552.

Choi, Soonwon, *et al.*, 2017, "Observation of discrete time-crystalline order in a disordered dipolar many-body system," Nature (London) **543**, 221–225.

Cooper, N. R., J. Dalibard, and I. B. Spielman, 2018, "Topological bands for ultracold atoms," arXiv:1803.00249.

D'Alessio, Luca, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol, 2016, "From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics," Adv. Phys. **65**, 239–362.

D'Alessio, Luca, and Marcos Rigol, 2014, "Long-time behavior of isolated periodically driven interacting lattice systems," Phys. Rev. X **4**, 041048.

Dasgupta, Chandan, and Shang-keng Ma, 1980, "Low-temperature properties of the random Heisenberg antiferromagnetic chain," Phys. Rev. B **22**, 1305–1319.

De Roeck, Wojciech, and François Huveneers, 2017, "Stability and instability towards delocalization in many-body localization systems," Phys. Rev. B **95**, 155129.

De Tomasi, Giuseppe, Soumya Bera, Jens H. Bardarson, and Frank Pollmann, 2017, "Quantum mutual information as a probe for many-body localization," Phys. Rev. Lett. **118**, 016804.

Deutsch, J. M., 1991, "Quantum statistical mechanics in a closed system," Phys. Rev. A **43**, 2046–2049.

Doherty, Marcus W., Neil B. Manson, Paul Delaney, Fedor Jelezko, Jörg Wrachtrup, and Lloyd C. L. Hollenberg, 2013, "The nitrogen-vacancy colour centre in diamond," Phys. Rep. **528**, 1–45.

Dumitrescu, Philipp T., Romain Vasseur, and Andrew C. Potter, 2017, "Scaling theory of entanglement at the many-body localization transition," Phys. Rev. Lett. **119**, 110604.

Eisert, J., M. Cramer, and M. B. Plenio, 2010, "Colloquium: Area laws for the entanglement entropy," Rev. Mod. Phys. **82**, 277–306.

Else, Dominic V., Bela Bauer, and Chetan Nayak, 2016, "Floquet time crystals," Phys. Rev. Lett. **117**, 090402.

Else, Dominic V., and Chetan Nayak, 2016, "Classification of topological phases in periodically driven interacting systems," Phys. Rev. B **93**, 201103.

Evers, Ferdinand, and Alexander D. Mirlin, 2008, "Anderson transitions," Rev. Mod. Phys. **80**, 1355–1417.

Fallani, L., J. E. Lye, V. Guarrera, C. Fort, and M. Inguscio, 2007, "Ultracold atoms in a disordered crystal of light: Towards a bose glass," Phys. Rev. Lett. **98**, 130404.

Finkelstein, A. M., 1983, "Influence of coulomb interaction on the properties of disordered metals," Zh. Eksp. Teor. Fiz. **84**, 168–189 [http://www.jetp.ac.ru/cgi-bin/dn/e_057_01_0097.pdf].

Fischer, Mark H., Mykola Maksymenko, and Ehud Altman, 2016, "Dynamics of a many-body-localized system coupled to a bath," Phys. Rev. Lett. **116**, 160401.

Fisher, Matthew P. A., Peter B. Weichman, G. Grinstein, and Daniel S. Fisher, 1989, "Boson localization and the superfluid-insulator transition," Phys. Rev. B **40**, 546–570.

Fleishman, L., and P. W. Anderson, 1980, "Interactions and the Anderson transition," Phys. Rev. B **21**, 2366–2377.

Friedman, A. J., R. Vasseur, A. C. Potter, and S. A. Parameswaran, 2017, "Localization-protected order in spin chains with non-Abelian discrete symmetries," arXiv:1706.00022.

Geraedts, Scott D., R. N. Bhatt, and Rahul Nandkishore, 2017, "Emergent local integrals of motion without a complete set of localized eigenstates," Phys. Rev. B **95**, 064204.

Giamarchi, T., and H. J. Schulz, 1988, "Anderson localization and interactions in one-dimensional metals," Phys. Rev. B **37**, 325–340.

Gopalakrishnan, Sarang, Markus Müller, Vedika Khemani, Michael Knap, Eugene Demler, and David A. Huse, 2015, "Low-frequency conductivity in many-body localized systems," Phys. Rev. B **92**, 104202.

Goremykina, A., R. Vasseur, and M. Serbyn, 2019, "Analytically solvable renormalization group for the many-body localization transition," Phys. Rev. Lett. **122**, 040601.

Gornyi, I. V., A. D. Mirlin, and D. G. Polyakov, 2005, "Interacting electrons in disordered wires: Anderson localization and low-T transport," Phys. Rev. Lett. **95**, 206603.

Griffiths, Robert B., 1969, "Nonanalytic behavior above the critical point in a random Ising ferromagnet," Phys. Rev. Lett. **23**, 17–19.

Gross, D., V. Nesme, H. Vogts, and R. F. Werner, 2012, "Index theory of one dimensional quantum walks and cellular automata," Commun. Math. Phys. **310**, 419–454.

Gutman, D. B., I. V. Protopopov, A. L. Burin, I. V. Gornyi, R. A. Santos, and A. D. Mirlin, 2016, "Energy transport in the Anderson insulator," Phys. Rev. B **93**, 245427.

Harper, Fenner, and Rahul Roy, 2017, "Floquet topological order in interacting systems of bosons and fermions," Phys. Rev. Lett. **118**, 115301.

Harris, A. B., 1974, "Upper bounds for the transition temperatures of generalized Ising models," J. Phys. C **7**, 3082.

Haviland, D. B., Y. Liu, and A. M. Goldman, 1989, "Onset of superconductivity in the two-dimensional limit," Phys. Rev. Lett. **62**, 2180–2183.

Hebard, A. F., and M. A. Paalanen, 1990, "Magnetic-field-tuned superconductor-insulator transition in two-dimensional films," Phys. Rev. Lett. **65**, 927–930.

Ho, Wen Wei, Soonwon Choi, Mikhail D. Lukin, and Dmitry A. Abanin, 2017, "Critical time crystals in dipolar systems," Phys. Rev. Lett. **119**, 010602.

Houck, Andrew A., Hakan E. Türeci, and Jens Koch, 2012, "On-chip quantum simulation with superconducting circuits," Nat. Phys. **8**, 292–299.

Huse, David A., Rahul Nandkishore, and Vadim Oganesyan, 2014, "Phenomenology of fully many-body-localized systems," Phys. Rev. B **90**, 174202.

Huse, David A., Rahul Nandkishore, Vadim Oganesyan, Arijeet Pal, and S. L. Sondhi, 2013, "Localization-protected quantum order," Phys. Rev. B **88**, 014206.

Hyatt, Katharine, James R. Garrison, Andrew C. Potter, and Bela Bauer, 2017, "Many-body localization in the presence of a small bath," Phys. Rev. B **95**, 035132.

Imbrie, John Z., 2016a, "Diagonalization and many-body localization for a disordered quantum spin chain," Phys. Rev. Lett. **117**, 027201.

Imbrie, John Z., 2016b, "On many-body localization for quantum spin chains," J. Stat. Phys. **163**, 998–1048.

Islam, Rajibul, Ruichao Ma, Philipp M. Preiss, M. Eric Tai, Alexander Lukin, Matthew Rispoli, and Markus Greiner, 2015, "Measuring entanglement entropy in a quantum many-body system," Nature (London) **528**, 77.

Iyer, Shankar, Vadim Oganesyan, Gil Refael, and David A. Huse, 2013, "Many-body localization in a quasiperiodic system," Phys. Rev. B **87**, 134202.

Jordan, P., and E. Wigner, 1928, "Über das paulische äquivalenzverbot," Z. Phys. **47**, 631–651.

Kennes, D. M., and C. Karrasch, 2015, "Entanglement scaling of excited states in large one-dimensional many-body localized systems," arXiv:1511.02205.

Khemani, V., F. Pollmann, and S. L. Sondhi, 2015, "Obtaining highly-excited eigenstates of many-body localized Hamiltonians by the density matrix renormalization group," arXiv:1509.00483.

Khemani, Vedika, Achilleas Lazarides, Roderich Moessner, and S. L. Sondhi, 2016, "Phase structure of driven quantum systems," Phys. Rev. Lett. **116**, 250401.

Khemani, Vedika, D. N. Sheng, and David A. Huse, 2017, "Two universality classes for the many-body localization transition," Phys. Rev. Lett. **119**, 075702.

Kim, Hyungwon, and David A. Huse, 2013, "Ballistic spreading of entanglement in a diffusive nonintegrable system," Phys. Rev. Lett. **111**, 127205.

Kitagawa, Takuya, Erez Berg, Mark Rudner, and Eugene Demler, 2010, "Topological characterization of periodically driven quantum systems," Phys. Rev. B **82**, 235114.

Kjäll, Jonas A., Jens H. Bardarson, and Frank Pollmann, 2014, "Many-body localization in a disordered quantum Ising chain," Phys. Rev. Lett. **113**, 107204.

Kondov, S. S., W. R. McGehee, W. Xu, and B. DeMarco, 2015, "Disorder-induced localization in a strongly correlated atomic Hubbard gas," Phys. Rev. Lett. **114**, 083002.

Lazarides, Achilleas, Arnab Das, and Roderich Moessner, 2014, "Equilibrium states of generic quantum systems subject to periodic driving," Phys. Rev. E **90**, 012110.

Lazarides, Achilleas, Arnab Das, and Roderich Moessner, 2015, "Fate of many-body localization under periodic driving," Phys. Rev. Lett. **115**, 030402.

Lenarčič, Zala, Ehud Altman, and Achim Rosch, 2018, "Activating many-body localization in solids by driving with light," arXiv:1806.04772.

Levi, Emanuele, Markus Heyl, Igor Lesanovsky, and Juan P. Garrahan, 2016, "Robustness of many-body localization in the presence of dissipation," Phys. Rev. Lett. **116**, 237203.

Levitov, L. S., 1990, "Delocalization of vibrational modes caused by electric dipole interaction," Phys. Rev. Lett. **64**, 547–550.

Li, Xiao, Xiaopeng Li, and S. Das Sarma, 2017, "Mobility edges in one-dimensional bichromatic incommensurate potentials," Phys. Rev. B **96**, 085119.

Li, Xiaopeng, Sriram Ganeshan, J. H. Pixley, and S. Das Sarma, 2015, "Many-body localization and quantum nonergodicity in a model with a single-particle mobility edge," Phys. Rev. Lett. **115**, 186601.

Lim, S. P., and D. N. Sheng, 2015, "Nature of many-body localization and transitions by density matrix renormaliztion group and exact diagonalization studies," arXiv:1510.08145.

Luitz, David J., François Huveneers, and Wojciech De Roeck, 2017, "How a small quantum bath can thermalize long localized chains," Phys. Rev. Lett. **119**, 150602.

Luitz, David J., Nicolas Laflorencie, and Fabien Alet, 2015, "Many-body localization edge in the random-field Heisenberg chain," Phys. Rev. B **91**, 081103.

Lukin, A., M. Rispoli, R. Schittko, M. E. Tai, A. M. Kaufman, S. Choi, V. Khemani, J. Léonard, and M. Greiner, 2018, "Probing entanglement in a many-body-localized system," arXiv:1805.09819.

Lüschen, H., 2018, Localization of ultracold atoms in quasi-periodic optical lattices, Ph.D. thesis (LMU Munich).

Lüschen, Henrik P., Pranjal Bordia, Sean S. Hodgman, Michael Schreiber, Saubhik Sarkar, Andrew J. Daley, Mark H. Fischer, Ehud Altman, Immanuel Bloch, and Ulrich Schneider, 2017, "Signatures of many-body localization in a controlled open quantum system," Phys. Rev. X **7**, 011034.

Lüschen, Henrik P., Sebastian Scherg, Thomas Kohlert, Michael Schreiber, Pranjal Bordia, Xiao Li, S. Das Sarma, and Immanuel Bloch, 2018, "Single-particle mobility edge in a one-dimensional quasiperiodic optical lattice," Phys. Rev. Lett. **120**, 160404.

McCoy, Barry M., 1969, "Incompleteness of the critical exponent description for ferromagnetic systems containing random impurities," Phys. Rev. Lett. **23**, 383–386.

Medvedyeva, Mariya V., Tomaz ž Prosen, and Marko Žnidarič, 2016, "Influence of dephasing on many-body localization," Phys. Rev. B **93**, 094205.

Meldgin, Carolyn, Ushnish Ray, Philip Russ, David Chen, David M. Ceperley, and Brian DeMarco, 2016, "Probing the bose glass–superfluid transition using quantum quenches of disorder," Nat. Phys. **12**, 646.

Mermin, N. D., and H. Wagner, 1966, "Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic Heisenberg models," Phys. Rev. Lett. **17**, 1133–1136.

Michal, V. P., B. L. Altshuler, and G. V. Shlyapnikov, 2014, "Delocalization of weakly interacting bosons in a 1d quasiperiodic potential," Phys. Rev. Lett. **113**, 045304.

Michal, Vincent P., Igor L. Aleiner, Boris L. Altshuler, and Georgy V. Shlyapnikov, 2016, "Finite-temperature fluid–insulator transition of strongly interacting 1d disordered bosons," Proc. Natl. Acad. Sci. U.S.A. **113**, E4455–E4459.

Mott, N. F., 1968, "Conduction in glasses containing transition metal ions," J. Non-Cryst. Solids **1**, 1–17.

Nandkishore, Rahul, 2015, "Many-body localization proximity effect," Phys. Rev. B **92**, 245141.

Nandkishore, Rahul, Sarang Gopalakrishnan, and David A. Huse, 2014, "Spectral features of a many-body-localized system weakly coupled to a bath," Phys. Rev. B **90**, 064203.

Nandkishore, Rahul, and David A. Huse, 2015, "Many-body localization and thermalization in quantum statistical mechanics," Annu. Rev. Condens. Matter Phys. **6**, 15–38.

Nandkishore, Rahul M., and S. L. Sondhi, 2017, "Many-body localization with long-range interactions," Phys. Rev. X **7**, 041021.

Nathan, F., D. Abanin, E. Berg, N. H. Lindner, and M. S. Rudner, 2017, "Stability of anomalous Floquet insulators," arXiv:1712.02789.

O'Brien, T. E., Dmitry A. Abanin, Guifre Vidal, and Z. Papić, 2016, "Explicit construction of local conserved operators in disordered many-body systems," Phys. Rev. B **94**, 144208.

Oganesyan, Vadim, and David A. Huse, 2007, "Localization of interacting fermions at high temperature," Phys. Rev. B **75**, 155111.

Ovadia, M., D. Kalok, I. Tamir, S. Mitra, B. Sacépé, and D. Shahar, 2015, "Evidence for a finite-temperature insulator," Sci. Rep. **5**, 13503.

Pal, Arijeet, and David A. Huse, 2010, "Many-body localization phase transition," Phys. Rev. B **82**, 174411.

Pekker, David, Bryan K. Clark, Vadim Oganesyan, and Gil Refael, 2017, "Fixed points of wegner-wilson flows and many-body localization," Phys. Rev. Lett. **119**, 075701.

Pekker, David, Gil Refael, Ehud Altman, Eugene Demler, and Vadim Oganesyan, 2014, "Hilbert-glass transition: New universality of temperature-tuned many-body dynamical quantum criticality," Phys. Rev. X **4**, 011052.

Po, Hoi Chun, Lukasz Fidkowski, Takahiro Morimoto, Andrew C. Potter, and Ashvin Vishwanath, 2016, "Chiral Floquet phases of many-body localized bosons," Phys. Rev. X **6**, 041070.

Polchinski, J., 1992, "Effective field theory and the Fermi surface," arXiv:hep-th/9210046.

Polkovnikov, A., 2011, "Microscopic diagonal entropy and its connection to basic thermodynamic relations," Ann. Phys. (Amsterdam) **326**, 486–499.

Pollmann, F., V. Khemani, J. I. Cirac, and S. L. Sondhi, 2015, "Efficient variational diagonalization of fully many-body localized Hamiltonians," arXiv:1506.07179.

Ponte, Pedro, Anushya Chandran, Z. Papić, and Dmitry A. Abanin, 2015, "Periodically driven ergodic and many-body localized quantum systems," Ann. Phys. (Amsterdam) **353**, 196–204.

Ponte, Pedro, C. R. Laumann, David A. Huse, and A. Chandran, 2017, "Thermal inclusions: how one spin can destroy a many-body localized phase," Phil. Trans. R. Soc. A **375**, 20160428.

Ponte, Pedro, Z. Papić, François Huveneers, and Dmitry A. Abanin, 2015, "Many-body localization in periodically driven systems," Phys. Rev. Lett. **114**, 140401.

Potter, Andrew C., Takahiro Morimoto, and Ashvin Vishwanath, 2016, "Classification of interacting topological Floquet phases in one dimension," Phys. Rev. X **6**, 041001.

Potter, Andrew C., and Romain Vasseur, 2016, "Symmetry constraints on many-body localization," Phys. Rev. B **94**, 224206.

Potter, Andrew C., Romain Vasseur, and S. A. Parameswaran, 2015, "Universal properties of many-body delocalization transitions," Phys. Rev. X **5**, 031033.

Prakash, Abhishodh, Sriram Ganeshan, Lukasz Fidkowski, and Tzu-Chieh Wei, 2017, "Eigenstate phases with finite on-site non-Abelian symmetry," Phys. Rev. B **96**, 165136.

Protopopov, Ivan V., Wen Wei Ho, and Dmitry A. Abanin, 2017, "Effect of SU(2) symmetry on many-body localization and thermalization," Phys. Rev. B **96**, 041122.

Rademaker, Louk, and Miguel Ortuño, 2016, "Explicit local integrals of motion for the many-body localized state," Phys. Rev. Lett. **116**, 010404.

Ros, V., M. Müller, and A. Scardicchio, 2015, "Integrals of motion in the many-body localized phase," Nucl. Phys. B **891**, 420–465.

Roushan, P., *et al.*, 2017, "Spectroscopic signatures of localization with interacting photons in superconducting qubits," Science **358**, 1175–1179.

Rudner, Mark S., Netanel H. Lindner, Erez Berg, and Michael Levin, 2013, "Anomalous edge states and the bulk-edge correspondence for periodically driven two-dimensional systems," Phys. Rev. X **3**, 031005.

Schirhagl, Romana, Kevin Chang, Michael Loretz, and Christian L. Degen, 2014, "Nitrogen-vacancy centers in diamond: Nanoscale

sensors for physics and biology," Annu. Rev. Phys. Chem. **65**, 83–105.

Schollwöck, U., 2005, "The density-matrix renormalization group," Rev. Mod. Phys. **77**, 259–315.

Schreiber, Michael, Sean S. Hodgman, Pranjal Bordia, Henrik P. Lüschen, Mark H. Fischer, Ronen Vosk, Ehud Altman, Ulrich Schneider, and Immanuel Bloch, 2015, "Observation of many-body localization of interacting fermions in a quasirandom optical lattice," Science **349**, 842–845.

Serbyn, M., M. Knap, S. Gopalakrishnan, Z. Papić, N. Y. Yao, C. R. Laumann, D. A. Abanin, M. D. Lukin, and E. A. Demler, 2014, "Interferometric probes of many-body localization," Phys. Rev. Lett. **113**, 147204.

Serbyn, Maksym, and Dmitry A. Abanin, 2017, "Loschmidt echo in many-body localized phases," Phys. Rev. B **96**, 014202.

Serbyn, Maksym, Alexios A. Michailidis, Dmitry A. Abanin, and Z. Papić, 2016, "Power-law entanglement spectrum in many-body localized phases," Phys. Rev. Lett. **117**, 160601.

Serbyn, Maksym, Z. Papić, and D. A. Abanin, 2014, "Quantum quenches in the many-body localized phase," Phys. Rev. B **90**, 174302.

Serbyn, Maksym, Z. Papić, and Dmitry A. Abanin, 2013a, "Local conservation laws and the structure of the many-body localized states," Phys. Rev. Lett. **111**, 127201.

Serbyn, Maksym, Z. Papić, and Dmitry A. Abanin, 2013b, "Universal slow growth of entanglement in interacting strongly disordered systems," Phys. Rev. Lett. **110**, 260601.

Serbyn, Maksym, Z. Papić, and Dmitry A. Abanin, 2015, "Criterion for many-body localization-delocalization phase transition," Phys. Rev. X **5**, 041047.

Setiawan, F., Dong-Ling Deng, and J. H. Pixley, 2017, "Transport properties across the many-body localization transition in quasi-periodic and random systems," Phys. Rev. B **96**, 104205.

Shankar, R., 1994, "Renormalization-group approach to interacting fermions," Rev. Mod. Phys. **66**, 129–192.

Sherrington, David, and Scott Kirkpatrick, 1975, "Solvable model of a spin-glass," Phys. Rev. Lett. **35**, 1792–1796.

Silevitch, D. M., G. Aeppli, and T. F. Rosenbaum, 2017, "Probing many-body localization in a disordered quantum magnet," arXiv:1707.04952.

Slagle, Kevin, Yi-Zhuang You, and Cenke Xu, 2016, "Disordered XYZ spin chain simulations using the spectrum bifurcation renormalization group," Phys. Rev. B **94**, 014205.

Smith, J., A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, 2016, "Many-body localization in a quantum simulator with programmable random disorder," Nat. Phys. **12**, 907–911.

Srednicki, Mark, 1994, "Chaos and quantum thermalization," Phys. Rev. E **50**, 888–901.

Srednicki, Mark, 1999, "The approach to thermal equilibrium in quantized chaotic systems," J. Phys. A **32**, 1163.

Sutherland, B., 2004, *Beautiful Models: 70 Years of Exactly Solved Quantum Many-body Problems* (World Scientific, Singapore).

Thiery, T., M. Müller, and W. De Roeck, 2017, "A microscopically motivated renormalization scheme for the MBL/ETH transition," arXiv:1711.09880.

Thomson, S. J., and M. Schiró, 2018, "Time evolution of many-body localized systems with the flow equation approach," Phys. Rev. B **97**, 060201.

Vasseur, R., S. A. Parameswaran, and J. E. Moore, 2015, "Quantum revivals and many-body localization," Phys. Rev. B **91**, 140202.

Vasseur, R., A. C. Potter, and S. A. Parameswaran, 2015, "Quantum criticality of hot random spin chains," Phys. Rev. Lett. **114**, 217201.

Vasseur, Romain, Aaron J. Friedman, S. A. Parameswaran, and Andrew C. Potter, 2016, "Particle-hole symmetry, many-body localization, and topological edge modes," Phys. Rev. B **93**, 134207.

Verstraete, F., V. Murg, and J. I. Cirac, 2008, "Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems," Adv. Phys. **57**, 143–224.

Vogtmann, K., A. Weinstein, and V. I. Arnol'd, 2013, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics (Springer, New York).

von Keyserlingk, C. W., and S. L. Sondhi, 2016a, "Phase structure of one-dimensional interacting Floquet systems. I. Abelian symmetry-protected topological phases," Phys. Rev. B **93**, 245145.

von Keyserlingk, C. W., and S. L. Sondhi, 2016b, "Phase structure of one-dimensional interacting Floquet systems. II. Symmetry-broken phases," Phys. Rev. B **93**, 245146.

Vosk, Ronen, and Ehud Altman, 2013, "Many-body localization in one dimension as a dynamical renormalization group fixed point," Phys. Rev. Lett. **110**, 067204.

Vosk, Ronen, David A. Huse, and Ehud Altman, 2015, "Theory of the many-body localization transition in one-dimensional systems," Phys. Rev. X **5**, 031032.

Wahl, T. B., A. Pal, and S. H. Simon, 2016, "Efficient representation of fully many-body localized systems using tensor networks," arXiv:1609.01552.

Wahl, T. B., A. Pal, and S. H. Simon, 2017, "Signatures of the Many-body Localized Regime in Two Dimensions," arXiv:1711.02678.

Wei, Ken Xuan, Chandrasekhar Ramanathan, and Paola Cappellaro, 2018, "Exploring localization in nuclear spin chains," Phys. Rev. Lett. **120**, 070501.

Wigner, Eugene P., 1951, "On the statistical distribution of the widths and spacings of nuclear resonance levels," Math. Proc. Cambridge Philos. Soc. **47**, 790–798.

Wu, Yang-Le, and S. Das Sarma, 2016, "Understanding analog quantum simulation dynamics in coupled ion-trap qubits," Phys. Rev. A **93**, 022332.

Xu, Kai, *et al.*, 2018, "Emulating many-body localization with a superconducting quantum processor," Phys. Rev. Lett. **120**, 050507.

Yao, N. Y., C. R. Laumann, S. Gopalakrishnan, M. Knap, M. Müller, E. A. Demler, and M. D. Lukin, 2014, "Many-body localization in dipolar systems," Phys. Rev. Lett. **113**, 243002.

Yu, Rong, *et al.*, 2012, "Bose glass and Mott glass of quasiparticles in a doped quantum magnet," Nature (London) **489**, 379–384.

Yu, X., D. Pekker, and B. K. Clark, 2015, "Finding matrix product state representations of highly-excited eigenstates of many-body localized Hamiltonians," arXiv:1509.01244.

Zhang, J., *et al.*, 2017, "Observation of a discrete time crystal," Nature (London) **543**, 217–220.

Zhang, Liangsheng, Bo Zhao, Trithep Devakul, and David A. Huse, 2016, "Many-body localization phase transition: A simplified strong-randomness approximate renormalization group," Phys. Rev. B **93**, 224201.

Znidarič, M., T. Prosen, and P. Prelovsek, 2008, "Many-body localization in the Heisenberg XXZ magnet in a random field," Phys. Rev. B **77**, 064426.

Znidarič, Marko, Antonello Scardicchio, and Vipin Kerala Varma, 2016, "Diffusive and subdiffusive spin transport in the ergodic phase of a many-body localizable system," Phys. Rev. Lett. **117**, 040601.